Combinatorial Approaches To The Jacobian Conjecture

by

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A thesis
presented to the University of Waterloo
in fulfilment of the
thesis requirement for the degree of
Master of Mathematics
in
Combinatorics and Optimization

Waterloo, Ontario, Canada, 2007

Abstract

The Jacobian Conjecture is a long-standing open problem in algebraic geometry. Though the problem is inherently algebraic, it crops up in fields throughout mathematics including perturbation theory, quantum field theory and combinatorics. This thesis is a unified treatment of the combinatorial approaches toward resolving the conjecture, particularly investigating the work done by Wright and Singer. Along with surveying their contributions, we present new proofs of their theorems and motivate their constructions. We also resolve the Symmetric Cubic Linear case, and present new conjectures whose resolution would prove the Jacobian Conjecture to be true.

Acknowledgements

I would like to thank Prof. Ian Goulden, my supervisor, for his advice throughout the research and writing process. His guidance and feedback has been invaluable. I also thank the thesis readers, Prof. Chris Godsil and Prof. David Wagner, for their assistance in fine tuning the document. I thank Eric Martin for his assistance in editing and writing. He has played a key role in developing my mathematical writing skills. I would like to thank my office mate Graeme Kemkes for his moral support and advice throughout the process. I would like to thank the Government of Ontario Ministry of Training, Colleges and Universities for providing financial support by means of a postgraduate scholarship. I especially thank the people who I credit for fostering my interest in mathematics, especially Dora DiRocco-Smith, who recognized my passion for mathematics and guided me to pursue it. Finally, and most importantly, I thank my parents Hanan and Abdissalam Omar for their love and support throughout my education.

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Chapter 1

The Jacobian Conjecture

1.1 Introduction

The Jacobian Conjecture is one of the most well known open problems in mathematics. The problem was originally formulated by Keller [5] in 1939. In the late 1960s, Zariski and his student Abhyankar were among the main movers of the conjecture, and motivated research in the area. Since then, hundreds of papers have been published on the subject, using approaches from many different areas of mathematics including analysis, algebra, combinatorics and complex geometry. This thesis focuses on the combinatorial approaches toward resolving the conjecture.

The first chapter provides an overview of the work done toward resolving the Jacobian Conjecture. In this chapter, we give an overview of the material covered throughout the rest of the thesis.

The second chapter is dedicated to the pioneering work of Bass, Connell and Wright [2] in finding a combinatorial means of presenting the Jacobian Conjecture. We first outline the work done by Abhyankar [1] in establishing an easily expressible formal inverse for any multivariable polynomial with complex coefficients. Using the formal expansion of the inverse, we detail how Bass, Connell, and Wright found a combinatorial interpretation of its summands. This led to a combinatorial formulation of the Jacobian Conjecture.

The third chapter investigates the combinatorial consequences of a reduction due to De Bondt and Van den Essen [3]. We then show how Wright [13] used this reduction to solve cases of the Jacobian

Conjecture. We also show how Wright's work was used to formulate the Jacobian Conjecture in a different light than the earlier approach by Bass, Connell and Wright [2].

In the fourth chapter, we see how Singer [9] used a more refined combinatorial structure than Wright [11] to express the formal inverse of a function combinatorially. We show that this approach provides a more systematic method for resolving cases of the Jacobian Conjecture than Wright's method [11].

1.2 Jacobian Conjecture

A function $F : \mathbb{C}^n \to \mathbb{C}^n$ is defined to be *polynomial* if it is of the form $F = (F_1, \ldots, F_n)$ where $F_i \in \mathbb{C}[x_1, \ldots, x_n]$ for all $1 \leq i \leq n$. We define JF to be the *Jacobian* of the function F. In other words, JF is the matrix in $M_{n\times n}(\mathbb{C}[x_1, \ldots, x_n])$ with $JF_{i,j} = D_iF_j$ where $D_i = \frac{\partial}{\partial x_i}$. The determinant of JF will be denoted $|JF|$. As an example, consider $E : \mathbb{C}^3 \to \mathbb{C}^3$ given by $E(x_1, x_2, x_3) = (2x_1 + x_2^2, x_2^3 - x_3^5, (1 + 3i)x_3^8)$. Then E is polynomial, and

$$JE = \begin{pmatrix} 2 & 2x_2 & 0 \\ 0 & 3x_2^2 & -5x_3^4 \\ 0 & 0 & 8(1+3i)x_3^7 \end{pmatrix},$$

so $|JE| = 48(1 + 3i)x_2^2x_3^7$.

Notice that for any polynomial function F, $|JF|$ is a function from $\mathbb{C}^n \to \mathbb{C}$. It is an elementary theorem in calculus that if $F : \mathbb{C}^n \to \mathbb{C}^n$ is invertible at a point $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$ then $|JF|(a)$ is non-zero. Using this, we can establish a necessary condition in order for a polynomial function F to be invertible on all of $\mathbb{C}^n$.

Theorem 1.2.1. [2] *Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map. If F is invertible everywhere on $\mathbb{C}^n$, then $|JF|$ is a non-zero constant.*

Proof. We prove the contrapositive, so we assume that $|JF|$ is either 0 or not constant. If $|JF| = 0$, then $|JF|(a) = 0$ for all $a \in \mathbb{C}^n$, so F is not invertible. If $|JF|$ is not constant, then $|JF|$ is a non-constant polynomial in $\mathbb{C}[x_1, \ldots, x_n]$. Since $\mathbb{C}$ is algebraically closed, there exists a solution $a \in \mathbb{C}^n$ to the polynomial equation $|JF| = 0$. Thus $|JF|(a) = 0$, contradicting the invertibility of F at the point a. $\square$

A natural question to then ask is whether or not the converse of Theorem 1.2.1 is true. This question is easily answered when F is a linear operator on $\mathbb{C}^n$. Our proof is self-contained.

Theorem 1.2.2. *A linear operator F on $\mathbb{C}^n$ is invertible if and only if $|JF|$ is a non-zero constant.*

Proof. Since F is linear, $JF \in M_{n\times n}(\mathbb{C})$ and $F(x_1,\ldots,x_n) = (JF)[x_1 \cdots x_n]^T$. Assume $|JF|$ is a non-zero constant. Then from elementary linear algebra, JF is invertible. Letting $G : \mathbb{C}^n \to \mathbb{C}^n$ be defined by $G(x_1,\ldots,x_n) = (JF)^{-1}[x_1 \cdots x_n]^T$, we see that G is the inverse of F. The converse follows by Theorem 1.2.1. □

If we change certain conditions on F in Theorem 1.2.1, the converse will not necessarily be true. Consider the following counterexamples:

Characteristic p $\neq$ 0: Instead of working in the algebraically closed field $\mathbb{C}$, consider working in $\mathbb{Z}_p$. It is not true that if $F : \mathbb{Z}_p^n \to \mathbb{Z}_p^n$ and $|JF|$ is a non-zero constant then F must be invertible. As an example, consider the function $F : \mathbb{Z}_p \to \mathbb{Z}_p$ given by $F(x) = -x^p + x$. Then we have $JF = -px^{p-1} + 1 = 1$, but $F(x)$ is 0 everywhere by Fermat's Little Theorem, and is therefore not invertible.

Analytic Functions: Instead of restricting to polynomial functions, consider working with any analytic function F. It is again not necessarily true that if $|JF|$ is a non-zero constant, then F is invertible. Consider $F : \mathbb{C}^2 \to \mathbb{C}^2$, $F_1 = e^{x_1}, F_2 = x_2e^{-x_1}$. Then $|JF| = 1$ but F is not surjective (it does not map to (0,y) for any $y \in \mathbb{C}$).

From these examples, it is natural to ask if polynomial functions on $\mathbb{C}^n$ are a class of functions satisfying the converse of Theorem 1.2.1. In other words, it is natural to ask if every polynomial function $F : \mathbb{C}^n \to \mathbb{C}^n$ satisfying $|JF| \in \mathbb{C}$ is globally invertible on $\mathbb{C}^n$. This problem, known as the Jacobian Conjecture, is the crux of this thesis.

Conjecture 1.2.3. *(Jacobian Conjecture)* [5] *Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial function. If $|JF|$ is a non-zero constant, then F is globally invertible on $\mathbb{C}^n$. That is, there exists a polynomial function $G : \mathbb{C}^n \to \mathbb{C}^n$ such that $F(G(x_1,\ldots,x_n)) = (x_1,\ldots,x_n)$ for all $1 \leq i \leq n$.*

The following is an example supporting the Jacobian Conjecture. Let $F : \mathbb{C}^2 \to \mathbb{C}^2$, $F = (F_1, F_2)$ where $F_1 = x_1 + (x_1 + x_2)^2$ and $F_2 = x_2 - (x_1 + x_2)^2$.

The determinant of the Jacobian of F is

$$\begin{vmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} \\ \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} \end{vmatrix} = \begin{vmatrix} 1+2x_1+2x_2 & 2x_1+2x_2 \\ \\ -2x_1-2x_2 & 1-2x_1-2x_2 \end{vmatrix} = 1.$$

Thus the Jacobian Conjecture predicts that F has an inverse on all of $\mathbb{C}^n$. Indeed it does. Notice that

$$F_1 - (F_1+F_2)^2 = x_1,$$

and

$$F_2 + (F_1+F_2)^2 = x_2.$$

It follows that the inverse of the map $F = (F_1, F_2)$ is the map $G = (G_1, G_2)$ defined by

$$G_1 = x_1 - (x_1+x_2)^2, \qquad G_2 = x_2 + (x_1+x_2)^2.$$

We now make some key remarks that will allow us to further restrict the set of functions we need to consider for the Jacobian Conjecture. If $T : \mathbb{C}^n \to \mathbb{C}^n$ is invertible, then deciding whether F is invertible is equivalent to deciding whether $T \circ F$ is invertible. Now let $T = I - F(0)$, the operator that translates each point in $\mathbb{C}^n$ by $F(0)$. Note that T is invertible. Moreover, we have that $(T \circ F)(0) = 0$. Thus, replacing F by $T \circ F$, we can assume that $F(0) = 0$, so F has no constant term. Furthermore, if we let $F_{(1)}$ be the linear term of F, then $F_{(1)} = JF(0)[x_1 \dots x_n]^T$. Under the conditions of the Jacobian Conjecture, $|JF|(0) \neq 0$, so $F_{(1)}$ is invertible. Thus if we let $T = (F_{(1)})^{-1}(F)$, we have that $T_{(1)} = I$. Thus $T_i = x_i - H_i$ where all terms in H_i have degree ≥ 2, $1 \leq i \leq n$. In conclusion, we can assume that $F_i = x_i - H_i$ where all terms in H_i have degree ≥ 2, $1 \leq i \leq n$. We now state this as a theorem.

Theorem 1.2.4. *[2] The Jacobian Conjecture holds if it is true for maps $F : \mathbb{C}^n \to \mathbb{C}^n$ satisfying $F = x - H$ where $x = (x_1, \dots, x_n)$, $H = (H_1, \dots, H_n)$, and each H_i has degree at least 2 for each $1 \leq i \leq n$.*

We have just seen that we can reduce the Jacobian Conjecture to a specific type of polynomial function. Many of the major results toward resolving the Jacobian Conjecture are of this type. Other results include the resolution of special cases. In the next section, we survey some of the early results.

1.3 History and Background

It is believed that the Jacobian Conjecture was first posed by O.H. Keller in 1939 [5]. One of the first major results toward its resolution was due to Moh [7]. He used methods in algebraic geometry combined with computer assistance to verify that the Jacobian Conjecture holds for the case when $n = 2$ and the degree of F is at most 100. Wang [10] generalized Moh's result by proving that the conjecture is true for all maps whose components have maximum degree 2, for every n. Oda and Yoshida [8] provided a very short proof of Wang's result. We outline Oda and Yoshida's proof, but first state a theorem that is essential to the proof.

Theorem 1.3.1. [2] *Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map with JF invertible. Then the following are equivalent:*

1. *F is invertible.*
2. *F is injective.*
3. *$\mathbb{C}[x_1, \ldots, x_n]$ is a finitely generated $\mathbb{C}[F_1, \ldots, F_n]$-module.*

We now state the theorem of Wang [10] but provide Oda and Yoshida's proof.

Theorem 1.3.2. [8] *Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map with JF a non-zero constant. Further assume that the degree of every component of F is at most 2. Then F is invertible.*

Proof. To show that F is invertible, we show that F is injective, and the result follows by Theorem 1.3.1. Assume otherwise for contradiction. Then there exists $a \neq b \in \mathbb{C}^n$ such that $F(b) = F(a)$. Let $c = b - a$. Consider the function $S : \mathbb{C}^n \to \mathbb{C}^n$ given by $S(x) = F(x + a) - F(x)$. Then S has degree at most 2 (since it is the difference of two quadratic functions) and $S(c) = F(b) - F(a) = 0$. Split S into its homogeneous degree 1 and homogeneous degree 2 parts, say $S_{(1)}$ and $S_{(2)}$ respectively. Then we have

$$\begin{aligned} 0 = S(c) &= S_{(1)}(c) + S_{(2)}(c) = S_{(1)}(c) + 2(\frac{1}{2})S_{(2)}(c) \\ &= \frac{d}{dt}(S_{(1)}(c)t + S_{(2)}(c)t^2)|_{t=\frac{1}{2}} = \frac{d}{dt}(S(tc))|_{t=\frac{1}{2}} \\ &= JS(\frac{c}{2}) \cdot c \end{aligned}$$

But $c \neq 0$ and $JS(\frac{c}{2}) \neq 0$, a contradiction. $\square$

Apart from the algebraic approaches we've seen thus far, Abhyankar attempted a different approach to the Jacobian Conjecture. In all the cases we are considering, $F(0) = 0$ and $JF(0)$ is invertible, so F has a formal inverse $G = (G_1, \dots, G_n)$ where G_i is in the formal power series ring $\mathbb{C}[[x_1, \dots, x_n]]$ for all $1 \leq i \leq n$. Thus the Jacobian Conjecture is the problem of whether or not the formal inverse of F is in fact polynomial in every component. In particular, assume $F = (F_1, \dots, F_n)$ where $F_i \in \mathbb{C}[x_1, \dots, x_n]$ for all $1 \leq i \leq n$. We can consider F_i as lying in $\mathbb{C}[[x_1, \dots, x_n]]$. We seek $G_1, \dots, G_n \in \mathbb{C}[[x_1, \dots, x_n]]$ such that $G_i(F) = x_i$ for all $1 \leq i \leq n$ and aim to prove that under the conditions of the Jacobian Conjecture, $G_i \in \mathbb{C}[x_1, \dots, x_n]$ for all $1 \leq i \leq n$. An advantage to this approach is that, in certain cases, classical theorems can be used to arrive at an inverse immediately. For instance, consider when $F_i = x_i H_i$, $H_i \in \mathbb{C}[[x_1, \dots, x_n]]$ for all $1 \leq i \leq n$. Then G can be determined explicitly by using the multivariable form of Lagrange's Implicit Function Theorem. Such results convinced Abhyankar and his student Gurjar [1] to seek a general inverse formula for F. They succeeded, and their findings led to the following result.

Theorem 1.3.3. *(Abhyankar's Inversion Formula) [1] Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial function, $F = x - H$. Let $G = (G_1, \dots, G_n)$ be the formal power series inverse of F. Then*

$$G_i = \sum_{\substack{p \in \mathbb{N}^n \\ p = (p_1, \dots, p_n)}} \frac{D_1^{p_1} \cdots D_n^{p_n}}{p_1! \cdots p_n!} (x_i \cdot |J(F)| \cdot H_1^{p_1} \cdots H_n^{p_n}).$$

Abhyankar's Inversion Formula sparked the movement toward a combinatorial approach to the Jacobian Conjecture. To start this movement, Bass, Connell and Wright used the formula to find an expression for the formal power series inverse of F. They showed that the inverse could be expressed as a sum of products of differential operators acting on F, indexed by vertex-coloured trees. Before introducing this formally, some notation and definitions are needed. We denote the set of rooted trees by $\mathbb{T}_{rt}$. If $T \in \mathbb{T}_{rt}$, we denote by $Aut(T)$ the automorphism group of T as a rooted ordered tree. If $v \in V(T)$ we denote by v^+ the set of *children* of v in T. That is, v^+ is the set of vertices adjacent to v whose distance from the root of T (which we denote by $rt(T)$) is greater than that of v. A *colouring* of $V(T)$ with n colours is a function $c : V(T) \to \{1, \dots, n\}$. We denote $c(v^+)$ to be the multiset of colours assigned to the vertices in v^+. Finally, given a polynomial map $F = x - H$ on $\mathbb{C}^n$ we say F is of *homogeneous type of degree d* if every component of H is homogeneous of degree exactly d. We have the following inverse formula by Bass, Connell and Wright [2].

Theorem 1.3.4. *(Bass-Connell-Wright Tree Inversion Formula)[2] Let $F = x - H$ be of homogeneous type of degree $d \geq 2$, and let $G = (G_1, \dots, G_n)$ be the formal inverse of F. Then $G = (x_1 + N_1, \dots, x_n +$*

N_n) where

$$N_i = \sum_{T \in \mathbb{T}_{rt}} \frac{1}{|Aut(T)|} \sum_{\substack{c:V(T)\to\{1,\ldots,n\} \\ c(rt_T)=i}} \prod_{v \in V(T)} D_{c(v^+)} H_{c(v)}.$$

In addition to the Bass-Connell-Wright Tree Inversion Formula, Bass, Connell, and Wright reduced the Jacobian Conjecture to maps of homogeneous type of degree 3.

Theorem 1.3.5. *(Cubic Reduction)[2] The Jacobian Conjecture is true if it holds for maps $F : \mathbb{C}^n \to \mathbb{C}^n$ of homogeneous type of degree 3, for all $n \geq 1$.*

Shortly after this, Druzkowski [4] showed that the reduction can be refined.

Theorem 1.3.6. *(Cubic Linear Reduction)[4] The Jacobian Conjecture is true if it holds for maps $F : \mathbb{C}^n \to \mathbb{C}^n$ of homogeneous type of degree 3 such that $F_i = x_i - L_i^3$ where L_i is linear for all $1 \leq i \leq n$.*

Only a few cases of the Cubic and Cubic Linear Reductions have been settled. One of these cases was established in 1993, when Wright [12] proved that the Cubic Reduction holds for maps from $\mathbb{C}^3$ to itself. Further work by Hubbers [6] established that the Cubic Reduction holds for maps from $\mathbb{C}^4$ to itself. Though only a handful of cases were settled, combinatorial approaches to the Jacobian Conjecture were still developing. In 2001, Singer [9] discovered an approach to the Jacobian Conjecture that expressed the formal inverse of a function in terms of a sum of weight functions applied to Catalan trees (see Chapter 4). Using this approach, he was able to find a different means of combinatorially attacking the conjecture than that of Bass, Connell and Wright. He also found stronger results for special cases. We summarize his results.

Theorem 1.3.7. *[9] Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map with $F = x + H$ and $|JF|$ a non-zero constant. Then*

1. *If the polynomials H_i are homogeneous of total degree 2 and $(JH)^3 = 0$, then $H \circ H \circ H = 0$ and F has inverse $G = (G_1, \ldots, G_n)$, $deg(G_i) \leq 6$ for all $1 \leq i \leq n$.*
2. *If the polynomials H_i are homogeneous of total degree at least 2 and $(JH)^2 = 0$, then $H \circ H = 0$, and the inverse of F is $G = x - H$.*

In 2003, De Bondt and Van den Essen [3] reduced the Jacobian Conjecture to the case when the Jacobian matrix is symmetric. Before introducing their theorem, we make a key observation. Recall we can assume that any function $F : \mathbb{C}^n \to \mathbb{C}^n$ is of the form $F = x - H$ where every term in H_i has degree at least 2, $1 \leq i \leq n$ (by Theorem 1.2.4). Now $JF = I_{n \times n} - JH$ where $I_{n \times n}$ is the $n \times n$ identity matrix, and JH is the Jacobian matrix of the map $H = (H_1, \ldots, H_n)$. Thus if JF is symmetric, JH is symmetric. Since JH is symmetric, for all $1 \leq i, j \leq n$, $\frac{\partial}{\partial x_i} H_j = \frac{\partial}{\partial x_j} H_i$. Thus JH is the Hessian of some polynomial $P \in \mathbb{C}[x_1, \ldots, x_n]$. Consequently, $H = \nabla P$. The result of De Bondt and Van den Essen can then be summarized as follows:

Theorem 1.3.8. *(Symmetric Reduction)[3] The Jacobian Conjecture is true if it holds for maps* $F : \mathbb{C}^n \to \mathbb{C}^n$ *with* $F = x - H$, H *homogeneous of degree 3, and* $H = \nabla P$ *for some polynomial* $P \in \mathbb{C}[x_1, \ldots, x_n]$ *of degree 4.*

In [13], Wright simplified the Bass-Connell-Wright Tree Inversion Formula for the symmetric case. The simplification is as follows:

Theorem 1.3.9. *(Symmetric Tree Formula)[13] Let* $F : \mathbb{C}^n \to \mathbb{C}^n$, $F = x - \nabla P$, *and let* $G = (G_1, \ldots, G_n)$ *be its inverse. Then* $G = x + \nabla Q$,

$$Q = \sum_{T \in \mathbb{T}} \sum_{c : E(T) \to \{1, \ldots, n\}} \prod_{v \in V(T)} D_{inc(v)} P,$$

where $inc(v)$ *is the set of edges* $\{e_1, \ldots, e_k\}$ *incident to* v, $\mathbb{T}$ *is the set of isomorphism classes of unrooted trees, and* $D_{inc(v)} = D_{c(e_1)} \ldots D_{c(e_k)}$.

Wright used the Symmetric Tree Formula to find combinatorial properties that emulate the conditions given in the Jacobian Conjecture. He did this by developing a relationship between the tree formula and a combinatorial algebra (the Grossman-Larson Algebra). By doing so, he was able to set up a systematic computational method for solving the homogeneous degree 3 symmetric case. Because of the Symmetric Reduction and the Cubic Reduction, the computational method provided a tractable means of resolving the entire conjecture.

Notice the extent to which combinatorial approaches to the Jacobian Conjecture have led to significant reductions and the resolution of special cases. We can further our understanding of the problem by studying these combinatorial approaches. This thesis does just that. In the second chapter, we look at the development of the Bass-Connell-Wright Tree Inversion Formula and its combinatorial implications. In the third chapter, we investigate Wright's contributions, particularly those that have resulted from the Symmetric Tree Formula and its relationship with the Grossman-Larson Algebra. In the fourth chapter, Singer's approach is detailed. We then look at extensions of these approaches, and conclude with conjectures that arise from them.

Chapter 2

Bass-Connell-Wright Tree Inversion

2.1 Introduction

In this chapter we investigate the pioneering contribution to the development of a combinatorial approach to the Jacobian Conjecture. This development was due to Bass, Connell and Wright in their paper "The Jacobian Conjecture: Reduction of Degree and Formal Expansion of the Inverse" [2]. In the paper, Bass, Connell and Wright made a significant reduction to the Jacobian Conjecture. This is discussed in Section 2.2. This reduction played a role in the development of the Bass-Connell-Wright Tree Inversion Formula for polynomial functions. The Bass-Connell-Wright Tree Inversion Formula led to the first successful presentation of the Jacobian Conjecture as a combinatorial problem. The remainder of the chapter concentrates on the development of the formula. To develop the Bass-Connell-Wright Tree Inversion Formula, Bass, Connell and Wright started by making direct use of Abhyankar's Inversion Formula [1]. In Section 2.3, we give a detailed the proof of Abhyankar's Inversion Formula. We then show how this was extended to a more detailed formal inverse formula. After this refinement, we show how Bass, Connell and Wright found certain labelled trees that encoded all its important information. We use this in Section 2.4 to develop the Bass-Connell-Wright Tree Inversion Formula. We end in Section 2.5 with some observations and computations.

2.2 Reduction Theorem

Bass, Connell and Wright are credited for being the first to make major breakthroughs toward the resolution of the Jacobian Conjecture. Their main contributions are presented in [2]. In this paper, they accomplish two feats. First, they significantly reduce the problem to a special case. They establish that in order to prove the Jacobian Conjecture, it suffices to prove it for maps of homogeneous type of degree exactly 3. They further reduce the problem by showing that one can assume that if F is of homogeneous type of degree 3 with $F = x - H$, then the map $H = (H_1, \ldots, H_n)$ has a *nilpotent* Jacobian matrix. That is, $(JH)^m = 0$ for some positive integer m (which is equivalent to $(JH)^n = 0$ since JH is an $n \times n$ matrix). The proof of this significant reduction has yet to provide combinatorial insight. As a consequence, we state the theorem but omit its proof.

Theorem 2.2.1. *(Reduction Theorem) [2] The Jacobian Conjecture is true if it holds for maps of homogeneous type of degree 3 with a nilpotent Jacobian matrix. That is, the Jacobian Conjecture is true if it holds for maps* $F : \mathbb{C}^n \to \mathbb{C}^n$ *with* $F = x - H$ *, where* H_i *is homogeneous of degree 3 for all* $1 \leq i \leq n$ *and* $H = (H_1, \ldots, H_n)$ *satisfies* $(JH)^n = 0$.

Though the proof of this theorem does not seem to provide any combinatorial insight into the Jacobian Conjecture, it is necessary for refining Abhyankar's Inversion Formula.

2.3 Abhyankar's Inversion Formula

Before developing Abhyankar's Inversion Formula, we need some notation. We define $\mathbb{N}$ to be the set $\{0, 1, 2, \ldots\}$. If $p \in \mathbb{N}^n$ with $p = (p_1, \ldots, p_n)$ we define

$$p! = p_1! \cdots p_n!$$

If $a = (a_1, \ldots, a_n)$ is an n-tuple of objects in any $\mathbb{Q}$-algebra, define

$$a^p = a_1^{p_1} \cdots a_n^{p_n}.$$

Similarly, if D_i denotes the *differential operator* $\frac{\partial}{\partial x_i}$ on $\mathbb{C}[x_1, \ldots, x_n]$, then we define

$$D^p = D_1^{p_1} \cdots D_n^{p_n}.$$

Now recall Abhyankar's Inversion Formula (Theorem 1.3.3).

Theorem 2.3.1. *(Abhyankar's Inversion Formula) [1] Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be polynomial, $F = x - H$. Let $H = (H_1, \ldots, H_n)$, and let $G = (G_1, \ldots, G_n)$ be the formal power series inverse of F. Then*

$$G_i = \sum_{\substack{p \in \mathbb{N}^n \\ p=(p_1,\ldots,p_n)}} \frac{1}{p_1! \cdots p_n!} D^p(x_i \cdot |J(F)| \cdot H^p).$$

In order to prove this theorem, we prove the following theorem, and show that Abhyankar's Inversion Formula is a corollary of it.

Theorem 2.3.2. *(Abhyankar) [1] Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be polynomial, $F = x - H$. For all $U \in \mathbb{C}[[x_1, \ldots, x_n]]$, define*

$$< U, F >= \sum_{\substack{p \in \mathbb{N}^n \\ p=(p_1,\ldots,p_n)}} \frac{1}{p_1! \cdots p_n!} D^p(U(F) \cdot |J(F)| \cdot H^p). \tag{2.1}$$

Then $< U, F >= U$.

Proof. (of Theorem 2.3.1) Assume that the inverse of F is $G = (G_1, \ldots, G_n)$ as in Theorem 2.3.1. Let $U = G_i$ in Theorem 2.3.2. Then $U(F) = G_i(F) = x_i$ since G is the inverse of F. Furthermore, $F = x - H$ implies $H = x - F$. The result follows immediately from these observations. ☐

We now prove Theorem 2.3.2.

Proof. (Theorem 2.3.2) First, we experiment with the formula $< U, F >$ in a very simple case. We assume that F is a function in only one variable, say x, and $U = x^m$ for some positive integer. By the Reduction Theorem (Theorem 2.2.1), we can assume that $F = x - H$ where H is a homogeneous polynomial. Then $U(F) = F^m = (x - H)^m$. Letting $D = \frac{\partial}{\partial x}$, we see that since F is a map in one variable, $|JF| = DF$. Thus $|JF| = D(x - H) = 1 - DH$. Using these observations, we can apply (2.1) to get

$$< x^m, F >= \sum_{p=0}^{\infty} \frac{D^p}{p!}((x - H)^m(1 - DH)H^p). \tag{2.2}$$

Our aim is to prove that the expression in (2.2) is in fact equal to x^m. This can be done directly. We see that

$$\begin{aligned}
\sum_{p=0}^{\infty}\frac{D^p}{p!}((x-H)^m(1-DH)H^p) &= \sum_{p=0}^{\infty}\frac{D^p}{p!}\sum_{i=0}^{m}(-1)^i\binom{m}{i}x^{m-i}(H^{p+i}-H^{p+i}DH)\\
&= \sum_{p=0}^{\infty}\sum_{i=0}^{m}\sum_{j=0}^{min(p,m-i)}(-1)^i\binom{m}{i}\frac{D^j}{j!}x^{m-i}\cdot\frac{D^{p-j}}{(p-j)!}(H^{p+i}-H^{p+i}DH)\\
&= \sum_{p=0}^{\infty}\sum_{i=0}^{m}\sum_{j=0}^{min(p,m-i)}(-1)^i\frac{m!(m-i)!}{i!(m-i)!j!(p-j)!(m-i-j)!}\\
&\cdot x^{m-i-j}\cdot D^{p-j}(H^{p+i}-H^{p+i}DH).
\end{aligned}$$

Now substitute $t=i+j$ and $q=p-j$. The sum can be re-written as

$$\begin{aligned}
&\sum_{t=0}^{m}\sum_{i=0}^{t}\sum_{q=0}^{\infty}(-1)^i\frac{m!}{i!(t-i)!q!(m-t)!}x^{m-t}D^q(H^{q+t}-H^{q+t}DH)\\
&=\sum_{t=0}^{m}\sum_{i=0}^{t}\sum_{q=0}^{\infty}(-1)^i\binom{t}{i}\binom{m}{t}x^{m-t}\frac{D^q}{q!}(H^{q+t}-H^{q+t}DH)\\
&=\sum_{t=0}^{m}\binom{m}{t}x^{m-t}\left(\sum_{i=0}^{t}(-1)^i\binom{t}{i}\right)\cdot\left(\sum_{q=0}^{\infty}\frac{D^q}{q!}(H^{q+t}-H^{q+t}DH)\right)
\end{aligned}$$

Two key observations are needed in order to establish the result we want. First, we know that $\sum_{t=0}^{m}\binom{m}{t}x^{m-t}\left(\sum_{i=0}^{t}(-1)^i\binom{t}{i}\right)=\sum_{t=0}^{m}\binom{m}{t}x^{m-t}(1-1)^t$. Indexed by t, the summands are 0 for $t>0$ and x^m for $t=0$. When $t=0$, the sum indexed by q becomes

$$\begin{aligned}
\sum_{q=0}^{\infty}\frac{D^q}{q!}(H^q-H^qDH) &= \sum_{q=0}^{\infty}\left(\frac{D^q}{q!}H^q-\frac{D^q}{q!}D\left(\frac{H^{q+1}}{q+1}\right)\right)\\
&= \sum_{q=0}^{\infty}\left(\frac{D^q}{q!}H^q-\frac{D^{q+1}}{(q+1)!}H^{q+1}\right)\\
&= 1.
\end{aligned}$$

Thus the only remaining term in the right hand side of (2.2) is x^m. We have thus proven the following lemma.

Lemma 2.3.3. [2] *Let $<U,F>$ be as defined in (2.1). Then $<x^m,F>=x^m$.*

Lemma 2.3.3 naturally generalizes to all functions. We establish this in steps. First, consider $F:\mathbb{C}^n\to\mathbb{C}^n$ where F only alters one variable, say x_1. That is, $F=(F_1,x_2,\ldots,x_n)$ for some function

F_1. We make some important observations about $< U, F >$ in this case. First, notice that JF is zero everywhere except possibly in the first column and on the diagonal. Thus JF is lower triangular, so $|JF|$ is the product of the diagonal entries of JF. But $(JF)_{1,1} = D_1F_1$ and $(JF)_{i,i} = D_iF_i = 1$ for all $2 \leq i \leq n$. Thus $|JF| = D_1F_1$. A second important observation is that for any $p \in \mathbb{N}^n$, $H^p = (x-F)^p = \prod_{i=1}^{n}(x_i - F_i)^{p_i}$ is zero unless p is of the form $(p_1, 0, \ldots, 0)$ since for every $i \geq 2$, $x_i - F_i = x_i - x_i = 0$. Using these two observations, we can simplify the expression $< U, F >$ to

$$\sum_{p_1=0}^{\infty} \frac{1}{p_1!} D_1^{p_1}(U(F) \cdot D_1F_1 \cdot (x_1 - F_1)^{p_1}). \tag{2.3}$$

Now factor out powers of x_1 in U. That is, write

$$U = \sum_{m=0}^{\infty} U_m(x_2, \ldots, x_n)x_1^m,$$

where each $U_m(x_2, \ldots, x_n) \in \mathbb{C}[[x_2, \ldots, x_n]]$. Then we have

$$\begin{aligned} < U, F > &= \sum_{m=0}^{\infty} U_m(x_2, \ldots, x_n) \sum_{p_1=0}^{\infty} \frac{1}{p_1!} D_1^{p_1}(F_1^m \cdot D_1F_1 \cdot (x_1 - F_1)^{p_1}) \\ &= \sum_{m=0}^{\infty} U_m(x_2, \ldots, x_n) < x_1^m, F_1 > \qquad \text{(by (2.1))} \\ &= \sum_{m=0}^{\infty} U_m(x_2, \ldots, x_n)x_1^m \qquad \text{(by (2.2))} \\ &= U. \end{aligned}$$

Thus we have established that $< U, F >= U$ for any function $F = (F_1, x_2, \ldots, x_n)$ that alters only the variable x_1. Notice that we could have chosen to alter x_i instead of x_1 for any i, $2 \leq i \leq n$, and the proof would remain the same. In other words, by changing indices, our proof shows that $< U, F >= U$ for any $F = (x_1, \ldots, x_{i-1}, F_i, x_{i+1}, \ldots, x_n)$, $1 \leq i \leq n$. We now state these conclusions in the following lemma.

Lemma 2.3.4. [2] *Let $F : \mathbb{C}^n \to \mathbb{C}^n$, $F = (x_1, \ldots, x_{i-1}, F_i, x_{i+1}, \ldots, x_n)$ for some $F_i \in \mathbb{C}[x_1, \ldots, x_n]$. Then using the notation from Theorem 2.3.2, $< U, F >= U$ for all $U \in \mathbb{C}[[x_1, \ldots, x_n]]$.*

To show that $< U, F >= U$ in general, we need two more steps. First, we show that the property $< U, F >= U$ is preserved under composition. That is, we show that if H and G satisfy $< U, H >= U$ and $< U, G >= U$ for all $U \in \mathbb{C}[[x_1, \ldots, x_n]]$, then so does $F = H(G)$. Secondly, we will show that every function F we are considering is the composition of functions of the form in Lemma 2.3.4. Combining these two results proves that $< U, F >= U$ in general.

The proof of the first step is a relatively straightforward computation. Suppose that $F = H(G)$ where $< U, H >= U$ and $< U, G >= U$ for all $U \in \mathbb{C}[[x_1, \ldots, x_n]]$. Then

$$\begin{aligned}
< U, F > &= \sum_{p \in \mathbb{N}^n} \frac{D^p}{p!} (U(H(G)) \cdot |J(H(G))| \cdot (x - H(G))^p) \\
&= \sum_{p \in \mathbb{N}^n} D^p \left(U(H(G)) \cdot |J(G)| \cdot |JH|(G) \cdot \frac{(x - G + G - H(G))^p}{p!} \right) \\
&= \sum_{p \in \mathbb{N}^n} D^p \left(U(H(G)) \cdot |J(G)| \cdot |JH|(G) \cdot \sum_{q+r=p} \frac{(G - H(G))^q}{q!} \frac{(x-G)^r}{r!} \right) \\
&= \sum_{q \in \mathbb{N}^n} D^q \left(\sum_{r \in \mathbb{N}^n} D^r \left(U(H(G)) \cdot |JH|(G) \cdot |J(G)| \cdot \frac{(G - H(G))^q}{q!} \frac{(x-G)^r}{r!} \right) \right) \\
&= \sum_{q \in \mathbb{N}^n} D^q \left\langle U(H) \cdot |J(H)| \cdot \frac{(x-H)^q}{q!}, G \right\rangle \\
&= \sum_{q \in \mathbb{N}^n} D^q \left(U(H) \cdot |J(H)| \cdot \frac{(x-H)^q}{q!} \right) \\
&=< U, H > \\
&= U.
\end{aligned}$$

For the second step, assume as usual that F is of the form $F = (F_1, \ldots, F_n)$. Since F is invertible in $\mathbb{C}[[x_1, \ldots, x_n]]$, we can uniquely define a function $T_i \in \mathbb{C}[[x_1, \ldots, x_n]]$ by the condition that

$$T_i(x_1, \ldots, x_i, F_{i+1}, \ldots, F_n) = F_i \qquad 1 \leq i \leq n.$$

Now define $H^{(i)} = (x_1, \ldots, x_{i-1}, T_i, x_{i+1}, \ldots, x_n)$. We see that $T_n = F_n$ by definition, and by induction on $n - i$, we have that for all $1 \leq i \leq n$,

$$H^{(i)} \circ \cdots \circ H^{(n)} = (x_1, \ldots, x_{i-1}, F_i, \ldots, F_n).$$

Thus $H^{(1)} \circ \cdots \circ H^{(n)} = F$, and so F is the composition of functions that alter only one variable. This completes the proof of Abhyankar's Inversion Formula. □

2.4 Bass-Connell-Wright Tree Inversion Formula

In this section we are concerned with furthering Abhyankar's Inversion Formula to establish a means of expressing the inverse as a tree sum. To do this, we first relate the inversion formula to sums indexed by functions on finite sets. These functions naturally give us our desired tree sums. To start off, we let G be the inverse of the polynomial map $F = x - H$ and recall that by Theorem 2.3.1,

$$G_i = \sum_{p \in \mathbb{N}^n} \frac{1}{p!} D^p(x_i \cdot H^p)$$

since we can assume $|JF| = 1$ by scaling F appropriately. Now define ${G_i}^{(d)}$ to be the homogeneous degree d component of G_i. Then it follows that

$${G_i}^{(d)} = \sum_{\substack{p \in \mathbb{N}^n \\ |p|=d}} \frac{1}{p!} D^p(x_i \cdot H^p).$$

Since $G_i = \sum_{d \geq 0} {G_i}^{(d)}$, G_i is polynomial if and only if ${G_i}^{(d)} = 0$ for sufficiently large d. We therefore have the following proposition.

Proposition 2.4.1. [2] *Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map. Let G be the inverse of F in $\mathbb{C}[[x_1, \ldots, x_n]]$, and let ${G_i}^{(d)}$ be the degree d component of G_i. Then F is invertible if and only if ${G_i}^{(d)} = 0$ for sufficiently large d, for all i, $1 \leq i \leq n$.*

The previous proposition motivates the in-depth study of the homogeneous polynomials ${G_i}^{(d)}$.

2.4.1 A Functionally Indexed Formula for ${G_i}^{(d)}$

As claimed in the introduction of this chapter, to work toward the development of a tree formula for the inverse G of a polynomial function F, we aim to express the expansion of G_i as a sum indexed by functions between finite sets. It will be useful to consider the functions ${G_i}^{(d)}$ separately when doing this. To begin this process, we start off with a definition motivated by our known expansion of ${G_i}^{(d)}$. For any function $L \in \mathbb{C}[[x_1, \ldots, x_n]]$ we define

$$L_{[d]} = d! \sum_{\substack{p \in \mathbb{N}^n \\ |p|=d}} \frac{1}{p!} D^p(L \cdot H^p) \tag{2.4}$$

As examples, consider $L_{[0]}$ and $L_{[1]}$. We have that

$$L_{[0]} = L, \tag{2.5}$$

$$\begin{aligned} L_{[1]} &= \sum_{p=1}^{n} D_p(LH_p) = \sum_{p=1}^{n}(D_pL)H_p + L \cdot \sum_{p=1}^{n} D_pH_p \\ &= \sum_{p=1}^{n}(D_pL)H_p + L \cdot \operatorname{Trace}(JH) = \sum_{p=1}^{n}(D_pL)H_p. \end{aligned}$$

We also have

$$G_i{}^{(d)} = \frac{1}{d!} x_{i[d]}. \tag{2.6}$$

From these equations, we deduce that

$$G_i{}^{(0)} = x_i, \tag{2.7}$$

and

$$G_i{}^{(1)} = H_i. \tag{2.8}$$

To continue toward our goal, we express $L_{[d]}$ as a sum indexed over functions from $\{1, \dots, d\}$ to $\{1, \dots, n\}$.

Lemma 2.4.2. *For $L \in \mathbb{C}[[x_1, \dots, x_n]]$ and $d \geq 0$,*

$$L_{[d]} = \sum_{r:\{1,\dots,d\}\to\{1,\dots,n\}} D_r(L \cdot H_r).$$

Here and in what follows, $D_r = D_{r_1} \cdots D_{r_d}$, $H_r = H_{r_1} \cdots H_{r_d}$ and $r_i = r(i)$.

Proof. For $r : \{1, \dots, d\} \to \{1, \dots, n\}$ define $p(r) = (|r^{-1}(1)|, \dots, |r^{-1}(n)|)$ where the i^{th} entry of $p(r)$ is the number of elements mapped to i under the function r. Note that $D_r = D^{p(r)}$ and $H_r = H^{p(r)}$, so D_r and H_r are defined uniquely by $p(r)$. It follows that the number of functions r that share the same sequences of preimages $p(r) = (p_1, \dots, p_n)$ is the multinomial coefficient $\frac{d!}{p(1)!\cdots p(n)!}$. Thus

$$\begin{aligned} \sum_{r:\{1,\dots,d\}\to\{1,\dots,n\}} D_r(L \cdot H_r) &= \sum_{\substack{p\in\mathbb{N}^n \\ |p|=d}} \frac{d!}{p_1!\cdots p_n!} D^p(L \cdot H^p) \\ &= \sum_{\substack{p\in\mathbb{N}^n \\ |p|=d}} d! \frac{D^p}{p!}(L \cdot H^p) = L_{[d]}. \end{aligned}$$

□

The summands in the new expression for $L_{[d]}$ are set up in such a way that it is natural to expand them using the product rule for derivatives. Before doing this, we need to introduce some notation. For any subset $S \subset \{1, \ldots, d\}$ we denote S^c to be the complement of S in $\{1, \ldots, d\}$. Furthermore, we use the natural notation $D_{r_S} = \prod_{i \in S} D_{r_i}$. We now expand the summands of $L_{[d]}$.

Lemma 2.4.3. [2] *Let* $L \in \mathbb{C}[[x_1, \ldots, x_n]]$*, and* $d > 0$*. Then*

$$L_{[d]} = \sum_{e=0}^{d} \binom{d}{e} \sum_{f:\{1,\ldots,e\}\to\{1,\ldots,n\}} (D_f L) \cdot (H_f)_{[d-e]}.$$

Proof. For any $d > 0$, we have by Lemma 2.4.2,

$$\begin{aligned}
L_{[d]} &= \sum_{r:\{1,\ldots,d\}\to\{1,\ldots,n\}} D_r(L \cdot H_r) \\
&= \sum_{r:\{1,\ldots,d\}\to\{1,\ldots,n\}} \sum_{S\subset\{1,\ldots,d\}} (D_{r_S} L)(D_{r_{S^c}} H_r) \\
&= \sum_{S\subset\{1,\ldots,d\}} \left(\sum_{f:S\to\{1,\ldots,n\}} \sum_{g:S^c\to\{1,\ldots,n\}} (D_f L) D_g(H_f H_g) \right) \\
&= \sum_{e=0}^{d} \binom{d}{e} \sum_{f:\{1,\ldots,e\}\to\{1,\ldots,n\}} (D_f L) \left(\sum_{g:\{1,\ldots,d-e\}} D_g(H_f H_g) \right) \\
&= \sum_{e=0}^{d} \binom{d}{e} \sum_{f:\{1,\ldots,e\}\to\{1,\ldots,n\}} (D_f L) \cdot (H_f)_{[d-e]}.
\end{aligned}$$

□

Notice that in Lemma 2.4.3, $L_{[d]}$ is defined in terms of expressions of the form $(H_f)_{[d-e]}$. These expressions can be further decomposed using the same lemma. The recursive nature of this decomposition leads to an easy inductive proof of the following lemma.

Lemma 2.4.4. [2] *Let* $L \in \mathbb{C}[[x_1, \ldots, x_n]]$ *and* $d > 0$*. Then*

$$L_{[d]} = \sum_{h=1} \sum_{\substack{e\in\{1,\ldots,d\}^h \\ |e|=d}} \binom{d}{e_1, \ldots, e_h} \sum_{\substack{f=(f_1,\ldots,f_h) \\ f_j:\{1,\ldots,e_j\}\to\{1,\ldots,n\}}} L_{e,f}$$

where $|e| = e_1 + \ldots + e_h$ *and*

$$L_{e,f} = (D_{f_1}L)(D_{f_2}H_{f_1})\cdots(D_{f_h}H_{f_{h-1}})H_{f_h}.$$

Proof. This immediately follows by induction on d, applying Lemma 2.4.3. $\square$

Applying Lemma 2.4.4 to ${G_i}^{(d)}$ we get the following result.

Proposition 2.4.5. [2] *We have* ${G_i}^{(0)} = x_i$, ${G_i}^{(1)} = H_i$, *and for* $d \geq 2$,

$$d!{G_i}^{(d)} = \sum_{\substack{e=(e_2,\ldots,e_h)\\ 1+e_2+\cdots+e_h=d}} \binom{d}{1, e_2, \ldots, e_h} \sum_{\substack{f=(f_2,\ldots,f_h)\\ f_j:\{1,\ldots,e_j\}\to\{1,\ldots,n\}}} (H_i)_{e,f},$$

where

$$(H_i)_{e,f} = (D_{f_2}H_i)(D_{f_3}H_{f_2})\cdots(D_{f_h}H_{f_{h-1}})H_{f_h}.$$

Maintaining the spirit of expressing ${G_i}^{(d)}$ completely in terms of sums indexed by functions, we aim to express $(H_i)_{e,f}$ in such a way. From the definition of $(H_i)_{e,f}$, we see that we can express D_fH_g as a sum indexed by functions. This can be done as follows. Assume $f : \{1, \ldots, e\} \to \{1, \ldots, n\}$ and $g : \{1, \ldots, e'\} \to \{1, \ldots, n\}$. Then we have that

$$D_fH_g = \sum_{u:\{1,\ldots,e\}\to\{1,\ldots,e'\}} (D_{f,u^{-1}(1)}H_{g(1)})\cdots(D_{f,u^{-1}(e')}H_{g(e')}) \tag{2.9}$$

where $D_{f,S} = \prod_{i\in S} D_{f(i)}$. Now substituting (2.9) into Lemma 2.4.4 gives us our final desired expression for ${G_i}^{(d)}$.

Lemma 2.4.6. [2] *For* $d > 0$,

$$d!({G_i}^{(d)}) = \sum_{h=2}^{d}\sum_{e}\sum_{f}\sum_{u} \binom{d}{1, e_2, \ldots, e_h} (H_i)_{e,f,u} \tag{2.10}$$

where the indices range as follows:

$$e = (e_2, \ldots, e_h), \qquad 1 + e_2 + \ldots + e_h = d$$

$$f = (f_2, \ldots, f_h), \qquad f_j : \{1, \ldots, e_j\} \to \{1, \ldots, n\}$$

$$u = (u_2, \ldots, u_h), \qquad u_j : \{1, \ldots, e_j\} \to \{1, \ldots, e_{j-1}\}$$

and

$$(H_i)_{e,f,u} = (D_{f_2}H_i) \cdot \big((D_{f_3,u_3{}^{-1}(1)}H_{f_2}(1)) \cdots (D_{f_3,u_3{}^{-1}(e_2)}H_{f_2}(e_2))\big)$$
$$\cdots \Big((D_{f_h,u_h{}^{-1}(1)}H_{f_{h-1}}(1)) \cdots (D_{f_h,u_h{}^{-1}(e_{h-1})}H_{f_{h-1}}(e_{h-1}))\Big) \cdot H_{f_h}.$$

We have now developed an inverse formula for ${G_i}^{(d)}$ whose summands are all indexed by functions. This expression for ${G_i}^{(d)}$ will serve as the key to developing the Bass-Connell-Wright Tree Inversion Formula, the focus of the next section.

2.4.2 Tree Inversion Formula

We aim to show that (2.10) can be expressed as a sum indexed by labelled rooted trees. First, notice that in (2.10), the index u only depends on the index e, so we can interchange the u-summation and the f-summation. Now given indices (e, u) in the two inner sums of this rearranged sum, we construct a vertex labelled rooted tree whose underlying structure is given by (e, u). The pair (e, u) gives rise to the following sequence of functions:

$$\{1, \ldots, e_h\} \xrightarrow{u_h} \{1, \ldots, e_{h-1}\} \xrightarrow{u_{h-1}} \cdots \xrightarrow{u_3} \{1, \ldots, e_2\} \xrightarrow{u_2} \{1, \ldots, e_1\} = \{1\}. \tag{2.11}$$

The sequence in (2.11) can naturally be identified with a rooted tree $T = T_{e,u}$ with d vertices. The vertex set $V(T)$ will be the disjoint union of the sets $\{1, \ldots, e_j\}$, $1 \leq j \leq n$. For any $i \in \{1, \ldots, e_j\}$, we create an edge between i and $u_j(i)$. Furthermore, given any f in the outer sum of the altered version of (2.10), we can use the maps $f_j : \{1, \ldots, e_j\} \to \{1, \ldots, n\}$ to colour the vertices in e_j, $1 \leq j \leq h$. The construction of the vertex-coloured tree T is best illustrated in the example in Figure 2.1. In this figure, the colour of a vertex is written inside the vertex. The number outside any given vertex is the element of e_j corresponding to that vertex. We see that $e_3 = 4$, $e_2 = 2$ and $e_1 = 1$. The functions $u_3 : \{1, 2, 3, 4\} \to \{1, 2\}$ and $u_2 : \{1, 2\} \to \{1\}$ map any vertex (except the root) to its parent. For instance, $u_3(4) = 2$.

We now construct our tree $T = T_{e,u}$ concretely with motivation from Figure 2.1. First, define the vertex set of T to be $V(T) = \bigcup_{j=1}^{h} V_j(T)$, $V_j(t) = \{v_{j,1}, \ldots, v_{j,e_j}\}$ for all $1 \leq j \leq h$. Here we see that any vertex $v_{j,r}$ is naturally associated with the r^{th} element of e_j. Also, $v_{1,1}$ naturally acts as the root of this tree. The edge set $E(T)$ consists of the pairs $\{v_{j,r}, v_{j-1,u_j(r)}\}$ where $2 \leq j \leq h, 1 \leq r \leq e_j$. Thus $E(T)$ is completely determined by the functions $\{u_2, \ldots, u_h\}$. Furthermore, $f = (f_2, \ldots, f_h)$ is a

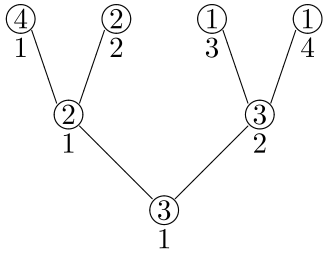


Figure 2.1: Tree construction based on functions u, e, f.

colouring of $V(T)$. In particular, $f_j : \{1, \ldots, e_j\} \to \{1, \ldots, n\}$, so we can consider f as a function from $V(T) \to \{1, \ldots, n\}$ given by $f(v_{j,r}) = f_j(r)$, $2 \leq j \leq h$, with the additional condition that $f(v_{1,1}) = i$. For simplicity, we write $f(v)$ as f_v for any vertex $v \in V(T)$. Recall by (2.10) that we have

$$(H_i)_{e,f,u} = (D_{f_2} H_i) \cdot \big((D_{f_3,u_3{}^{-1}(1)} H_{f_2}(1)) \cdots (D_{f_3,u_3{}^{-1}(e_2)} H_{f_2}(e_2))\big) \\ \cdots \Big((D_{f_h,u_h{}^{-1}(1)} H_{f_{h-1}}(1)) \cdots D_{f_h,u_h{}^{-1}(e_{h-1})} H_{f_{h-1}}(e_{h-1})\Big) \cdot H_{f_h}.$$

Each element of the form $(D_{f_j,u_j{}^{-1}(k)} H_{f_{j-1}}(k))$ in the product can be re-written as $\left(\prod_{w \in k^+} D_{f_w}\right) H_{f_k}$. Thus if we define

$$D_{f_{v^+}} = \prod_{w \in v^+} D_{f_w}$$

and

$$P_{T,f} = \prod_{v \in V(T)} \left(D_{f_{v^+}} H_{f_v}\right)$$

then we have that

$$(H_i)_{e,f,u} = P_{T,f} \qquad (T = T_{e,u}). \tag{2.12}$$

We now have a tree formula parallel to that of (2.10). Using the same notation for indices in (2.10) we have

Lemma 2.4.7. *For $d > 0$,*

$$d! {G_i}^{(d)} \sum_{h=2}^{d} \sum_{e} \sum_{u} \sum_{f} \binom{d}{e_1, \ldots, e_h} P_{T_{e,u},f}$$

where the ranges of indices are the same as those of Lemma 2.4.6 and

$$P_{T,f} = \prod_{v \in V(T)} \left(D_{f_{v^+}} H_{f_v}\right).$$

We would like to write the sum in Lemma 2.4.7 as a sum indexed strictly by vertex-coloured trees T with root labelled i and vertex-colouring given by the functions f. Thus, we need a way to eliminate the dependence of our trees on pairs (e, u). To do this, we consider the tree $T = T_{e,u}$, and ask for the number of pairs (e', u') such that $T_{e,u}$ and $T_{e',u'}$ are isomorphic as rooted trees. Since e determines $V(T)$, we must have that $e = e'$. Furthermore, any isomorphism between $T_{e,u}$ and $T_{e',u'}$ must induce a bijection from $V_j(T_{e,u})$ to $V_j(T_{e',u'})$, so that the children of a vertex in one is a permutation of the children of its corresponding vertex in the other. The number of such isomorphisms is simply the size of the automorphism group of T, $|Aut(T)|$. Since there are $e!$ total possible trees given by any e, we arrive at the following:

$$d!{G_i}^{(d)} = \sum_{h=2}^{d}\sum_{T}\sum_{f}\binom{d}{e_1, \ldots, e_h}\frac{e!}{|Aut(T)|}P_{T,f}.$$

Rearranging and simplifying, we have our final expression for ${G_i}^{(d)}$. We state this as a theorem in its full generality. The conditions on the function F in the theorem will be those given by the Reduction Theorem (Theorem 2.2.1).

Theorem 2.4.8. *(Bass-Connell-Wright Tree Inversion Formula)* [2] *Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be polynomial, $F = x - H$, each H_i homogeneous of a fixed degree, and $|JF|$ a non-zero constant. Let $G = (G_1, \ldots, G_n)$ be the formal inverse of F. That is, $G_i \in \mathbb{C}[[x_1, \ldots, x_n]]$ such that $G_i(F_1, \ldots, F_n) = x_i$ for all $1 \leq i \leq n$. Then $G_i = \sum_{d \geq 0} {G_i}^{(d)}$ where ${G_i}^{(0)} = x_i$, ${G_i}^{(1)} = H_i$ and*

$$ {G_i}^{(d)} = \sum_{T \in \mathbb{T}_d} \frac{1}{|Aut(T)|} \sum_{\ell} P_{T,\ell}. \tag{2.13}$$

Here, $\mathbb{T}_d$ is the set of isomorphism classes of rooted trees with d vertices, ℓ varies over vertex-coloured trees T with root labelled i, and

$$P_{T,f} = \prod_{v \in V(T)} \left(D_{f_{v^+}} H_{f_v} \right).$$

Furthermore, G_i is polynomial if and only if ${G_i}^{(d)} = 0$ for sufficently large d, for each i, $1 \leq i \leq n$.

This concludes the development of the Bass-Connell-Wright Tree Inversion Formula (Theorem 2.4.8). The formula will be the basis of the material in the chapter to follow. In the next section, we focus on computations involving the Bass-Connell-Wright Tree Inversion Formula in order to gain insight on applying it.

2.5 Computations

This section focuses on computational results arising from the Bass-Connell-Wright Tree Inversion Formula. We look at evaluations of $P_{T,f}$ and ${G_i}^{(d)}$, and some properties of them. These results are used throughout Chapter 3.

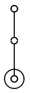
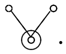
We start off by computing $P_{T,f}$ for some small trees. The simplest non-trivial tree to work with is K_2, the complete graph on 2 vertices. Let T be this tree with vertex set $\{v_1, v_2\}$, v_1 being the root vertex coloured i . Then we have that

$$\sum_f P_{T,f} = \sum_f \prod_{v \in V(T)} D_{f_{v+}} H_{f_v} = \sum_{\substack{f:V(T) \to \{1,\dots,n\} \\ f(v_1)=i}} \left(D_{f(v_2)} H_i\right) H_{f(v_2)} = \sum_{j=1}^{n} (D_j H_i) H_j.$$

Since K_2 is the only tree on two vertices up to isomorphism, and the size of its automorphism group is 1, we conclude that

$${G_i}^{(2)} = \sum_{j=1}^{n} (D_j H_i) H_j.$$

We can similarly find an explicit expression for ${G_i}^{(3)}$. There are two rooted trees on three vertices up to isomorphism. These trees are

$T_1 =$  $\quad T_2 =$  .

We see that

$$\sum_f \left(\frac{1}{|Aut(T_1)|} P_{T_1,f} + \frac{1}{|Aut(T_2)|} P_{T_2,f} \right) = \sum_{j=1}^{n} \sum_{k=1}^{n} (D_j H_i)(D_k H_j) H_k + \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} (D_j D_k H_i) H_j H_k.$$

We therefore conclude that

$${G_i}^{(3)} = \sum_{j=1}^{n} \sum_{k=1}^{n} (D_j H_i)(D_k H_j) H_k + \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} (D_j D_k H_i) H_j H_k.$$

As we can see, computing ${G_i}^{(d)}$ involves many sums and products of differential operators. In the next chapter, we establish a compact method for computing these polynomials.

We can also make some observations on $P_{T,f}$ based on the structure of T. A particular observation is frequently used throughout Chapter 3, and is thus stated here as a theorem. The theorem is due to Wright [13] but appears without proof.

Theorem 2.5.1. [*13*] *Let T be a rooted tree. Assume there exists a vertex $w \in V(T)$ with up-degree at least 4. Then $P_{T,f} = 0$.*

Proof. Recall that we can assume that each H_i is homogeneous of degree 3 by the Reduction Theorem (Theorem 2.2.1). Now assume T has a vertex w with at least 4 children. Then for any function f, $D_{f_{w+}} H_{f_w} = 0$ since $D_{f_{w+}}$ is the product of at least 4 differential operators acting on the degree 3 polynomial H_{f_w}. Thus we have $P_{T,f} = 0$. $\square$

This concludes our in-depth look at the development of the Bass-Connell-Wright Tree Inversion Formula. In the next chapter, we see how this combinatorial development allows for the resolution of special cases of the Jacobian Conjecture.

Chapter 3

Symmetric Reduction

This chapter focuses on the influence of a recent reduction by De Bondt and Van den Essen [3] that has led to the resolution of several cases of the Jacobian Conjecture. They proved that in addition to the conditions of the Bass-Connell-Wright Reduction Theorem (Theorem 2.2.1), one can also assume that the Jacobian matrix of the function in question is symmetric. In Section 3.1 we present the proof in full detail. In Section 3.2, we show how Wright used the symmetric condition to refine the Bass-Connell-Wright Tree Inversion Formula. Using this refinement we provide a proof that a certain class of functions are invertible. In Section 3.3, we use the refined tree inversion formula to annihilate the sums indexed by certain classes of trees. This naturally leads to the introduction of a tree algebra that will allow us to carry out calculations with the tree formulae. We use these developments to prove a special case of the Jacobian Conjecture. To establish this special case, Wright used a theorem due to Zhao in [14]. We provide a different proof that is independent of Zhao's Theorem. We conclude the chapter in Section 3.4 by formulating the Jacobian Conjecture in terms of the Grossman-Larson algebra as a means of establishing a computational approach to the problem.

3.1 Symmetric Reduction

In 2005, De Bondt and Van Den Essen [3] discovered the following reduction to the Jacobian Conjecture.

Theorem 3.1.1. *(Symmetric Reduction) [3] The Jacobian Conjecture is true if it holds for all polynomial maps* $F = x - H$ *where* H *is homogeneous of degree 3,* JH *is nilpotent, and* JH *is symmetric.*

Note that in Theorem 3.1.1, every condition except the symmetry of JH follows from the Bass-Connell-

Wright Reduction Theorem (Theorem 2.2.1), so De Bondt and Van Den Essen essentially proved that one can assume that JH is symmetric. In this section, we will give a detailed presentation of the proof of Theorem 3.1.1.

Before we begin the proof of Theorem 3.1.1, we make a few remarks and introduce some notation and definitions. Firstly, JH is symmetric if and only if H is the gradient of some polynomial in $\mathbb{C}[x_1, \ldots, x_n]$. This is known as Poincare's Lemma (see [3]). Thus there exists $f \in \mathbb{C}[x_1, \ldots, x_n]$ such that $H = \nabla f = (f_{x_1}, \ldots, f_{x_n})$ where $f_{x_i} = \frac{\partial}{\partial x_i} f$ for each i, $1 \leq i \leq n$. In other words, $JH = \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)$, the Hessian of f. We use the notation $h(f)$ to denote the *Hessian* of f. We now introduce the following conjecture which is analogous to the Jacobian Conjecture.

Conjecture 3.1.2. *(Hessian Conjecture) [3] Let $f \in \mathbb{C}[x_1, \ldots, x_n]$. If $h(f)$ is nilpotent, then $F = x - \nabla f$ is invertible.*

Notice that if we can prove that the Hessian Conjecture and the Jacobian Conjecture are equivalent, then we have reduced the Jacobian Conjecture to the Symmetric Case. We show that these two conjectures are in fact equivalent.

If the Jacobian Conjecture holds, it is immediate that the Hessian Conjecture holds as well. To see this, let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be such that $h(f)$ is nilpotent, and set $F = (x_1, \ldots, x_n) + (f_{x_1}, \ldots, f_{x_n})$. Then we have that $h(f) = J(\nabla f)$ is nilpotent so by the Jacobian Conjecture, F is invertible. It remains to prove that the Hessian Conjecture implies the Jacobian Conjecture. We do this by proving the following theorem.

Theorem 3.1.3. *[3] The Jacobian Conjecture and the Hessian Conjecture are equivalent. That is, if the Hessian Conjecture holds for $2n$-dimensional maps, then every n-dimensional map $F = x - H$ with JH nilpotent is invertible.*

Proof. We prove Theorem 3.1.3 in two steps. First, we consider the function $f_H \in \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ given by

$$f_H = (-i)H_1(x_1 + iy_1, \ldots, x_n + iy_n)y_1 + \ldots + (-i)H_n(x_1 + iy_1, \ldots, x_n + iy_n)y_n \tag{3.1}$$

and show that the assumption that JH is nilpotent implies the nilpotency of $h(f_H)$. We then directly show that the nilpotency of $h(f_H)$ implies the invertibility of F.

To start, we construct an invertible linear map $S : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ given by

$$S(x_1, \dots, x_n, y_1, \dots, y_n) = (x_1 - iy_1, \dots, x_n - iy_n, y_1, \dots, y_n) \tag{3.2}$$

and define $g_H = f_H \circ S = \sum_{j=1}^{n} (-i) H_1(x_1, \dots, x_n) y_j$. If we partially differentiate g_H twice with respect to two y variables, the result is 0. If we partially differentiate with respect to an x variable and then a y variable (or y then x), we get an entry from JH (or $(JH)^T$) with an extra factor of $-i$. Thus

$$h(g_H) = \begin{pmatrix} * & (-i)(JH)^T \\ \\ (-i)JH & 0 \end{pmatrix}. \tag{3.3}$$

Now JH is nilpotent, and H is a function on n variables, so $(JH)^n = 0$. Thus the characteristic polynomial of the matrix JH as a variable in z must be z^n. In other words, $|zI_n - JH| = z^n$. Similarly, $|zI_n - (JH)^T| = z^n$. To show that $h(f_H)$ is nilpotent, we must show that $|zI_{2n} - h(f_H)| = z^{2n}$. Introduce a new function $p = \frac{1}{2} \sum_{j=1}^{n} (x_j^2 + y_j^2)$. Then $h(zp) = zI_{2n}$, so by linearity of the Hessian,

$$h(zp - f_H) = zI_{2n} - h(f_H). \tag{3.4}$$

Now recall the invertible linear map S from (3.2). Its matrix representation with respect to the standard basis is upper triangular with 1's along the diagonal, so $|S| = 1$. Thus if we compose the function $zp - f_H$ with S we have that

$$|h((zp - f_H) \circ S)| = |h(zp \circ S - g)| = |S^T| |h(zp - f)|_{|S(x,y)}|S| = |h(zp - f)|_{|S(x,y)}. \tag{3.5}$$

We can compute $zp \circ S$:

$$zp \circ S = z \left(\frac{1}{2} \sum_{j=1}^{n} \left((x_j - iy_j)^2 + y_j^2 \right) \right) = z \left(\frac{1}{2} \sum_{j=1}^{n} x_j^2 - \sum_{j=1}^{n} i x_j y_j \right). \tag{3.6}$$

We deduce that $h(zp \circ S - g_H) = h(zp \circ S) - h(g_H)$ which from (3.3) and (3.6) implies

$$h(zp \circ S - g_H) = \begin{pmatrix} * & -izI_n + i(JH)^T \\ \\ -iz + iJH & 0 \end{pmatrix}.$$

Consequently we have that

$$|h(zp \circ S - g_H)| = |zI_n - JH||zI_n - (JH)^T|. \tag{3.7}$$

Combining (3.5), (3.7) and (3.6) we get

$$|zI_{2n} - h(f_H)|_{S(x,y)} = |zI_n - JH||zI_n - (JH)^T| = z^{2n}.$$

Hence $h(f_H)$ is nilpotent. This completes step 1.

In step 2, we show that $F = x - H$ is invertible if $h(f_H)$ is nilpotent, under the hypothesis that the Hessian Conjecture is true. To do this, consider the function $R = (x_1 - (f_H)_{x_1}, \ldots, x_n - (f_H)_{x_n}, y_1 - (f_H)_{y_1}, \ldots, y_n - (f_H)_{y_n})$. Since $h(f_H)$ is nilpotent, F is invertible by the Hessian Conjecture. We also know that the map S defined in (3.2) is invertible. Thus in particular $S^{-1} \circ R \circ S$ must be invertible. A straightforward computation shows that $S^{-1} \circ R \circ S = (x_1 - H_1(x_1, \ldots, x_n), \ldots, x_n - H_n(x_1, \ldots, x_n), *, \ldots, *)$. Since the restriction of $S^{-1} \circ R \circ S$ to the first n variables is invertible, F is invertible. □

Wright discovered direct consequences of the Symmetric Reduction on the role of combinatorics in resolving the Jacobian Conjecture. In the next section we detail Wright's first major step in accomplishing this, a refinement of the Bass-Connell-Wright Tree Inversion Formula.

3.2 The Symmetric Tree Inversion Formula

Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map whose Jacobian matrix is symmetric. By the Symmetric Reduction (Theorem 3.1.1), we can assume $F = x - \nabla P$ where $P \in \mathbb{C}[x_1, \ldots, x_n]$. If $G \in \mathbb{C}[[x_1, \ldots, x_n]]$ is the formal inverse of F, then we have the following theorem.

Theorem 3.2.1. *(Symmetric Tree Inversion Formula)* [13] *$G = x + \nabla Q$ where*

$$Q = \sum_{T \in \mathbb{T}_m} \frac{1}{|Aut(T)|} Q_{T,P}$$

and

$$Q_{T,P} = \sum_{l:E(T) \to \{1,\ldots,n\}} \prod_{v \in V(T)} D_{inc(v)} P.$$

We frequently write Q as $Q = Q^{(1)} + Q^{(2)} + Q^{(3)} + \cdots$ where $Q^{(m)}$ is the homogeneous degree m summand in Q. That is, $Q^{(m)} = \sum_{T \in \mathbb{T}_m} \frac{1}{|Aut(T)|} Q_{T,P}$.

Proof. To prove Theorem 3.2.1, it suffices to show that for each i, $1 \leq i \leq n$, $G_i = x_i + D_i Q$. We have that:

$$\begin{aligned} D_i Q &= D_i \left(\sum_{T \in \mathbb{T}} \frac{1}{|Aut(T)|} \sum_{l: E(T) \to \{1,\ldots,n\}} \prod_{v \in V(T)} D_{inc(v)} P \right) \\ &= \sum_{T \in \mathbb{T}} \frac{1}{|Aut(T)|} \sum_{l: E(T) \to \{1,\ldots,n\}} D_i \left(\prod_{v \in V(T)} D_{inc(v)} P \right). \end{aligned}$$

Now

$$D_i \left(\prod_{v \in V(T)} D_{inc(v)} P \right) = \sum_{w \in V(T)} D_i \left(D_{inc(w)} P \right) \prod_{\substack{v \in V(T) \\ v \neq w}} D_{inc(v)} P.$$

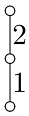
We can write the expression $D_i(D_{inc(w)}P)$ as a sequence of differential operators indexed by the edge labels of a new tree in the following way. We create a tree by adding an edge to the tree T so that this edge is incident with w, and incident to no other vertex. In other words, the end of this edge opposite to w is exposed. We label this edge with the label i. Our sum now becomes

$$\sum_{T \in \mathbb{T}} \frac{1}{|Aut(T)|} \sum_{l: E(T) \ldots \{1,\ldots,n\}} \sum_{w \in V(T)} \prod_{v \in V(T)} D_{inc(v) + \delta_{v,w} e_i} P \tag{3.8}$$

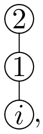
where $inc(v) + \delta_{v,w} e_i$ adds the exposed edge e_i to the vertex w and to no other vertex. Now given any $T \in \mathbb{T}$, $l : E(T) \to \{1, \ldots, n\}$, and $w \in V(T)$, we create a vertex-labelled rooted tree T_w by modifying T as follows: declare w to be the root, label w with the label i, and label each vertex v by the label $l(e)$ of the edge e that is immediately before v on the unique wv-path in T. An intuitive way to understand this process is to think of taking the tree $T \in \mathbb{T}$ with edge-labelling $l : E(T) \to \{1, \ldots, n\}$, choosing a vertex $w \in V(T)$, adding an edge incident to w labelled with i having an exposed vertex on the end opposite w (as described just before (3.8)), and pushing each edge-label to the vertex incident to it that is furthest from w. From this observation, if we let $l_w : V(T) \to \{1, \ldots, n\}$ be the vertex-labelling of T_w, and $k_w(v)$ be the multiset of labels of the parent vertices of v in T_w, we have that

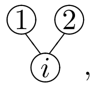
$$\prod_{v \in V(T)} D_{inc(v) + \delta_{v,w} e_i} P = \prod_{v \in V(T)} D_{k_w(v)} D_{l(v)} P, \tag{3.9}$$

since the multiset of labels on the edges incident to w in the edge-labelling of T is precisely the same as the multiset of vertex labels on the children of w, and w itself in T_w. We use an example to illustrate these observations. Consider the edge-labelled tree

$T =$ 

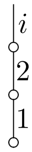
Since T has 3 vertices, there are 3 candidates for T_w, precisely

$T_1 =$ 

, $T_2 =$ 

, $T_3 =$ 

Using the process described above, the tree T_3 is derived from the tree T' by adding the following exposed edge to T,

$T' =$ 

We then have

$$\prod_{v\in V(T')} D_{inc(v)+\delta_{v,w}e_i}P = (D_1P)(D_1D_2P)(D_2D_iP)$$

and

$$\prod_{v\in V(T_3)} D_{k_w(v)}D_{l(v)}P = (D_2D_iP)(D_2D_1P)(D_1P)$$

which are equal. This can be similarly verified for T_1 and T_2.

Therefore, using (3.9), we can extend (3.8) to become

$$\begin{aligned}
&\sum_{T\in\mathbb{T}} \frac{1}{|Aut(T)|} \sum_{\substack{l:V(T)\to\{1,\ldots,n\}\\ l(w)=i}} \sum_{w\in V(T)} \prod_{v\in V(T)} D_{k_w(v)}D_{l(v)}P \\
&= \sum_{T\in\mathbb{T}} \frac{1}{|Aut(T)|} \sum_{\substack{l:V(T)\to\{1,\ldots,n\}\\ l(w)=i}} \sum_{w\in V(T)} \prod_{v\in V(T)} D_{k_w(v)}(\nabla P)_{l(v)} \\
&= \sum_{T\in\mathbb{T}} \frac{1}{|Aut(T)|} \sum_{w\in V(T)} \sum_{\substack{l:V(T)\to\{1,\ldots,n\}\\ l(w)=i}} \prod_{v\in V(T)} D_{k_w(v)}(\nabla P)_{l(v)} \\
&= \sum_{T\in\mathbb{T}} \frac{1}{|Aut(T)|} \sum_{w\in V(T)} \sum_{\substack{l:V(T)\to\{1,\ldots,n\}\\ l(w)=i}} \mathbf{P}_{T,\nabla P,i} \\
&= \sum_{S\in\mathbb{T}_{rt}} \sum_{T\in\mathbb{T}} \sum_{\substack{w\in V(T)\\ T_w\cong S}} \frac{1}{|Aut(T)|}\mathbf{P}_{S,\nabla P,i}
\end{aligned}$$

where $\mathbf{P}_{T,F,i}$ is as defined in Theorem 2.4.8. Now fix an $S \in \mathbb{T}_{rt}$. We are then summing over trees T which have a vertex w so that T rooted at w is isomorphic to S. If we let S' be the tree obtained from S by ignoring the root, then the only such unrooted tree T is S' itself. Thus our sum becomes

$$\sum_{S \in \mathbb{T}_{rt}} \sum_{\substack{w \in V(S') \\ S'_w \cong_{\mathbb{T}_{rt}} S}} \frac{1}{|Aut(S')|} \mathbf{P}_{S,\nabla P,i} = \sum_{S \in \mathbb{T}_{rt}} \frac{|\{w \in V(S') | S'_w \cong_{\mathbb{T}_{rt}} S\}|}{|Aut(S')|} \mathbf{P}_{S,\nabla P,i}.$$

The automorphism group of S' defines an action on $V(S')$. The orbit of the root r is precisely the set of vertices $w \in V(S')$ so that S'_w and S are isomorphic as rooted trees. The stabilizer of r is the set of automorphisms that fix r as a root, which is precisely the set of automorphisms of S. Thus by the Orbit-Stabilizer Theorem, our sum becomes

$$\sum_{S \in \mathbb{T}_{rt}} \frac{\frac{|Aut(S')|}{|Aut(S)|}}{|Aut(S')|} \mathbf{P}_{S,\nabla P,i} = \sum_{S \in \mathbb{T}_{rt}} \frac{1}{|Aut(S)|} \mathbf{P}_{S,\nabla P,i}.$$

By the Bass-Connell-Wright Tree Inversion Formula, the final expression is precisely G_i. $\square$

One consequence of Theorem 3.2.1 that is needed throughout our discussion is the following theorem by Zhao.

Theorem 3.2.2. *(Zhao's Formula) [14] Let $Q^{(m)}$, $m \geq 1$, be the homogeneous summands of the potential function Q as in Theorem 3.2.1. Then $Q^{(1)} = P$ and for $m \geq 2$,*

$$Q^{(m)} = \frac{1}{2(m-1)} \sum_{\substack{k+l=m \\ k,l \geq 1}} \left(\nabla Q^{(k)} \cdot \nabla Q^{(l)} \right).$$

From this theorem we have the following immediate corollary.

Corollary 3.2.3. *(Gap Theorem) [14] Let F be a polynomial function with symmetric Jacobian matrix. Using the notation in Theorem 3.2.1, F is invertible (that is, G is a polynomial) if there exists a positive integer M such that*

$$Q^{(M+1)} = Q^{(M+2)} = \cdots = Q^{(2M)} = 0.$$

We end this section by showing an example of the use of the Symmetric Tree Inversion Formula (Theorem 3.2.1). Consider any function $F = x - \nabla P$ where $P = L^k$, $L = \sum_{i=1}^{n} a_i x_i$ for some positive integer k, and

$$\sum_{i=1}^{n} {a_i}^2 = 0. \tag{3.10}$$

Since the coefficients $a_1, \ldots, a_n$ are complex numbers, there are many maps that satisfy (3.10). Recall now that we can assume we are working under the conditions in the Reduction Theorem (Theorem 2.2.1) so that ∇P is homogeneous of degree 3. Thus we can assume k is 4.

Before applying the Symmetric Tree Inversion Formula to our example we need a few definitions. For any vertex v in a tree T, we denote the degree of v in T by $deg(v)$. We define $a_{inc(v)}$ to be the product $a_{l_1} \cdots a_{l_r}$ where $\{l_1, \ldots, l_r\}$ is the multiset of labels in $inc(v)$. Now consider a tree T with $|V(T)| \geq 2$. We can assume T has no vertex of degree more than 4, so $(4)_{deg(v)} = 4(4-1)\cdots(4-deg(v)+1)$ is a positive integer for all $v \in V(T)$. We then have

$$\begin{aligned}
Q_{T,P} &= \sum_{l:E(T)\to\{1,\ldots,n\}} \prod_{v\in V(T)} D_{inc(v)}P \\
&= \sum_{l:E(T)\to\{1,\ldots,n\}} \prod_{v\in V(T)} (4)_{deg(v)} a_{inc(v)} L^{4-deg(v)} \\
&= \sum_{l:E(T)\to\{1,\ldots,n\}} L^{4|V(T)|-\sum_{v\in V(T)} deg(v)} \prod_{v\in V(T)} (4)_{deg(v)} a_{inc(v)} \\
&= \sum_{l:E(T)\to\{1,\ldots,n\}} L^{2|V(T)|+2} \prod_{v\in V(T)} (4)_{deg(v)} a_{inc(v)} \\
&= L^{2|V(T)|+2} \sum_{l:E(T)\to\{1,\ldots,n\}} \prod_{v\in V(T)} (4)_{deg(v)} a_{inc(v)} \\
&= L^{2|V(T)|+2} \prod_{v\in V(T)} (4)_{deg(v)} \left(\sum_{l:E(T)\to\{1,\ldots,n\}} \prod_{v\in V(T)} a_{inc(v)} \right) \\
&= L^{2|V(T)|+2} \prod_{v\in V(T)} (4)_{deg(v)} \left(\sum_{i=1}^{n} {a_i}^2 \right)^{|E(T)|} \\
&= 0
\end{aligned}$$

by (3.10). Thus $Q_{T,P} = 0$ for all trees T with more than one vertex. If T has exactly one vertex, $Q_{T,P}$ is trivially equal to P. This implies the following theorem which is not found in the literature.

Theorem 3.2.4. *Let $F = x - \nabla P$ be such that $P = (\sum_{i=1}^{n} a_i x_i)^4$ and $\sum_{i=1}^{n} a_i^2 = 0$. Then F is invertible with inverse $G = x + \nabla P$.*

Along the lines of this example, we continue by looking at consequences of the Symmetric Tree

Inversion Formula in the next section. We focus on those examples that lead toward the resolution of special cases of the Jacobian Conjecture.

3.3 Consequences of the Symmetric Tree Inversion Formula

3.3.1 The Symmetric Case when $(JH)^3 = 0$

We use the Symmetric Tree Inversion Formula by Wright [13] to solve cases of the Jacobian Conjecture. The first example of this is the resolution of the symmetric case when $(JH)^3 = 0$. By the Symmetric Reduction (Theorem 3.1.1), we then have that $F = x - \nabla P$ where $(h(P))^3 = 0$. In order to prove this case, we first prove a theorem that gives us a class of trees T such that $Q_{T,P} = 0$. This class of trees is a generalization of the trees we encountered in Section 2.5 when computing ${G_i}^{(2)}$ and ${G_i}^{(3)}$.

We say that a tree T has a *naked r chain* if it contains a path on r vertices whose internal vertices have degree exactly 2 and whose endpoints have degree at most 2. The following theorem characterizes the relationship between trees containing a naked r chain and $Q_{T,P}$.

Theorem 3.3.1. *(Chain Vanishing Theorem) [13] Let* $P \in \mathbb{C}[[x_1, \ldots, x_n]]$ *be homogeneous with* $(h(P))^r = 0$ *for some* $r \geq 1$*. Let* T *be a tree which contains a naked* r *chain. Then* $Q_{T,P} = 0$.

Proof. Let $V(T)$ and $E(T)$ denote the vertex set and edge set of T respectively. Let R be a naked r chain in T. First, assume that the endpoints of R both have degree 2. Then R can be written as an alternating sequence of vertices and edges, say $R = v_1 e_1 \cdots e_{r-1} v_r$, where $v_i \in V(T)$ for all $1 \leq i \leq r$ and $e_j \in E(T)$ for all $1 \leq j \leq r-1$. We also let e_0 and e_r be the other edges incident to v_1 and v_r respectively. Partition $E(T)$ into the disjoint union of $\{e_1, \ldots, e_{r-1}\} \cup E'$ and $V(T)$ into the disjoint union of $\{v_1, \ldots, v_r\} \cup V'$. We then have that

$$\begin{aligned}
Q_{T,P} &= \sum_{l:E(T)\to\{1,\ldots,n\}} \prod_{v \in V(T)} D_{inc(v)} P \\
&= \sum_{l':E'\to\{1,\ldots,n\}} \sum_{l:\{e_1,\ldots,e_{r-1}\}\to\{1,\ldots,n\}} \prod_{v\in V'} D_{inc(v)} P \prod_{v\in\{v_1,\ldots,v_r\}} D_{inc(v)} P \\
&= \sum_{l':E'\to\{1,\ldots,n\}} \prod_{v\in V'} D_{inc(v)} P \sum_{l:\{e_1,\ldots,e_{r-1}\}\to\{1,\ldots,n\}} \prod_{v\in\{v_1,\ldots,v_r\}} D_{inc(v)} P \\
&= \sum_{l':E'\to\{1,\ldots,n\}} \prod_{v\in V'} D_{inc(v)} P \sum_{i_1,\ldots,i_{r-1}} (D_{l'(e_0)i_1} P)(D_{i_1 i_2} P) \cdots (D_{i_{r-2} i_{r-1}} P)(D_{i_{r-1} l'(e_r)} P). \qquad (3.11)
\end{aligned}$$

The final summation is the $(l'(e_0)l'(e_r))^{th}$ entry of $(h(P))^r$, which is 0 by assumption. Thus $Q_{T,P} = 0$. Now assume the endpoints of R do not necessarily both have degree 2. Without loss of generality we assume that e_r is present but e_0 isn't. Then the inner summation of (3.11) is

$$\sum_{i_1,\ldots,i_{r-1}} (D_{i_1}P)(D_{i_1 i_2}P)\cdots(D_{i_{r-2}i_{r-1}}P)(D_{i_{r-1}l'(e_r)}P).$$

Since P is homogeneous, Euler's formula says $D_{i_1}P = \frac{1}{d-1}\sum_{i_0=1}^{n} D_{i_0 i_1}P$. We then have that the inner summation of (3.11) is equal to

$$\frac{1}{d-1}\sum_{i_0,i_1,\ldots,i_{r-1}} (D_{i_0 i_1}P)(D_{i_1 i_2}P)\cdots(D_{i_{r-2}i_{r-1}}P)(D_{i_{r-1}i_{l'(e_r)}}P),$$

which is 0 since $(h(P))^r = 0$. Finally if both v_0 and v_r have degree 1 (that is, both e_0 and e_r are absent), then (3.11) becomes

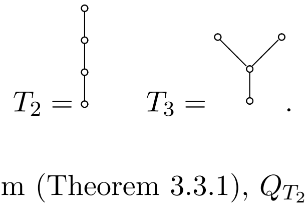
$$\sum_{i_1,\ldots,i_{r-1}} (D_{i_1}P)(D_{i_1 i_2}P)\cdots(D_{i_{r-2}i_{r-1}}P)(D_{i_{r-1}}P),$$

and applying Euler's formula to $D_{i_1}P$ and $D_{i_{r-1}}P$ finishes the proof. ☐

We are now prepared to resolve the symmetric case when $(JH)^3 = 0$.

Theorem 3.3.2. *(Symmetric $(JH)^3 = 0$ Case) Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be such that $F = x - H$ where H is symmetric and homogeneous of degree 3 with $(JH)^3 = 0$. Then F is invertible.*

Proof. Since H is symmetric and homogeneous of degree 3, then we can assume $F = x - \nabla P$ where P is homogeneous of degree 4. Let $G = x + \nabla Q$ be the formal inverse of F. Then by Corollary 3.2.3, letting $M = 3$, it suffices to prove that $Q^{(3)} = Q^{(4)} = 0$. We know by Theorem 3.2.1 that $Q^{(3)} = \sum_{T\in\mathbb{T}_3}\frac{1}{|Aut(T)|}Q_{T,P}$. Up to isomorphism, there is only one unrooted tree on 3 vertices, a path on 3 vertices. Let T_1 be this tree. Since $JH = h(P)$, $(h(P))^3 = 0$, so by the Chain Vanishing Theorem (Theorem 3.3.1), we have $Q_{T_1,P} = 0$. Thus $Q^{(3)} = 0$. Now $Q^{(4)} = \sum_{T\in\mathbb{T}_4}\frac{1}{|Aut(T)|}Q_{T,P}$. Up to isomorphism there are 2 trees on 4 vertices, T_2 and T_3 in the diagram below:



Again, by the Chain Vanishing Theorem (Theorem 3.3.1), $Q_{T_2,P} = 0$. To show that $Q^{(4)} = 0$, it then suffices to show that $Q_{T_3,P} = 0$. To do this, apply the operator $\sum_{i=1}^{n}(D_iP)D_i$ to $Q_{T_1,P}$. We get

$$\begin{aligned}0 = \sum_{i=1}^{n}(D_iP)D_iQ_{T_1,P} &= \sum_{i=1}^{n}(D_iP)\left(D_i\sum_{j,k}(D_jP)(D_jD_kP)(D_kP)\right)\\ &= \sum_{i,j,k}(D_iP)(D_iD_jP)(D_jD_kP)(D_kP) + \sum_{i,j,k}(D_iP)(D_jP)(D_iD_jD_kP)(D_kP)\\ &\quad + \sum_{i,j,k}(D_jP)(D_jD_kP)(D_kD_iP)(D_iP).\end{aligned}$$

From the last line we have $0 = Q_{T_2,P} + Q_{T_3,P} + Q_{T_2,P}$. But $Q_{T_2,P} = 0$, so it follows that $Q_{T_3}P = 0$. Thus $Q^{(4)} = 0$ and the result follows. $\square$

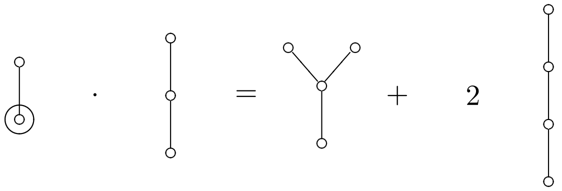
To prove Theorem 3.3.2, we wanted to find a positive integer M such that $Q^{(M+1)} = Q^{(M+2)} = \cdots = Q^{(2M)} = 0$. To do this, we chose a value of M such that for all $M+1 \leq m \leq 2M$, $Q_{T,P} = 0$ for all $T \in \mathbb{T}_m$. For $T \in \{T_1, T_2\}$, $Q_{T,P} = 0$ immediately by the Chain Vanishing Theorem (Theorem 3.3.1). For T_3, we tried to express $Q_{T_3,P}$ as a linear combination of the other trees, all of which satisfied the conditions of the Chain Vanishing Theorem (Theorem 3.3.1). In order to find this linear combination, we applied a differential operator to T_1, a tree satisfying the conditions of the Chain Vanishing Theorem (Theorem 3.3.1). This is the approach we use in more general cases. First, we find a set of trees that satisfy the Chain Vanishing Theorem (Theorem 3.3.1), and aim to express all other trees as linear combinations of these trees. In order to do this systematically, we invoke the use of an algebra that mimics our calculations. In the next section, we introduce the algebra and show how computations in it relate to the computations we need.

3.3.2 Grossman-Larson Algebra

In this section, we introduce the Grossman-Larson Algebra and establish its relationship to the ring of differential operators on $\mathbb{C}[x_1, \ldots, x_n]$. We use this relationship to establish special cases of the Jacobian Conjecture. Before doing this, some notation and definitions are required. First, let $\{T_1, \ldots, T_r\}$ be a multiset of trees in $\mathbb{T}_{rt}$ with roots $rt_{T_1}, \ldots, rt_{T_r}$ respectively, and let S be a tree in $\mathbb{T} \cup \mathbb{T}_{rt}$. For any sequence of vertices $(v_1, \ldots, v_r) \in V(S)^r$, we denote by $(T_1, \ldots, T_r) - \circ_{(v_1,\ldots,v_r)} S$ the tree formed by joining each T_i to S by adding an edge between rt_{T_i} and v_i. Let $\mathfrak{H}$ be the $\mathbb{Q}-$vector space spanned by all rooted trees, where addition is formal. Let $\mathfrak{M}$ be the $\mathbb{Q}-$vector space spanned by all unrooted trees. We can define actions of $\mathfrak{H}$ on $\mathfrak{M}$ and $\mathfrak{H}$ itself in the following way. Let $T \in \mathbb{T}_{rt}$ with root rt_T, and let $S \in \mathbb{T} \cup \mathbb{T}_{rt}$. Define $DelRoot(T) = \{T_1, \ldots, T_r\}$ to be the multiset of trees in $T \backslash \{rt_T\}$. We define the action

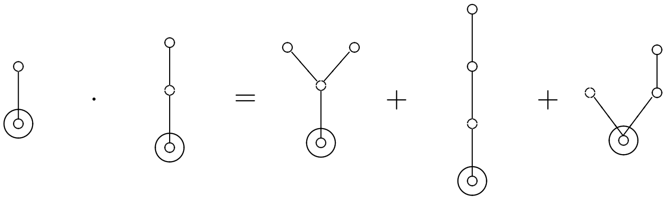
$$T \cdot S = \sum_{(v_1,\dots,v_r)\in V(S)^r} [(T_1,\dots,T_r) {-\!\circ}_{(v_1,\dots,v_r)} S]. \tag{3.12}$$

and extend this linearly on $\mathfrak{M}$ and $\mathfrak{H}$. The following example illustrates the action of $\mathfrak{H}$ on $\mathfrak{M}$.



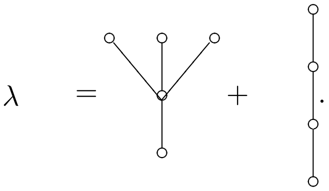
(3.13)

Another example illustrates the action of $\mathfrak{H}$ on itself.



(3.14)

Notice that $T \cdot S$ is an element in $\mathfrak{M}$ if $S \in \mathbb{T}$ and $T \cdot S$ is an element of $\mathfrak{H}$ if $S \in \mathbb{T}_{rt}$. As a consequence, $\mathfrak{H}$ has a ring structure on it whose product is defined by the action in (3.12), and $\mathfrak{M}$ is endowed with an $\mathfrak{H}$-module structure defined by the same action. With this observation, the Chain Vanishing Theorem (Theorem 3.3.1), and the Reduction Theorem (Theorem 2.2.1), it is natural to consider the following $\mathfrak{H}$-submodules of $\mathfrak{M}$. First, denote by $C(r)$ the $\mathfrak{H}$-submodule of $\mathfrak{M}$ generated by all trees containing a naked r chain. Second, denote by $V(s)$ the subspace of $\mathfrak{M}$ generated by all trees containing a vertex of degree at least $s+1$. Notice in particular that for any $T \in \mathfrak{H}$ and $S \in V(s)$, every tree in the sum $T \cdot S$ contains a vertex of degree $s+1$. Thus $V(s)$ is an $\mathfrak{H}$-submodule of $\mathfrak{M}$. For positive integers r, s, we define the $\mathfrak{H}$-submodule $\mathfrak{M}(r,s) = C(r) + V(s)$, and finally define the quotient module $\overline{\mathfrak{M}}(r,s)$ $= \mathfrak{M}/\mathfrak{M}(r,s)$. We also use the notation $\overline{\mathfrak{M}}(r,\infty)$ to denote $\mathfrak{M}/C(r)$. For example, consider $\lambda \in \mathfrak{M}$ given by



(3.15)

Now consider $\overline{\lambda}$, the image of λ after projection into $\overline{\mathfrak{M}}(4,3)$. Any tree containing a vertex with degree at least 4 or containing a naked 4 chain is annhilated, so $\overline{\lambda} = 0$ in $\overline{\mathfrak{M}}(4,3)$. Note that if we replace each occurrence of a tree T with $Q_{T,P}$ in the sum (3.15), we get 0 as well by the Chain Vanishing Theorem

(Theorem 3.3.1) and the Reduction Theorem (Theorem 2.2.1). This gives strong evidence toward a relationship between terms in $\mathfrak{M}$ and sums with summands $Q_{T,P}$. We now make this relationship precise.

First, we define the homomorphism $\rho_p : \mathfrak{M} \to \mathbb{C}[x_1, \ldots, x_n]$ by sending an unrooted tree T to $Q_{T,P}$ and extending linearly. Now let $\mathfrak{D}[x_1, \ldots, x_n]$ be the ring of differential operators on $\mathbb{C}[x_1, \ldots, x_n]$. Given a rooted tree S let $e_1, \ldots, e_r$ be the edges adjacent to rt_S. Now define the homomorphism $\phi_p : \mathfrak{H} \to \mathfrak{D}[x_1, \ldots, x_n]$ where for each rooted tree S,

$$\phi_p(S) = \sum_{l:E(S)\to\{1,\ldots,n\}} \left(\prod_{v\in V(S)\setminus\{rt_S\}} D_{inc(v)}P \right) D_{l(e_1)l(e_2)\cdots l(e_r)}.$$

Notice that $\phi_p(S)$ mimics the definition of $Q_{T,P}$ for trees in $\mathbb{T}$. The maps ϕ_p and ρ_p are easily seen to be compatible with the structures of $\mathfrak{M}$ as an $\mathfrak{H}$-module and $\mathbb{C}[x_1, \ldots, x_n]$ as a $\mathfrak{D}[x_1, \ldots, x_n]$-module. In other words, the following diagram commutes

$$\begin{tikzcd}
\mathfrak{H} \times \mathfrak{M} \arrow[r] \arrow[d, "\phi_p \times \rho_p"] & \mathfrak{M} \arrow[d, "\rho_p"] \\
\mathfrak{D}[x_1, \ldots, x_n] \times \mathbb{C}[x_1, \ldots, x_n] \arrow[r] & \mathbb{C}[x_1, \ldots, x_n]
\end{tikzcd}$$

where horizontal arrows are given by the module action. Thus we have established an explicit correspondence between the structure of $\mathfrak{M}$ as a $\mathfrak{H}$-module and $\mathbb{C}[x_1, \ldots, x_n]$ as a $\mathfrak{D}[x_1, \ldots, x_n]$-module. One immediate consequence of this correspondence is the following:

Proposition 3.3.3. *[13] Let r, s be positive integers and $P \in \mathbb{C}[x_1, \ldots, x_n]$ be homogeneous of degree s with $(h(P))^r = 0$. Then $\rho_p(V(s)) = \rho_p(C(r)) = 0$.*

Proof. Let T be a tree with a vertex of degree at least $s+1$. Then $\rho_p(T) = Q_{T,P} = 0$ since $D_{inc(s)}P = 0$. By the compatibility of ϕ_p and ρ_p, this extends to the entire module $V(s)$. Thus $\rho_p(V(s)) = 0$. Now assume that T has a naked r chain. Then $\rho_p(T) = Q_{T,P} = 0$ by the Chain Vanishing Theorem (Theorem 3.3.1). Again by the compatibility of ϕ_p and ρ_p, this extends to the entire module $C(r)$, so $\rho_p(C(r)) = 0$. ☐

By Proposition 3.3.3, ρ_p induces a homomorphism $\overline{\rho_p}(r, s) : \overline{\mathfrak{M}}(r, s) \to \mathbb{C}[x_1, \ldots, x_n]$ that is compatible with ϕ_p. That is, the following diagram commutes

$$\begin{tikzcd}
\mathfrak{H} \times \overline{\mathfrak{M}}(r,s) \arrow[r] \arrow[d, "\phi_p \times \overline{\rho_p}(r,s)"] & \overline{\mathfrak{M}}(r,s) \arrow[d, "\overline{\rho_p}(r,s)"] \\
\mathfrak{D}[x_1,\ldots,x_n] \times \mathbb{C}[x_1,\ldots,x_n] \arrow[r] & \mathbb{C}[x_1,\ldots,x_n]
\end{tikzcd}$$

where horizontal arrows are given by the module action. Using this, we can state sufficient conditions for a function F to be invertible in the case that F has a symmetric Jacobian matrix. Let $F = x - \nabla P$, and $G = x + \nabla Q$ be its formal power series inverse (we can assume it has this form by Theorem 3.2.1). Now define $v_m \in \mathfrak{M}$ by

$$v_m = \sum_{T \in \mathbb{T}_m} \frac{1}{|Aut(T)|} T.$$

Notice that $\rho_p(v_m) = Q^{(m)}$. Thus $\overline{\rho_p}(\overline{v_m}) = Q^{(m)}$ in $\overline{\mathfrak{M}}(r,s)$. We conclude that if $\overline{v_m} = 0$ in $\overline{\mathfrak{M}}(r,s)$, then $Q^{(m)} = 0$. The following proposition follows from these observations.

Proposition 3.3.4. [*13*] *Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial function with symmetric Jacobian matrix and formal inverse $G = x + \nabla Q$. Further assume $(h(P))^n = 0$. Then using the notation from Theorem 3.2.1, if there exists a positive integer M such that $v_m = 0$ in $\overline{\mathfrak{M}}(n,4)$ for all $m \geq M$, then F is invertible.*

Using the Gap Theorem (Corollary 3.2.3), we can state a weaker version of Proposition 3.3.4 that implies the Jacobian Conjecture is true.

Proposition 3.3.5. [*13*] *Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial function with symmetric Jacobian matrix and formal inverse $G = x + \nabla Q$. Further assume $(h(P))^n = 0$. Then using the notation from Theorem 3.2.1, if there exists a positive integer M such that $v_m = 0$ in $\overline{\mathfrak{M}}(n,4)$ for all $M+1 \leq m \leq 2M$, then F is invertible.*

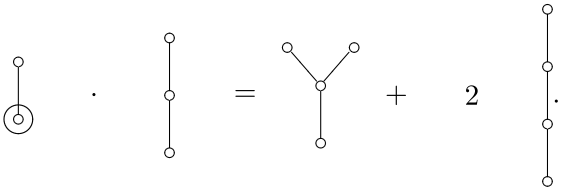
Using Proposition 3.3.5, we present an alternate proof of the Symmetric $(JH)^3 = 0$ Case of the Jacobian Conjecture using the Grossman-Larson Algebra. This proof is adapted from the original proof by Wright [13].

Theorem 3.3.6. *(Symmetric $(JH)^3 = 0$ Case Revisited) [13] Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be such that $F = x - H$ where H is symmetric and homogeneous with $(JH)^3 = 0$. Then F is invertible.*

Proof. We are given that F has a symmetric Jacobian matrix so we can assume $F = x - H$ where $H = \nabla P$ for some $P \in \mathbb{C}[x_1, \ldots, x_n]$. Now $(JH)^3 = 0$ implies $(h(P))^3 = 0$, so $\rho_p(C(3)) = 0$ by Proposition 3.3.3. By the same proposition, we also know that $\rho_p(V(4)) = 0$ since P is homogeneous of degree at most 4. Thus ρ_p induces a homomorphism $\overline{\rho_p}$ on $\overline{\mathfrak{M}}(3,4)$ and $\overline{\rho_p}(\overline{v_m}) = Q^{(m)}$ for all $m \geq 1$. By the Gap Theorem (Corollary 3.2.3), it is sufficient to show that $Q^{(3)} = Q^{(4)} = 0$. Thus, in the Grossman-Larson Algebra, it suffices to show that $\overline{\rho_p}(\overline{v_3}) = \overline{\rho_p}(\overline{v_4}) = 0$ in $\overline{\mathfrak{M}}(3,4)$. Consider the trees T_1, T_2 and T_3 defined in Theorem 3.3.2. By the definition of v_m,

$$v_3 = T_1, \qquad v_4 = T_2 + \frac{1}{6}T_3.$$

We know that T_1 is a path on 3 vertices, so $v_3 \in C(3)$. Thus $\overline{v_3} = 0$ in $\overline{\mathfrak{M}}(3,4)$. Consider the product



(3.16)

Let T be the rooted tree in this product. Then $T \cdot T_1 = 2T_2 + T_3$. Since $\overline{T_1} = 0$, $\overline{T \cdot T_1} = 0$, so $2\overline{T_2} + \overline{T_3} = 0$. But $T_2 \in C(3)$ so $\overline{T_2} = 0$. We conclude that $\overline{T_3} = 0$ and hence

$$\overline{v_4} = \overline{T_2} + \frac{1}{6}\overline{T_3} = 0.$$

Thus $\overline{v_3} = \overline{v_4} = 0$ in $\overline{\mathfrak{M}}(3,4)$, implying $\overline{\rho_p}(\overline{v_3}) = \overline{\rho_p}(\overline{v_4}) = 0$. ☐

The proof of Theorem 3.3.6 invoked the Gap Theorem (Corollary 3.2.3). We present a stronger statement than that of Theorem 3.3.6 that proves Theorem 3.3.6 independent of the Gap Theorem (Corollary 3.2.3). The proof of this theorem is not found in the literature.

Theorem 3.3.7. [*13*] *$T \in C(3) + V(4)$ for all trees T with at least 3 vertices.*

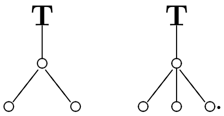
An immediate corollary of Theorem 3.3.7 is the following:

Corollary 3.3.8. [*13*] *Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be such that $F = x - H$ where H is symmetric and homogeneous with $(JH)^3 = 0$. Then F is invertible.*

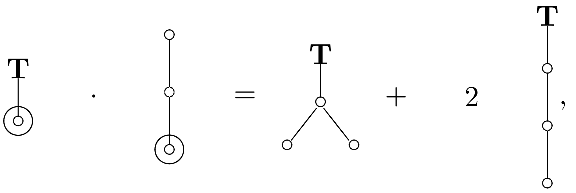
Proof. (Corollary 3.3.8) Let $G = x + \nabla Q$ be the formal inverse of F. Since $T \in C(3) + V(4)$ for all trees T with at least 3 vertices, $\overline{v_m} = 0$ for all $m \geq 3$, so $Q^{(m)} = 0$ for all $m \geq 3$, implying F is invertible. ☐

We now prove Theorem 3.3.7.

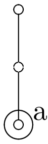
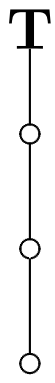
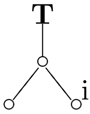
Proof. (Theorem 3.3.7) Consider any tree of either of the forms

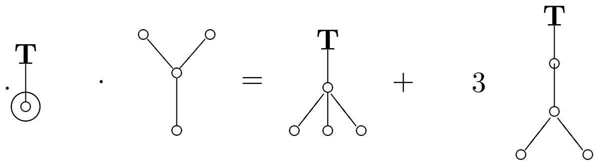


where $\mathbf{T} \in \mathbb{T}$ is arbitrary. We have that

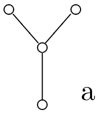
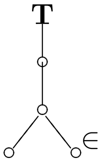
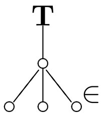


(3.17)

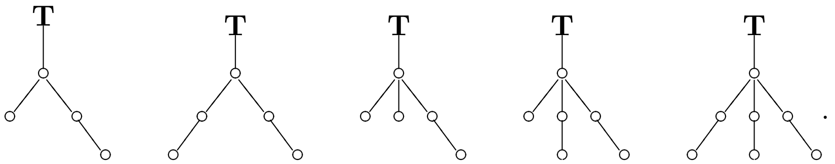
but we know that  and  are both in $C(3)$. It thus follows by (3.17) that  is in $C(3)$ as well. Moreover,



(3.18)

We know that  and  $\in C(3)$ by our result from (3.17). Thus  $\in C(3)$ as well.

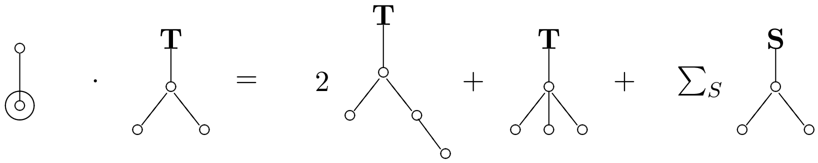
Now consider any tree T' with $|V(T')| \geq 3$. By Theorem 2.5.1, we can assume vertices in T' have degree at most 4. Start a breadth-first tree for T' at any leaf that is the end of a longest path and consider the structure of T' looking 2 levels into a breadth-first search. By our conclusions from (3.17) and (3.18), T' will be of one of the following forms for some tree $T \in \mathbb{T}$



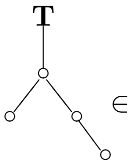
(3.19)

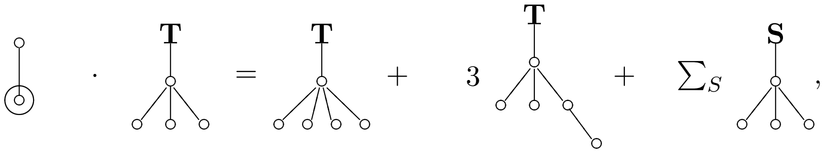
It then suffices to show that all the trees in (3.19) lie in $C(3) + V(4)$.

Firstly, we have that

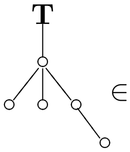


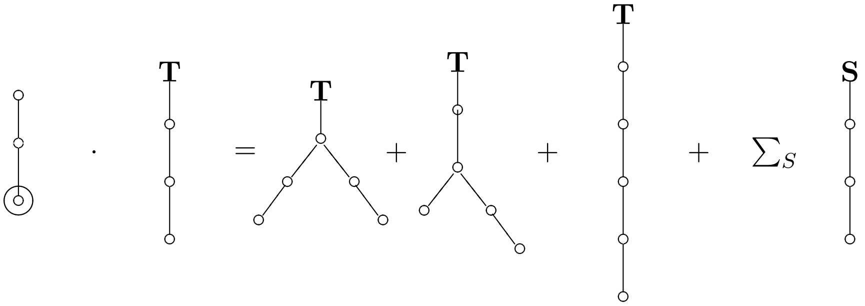
(3.20)

where the sum ranges over some set of trees. The unrooted tree being multiplied on the left hand side and the trees in the summation on the right hand side of the (3.20) are in $C(3)$ by our conclusion from (3.17). The middle tree on the right hand side of (3.20) is in $C(3)$ as well by our conclusion from (3.18). Thus  $\in C(3)$. We make a similar observation again from the equation

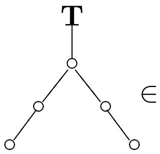


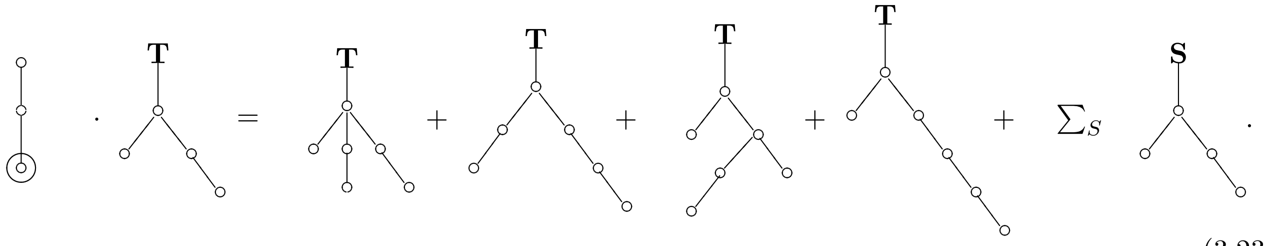
(3.21)

which implies  $\in C(3) + V(4)$ by the conclusion from (3.18) and the definition of $V(4)$. We also have

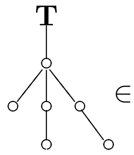


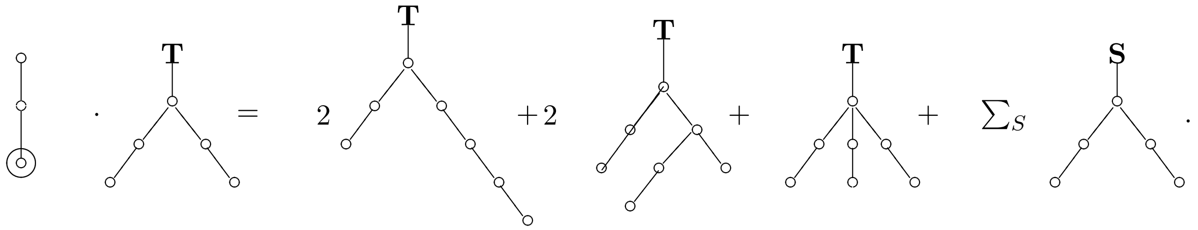
(3.22)

which by (3.18) and the definition of $C(3)$ implies  $\in C(3)$. Yet another product gives us more information.

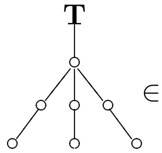


(3.23)

Using the conclusion from (3.20) and the definition of $C(3)$, we get  $\in C(3)$. Finally, we have



(3.24)

From (3.20), (3.22), and the definition of $C(3)$, we get  $\in C(3)$. Thus all the trees in (3.19) belong to $C(3) + V(4)$, and we have our result. □

3.4 Computational Approach

Using the Grossman-Larson Algebra, we can set up the Jacobian Conjecture in a computational framework. To do this, recall that to prove the Jacobian Conjecture, it suffices to find a positive integer M_n such that for all $M_n + 1 \leq m \leq 2M_n$, $\overline{v_m} = 0$ in $\overline{\mathfrak{M}}(n, 4)$ (by Theorem 3.3.5). Now fix a positive integer m. Let k be a positive integer with $1 \leq k \leq n$. Assume T is a rooted tree with k vertices excluding the root, and $S \in \mathbb{T}_{m-k}$. Then $T \cdot S$ is a sum of trees in $\mathbb{T}_m$. Thus if $S \in C(n) + V(4)$, $T \cdot S$ is a linear combination of trees on m vertices, the linear combination being 0 in $\overline{\mathfrak{M}}(n, 4)$. We can generate many linear combinations in this way by choosing T arbitrarily and S to have a naked n chain or a vertex with degree at least 5 (in order to ensure $S = 0$ in $\overline{\mathfrak{M}}(n, 4)$). It then suffices to check if $\overline{v_m}$ is in the span of the linear combinations. As an example of how this computational process works, we switch to looking at $\overline{\mathfrak{M}}(4, 3)$ and consider v_6 in this quotient module.

Let A_1 and A_2 be paths on 4 and 5 vertices respectively. Consider these paths along with the trees

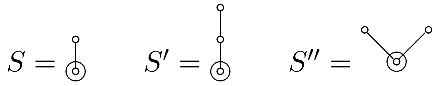


Figure 3.1: Rooted Trees

from Figure 3.1 and Figure 3.2. We see that $A_1, A_2, B_1 \in C(4)$ and $B_4, B_6 \in V(3)$. Thus A_1, A_2, B_1, B_4 and B_6 are all 0 in $\overline{\mathfrak{M}}(4,3)$. It follows that,

$$\begin{aligned}
0 &= S' \cdot A_1 = 2B_1 + 2B_3 = 2B_3 \\
0 &= S \cdot A_2 = 2B_1 + 2B_2 + B_3 = 2B_2 + B_3 \\
0 &= S'' \cdot A_1 = 2B_1 + 6B_2 + 4B_3 + 2B_4 + 2B_5 = 6B_2 + 4B_3 + 2B_5.
\end{aligned}$$

Since $B_1 = B_4 = B_6 = 0$ in $\overline{\mathfrak{M}}(4,3)$, $\overline{v_6}$ is a $\mathbb{Q}$-linear combination of $\{B_2, B_3, B_5\}$. Now any $\mathbb{Q}$-linear combination of $\{B_2, B_3, B_5\}$ can be written as a $\mathbb{Q}$-linear combination of $\{B_3, 2B_2+B_3, 6B_2+4B_3+2B_5\}$ since the transition matrix between the two sets of vectors is triangular with no zeroes on the diagonal. Thus $\overline{v_6}$ is a $\mathbb{Q}$-linear combination of $\{B_3, 2B_2 + B_3, 6B_2 + 4B_3 + 2B_5\}$, all of which are 0 in $\overline{\mathfrak{M}}(4,3)$. It follows that $\overline{v_6} = 0$ in $\overline{\mathfrak{M}}(4,3)$.

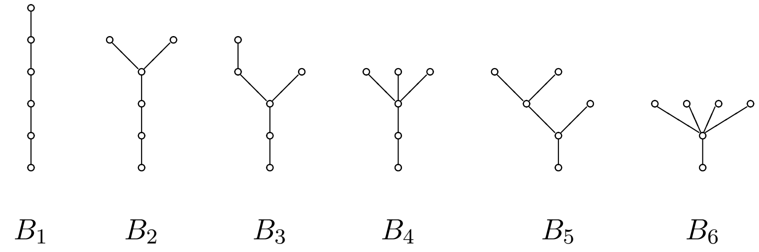


Figure 3.2: Trees with 6 vertices

Li-Yang Tan [13] created a computer program to assist Wright in expressing the values v_m as linear combinations that are 0 in $\overline{\mathfrak{M}}(n,4)$ for various n. Using this computational method, another case of the Jacobian Conjecture was resolved.

Theorem 3.4.1. [*13*] *The Jacobian Conjecture is true for all maps $F : \mathbb{C}^n \to \mathbb{C}^n$ with $F = x - H$, JH symmetric, and $(JH)^4 = 0$.*

The computer program in fact showed that all trees $T \in \mathbb{T}_m$ are 0 in $\overline{\mathfrak{M}}(4,4)$ for $m = 8, 9, 10, 11, 12, 14$. It turns out this is not true when $m = 13$, but $\overline{v_{13}} = 0$, and so $\overline{v_m} = 0$ for $m = 8, 9, 10, 11, 12, 13, 14$ in

$\overline{\mathfrak{M}}(4,4)$. Thus Theorem 3.4.1 follows from Proposition 3.3.4.

This concludes our investigation of the Symmetric Reduction and its influence on resolving the Jacobian Conjecture. In the next chapter, we look at Singer's approach to the conjecture which parallels the work of Bass, Connell and Wright but uses a slightly different combinatorial setting to approach the problem.

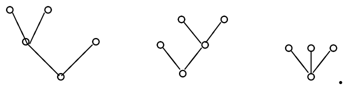
Chapter 4

Catalan Tree Inversion

In this chapter, we detail the developments of Singer in [9]. In Section 4.1, we show how Singer used Catalan trees to determine the formal power series inverse of polynomial functions. In Section 4.2, we illustrate Singer's combinatorial interpretation of the nilpotency condition. We also investigate how Singer used this interpretation to pose the Jacobian Conjecture combinatorially. Using these discoveries, in Section 4.3 we show how Singer developed a systematic method for approaching certain cases of the Jacobian Conjecture, and how he resolved some of these cases.

4.1 Catalan Tree Inversion Formula

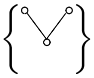
A *Catalan tree* is an ordered rooted tree such that every non-leaf vertex has up-degree at least 2. We denote the set of Catalan trees by C and the set of Catalan trees with p leaves by C_p. For example, C_3 consists of the trees



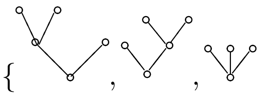
In particular we have that

$$C_p = \bigcup_{\substack{k\geq 2\\ i_1+\ldots+i_k=p}} \{ \overset{T_1T_2\cdots T_k}{\circ} : T_j \in C_{i_j}, 1 \leq j \leq k\}. \tag{4.1}$$

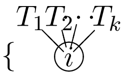
For example,

$C_1 = \{\circ\} \qquad C_2 = \Big\{$  $\Big\}$.

Thus by (4.1),

$$C_3 = \{$$  $$\}.$$

Moreover, we can vertex-colour trees in C. Consider the set of vertex-coloured Catalan trees with root coloured i. We denote by $C^{(i)}$ the subset of vertex-coloured Catalan trees with root labelled i where the colours of the children of any vertex are weakly increasing from left to right. In other words, we recursively define $C^{(i)}$ as follows:

$$C_p^{(i)} = \bigcup_{\substack{k\geq 2\\ i_1+\ldots+i_k=p\\ 1\leq l_1\leq\cdots\leq l_k\leq n}} \{$$  $$: T_j \in C_{i_j}{}^{(l_j)}, 1 \leq i \leq k\}. \tag{4.2}$$

An example of a tree in $C_7^{(2)}$ is given in Figure 4.1.

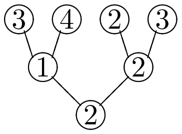


Figure 4.1: A coloured Catalan tree in $C_7^{(2)}$

Given a polynomial function $F : \mathbb{C}^n \to \mathbb{C}^n$, $F = x - H$, we can define a weight function on C. Recall that by Theorem 1.2.4 we can assume H has no constant or linear terms, so for each $1 \leq i \leq n$ we can write

$$H_i = \sum_{\substack{k\geq 2\\ 1\leq i_1\leq\cdots\leq i_k\leq n}} h^{(i)}_{i_1,i_2,\cdots,i_k} x_{i_1}\cdots x_{i_k}.$$

Let $V_L(T)$ denote the set of leaves in $V(T)$. Define the *weight function* $\omega : \bigcup_{i=1}^n C^{(i)} \to \mathbb{C}[x_1,\ldots,x_n]$ given by

$$\omega(T) = \prod_{v\in V(T)\setminus V_L(T)} h^{(l(v))}_{l(v^+)} \prod_{v\in V_L(T)} x_{l(v)}. \tag{4.3}$$

As an example, if T is the tree from Figure 4.1, then

$$\omega(T) = h^{(2)}_{1,2} h^{(1)}_{3,4} h^{(2)}_{2,3} x_2 x_3^2 x_4.$$

We can equivalently define (4.3) recursively as follows:

$$\omega(\textcircled{i}) = x_i$$

$$\omega\Big(\overset{T_1T_2\cdots T_k}{\textcircled{i}}\Big) = h^{(i)}_{i_1,i_2,\cdots,i_k}\prod_{j=1}^{k}\omega(T_j)$$

where $T_j \in C^{(i_j)}$ for all $1 \leq j \leq k$. We now state and prove the Catalan Tree Inversion Formula which gives us the inverse of F in terms of ω.

Theorem 4.1.1. *(Catalan Tree Inversion Formula) [9] Let $F : \mathbb{C}^n \to \mathbb{C}^n$, $F = x - H$ (we can assume H has no constant or linear terms). Let ω be the weight function defined in (4.3). Let G be the formal inverse of F. Then for each $1 \leq i \leq n$,*

$$G_i = \sum_{T\in C^{(i)}} \omega(T).$$

Proof. We use the definition of ω recursively on G_i.

$$\begin{aligned}
G_i = \sum_{T\in C^{(i)}} \omega(T) &= x_i + \sum_{\substack{k\geq 2\\ 1\leq i_1\leq\cdots\leq i_k\leq n\\ T_1\in C^{(i_1)},\ldots,T_k\in C^{(i_k)}}} \omega\Big(\overset{T_1T_2\cdots T_k}{\textcircled{i}}\Big)\\
&= x_i + \sum_{\substack{k\geq 2\\ 1\leq i_1\leq\cdots\leq i_k\leq n}} h^{(i)}_{i_1,i_2,\cdots,i_k} \sum_{T_1\in C^{(i_1)},\ldots,T_k\in C^{(i_k)}} \prod_{j=1}^{k}\omega(T_j)\\
&= x_i + \sum_{\substack{k\geq 2\\ 1\leq i_1\leq\cdots\leq i_k\leq n}} h^{(i)}_{i_1,i_2,\cdots,i_k} \sum_{T_1\in C^{(i_1)},\ldots,T_k\in C^{(i_k)}} \prod_{j=1}^{k}\omega(T_j)\\
&= x_i + \sum_{\substack{k\geq 2\\ 1\leq i_1\leq\cdots\leq i_k\leq n}} \prod_{j=1}^{k}\left(\sum_{T_j\in C^{(i_j)}}\omega(T_j)\right)\\
&= x_i \sum_{\substack{k\geq 2\\ 1\leq i_1\leq\cdots\leq i_k\leq n}} h^{(i)}_{i_1,i_2,\cdots,i_k}\prod_{j=1}^{k} G_{i_j}\\
&= x_i + H_i(G_1,\ldots,G_n).
\end{aligned}$$

Thus $G_i = x_i - H_i(G_1, \ldots, G_n) = F_i(G_1, \ldots, G_n)$. □

At times it will be convenient to ignore the vertex colours of a coloured Catalan tree $T \in C^{(i)}$ and consider only its underlying tree in C, which we denote by *shape*(T). This leads to the definition of the weight function ω_i on C given by

$$\omega_i(T) = \sum_{\substack{S \in C^{(i)} \\ shape(S)=T}} \omega(S).$$

We can therefore express G_i in terms of ω_i by

$$G_i = \sum_{T \in C^{(i)}} \omega(T) = \sum_{T \in C} \sum_{\substack{S \in C^{(i)} \\ shape(S)=T}} \omega(S) = \sum_{T \in C} \omega_i(T).$$

Using the weight function ω_i, we can state the Jacobian Conjecture in terms of Catalan trees.

Theorem 4.1.2. *(Jacobian Conjecture - Catalan Tree Version)* [9] *Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial function. Let $G = (G_1, \ldots, G_n)$ be the formal inverse of F. Then G is polynomial if and only if for each $1 \leq i \leq n$,*

$$\sum_{T \in C_p} \omega_i(T) = 0 \tag{4.4}$$

for sufficiently large p.

By the Reduction Theorem we assume H is homogeneous of degree d for some positive integer d (we can further assume $d = 3$ but we consider general d for later arguments). From this we can conclude that the weight of most Catalan trees is zero. The proof of this is not in the literature.

Proposition 4.1.3. [9] *Assume H is homogeneous of degree d. Let $T \in \mathbb{T}_{rt}$. If there exists a vertex $v \in V(T) \backslash V_L(T)$ such that the up-degree of v is not d, then $\omega_i(T) = 0$.*

Proof. We are given a tree T with a vertex v such that $v^+ = \{v_1, \ldots, v_k\}$, $k \neq d$. Let l be a colouring of $V(T)$ with colours in $\{1, \ldots, n\}$ and root coloured i. The contribution of the vertex v to $\omega(T)$ is

$$h^{(i)}_{l(v_1),\ldots,l(v_k)} \prod_{i=1}^{k} \omega(T_k)$$

where T_j is the subtree of T rooted at v_j for all $1 \leq j \leq k$. Since H_i is homogeneous of degree $d \neq k$, $h^{(i)}_{l(v_1),\ldots,l(v_k)} = 0$. It follows from (4.3) and the definition of ω_i that $\omega_i(T) = 0$. □

In the next section, we look at conclusions that can be made by assuming the nilpotency of the Jacobian matrix.

4.2 Jacobian Nilpotency and Catalan Trees

The condition that $(JH)^n = 0$ can be translated into a combinatorial property of a certain class of Catalan trees. In order to establish this property, we need to define a new type of Catalan tree, and introduce a formal multiplication between such trees.

4.2.1 Marked Catalan Trees

A *marked Catalan tree* is a pair (T, v) where T is a Catalan tree and v is a leaf of T. We denote the set of marked Catalan trees by $(C, *)$. This naturally leads to defining $(C_p, *)$ and $(C^{(i)}, *)$ as the marked versions of C_p and $C^{(i)}$ respectively. We additionally define $C^{(i,j)}$ to be the set of trees in $(C^{(i)}, *)$ with marked vertex coloured j. Figure 4.2 gives an example of such a tree in $C_7^{(2,4)}$.

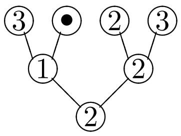
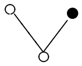


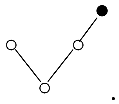
Figure 4.2: A coloured Catalan tree in $C_7^{(2,4)}$, bulleted vertex coloured 4

We can naturally define a product on $(C, *)$. Let $(S, u), (T, v) \in (C, *)$. We define the *Catalan product* $(S, u)(T, v)$ to be the marked Catalan tree obtained by replacing u in S by (T, v). As an example, if

$(S, u) =$  $(T, v) =$ 

then it follows that

$$(S, u)(T, v) = \qquad . \tag{4.5}$$



We can similarly define the product of $(S, u) \in (C, *)$ and $T \in C$ to be the tree in C obtained by replacing u in S by T. Given a tree S with n leaves, we also define the ordered product $S \circ (T_1, \ldots, T_n)$ to be the tree obtained by replacing the i^{th} (in depth-first order) leaf of S by T_i.

There is a certain class of marked trees that will be of particular interest. We say that $(T, v) \in (C, *)$ has a *chain of height k* if $T \backslash V_L(T)$ is a path on k vertices. We denote by CH_k the set of marked trees with a chain of height exactly k. As an example, the tree from (4.5) is in CH_2. We extend our weight function ω to the class $C^{(i,j)}$ by setting

$$\omega_{i,j}(T,v) = \frac{1}{x_j} \sum_{\substack{(S,v)\in C^{(i,j)} \\ shape(S,v)=(T,v)}} \omega(S) \tag{4.6}$$

In other words, we restrict ω_i to marked Catalan trees whose marked vertex is coloured j, and remove the effect of the marked vertex on the weight. The motivation for this definition of $\omega_{i,j}$ is its compatibility with the product on $(C,*)$ that allows for the following matrix-like identity, which is not proven in the literature.

Theorem 4.2.1. [9] *Let* $(S,u),(T,v) \in (C,*)$. *Then*

$$\omega_{i,j}((S,u)(T,v)) = \sum_{k=1}^{n} \omega_{i,k}(S,u)\omega_{k,j}(T,v). \tag{4.7}$$

Proof. If $(S,u),(T,v) \in (C,*)$, we have

$$\omega_{i,j}((S,u)(T,v)) = \frac{1}{x_j} \sum_{\substack{(R,v)\in C^{(i,j)} \\ shape(R,v)=(S,u)(T,v)}} \omega(R). \tag{4.8}$$

Given any (R,v) on the right hand side of (4.8), the vertex u may be coloured with any colour in $\{1,\ldots,n\}$. If u is coloured k, then split (R,v) into the product $(S',u)(T',v)$ where $(S',u)\in C^{(i,k)}$ has shape (S,u) and $(T',v)\in C^{(k,j)}$ has shape (T,v). We can therefore write (4.8) as

$$\frac{1}{x_j}\sum_{k=1}^{n} \sum_{(R,v)=(S',u)(T',v)} \omega(R), \tag{4.9}$$

where the inner sum runs over all $(S',u)\in C^{(i,k)}, (T',v)\in C^{(k,j)}$ with $shape(S',u)=(S,u)$ and $shape(T',v)=(T,v)$. By the definition of $\omega(R)$ we see that if $(R,v)=(S',u)(T',v)$ where $(S',u)\in C^{(i,k)}$ and $(T',v)\in C^{(k,j)}$ then $\omega(R)=\omega(S')\omega(T')\frac{1}{x_k}$. Thus (4.9) becomes

$$\frac{1}{x_j}\sum_{k=1}^{n}\frac{1}{x_k}\sum_{(R,v)=(S',u)(T',v)} \omega(S')\omega(T') = \sum_{k=1}^{n}\omega_{i,k}(S,u)\omega_{k,j}(T,v).$$

In conclusion,

$$\omega_{i,j}((S,u)(T,v)) = \sum_{k=1}^{n} \omega_{i,k}(S,u)\omega_{k,j}(T,v).$$

□

When one of the trees is unmarked we have a similar theorem.

Theorem 4.2.2. [9] *Let* $(S,u)\in(C,*)$ *and* $T\in C$. *Then*

$$\omega_i((S,u)T) = \sum_{k=1}^{n}\omega_{i,k}(S,u)\omega_k(T). \tag{4.10}$$

4.2.2 The Interpretation of $(JH)^n = 0$

We start our investigation of the relationship between the nilpotency of the Jacobian matrix and the weight function $\omega_{i,j}$ on marked Catalan trees with the following lemma, whose proof is not contained in the literature.

Lemma 4.2.3. *[9]*

$$\frac{\partial H_i}{\partial x_j} = \sum_{(T,v)\in CH_1} \omega_{i,j}(T,v).$$

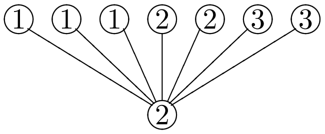
Proof. We show that there is a correspondence between monomials in $\frac{\partial H_i}{\partial x_j}$ and summands on the right hand side of the equation. We introduce this first with an example. Consider the monomial $h^{(3)}_{1,1,1,2,2,3,3}x_1^3x_2^2x_3^2$ in H_3. Its corresponding monomial in $\frac{\partial H_3}{\partial x_2}$ is

$$2\left(h^{(3)}_{1,1,1,2,2,3,3}x_1^3x_2x_3^2\right).$$

This suggests there are exactly 2 trees $(T,v) \in CH_1$ also lying in $C^{(3,2)}$ with

$$\omega(T) = h^{(3)}_{1,1,1,2,2,3,3}x_1^3x_2^2x_3^2.$$

In order for $\omega(T) = h^{(3)}_{1,1,1,2,2,3,3}x_1^3x_2^2x_3^2$, T must be the tree



(4.11)

Furthermore, the only vertices in v that can be marked are the leaves coloured with the colour 2, of which there are exactly two.

In general, any monomial in H_i is of the form $h^{(i)}_{1^{i_1},\ldots,n^{i_n}}x_1^{i_1}\cdots x_n^{i_n}$. The only unmarked tree in CH_1 with this as a weight is the tree $T \in C^{(i)}$ whose non-root vertices are all leaves, and whose root rt satisfies $rt^+ = \{1^{i_1},\ldots,n^{i_n}\}$. There are i_j vertices in T coloured j. Thus T induces exactly i_j trees $\{T_1,\ldots,T_{i_j}\}$ in $C^{(i,j)}$ that are also in CH_1 with weight $h^{(i)}_{1^{i_1},\ldots,n^{i_n}}x_1^{i_1}\cdots x_n^{i_n}$. Summing the weights of these trees we have

$$\omega(T_1) + \ldots + \omega(T_{i_j}) = i_j h^{(i)}_{1^{i_1},\ldots,n^{i_n}}x_1^{i_1}\cdots x_n^{i_n} = \frac{1}{x_j}\frac{\partial}{\partial x_j}\left(h^{(i)}_{1^{i_1},\ldots,n^{i_n}}x_1^{i_1}\cdots x_n^{i_n}\right).$$

The result follows by extending this process to all monomials in H_i. □

From Lemma 4.2.3 we arrive at a connection between the nilpotency of the Jacobian matrix and Catalan trees. This relationship is established in the next theorem. The proof is not in the literature.

Theorem 4.2.4. [9] *Let $F : \mathbb{C}^n \to \mathbb{C}^n$, $F = x - H$ satisfy the conditions of the Reduction Theorem (Theorem 2.2.1). Then*

$$\sum_{(T,v)\in CH_n} \omega_{i,j}(T,v) = (JH)^n_{i,j} = 0.$$

Proof. Denote by W the matrix defined element-wise by

$$W_{i,j} = \sum_{(T,v)\in CH_1} \omega_{i,j}(T,v).$$

We claim that

$$(W^n)_{i,j} = \sum_{(T,v)\in CH_n} \omega_{i,j}(T,v).$$

Any $(T,v) \in CH_n$ can be decomposed as the product of two trees, one in CH_{n-1} and one in CH_1. To see how this works, refer to (4.5). Thus, by induction on n, using (4.7),

$$(W^n)_{i,j} = \sum_{(T,v)\in CH_n} \omega_{i,j}(T,v).$$

From Lemma 4.2.3, we know that $W_{i,j} = \frac{\partial H_i}{\partial x_j}$, so $W = JH$. It follows that

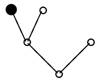
$$0 = ((JH)^n)_{i,j} = \sum_{(T,v)\in CH_n} \omega_{i,j}(T,v).$$

□

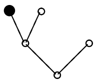
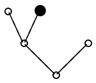
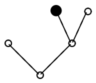
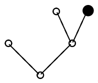
4.2.3 The Degree 2 Case when $(JH)^2 = 0$

In this section, we settle the degree 2, $(JH)^2 = 0$ case of the Jacobian Conjecture. Though the development of Wright [13] (see Chapter 3) gives a shorter proof, we provide a proof using the weight functions on Catalan trees that motivates definitions and approaches needed for more general arguments beyond this specific case. The proof requires us to consider isomorphism classes of Catalan trees, which we now introduce.

Let T be a Catalan tree. We denote by $[T]$ the set of Catalan trees isomorphic to T as a rooted tree. Any two trees in $[T]$ are said to be *equivalent*. The number of trees in $[T]$ is denoted $sym(T)$. For $(T,v) \in (C,*)$, we similarly denote by $[T,v]$ the set of trees in $(C,*)$ isomorphic to T as a rooted tree where the isomorphism sends a marked vertex to a marked vertex, and denote by $sym(T,v)$ the number of trees in $[T,v]$. As an example, the four trees isomorphic to

$T =$ 

are

$T_1 =$  $T_2 =$  $T_3 =$  $T_4 =$  , (4.12)

so $[T] = \{T_1, T_2, T_3, T_4\}$. We naturally extend the weight functions ω_i and $\omega_{i,j}$ to the isomorphism classes $[T]$ and $[T, v]$ respectively by setting

$$\omega_i[T] = \sum_{S \in [T]} \omega_i(T)$$

and

$$\omega_{i,j}[T, v] = \sum_{(S,u) \in [T,v]} \omega_{i,j}(S, u).$$

Now that we have introduced equivalence classes of Catalan trees, we are prepared to prove the specific case in question. The proof is adapted from the proof by Singer [9].

Theorem 4.2.5. *[9] Let $F : \mathbb{C}^n \to \mathbb{C}^n$, $F = x - H$, where H_i is homogeneous of degree 2 for each $1 \leq i \leq n$, and $(JH)^2 = 0$. Then $H \circ H = 0$, so F is invertible with inverse $G = x + H$.*

Proof. Fix $i \in \{1, \ldots, n\}$. Since H_i is homogeneous of degree 2, we can write H_i as

$$H_i = \sum_{1 \leq i_1 \leq i_2 \leq n} h^{(i)}_{i_1,i_2} x_{i_1} x_{i_2}.$$

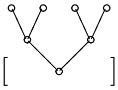
Thus we have that

$$H_i(H_1, \ldots, H_n) = \sum_{1 \leq i_1 \leq i_2 \leq n} h^{(i)}_{i_1,i_2} H_{i_1} H_{i_2}.$$

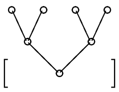
By substituting H_{i_1} and H_{i_2}, we see that monomials in $(H \circ H)_i$ are of the form

$$h^{(i)}_{i_1,i_2} h^{(i_1)}_{i_3,i_4} h^{(i_2)}_{i_5,i_6} x_{i_3} x_{i_4} x_{i_5} x_{i_6}$$

where $1 \leq i_1 \leq i_2 \leq n$, $1 \leq i_3 \leq i_4 \leq n$, $1 \leq i_5 \leq i_6 \leq n$. Consequently,

$$(H \circ H)_i = \sum_{\substack{1 \leq i_1 \leq i_2 \leq n \\ 1 \leq i_3 \leq i_4 \leq n \\ 1 \leq i_5 \leq i_6 \leq n}} h^{(i)}_{i_1,i_2} h^{(i_1)}_{i_3,i_4} h^{(i_2)}_{i_5,i_6} x_{i_3} x_{i_4} x_{i_5} x_{i_6} = \omega_i[$$  $$].$$

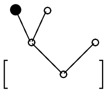
Thus, in order to prove $H \circ H = 0$, it suffices to show that

$\omega_i[$  $] = 0.$

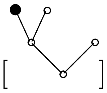
By Lemma 4.2.3 we know that $(JH)^2 = 0$ implies

$$\sum_{(T,v) \in CH_2} \omega_{i,j}(T, v) = 0$$

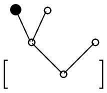
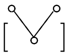
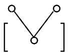
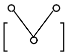
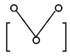
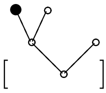
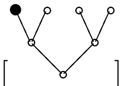
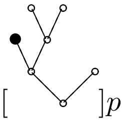
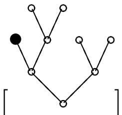
for all $1 \leq i, j \leq n$. Since H is homogeneous of degree 2, $\omega_{i,j}(T, v) = 0$ unless every non-leaf vertex in T has up-degree exactly 2 (by Proposition 4.1.3). There are only 4 marked Catalan trees in CH_2 in which every non-leaf vertex has up-degree exactly 2. These are precisely the 4 trees in (4.12). Thus

$$\sum_{(T,v)\in CH_2} \omega_{i,j}(T,v) = \omega_{i,j}[$$

$$] = 0. \tag{4.13}$$

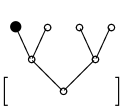
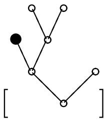
Now by definition,

$$\omega_{i,j}[$$

$$] = \sum_{\substack{1\leq i_1\leq i_2\leq n\\ 1\leq j\leq i_4\leq n}} \frac{1}{x_j}\left(h^{(i)}_{i_1,i_2} h^{(i_1)}_{j,i_4} x_{i_2} x_{i_4} x_j\right)$$
$$= \sum_{\substack{1\leq i_1\leq i_2\leq n\\ 1\leq j\leq i_4\leq n}} h^{(i)}_{i_1,i_2} h^{(i_1)}_{j,i_4} x_{i_2} x_{i_4}.$$

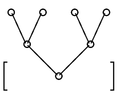
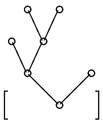
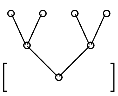
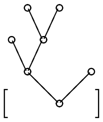
Let p, q be indeterminates. We have

$$0 = \omega_{i,j}[$$

$$]\Big(\omega_1[\circ]p + \omega_1[$$

$$]q, \ldots, \omega_n[\circ]p + \omega_n[$$

$$]q\Big)$$
$$= \sum_{\substack{1\leq i_1\leq i_2\leq n\\ 1\leq j\leq i_4\leq n}} h^{(i)}_{i_1,i_2} h^{(i_1)}_{j,i_4}\Big(\omega_{i_2}[\circ]p + \omega_{i_2}[$$

$$]q\Big)\Big(\omega_{i_4}[\circ]p + \omega_{i_4}[$$

$$]q\Big)$$
$$= \sum_{\substack{1\leq i_1\leq i_2\leq n\\ 1\leq j\leq i_4\leq n}} h^{(i)}_{i_1,i_2} h^{(i_1)}_{j,i_4}\left(x_{i_2}p + \sum_{1\leq i_5\leq i_6\leq n} h^{(i_2)}_{i_5,i_6} x_{i_5} x_{i_6} q\right)\left(x_{i_4}p + \sum_{1\leq i_5\leq i_6\leq n} h^{(i_4)}_{i_5,i_6} x_{i_5} x_{i_6} q\right)$$
$$= \omega_{i,j}[$$

$$]p^2 + \omega_{i,j}[$$

$$]pq + \omega_{i,j}[$$

$$]pq + \omega_{i,j}[$$

$$]q^2.$$

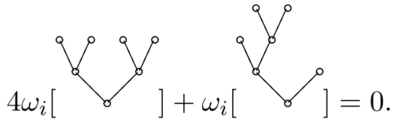
So in particular, the pq coefficient is 0. That is,

$$\omega_{i,j}[$$

$$] + \omega_{i,j}[$$

$$] = 0. \tag{4.14}$$

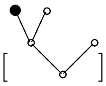
Let M be the $n\times n$ matrix whose $(i,j)^{th}$ entry is the expression in (4.14). By (4.10) we have the matrix equation

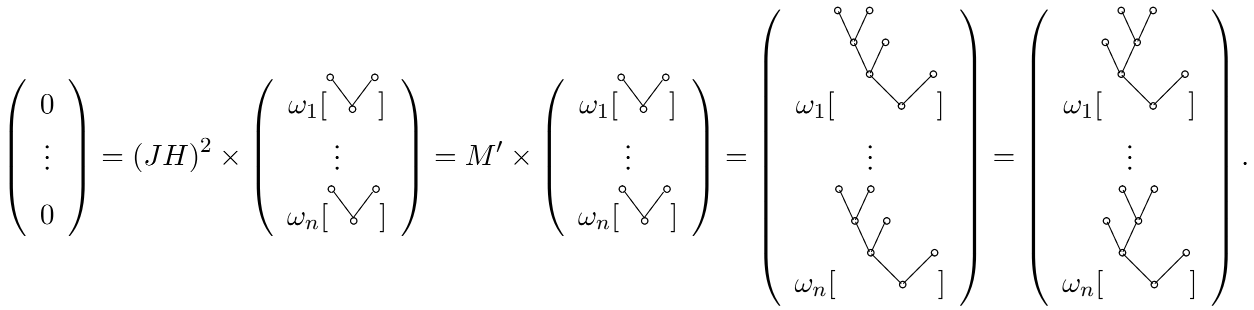
$$M \times \begin{pmatrix} \omega_1[\circ] \\ \vdots \\ \omega_n[\circ] \end{pmatrix} = \begin{pmatrix} 4\omega_1[\;\text{(tree)}\;] + \omega_1[\;\text{(tree)}\;] \\ \vdots \\ 4\omega_n[\;\text{(tree)}\;] + \omega_n[\;\text{(tree)}\;] \end{pmatrix}$$





and so by (4.14), for all $1 \leq i \leq n$,

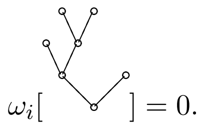


(4.15)

Let M' be the $n \times n$ matrix whose entry $(i,j)^{th}$ entry is $\omega_{i,j}[$  $]$. Then we also have

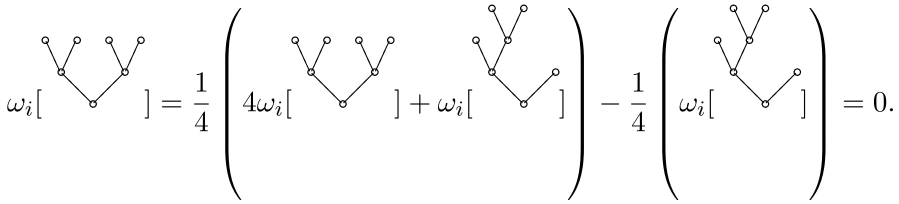


Thus we get that



(4.16)

Using both (4.15) and (4.16) we conclude that



□

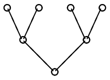
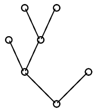
The proof of Theorem 4.2.5 suggests a strategy for approaching other cases of the Jacobian Conjecture. Recall from Theorem 4.1.2 that to establish any case of the Jacobian Conjecture, it suffices to show that

$$\sum_{T \in C_p} \omega_i(T) = 0$$

for sufficiently large p. In order to prove this, we can equivalently prove $\omega_i[T] = 0$ for all $T \in C_p$, for p sufficiently large. As motivated by the proof of Theorem 4.2.5, we can do this by finding a set of linear combinations L of the form

$$\sum_j c_j \omega_i[T_j],$$

showing that $\omega_i[T]$ is a summand in each linear combination, and finally showing that members of L are zero when $(JH)^n = 0$. For instance, in Theorem 4.2.5, we had

$T_1 =$  $T_2 =$ 

and

$$L = \{4\omega_i[T_1] + \omega_i[T_2], \omega_i[T_2]\}.$$

All members of L were proven to be zero when $(JH)^2 = 0$ (by (4.15) and (4.16)) and

$$\omega_i[T_1] \in span_{\mathbb{Q}} L.$$

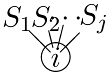
We therefore need a systematic method for performing Gaussian elimination on linear combinations of trees. To do this, we need an ordering on Catalan trees and a characterization of leading terms in linear combinations of these trees. This is the focus of the next section.

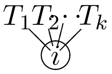
4.2.4 Linear Combinations of Catalan Trees

In this section we define a partial order on Catalan trees, and find leading terms of linear combinations of them with respect to this partial order. As mentioned in the previous section, the motivation for this is to find a systematic method of performing Gaussian elimination on linear combinations of trees. We also introduce definitions and constructs to deal with chains in trees, in order to exploit the nilpotency condition.

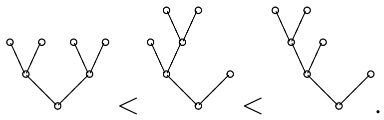
Orderings on $C \cup (C, *)$

We define a total ordering $<$ on $C \cup (C, *)$ as follows. Let $S, T \in C \cup (C, *)$. If S has fewer leaves than T (or vice-versa), then $S < T$ ($S > T$). Otherwise, S and T have the same number of leaves. In this case, we define $<$ recursively as follows. As a base case, an unmarked tree with one vertex is defined to be smaller than a marked tree with one vertex. Otherwise, if

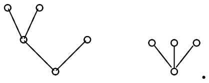
$S =$ 

$T =$ 

then $S < T$ if and only if $S_1S_2\dots S_j$ is less than $T_1T_2\dots T_k$ in lexicographic order. For example, we have that



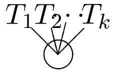
Trees that are the largest in their equivalence class are called *standard trees*. These trees are used as equivalence class representatives. For example, the standard trees in C_3 are



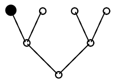
Given a linear combination of trees $\{T_1, \dots, T_k\}$ where $T_1 < \dots < T_k$, we define T_1 to be the *leading term* of the linear combination. We also define two orderings on multisets of trees. If M_1 and M_2 are multisets of trees, we say that $M_1 \leq M_2$ if there is an injection $\phi : M_1 \to M_2$ so that $T < \phi(T)$ for each $T \in M_1$. We also define the ordering $\preceq$ by setting $M_1 \preceq M_2$ if and only if $S \leq T$ for all $S \in M_1$ and $T \in M_2$.

Branch Words and Catalan Sums

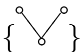
Let $(T, v) \in (C, *)$. We recursively define the *branch word* $B_v(T)$ of (T, v) as follows. If (T, v) is a single marked vertex, $B_v(T)$ is the empty word. Otherwise, we have

$T =$ 

with v being a leaf of T_i for some i, $1 \leq i \leq k$. We define $B_v(T)$ recursively by setting $B_v(T) = B_v(T_i)M$ where M is the ordered multiset $T_1T_2 \cdots T_{i-1}T_{i+1}T_{i+2} \cdots T_k$. As an example, if T is the tree



where v is the marked vertex, then

$B_v(T) = \{\circ\}\{$  $\}$

Two branch words $B_1 = M_1M_2\dots M_j$, $B_2 = N_1N_2\dots N_k$ are said to be equivalent if $j = k$ and there is a bijection $\phi_i : M_i \to N_i$ so that T is equivalent to $\phi_i(T)$ for all $T \in M_i$. In other words, M_i is a rearrangement of N_i for each i, $1 \leq i \leq k$. The following theorem characterizes the equivalence of marked Catalan trees based on their branch word. We omit its proof.

Theorem 4.2.6. [9] *Let $(S, u), (T, v) \in (C, *)$. Then $(S, u) \equiv (T, v)$ if and only if $B_u(S) \equiv B_v(T)$.* $\square$

We now define sums of Catalan trees and establish properties of products of these sums. Let $T \in C \cup (C, *)$. We denote by $sum(T)$ the formal sum

$$sum(T) = \sum_{T' \equiv T} T'.$$

We multiply such sums in a natural way. If $(S, v) \in (C, *)$, $T \in C \cup (C, *)$, we set

$$sum(S, v)sum(T) = \sum_{\substack{(S',v') \equiv (S,v) \\ T' \equiv T}} (S', v')T'.$$

The following two lemmas establish product rules for these sums.

Lemma 4.2.7. [9] *Let* $(R, u), (S, v) \in (C, *)$ *and* $(T, v) = (R, u)(S, v)$. *Then*

$$sum(R, u)sum(S, v) = sum(T, v).$$

Proof. Any term in the product is of the form $(R', u')(S', v')$ where $(R', u') \equiv (R, u)$ and $(S', v') \equiv (S, v)$. By Theorem 4.2.6, $B_{u'}(R') \equiv B_u(R)$ and $B_{v'}(S') \equiv B_v(S)$. Thus,

$$B_{v'}(T') = B_{v'}(S')B_{u'}(R') \equiv B_v(S)B_u(R) = B_v(T).$$

Thus $(T', v') \equiv (T, v)$, so any term in the product $sum(R, u)sum(S, v)$ is equivalent to (T, v). Moreover, any tree $(T', v') \equiv (T, v)$ can be uniquely decomposed as $(T', v') = (R', u')(S', v')$ where $(R', u') \equiv (R, u)$ and $(S', v') \equiv (S, v)$. To prove this, first note that the height of v' from the root of T' must be the height of v from the root of T. Now choose the unique vertex u' in (T', v') on the path from its root to v' such that its height is the same as the height of u from the root of T. This factors (T', v') into a product $(R', u')(S', v')$. We have that

$$B_{v'}(S')B_{u'}(R') = B_{v'}(T') \equiv B_v(T) = B_v(S)B_u(R).$$

Thus $B_{u'}(R') \equiv B_u(R)$ and $B_{v'}(S') \equiv B_v(S)$ so again by Theorem 4.2.6, $(R', u') \equiv (R, u)$ and $(S', v') \equiv (S, v)$ ☐

A similar rule holds if we assume $S \in C$ instead.

Lemma 4.2.8. [9] *Let* $(R, v) \in (C, *)$, $S \in C$ *and* $T = (R, v)S$. *Then there exists a constant* l_T *such that*

$$sum(R, v)sum(S) = l_T sum(T).$$

The proof of Lemma 4.2.8 is similar to that of Lemma 4.2.7 so we omit its proof. We can summarize Lemma 4.2.8 and Lemma 4.2.7 in the following proposition.

Proposition 4.2.9. [9] *Let* $(R, v) \in (C, *)$, $S \in C \cup (C, *)$, $T = (R, v)S$. *Then*

$$sum(R, v)sum(S) = \frac{sym(R, v)sym(S)}{sym(T)} sum(T)$$

Proof. From Theorem 4.2.8 and Theorem 4.2.7, there exists a constant α such that

$$sum(R, v)sum(S) = \alpha sum(T)$$

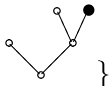
and from this α must satisfy

$$sym(R, v)sym(S) = \alpha sym(T).$$

$\square$

Chain Compositions

We refine CH_k by setting $CH_{(i_1,\ldots,i_k)}$ to be the equivalence class of all trees in CH_k having branch word $M_1 M_2 \cdots M_k$ where M_j consists of i_j single vertices. For example,

$CH_{(1,1)} = \{(C, v) \in CH_2 : (C, v) \equiv$  $\}.$

Now let $M = \{T_1, \ldots, T_r\}$ be a multiset of standard Catalan trees. Let $i_1, \ldots, i_k$ be a collection of positive integers with $i_1 + \ldots + i_k = r$. Then $CH_{(i_1,\ldots,i_k)}$ has exactly r leaves. We denote by $CH_{(i_1,\ldots,i_k)} \circ M$ the multiset of trees formed in the following way: Choose a multiset of trees $A = \{T_1^{'}, \ldots, T_r^{'}\}$ such that $T_j^{'} \equiv T_j$ for all $1 \leq j \leq r$ and replace each leaf of CH_k by exactly one member of A. We set $B_{(i_1,\ldots,i_k)}(M)$ to be the set of branch words $M_1 \cdots M_k$ which are multiset partitions of M with $|M_j| = i_j$. The following theorem establishes an expression for the sum of weights of trees in $CH_{(i_1,\ldots,i_k)} \circ M$. The proof is not found in the literature.

Proposition 4.2.10. [9] *With the above notation,*

$$\sum_{(T,v) \in CH_{(i_1,\ldots,i_k)} \circ M} (T, v) = \frac{r!}{\alpha} \sum_{\substack{(T,v) \in standard(C,*) \\ B_v(T) \in B_{(i_1,\ldots,i_k)}(M)}} sum(T, v)$$

where α *is the number of distinct rearrangements of* $\{T_1, \ldots, T_r\}$.

Proof. Let $(T,v) \in CH_{(i_1,\ldots,i_k)}$. There are exactly $\frac{r!}{\alpha}$ trees in $(T',v) \in (C,*)$ such that $B_v(T') = B_v(T)$ (and consequently $(T',v) = (T,v)$ since all these trees are standard). These are precisely the trees obtained by rearranging trees in the set A which are the same. Moreover, we can create a tree $(T,v) \in CH_{(i_1,\ldots,i_k)} \circ M$ whose branch word is any given word in $B_{(i_1,\ldots,i_k)}(M)$. To show this, let $M' = M_1 \cdots M_k$ be a word in $B_{(i_1,\ldots,i_k)} \circ M$ so that $|M_j| = i_j$. Choose a chain $C \in CH_{(i_1,\ldots,i_k)}$. Create (T,v) by replacing each of the i_j non-leaf vertices of height j by exactly one tree in M_j. Then the branch word of (T,v) is M'. It follows that

$$\begin{aligned}\sum_{(T,v)\in CH_{(i_1,\ldots,i_k)}\circ M} (T,v) &= \frac{r!}{\alpha} \sum_{B_v(T)\in B_{(i_1,\ldots,i_k)}(M)} (T,v) \\ &= \frac{r!}{\alpha} \sum_{\substack{(T,v)\in standard(C,*) \\ B_v(T)\in B_{(i_1,\ldots,i_k)}(M)}} sum(T,v).\end{aligned}$$

□

From Proposition 4.2.10 and Proposition 4.2.9 we have the following theorem.

Theorem 4.2.11. [9] *Let $(R,u) \in (C,*)$, $T \in C \cup (C,*)$. Then*

$$\begin{aligned}&sum(R,u)\left(\sum_{(S,v)\in CH_{(i_1,\ldots,i_k)}\circ M}(S,v)\right)sum(T) \\ &\quad = \frac{r!}{\alpha} \sum_{\substack{(S,v)\in standard(C,*) \\ B_v(S)\in B_{(i_1,\ldots,i_k)}(M)}} \frac{sym(R,u)sym(S,v)sym(T)}{sym((R,u)(S,v)T)} sum((R,u)(S,v)T)\end{aligned}$$

The main purpose of Theorem 4.2.11 is that it can be used to generalize (4.14). As in the arguments leading up to (4.14), we consider $\sum_{T\in CH_k} \omega_{a,b}(T)$ (which we denote by $C^{(k)}_{a,b}(x_1,\ldots,x_n)$ for simplicity) as a function of $x_1,\ldots,x_n$. By Theorem 4.2.4 we have that

$$C^{(k)}_{a,b}(x_1,\ldots,x_n) = (JH)^k_{a,b}.$$

Let M be a multiset of trees. As in the computations leading to (4.14), given indeterminates $q_1,\ldots,q_r$, we are interested in determining

$$[q_1\cdots q_r]C^{(k)}_{a,b}\left(\sum_{T_j\in M}\omega_1[T_j]q_j,\ldots,\sum_{T_j\in M}\omega_n[T_j]q_j\right).$$

Using Theorem 4.2.11, we find a slightly stronger result. The proof of this result is not in the literature.

Theorem 4.2.12. *[9]*

$$[q_1 \cdots q_r] \sum_{1 \le a,b \le n} \omega_{i,a}[R,v] C_{a,b}^{(k)} \left(\sum_{T_j \in M} \omega_1[T_j] q_j, \ldots, \sum_{T_j \in M} \omega_n[T_j] q_j \right) \omega_b[T] \tag{4.17}$$

$$= \frac{r!}{\alpha} \sum_{\substack{i_1+\cdots+i_k=r \\ (S,v)\in standard(C,*) \\ B_v(S)\in B_{(i_1,\ldots,i_k)}(M)}} \frac{sym(R,u)sym(S,v)sym(T)}{sym((R,u)(S,v)T)} \omega_i[(R,u)(S,v)T]. \tag{4.18}$$

Proof. We first find an expression for

$$[q_1 \cdots q_r] C_{a,b}^{(k)} \left(\sum_{T_j \in M} \omega_1[T_j] q_j, \ldots, \sum_{T_j \in M} \omega_n[T_j] q_j \right) \tag{4.19}$$

$$= [q_1 \cdots q_r] \sum_{(S,v)\in CH_k} \omega_{a,b}(S) \left(\sum_{T_j \in M} \omega_1[T_j] q_j, \ldots, \sum_{T_j \in M} \omega_n[T_j] q_j \right). \tag{4.20}$$

The degree of any term in $\omega_{a,b}(S)$ is the number of unmarked leaves in S. It follows that the only terms in the sum potentially having a non-zero $q_1 \cdots q_r$ coefficient are those that have r unmarked leaves. Thus (4.20) can be restricted to

$$[q_1 \cdots q_r] \sum_{\substack{(S,v)\in CH_{(i_1,\ldots,i_k)} \\ i_1+\ldots+i_k=r}} \omega_{a,b}(S) \left(\sum_{T_j \in M} \omega_1[T_j] q_j, \ldots, \sum_{T_j \in M} \omega_n[T_j] q_j \right).$$

Expanding the weight functions as polynomials and making the substitutions, we see as in (4.14) that

$$\begin{aligned}
&[q_1 \cdots q_r] \sum_{\substack{(S,v)\in CH_{(i_1,\ldots,i_k)} \\ i_1+\ldots+i_k=r}} \omega_{a,b}(S) \left(\sum_{T_j \in M} \omega_1[T_j] q_j, \ldots, \sum_{T_j \in M} \omega_n[T_j] q_j \right) \\
&= \sum_{\substack{(S,v)\in CH_{(i_1,\ldots,i_k)} \\ i_1+\ldots+i_k=r}} [q_1 \cdots q_r] \omega_{a,b}(S) \left(\sum_{T_j \in M} \omega_1[T_j] q_j, \ldots, \sum_{T_j \in M} \omega_n[T_j] q_j \right) \\
&= \sum_{\substack{(S,v)\in CH_{(i_1,\ldots,i_k)} \\ i_1+\ldots+i_k=r}} \sum_{\sigma \in \mathfrak{S}_r} \sum_{{T_1}' \equiv T_{\sigma(1)},\ldots,{T_n}' \equiv T_{\sigma(n)}} \omega_{a,b}(S \circ (T_1, \ldots, T_n)) \\
&= \sum_{\substack{(S,v)\in CH_{(i_1,\ldots,i_k)} \circ M \\ i_1+\ldots i_k=r}} \omega_{a,b}[S,v].
\end{aligned}$$

Thus we have that

$$[q_1 \cdots q_r] C_{a,b}^{(k)} \left(\sum_{T_j \in M} \omega_1[T_j] q_j, \ldots, \sum_{T_j \in M} \omega_n[T_j] q_j \right) = \sum_{\substack{(S,v)\in CH_{(i_1,\ldots,i_k)} \circ M \\ i_1+\ldots+i_k=r}} \omega_{a,b}[S,v]. \tag{4.21}$$

Consequently,

$$
\begin{aligned}
&[q_1 \cdots q_r] \sum_{1 \leq a,b \leq n} \omega_{i,a}[R,v] C^{(k)}_{a,b} \left(\sum_{T_j \in M} \omega_1[T_j] q_j, \ldots, \sum_{T_j \in M} \omega_n[T_j] q_j \right) \omega_b[T] \\
&= \sum_{1 \leq a,b \leq n} \omega_i[R,u] \left(\sum_{\substack{(S,v) \in CH_{(i_1,\ldots,i_k)} \circ M \\ i_1 + \ldots i_k = r}} \omega_{a,b}[S,v] \right) \omega_b[T] \qquad \text{(by 4.21)} \\
&= \omega_i \left[sum(R,u) \left(\sum_{\substack{(S,v) \in CH_{(i_1,\ldots,i_k)} \circ M \\ i_1 + \ldots + i_k = r}} (S,v) \right) sum(T) \right] \\
&= \omega_i \left[\frac{r!}{\alpha} \sum_{\substack{i_1 + \ldots + i_k = r \\ (S,v) \in standard(C,*) \\ B_v(S) \in B_{(i_1,\ldots,i_k)}(M)}} \frac{sym(R,u) sym(S,v) sym(T)}{sym((R,u)(S,v)T)} (R,u)(S,v)T \right] \text{(by Theorem 4.2.11)} \\
&= \frac{r!}{\alpha} \sum_{\substack{i_1 + \ldots + i_k = r \\ (S,v) \in standard(C,*) \\ B_v(S) \in B_{(i_1,\ldots,i_k)}(M)}} \frac{sym(R,u) sym(S,v) sym(T)}{sym((R,u)(S,v)T)} \omega_i[(R,u)(S,v)T].
\end{aligned}
$$

□

The following is an immediate corollary of Theorem 4.2.12, whose proof is not in the literature.

Corollary 4.2.13. [9] *If H_i is homogeneous of degree $d+1$ for each i and $(JH)^k = 0$, then*

$$
\sum_{\substack{(S,v) \in standard(C,*) \\ B_v(S) = B_{d^k}(M)}} \frac{sym(S,v)}{sym((R,u)(S,v)T)} \omega_i[(R,u)(S,v)T] = 0.
$$

Proof. If $(JH)^k = 0$ then by Theorem 4.2.4, $C^{(k)}_{a,b}(x_1, \ldots, x_n) = 0$ for all $1 \leq a, b \leq n$. Thus by Theorem 4.2.12,

$$
\frac{r!}{\alpha} \sum_{\substack{(S,v) \in standard(C,*) \\ B_v(S) \in B_{(i_1,\ldots,i_k)}(M)}} \frac{sym(R,u) sym(S,v) sym(T)}{sym((R,u)(S,v)T)} \omega_i[(R,u)(S,v)T] = 0. \tag{4.22}
$$

Pick any tree $(R', u')(S', v')T$ from a summand of (4.22). Then

$$
\omega_i((R', u')(S', v')T) = \sum_{1 \leq a,b \leq n} \omega_i(R', u') \omega_a(S', v') \omega_b(T).
$$

If S has a non-leaf vertex of up-degree other than $d+1$, then $\omega_a(S', v') = 0$ for all $1 \leq a \leq n$, so $\omega_i((R', u')(S', v')T) = 0$. Thus we can restrict (4.22) to trees where each non-leaf vertex has up-degree

$d+1$. This implies $i_1 = i_2 = \ldots = i_k = (d+1) - 1 = d$, so $B_v(S) \in B_{d^k}(M)$. Together with factoring out $sym(R,u)$ and $sym(T)$ from (4.22), we have our result. $\square$

Corollary 4.2.13 presents a collection of linear combinations all of which are annihilated by ω_i. We now need a method to find the leading term in such linear combinations in order to perform Gaussian elimination as we did at the end of Section 4.2.3. In (4.22), we summed over weights on equivalence classes of trees. Thus, if (S', v') is the smallest standard tree with branch word in $B_{(i_1,\ldots,i_k)}(M)$, then the leading terms in the sum are members of $[(R,u)(S',v')T]$. It therefore suffices to find the smallest standard tree (S',v') with branch word in $B_{(i_1,\ldots,i_k)}(M)$. Singer does this in [9]. We omit the proof as it is removed from the combinatorial focus of this thesis.

Theorem 4.2.14. [9] *Let $i_1, \ldots, i_k$ be a collection of positive integers and M be a multiset of standard trees of cardinality $i_1 + \ldots + i_k$. Then the smallest standard tree in $B_{(i_1,\ldots,i_k)}(M)$ is the tree (S',v') with $B_{v'}(S') = M_1 \cdots M_k$ and $M_1 \preceq M_2 \preceq \ldots \preceq M_k$.*

Combining Theorem 4.2.11 and Theorem 4.2.14 we have the following result.

Theorem 4.2.15. [9] *Let M be a multiset of standard Catalan trees of cardinality r and $(i_1, \ldots, i_k)$ be a partition of r. Let $(R,u) \in (C,*)$, $T \in C$. Then*

$$sum(R,u)\left(\sum_{(S,v)\in CH_{(i_1,\ldots,i_k)}\circ M} (S,v)\right) sum(T)$$

is a linear combination over equivalence classes of Catalan trees with leading terms from the equivalence class $[(R,u)(S',v')T]$, where (S',v') is the tree satisfying $B_v(S) = M_1 \cdots M_k$ and $M_1 \preceq M_2 \preceq \ldots \preceq M_k$.

We have set up linear combinations of trees that are annihilated by ω_i, and have found leading terms in such combinations. This enables us tackle more cases of the Jacobian Conjecture in the next section.

4.3 Applications to the Jacobian Conjecture

In this section we use the developments from the previous section with regards to leading terms of linear combinations to resolve a case of the Jacobian Conjecture. Before doing so, we identify certain trees that are easily seen to be leading terms of linear combinations.

Let $T \in C$ be a standard tree and let v be a leaf of T such that $B_v(T) = M_1 \cdots M_j$. If there is a positive integer a such that $M_a \preceq M_{a+1} \preceq \ldots \preceq M_{a+k-1}$ then we call T a *k-good tree*. Any standard tree that is not k-good is said to be a *k-bad tree*. If (T,v) is a marked tree and $B_v(T) = M_1 \cdots M_k$ with $M_1 \preceq M_2 \preceq \ldots \preceq M_k$, then (T,v) is said to be an *especially k-good tree*. Note that any k-good tree T can be factored as $(Q,u)(R,v)S$ where (R,v) is especially k-good. Given this decomposition of T, it immediately follows that T is the leading term of

$$sum(Q,u)\left(\sum_{(R',v')\in CH_{(i_1,\ldots,i_k)}\circ M}(R',v')\right)sum(S).$$

by Theorem 4.2.15. We use this property to prove the following theorem.

Theorem 4.3.1. [9] *The set of linear combinations of the form*

$$sum(Q,u)\left(\sum_{\substack{(R,v)\in CH_{(1,1,1)}\circ M\\(Q,u)(R,v)S\in B_p}}(R,v)\right)sum(S)$$

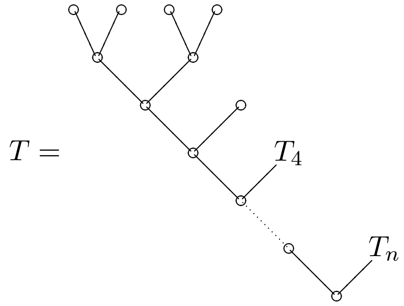
spans the set $\{sum(T) : T \in B_p\}$ *for* $p \geq 7$ *where* B_p *is the set of binary Catalan trees with* p *leaves.*

To prove Theorem 4.3.1, we must show that every tree $T \in \bigcup_{p=7}^{\infty} B_p$ is a leading term in sums of the form

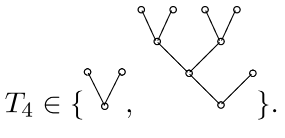
$$sum(Q,u)\left(\sum_{\substack{(R,v)\in CH_{(1,1,1)}\circ M\\(Q,u)(R,v)S\in B_p}}(R,v)\right)sum(S).$$

Any 3-good tree is a leading term of some linear combination of this type since such a tree can be factored as $(Q',u')(R',v')S$ where (R',v') is especially 3-good and hence is in $CH_{(1,1,1)} \circ M$. We refer to such linear combinations as 3-good combinations. Thus it suffices to show that every standard 3-bad binary tree having at least seven leaves is the leading term of a linear combination of 3-good combinations. Our first step is to characterize 3-bad standard binary trees having at least seven leaves.

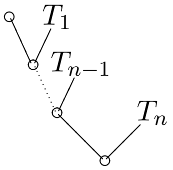
Lemma 4.3.2. [9] *Let T be a 3-bad standard binary tree with at least seven leaves. Then*



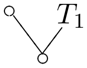
where



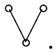
Proof. Since T is standard,

$T =$ 

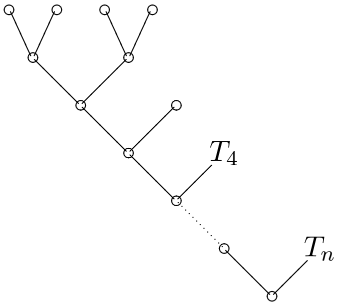
where $T_1 = \circ$ and each T_i is standard. We are given that T is 3-bad, so T_2 can not be equal to $\circ$. Again, since T is 3-bad, we must have that

$T_2 \leq$  .

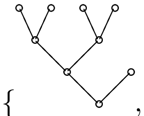
Hence

$T_2 =$  .

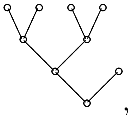
If $T_3 \geq T_2$, then together with the chain in T, T_1, T_2, T_3 form an especially 3-good subtree, implying T is 3-good which contradicts that it is 3-bad. Thus $T_3 < T_2$ implying that $T_3 = \circ$. Since T has at least seven leaves, n must be at least 4. It follows that

$T =$  .

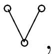
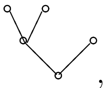
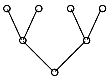
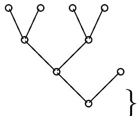
Consider the two subtrees of T rooted at the root of T_4 and its sibling. These two trees are

$\{$  $, T_4\}$.

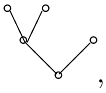
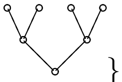
In order for T to be standard,

$T_4 \leq$  ,

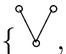
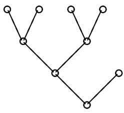
otherwise we can switch these two trees to obtain a larger tree in the equivalence class of T. For this inequality to hold and for T_4 to not be especially 3-good itself, we must have that

$T_4 \in \{\circ,$  ,  ,  ,  $\}$.

If we choose

$T_4 \in \{\circ,$  ,  $\}$

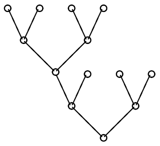
then T itself will be 3-good. We therefore conclude that

$T_4 \in \{$  ,  $\}$.

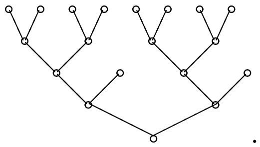
□

We are now prepared to prove Theorem 4.3.1.

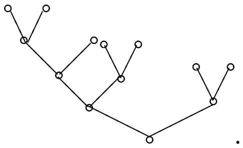
Proof. (Theorem 4.3.1) By Lemma 4.3.2, every 3-bad standard binary tree having at least 7 leaves can be factored as either $(R, v)T_1$ or $(R, v)T_2$ where

$T_1 =$ 

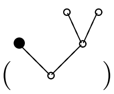
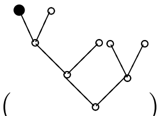
and

$T_2 =$  .

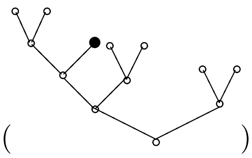
If we show that T_1 and T_2 are leading terms of linear combinations of 3-good combinations, then any tree of the form $(R, v)T_1$ and $(R, v)T_2$ will be a leading term of linear combinations of 3-good combinations for any $(R, v) \in (C, *)$. Since every 3-bad binary tree is of one of the these two forms, it suffices to show that T_1 and T_2 are leading terms of linear combinations of 3-good combinations. We first show that T_1 is a leading term of a linear combination of 3-good combinations. Let

$S_1 =$  .

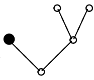
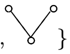
S_1 is 3-good and can be factored in two different ways as $(Q, u)(S, v)R$ where (S, v) is especially 3-good. These factorizations are

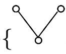
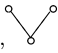
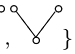
$S_1 = ($  $)($  $)(\circ)$

and

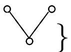
$S_1 = (\bullet)($  $)(\circ)$.

Thus S_1 is the leading term in both of the following linear combinations:

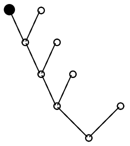
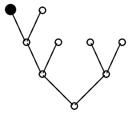
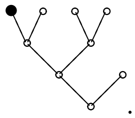
$L_1 = sum($  $)\Big(\sum_{(R,v)\in CH_{(1,1,1)}\circ\{\circ,\circ,}$  $\}$ $(R,v)\Big)\, sum(\circ)$

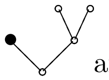
$L_2 = sum(\bullet)\Big(\sum_{(R,v)\in CH_{(1,1,1)}\circ\{}$  $,$  $,$  $\}$ $(R,v)\Big)\, sum(\circ)$

We can now use Theorem 4.2.11 to simplify these linear combinations. Let

$M = \{\circ,\circ,$  $\}.$

There are precisely 3 trees in $(S', v) \in standard(C, *)$ with $B_v(S') \in B_{(1,1,1)}(M)$. These trees are

$T_3 =$  $T_4 =$  $T_5 =$  .

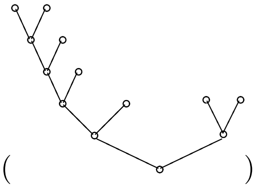
For simplicity let $Q =$  and $Q' = \circ$. The symmetry numbers of these trees are

$$sym(Q) = 2 \qquad sym(Q') = 1 \qquad sym(T_3) = 16 \qquad sym(T_4) = 8 \qquad sym(T_5) = 8.$$

Moreover, we also know that

$$sym(QT_3Q') = 32 \qquad sym(QT_4Q') = sym(S_1) = 8 \qquad sym(QT_5Q') = sym(T_1) = 4.$$

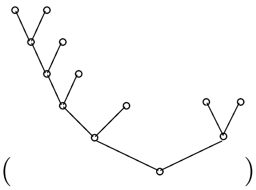
Applying Theorem 4.2.11, we have

$L_1 = \frac{3!}{3}\Big(sum($  $) + 2sum(S_1) + 4sum(T_1)\Big).$

Applying the same process to L_2 we have

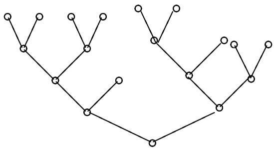
$$L_2 = \frac{3!}{1} sum(S_1).$$

Hence

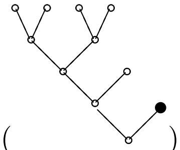
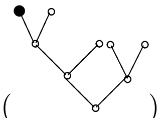
$6L_1 - 8L_2 = 24sum(T_1) + 12sum($  $)$

is a linear combination of 3-good combinations with leading term T_1.

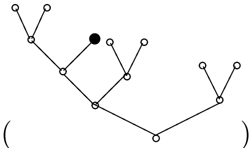
We repeat the process for the tree T_2. Let

$S_2 =$  .

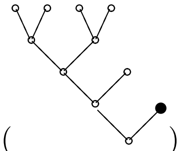
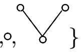
The tree S_2 is 3-good and can be factored in the following ways:

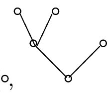
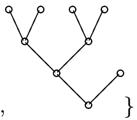
$S_2 = ($  $)($  $)(\circ)$

and

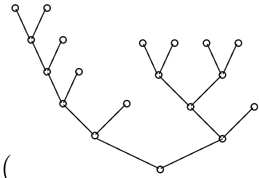
$S_2 = (\bullet)($  $)(\circ)$.

Thus S_2 is the leading term in both L_3 and L_4 given below.

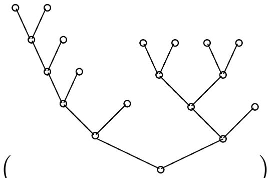
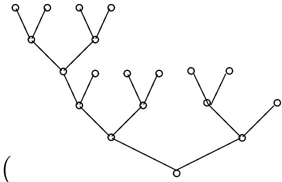
$L_3 = sum($  $)\Bigg($ $\sum_{(R,v)\in CH_{(1,1,1)}\circ\{\circ,\circ,}$  $\}$ $(R,v)\Bigg)\, sum(\circ)$

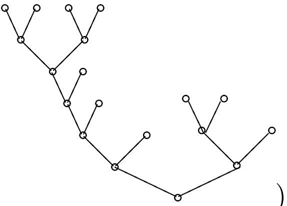
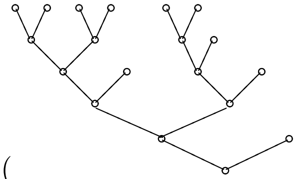
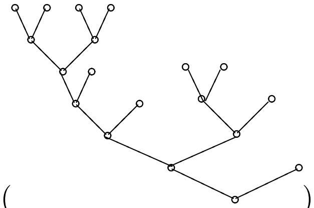
$L_4 = sum(\bullet)\Bigg($ $\sum_{(R,v)\in CH_{(1,1,1)}\circ\{\circ,}$  $,$  $\}$ $(R,v)\Bigg)\, sum(\circ)$.

Applying Theorem 4.2.11 we have

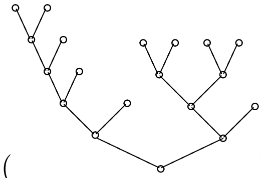
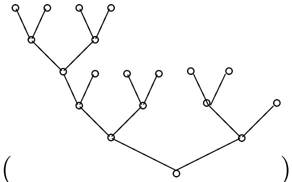
$L_3 = \frac{3!}{3}(2sum(S_2) + 4sum(T_2) + sum($  $))$

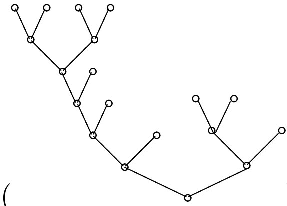
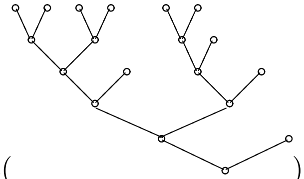
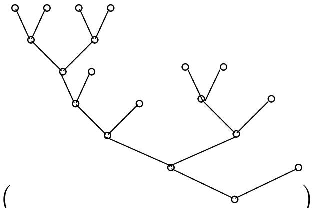
and

$L_4 = \frac{3!}{3!}(sum(S_2) + sum($  $) + sum($  $)+$

$sum($  $) + sum($  $) + sum($  $))$

Thus

$L_3 - 4L_4 = 8sum(T_2) - 2sum($  $) - 4sum($  $)$

$- 4sum($  $) - 4sum($  $) - 4sum($  $)$

is a linear combination of 3-good combinations with leading term T_2. This concludes the proof. $\square$

The following is an immediate corollary of Theorem 4.3.1.

Corollary 4.3.3. [9] *If* H_i *is homogeneous of degree* 2 *for all* $1 \leq i \leq n$ *and* $(JH)^3 = 0$*, then* $\omega_i[T] = 0$ *for all* $T \in \bigcup_{p=7}^{\infty} B_p$*. Thus, the inverse* F *is a polynomial system of degree at most* 6*.*

Proof. If the set of linear combinations in Theorem 4.3.1 span the set $\{sum(T) : T \in B_p\}$ for $p \geq 7$, then each $T \in B_p$ $(p \geq 7)$ is a linear combination of 3-good combinations. Since $(JH)^3 = 0$ and

$deg(H_i) \leq 2$ for all $1 \leq i \leq n$, then by both Corollary 4.2.13 and Theorem 4.2.11, $\omega_i[T] = 0$ for all $T \in \bigcup_{p=7}^{\infty} C_p$. Thus if G_i is the i^{th} component of G,

$$G_i = \sum_{p \geq 0} \sum_{T \in C_p} \omega_i[T] = \sum_{p=0}^{6} \sum_{T \in C_p} \omega_i[T].$$

Thus $deg(G_i) \leq 6$. □

The degree bound of 6 is independent of the number of variables in F. This improves the degree bound of 2^{n-1} for n-dimensional polynomial functions due to Bass, Connell and Wright in [2].

Theorem 4.3.1 gives insight on how to pose the Jacobian Conjecture using information about linear combinations of Catalan trees. Since we have the Reduction Theorem (Theorem 2.2.1) at our disposal, we can assume $F = x - H$ where H is homogeneous of degree 3. It follows that the weight of any tree with a vertex having up-degree other than 3 is zero. Consequently, we would restrict our attention to ternary trees. Let $\mathfrak{T}$ denote the set of ternary Catalan trees, and $\mathfrak{T}_p$ be the set of ternary Catalan trees having p leaves. The following is a conjecture about linear combinations of ternary trees that parallels Theorem 4.3.1, and whose resolution would prove the Jacobian Conjecture to be true. This conjecture is not presented in the literature.

Conjecture 4.3.4. *Let k be a positive integer. The set of linear combinations of the form*

$$sum(Q,u) \left(\sum_{\substack{(R,v) \in CH_{2k} \circ M \\ (Q,u)(R,v)S \in \mathfrak{T}_p}} (R,v) \right) sum(S)$$

spans the set $\{sum(T) : T \in \mathfrak{T}_p\}$ whenever $p \geq f(k)$ (for some value $f(k)$ dependent on k)

Proposition 4.3.5. *If Conjecture 4.3.4 is true for every positive integer k, then the Jacobian Conjecture is true.*

Proof. Let $F : \mathbb{C}^k \to \mathbb{C}^k$ be a polynomial function. By the Reduction Theorem, we can assume H_i is homogeneous of degree 3 for each $1 \leq i \leq k$ and that $(JH)^k = 0$. By Theorem 4.2.11 and Corollary 4.2.13, $\omega_i[T] = 0$ for all $T \in \mathfrak{T}_p$ with $p \geq f(k)$. Thus

$$G_i = \sum_{p=0}^{f(k)} \sum_{T \in \mathfrak{T}_p} \omega_i[T]$$

which is polynomial. □

We have now seen various combinatorial formulations of the Jacobian Conjecture using different combinatorial structures. In the next and final chapter, we present ideas for potentially unifying these approaches. We also ask various questions whose answers would give more insight on resolving the Jacobian Conjecture.

Chapter 5

Further Directions

In this thesis, we have seen that combinatorial properties of trees model the algebraic properties of the formal power series inverse of certain polynomial functions. The main approaches we investigated were due to Wright and Singer. In this concluding chapter, we pose questions in the context of these approaches. In doing so, we resolve the Symmetric Cubic Linear case. Along with asking questions that would resolve the Jacobian Conjecture, we pose questions whose answers would give us further insight on combinatorial approaches to the conjecture.

5.1 Combinatorial Interpretation of Reductions

The first question we consider is whether or not other reductions of the Jacobian Conjecture have combinatorial interpretations. In particular, we saw in Chapter 1 (Theorem 1.3.6) that the conjecture has been reduced to the case where $H_i = L_i^3$ where L_i is linear for all $1 \leq i \leq n$. What are the combinatorial consequences of restricting ourselves to such functions? Can the approaches of Singer [9] or Wright [13] help us find this out? Since Wright's approach uses the Symmetric Reduction (Theorem 3.1.1), incorporating his work would restrict us to the case when F is symmetric and cubic linear. Can we solve this case of the Jacobian Conjecture? In fact we can, and we establish this in the proof of the following theorem. The statement and proof of the theorem do not appear in the literature.

Theorem 5.1.1. *(Symmetric Cubic Linear Case) The Jacobian Conjecture is true when F is symmetric and cubic linear.*

Proof. If JH is nilpotent and symmetric, then there exists some polynomial $P \in \mathbb{C}[x_1, \ldots, x_n]$ such that $F = x - \nabla P$. Since $H_i = L_i^3$, $D_i P = L_i^3$ for each $1 \leq i \leq n$. Since L_i is linear, P must be of the form $(a_1 x_1 + \ldots + a_n x_n)^4$ for some constants $a_1, \ldots, a_n \in \mathbb{C}$. For simplicity, let $P_L = a_1 x_1 + \ldots + a_n x_n$ Now the Jacobian of H is

$$JH = 12P_L^2 \begin{pmatrix} a_1 a_1 & \cdots & a_1 a_n \\ \vdots & \ddots & \vdots \\ a_n a_1 & \cdots & a_n a_n \end{pmatrix}.$$

Let $v = [a_1, a_2, \ldots, a_n]$ and $M = v^T v$. Then $JH = 12P_L^2 M$. Thus $(JH)^m = (12P_L^2)^m M^m = (12P_L^2)^m (a_1^2 + \ldots + a_n^2)^{(m-1)} M$ which is not zero unless $a_1^2 + \ldots + a_n^2 = 0$. In this case, F is invertible by Theorem 3.2.4. □

Another question about reductions arises naturally from the thesis. After the Symmetric Reduction in 2005, Wright [13] refined the Tree Inversion Formula as detailed in Chapter 3. Singer's Catalan Inversion Formula [9] was developed in 2001, before the Symmetric Reduction was proven. One would therefore expect that there is a Symmetric Catalan Inversion Formula. As suggested by Wright's work, such an inversion formula could lead to more combinatorial insight on the problem. It could also simplify the situation computationally, since Singer has a direct method for performing Gaussian elimination on linear combinations. We suggest that in order to find a Symmetric Tree Inversion Formula, the relationship between the Bass-Connell-Wright Tree Inversion Formula and Singer's Catalan Inversion Formula should be investigated.

5.2 Combinatorial Questions

As we have seen in both Singer and Wright's approaches, there are direct combinatorial questions that can give us more insight on the Jacobian Conjecture. We discuss such questions that have yet to be resolved.

In Singer's approach from Chapter 1, we saw that proving the Jacobian Conjecture could be reduced to proving that standard *k-bad* ternary trees are leading terms of linear combinations of *k-good* combinations. In order to do this, one must characterize *k-bad* ternary trees. Is there a combinatorial characterization of these trees? Are these trees abundant in the set of ternary trees? Singer [9] found numerical evidence to suggest that *k-bad* standard binary trees decrease in density very rapidly in B_p as p increases. The author notes that through computation there are exactly 29 3-bad ternary trees.

Along these lines, one can similarly ask for a complete characterization of sets of unmarked ternary trees whose marked counterparts have branch words that are permutations of each other. This would be beneficial in making the Gaussian elimination process more systematic.

In Wright's approach from Chapter 3, we saw that certain submodules of $\mathfrak{M}$ defined by combinatorial properties are intimately tied with algebraic properties. For instance, if JH is symmetric and $(JH)^r = 0$, then the submodule $C(r)$ is the zero module. Furthermore, if H has degree 3, the submodule $V(4)$ is the zero module. A natural question to ask is whether or not there are other combinatorially generated submodules of $\mathfrak{M}$ that are annihilated by algebraic properties. This would restrict the set of trees we must annihilate in the Grossman-Larson Algebra.

Proceeding with any of these suggested approaches is bound to uncover more of the rich combinatorial information hidden within the Jacobian Conjecture.

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