Maximal Operators in $\mathbb{R}^2$

by

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A thesis
presented to the University of Waterloo
in fulfilment of the
thesis requirement for the degree of
Master of Mathematics
in
Pure Mathematics

Waterloo, Ontario, Canada, 2007

Author's declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

A maximal operator over the bases $\mathcal{B}$ is defined as

$$Mf(x) = \sup_{x \in B \in \mathcal{B}} \frac{1}{|B|} \int_B |f(y)| dy.$$

The boundedness of this operator can be used in a number of applications including the Lebesgue differentiation theorem. If the bases are balls or rectangles parallel to the coordinate axes, the associated maximal operator is bounded from L^p to L^p for all $p > 1$. On the other hand, Besicovitch showed that it is not bounded if the bases consists of arbitrary rectangles. In $\mathbb{R}^2$ we associate a subset Ω of the unit circle to the bases of rectangles in direction $\theta \in \Omega$. We examine the boundedness of the associated maximal operator M_Ω when Ω is lacunary, a finite sum of lacunary sets, or finite sets using the Fourier transform and geometric methods. The results are due to Nagel, Stein, Wainger, Alfonseca, Soria, Vargas, Karagulyan and Lacey.

Acknowledgments

I would like to thank my supervisor Professor Kathryn Hare for suggesting this topic and helping me write this thesis. I also thank my readers Professor Nicolaas Spronk and Professor Yu-Ru Liu for their helpful comments.

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Chapter 1

Introduction

1.1 Maximal operators

Hardy and Littlewood introduced the one-dimensional maximal operator in 1930:

$$Mf(x) = \sup_h \frac{1}{2h} \int_{x-h}^{x+h} |f(x)| dx.$$

They showed that M is strong type (p,p) for $p > 1$ and Mf is in L^1 if $f \log_+ f$ is integrable ($\log_+ t = \max\{0, \log t\}$) [14]. The maximal operator has a number of applications, including the Lebesgue differentiation theorem and it has been generalized to higher dimensions where the integration is performed over various types of bases.

More precisely, we define the maximal operator over bases $\mathcal{B}$ as:

Definition 1.1.1. *For a family $\mathcal{B}$ of subsets of $\mathbb{R}^n$, we define a* maximal operator *$M_{\mathcal{B}}$ by:*

$$M_{\mathcal{B}} f(x) = \sup_{x \in S \in \mathcal{B}} \frac{1}{|S|} \int_S |f(y)| dy.$$

We can take $\mathcal{B}$ to be:

- balls
- rectangles with sides parallel to coordinate axes
- rectangles with fixed eccentricity
- rectangles in directions $\Omega \subset \mathbb{S}^{n-1}$

or any arbitrary figures but in all cases we will assume that $\mathcal{B}$ is translation invariant. Note that we use the term *rectangle* to mean right angled parallelepipeds in n-dimensions. Also, a rectangle is in direction $\theta \in \Omega$ if its longest side is in direction θ.

When the bases are balls, we call the associated maximal operator the *classical Hardy-Littlewood maximal operator*. When the bases are rectangles with sides parallel to coordinate axes, we call it the *strong maximal operator*. Another famous maximal operator, and an example where the rectangles have fixed eccentricity, is the Kakeya maximal operator K_N with

$$\mathcal{B} = \{\text{rectangles of dimension } a \times \cdots \times a \times aN, a > 0\}.$$

As we will show later, it is known that the classical Hardy-Littlewood maximal operator and the strong maximal operator are bounded from L^p to L^p for all $p > 1$. On the other hand, Besicovitch showed that it is not bounded for any p if the bases consists of arbitrary rectangles. Hence, it is interesting to consider what type of restriction is necessary and sufficient for the associated maximal operator to be bounded. In $\mathbb{R}^2$, we associate a subset Ω of the unit circle to the bases of rectangles in direction $\theta \in \Omega$. We will discuss results of many papers that deal with the boundedness of M_Ω according to the "size" of Ω.

1.2 History of maximal operators in directions

Here we give some historical background on maximal operator over rectangles, mainly taken from a paper by Karagulyan and Lacey [17]. In 1977, Cordoba [6] considered the Kakeya maximal operator in $\mathbb{R}^2$ while studying the Bochner-Riesz conjecture - one of the central problems in harmonic analysis [22]. He obtained a slow increase in the norm on L^2 using a geometric method to prove a covering lemma, as described in Cordoba and R. Fefferman [8].

Equivalent to the Kakeya maximal operator K_N is the directional maximal operator in N uniformly distributed directions. The sharpest bound for this case, which grows logarithmically in N, was proved by Stromberg in 1978 [26]. An estimate for N arbitrary directions was first obtained by Barrionuevo [4, 5]. The sharpest bound without the uniformity condition, which also turns out to be logarithmic in N, was obtained by Katz in 1999, using a duality argument [19].

The set of lacunary directions was also investigated by Stromberg [27] and in 1978, Nagel, Stein and Wainger [23] established the boundedness on L^p for all $p > 1$ using the Fourier transform method. These results are related to interesting results on multipliers as shown by Cordoba and R. Fefferman [7]. In 1981, Sjögren and Sjölin [24] showed that a finite sum of lacunary sets still gives a bounded maximal operator. The proof of this is included as a corollary to Alfonseca, Soria and Vargas' result in Section 4.3.

In 1995, Vargas [28] considered maximal operators associated with truncated Cantor set of directions. The unboundedness of the maximal operator in Cantor set of directions on L^2 was shown by Katz [18] in 1997. This result is extended by Hare for more general Cantor type sets [15]. However, for L^p with $p > 2$, the problem remains unsolved. More sets that give unbounded maximal operators include the countable set $\{1/n\}$ and any dense subset of a set of positive measure [9, 12].

It is a challenging problem to identify the necessary and sufficient conditions on a set of directions Ω which make the associated maximal operator M_Ω bounded. When the domain is restricted to radial functions in $L^p(\mathbb{R}^n)$, Duoandikoetxea and Vargas showed that M_Ω is bounded if $p-1$ is greater than the box dimension of Ω and is unbounded if $p-1$ is less than the box dimension of Ω [11]. However, when the domain is all of L^p, Hare and Rönning give examples of Cantor sets with box dimension zero for which the associated maximal operators are not bounded [16]. It remains open whether the boundedness of M_Ω implies that Ω has dimension zero and whether it implies that Ω is countable [16]. We also do not have an example of Ω whose associated maximal operator is bounded on L^p for some p and unbounded for another p [10].

In this paper, we will focus on the boundedness results on maximal operators in $\mathbb{R}^2$. The sets of directions we consider will include lacunary sets, sums of lacunary sets and finite sets. The two basic strategies are the Fourier transform method and the geometric method. For the Fourier transform method, we start with the classical paper by Nagel, Stein and Wainger [23] and show a result by Alfonseca [1] using a similar technique. We also discuss a different Fourier analytic approach by Karagulyan and Lacey [17]. The geometric methods discussed in this paper are due to Alfonseca, Soria and Vargas [2, 3].

Chapter 2

Preliminaries

This section is based on the text *Classical and Modern Fourier Analysis* by Grafakos [13]. The Riesz-Thorin interpolation follows the treatment in *An Introduction to Harmonic Analysis* by Katznelson [21].

2.1 L^p and weak L^p spaces

Definition 2.1.1. *Let (X, μ) be a measure space. For a measurable function f, we define the* distribution function d_f *on $[0, \infty)$ as follows:*

$$d_f(\alpha) = \mu(\{x \in X : |f(x)| > \alpha\})$$

Note that

$$d_{f+g}(\alpha + \beta) \leq d_f(\alpha) + d_g(\beta).$$

We also have the following useful formula:

Proposition 2.1.2. *For $f \in L^p(X, \mu)$ and $1 \leq p < \infty$, we have*

$$\|f\|_{L^p}^p = p \int_0^\infty \alpha^{p-1} d_f(\alpha) d\alpha.$$

Proof. This is a straightforward calculation:

$$\begin{aligned}
p\int_0^\infty \alpha^{p-1}d_f(\alpha)d\alpha &= p\int_0^\infty \alpha^{p-1}\int_X \chi_{\{x:|f(x)|>\alpha\}}d\mu(x)d\alpha \\
&= \int_X\int_0^{|f(x)|} p\alpha^{p-1}d\alpha d\mu(x) \\
&= \int_X |f(x)|^p d\mu(x) \\
&= \|f\|_{L^p}^p.
\end{aligned}$$

□

Definition 2.1.3. *For $1 \leq p < \infty$, we define the space* weak-$L^p(X,\mu)$ *as the set of all μ-measurable functions f such that*

$$\|f\|_{L^{p,\infty}} = \inf\{C > 0 : d_f(\alpha) \leq \frac{C^p}{\alpha^p} \text{ for all } \alpha > 0\}$$

is finite. By definition, the space weak-$L^\infty(X,\mu)$ is $L^\infty(X,\mu)$.

The weak-L^p space is a quasi-normed linear space and is complete with respect to $\|\cdot\|_{L^{p,\infty}}$ (refer to Section 1.1 of [13]). We also observe that

$$\alpha^p d_f(\alpha) \leq \int_{\{x:|f(x)|>\alpha\}} |f(x)|^p d\mu(x) \leq \|f\|_{L^p}^p.$$

Hence $\|f\|_{L^{p,\infty}} \leq \|f\|_{L^p}$ and $L^p(X) \subset L^{p,\infty}(X)$. By considering $h(x) = |x|^{-n/p}$ in $\mathbb{R}^n$, we see that the inclusion is strict .

Definition 2.1.4. *We say that an operator is* strong type (p,p) *(resp.* weak type (p,p)*) if it maps L^p to L^p (resp. L^p to $L^{p,\infty}$).*

2.2 Convolution

Recall that a locally compact group G has a left invariant Haar measure λ on G. Let f, g be in $L^1(G) = L^1(G,\lambda)$.

Definition 2.2.1. *The* convolution $f * g$ *is defined by*

$$(f * g)(x) = \int_G f(y)g(y^{-1}x)d\lambda(y).$$

A useful inequality involving convolutions is the Minkowski's inequality.

Theorem 2.2.2. *(Minkowski's inequality) Let $1 \leq p \leq \infty$. For f in $L^p(G)$ and g in $L^1(G)$ we have*

$$\|g * f\|_{L^p(G)} \leq \|g\|_{L^1(G)} \|f\|_{L^p(G)}.$$

Proof. The cases $p = 1$ and $p = \infty$ are trivial. For $1 < p < \infty$,

$$|(g*f)(x)| \leq \int_G |f(y^{-1}x)||g(y)|d\lambda(y) \leq \left(\int_G |f(y^{-1}x)|^p |g(y)|d\lambda(y) \right)^{1/p} \left(\int_G |g(y)|d\lambda(y) \right)^{1/p'}$$

by Holder's inequality. Taking the L^p norms of both sides,

$$\begin{aligned}
\|g * f\|_{L^p}^p &\leq \|g\|_{L^1}^{p-1} \int_G \int_G |f(y^{-1}x)|^p |g(y)|d\lambda(y) d\lambda(x) \\
&= \|g\|_{L^1}^{p-1} \int_G \int_G |f(y^{-1}x)|^p d\lambda(x) |g(y)| d\lambda(y) \\
&= \|g\|_{L^1}^{p-1} \int_G \int_G |f(x)|^p d\lambda(x) |g(y)| d\lambda(y) \\
&= \|f\|_{L^p}^p \|g\|_{L^1} \|g\|_{L^1}^{p-1} = \|f\|_{L^p}^p \|g\|_{L^1}^p
\end{aligned}$$

where we use the Fubini's theorem for the first equality and the translation invariance of the Haar measure for the second. □

2.3 Interpolation

In this section, we prove two interpolation theorems - the Marcinkiewicz interpolation theorem and the Riesz-Thorin interpolation theorem.

2.3.1 Marcinkiewicz interpolation theorem

Theorem 2.3.1. *(Marcinkiewicz Interpolation theorem) Let (X, μ) and (Y, ν) be two measure spaces and let $1 \leq p_0 < p_1 \leq \infty$. Let T be a sublinear operator (i.e. $|T(f + g)| \leq |Tf| + |Tg|$) defined on the space $L^{p_0}(X) + L^{p_1}(X)$ and taking values in the space of measurable functions on Y. Assume that there exist two positive constants A_0 and A_1 such that*

$$\|T(f)\|_{L^{p_0,\infty}(Y)} \leq A_0 \|f\|_{L^{p_0}(X)} \qquad \text{for all } f \in L^{p_0}(X), \tag{2.3.1}$$

$$\|T(f)\|_{L^{p_1,\infty}(Y)} \leq A_1 \|f\|_{L^{p_1}(X)} \qquad \text{for all } f \in L^{p_1}(X). \tag{2.3.2}$$

Then for all $p_0 < p < p_1$ and for all f in $L^p(X)$ we have

$$\|T(f)\|_{L^p(Y)} \leq A \|f\|_{L^p(X)},$$

where

$$A = 2\left(\frac{p}{p-p_0}+\frac{p}{p_1-p}\right)^{1/p} A_0^{\frac{\frac{1}{p}-\frac{1}{p_1}}{\frac{1}{p_0}-\frac{1}{p_1}}} A_1^{\frac{\frac{1}{p_0}-\frac{1}{p}}{\frac{1}{p_0}-\frac{1}{p_1}}}.$$

Proof. Assume that $p_1 < \infty$ first. Fix a function f in $L^p(X)$ and $\alpha > 0$. We split $f = h_\alpha + l_\alpha$ by $h_\alpha = f \cdot \chi_{\{|f(x)|>\delta\alpha\}}$ for $\delta > 0$ to be determined later. We note that h_α is in L^{p_0} and l_α in L^{p_1}. By sublinearity of T,

$$|T(f)| \le |T(h_\alpha)| + |T(l_\alpha)|,$$

which implies

$$d_{T(f)}(\alpha) \le d_{|T(h_\alpha)|+|T(l_\alpha)|}(\alpha) \le d_{T(h_\alpha)}(\alpha/2) + d_{T(l_\alpha)}(\alpha/2).$$

From (2.3.1) and (2.3.2), we have

$$d_{T(f)}(\alpha) \le \frac{A_0^{p_0}}{(\alpha/2)^{p_0}} \int_{\{|f|>\delta\alpha\}} |f(x)|^{p_0} d\mu(x) + \frac{A_1^{p_1}}{(\alpha/2)^{p_1}} \int_{\{|f|\le\delta\alpha\}} |f(x)|^{p_1} d\mu(x).$$

Using Proposition 2.1.2

$$\begin{aligned}
\|T(f)\|_{L^p}^p &\le p(2A_0)^{p_0} \int_0^\infty \alpha^{p-1}\alpha^{-p_0} \int_{\{|f|>\delta\alpha\}} |f(x)|^{p_0} d\mu(x) d\alpha + \\
&\quad p(2A_1)^{p_1} \int_0^\infty \alpha^{p-1}\alpha^{-p_1} \int_{\{|f|\le\delta\alpha\}} |f(x)|^{p_1} d\mu(x) d\alpha \\
&\le p(2A_0)^{p_0} \int_X |f(x)|^{p_0} \int_0^{|f(x)|/\delta} \alpha^{p-1-p_0} d\alpha d\mu(x) + \\
&\quad p(2A_1)^{p_1} \int_X |f(x)|^{p_1} \int_{|f(x)|/\delta}^\infty \alpha^{p-1-p_1} d\alpha d\mu(x) \\
&= \frac{p(2A_0)^{p_0}}{(p-p_0)\delta^{p-p_0}} \int_X |f(x)|^{p_0} |f(x)|^{p-p_0} d\mu(x) + \\
&\quad \frac{p(2A_1)^{p_1}}{(p_1-p)\delta^{p-p_1}} \int_X |f(x)|^{p_1} |f(x)|^{p-p_1} d\mu(x) \\
&= p\left(\frac{(2A_0)^{p_0}}{p-p_0}\delta^{p_0-p} + \frac{(2A_1)^{p_1}}{p_1-p}\delta^{p_1-p}\right) \|f\|_{L^p}^p,
\end{aligned}$$

and the convergence of the integrals in α is justified from $p_0 < p < p_1$. Pick δ so that

$$(2A_0)^{p_0}\delta^{p_0-p} = (2A_1)^{p_1}\delta^{p_1-p},$$

and that completes the proof.

If $p_1 = \infty$, write $f = h_\alpha + l_\alpha$ as above. Then, $\|T(l_\alpha)\|_{L^\infty} \le A_1\|l_\alpha\|_{L^\infty} \le A_1\delta\alpha = \alpha/2$ provided we choose $\delta = (2A_1)^{-1}$. Hence,

$$d_{T(f)}(\alpha) \le d_{|T(h_\alpha)|}(\alpha/2) + d_{|T(l_\alpha)|}(\alpha/2)$$

where the second term is 0. Similarly to above, we have that

$$d_{T(h_\alpha)}(\alpha/2) \leq \frac{(2A_0)^{p_0}\|h_\alpha\|_{L^{p_0}}^{p_0}}{\alpha^{p_0}} = \frac{(2A_0)^{p_0}}{\alpha^{p_0}} \int_{\{|f|>\delta\alpha\}} |f(x)|^{p_0} d\mu(x).$$

and thus,

$$\begin{aligned}
\|T(f)\|_{L^p}^p &= p\int_0^\infty \alpha^{p-1} d_{T(f)}(\alpha) d\alpha \\
&\leq p\int_0^\infty \alpha^{p-1} d_{T(h_\alpha)}(\alpha/2) d\alpha \\
&\leq p\int_0^\infty \alpha^{p-1}\frac{(2A_0)^{p_0}}{\alpha^{p_0}} \int_{\{|f|>\alpha/(2A_1)\}} |f(x)|^{p_0} d\mu(x) d\alpha \\
&= p(2A_0)^{p_0} \int_X |f(x)|^{p_0} \int_0^{2A_1|f(x)|} \alpha^{p-p_0-1} d\alpha d\mu(x) \\
&= \frac{p(2A_1)^{p-p_0}(2A_0)^{p_0}}{p-p_0} \int_X |f(x)|^p d\mu(x).
\end{aligned}$$

This proves the theorem with the constant

$$A = 2\left(\frac{p}{p-p_0}\right)^{1/p} A_1^{1-\frac{p_0}{p}} A_0^{\frac{p_0}{p}},$$

which is the desired value. $\square$

2.3.2 Riesz-Thorin interpolation theorem

We present another interpolation theorem called the Riesz-Thorin interpolation theorem. We will follow the proof in "An Introduction to Harmonic Analysis" by Y. Katznelson. This theorem is based on complex methods which were developed before real methods. While this theorem can be used for a more general definition of interpolation spaces, it requires the operator to be linear. Before showing the theorem, we need the following definitions.

Let B be a normed linear space and let F be a function from some domain $\Omega \in \mathbb{C}$ to B. We say that F is holomorphic in Ω if, for every continuous linear functional μ on B, $h(z) = \langle F(z), \mu\rangle$ is holomorphic in Ω.

Assume that B is equipped with two norms $\|\cdot\|_0$ and $\|\cdot\|_1$. Let $\mathcal{B}$ denote the family of all B-valued functions which are holomorphic and bounded with respect to both norms, in a neighborhood of the strip $\Omega = \{z : 0 \leq \Re(z) \leq 1\}$. Then $\mathcal{B}$ is a linear space with norm as follows: for $F \in \mathcal{B}$, put

$$\|F\| = \sup_y\{\|F(iy)\|_0, \|F(1+iy)\|_1\}.$$

For $0 < \alpha < 1$, the set $\mathcal{B}_\alpha = \{F \in \mathcal{B} : F(\alpha) = 0\}$ is a linear subspace of $\mathcal{B}$. We say that $\|\cdot\|_0$ and $\|\cdot\|_1$ are *consistent* if $\mathcal{B}_\alpha$ is closed in $\mathcal{B}$ for all $0 < \alpha < 1$.

We interpolate two consistent norms on B as follows: for $0 < \alpha < 1$, the quotient space $\mathcal{B}/\mathcal{B}_\alpha$ is algebraically isomorphic to B via the canonical mapping $F \to F(\alpha)$. Since $\mathcal{B}_\alpha$ is closed, we give $\mathcal{B}/\mathcal{B}_\alpha$ the quotient norm which induces a norm on B denoted by $\|\cdot\|_\alpha$.

Before presenting the interpolation theorem, we provide a convenient criterion for checking the consistency of two norms.

Lemma 2.3.2. *Assume that for every non-zero $f \in B$, there exists a functional μ continuous with respect to both $\|\cdot\|_0$ and $\|\cdot\|_1$, such that $\langle f, \mu\rangle \neq 0$. Then $\|\cdot\|_0$ and $\|\cdot\|_1$ are consistent.*

Proof. Let $0 < \alpha < 1$ and let $F_n \in \mathcal{B}_\alpha, F_n \to F$ in $\mathcal{B}$. Let μ be an arbitrary linear functional continuous with respect to both norms.

The functions $\langle F_n(z), \mu\rangle$ are bounded on the strip Ω and tend to $\langle F(z), \mu\rangle$ uniformly on the lines $z = iy$ and $z = 1 + iy$ by the definition of norm on $\mathcal{B}$. By the theorem of Phragmèn-Lindelöf, the convergence is uniform throughout Ω and, in particular, $\langle F(\alpha), \mu\rangle = \lim_{n\to\infty}\langle F_n(\alpha), \mu\rangle = 0$. But by the assumption, if $F(\alpha) \neq 0$, the previous equality fails for at least one μ. Hence $F(\alpha) = 0$ and $\mathcal{B}_\alpha$ is closed. $\square$

Theorem 2.3.3. *(Interpolation theorem for general interpolation spaces) Let B (resp. B') be a linear space with two consistent norms $\|\cdot\|_0$ and $\|\cdot\|_1$ (resp. $\|\cdot\|'_0$ and $\|\cdot\|'_1$.) Denote the interpolating norms by $\|\cdot\|_\alpha$ (resp. $\|\cdot\|'_\alpha$), $0 < \alpha < 1$. Let S be a linear map from B to B' which is bounded as*

$$S : (B, \|\cdot\|_j) \to (B', \|\cdot\|'_j), \quad j = 0, 1. \tag{2.3.3}$$

Then S is bounded as

$$S : (B, \|\cdot\|_\alpha) \to (B', \|\cdot\|'_\alpha),$$

and its norm $\|S\|_\alpha$ satisfies

$$\|S\|_\alpha \leq \|S\|_0^{1-\alpha}\|S\|_1^\alpha.$$

Proof. We denote by $\mathcal{B}'$ the space of holomorphic B'-valued functions which is used in defining $\|\cdot\|'_\alpha$. The map $S : B \to B'$ can be extended to a map $S : \mathcal{B} \to \mathcal{B}'$ via $SF(z) = S(F(z))$. To show that SF so defined is holomorphic with respect to either norms, we consider an arbitrary functional μ continuous with respect to $\|\cdot\|'_0$ or $\|\cdot\|'_1$ and notice that $\langle SF(z), \mu\rangle = \langle F(z), S^*\mu\rangle$. Also $SF(z)$ is bounded from (2.3.3). Thus $SF \in \mathcal{B}'$.

Let $f \in B, \|f\|_\alpha = 1$. Then there exists an $F \in \mathcal{B}$ such that $F(\alpha) = f$ and such that $\|F\| < 1 + \epsilon$. Since $e^{a(z-\alpha)}F(z)$ with $e^a = \|S\|_0\|S\|_1^{-1}$ is also a holomorphic function, applying S to $e^{a(z-\alpha)}F(z)$

$$\begin{aligned}
\|Sf\|'_\alpha &\leq \|S(e^{a(z-\alpha)}F(z))\|' \\
&= \sup_t\{e^{-a\alpha}\|SF(it)\|'_0, e^{a(1-\alpha)}\|SF(1+it)\|'_1\} \\
&\leq (1+\epsilon)\sup\{e^{-a\alpha}\|S\|_0, e^{a(1-\alpha)}\|S\|_1\} \\
&= (1+\epsilon)\|S\|_0^{1-\alpha}\|S\|_1^\alpha.
\end{aligned}$$

□

Now we export this theorem to L^p settings.

Theorem 2.3.4. *(Riesz-Thorin interpolation theorem for L^p) Let (X,μ) and (Y,ν) be two measure spaces. Let T be a linear operator defined on the set $\operatorname{span}\{\chi_A : A \subset X, \mu(A) < \infty\}$ and taking values in the set of measurable functions on Y. Let $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ and assume that*

$$\begin{aligned}
\|T(f)\|_{L^{q_0}} &\leq M_0\|f\|_{L^{p_0}} \\
\|T(f)\|_{L^{q_1}} &\leq M_1\|f\|_{L^{p_1}}
\end{aligned}$$

for all f simple functions on X. Then for all $0 < \theta < 1$ we have

$$\|T(f)\|_{L^q} \leq M_0^{1-\theta}M_1^\theta\|f\|_{L^p}$$

for all f simple functions on X, where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

By density, T has a unique extension as a bounded operator from $L^p(X,\mu)$ into $L^q(Y,\nu)$ for all p and q as above.

Proof. We show that the interpolation norm on simple functions on X as above with $\alpha = \theta$ coincides with $L^p(X,\mu)$ norm with

$$\frac{1}{p} = \frac{\theta}{p_0} + \frac{1-\theta}{p_1}.$$

The same argument holds for L^q and we can use Theorem 2.3.3 to finish the proof.

Let B denote all the simple functions on X. First we show that L^{p_0} and L^{p_1} are consistent. For $0 \neq f = |f|e^{i\psi}$, we have that $D_n = \{|f| > 1/n\}$ has finite non-zero measure for some n_0. Choose μ so that $\langle f, \mu\rangle = \int f\overline{g}dx$ where $g = e^{i\psi}\chi_{D_{n_0}}$, and then apply Lemma 2.3.2. By Holder's inequality, μ is bounded with respect to both norms.

Let $f \in B$ and $\|f\|_{L^p} \leq 1$. Consider $F(z) = |f|^{a(z-\theta)+1}e^{i\psi}$ where $f = |f|e^{i\psi}$ for a real-valued ψ and

$$a = \frac{p_0 - p_1}{p_0\theta + p_1(1-\theta)} \qquad \big(= -(1-\theta)^{-1} \quad \text{if } p_1 = \infty\big).$$

We have $F(\theta) = f$ and consequently $\|f\|_\theta \leq \|F\|$. Now, $|F(iy)| = |f|^{1-a\theta} = |f|^{p/p_0}$ so that

$$\|F(iy)\|_0 = \left(\int |f|^p dx \right)^{1/p_0} \leq 1$$

and similarly, $\|F(1+iy)\|_1 \leq 1$. Hence, $\|f\|_\theta \leq 1$.

To prove the reverse inequality, observe that the conjugates p_0' and p_1' satisfy

$$\frac{1}{p'} = \frac{\theta}{p_0'} + \frac{1-\theta}{p_1'}.$$

Let $f \in B$ and assume $\|f\|_{L^p} > 1$. Then since B is dense in $L^{p'}$, there exists a $g \in B$ such that $\|g\|_{L^{p'}} < 1$ and $\int fg d\mu > 1$. From the above result, there is a function $G \in \mathcal{B}$ such that $G(\theta) = g$ and $\|G\|$ with respect to p_0' and p_1' is bounded by 1. Let $F \in \mathcal{B}$ such that $F(\theta) = f$. The function $h(z) = \int F(z)G(z)d\mu$ is holomorphic. Using Holder's inequality, we also have that on the boundary,

$$|h(z)| \leq \|F\|\|G\| \leq \|F\|.$$

Now since $h(\theta) > 1$, by the Phragmèn-Lindelöf theorem, $|h(z)|$ must exceed 1 on the boundary implying that $\|F\| > 1$. Thus, $\|f\|_\theta \geq 1$ as required. $\square$

We also need the following version of the theorem. Before presenting the theorem, we extend the definition of the L^p space as follows. We say that an l^r-valued function f is measurable if, for any $u^* \in (l^r)^*$, the complex-valued map $x \mapsto \langle u^*, f(x) \rangle$ is measurable. As a consequence, for each $\alpha > 0$, $\{x : \|f(x)\|_{l^r} > \alpha\}$ is measurable. We define a norm on the measurable functions as $\|f\| = (\int_X \|f(x)\|_{l^r}^p d\mu)^{1/p}$ and denote the space of functions with bounded such norm as $L^p(X, l^r)$.

Theorem 2.3.5. *(vector valued Riesz-Thorin interpolation theorem) Let*

$$1 \leq p_0, q_0, p_1, q_1, r_0, s_0, r_1, s_1 \leq \infty$$

and $0 < \theta < 1$ *satisfy*

$$\frac{1-\theta}{p_0} + \frac{\theta}{p_1} = \frac{1}{p} \qquad \frac{1-\theta}{q_0} + \frac{\theta}{q_1} = \frac{1}{q}$$
$$\frac{1-\theta}{r_0} + \frac{\theta}{r_1} = \frac{1}{r} \qquad \frac{1-\theta}{s_0} + \frac{\theta}{s_1} = \frac{1}{s}.$$

Suppose that $\vec{T}$ is a linear operator that maps $L^{p_0}(\mathbb{R}^n, l^{r_0})$ into $L^{q_0}(\mathbb{R}^n, l^{s_0})$ with norm A_0 and $L^{p_1}(\mathbb{R}^n, l^{r_1})$ into $L^{q_1}(\mathbb{R}^n, l^{s_2})$ with norm A_1. Then $\vec{T}$ maps $L^p(\mathbb{R}^n, l^r)$ into $L^q(\mathbb{R}^n, l^s)$ with norm at most $A_0^{1-\theta} A_1^\theta$.

Proof. Similarly to the previous theorem, we just need to show that $L^p(\mathbb{R}^n, l^r)$ coincides with the interpolation norm between $L^{p_0}(\mathbb{R}^n, l^{r_0})$ and $L^{p_1}(\mathbb{R}^n, l^{r_1})$. We let the underlying space be

$$B = \{(g_j) : g_j \text{ is simple with compact support and } g_j = 0 \text{ except finitely many } j\}$$

and extend the result to $L^p(\mathbb{R}^n, l^r)$ by density. We check the consistency of the two norms as in the previous theorem. For $0 \neq f \in L^{p_0}(l^{r_0}) \cap L^{p_1}(l^{r_1})$, we have that $D_{n,j} = \{|f_j| > 1/n\}$ has some finite non-zero measure for some j_0, n_0 since $\cup D_{n,j}$ has positive measure. Choose μ so that $\langle f, \mu\rangle = \int \sum_j f_j \overline{g}_j dx$ where

$$g_j = 0 \qquad \text{if } j \neq j_0$$

$$g_{j_0} = e^{i\psi_{j_0}} \chi_{D_{n_0,j_0}}$$

where $f_j = |f_j|e^{i\psi_j}$. Apply Lemma 2.3.2. By Holder's inequality,

$$\left|\int \sum_j f_j g_j\right| \leq \int \Big(\sum_j |f_j|^r\Big)^{1/r} \Big(\sum_j |g_j|^{r'}\Big)^{1/r'} \leq \left\|\Big(\sum_j |f_j|^r\Big)^{1/r}\right\|_{L^p} \left\|\Big(\sum_j |g_j|^{r'}\Big)^{1/r'}\right\|_{L^{p'}}$$

and the fact that (g_j) is in $L^p(l^r)$ for all p, r, it is bounded with respect to both norms.

For $\|(f_j)\|_{L^p(l^r)} \leq 1,\ f_j = |f_j|e^{i\psi_j}$, let

$$F(z) = \left(|f_j|^{P(z)} \Big(\sum |f_j|^r\Big)^{\frac{1}{r}Q(z)} e^{i\psi_j}\right)(x)$$

which is a B-valued function where

$$P(z) = \frac{r}{r_0}(1-z) + \frac{r}{r_1}z,$$

$$Q(z) = \left(\frac{p}{p_0} - \frac{r}{r_0}\right)(1-z) + \left(\frac{p}{p_1} - \frac{r}{r_1}\right)(z)$$

are analytic and

$$f_j = |f_j|e^{i\psi_j}.$$

Note that when $z = \theta$, we have $P(z) = 1$ and $Q(z) = 0$ so that $F(z)$ is exactly (f_j). When $z = iy$, we get that

$$\begin{aligned}
\|F(iy)\|_0 &= \left(\int \left|\Big(\sum |f_j|^{P(0)r_0}\Big)^{1/r_0} \Big(\sum |f_j|^r\Big)^{Q(0)/r}\right|^{p_0} dx\right)^{1/p_0} \\
&= \left(\int \left|\Big(\sum |f_j|^r\Big)^{1/r_0 + Q(0)/r}\right|^{p_0} dx\right)^{1/p_0} \\
&= \left(\int \left|\Big(\sum |f_j|^r\Big)\right|^{p/r} dx\right)^{1/p_0} \\
&= \left(\left(\int \left|\Big(\sum |f_j|^r\Big)^{1/r}\right|^p dx\right)^{1/p}\right)^{p/p_0} \\
&= \|(f_j)\|_\alpha^{p/p_0}.
\end{aligned}$$

Similarly, we get the norm $\|(f_j)\|_\alpha^{p/p_1}$ when $z = 1 + iy$.

For the reverse inequality, let $(f_j) \in B$ with $\|(f_j)\|_{L^p(l^r)} > 1$. Since B is dense in $L^{p'}(l^{r'})$, there exists a $(g_j) \in B$ such that $\|(g_j)\|_{L^{p'}(l^{r'})} < 1$ and $\int \sum_j f_j g_j > 1$. Also, there exists a function $G \in \mathcal{B}$ such that $G(\theta) = (g_j)$ and $\|G\|$ with respect to $L^{p_0}(l^{r_0})$ and $L^{p_1}(l^{r_1})$ is bounded by 1. Let $F \in \mathcal{B}$ such that $F(\theta) = (f_j)$. The function $h(z) = \int F(z) \cdot G(z) dx$ is holomorphic and bounded in Ω. Also $h(\theta) > 1$ so by the Phragmèn-Lindelöf theorem, $|h(z)|$ exceeds 1 on the boundary. However on the boundary, $|h(z)| \leq \|F\|\|G\| \leq \|F\|$ so that $\|F\| > 1$. Hence $\|(f_j)\|_\theta \geq 1$ and we are done. $\square$

2.4 Schwartz Class and Fourier transform

Schwartz functions are C^∞ functions each of whose derivatives decay super-polynomially. By using suitable Schwartz functions, we can work with Fourier transforms and inverse Fourier transforms without worrying about the convergence issues.

Given $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we denote the partial derivative (resp. mth partial derivative) with respect to the jth variable by $\partial_j f$ (resp. $\partial_j^m f$.) A multiindex α is an ordered n-tuple of nonnegative integers. For a multiindex $\alpha = (\alpha_1, \ldots, \alpha_n), \partial^\alpha f$ denotes the derivative $\partial_1^{\alpha_1} \ldots \partial_n^{\alpha_n} f$. Also, $|\alpha|$ denotes its size $\alpha_1 + \cdots + \alpha_n$.

We have the Leibniz rule

$$\partial^\alpha (fg) = \sum_\beta \binom{\alpha_1}{\beta_1} \ldots \binom{\alpha_n}{\beta_n} (\partial^\beta f)(\partial^{\alpha-\beta} g). \tag{2.4.1}$$

Definition 2.4.1. *A C^∞ complex-valued function f on $\mathbb{R}^n$ is called a* Schwartz function *if for all multiindices α and β there exist positive constants $C_{\alpha,\beta}$ such that*

$$\rho_{\alpha,\beta}(f) = \sup_{x \in \mathbb{R}^n} \left| x^\alpha \partial^\beta f(x) \right| = C_{\alpha,\beta} < \infty.$$

The quantities $\rho_{\alpha,\beta}(f)$ are called the *Schwartz seminorms* of f. The set of all Schwartz functions on $\mathbb{R}^n$ is denoted by $\mathcal{S}(\mathbb{R}^n)$. Note that, alternatively, f is a Schwartz function if and only if

$$\sup_{x \in \mathbb{R}^n} \left| \partial^\alpha (x^\beta f(x)) \right| < \infty$$

for all multiindices α, β. Another characterization that will be useful in subsequent arguments is that f is Schwartz if and only if, for all multiindex β and $N \geq 0$,

$$|\partial^\beta f(x)| \leq C(1 + |x|)^{-N} \tag{2.4.2}$$

for some constant C. It is also easy to see that if f is in $\mathcal{S}$ then for all multiindices α and β, $x^\alpha \partial^\beta f$ is in $\mathcal{S}$ as well.

These seminorms give a metric on $\mathcal{S}(\mathbb{R}^n)$ by

$$d(f,g) = \sum_{j=1}^{\infty} 2^{-j} \frac{\rho_j(f-g)}{1+\rho_j(f-g)}$$

where ρ_j is an enumeration of $\rho_{\alpha,\beta}$. Equipped with the topology induced by this norm, $\mathcal{S}(\mathbb{R}^n)$ is a *Fréchet space* (complete metrizable locally convex space). We have that $\mathcal{S} \subset L^p$ and that convergence in $\mathcal{S}$ is stronger than convergence in all L^p since:

$$\begin{aligned} \|g\|_{L^p}^p = \quad &\leq \int_{|x|\leq 1} \|g\|_{L^\infty}^p dx + \int_{|x|>1} |x|^{n+1} |g(x)|^p |x|^{-(n+1)} dx \\ &\leq |B(0,1)| \|g\|_{L^\infty}^p + \sup_{|x|>1} |x|^{n+1} |g(x)|^p \int_{|x|>1} |x|^{-(n+1)} dx \end{aligned}$$

where each of the two terms are controlled by the Schwartz seminorms.

Proposition 2.4.2. *If f, g are in $\mathcal{S}(\mathbb{R}^n)$ then so are fg and $f * g$. Moreover,*

$$\partial^\alpha (f * g) = (\partial^\alpha f) * g = f * (\partial^\alpha g) \tag{2.4.3}$$

for all multiindices α.

Proof. The fact that fg is C^∞ and all of its derivatives decay super-polynomially follows easily from Leibniz rule (2.4.1).

It is enough to show (2.4.3) for $\alpha = (1, 0, \ldots 0)$ as the remaining cases follow from symmetry and induction. Letting $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^n$, we have

$$\frac{f(y + he_1) - f(y)}{h} - (\partial_1 f)(y) \to 0. \tag{2.4.4}$$

By the mean value theorem, we have that the first term above is bounded by $\sup_{\delta<1} |\partial_1 f(y + \delta e_1)| \leq \|\partial_1 f\|_{L^\infty}$. Hence, the left hand side of (2.4.4) is bounded by a constant, which is integrable with respect to the measure $g(x-y)dy$. By the Lebesgue dominated convergence theorem, (2.4.4) integrated with respect to the measure $g(x-y)dy$ converges to zero as $h \to 0$. This shows (2.4.3) and that $f * g$ is C^∞.

Next we show that $f*g$ decays rapidly. Since $\|f*g\|_{L^\infty}$ is bounded by Minkowski's inequality Theorem 2.2.2, we only need to show $|(f*g)(x)| \leq C_N(1+|x|)^{-N}$ for arbitrarily large N. For each $N > n$ we have

$$|(f*g)(x)| \leq C_N \int_{\mathbb{R}^n} (1+|x-y|)^{-N} (1+|y|)^{-N} dy.$$

By breaking up the above integral over regions where $|y-x| \geq |x|/2$ and where $|y-x| < |x|/2$, we have

$$\begin{aligned} &\int_{|y-x|\geq|x|/2} (1+|x-y|)^{-N} (1+|y|)^{-N} \\ \leq \quad &\int_{|y-x|\geq|x|/2} (1+|x|/2)^{-N} (1+|y|)^{-N} dy \leq C_N' (1+|x|)^{-N} \end{aligned}$$

and

$$\begin{aligned}
&\int_{|y-x|<|x|/2} (1+|x-y|)^{-N}(1+|y|)^{-N} \\
\leq\ &\int_{|y-x|<|x|/2} (1+|x-y|)^{-N}(1+|x|/2)^{-N} dy \\
\leq\ &|B(x,|x|/2)|(1+|x|/2)^{-N} \leq C(1+|x|)^{-(N-n)}
\end{aligned}$$

where we use the fact that $|y-x|<|x|/2$ implies $|y|>|x|/2$.

By (2.4.3), all the derivatives of $f*g$ also decay rapidly. $\square$

Definition 2.4.3. *Given f in $\mathcal{S}(\mathbb{R}^n)$ we define the* Fourier transform *$\widehat{f}$ of f as*

$$\widehat{f}(\xi)=\int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}dx.$$

Lemma 2.4.4. *For $f(x)=e^{-\pi|x|^2}\in\mathcal{S}(\mathbb{R}^n)$, we have that $\widehat{f}(\xi)=f(\xi)$.*

Proof. To see this for $n=1$, observe that

$$s\to\int_{-\infty}^{\infty} e^{-\pi(x+is)^2}dx,\, s\in\mathbb{R}$$

is constant since its derivative is

$$\int_{-\infty}^{\infty} -2\pi i(x+is)e^{-\pi(x+is)^2}dx=\int_{-\infty}^{\infty} i\frac{d}{dx}\big(e^{-\pi(x+is)^2}\big)dx=0,$$

where the exchange of the integral and partial derivative is justified by Lebesgue dominated convergence theorem (see (2.4.4)).

Also when $s=0$, $\int_{-\infty}^{\infty} e^{-\pi x^2}dx=1$ is a well known equality. Then

$$\begin{aligned}
\widehat{f}(\xi) &= \int_{\mathbb{R}} e^{-\pi x^2}e^{-2\pi i x\xi}dx \\
&= \int_{\mathbb{R}} e^{-\pi(x+i\xi)^2}e^{-\pi\xi^2}dx \\
&= e^{-\pi\xi^2}\left(\int_{\mathbb{R}} e^{-\pi(x+i\xi)^2}dx\right)
\end{aligned}$$

where the last term in brackets is constantly 1 from the remark above.

For $n\geq 2$, we observe from the definition of the Fourier transform that if $f\in\mathcal{S}(\mathbb{R}^n)$ and $g\in\mathcal{S}(\mathbb{R}^m)$, then

$$(f(x_1,\dots,x_n)g(x_{n+1},\dots,x_{n+m}))^\wedge=\widehat{f}(\xi_1,\dots\xi_n)\widehat{g}(\xi_{n+1},\dots,\xi_{n+m}).$$

Simply apply 1-dimensional result to the above n times. $\square$

We also introduce the following notation: for $x, y \in \mathbb{R}^n, a > 0$ and $v = (v_1, \dots, v_n) \in \mathbb{R}^n_+$,

$$\begin{aligned} \widetilde{f}(x) &= f(-x) \\ \tau^y(f)(x) &= f(x-y) \\ \delta^a(f)(x) &= f(ax) \\ f_a &= a^{-n}\delta^{1/a}(f) \end{aligned} \tag{2.4.5}$$

and analogously,

$$\begin{aligned} vx &= (v_1x_1, \dots, v_nx_n) \\ \delta^v(f)(x) &= f(vx) \\ f_v &= |v|^{-1}\delta^{v^{-1}}(f) \end{aligned}$$

where $v^{-1} = (v_1^{-1}, \dots, v_n^{-1})$ and $|v| = |v_1 \dots v_n|$.

Then we have the following properties of the Fourier transform:

Proposition 2.4.5. *Given f, g in $\mathcal{S}(\mathbb{R}^n), \alpha$ a multiindex, A an orthogonal matrix which acts on $\xi \in \mathbb{R}^n$ as a column vector, and a, y, v as above, we have*

1. $\|\widehat{f}\|_{L^\infty} \leq \|f\|_{L^1}$
2. $\widehat{\widetilde{f}} = \widetilde{\widehat{f}}$, $\widehat{\overline{f}} = \overline{\widetilde{\widehat{f}}}$ *and* $\widehat{f \circ A}(\xi) = \widehat{f}(A\xi)$
3. $\widehat{\tau^y(f)}(\xi) = e^{-2\pi i y\cdot\xi}\widehat{f}(\xi)$ *and* $(e^{2\pi i x\cdot y}f(x))^\wedge(\xi) = \tau^y(\widehat{f})(\xi)$
4. $(\partial^\alpha f)^\wedge(\xi) = (2\pi i\xi)^\alpha \widehat{f}(\xi)$ *and* $(\partial^\alpha \widehat{f})(\xi) = ((-2\pi i x)^\alpha f(x)(\xi)$
5. $\widehat{\delta^a(f)} = a^{-n}\delta^{1/a}(\widehat{f}) = (\widehat{f})_a$ *and more generally,* $(\delta^v(f))^\wedge = |v|^{-1}\delta^{v^{-1}}(\widehat{f}) = (\widehat{f})_v$
6. $\widehat{f} \in \mathcal{S}$
7. $\widehat{f * g} = \widehat{f}\,\widehat{g}$.

Proof. Proof of (1) - (3), (5) and (7) follows directly from the definition of Fourier transform. For (4), it is enough to show for $\alpha = (1, 0, \dots, 0)$. We have

$$\begin{aligned} (\partial_1 f)^\wedge(\xi) &= \int_{\mathbb{R}^n} (\partial_1 f)(x)e^{-2\pi i x\cdot\xi}dx \\ &= -\int_{\mathbb{R}^n} f(x)(-2\pi i\xi_1)e^{-2\pi i x\cdot\xi}dx \\ &= (2\pi i\xi_1)\widehat{f}(\xi) \end{aligned}$$

where the second equality comes from integration by parts. Also, similarly to (2.4.4)

$$\frac{e^{-2\pi i x\cdot(\xi+he_1)} - e^{-2\pi i x\cdot\xi}}{h} - (-2\pi i x_1)e^{-2\pi i x\cdot\xi} \to 0$$

and is bounded by $C|x|$ which is integrable with respect to $f(x)dx$. Use the Lebesgue dominated convergence theorem to obtain (10). Finally, for (6), we use (4) and (1) to get

$$\begin{aligned} \left\| x^\alpha (\partial^\beta \widehat{f})(x) \right\|_{L^\infty} &= (2\pi)^{|\beta|} \left\| x^\alpha (x^\beta f(x))^\wedge \right\|_{L^\infty} \\ &= \frac{(2\pi)^{|\beta|}}{(2\pi)^{|\alpha|}} \left\| (\partial^\alpha (x^\beta f(x)))^\wedge \right\|_{L^\infty} \\ &\leq \frac{(2\pi)^{|\beta|}}{(2\pi)^{|\alpha|}} \left\| \partial^\alpha (x^\beta f(x)) \right\|_{L^1} < \infty. \end{aligned}$$

□

Definition 2.4.6. *Given a Schwartz function f, we define*

$$f^\vee(x) = \widehat{f}(-x),$$

for all $x \in \mathbb{R}^n$. The operation

$$f \to f^\vee$$

is called the inverse Fourier transform*.*

We note that the Fourier transform and the inverse Fourier transform are indeed inverse operations of each other from the following theorem.

Theorem 2.4.7. *Given f, g, and h in $\mathcal{S}(\mathbb{R}^n)$ we have*

1. $\int_{\mathbb{R}^n} f(x)\widehat{g}(x)dx = \int_{\mathbb{R}^n} \widehat{f}(x)g(x)dx$
2. $(f^\wedge)^\vee = f = (f^\vee)^\wedge$
3. $\|f\|_{L^2} = \|f^\wedge\|_{L^2} = \|f^\vee\|_{L^2}$ *(Plancherel's identity).*

Proof. (1) follows immediately from the definition of the Fourier transform and Fubini's theorem. For (2), use (1) with $g(x) = e^{2\pi i x \cdot t} e^{-\pi|\epsilon x|^2}$ for some $t \in \mathbb{R}^n$. From the properties of Fourier transform and Lemma 2.4.4

$$\widehat{g}(x) = \frac{1}{\epsilon^n} e^{-\pi|(x-t)/\epsilon|^2}$$

which is an approximate identity as $\epsilon \to 0^+$. Thus,

$$\int_{\mathbb{R}^n} f(x)\epsilon^{-n} e^{-\pi\epsilon^{-2}|x-t|^2} dx = \int_{\mathbb{R}^n} \widehat{f}(x) e^{2\pi i x \cdot t} e^{-\pi|\epsilon x|^2} dx.$$

The left hand side converges to $f(t)$ by the property of an approximate identity and the right hand side converges to $(\widehat{f})^\vee(t)$ by the Lebesgue Dominated Convergence theorem. The second equality follows similarly.

We obtain (3) from (1) as:

$$\int f(x)\overline{f}(x)dx = \int \widehat{f}(\xi)\overline{\widehat{f}}(\xi)d\xi.$$

□

Consider the Fourier transform defined only on $\mathcal{S}$. By the density of $\mathcal{S}$ in L^2 and Plancherel's identity, we can extend the Fourier transform $\widehat{g}$ to all $g \in L^2$. We can also extend it to L^1 by Proposition 2.4.5 (1) and note that this extension coincides with the initial definition of Fourier transform defined on L^1. Together, we extend the Fourier transform to all functions in $L^1 + L^2$.

2.5 Marcinkiewicz multiplier theorem

A multiplier m is a mapping $f \mapsto (m\widehat{f})^\vee$. We will need the following theorem which specifies a sufficient condition for a multiplier to be bounded from L^p to L^p.

We use the notation $I_j = (-2^{j+1}, -2^j) \cup (2^j, 2^{j+1})$ and $R_{\mathbf{j}} = I_{j_1} \times \cdots \times I_{j_n}$ whenever $\mathbf{j} = (j_1, \ldots, j_n) \in \mathbb{Z}^n$.

Theorem 2.5.1. *(Marcinkiewicz Multiplier theorem on $\mathbb{R}^n$) Let m be a bounded function on $\mathbb{R}^n$ that is C^n in all regions $R_{\mathbf{j}}$, as defined above, for $\mathbf{j} = (j_1, \ldots, j_n) \in \mathbb{Z}^n$. Assume that there is a positive constant A such that for all $k \in \{1, \ldots, n\}$, all permutations $j_1, \ldots j_n$ of $\{1, \ldots, n\}$, all $l_{j_1}, \ldots l_{j_n} \in \mathbb{Z}$ and all $\xi_s \in I_{l_s}$ for $s \notin \{j_1, \ldots, j_k\}, 1 \leq s \leq n$, we have*

$$\int_{I_{l_{j_1}}} \cdots \int_{I_{l_{j_k}}} |(\partial_{j_1} \ldots \partial_{j_k} m)(\xi_1, \ldots, \xi_n)| d\xi_{j_1} \ldots d\xi_{j_k} \leq A. \tag{2.5.1}$$

Then m is a bounded multiplier from L^p to L^p for $1 < p < \infty$ and there is a constant $C_n < \infty$ such that

$$\|m\|_{\mathcal{M}_p(\mathbb{R}^n)} \leq C_n(A + \|m\|_{L^\infty}) \max\{p, (p-1)^{-1}\}^{6n}. \tag{2.5.2}$$

The proof is based on the Littlewood-Paley theory which we will not present here. We have a simpler criterion for satisfying (2.5.1).

Corollary 2.5.2. *Let m be a bounded function defined away from the coordinate axes on $\mathbb{R}^n$ that is C^n in that region. Assume furthermore that for all $k \in \{1, \ldots, n\}$ all $j_1, \ldots, j_k \in \{1, 2, \ldots, n\}$ and all $\xi_s \in \mathbb{R} \setminus \{0\}$ for each $s \in \{1, \ldots, n\}$ we have*

$$|(\partial_{j_1} \ldots \partial_{j_k} m)(\xi_1, \ldots, \xi_n)| \leq A|\xi_{j_1}|^{-1} \ldots |\xi_{j_k}|^{-1}. \tag{2.5.3}$$

Then m satisfies (2.5.2)

Proof. Observe that condition (2.5.3) implies (2.5.1). □

2.6 Rademacher functions and Khintchine's inequalities

The Rademacher functions are defined on $[0,1]$ as

$$r_j(t) = \operatorname{sgn}(\sin(2^j \pi t))$$

for $j = 0, 1, 2$, and so on. For example, we have

$$r_0(t) = 1;$$

$$r_1(t) = 1 \text{ if } t \in [0, 1/2] \quad \text{and} \quad r_1(t) = -1 \text{ if } t \in (1/2, 1];$$

$$r_2(t) = 1 \text{ if } t \in [0, 1/4] \cup (1/2, 3/4] \quad \text{and} \quad r_2(t) = -1 \text{ if } t \in (1/4, 1/2] \cup (3/4, 1].$$

We have that r_j are mutually independent random variables on $[0,1]$ as, for all integrable functions f_j we have

$$\int_0^1 \prod_{j=0}^n f_j(r_j(t))dt = \prod_{j=0}^n \int_0^1 f_j(r_j(t))dt.$$

To show this equality, we simply observe that both sides are equal to

$$\prod_{j=0}^n \frac{f_j(-1) + f_j(1)}{2}.$$

We also have the following inequalities due to Khintchine.

Theorem 2.6.1. *For any $1 \le p < \infty$ and for any real-valued square summable sequences $\{a_j\}$ and $\{b_j\}$ we have*

$$B_p \Big(\sum_j |a_j + ib_j|^2 \Big)^{1/2} \le \Big\| \sum_j (a_j + ib_j) r_j \Big\|_{L^p([0,1])} \le A_p \Big(\sum_j |a_j + ib_j|^2 \Big)^{1/2}$$

for some constants $0 < A_p, B_p < \infty$ that only depend on p.

Proof. Let

$$F(t) = \sum_j (a_j + ib_j) r_j(t), \quad t \in [0,1].$$

First, we show the right inequality. If we have proved it for $b_j = 0$, then

$$\begin{aligned} \Big\| \sum_j (a_j + ib_j) r_j \Big\|_{L^p} &\le \Big\| \sum_j a_j r_j \Big\|_{L^p} + \Big\| \sum_j b_j r_j \Big\|_{L^p} \\ &\le A_p \Big[\Big(\sum_j |a_j|^2 \Big)^{1/2} + \Big(\sum_j |b_j|^2 \Big)^{1/2} \Big] \\ &\le \sqrt{2} A_p \Big(\sum_j |a_j + ib_j|^2 \Big)^{1/2}, \end{aligned}$$

so we can assume $b_j = 0$. We can also assume that only finitely many terms of the sequence $\{a_j\}$ are non-zero by using a limiting argument. Finally, rescale the constants so that we have $\sum_j |a_j|^2 = 1$.

Now let $\rho > 0$. We have

$$\begin{aligned} \int_0^1 e^{\rho \sum_j a_j r_j(t)} dt &= \prod_j \int_0^1 e^{\rho a_j r_j(t)} dt \\ &= \prod_j \frac{e^{\rho a_j} + e^{-\rho a_j}}{2} \\ &\leq \prod_j e^{\rho^2 a_j^2/2} = e^{\rho^2 \sum_j a_j^2/2} = e^{\rho^2/2} \end{aligned}$$

where we used the inequality $(e^x + e^{-x})/2 \leq e^{x^2/2}$ for all real x from the power series expansion. The same upper bound holds for $\int_0^1 e^{-\rho \sum_j a_j r_j(t)} dt$ so we have

$$\int_0^1 e^{\rho |F(t)|} dt \leq 2e^{\rho^2/2}.$$

It follows that

$$e^{\rho\alpha} |\{t \in [0,1] : |F(t)| > \alpha\}| \leq \int_0^1 e^{\rho |F(t)|} dt \leq 2e^{\rho^2/2}.$$

By picking $\rho = \alpha$, we obtain

$$d_F(\alpha) = \big|\{t \in [0,1] : |F(t)| > \alpha\}\big| \leq 2e^{\rho^2/2 - \rho\alpha} = 2e^{-\alpha^2/2}.$$

Formula 2.1.2 gives

$$\|F\|_{L^p}^p = \int_0^\infty p\alpha^{p-1} d_F(\alpha) d\alpha \leq \int_0^\infty p\alpha^{p-1} 2e^{-\alpha^2/2} d\alpha = 2^{p/2} p\Gamma(p/2).$$

Now show the left inequality. Pick an s such that $1 < s < \infty$ and find $0 < \theta < 1$ such that

$$\frac{1}{2} = \frac{1-\theta}{p} + \frac{\theta}{s}.$$

Then, from the above

$$\|F\|_{L^2} \leq \|F\|_{L^p}^{1-\theta} \|F\|_{L^s}^{\theta} \leq \|F\|_{L^p}^{1-\theta} A_s^\theta \|F\|_{L^2}^\theta.$$

It follows that

$$\|F\|_{L^2} \leq A_s^{\theta/(1-\theta)} \|F\|_{L^p}.$$

$\square$

Chapter 3

Classical Results

3.1 Formulations of maximal operator

We have many equivalent formulations of maximal operators. For bases $\mathcal{B}$ as in Definition 1.1.1, consider $M^1_{\mathcal{B}}$ and $M^2_{\mathcal{B}}$ defined by

$$M^1_{\mathcal{B}} f = \sup_{x \in B \in \mathcal{B}} \frac{1}{|B|} \int_B |f| dx$$

$$M^2_{\mathcal{B}} f = \sup_{x \in B \in \mathcal{B}} \left| \frac{1}{|B|} \int_B f dx \right|$$

we have that

$$\left\| M^1_{\mathcal{B}} \right\|_{L^p \to L^p} = \left\| M^2_{\mathcal{B}} \right\|_{L^p \to L^p} . \tag{3.1.1}$$

The norm $\|M^1_{\mathcal{B}} f\|$ clearly dominates $\|M^2_{\mathcal{B}} f\|$ for all f. The converse equality follows from the fact that $\left\|M^1_{\mathcal{B}} f\right\| = \left\|M^2_{\mathcal{B}}(|f|)\right\|$.

Notice that in Definition 1.1.1, we are taking the supremum over all $S \in \mathcal{B}$ which contain the point x. When the bases are balls or parallelepipeds, we can strengthen this requirement and take the supremum over all S which are centered at x. We will call such a maximal operator a *centered* maximal operator as opposed to an *uncentered* maximal operator. These are equivalent as we have

$$(M^c_{\mathcal{B}} f)(x) \leq (M^u_{\mathcal{B}} f)(x) \leq 2^n (M^c_{\mathcal{B}} f)(x).$$

where $M^c_{\mathcal{B}}$ and $M^u_{\mathcal{B}}$ denote the centered and uncentered maximal operator, respectively. The first inequality is obvious and the second inequality holds because for any $x \in B$ with B either a ball or a rectangle, $B \subset B'$ where B' is centered at x and twice as large. Then,

$$\frac{1}{|B|} \int_B f dx \leq \frac{2^n}{|B'|} \int_{B'} f dx.$$

For the remainder of this paper, we use the above equivalent formulations interchangeably.

3.2 Hardy-Littlewood maximal operator

The classical Hardy-Littlewood maximal operator M in n-dimensions has bases

$$\mathcal{B} = \{\text{balls of radius } r > 0\}.$$

Since a ball can be included in a cube of comparable size and vice versa, we can equivalently take

$$\mathcal{B} = \{\text{cubes of sides } l > 0\}.$$

We have the following boundedness results.

Theorem 3.2.1. *M is of weak type (1,1) and strong type (p, p) for all $1 < p \leq \infty$.*

Proof. We have a trivial estimate for L^∞. Hence if we show the weak type (1,1) estimate, the rest follows from Marcinkiewicz interpolation theorem. To show the weak type (1,1) estimate, we need the following covering lemma.

Lemma 3.2.2. *Let $\{B_1, B_2, \ldots, B_k\}$ be a finite collection of open balls in $\mathbb{R}^n$. Then there exists a finite subcollection $\{B_{j_1}, \ldots, B_{j_l}\}$ of pairwise disjoint balls such that*

$$\left|\bigcup_{i=1}^{l} B_{j_i}\right| \geq 3^{-n}\left|\bigcup_{i=1}^{k} B_i\right|.$$

Proof. Without loss of generality, $|B_1| \geq |B_2| \geq \cdots \geq |B_k|$. Let $j_1 = 1$. Having chosen $j_1, \ldots, j_i$, let j_{i+1} be the least index $s > j_i$ such that $\cup_{m=1}^{i} B_{j_m}$ is disjoint from B_s. This process will terminate after finite number of steps since we started with finite collection. By the selection rule, we have pairwise disjoint $B_{j_1}, \ldots, B_{j_l}$. If some B_m was not selected, then B_m must intersect some ball B_{j_r} for some $j_r < m$. Since B_m has smaller radius than B_{j_r}, $B_m \subset 3B_{j_r}$. Thus,

$$\left|\bigcup_{i=1}^{k} B_i\right| \leq \left|\bigcup_{r=1}^{l} 3B_{j_r}\right|$$

and this quantity is dominated by

$$\sum_{r=1}^{l} |3B_{j_r}| = 3^n \sum_{r=1}^{l} |B_{j_r}| = 3^n \left|\bigcup_{r=1}^{l} B_{j_r}\right|.$$

□

We return to the proof of the weak type (1,1) estimate. It suffices to show that

$$\left|\{x \in \mathbb{R}^n : |Mf(x)| > \alpha\}\right| \leq 3^n \frac{\|f\|_{L^1}}{\alpha}.$$

Since the centered maximal operator is $Mf(x) = \sup_r T_r f(x)$ where $T_r f(x) = \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)dy$ is a supremum of continuous functions $T_r f$, we have that

$$E_\alpha = \{x \in \mathbb{R}^n : |Mf(x)| > \alpha\}$$

is open. Let K be a compact subset of E_α. For each $x \in K$ there exists an open ball B_x containing the point X such that

$$\int_{B_x} |f(y)|dy > \alpha |B_x|.$$

By compactness there exists a finite subcover $\{B_{x_1}, \ldots, B_{x_k}\}$. Using Lemma 3.2.2 we find a subcollection of pairwise disjoint balls. Then,

$$|K| \leq \sum_{i=1}^{k} |B_{x_i}| \leq 3^n \sum_{i=1}^{l} |B_{x_{j_i}}| \leq \frac{3^n}{\alpha} \sum_{i=1}^{l} \int_{B_{x_{j_i}}} |f(y)|dy \leq \frac{3^n}{\alpha} \int_{\mathbb{R}^n} |f(y)|dy.$$

Taking the supremum over all compact $K \subset E_\alpha$, we get the desired weak type estimate. ☐

3.3 Lebesgue Differentiation theorem

As an application of the maximal operator, we present the following theorem.

Theorem 3.3.1. *(Lebesgue Differentiation theorem) For any locally integrable function f on $\mathbb{R}^n$ we have*

$$\lim_{r \to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)dy = f(x)$$

for almost all x in $\mathbb{R}^n$. Consequently we have $|f| \leq M(f)$ almost everywhere.

Note that this is a standard calculus result for a continuous function f, where equality holds for all x. We will use the boundedness of the maximal operator to extend this result to all locally integrable functions.

We first need the following theorem in a more general setting. Let $(X, \mu), (Y, \nu)$ be measure spaces and let $1 \leq p \leq \infty, 1 \leq q \leq \infty$. Suppose that D is a dense subspace of $L^p(X, \mu)$, i.e. for all $f \in L^p$ and all $\delta > 0$ there exists a $g \in D$ such that $\|f - g\|_{L^p} < \delta$. Suppose that for every $\epsilon > 0, T_\epsilon$ is a linear operator defined on $L^p(X, \mu)$ with values in the set of measurable functions on Y. Define a sublinear operator

$$T_*(f)(x) = \sup_{\epsilon > 0} |T_\epsilon(f)(x)|.$$

Theorem 3.3.2. *Let $1 \leq p \leq \infty, 1 \leq q < \infty$ and T_ϵ and T_* as above. Suppose that for some $B > 0$ and for all $f \in L^p(X)$ we have*

$$\|T_*(f)\|_{L^{q,\infty}} \leq B\|f\|_{L^p}$$

and that for all $f \in D$

$$\lim_{\epsilon \to 0} T_\epsilon(f) = T(f)$$

exists, is finite ν-almost everywhere and defines a linear operator on D. Then for all $f \in L^p(X, \mu)$ the above limit exists and is finite ν-almost everywhere and defines a linear operator T on $L^p(X)$ uniquely extending T defined on D and satisfying

$$\|T(f)\|_{L^{q,\infty}} \leq B\|f\|_{L^p}.$$

Proof. Given f in L^p we define the *oscillation* of f

$$O_f(y) = \limsup_{\epsilon \to 0} \limsup_{\theta \to 0} |T_\epsilon(f)(y) - T_\theta(f)(y)|.$$

Let $S_{f,\delta} = \{y \in Y : O_f(y) > \delta\}$. If we show that $\nu(S_{f,\delta}) = 0$ then for any $y \notin \cup_n S_{f,1/n}$, which is a null set, $T_\epsilon(f)(y)$ is Cauchy and hence converges to some $T(f)(y)$ as $\epsilon \to 0$. Since T_ϵ is linear, so is the limit T and this extends the original T defined on D.

We use density of D to approximate O_f. Given $\eta > 0$, find a $g \in D$ such that $\|f - g\|_{L^p} < \eta$. Since $T_\epsilon(g) \to T(g)$ ν-almost everywhere, it follows that $O_g = 0$ ν-almost everywhere. Hence,

$$O_f(y) \leq O_g(y) + O_{f-g}(y) = O_{f-g}(y) \qquad \nu - \text{almost everywhere}$$

where we used the linearity of T_ϵ and triangular inequality. Then for any $\delta > 0$ we have

$$\begin{aligned} \nu(\{y \in Y : O_f(y) > \delta\}) &\leq \nu(\{y \in Y : O_{f-g}(y) > \delta\}) \\ &\leq \nu(\{y \in Y : 2T_*(f - g)(y) > \delta\}) \\ &\leq (2B\|f - g\|_{L^p}/\delta)^q \\ &\leq (2B\eta/\delta)^q. \end{aligned}$$

Let $\eta \to 0$ to finish the proof. The last claim of the theorem is a direct consequence of $|T(f)| \leq |T_*(f)|$. $\square$

Now we return to the proof of the Lebesgue differentiation theorem.

Proof. (of Theorem 3.3.1) Since the above is a local result, we only need to show for all $f \in L^1$. Let $T_\epsilon(f) = k_\epsilon * f$ where $k = \frac{1}{|B(0,1)|}\chi_{B(0,1)}$ (recall the notation (2.4.5)). Since each T_ϵ is bounded by the centered Hardy-Littlewood maximal operator, we have that T_* is also weak type (1,1). Take D to be all continuous functions with compact support, which is dense in L^1. From Theorem 3.3.2, we extend the result to all $f \in L^1$. $\square$

3.4 Strong maximal operator

The strong maximal operator, denoted M_s, in n-dimensions takes

$$\mathcal{B} = \{\text{rectangles with sides parallel to the coordinate axes}\}.$$

Theorem 3.4.1. *M_s is of strong type (p,p) but not of weak type (1,1) for $n \geq 2$.*

Proof. To prove strong type (p,p), we repeat the 1-dimensional estimate n times. More specifically, for $f \in L^p \cap L^\infty$ with compact support and for R_x centered at x,

$$\begin{aligned}
&\left|\frac{1}{|R_x|}\int_{R_x} f(y)dy\right| \\
\leq\ & \frac{1}{2^n h_1 \dots h_n}\int_{-h_1}^{h_1}\dots\int_{-h_n}^{h_n} |f(x+y)|dy_n \dots dy_1 \\
=\ & \frac{1}{2^{n-1} h_1 \dots h_{n-1}}\int_{-h_1}^{h_1}\dots\int_{-h_{n-1}}^{h_{n-1}} \left[\frac{1}{2h_n}\int_{-h_n}^{h_n} |f(x_1+y_1,\dots,x_n+y_n)|dy_n\right] dy_{n-1}\dots dy_1 \\
\leq\ & \frac{1}{2^{n-1} h_1 \dots h_{n-1}}\int_{-h_1}^{h_1}\dots\int_{-h_{n-1}}^{h_{n-1}} M_n f(x_1+y_1,\dots,x_{n-1}+y_{n-1},x_n)dy_{n-1}\dots dy_1 \\
\dots\ & \\
\leq\ & M_1 \circ \dots \circ M_n f(x)
\end{aligned}$$

where M_i is a 1-dimensional maximal operator in x_i. Hence,

$$\begin{aligned}
\|M_s f\|_{L^p(\mathbb{R}^n)}^p &\leq \int_{\mathbb{R}}\dots\int_{\mathbb{R}} |M_1\circ\dots\circ M_n f(x)|^p dx_1\dots dx_n \\
&= \int_{\mathbb{R}}\dots\int_{\mathbb{R}}\left[\int_{\mathbb{R}} |M_1(M_2\circ\dots\circ M_n f(x))|^p dx_1\right] dx_2\dots dx_n \\
&\leq \int_{\mathbb{R}}\dots\int_{\mathbb{R}}\left[C_p\int_{\mathbb{R}} |M_2\circ\dots\circ M_n f(x)|^p dx_1\right] dx_2\dots dx_n \\
&= C_p\int_{\mathbb{R}}\int_{\mathbb{R}}\dots\int_{\mathbb{R}} |M_2\circ\dots\circ M_n f(x)|^p dx_2\dots dx_n dx_1 \\
&= C_p^n\int_{\mathbb{R}}\dots\int_{\mathbb{R}} |f(x)|^p dx_1\dots dx_n
\end{aligned}$$

where $C_p = \|M_i\|_{L^p\to L^p}$ and the use of Fubini's theorem is justified since $(M_1\circ\dots\circ M_n)f \leq \frac{C}{|x_1|\dots|x_n|}$ is in $L^p(\mathbb{R}^n)$. Since compactly supported functions in $L^p \cap L^\infty$ are dense in L^p, we can extend the result to all $f \in L^p$.

For $n \geq 2$, the counterexample for the weak type (1,1) inequality is provided by the characteristic function of a unit cube U centred at the origin. We only show this for $n = 2$. Note that

$$\{x : M\chi_U(x) > 1/4\alpha\} \supset \{x_1, x_2 \geq 1, \text{ and } x_1 x_2 < \alpha\}$$

by taking average over the rectangle formed by $(0,0),(x_1,0),(x_1,x_2)$ and $(0,x_2)$. But this set has measure

$$\int_1^\alpha \frac{\alpha}{x}dx = \alpha \log \alpha.$$

By letting α be arbitrarily large, we violate the weak L^1 norm condition.

For higher dimensions, we argue that when $x_3,\ldots,x_n$ are fixed between -1/2 and 1/2, we get that $\{x : M\chi_U(x) > 1/4\alpha\} \supset \{x : -1/2 < x_3,\ldots,x_n < 1/2, \text{ and } x_1,x_2 \geq 1, \text{ and } x_1x_2 < \alpha\}$ which has the same measure as above. $\square$

3.5 Besicovitch's construction

We saw in the previous section that the maximal operator is bounded when the bases consist of rectangles of arbitrary dimensions, but sides parallel to the axes. If we drop the condition that the sides are parallel to the axes, then the maximal operator is unbounded for any L^p. It is easy to show this for $p \leq n$ by considering a characteristic function of the unit ball $\chi_{B(0,1)}$. Then, $M\chi_{B(0,1)} \sim \min\{1, 1/|x|\}$ (where, for functions f and g, $f \sim g$ means that there are constants A and B such that $Af \leq g \leq Bf$). Thus,

$$\|M\chi_{B(0,1)}\|_{L^p}^p \sim \int_{|x|\geq 1} \frac{1}{|x|^p}dx = \int_1^\infty \frac{dr}{|x|^{p-(n-1)}}$$

which is infinite if $p \leq n$. However, by constructing a so-called Besicovitch set, we can extend this result to all p.

A *Besicovitch set* is a subset of R^n which contains a unit line segment in each direction. Besicovitch sets are also known as *Kakeya sets*. In 1917 the Russian mathematician Besicovitch constructed a planar set of measure 0 which contains a line segment in each direction while working on a problem in Riemann integration. In 1917 the Japanese mathematician Kakeya independently asked for the smallest area of a convex set within which one could rotate a needle by 180 degrees in the plane. While the original question was resolved by Pal in 1921, the question without the convexity condition remained open until 1928 when Besicovitch showed that it could be arbitrarily small [22].

Perron, Schoenberg, Kahane, etc. (see [25]) discovered many ways to construct a Besicovitch set of measure zero. The main idea is that we can slice a triangle into many thin sub-triangles and rearrange them so they overlap a lot. This new figure will contain the same directions as the triangle, but with considerably smaller area. We can generalize this planar set to higher dimensions.

Here we present the Besicovitch set of arbitrarily small size in dimension 2. We will obtain the final figure starting from a triangle and repeating an "overlapping" operation. We describe this operation first. Fix a constant α with $1/2 < \alpha < 1$. Suppose that T is a

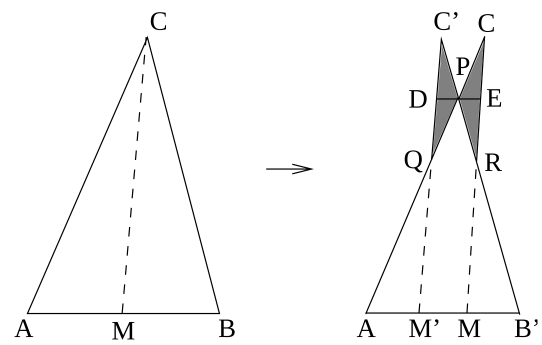

Figure 3.1: "Overlapping" operation.

triangle ABC whose base AB lies along the x-axis. We bisect the base AB at M, obtaining two sub-triangles: the "left" triangle AMC and the "right" triangle MBC. We translate the "right" triangle leftwards to obtain the overlapping figure $\Phi(T)$ (see Figure 3.1). Here $M'B'C'$ is the translate of MBC; the left triangle AMC remained fixed. The figure $\Phi(T)$ is the union of the smaller triangle called the "heart" $\Phi_h(T) = AB'P$ (which is similar to the original triangle ABC) and the union $\Phi_a(t)$ of the two small "arms" (shaded above.) The constant α is the length ratio between $\Phi_h(T)$ and T. We have that $|\Phi_h(T)| = \alpha^2|T|$. We also have that triangle $C'DP$ and REP and $C'M'B'$ are similar with ratio $1-\alpha : 1-\alpha : 1$. The same is true for triangles CEP, QDP and CAM. Thus, $|\Phi_a(T)| = 2(1-\alpha)^2|T|$. Together we get that

$$|\Phi(T)| = [\alpha^2 + 2(1-\alpha)^2]|T|. \tag{3.5.1}$$

We will repeat this process to "generate our monster, which will have a tiny heart and many arms" [25].

We will imagine that we have a piece of paper that we "cut up" and "glue together". We also fix $1/2 < \alpha < 1$. We start with paper with the shape of triangle ABC. Subdivide the base AB into 2^n equal subintervals, with division points $A = A_0, A_1, \ldots, A_{2^n} = B$. We "cut it up" to 2^n disjoint smaller triangles $A_jA_{j+1}C$ for $0 \le j < 2^n$.

For each j, we overlap $A_{2j}A_{2j+1}C$ and $A_{2j+1}A_{2j+2}C$ to form $\Phi(A_{2j}A_{2j+2}C)$ as above and "glue each pair together". Now we have 2^{n-1} pieces to move around. Note that $\Phi_h(A_{2j}A_{2j+2}C)$, when put together so that the parallel sides coincide, makes a triangle similar to the original triangle ABC but of size $\alpha^2|ABC|$. Also the sum of the size of "arms" created by "gluing" is $2(1-\alpha)^2|ABC|$.

We are now ready to glue $\Phi(A_{4j}A_{4j+2}C)$ and $\Phi(A_{4j+2}A_{4j+4}C)$ together. We glue them so that $\Phi_h(A_{4j}A_{4j+2}C)$ and $\Phi_h(A_{4j+2}A_{4j+4}C)$ overlap as above. We now created the second generation of "arms" and the sum of area of these is $2(1-\alpha)^2(\alpha^2|ABC|)$. Again, newly created 2^{n-2} hearts, when put together so that the parallel sides coincide, makes a triangle similar to the original triangle ABC but of size $\alpha^4|ABC|$. At the end of this step, we have 2^{n-2} pieces to move around.

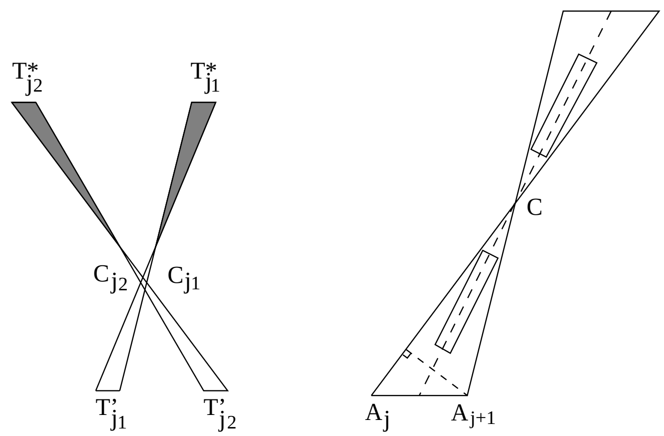

Figure 3.2: Disjointness of T^* and the rectangles R_j.

We repeat this process n times and create one "glued" piece $\Phi_n(ABC)$. The area of "arms", which may overlap, does not exceed the sum

$$\begin{aligned}\sum_{i=1}^{n} 2(1-\alpha)^2\alpha^{2(i-1)}|ABC| &\le 2(1-\alpha)^2|ABC|\sum_{i=0}^{\infty}\alpha^{2i} \\ &= \frac{2(1-\alpha)^2}{1-\alpha^2}|ABC| \le 2(1-\alpha)|ABC|\end{aligned}$$

and the area of the newly created heart is $\alpha^{2n}|ABC|$. Hence, the area of $\Phi_n(ABC)$ is at most $(\alpha^{2n} + 2(1-\alpha))|ABC|$. By making α close to 1 and making n large enough, we can make this arbitrarily small. Note that the glued piece still contains all the line segments parallel to the line CX where X is any point on AB. Hence, assuming we started off with an equilateral triangle with unit length height, six of these figures will contain a unit line segment of any arbitrary direction.

The Kakeya set is related to many problems in harmonic analysis. For example, C. Fefferman used it to prove the ball multiplier theorem and it is also related to the restriction problem, Bochner-Riesz conjecture and the Kakeya conjecture.

It is related to the maximal operator in arbitrary directions, denoted by M^*, as it provides a characteristic function f with arbitrarily small L^p norm but with $\|M^*f\|_{L^p}$ at least $C > 0$. In order to see how such a characteristic function is constructed, let $T_j, 0 \le j < 2^n$ denote the triangles $A_jA_{j+1}C$ that make up ABC and let T'_j denote the corresponding translated triangles comprising $\Phi_n(ABC)$. While T_j have common vertex C, T'_j will have top vertices C_j. Denote by T^*_j the triangles obtained by reflecting the T'_j through C_j. While we moved T_j closer together to overlap, T^*_j in fact separated further and remained disjoint as in Figure 3.2. To complete the argument, we start with ABC being an equilateral triangle with height

2 and consider $T_j = CA_jA_{j+1}$. Note that

$$\angle A_jCA_{j+1} \geq \frac{|A_jA_{j+1}| \sin \angle CA_{j+1}A_j}{|CA_{j+1}|}$$

since the arc length is bounded below by the length of the perpendicular (see Figure 3.2.) We have that $|CA_{j+1}| \leq 2\sqrt{3}, |A_jA_{j+1}| = \frac{4}{\sqrt{3}\cdot 2^n}$ and $|\sin \angle CA_{j+1}A_j| \geq \sqrt{3}/2$ so that

$$\angle A_jCA_{j+1} \geq \frac{1}{\sqrt{3}\cdot 2^n}.$$

Now draw a line from C to the midpoint of A_jA_{j+1} and mark off points P_1 and P_2 on it at distances $1/2$ and $3/2$ from the vertex. If we draw a rectangle whose major axis is P_1P_2 and whose side lengths are 1 and $\frac{1}{\sqrt{3}\cdot 2^{n+1}}$, then we can guarantee that this rectangle R_j is contained inside T_j. Let R_j' be the translation of R_j inside T_j' and let R_j^* be the reflection of R_j' through C_j.

We now have 2^n rectangles R_j' of dimension $1 \times c2^n$ with R_j^* disjoint. Then,

$$\|\chi_{\cup R_j'}\|_{L^p} \leq |\Phi_n(|ABC|)|^{1/p} \leq \left(\frac{4}{\sqrt{3}}(\alpha^{2n} + 2(1-\alpha))\right)^{1/p}.$$

We also get that for $x \in R_j^*, M^*(\chi_{\cup R_j'})(x) \geq 1/3$ so

$$\|M^*(\chi_{\cup R_j'})\|_{L^p} \geq \frac{1}{3}c^{1/p}.$$

By making α close to 1 and n larger, we see that M^* is not strong type (p,p). For higher dimensions, just use the characteristic function of the set $B \times [0,1] \times \cdots \times [0,1]$.

This type of construction is called the *Perron Tree construction*. It can be generalized to identify more sets associated with unbounded maximal operators [15, 16].

As an aside, we note that Besicovitch set can have measure 0. For any triangle XYZ with X on the line $y = 1$ and Y, Z on the line $y = 0$, consider the reflection Y' of Y about the midpoint of XZ (we will always take a reflection of the "lower-left" corner). We will denote the parallelogram $XYZY'$ by $P(XYZ)$. In the previous argument, consider each translated triangle T_j' comprising $\Psi_n(ABC)$. Suppose that $\Psi_n(ABC)$ is translated so that the rightmost component T_{2^n-1}' stays unmoved, i.e. T_{2^n-1}' coincide with T_{2^n-1}. Then we can show that each $P(T_j')$ is contained in $P(ABC)$ and $\cup P(T_j')$ has arbitrarily small area.

Hence, we have created, from a parallelogram π with two horizontal sides on $y = 0$ and $y = 1$, an arbitrarily small union of parallelograms π_i with $\pi_i \subset \pi$ and $\cup \pi_i$ still containing a translation of each line segment in π joining the lines $y = 0$ and $y = 1$. We repeat this process creating $K_1 \supset K_2 \supset \cdots \supset K_j \supset \ldots$ and note that $\cap_j K_j$ is a Besicovitch set with measure 0.

3.6 More properties of maximal operators

While the maximal operator itself is sublinear, we can often use a related linear operator to find a norm estimate. For example, consider a pre-determined set of rectangles B_x and integrate over them as opposed to taking the supremum:

$$Tf(x) = \frac{1}{|B_x|}\int_{B_x} f dy. \tag{3.6.1}$$

We clearly have $\|Tf\| \leq \|Mf\|$. Conversely, for $f \geq 0$, we will achieve $\|Tf\| \geq \|Mf\| - \epsilon$ for certain choice of B_x.

We can also regard the maximal operator as a linear mapping from $L^p(X)$ to $L^p(X, l^\infty)$ (the space of l^∞-valued functions which are p-integrable) as opposed to mapping $L^p(X)$ to itself. Let us select a countable subset $\mathcal{B}'$ of the bases $\mathcal{B}$. In case the bases are balls, we can choose balls with rational center and rational radius. In case of rectangles, we choose rectangles with rational vertices. Then, since we can approximate each $B \in \mathcal{B}$ with $B' \in \mathcal{B}'$ arbitrarily close, we have:

$$\|M_{\mathcal{B}'}\| = \|M_{\mathcal{B}}\|. \tag{3.6.2}$$

By enumerating $0 \in B \in \mathcal{B}'$ by B_j, we get

$$Tf(x) = \Big(\frac{1}{|B_j|}\int_{x+B_j} f dy\Big)_{j=1}^{\infty} \tag{3.6.3}$$

We clearly have $\|Tf\|_{L^p(l^\infty)} = \|Mf\|_{L^p}$ where we used the version $Mf = M^2_{\mathcal{B}'}$ in (3.1.1).

There are several advantages to linearizing the maximal operator. First, it enables us to use the Riesz-Thorin interpolation theorem. Second, the "averaging" operation as in (3.6.1) is of convolution type and hence multiplier theorems can applied to study it.

The maximal operator is also scalar invariant, which means that the norm of the maximal operator does not change when we scale the bases from $\mathcal{B}$ to $c\mathcal{B}$ for any $c > 0$. To see this, we show the pointwise equality

$$M_{c\mathcal{B}}(\delta^{1/c} f)(x) = (M_{\mathcal{B}} f)(x/c).$$

We show this for a more general dilation $\delta^{v^{-1}} f$ and

$$v\mathcal{B} = \Big\{vB = \{z \in \mathbb{R}^n : v^{-1}z \in B\}\Big\}_{B\in\mathcal{B}}.$$

We have for $y' = v^{-1}y$ (recall notation (2.4.5))

$$\begin{aligned} M_{v\mathcal{B}}(\delta^{v^{-1}} f)(x) &= \sup_{x\in vB\in v\mathcal{B}} \frac{1}{|vB|}\int_{vB} f(y_1/v_1, \dots, y_n/v_n) dy \\ &= \sup_{v^{-1}x\in B\in\mathcal{B}} \frac{1}{|v|\cdot|B|}\int_{B} f(y')|v| dy' \\ &= M_{\mathcal{B}} f(v^{-1}x). \end{aligned} \tag{3.6.4}$$

Thus,

$$\begin{aligned}
\|M_{c\mathcal{B}}(\delta^{1/c}f)\|_{L^p} &= \|\delta^{1/c}(M_{\mathcal{B}}f)\|_{L^p} \\
&= c^{n/p}\|M_{\mathcal{B}}f\|_{L^p} \\
&\leq \|M_{\mathcal{B}}\|\big(c^{n/p}\|f\|_{L^p}\big) \\
&= \|M_{\mathcal{B}}\| \cdot \|\delta^{1/c}f\|_{L^p}.
\end{aligned}$$

This, combined with the monotone convergence theorem, can give ways to restrict the bases further, while maintaining the norm. For example, for

$$\mathcal{B}_\Omega = \{\text{rectangles in direction } \theta \in \Omega\},$$

consider the maximal operator $M_{\mathcal{B}^t_\Omega}$ for $t > 0$, associated with

$$\mathcal{B}^t_\Omega = \{B \in \mathcal{B}_\Omega : \text{ shorter side of } B \text{ is at least t }\}.$$

Then, by the monotone convergence theorem,

$$\|M_{\mathcal{B}_\Omega}f\|_{L^p} = \lim_n \big\|M_{\mathcal{B}^{1/n}_\Omega}f\big\|_{L^p}$$

and we have $\|M_{\mathcal{B}_\Omega}\|_{L^p\to L^p} = \sup_n \big\|M_{\mathcal{B}^{1/n}_\Omega}\big\|_{L^p\to L^p}$. But by the scalar invariance, $\big\|M_{\mathcal{B}^{1/n}_\Omega}\big\| = \|M_{\mathcal{B}^1_\Omega}\|$ so we have

$$\|M_{\mathcal{B}^1_\Omega}\| = \|M_{\mathcal{B}_\Omega}\|. \tag{3.6.5}$$

We also have the following useful pointwise estimate using maximal operators.

Theorem 3.6.1. *If $K(x) = k(|x|)$ in $\mathbb{R}^n$ and $k \geq 0$ is a decreasing function on $[0, \infty)$ which is continuous except at a finite number of points, then for any locally integrable f,*

$$\sup_\epsilon(|f| * K_\epsilon)(x) \leq \|K\|_{L^1}Mf(x)$$

where M denotes the Hardy-Littlewood maximal operator.

Proof. Note that since $\|K\|_{L^1} = \|K_\epsilon\|_{L^1}$, we only need to show

$$(|f| * K)(x) \leq \|K\|_{L^1}Mf(x).$$

Also if the above is true at $x = 0$ for all f, then by translation of f, it is true for all $x = x_0$ by letting $f' = f(x - x_0)$.

First assume that K is continuous and compactly supported. Then, letting $e_1 = (1, 0, \ldots, 0)$

$$\int_{\mathbb{R}^n} |f(y)|K(-y)dy = \int_0^\infty \int_{S^{n-1}} |f(r, \theta)|K(re_1)r^{n-1}d\theta dr.$$

Set

$$F(r) = \int_{S^{n-1}} |f(r,\theta)|d\theta \quad \text{and} \quad G(r) = \int_0^r F(s)s^{n-1}ds,$$

where $d\theta$ denotes surface measure on S^{n-1}. Using integration by parts,

$$\begin{aligned} \int_{\mathbb{R}^n} |f(y)|K(-y)dy &= \int_0^\infty F(r)r^{n-1}K(re_1)dr \\ &= \lim_{R\to\infty} G(R)K(Re_1) - G(0)K(0) - \int_0^\infty G(r)dK(re_1) \quad (3.6.6) \\ &= \int_0^\infty G(r)d(-K(re_1)) \end{aligned}$$

where the integrals are of Lebesgue-Stieltjes type and we used the fact that $G(0) = 0$ and $K(Re_1) = 0$ for large enough R. Then, since

$$G(r) = \int_0^r F(s)s^{n-1}ds = \int_{|y|\le r} |f(y)|dy \le |B(0,r)|(Mf(0)),$$

and $d(-K(re_1))$ is positive, we have (3.6.6) dominated by

$$|B(0,1)|(Mf(0)) \int_0^\infty r^n d(-K(re_1)) = Mf(0) \int_0^\infty K(re_1)n|B(0,1)|r^{n-1}dr = Mf(0)\|K\|_{L^1},$$

where $n|B(0,1)|r^{n-1}$ is the surface area of sphere of radius r.

Now for a general K, we find a sequence K_j, compactly supported and continuous that increases to K pointwise, which is possible since K is continuous except at a finite number of points. Then by the monotone convergence theorem,

$$|f| * K = \sup |f| * K_j \le \|K_j\|_{L^1} Mf(x) \le \|K\|_{L^1} Mf(x).$$

□

Corollary 3.6.2. *If $\phi \in \mathcal{S}(\mathbb{R}^n)$, then for any locally integrable f*

$$\left|(\phi)^\vee * f(x)\right| \le CMf(x)$$

for some C depending on ϕ.

Proof. Since $(\phi)^\vee$ is also a Schwartz function, we can bound it by $P(x) = 1/(|x|+1)^{n+1} \in L^1$. Then, by Theorem 3.6.1,

$$\left|(\phi)^\vee * f(x)\right| \le \left||(\phi)^\vee| * |f|(x)\right| \le \left|P * |f|(x)\right| \le \|P\|_{L^1} Mf(x).$$

□

We have the following analogous theorem for the strong maximal operator.

Theorem 3.6.3. *If $K(x) = k_1(|x_1|)\dots k_n(|x_n|)$ in $\mathbb{R}^n$ and each $k_i \geq 0$ is a decreasing integrable function on $[0,\infty)$ which is continuous except at a finite number of points, then for locally integrable f*

$$\sup_{v\in\mathbb{R}^n_+} (|f| * K_v)(x) \leq \|K\|_{L^1} M_s f(x)$$

where M_s denotes the strong maximal operator.

Proof. The proof is very similar to Theorem 3.6.1. Since $\|K\|_{L^1} = \|K_v\|_{L^1}$, we only need to show

$$(|f| * K)(x) \leq \|K\|_{L^1} M_s f(x).$$

Again we assume $x = 0$ and K is continuous and compactly supported. Then,

$$\begin{aligned}
\int_0^\infty |f(y)|K(-y)dy_1 &= \lim_{R\to\infty} \Big[F_1(y)K(y)\Big]_{y_1=0}^R - \int_0^R F_1(y)\partial_1 K(y) \\
&= \int_0^\infty F_1(y_1,\dots,y_n)k_2(|y_2|)\dots k_n(|y_n|)d(-k_1(y_1))
\end{aligned}$$

where $F_1(y) = \int_0^{y_1} |f(r, y_2,\dots,y_n)|dr$ and

$$\Big[F_1(y)K(y)\Big]_{y_1=0}^R = F_1(R, y_2,\dots,y_n)K(R, y_2,\dots,y_n) - F_1(0, y_2,\dots y_n)K(0, y_2,\dots,y_n) = 0$$

for large enough R.

Repeating this argument with Fubini's Theorem, we get that

$$\int_0^\infty \dots \int_0^\infty |f(y)|K(-y)dy_1\dots dy_n = \int_0^\infty \dots \int_0^\infty F_n(y)d(-k_1(y_1))\dots d(-k_n(y_n))$$

where

$$F_n(y) = \int_0^{y_n} \dots \int_0^{y_1} |f(r)|dr_1\dots dr_n.$$

Putting together similar results for each "quadrant",

$$\int_{\mathbb{R}^n} |f(y)|K(-y)dy = \int_0^\infty \dots \int_0^\infty G(y)d(-k_1(y_1))\dots d(-k_n(y_n))$$

where

$$\begin{aligned}
G(y) &= \int_{-y_1}^{y_1} \dots \int_{-y_n}^{y_n} |f(r)|dr_n\dots dr_1 \\
&\leq (2^n|y|M_s f)(0).
\end{aligned}$$

Hence,

$$\begin{aligned}(|f|*K)(0) &\leq (M_s|f|)(0)\int_0^\infty 2y_1 d(-k_1(y_1))\ldots\int_0^\infty 2y_n d(-k_n(y_n))\\ &= (M_s|f|)(0)\int_0^\infty k_1(y_1)dy_1\ldots\int_0^\infty k_n(y_n)dy_n\\ &= (M_s|f|)(0)\|K\|_{L^1}\end{aligned}$$

where we again used the fact that K is compactly supported after integration by parts. $\square$

Another useful result in considering directional maximal operator is the equivalence between the following two maximal operators.

Theorem 3.6.4. *For $\Omega \subset \mathbb{S}^1$ and*

$$\mathcal{B}_\Omega = \{\text{rectangles in direction } \theta \in \Omega\},$$

we have that

$$M_\Omega f(x) = \sup_{x\in R\in\mathcal{B}_\Omega} \frac{1}{|R|}\int_R |f(y)|dy \tag{3.6.7}$$

and

$$M_\Omega^l f(x) = \sup_{h>0, v\in\Omega} \frac{1}{2h}\int_{-h}^h |f(x-tv)|dt \tag{3.6.8}$$

are equivalent.

The first version has the rectangular bases in direction Ω while the second version only takes integration over a line. Regarding Ω as a subset of $[0, 2\pi)$ via

$$\theta \leftrightarrow (\cos\theta, \sin\theta) \in \mathbb{S}^1, \tag{3.6.9}$$

we may rewrite (3.6.8) as

$$M_\Omega^l f(x) = \sup_{h>0,\theta\in\Omega} \frac{1}{2h}\int_{-h}^h |f(x_1 - y_1, x_2 - y_1\tan\theta)|dy_1.$$

We also note that this theorem can be generalized to higher dimensions.

Proof. Without loss of generality, we assume that $\Omega \subset [0, \pi/4)$. First, we consider the maximal operator M_Ω^P over the base

$$\begin{aligned}\mathcal{B}_\Omega^P = \{&\text{all translations of parallelograms with vertices}\\ &(0,0), (0,b), (a, a\tan\theta) \text{ and } (a, b + a\tan\theta) : a \geq b > 0, \theta\in\Omega\}.\end{aligned} \tag{3.6.10}$$

Since $0 \leq \theta < \pi/4$, any rectangle in direction θ with $0 \leq \theta < \pi/4$ can be included in such a parallelogram of size at most twice as big, and vice versa. Hence, M_Ω^P is equivalent to M_Ω.

For a parallelogram P centered at $x = (x_1, x_2)$ and in direction θ,

$$\begin{aligned}
\frac{1}{|P|}\int_P |f(y)|dy &= \frac{1}{2h_1 \cdot 2h_2}\int_{-h_1}^{h_1}\int_{-h_2}^{h_2} |f(x_1 - y_1, x_2 - y_1\tan\theta + y_2)|dy_2 dy_1 \\
&\leq \frac{1}{2h_1}\int_{-h_1}^{h_1} M_2 f(x_1 - y_1, x_2 - y_1\tan\theta)dy_1 \\
&\leq M_\theta^l M_2 f(x_1, x_2),
\end{aligned}$$

where M_2 is a one-dimensional Hardy-Littlewood maximal operator in x_2. Combining with Fubini's theorem and boundedness in $L^p(\mathbb{R})$, we can show that it maps $L^p(\mathbb{R}^2)$ to itself. Hence,

$$\|M_\Omega^P f\|_{L^p} \leq \|M_\Omega^l(M_2 f)\|_{L^p} \leq C\|M_\Omega^l\|_{L^p\to L^p}\|f\|_{L^p}.$$

Conversely, by the Lebesgue differentiation theorem, for almost all x,

$$\begin{aligned}
&\frac{1}{2h}\int_{-h}^{h} |f(x_1 - y_1, x_2 - y_1\tan\theta)|dy_1 \\
= \;& \lim_{h_2\to 0}\frac{1}{2h_2}\int_{-h_2}^{h_2}\left(\frac{1}{2h}\int_{-h}^{h} |f(x_1 - y_1, x_2 - y_1\tan\theta + y_2)|dy_1\right)dy_2 \\
\leq \;& (M_\Omega^P f)(x_1, x_2),
\end{aligned}$$

which implies

$$\|M_\Omega^l f\|_{L^p} \leq \|M_\Omega^P f\|_{L^p}.$$

□

In order to use Fourier transform methods, it is useful to consider an equivalent variant of the maximal operator using Schwartz functions so that the Fourier transform is also nice to deal with. Precisely, we choose $\psi \in C_0^\infty(\mathbb{R})$ with $\psi(t) \geq 0$ and $\psi(0) > 0$. Assume that ψ is supported on $|x| < C_1$ with $\|\psi\|_\infty < C_2$ and that $\psi(x) > C_3 > 0$ for $|x| < C_4$. Then, for any $v \in \mathbb{S}^{n-1}$,

$$\begin{aligned}
\frac{1}{2h}\int |f(x - tv)| \cdot \chi_{[-1,1]}(t/h)dt &\geq \frac{1}{2hC_2}\int |f(x - tv)|\psi(C_1 t/h)dt \\
&= \frac{C}{2h'}\int |f(x - tv)|\psi(t/h')dt
\end{aligned}$$

where $h' = h/C_1$ and $C = C_1/C_2$. Also,

$$\begin{aligned}
\frac{1}{2h}\int |f(x - tv)| \cdot \chi_{[-1,1]}(t/h)dt &\leq \frac{1}{2hC_3}\int |f(x - tv)|\psi(C_4 t/h)dt \\
&= \frac{C}{2h'}\int |f(x - tv)|\psi(t/h')dt
\end{aligned}$$

where $h' = h/C_4$ and $C = C_4/C_3$. We set

$$N_{h,j}f(x) = \frac{1}{h}\int \psi(t/h)f(x - tv_j)dt, \tag{3.6.11}$$

$$N_\Omega f = \sup_{h>0, v_j \in \Omega} |N_{h,j} f(x)|.$$

We note that, using a similar argument to (3.1.1), we get an equivalent norm whether we take the absolute value before or after the integration. Using the version (3.6.8) of the maximal operator,

$$\|N_\Omega\|_{L^p} \sim \|M_\Omega\|_{L^p}. \tag{3.6.12}$$

In view of (3.6.2), when Ω is considered as a subset of angles in $[0, \pi/4)$ as in (3.6.9),

$$\|M_\Omega\| = \|M_{\overline{\Omega}}\| \tag{3.6.13}$$

where $\overline{\Omega}$ is the closure of $\Omega \subset [0, \pi/4)$ in the usual topology. From this observation follows that it is enough to consider countable set Ω when studying maximal operators in directions. Also, any set whose closure contains an interval has an unbounded maximal operator. Note that, in fact, if the closure is a set of positive measure, then the associated maximal operator is unbounded [9, 12].

Chapter 4

Main Results

In this chapter we consider the maximal operator where the bases are

$$\mathcal{B}_\Omega = \{\text{rectangles in direction } \theta \in \Omega\}$$

for $\Omega \subset \mathbb{S}^n$. We denote such an operator by $M_{\mathcal{B}_\Omega}$ or M_Ω in accordance with previous chapters. Also, as in Theorem 3.6.4 we may give Ω as a subset of $[0, 2\pi)$.

Our discussion will focus on bounded maximal operators in 2 dimensions.

4.1 Lacunary Directions

This section follows the paper "Differentiation in lacunary directions" by Nagel, Stein and Wainger in 1978 [23]. Although the paper deals with the n dimensional case, we shall only deal with the 2 dimensional case which captures most of the ideas. The main theorem is as follows:

Definition 4.1.1. *Let $\{\theta_j\}$ be a decreasing sequence of positive numbers. $\{\theta_j\}$ is said to be* lacunary *if there exists a $\lambda < 1$ such that $0 \leq \theta_{j+1} \leq \lambda\theta_j$ for all j.*

Theorem 4.1.2. *Let $\Omega = \{\theta_j\}$ be a lacunary sequence going to zero. Then*

$$\|M_\Omega f\|_{L^p} \leq C_p \|f\|_{L^p}$$

for all $1 < p \leq \infty$ and some constant C_p that only depends on p.

Here and in the $n-$dimensional generalization, the sequence being lacunary quantifies that the set is "thin" enough. In view of (3.6.12), we consider each $N_{h,j}f$

$$N_{h,j}f(x) = \frac{1}{h}\int \psi(t/h) f(x - tv_j)dt.$$

We let $\mathbf{1} = (1,1) \in \mathbb{R}^2$ and let $\omega(\xi)$ denote a function which is homogeneous of degree zero and C^∞ away from the origin in $\mathbb{R}^2$ which satisfies:

$$\omega(\xi) = \begin{cases} 1 & \text{if} \quad |\xi \cdot \mathbf{1}| < \frac{c}{2}|\xi| \\ 0 & \text{if} \quad |\xi \cdot \mathbf{1}| \geq c|\xi| \end{cases}$$

where $c > 0$ is a small constant. We also define for $j \geq 1$,

$$\omega_j(\xi) = \omega(\xi_1, \theta_j \xi_2).$$

By making c small enough, we can make the support of each ω_j disjoint. More precisely, if we make c small enough so that $\omega(\xi)$ and $\omega^\lambda(\xi) = \omega(\xi_1, \lambda\xi_2)$ are disjoint, then by scaling in the y direction, we get that ω_j and ω_j^λ are disjoint. Since $\{\theta_j\}$ is lacunary, we have that ω_j and ω_{j+1} are disjoint.

The proof of 4.1.2 consists of repeatedly applying the following two lemmas.

Lemma 4.1.3. *If for some $p > 1$ and for any sequence of functions $\{g_j\}$*

$$\Big\| \Big(\sum_{j=1}^{\infty} \sup_{h>0} |N_{h,j} g_j|^2 \Big)^{1/2} \Big\|_p \leq C_p \Big\| \Big(\sum_{j=1}^{\infty} |g_j|^2 \Big)^{1/2} \Big\|_p \tag{4.1.1}$$

holds, then $\|Nf\|_p \leq A_p \|f\|_p$.

In particular, (4.1.1) holds for $p = 2$.

Proof. We let $\phi_1(\xi)$ be a C^∞ function with compact support that is identically one in a neighborhood of the origin. Let $\phi_2(\xi) = 1 - \phi_1(\xi)$. Then if v_j is a unit vector in direction θ_j and

$$m(s) = \int_{-\infty}^{\infty} e^{-ixs} \psi(x) dx,$$

we have

$$\begin{aligned} \widehat{N_{h,j} f}(\xi) &= \int_{\mathbb{R}^2} \Big(\int_{\infty}^{\infty} \psi_h(t) f(x - t v_j) dt \Big) e^{-ix\cdot\xi} dx \\ &= \int_{\infty}^{\infty} \int_{\mathbb{R}^2} \psi_h(t) f(x') e^{-ix'\cdot\xi - itv_j\cdot\xi} dx' dt \\ &= m(hv_j \cdot \xi) \widehat{f}(\xi). \end{aligned}$$

We break this up into

$$\begin{aligned} & m(hv_j \cdot \xi) \phi_1(h\xi_1, h\theta_j\xi_2) \widehat{f}(\xi) \\ & + m(hv_j \cdot \xi) \phi_2(h\xi_1, h\theta_j\xi_2)(1 - \omega(h\xi_1, h\theta_j\xi_2)) \widehat{f}(\xi) \\ & + m(hv_j \cdot \xi) \phi_2(h\xi_1, h\theta_j\xi_2) \omega(h\xi_1, h\theta_j\xi_2) \widehat{f}(\xi) \\ = \; & \widehat{I_{h,j}(f)}(\xi) + \widehat{II_{h,j}(f)}(\xi) + \widehat{III_{h,j}(f)}(\xi). \end{aligned}$$

Note that $K = (m(\xi_1 + \xi_2)\phi_1(\xi))^\vee$ is a Schwartz function since it the inverse Fourier transform of a product of two C^∞ functions which is compactly supported. We can bound it above with

$$p(x) = \frac{C}{(|x_1|+1)^2(|x_2|+1)^2}$$

which means that each $K_j = (m(v_j \cdot \xi)\phi_1(\xi_1, \theta_j \xi_2))^\vee$ is bounded by $\frac{1}{\theta_j}\delta^{(1,\theta_j)}p(x)$ using Proposition 2.4.5 (5). Since each $\frac{1}{\theta_j}\delta^{(1,\theta_j)^{-1}}p(x)$ has the same L^1 norm $C = \|p\|_{L^1}$, we use Theorem 3.6.3 with $\epsilon = h$ and take the L^p norm to get

$$\left\| \sup_{h,j} |I_{h,j}f| \right\|_{L^p} \leq C\|M_s f\|_{L^p} \leq C'\|f\|_{L^p}.$$

For $II_{h,j}$, we choose K such that

$$\widehat{K} = m(\xi_1 + \xi_2)\phi_2(\xi)(1 - \omega(\xi)).$$

Since $II_{h,j}$ multiplies the Fourier transform of f by $\delta^{(h,h\theta_j)}\widehat{K}$, it is a convolution with $(\frac{1}{\theta_j}\delta^{(1,\theta_j)^{-1}}K)_h$. If we show that $\widehat{K}$ is a Schwartz function, then we can proceed similarly to the above: bound K by $p(x) = \frac{c}{(|x_1|+1)^2(|x_2|+1)^2}$, so that each $(\frac{1}{\theta_j}\delta^{(1,\theta_j)^{-1}}K)_h$ is bounded by $(\frac{1}{\theta_j}\delta^{(1,\theta_j)^{-1}}p)_h$, and invoke Theorem 3.6.3.

We need to show $\widehat{K}$ decays rapidly, as it is clearly in C^∞. By the definition of ω and ϕ_2 we only need to check the decay condition of $\partial^\alpha m(\xi_1 + \xi_2)(1 - \omega(\xi))$ far away from the origin when $|\xi \cdot \mathbf{1}| \geq \frac{c}{2}|\xi|$. Since $\partial^\beta m(\xi_1 + \xi_2) = (d^{|\beta|}m)(\xi_1 + \xi_2)$ is a one dimensional Schwartz function, we have

$$|(d^{|\beta|}m)(\xi_1 + \xi_2)| \leq \frac{C}{|\xi_1 + \xi_2|^n} \leq \frac{C}{(c/2)^n \cdot |\xi|^n}$$

for all n. To bound $\partial^\beta \omega$, we note that if $f \in C^\infty(\mathbb{R}^n)$ is homogeneous of degree m, i.e. $f(rx) = r^m f(x)$, then $\partial_{x_i} f$ is homogeneous of degree $m-1$. Indeed,

$$\begin{aligned} (\partial_{x_i} f)(rx) &= \lim_{h\to 0} \frac{f(rx + rhe_i) - f(rx)}{rh} \\ &= \lim_{h\to 0} r^{m-1} \frac{f(x + he_i) - f(x)}{h} \\ &= r^{m-1}\partial_{x_i} f(x). \end{aligned} \tag{4.1.2}$$

Since ω is homogeneous of degree 0, $\partial^\beta \omega$ is bounded by a constant away from the origin as it is homogeneous of degree $-|\beta|$.

Now it remains to bound $\sup_{h,j} III_{h,j}f(x)$. Define $IV_{h,j}$ by

$$(\widehat{IV_{h,j}f})(\xi) = m(hv_j \cdot \xi)\omega(h\xi_1, h\theta_j \xi_2)\widehat{f}(\xi).$$

Suppose we show that $\| \sup_{h,j} IV_{h,j}\|_{L^p\to L^p}$ is bounded. Then, for $\phi_{h,j} = (\delta^{(1,\theta_j)}\phi_1)_h$,

$$\begin{aligned} |(1-\phi_{h,j})IV_{h,j}f(x)| &\leq |IV_{h,j}f(x)| + |\phi_{h,j}IV_{h,j}f(x)| \\ &\leq |IV_{h,j}f(x)| + (|(\phi_{h,j})^\vee * |IV_{h,j}f|)(x) \\ &\leq |IV_{h,j}f(x)| + C|M_s(IV_{h,j}f)(x)| \end{aligned}$$

where we again bound $\phi_{h,j}$ by $(\delta^{(1,\theta-j)}p)_h$, each with the same L^1 norm. Since $M_s|f| \geq M_s|g|$ if $|f| \geq |g|$, we get

$$\sup_{h,j} |(1-\phi_{h,j})IV_{h,j}f(x)| \leq \sup_{h,j} |IV_{h,j}f(x)| + CM_s(\sup_{h,j} |IV_{h,j}f|)(x)$$

so that,

$$\begin{aligned} \|\sup_{h,j} III_{h,j}f\|_{L^p} &= \|\sup_{h,j}(1-\phi_{h,j})IV_{h,j}f\|_{L^p} \\ &\leq \|\sup_{h,j}|IV_{h,j}f|\|_{L^p} + C\|M_s(\sup_{h,j}|IV_{h,j}f|)\|_{L^p} \\ &\leq \|\sup_{h,j}|IV_{h,j}f|\|_{L^p} + C'\|\sup_{h,j}|IV_{h,j}f|\|_{L^p}. \end{aligned}$$

Hence, let us prove that $\sup_{h,j}|IV_{h,j}|$ is bounded. Let us denote

$$\widehat{S_j f}(\xi,\nu) = \omega(\xi_1, \theta_j\xi_2)\widehat{f}(\xi).$$

We note that

$$\sup_{h,j}|IV_{h,j}f| \leq \Big(\sum_{j=1}^{\infty} \sup_{h>0} |IV_{h,j}f(x)|^2\Big)^{1/2},$$

and since $\omega(h\xi) = \omega(\xi)$, by letting $g_j = S_j f$ and applying the assumption (4.1.1), we have

$$\left\|\sup_{h,j}|IV_{h,j}f|\right\|_{L^p} \leq C_p \left\|\Big(\sum |S_j f(x)|^2\Big)^{1/2}\right\|_{L^p}.$$

It suffices to show

$$\left\|\Big(\sum (S_j f)^2\Big)^{1/2}\right\|_{L^p} \leq A_p\|f\|_{L^p}, \qquad 1 < p < \infty. \tag{4.1.3}$$

By (4.1.2), we have for $\alpha = (\alpha_1, \alpha_2) \in \{(0,0),(0,1),(1,0),(1,1)\}$

$$\partial^\alpha \omega(\xi) \leq \frac{1}{|\xi|^{|\alpha|}}\partial^\alpha\omega\Big(\frac{\xi}{|\xi|}\Big) \leq \frac{A_\alpha}{|\xi_1|^{\alpha_1}|\xi_2|^{\alpha_2}}$$

for $A_\alpha = \sup_{\mathbb{S}^1}\partial^\alpha\omega$. Also, for each $\omega_j(\xi) = \omega(\xi_1, \delta_j\xi_2)$,

$$\partial^\alpha\omega_j(\xi) \leq \delta_j^{\alpha_2}(\partial^\alpha\omega)(\xi_1,\delta_j\xi_2) \leq \delta_j^{\alpha_2}\frac{A}{|\xi_1|^{\alpha_1}|\delta_j\xi_2|^{\alpha_2}} = \frac{A}{|\xi_1|^{\alpha_1}|\xi_2|^{\alpha_2}}.$$

Since ω_j have disjoint support away from 0, we have

$$\left|\partial^\alpha\Big(\sum_j \pm\omega_j(\xi)\Big)\right| \le \sup_j |(\partial^\alpha \omega_j)(\xi)| \le \frac{A}{|\xi_1|^{\alpha_1}|\xi_2|^{\alpha_2}} \tag{4.1.4}$$

for each $\alpha \in \{(0,0),(0,1),(1,0),(1,1)\}$. This condition satisfies (2.5.3) with $A = \max_\alpha A_\alpha$ so by the corollary to the Marcinkiewicz multiplier theorem 2.5.2, we have that $\sum_j \pm\omega_j$ is a bounded multiplier from $L^p(\mathbb{R}^2)$ to itself.

By the standard argument with Rademacher functions (Theorem 2.6.1), we get that

$$\begin{aligned}
\Big\|\Big(\sum |S_j f|^2\Big)^{1/2}\Big\|_{L^p} &\le A_p\left(\int_{\mathbb{R}^2}\left(\int_0^1 \Big|\sum r_j(t)S_j f\Big|^p dt\right)^{\frac{1}{p}p} dx\right)^{1/p} \\
&= A_p\left(\int_0^1\int_{\mathbb{R}^2}\Big|\sum r_j(t)S_j f\Big|^p dxdt\right)^{1/p} \\
&= A_p\left(\int_0^1 \Big\|\sum r_j(t)S_j f\Big\|^p_{L^p(\mathbb{R}^2)} dt\right)^{1/p} \\
&\le A_p\cdot C\left(\int_0^1 \|f\|^p_{L^p} dt\right)^{1/p} = A_p'\|f\|_{L^p}.
\end{aligned}$$

□

We need the following lemma, called the "boot strapping" argument, to extend the range of p where the assumption (4.1.1) holds.

Lemma 4.1.4. *If*

$$\|Nf\|_r \le A_r\|f\|_r \tag{4.1.5}$$

for every f in L^r, then (4.1.1) *holds for every p such that*

$$\frac{1}{2} < \frac{1}{p} < \frac{1}{2}\Big(1+\frac{1}{r}\Big).$$

Proof. We are interested in showing, for $q = 2$ and for p in the above range,

$$\Big\|\Big(\sum_j |\sup_h N_{h,j}g_j|^q\Big)^{1/q}\Big\|_{L^p} \le A_p\Big\|\Big(\sum_j |g_j|^q\Big)^{1/q}\Big\|_{L^p}. \tag{4.1.6}$$

If $p = q > 1$,

$$\begin{aligned}
\Big\|\Big(\sum_j |\sup_h N_{h,j}g_j|^q\Big)^{1/q}\Big\|^q_{L^q} &= \int\sum_j |\sup_h N_{h,j}g_j|^q dx \\
&\le \sum_j \int \Big|N_j g_j\Big|^q dx \\
&\le A_q\sum_j \|g_j\|^q_{L^q}
\end{aligned}$$

by the boundedness of maximal operator in one direction $N_j f = \sup_h N_{j,h} f$. But the above is equal to

$$A_q \sum_j \int |g_j|^q dx = A_q \Big\| \Big(\sum_j |g_j|^q \Big)^{1/q} \Big\|_{L^q}^q.$$

If $p = r$ and $q = \infty$, then (4.1.6) is just the hypothesis of the lemma with $g_j = g$ for all j.

Now the result follows from the vector-valued version of the Riesz-Thorin interpolation theorem (Theorem 2.3.5). To apply this theorem let $p_0 = q_0 = r, r_0 = s_0 = \infty$ and $p_1 = r_1 = q_1 = s_1 = \alpha$ with $1 < \alpha < 2$. We want to linearize the operator

$$(\sup_h N_{h,j})_j : L^r(l^\infty) \cap L^\alpha(l^\alpha) \to L^r(l^\infty) \cap L^\alpha(l^\alpha),$$

$$(f_j(x)) \mapsto (\sup_h N_{h,j} f_j(x)).$$

Let $H(x) \in \mathbb{R}^+$ be predetermined for each x and define a linear operator N_H via

$$(f_j(x)) \mapsto (N_{H(x),j} f_j(x)).$$

Choose q by

$$\frac{1}{q} = \frac{1}{2} + \frac{1 - \alpha/2}{r}$$

and observe that $\Big\|N_H((f_j))\Big\|_{L^q(l^2)} \geq \Big\|(\sum_j (\sup_h N_{h,j} f_j)^2)^{1/2}\Big\|_{L^q} - \epsilon$ for a certain choice of $H(\cdot)$ depending on f_j and ϵ. Hence, by letting $\theta = \alpha/2$ and using Theorem 2.3.5,

$$\begin{aligned}
\Big\| \sup_h (\sum_j (N_{h,j} f_j)^2)^{1/2} \Big\|_{L^q} &\leq \sup_H \|N_H((f_j))\|_{L^q(l^2)} \\
&\leq \sup_H \Big\|N_H\Big\|^\theta_{L^r(l^\infty) \to L^r(l^\infty)} \Big\|N_H\Big\|^{1-\theta}_{L^\alpha(l^\alpha) \to L^\alpha(l^\alpha)} \|(f_j)\|_{L^q(l^2)} \\
&\leq \Big\|(\sup_h N_{h,j})_j\Big\|^\theta_{L^r(l^\infty) \to L^r(l^\infty)} \Big\|(\sup_h N_{h,j})_j\Big\|^{1-\theta}_{L^\alpha(l^\alpha) \to L^\alpha(l^\alpha)} \|(f_j)\|_{L^q(l^2)} \\
&\leq C \|(f_j)\|_{L^q(l^2)}
\end{aligned}$$

as q ranges from $\frac{2}{1+1/r}$ to 2. □

Proof. (of Theorem 4.1.2) We start by using Lemma 4.1.3 to prove (4.1.5) for $p = 2$. Then, Lemma 4.1.4 shows that (4.1.1) is true for all $4/3 < p < 2$. Hence, (4.1.5) is true for all $4/3 < p < 2$. By repeating this argument, we can show that (4.1.5) is true for all $2 > p > 4/3, > 8/7, > 16/15 \dots$ and so on. Therefore, it is true for all $1 < p < 2$. By interpolating this with the trivial estimate for L^∞, we get (4.1.5) for all $p > 1$. □

4.2 Alfonseca, Soria and Vargas's Extension using Fourier method

The paper we discuss in this section is "Strong type inequalities and an almost-orthogonality principle for families of maximal operators along directions in $\mathbb{R}^n$" by Alfonseca [1], which uses the Fourier transform method to obtain strong type (p,p) almost-orthogonality. First we need some definitions.

Let $\Omega_0 = \{\theta_i\}$ be a finite or infinite, strictly decreasing subset of $[0, \pi/4)$ with $\theta_0 = \pi/4$. We refer to Ω_0 as the 'separating' set and θ_j as the 'separators'. For each $j \geq 1$, we have a set $\Omega_j \subset [\theta_j, \theta_{j-1})$, with $\theta_j \in \Omega_j$. The maximal operators associated to these sets are $M_{\Omega_0}, M_{\Omega_j}$ and $M_\Omega = M_{\cup_{j\geq 1}\Omega_j} = \sup_{j\geq 1} M_{\Omega_j}$.

We also introduce a certain square function associated with Ω_0. For each $j \geq 1$, set $\delta_j = |\theta_{j-1} - \theta_j|$. Let us consider the angular sectors

$$\Delta_j = \left\{(x,y) \in \mathbb{R}^2 : \theta_j - \frac{1}{20}\delta_j \leq \arctan\left(\frac{x}{-y}\right) < \theta_{j-1} + \frac{1}{20}\delta_j\right\}$$

and the wider sectors

$$\widetilde{\Delta_j} = \left\{(x,y) \in \mathbb{R}^2 : \theta_j - \frac{1}{10}\delta_j \leq \arctan\left(\frac{x}{-y}\right) < \theta_{j-1} + \frac{1}{10}\delta_j\right\}.$$

We define Schwartz functions ω_j similar to the previous section. Given $j \geq 1$, we pick a function ω_j homogeneous of degree zero, C^∞ away from the origin, identically equal to 1 in Δ_j and vanishing outside $\widetilde{\Delta_j}$.

As a prototypical example, we can define ω to be a homogeneous function of degree 0, C^∞ away from the origin with

$$\omega(\xi) = 1 \quad \text{if } |x| \leq (1/2 + 1/11)|y|$$

$$\omega(\xi) = 0 \quad \text{if } |x| \geq (1/2 + 1/10)|y|.$$

Check that $\tan((1/2 + 1/20)\theta) \leq (1/2 + 1/11)\theta$ for $0 < \theta < (1/2 + 1/10)\pi/4$ by convexity. We also have that $\theta \leq \tan\theta$. By denoting the θ degree rotation of f by $\rho_\theta f$, we can show $\omega_j = (\rho_{[(\theta_j+\theta_{j-1})/2]} \circ \delta^{(1,\delta_j)})\omega$ satisfies the above condition.

We define a multiplier operator S_j and square function S in the same way as in the previous section, namely

$$\widehat{(S_j f)} = \omega_j \widehat{f}$$

and

$$Sf(x) = \Big(\sum_{j\geq 1} |S_j f(x)|^2\Big)^{1/2}.$$

Then the following two results hold.

Theorem 4.2.1. *For $2 \leq p < \infty$, there exists a finite constant C_p such that*

$$\|M_\Omega f\|_{L^p} \leq C_p \Big[\|M_{\Omega_0} f\|_{L^p} + \Big(\sup_{j\geq 1} \|M_{\Omega_j}\|_{L^p \to L^p} \Big) \|Sf\|_{L^p} \Big]. \tag{4.2.1}$$

Theorem 4.2.2. *For $1 < p < 2$, there exists a finite constant C_p such that*

$$\|M_\Omega f\|_{L^p} \leq C_p \Big[\|M_{\Omega_0}\|_{L^p \to L^p} + \Big(\sup_{j\geq 1} \|M_{\Omega_j}\|_{L^p \to L^p} \Big) \|S\|_{L^p \to L^p}^{2/p} \Big] \|f\|_{L^p}. \tag{4.2.2}$$

Theorem 4.2.1 follows from the two lemmas below. Here, M_θ denotes M_Ω with $\Omega = \{\theta\}$.

Lemma 4.2.3. *For each $j \geq 1$ and for all $\theta \in \Omega_j$,*

$$M_\theta f(x) \leq C[M_{\theta_j} f(x) + M M_\theta (S_j f)(x)],$$

where M is the ordinary Hardy-Littlewood maximal operator.

Proof. As in the previous section, choose $\psi \in \mathcal{S}(\mathbb{R})$ with $m = \widehat{\psi}$. We modify the notation in (3.6.11) slightly and write $N_{h,j,\theta}$ where θ is in Ω_j. We have

$$M_\theta f(x) \quad \leq \quad C \sup_{h>0} N_{h,j,\theta} f(x).$$

Let $\phi \in \mathcal{S}(\mathbb{R}^2)$ be a radial C^∞ function with compact support, that is identically one in a neighbourhood of the origin. We have

$$\begin{aligned}
(\widehat{N}_{h,j,\theta} f)(\xi) &= \int_{\mathbb{R}^2} \Big(\int_{-\infty}^{\infty} \psi_h(t) f(x_1 - t\cos\theta, x_2 - t\sin\theta) dt \Big) e^{-x_1\xi_1 - x_2\xi_2} dx \\
&= \int_{-\infty}^{\infty} \int_{\mathbb{R}^2} \psi_h(t) f(x_1', x_2') e^{-x_1'\xi_1 - x_2'\xi_2 - t(\xi_1 \cos\theta + \xi_2 \sin\theta)} dx' dt \\
&= m(h\xi_1 \cos\theta + h\xi_2 \sin\theta) \widehat{f}(\xi),
\end{aligned}$$

and we break it up into

$$\begin{aligned}
& m(h\xi_1 \cos\theta + h\xi_2 \sin\theta) \phi(h\delta_j \xi) \widehat{f}(\xi) \\
+ \;& m(h\xi_1 \cos\theta + h\xi_2 \sin\theta)(1 - \phi(h\delta_j\xi))(1 - \omega_j(h\xi)) \widehat{f}(\xi) \\
+ \;& m(h\xi_1 \cos\theta + h\xi_2 \sin\theta)(1 - \phi(h\delta_j\xi)) \omega_j(h\xi) \widehat{f}(\xi) \\
= \;& \widehat{I_{h,j,\theta}(f)}(\xi) + I\widehat{I_{h,j,\theta}(f)}(\xi) + II\widehat{I_{h,j,\theta}(f)}(\xi).
\end{aligned}$$

For $I_{h,j,\theta}$, we use new coordinates which makes angle θ with (ξ_1, ξ_2), namely, $\nu = (\nu_1, \nu_2)$ with

$$\begin{aligned}
\nu_1 &= \xi_1 \cos\theta + \xi_2 \sin\theta \\
\nu_2 &= -\xi_1 \sin\theta + \xi_2 \cos\theta.
\end{aligned}$$

Since ϕ is radial, we need to bound

$$m(h\xi_1 \cos\theta + h\xi_2 \sin\theta)\phi(h\delta_j\xi) = m(h\nu_1)\phi(h\delta_j\nu).$$

Let us consider the Fourier transform $K_1 = K_1(z_1, z_2)$ of $m(\nu_1)\phi(\delta_j\nu)$. From 2.4.5 (2), we have that (z_1, z_2) also makes angle θ with (x_1, x_2). From Proposition 2.4.5 (5),

$$(m(h\nu_1)\phi(h\delta_j\nu))^\wedge = (K_1)_h.$$

By differentiating with respect to ν_1 and ν_2 and denoting

$$\frac{\partial^{a+b} f}{(\partial\nu_1)^a(\partial\nu_2)^b}$$

by $f_{(a,b)}$, we have

$$(m(\nu_1)\phi(\delta_j\nu))_{(a,b)} \quad = \quad \sum_{k=0}^{a} \binom{a}{k} m_{(k,0)}(\nu_1)(\delta_j)^{(a-k)+b}\phi_{(a-k,b)}(\delta_j\nu)$$

and hence, we get that K_1 satisfies

$$\begin{aligned}
|z_1|^a|z_2|^b|K_1(z)| &\leq \left\| (m(\nu_1)\phi(\delta_j\nu))_{(a,b)} \right\|_{L^1} \\
&= \sum_{k=0}^{a} (\delta_j)^{(a-k)+b} \binom{a}{k} \left\| m_{(k,0)}(\nu_1)\phi_{(a-k,b)}(\delta_j\nu) \right\|_{L^1}.
\end{aligned}$$

If we let $R > 0$ be such that $\mathrm{supp}\phi \subset [-R, R] \times [-R, R]$, the above is bounded by

$$\begin{aligned}
& \sum_{k=0}^{a} (\delta_j)^{(a-k)+b} A_{a,b,k} \left\| m_{(k,0)}(\nu_1)\chi_{[-R,R]\times[-R,R]}(\delta_j\nu) \right\|_{L^1(\mathbb{R}^2)} \\
= & \sum_{k=0}^{a} (\delta_j)^{(a-k)+b} A'_{a,b,k} \left\| m_{(k,0)}(\nu_1) \right\|_{L^1(\mathbb{R}\times[-\delta_j^{-1}R, \delta_j^{-1}R])} \\
= & \sum_{k=0}^{a} (\delta_j)^{(a-k)+b} A'_{a,b,k} \delta_j^{-1}(2R) \left\| m_{(k,0)}(\nu_1) \right\|_{L^1(\mathbb{R})} \\
= & \sum_{k=0}^{a} (\delta_j)^{(a-k)+b-1} A''_{a,b,k} \\
\leq & \ C_{a,b}\delta_j^{b-1} \quad (\delta_j \leq 1)
\end{aligned}$$

where the constants $A_{a,b,k}, A'_{a,b,k}$ and $A''_{a,b,k}$ only depend on a, b, k, ϕ and m. The last equality holds for all $\delta_j \leq 1$, for some constant $C_{a,b}$ which only depend on a, b, ϕ and m. Equivalently,

$$|z_1|^a|z_2/\delta_j|^b|\delta_j K_1(z)| \leq C_{a,b}.$$

By choosing a suitable C which only depends on ϕ and m,

$$(|z_1|^2 + |z_2/\delta_j|^2 + 1)^2|\delta_j K_1(z)| \leq C.$$

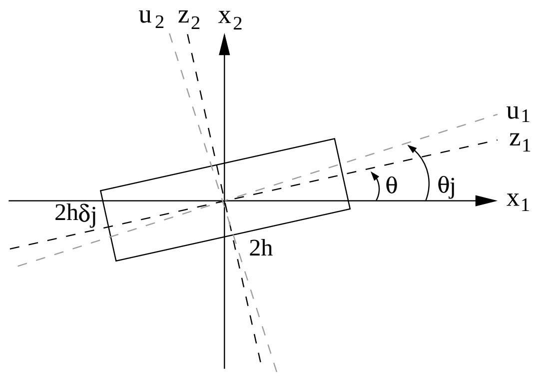

Figure 4.1: Rectangle of eccentricity δ_j in direction θ.

Now let $K'(z_1, z_2) = \delta_j K_1(z_1, \delta_j z_2)$ so that

$$|K'(z)| \leq \frac{C}{(|z|^2+1)^2} = p(z).$$

Hence using Theorem 3.6.1,

$$|f| * |(K')_h(z)| \leq |f| * |p_h(z)| \leq \|p\|_{L^1} Mf(z).$$

Recalling the notation (2.4.5) and (3.6.4) and noting

$$\begin{aligned} ((\delta^v f) * (\delta^v g))(x) &= \int (\delta^v f)(y)(\delta^v g)(-y+x)dy \\ &= \int f(vy)g(-vy+vx)\frac{d(vy)}{|v|} = \frac{1}{|v|}(f*g)(vx), \end{aligned}$$

we have for $v = (1, \delta_j)$,

$$\begin{aligned} |f| * |(K_1)_h(z)| &= \delta_j^{-1}|f| * (\delta^{v^{-1}} K')_h(z) \\ &= (\delta^v f) * (K')_h(v^{-1}z) \\ &\leq \|p\|_{L^1} M(\delta^v f)(v^{-1}z) \\ &= \|p\|_{L^1}(M_{v\mathcal{B}} f)(z). \end{aligned}$$

Since we can consider the Hardy-Littlewood maximal operator as having bases of squares, we have that the operator $(K_1)_h * f$ is bounded by the maximal operator over rectangles of eccentricity δ_j and sides parallel to the coordinate axes in (z_1, z_2) (see Figure 4.1.) Since $|\theta - \theta_j| \leq \delta_j$, we claim that such rectangles can be included in a rectangle with sides three times as large and parallel to θ_j (see Figure 4.2.) This can be easily checked: we view with respect to coordinate (u_1, u_2) and consider the smaller rectangle is rotated by angle $\theta - \theta_j$. Since $|\theta - \theta_j| < \delta_j$, the u_2-coordinate of the vertex of the smaller rectangle rises by Δ which is no more than $\delta_j\sqrt{1+\delta_j^2} < 2\delta_j$. Hence, it stays within the larger rectangle. Thus, we have shown that

$$|\sup_{h>0} I_{h,j,\theta} f(x)| \leq CM_{\theta_j} f(x). \tag{4.2.3}$$

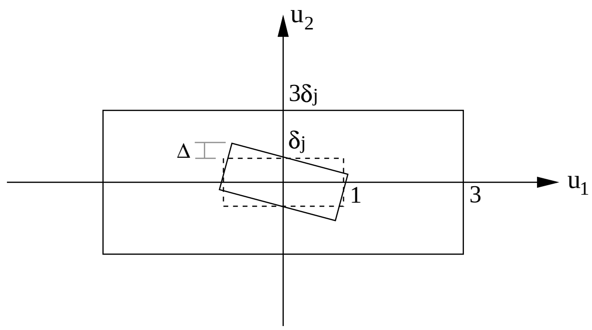

Figure 4.2: Rectangle in direction θ_j is included in a rectangle in θ of comparable size.

For the second term, consider the Fourier transform K_2 of $m(\nu_1)(1-\phi(\delta_j\nu))(1-\omega_j(\nu))$. We already know that

$$(1-\phi(\delta_j\nu))_{(a,b)} = \delta_j^{a+b}(1-\phi)_{(a,b)}(\delta_j\nu). \tag{4.2.4}$$

Also we have that, for any $R>0$,

$$\sup_{|\nu|>R/\delta_j} |(1-\omega_j(\nu_1,\nu_2))_{(m,n)}| \leq B\delta_j^n \tag{4.2.5}$$

for some B independent of δ_j. Let us verify this for our model ω_j. It suffices to show

$$\sup_{|\nu|>R/\delta_j} |(\omega_j(\nu_1,\nu_2))_{(m,n)}| \leq B\delta_j^n.$$

We start with the fact that

$$\sup_{|\nu|>R} |(\omega(\nu_1,\nu_2))_{(m,n)}| \leq B.$$

This implies that

$$\sup_{|\nu|>R/\delta_j} |(\delta^{(1,\delta_j)}\omega(\nu_1,\nu_2))_{(m,n)}| \quad \leq \quad \sup_{|\delta^{(1,\delta_j)}\nu|>R} |(\delta^{(1,\delta_j)}\omega(\nu_1,\nu_2))_{(m,n)}| \leq B\delta_j^n. \tag{4.2.6}$$

Recall that ω_j is

$$\delta^{(1,\delta_j)}\omega(\nu_1\cos\theta - \nu_2\sin\theta, \nu_1\sin\theta + \nu_2\cos\theta)$$

for some $|\theta| \leq \delta_j/2$. If $\theta = 0$, then the claim is just (4.2.6). If not, for $\nu_1' = \nu_1\cos\theta - \nu_2\sin\theta$ and $\nu_2' = \nu_1\sin\theta + \nu_2\cos\theta$,

$$\begin{aligned}
\sup_{|\nu|>R/\delta_j} |(\omega_j(\nu_1,\nu_2))_{(m,n)}| \quad &\leq \quad \sum_{\substack{a+b=m\\ c+d=n}} \sup_{|\nu'|>R/\delta_j} |(\delta^{(1,\delta_j)}\omega(\nu_1',\nu_2'))_{a+c,b+d}||\cos\theta|^{a+d}|\sin\theta|^{b+c} \\
&\leq \quad \sum_{\substack{a+b=m\\ c+d=n}} |B\delta_j^{b+d}||\delta_j|^{b+c} \\
&\leq \quad \sum_{\substack{a+b=m\\ c+d=n}} |B\delta_j^{2b+n}| \leq B'\delta_j^n \qquad (\delta_j \leq 1)
\end{aligned}$$

where we used that $|\cos\theta| \leq 1$ and $|\sin\theta| \leq |\theta| \leq \delta_j$. Hence we have obtained (4.2.5) for our ω_j.

Using (4.2.4) and (4.2.5) with R such that ϕ is identically 1 on $B(0, R)$,

$$
\begin{aligned}
& |z_1|^a |z_2|^b |K_2(z)| \\
\leq\ & \|(m(\nu_1)(1-\phi(\delta_j\nu))(1-\omega_j(\nu)))_{(a,b)}\|_{L^1} \\
\leq\ & \Big\| \sum_{k_1,k_2,k_3} \binom{a}{k_1}\binom{a-k_1}{k_2}\binom{b}{k_3} (m(\nu_1))_{(k_1,0)}(1-\phi(\delta_j\nu))_{(k_2,k_3)}(1-\omega_j(\nu))_{(a-(k_1+k_2),b-k_3)} \Big\|_{L^1} \\
\leq\ & \sum_{k_1,k_2,k_3} C_{a,b,k_1,k_2,k_3} \delta_j^{k_2+k_3} \|m(\nu_1)_{(k_1,0)}(1-\phi)_{(k_2,k_3)}(\delta_j\nu)(1-\omega_j)_{(a-(k_1+k_2),b-k_3)}(\nu)\|_{L^1} \\
\leq\ & C_{a,b,\phi,\omega} \sum_{k_1,k_2,k_3} \delta_j^{k_2+k_3} \|m_{(k_1,0)} B \delta_j^{b-k_3} \chi_{\Delta'_j}(\nu)\|_{L^1} \\
\leq\ & C'_{a,b,\phi,\omega} \sum_{k_1,k_2,k_3} \delta_j^{k_2+b} \|m_{(k_1,0)} \chi_{\Delta'_j}(\nu)\|_{L^1} \qquad (4.2.7)
\end{aligned}
$$

where the support of $(1-\phi)_{(k_2,k_3)}(\delta_j\nu)(1-\omega_j)_{(a-(k_1+k_2),b-k_3)}(\nu)$ is contained in the sector $\Delta'_j = \{(\nu_1,\nu_2) : |\nu_1| \geq \frac{1}{20}\delta_j|\nu_2|\}$. But,

$$
\begin{aligned}
\|m_{(k_1,0)}\chi_{\Delta'_j}(\nu)\|_{L^1} &\leq 4\int_0^\infty \int_{\delta_j\nu_2/20}^\infty |m(\nu_1)| d\nu_1 d\nu_2 \\
&= 80\delta_j^{-1} \int_0^\infty \int_{\nu_2}^\infty |m(\nu_1)| d\nu_1 d\nu_2
\end{aligned}
$$

where the last double integral is some fixed finite value since m is a Schwartz function. Hence (4.2.7) is bounded by a constant multiple of

$$\sum_{k_1,k_2,k_3} \delta^{k_2+b} C_{m,\phi,\omega} \delta_j^{-1} \leq C_{m,\phi,\omega} \delta_j^{b-1} \qquad (\delta_j \leq 1).$$

Thus,

$$\Big| \sup_{h>0} II_{h,j,\theta} f(x) \Big| \leq C M_{\theta_j} f(x). \qquad (4.2.8)$$

as above.

For the third term, we apply the multipliers ω_j, m and $(1-\phi)$ in succession. Multiplier m is bounded by M_θ by definition of one dimensional Hardy-Littlewood maximal operator. Multiplier ϕ is dominated by the classical Hardy-Littlewood maximal operator by Corollary 3.6.2. Hence,

$$
\begin{aligned}
\big|((1-\phi)m\omega_j f)(x)\big| &\leq \big|(m(S_j f)^\wedge)^\vee(x)\big| + \big|(\phi m(\omega_j f)^\wedge)^\vee(x)\big| \\
&\leq 2M(m(S_j f)^\wedge)^\vee(x) \\
&\leq 2MM_\theta S_j(f)(x).
\end{aligned}
$$

$\square$

Lemma 4.2.4. *Assume that for some $p > 1, q \geq 2$ and for any sequence of functions $\{f_j\}$, one has*

$$\Big\| \Big(\sum_{j=1}^{\infty} |M_{\Omega_j} f_j|^q \Big)^{1/q} \Big\|_{L^p} \leq B \Big\| \Big(\sum_{j=1}^{\infty} |f_j|^q \Big)^{1/q} \Big\|_{L^p}.$$

Then

$$\|M_\Omega f\|_{L^p} \leq C_p \Big[\|M_{\Omega_0} f\|_{L^p} + B\|Sf\|_{L^p} \Big]$$

for some constant C_p.

Proof. From the pointwise estimate in Lemma 4.2.3, we get

$$M_\Omega f(x) = \sup_{j\geq 1, \theta \in \Omega_j} M_\theta f(x) \leq C \Big[M_{\Omega_0} f(x) + \sup_{j \geq 1, \theta \in \Omega_j} M M_\theta (S_j f)(x) \Big].$$

Notice that

$$\sup_{j\geq 1, \theta\in\Omega_j} M_\theta(S_j f) \leq \left(\sum_{j=1}^{\infty} \sup_{\theta \in \Omega_j} (M_\theta(S_j f))^q \right)^{1/q} = \left(\sum_{j=1}^{\infty} (M_{\Omega_j}(S_j f))^q \right)^{1/q}.$$

By the hypothesis of the lemma,

$$\begin{aligned} \Big\| \sup_{j\geq 1, \theta\in\Omega_j} M M_\theta(S_j f) \Big\|_{L^p} &\leq \Big\| M \Big(\sup_{j\geq 1, \theta\in\Omega_j} M_\theta(S_j f) \Big) \Big\|_{L^p} \\ &\leq A \Big\| \sup_{j\geq 1, \theta\in\Omega_j} M_\theta(S_j f) \Big\|_{L^p} \\ &\leq B \Big\| \Big(\sum_{j=1}^{\infty} |S_j f|^q \Big)^{1/q} \Big\|_{L^p} \\ &\leq B \Big\| \Big(\sum_{j=1}^{\infty} |S_j f|^2 \Big)^{1/2} \Big\|_{L^p} \\ &= B\|Sf\|_{L^p} \end{aligned}$$

where we simply use the fact that $q \geq 2$ to obtain the second inequality. $\square$

Proof. (of Theorem 4.2.1) This follows from Lemma 4.2.4 using $B = \sup \|M_{\Omega_j}\|_{L^p \to L^p}$ and $p = q$ as

$$\int \sum |M_{\Omega_j} f_j|^p = \sum \int |M_{\Omega_j} f_j|^p \leq \sum B^p \int |f_j|^p = B^p \int \sum |f_j|^p.$$

$\square$

Proof. (of Theorem 4.2.2) For technical reasons, we prove (4.2.2) for for an adjust the square function $\widetilde{S}$ associated with $\widetilde{\omega_j}$ where $\widetilde{\omega_j}$ is identically equal to 1 in $\widetilde{\Delta_j}$ and vanishing outside a slightly wider sector.

From the pointwise estimate (4.2.3) and (4.2.8),

$$\sup_{h,j\geq 1,\theta\in\Omega_j}(I_{h,j,\theta}(f)+II_{h,j,\theta}(f))\leq AM_{\Omega_0}f.$$

for some constant A.

From the observation (3.6.13), we assume that Ω is countable. In addition, suppose we prove that (4.2.2) is true for any finite set Ω, with C_p independent of Ω. For any countable set Ω, we let

$$\Lambda^1\subset\cdots\subset\Lambda^n\subset\cdots\subset\Omega$$

where $|\Lambda^n|=n$. Also let $\Lambda_j^n=\Omega_j\cap\Lambda^n$ for $j\geq 1$. By Monotone convergence theorem, we have

$$\begin{aligned}\|M_\Omega f\|_{L^p} &= \sup_n\|M_{\Lambda^n}f\|_{L^p}\\ &\leq \sup_n C_p\Big[\|M_{\Omega_0}\|_{L^p\to L^p}+\big(\sup_{j\geq 1}\|M_{\Lambda_j^n}\|_{L^p\to L^p}\big)\|S\|_{L^p\to L^p}^{2/p}\Big]\|f\|_p \qquad (4.2.9)\\ &\leq C_p\Big[\|M_{\Omega_0}\|_{L^p\to L^p}+\big(\sup_{j\geq 1}\|M_{\Omega_j}\|_{L^p\to L^p}\big)\|S\|_{L^p\to L^p}^{2/p}\Big]\|f\|_p.\end{aligned}$$

Hence, we can assume Ω is finite. Then there is a minimal constant $C(\Omega)$ such that for all f we have

$$\Big\|\sup_{h,j\geq 1,\theta\in\Omega_j}|III_{h,j,\theta}(f)|\Big\|_{L^p}\leq C(\Omega)\|f\|_{L^p}.$$

Let us take a sequence of functions $\{g_j\}$. Then,

$$\begin{aligned}\sup_{h,j\geq 1,\theta\in\Omega_j}|III_{h,j,\theta}(g_j)| &\leq \sup_{h,j\geq 1,\theta\in\Omega_j}|N_{h,j,\theta}(g_j)|+\sup_{h,j\geq 1,\theta\in\Omega_j}|I_{h,j,\theta}(g_j)+II_{h,j,\theta}(g_j)|\\ &\leq \sup_{h,j\geq 1,\theta\in\Omega_j}\Big|N_{h,j,\theta}\Big(\sup_{j\geq 1}|g_j|\Big)\Big|+AM_{\Omega_0}\Big(\sup_{j\geq 1}|g_j|\Big)\\ &\leq \sup_{h,j\geq 1,\theta\in\Omega_j}\Big|III_{h,j,\theta}\Big(\sup_{j\geq 1}|g_j|\Big)\Big|+2AM_{\Omega_0}\Big(\sup_{j\geq 1}|g_j|\Big).\end{aligned}$$

Therefore,

$$\Big\|\sup_{h,j\geq 1,\theta\in\Omega_j}|III_{h,j,\theta}(g_j)|\Big\|_{L^p}\leq\big(C(\Omega)+2A\|M_{\Omega_0}\|_{L^p\to L^p}\big)\cdot\Big\|\sup_{j\geq 1}|g_j|\Big\|_{L^p}.$$

On the other hand, we have

$$\begin{aligned}\Big\|\Big(\sum_{j\geq 1}\sup_{h,\theta\in\Omega_j}|III_{h,j,\theta}(g_j)|^p\Big)^{1/p}\Big\|_{L^p} &= \Big(\sum_{j\geq 1}\int\sup_{h,\theta\in\Omega_j}|III_{h,j,\theta}(g_j)|^p\Big)^{1/p}\\ &\leq \Big(\sum_{j\geq 1}\Big\|\sup_{h,\theta\in\Omega_j}III_{h,j,\theta}\Big\|_{L^p\to L^p}^p\int|g_j|^p\Big)^{1/p}\\ &\leq \Big(\sup_{j\geq 1}\Big\|\sup_{h,\theta\in\Omega_j}|III_{h,j,\theta}|\Big\|_{L^p\to L^p}\Big)\Big\|\Big(\sum_{j\geq 1}|g_j|^p\Big)^{1/p}\Big\|_{L^p}.\end{aligned}$$

As in the proof of Lemma 4.1.4, we need to linearize the operator

$$\sup_{h,\theta} III : L^p(\mathbb{R}^2, l^r) \to L^p(\mathbb{R}^2, l^r)$$

in order to apply Theorem 2.3.5. Let $H(x,j) \in \mathbb{R}_+$ and $\Theta(x,j) \in \Omega_j$ be predetermined for each x, j. Then

$$III_{H,\Theta} : (g_j(x)) \mapsto (III_{H(x,j),\Theta(x,j)} g_j)(x)$$

is a linear operator which satisfies

$$\Big\| \Big(\sum_{j\geq 1} \sup_{h,\theta\in\Omega_j} |III_{h,j,\theta}(g_j)|^2 \Big)^{1/2} \Big\|_{L^p} - \epsilon \leq \Big\| III_{H,\Theta}(g_j) \Big\|_{L^p(\mathbb{R}^2,l^2)}$$

for some particular choice of H and Θ depending on g_j and ϵ. By Theorem 2.3.5 with $\theta = p/2$,

$$\begin{aligned}
\Big\| III_{H,\Theta}(g_j) \Big\|_{L^p(l^2)} &\leq \Big\| III_{H,\Theta} \Big\|_{L^p(l^2)\to L^p(l^2)} \Big\| \Big(\sum_{j\geq 1} |g_j|^2 \Big)^{1/2} \Big\|_{L^p} \\
&\leq \Big\| III_{H,\Theta} \Big\|^{1-p/2}_{L^p(l^\infty)\to L^p(l^\infty)} \Big\| III_{H,\Theta} \Big\|^{p/2}_{L^p(l^p)\to L^p(l^p)} \Big\| \Big(\sum_{j\geq 1} |g_j|^2 \Big)^{1/2} \Big\|_{L^p} \\
&\leq (C(\Omega) + 2A\|M_{\Omega_0}\|_{L^p\to L^p})^{1-p/2} \\
&\quad \times \Big(\sup_{j\geq 1} \Big\| \sup_{h,\theta\in\Omega_j} |III_{h,j,\theta}| \Big\|_{L^p\to L^p} \Big)^{p/2} \Big\| \Big(\sum_{j\geq 1} |g_j|^2 \Big)^{1/2} \Big\|_{L^p}.
\end{aligned}$$

Together with the fact $III_{h,j,\theta}(1 - \widetilde{S}_j) = 0$, the above implies that

$$\begin{aligned}
&\Big\| \sup_{h,j\geq 1,\theta\in\Omega_j} |III_{h,j,\theta}(f)| \Big\|_{L^p} \\
\leq\ & \Big\| \sup_{h,j\geq 1,\theta\in\Omega_j} |III_{h,j,\theta}(\widetilde{S}_j f)| \Big\|_{L^p} \\
\leq\ & \Big\| \Big(\sum_{j\geq 1} \sup_{h,\theta\in\Omega_j} |III_{h,j,\theta}(\widetilde{S}_j f)|^2 \Big)^{1/2} \Big\|_{L^p} \\
\leq\ & (C(\Omega) + 2A\|M_{\Omega_0}\|_{L^p\to L^p})^{1-p/2} \Big(\sup_{j\geq 1} \Big\| \sup_{h,\theta\in\Omega_j} |III_{h,j,\theta}| \Big\|_{L^p\to L^p} \Big)^{p/2} \|\widetilde{S} f\|_{L^p}.
\end{aligned}$$

Using the minimality of $C(\Omega)$ and the fact that $a^\lambda b^{1-\lambda} \leq \lambda a + (1-\lambda) b$ for $0 < \lambda < 1$,

$$\begin{aligned}
C(\Omega) &\leq (C(\Omega) + 2A\|M_{\Omega_0}\|_{L^p\to L^p})^{1-p/2} \Big(\sup_{j\geq 1} \Big\| \sup_{h,\theta\in\Omega_j} |III_{h,j,\theta}| \Big\|_{L^p\to L^p} \Big)^{p/2} \|\widetilde{S}\|_{L^p\to L^p} \\
&\leq \Big(1 - \frac{p}{2}\Big)(C(\Omega) + 2A\|M_{\Omega_0}\|_{L^p\to L^p}) + \frac{p}{2} \sup_{j\geq 1} \Big\| \sup_{h,\theta\in\Omega_j} |III_{h,j,\theta}| \Big\|_{L^p\to L^p} \|\widetilde{S}\|^{2/p}_{L^p\to L^p}.
\end{aligned}$$

By isolating $C(\Omega)$ above, we get

$$C(\Omega) \leq 2A\Big(\frac{2}{p}-1\Big)\|M_{\Omega_0}\|_{L^p\to L^p} + \sup_{j\geq 1}\Big\|\sup_{h,\theta\in\Omega_j}|III_{h,j,\theta}|\Big\|_{L^p\to L^p}\|\widetilde{S}\|^{2/p}_{L^p\to L^p}$$

and the theorem follows from observing

$$\begin{aligned}
&\Big\|\sup_{h,\theta\in\Omega_j}|III_{h,j,\theta}f|\Big\|_{L^p}\\
\leq\ &\Big\|\sup_{h,\theta\in\Omega_j}|N_{h,j,\theta}|\Big\|_{L^p\to L^p}\|f\|_{L^p} + \Big\|\sup_{h,\theta\in\Omega_j}|I_{h,j,\theta}f| + \sup_{h,\theta\in\Omega_j}|II_{h,j,\theta}f|\Big\|_{L^p}\\
\leq\ &\Big(\Big\|\sup_{h,\theta\in\Omega_j}|N_{h,j,\theta}|\Big\|_{L^p\to L^p} + C\|M_{\theta_j}\|_{L^p\to L^p}\Big)\|f\|_{L^p}\\
\leq\ &C\Big\|\sup_{h,\theta\in\Omega_j}|N_{h,j,\theta}|\Big\|_{L^p\to L^p}\|f\|_{L^p}.
\end{aligned}$$

□

Corollary 4.2.5. *Let Ω_0 be a lacunary sequence. Then*

$$\|M_\Omega f\|_p \leq C_p\Big(\sup_{j\geq 1}\|M_{\Omega_j}\|_{L^p\to L^p}\Big)\|f\|_p, \qquad 1<p<\infty.$$

Proof. Without loss of generality, we can assume that $1/2<\lambda<1$. We can also assume that $\theta_{j+1}\geq\lambda^2\theta_j$ for all j. This can be achieved by adding some terms to the initial sequence. This ensures that the sectors $\widetilde{\Delta}_j$ and $\widetilde{\Delta}_{j+k}$ are disjoint for all j for some k depending on λ. As a consequence of the Littlewood-Paley theory, we get the boundedness for S similarly to the proof of Lemma 4.1.4. The rest follows directly by applying the result of Theorem 4.1.2 to Theorem 4.2.1 and 4.2.2. □

Note that Corollary 4.2.5 implies that any finite sum of lacunary directions will give bounded maximal operator. This is not true for an "infinite sum" of lacunary directions. This type of result is proved originally on Cantor sets by Katz [18]. It was later extended by Hare to general Cantor type sets [15].

4.3 Alfonseca, Soria and Vargas's Geometric argument

We discuss two papers by Alfonseca, Soria and Vargas [2, 3]. They share the main geometric argument as presented in 4.3.2. The first paper uses covering lemma similar to the proof of the classical Hardy-Littlewood maximal operator to obtain weak type $(2,2)$ result, while the second paper checks an equivalent condition of strong type $(2,2)$ boundedness due to Carbery (Theorem 4.3.9.)

4.3.1 Weak-type $(2, 2)$ estimate

This section follows the paper "A remark on maximal operators along directions in $\mathbb{R}^2$" by Alfonseca, Soria, and Vargas [3]. We use the same terminology as the previous section and obtain the following weak-type (2,2) almost-orthogonality:

Theorem 4.3.1. *There exist constants C_1 and C_2 independent of the set Ω, such that*

$$\|M_\Omega\|^2_{L^2\to L^{2,\infty}} \le C_1 \sup_{j\ge 1} \|M_{\Omega_j}\|^2_{L^2\to L^{2,\infty}} + C_2\|M_{\Omega_0}\|^2_{L^2\to L^{2,\infty}}.$$

As a corollary, we get the result by Katz [19] which solved a conjecture that was open for many years: if Ω has cardinality $N > 1$, then

$$\|M_\Omega\|_{L^2\to L^{2,\infty}} \le C(\log N)^\alpha,$$

for some constants C and α independent of Ω. Note that this method does not give the sharp exponent $\alpha = 1/2$ as given by Katz's original proof. Another direct consequence is that if Ω_0 is a lacunary sequence, then

$$\| \sup M_{\Omega_j}\|_{L^2\to L^{2,\infty}} \le C \sup \|M_{\Omega_j}\|_{L^2\to L^{2,\infty}}.$$

Proof. In view of (3.6.10), we consider the maximal operators

$$M^P_\Omega f(x) = \sup_{x\in R\in \mathcal{B}^P_\Omega} \frac{1}{|B|}\int_R |f|dy.$$

Given $S \subset \mathbb{R}^2$, $P_1(S)$ and $P_2(S)$ will denote the projection of R onto the x-axis and y-axis, respectively. For $R \in \mathcal{B}^P_\Omega$ with $P_1(R) = [a^1_R, a^2_R]$, we also define $P_{2,1}(R) = \{y : (a^1_R, y) \in R\}$ and $P_{2,2}(R) = \{y : (a^2_R, y) \in R\}$ to be the projections of the left and right sides of R onto the y-axis, respectively. Note that $|P_{2,1}(R)| = |P_{2,2}(R)|$ and $|R| = |P_1(R)| \cdot |P_{2,1}(R)|$.

Since $|\{M_\Omega f(x) > \lambda\}| = |\{\cup_n M_{\Omega_n} f(x) > \lambda\}| = \lim_n |\{M_{\Omega_n} f(x) > \lambda\}|$ for $\Omega_{n-1} \subset \Omega_n \subset \Omega$ with $|\Omega_n| = n$, it suffices to consider the case when Ω is finite, similarly to 4.2.9. Furthermore, when Ω is finite, we can assume without loss of generality that Ω_0 is finite as well.

If $x \in \{M^P_\Omega f(x) > \lambda\}$, there is $R_x \in \mathcal{B}^P_\Omega$ containing x such that

$$\frac{1}{|R_x|}\int_{R_x} |f(y)|dy > \lambda, \tag{4.3.1}$$

and therefore

$$\{M_\Omega f(x) > \lambda\} \subset \bigcup_{x\in\{M_\Omega f(x)>\lambda\}} R_x.$$

Hence, if we consider R_x without their boundaries momentarily so that they are open, then for any compact set $K \subset \{M_\Omega f(x) > \lambda\}$, $K \subset \bigcup_{j=1}^{s} R_{x_j}$ for some finite family of parallelograms $\mathcal{F} = \{R_{x_j}\}_{j=1}^{s}$ satisfying (4.3.1). From the family $\mathcal{F}$ we select a subfamily $\overline{\mathcal{F}} = \{B_k\}$ in the following way: we take B_1 to have the longest projection on the x-axis. Assuming we have already chosen $B_1, \ldots, B_{n-1}$, we take B_n from the remaining collection $\mathcal{F} \setminus \{B_k\}_{j=1}^{n-1}$ such that $|P_1(R)|$ is maximal among the parallelograms satisfying

$$\sum_{k=1}^{n-1} |R \cap B_k| \leq \frac{1}{2}|R|. \tag{4.3.2}$$

This process will terminate after N steps for some finite N. Since (4.3.2) implies

$$\left| R \setminus \left(\bigcup_{k=1}^{n-1} B_k \right) \right| \geq \frac{1}{2}|R|,$$

we sum it over all k to get

$$\sum |B_k| \leq 2|\cup B_k|.$$

In addition,

$$\begin{aligned} \int \left(\sum \chi_{B_k} \right)^2 &= \int \sum \chi_{B_k} + 2 \sum_{i=2}^{N} \sum_{j=1}^{i-1} \chi_{B_i \cap B_j} \\ &\leq \sum |B_k| + 2 \sum_{i=2}^{N} \frac{1}{2}|B_i| \\ &\leq 2 \sum |B_k|. \end{aligned}$$

In order to establish the weak type $(2,2)$ norm of M_Ω^P, we observe that

$$\sum |B_k| \leq \frac{1}{\lambda} \sum \int_{B_k} |f| \leq \frac{1}{\lambda} \|f\|_{L^2} \left\| \sum \chi_{B_k} \right\|_{L^2} \leq \frac{\sqrt{2}}{\lambda} \|f\|_{L^2} \left(\sum |B_k| \right)^{1/2}.$$

Hence,

$$\left(\sum |B_k| \right)^{1/2} \leq \frac{\sqrt{2}}{\lambda} \|f\|_{L^2}.$$

Thus, if we show that

$$\left| \bigcup_{R \in \mathcal{F} \setminus \overline{\mathcal{F}}} R \right| \leq c_0 \sum |B_k|, \tag{4.3.3}$$

for some $c_0 \geq 1$, then

$$\begin{aligned} |K| &\leq \left| \bigcup B_k \right| + \left| \bigcup R_{x_j} \setminus \bigcup B_k \right| \\ &\leq \left| \bigcup B_k \right| + \left| \bigcup_{R \in \mathcal{F} \setminus \overline{\mathcal{F}}} R \right| \\ &\leq (1 + c_0) \left(\sum |B_k| \right) \leq \frac{2(1 + c_0)}{\lambda^2} \|f\|_{L^2}^2, \end{aligned}$$

which proves the theorem with $\|M_\Omega\|_{L^2}^2 = 4c_0$.

It remains to show (4.3.3). Let R be one of the parallelograms in $\mathcal{F} \setminus \overline{\mathcal{F}}$. Then, we claim

$$\sum_{\substack{B_k:\\ |P_1(B_k)|\geq|P_1(R)|}} |R \cap B_k| > \frac{1}{2}|R|.$$

Suppose otherwise. From the way B_k are selected, $|P_1(B_1)| \geq |P_1(B_2)| \geq \cdots \geq |P_1(B_N)|$. Hence, if k_0 is the largest such that $|P_1(B_{k_0})| \geq |P_1(R)|$, then R must be chosen as B_{k_0+1} which is a contradiction.

For notational convenience, we will write $\mathcal{B}_l$ instead of $\mathcal{B}_{\Omega_l}$. For $R \in \mathcal{B}_l$, we have either

$$\sum_{\substack{B_k\in\mathcal{B}_l:\\ |P_1(B_k)|\geq|P_1(R)|}} \frac{|R \cap B_k|}{|R|} > \frac{1}{4},$$

or

$$\sum_{\substack{B_k\notin\mathcal{B}_l:\\ |P_1(B_k)|\geq|P_1(R)|}} \frac{|R \cap B_k|}{|R|} > \frac{1}{4}.$$

Let us denote,

$$\mathcal{F}_1 = \bigcup_l \Big\{ R \in (\mathcal{F} \setminus \overline{\mathcal{F}}) \cap \mathcal{B}_l : \sum_{\substack{B_k\in\mathcal{B}_l:\\ |P_1(B_k)|\geq|P_1(R)|}} \frac{|R \cap B_k|}{|R|} > \frac{1}{4} \Big\}$$

and

$$\mathcal{F}_2 = \bigcup_l \Big\{ R \in (\mathcal{F} \setminus \overline{\mathcal{F}}) \setminus \mathcal{B}_l : \sum_{\substack{B_k\in\mathcal{B}_l:\\ |P_1(B_k)|\geq|P_1(R)|}} \frac{|R \cap B_k|}{|R|} > \frac{1}{4} \Big\}.$$

If $R \in \mathcal{F}_1$, $R \subset \{x : M_{\Omega_l}(\sum_{B_k\in\mathcal{B}_l} \chi_{B_k}) > 1/4\}$. Hence,

$$\begin{aligned}
\Big|\bigcup_{\mathcal{F}_1} R\Big| &\leq \sum_l \Big|\big\{M_{\Omega_l}\big(\sum_{B_k\in\mathcal{B}_l} \chi_{B_k}\big) > 1/4\big\}\Big| \\
&\leq 16 \sum_l \Big(\|M_{\Omega_l}\|_{L^2\to L^{2,\infty}}^2 \cdot \Big\|\sum_{B_k\in\mathcal{B}_l} \chi_{B_k}\Big\|_{L^2}^2\Big) \\
&\leq 16 \sup_l \|M_{\Omega_l}\|_{L^2\to L^{2,\infty}}^2 \cdot \Big\|\sum_{B_k\in\overline{\mathcal{F}}} \chi_{B_k}\Big\|_{L^2}^2 \qquad (4.3.4)\\
&\leq 32 \sup_l \|M_{\Omega_l}\|_{L^2\to L^{2,\infty}}^2 \Big(\sum |B_k|\Big).
\end{aligned}$$

To estimate $|\cup_{\mathcal{F}_2} R|$, we need the following lemma whose proof we defer until the end of the section.

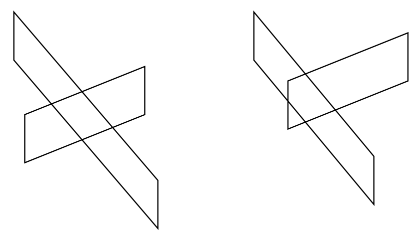

Figure 4.3: The left pair of parallelograms cross entirely. The right pair do not.

Lemma 4.3.2. *There is a constant c with the property that for any $R \in \mathcal{B}_l$, there is $\widehat{R} \in \mathcal{B}_0$ such that*

$$\frac{|B \cap R|}{|R|} \leq c \frac{|B \cap \widehat{R}|}{|\widehat{R}|} \tag{4.3.5}$$

for all $B \in \mathcal{B}_k$ with $k \neq l$ and $|P_1(B)| \geq |P_1(R)|$.

Using the lemma, we have that for any $R \in \mathcal{F}_2 \cap \mathcal{B}_l$, there is $\widehat{R} \in \mathcal{B}_0$ such that (4.3.5) holds for all $B = B_k \notin \mathcal{B}_l$ with $|P_1(B_k)| \geq |P_1(R)|$. Hence, by the definition of $\mathcal{F}_2$

$$\begin{aligned}
\Big|\bigcup_{\mathcal{F}_2} R\Big| &\leq \Big|\{M_{\Omega_0}\Big(\sum \chi_{B_k}\Big) > \frac{1}{4c}\}\Big| \\
&\leq 16c^2 \|M_{\Omega_0}\|^2_{L^2 \to L^{2,\infty}} \cdot \Big\|\sum \chi_{B_k}\Big\|^2_{L^2} \qquad (4.3.6)\\
&\leq 32c^2 \|M_{\Omega_0}\|^2_{L^2 \to L^{2,\infty}} \cdot \Big(\sum |B_k|\Big).
\end{aligned}$$

Combining (4.3.4) and (4.3.6), we get (4.3.3) with $c_0 = 32 \sup \|M_{\Omega_l}\|^2 + 32c^2 \|M_{\Omega_0}\|^2$. □

Before proving Lemma 4.3.2, we give the following terminology and prove two geometric properties. Given $U, V \in \mathcal{B}_\Omega$, we say that U crosses V entirely if, for $J = P_1(U) \cap P_1(V)$ and $S = \{(x,y) : x \in J\}$, $\widetilde{U} = U \cap S, \widetilde{V} = V \cap S$, we have

$$\widetilde{U} \cap \widetilde{V} \neq \emptyset,$$

$$P_{2,i}(\widetilde{U}) \cap P_{2,i}(\widetilde{V}) = \emptyset \text{ for } i = 1, 2.$$

See Figure 4.3. Note that U crosses entirely V if and only if V crosses entirely U.

Lemma 4.3.3. *If V_1, V_2 cross entirely U, with $|P_{2,1}(V_1)| = |P_{2,1}(V_2)|$ and $\angle(V_2, U) = \alpha_2 \leq \alpha_1 = \angle(V_1, U)$, then*

$$|V_1 \cap U| \leq |V_2 \cap U|.$$

Proof. Since the shear transformation $\begin{pmatrix} 1 & 0 \\ -\tan\theta & 1 \end{pmatrix}$ from $\mathbb{R}^2$ to $\mathbb{R}^2$ preserves area, we can assume without loss of generality that U has sides parallel to the axis. Then, if $a = |P_{2,1}(U)|$

and $b = |P_{2,1}(V_j)|$,

$$|V_j \cap U| = \frac{ab}{\tan \alpha_j},$$

so if $\alpha_2 \leq \alpha_1$, then $|V_1 \cap U| \leq |V_2 \cap U|$. $\square$

Lemma 4.3.4. *If V_1, V_2 are parallel, cross entirely U and $|P_1(V_1)| = |P_1(V_2)| = L$, then*

$$\frac{|U \cap V_1|}{|V_1|} = \frac{|U \cap V_2|}{|V_2|}.$$

Proof. Again, we may assume that U has sides parallel to the axis. Let $a = |P_{2,1}(U)|, \alpha = \angle(V_j, U)$ and $b_j = |P_{2,1}(V_j)|$ for $j = 1, 2$. Then, we have

$$\frac{|U \cap V_j|}{|V_j|} = \frac{a \cdot b_j}{\tan \alpha} \cdot \frac{1}{b_j \cdot L} = \frac{a}{L \tan \alpha},$$

which does not depend on j. $\square$

Proof. (of Lemma 4.3.2) We assume that $B \cap R \neq \emptyset$ since the inequality is trivial otherwise. Let α_R and α_B be the angles that R and B form with the x-axis respectively. Assume that $\alpha_B > \alpha_R$ only for notational convenience as the proof proceeds identically in either case. Then

$$\alpha_B \geq \theta_k > \alpha_R.$$

Let $\widetilde{R}$ be the smallest rectangle in the direction of θ_k containing R. We will prove that (4.3.5) is satisfied if we choose $\widehat{R}$ to be the dilation of $\widetilde{R}$ by 3 about its center (see Figure 4.4.)

We need to introduce a few more notations. For any rectangle D, let D^∞ denote the smallest infinite strip in the long direction containing D and let

$$\widehat{R}_{mid} = (\widetilde{R})^\infty \cap \widehat{R}$$

$$B^* = B^\infty \cap [P_1(\widehat{R}) \times \mathbb{R}]$$

$$R^* = R^\infty \cap [P_1(\widehat{R}) \times \mathbb{R}].$$

We deal with a few cases.

Case 1. $|P_{2,1}(R)| \geq \frac{1}{3}|P_{2,1}(\widetilde{R})|$

This is simple since $\widehat{R} \supset R$ and $|\widehat{R}| = 9|\widetilde{R}|$ implies

$$\frac{|B \cap R|}{|R|} \leq \frac{|B \cap \widehat{R}|}{|R|} \leq \frac{|B \cap \widehat{R}|}{|\widetilde{R}|/3} = 27\frac{|B \cap \widehat{R}|}{|\widehat{R}|}.$$

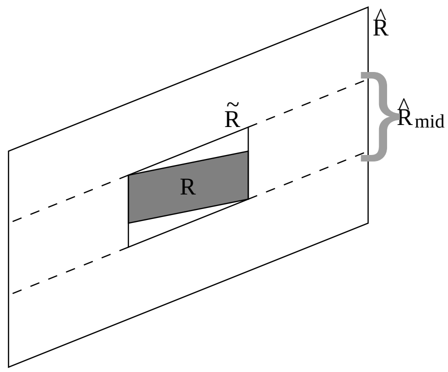

Figure 4.4: Definition of parallelograms used in the proof.

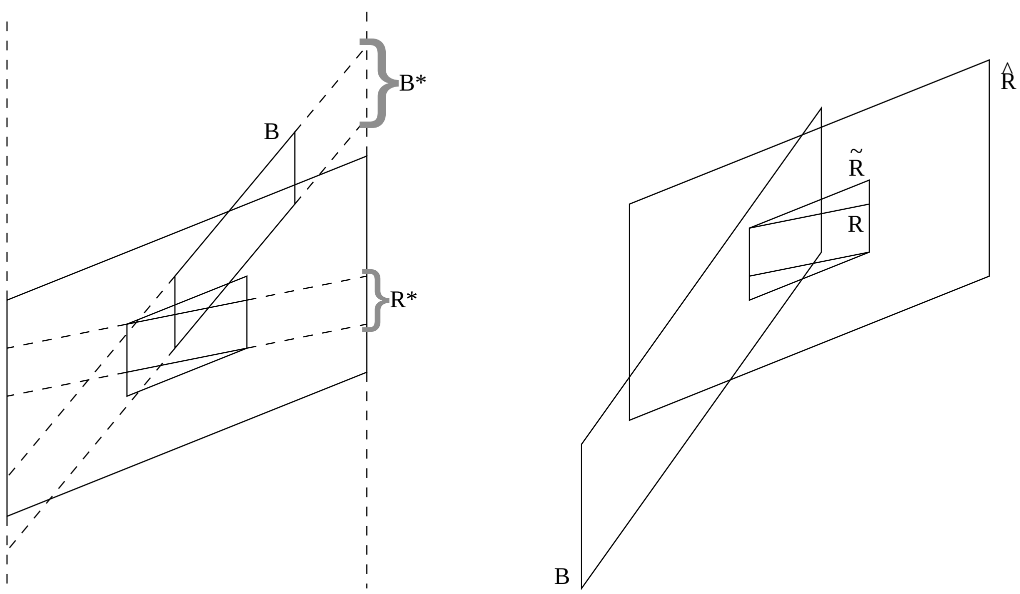

Figure 4.5: Possible configurations.

Case 2. $|P_{2,1}(B)| \geq \frac{1}{3}|P_{2,1}(\widetilde{R})|$

Let $d = P_1(B \cap \widehat{R}) \subset \mathbb{R}$ and $D = d \times \mathbb{R}$. Then,

$$B \cap R \subset D \cap R^\infty$$

so

$$|B \cap R| \leq |d| \cdot P_{2,1}(R).$$

Also, $B \cap \widehat{R}$ contains a vertical line segment of length at least $|P_{2,1}(\widetilde{R})|/3$ since B and $\widetilde{R}$ intersect and B has "height" at least $|P_{2,1}(\widetilde{R})|/3$. Note that any convex figure which contains a vertical line segment of length a and has horizontal projection of length b has area at least $ab/2$. Hence,

$$|B \cap \widehat{R}| \geq \frac{1}{2}(|d| \cdot |P_{2,1}(\widetilde{R})|/3).$$

Together, we get that

$$\frac{|B \cap \widehat{R}|}{|\widehat{R}|} \geq \frac{|d| \cdot |P_{2,1}(\widetilde{R})|}{6 \cdot 9|\widetilde{R}|} \geq \frac{|d| \cdot |P_{2,1}(R)|}{54|R|} \geq \frac{1}{54}\frac{|B \cap R|}{|R|}.$$

Case 3. Suppose neither of the above conditions satisfy. In other words,

$$|P_{2,1}(R)| < \frac{1}{3}|P_{2,1}(\widetilde{R})| \tag{4.3.7}$$

$$|P_{2,1}(B)| < \frac{1}{3}|P_{2,1}(\widetilde{R})|. \tag{4.3.8}$$

In this case, it will be helpful to introduce another parallelogram $\frac{5}{3}\widehat{R}_{mid}$ which is the dilation of $\widehat{R}_{mid}$ by $5/3$ in y-direction about its center (see Figure 4.6.)

We claim that $|B^* \cap \widehat{R}| \leq 6|B \cap \widehat{R}|$. We assume without loss of generality that $\widehat{R}$ is horizontal and that $|P_1(B)| = |P_1(R)|$. If $B \subset \widehat{R}$, then the result is trivial. If not, from the assumption that $|P_1(B)| = |P_1(R)|$ and the fact that $B \cap \widetilde{R} \neq \emptyset$, we have $P_2(B \setminus \widehat{R}) \neq \emptyset$ and hence, $|P_2(B \cap \widehat{R})| \geq |P_2(\widehat{R})|/3$. Again without loss of generality, assume that at least one of the vertices of B is "above" $\widehat{R}$. The upper side of $\frac{5}{3}\widehat{R}_{mid}$ cuts B with length $|P_{2,1}(B)|/|\tan(\alpha_B - \alpha_R)|$ (the two "left" corners of B are below this line and the two "right" corners are above this line.) Since $B \cap \widehat{R}$ is convex, we have

$$|B \cap \widehat{R}| \geq \frac{1}{2}\frac{|P_{2,1}(B)|}{|\tan(\alpha_B - \alpha_R)|} \cdot \frac{|P_2(\widehat{R})|}{3} = \frac{1}{6}|B^\infty \cap (\widehat{R})^\infty| \geq \frac{1}{6}|B^* \cap \widehat{R}|.$$

Note also that (4.3.7) and (4.3.8) imply that R^* and $\frac{5}{3}\widehat{R}_{mid}$ cross entirely. Since B^* must intersect with $\widetilde{R}$ and B^* is "skinny" as in (4.3.8), R^* and B^* necessarily cross entirely. Also, since $\widehat{R}_{mid} \subset \frac{5}{3}\widehat{R}_{mid}$, R^* crosses $\widehat{R}_{mid}$ entirely as well.

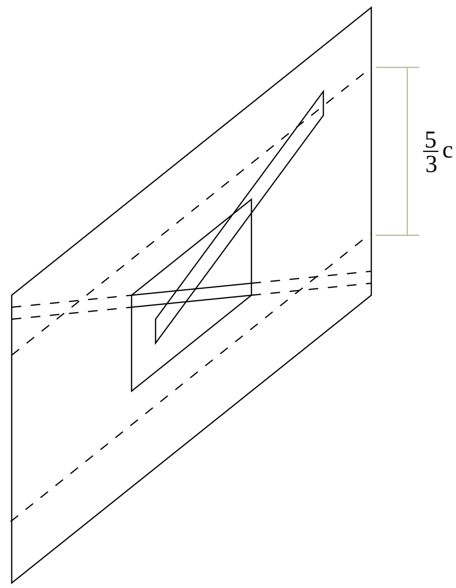

Figure 4.6: R^* and $\frac{5}{3}\widehat{R}_{mid}$ cross entirely.

Case 3a. Suppose that B^* crosses entirely $\widehat{R}_{mid}$.

Let R^{rot} be a rectangle in the direction of θ_k such that

$$P_1(R^{rot}) = P_1(\widehat{R})$$

$$R^{rot} \subset \widehat{R}_{mid}$$

$$|P_{2,1}(R^{rot})| = |P_{2,1}(R)|.$$

Since $R^{rot} \subset \widehat{R}_{mid}$, B^* crosses R^{rot} entirely and by Lemma 4.3.3,

$$\frac{|B \cap R|}{|R|} \leq \frac{|B^* \cap R^*|}{|R|} \leq \frac{|B^* \cap R^{rot}|}{|R|}.$$

By Lemma 4.3.4, and using $|B^* \cap \widehat{R}_{mid}| \leq 6|B \cap \widehat{R}|$, the above is equal to

$$\frac{3|B^* \cap R^{rot}|}{|R^{rot}|} = \frac{3|B^* \cap \widehat{R}_{mid}|}{|\widehat{R}_{mid}|} \leq \frac{18|B \cap \widehat{R}|}{|\widehat{R}|/3} = \frac{54|B \cap \widehat{R}|}{|\widehat{R}|}.$$

Case 3b. Suppose B^* does not cross entirely $\widehat{R}_{mid}$. Notice that this means that $P_{2,1}(B^*) \subset P_{2,1}(\widehat{R})$ or $P_{2,2}(B^*) \subset P_{2,2}(\widehat{R})$. Hence, $|B^*| \leq 3|B^* \cap \widehat{R}|$.

Let B^{rot} be a rectangle in the direction of θ_k such that

$$P_1(B^{rot}) = P_1(\widehat{R})$$

$$B^{rot} \subset \widehat{R}_{mid}$$

$$|P_{2,1}(B^{rot})| = |P_{2,1}(B)|.$$

Then, using that B^{rot} crosses R^* entirely, we have

$$\frac{|B^* \cap R|}{|B^*|} \le \frac{|B^* \cap R^*|}{|B^*|} \le \frac{|B^{rot} \cap R^*|}{|B^{rot}|} = \frac{|\widehat{R}_{mid} \cap R^*|}{|\widehat{R}_{mid}|} \le \frac{|R^*|}{|\widehat{R}|/3} = \frac{9|R|}{|\widehat{R}|}$$

or, in other words,

$$\frac{|B^* \cap R|}{|R|} \le \frac{9|B^*|}{|\widehat{R}|}.$$

Hence,

$$\frac{|B \cap R|}{|R|} \le \frac{|B^* \cap R|}{|R|} \le \frac{9|B^*|}{|\widehat{R}|} \le \frac{27|B^* \cap \widehat{R}|}{|\widehat{R}|} \le \frac{27 \cdot 6|B \cap \widehat{R}|}{|\widehat{R}|}.$$

Thus, $c = 27 \cdot 6$ is the constant we were looking for. □

As a corollary, we get the weak type $(2,2)$ version of Katz's result stated as follows:

Corollary 4.3.5. *There exist positive constants C and α such that for any set $\Omega \subset [0, \pi/4)$ of cardinality $N > 1$ one has*

$$\|M_\Omega\|_{L^2 \to L^{2,\infty}} \le C(\log N)^\alpha.$$

Proof. First fix a $1/2 < T < 1$. It is clear that the above holds for small N. In particular, for any choice of α, we can choose C such that the above is true for all N that satisfies $\sqrt{N} + 1 > N^T$.

Now we use induction on N for larger values. Suppose that it is true for all $|\Omega| < N$. Now, if $|\Omega| = N$, we choose a subset Ω_0 of cardinality $\lceil \sqrt{N} \rceil = m$ such that the corresponding subsets $\Omega_l, l = 1, 2, \ldots, m$ have all cardinality at most m. Then by the previous theorem,

$$\begin{aligned}
\|M_\Omega\|_{L^2 \to L^{2,\infty}} &\le C_1 \sup_{1 \le l \le m} \|M_{\Omega_l}\|^2_{L^2 \to L^{2,\infty}} + C_2 \|M_{\Omega_0}\|^2_{L^2 \to L^{2,\infty}} \\
&\le C_1 C^2 (\log m)^{2\alpha} + C_2 C^2 (\log m)^{2\alpha} \\
&\le C_1 C^2 (\log N^T)^{2\alpha} + C_2 C^2 (\log N^T)^{2\alpha} \\
&\le C^2 (C_1 + C_2) T^{2\alpha} (\log N)^{2\alpha}.
\end{aligned}$$

Hence choose α so that $(C_1 + C_2)T^{2\alpha} \le 1$ and we are done. □

We also make a comment that this does not give a sharp bound of $\alpha = 1/2$ as proved by Katz. If we had $C_1 = 1$ then we can obtain the sharp exponent by partitioning Ω with $|\Omega_0| = 2$ repeatedly. Note that the above inequality also implies the strong type estimate as follows:

Corollary 4.3.6. *For the set of directions Ω with $|\Omega| = N$, we have*

$$\|M_\Omega\|_{L^2\to L^2} \le C(\log N)^{\alpha+1/2}.$$

Note that Katz obtained this result with the sharpest exponent $\alpha = 1$.

Proof. This follows from a well known interpolation arguments presented below. We have that M_Ω with $|\Omega| = N$ is a sublinear operator with weak type (1,1) norm of AN (where A is a constant independent of N) by the trivial estimate and (∞, ∞) norm of 1. Hence by the Marcinkiewicz interpolation theorem, we have (p, p) norm of $A_p N^{1/p}$. Setting $p = 3/2$, we have

$$\|M_\Omega\|_{L^{3/2}\to L^{3/2,\infty}} \le \|M_\Omega\|_{L^{3/2}\to L^{3/2}} \le AN^{2/3}.$$

Split $f = f_1 + f_2 + f_3$ where

$$\begin{aligned} f_1 &= f\chi_{\{|f|\le\lambda/4\}} \\ f_2 &= f\chi_{\{\lambda/4<|f|\le N^2\lambda\}} \\ f_3 &= f\chi_{\{N^2\lambda<|f|\}}. \end{aligned}$$

It follows that

$$|\{M_\Omega(f) > \lambda\}| \le |\{M_\Omega(f_2) > \lambda/3\}| + |\{M_\Omega(f_3) > \lambda/3\}|.$$

By using $L^{2,\infty}$ estimate for f_2 and $L^{3/2,\infty}$ estimate for f_3, we get

$$\begin{aligned} \|M_\Omega(f)\|_{L^2}^2 &= 2\int_0^\infty \lambda|\{M_\Omega(f) > \lambda\}|d\lambda \\ &\le \int_0^\infty 2\lambda|\{M_\Omega(f_2) > \lambda/3\}|d\lambda + \int_0^\infty 2\lambda|\{M_\Omega(f_3) > \lambda/3\}|d\lambda \\ &\le \int_0^\infty \frac{2\lambda C^2(\log N)^{2\alpha}}{\lambda^2/9}\int |f|^2\chi_{\{\lambda/4<|f|\le N^2\lambda\}}dxd\lambda \\ &\quad + \int_0^\infty \frac{2\lambda A^{3/2}N}{\lambda^{3/2}/3^{3/2}}\int |f|^{3/2}\chi_{\{|f|>N^2\lambda\}}dxd\lambda \\ &\le 18C^2(\log N)^{2\alpha}\int_{\mathbb{R}^2}|f(x)|^2\int_{\frac{|f(x)|}{N^2}}^{4|f(x)|}\frac{d\lambda}{\lambda}dx \\ &\quad +2\cdot 3^{3/2}A^{3/2}N\int_{\mathbb{R}^2}|f(x)|^{3/2}\int_0^{\frac{|f(x)|}{N^2}}\frac{d\lambda}{\lambda^{1/2}}dx \\ &= \left(36C^2(\log N)^{2\alpha}(\log 2N) + 4\cdot 3^{3/2}A^{3/2}\right)\|f\|_{L^2}^2 \\ &\le C'(\log N)^{2\alpha+1}\|f\|_{L^2}^2. \end{aligned}$$

$\square$

Another corollary that makes use of the boundedness of maximal operator in lacunary directions extends the result by Sjögren and Sjölin [24], as follows:

Theorem 4.3.7. *Let $\Omega_0 \subset [0, \pi/4)$ denote the elements of a lacunary sequence $\{\theta_l\}$ and consider $\Omega_l, l = 1, 2, \dots$ arbitrary sets with $\Omega_l \subset [\theta_l, \theta_{l-1})$. Set $\Omega = \cup_{l \geq 0} \Omega_l$. Then the maximal operator M_Ω has the property*

$$\|M_\Omega\|_{L^2 \to L^{2,\infty}} \leq C \sup_l \|M_{\Omega_l}\|_{L^2 \to L^{2,\infty}}.$$

Proof. This follows directly from the theorem. □

4.3.2 Strong type $(2, 2)$ estimate

We follow another work of Alfonseca, Soria and Vargas [2], "An almost-orthogonality principle in L^2 for directional maximal functions", where a strong type (2,2) almost-orthogonality is obtained.

Theorem 4.3.8. *There exists a constant C independent of Ω such that*

$$\|M_\Omega\|_{L^2 \to L^2} \leq \sup_{j \geq 1} \|M_{\Omega_j}\|_{L^2 \to L^2} + C \|M_{\Omega_0}\|_{L^2 \to L^2}. \tag{4.3.9}$$

Notice that we have the constant 1 in front of the supremum, which will give a sharp bound of Katz's result as a corollary, as pointed out earlier.

Proof. We linearize the maximal operator in the following way. For any $\alpha \in \mathbb{Z}^2$, let Q_α denote the unit cube centered at α. Given a set $\Lambda \subset [0, \pi/4)$, for each $\alpha \in \mathbb{Z}^2$ we choose a parallelogram $R_\Lambda(\alpha) \in \mathcal{B}_\Lambda^P$ such that $R_\Lambda(\alpha) \supset Q_\alpha$ (recall the notation (3.6.10).) We define the operator T_{R_Λ} as

$$T_{R_\Lambda} f(x) = \sum_{\alpha \in \mathbb{Z}^2} \frac{1}{|R_\Lambda(\alpha)|} \left(\int_{R_\Lambda(\alpha)} f \right) \chi_{Q_\alpha}(x).$$

We have that this is dominated by the centered maximal operator M'_Λ whose bases consist of

$$\mathcal{B}'_\Lambda = \{R : R \in \mathcal{B}_\Lambda^P \text{ and vertical side of } R \text{ is at least } 2\}.$$

Also for certain choice of $R_\Lambda(\cdot)$, we have $\|M'_\Lambda\| - \epsilon \leq \|T_{R_\Lambda}\|$. Hence, $\|M'_\Lambda\| = \sup_{R_\Lambda(\cdot)} \|T_{R_\Lambda}\|$. By scalar invariance (3.6.5), M'_Λ is equivalent to M_Λ^P which is equivalent to M_Λ. Thus, it is enough to prove (4.3.9) where M_Ω is replaced by T_Ω.

We fix a specific $R_\Lambda(\cdot)$ and write T_Λ and R_α instead of T_{R_Λ} and $R_\Lambda(\alpha)$. We will use the following theorem due to Carbery which appears within paper [2].

Theorem 4.3.9. *Let T_Λ be as above. Then T_Λ is of strong type (p,p) if and only if there exist a constant $C_{p'}$ such that for any sequence $\{\lambda_\alpha\} \subset \mathbb{R}_+$, we have,*

$$\int \Big(\sum_\alpha \lambda_\alpha \frac{1}{|R_\alpha|} \chi_{R_\alpha} \Big)^{p'} \leq C_{p'} \sum_\alpha |\lambda_\alpha|^{p'}.$$

Moreover, the infimum of the constants $(C_{p'})^{1/p'}$ satisfying the above is $\|T_\Lambda\|_{L^p \to L^p}$.

Proof. If T_Λ is of strong type (p,p), then its adjoint T^*_Λ is of strong type (p',p') with the same norm. We can calculate T^*_Λ by

$$\begin{aligned}
\langle T^* g, h \rangle &= \langle g, Th \rangle \\
&= \int g(x) (\sum_\alpha \frac{1}{|R_\alpha|} \int h(y) \chi_{R_\alpha}(y) dy \chi_{Q_\alpha}(x)) dx \\
&= \int \int \sum_\alpha \frac{1}{|R_\alpha|} g(x) h(h) \chi_{R_\alpha}(y) \chi_{Q_\alpha}(x) dx dy \\
&= \int h(y) \sum_\alpha \frac{1}{|R_\alpha|} \chi_{R_\alpha}(y) (\int g(x) \chi_{Q_\alpha}(x) dx) dy,
\end{aligned}$$

obtaining,

$$T^*_\Lambda g(x) = \sum_\alpha \Big(\int_{Q_\alpha} g \Big) \frac{1}{|R_\alpha|} \chi_{R_\alpha}(x),$$

Taking $g = \sum_\alpha \lambda_\alpha \chi_{Q_\alpha}$, we get the required result with $C_{p'} = \|T^*_\Lambda\|^{p'}_{L^{p'} \to L^{p'}} = \|T_\Lambda\|^{p'}_{L^p \to L^p}$.

Conversely, if we have the above inequality, then for all $h \in L^{p'}$, let $\lambda_\alpha = \Big| \int_{Q_\alpha} h \Big|$ to get

$$\int |T^*_\Lambda h|^{p'} \leq C_{p'} \sum_\alpha \Big| \int_{Q_\alpha} h \Big|^{p'} \leq C_{p'} \sum_\alpha \int_{Q_\alpha} |h|^{p'} \cdot (\int_{Q_\alpha} 1^p)^{p'/p} \leq C_{p'} \int |h|^{p'},$$

Hence, T_Λ is of strong type (p,p) and its norm is bounded by $(C_{p'})^{1/p'}$. □

Let us return to the proof of our main theorem. Fix a choice of parallelograms $\{R_\alpha\}$. We need to show that the inequality in the above theorem is satisfied with $p = 2$ and

$$C_2^{1/2} = \sup_{j \geq 1} \|M_{\Omega_j}\|_{L^2 \to L^2} + C \|M_{\Omega_0}\|_{L^2 \to L^2}.$$

Set

$$\begin{aligned}
I^2 &= \int \Big(\sum_\alpha \lambda_\alpha \frac{1}{|R_\alpha|} \chi_{R_\alpha} \Big)^2 = \int \Big(\sum_l \sum_{\alpha : R_\alpha \in \Omega_l} \lambda_\alpha \frac{1}{|R_\alpha|} \chi_{R_\alpha} \Big)^2 \\
&= \int \sum_l \Big(\sum_{\alpha : R_\alpha \in \Omega_l} \lambda_\alpha \frac{1}{|R_\alpha|} \chi_{R_\alpha} \Big)^2 + 2 \sum_l \sum_{j<l} \int \sum_{R_\alpha \in \Omega_l} \sum_{R_\beta \in \Omega_j} \lambda_\alpha \lambda_\beta \frac{1}{|R_\alpha||R_\beta|} \chi_{R_\alpha} \chi_{R_\beta} \\
&= A + B.
\end{aligned}$$

For the first term, we use Theorem 4.3.9 with $p = 2$ and $\Lambda = \Omega_l$ to get

$$\begin{aligned}
A &\leq \sum_l \|T_{\Omega_l}\|^2_{L^2\to L^2}\Big(\sum_{\alpha:R_\alpha\in\Omega_l}|\lambda_\alpha|^2\Big)\\
&\leq \Big(\sup_l \|M_{\Omega_l}\|^2_{L^2\to L^2}\Big)\Big(\sum_l\sum_{\alpha:R_\alpha\in\Omega_l}|\lambda_\alpha|^2\Big)\\
&\leq \Big(\sup_l \|M_{\Omega_l}\|^2_{L^2\to L^2}\Big)\Big(\sum_\alpha|\lambda_\alpha|^2\Big).
\end{aligned}$$

Hence it remains to bound B. If $R_\alpha \in \Omega_l$ and $R_\beta \in \Omega_j$ with $j < l$, then we claim that we can find parallelograms $\widetilde{R}_\alpha$ and $\widetilde{R}_\beta$ containing R_α and R_β respectively, pointing in the direction θ_j and satisfying

$$\frac{|R_\alpha\cap R_\beta|}{|R_\alpha||R_\beta|}\leq C\frac{|\widetilde{R}_\alpha\cap R_\beta|}{|\widetilde{R}_\alpha||R_\beta|}+C\frac{|R_\alpha\cap \widetilde{R}_\beta|}{|R_\alpha||\widetilde{R}_\beta|}.$$

Indeed, if $P_1(R_\alpha) \leq P_1(R_\beta)$, then we can find $\widetilde{R}_\alpha$ such that

$$\frac{|R_\alpha\cap R_\beta|}{|R_\alpha||R_\beta|}\leq C\frac{|\widetilde{R}_\alpha\cap R_\beta|}{|\widetilde{R}_\alpha||R_\beta|}$$

using Lemma 4.3.2. If $P_1(R_\alpha) \geq P_1(R_\beta)$, then we find $\widetilde{R}_\beta$ such that

$$\frac{|R_\alpha\cap R_\beta|}{|R_\alpha||R_\beta|}\leq C\frac{|R_\alpha\cap \widetilde{R}_\beta|}{|R_\alpha||\widetilde{R}_\beta|}.$$

Thus, the claim is proven.

Then

$$\begin{aligned}
B \leq 2C\sum_l\sum_{j<l}\int\sum_{R_\alpha\in\Omega_l}\sum_{R_\beta\in\Omega_j}\lambda_\alpha\lambda_\beta\frac{1}{|\widetilde{R}_\alpha||R_\beta|}\chi_{\widetilde{R}_\alpha}\chi_{R_\beta}\\
+2C\sum_l\sum_{j<l}\int\sum_{R_\alpha\in\Omega_l}\sum_{R_\beta\in\Omega_j}\lambda_\alpha\lambda_\beta\frac{1}{|R_\alpha||\widetilde{R}_\beta|}\chi_{R_\alpha}\chi_{\widetilde{R}_\beta}\\
= B^- + B^+.
\end{aligned}$$

We only consider B^- as B^+ works the same way. We have

$$\begin{aligned}
B^- &\leq 2C\int\Big(\sum_l\sum_{R_\alpha\in\Omega_l}\lambda_\alpha\frac{\chi_{\widetilde{R}_\alpha}}{|\widetilde{R}_\alpha|}\Big)\Big(\sum_j\sum_{R_\beta\in\Omega_j}\lambda_\beta\frac{\chi_{R_\beta}}{|R_\beta|}\Big)\\
&\leq 2C\bigg(\int\Big(\sum_l\sum_{R_\alpha\in\Omega_l}\lambda_\alpha\frac{\chi_{\widetilde{R}_\alpha}}{|\widetilde{R}_\alpha|}\Big)^2\bigg)^{1/2}\bigg(\int\Big(\sum_j\sum_{R_\beta\in\Omega_j}\lambda_\beta\frac{\chi_{R_\beta}}{|R_\beta|}\Big)^2\bigg)^{1/2}
\end{aligned}$$

by the Cauchy-Schwartz inequality.

Since $\widetilde{R}_\alpha \in \Omega_0$ for all α, using Theorem 4.3.9 again, we obtain

$$B^- \leq 2C\|M_{\Omega_0}\|_{L^2\to L^2}\left(\sum_\alpha |\lambda_\alpha|^2\right)^{1/2} I,$$

Combining the two estimates, we get

$$I^2 \leq \left(\sup_l \|M_{\Omega_l}\|^2_{L^2\to L^2}\right)\left(\sum_\alpha |\lambda_\alpha|^2\right) + 4C\|M_{\Omega_0}\|_{L^2\to L^2}\left(\sum_\alpha |\lambda_\alpha|^2\right)^{1/2} I.$$

Note that by the quadratic formula,

$$I^2 - ALI - B^2L^2 \leq 0 \Rightarrow I \leq \frac{A+\sqrt{A^2+4B^2}}{2}L \leq \frac{A+(A+2B)}{2}L = (A+B)L.$$

Hence, by letting $A = C\|M_{\Omega_0}\|_{L^2\to L^2}, B = \sup_l \|M_{\Omega_l}\|_{L^2\to L^2}, L = \left(\sum_\alpha |\lambda_\alpha|^2\right)^{1/2}$, we get

$$I \leq \left(\sup_l \|M_{\Omega_l}\|_{L^2\to L^2} + C\|M_{\Omega_0}\|_{L^2\to L^2}\right)\left(\sum_\alpha |\lambda_\alpha|^2\right)^{1/2}$$

from which we get our result by Theorem 4.3.9. □

As a corollary, we get the following result due originally to Katz [19].

Corollary 4.3.10. *There exists a constant K such that for any set $\Omega \subset [0,\pi/4)$ with cardinality $N > 1$, one has*

$$\|M_\Omega\|_{L^2\to L^2} \leq K(\log N).$$

Proof. If we show this for all $N = 2^M$, then for any integer N and $|\Omega| = N$, we have $\|M_\Omega\| \leq K(\log N') \leq 2K(\log N)$, where N' is the smallest power of 2 greater than N. Hence let $N = 2^M$. We use induction on M. For $M = 2$ the inequality follows from the boundedness of the strong maximal operator. Suppose that it is true for all $|\Omega| = 2^k$ with $k < M$. Also we can assume that K is at least $2C/\log 2$. If we define Ω_0 to be consisting of just the first element and the middle element, we get that

$$\|M_\Omega\|_{L^2\to L^2} \leq K\log N/2 + 2C = K\log N - K\log 2 + 2C \leq K\log N.$$

□

4.4 Karagulyan and Lacey's approach

Here we follow G. A. Karagulyan and M. T. Lacey's paper "An estimate of the maximal operators associated with generalized lacunary sets" [17]. This approach produces results that are consequences of Alfonseca, Soria and Vargas's work, but is self-contained and does not require the advanced machineries developed in the preliminary section. We will use definitions slightly different from the original paper.

Fix constants $0 < C_1 < C_2$. For $I = \{1, \ldots, n\}$ $(n \geq 2)$, $\mathbb{N}$, or $\mathbb{Z}$, we say that a monotonic sequence $\{v_k\}_{k \in I} \subset [a, b]$ is *well-behaving* in $[a, b]$ if, for each $A_k = [v_k, v_{k+1}]$ and $B = [a, b]^c = (-\infty, a) \cup (b, \infty)$, we have

$$C_1|A_k| \leq d(A_k, B) \leq C_2|A_k|$$

where, for sets $A, B \subset \mathbb{R}$, $d(A, B)$ denotes the distance between A and B. We recursively define an Nth degree well-behaving set. A first degree well-behaving set is a well-behaving set. An $N+1$th degree well-behaving set in $[a, b]$ is created from a well-behaving set $\{v_k\}$ in $[a, b]$ by adding some points Ω_k between neighbouring points v_k, v_{k+1}: precisely, we require that each Ω_k is Nth degree well-behaving in $[v_k, v_{k+1}]$.

The choice of such definition will be apparent in its usage in Theorem 4.4.1. Also note that, alternatively, given an Nth degree well-behaving set Ω, one can find $\Omega = \Omega_N \supset \Omega_{N-1} \supset \ldots \Omega_1$ such that each Ω_k is a kth degree well-behaving set and Ω_{k+1} is obtained by adding a well-behaving set between some neighbouring points of Ω_k. Note, however, that not all the neighbouring points of Ω_k can "host" a well-behaving set: both the neighbouring points must be from $\Omega_k \setminus \Omega_{k-1}$.

We note that certain finitely nested lacunary sequences are Nth degree well-behaving sets. As in the original paper, we say that a sequence $\{v_k\}$ is 1-lacunary if there exists v_∞ such that

$$\frac{1}{2}|v_k - v_{k+1}| < |v_{k+1} - v_\infty| < |v_k - v_{k+1}|.$$

Alternatively, the sequence $\{|v_k - v_{k+1}|\}$ forms a lacunary sequence with each ratio $(v_{k+1} - v_\infty)/(v_k - v_\infty)$ lying between $1/3$ and $1/2$. An example of a 1-lacunary set is $v_k = (2.5)^{-k}$ with $v_\infty = 0$. By nesting 1-lacunary sequences N times (to form what is called an N-lacunary set), we can create an Nth degree well-behaving set with $C_1 = 1/2$ and $C_2 = 1$.

Theorem 4.4.1. *For all integers N and all Nth degree well-behaving sets Ω in $[0, 1]$, we have*

$$\|M_\Omega f(x)\|_2 \leq CN\|f\|_2,$$

for some constant C independent of N.

Note that in this section $\alpha \in \Omega$ represents a direction of a line with slope α. Since α and $\tan \alpha$ are comparable, this interpretation does not affect the type of result.

Similarly to the previous section, we reduce the bases to the parallelograms with one side parallel to the y-axis and the other side forming an angle of slope α with the x-axis. We have that

$$P_\alpha f(x) = \sup_{\delta_1,\delta_2} \frac{1}{4\delta_1\delta_2} \int_{x_1-\delta_1}^{x_1+\delta_1} \int_{x_2+(t_1-x_1)\alpha-\delta_2}^{x_2+(t_1-x_1)\alpha+\delta_2} |f(t_1,t_2)| dt_2 dt_1.$$

Note that we are not forcing the angled side to be longer than the vertical side. This change is not significant as the above is comparable to $M_{\Omega\cup\{\pi/2\}}$, which is comparable to M_Ω. We want to prove

$$\| \sup_{\alpha\in\Omega} P_\alpha f\|_2 \le CN\|f\|_2$$

where Ω is any Nth degree well-behaving set from (0,1).

We will use the Fejer kernel

$$K_r(x) = \int_{-r}^{r} \Big(1 - \frac{|t|}{r}\Big) e^{-itx} dt = \frac{4\sin^2 \frac{rx}{2}}{rx^2}. \tag{4.4.1}$$

For any r, R with $0 \le r < R/2$ we define,

$$\psi_r(x) = 2K_{2r}(x) - K_r(x),$$

$$\psi_{r,R}(x) = \psi_R(x) - \psi_r(x)$$

where $\psi_{0,R}(x)$ means $\psi_R(x)$. Since

$$\widehat{K}_r(\xi) = \max\Big\{1 - \frac{|\xi|}{r}, 0\Big\}$$

we have,

$$\widehat{\psi}_{r,R}(\xi) = \begin{cases} 1, & \text{if } |\xi| \in [2r, R] \\ 0, & \text{if } 0 \le |\xi| \le r \text{ or } |\xi| > 2R \\ \text{linear} & \text{on each } \pm[r,2r], \pm[R,2R]. \end{cases}$$

Using (4.4.1) we see that

$$|\psi_R(x)| \le C\Big(\min\Big\{\frac{1}{Rx^2}, R\Big\}\Big).$$

Thus, for

$$\gamma_k = \frac{1}{2^k}$$

$$\omega_k = \Big[-\frac{2^k}{R}, \frac{2^k}{R}\Big],$$

we have

$$|\psi_R(x)| \le C\sum_k \gamma_k \frac{\chi_{\omega_k}(x)}{|\omega_k|} = \zeta_R(x)$$

$$\gamma_k > 0, \quad \sum_k \gamma_k < 1, \quad \omega_k \supset (-1/R, 1/R).$$

Also, choose a Schwartz function ϕ with

$$\phi \geq 0, \ \operatorname{supp}\widehat{\phi} \subset [-1,1].$$

Recalling (2.4.5), define

$$\Gamma^{\alpha}_{r,R,h} f(x) = \Big(|\psi_{r,R}(x_2 - x_1\alpha)|\phi_h(x_1)\Big) * f(x).$$

Since $\psi_{0,R} = 2K_{2R} - K_R$ satisfies that $\widehat{\psi}_R(x) = \widehat{\psi}_1(x/R)$,

$$\psi_{0,R} = (\psi_{0,1})_R$$

and also, $\psi_{0,1}(0) = 1$. Hence, we have that

$$P_\alpha f(x) = P_\alpha |f|(x) \leq C \sup_{R,h} \Gamma^{\alpha}_{R,h}|f|(x)$$

where $\Gamma^{\alpha}_{R,h} = \Gamma^{\alpha}_{0,R,h}$.

Therefore, it suffices to prove for $f \geq 0$

$$\Big\| \sup_{R,h,\alpha\in\Omega} \Gamma^{\alpha}_{R,h} f(x) \Big\|_2 \leq CN\|f\|_2.$$

We take the Fourier transform of both sides to get

$$\begin{aligned}
\widehat{\Gamma}^{\alpha}_{r,R,h} f(\xi) &= \Big(\int \psi_{r,R}(x_2 - x_1\alpha)\phi_h(x_1) e^{-x_1\xi_1 - x_2\xi_2} dx\Big)\widehat{f}(\xi) \\
&= \Big(\int \psi_{r,R}(x_2 - x_1\alpha)\phi_h(x_1) e^{-(\xi_1+\alpha\xi_2)x_1 - (x_2 - x_1\alpha)\xi_2} dx\Big)\widehat{f}(\xi) \\
&= \Big(\int\int \psi_{r,R}(x_2')\phi_h(x_1) e^{-(\xi_1+\alpha\xi_2)x_1 - \xi_2 x_2'} dx_2' dx_1\Big)\widehat{f}(\xi) \\
&= \Big(\int \psi_{r,R}(x_2') e^{-\xi_2 x_2'} dx_2' \int \phi_h(x_1) e^{-(\xi_1+\alpha\xi_2)x_1} dx_1\Big)\widehat{f}(\xi) \\
&= \widehat{\psi}_{r,R}(\xi_2)\widehat{\phi}(h(\xi_1+\alpha\xi_2))\widehat{f}(\xi).
\end{aligned}$$

Now the proof requires the following two lemmas.

Lemma 4.4.2. *Let $\alpha, \beta \in (0,1)$ be any numbers and $h, R > 0$. Then*

$$|\Gamma^{\alpha}_{R,h} f| = |\Gamma^{\alpha}_{0,R,h} f| \leq C(hR|\alpha - \beta| + 1)P_\beta f(x), x \in \mathbb{R}^2.$$

This implies that for $0 < r < R/2, h > 0$, we have

$$|\Gamma^{\alpha}_{r,R,h} f(x)| \leq |\Gamma^{\alpha}_{R,h} f(x)| + |\Gamma^{\alpha}_{r,h} f(x)| \leq 2C(hR|\alpha - \beta| + 1)P_\beta f(x), x \in \mathbb{R}^2.$$

Proof. As noted earlier,

$$\psi_R(x_2 - x_1\alpha) \le C\sum_k \frac{\gamma_k}{|\omega_k|}\chi_{\omega_k}(x_2 - x_1\alpha)$$

where we have $|\omega_k| > 2/R$. Denote $\lambda(x_1) = 2R|x_1||\alpha - \beta| + 2$ and assume $x_2 - x_1\alpha \in \omega_k$ for some k. We have

$$\begin{aligned} \left|\frac{x_2 - x_1\beta}{\lambda(x_1)}\right| &= \left|\frac{x_2 - x_1\alpha + x_1(\alpha-\beta)}{\lambda(x_1)}\right| \\ &\le \left|\frac{x_2 - x_1\alpha}{2}\right| + \frac{1}{2R} \le \frac{|\omega_k|}{2}, \end{aligned}$$

and hence,

$$\frac{x_2 - x_1\beta}{\lambda(x_1)} \in \omega_k.$$

Thus,

$$\chi_{\omega_k}(x_2 - x_1\alpha) \le \chi_{\omega_k}\left(\frac{x_2 - x_1\beta}{\lambda(x_1)}\right)$$

which means

$$|\psi_R(x_2 - x_1\alpha)| \le C\sum_k \frac{\gamma_k}{|\omega_k|}\chi_{\omega_k}\left(\frac{x_2 - x_1\beta}{\lambda(x_1)}\right) \le \zeta_R\left(\frac{x_2 - x_1\beta}{\lambda(x_1)}\right).$$

Thus, letting $\phi'(x) = \max\{\phi(x), |x|\phi(x)\}$,

$$\begin{aligned} \frac{1}{h}\phi\left(\frac{x_1}{h}\right)\zeta_R\left(\frac{x_2 - x_1\beta}{\lambda(x_1)}\right) &\le C\frac{1}{h}\left(R|\alpha-\beta||x_1|\phi\left(\frac{x_1}{h}\right) + \phi\left(\frac{x_1}{h}\right)\right)\frac{1}{\lambda(x_1)}\zeta_R\left(\frac{x_2 - x_1\beta}{\lambda(x_1)}\right) \\ &\le C\frac{1}{h}(Rh|\alpha-\beta| + 1)\phi'\left(\frac{x_1}{h}\right)\frac{1}{\lambda(x_1)}\zeta_R\left(\frac{x_2 - x_1\beta}{\lambda(x_1)}\right). \end{aligned}$$

But we know that since ϕ' is also Schwartz,

$$\phi' \le C\sum_{m\ge1}\frac{1}{2^{2m}}\chi_{[-2^m,2^m]},$$

and hence, the above is bounded by

$$\begin{aligned} &C\frac{1}{h}(Rh|\alpha-\beta| + 1)\sum_{m\ge1}\frac{1}{2^m}\frac{1}{2^m}\chi_{[-2^m,2^m]}\left(\frac{x_1}{h}\right)\frac{1}{\lambda(x_1)}\sum_k \gamma_k\frac{1}{|\omega_k|}\chi_{\omega_k}\left(\frac{x_2 - x_1\beta}{\lambda(x_1)}\right) \\ &\le C\frac{1}{h}(Rh|\alpha-\beta| + 1)\sum_{m\ge1}\sum_k \frac{\gamma_k}{2^m}\frac{1}{2^m}\chi_{[-2^m,2^m]}\left(\frac{x_1}{h}\right)\frac{1}{\lambda(x_1)}\frac{1}{|\omega_k|}\chi_{\omega_k}\left(\frac{x_2 - x_1\beta}{\lambda(x_1)}\right). \end{aligned}$$

Since $\sum_{m\ge1}\sum_k \frac{\gamma_k}{2^m} < 1$, we only need to show that convolving with each summand

$$\frac{1}{2^m}\frac{1}{h}\chi_{[-2^m,2^m]}\left(\frac{x_1}{h}\right)\frac{1}{\lambda(x_1)}\frac{1}{|\omega_k|}\chi_{\omega_k}\left(\frac{x_2 - x_1\beta}{\lambda(x_1)}\right)$$

is bounded by P_β. Indeed, the above is bounded by

$$
\begin{aligned}
&\sum_{l\ge 1}\frac{1}{2^m}\frac{1}{h}\chi_{[2^{m-l}h,2^{m-l+1}h]}(|x_1|)\frac{1}{\lambda(x_1)}\frac{1}{|\omega_k|}\chi_{\omega_k}\left(\frac{x_2-x_1\beta}{\lambda(x_1)}\right)\\
\le\ &\sum_{l\ge 1}\frac{1}{2^m}\frac{1}{h}\chi_{[2^{m-l}h,2^{m-l+1}h]}(|x_1|)\frac{1}{\lambda(2^{m-l}h)}\frac{1}{|\omega_k|}\chi_{\omega_k}\left(\frac{x_2-x_1\beta}{\lambda(2^{m-l+1}h)}\right)\\
\le\ &C\sum_{l\ge 1}\frac{1}{2^m}\frac{1}{h}\chi_{[2^{m-l}h,2^{m-l+1}h]}(|x_1|)\frac{1}{\lambda(2^{m-l+1}h)}\frac{1}{|\omega_k|}\chi_{\omega_k}\left(\frac{x_2-x_1\beta}{\lambda(2^{m-l+1}h)}\right)\\
\le\ &C\sum_{l\ge 1}2^{-l}\frac{1}{2^{m-l}h}\chi_{[-2^{m-l+1}h,2^{m-l+1}h]}(x_1)\frac{1}{\lambda(2^{m-l+1}h)}\frac{1}{|\omega_k|}\chi_{\omega_k}\left(\frac{x_2-x_1\beta}{\lambda(2^{m-l+1}h)}\right)
\end{aligned}
$$

since $\lambda(2^{m-l}h)$ and $\lambda(2^{m-l+1}h)$ are comparable. Hence the above is bounded by the maximal operator over parallelograms in the direction β. $\square$

For any interval $J=(a,b)$, we denote by $S(J)$ the sector $\{(x_1,x_2): ax_2\le -x_1\le bx_2\}$ and by S' a slightly wider sector $\{(x_1,x_2): (a-C_1|b-a|/2)x_2\le -x_1\le (b+C_1|b-a|/2)x_2\}$. Denote by $\omega_S f$ the multiplier operator $\widehat{\omega_S f}=\chi_S\widehat{f}$.

Lemma 4.4.3. *Let $[0,1]=J_0\supset J_1\supset J_2\supset\cdots\supset J_n$ be some sequence of intervals with*

$$J_k=[\alpha_k,\beta_k]$$

$$d((J_k)^c,J_{k+1})\le C_2|J_{k+1}| \tag{4.4.2}$$

for $0\le k<n$. Then for any $\theta\in J_n$ and any function $f\in L^2(\mathbb{R}^2)$ we have

$$P_\theta f\le C\Big[P_0 f+P_\theta\Big(\omega_{S'(J_n)}f\Big)+\sum_{k=1}^{n-1}P_{\alpha_k}\Big(\omega_{S'(J_k)}f\Big)+P_{\beta_k}\Big(\omega_{S'(J_k)}f\Big)\Big]$$

where P_0 is P_{α_0} with $\alpha_0=0$.

Proof. Let θ be fixed. For any R,h, we have

$$\widehat{\Gamma^\theta_{R,h}f}(\xi)=\widehat{\psi}_R(\xi_2)\widehat{\phi}(h(\xi_1+\xi_2\theta))\widehat{f}(x).$$

Denote

$$r_0=0\quad\text{ and }\quad r_k=\frac{2}{C_1h|J_k|},\quad 1\le k\le n.$$

We have that

$$\widehat{\psi}_R(\xi_2)=\sum_{k=1}^{m}\widehat{\psi}_{r_{k-1},r_k}(\xi_2)+\widehat{\psi}_{r_m,R}(\xi_2)$$

where $m = \max\{k \le n : 2r_k < R\}$. Denote

$$\Gamma_k f(x) = \Gamma^{\theta}_{r_k, r_{k+1}, h} f(x), \qquad 0 \le k < m,$$
$$\Gamma_m f(x) = \Gamma^{\theta}_{r_m, R, h} f(x)$$

so that

$$\Gamma^{\theta}_{R,h} f \le \sum_{k=0}^{m} \Gamma_k f.$$

If we show that

$$\operatorname{supp} \widehat{\psi}_{r_k, r_{k+1}}(\xi_2) \widehat{\phi}(h(\xi_1 + \xi_2 \theta)) \subset S'(J_k), \qquad 1 \le k < m,$$
$$\operatorname{supp} \widehat{\psi}_{r_m, R}(\xi_2) \widehat{\phi}(h(\xi_1 + \xi_2 \theta)) \subset S'(J_m),$$

then we have

$$\Gamma_k f = \Gamma_k \Big(\omega_{S'(J_k)} f\Big), \qquad 1 \le k \le m.$$

Indeed, we have

$$\operatorname{supp} \widehat{\psi}_{r_k, r_{k+1}}(\xi_2) \widehat{\phi}(h(\xi_1 + \xi_2 \theta)) = \Big\{(\xi_1, \xi_2) : r_k \le |\xi_2| \le 2r_{k+1}, |\xi_1 + \xi_2 \theta| < 1/h\Big\}$$

where the last set is a parallelogram with vertices $(-r_k\theta \pm 1/h, r_k)$ and $(-2r_{k+1}\theta \pm 1/h, 2r_{k+1})$. These vertices are in $S'(J_k)$ since

$$\frac{r_k\theta \pm 1/h}{r_k} = \theta \pm \frac{C_1 |J_k|}{2}$$

and similarly for the next pair and for m.

Using 4.4.2, we have for $1 \le k \le m-1$,

$$|\Gamma_k f| \le C(hr_{k+1} \min\{|\theta - \alpha_k|, |\theta - \beta_k|\} + 1) \times$$
$$\Big(P_{\alpha_k}\Big(\omega_{S'(J_k)} f\Big) + P_{\beta_k}\Big(\omega_{S'(J_k)} f\Big)\Big).$$

Since $\theta \in J_{k+1} \subset J_k$, $\min\{|\theta - \alpha_k|, |\theta - \beta_k|\} \le 2|J_{k+1}|$. Hence we get that

$$hr_{k+1} \min\{|\theta - \alpha_k|, |\theta - \beta_k|\} \le 6$$

which implies

$$|\Gamma_k f| \le \frac{4C}{C_1}\Big(P_{\alpha_k}\Big(\omega_{S'(J_k)} f\Big) + P_{\beta_k}\Big(\omega_{S'(J_k)} f\Big)\Big). \qquad (4.4.3)$$

For Γ_0, we note that to satisfy (4.4.2), we must have $|J_1| \ge 1/(1 + 2C_2)$. Since, $|\theta| \le 1$, we have that $Chr_1|\theta - 0|$ is bounded by some constant C. Hence, the lemma implies

$$|\Gamma_0 f| \le C P_0 f.$$

Finally, to bound $|\Gamma_m f|$ there are two cases. If $m = n$ then

$$|\Gamma_m f| = |\Gamma_n f| \leq P_\theta \omega_{S'(J_n)} f \tag{4.4.4}$$

using $\alpha = \beta = \theta$ in the lemma. Otherwise, we have $R \leq 2r_{m+1}$ so that $hR(2|J_{m+1}|) \leq 2h|J_{m+1}|\frac{4}{C_1 h|J_{m+1}|} = \frac{8}{C_1}$. Hence,

$$\begin{aligned}|\Gamma_m f| &\leq ChR\min\{|\theta - \alpha_{m+1}|, |\theta - \beta_{m+1}|\} + 1) \times \\ &\qquad (P_{\alpha_m}(\omega_{S'(J_m)} f) + P_{\beta_m}(\omega_{S'(J_m)} f)) \\ &\leq C'(P_{\alpha_m}(\omega_{S'(J_m)} f) + P_{\beta_m}(\omega_{S'(J_m)} f)).\end{aligned} \tag{4.4.5}$$

In either case, putting (4.4.3), (4.4.4) and (4.4.5) together,

$$\begin{aligned}|\Gamma^\theta_{R,h} f| &\leq \sum_{k=0}^{m} |\Gamma_k f| \\ &\leq C\Big(|P_0 f| + \sum_{k=0}^{m-1} |\Gamma_k f| + |\Gamma_m f|\Big) \\ &\leq C\Big(|P_0 f| + \sum_{k=0}^{m-1} \big(P_{\alpha_k}(\omega_{S'(J_k)} f) + P_{\beta_k}(\omega_{S'(J_k)} f)\big) + |\Gamma_m f|\Big) \\ &\leq C\Big(|P_0 f| + \sum_{k=0}^{n-1} \big(P_{\alpha_k}(\omega_{S'(J_k)} f) + P_{\beta_k}(\omega_{S'(J_k)} f)\big) + |P_\theta \omega_{S'(J_n)} f|\Big).\end{aligned}$$

□

Proof. (of Theorem 4.4.1) Let $\Omega \subset [0,1]$ be any Nth degree well-behaving set in $[0,1]$. Find Ω_k such that $\Omega = \Omega_N \supset \ldots \Omega_{N-1} \supset \cdots \supset \Omega_1$, as noted before. Fix an angle $\theta \in \Omega$ and $R, h > 0$. Suppose $\theta \in \Omega_m \setminus \Omega_{m-1}$ for some $m \leq N$. Denote by G_k the set of all intervals whose vertices are from $\Omega_k \setminus \Omega_{k-1}$ and are neighbouring points in Ω_k.

For $1 \leq k \leq m$, let $J_k = [\alpha_k, \beta_k] \in G_k$ be the interval that contains θ. By the definition of well-behaving sets, J_k satisfies the conditions of Lemma 4.4.3. Hence,

$$\begin{aligned}|M_\theta f|^2 &\leq C\left(M_0 f + \sum_{k=1}^{m}\Big(M_{\alpha_k}\big(\omega_{S'(J_k)} f\big) + M_{\beta_k}\big(\omega_{S'(J_k)} f\big)\Big)\right)^2 \\ &\leq C\left(|M_0 f|^2 + m\sum_{k=1}^{m}\big|M_{\alpha_k}\big(\omega_{S'(J_k)} f\big)\big|^2 + \big|M_{\beta_k}\big(\omega_{S'(J_k)} f\big)\big|^2\right)\end{aligned}$$

using arithmetic-quadratic mean inequality twice. Hence summing over every interval $J = (\alpha, \beta) \in G_j$,

$$\sup_{\theta\in\Omega} |M_\theta f|^2 \leq |M_0 f|^2 + N\sum_{k=1}^{N}\sum_{J=(\alpha,\beta)\in G_j} \big|M_\alpha\big(\omega_{S'(J)} f\big)\big|^2 + \big|M_\beta\big(\omega_{S'(J)} f\big)\big|^2. \tag{4.4.6}$$

We can bound the right hand side by the (2,2) bound of strong maximal operator for each $1 \le k \le N$,

$$\begin{aligned}
& \int_{\mathbb{R}^2} \sum_{J=(\alpha,\beta)\in G_j} \left|M_\alpha\big(\omega_{S'(J)}f\big)\right|^2 + \left|M_\beta\big(\omega_{S'(J)}f\big)\right|^2 dx \\
\le\ & C\left(\int \sum_{J=(\alpha,\beta)\in G_j} \chi_{S'(J)}|\widehat{f}|^2 d\xi\right) \qquad (4.4.7) \\
\le\ & 2C\int |\widehat{f}|^2 d\xi = 2C\int |f|^2 dx.
\end{aligned}$$

since the sum of the characteristic functions of $S'(J)$ is bounded by some constant C by the definition of well-behaving set. Integrating (4.4.6) and applying (4.4.7) we get $\|M_\Omega f\|_{L^2}^2 \le C(1+N^2)\|f\|_{L^2}$. □

We again get Katz's result 4.3.10 as a corollary. We only need to show that the set of directions of cardinality 2^N is an Nth degree well-behaving set. We fix $C_1 = 1/2$ and $C_2 = 1$ and use induction. Assume, without loss of generality, that the set of directions is contained in $[0,1]$. Suppose the middle point (2^{N-1}th point) is x. Simply consider the well-behaving set $v_k = (2.5)^k x$ for $k \le 0$ and $v_k = 1-(1-x)(2.5)^k$ for $k > 0$. Between each neighbouring point v_k and v_{k+1} sits a set of cardinality at most 2^{N-1} and so by induction, the set is Nth degree well-behaving. Since Katz's result is sharp as to the power of $(\log|\Omega|)$, we know that Theorem 4.4.1 is sharp as to the power of N.

Chapter 5

Further Discussion

We give a brief account of another very interesting variant of maximal operators - the Kakeya maximal operator.

5.1 Kakeya conjecture

As introduced earlier, the Kakeya maximal operator K_N in n dimensions has bases consisting of rectangles with lengths

$$a \times \cdots \times a \times aN \qquad N > 2. \tag{5.1.1}$$

As discovered by A. Cordoba [6], this problem is related to the study of the boundedness of Bochner-Riesz multipliers and it is also related to other important problems in harmonic analysis such as the restriction problem.

Since a rectangle with the above eccentricity can be included in a ball of radius $\sqrt{n}(aN)$, there is a dimensional constant C_n with the pointwise inequality

$$K_N f(x) \leq C_n N^{n-1} M f(x)$$

where M is the classical Hardy-Littlewood maximal operator. Using this inequality for the weak-type (1,1) estimate and interpolating with the trivial bound for L^∞, we get that $\|K_N\|_{L^p \to L^p}$ is proportional to $N^{(n-1)/p}$.

A conjecture claims that K_N is bounded on $L^n(\mathbb{R}^n)$ with a constant given by $C_n(\log N)^{\alpha_n}$ for some $C_n, \alpha_n > 0$ [28]. From this, a much sharper estimate follows by interpolation:

Conjecture 5.1.1. *For $1 < p < \infty$ we have $\|K_N f\|_{L^p} \leq C(p,N)\|f\|_{L^p}$ for*

$$C(p,N) = \begin{cases} C(\log N)^{\alpha(p)}, & \text{if } p \geq n; \\ CN^{n/p-1}(\log N)^{\alpha(p)}, & \text{if } 1 < p < n \end{cases}$$

where the exponent $\alpha(p)$ is nonnegative and it is strictly positive for $p \geq n$.

We note that the problem for the Kakeya maximal operator is equivalent to considering a directional maximal operator in some finite equidistributed set $\Omega \subset S^{n-1}$ [10]. Here, the term "equidistributed" means that for any $u \in S^{n-1}$, there is a direction $v \in \Omega$ such that the distance between u and v is less than $1/N$. We can replace each of the rectangles as in (5.1.1) in direction u with rectangles in direction $v \in \Omega$ with comparable size and vice versa. Note that the number of directions needed for such Ω is in order of N^{n-1}. Hence, the Kakeya conjecture for K_N is controlled by the similar conjecture for M_Ω with equidistributed $|\Omega| = N^{n-1}$. Via this equivalence, we have proven, as a corollary to 4.3.10, that Conjecture 5.1.1 is true for $n = 2$.

Recall that the Besicovitch set we constructed in Section 3.5 could have measure 0. However, it is believed that the fractal dimensions of the Besicovitch set is large. Indeed, if the norm conjecture for the Kakeya maximal operator holds true for some $p_0 \leq n$, then the Hausdorff dimension of a Besicovitch set is at least p_0 [10]. By adding two dimensional formulations of the Kakeya conjecture, we get the following Kakeya conjecture.

Conjecture 5.1.2. *(the Kakeya conjecture) We have that*
(a) the box dimension of the Besicovitch set in $\mathbb{R}^n$ is n;
(b) the Hausdorff dimension of the Besicovitch set in $\mathbb{R}^n$ is n;
(c) the norm conjecture (Conjecture 5.1.1) is true for $p = n$.

Note that (c) $\Rightarrow$ (b) $\Rightarrow$ (a) [20]. All three versions are true for $n = 2$ and all three are still open for $n \geq 3$.

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