Signatures of New Physics from the Primordial Universe

by

Amjad Ashoorioon

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AUTHOR'S DECLARATION FOR ELECTRONIC SUBMISSION OF A THESIS

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

During inflation quantum fluctuations of the field driving inflation, known as the inflaton, were stretched by inflationary expansion to galactic size scales or even larger. A possible implication of inflation – if it is correct – is that our observable universe was once of sub-Planckian size. Thus inflation could act as a magnifier to probe the short distance structure of space-time. General arguments about the quantum theory of gravity suggest that the short distance structure of space-time can be modeled as arising from some corrections to the well-known uncertainty relation between the position and momentum operators. Such modifications have been predicted by more fundamental theories such as string theory. This modified commutation relation has been implemented at the first quantized level to the theory of cosmological perturbations. In this thesis, we will show that the aforementioned scenario of implementing the minimal length to the action has an ambiguity: total time derivatives that in continuous space-time could be neglected and do not contribute to the equations of motion, cease to remain total time derivatives as we implement minimal length. Such an ambiguity opens up the possibility for trans-Planckian physics to leave an imprint on the ratio of tensor to scalar fluctuations. In near de-Sitter space, we obtain the explicit dependence of the tensor/scalar on the minimal length. Also the first consistency relation is examined in a power-law background, where it is found that despite the ambiguity that exists in choosing the action, Planck scale physics modifies the consistency relation considerably as it leads to large oscillations in the scalar spectral index in the observable range of scales. In the second part of the thesis, I demonstrate how the assumption of existence of invariant minimal length can assist us to explain the origin of cosmic magnetic fields. The third part of the thesis is dedicated to the study of signatures of M-theory Cascade inflation.

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To My Mother & Father

&

To My Wife

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Chapter 1

Introduction

It has been understood that the answer to some of the fundamental questions in cosmology involve events that took place during the first 10^{-2} sec of the history of the universe. These questions include the origin of nucleons, the origin of matter-antimatter asymmetry, the nature of the dark matter, the origin of smoothness and flatness of the universe, the origin of the density inhomogeneities of the cosmic microwave background radiation (CMBR) that initiated structure formation, the origin of the expansion, and even the ultimate fate of the universe. Mingling ideas borrowed from particle physics, which are valid up to the weak scale, with cosmology, has led to a few successes in explaining some of these cosmological events. People hope that they can test more speculative ideas of particle physics that are relevant at higher energies with cosmological probes. The urge for such investigations has become pressing when it is observed that such exciting and fundamental ideas in particle physics involve energy scales well beyond reach of terrestrial accelerators. Current accelerators can hardly achieve the Tev scale, whereas the energy scale needed to test the grand unification ideas is in excess of 10^{14} Gev, and the unification that involves gravity is 10^{19} Gev. Such usage of astrophysical and cosmological observations to confine theoretical speculations has ensued in an explosion of activities and progress in theoretical cosmology.

During the past two decades we have witnessed a flurry of activity in observational cosmology too: First COBE [1] and then WMAP I & II [2–6] measured precisely the spectrum and anisotropies of the CMBR and revealed important information about the nature of primordial inhomogeneities as well as the constituents of our universe. Red shift surveys such as 2dFGRS [7–9], 2dFQSO [10–12] and Canada-France redshift survey [13–16] have revealed the nature of the large scale structure of the universe and shed light upon the distribution of matter. The high-Z supernovae exposed us to the fact that our universe has gone through a recent phase of acceleration [17–19]. New experiments, such as Planck [20], are going to probe the cosmos in further detail and discover other signatures that might be left from the first second of the history of universe. There is a hope that observation will be able to test the high energy physicists speculations that seem to be unverifiable at ground-based accelerators.

Our understanding of the universe from $t \sim 10^{-2}$ second to $t \sim 15$ Gyr is based on hot big bang cosmology, which is the Friedmann-Robertson-Walker (FRW) cosmological model. The model is quite robust and there are no observational data that clashes with the model. It can explain the current expansion of the universe, the primordial abundance of the light elements and the origin of the CMBR. However this simple elegant model suffers from some shortcomings that point to some grander theory. This model cannot explain the origin of structures in the universe. It cannot get rid of unwanted relics that are produced in the context of grand unified theories. It cannot also explain why the universe is flat, homogeneous and isotropic. The latter problem is basically an initial data problem and is related to the fact that entropy in the universe is so large, $S \approx 10^{88}$. One expects this number to be of order unity as it is a dimensionless number. Below we will explain these problems in more quantitative way.

1.1 Deficiencies of Hot Big Bang model

The dynamics of an FRW universe containing matter with density ρ and pressure p is determined by the Einstein acceleration equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3m_{\rm Pl}^2}(\rho + 3p), \tag{1.1}$$

the Friedmann equation

$$H^2 = \frac{8\pi}{3m_{\rm Pl}^2}\rho - \frac{k}{a^2}, \tag{1.2}$$

and the continuity equation

$$\dot{\rho} + 3H(\rho + p) = 0 \tag{1.3}$$

where $a(t)$ is the scale factor of the universe, $H \equiv \frac{\dot{a}}{a}$ is the Hubble expansion parameter, a dot denotes differentiation with respect to the cosmic time t, $m_{\rm Pl}$ is the Planck mass, defined as $m_{\rm Pl} \equiv (\hbar c/G)^{1/2}$ in terms of fundamental constants, and $k = 0, -1, +1$ for spatially flat, open or closed cosmologies respectively. Units are chosen such that $c = \hbar = 1$. One can express the Friedmann equation 1.2 in terms of $\Omega \equiv \rho/\rho_c$, which is the ratio of energy density of the universe to the critical density, ρ_c defined as:

$$\rho_c \equiv \frac{3m_{\rm Pl}^2 H_0^2}{8\pi}. \tag{1.4}$$

The current observational value of ρ_c is $1.88h^2 \times 10^{-29}{\rm gcm}^{-3}$, where the Hubble parameter is parameterized such that $H_0 = 100h{\rm kms}^{-1}{\rm Mpc}^{-1}$ and $0.4 \leq h \leq 1$. h parameterizes the experimental uncertainty in the value of Hubble parameter today. Now the Friedmann equation takes the following form:

$$\Omega - 1 = \frac{k}{a^2 H^2}, \tag{1.5}$$

or equivalently

$$\frac{\Omega - 1}{\Omega} = \frac{3m_{\rm Pl}^2 k}{8\pi\rho a^2} \tag{1.6}$$

For a radiation dominated universe $\rho = 3p$ where $\rho = \pi^2 g_* T^4/30$ where $g_* = \mathcal{O}(10^2)$ is the number of relativistic degrees of freedom during that time, see [21] and T is the temperature of the universe. For a spatially flat radiation dominated universe $a(t) \propto t^{1/2}$ and so the Hubble parameter is given by

$$H = 1.66 g_* \left(\frac{T^2}{m_{\rm Pl}}\right) = \frac{1}{2t}. \tag{1.7}$$

The above equation yields a simple relation between time and temperature during the radiation dominated phase:

$$\left(\frac{t}{\rm sec}\right) \approx \left(\frac{T}{\rm Mev}\right)^{-2}. \tag{1.8}$$

The above relation with eqs. 1.7 and 1.6

$$\left|\frac{\Omega - 1}{\Omega}\right| \approx \frac{10^{43}}{S^{2/3}}\left(\frac{t}{\rm sec}\right) \approx \frac{10^{37}}{S^{2/3}}\left(\frac{\rm GeV}{T}\right)^2, \tag{1.9}$$

where $S \approx 10^{88}$ is the entropy contained with the present Hubble radius. The large amount of entropy in the current horizon patch implies that $\Omega = 1 \pm 10^{-60}$ at the Planck time, $t = 10^{-43}$ second. In addition, it implies that the radius of curvature of the universe was huge at the Planck time:

$$R_{\rm curv} \geq 10^{30} H^{-1}. \tag{1.10}$$

While no law of physics precludes such initial data, this suggests that our FRW model was very special indeed. If the above quantity was of order unity at the above time, we would have recollapsed after a few Planck times for $\Omega > 1$ or would have reached a temperature 3K at the ripe age of 10^{-11} second for $\Omega < 1$. This problem is called the *flatness* problem. This problem is closely related to the *entropy* problem which is as follows: The CMBR, the relic radiation from Big Bang, has a uniform temperature across the sky, on angular scales from $10''$ to 180°, to about one part in 10^4. The universe could have reached such smoothness if the entire universe was in causal contact at the time of last scattering, $t_{\rm ls}$. The particle horizon after the matter-radiation equality can be expressed in terms of entropy within the horizon volume [21]:

$$S_{\rm hor} = 3 \times 10^{87} (\Omega h^2)^{-3/2} (1+z)^{-3/2}, \tag{1.11}$$

where $1 + z \equiv a(t_0)/a(t)$ is the redshift and for $t = t_{\rm ls}$, $z \approx 1100$. The entropy within the horizon at the last scattering surface is $S_{\rm hor} \simeq 10^{83}$. In another words, within the Big Bang scenario, the particle horizon at the last scattering epoch was roughly about 10^{-5} of horizon today and would subtend an angle of only about 0.8° in the sky. This is also referred to as *horizon* problem. Equivalently, the flatness problem arises because the entropy in a comoving volume is conserved. It seems that it would be possible to solve this problem if the cosmic expansion was non-adiabatic for some finite interval of time:

$$S_f = Z^3 S_i, \tag{1.12}$$

where Z is a numerical factor. Misner and others advocated [22] the *chaotic cosmological model*, in which more generic, anisotropic and inhomogeneous initial space-time would smooth itself out through dissipative processes, producing the enormous entropy of our universe. However a more compelling scenario of entropy production was put forward by Guth [23] and others [24, 25, 27, 28] who called it *inflation*. We will come back to this scenario in detail later.

The other problem that standard cosmology is faced with is its inability to explain the origin of *primordial inhomogeneities*. The Big Bang scenario can explain the growth of structures from small-scale inhomogeneities in a matter-dominated universe via the Jeans instability. Since non-linear structures exist today on scales from 1 Mpc to 10 Mpc or so, and as the fluctuations grow linearly with the scale factor during the matter-dominated era, we can deduce that perturbations of order 10^{-5} must have existed on these scales at the time of matter-radiation equality. Following such scales during the radiation-dominated epoch, one would observe that, at early times, these scales were outside the horizon. Therefore, in the context of Big Bang cosmology it is not possible to justify the origin of large scale structures with causal microphysical processes. The other shortcoming that standard cosmology suffers from is the problem of unwanted relics. In the context of grand unified gauge theories, a variety of stable heavy particles are produced that survive the annihilation and contribute to the energy of the universe such that they may overclose the universe. Standard Big Bang cosmology does not have any mechanism to obviate these problems.

1.2 Inflation

Inflation is a scenario suggested by Alan Guth in his original paper of 1981 [23]. In this scenario the scale factor of the universe grows by a huge factor of Z in a finite interval of time, from t_i to t_f. In Guth's original paper, inflation happened at or below the GUT energy scale which corresponds to a time-scale 10^{-40} second. From equation 1.9, the quantity $(\Omega^{-1}-1)\rho a^2$ is conserved and therefore we have:

$$(\Omega_f^{-1}-1)\rho_f a_f^2 = (\Omega_i^{-1}-1)\rho_i a_i^2 \tag{1.13}$$

On the other hand, in the standard model we have [29]:

$$(\Omega_i^{-1}-1)\rho_i a_i^2 \approx 10^{-56}(\Omega_0^{-1}-1)\rho_f a_f^2 \tag{1.14}$$

In order to have $\left|\Omega_0^{-1}-1\right| \sim \mathcal{O}(1)$, as required from observation, and from 1.13 & 1.14 one must have $\rho_f a_f^2 \gg \rho_i a_i^2$. From Friedmann equation 1.2, we also have

$$3\dot{a}^2 - \frac{8\pi}{m_{\rm pl}^2}\rho a^2 = \text{const.} \tag{1.15}$$

Consequently, the above inequality is satisfied if $\dot{a}_f > \dot{a}_i$. Hence the necessary condition for inflation to proceed is that the scale factor accelerates:

$$\ddot{a} > 0 \tag{1.16}$$

The previous inequality 1.16, could be written in terms of H and $\dot{H}$ as:

$$\dot{H} + H^2 > 0. \tag{1.17}$$

Three categories of inflation can be singled out in 1.17: The case $\dot{H} < 0$, which we call sub-inflation, the usual de-Sitter inflation, $\dot{H} = 0$, and the super-inflation which is designated by $\dot{H} > 0$. Defining $w \equiv p/\rho$, the three cases correspond to $-1 < w < -1/3$, $w = -1$ and $w < -1$ respectively.

The next question is about the nature of the energy source that drives this accelerated expansion. It follows from Einstein acceleration equation 1.1 and 1.16 that $\rho+3p < 0$ or equivalently $w < -1/3$. Such an energy source violates the strong energy condition [30]. The simplest way to realize inflation is by some homogeneous scalar field, ϕ, with positive potential, $V(\phi)$. Such a scalar field is equivalent to a perfect fluid with the following energy density and pressure:

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi), \tag{1.18}$$

and

$$p = \frac{1}{2}\dot{\phi}^2 - V(\phi). \tag{1.19}$$

One can achieve the requirement $\rho + 3p < 0$, and equivalently $\ddot{a} > 0$, if

$$\dot{\phi}^2 < V(\phi). \tag{1.20}$$

Inflation is thus achieved if the potential energy of the scalar field is dominant over its kinetic energy.

One should here note that there are alternative scenarios of inflation driven by the kinetic energy of scalar field [31, 32]. Such models use a general class of non-quadratic kinetic terms for the scalar field to obtain inflationary evolution. The non-standard kinetic term is motivated by appealing to the existence of higher-order corrections to the effective action for ϕ in string theory. Such higher order corrections are produced as α'-corrections due to the massive modes of string [33]. To show how this scenario works, let us consider a single scalar field interacting with gravity through non-standard kinetic term,

$$S = \int d^4x\sqrt{g}\left(-\frac{R}{16\pi G} + p(X)\right), \tag{1.21}$$

with $X = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi$. The energy density for such fluid, only depends on X:

$$\varepsilon(X) = 2X\frac{\partial p}{\partial X} - p(X). \tag{1.22}$$

If in the function $p(X) = KX + \frac{1}{2}LX^2 + \cdots$, all the coefficients $K, L, \cdots$ are positive, p & ε are always positive. Then, it is easy to show that ε will monotonically decrease toward zero, and the evolution will be driven to the "attracting solution "

$$\varepsilon = H^2 \approx \frac{1}{9t^2}, \qquad a \approx a_0 t^{1/3}, \tag{1.23}$$

which corresponds to the asymptotic equation of state $p \approx \varepsilon$ valid near $\varepsilon = 0$ where the usual kinetic term $p = \frac{1}{2}KX$ dominates. On the other hand, if the expansion coefficients $K, L, \cdots$ may take negative values, the graph $p = p(X)$ may look more complicated and can allow for exponential-type inflationary behavior. From equation 1.22, one can see that the extrema of the function $p(X)$, i.e. $\frac{\partial p}{\partial X} = 0$, corresponds to values where $p = -\varepsilon$, and therefore allows for de-Sitter expansion. One can show that all the intersection points with the $p = -\varepsilon$ are attractors of the (future) evolution. Models that exploit non-standard forms of kinetic terms are called k-inflationary scenarios [31, 32]. However, hereafter, we shall restrict ourselves to potential-driven models.

In this work, we will mainly focus on a chaotic inflationary scenario [34]. To illustrate the main idea of this scenario, let us first consider a theory of a scalar field ϕ with a degenerate effective potential $V(\phi)$, minimized at $\phi = \phi_0$. It is clear that in such a theory there are no reasons to expect that the classical field ϕ is equal to ϕ_0 in the whole universe. On the contrary, one may expect that all values of ϕ may appear in different regions of space, sufficiently far removed from each other, with equal probability. This means that in such a theory the field displaced from its minimum, evolves to its true vacuum. Especially its equation of motion is classical and is governed by:

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0. \tag{1.24}$$

This is similar to the equation of ball rolling down a hill with friction. The crucial aspect of the evolution of ϕ is the time required for ϕ to roll to its minimum, Δt. If the scalar potential is sufficiently flat, i.e.

$$\left|\ddot{\phi}\right| \ll H \left|\dot{\phi}\right|, \tag{1.25}$$

this time can be long compared to the time-scale of expansion of the universe, i.e. $\Delta t \gg H^{-1}$. The above condition, with the condition 1.20, are known as *slow-roll* conditions. During this slow-roll evolution, the universe is dominated by the vacuum of energy of the scalar field and the universe expands quasi-exponentially:

$$a(t) = a_i \exp\left(\int_{t_i}^{t} dt' H(t')\right) \tag{1.26}$$

The expansion is quasi-exponential since $H(\phi) \approx 8\pi V(\phi)/3m_{\rm Pl}^2$ is almost constant. During the inflationary expansion, the curvature term, k/a^2, rapidly redshifts away. As the field rolls down the potential, gradually its kinetic energy increases. Eventually, its kinetic energy dominates over its potential energy and inflation comes to an end when $\dot{\phi}^2 \approx V(\phi)$. The field then rapidly oscillates about the minimum of the potential. The spatially coherent oscillations of the field ϕ around its minimum corresponds to a condensate of zero-momentum ϕ particles of mass $m_\phi^2 = V''(\phi)$. At this stage the coupling of ϕ to other field becomes important. The condensate of ϕ particles decay to other fields that couple to ϕ due to quantum particle creation. It is these oscillations that result in particle production and a reheating of the universe [35–37].

During the slow-roll evolution, the equation of motion reduces to

$$3H\dot{\phi} = -V'(\phi). \tag{1.27}$$

Thus, the number of e-foldings of growth in the scale factor could be calculated as

$$\ln a_2/a_1 = N_e = \int_{t_1}^{t_2} H dt = -\frac{8\pi}{m_{\rm Pl}^2}\int_{\phi_1}^{\phi_2} \frac{V(\phi)}{V'(\phi)} d\phi. \tag{1.28}$$

For the simplest chaotic inflationary model with a quadratic potential, $V(\phi) = m^2\phi^2/2$, where m is the mass of the field ϕ, hereafter called the *inflaton*, the number of e-foldings is given by:

$$N_e = \frac{2\pi}{m_{\rm Pl}^2}(\phi_i^2 - \phi_f^2). \tag{1.29}$$

Inflation starts at super-Planckian values of $\phi_i \gg m_{\rm Pl}$, where $V(\phi_i) \approx m_{\rm Pl}^4$ and ends when $\phi_f \approx \mathcal{O}(m_{\rm Pl})$. Thus N_e is approximately given by $\exp(4\pi m_{\rm Pl}^2/m^2)$. On the other hand, having a perturbation amplitude consistent with the maximum anisotropy of the CMBR requires that $m \approx 10^{-6} m_{\rm Pl}$ [80]. This implies that the scale factor of the universe increases by a factor of $10^{10^{12}}$. Indeed, if the original domain is one Planck length in extent, its final size will be of order $10^{10^{12}}$ cm; for comparison the size of the observable universe is approximately 10^{28} cm. Here lies the origin of the trans-Planckian problem of inflationary cosmology [38]. As we will see in the next three chapters, the assumptions underlying calculation of perturbations are the validity of quantum field theory (QFT) and general relativity. However, if such models of inflation are correct, the modes that have contributed to structure formation in our observable universe have been stretched from sub-Planckian scales where both of these two theories break down. In some particular framework of short-distance physics, we investigate how the predictions of standard cosmological perturbation theory get modified.

1.3 Fluctuations in standard inflation

In inflation, see [39, 40], we consider the action of the scalar inflaton field, minimally coupled to gravity:

$$S = \frac{1}{2}\int (\partial_\mu\phi\partial^\mu\phi - V(\phi))\sqrt{-g}\, d^4x - \frac{1}{16\pi G}\int R\sqrt{-g}\, d^4x \tag{1.30}$$

One assumes the background to be a homogenous isotropic Friedmann universe with zero spatial curvature. In comoving coordinates y and comoving time τ, the metric reads $ds^2 = a^2(\tau)\left(d\tau^2 - \delta_{ij}dy^i dy^j\right)$. The perturbations of the metric tensor can be decomposed into scalar, vector and tensor modes according to their transformation properties under spatial coordinate transformations on the constant-time hypersurfaces, namely $ds^2 = ds_S^2 + ds_V^2 + ds_T^2$, where:

$$ds_S^2 = a^2(\tau)\left((1+2\Phi)d\tau^2 - 2\partial_i B dy^i d\tau - [(1-2\Psi)\delta_{ij} + 2\partial_i\partial_j E]dy^i dy^j\right) \tag{1.31}$$

$$ds_V^2 = a^2(\tau)\left(d\tau^2 + 2V_i dx^i d\tau - [\delta_{ij} + W_{i,j} + W_{j,i}]dx^i dx^j\right) \tag{1.32}$$

$$ds_T^2 = a^2(\tau)\left(d\tau^2 - [\delta_{ij} + h_{ij}]dx^i dx^j\right) \tag{1.33}$$

This generalizes the decomposition of vector fields into a curl and a gradient field. Here, Φ, B, Ψ and E are scalar fields, V_i and W_i are 3-vector fields satisfying $V_{i,i} = W_{i,i} = 0$

and h_{ij} is a symmetric three-tensor field satisfying $h_i^i = 0 = h_{ij}{}^{,j}$. The repetition of indices does not indicate summation. The inflaton field fluctuates about its spatially homogeneous background $\phi(\mathbf{y},\tau) = \phi_0(\tau) + \delta\phi(\mathbf{y},\tau)$. Here, $\phi_0(\tau)$ is the homogenous part of the scalar field which is driving the background expansion and the perturbation is assumed small: $|\delta\phi| \ll \phi_0$. In standard inflation, vector fluctuations are not amplified by the expansion but it should be interesting to reconsider if this still holds true in inflation with a minimum length. Here, we will focus on scalar and tensor fluctuations.

1.3.1 Scalar perturbations

It is the quantum fluctuations of the intrinsic curvature perturbations of the comoving hypersurface $\Re$ which are thought to have seeded what later became the dominant perturbations in the CMBR. The intrinsic curvature $\Re$, which is gauge invariant, can be expressed as:

$$\Re = -\frac{a'}{a}\frac{\delta\phi}{\phi_0'} - \Psi, \tag{1.34}$$

The prime denotes differentiation with respect to conformal time, τ. Expanding the action to second order yields for the action of $\Re$

$$S_S^{(1)} = \frac{1}{2}\int d\tau\; d^3\mathbf{y}\; z^2\left((\partial_\tau\Re)^2 - \delta^{ij}\;\partial_i\Re\partial_j\Re\right) \tag{1.35}$$

where:

$$z = \frac{a\phi_0'}{\alpha}, \quad \alpha = a'/a \tag{1.36}$$

In the vast literature on standard inflationary theory, however, a slight reformulation of the action is usually preferred as the starting point for quantization. Namely, one often introduces an auxiliary field variable, u, through

$$u = -z\Re = a\left(\delta\phi + \frac{\phi_0'\Psi}{\alpha}\right) \tag{1.37}$$

whose dynamics follows from the action:

$$S_S^{(2)} = \frac{1}{2}\int d\tau d^3\mathbf{y}\left((\partial_\tau u)^2 - \delta^{ij}\;\partial_i u\;\partial_j u + \frac{z''}{z}u^2\right) \tag{1.38}$$

As long as we do not introduce a minimum length, the two actions $S_S^{(1)}$ and $S_S^{(2)}$ are equivalent. More precisely, they differ by a boundary term:

$$S_S^{(1)} - S_S^{(2)} = \int d\tau \, d^3\mathbf{y} \, \frac{d}{d\tau}\left(\frac{z'}{z}u^2\right) \tag{1.39}$$

The reason why one often prefers to quantize starting from the action $S_S^{(2)}$ rather than from the action $S_S^{(1)}$ is that $S_S^{(2)}$ possesses no overall time-dependent factor, and this gives it the appearance of an action of a free field theory on flat space. Its only nontrivial aspect is that the field $u(\mathbf{y}, \tau)$ has a time-varying "mass" z''/z. The similarity to a Minkowski space theory suggests that in this formulation the field can be quantized in the same way that one would quantize a field on flat space. This suggests that one can identify the vacuum state in the same way as one does in the case of Minkowski space theories. Concretely, the Euler Lagrange field equation reads:

$$\hat{u}'' - \nabla^2 \hat{u} - \frac{z''}{z}\hat{u} = 0. \tag{1.40}$$

The momentum conjugate to $u(\mathbf{y}, \tau)$ is given by $\pi(\mathbf{y}, \tau) = \frac{\partial \mathcal{L}_s}{\partial u'} = u'(\mathbf{y}, \tau)$. To quantize, one promotes u and π to operators, $\hat{u}$ and $\hat{\pi}$, which satisfy canonical commutation relations on hypersurfaces of constant τ:

$$[\hat{u}(\tau, \mathbf{y}), \hat{u}(\tau, \mathbf{y}')] = [\hat{\pi}(\tau, \mathbf{y}), \hat{\pi}(\tau, \mathbf{y}')] = 0 \tag{1.41}$$

$$[\hat{u}(\tau, \mathbf{y}), \hat{\pi}(\tau, \mathbf{y}')] = i\delta^3(\mathbf{y} - \mathbf{y}') \tag{1.42}$$

Employing the plane wave expansion

$$\hat{u}(\tau, \mathbf{y}) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \left[u_k(\tau)\hat{a}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{y}} + u_k^*(\tau)\hat{a}_{\mathbf{k}}^\dagger e^{-i\mathbf{k}\cdot\mathbf{y}}\right] \tag{1.43}$$

the fields will obey the commutation relations Eqs.1.41,1.42 if the operators $\hat{a}_{\mathbf{k}}$ obey the Fock commutation relations $[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}] = [\hat{a}_{\mathbf{k}}^\dagger, \hat{a}_{\mathbf{k}'}^\dagger] = 0, [\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}^\dagger] = i\delta^3(\mathbf{k} - \mathbf{k}')$ and if the mode functions u_k obey the Wronskian condition:

$$u_k^* \frac{du_k}{d\tau} - u_k \frac{du_k^*}{d\tau} = -i. \tag{1.44}$$

Further, the field equation will be obeyed if the number-valued functions $u_k(\tau)$ obey the mode equation:

$$u_k'' + \left(k^2 - \frac{z''}{z}\right) u_k = 0. \tag{1.45}$$

At this point, initial conditions must be chosen for the solution of Eq.1.45. This choice is crucial because it implies the identification of the vacuum state and this affects all predictions of the theory. Intuitively, one expects that if a mode can be followed back to when its proper wavelength was infinitesimally short then one sees the mode when it was virtually unaffected by curvature, i.e. here by the expansion. One should therefore be able to identify the correct solution of the mode equation at those early times, which then sets the initial conditions of the mode for all time. Indeed, in Eq.1.45, one observes that $z''/z \to 0$ at early times, $\tau \to -\infty$, i.e. when the mode's proper wavelength was arbitrarily short. In this limit, Eq.1.45 formally turns into $u_k'' + k^2 u_k = 0$ which is the zero mass wave equation for a Minkowski space theory. For such a theory the correct solution of the wave equation is known and one proceeds, therefore, to impose as the initial condition for Eq.1.45:

$$u_k(\tau) \to \frac{1}{\sqrt{2k}}\, e^{-ik\tau} \qquad \text{for } \tau \to -\infty \tag{1.46}$$

This identifies the initial vacuum of each mode as the incoming lowest energy vacuum. The mathematical problem for calculating u is now well-posed. Finding u then yields the mode function for the intrinsic curvature $\Re = -u/z$ and from it we finally obtain the observationally relevant power spectrum $P_S^{1/2}(k)$, of the intrinsic curvature's quantum fluctuations after horizon crossing:

$$P_s^{1/2}(k) = \sqrt{\frac{k^3}{2\pi^2}}\, |\Re_k| \Big|_{\frac{k}{aH} \ll 1} \tag{1.47}$$

To conclude: before introducing a minimum length the actions $S_S^{(1)}$ and $S_S^{(2)}$ differ merely by a boundary term. Thus, re-expressing the mode equation Eq.1.45 that followed from $S_S^{(2)}$ in terms of the intrinsic curvature yields

$$\Re_{\tilde{k}}'' + \frac{2z'}{z}\Re_{\tilde{k}}' + k^2 \Re_{\tilde{k}} = 0 \tag{1.48}$$

which is of course the same mode equation that one obtains as Euler Lagrange equation directly from the action $S_S^{(1)}$. The rationale for taking the detour via the action $S_S^{(2)}$ is that this route exhibits a similarity with QFT on Minkowski space which suggest a particular choice of initial condition and thus of the vacuum.

1.3.2 Tensor perturbations

The situation for the tensor modes h is similar. Their dynamics is determined by expanding the Einstein-Hilbert action to second order:

$$S_T^{(1)} = \frac{m_{Pl}^2}{64\pi} \int d\tau d^3\mathbf{y}\, a^2(\tau)\, \partial_\mu h^i{}_j\, \partial^\mu h_i{}^j \tag{1.49}$$

The aim is to calculate the spectrum of the quantum fluctuations of h after horizon crossing. This spectrum is hoped to become testable through measurements of the CMB's B-polarization spectrum, the first measurements of which may come from the upcoming PLANCK satellite telescope [20].

It is clear that the action $S_T^{(1)}$ is of precisely the same form as $S_S^{(1)}$, up to constants and the replacement of $z(\tau)$ by $a(\tau)$. This means that it is possible to reformulate also the tensor action to give it the appearance of a Minkowski space theory with variable mass term, thereby obtaining a criterion for picking the initial conditions. Therefore, instead of quantizing directly from $S_T^{(1)}$, one often prefers to introduce the re-scaled variable, $P^i{}_j$

$$P^i{}_j(y) = \sqrt{\frac{m_{Pl}^2}{32\pi}}\, a(\tau) h^i{}_j(y) \tag{1.50}$$

whose dynamics follows from the action:

$$S_T^{(2)} = \frac{1}{2} \int d\tau d^3\mathbf{y} \left(\partial_\tau P_i^j \partial^\tau P^i{}_j - \delta^{rs} \partial_r P_i^j \partial_s P^i{}_j + \frac{a''}{a} P_i^j P^i{}_j \right) \tag{1.51}$$

Analogously to the case of scalar fluctuations, the two actions $S_T^{(1)}, S_T^{(2)}$ merely differ by a term which is a total time derivative

$$\triangle S_T = S_T^{(2)} - S_T^{(1)} = \frac{32\pi}{m_{Pl}^2} \int d\tau d^3\mathbf{y} \left(\alpha P_i^j\, P^i{}_j \right)' \tag{1.52}$$

and therefore lead to the same equation of motion.

One proceeds by decomposing $P^i{}_j$ into its Fourier components

$$P^i{}_j = \sum_{\lambda=+,\times} \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\, p_{\mathbf{k},\lambda}(\tau)\, \epsilon^i{}_j(\mathbf{k};\lambda)\, e^{i\mathbf{k}\cdot\mathbf{y}} \tag{1.53}$$

where $\epsilon^i{}_j(\mathbf{k};\lambda)$ is the polarization tensor, satisfying the conditions: $\epsilon_{ij} = \epsilon_{ji}$, $\epsilon^i{}_i = 0$, $k^i\epsilon_{ij} = 0$ and $\epsilon^i{}_j(\mathbf{k};\lambda)\epsilon^{j*}{}_i(\mathbf{k};\lambda') = \delta_{\lambda\lambda'}$. There are two independent polarization states, usually denoted $\lambda = +, \times$. It is convenient to choose $\epsilon_{ij}(-\mathbf{k};\lambda) = \epsilon^*_{ij}(\mathbf{k};\lambda)$ which implies that $p_{\mathbf{k},\lambda} = p^*_{-\mathbf{k},\lambda}$. The action for tensor perturbations then takes the form:

$$S_T^{(2)} = \sum_{\lambda=+,\times} \int d\tau d^3\mathbf{k} \left((\partial_\tau |p_{\mathbf{k},\lambda}|)^2 - \left(k^2 - \frac{a''}{a}\right) |p_{\mathbf{k},\lambda}|^2 \right) \tag{1.54}$$

To quantize, one promotes $p_{\mathbf{k},\lambda}$ to an operator $\hat{p}_{\mathbf{k},\lambda}$ and expands it in terms of creation and annihilation operators, $\hat{p}_{\mathbf{k},\lambda} = p_k(\tau)\hat{a}_{\mathbf{k},\lambda} + p^*_k(\tau)\hat{a}^\dagger_{\mathbf{k},\lambda}$ to obtain for the mode functions $p_k(\tau)$ (omitting the index λ) the wave equation:

$$p''_k + \left(k^2 - \frac{a''}{a}\right) p_k = 0, \tag{1.55}$$

Analogously to scalar fluctuations, also the $p_k(\tau)$ must obey the Wronskian condition (1.44). As for scalar modes, the similarity with the zero mass Minkowski space wave equation at early times suggests to impose the initial condition that the field takes the form given in Eq.1.46 for $k/aH \to \infty$. The mathematical problem is then well defined and p can be calculated. From p one obtains the tensor mode $h^i{}_j = \sqrt{\frac{32\pi}{m_{Pl}^2 a^2}} p^i{}_j$ and finally the spectrum of tensor quantum fluctuations h_k after horizon crossing:

$$P_T^{1/2} = \sqrt{\frac{k^3}{2\pi^2}} |h_k| \Big|_{\frac{k}{aH} \ll 1} \tag{1.56}$$

To summarize, as in the case of scalar fluctuations, one starts quantization from the action $S_T^{(2)}$ so as to exploit the similarity with Minkowski space QFT for identifying the initial conditions and thus the vacuum state for tensor modes.

1.4 Some other models of inflation

1.4.1 Old inflation

The first potential-driven inflationary model was not based on slow-rolling, but on false vacuum decay [23, 41]. This model was formulated in the context of a scalar field with potential

$V(\phi)$ that undergoes a first order phase transition. At temperatures well below the critical temperature, $T \ll T_c$, the potential has a false vacuum at $\phi = 0$ with $V(0) = V_0 > 0$, whereas the true vacuum is at $\phi = \phi_0$ with $V(\phi_0) = 0$. At temperatures above the critical temperature, $T \gg T_c$, the potential gets some finite temperature corrections that are proportional to $\phi^2 T^2$ [42–45] and the only true vacuum state is at $\phi = 0$. As the universe cools below the critical temperature, ϕ gets trapped in the false vacuum, $\phi = 0$ and the universe starts expanding with the energy of the false vacuum. After a period of Γ^{-1}, where Γ is the tunneling rate, bubbles of $\phi = \phi_0$ begin to nucleate [46, 47] in the sea of false vacuum $\phi = 0$. It was immediately realized that this inflationary scenario suffers from the graceful exit problem [48]. Achieving successful inflation required that the tunneling transition rate be small, implying that the nucleation of the true vacuum bubbles was rare. Also, the latent heat needed for reheating was stored in the kinetic energy of the bubble walls, so that reheating had to be provided via bubble collisions. The small tunneling rate required for sufficient inflation precludes bubble collisions from reheating the universe. At the end, the phase transition is never complete and most of the universe continues to inflate forever [49]. Some ways to get around these problems have been suggested in [50–52], in the context of Brans-Dicke gravity and in [53] using a spectator scalar field coupled non-minimally to gravity. The variant of old inflation suggested in the context of Brans-Dicke theory is called ”extended inflation”. It is known, however, that extended inflation suffers from some problems too [54]. In this model, bubbles that nucleate very early, for example 60-efoldings before the end of inflation can be swept up to very large sizes, comparable to our observable universe. Such bubbles are unable to thermalize properly and can lead to substantial inhomogeneities that are observable in the CMBR.

1.4.2 New inflation

Soon after the seminal paper of Guth, Linde [27] and Albrecht and Steinhardt [28] independently put forward a modified scenario called ”New inflation”. The starting point is a scalar field with a double well potential that undergoes a second order phase transition. For temperatures greater than the critical temperature, $V_T(\phi)$ is minimized at $\phi = 0$ and therefore $\phi(x)$ is confined at $\phi = 0$. At zero temperature, $T \ll T_c$, the potential has a local maximum at $\phi = 0$.

The potential $V(\phi)$ is very flat near the false vacuum at $\phi = 0$. Therefore, when temperature drops below the critical temperature, thermal fluctuations trigger the instability of $\phi = 0$ and ϕ slowly rolls toward the global minima of potential at ϕ_0. There is no graceful exit problem in the new inflationary universe, since the fluctuation domains are established at the beginning of inflation, i.e. any boundary walls will be inflated away outside the present horizon.

1.4.3 Hybrid inflation

In this model, which was originally suggested by Linde [55], the slowly rolling field, ϕ, is not the one responsible for the energy density of the universe during inflation. It is coupled to another field, ψ, which is held in its place by interacting with the inflaton. The slow-roll inflation continues until the ϕ falls below a critical value ϕ_c. When that happens, ψ is destabilized, rolling to its true vacuum and inflation ends. The potential for the original model of hybrid inflation is:

$$V(\phi, \psi) = \frac{1}{2}m^2\phi^2 + \frac{1}{4}\lambda(\psi^2 - M^2)^2 + \frac{1}{2}\lambda'\psi^2\phi^2. \tag{1.57}$$

The field ψ is fixed at the origin if $\phi^2 > \phi_c^2$, where

$$\phi_c^2 = \frac{m_\psi^2}{\lambda'} = \frac{\lambda}{\lambda'}M^2, \tag{1.58}$$

whereas at smaller values of ϕ, the ground state of ψ tends to $\pm M$. The idea of hybrid inflation is that ϕ is slowly rolling, like the inflaton field in slow-roll inflation, but that the energy density of the universe is dominated by ψ, $V_0 = \frac{1}{4}\lambda M^4$, to the potential. In hybrid inflation, ϕ_c can be made smaller than $m_{\rm Pl}$ and thus inflation without super-Planck values for the inflaton. As hybrid inflation ends with a phase transition, there is always the possibility formation of topological defects [56].

1.4.4 Multi-Component Inflation

So far we have considered only one possible inflationary trajectory in the space of scalar fields. This is automatically true, if we have a single scalar field. However, in general there will be a whole family of possible inflationary trajectories. There are trivial examples when, in addition to the inflaton, there are massless fields that do not make any contribution to the potential during the inflation. Unless the fields survive and become important after inflation, all infinite number of trajectories are equivalent. Another trivial case happens when a symmetry ensures the equivalence of trajectories. Then the transition from inflation to matter and radiation dominated universe is the same for each trajectory.

Here, we are mainly interested in models in which the trajectories that are not equivalent. In such models, the inflaton field is a multi-component object with components ϕ_a. Mainly in these models, it is assumed that while cosmological scales are leaving the horizon, all component fields roll slowly:

$$3H\dot{\phi}_a = -\frac{\partial V}{\partial \phi_z}. \tag{1.59}$$

Differentiating this and comparing it with the exact equation $\ddot{\phi}_a + 3H\dot{\phi}_a = \partial V/\partial\phi_a$, gives:

$$M_{\rm Pl}^2 \left(\frac{\partial V/\partial\phi_a}{V}\right)^2 \ll 1 \quad M_{\rm Pl}^2 \left|\frac{\partial V/\partial\phi_a\partial\phi_b}{V}\right| \ll 1 \tag{1.60}$$

One calculate the adiabatic density perturbations generated by multi-component inflaton using the δN formalism developed by Stewart and Sasaki [57]. In general, one can show that:

$$\delta N = \Re, \tag{1.61}$$

where δN is the difference between the number of e-foldings in the presence and absence of curvature perturbations. We have chosen a foliation such that the initial hypersurface is flat and the final one is comoving. Assuming slow-roll, one can write δN as

$$\delta N = \frac{\partial N}{\partial \phi_a}\delta\phi_a \tag{1.62}$$

where repeated indices are summed over. The perturbations $\delta\phi_a$ are Gaussian random fields generated by the vacuum fluctuations and thus we have:

$$P_s(k) \equiv \frac{k^3}{2\pi^2}\langle \Re_k \Re_l\rangle \delta^3(\mathbf{k}-\mathbf{l}) = \frac{V}{75\pi^2 M_{\rm Pl}^2}\frac{\partial N}{\partial\phi_a}\frac{\partial N}{\partial\phi_a}. \tag{1.63}$$

Although we have so far assumed that the potential for each of these components are flat enough to sustain slow-roll inflation, in inflationary scenarios with more than one scalar field, inflation may proceed even if each of the individual fields has a potential too steep for that field to sustain inflation on its own. This is the idea of assisted inflation [26, 58]. The original idea was proposed using steep exponential potentials. For simplicity, we begin by considering m scalar fields, ϕ_i, which each have an identical potential

$$V(\phi_i) = V_0 \exp\left(-\sqrt{\frac{2}{p_i}}\frac{\phi_i}{m_{\rm Pl}}\right). \tag{1.64}$$

These fields only affect each other via expansion and do not have any direct interaction with each other. The equations of motion are:

$$H^2 = \frac{8\pi}{m_{\rm Pl}^2}\sum_{i=1}^{i=m}\left[V(\phi_i) + \frac{1}{2}\dot{\phi}_i^2\right], \tag{1.65}$$

$$\ddot{\phi}_i = -3H\dot{\phi}_i - \frac{dV(\phi_i)}{d\phi_i}, \tag{1.66}$$

One can show that a particular solution where all scalars are equal is a unique late-time attractor for the system

$$\phi_1 = \phi_2 = \cdots = \phi_m \tag{1.67}$$

The above equations of motion then can be mapped to a single scalar field $\tilde{\phi}$ defined as

$$\tilde{\phi_1}^2 = m\phi_1^2, \tag{1.68}$$

with the following potential:

$$\tilde{V}(\tilde{\phi}) = \tilde{V}_0 \exp\left(-\sqrt{\frac{2}{\tilde{p}}}\frac{\tilde{\phi}}{m_{\rm Pl}}\right), \quad \tilde{V}_0 = mV_0, \quad \tilde{p} = mp. \tag{1.69}$$

For an exponential potential, above one can show that scale factor behaves as $a(t) \sim t^{\tilde{p}}$, if $\tilde{p} > 1/3$ [59]. The expansion becomes quicker as the number of scalar fields increase. In particular, potentials with $p < 1$, which for a single field are unable to support inflation, can do so as long as there are enough scalar fields to make $mp > 1$. We will use this scenario in the last chapter to construct an inflationary model in the context of M-theory.

1.5 Building inflationary models in string theory

Inflation is likely to be associated with physics at energies considerably higher than the weak scale, for which string theory is arguably our most promising candidate. Due to the proximity of the energy scale of inflation to the string scale, inflation can play the role of a bridge between string theory and observation. This provides motivation for looking for stringy setups that give rise to inflation.

So far we had focused our attention on phenomenological models that can realize the idea and goals of inflation. However, we have not mentioned what inflaton might be in the context of a fundamental theory such as string theory. The search for inflation in string theory has not been easy due to two problems. The first problem is due to moduli. Supersymmetric vacua of string theory have many massless scalar fields with strictly tree level flat potentials. These scalar fields are known as moduli. Basically one of these fields could play the role of inflaton, if one could manage to give it a small slope by invoking SUSY breaking. However, once the supersymmetry is broken, the moduli usually acquire a mass. Since the inflaton potential is – by design – very shallow, its slope can be dominated by small contributions to the potential. Especially, when supersymmetry is broken, moduli find masses proportional to Hubble parameter, which spoils the slow-roll inflation. Further studying had revealed that this supersymmetry breaking would not occur within a perturbative regime and required non-perturbative string theory. The second obstacle in building inflation within string theory was the belief that the string scale is $m_{\rm Pl}$. This made it difficult to achieve inflationary potentials which produce the right amplitude for fluctuations.

The advent of D-branes [60] opened up news way for the semi-classical study of supersymmetry breaking which produces potential for the moduli. The most actively pursued direction along this line has been to have supersymmetry break due to the presence of both branes and anti-branes. Also the existence of D-branes opens up the possibility to bring the string scale to scales much below the Planck scale. This could happen in a brane-world scenario where all low-energy particles except for the graviton are trapped on the brane [61]. Gravitons can propagate in the extra dimensions and make the string scale well below the Planck scale. The other way to lower the string scale is by using warped compactifications [62]. These two methods allow the scale of inflation to become right for the generation of density perturbations.

The first idea for obtaining inflation from string theory is based on the relative motion of branes and anti-branes, which plays the role of inflaton, through a background geometry under their mutual attraction [63]. This scenario is based on the long-range interactions that occur due to the mediation of massless fields like gravity in the bulk. In the first glance, the potential seems to be sufficiently flat to sustain slow-roll inflation [64]. However, as it is not possible to separate the branes more than the size of the compact space, this expectation seems to be incorrect [65]. For $d_\perp = k + 2$ transverse dimensions to the brane, the potential behaves as $V(r) = A + B/r^k$ [64]. Thus the second slow-roll parameter, η, is:

$$\eta = \frac{m_{\rm Pl}^2 V''}{V} \approx \frac{ck(k+1)B}{A}\left(\frac{R}{r}\right)^{k+2} \qquad (1.70)$$

where, A, B and c are set by the appropriate powers of the string scales, ℓ_s. The constant, c, and the volume of transverse dimensions, R^{k+2},enters through their appearance in the 4D Planck mass, $m_{\rm Pl}^2 = cR^{k+2}$. As the inter-brane separation cannot exceed the size of extra-dimension, $r \leq R$, η is not generically small. The best approach to solve this problem is proposed in [66] which uses the stabilization construction of [67]. The fluxes are adjusted in the Calabi-Yau sector to assure the presence of a long warped throat, whose tip is a smoothed-off singularity. According to [67] one or more anti-D3 branes is imagined to reside at the tip of the throat. The five dimensional space-time consisting of large four dimensional space-time and the radial dimension of the throat is anti-de Sitter. The presence of the anti D3 brane breaks the 4D $N = 1$ supersymmetry. Finally a D3 brane is imagined to slide down the throat toward the $\bar{D}_3$ brane. The relative motion of D-branes toward the anti-branes plays the role of the inflaton. Due to the interplay between the moduli-stabilizing potential and the inflaton potential, inflation can be obtained in this picture at the price of adjusting the parameters to the level of $(0.1 - 1)\%$ level. The spectral index obtained in this sort of inflationary models is slightly blue, with $n_s \sim 1.03 - 1.08$ [68]. It is possible to embed the low-energy world in this scenario by locating the standard model sector onto a system of intersecting D3 and D7 branes somewhere within the six internal dimension. One would be able then to pose post-inflationary questions, like the nature of reheating. Another exciting possibility which emerges with the brane-antibrane inflationary scenarios is the production of cosmic strings at the end of inflation

[69]. Such cosmic strings are long-lived [70] and have a considerable energy density which is detectable through its effect on the CMB configuration.

The last chapter of this thesis has been allocated to the study of an alternative M-theory scenario in which the cumulative effect of so many of the moduli can cause inflation, although an individual moduli potential is too steep to sustain inflation. The scenario uses the assisted inflationary model to realize inflation within M-theory.

1.6 Outline of the thesis

1.6.1 Fingerprints of Planck Scale Physics in the CMB

The first three chapters of this thesis investigate possible Planck scale signatures that might be left on the CMB. As mentioned earlier, some chaotic inflationary models suggest that quantum fluctuations of the inflaton were stretched from Planck scale by inflationary expansion to galactic-size scales or even larger. Vacuum fluctuations of the inflaton produce both scalar and tensor perturbations, both of which contribute to the anisotropy of the cosmic microwave background radiation. For any inflationary model one can calculate the ratio of tensor to scalar fluctuations, r, which multiplies the upper bound on the scalar density perturbations by a factor of $(1+r)^{1/2}$. By knowing it one can tighten the bounds on the scalar spectral index. It is therefore important to know r in as much detail as possible in order to extract cosmological parameters with more precision. Also since scalar and tensor perturbations originate from a single inflaton potential they are not independent. A hierarchy of consistency conditions links them together. It has been argued that such conditions – if empirically verified – would offer strong support for the idea of inflation. Observational difficulties will probably render only the first consistency condition useful. The first of the consistency relations states that the ratio of the amplitude of tensor to scalar perturbations is a constant known as the tensor spectral index.

General arguments about the quantum theory of gravity and some studies in string theory suggest that the short distance structure of space-time at Planck-scale can be modelled as arising from some corrections to the well-known uncertainty relation between the position and momentum operators. This modified commutation relation was implemented at the first quantized level to the theory of cosmological perturbations [71]. In the second chapter, I discuss a property of this approach that total time derivatives, which in continuous space-time could be neglected and do not contribute to the equations of motion, cease to remain total time derivatives as minimal length is implemented [72]. Since in the Lagrangian formulation of quantum field theory, the initial conditions are introduced as boundary terms, this discovery indicates that the choice of initial conditions affects our estimate of the contribution of Quantum Gravity to the total spectrum of CMBR. This finding was similar in spirit to the finding of Prof. Veneziano's group at CERN who, around the same time but using different techniques, discovered that trans-Planckian effects depend crucially on the definition of the vacuum state, in particular on which Hamiltonian is minimized on the new physics hypersurface in order to select such a state [73]. In the third chapter, I study the consequences of this discovery for the tensor/scalar ratio [74]. I find the dependence of the ratio on the minimal length for the near-de-Sitter background and showed that r can acquire an oscillatory behavior if the choice of boundary term in the action is non-minimal [74].

In the fourth chapter, I examine the first consistency relation in a power-law background, where I find that – despite the ambiguity that exists in choosing the action – Planck scale physics modifies the consistency relation considerably. It also leads to the running of the spectral index. For modes that are larger than our current horizon, the tensor spectral index is positive. For a window of k values with amplitudes of the same order of the modes which are the precursor to structure formation, the behavior of the tensor spectral index is oscillatory about the standard Quantum Field theory result, taking both positive and negative values, [75].

1.6.2 Cosmological Magnetic Fields

Magnetic fields are present throughout the universe and play an important role in many astrophysical situations. It has been suggested that inflation can be a prime candidate for the production of primeval magnetic fields. Quantum fluctuations of the massless scalar and tensor (gravitational) fields are very much amplified at the inflationary epoch. This is closely related to the fact that these fields are not conformally invariant. However, electromagnetic waves are not produced in such conditions since classical electrodynamics is conformally invariant in the limit of vanishing masses of fermions. As the existence of a preferred minimal length breaks the conformal invariance of the background geometry, it is plausible that this effect could generate some electromagnetic field amplification. In the fifth chapter, I show that this scenario is equivalent to endowing the photon with a large negative mass during inflation. The effective mass vanishes in a radiation and matter dominated universe. Tuning the free parameter of the theory, I show that the seed required by the dynamo mechanism can be generated. I also showed that this mechanism can produce the requisite galactic magnetic field without resorting to a dynamo mechanism [76].

1.6.3 M-theory Signatures in the CMB

In the context of M-theory, K. Becker *et. al.* have realized assisted inflation [77]. Assisted inflation is a multi-scalar version of power-law inflation where cooperative behavior of several scalar fields with exponential potentials give rise to power-law inflation, $a \propto t^p$, even though the individual single-field potential is too steep to sustain inflation on its own. In this model the non-perturbative dynamics of N M5-branes on $\mathbf{S}^1/\mathbf{Z}_2$ produces assisted inflation. The open membrane instanton interactions between the M5-branes each give rise to an exponential potential that is too steep to produce inflation but leads to inflation if they are combined together. During inflation the distance between the M5-branes grow until they reach the size of the orbifold. At this stage the two outermost M5-branes coalesce with the boundaries through small instanton transitions. Hence the number of M5-branes and also the effective potential for the inflaton jumps instantaneously. Since parameter p in the scale factor depends on the number of M5-branes, p also jumps to a lower value. At the end, we have a cascade of power-law inflations where power decreases stepwise.

In the sixth chapter, I calculate the density perturbations in this model and show that it possesses three distinctive signatures [78]: a decisive power suppression at small scales, oscillations around the scales that cross the horizon when the inflaton potential jumps and stepwise decrease in the scalar spectral index. All three properties result from features in the inflaton potential. The features in the inflaton potential are generated whenever two M5-branes collide with the boundaries. The derived small-scale power suppression serves as a possible explanation for the dearth of observed dwarf galaxies in the Milky Way halo. The oscillations, furthermore, allow to directly probe M-theory by measurements of the spectral index and to distinguish cascade inflation observationally from other string inflation models.

Chapter 2

Minimum Length Cutoff in Inflation and Uniqueness of the Action

Some of the predictions of fundamental theories of physics can only be observed on energy scales as high as the Planck scale. The availability of such high energies in the early universe and the huge separation between conventional accelerator experiments and the Planck scale has led many to turn from accelerator-based experiments to cosmological observations in order to test such theories. Inflationary cosmology [23, 24, 27, 28, 39, 40, 79] is one of the paradigms that may serve this purpose. There, it is assumed that quantum fluctuations of the inflaton are stretched by inflationary expansion to cosmological scales. About 60 e-folds of inflationary expansion are necessary to solve many of the puzzles of big bang cosmology but in most inflationary models the expansion is much larger. In models proposed by Linde [80] the universe has expanded by a factor of $10^{10^{12}}$. An implication of these models is, therefore, that our observable universe was of *sub-Planckian* size at the beginning of the (last) inflationary period. This suggests that inflation could act as a magnifying glass for probing the short distance structure of space-time.

A similar question had arisen concerning a possible sensitivity of black hole radiation to trans-Planckian physics. There, it has been found that Hawking radiation is largely immune to trans-Planckian effects, see e.g. [81]. In the case of inflation, however, it has been found that the inflationary predictions for the cosmic microwave background (CMB) do possess a small and possibly even observable sensitivity to modifications of quantum field theory in the ultraviolet. To this end, various examples of ultraviolet-modified dispersion relations, some motivated by solid state analogs, have been tested for their effects on inflation, see [83–88]. In particular, and this will be our interest here, the ultraviolet cutoff described by a lower bound in the formal uncertainty in position, Δx_{min}, has been investigated for its implications in inflation, see [71, 89, 91, 92].

To model the small scale structure of space through a finite minimum position uncertainty Δx_{min} is of interest because the corresponding modified uncertainty principle has been motivated to arise from quite general quantum gravity arguments as well as from string theory, see e.g. [93–100]. In fact, any theory with this type of ultraviolet cutoff can be written, equivalently, as a continuum theory and as a lattice theory, see [101, 102]. While in the continuum formulation the theory displays unbroken external symmetries, the theory's ultraviolet regularity is displayed in its lattice formulation.

Indeed, it has been found that inflationary predictions for the CMB are sensitive to the natural ultraviolet cutoff if the cutoff is modelled through a finite minimum uncertainty in position, Δx_{min}. The magnitude by which the cutoff affects the predicted scalar and tensor spectra in the CMB was found to depend crucially on the initial conditions when a mode's evolution begins, which is when its proper wavelength is the minimum length. These initial conditions determine how close the modes' state is to the adiabatic vacuum during the period of adiabatic evolution before the mode crosses the Hubble horizon. If the modes are in the adiabatic vacuum during the phase of adiabatic evolution then the effects of Planck scale physics on inflationary predictions should be no bigger than of the order of σ^2, see [89, 90], where:

$$\sigma = \frac{\Delta x_{min}}{L_{Hubble}} \tag{2.1}$$

Here, L_{Hubble} is the Hubble length during inflation. Thus, for GUT scale inflation, we have approximately $\sigma \approx 10^{-3}$ if the cutoff length, Δx_{min}, is at the string scale and $\sigma \approx 10^{-5}$ if the cutoff length is at the Planck scale. In principle, however, the cutoff can lead to arbitrarily large effects, for example if the modes' state during the adiabatic phase differs strongly from the adiabatic vacuum (see also [103]). In this case, the modulus of the mode functions oscillates at horizon crossing and these oscillations translate into characteristic oscillations in the CMB spectra. This possibility is restricted, however, by the need to keep the back-reaction small, see [104, 105]. Interestingly, it was found that this constraint still allows nontrivial vacua with effects as large as of order σ, see Easther *et.al.* [91, 92]. Effects of this magnitude might reach the threshold of observability.

So far, initial conditions have been proposed based on analyticity arguments [89] and based on similarity to the Bunch Davies vacuum [91, 92]. A further suggestion is to minimize the field uncertainties [106, 107]. Still, however, the crucial question of how to determine initial conditions for the new comoving modes that are continually being created during an expansion has not been conclusively answered. The problem is of course equivalent to identifying the vacuum state.

Here, we address this problem by reconsidering how the vacuum state is usually identified within inflationary QFT without a minimum length. Namely, the usual strategy is to make use of the fact that the action can be rewritten so as to resemble the familiar action of a field on Minkowski space with time-variable mass term. When quantizing, one then chooses the vacuum as one does for Minkowski space theories. We will find that this method is no longer reliable when there is a minimum length. The reason is that the reformulation of the action requires the neglect of a boundary term and we will see that this terms ceases to be a boundary term when the the minimum length is introduced. We find that the differences are small but noticeable both in the initial conditions and in the evolution equations. This shows that in any approach to introducing a minimum length into inflation will have to take into account that reformulations of an action that appear to be harmless because requiring merely the neglect of a boundary term can lead to an unintended modification of the theory.

To see how this phenomenon can arise, let us recall that the particular model of a natural ultraviolet cutoff that we are considering is described by quantum mechanical uncertainty relations with correction terms in the ultraviolet, of the form

$$\Delta x \Delta p \geq \frac{\hbar}{2} \left(1 + \beta \left(\Delta p\right)^2 + ...\right) \tag{2.2}$$

where $\beta > 0$ is a positive constant. In the simplest case, such an uncertainty relation arises from the modified commutation relation:

$$[\mathbf{X}, \mathbf{P}] = i\hbar(\mathbf{1} + \beta\, \mathbf{P}^2) \tag{2.3}$$

It is not difficult to show that the uncertainty relation then implies a finite lower bound to the position uncertainty Δx:

$$\Delta x_{min} = \hbar\sqrt{\beta} \tag{2.4}$$

By choosing β appropriately we obtain a cutoff at the string or at the Planck scale. This type of ultraviolet cutoff was introduced into quantum field theory in [108] and then into inflationary cosmology in [71].

It is clear that similar to quantization, which changes the commutativity properties and therefore comes with a well-known ordering ambiguity, the introduction of a minimum length through an equation such as Eq.2.3 changes the commutativity properties and therefore comes with an ordering ambiguity. In principle, of course, ordering ambiguities can be of arbitrary magnitude. For example, a classical system is unchanged by adding terms of the form $(xp - px)f(x, p)$ to its Hamiltonian H. When promoting the x and p to operators the resulting terms become proportional to $\hbar$ and could be arbitrarily large and significant to the evolution. Similarly here, normally vanishing terms of the form $(xp - px - i\hbar)g(x, p)$ become nonzero when $\beta \neq 0$. Here, the new Hamiltonian is determined only up to terms that vanish when setting the minimum length to zero. Those terms can be arbitrarily large and, in principle, only experiments could decide which choice is correct. This is to be expected in any approach to introducing some form of a natural minimum length.

Of course, in the case of quantization it has proven to be a very reliable strategy to adopt the minimalist approach to resolving the ordering ambiguity: write the Hamiltonian in its most simple and symmetric form and leave it unchanged when introducing $\hbar$, i.e. do not introduce terms by hand. The same minimalist approach has tacitly been applied in the literature when the minimum length uncertainty relation has been used in inflationary quantum field theory. We will now review this procedure, thereby uncovering potential pitfalls with implications for the determination of the vacuum.

2.1 Inflation with the minimum length uncertainty relation

The minimum length uncertainty principle was first introduced into inflation in [71]. One starts by implementing the uncertainty relations in first quantization through modifications of the canonical x, p commutation relations, as in Eq.2.3. The first quantization commutation relations then carry over to quantum field theory. Note that since momentum space is unaffected by the minimum length uncertainty relations the field commutators in momentum space remain unchanged by the procedure.

In [71], this program was carried out explicitly for an action of the form of the tensor action $S_T^{(1)}$. This showed how $\beta > 0$ generalizes the action $S_T^{(1)}$ to a new action $S_{T,\beta}^{(1)}$ and how the tensor fluctuations' equation of motion changes correspondingly. Those results immediately also translate to the case of the scalar action $S_S^{(1)}$ since this action differs merely by overall constants and by suitably replacing $a(\tau)$ with $z(\tau)$. These equations of motion have been further investigated in [91, 92]. We will explicitly list those equations of motion below.

Our aim now is to carry out the same program for introducing the minimum length, starting, however, from the often-used actions $S_S^{(2)}$ and $S_T^{(2)}$ to derive the generalized actions $S_{S,\beta}^{(2)}$ and $S_{T,\beta}^{(2)}$ and the correspondingly generalized equations of motion for scalar and tensor fluctuations. We will find that the generalized actions $S_{S,\beta}^{(1)}$ and $S_{S,\beta}^{(2)}$ as well as the generalized actions $S_{T,\beta}^{(1)}$ and $S_{T,\beta}^{(2)}$ no longer differ merely by boundary terms, are therefore not equivalent and lead to slightly different equations of motion.

2.1.1 Scalar fluctuations with minimum length

The minimum length is to be introduced as a minimum proper length in the CMB rest frame. To this end, we transform the action $S_S^{(2)}$ as given in Eq.1.38 from comoving coordinates y^i and time τ to proper coordinates x^i and time τ, where $x^i = a(\tau)y^i$. Since the transformation is time-dependent, the chain rule leads to a nontrivial transformation of the derivative ∂_τ on fields and we obtain:

$$S_S^{(2)} = \int d\tau \frac{d^3\mathbf{x}}{2a^3} \left\{ \left[\left(\partial_\tau + \frac{a'}{a} \sum_{i=1}^{3} \partial_{x^i} x^i - \frac{3a'}{a} \right) u \right]^2 - a^2 \sum_{i=1}^{3} (\partial_{x^i} u)^2 + \frac{z''}{z} u^2 \right\} \tag{2.5}$$

We identify $-i\partial_{x^i}$ as the momentum operator, $\mathbf{P}^i$, and x^i as the position operator $\mathbf{X}^i$. These operators are defined on a Hilbert space of fields (not states) with:

$$\begin{aligned}(u_1, u_2) &= \int d^3\mathbf{x}\, u_1^*(x)u_2(x)\\ \mathbf{X}^i u(x) &= x^i u(x)\\ \mathbf{P}^i u(x) &= -i\partial_{x^i} u(x).\end{aligned} \tag{2.6}$$

The fields thus form a Hilbert space representation of the commutation relations:

$$[\mathbf{X}^i, \mathbf{P}^j] = i\delta^{ij},\ \ [\mathbf{X}^i, \mathbf{X}^j] = 0,\ \ [\mathbf{P}^i, \mathbf{P}^j] = 0 \tag{2.7}$$

This merely expresses that the canonical commutation relations of first quantization are present also in second quantization. For example, in quantum field theory the $\hbar$ of the Fourier factor $e^{ixp/\hbar}$ directly derives from the $\hbar$ in the commutation relations of first quantization. Of course, the operators $\mathbf{X}^i$ and $\mathbf{P}^j$ no longer possess a simple interpretation as observables. We see from Eqs.1.38 and 2.5 that under the time-dependent mapping from comoving to proper positions the chain rule makes the action of ∂_τ on fields in comoving coordinates transform into a new action on fields in proper coordinates, namely $\partial_\tau \to A(\tau)$, where:

$$A(\tau) = \left(\partial_\tau + i\frac{a'}{a}\sum_{i=1}^{3}\mathbf{P}^i\mathbf{X}^i - 3\frac{a'}{a}\right) \tag{2.8}$$

Using the operators $\mathbf{X}$ and $\mathbf{P}$ we can write the action (2.5) in representation-independent form

$$S_S^{(2)} = \int \frac{d\tau}{2a^3}\left(\left(u, A^\dagger(\tau)A(\tau)u\right) - a^2\left(u, \mathbf{P}^2 u\right) + \frac{z''}{z}\left(u, u\right)\right), \tag{2.9}$$

meaning that Eq.2.9 is true without referring to the position representation, momentum representation or any other representation of the fields. Following [71], our strategy now is to maintain this representation-independent form of the action while introducing a cutoff by modifying the underlying position-momentum commutation relations Eqs.2.7. The modified commutation relations should break neither translation nor rotation invariance and should introduce a finite minimum position uncertainty Δx_{min} in all three position variables. This breaks the Lorentz Invariance. One should note that it is not clear that a complete theory of quantum gravity violates the Lorentz invariance or not. Several studies in Loop Quantum Gravity have claimed possible violations of Lorentz symmetry in some semi-classical models of extended matter dynamics [109, 110]. On the other hand, string theory does not break the Lorentz invariance, however see [111] for a different point of view. It has been shown [112] that all such commutation relations must have the following form

$$[\mathbf{X}^i, \mathbf{P}^j] = i \left(\frac{2\beta p^2}{\sqrt{1+4\beta p^2} - 1}\, \delta^{ij} + 2\beta\, \mathbf{P}^i \mathbf{P}^j \right), \quad [\mathbf{X}^i, \mathbf{X}^j] = 0, \quad [\mathbf{P}^i, \mathbf{P}^j] = 0 \tag{2.10}$$

to first order in the parameter β, which is chosen positive, see [112]. The minimum position uncertainty Δx_{min} in every coordinate is given by

$$\Delta x_{min} = \frac{\sqrt{\beta}}{2} (1 + d/2)^{1/4} \left(\sqrt{1 + d/2} + 1 \right) \tag{2.11}$$

where d is the number of space dimensions, see [113]. Here $d = 3$, so that $\Delta x_{min} \approx 1.62\, \sqrt{\beta}$. Correspondingly, $\sigma \approx 1.62\, \sqrt{\beta} H$, where H is the Hubble parameter. A convenient Hilbert space representation of the modified commutation relations, Eqs.2.10, is given by

$$\mathbf{X}^i \psi(\rho) = i \partial_{\rho^i} \psi(\rho) \tag{2.12}$$

$$\mathbf{P}^i \psi(\rho) = \frac{\rho^i}{1 - \beta\rho^2} \psi(\rho) \tag{2.13}$$

with the scalar product:

$$(\psi_1, \psi_2) = \int_{\rho^2 < \beta^{-1}} d^3\rho\, \psi_1^*(\rho) \psi_2(\rho) \tag{2.14}$$

Thus, the operator $A(\tau)$ changes as the minimum length is introduced, i.e. when $\beta > 0$. The action for scalars, Eq.2.9, then also changes to become:

$$S^{(2)}_{S,\beta} = \int d\tau \int_{\rho^2 < \beta^{-1}} d^3\rho\, \frac{1}{2a^3} \left\{ \left| \left(\partial_\tau - \frac{a'}{a} \frac{\rho^i}{1 - \beta\rho^2} \partial_{\rho^i} - \frac{3a'}{a} \right) u \right|^2 - \frac{a^2 \rho^2 |u|^2}{(1 - \beta\rho^2)^2} + \frac{z''}{z} |u|^2 \right\} \tag{2.15}$$

The presence of ρ derivatives means that the ρ modes are coupled. Conveniently, in the new variables $(\tilde{\tau}, \tilde{k})$

$$\tilde{\tau} = \tau,$$

$$\tilde{k}^i = a \rho^i e^{-\beta\rho^2/2} \tag{2.16}$$

the $\tilde{k}$ modes decouple. To see this, note that:

$$\partial_\tau - \frac{a'}{a}\frac{\rho^i}{1-\beta\rho^2}\,\partial_{\rho^i} = \partial_{\tilde{\tau}}. \tag{2.17}$$

We will use the common index notation $\bar{u}_{\tilde{k}}$ for the decoupling modes. The $\tilde{k}$ modes only coincide with the usual comoving modes on large scales, i.e., only for small ρ^2. This means that the precisely comoving k modes, obtained by scaling $k^i = ap^i$, are decoupling only at large distances while they do couple at distances close to the cutoff scale. The action now takes a decoupled form (i.e. there are no $\tilde{k}$ derivatives):

$$S^{(2)}_{S,\beta} = \int d\tilde{\tau} \int_{\tilde{k}^2<a^2/e\beta} d^3\tilde{k}\; a^{-6}\,\frac{\kappa}{2}\left(\left|\left(\partial_\tau - \frac{3a'}{a}\right)\bar{u}_{\tilde{k}}(\tau)\right|^2 - \mu\,|\bar{u}_{\tilde{k}}|^2 + \frac{z''}{z}\,|\bar{u}_{\tilde{k}}|^2\right) \tag{2.18}$$

Here, the functions μ and κ are defined through

$$\mu(\tau,\tilde{k}) = -\frac{a^2}{\beta}\frac{W(-\beta\tilde{k}^2/a^2)}{(1+W(-\beta\tilde{k}^2/a^2))^2} \tag{2.19}$$

$$\kappa(\tau,\tilde{k}) = \frac{e^{-\frac{3}{2}W(-\beta\tilde{k}^2/a^2)}}{1+W(-\beta\tilde{k}^2/a^2)} \tag{2.20}$$

where W is the Lambert W function (see e.g. [114]), which is defined so that $W(x)e^{W(x)} = x$. As expected, each comoving mode $\tilde{k}$ has a starting time, τ_c, namely the time at which $a(\tau_c) = \sqrt{e\beta\tilde{k}^2}$, which is when $\rho^2 = 1/\beta$, which is when the mode's proper wavelength is the cutoff length. The equation of motion that follows from the action $S^{(2)}_{S,\beta}$ is:

$$\bar{u}''_{\tilde{k}} + \left(\frac{\kappa'}{\kappa} - 6\frac{a'}{a}\right)\bar{u}'_{\tilde{k}} + \left(\mu - 3\,\frac{\kappa' a'}{\kappa a} - 3\left(\frac{a'}{a}\right)' + 9\left(\frac{a'}{a}\right)^2 - \frac{z''}{z}\right)\bar{u}_{\tilde{k}} = 0 \tag{2.21}$$

The equation of motion contains a number of terms that involve the scale factor a and appears rather complicated. This is not a consequence of the introduction of the minimum length. Instead, it is merely due to our choice of variables. To see this, note first that the functions μ and κ are simpler in the variables τ and ρ_i:

$$\mu(\tau,\rho) = \frac{a^2\rho^2}{(1-\beta\rho^2)^2} \tag{2.22}$$

$$\kappa(\rho) = \frac{e^{3\beta\rho^2/2}}{1-\beta\rho^2} \tag{2.23}$$

Thus, as the cutoff is removed, $\beta \to 0$, we have that $\mu \to k^2$ and $\kappa \to 1$. The action Eq.2.18 thus turns into a conventional-looking action, except for an overall factor of a^{-6}. The many terms of a and a' in the equation of motion Eq.2.21 trace back to this pre-factor a^{-6} in the action in Eq.2.18. The occurrence of the factor a^6 might be surprising since we had started with the action $S_S^{(2)}$ as given in Eq.1.38 which of course does not possess a time-dependent pre-factor. The reason for the occurrence of this pre-factor is that the operations of Fourier transforming and of scaling do not commute: We did not directly Fourier transform the starting action Eq.1.38 from comoving positions to comoving momenta, as is usually done. Instead, we first scaled the comoving position coordinates to proper coordinates (where we introduced the minimum length), then Fourier transformed to proper momenta, and finally scaled to comoving momenta. The field variable $\bar{u}_{\tilde{k}}$ therefore differs from the usual field variable $u_{\tilde{k}}$ by a factor of a^{-3}:

$$u_{\tilde{k}} = a^{-3}\bar{u}_{\tilde{k}} \tag{2.24}$$

In this commonly used field variable, the action $S_{S,\beta}^{(2)}$ for scalar fluctuations, Eq.2.18, then takes the more familiar-looking form

$$S_{S,\beta}^{(2)} = \int d\tilde{\tau} \int_{\tilde{k}^2 < a^2/e\beta} d^3\tilde{k}\ \frac{1}{2}\kappa \left(u_{\tilde{k}}'^{*} u_{\tilde{k}}' - \left(\mu - \frac{z''}{z}\right) u_{\tilde{k}}^{*} u_{\tilde{k}} \right) \tag{2.25}$$

and also yields the equation of motion in a simpler form:

$$u_{\tilde{k}}'' + \frac{\kappa'}{\kappa} u_{\tilde{k}}' + \left(\mu - \frac{z''}{z}\right) u_{\tilde{k}} = 0, \qquad \left(\text{derived from } S_{S,\beta}^{(2)}\right) \tag{2.26}$$

Note that the introduction of the minimum length did leave us with a time-dependent pre-factor $\kappa(\tau, \tilde{k})$ in the action Eq.2.25, a fact that we will return to. The mode equation Eq.2.26 generalizes Eq.1.45 in the presence of the minimal length cutoff - when starting from the action $S_S^{(2)}$. We need to add that the canonical commutation relations between $u_{\tilde{k}}$ and its conjugate momentum, $\pi_{\tilde{k}} = \kappa u_k'$, namely

$$[u_{\tilde{k}}, \pi_{\tilde{k}'}] = i\delta^3(\tilde{k} - \tilde{k}'), \tag{2.27}$$

require that the solutions to equation (2.27) also obey the slightly generalized Wronskian condition

$$u_{\tilde{k}}(\tau) u_{\tilde{k}}^{*\prime}(\tau) - u_{\tilde{k}}^{*}(\tau) u_{\tilde{k}}'(\tau) = i\kappa^{-1}. \tag{2.28}$$

Expressing the equation of motion Eq.2.26 in terms of the intrinsic curvature, $\Re = -u/z$, we obtain:

$$\Re''_{\tilde{k}} + \left(\frac{\kappa'}{\kappa} + \frac{2z'}{z}\right)\Re'_{\tilde{k}} + \left(\mu + \frac{z'\kappa'}{z\kappa}\right)\Re_{\tilde{k}} = 0 \qquad \left(\text{derived from } S^{(2)}_{S,\beta}\right) \tag{2.29}$$

It is straightforward to show that the wave equation and Wronskian equation reduce to the usual wave equation Eq.1.45 and Wronskian condition Eq.1.44 in the limit $\beta \to 0$, i.e. when the minimum length cutoff is removed. To summarize, we calculated the generalization of the action $S^{(2)}_{S}$ to the action $S^{(2)}_{S,\beta}$ and we have found the corresponding equation of motion in Eq.2.26.

Let us now compare with the result of introducing the minimum length uncertainty relation into the action $S^{(1)}_{S}$ to obtain $S^{(1)}_{S,\beta}$. We read off from [71] that the action $S^{(1)}_{S,\beta}$ yields the wave equation:

$$\Re''_{\tilde{k}} + \left(\frac{\kappa'}{\kappa} + \frac{2z'}{z}\right)\Re'_{\tilde{k}} + \mu\Re_{\tilde{k}} = 0 \qquad \left(\text{derived from } S^{(1)}_{S,\beta}\right) \tag{2.30}$$

It is expressed in terms of the intrinsic curvature. In order to better compare with the equation of motion Eq.2.26 which followed from $S^{(2)}_{S,\beta}$, we rewrite Eq.2.30 in terms of the field variable u, to obtain:

$$u''_{\tilde{k}} + \frac{\kappa'}{\kappa}u'_{\tilde{k}} + \left(\mu - \frac{z''}{z} - \frac{z'}{z}\frac{\kappa'}{\kappa}\right)u_{\tilde{k}} = 0 \qquad \left(\text{derived from } S^{(1)}_{S,\beta}\right) \tag{2.31}$$

The results of [71] also show that this field $u_{\tilde{k}}$ satisfies the same Wronskian condition Eq.2.28. Clearly, the equations of motion Eq.2.26 and Eq.2.31 differ and we will need to investigate the origin and extent of the difference.

2.1.2 Tensor fluctuations with minimum length

From the case of scalar fields we find the corresponding two actions $S^{(1)}_{T,\beta}$ and $S^{(2)}_{T,\beta}$ for tensor perturbations, namely by inserting suitable constants and by replacing occurrences of z by a. For $\beta = 0$ the two actions are of course equivalent since differing merely by a boundary term.

For $\beta > 0$, however, we find that they yield slightly different equations of motion:

$$h''_{\tilde{k}} + \left(\frac{\kappa'}{\kappa} + \frac{2a'}{a}\right) h'_{\tilde{k}} + \mu h_{\tilde{k}} = 0 \qquad \left(\text{derived from } S^{(1)}_{T,\beta}\right) \tag{2.32}$$

$$p''_{\tilde{k}} + \frac{\kappa'}{\kappa} p'_{\tilde{k}} + \left(\mu - \frac{a''}{a} - \frac{a'}{a}\frac{\kappa'}{\kappa}\right) p_{\tilde{k}} = 0 \qquad \left(\text{derived from } S^{(1)}_{T,\beta}\right) \tag{2.33}$$

$$h''_{\tilde{k}} + \left(\frac{\kappa'}{\kappa} + \frac{2a'}{a}\right) h'_{\tilde{k}} + \left(\mu + \frac{a'\kappa'}{a\kappa}\right) h_{\tilde{k}} = 0 \qquad \left(\text{derived from } S^{(2)}_{T,\beta}\right) \tag{2.34}$$

$$p''_{\tilde{k}} + \frac{\kappa'}{\kappa} p'_{\tilde{k}} + \left(\mu - \frac{a''}{a}\right) p_{\tilde{k}} = 0 \qquad \left(\text{derived from } S^{(2)}_{T,\beta}\right) \tag{2.35}$$

2.1.3 Origin of the differences in the mode equations

In order to trace the inequivalence of the obtained equations of motion, we begin by noting that we encountered an ordering ambiguity in Eqs. 2.8 and 2.10 when modifying the commutation relations: consider the formal position and momentum operators in the operator $A(\tau)$. We could have used the first quantization's canonical commutation relations to bring the positions and momenta in arbitrary order. Clearly, it does matter, however, whether we do this before or after we change the first quantization's commutation relations. In this way, for example by adding terms of the form $(xp - px - i\hbar)f(x,p)$ before changing the commutation relations, we could have introduced into the action arbitrary terms that vanish as the minimum length is set to zero, i.e. as $\beta \to 0$.

Of course, in any theory that generalizes quantum field theory by introducing a minimum length parameter one can guess Hamiltonians and actions etc. only up to terms which vanish as the minimum length parameter vanishes - much like quantum Hamiltonians can be guessed from classical Hamiltonians only up to terms that vanish as $\hbar \to 0$. We encountered essentially an instance of Dirac's observation that quantization removes degeneracy. As in the case of quantization, the minimalist approach to dealing with the ambiguity is to bring the action into a simple form and not to use the ambiguity to introduce any such terms by hand that would vanish as the minimum length is set to zero. This was the approach tacitly adopted in [71] and we here also adopted the same minimalist approach when we introduced the minimum length into $S^{(1)}_S$, $S^{(2)}_S$, $S^{(1)}_T$ and $S^{(2)}_T$. We then found that actions $S^{(1)}_{S,\beta}$ and $S^{(2)}_{S,\beta}$ (and similarly $S^{(1)}_{S,\beta}$, $S^{(2)}_{S,\beta}$)

yield differing equations of motion. How could this happen, given that the two actions $S_S^{(1)}$ and $S_S^{(2)}$ (and similarly $S_{S,\beta}^{(1)}, S_{S,\beta}^{(2)}$) are equivalent?

We already indicated that the answer traces back to the fact that the actions in the two formulations of types $S^{(1)}$ and $S^{(2)}$ are equivalent only up to a boundary term. After the minimum length is introduced these terms are no longer boundary terms. To see that this is the case, consider the scalar actions $S_{S,\beta}^{(2)}$ as expressed in terms of the field $u_{\tilde{k}}$, see Eq.2.25. We notice that it possesses in its integration measure a time-dependent factor:

$$\int d\tau \; d^3\tilde{k} \; \kappa(\tau, \tilde{k}) \tag{2.36}$$

If we remove the minimum length, $\beta \to 0$, we obtain of course $\kappa \to 1$. In the case $\beta > 0$, however, if

$$\int d\tau d^3\tilde{k} \; \Delta\mathcal{L} = \int d\tau d^3\tilde{k} \; \frac{d}{d\tau} f(\tau) \tag{2.37}$$

is a negligible boundary term arising from a total time derivative, then in the presence of the minimum length uncertainty relation

$$\int d\tau d^3\tilde{k} \; \kappa(\tau, \tilde{k}) \; \Delta\mathcal{L} = \int d\tau d^3\tilde{k} \; \kappa(\tau, \tilde{k}) \; \frac{d}{d\tau} f(\tau) \tag{2.38}$$

is not a boundary term. The same phenomenon occurs for tensor fluctuations: the two actions which are normally equivalent because differing merely by the total time derivative $\triangle S_T$ given in Eq.1.52 now yield different equations of motion. Indeed, as expected, when the minimal length is introduced the two actions differ by:

$$S_{T,\beta}^{(2)} - S_{T,\beta}^{(1)} = \int d\tau d^3\tilde{\mathbf{k}} \; \left(a' a h_{\tilde{\mathbf{k}}}^2\right)' \; \kappa(\tau, \tilde{k}) \tag{2.39}$$

The integrand is generally not a total time derivative due to the presence of the function $\kappa(\tau, \tilde{k})$.

2.2 Conclusions

While the framework of quantum field theory is well-tested down to distances of about $10^{-18}m$, it is generally expected that there are corrections due to quantum gravity when approaching the

Planck length of about $10^{-35}m$ which may well constitute a fundamental smallest length in nature.

If, therefore, there exists a finite minimum wavelength then, during inflation, comoving modes are continually being created. Initially, a new comoving mode will evolve under the influence of Planck scale effects but it is clear that at late times a comoving mode's equation of motion will reduce to the usual low-energy mode equation, namely when the mode's proper wavelength becomes much larger than the minimum length. Thus, as was pointed out in [105], the effects of the Planck scale can propagate into the observable low energy realm essentially only by selecting a solution of the mode equation which at late times differs from the usually assumed solution for the usual mode equation.

This suggests a simple technique for exploring possible effects that Planck scale physics could have on inflationary predictions for the CMB: Assume that standard quantum field theory holds *unchanged* down to the minimum wavelength where modes are being created. Then, consider a variety of possible initial conditions for the newly created modes by applying candidate criteria for identifying the vacuum state. It is clear that in the time translation invariant de Sitter case all effects reduce to merely an overall re-normalization of the flat spectrum (if each mode's initial condition is chosen by applying the same criterion). When the Hubble parameter is varying, however, then modes oscillate a varying number of times before crossing the horizon. Thus, generically, a mode's amplitude will be alternatingly large and small when crossing the horizon. This can lead to potentially observable characteristic oscillations in the spectrum, see [107]. In this approach quantum field theory is assumed to hold unchanged down to the Planck scale and therefore Planck scale physics is modelled so as to affect the predictions of inflation, for any given evolution of the scale factor $a(\tau)$, merely through the initial conditions. In any realistic model, of course, the quantum field theoretic mode equations will be modified when approaching the Planck scale. This too will have an effect on the number of oscillations that a mode undergoes before horizon crossing and it will therefore contribute to the predicted oscillations of the CMB spectra.

Here, we considered a concrete model for how quantum field theory is modified when approaching the Planck length, namely by introducing the minimum length uncertainty principle. The equations of motion then indeed became modified at scales close to the Planck scale. As was shown in [89, 91, 92], the inflationary predictions for the CMB are to some extent affected, possibly leading to observable oscillations in the fluctuation spectra. However, while the equations of motion are known, the details of the predictions still significantly depend on precisely which initial condition is chosen, i.e. on the identification of the vacuum state.

As yet, it is not fully understood in any model how Planck scale physics determines the initial conditions of modes as they are being created, i.e. when their proper wavelength is the minimum length. Within our model of spacetime as obeying a minimum length uncertainty relation the problem of determining the initial conditions for new comoving modes is further complicated by the fact that the mode equation possesses an irregular singular point at the initial time, see [71, 89, 91, 92]. So far, in the literature, a mathematical argument based on analyticity [89] and a physical argument based on similarity to the Bunch Davies vacuum [91, 92] have been discussed and the implications for the CMB have been investigated. Nevertheless, the crucial problem of determining the initial state of modes when they emerge from the Planck scale in an expansion is still essentially unsolved.

Therefore, we here reconsidered the conventional approach to fixing the vacuum: Introduce new variables in terms of which the action resembles that of a Minkowski space theory with variable mass - a theory for which the correct vacuum is known. Interestingly, we found that introducing the minimum length into this reformulated action does not yield the same theory - the action and the equations of motion differ slightly. This means that the initial conditions are slightly affected and starting from the actions of type $S^{(2)}$ would therefore affect how much the mode's state during the adiabatic phase deviates from the adiabatic vacuum. Thus, the predicted magnitude of the effect of the minimum length on the CMB is correspondingly affected a little bit. In the next two chapters we will investigate the implications of such discovery for the tensor/scalar ratio and consistency relation between tensor and scalar spectra.

Chapter 3

On the Tensor/Scalar Ratio in Inflation with UV Cutoff

Anisotropy of the cosmic microwave background radiation (CMB) originates from both tensor and scalar perturbations. To study the characteristics of each of these two kinds of perturbations, one has to determine the contribution of each to the anisotropy of CMB. For example, the ratio of the power spectra of tensor/scalar perturbations can be used to tighten bounds on the scalar spectral index. We investigate here the implications for the tensor/scalar ratio of the discovery made in the last chapter that the introduction of a minimal length cutoff in the structure of space-time does not leave boundary terms invariant. Such a cutoff introduces an ambiguity in the choice of action for tensor and scalar perturbations, which in turn can affect this ratio.

As we reviewed in the last chapter, the quantum theory of gauge invariant cosmological perturbations is based on the validity of general relativity and quantum field theory. Both of these theories break down at Planckian scales. However if inflation lasts a little bit longer than what is required to solve the problems of standard cosmology- as predicted by most inflationary models [80]- many scales of cosmological size today have been sub-Planckian at the onset of inflation. So it is natural to ask if the present cosmic microwave spectrum carries any thumbprint of physics at such small scales.

Vacuum fluctuations of the inflaton, ϕ_0 – the field that drives inflation – produce both scalar and tensor perturbations, both of which contribute to the anisotropy of the cosmic microwave background radiation. For any inflationary model one can calculate r, the ratio of tensor to scalar amplitudes. r multiplies the upper bound on the scalar density perturbations by a factor of $(1+r)^{-1/2}$. By knowing it one can tighten the bounds on the scalar spectral index [115, 116]. A more important consequence of measuring r exactly is determining the contribution of tensor perturbations to the anisotropy of the CMB. As the amplitude of tensor perturbations is proportional to the Hubble scale during inflation, by knowing r, one can nail down the energy scale of inflation. It is therefore important to know r in as much detail as possible in order to extract cosmological parameters with more precision.

The effect of trans-Planckian physics on the tensor/scalar ratio was addressed for the first time in [117], where the authors discovered that the ratio will be influenced by the short distance physics, if trans-Planckian physics does not lead to the same vacuum for scalar and tensor fluctuations. In this chapter, following the discovery of last chapter, we explore how the non-minimal choices of the boundary term for tensor and scalar fluctuations affect the tensor/scalar ratio. The structure of this chapter is as follows: first, we present the equations that scalar and tensor fluctuations satisfy in the presence of a UV cut off, categorizing various cases for which the ratio can change. Following ref. [91] we then solve these equations for scalar and tensor perturbations numerically in a near de-Sitter background. We compute how the scalar power spectrum varies as a function of σ, the ratio of noncommutative and Hubble lengths. In the fourth section we ultimately find the ratio of tensor to scalar fluctuations.

3.1 Ratio of tensor/scalar fluctuations with a cutoff

As mentioned in the last chapter, using the gauge-invariant intrinsic curvature perturbations of the comoving hypersurface, $\Re$, the action for scalar perturbations can be written as

$$S_S^{(1)} = \frac{1}{2}\int d\tau\, d^3\mathbf{y}\, z^2 \left((\partial_\tau \Re)^2 - \delta^{ij}\, \partial_i \Re \partial_j \Re\right). \tag{3.1}$$

or alternatively as

$$S_S^{(2)} = \frac{1}{2}\int d\tau d^3\mathbf{y} \left((\partial_\tau u)^2 - \delta^{ij}\, \partial_i u\, \partial_j u + \frac{z''}{z} u^2\right). \tag{3.2}$$

where $\Re = -u/z$ [39, 40] and z is given by equation 1.36.

These two actions for scalar fluctuations are equivalent up to a boundary term in absence of minimal length, with $S_S^{(2)}$ more commonly used in the literature because of its similarity with the action of a massive free scalar field in Minkowskian space-time. However the effective mass, z''/z is time dependent.

When the generalized uncertainty principle (2.3) is employed, $S_S^{(1)}$ and $S_S^{(2)}$ are no longer equivalent. Instead, they respectively yield the following equations of motion for the Fourier components of u, [72]:

$$u''_{\tilde{k}} + \frac{\kappa'}{\kappa} u'_{\tilde{k}} + \left(\mu - \frac{z''}{z} - \frac{z'}{z}\frac{\kappa'}{\kappa}\right) u_{\tilde{k}} = 0, \tag{3.3}$$

$$u''_{\tilde{k}} + \frac{\kappa'}{\kappa} u'_{\tilde{k}} + \left(\mu - \frac{z''}{z}\right) u_{\tilde{k}} = 0, \tag{3.4}$$

where $\mu(\tau, \rho)$ and $\kappa(\rho)$ are given by equation 2.22. ρ is a parameter that plays the role of inverse wavelength. The difference in the above equations of motion was attributed to the non-triviality of the manner in which minimal length affects the boundary terms. The scalar fluctuation amplitude is then defined as [118]

$$A_S(k) \equiv \frac{2}{5} P_S^{1/2} = \frac{2}{5}\sqrt{\frac{k^3}{2\pi^2}} \left|\frac{u_{\tilde{k}}}{z}\right|_{\tilde{k}/aH \to 0}. \tag{3.5}$$

Similarly, for tensor perturbations, among an infinite number of actions that are equivalent in the absence of minimal length we *choose* [72] to start from

$$S_T^{(1)} = \frac{m_{Pl}^2}{64\pi} \int d\tau d^3\mathbf{y}\; a^2(\tau)\, \partial_\mu h^i{}_j\, \partial^\mu h_i{}^j \tag{3.6}$$

and

$$S_T^{(2)} = \frac{1}{2}\int d\tau d^3\mathbf{y} \left(\partial_\tau P_i{}^j \partial^\tau P^i{}_j - \delta^{rs}\partial_r P_i{}^j \partial_s P^i{}_j + \frac{a''}{a} P_i{}^j P^i{}_j\right), \tag{3.7}$$

that again differ from each other by a boundary term. As we will see, one of these actions has a minimal effect on tensor/scalar ratio whereas the other one has a maximal effect. Also, the $S_T^{(1)}$ is the action one obtains by directly expanding the Einstein-Hilbert action. Here

$$P^i{}_j(y) = \sqrt{\frac{m_{Pl}^2}{32\pi}}\, a(\tau) h^i{}_j(y) \tag{3.8}$$

and h_{ij} is the transverse traceless part of tensor perturbations of the metric (1.33).

The $\tilde{k}$-Fourier component of P_{ij} (denoted $p_{\tilde{k}}$), satisfies the following equation of motion using the cutoff modified $S_T^{(1)}$

$$p''_{\tilde{k}} + \frac{\kappa'}{\kappa} p'_{\tilde{k}} + \left(\mu - \frac{a''}{a} - \frac{a'}{a}\frac{\kappa'}{\kappa} \right) p_{\tilde{k}} = 0, \tag{3.9}$$

whereas it satisfies

$$p''_{\tilde{k}} + \frac{\kappa'}{\kappa} p'_{\tilde{k}} + \left(\mu - \frac{a''}{a} \right) p_{\tilde{k}} = 0. \tag{3.10}$$

if we employ the variational principle on the cutoff modified $S_T^{(2)}$.

We define the tensor amplitude as [118]

$$A_T(k) \equiv \frac{1}{10} P_T^{1/2} = \frac{1}{10} \sqrt{\frac{k^3}{2\pi^2}} \, |p_{\tilde{k}}|_{\tilde{k}/aH \to 0} \tag{3.11}$$

The ratio of tensor to scalar fluctuations and scalar spectral index are respectively given by

$$r \equiv \frac{A_T^2}{A_S^2}, \tag{3.12}$$

$$n(k) - 1 \equiv \frac{d \ln A_S^2(k)}{d \ln k}. \tag{3.13}$$

The effect of r is to multiply the upper bound on the density perturbations by a factor of $(1+r)^{-1/2}$ which in turn affects our estimation of the scalar spectral index [115, 116].

In the absence of minimal length, one can expand the ratio of tensor/scalar fluctuations in terms of the slow roll parameters. To first order it is [115, 118, 119]

$$\frac{A_T^2}{A_S^2} = \epsilon \tag{3.14}$$

where

$$\epsilon \equiv \frac{3\dot{\phi}_0^2}{2} \left(V(\phi_0) + \frac{1}{2}\dot{\phi}_0^2 \right)^{-1} = \frac{m_{Pl}^2}{4\pi} \left(\frac{H_\phi}{H} \right)^2. \tag{3.15}$$

is the first slow-roll parameter [118]. Here, ϕ subscript denotes differentiation with respect to ϕ. In presence of minimal length the relation (3.14), takes the following form

$$\frac{A_T^2}{A_S^2} = \epsilon \left| \frac{p_k}{u_k} \right|^2_{k/aH \to 0} \tag{3.16}$$

The ambiguity in choosing the actions for scalar and tensor fluctuations in the presence of the minimum length is a new source of trans-Planckian effects that can modify the tensor/scalar ratio r. In general we have four possibilities:

I,II If we choose either $(S_S^{(1)}, S_T^{(1)})$ or $(S_S^{(2)}, S_T^{(2)})$ as the actions describing scalar and tensor fluctuations, the scalar modes $u_{\tilde{k}}$, and tensor modes $p_{\tilde{k}}$, satisfy differential equations that are as similar as possible. In particular this implies that in the special cases of near-de-Sitter and power-law inflation where $z''/z = a''/a$ (see [118]) and $z'/z = a'/a$ (see Appendix to this chapter), the equations governing both scalar and tensor perturbations are identical. Since metric and inflaton perturbations cannot be fully distinguished in a gauge invariant manner, scalar and tensor modes should also obey the same initial conditions, yielding $\left|\frac{p_k}{u_k}\right| = 1$. The distinction between cases I and II becomes apparent when the inflating background deviates from the power-law and near-de-sitter backgrounds.

III,IV The other extreme is to select either of the pairs $(S_S^{(1)}, S_T^{(2)})$ or $(S_S^{(2)}, S_T^{(1)})$ to describe the situation. In these cases the modes $u_{\tilde{k}}$ and $p_{\tilde{k}}$, satisfy differential equations of differing form even in near-de-Sitter and power-law backgrounds. In particular the tensor amplitude is not just ϵ times the scalar amplitude in power-law backgrounds. In the next section we present a complete analysis of the scalar and tensor spectra in near-de-Sitter space. We will investigate how the ratio of tensor to scalar perturbations varies as a function of σ, the ratio of minimal length to Hubble length during inflation. If general covariance holds at short distances, one would not expect that scalar and tensor perturbations satisfy different equations of motion at Planckian epochs. While this expectation is reasonable, it is by no means guaranteed: scalar modes are a mix of field and metric fluctuations in an arbitrary gauge, whereas the tensor modes are purely metric. It is not clear that a short-distance cutoff will affect fluctuations of the metric in the same way as fluctuations in an arbitrary scalar field. By taking into account this possibility seriously, we implicitly assume that general covariance could have been broken during inflation.

3.2 Scalar perturbations with minimum length in near-de-Sitter space

Curvature fluctuations arise because the value of the inflaton field is coupled to the energy density of the vacuum energy driving inflation, i.e. fluctuations in the inflaton field result in fluctuations in the expansion rate at linear order in perturbation theory. This coupling is what creates fluctuations in the intrinsic curvature scalar, which are then manifest as density fluctuations. Since in de Sitter space fluctuations in the inflaton field, ϕ, are not coupled to fluctuations in the energy density, the amplitude of density fluctuations is zero. A naive exploitation of the formalism of refs. [39, 118] implies that the expression for density fluctuations is singular for de Sitter space. The reason that the expression is singular is not because the density fluctuation amplitude is singular, but because the foliation of space-time implicit in the choice of gauge becomes singular.

Nevertheless, we can proceed in this manner by assuming that ϵ is close to zero, i.e. that the background is arbitrarily close to the de Sitter limit. Note that we are taking H, the Hubble parameter, to be very small, since it is known from COBE that $P_S = H^2/\epsilon \simeq const. \times 10^{-5}$.

We begin with an analysis of the scalar power spectrum, tracking the normalized modes which are inside the horizon until they are far outside the horizon, where their amplitude determines the perturbation spectrum. To this end, we will solve the mode equation (3.3) numerically. As in Ref.[91, 92], we describe the initial evolution by an approximate analytic solution, which we then evolve numerically to late times.

In this section and in what follows, we first analyze the action $S_S^{(2)}$ and $S_T^{(1)}$ for scalar and tensor perturbations respectively. In de Sitter space $a = -1/(H\tau)$ and $z''/z = 2/\tau^2$. A mode with a fixed comoving wave number $\tilde{k}$ corresponds over time to increasing proper wavelengths. Each mode's proper wavelength corresponds to the Planck length at some time τ that depends on $\tilde{k}$ and this is when the evolution of that mode begins. This time is when $a^2(\tau_{\tilde{k}}) \simeq \beta\tilde{k}^2$ and $\rho^2 = 1/\beta$. At this initial time equation (3.3) has an irregular singular point.

Since de Sitter space is time-translation invariant, the equation can be written in terms of the dimensionless parameter $w = \tilde{k}\tau$, in terms of which all the modes evolve jointly:

$$\frac{d^2 u_{\tilde{k}}}{dw^2} + n(w)\frac{du_{\tilde{k}}}{dw} + \Omega^2(w) u_{\tilde{k}}(w) = 0, \tag{3.17}$$

where

$$n(w) = \frac{1}{\theta(\zeta(w))}\frac{d\theta(\zeta(w))}{dw} \tag{3.18}$$

$$\Omega^2(w) = -\left(\frac{1}{\sigma^2 w^2}\frac{W(\zeta(w))}{(1+W(\zeta(w)))^2} + \frac{2}{w^2}\right) \tag{3.19}$$

and in de Sitter space $\zeta(w) = -\sigma^2 w^2$. Here we define

$$\sigma = \sqrt{\beta}/H^{-1}, \tag{3.20}$$

which is the ratio of the minimal length scale and the Hubble length scale during inflation. The function $W(x)$ is the Lambert W-function, defined by the relation $W(x)e^{W(x)} = x$ [114].

Equation (3.17) has a singular point at $w_{crit} = \varpi = -\frac{1}{\sigma\sqrt{e}}$. The singular point at ϖ is an irregular singular point because the coefficients of $du_{\tilde{k}}/dw$ and $u_{\tilde{k}}$ are not analytic in $v = w - \varpi$

$$n(v) = -\frac{1}{2v} - \frac{7}{12}\frac{e^{1/2}}{\sqrt{Av}} + \frac{67}{144}\frac{e}{A} + \cdots$$

$$\Omega^2(v) = \frac{A}{v} - \frac{e^{1/2}\sqrt{A}}{\sqrt{v}} - \frac{37}{72}e + \cdots \tag{3.21}$$

where

$$A = \frac{e^{1/2}}{4\sigma}. \tag{3.22}$$

Proceeding along the lines given in ref. [91, 92], we solve for the leading behavior of $u_{\tilde{k}}$ by extracting the most singular terms of the equation of motion

$$\ddot{u}_{\tilde{k}} - \frac{1}{2v}\dot{u}_{\tilde{k}} + \frac{A}{v}u_{\tilde{k}} = 0, \tag{3.23}$$

where in the overdot now denotes the derivative with respect to v. Ignoring the $\dot{u}_{\tilde{k}}$ term, this equation is similar to the high frequency limit of the mode equation:

$$u_k''(\tau) + \Omega_k^2(\tau) u_k(\tau) = 0 \tag{3.24}$$

whose solution is approximated by the WKB form

$$u_k(\tau) = \frac{1}{\sqrt{2\Omega_k}} \exp(-i \int^{\tau} \Omega_k(\tau') d\tau') \tag{3.25}$$

if the adiabatic condition $|\Omega_k'/\Omega_k^2| \ll 1$ is satisfied. This choice of vacuum, which is called Bunch-Davies vacuum, reduces to the Minkowskian vacuum for wavelengths smaller than the Hubble scale. Inspired by this similarity, one can suggest a Bunch-Davies-like vacuum of the form:

$$u_{\tilde{k}}(v) = (\frac{v}{4\tilde{k}^2 A})^{1/4} \exp(-2i\sqrt{Av}) \tag{3.26}$$

with

$$\Omega_{\tilde{k}} = \tilde{k}\sqrt{A/v}. \tag{3.27}$$

This vacuum does not satisfy the adiabaticity conditions in the vicinity of its creation time, $v = 0$. To be specific:

$$\frac{\Omega_{\tilde{k}}'}{\Omega_{\tilde{k}}^2} = \frac{\tilde{k}}{2} \frac{1}{\sqrt{Av}} \tag{3.28}$$

For $v \sim 0$ the adiabatic condition is violated. It means that each mode is born in an excited state. In the model of trans-Planckian physics proposed in ref.[85], each mode undergoes three phases in its evolution. In the first phase, the wavelength of the given mode is much smaller than the Planck length: $\lambda \ll l_p$. Each mode is born into the vacuum state that minimizes the Hamiltonian and satisfies the adiabaticity condition. In the second phase, the wavelength of the mode is larger than the Planck length but still smaller than the Hubble radius, $l_p \ll \lambda \ll l_H$. In the third phase the mode is outside the Hubble radius: $\lambda \gg l_H$. In our version of this scenario, the first phase is removed and replaced by an excited initial state which violates the adiabaticity conditions.

In fact, equation (3.23) is solved exactly by

$$u(y) = C_+ F(v) + C_- F^*(v) \tag{3.29}$$

where

$$F(y) = (\frac{\sqrt{A}}{2} + iA\sqrt{v}) \exp(-2i\sqrt{Av}) \tag{3.30}$$

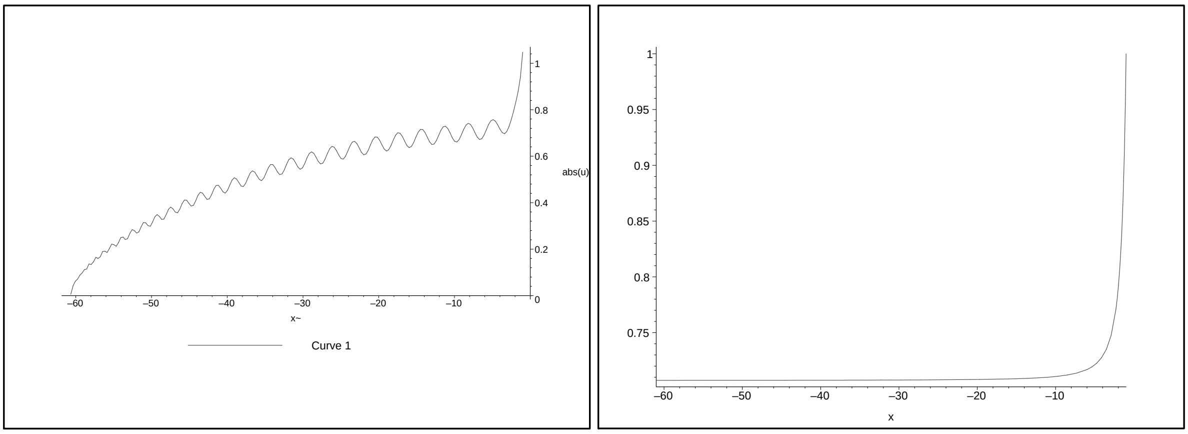


Figure 3.1: The left figure shows how each mode evolves in the presence of a cutoff as a function of $w = \tilde{k}\tau$. Each mode is created at a finite conformal time and the amplitude of the modes is modulated on a monotonously increasing curve. C_- is assumed to be zero. The right figure shows how in absence of minimal length each mode is created at infinite conformal time and its amplitude monotonously increases until it leaves the horizon.

and where the coefficients $C_\pm$ are constrained through the Wronskian condition 2.28 that reduces to

$$|C_+|^2 - |C_-|^2 = \frac{e^{-1}}{2\tilde{k}A^3}. \tag{3.31}$$

This equation will lead to a one parameter family of solutions. Similarity with the Bunch-Davies vacuum suggests that $C_- = 0$. In addition, in this case, it is possible to obtain conventional QFT result when $\sigma \to 0$. However there exist other legitimate choices of the vacuum. Specifically, it is possible to recover the standard QFT result in the limit $\sigma \to 0$, if C_- approaches zero faster than $\sigma^{3/2}$, as we shall subsequently demonstrate. However we will first assume that $C_- = 0$.

Equation (3.17) has been solved to order $1/v$. As in Ref.[91] we will extract the subleading behavior of $u_{\tilde{k}}$ by the method of dominant balance [120]. We solve the differential equation (3.17) up to $1/\sqrt{v}$ by defining $u_{\tilde{k}}(v) = F(v)(1 + \epsilon_1(v))$, extracting the most singular terms. The equation of motion for ϵ_1 is:

$$\frac{d^2\epsilon_1}{dv^2} - \frac{1}{2v}\frac{d\epsilon_1}{dv} - \frac{3}{2}e^{1/2}\sqrt{\frac{A}{v}} = 0 \tag{3.32}$$

which has the solution

$$\epsilon_1(v) = \frac{1}{3}\sqrt{A}e^{1/2}v^{3/2}(3\ln(v) - 2). \tag{3.33}$$

The solution is improved by extracting the residual $\ln(v)$ terms. To do so, we replace $u_{\tilde{k}}$ by $F(v)(1 + \epsilon_1(v))(1 + \epsilon_2(v))$ and extract the most singular terms to obtain the following differential equation for $\epsilon_2(v)$

$$\frac{d^2\epsilon_2}{dv^2} - \frac{1}{2v}\frac{d\epsilon_2}{dv} - \frac{7}{8}e\ln(v) = 0. \tag{3.34}$$

whose solution is

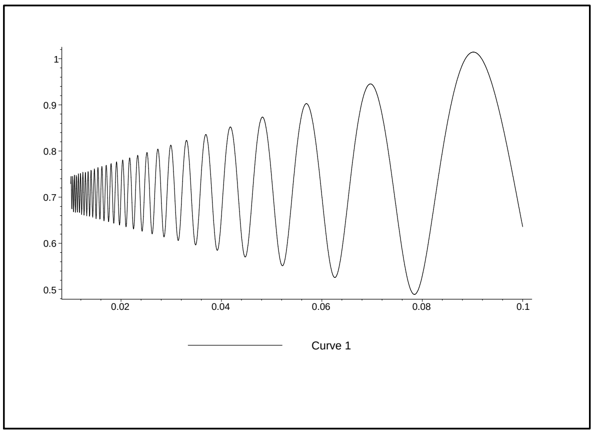
$$\epsilon_2(v) = \frac{7}{16}ev^2(2\ln(v) - 5). \tag{3.35}$$

We have solved the differential equation (3.17) up to terms that vanish as $v \to 0$. We glue this analytic solution, which is valid when the mode is in the vicinity of the irregular singular point and inside the horizon, to the full numerical evaluation of the mode equation. As the coefficients of $u_{\tilde{k}}$ and $u'_{\tilde{k}}$ are infinite at $v = 0$, this junction is done at a finite nonzero value of v_0. By varying v_0 we have checked that our results do not depend on the choice of starting point. We evolved the mode equation using Fehlberg fourth-fifth order Runge-Kutta method implemented in Maple 9. In Figure 3.1 we have shown how each fluctuation mode evolves as a function of $w = \tilde{k}\tau$. In the absence of minimal length there is no birth time for the modes and $|u_k|$ increases monotonically as it evolves. As we incorporate minimal length into the problem, each mode is created at a definite $\tilde{k}$-dependent time and $|u_{\tilde{k}}|$ is modulated on a monotonically increasing function until it crosses the horizon. At that time $|u_k|$ stops oscillating and goes to infinity as we approach the present time. One should notice that parameter $\tilde{k}$ is different from the comoving momentum at large momenta. This difference modifies the condition of horizon crossing in terms of the parameter $\tilde{k}$. Note that ρ plays the role of inverse wavelength in our model. Using the relation between ρ and $\tilde{k}$ [72],

$$\tilde{k}^i = a\rho^i e^{-\beta\rho^2/2}, \tag{3.36}$$

we can express the criterion of horizon crossing , $\rho = H$, in terms of a parameter $\tilde{k}$

$$\tilde{k} = aH\exp(-\beta H^2/2) \tag{3.37}$$



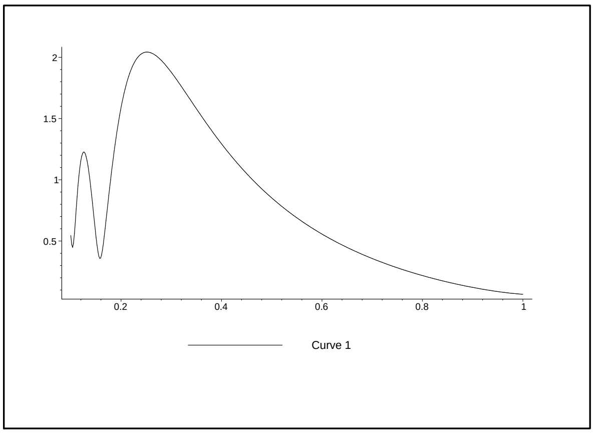


Figure 3.2: The dependence of $\frac{\sqrt{2}\pi\dot{\phi}}{H^2} P_S^{1/2}$ is plotted against σ. As σ goes to zero, the standard result of $\frac{1}{\sqrt{2}}$ is obtained. It is assumed that $C_- = 0$.

which takes the following form in de Sitter space

$$w_{\text{horizon}} = -\exp(-\sigma^2/2) \tag{3.38}$$

where $w = \tilde{k}\tau$. In absence of minimal length this criterion reduces to the familiar one in de Sitter space, $w = -1$.

For values of σ close to 1, i.e. when the energy scale of inflation is of the order of the minimal length, the horizon-crossing condition is considerably modified. Of course, we are really interested in the asymptotic values of $|u_{\tilde{k}}|$, when $\rho/H \to 0$. To implement it numerically, we have assumed this condition is satisfied when $\rho/H = 0.01$. We can express this condition in terms of a parameter w:

$$w_{\text{asymp}} = -0.01 \exp(-\sigma^2/20000). \tag{3.39}$$

The general answer to Equation(3.17), has the form of $u_{\tilde{k}}(\tau) = N(\tilde{k})U_{\tilde{k}}(w)$. Comparison with Equation (3.31) yields $N(\tilde{k}) = 1/\sqrt{\tilde{k}}$. So, the power spectrum in near-de Sitter space is:

$$P_S^{1/2} = \sqrt{\frac{\tilde{k}^3}{2\pi^2}} \left| \frac{u_{\tilde{k}}}{z} \right| = \frac{H^2}{\dot{\phi}} \frac{|-wU_{\tilde{k}}(w)|}{\pi\sqrt{2}} |_{w=w_{\text{asymp}}} \tag{3.40}$$

On the other hand, quantum field theory yields the following result for the near-de Sitter space

$$P_S^{1/2}(\sigma = 0) = \frac{H^2}{2\pi\dot{\phi}} \tag{3.41}$$

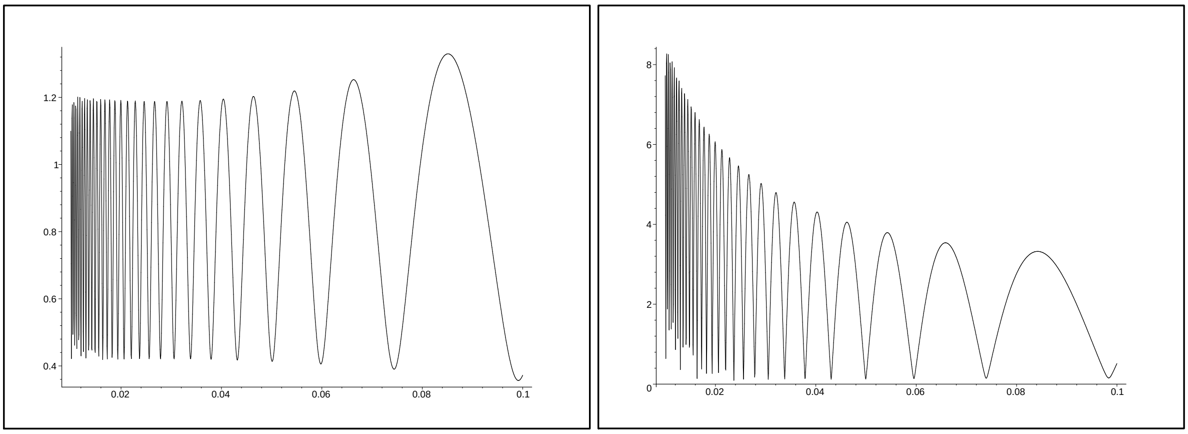


Figure 3.3: The left figure shows the dependence of $\frac{\sqrt{2}\pi\dot{\phi}}{H^2}P_S^{1/2}$ against σ when $C_-/C_+ = 0.5$. The right figure graphs the dependence of $\frac{\sqrt{2}\pi\dot{\phi}}{H^2}P_S^{1/2}$ on σ when $C_- \propto \sigma$. In neither of these cases do we recover the standard field theory result when $\sigma \to 0$.

In Fig. 3.2 we have displayed the σ dependence of the scalar power spectrum when $C_- = 0$. For small values of σ, the power spectrum has an oscillatory behavior around its standard result. For sufficiently large values of σ the power spectrum is significantly suppressed. This could be used as a mechanism for solving the fine tuning problem of inflationary models [91].

Next we relax the condition $C_- = 0$. Equation (3.31) will lead to a one parameter family of solutions. The criterion of approaching the result of conventional quantum field theory when $\sigma \to 0$ can be used to constrain our space of solutions. It has been pointed out [91] that if the ratio of C_-/C_+ is a non-zero constant then the tensor power spectrum does not approach its standard result in the limit $\sigma \to 0$. In Fig. 3.3, we have examined this statement for scalar perturbations and noticed that the same thing happens for scalar perturbations too. However, in general, C_- can be a function of σ. Since we know that when $C_- = 0$ and $C_+ \propto \sigma^{3/2}$, we obtain the standard result in the limit of $\sigma \to 0$, we expect that if C_- approaches zero faster than $\sigma^{3/2}$ when $\sigma \to 0$ the criterion of recovering the standard QFT result is satisfied. We have verified this statement for $C_- \propto \sigma$ and $C_- \propto \sigma^2$ respectively as shown in Figures 3.3 and 3.4. Hence we conjecture that it is possible that C_- be proportional to σ^n, $n > \frac{3}{2}$ whilst obtaining the standard QFT result in the limit $\sigma \to 0$.

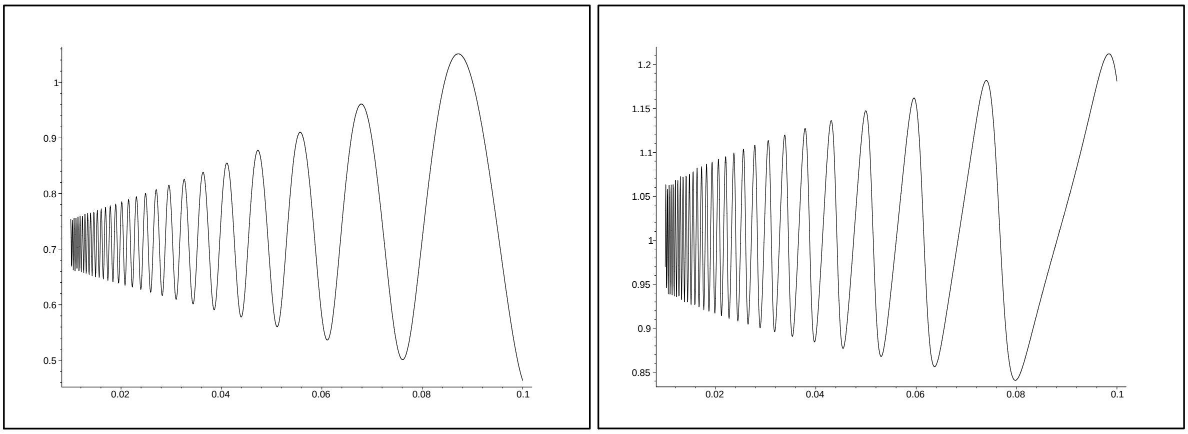


Figure 3.4: The left figure shows the dependence of $\frac{\sqrt{2}\pi}{H} P_S^{1/2}$ against σ when $C_- \propto \sigma^2$. The right figure shows the ratio of scalar power spectrum when $C_- = 0$ to scalar power spectrum when $C_- \propto \sigma^2$. In this case the scalar power spectrum approaches the standard result when $\sigma \to 0$

In most inflationary models the expansion rate is slower than de Sitter space; the ratio of minimum length to physical horizon is not constant and decreases towards the end of inflation. Our study of de Sitter space suggests that the amplitude of the longer modes will be affected more. However, one should also note that observation constrains the energy scale of inflation to change very slowly during inflation [129], so any change of minimal length over the Hubble radius ratio should be very small.

3.3 Tensor/Scalar ratio with minimum length in near-de Sitter space

As explained above, if the action of the tensor perturbations, in absence of σ, is given in eq.(3.7) then the tensor power spectrum will be ϵ times that of the scalar perturbations. Otherwise, if the action is given by eq. (3.6), its power spectrum will not be a simple multiple of the scalar perturbations. Although a complete analysis of the equation of motion derived from this action has been done once in de Sitter space [91], we recapitulate those calculations in the present

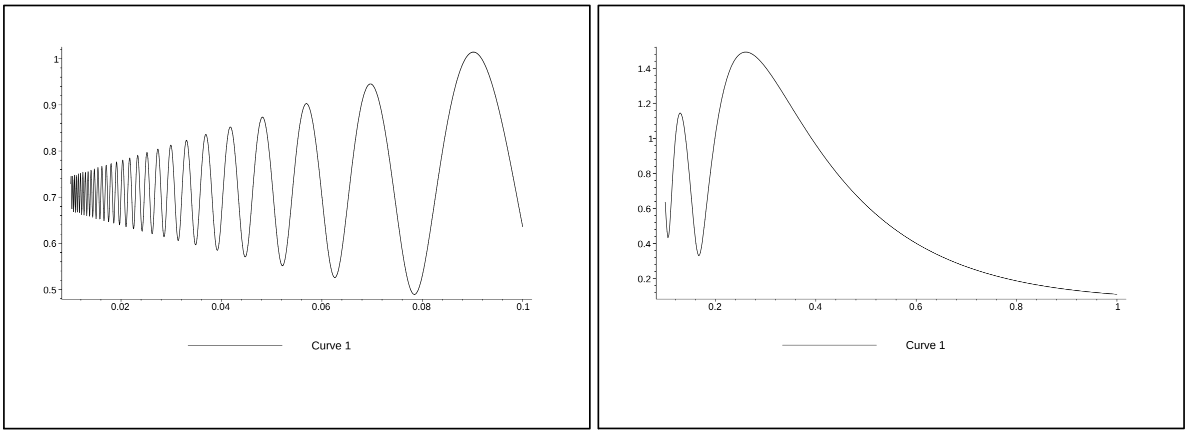


Figure 3.5: The dependence of $\frac{\sqrt{\pi}m_P}{4H}P_T^{1/2}$ is plotted against σ. $\frac{\sqrt{\pi}m_P}{4H}P_T^{1/2}$ approaches the standard result of $\frac{1}{\sqrt{2}}$ as σ goes to zero. We have assumed that $D_- = 0$.

context to find the ratio of tensor to scalar fluctuations. The outline of the calculations is completely similar to what was done for scalar perturbations: we tailor the solution that is valid in vicinity of the irregular singular point to the numerically integrated solution. The exact analytic solution in the neighborhood of the singular point is:

$$u_{\tilde{k}} = D_+ G(v)(1+\xi_1(v))(1+\xi_2(v)) + D_- G^*(v)(1+\xi_1(v))(1+\xi_2(v)) \tag{3.42}$$

where

$$\begin{aligned} G(y) &= (\frac{\sqrt{B}}{2} + iB\sqrt{v})\exp(-2i\sqrt{Bv}), \\ \xi_1(v) &= \frac{1}{3}e^{1/2}v^{3/2}\sqrt{A}(3\ln(v)-2), \\ \xi_2(v) &= \frac{7}{8}ev^2\ln(v) - \frac{35}{16}ev^2, \end{aligned} \tag{3.43}$$

$$B = \frac{1}{8}\frac{8A^2+e}{A} \tag{3.44}$$

D_+ and D_- are constrained by the following wronskian condition

$$|D_+|^2 - |D_-|^2 = \frac{e^{-1}}{2\tilde{k}B^3} \tag{3.45}$$

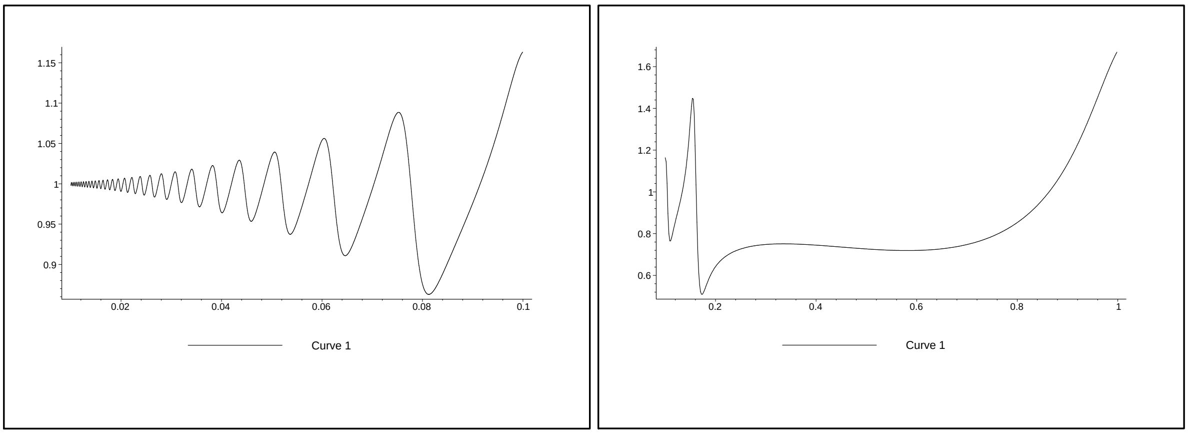


Figure 3.6: The dependence of $\frac{P_T^{1/2}}{4\sqrt{\epsilon}P_S^{1/2}}$ is plotted against σ. It approaches the standard result of unity as σ goes to zero. C_- and D_- are set to zero.

Again, we have considerable freedom in choosing our vacuum. Since the right-hand side of eq.(89) approaches zero like σ^3 when $\sigma \to 0$, if D_- tends to zero faster than $\sigma^{3/2}$ we can recover the standard QFT result. Hereafter we restrict ourselves to $D_- = 0$ so as to have a Bunch-Davies-like vacuum. We use equation (3.45) with $D_- = 0$ at a point close to the singularity to integrate the differential equation. In Fig. 3.5 we have displayed the σ-dependence of the tensor power spectrum. The oscillationary behavior in vicinity of $\sigma = 0$ and decaying behavior for larger values of σ has repeated.

Fig. 3.6 shows how the tensor to scalar perturbations ratio varies as a function of σ when $C_- = D_- = 0$. For small values of σ, it oscillates about its standard value, ϵ. For intermediate values of σ it remains almost constant on a value that is less than its standard result. Although the tensor and scalar fluctuations both decrease as σ increases, their ratio gradually increases by increasing σ and even becomes larger than its standard value. This means that the tensor fluctuations decrease more slowly than do the scalar fluctuations.

We can derive some qualitative features of the same study for power-law backgrounds from what we derived in near-de Sitter space. At the beginning of inflation the expansion is faster than it is at the end of inflation and so the Hubble parameter is larger. Hence the effect is much more profound for modes that leave the horizon at that time. For such modes, the ratio will be much more distorted from standard predictions. We plan to return to this problem in greater

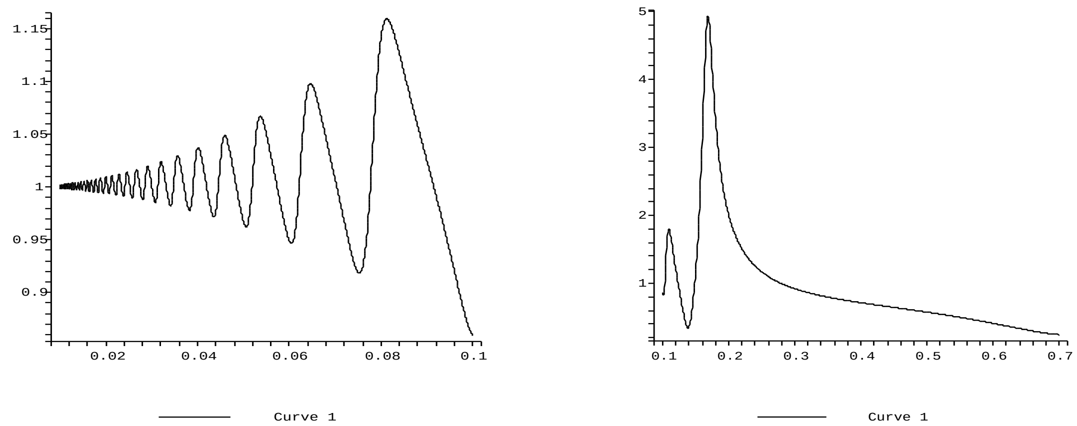


Figure 3.7: The dependence of $\frac{P_T^{1/2}}{4\sqrt{\epsilon}P_S^{1/2}}$ is plotted against σ. It approaches the standard result of unity as σ goes to zero. C_- and D_- are set to zero. It is assumed that $S_S^{(1)}$ and $S_T^{(2)}$ describes the behavior of scalar and tensor perturbation, respectively.

detail in next chapter.

Now we assume that $S_S^{(1)}$ and $S_T^{(2)}$ describes the behavior of scalar and tensor perturbation. In near-De-sitter and power-law backgrounds the equations derived from $S_S^{(1)}$ and $S_T^{(2)}$ are the same as the ones derived from $S_T^{(1)}$ and $S_S^{(2)}$ respectively. Therefore the ratio $\frac{A_T^2}{\epsilon A_S^2}$ will be reversed. Fig. 3.7 shows that how the ratio of tensor to scalar perturbations varies as a function of σ. The same oscilationary behavior in vicinity of $\sigma = 0$ has repeated. However in this case the tensor/scalar ratio decreases as the ratio of minimal length approaches the Hubble length during the inflation. This mechanism might be used to dampen the contribution of tensor amplitudes to the anisotropy of the microwave background radiation.

3.4 Approximate dependence of amplitude of r on σ

The equations of motion that arise from the scalar action $S_S^{(1)}$ and tensor action $S_T^{(2)}$ (the situation is the same for $S_S^{(2)}$ and $S_T^{(1)}$) differ merely in their "mass terms", i.e. in the terms that multiply the undifferentiated field. In near-de-Sitter and power-law backgrounds, the mass

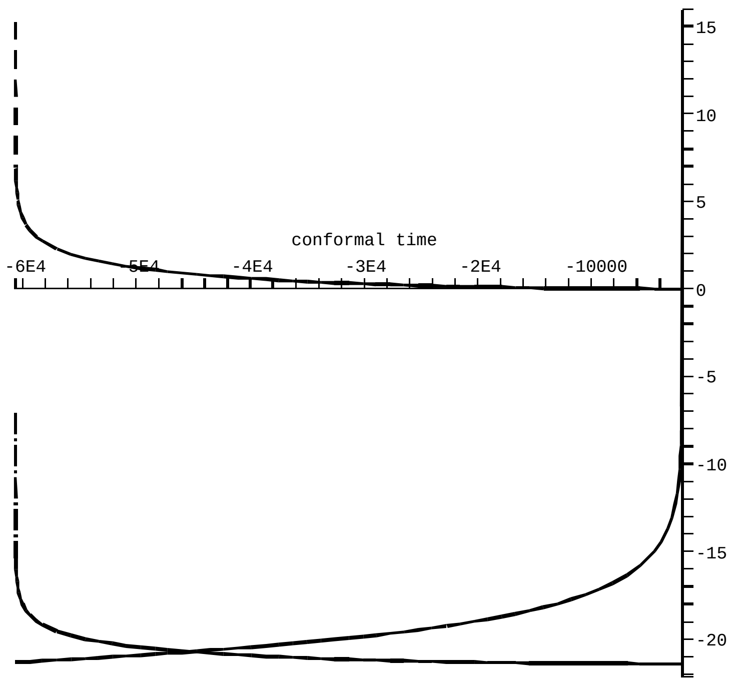


Figure 3.8: Comparison of the three "mass terms" $\ln(\mu)$ (dashed), $\ln(|d|)$ (dotted) and $\ln(a''/a)$ (solid) versus conformal time. The term d of the ambiguity is dominated by μ and a''/a throughout the evolution.

terms differ by the term

$$d(\tau, \tilde{k}) = \frac{a'\kappa'}{a\kappa}. \tag{3.46}$$

Although above we obtained numerically the effect of presence of such a term in the equation of motion, let us obtain an estimation for the amplitude of dependence of ratio, r, on σ. The term d competes with the two other "mass" terms, a''/a and μ. In Fig. 3.8, the magnitudes of the terms, d, $\frac{a''}{a}$ and μ are compared by plotting the logarithm of their absolute values against conformal time. We chose the de-Sitter background with a realistic Planck length to Hubble length ratio of $\sigma = 10^{-5}$. In de Sitter space all $\tilde{k}$ modes evolve in the same way and we arbitrarily chose $\tilde{k} = 1$. The curves start at the creation time $\tau_{\tilde{k}}$ of the mode and end at future infinity $\tau = 0$. There are three distinct phases in a mode's evolution:

A) In the initial phase close to the creation time, the behavior of the differential equation is dominated by the terms μ and d which both appear to diverge. (The function a'/a is regular at $\tau_{\tilde{k}}$ since the creation time of a particular mode is not a special time for the scale factor). We need to determine, therefore, the relative magnitudes of d and μ as $\tau \rightarrow \tau_{\tilde{k}}^{+}$. Now a straightforward calculation yields for the ratio of the functions d and μ:

$$\frac{d(\tau,\tilde{k})}{\mu(\tau,\tilde{k})} \;=\; -\,\sigma^2\left(5+3\;W(-k^2\sigma^2\tau^2)\right) \tag{3.47}$$

The range of the Lambert W function is the finite interval $[-1,0]$. Thus, since μ is divergent at $\tau_{\tilde{k}}$, also d is divergent at $\tau_{\tilde{k}}$. However, Eq.3.47 also implies that at all times the term d is much smaller than the term μ, namely by a factor σ^2, up to a pre-factor of order one. Thus, any criterion that one might adopt for determining the mode's initial condition at $\tau_{\tilde{k}}$ is, therefore, affected by the presence or absence of the term d, though only relatively weakly because μ dominates d by a factor of order σ^2.

B) The initial period is followed by an adiabatic period in which all three terms μ, d and a''/a are slowly varying. This phase lasts until horizon crossing. In order to estimate the effect of the term d on the time evolution in this phase we compare the oscillation frequencies $\Omega_d = \sqrt{\mu - a''/a - d}$ and $\Omega_0 = \sqrt{\mu - a''/a}$ in the adiabatic phase with and without the term d respectively. During the adiabatic phase we can neglect the term a''/a and we can set $\mu \approx \tilde{k}^2$ to obtain: $\Omega_d/\Omega_0 = (1 - d/(2\mu)) + \mathcal{O}(d^2/\mu^2)$, i.e. $\Omega_d \approx \Omega_0(1 + b\sigma^2)$ and thus $\Delta\Omega \approx b\tilde{k}\sigma^2$ where b is of order one. In principle, this frequency shift alters the number N of oscillations during the duration of the second phase. In order to estimate N, we recall that the $\tilde{k}$ mode is created at the time $\tau_{\tilde{k}} = -\frac{1}{H\tilde{k}\sqrt{e\beta}}$ and that therefore the duration of the second phase is of the order of $T \approx \frac{1}{H\tilde{k}\sqrt{e\beta}}$. Recall that $\sigma = \sqrt{\beta}H$, which implies $T \approx 1/(\tilde{k}\sigma\sqrt{e})$. Thus, in the course of the adiabatic second phase, the presence or absence of the term d implies approximately N more or fewer oscillations, where

$$N \approx \Delta\Omega\; T \approx \frac{b\;\tilde{k}\;\sigma^2}{\tilde{k}\sigma\sqrt{e}} \approx \sigma \tag{3.48}$$

C) The last period, from horizon crossing to the infinite future $\tau \rightarrow 0$ is not clearly resolved in our plot. It is the period when the term a''/a diverges and entirely dominates the μ and d terms since they stay finite.

Overall, the presence of factor d changes the number of e-folds only during the second phase. Taking into account the relation 1.61, one realizes that the presence of factor d in the mode equation introduces a modification of order σ in the curvature perturbations of comoving hypersurface. As the power spectrum, $P(k)$, is proportional to $\Re^2$, one can conclude that the presence of factor d in the mass term induces a factor σ^2 modification in the calculated power spectrum from $S_S^{(1)}$. This chain of reasoning leads us to believe that $\triangle r/r$ is proportional to σ^2.

3.5 Conclusion

Both tensor and scalar perturbations are responsible for the anisotropy of the CMB. Knowledge of the ratio of tensor/scalar perturbations provides an important constraint on related cosmological parameters.

We have investigated the implications of implementing a minimal length hypothesis from the generalized uncertainty principle (2.3) for the tensor/scalar ratio r in inflationary scenarios. Specifically, we have studied how an ambiguity generically present in this hypothesis [72] leads to different conclusions about how and whether trans-Planckian physics alters r. In two of the cases the ratio remains constant, unless the background deviates from a power-law expansion during inflation. In the other two cases, the ratio is modified even in a simple de Sitter or power-law background. We also found the dependence of the ratio on the minimal length for the near-de-Sitter background in these two cases.

The tensor fluctuations are expected to contribute to the CMB's B -polarization. This effect may be observable with the upcoming PLANCK satellite. One can then differentiate the contribution of tensor fluctuations from scalar ones to check the above scenario.

3.6 Supplementary material for Chapter 3

The scalar gauge invariant parameter, u, is proportional to $\Re$, the intrinsic curvature perturbations of the spatial hypersurface through a factor z see eq. (1.36) which is equivalently can be defined as:

$$z = \frac{a\dot{\phi}_0}{H} \tag{3.49}$$

where H is the Hubble parameter, $\dot{a}/a$ (dot denotes differentiation with respect to the physical time). So one obtains:

$$\frac{z'}{z} = \frac{a'}{a} + \frac{\dot{\phi_0}'}{\dot{\phi}_0} - \frac{H'}{H}. \tag{3.50}$$

$\dot{\phi_0}'/\dot{\phi}_0$ can be written as $a\ddot{\phi}_0/\dot{\phi}_0$. Using the definition of η

$$\eta(\phi) \equiv -\frac{\ddot{\phi}_0}{H\dot{\phi}_0} = \frac{m_{Pl}^2}{4\pi}\left(\frac{H_{\phi\phi}}{H}\right) = \epsilon - \frac{m_{Pl}\epsilon_\phi}{\sqrt{16\pi\epsilon}}, \tag{3.51}$$

this can be written in terms of the slow roll parameters:

$$\frac{\dot{\phi_0}'}{\dot{\phi}_0} = -\eta\frac{a'}{a}. \tag{3.52}$$

H'/H is equal to $a\dot{\phi}_0 H_\phi/H$. Choosing the convention that $\dot{\phi}_0 > 0$, from Eq. (3.15) one derives:

$$\frac{H_\phi}{H} = -\frac{2\sqrt{\pi\epsilon}}{m_{Pl}} \tag{3.53}$$

and

$$\dot{\phi_0}^2 = \frac{2\epsilon V}{3-\epsilon}. \tag{3.54}$$

The inflaton energy density is $\frac{\dot{\phi_0}^2}{2} + V$ and the first Friedmann equation

$$H^2 = \frac{8\pi}{3m_{Pl}^2}(\frac{\dot{\phi_0}^2}{2} + V), \tag{3.55}$$

combined with (3.54), yields:

$$V = \frac{m_{Pl}^2 H^2(3-\epsilon)}{8\pi}. \tag{3.56}$$

From Equations (3.54)and (3.56) one concludes:

$$\dot{\phi}_0 = \frac{m_{Pl} H}{2} \sqrt{\frac{\epsilon}{\pi}}. \tag{3.57}$$

Using the above equation one obtains:

$$\frac{H'}{H} = -\epsilon \frac{a'}{a}. \tag{3.58}$$

Inserting equations (3.58) & (3.52) back into eq.(3.50), we obtain the following expansion for z'/z in terms of the slow roll parameters:

$$\frac{z'}{z} = \frac{a'}{a}(1 + \epsilon - \eta). \tag{3.59}$$

In power-law and near-De-sitter space $\epsilon = \eta$ and so $z'/z = a'/a$.

Chapter **4**

Running of the Spectral Index and Violation of the Consistency Relation Between Tensor and Scalar Spectra from trans-Planckian

4.1 Introduction

One of the intriguing properties of inflationary cosmology, which could be used to test the fundamental theories of quantum gravity, is its capacity to accommodate sub-Planckian fluctuations that were redshifted exponentially during a quasi-de-Sitter expansion of the universe [38]. Many realizations of inflation predict several more e-foldings than are required to solve the problems of standard cosmology [80]. Assuming that these inflationary models are correct, all scales of cosmological interest today originate inside the Planck scale at the early stages of inflation. These fluctuations would be manifest in the temperature anisotropy of the cosmic microwave background radiation (CMBR), which can be regarded as a fossil record of primeval inhomogeneities. It is therefore reasonable to expect that by studying the cosmic microwave background radiation one can extract information about physics at very small distance scales [121].

A beautiful mechanism was suggested in [71] to incorporate minimum length into the inflationary formalism. The only assumption underlying this formalism is that the fundamental theory of quantum gravity possesses a linear operator X^i for every space-time coordinate and that its expectation value $\langle X^i \rangle$ is real. One can then show that the short distance structure of any such coordinate could not only be continuous or discrete, but could also be unsharp in one of two ways [93]. The two unsharp cases are distinguished by the so-called deficiency indices of X^i being either nonzero and equal (Fuzzy type A) or unequal (Fuzzy type B) [93]. In the case of fuzzy type B, sequences of vectors in the physical domain exist such that $\triangle X^i$ converges to zero. They are fuzzy in the sense that vectors of increasing localization around different expectation values in general do not become orthogonal. Fuzzy type A behavior has appeared in a number of studies in quantum gravity and string theory where the uncertainty in $\triangle X^i$ has a finite lower bound at Planck scale [94]. As mentioned in the previous chapters, this short distance structure can be modelled as quantum gravitational correction to the commutation relation between the position and momentum operators

$$[\mathbf{X}, \mathbf{P}] = i(1 + \beta \mathbf{P}^2), \tag{4.1}$$

where $\beta^{1/2}$ parameterizes the minimum length. The equations for tensor modes were later analyzed numerically in [91, 92], and it was predicted that the effect on the CMBR can be as large as σ, where σ is the ratio of the minimum length to the Hubble length during inflation, $\beta^{1/2} H \equiv \sigma$.

As discussed in the previous chapters, this mechanism of implementing minimal length in the action has an ambiguity: the usual strategy for determining the initial condition requires reformulating the action and discarding a boundary term. In the absence of minimal length, two actions that differ by a boundary term are equivalent. However, the introduction of a minimal length scale renders two actions that normally differ by a boundary term inequivalent, yielding different equations of motion. One has an infinite set of actions that are equivalent when the minimal length is set to zero. Only experiment can adjudicate which choice of action is preferable. Nevertheless, in the last chapter, from the infinite number of actions that are equivalent in the absence of minimal length, we adopt two actions for each of tensor and scalar fluctuations. The first one is chosen by a minimalist criterion: we select the action that is derived directly by expanding the action of a scalar field minimally coupled to gravity without introducing any additional terms by hand. The second action, which differs from the first by a boundary term, is chosen by the criterion of similarity with the action of a free massive scalar field in a Minkowskian background. Such a similarity simplifies the task of choosing the vacuum. Basically, one can choose the vacuum as one does in Minkowskian space-time.

We will examine, in the context of the minimal length hypothesis as implemented in the second chapter, a firm prediction of inflationary cosmology, the consistency relation between scalar and tensor perturbations. In the case of single-field slow-roll inflation the consistency conditions are given in terms of equality relations, whereas for multiple-field models of inflation these are weakened to inequalities. The first of the consistency relations states that the ratio of the amplitude of tensor to scalar perturbations is a constant known as the tensor spectral index [119, 122]. We investigate how the effects of trans-Planckian physics alter this ratio and the tensor spectral index under the considerations noted in the second chapter. This is in contrast to recent work in this area in which this possibility was investigated without focusing on any specific model of short distance physics, instead assuming that the trans-Planckian energies result in a vacuum state that is different from the standard Bunch-Davies vacuum [117]. We shall restrict ourselves to single-field inflation for the rest of the discussion, although our results could straightforwardly be generalized to multiple-field inflation.

This chapter is structured as follows: first we recapitulate our results from chapter 2 for both tensor and scalar fluctuations and also remind the reader the form of first consistency relation. Next, we study numerically the equations of motion for scalar and tensor modes in a power-law background and derive the tensor/scalar fluctuations and the tensor spectral index in each case. As mentioned earlier, actions that differ by a boundary term are rendered inequivalent once one implements the minimal length hypothesis. Although this implies an infinite amount of freedom in choosing the action for both scalar and tensor fluctuations, there are only a few actions that have reasonable physical motivation, and we shall confine our considerations to these cases. Specifically, we shall discuss how these physically well-motivated but distinct actions modify the consistency relation between tensor and scalar spectra.

4.2 Tensor/Scalar Ratio and the Violation of the Consistency Relation

As derived in the second chapter, incorporating the minimal length hypothesis to the scalar action, $S_S^{(1)}$, written in terms of gauge invariant curvature perturbations of the comoving hypersurface, $\Re$, yields

$$u''_{\tilde{k}} + \frac{\kappa'}{\kappa} u'_{\tilde{k}} + \left(\mu - \frac{z''}{z} - \frac{z'}{z}\frac{\kappa'}{\kappa}\right) u_{\tilde{k}} = 0, \quad (4.2)$$

where $u = -z\Re$ [39, 40] and z is defined in eq. 1.36.

However the cutoff modified equation of motion for the fluctuation mode $u_{\tilde{k}}$, derived from $S_S^{(2)}$ is:

$$u''_{\tilde{k}} + \frac{\kappa'}{\kappa} u'_{\tilde{k}} + \left(\mu - \frac{z''}{z}\right) u_{\tilde{k}} = 0, \quad (4.3)$$

where

$$\mu(\tau, \tilde{k}) = -\frac{a^2}{\beta} \frac{W(-\beta \tilde{k}^2/a^2)}{(1 + W(-\beta \tilde{k}^2/a^2))^2} \quad (4.4)$$

$$\kappa(\tau, \tilde{k}) = \frac{e^{-\frac{3}{2}W(-\beta \tilde{k}^2/a^2)}}{1 + W(-\beta \tilde{k}^2/a^2)}. \quad (4.5)$$

As before, $\tilde{k}^i = a\rho^i e^{-\beta\rho^2/2}$ where ρ^i is the Fourier transform of the physical coordinate x^i. $\tilde{k}^i$ is a variable that is equivalent to comoving momentum at large wavelengths.

Following [118], we define the scalar amplitude as

$$A_S(k) \equiv \frac{2}{5} P_S^{1/2} = \frac{2}{5}\sqrt{\frac{k^3}{2\pi^2}} \left|\frac{u_{\tilde{k}}}{z}\right|_{\tilde{k}/aH \to 0}. \quad (4.6)$$

Similarly, implementing the minimal length to the actions of tensor perturbations, $S_T^{(1)}$ & $S_T^{(2)}$ yields respectively

$$p''_{\tilde{k}} + \frac{\kappa'}{\kappa} p'_{\tilde{k}} + \left(\mu - \frac{a''}{a} - \frac{a'}{a}\frac{\kappa'}{\kappa}\right) p_{\tilde{k}} = 0, \quad (4.7)$$

where $S_T^{(2)}$ differs from $S_T^{(1)}$ by a boundary term. $P^i{}_j(y) \equiv \sqrt{\frac{m_{Pl}^2}{32\pi}}\, a(\tau) h^i{}_j(y)$ where $h^i{}_j(y)$ is the tensor perturbations of the metric. Following [118], we define the tensor amplitude as

$$A_T(k) \equiv \frac{1}{10} P_T^{1/2} = \frac{1}{10}\sqrt{\frac{k^3}{2\pi^2}}\, |p_{\tilde{k}}|_{\tilde{k}/aH \to 0}\ . \quad (4.8)$$

One can expand the ratio of tensor/scalar fluctuations in terms of the slow roll parameters in the absence of a cut-off. To first order it is [118, 119, 122]

$$r \equiv \frac{A_T^2}{A_S^2} = \epsilon \tag{4.9}$$

where

$$\epsilon \equiv \frac{3\dot{\phi}_0^2}{2}\left(V(\phi_0) + \frac{1}{2}\dot{\phi}_0^2\right)^{-1} = \frac{m_{Pl}^2}{4\pi}\left(\frac{H_\phi}{H}\right)^2 \tag{4.10}$$

with the ϕ subscript and over-dot respectively denoting differentiation with respect to ϕ and the cosmic time, t, related to conformal time τ by $t = \int a d\tau$. Both tensor and scalar fluctuations contribute to the anisotropy of the CMBR. Hence, to extract the characteristics such as spectral indices for each type of fluctuation we need to know r [115, 116].

Since scalar and tensor perturbations originate from a single inflaton potential they are not independent. A hierarchy of consistency conditions links them together [118]. It has been argued that such conditions – if empirically verified – would offer strong support for the idea of inflation. Observational difficulties will probably render only the first consistency condition useful. The first of these consistency relations relates r to the tensor spectral index, n_T, defined as

$$n_T(k) \equiv \frac{d \ln A_T^2(k)}{d \ln k}. \tag{4.11}$$

To first order in slow-roll parameters n_T can expanded, yielding

$$n_T = -2\epsilon, \tag{4.12}$$

and so the first-order consistency relation takes the following form

$$r \equiv \frac{A_T^2}{A_S^2} = -\frac{n_T}{2} \tag{4.13}$$

in the absence of a cut-off.

In presence of minimal length the relation (3.14) is modified

$$\frac{A_T^2}{A_S^2} = \epsilon \left|\frac{p_k}{u_k}\right|^2_{k/aH \to 0} \tag{4.14}$$

where u_k and p_k will satisfy different differential equations contingent upon the choice of action in the presence of a cutoff. Furthermore, eq.(4.12) no longer holds true[1].

Hence one expects that Planck scale physics will modify the consistency relation. Our predictions of course depend on the choice of the action for tensor and scalar perturbations. As noted above, for both tensor and scalar spectra there are two physically motivated actions, yielding four cases of interest that we will separately analyze below.

4.2.1 The Mode Equations in a Power Law Background

Before presenting our numerical results for the power spectra, it will be instructive to consider the explicit form of the mode equations (2.26, 2.31, 2.33 and 2.35). A power-law inflationary background is described by

$$a(t) = t^p, \qquad a(\tau) = \left(\frac{\tau}{\tau_0}\right)^q, \qquad q = \frac{p}{1-p}, \tag{4.15}$$

where $t(\tau)$ is the cosmic (conformal) time and $p > 1$. Assuming that at $t = 1$, $a(t) = 1$ then $\tau_0 = 1/(p-1)$. We will track the evolution of the modes numerically from when they are created at the time $\tau_{\tilde{k}} \equiv \tau_0 \left(e\beta\tilde{k}^2\right)^{1/2q}$, until $\tau \to 0$ at which point we calculate the power spectrum. To this end, we define a new variable, y, so that $\tau = \tau_{\tilde{k}}(1-y)$. It will prove convenient to work with the rescaled quantity $k = \tilde{k}e^{p/2}/\tilde{k}_{crit}$., where $\tilde{k}_{crit}$ corresponds to the mode that crosses the horizon just before the Hubble radius reaches the minimal length scale, $\sqrt{\beta}H = 1$. Explicitly it is given by

$$\tilde{k}_{crit} = e^{-1/2}p(\beta p^2)^{(p-1)/2}. \tag{4.16}$$

With these definitions, the mode equation (2.31) becomes

$$\begin{aligned}\ddot{u}_{\tilde{k}} - \frac{q}{1-y}\frac{W(5+3W)}{(1+W)^2}\dot{u}_{\tilde{k}} - \Bigg(&\frac{eq^2k^{2/p}W}{(1-y)^{-2q}(1+W)^2} \\ &+\frac{q(q-1)}{(1-y)^2} + \frac{q^2}{(1-y)^2}\frac{W(5+3W)}{(1+W)^2}\Bigg)u_{\tilde{k}} = 0,\end{aligned} \tag{4.17}$$

[1] We are grateful to A. Kempf for bringing this to our attention

where an overdot now denotes a derivative with respect to y, and the argument of the Lambert W function is $-e^{-1}(1-y)^{-2q}$. The other mode equations (2.26),(2.33) and (2.35) may be obtained by dropping the final term in the parentheses and replacing $u_{\tilde{k}}$ with $p_{\tilde{k}}$ as necessary. The definition of k was chosen to remove the explicit dependence upon the minimal length, but we now see that there is an added benefit to the choice of these variables. For large p, to very good approximation q is -1 . In fact, actually setting $q=-1$ changes the equations very little. To a very good approximation then, all the important dependence upon p occurs through the factor $k^{2/p}$. When written using the k variable the behavior of the modes is independent of β, but following all the factors through we find that the normalization of the power spectrum varies as $\beta^{-1/2}$.

We start tracking the mode numerically just after it is created by solving the mode equations approximately for small y. The approximate solution enables us to choose our vacuum, and fix the normalization of the mode function. There is a branch cut in the Lambert W function when its argument is $-e^{-1}$, but since we will start following the mode numerically from some small but positive y we may treat it as a removable singularity when we determine the approximate solution, since $W \sim -1+2\sqrt{-qy}+O(y)$ as $y \to 0^+$. In general the asymptotic solution is expressible in terms of Hankel functions. The explicit form is dependent on the exact mode equation since it is the terms having the factor $(1+W)^{-2}$ that dominate as $y \to 0$.

4.2.2 $(S_T^{(1)}, S_S^{(2)})$

Let us first assume that the actions for tensor and scalar fluctuations are $S_T^{(1)}$ and $S_S^{(2)}$, respectively. Since in this case $u_{\tilde{k}}$ and $p_{\tilde{k}}$ satisfy different equations, the tensor/scalar ratio differs from the standard quantum field theory prediction. Solving equation (4.17) near $y=0$ with the method of dominant balance [120] (previously employed in other studies [74, 91, 92]); we find

$$p_k(y) = D_+\, G(k,y)(1+\xi_1(k,y))(1+\xi_2(k,y)) + D_-\, G^*(k,y)(1+\xi_1^*(k,y))(1+\xi_2^*(k,y)) \tag{4.18}$$

where

$$\begin{aligned} G(k,y) &= y^{3/4}H_{-3/4}(2\sqrt{A_k y}) \\ \xi_1(k,y) &= -\frac{ek^{2/p}}{6}(-qy)^{3/2}(2-3\log y) \\ \xi_2(k,y) &= \frac{qy^2}{48}\left(3q(16-59ek^{2/p})+4i(2+ek^{2/p})(3i+7q^2\sqrt{2+ek^{2/p}})+42qek^{2/p}\log y\right) \end{aligned} \quad (4.19)$$

and

$$A_k = -\frac{q}{4}\left(2+ek^{2/p}\right). \quad (4.20)$$

The quantities D_- and D_+ are constrained by the Wronskian condition which implies:

$$|D_+|^2 - |D_-|^2 = -\eta_{\tilde{k}}\pi\sqrt{-q}e^{-3/2}. \quad (4.21)$$

In a near de-Sitter background if D_-/D_+ goes to zero when $\beta \to 0$, then the standard QFT result can be recovered [74]. However if D_-/D_+ is constant in this limit, one cannot recover the standard QFT result as $\beta \to 0$ [92]. We conjecture that the same type of reasoning is valid in a power-law background, and therefore we still have freedom in choosing the vacuum. We shall proceed with the choice $D_- = 0$, which corresponds to a Bunch-Davies-like vacuum.

This analytic solution can then be used as an initial condition to numerically integrate the differential equation from a point in the vicinity of the singular point until $\tau \approx 0$. At this point, we extract the late time amplitude of $u_{\tilde{k}}$. Figure 4.1 illustrates our results for the tensor amplitude for $p = 500$. This value of p is consistent with recent observations indicating that, the scalar spectral index, n_S, which for a power-law background in the absence of minimal length happens to be $1 - 2/p$, is greater than 0.95 [4]. We also assume that $\beta = 100^2$, which corresponds to a minimal length 100 times larger than Planck scale. This is a reasonable assumption in the framework of scenarios of large extra dimensions [61]. We see that the standard tensor power spectrum is modulated by oscillations, corresponding to a slow decrease in H as the universe expands. Increasing p (though still working with the rescaled variable k) does not change the qualitative features of the power spectrum, it only results in a shift of the $\log k$ axis to the left. Since k appears in the mode equation as $k^{2/p}$ and $\tilde{k}_{crit}$ scales as p^p this rescaling of the axis can have a significant effect on the spectrum when we compare power spectra for different p with a common set of units for $\tilde{k}$. As p increases, the wavelength of the oscillations increases [92]. Also, as k increases the frequency of the oscillation increases, though the amplitude decreases. As expected, when $k \to \infty$, we recover the standard field theory result. The left graph in Figure 4.1 illustrates the power spectrum for the modes that have a larger amplitude than those seeding the structure formation in Hubble patch. In the right graph we plot a window of k where the amplitudes are of the same order as the modes that are precursors to structure formation, $10^{-5} \leq P_S^{1/2} \leq 10^{-4}$ [92], or equivalently with $p = 500$, $10^{-7} \leq A_T \leq 10^{-6}$.

On the left in Figure 4.2 we plot the tensor spectral index for the range of wavelengths that lie outside our horizon. The existence of minimal length results in running from a blue to a red spectrum on such scales. This happens despite the fact that ϵ, the first slow roll parameter, does not have a local minimum. This is a counterexample to generic result of [123], which claims that if the spectral index is to run from a blue to a red spectrum there must be a local minimum in the slope of the potential. On the right in Figure 4.2, we graph n_T in the observable range of k. While we see the expected oscillations about the standard value, the large k behavior is now more difficult to understand. The increasing frequency of oscillations for the tensor power spectrum results in a growth of both the amplitude and frequency of the oscillations. Current measurements put a lower bound of 40 on p [4]. With such a weak lower bound, the frequency of the oscillations is very small. As any measurement of the spectral index is taken over a finite range of k, one would not be able to detect such oscillations. Taking small intervals centered at successively larger values of k we would find that the average value of n_T over the interval approaches the standard field theory result. To be able to detect these oscillations one needs extremely precise measurements. This oscillationary behavior of the spectral index is quite distinct from another model of trans-Planckian physics based on the non-commutativity of physical time and space coordinates [124]. For such such a model, it was shown that the spectral index runs from $n > 1$ on large scales to $n < 1$, where transition happens on scales close to H_0^{-1} [125–128].

Assuming that the action for scalar perturbations is described by $S_S^{(2)}$, the scalar modes satisfy eq.(2.35). Exploiting the dominant balance technique, we again extract the most singular terms in the mode equation in the vicinity of the irregular singular point with the approximate

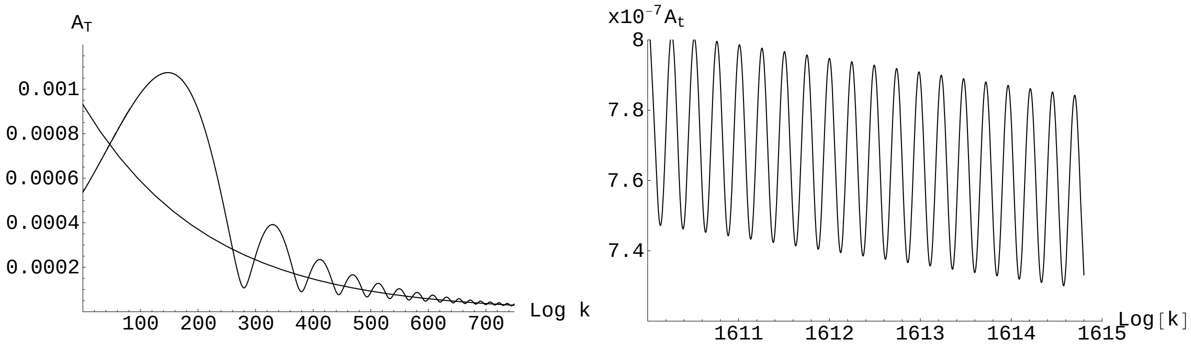


Figure 4.1: In these figures we assume that tensor perturbations are described by $S_T^{(1)}$. The left graph shows the dependence of A_T on $\log k$ for $p = 500$ and $\beta = 10^4$. $k = 1$ corresponds to $k_{crit}(500)$. The large modulation corresponds to physical scales much larger than our horizon. On the right, we plot the tensor amplitude with a window of k whose amplitudes are of the same order as the modes that originated the structure formation in our universe.

solution:

$$u_k(y) = C_+ \; F(k,y)(1+\epsilon_1(k,y))(1+\epsilon_2(k,y)) + C_- \; F^*(k,y)(1+\epsilon_1^*(k,y))(1+\epsilon_2^*(k,y)), \tag{4.22}$$

where

$$\begin{aligned} F(k,y) &= y^{3/4} H_{-3/4}(2\sqrt{B_k y}) \\ \epsilon_1(k,y) &= -\frac{e k^{2/p}}{6}(-qy)^{3/2}(2 - 3\log y) \\ \epsilon_2(k,y) &= -\frac{qy^2}{48}\left(48(1-q) + 28iqe^{3/2}k^{3/p} + 3ek^{2/p}(4 + 59q - 14q\log y\right). \end{aligned} \tag{4.23}$$

and B_k is given by

$$B_k = -\frac{q}{4} e k^{2/p}. \tag{4.24}$$

Again, we have a constraint on the integration constants C_+ and C_- from the Wronskian condition:

$$|C_+|^2 - |C_-|^2 = -\eta_{\tilde{k}} \pi \sqrt{-q} e^{-3/2}. \tag{4.25}$$

In the rest of the analysis, we choose $C_- = 0$, to have a Bunch-Davies-like vacuum. However, we emphasize that this choice is not unique and there is still a considerable amount of freedom in the choice of C_-. Specifically, inspired by our analysis in near-de-Sitter space [74], we conjecture that if

$$\lim_{\beta \to 0} \frac{C_-}{C_+} = 0, \tag{4.26}$$

we recover the standard result.

This approximate solution is again used to set the initial conditions for a numerical integration of the mode equation. The qualitative behavior of the scalar power spectrum is found to be similar to that found for the tensor modes. In Figure 4.3, we have plotted the tensor/scalar ratio for $p = 500$ and $\beta^{1/2} = 100$. The main effect of the different action for the scalar perturbations in this case appears to be a slight "compression" of the oscillations to smaller k. This compression causes the tensor/scalar ratio to oscillate in the observable window of k about the constant value we expect to find when there is no minimum length. This is depicted on the right graph in Figure 4.3. Knowing this ratio is important if one is to understand the contribution that each of these types of perturbation makes to the anisotropy of the CMBR [116]. In Fig.4.3 we see from the left graph that the ratio stays constant at a value less than the standard QFT result for small values of k that correspond to wavelengths outside our horizon. For increasing k it attenuates until reaches a minimum, after which it increases to a value much higher than the standard result. Thereafter it starts oscillating about the standard QFT predictions. The amplitude of the oscillations dies off as k increases. Notice also that A_T^2/A_S^2 is suppressed relative to n_T by an order of magnitude, implying in general a violation of the consistency relation (4.13).

This behavior for the tensor/scalar ratio was anticipated from our earlier calculations in near de-Sitter background [74], using $S_T^{(1)}$ for tensor and $S_S^{(2)}$ for scalar perturbations. In near de-Sitter space for small values of $\sigma \equiv \sqrt{\beta}H$, the ratio oscillates around the quantum field-theoretic prediction. For a fixed value of minimal length, this corresponds to small values of the Hubble parameter. As in a power-law background, short wavelength modes (large k) experience a slower rate of expansion, and so we expect oscillationary behavior in this region. Larger wavelengths are generated at the beginning of inflation, when the Hubble parameter (and in turn σ) were larger. For such wavelengths, this ratio is almost constant in a near de-Sitter background. In a power-law background for such values of k we see (Fig. 4.2) that the ratio is constant.

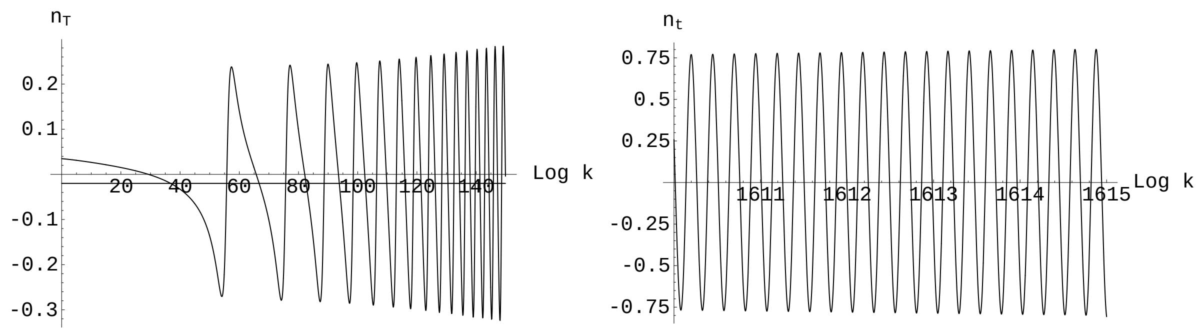


Figure 4.2: $S_T^{(1)}$ is assumed to be the action of tensor fluctuations. Left graph shows the dependence of n_T on $\log k$ for wavelengths far bigger than our horizon. β and p are assumed to be 10^4 and 500, respectively. The horizontal line represents the result when there is no minimum length. On the right we have graphed n_T in the observable range of k.

4.2.3 $(S_T^{(2)}, S_S^{(1)})$

In this section, we assume that tensor and scalar perturbations satisfy eqs.(2.35) and (2.31) respectively. In a power-law background, $z''/z = a''/a$ [118] and $z'/z = a'/a$ [74] so the equation describing scalar (tensor) perturbations is the same as the one describing tensor (scalar) perturbations in section 4.1. From equation (3.16), one can deduce that r/ϵ now is just the inverse of r/ϵ from the last section.

In Figure 4.4 we show the tensor spectral index derived from the action $S_T^{(2)}$ overlaid on that found from $S_T^{(1)}$. Again, we note that the removal of the third term in parentheses of (4.17) causes a compression of the oscillations to smaller values of k, but the magnitude of oscillations is still larger than those of A_T^2/A_S^2 by an order of magnitude, indicating in general that the consistency condition is still violated.

4.2.4 $(S_T^{(1)}, S_S^{(1)})$ and $(S_T^{(2)}, S_S^{(2)})$

For both of these cases the mode equations for tensor and scalar fluctuations are identical. We therefore recover the standard field theory result for the ratio A_T^2/A_S^2. The tensor spectral index

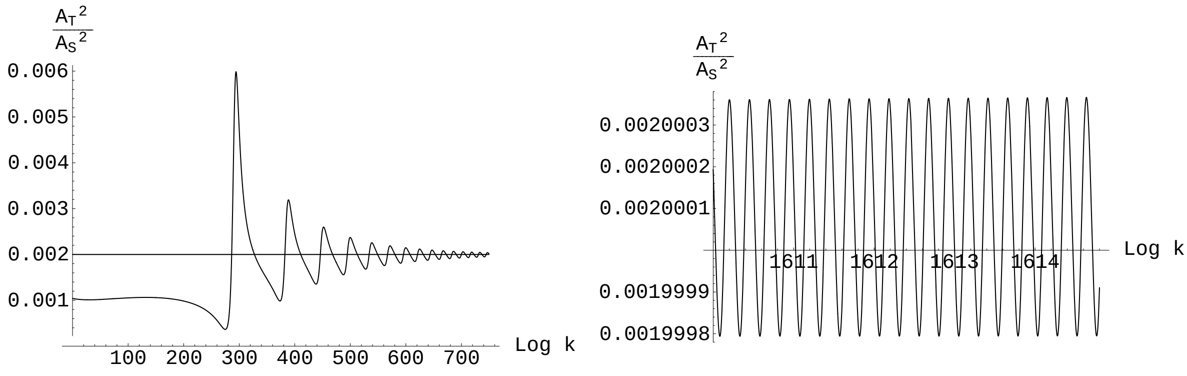


Figure 4.3: We assume $(S_T^{(1)}, S_S^{(2)})$ respectively describe tensor and scalar fluctuations. The left figure shows r for $p = 500$ and $\beta = 100^2$ for wavelengths far bigger than our horizon. On the right we plot the ratio of tensor to scalar fluctuations, in the observable window of k.

has already been presented in Figures 4.1 and 4.3. Again the oscillations about the standard result indicate there are violations of the consistency condition.

4.2.5 β dependence of fluctuations

Up until now we have been working with a rescaled variable that allows us to study the generic behavior for any β. Recall that the definition of our variable k involves β dependence of the form $k \sim \beta^{(p-1)/2}\tilde{k}$ and there is an overall factor of $\beta^{-1/2}$ in the normalization of the power spectrum. One may then qualitatively compare our results for different values of β by noting that, up to normalization, changing β just corresponds to a shifting of the $\log k$ axis. For example, a value of $\beta = 100^2$, causes the spectra of Figures 4.1, 4.2 and 4.3 to shift to the left relative to the $\beta = 500^2$ results. For a given $\tilde{k}$ the net result is that the size of the fluctuations about the standard field theory result are suppressed.

To be more exact, we may parameterize the tensor power spectrum as $A_T^2 = A_{T,\text{qft}}^2(1 + \delta A_T(\beta, k))$, where $A_{T,\text{qft}}$ is the standard quantum field theory result for the tensor power spectrum. In Figure 4.4, for action $S_T^{(2)}$, we plot $\delta A_T(\beta, k)$ for $\beta = 500^2$ and $\beta = 100^2$ written in units where $k = 1$ corresponds to $\tilde{k} = e^{p/2}/\tilde{k}_{crit}$ for $\beta = 100^2$. We find that the size of the oscillations and their wavelength both appear to vary as $\beta^{1/2}$, the only dimensionful parameter

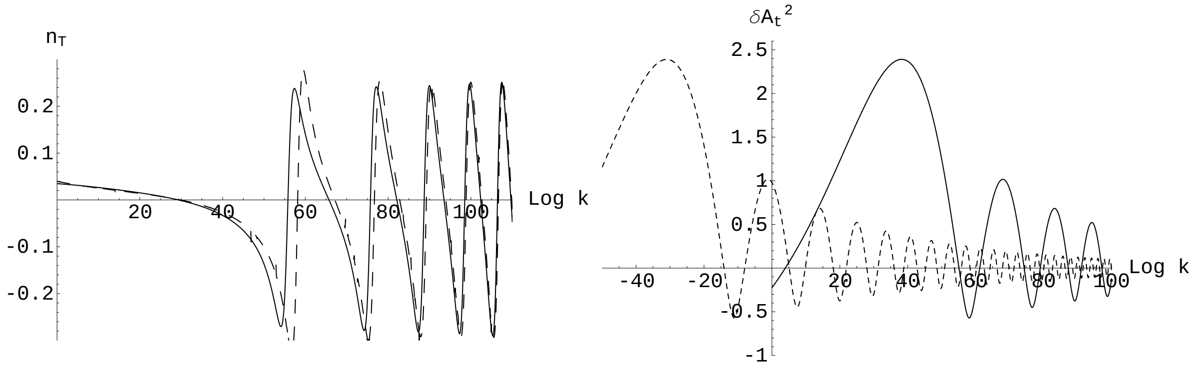


Figure 4.4: In the figure on the left, we overlay the tensor index for the power spectra obtained from $S_T^{(1)}$ (solid) and $S_T^{(2)}$ (dashed). On the right, we have graphed δA_T, modifications to the standard quantum field theory prediction due to the presence of minimal length, for $\beta = 500^2$ (solid line) and $\beta = 100^2$ (dashed). Here, we have assumed that $S_T^{(2)}$ describes the action for tensor perturbations and $p = 100$.

in the problem.

4.3 Observability of Trans-Planckian Signatures

As argued above, the effect of of trans-Planckian physics could be proportional to the ratio of minimal length over Hubble radius during inflation, $\sqrt{\beta}H$. Identifying $1/\sqrt{\beta}$ with Λ, equivalently one can express the magnitude of the effect as H/Λ. It is interesting to know to what extent the minimal length can be constrained within the given observational dataset. As there are fundamental limits from cosmic variance on how accurately the primordial spectrum can be measured, one can argue that the effect should be observable with the proviso that it is larger than the limits of cosmic variance. The lower limits of cosmic variance are set at small scales. The intrinsic uncertainty in knowing the cosmic multipoles, C_ℓs, could be expressed as:

$$\frac{\triangle C_\ell}{C_\ell} = \sqrt{\frac{2}{2\ell+1}}. \tag{4.27}$$

CMB probes up to $\ell_{max} = 2000$ and so the lowest value of cosmic variance is around 0.022. One may conclude that the effect of trans-Planckian physics should be larger than the above limit, in order to be observable for the CMB experiments. However, one should emphasize that the above lower bound should be used cautiously as the signal would involve all of the above C_ℓs and the ripples in the power spectrum are convolved with the intrinsic structure in the spectrum. Easther et. al. [130] tried to answer this question more accurately, using various data analysis techniques and they concluded that the the amplitude of perturbations is directly correlated with the detectability of any trans-Planckian signature. The amplitude of tensor perturbations is proportional to $H/m_{\rm Pl}$, which should be less than 10^{-4} to avoid smearing the CMB temperature beyond its limit of anisotropy. For $r \approx 0.15$, they concluded that H/Λ should be greater than 0.004 to leave detectable modulations. If $r \approx 0.00013$, H/Λ bigger than 0.02 could be recovered.

As stated in section 3.4, the effect of trans-Planckian signatures on $\triangle r/r$ is proportional to $(H/\Lambda)^2$. Verde et. al., [131], have considered all experiments that try to measure the CMB polarization, including the space and ground-based ones, and found that foreground contamination and residuals from foreground subtraction are the limiting factors in detecting a primordial gravity wave signal. An ideal experiment, with no lensing and no foreground, could have reached the sensitivity of $\triangle r/r \approx 10^{-3}$ at 1σ error. This means that an ideal experiment could detect possible trans-Planckian modulations in r, if $H/\Lambda > \text{few} \times 0.01$. However, lensing and foreground, reduces the sensitivity of $\triangle r/r$ to 0.1 which means that any trans-Planckian signature is observable in the ratio, if $H/\Lambda \geq 0.3$.

4.4 Conclusion

In this chapter, we investigated the consistency relation between tensor and scalar fluctuations in the framework of power-law inflation with a cut-off due to minimal length. Since the method of implementing the minimal length hypothesis (4.1) depends on the action one starts from, there is a choice amongst an infinite number of actions that in the absence of minimal length are otherwise equivalent. However there are only two physically reasonable cases for both scalar and tensor perturbations: that of minimality (add no boundary terms to the original action) and that of simplicity (add terms such that the modified action most closely resembles the action of a free massive scalar field in a Minkowski background). This yields four distinct cases and we investigated each for a choice background consistent with recent observations that constrain the magnitude of the scalar spectral index.

Confining our attention to these cases, we found that Planck scale physics can considerably modify the consistency condition (4.13) and can lead to the running of spectral indices regardless of which action one employs. Depending on the choice of action for tensor and scalar perturbations, we may find that the tensor/scalar ratio oscillates (in the observable window of k) about the constant value we expect to find in the absence of minimal length. However the magnitude of the modifications depends upon the choice of action. Constraining this choice – both observationally and theoretically – remains a challenge for future studies.

Chapter 5

Generation of Cosmological Seed Magnetic Fields from Inflation with Cutoff

5.1 Introduction

Cosmic magnetic fields are ubiquitous at all large intragalactic scales. It is a well-known observational fact that our galaxy and many other spiral galaxies are endowed with coherent magnetic fields of μG (microgauss) strength [132–136], having approximately the same energy density as the cosmic microwave background radiation (CMBR). There is also evidence for larger magnetic fields of similar strength within clusters [137, 138]. The presence of magnetic fields at larger scales has also been confirmed [139, 140]. These magnetic fields play an important role in various astrophysical processes, such as the confinement of cosmic rays and the transfer of angular momentum away from protostellar clouds so that they can collapse and become stars. Magnetic fields are also present in the intracluster gas of rich clusters of galaxies, in quasistellar objects (QSO's) and in active galactic nuclei. They may influence the formation process of large-scale structure [141–143].

It is widely believed that galactic magnetic fields are amplified and sustained by a dynamo mechanism [134–136, 144, 146], in which the cyclonic turbulent motion of ionized gas combined with the differential rotation of the galaxy exponentially amplifies a "seed" magnetic field. This continues until the backreaction of the motion of the plasma offsets the growth of the field, stabilizing it to dynamical equipartition strength. However, while the dynamo mechanism provides an amplification mechanism, it does not explain the origin of galactic magnetic fields, and requires a "coherent" seed magnetic field for it to be effective. Indeed, it has been shown that seed magnetic fields that are too incoherent may undermine the action of the dynamo [148]. Most dynamo scenarios require a minimum coherence length equal to the dimension of the largest turbulent eddy, usually around ~ 100 pc. If the mechanism has functioned over the whole age of the galaxy($\sim$ 10G yr) a seed field of 10^{-19}G is required. If recent observations are correct and the universe is dominated by a dark-energy density component [18, 150–152], then galaxies are older than previously thought and the seed magnetic field may be as low as $B_{seed} \sim 10^{-30}$G [153]. This happens since the presence of dark energy increases the age of the universe significantly, but does not change the time of galaxy formation [153]. Dynamo mechanism amplifies the seed magnetic field exponentially, $B(t) = B_{seed} \exp\left[\Gamma(t - t_{gf})\right]$, where $0.3 < \Gamma^{-1} < 0.8$ [Gyr]. Presence of dark energy changes the age of the universe by a 7 Gyr. For $\Gamma^{-1} \approx 0.3$ Gyr, the required seed decreases by an eleven orders of magnitude.

A contrasting view is that the primeval magnetic flux trapped in the gas that collapsed to form the galaxy is responsible for the existence of galactic magnetic fields. This hypothesis also requires the existence of a seed magnetic field, one that is as great as the field observed today [154, 155]. Several scenarios have been suggested for creation of the required seed magnetic field, the most important of which involve battery [148] or vorticity [156–158] effects. The battery mechanism requires a large-scale misalignment of density and pressure gradients usually related to active galactic nuclei (AGN) or starburst activity. Therefore, it is difficult to realize in the majority of galaxies. The vorticity mechanism is based on the relative motions of photons and electrons induced by vorticity that was present before decoupling. Of course this mechanism assumes the existence of primeval vorticity. In addition, large-scale vortical motions can be effective only if ionization of the plasma is considerable, which does not occur at the galaxy-formation epoch.

Throughout most of the history of the universe the average time τ between particle interactions has been much smaller than the expansion time scale, $\tau \ll t_{Hubble}$. Consequently the universe has been a good conductor [159], and any primeval cosmic magnetic field would have evolved in a manner that preserved magnetic flux: $Ba^2 \sim$ constant, where a is the scale factor. Hence the dimensionless ratio $r = B^2/(8\pi\rho_\gamma)$, where ρ_γ is the radiation energy density, is almost constant and provides a convenient measure of magnetic field strength. If there had been a pregalactic cosmic magnetic field that collapsed with the gas that formed the galaxy, its strength must have increased as $[\rho_{gal}/\rho_{tot}(t)]^{2/3}$, where $\rho_{tot}(t)$ is the average cosmic mass density at time t. As $\rho_{tot} \propto a^{-3}$ and $\rho_{gal}/\rho_{tot} = 10^6$ today ($t_0 = 0$), it follows that the strength of the magnetic field at the time of formation $t_{form.}$ must have been $10^4[a(t_{form.})/a(t_0)]^2 B_{cosmic}$ or $B_{gal.} \simeq 3r^{1/2}10^{-2}$G. This yields $r \simeq 10^{-34}$ for initiating the galactic dynamo, or alternatively $r \simeq 10^{-8}$ for seeding the galactic magnetic field itself while avoiding the necessity of a galactic dynamo. If the existence of dark energy in the universe is confirmed, the minimum r required to seed the dynamo mechanism reduces to 10^{-56}.

Inflation offers the hope of furnishing a mechanism for kinematically and dynamically producing the seed for cosmic magnetic fields. It provides the kinematic means for producing long-wavelength effects at very early times through microphysical processes operating on scales less than the Hubble radius. Since an electromagnetic wave with $\lambda_{phys} \geq H^{-1}$ has the appearance of static $\mathbf{E}$ and $\mathbf{B}$ fields, very long wavelength photons ($\lambda_{phys} \gg H^{-1}$) can lead to large-scale magnetic fields (which then become supported by currents). Of course the electric field generated during the inflationary stage not only is not amplified, but is actually damped down due to the large conductivity of the primeval plasma. Another reason inflation is considered to be a prime candidate for field amplification is the fact that during inflation the universe is devoid of charged particles. Hence, the magnetic flux is not necessarily conserved and r can increase. Furthermore, inflation can superadiabatically amplify the energy density ($\simeq k d\rho/dk$) of the minimally coupled field [159]. Then the energy density decays as a^{-2}, rather than the usual result a^{-4} (“adiabatic result”).

However the conformal flatness of the Robertston-Walker metric prevents the background gravitational field from producing particles, provided the underlying matter theory is conformally invariant. A pure U(1) gauge theory with the standard Lagrangian $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ is conformally invariant, from which it follows that $\rho_B \propto B^2$ always decreases as a^{-4}. During the inflationary epoch, the total energy density in the universe is dominated by vacuum energy and therefore the energy density in any magnetic field during inflation is significantly suppressed.

In fact it can be shown that $r = 10^{-104}\lambda_{Mpc}$ [159], where $\lambda_{Mpc} \equiv \lambda/1Mpc$.

Several proposals have been given to break the conformal invariance of the theory: (i) coupling the electromagnetic field to a non-conformally-covariant charged field [161], (ii) coupling the electromagnetic field to gravity via either gauge non-invariant terms such as $RA_\mu A^\mu$, $R_{\mu\nu}A^\mu A^\nu$ or gauge invariant ones like $R_{\mu\nu\lambda\kappa}F^{\mu\nu}F^{\lambda\kappa}/m^2$, $R_{\mu\nu}F^{\mu\kappa}F_\kappa{}^\nu/m^2$ or $RF^{\mu\nu}F_{\mu\nu}/m^2$ [159], (iii) invoking effects due to the quantum conformal anomaly [161, 162], (iv) creating primordial magnetic fields at either the QCD transition epoch [163] or the electroweak transition [164], (v) breaking conformal invariance via nonzero vacuum expectation values of flat directions in minimally supersymmetric standard models (MSSM) [165].

Attempts to realize the first possibility were carried out by coupling the electromagnetic field to the scalar field Φ responsible for inflation via a term $\propto e^\Phi F^{\mu\nu}F_{\mu\nu}$ [160]. This investigation showed that in the exponential potential for the inflaton

$$V(\Phi) = (\frac{6-q}{3})\frac{16\pi}{{m_{Pl}}^2}\rho^0_{b\Phi}e^{(-\sqrt{\frac{q}{2}}(\Phi-\Phi_0))}, \tag{5.1}$$

it is possible to generate an intergalactic magnetic field whose present strength (depending on values of parameters of the model) lies between 10^{-65} to 10^{-10} on a scale of $1/1000$ that of the Hubble scale. In (5.1) m_{Pl} is the planck mass, Φ^0 is the value of the scalar field and $\rho^0_{b\Phi}$ is the homogeneous scalar field energy density when the scale factor is a_0. Thus in this scenario one can have the desired galactic magnetic field by resorting to the dynamo mechanism.

Considering next possibility (ii), if we add gauge non-invariant terms to the action, the $U(1)$ gauge invariance will be broken. To avoid the phenomenological disasters this can cause one can endow the photon with a mass squared of the order of H^2 (well below present limits of detectability). It has been shown [159] that such a term can create primeval fields with strength as large as $r \sim 10^{-8}$.

Gauge invariant modifications to the action have much better theoretical motivation. For example, all RF^2 terms can be obtained by calculating the effective Lagrangian for QED in curved space-time to one loop order [166]. At early times, when $R^{1/2} \sim H \sim \rho^{1/2}_{tot}/m_{pl} \gg 10^{-11}m_{pl}$, these terms govern the behavior of the electromagnetic field. However at later times, when $R^{1/2} \ll 10^{-11}m_{pl}$, they are negligible compared to the standard $-\frac{1}{4}F^2$ term. In a power-law inflationary background the amplitude of large-scale fields is not large enough to be astrophysically interesting [159].

The third possibility has proven to be promising for gauge theories with large groups and a greater number of bosons than fermions. In such theories, it has been shown that this mechanism for breaking conformal invariance in quantum electrodynamics can create a sufficient amount of primordial magnetic field [161]. However, in the simplest version of the grand unified $SU(5)$ model with three generations of fermions the magnetic field produced is below the requirement of the dynamo mechanism.

Phase transitions at different cosmological epochs (grand unification, the electroweak transition [164] or the quark confinement epoch [163]) have also been considered. However, since the generating mechanisms are causal, the coherence of the created magnetic field cannot be larger than the particle horizon at the time of the phase transition. Because all the above transitions occurred very early in the universe's history, the comoving size of the horizon is rather small. The best case is the QCD transition, for which the horizon corresponds to ~ 1 a.u. Consequently the real magnetic fields generated lack sufficient coherence.

The MSSM flat directions, made up of gauge invariant combinations of squarks and sleptons, acquire non-vanishing vacuum expectation values (vev) during inflation. These flat directions endow the standard model gauge fields with mass and break the conformal invariance. The quantum fluctuations of these flat directions, in contrast to the their classical vevs, induce fluctuations in the gauge degrees of the freedom that cannot be gauged away. The gauge field fluctuations that are stretched outside the horizon during inflation, provide us with a seed (hyper)magnetic field after they re-enter the horizon. They give rise to $U(1)_{em}$ magnetic field with strength of 10^{-30} G, as required by the dynamo mechanism [165].

Here we consider an alternative mechanism that is based on a hypothesis of a minimal fundamental length scale. Minimal length breaks conformal invariance and so it might be expected that primordial magnetic fields can be produced during inflation. One suggestion [71] for implementing minimal length into the inflationary scenario in the context of trans-Planckian physics [85] is based on the hypothesis of a generalized uncertainty principle:

$$\Delta x \Delta p \geq \frac{1}{2}\left(1 + \beta\,(\Delta p)^2\right), \tag{5.2}$$

where $\sqrt{\beta}$ is the ultraviolet cutoff on the order of the Planck or string length. In this section we employ this formalism to implement minimal length into the action of electrodynamics. This translates into a UV cutoff which, once implemented, has the sole effect of modifying the evolution of the electromagnetic field. As we will demonstrate, the formalism is not able to create a squeezing effect for the electromagnetic field. Therefore the energy density of the electromagnetic field attenuates adiabatically, $\rho_B \propto a^{-4}$.

However, it has recently been shown [72] that terms in the action that are total time derivatives are not invariant under the influence of the minimal length hypothesis [71]. Consequently such terms contribute to the equations of motion of the matter fields. We consider in this thesis an example of a total time derivative that, under the influence of the UV cutoff, causes the photon to gain a large negative mass during inflation. This effect goes to zero at the end of inflation and so such a mass is undetectable today. We shall show that this approach is successful in providing the dynamo mechanism with sufficient primordial seed magnetic field. Even in absence of the dynamo mechanism, one can adjust a free parameter in the action to account for the observed magnetic field of galaxies today.

5.2 Cutoff Breaking of Conformal Invariance

The existence of a preferred minimal length breaks the conformal invariance of the background geometry. Here we will examine the effect of this conformal breaking on the evolution of electromagnetic fields.

We introduce a fundamental length (i.e. the presence of a cut-off) in the inflationary scenario via generalization of the quantum mechanical commutation relation [71]

$$[\mathbf{X}, \mathbf{P}] = i \longrightarrow [\mathbf{X}, \mathbf{P}] = i \left(f\left(\beta\right) \mathbf{1} + g(\beta) \mathbf{P}^i \mathbf{P}^j \right) \tag{5.3}$$

where $f\left(\beta\right), g\left(\beta\right)$ are functions such that $f\left(0\right) = 1$ and $g\left(0\right) = 0$; their actual form is determined below. This generalization significantly modifies trans-Planckian physics, whose effects are then manifest in the CMBR. Here we employ the above formalism to find the effect this cutoff has on the evolution of magnetic fields.

We begin with the action of electromagnetism in an expanding curved background

$$S = -\int \frac{1}{4}\sqrt{-g}g^{\mu\nu}g^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta}d^3\mathbf{y}d\tau \tag{5.4}$$

where the y^i's are comoving spatial coordinates related to the proper ones by $x^i = a(\tau)y^i$ and τ is the conformal time. Assuming that the background is flat Friedmann Robertson Walker, with the metric

$$ds^2 = \begin{cases} -dt^2 + a^2(t)\sum_{i=1}^{i=3} dy^{i^2} \\ a^2(\tau)(-d\tau^2 + \sum_{i=1}^{i=3} dy^{i^2}), \end{cases} \tag{5.5}$$

one can write down the action in the following form:

$$S = \frac{1}{4}\int [2F_{oi}F_{oi} - F_{ik}F_{ik}]d^3\mathbf{y}d\tau \tag{5.6}$$

Roman indices i and k run from 1 to 3 and repeated indices are summed over. The disappearance of the scale factor $a(\tau)$ is a consequence of the conformal invariance of electromagnetism. By imposing the radiation gauge $A_0 = \partial_i A_i = 0$, the above action can be rewritten in the following form

$$S = \frac{1}{2}\int [(\partial_0\mathbf{A})^2 - (\nabla\times\mathbf{A})^2]d^3\mathbf{y}d\tau. \tag{5.7}$$

in terms of the electromagnetic potential A_i. This action is the familiar electrodynamic action,

$$S = \frac{1}{2}\int (\mathbf{E}^2 - \mathbf{B}^2)d^3\mathbf{y}d\tau, \tag{5.8}$$

written in radiation gauge.

The most general form of the modified commuation relation (1.30) that breaks Lorentz invariance (see also [167]) while preserving the translational and rotational symmetry is given by (2.10). We still assume that $[\mathbf{X}^i, \mathbf{X}^j] = [\mathbf{P}^i, \mathbf{P}^j] = 0$. To impose this modified commuation relation, we rewrite the action using proper spatial coordinates $x = a(\tau)y$ in the form:

$$S = \frac{1}{2}\int \frac{d\tau d^3\mathbf{x}}{2a^3}\left\{\left(\left[\partial_\tau + \frac{a'}{a}\partial_{x^i}x^i - \frac{3a'}{a}\right]\mathbf{A}\right)^2 - a^2\left(\nabla\times\mathbf{A}\right)^2\right\} \tag{5.9}$$

We can identify $-i\partial_{x^i}$ as the momentum operator, $\mathbf{P}^i$, and x^i as the position operator, $\mathbf{X}^i$. We can cast the action (5.9) to a simpler form:

$$S = \int \frac{d\tau}{2a^3} \left\{ \left(\mathbf{A}, B^\dagger(\tau) B(\tau) \mathbf{A}\right) + a^2 \left(\mathbf{P} \times \mathbf{A}\right)^2 \right\}, \tag{5.10}$$

where we have consolidated $(\partial_\tau + i\frac{a'}{a}\sum_{i=1}^{3} \mathbf{P}^i \mathbf{X}^i - 3\frac{a'}{a})$ into a new operator $B(\tau)$. Since $\partial_i A^i = 0$, it means that $(\mathbf{P} \times \mathbf{A})^2 = \mathbf{P}^2 \mathbf{A}^2$. A suitable vectorial Hilbert space representation of the new commutation relation can be defined by using auxiliary variables ρ^l:

$$\mathbf{X}^l \mathbf{A}(\rho) = i\partial_{\rho^l} \mathbf{A}(\rho) \tag{5.11}$$

$$\mathbf{P}^l \mathbf{A}(\rho) = \frac{\rho^l}{1 - \beta\rho^2} \mathbf{A}(\rho) \tag{5.12}$$

$$(A_i(\rho), A'_j(\rho)) = \int_{\rho^2 < \beta^{-1}} d^3\rho A_i^*(\rho) A'_j(\rho) \tag{5.13}$$

Ultimately the action takes the following form:

$$S = \int d\tau \int_{\rho^2 < \beta^{-1}} d^3\rho \frac{1}{2a^3} \left\{ |(\partial_\tau - \frac{a'}{a} \frac{\rho^i}{1 - \beta\rho^2} \partial_{\rho^i} - \frac{3a'}{a}) \mathbf{A}|^2 - \frac{a^2 \rho^2 |\mathbf{A}|^2}{(1 - \beta\rho^2)^2} \right\} \tag{5.14}$$

As before, the presence of ρ derivatives means that the ρ modes are coupled. However we can define variables $(\tilde{\tau}, \tilde{k})$ as in (2.16). The $\tilde{k}$ modes decouple because of relation (2.17)

We will use the common index notation $\bar{A}_{\tilde{k}}$ for those decoupling modes. The $\tilde{k}$ modes coincide with the usual comoving modes on large scales, i.e., only for small ρ^2. This means that the comoving k modes that are obtained by scaling, $k^i = ap^i$, decouple at large distances and couple at small distances. The action now takes the form

$$S = \int d\tilde{\tau} \int_{\tilde{k} < a^2/e\beta} d^3\tilde{k} \mathcal{L} \tag{5.15}$$

where

$$\mathcal{L} = \frac{1}{2}\nu \left\{ \left| \left(\partial_\tau - 3\frac{a'}{a}\right) \bar{\mathbf{A}}_{\tilde{k}} \right|^2 - \mu \left|\bar{\mathbf{A}}_{\tilde{k}}\right|^2 \right\}$$

μ is as in (2.22) and ν is defined as κ/a^6 where κ is defined in (2.23). It is convenient to express these functions in terms of the Lambert W function

$$\mu(\tau, \tilde{k}) = -\frac{a^2}{\beta} \frac{W(\zeta)}{(1 + W(\zeta))^2} \tag{5.16}$$

$$\nu(\tau,\tilde{k}) = \frac{e^{-3W(\zeta)/2}}{a^6(1+W(\zeta))} \tag{5.17}$$

where $\zeta = -\beta\tilde{k}^2/a^2$. The equation of motion for the action (1.47) is:

$$\bar{\mathbf{A}}''_{\tilde{k}} + \frac{\nu'}{\nu}\bar{\mathbf{A}}'_{\tilde{k}} + \left(\mu - 3\frac{\nu'}{\nu}(\frac{a'}{a}) - 3(\frac{a'}{a})' - 9(\frac{a'}{a})^2\right)\bar{\mathbf{A}}_{\tilde{k}} = 0 \tag{5.18}$$

The operations of Fourier transforming and of scaling from proper position coordinates do not commute [71]. Hence the field variable $\bar{\mathbf{A}}_{\tilde{k}}$ is different from that commonly employed in the literature, $\mathbf{A}_{\tilde{k}}$ by a factor of a^3:

$$\bar{\mathbf{A}}_{\tilde{k}} = a^3\mathbf{A}_{\tilde{k}} \tag{5.19}$$

Taking into account eq.(5.19), we obtain

$$\mathbf{A}''_{\tilde{k}} + \frac{\kappa'}{\kappa}\mathbf{A}'_{\tilde{k}} + \mu\mathbf{A}_{\tilde{k}} = 0; \tag{5.20}$$

as the equation of motion for scalar perturbations in presence of a minimal length cutoff.

The solutions to equation (5.20) are constrained by the Wronskian condition which follows from the canonical commutation relation between $\mathbf{A}_{\tilde{k}}$ and its conjugate momentum, $\mathbf{\Pi}_{\tilde{k}} = \kappa\mathbf{A}'_{\tilde{k}}$

$$[A^i_{\tilde{k}}, \Pi^j_{\tilde{k}'}] = i\delta^{ij}\delta^3(\tilde{k}-\tilde{k}') \tag{5.21}$$

or equivalently

$$A^i_{\tilde{k}}A'^{j*}_{\tilde{k}} - A^{i*}_{\tilde{k}}A'^{j}_{\tilde{k}} = i\kappa^{-1}\delta^{ij} \tag{5.22}$$

During the de Sitter phase, $a = -1/H\tau$ and so $\zeta = -\beta H^2\tilde{k}^2\tau^2$. $\sigma = \sqrt{\beta}H$ is the ratio of the cutoff to the Hubble parameter during inflation. The factors μ and κ'/κ have the following expansions in the limit in which the mode is outside the horizon ($\tilde{k}\tau \ll 1$):

$$\mu(\tau,\tilde{k}) = \tilde{k}^2 + 3\beta^2H^2\tilde{k}^4\tau^2 + \cdots \tag{5.23}$$

$$\frac{\kappa'(\tau,\tilde{k})}{\kappa(\tau,\tilde{k})} = 5\beta H^2\tilde{k}^2\tau + 12\beta^2H^4\tilde{k}^4\tau^3 + \cdots \tag{5.24}$$

In that regime, the modes satisfy the following equation:

$$\mathbf{A}''_{\tilde{k}} + \tilde{k}^2\mathbf{A}_{\tilde{k}} = 0 \tag{5.25}$$

Thus $\mathbf{A}_{\tilde{k}} \propto e^{i\tilde{k}\tau}$ and $\mathbf{B}_i = \epsilon_{ilm}F_{lm}/a^2$ where $F_{lm} = \partial_l A_m - \partial_m A_l$. As a result ρ_B varies like a^{-4}, which is the adiabatic result.

Superficially this mechanism is unable to amplify the cosmic magnetic fields. However it has been shown that this method of implementing the cut-off in the action has an ambiguity: total time derivatives no longer reduce to pure boundary terms [72]. In continuous space-time, the presence of such a boundary term does not affect the evolution of the electromagnetic potential. However the operator ∂_τ that acts on the electromagnetic potential inside the total time derivative transforms to $B(\tau)$ in the proper spatial coordinates. Since the modification of the commutation relation between X^i and P^j affects how this operator acts upon A_μ, this procedure of implementing minimal length will not keep such total time derivatives invariant.

Fortunately another option is available. It can be shown that any boundary term in physical space transforms to a non-boundary term in momentum space in the following manner:

$$\int (f(a,A))' d^3\mathbf{y} d\tau \rightarrow \int \kappa(\tau,\tilde{k})(f(a,A_{\tilde{k}}))' d^3\tilde{k} d\tau \tag{5.26}$$

and so it is possible that they may contribute to the equation of motion in such a way that the above behavior of the magnetic field is modified. Since we wish to break the conformal invariance, we endow the photon with a mass term [168–171]. This implies that $f(a,A) = g(a,a',\ldots)A^\mu A_\mu$. However we also do not want to modify the behavior of the photon in the well-understood part of the history of the universe, namely the radiation and matter dominated eras. Since during these eras the scale factor respectively behaves as τ and τ^2, we assume that $g(a) \propto a'''$. So far the proposed boundary term adds to Eq.(5.25) a term $\propto -\frac{a'''\kappa'}{a^2\kappa}\mathbf{A}_{\tilde{k}}$ which looks like $\mathbf{A}_{\tilde{k}}/\tau$ as the mode crosses outside the horizon. Recalling the equation of motion for scalar fluctuations, u_k, (1.45), the term that creates the amplification is $z''u_k/z$ which behaves like u_k/τ^2 as the mode is far outside the horizon. Hence we multiply the previous term with another factor of a to produce the desired behavior.

Summarizing, the proposed boundary term is:

$$\triangle S = \frac{1}{M_1^2}\int (A^\mu A_\mu a''' a)' d^3\mathbf{y} d\tau \tag{5.27}$$

where μ runs over space-time indices $0\cdots 3$ and M_1 is an arbitrary constant with dimensions of mass whose presence keeps $\triangle S$ dimensionless. We can write this covariantly as

$$\triangle S = \frac{1}{M_1^2} \int \nabla_\alpha (\Xi \zeta^\alpha) \sqrt{g} d^3 \mathbf{y} d\tau$$

where Ξ is the scalar function

$$\Xi = \frac{1}{3} (A^\mu A_\mu) \left(\nabla^2 K - 2 \nabla^\mu \nabla^\nu K_{\mu\nu} \right) \sqrt{\zeta \cdot \zeta}$$

where $K_{\mu\nu}$ is the extrinsic curvature of the boundary surface whose normal is $n_\mu = \left(a(\tau), \vec{0} \right)$ and $\zeta^\alpha = \left(1, \vec{0} \right)$ is the conformal Killing vector of the spacetime. Also, one can express a''' in the following form:

$$a''' = a^4 H^3 (1 - 2q + j) \tag{5.28}$$

where q and j are respectively the deceleration and jerk parameters defined as [172]

$$q = -\frac{\ddot{a}}{aH^2}, \tag{5.29}$$

$$j = \frac{\dddot{a}}{aH^3}, \tag{5.30}$$

where dot denotes differentiation with respect to the physical time. The presence of such a term modifies the propagator of the photon only during inflation. The vertices and propagator of the electron do not get modified at any time. Therefore, the amplitude for the diagrams that describe photon splitting [173–175], $\gamma \to n\gamma$, remain intact and hence abide with the current bounds that exist on photon splitting [176].

The equation of motion for $\mathbf{A}_{\tilde{k}}$ derived from the variation of the cutoff-modified action $S + \triangle S$ is

$$\mathbf{A}''_{\tilde{k}} + (\tilde{k}^2 - \frac{1}{M_1^2} \frac{a''' \kappa'}{a\kappa}) \mathbf{A}_{\tilde{k}} = 0, \tag{5.31}$$

During the de Sitter expansion, $a = -1/H\tau$. For modes outside the horizon, $\tilde{k}\tau \ll 1$, and eq.(5.31) reduces to

$$\mathbf{A}''_{\tilde{k}} - \frac{n}{\tau^2} \mathbf{A}_{\tilde{k}} = 0 \tag{5.32}$$

where $n = 30\sigma^2 \tilde{k}^2 / M_1^2$. In this limit we have $|\mathbf{A}_{\tilde{k}}| \propto \tau^{m_\pm}$ where $m_\pm = \frac{1}{2}(1 \pm \sqrt{1 + 4n})$. Here $\sigma = \sqrt{\beta}/H^{-1}$, where $\sqrt{\beta}$ is the minimal length associated with the ultraviolet cutoff and

H is the Hubble constant during inflation. The fastest growing solution during the de Sitter phase is proportional to τ^p or equivalently a^{-p}, where we set $p = m_-$. Note that for $p = -1$ ($n = 2$), $|\mathbf{A}_{\tilde{k}}|$ varies like a and $\rho_B \propto a^{-2}$, which is the superadiabatic result. The evolution of electromagnetic waves during reheating and the matter-dominated (MD) era is described by the same equation as (2.18) with $a \propto \tau^2$, whereas in the radiation dominated (RD) epoch $a \propto \tau$. In these three epochs the effective mass of the photon vanishes and $\rho_B \propto a^{-4}$. During RD, MD, and reheating, the electromagnetic field behaves as it does in absence of the cut-off.

The ratio of the energy stored in the k-th mode of quantum fluctuations, $\rho_B(k)$, to the total energy density of the universe, ρ_{tot}, at first horizon-crossing, $a = a_1$, is approximately equal to $[M/m_{Pl}]^4$. Here M^4 is the vacuum energy density during inflation. Such a quantum fluctuation will be excited during the de Sitter expansion, ref.[178], and can be treated as a classical fluctuation in the electromagnetic field when it crosses outside the horizon. After horizon-crossing $\rho_B(k)$ varies as $a^{-2(p+2)}$ while the total energy density of the universe remains constant, $\rho_{tot} \propto M^4$. Since the extra term added to the equation of motion is zero during reheating, in the MD and RD epochs the stored energy density in the k-th mode magnetic fluctuation attenuates adiabatically, $\rho_B \propto a^{-4}$. In reheating and the MD era, the energy density of the universe decreases as a^{-3} whereas in the RD epoch the total energy density of the universe diminishes as a^{-4}. Therefore the invariant ratio, $\rho_B(k)/\rho_\gamma$ on the scale λ is:

$$r \simeq e^{-2N(\lambda)(p+2)} \left[\frac{M}{m_{Pl}}\right]^{8/3} \left[\frac{T_{RH}}{m_{Pl}}\right]^{4/3}, \tag{5.33}$$

where $N(\lambda)$ is the number of e-folds the universe expands between the first horizon crossing of the comoving scale λ and the end of inflation. It is given by the following equation [178]:

$$N(\lambda) = 45 + \ln \lambda_{Mpc} + \frac{2}{3}\ln(M_{14}) + \frac{1}{3}\ln(T_{10}) \tag{5.34}$$

and $M = M_{14} 10^{14}$ GeV, $T_{RH} = T_{10} 10^{10}$ GeV. Plugging this equation back into Eq.(5.33), one obtains

$$r \simeq (7 \times 10^{25})^{-2(p+2)} \left[\frac{M}{m_{Pl}}\right]^{-4p/3} \left[\frac{T_{RH}}{m_{Pl}}\right]^{-2p/3} \lambda_{Mpc}^{-2(p+2)} \tag{5.35}$$

The above formula is correct regardless of whether horizon re-crossing takes place at the RD or MD eras. Note that we have normalized our comoving scales such that today physical scales are equal to comoving scales, i.e. $a_{today} = 1$.

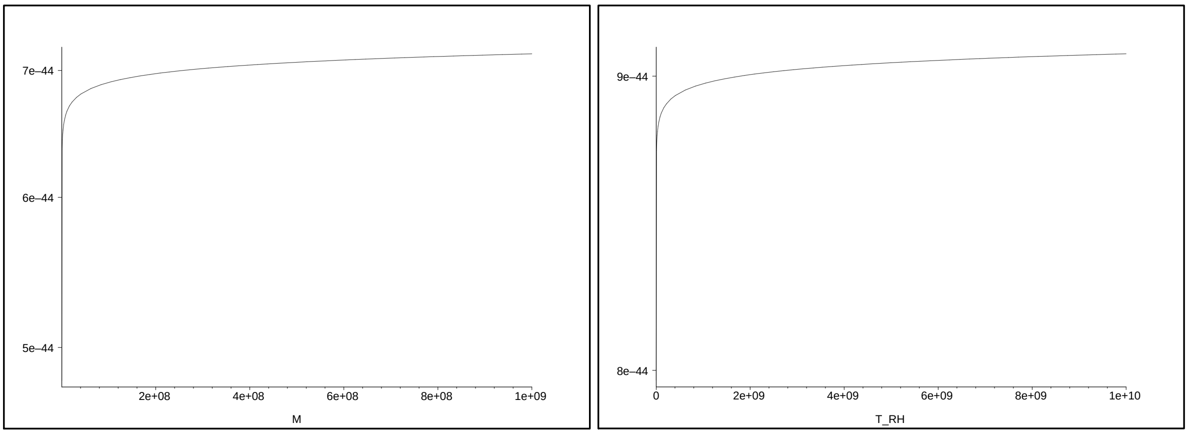


Figure 5.1: The left figure shows the dependence of M_1 on M, the energy scale of inflation. The right figure shows how M_1 varies as T_{RH} changes. For all physically acceptable values of M and T_{RH}, $M_1 \sim 10^{-43}$GeV.

Two constraints on M and T_{RH} should hold in any viable scenario of inflation. First, to prevent the production of long-wavelength gravitons that distort the microwave background radiation beyond its upper limit of anisotropy, $M < 10^{-2} m_{Pl}$. Second, M and T_{RH} should be greater than 1 GeV so that radiation domination takes place before nucleosynthesis.

To trigger the dynamo mechanism, there must be sufficient seed magnetic field at cosmologically interesting scales. This condition could be used to determine the value of M_1. Assuming that the required seed magnetic field has been substantial on galaxy-scales, $\lambda \sim$ 1Mpc, one can obtain a relation between M_1 and other relevant parameters of the problem:

$$M_1 \simeq \frac{1.6 \times 10^{-38} \big[\ln(T_{RH}/m_{Pl}) + 3 \ln b + 2 \ln(M/m_{Pl}) \big] \sigma}{\Big(\big[\ln r + 4 \ln b \big] \big[3 \ln r + 18 \ln b + 2 \ln(T_{RH}/m_{Pl}) + 4 \ln(M/m_{Pl}) \big] \Big)^{1/2}} Gev \quad (5.36)$$

where $b = 7 \times 10^{-25}$. If M, T_{RH} and σ are specified, one can obtain the corresponding values of M_1. In table 1, we have tabulated the results for different values of M, T_{RH} corresponding to different scenarios of inflation and some values of r required to initiate astrophysically interesting phenomena. As Fig.(1) and (2) show, for a fixed value of σ and all physically relevant values of M and T_{RH}, M_1 does not vary too much. For $\sigma \sim 10^{-5}$[72] we find $M_1 \sim 10^{-43}$GeV. Since the coupling of the added term is proportional to M_1^{-2}, the smallness of M_1 indicates that the coupling of electromagnetic field to the curvature of the expanding background, due to the existence of minimal length, has been enormous during the inflationary era. However the coupling is extinguished in all other epochs due to the special form of the interaction. One may wonder if the term added to the equation of motion for photon has any observable effect on the behavior of electromagnetic field at present time. The added term will be proportional to $\frac{\beta H_0^2}{\lambda^2 M_1^2}$, where H_0 is today Hubble parameter. Above, we calculated the value of M_1 to have the required seed to account for the cosmic magnetic fields and realized that $H_0 \approx M_1$. Thus, the above term becomes observable, if $\lambda \approx \sqrt{\beta}$, or equivalently when the wavelength of electromagnetic field becomes comparable with the scale of new physics. Experiments with very short wavelength photons should be able to put bounds on the value of β or even verify this scenario.

Although we should await a unified theory to determine how gravity is coupled to the other fields of nature, this phenomenological scenario suggests that the enigmatic primordial magnetic fields might have their origin in the special characteristics of space-time at high energies (See also ref.[179] on how non-commutativity of the space-time might help us account for primeval magnetic fields).

5.3 Conclusion

The origin of magnetic fields with μG strength that are observed on intragalactic scales remains an intriguing mystery. As the observed magnetic field is coherent on such cosmological scales, the first cosmological process one might think of as being able to produce such prevalent fields is inflation. However the conformal invariance of the electromagnetic field prohibits the quantum fluctuations of the electromagnetic field from squeezing and amplifying during inflation.

The existence of a minimal length breaks this conformal invariance. We have proposed a scenario based on this observation that can provide the requisite initial magnetic seed for the astrophysical dynamo mechanism. With a proper choice of the free parameter within the theory one can avoid the need for the dynamo mechanism.

T_{RH}(GeV)	$r\|_{\lambda=1Mpc}$	$M_1 \times 10^{43}$(GeV)
10^{17}	10^{-8}	0.5639
10^{9}	10^{-8}	0.5103
10^{17}	10^{-34}	0.7311
10^{9}	10^{-34}	0.6638
10^{17}	10^{-56}	0.9819
10^{9}	10^{-56}	0.8973

Table 5.1: Values for M_1 corresponding to different inflationary scenarios and different values of $r = (\rho_B)/\rho_\gamma|_{1Mpc}$. Here σ is assumed to be 10^{-5} and M is held constant at 10^{17} GeV. M_1 does not vary significantly for all interesting values of M, T_{RH} and r.

The scenario is based on the observation that incorporating minimal length at the level of first quantization, as was done for the first time in [71], does not render total time derivatives invariant under the influence of minimal length. Therefore one can have actions that are equivalent at the continuous space-time level, but are distinct from one another once the presence of minimal length is introduced. We added a prototype for such a total time derivative term to the action of electromagnetism that respects the behavior of the photon throughout the history of the universe except for the inflationary era. During inflation this term induces a huge mass for the photon. We found that to match this model with observation we must tune the free parameter of the model, M_1, to be extremely small. Since M_1^{-2} is proportional to the coupling of electromagnetism to the background geometry during inflationary epoch, the small size of M_1 is indicative of gravity and electromagnetism being strongly coupled at that time. We note that the numerical value of M_1 is approximately the inverse Hubble length, but we have found no deeper explanation for this coincidence at the level of the model presented here.

Of course it is conceivable that other gauge bosons of the standard model can inherit the same tachyonic instability that we have considered for the photon. However all other gauge bosons are non-Abelian and so will experience screening effects that we expect will tend to dampen out this instability [180, 181]. A detailed calculation of this effect remains an interesting subject for future study.

Of course the main drawback for this model is its arbitrariness in the choice of total time derivative. It would be really interesting if one were able to find candidates from existing models of fundamental physics. Our main goal here was that of demonstrating that specific characteristics of space-time at Planckian epochs can create observable phenomena in the universe at much later cosmological times.

Chapter 6

Power Spectrum and Signatures for M-theory Cascade Inflation

6.1 Introduction

For a long time it seemed difficult to connect inflation to string-theory. In its low-energy approximation string-theory is described by supergravity. Inflation based on the F-term potentials of 4-dimensional $\mathcal{N} = 1$ supergravities resulting from string/M-compactifications suffers from a large slow-roll parameter η. The origin of this problem traces back to the appearance of the Kähler-factor $\exp(K)$ in the F-term potential. New possibilities to address this problem arose with the advent of D-branes [60]. They allowed to identify the inflaton with open string modes such as the geometrical distance between two D-branes [63].

An inflaton requires a very shallow potential. Hence, a priori, moduli serve as natural candidates. To provide them with a non-trivial potential, supersymmetry needs to be broken, which can be done in various ways. One might add anti D-branes to the open string sector [66] or supersymmetry breaking fluxes to the closed string sector [182]. Also the inclusion of non-perturbative instanton effects leads to spontaneous supersymmetry breaking in the low-energy supergravity [183]. Assuming just a single inflaton, the task for deriving inflation from string-theory then becomes finding a way of breaking supersymmetry which leaves the inflaton with a sufficiently flat potential while endowing all other moduli with steep stabilizing potentials. All standard methods of breaking supersymmetry generate, however, steep potentials, not flat ones. One way to generate a flat inflaton potential nevertheless is to study brane-antibrane inflation in warped backgrounds with the inflaton being identified with the brane-antibrane distance [66]. Warped geometries arise in the presence of branes and fluxes. The eventual stabilization of the volume modulus, however, modifies the inflaton potential and renders it too steep for inflation unless fine-tuning is applied [184].

Here, we focus on an alternative mechanism to generate inflation in M/string-theory, the multi brane inflation proposal [77] (see also [185]). One starts with a multi inflaton scenario associating one inflaton with each inter-brane separation. The presence of several branes is indeed generically enforced by tadpole cancellation conditions. The interesting advantage of this mechanism lies in the fact that the potentials for the individual inflatons need no longer be flat. The reason is that the Hubble friction experienced by every inflaton becomes large – simply by increasing the number of inflatons – regardless of the steepness of the potentials. This had first been pointed out in [26] in the context of 4-dimensional Friedmann-Robertson-Walker (FRW) cosmologies based on exponential potentials which generate power-law inflation. The premise under which this mechanism operates is the suppression of strong cross-couplings among the inflatons. This suppression is given in multi brane inflation models since interactions between non-neighboring branes which could generate cross-couplings are suppressed by longer distances. In M-theory cascade inflation, there is an exponential suppression of such cross-couplings since interactions between the relevant M5-branes arise from non-perturbative open M2-instantons.

In this chapter, after highlighting the needed ingredients of M-theory cascade inflation, we focus on the determination of its power spectrum and the resulting observable signatures. Beyond demonstrating the compatibility of the power spectrum with present cosmological constraints, we find that it exhibits *three distinctive signatures – power suppression at small distances, stepwise decrease in the spectral index and oscillations in the spectrum.* The power suppression which follows in cascade inflation from M-theory dynamics might serve as an explanation for the scarceness of observed dwarf galaxies in the Milky Way halo, as suggested in [186], [187]. This is not explained by standard cosmology which overpredicts their abundance by an order of magnitude. The oscillations and stepwise decreases, on the other hand, provide a unique signature which allow one to probe M-theory observationally by measuring the spectral index. It furthermore clearly distinguishes M-theory cascade inflation observationally from other string inflation models.

6.2 Multiple M5-brane Assisted Inflation

To realize assisted inflation within heterotic M-theory [188, 189], one should find a setup with several scalar fields each having the same exponential potential. The setup suggested in [77] contains N parallel M5-branes distributed along the $\mathbf{S}^1/\mathbf{Z}_2$ interval. Compactifying M-theory on a six dimensional Calabi-Yau manifold and preserving $N = 1$ supersymmetry in four dimensions, one obtains N M5-branes which fill the 4-dimensional non-compact space-time and wrap the same two-cycle Σ_2 on the Calabi-Yau. It is a warped compactification due to the presence of G-flux sourced by the 10-dimensional boundaries and additional spacetime-filling M5-branes [190], [194]. For simplicity it is assumed that each M5-brane have wrapped the basis two-cycle Σ_2 once. Assuming the vanishing of expectation values for charged matter fields, the main contributions to the superpotential come from both open membrane instantons wrapping each the same Σ_2 on the Calabi-Yau and stretching between both boundaries (99), between two of the M5-branes (55), between the visible boundary and an M5-brane (95) or between an M5-brane and the hidden boundary (59), and also the gaugino condensation [182, 191, 192] on the hidden boundary [193], W_{GC}:

$$W = W_{99} + W_{55} + W_{59} + W_{95} + W_{GC}. \tag{6.1}$$

By grouping all N M5-branes together but away from both boundaries, one can neglect the inter-boundary interaction, W_{99}, and the interaction between M5-branes and either boundaries, W_{59} and W_{95}. The gaugino condensation on the hidden boundary, W_{GC}, could be neglected by placing the hidden boundary away from the singularity where the warp factor of the background geometry of vanishes [194–196]. Although having hidden boundary at the singularity leads to interesting particle phenomenology [197–199], the situation during inflation could be quite different. Particulary, it is possible that at the beginning of inflation one starts off with a subcritical-size orbifold which expands gradually to its critical value toward the end of inflation. By bringing the hidden boundary at subcritical distance, the hidden gauge theory becomes perturbative, as the Calabi-Yau volume on the hidden boundary quickly grows when the orbifold size shrinks [195]. Therefore we neglect the gaugino condensation and the H-flux superpotential induced by it during inflation. The corresponding flux, however, becomes important at the end of inflation. Taking all the above into account, among all the open membrane interactions in W, we only consider those between the M5-branes, W_{55}:

$$W = W_{55}. \tag{6.2}$$

Suppressing the gauge bundle moduli related to the E_8 Yang-Mills sector, the effective four dimensional $N = 1$ supergravity theory is described in terms of the volume modulus of the Calabi-Yau, S, the modules associated to the length of the orbifold, T and the M5-brane chiral superfields Y_i:

$$S = \mathcal{V} + \mathcal{V}_{OM} \sum_{i=1}^{N} \left(\frac{x_i^{11}}{L} \right) + i\sigma_S \tag{6.3}$$

$$T = \mathcal{V}_{OM} + i\sigma_T \tag{6.4}$$

$$Y_i = \mathcal{V}_{OM} \left(\frac{x_i^{11}}{L} \right) + i\sigma_i \tag{6.5}$$

Here $\mathcal{V}$ denotes the Calabi-Yau volume averaged over S^1/Z_2 and $\mathcal{V}_{OM}$ is the averaged volume of an open membrane instanton wrapping Σ_2 and stretching from one boundary to the other. L is the length of the S^1/Z_2 interval and the position modulus of the i-th M5-brane ranges over $0 \leq x_i^{11} \leq L$. The axions $\sigma_S, \sigma_T, \sigma_i$ arise from various components of the three-form potential C of eleven-dimensional supergravity, see [183]. There are also $h^{2,1}$ complex structure moduli Z^α. The superpotential takes the following form in term of the above moduli:

$$W = h \sum_{i<j} e^{-(Y_j - Y_i)}. \tag{6.6}$$

It is useful to define the following additional real moduli

$$s = S + \bar{S}, \qquad t = T + \bar{T}, \qquad y_i = Y_i + \bar{Y}_i, \qquad y = \left(\sum_{i=1}^{N} {y_i}^2 \right)^{1/2}, \tag{6.7}$$

and their *positive* functions

$$Q = s - \frac{y^2}{t}, \qquad R = 3Q^2 - 2\frac{y^4}{t^2}. \tag{6.8}$$

Kähler-potential for these moduli is given by [200–202]:

$$K = K_{(S)} + K_{(T)} + K_{(Y)} + K_{(Z)}, \tag{6.9}$$

where

$$K_{(S)} + K_{(Y)} = -\ln Q, \tag{6.10}$$

$$K_{(T)} = -\ln\left(\frac{d}{6}t^3\right) \tag{6.11}$$

$$K_{(Z)} = -\ln\left(i\int \Omega\wedge\bar{\Omega}\right). \tag{6.12}$$

Above, d and Ω are respectively the Calabi-Yau intersection number and Kähler form. The $N = 1$ supergravity expression for F-terms yields the following *positive* potential

$$U = M_{Pl}^4 e^K \left(\sum K^{\bar{I}J} D_{\bar{I}}\bar{W}_{55} D_J W_{55} - 3|W_{55}|^2\right), \tag{6.13}$$

where the Kähler covariant derivative and e^K are respectively defined as

$$D_i W_{55} \equiv \frac{\partial W_{55}}{\partial Y_i} + W_{55}\frac{\partial K}{\partial Y_i} \tag{6.14}$$

$$e^K = \frac{6}{\left(i\int \Omega\wedge\bar{\Omega}\right) Qt^3 d}. \tag{6.15}$$

To guarantee the partial minimization of the potential energy, one has to impose the following constraints:

$$D_\alpha W_{55} = 0 \tag{6.16}$$

$$D_i W_{55} = 0 \tag{6.17}$$

Applying the above constraints, the potential looks like the following

$$\frac{d\left(i\int \Omega\wedge\bar{\Omega}\right)}{6M_{Pl}^4}U = \left(\frac{3Q}{Rt^3} - \frac{2y^2}{Q^2t^4}\right)|W_{55}|^2 \tag{6.18}$$

To map the above potential to assisted inflation, the multiplicative factor of $|W_{55}|^2$ should be independent of y. To achieve this goal, one has to impose the following constraint:

$$Qt \gg y^2 \tag{6.19}$$

The above constraints will lead to an upper bound on N, the number of M5-branes. The Käler derivative in $D_iW_{55} = 0$ reduces to normal derivative, $\partial_i W_{55} = 0$, in the regime of validity of supergravity where both s and t are considerably larger than one. Consequently, if one concentrates on the dominant nearest neighbor interaction between adjacent M5-branes, such a constraint dictates that the M5-branes should be equidistantly distributed. Calling the real part of the separation of adjacent M5-branes $\Delta y/2$, the potential could be cast to the following form

$$U = \frac{6M_{Pl}^4 \left(i\int \Omega \wedge \bar{\Omega}\right)(N-1)^2}{st^3} e^{-\Delta y} \tag{6.20}$$

The kinetic term for Y_i is not canonical and takes the form

$$S_{kin} = -M_{Pl}^2 \int d^4x \sqrt{-g} K_{i\bar{j}} \partial_\mu Y_i \partial^\mu \bar{Y}_{\bar{j}} \tag{6.21}$$

where

$$K_{i\bar{j}} = \frac{4y_i y_j + 2Qt\delta_{ij}}{Q^2t^2} \tag{6.22}$$

and the repetition of indices indicates summation. In the limit where the constraint (6.19) is satisfied, one obtains $K_{i\bar{j}} = 2\delta_{ij}/Qt$. To map the dynamics of M5-branes to assisted inflation the fields must have canonical kinetic terms and therefore one can define canonically normalized real M5-brane position and difference fields

$$\phi_i = \frac{2M_{Pl}}{\sqrt{Qt}} y_i, \qquad \Delta\phi = \frac{2M_{Pl}}{\sqrt{Qt}} \Delta y. \tag{6.23}$$

The position fields, ϕ_i, are related to the center of mass field

$$\phi_{cm} = \frac{1}{N} \sum_{i=1}^{N} \phi_i \tag{6.24}$$

in the following manner

$$\phi = \phi_{cm} + \left(i - \frac{N+1}{2}\right) \Delta\phi. \tag{6.25}$$

Now if we switch from the set of fields $\{\phi_i, \Delta\phi\}$ to $\{\phi_{cm}, \Delta\phi\}$, the potential for ϕ_{cm} will be identically zero and the sum of ϕ_i kinetic terms becomes

$$\frac{1}{2}\sum_{i=1}^{N}\partial_\mu\phi_i\partial^\mu\phi_i = \frac{N(N^2-1)}{12}\partial_\mu\Delta\phi\partial^\mu\Delta\phi. \tag{6.26}$$

Thus one can define canonically normalized difference field, φ,

$$\varphi \equiv \sqrt{\frac{N(N^2-1)}{6}}\Delta\phi = M_{Pl}\sqrt{\frac{2N(N^2-1)}{3Qt}}\Delta y \tag{6.27}$$

such that the kinetic term takes the standard form, $\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi$. Ultimately the potential for the field, φ, takes the form:

$$U_N(\varphi) = U_0 e^{-\sqrt{\frac{2}{p_N}}\frac{\varphi}{M_{Pl}}} \tag{6.28}$$

where

$$U_0 = \tilde{U}_0(N-1)^2 = \frac{6M_{Pl}^4\left(i\int\Omega\wedge\bar{\Omega}\right)(N-1)^2}{st^3d} \tag{6.29}$$

and

$$p_N = \frac{4N(N^2-1)}{3Qt}. \tag{6.30}$$

The potential (6.28) can lead to power-law inflation, if

$$p_N > 1 \Leftrightarrow 4N(N^2-1) > 3Qt \tag{6.31}$$

The above constraint along with the one in (6.16) limit the viable range of N. Adopting typical values of $\mathcal{V} = 341$ and $\mathcal{V}_{OM} = 7$ and $x_i^{11}/L = \mathcal{O}(1/2)$ for the relevant case of unbroken E_8 (see table 1 of [203]) one obtains

$$s = 682 + 3.5N, \qquad t = 14, \qquad y^2 \simeq 49N \tag{6.32}$$

Then constraints (6.16) & (6.31) deliver the following bounds on N

$$19 < N \ll 195 \tag{6.33}$$

6.3 Cascade Inflation

Let us now describe the cascade inflation phase. The repulsive M2-interactions between the M5-branes cause them to spread over the $\mathbf{S}^1/\mathbf{Z}_2$ interval until the two outermost M5-branes hit the boundaries. The ensuing non-perturbative small instanton transition transforms the outermost M5-branes into small instantons on the boundaries [204]. More precisely, the small instantons are described by a torsion free sheaf, a singular bundle. The singular torsion free sheaf can then be smoothed out to a non-singular holomorphic vector bundle by moving in moduli space [205]. This process changes the topological data on the boundaries while the number N of M5-branes participating in the inflationary bulk dynamics drops to $N-2$. The small instanton transitions can be either chirality or gauge group changing [205]. We are considering the first case in which a change in the third Chern class of the visible boundary's vector bundle changes the number of fermion generations during the transition. This opens up the attractive possibility of reducing dynamically the number of generations during the cascade inflation phase, given that most compactifications exhibit a large number of generations far greater than three. Notice that for the chirality changing transition the gauge group will not change during the transition and unwanted relics are not produced.

The cascading process starts when the first two outermost M5-branes hit the boundaries and no longer participate in the bulk dynamics. We assume that the energy of the M5-branes that collide with the boundary redshifts as matter or energy. The remaining $N-2$ M5-branes will continue to spread until the second most outermost M5-branes hit the boundaries in a second transition and so on. The successive stepwise drop of the number of M5-branes by two marks the cascade inflation phase. Between each of these transitions we have a potential of the form (6.28) giving power-law inflation but with stepwise decreasing values for N and thus different parameters p_N and U_N after each transition. The cascade inflation process comes to an end when the number of M5-branes, given by

$$N_m = N - 2m \tag{6.34}$$

in the m-th phase, drops below a critical value N_K in the K-th phase determined by the exit condition

$$\text{exit from inflation}: \quad p_{N_K} = 1 \ . \tag{6.35}$$

Throughout the cascading process the inflaton will always be identified with the M5-brane separation and grows continuously. Since the energy of the M5-branes that coalesce with the boundaries redshifts as matter or radiation, their contribution will always remain subdominant to the M5-branes in the bulk. Therefore, evolution during cascade inflation could be approximated by a series of consecutive power-law inflation phases

$$a_m(\mathrm{t}) = a_m \mathrm{t}^{p_{N_m}}, \qquad \mathrm{t}_{m-1} \leq \mathrm{t} \leq \mathrm{t}_m, \qquad m = 1, ..., K \ . \tag{6.36}$$

Matching the scale factor at the transition times t_m determines the prefactors to be

$$a_m = a_1 \mathrm{t}_1^{p_{N_1}} \left(\frac{\mathrm{t}_2}{\mathrm{t}_1}\right)^{p_{N_2}} \left(\frac{\mathrm{t}_3}{\mathrm{t}_2}\right)^{p_{N_3}} \cdots \left(\frac{\mathrm{t}_{m-1}}{\mathrm{t}_{m-2}}\right)^{p_{N_{m-1}}} \frac{1}{(\mathrm{t}_{m-1})^{p_{N_m}}} \tag{6.37}$$

The scale factor, but not the Hubble parameter, is therefore continuous at the transition times t_m. The onset time of inflation, t_0, is determined by inverting the exact power-law inflation solution for $\varphi(\mathrm{t})$ in the initial phase and noting that $\Delta x(\mathrm{t}_0)/L \ll 1$. The result is

$$\mathrm{t}_0 \simeq \frac{2N^2}{3M_{Pl}} \sqrt{\frac{2td}{s}} \ . \tag{6.38}$$

Similarly, by inverting the solution for $\varphi(\mathrm{t})$ one obtains for the transition times

$$\mathrm{t}_m - \mathrm{t}_0 = \frac{1}{M_{Pl}} \sqrt{\frac{st^3 d}{6}} \left(\sum_{k=2}^{m} \frac{p_{N_k}(3p_{N_k} - 1)}{N_k - 1} e^{t\left(\frac{1}{N_k - 1} - \frac{1}{N_{k-1} - 1}\right)} + \frac{p_{N_1}(3p_{N_1} - 1)}{N_1 - 1} e^{\frac{t}{N_1}} \right) \tag{6.39}$$

from which the number of e-foldings generated during cascade inflation follows

$$N_e \equiv \ln\left(\frac{a(\mathrm{t}_f)}{a(\mathrm{t}_0)}\right) = \sum_{m=1}^{K} p_{N_m} \ln\left(\frac{\mathrm{t}_m}{\mathrm{t}_{m-1}}\right) \ . \tag{6.40}$$

WMAP three-year results indicate that the scalar spectral index, n_s, is $0.951^{+0.015}_{-0.019}$ [5]. For power-law inflation one has $n_s = 1 - 2/(p_N - 1)$ in the initial phase. Adopting typical values of $s = 682$ and $t = 14$ for the case of an unbroken hidden E_8 [77], N has to lie within the interval $61 \leq N \leq 75$ to satisfy the spectral index constraint. Of course, the initial number of M5-branes can be larger than this upper bound, with the proviso that the resulting n_s, at the scales of our Hubble radius, lies within the interval given by the WMAP data set. Taking the central value, $n_s \sim 0.951$, one finds $N = 66$ M5-branes.

The scale of inflation, M, can be at most of order the grand unified (GUT) scale to have gravitational waves under control. Assuming instant reheating one needs about 60 e-foldings to solve the problems of standard Big-Bang (SBB) cosmology. Mapping the cascade inflation model, with the above values for s and t, to GUT-scale inflation, requires $d \sim 4 \times 10^5$. One might lower the required minimal number of e-foldings by either lowering the reheating temperature, $T_{\rm RH}$, or M. However, for the above choices of s and t, lowering M requires larger values for d which seem to be non-generic. The details of reheating have yet to be worked out for cascade inflation, nonetheless, we assume instant reheating and therefore $T_{\rm RH} \sim M$. We should note here that the above values of s, t, d and N are not the only values that lead to GUT scale inflation which satisfy the theoretical and observational constraints. Surfing the landscape of parameters allows us to choose different sets of parameters. For example, one can also achieve a GUT-scale inflation by choosing $(s, t, d, N) = (6000, 11.4, 50000, 129)$ or (3000, 20, 1000, 123). As we will see, different values for these parameters determine the location of the resulting oscillations in the power spectrum. Henceforth, we will proceed with the initial values $(s, t, d, N) = (682, 14, 4 \times 10^5, 66)$ although the qualitative features do not change with other choices of parameters.

Starting initially with $N = 66$ M5-branes in the bulk, we find

$$t_0 = 3.21 \times 10^5 \tag{6.41}$$

in Planckian units. The total number of e-foldings is

$$N_e = 237.83, \tag{6.42}$$

which is much larger than the number of e-foldings required to solve the horizon and flatness problems of SBB. Most of the inflationary expansion takes place within the first power-law phase in which none of the M5-branes has yet collided with the boundaries. However, we are interested in the last 60 e-foldings of expansion which are within our observable horizon.

6.4 Power Spectrum of Cascade Inflation

Inflation, besides solving the flatness and horizon problems of standard cosmology, provides a causal mechanism to generate the seed for large scale structures of the universe. Temperature fluctuations of the cosmic microwave background radiation (CMBR) – the afterglow of the BigBang – are believed to be generated by quantum fluctuations of the field(s) responsible for inflation. WMAP alone indicates a flat Λ-dominated universe with nearly scale-invariant power spectrum with $n_s(0.05\mathrm{Mpc}^{-1}) \sim 0.95$ [5]. Any viable inflationary model should be able to produce a power spectrum compatible with these observations.

As explained in the introduction, during inflation two types of perturbations are produced: scalar (density) perturbations and tensor perturbations (gravitational waves). These two types of perturbations are both responsible for the temperature anisotropy of the CMBR. Let us focus on scalar perturbations. The evolution of Fourier components of scalar perturbations, u_k, is known to be governed by the equation (1.45). u_k is the Fourier component of the gauge invariant Mukhanov variable u and is proportional to the curvature perturbation $\mathcal{R}$ of the comoving hypersurface [118]

$$u = -z\mathcal{R} \ . \tag{6.43}$$

The solutions to the mode equation (1.45) are normalized so that they satisfy the Wronskian condition 1.44. Ultimately the scalar power spectrum is defined as

$$P_s^{1/2} = \sqrt{\frac{k^3}{2\pi^2}} \left|\frac{u_k}{z}\right|_{k/aH\to 0} \tag{6.44}$$

The power spectrum should be evaluated in the limit where the mode goes well outside the horizon. To recover the ordinary quantum field theory result at very short distances much smaller than the curvature scale, we require that the mode approaches the Bunch-Davies vacuum when $k/aH \to \infty$

$$u_k(\tau) \to \frac{1}{\sqrt{2k}} e^{-ik\tau} \ . \tag{6.45}$$

To solve the scalar mode equation (1.45), we parameterize the scale factor in terms of conformal time

$$a_m(\tau) = b_m(\tau - c_m)^{q_{N_m}}, \qquad \tau_{m-1} \le \tau \le \tau_m, \qquad m = 1, ..., K \ , \tag{6.46}$$

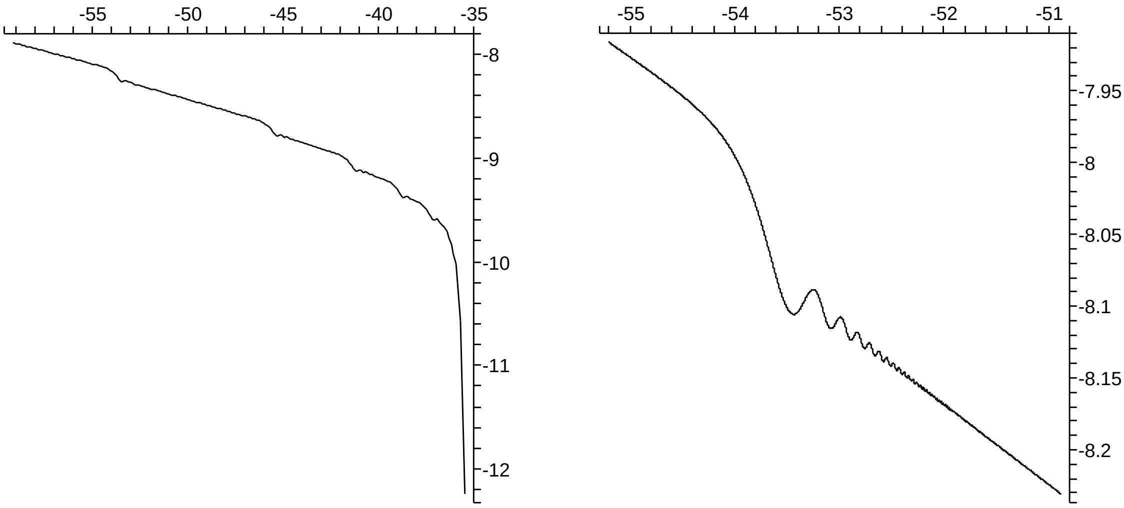


Figure 6.1: The left graph shows the dependence of $\log P_s(k)$ on $\log k$ for the scales that have crossed the horizon at least once during the last 60 e-foldings and are still outside the horizon at the end of inflation. It clearly displays the stepwise decrease in the amplitude of the power spectrum. The right graph shows $\log P_s(k)$ vs. $\log k$ around the first transition. The amplitude of oscillations decreases as k increases.

where

$$q_{N_m} = \frac{p_{N_m}}{1 - p_{N_m}} \ , \quad c_m = \tau_{m-1} - \frac{\mathrm{t}_{m-1}{}^{1-p_{N_m}}}{a_m(1 - p_{N_m})} \ , \quad b_m = a_m{}^{1/(1-p_{N_m})}(1 - p_{N_m})^{q_{N_m}}, \tag{6.47}$$

and τ_m is the conformal time corresponding to t_m. It can be found by the following recursive relation

$$\tau_m = t_m{}^{(1-p_{N_m})} - \frac{t_{m-1}{}^{(1-p_{N_m})}}{a_m(1 - p_{N_m})} + \tau_{m-1} \ . \tag{6.48}$$

Thus during each power-law phase, equation (1.45) simplifies to a Bessel equation

$$u_k'' + \left(k^2 - \frac{\nu_m^2 - \frac{1}{4}}{(\tau - c_m)^2}\right) u_k = 0, \qquad \tau_{m-1} \le \tau \le \tau_m \tag{6.49}$$

where

$$\nu_m = \frac{3}{2} + \frac{1}{p_{N_m} - 1} \ . \tag{6.50}$$

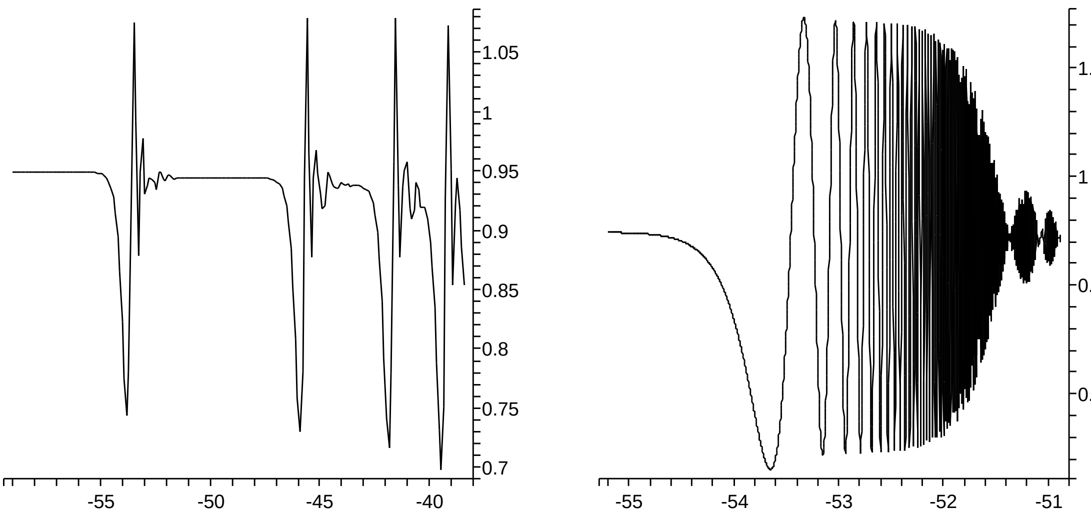


Figure 6.2: The left graph shows the dependence of n_s on $\log k$ for the first five inflationary bouts. It clearly displays the stepwise decrease in the spectral index. The right graph shows n_s vs. $\log k$ around the first transition. The period of oscillations decreases as k increases.

The Bessel equation has the following general solution

$$u_k(\tau) = C_m(k)(c_m - \tau)^{1/2} H^{(1)}_{\nu_m}(kc_m - k\tau) + D_m(k)(c_m - \tau)^{1/2} H^{(2)}_{\nu_m}(kc_m - k\tau) \quad (6.51)$$

where $H^{(1)}_{\nu_m}$ and $H^{(2)}_{\nu_m}$ are the first and second Hankel functions of order ν_m. Starting from the first power-law phase and demanding that the mode satisfies equation (6.45) at the beginning of cascade inflation, one can determine $C_1(k)$ and $D_1(k)$

$$C_1(k) = \frac{\pi}{2} e^{i(\nu_1 + \frac{1}{2})\pi/2} \, , \qquad D_1(k) = 0 \, . \quad (6.52)$$

Through the transition from one power-law phase to the next, the scale factor is continuous but the Hubble parameter is not. This happens because the potential has steps at the transitions. Potentials with steps occur in supergravity motivated models of inflation where the inflaton lies within the hidden sector and is gravitationally coupled to a visible sector which contains the standard model [206]. Spontaneous supersymmetry breaking that occurs in the visible sector changes the mass of the inflaton and leads to sudden downward steps in the inflaton potential.

During inflation, the resulting change in the potential is compensated by an increase in the inflaton kinetic energy. Therefore the slow-roll approximations become unreliable around the transition points [207]. Despite some similarity that exists between the cascade inflation model and these supergravity motivated inflation models, there is an eminent difference. In cascade inflation, the differences in potential energy after each step are transferred to the boundaries as the two outermost M5-branes dissolve into them via small instanton transitions. Thus the kinetic energy of the inflaton fields, whose role is played by the separations of the M5-branes in the bulk, will not get modified by the existence of such jumps in the potential. That is why we can still approximate the evolution by a power-law, even instantaneously after the transitions. Since the Hubble parameter decreases whereas the scale factor remains continuous through the transitions, the size of the Hubble radius increases slightly. Therefore some modes that have just gone outside the horizon, are recaptured and start oscillating again. As we will see these oscillations will be translated to oscillations in the power spectrum later.

Focusing on adiabatic perturbations, the 3-curvature perturbations of the comoving hypersurface, $\mathcal{R}$, are continuous and differentiable through the transitions [208]. This allows us to determine the i-th Bogoliubov coefficients in terms of $(i-1)$-th ones and calculate the power spectrum in the limit when all modes are far outside the horizon. We also note that [74]

$$z(\tau) = a(\tau) m_{Pl} \sqrt{\frac{\epsilon_m}{4\pi}}, \tag{6.53}$$

where

$$\epsilon_m = \frac{1}{p_{N_m}} \,. \tag{6.54}$$

The left graph in fig. 6.1 displays the power spectrum for the modes that have crossed the horizon at least once during the last 60 e-folds of inflation and are *still* outside the horizon at the end of inflation. Since the Hubble parameter and p_m, drops in each step, the total amplitude of perturbations decreases as well. The modes that cross the horizon twice during the transitions display oscillatory behavior. Actually, the oscillations last for an interval of k much larger than the interval crossed by the horizon twice. As the right graph in fig.6.1 shows, for the first step with approximately 7% drop in amplitude of the potential, the oscillations last for as much as three decades of k. The left graph in (6.2) presents n_s vs. $\log k$ for the first five inflationary bouts. Aside from the superimposed oscillations, the modes pick up the value of the spectral index of the bout during which they cross the horizon. Of course, this is only true for the first few bouts that last long enough to let the oscillations fade away. For the last inflationary bouts which last much less than an e-folding, this inference breaks down. The perturbation amplitudes are suppressed significantly and we have a very red spectrum at such scales. As the right graph in fig. (6.2) demonstrates, the period of oscillations in the power spectrum decreases as k increases.

6.5 Discussion

Although the standard $\Lambda\mathrm{CDM}$ inflationary model has been successful in describing the results of the WMAP anisotropy probe, the statistics for the data is rather poor. In fact, the probability that the best fit to the TT spectrum is correct is only about 3%. The poor statistics stems mainly from the features or "glitches" in the power spectrum that the model is unable to fit [4]. Although these glitches may have been caused by beam asymmetry, gravitational lensing of the cosmic microwave background or the non-Gaussianity in the noise maps, it is quite possible that these glitches are due to features in the underlying primordial curvature perturbation spectrum. Particularly, attempts that have been made to reconstruct the curvature perturbation from the WMAP data, noticed possible features in the primordial spectrum [209, 210]. The possibility that such features are signatures of trans-Planckian physics [72, 75], was investigated in several papers, see for e.g. [212]. They could also be generated by resonant particle production *during* inflation [213]. Mathews *et. al.* adopted this scenario and used the features observed in the matter power spectrum deduced from galaxy surveys and damped Lyman-α systems at high redshift to put some constraints on the mass of produced fermion species and their coupling to the inflaton [214]. The features may also be caused by steps in the underlying inflaton potential [206]. Covi *et. al.* have recently used this scenario to explain measured deviations of WMAP three-year data from a featureless power spectrum [211], using potentials with steps.

However the signatures of cascade inflation are in fact distinct from all three above scenarios. In contrast to trans-Planckian superimposed oscillations, that should be present across all scales, the oscillations in the cascade inflation power spectrum occur only around some particular wavelengths. Also as noted above, the dynamics of cascade inflation is different from supergravity inflationary models with transitions in the hidden sector. This difference, in turn, results in different features in the power spectrum: in cascade inflation, the scalar spectral index changes before and after the transitions, whereas in supergravity motivated models of inflation the scalar spectral index remains the same [211]. Finally, resonant particle production during inflation, rather than decreasing, *increases* the amplitude of density perturbations, since resonant extraction of inflaton field energy decreases the kinetic energy of the inflaton. Also, because of the resonant nature of the process, the produced feature is rather sharp, extending less than a decade in wavenumber. In cascade inflation, the number of M5-branes and other M-theory parameters such as the stabilized volume and orbifold moduli, can change the location and magnitude of the resultant oscillations but not its qualitative properties. It will be very interesting to constrain these M-theory parameters using observational data. One should also note that in the generation of oscillations in the power spectrum from supergravity motivated models with hidden sector inflation, one has to assume that symmetry breaking phase transitions happen during inflation [206]. In cascade inflation this assumption will necessarily be true as the small instanton transitions are generated by the same M5-branes whose dynamics drives inflation. Hence, features in the potential are an inevitable consequence.

Let us finally focus on another implication of cascade inflation for structure formation. N-body simulations of structure formation use the assumption of a scale-invariant power spectrum and predict the number of dwarf galaxies in the Milky-Way halo an order of magnitude larger than observed [215]. The simulations also predict cuspy central core densities, while galaxy rotation curves are often better fit with constant density cores [216, 217]. Solutions to these shortcomings range from those which alter baryonic physics (see for e.g. [218]) to those that modify the nature of dark matter [187, 219]. Furthermore, in [186] it was suggested that a sudden downswing in the power spectrum at small scales could explain the discrepancy and ameliorate the disagreement between cuspy simulated halos and smooth observed halos. Zentner and Bullock [220, 221] later argued, however, that the estimate used in [186] to determine the effect of the primordial spectrum on the subhalo population of the halo to be incorrect. In cascade inflation, the power spectrum and the scalar spectral index drop in each step. Its value at small scales is therefore necessarily smaller than what a simple extrapolation of the spectrum at large scales predicts. With our above choice of parameters, the first downturn occurs at at about 0.012 Mpc. A suppression at this scale ameliorates the problem of dearth dwarf galaxies without violating constraints from the Lyman-alpha forest. Thus, cascade inflation provides us with an M/string-theory inflation model that can naturally obviate this tension between theory and observation.

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