A k-Conjugacy Class Problem

by

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Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

In any group G, we may extend the definition of the conjugacy class of an element to the conjugacy class of a k-tuple, for a positive integer k. When $k = 2$, we are forming the conjugacy classes of ordered pairs, when $k = 3$, we are forming the conjugacy classes of ordered triples, etc.

In this report we explore a generalized question which Professor B. Doug Park has posed (for $k = 2$). For an arbitrary k, is it true that:

$$(G\ has\ finitely\ many\ k-conjugacy\ classes) \Longrightarrow (G\ is\ finite)\ ?$$

Supposing to the contrary that there exists an infinite group G which has finitely many k-conjugacy classes for all $k = 1, 2, 3, \ldots$, we present some preliminary analysis of the properties that G must have. We then investigate known classes of groups having some of these properties: universal locally finite groups, existentially closed groups, and Engel groups.

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Dedication

This thesis is dedicated to my parents, Donald and Janet and to my brother and sister-in-law, Douglas and Megan.

Contents

Chapter 1

Motivation

1.1 The Origin of the Problem

This problem was posed by Professor B. Doug Park to Professor John Lawrence during a discussion in the University of Waterloo's Pure Math lounge. He is interested in the answer because of its application to a topological problem which he is pursuing.

1.2 Definitions and Notation

1.2.1 Lie Groups

Definition (Hausdorff Space): A *Hausdorff space* is a topological space X in which $\forall\ p, q \in X$ with $p \neq q$, there exist open neighbourhoods U_p and U_q of p, q respectively such that $U_p \cap U_q = \emptyset$.

Definition (n-Manifold): An *n-dimensional topological (differentiable) manifold* (or *n-manifold*) is a Hausdorff space in which each point has an open neighbourhood which is homeomorphic (diffeomorphic) to an open neighbourhood of $\mathbb{R}^n$.

Definition (Lie Group): A *Lie group* is a differentiable manifold obeying the group axioms and that satisfies the additional condition that the group operations are differentiable.

Examples of Lie Groups:

1. $\mathbb{R}^n$
2. $GL(n)$
3. $SL(n)$
4. $U(n)$

5. $SU(n)$

1.2.2 Principal Bundles

Definition (Diffeomorphism): Let P and X be manifolds. Then $\pi : P \to X$ is a *diffeomorphism* $\iff$ π is differentiable and has a differentiable inverse.

Definition (C^∞ Map): Let P and X be topological spaces. Then $\pi : P \to X$ is a C^∞ *map* $\iff$ π has an nth derivative $\forall n \in \mathbb{P}$ (where $\mathbb{P} = \{1, 2, 3, \ldots\}$ denotes the positive integers).

Definition (Principal Bundle): Let X be an m-manifold. Let $x \in X$ be arbitrary. Let P be an $(m+n)$-manifold (N.B. m = base-dimension, n = fibre-dimension). Then P is a *G-principal bundle over X* $\iff$ there exists a C^∞ map $\pi : P \to X$ such that $\pi^{-1}(U_x) \underbrace{\cong}_{\leftarrow \Phi_{U_x}} U_x \times G$ for all "small enough" open contractible neighbourhoods U_x of x, for some diffeomorphism Φ_{U_x}.

Example of Principal Bundle: $P = G \times X$, with π = projection of second co-ordinate onto X.

Definition (Continuous Group): A group G is *continuous* $\iff$ the group operation is continuous. A nice example is $(\mathbb{R}, +)$. Note that a continuous group is necessarily infinite. Also note that conversely, infinite groups need not be continuous (e.g. $(\mathbb{Z}, +)$, $(\mathbb{Q}, +)$).

Definition (Topological Group): A group G is a *topological group* $\iff$ G is continuous and has a Hausdorff topology. A nice example is $(\mathbb{R}, +)$. The homeomorphism group of any compact Hausdorff space is a topological group when given the compact-open topology. Also, any Lie group is a topological group.

Definition (Discrete Group): The group G is *discrete* $\iff$ G is a topological group with the discrete topology.

A Generalization of Topological Covering Spaces: Principal bundles generalize the notion of topological covering spaces. In particular, G is discrete $\implies$ P is a covering space over X.

1.2.3 Connections on Principal Bundles

Remark: We do not present full details of the definition of a connection here. The interested reader is referred to [13].

Definition (Section of a Fibre Bundle): A *section of a fibre bundle* gives an element of the fibre over every point in X. Usually it is described as a map $s : X \to P$ such that $(\pi \circ s)$ is the identity on X.

Definition (Connection): Let P be a principal bundle. Fix $g \in G$. This determines a *section.* i.e. $\Phi(U_x \times \{g\})$ is a copy of U_x "upstairs". Note that this is a local property and not a global property. In general there is no global section.

Fix another point $y \in X$. Fix a path α from x to y in X. Then by hypothesis we have that $\pi^{-1}(y) \underbrace{\cong}_{\leftarrow \Psi} G$.

We wish to be able to do calculus globally on our manifold X. We are given co-ordinates locally on the manifold, not globally. So we need a consistent way to identify the fibres G lying above each point of X. This identification must agree on all the intersections of our open sets. This identification then permits us to "translate horizontally" the fibres G lying above any 2 distinct points of our manifold, even when the points do not use the same chart for their respective co-ordinates.

This consistent identification of $\pi^{-1}(x) \cong G,\ \forall x \in X$, is our *connection.* Here we denote our connection by A.

To better describe the behaviour on the overlap of 2 open sets, consider the following situation. Suppose that we have $x \in U_x \setminus U_y$, $y \in U_y \setminus U_x$ and $z \in U_x \cap U_y$. Then z lies in the overlap, so it is possible to use either Φ_{U_x} or Φ_{U_y} to identify the fibre lying above z. We therefore have the following diagram:

$$\begin{tikzcd}
 & G \arrow[dl, "\Phi_{U_x}|_{\{x\}\times G}"'] \arrow[dd, "\textit{multiplication by } g_z \in G"] \\
\pi^{-1}(z) & \\
 & G \arrow[ul, "\Phi_{U_y}|_{\{y\}\times G}"]
\end{tikzcd}$$

where the g_z comes from the map:

$$\begin{array}{cccc} g_{U_x \cap U_y} & : & U_x \cap U_y & \to & G \\ & & z & \to & g_z \end{array}$$

Remark: For a given manifold, there are uncountably many connections.

Definition (Curvature of a Connection): Let A be a connection on a manifold X. Then the *curvature of* A is analogous to the "derivative" of the connection.

Definition (Flat Connection): A is a *flat connection* $\Longleftrightarrow$ the curvature of A is 0. In terms of the above diagram, the connection is flat if all the $g_{U_x \cap U_y}$ are "close" to constant functions. In other words, the connection is flat if the fibre identification is "not too wild".

1.3 Motivation for the Group Theory Problem

1.3.1 A Deep Theorem

Theorem: 1.3.1 *There is a 1:1 correspondence between:*

$$\frac{\{flat\ connections\ on\ P\}}{(gauge)\ equivalence} \underbrace{\leftrightarrow}_{1:1} \frac{\{group\ homomorphisms\ \rho : \pi_1(X) \to G\}}{(conjugacy\ i.e.\ \rho_1 \sim g^{-1}\rho_2 g, for\ fixed\ g)}$$

Sketch of Proof: **($\Longrightarrow$)** Let A be a flat connection on P. Let $\pi_1(X)$ denote the fundamental group of X, i.e. homotopy equivalence classes of loops in X. Let $\pi_1(X)$ have the base point $x \in X$. Take $\alpha \in \pi_1(X)$. The goal is to use A to construct a homomorphism ρ_A.

Note that A gives an identification of the fibres above each point of α. Take $y, z \in \alpha,\ y \neq x, z \neq x, z \neq y$. Then:

$$A = \Phi^{-1}(x) \underbrace{\cong}_{\rightarrow} \Phi^{-1}(y) \underbrace{\cong}_{\rightarrow} \Phi^{-1}(z) \underbrace{\cong}_{\rightarrow} \Phi^{-1}(x)$$

By going all the way around the loop α, we induce an automorphism (N.B. not necessarily identity) on G. We claim that this automorphism is induced by multiplication by some $g \in G$. Then take $\rho_A(\alpha) = g$.

Sketch of Proof: **($\Longleftarrow$)** This proof requires the details of the definition of connection which we have omitted, so we won't attempt to sketch it here.

1.3.2 An Interesting Special Case

Riemann Surface of Genus g Take:

$$X = Riemann\ surface\ of\ genus\ g = 2 - manifold\ with\ g\ holes = \Sigma_g$$

We know that $\pi_1(X) = \langle \alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g \mid \prod_i [\alpha_i, \beta_i] = 1 \rangle$. For the proof, refer to [7].

We want to apply the theorem to assert that we get infinitely many flat connections $\iff$ we get infinitely many $\frac{\{\rho:\pi_1(X)\to G\}}{conjugacy\ equivalence}$. Note that this problem depends completely on G. If G is "bad", then we may only get the trivial homomorphism.

How to generate group homomorphisms We are interested in generating homomorphisms ρ. If our fundamental group was free, we could map our generators anywhere we like (by the universal property of free groups). Since we have a relation to satisfy, we have to be a bit more clever to make our map well-defined. A cheap way to do it is to map:

$$\begin{array}{rcl} \alpha_i & \mapsto & 1 \\ \beta_i & \mapsto & anything \ \in \ G \end{array}$$

It is easy to see that this satisfies the above relation, so the mapping is well-defined. So in essence we can throw away half of the generators, then map the other half anywhere we like. It is *good* if we can throw away half of the generators and still obtain infinitely many conjugacy classes.

Original Motivation Restated: As long as there exists a group homomorphism $\rho : \pi_1(X) \to$ Free Group, for example, $\rho : \frac{\pi_1(\Sigma_g)}{\alpha_1=\cdots=\alpha_g=1} \to \mathbb{F}_g$ then the k-conjugacy class problem says something about the number of gauge equivalence classes of flat connections on a G-principal bundle.

An Answer Would Give an Elegant Solution to the Original Problem: Since the problem was first posed, Professor Park and Chris Hays got around the problem in a less elegant manner. If the k-conjugacy class problem could be solved, it would yield a more elegant solution to the original problem.

The Group Theory Question: Let G denote a group. Is it true that:

$(The\ number\ of\ 2-conjugacy\ classes\ of\ G\ is\ finite) \Longrightarrow (G\ is\ finite)$?

This is the $k = 2$ case of the general analysis that follows for any k.

For $k = 1$, Theorem 6.4.6 on p. 189 of [17] shows that there exist groups of arbitrary infinite cardinal with only 2 1-conjugacy classes. So far we do not have the result proved for $k = 2, 3, 4, \ldots$. This suggests to us that we should examine the general case further.

Chapter 2

Preliminary Results

2.1 Definitions and Notation

2.1.1 k-Conjugacy Class

Definition (k-Conjugacy Class): Let G denote any group. The *k-conjugacy class of G containing the k-tuple* $(a_1, \ldots, a_k) \in G^k$ is the (non-empty) set:

$$S = \{(g^{-1}a_1 g, \ldots, g^{-1}a_k g) : g \in G\} \subseteq G^k$$

Observe that from this definition, we have that two k-tuples

$$(a_1, \ldots, a_k),\ (b_1, \ldots, b_k)\ \in\ G^k$$

lie in the same k-conjugacy class of G if and only if there exists $g \in G$ such that:

$$(a_1, \ldots, a_k) = (g^{-1}b_1 g, \ldots, g^{-1}b_k g)$$

2.1.2 Conventions

1. We adopt the convention that a group G *answers our question positively* if:

   (a) G is infinite, and

   (b) G has finitely many k-conjugacy classes, for all $k = 1, 2, 3, \ldots$.

2. We use the notation $\mathbb{P}$ to denote the positive integers, i.e.

$$\mathbb{P} = \{1, 2, 3, \ldots\}$$

2.1.3 Locally Finite Groups

Definition (Locally Finite): Let G denote a group. Then G is *locally finite* $\iff$ any finite subset of G generates a finite subgroup of G.

Examples of Locally Finite Groups:

1. Let $\{H_n\}_{n\in\mathbb{N}}$ be a sequence of finite groups. Define a new sequence of groups $\{G_n\}_{n\in\mathbb{N}}$ inductively as follows:

   (a) $G_1 = H_1$.

   (b) $G_{n+1} = H_{n+1} \wr G_n$, the standard wreath product.

   In this construction there is an obvious embedding of the group G_n into G_{n+1}. Thus $\{G_n\}_{n\in\mathbb{N}}$ is an ascending sequence of finite groups. Then $G = \bigcup_{n=1}^{\infty} G_n$ is a locally finite group and the subgroups $\{G_n\}$ form a *local system* of G.

2. Let Ω be any infinite set. Let $\overline{S}$ be the full symmetric group on the set Ω. Let G be the subgroup of $\overline{S}$ of all permutations of Ω which fix all but finitely many elements of Ω. (We often denote this G by S_∞.)

   Notice that any finite subset of G consists of finitely many permutations, each of which moves only finitely many elements of Ω. Therefore any finite subset of G generates a finite subgroup of G. In other words, G is locally finite. G has a local system of finite symmetric groups, one for each finite subset of Ω.

   The group $G = S_\infty$ is usually called the *restricted symmetric group* on Ω. G has a simple subgroup of index 2, the *alternating group on* Ω, consisting of all even permutations on Ω.

3. Let F be an infinite algebraic extension of a finite prime field. Then every finite set of elements of F is contained in a finite subfield of F. Thus the field F is locally finite. (Conversely, every locally finite field is an algebraic extension of a finite prime field.) The group $GL(n, F)$ of all invertible $n \times n$ matrices over F is locally finite. It has a local system consisting of $GL(n, F_i)$, one for each finite subfield F_i of F.

2.1.4 Local System of a Group

Definition (Local System): Let G be a group. Let Σ be a set of subgroups of G. Then Σ is a *local system of* $G \iff G = \bigcup_{S\in\Sigma} S$ and for every pair $S,\ T \in \Sigma$, there is a subgroup $U \leq G,\ U \in \Sigma$ such that $S,\ T \leq U$.

Example of a Local System: Let Ω be a countably infinite set. Take $G = S_\infty$, the restricted symmetric group on Ω. Take $\Sigma = \{G_i\}$, where G_i is the symmetric group on Ω_i, for each finite subset Ω_i of Ω. This local system has the nice property that it is composed of finite subgroups of G.

2.1.5 Characterization of Countable Locally Finite Groups

Lemma: 2.1.5.1 *Let G be a group. Then G is a countable locally finite group $\iff$ there is a local system Σ of G consisting of finite groups and linearly ordered by inclusion.*

Proof ($\Longrightarrow$): Let G be a countable locally finite group. Enumerate the elements:

$$G = \{g_1, g_2, \ldots\}$$

and put:

$$G_n = \langle g_i : 1 \leq i \leq n \rangle$$

The system of (distinct) subgroups in this sequence is a local system of G consisting of finite groups. By construction it is linearly ordered by inclusion.

Proof ($\Longleftarrow$): Let Σ be a local system of G consisting of finite subgroups of G and linearly ordered by inclusion. Then G is a locally finite group. Since Σ is, in fact, a well-ordered sequence of finite groups, G is countable. ■

2.1.6 Group Properties

1. Definition (Property 1): If $|H| < \infty$ then there exists $H^* \leq G$ such that $H^* \cong H$. In other words, G contains a copy of every finite group H.

   Remark: Phillip Hall's universal locally finite group has this property (3.2).

2. Definition (Property 2): Let $H_1, H_2 \leq G$ with $|H_1|, |H_2| < \infty$ and $H_1 \cong H_2$. Then there exists $g \in G$ such that $H_1 = g^{-1}H_2g$. In other words, any two finite isomorphic subgroups of G are conjugate in G.

   Remark: Phillip Hall's universal locally finite group has this property (3.2).

3. Definition (Property 3): $\forall k \in \mathbb{P}$ there exists a uniform bound on $|H|$ where H is a k-generated subgroup of G.

   Remark: We shall soon prove (Theorem (2.4.1)) that if G has finitely many $(k+1)$-conjugacy classes, then G satisfies Property 3.

2.1.7 Extended Commutator Notation

Definition: Let G denote any group. Let $x_1, \ldots, x_n \in G$. Define:

$$[x_1,\ x_2] \quad = \quad x_1^{-1}x_2^{-1}x_1x_2 \text{ (\textit{i.e. usual commutator})}$$

Then define inductively:

$$[x_1, \ldots, x_n] = [[x_1, \ldots, x_{n-1}], x_n]$$

If the second argument repeats, we use the following convenient shorthand:

$$\begin{aligned}
[x_1,\ \underbrace{x_2, \ldots, x_2}_{n\ \textit{times}}] &= [[x_1,\ \underbrace{x_2, \ldots, x_2}_{n-1\ \textit{times}}],\ {}_1x_2] \\
&= [[x_1,\ \underbrace{x_2, \ldots, x_2}_{n-2\ \textit{times}}],\ {}_2x_2] \\
&\ \vdots \\
&= [x_1,\ {}_nx_2]
\end{aligned}$$

2.2 Facts

Fact: 2.2.1 *Let G denote any group. Let $a, b, g \in G$. Then:*

$$g^{-1}[b,\ a]g \quad = \quad [g^{-1}bg,\ g^{-1}ag]$$

Proof: This is clear from the definition of the commutator. ■

Fact: 2.2.2 *Let G denote any group. Let $a, b, g \in G$ be such that $ga = ag$ (i.e. a and g commute). Then:*

$$g^{-1}[b,\ {}_ka]g \quad = \quad [g^{-1}bg,\ {}_ka]$$

Proof: This is clear from Fact (2.2.1) the definition of the extended commutator. ■

2.3 A Useful Lemma

Lemma (Roberts): 2.3.1 *Let G be a torsion group, with finitely many (say N) 2-conjugacy classes. Let $g \in G$ be any element. Let $t = |g|$. Then:*

$$t \quad \leq \quad N$$

Outline of Proof: Use the hypothesis of finitely many 2-conjugacy classes to obtain a contradiction from assuming that $t > N$.

Proof: Construct the following set of ordered pairs:

$$\{(g,\ g^i) \quad : \quad i = 0, \ldots, t-1\}$$

Since $|g| = t$, we have that these ordered pairs will all be distinct. Proceed by contradiction. Assume that $t > N$. Then, since G has only N 2-conjugacy classes, we must have 2 ordered pairs in the same conjugacy class. Say (g, g^e) and (g, g^f) are in the same conjugacy class with $0 \leq e < f \leq t-1$. Then there exists $h \in G$ such that:

$$(g,\ g^e) \quad = \quad (h^{-1}gh,\ h^{-1}g^f h)$$

Focusing on the first co-ordinate, we obtain:

$$\begin{aligned} g &= h^{-1}gh \\ hg &= gh \end{aligned}$$

So we have that g and h commute. Since g and h commute, we obtain from the second co-ordinate that:

$$\begin{aligned} g^e &= h^{-1}g^f h \\ &= h^{-1}hg^f (\textit{since } g \textit{ and } h \textit{ commute}) \\ &= g^f \\ g^{-e}g^e &= g^{-e}g^f \\ 1 &= g^{f-e} \end{aligned}$$

Recall:

$$\begin{array}{ccccccc} 0 & \leq & e & < & f & \leq & t-1 \\ 1 & \leq & f-e & \leq & t-1-e & < & t \\ 1 & \leq & f-e & < & t & & \end{array}$$

This contradicts the hypothesis that $|g| = t$. The assumption that $t > N$ led to this contradiction. Therefore this assumption was false, and we get the desired result. ■

2.4 Properties of a Positive Answer

Theorem (Lawrence): 2.4.1 *Suppose that a group G has finitely many $(k+1)$-conjugacy classes. Then there exists a positive integer N such that if H is a k-generated subgroup of G, then $|H| < N$.*

Outline of Proof: Take the k generators of a subgroup H. Form all $(k+1)$-tuples using these generators in the first k positions, and every element of H in the last position. Use the hypothesis of finitely many $(k+1)$-conjugacy classes to demonstrate that there is a uniform bound on the number of elements of H.

Proof: Let H be any k-generated subgroup of G. Write $H = \langle a_1, \ldots, a_k \rangle$. List all the elements of H, i.e. $H = \{h_1, h_2, \ldots\}$. Our goal is to prove that this list is finite, and to exhibit an N which uniformly bounds the length of the list.

Construct all $(k+1)$-tuples of the form $(a_1, a_2, \ldots, a_k, h_i) \in H^{(k+1)}$, where h_i runs through all the elements of H. For each $(k+1)$-tuple, construct the $(k+1)$-conjugacy class to which it belongs. Each $(k+1)$-tuple is then a representative of its class. Let m denote the number of $(k+1)$-conjugacy classes of G. By hypothesis, m is finite.

Consider the set of $(k+1)$-conjugacy classes described above. Note that they are not necessarily all distinct. We know that we have at most m distinct classes by hypothesis. Write a (possibly shorter) list of representatives from the distinct classes. Each representative has the form $(a_1, a_2, \ldots, a_k, h_i)$ for some $i \in \{1, 2, \ldots m\}$. We may not need all m of them. Since we seek a uniform bound, we treat the most pessimistic case possible.

Pick another arbitrary $(k+1)$-tuple $(a_1, a_2, \ldots, a_k, h_l)$ for any $h_l \in H$. By construction this new $(k+1)$-tuple must belong to one of the $(k+1)$-conjugacy classes constructed above, say the jth one. Then by definition there exists $g \in G$ such that:

$$(a_1, a_2, \ldots, a_k, h_l) = (g^{-1}a_1g, g^{-1}a_2g, \ldots, g^{-1}a_kg, g^{-1}h_jg)$$

Now observe that by the original construction of the $(k+1)$-tuples, conjugation must fix the a_is. Therefore, we have that:

$$\begin{aligned} a_1 &= g^{-1}a_1g \\ a_2 &= g^{-1}a_2g \\ &\vdots \\ a_k &= g^{-1}a_kg \\ h_l &= g^{-1}h_jg \\ \Longrightarrow h_l &= h_j \end{aligned}$$

Explanation of last equality: Conjugation by g must fix all the a_is. The a_is generate all of H. Therefore we can write $h_j = a_{j_1} \ldots a_{j_s}$. Then:

$$\begin{aligned} g^{-1}h_jg &= g^{-1}(a_{j_1} \ldots a_{j_s})g \\ &= (g^{-1}a_{j_1}g) \ldots (g^{-1}a_{j_s}g) \\ &= (a_{j_1}) \ldots (a_{j_s}) \ (since\ conjugation\ by\ g\ fixes\ the\ a_is) \\ &= h_j \end{aligned}$$

Since there are at most m distinct $(k+1)$-tuples, there are at most m choices for j. Therefore there are at most m elements in H. Take $N = k+1$ and the proof is completed. ■

Remark: A group G which satisfies the hypothesis of Theorem (2.4.1) also satisfies Property 3.

Corollary (Lawrence): 2.4.2 *A group G which satisfies the hypotheses of Theorem (2.4.1) for all $n = 1, 2, 3, \ldots$ is locally finite.*

Proof: Let $H = \langle g_1, \ldots, g_n \rangle$ be any finitely generated subgroup of G. By hypothesis, G has finitely many $(n+1)$-conjugacy classes. By Theorem (2.4.1), there is a uniform bound N on the order of H. In particular, H is finite. Therefore G is locally finite. ■

Remark: To answer our original problem positively, we require a bound on the number of k-conjugacy classes of our candidate group G. One way we could achieve this would be to require a bound on the number of isomorphism classes of k-generated subgroups. Further, we could require that all automorphisms of G be inner, i.e. given by conjugation.

Having done all of this leads us naturally to investigate Phillip Hall's universal locally finite group in Chapter 3.

Theorem (Lawrence): 2.4.3 *Let G be a torsion group, with finitely many 2-conjugacy classes. Then there exist positive integers $c < d$ such that $[b,\ {}_c a] = [b,\ {}_d a]$, $\forall\ a, b \in G$. The choice of $c,\ d$ does not depend on the choice of $a,\ b$.*

Outline of Proof: We break the proof into a series of simpler claims. At each stage we keep careful track of the bounds established in the previous step. We build toward explicit choices of $c,\ d$ which depend only on the finite number of conjugacy classes of G, not on the choice of $a,\ b$.

Proof: Let N denote the finite number of 2-conjugacy classes of G.

Claim 1: $\forall\ a,\ b \in G$, There exist $g \in G$ and positive integers $1 \le k < l \le N+1$ such that $(a,\ [b,\ {}_l a]) = (a,\ [g^{-1}bg,\ {}_k a])$.

Proof of Claim 1: Fix $a, b \in G$. Construct all pairs of the form $\{(a,\ [b,\ {}_i a]) : i \ge 1\}$. Construct the 2-conjugacy classes for each pair. By hypothesis, G has only N 2-conjugacy classes. Therefore letting i run to at least $N+1$ guarantees that at least two of the above pairs lie in the same 2-conjugacy class. Without loss of generality, say $i = k$ and $i = l$ are in the same 2-conjugacy class, with $1 \le k < l \le N+1$. Then by the definition of 2-conjugacy class, we have for some $g \in G$ that $(a,\ [b,\ {}_l a]) = (g^{-1}ag,\ g^{-1}[b,\ {}_k a]g)$.

By construction of the pairs, from the first co-ordinate, we obtain that:

$$\begin{aligned} a &= g^{-1}ag \\ \iff ga &= ag \end{aligned}$$

So g and a commute. Then we focus on the second co-ordinate to obtain that:

$$\begin{aligned} [b,\ _l a] &= g^{-1}[b,\ _k a]g \\ &= [g^{-1}bg,\ _k a]\ (by\ Fact\ (2.2.2)) \end{aligned}$$

So we have shown that given $a,\ b$, we can find $g \in G$ and $1 \leq k < l \leq N+1$ (depending on $a,\ b$) such that:

$$(a,\ [b,\ _l a]) = (a,\ [g^{-1}bg,\ _k a])$$

$\dashv$ (Claim 1)

Claim 2: $\forall\ a, b \in G$, there exist positive integers $1 \leq k < m$ (with $k < N+1$) such that $[b,\ _k a] = [b,\ _m a]$.

Proof of Claim 2: Take $a,\ b$ and $g,\ k,\ l$ to be the same as in the previous claim. Observe that, by definition:

$$[b,\ _l a] = [[b,\ _k a], \underbrace{a, \ldots, a}_{l-k\ copies}]$$

We proved above that:

$$[b,\ _l a] = [g^{-1}bg,\ _k a]$$

Equating the two RHS expressions then gives:

$$[g^{-1}bg,\ _k a] = [[b,\ _k a], \underbrace{a, \ldots, a}_{1(l-k)\ copies}]$$

Conjugate both sides by g to obtain:

$$\begin{aligned} g^{-1}[g^{-1}bg,\ _k a]g &= g^{-1}[[b,\ _k a], \underbrace{a, \ldots, a}_{1(l-k)\ copies}]g \\ [g^{-2}bg^2,\ _k a] &= [[g^{-1}bg,\ _k a], \underbrace{a, \ldots, a}_{1(l-k)\ copies}] \\ &= [[b,\ _l a], \underbrace{a, \ldots, a}_{1(l-k)\ copies}] \\ &= [[[b,\ _k a], \underbrace{a, \ldots, a}_{1(l-k)\ copies}], \underbrace{a, \ldots, a}_{1(l-k)\ copies}] \\ &= [[b,\ _k a], \underbrace{a, a, \ldots, a}_{2(l-k)\ copies}] \end{aligned}$$

By induction,

$$[g^{-t}bg^t,\ {}_ka] \quad = \quad [[b,\ {}_ka], \underbrace{a, a, \ldots, a}_{t(l-k)\ copies}]$$

G is a torsion group, therefore $g \in G$ has finite order. Say $|g| = t$. Then $g^t = g^{-t} = 1$, and:

$$\begin{aligned} [b,\ {}_ka] \quad &= \quad [[b,\ {}_ka], \underbrace{a, a, \ldots, a}_{t(l-k)\ copies}] \\ &= \quad [b,\ {}_{k+t(l-k)}a] \end{aligned}$$

Now take $m = k + t(l-k)$. The k appearing at the end of this proof is the same k from the previous claim. Therefore we have $k < N+1$, as required. Also, since in the earlier claim, $k < l$, we have that $(l-k) > 0$. The order t must be non-negative, therefore:

$$\begin{aligned} 0 \quad &< \quad t(l-k) \\ k \quad &< \quad k + t(l-k) \\ k \quad &< \quad m \end{aligned}$$

$\dashv$ (Claim 2)

Claim 3: The t that appears in the above claim must satisfy:

$$t \quad \leq \quad N$$

Proof of Claim 3: This is immediate by Lemma (2.3.1). $\dashv$ (Claim 3)

Claim 4: The k and m in the previous claim must satisfy:

$$\begin{aligned} k \quad &< \quad N+1 \ (\textit{N.B. we proved this in Claim } 1) \\ m \quad &\leq \quad (N+1)^2 \end{aligned}$$

Proof of Claim 4: From the proof of the above claim, we know that $\forall\ a, b \in G$, there exist positive integers $k < m$ with $k < N+1$ such that:

$$[b,\ {}_ka] \quad = \quad [b,\ {}_ma]$$

Rewrite the first line:

$$[b,\ {}_ka] \quad = \quad [[b,\ {}_ka],\ {}_{(m-k)}a]$$

Then, by induction on n:

$$\begin{aligned} [b,\ {}_ka] \quad &= \quad [[b,\ {}_ka],\ {}_{n(m-k)}a],\ \forall n \geq 0 \\ &= \quad [b,\ {}_{k+n(m-k)}a],\ \forall n \geq 0 \\ Moreover,\ [b,\ {}_{k+p}a] \quad &= \quad [b,\ {}_{k+p+n(m-k)}a],\ \forall n \geq 0,\ \forall p \geq 0 \qquad (2.1) \end{aligned}$$

The l and k in Claim 2 have to satisfy:

$$l - k \;<\; N + 1$$

By Claim 3 we have that:

$$t \;\leq\; N$$

Thus we have that:

$$t(l-k) \;\leq\; N(N+1)$$

Now, since $m = k + t(l-k)$:

$$\begin{aligned} m &\leq (N+1) + N(N+1) \\ &\leq (N+1)(N+1) \\ &\leq (N+1)^2 \end{aligned}$$

$\dashv$ (Claim 4)

Claim 5: $\forall\, a, b \in G$, $[b, {}_{N+1}a] = [b, {}_{(N+1)+((N+1)^2)!}a]$.

Proof of Claim 5: Take m, k to be the same as in the previous claim. Observe that:

$$\begin{aligned} m &> k \\ \Longrightarrow m - k &\geq 1 \end{aligned}$$

Also:

$$\begin{aligned} m &\leq (N+1)^2 \\ \Longrightarrow m - k &< (N+1)^2 \end{aligned}$$

Putting the above facts together, write: $1 \leq m - k < (N+1)^2$. Also observe that:

$$\begin{aligned} m - k &< (N+1)^2 \\ \Longrightarrow m - k &\mid ((N+1)^2)! \end{aligned}$$

Then in equation (2.1), take:

$$\begin{aligned} k + p &= N + 1 \; (\textit{for the correct choice of a positive integer } p) \\ n &= \frac{((N+1)^2)!}{m-k} \end{aligned}$$

Then we get that $\forall\ a, b \in G,\ [b,\ _{N+1}a] = [b,\ _{(N+1)+((N+1)^2)!}a]$. $\dashv$ (Claim 5)

Now to complete the proof of the theorem, take:

$$\begin{aligned} c &= (N+1) \\ d &= (N+1) + ((N+1)^2)! \ \blacksquare \end{aligned}$$

Remark: This result suggests to us that we should further investigate extended commutators and Engel groups. We do this in Chapter 6.

2.5 Conjecture that a Positive Answer Exists, and More Properties

Conjecture (Lawrence): 2.5.1 *There exists an infinite, locally finite group G, satisfying Property 2 and Property 3.*

Theorem (Lawrence): 2.5.2 *If Conjecture (2.5.1) is true, then we would have an example of an infinite group G with finitely many k-conjugacy classes, for all $k = 1, 2, 3, \ldots$.*

Outline of Proof: Use the uniform bound on the order of k-generated subgroups to argue that there can be only finitely many isomorphism classes of k-generated subgroups. Then argue that each isomorphism class can yield only finitely many k-conjugacy classes.

Proof: Let G satisfy the above hypothesis. In other words, let G be an infinite, locally finite group which satisfies property 2 and property 3. We shall demonstrate that this implies G has only finitely many k-conjugacy classes, for all $k = 1, 2, 3, \ldots$.

Fix a positive integer k.

Consider the set of all k-tuples $(g_1, \ldots, g_k)$. Form the k-generated subgroups using the k-tuples as generators: $\langle g_1, \ldots, g_k \rangle$. Since G satisfies property 3, we have a uniform bound on the size of any k-generated subgroup $\langle g_1, \ldots, g_k \rangle \leq G$. Let $\Psi(k)$ denote this uniform bound on the order of the k-generated subgroups.

Let $\alpha(k)$ denote the number of isomorphism classes of groups of order $\leq \Psi(k)$. Then $\alpha(k)$ is finite since $\Psi(k)$ is finite.

A Note on the Bound on $\alpha(k)$: In [10], Holt proved the following bound on the number of isomorphism classes for groups of a fixed order. Let $\alpha(k)$ denote the number of isomorphism classes for groups of order k. Let $k = \prod_{i=1}^{l} p_i^{g_i}$ be the prime factorization of k. Let $\lambda = \lambda(k) = \sum_{i=1}^{l} g_i$. Let $\mu = \mu(k) = max_{i=1}^{l} g_i$. Then:

$$\alpha(k) \leq k^{\lambda} \cdot \prod_{i=1}^{l} p_i^{(\frac{g_i^3}{6})}$$

The subgroup generated by any k-tuple falls into one of these finitely many isomorphism classes. Focus on any single isomorphism class. We will show that this isomorphism class must yield only finitely many conjugacy classes.

Let any two subgroups in the isomorphism class be denoted as follows:

$$\begin{aligned} G_1 &= \langle g_{11}, \ldots, g_{1k} \rangle \\ G_2 &= \langle g_{21}, \ldots, g_{2k} \rangle \end{aligned}$$

Fix a subgroup $H = \langle h_1, \ldots, h_k \rangle$ in the same isomorphism class as G_1, G_2. Then,

$$G_1 \cong H \cong G_2$$

Since G satisfies property 2), we can write:

$$g_1^{-1} G_1 g_1 = H = g_2^{-1} G_2 g_2$$

for some $g_1, g_2 \in G$.

Then, since $h_1, \ldots, h_k \in H$,

$$\begin{aligned} h_1 &= (g_1^{-1})(g_{11}^*)(g_1), \ for\ some\ g_{11}^* \in G_1 \\ h_2 &= (g_1^{-1})(g_{12}^*)(g_1), \ for\ some\ g_{12}^* \in G_1 \\ &\vdots \\ h_k &= (g_1^{-1})(g_{1k}^*)(g_1), \ for\ some\ g_{1k}^* \in G_1 \end{aligned}$$

Similarly,

$$\begin{aligned} h_1 &= (g_2^{-1})(g_{21}^*)(g_2), \ for\ some\ g_{21}^* \in G_2 \\ h_2 &= (g_2^{-1})(g_{22}^*)(g_2), \ for\ some\ g_{22}^* \in G_2 \\ &\vdots \\ h_k &= (g_2^{-1})(g_{2k}^*)(g_2), \ for\ some\ g_{2k}^* \in G_2 \end{aligned}$$

Now equate the 2 sets of RHS expressions:

$$\begin{aligned}
(g_1^{-1})(g_{11}^*)(g_1) &= (g_2^{-1})(g_{21}^*)(g_2) \\
(g_1^{-1})(g_{12}^*)(g_1) &= (g_2^{-1})(g_{22}^*)(g_2) \\
&\vdots \\
(g_1^{-1})(g_{1k}^*)(g_1) &= (g_2^{-1})(g_{2k}^*)(g_2)
\end{aligned}$$

Now observe that since G has property 3, we have a uniform bound on $|G_1|$ and $|G_2|$. Since:

$$\begin{aligned}
g_{11}^*, \ldots, g_{1k}^* &\in G_1, \ and \\
g_{21}^*, \ldots, g_{2k}^* &\in G_2,
\end{aligned}$$

there can be only finitely many choices for $g_{11}^*, \ldots, g_{1k}^*$ and $g_{21}^*, \ldots, g_{2k}^*$. Fix one choice. Then for this fixed choice, we obtain:

$$\begin{aligned}
g_{11}^* &= (g_1 g_2^{-1})(g_{21}^*)(g_2 g_1^{-1}) \\
g_{12}^* &= (g_1 g_2^{-1})(g_{22}^*)(g_2 g_1^{-1}) \\
&\vdots \\
g_{1k}^* &= (g_1 g_2^{-1})(g_{2k}^*)(g_2 g_1^{-1})
\end{aligned}$$

Re-write the above as:

$$\begin{aligned}
g_{11}^* &= (g_2 g_1^{-1})^{-1}(g_{21}^*)(g_2 g_1^{-1}) \\
g_{12}^* &= (g_2 g_1^{-1})^{-1}(g_{22}^*)(g_2 g_1^{-1}) \\
&\vdots \\
g_{1k}^* &= (g_2 g_1^{-1})^{-1}(g_{2k}^*)(g_2 g_1^{-1})
\end{aligned}$$

In other words, $(g_{11}^*, \ldots, g_{1k}^*)$ and $(g_{21}^*, \ldots, g_{2k}^*)$ lie in the same conjugacy class. To summarize, for the choice we fixed above, we obtain only one conjugacy class. We argued earlier that for a given isomorphism class, there are only finitely many choices. Thus, we have established that for a given isomorphism class, we obtain only finitely many k-conjugacy classes.

Now since the number of isomorphism classes also has to be finite, we have the desired result: G has finitely many k-conjugacy classes. We have exhibited an infinite group G with only finitely many k-conjugacy classes. As k was chosen arbitrarily, the result holds for all $k = 1, 2, 3, \ldots$. ■

If we do not try to find a group G which answers the question for all $k = 1, 2, 3, \ldots$, but for the moment focus on a particular fixed k, then we can get some control over the number of k-conjugacy classes of G using HNN extensions and the property of G being existentially closed. If we use these means to pursue a group G which is a positive answer to our question for this fixed k, then we would like a method to determine whether two candidate groups are really the same up to isomorphism. We need one new definition first.

Definition (Skeleton): For any group G, the *skeleton* of G, denoted by $Sk\ G$, is the class of all finitely generated groups that can be embedded in G.

Theorem: 2.5.3 *Let G and H be groups which are:*

1. *locally finite*
2. *countably infinite*
3. *satisfy property 2*
4. *have identical skeletons*

Then $G \cong H$.

Outline of Proof: This is a standard "back-and-forth" argument.

Proof: The goal of the proof is to build up G and H as infinite unions of their subgroups, with isomorphisms between the subgroups, as follows:

$$\begin{array}{ccccccccc} A_1 & \leq & A_2 & \leq & A_3 & \leq & \cdots & \leq & G \\ \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_3} & & & & \downarrow{\scriptstyle \phi} \\ B_1 & \leq & B_2 & \leq & B_3 & \leq & \cdots & \leq & H \end{array}$$

We proceed in a "zig-zag" fashion to ensure that we pick up all the elements of G and H in the infinite unions.

Since G and H are countable, we can write:

$$\begin{aligned} G &= \{g_1, g_2, \ldots\} \\ H &= \{h_1, h_2, \ldots\} \end{aligned}$$

Let $\langle g_1 \rangle = A_1 \leq G$. Since G and H have identical skeletons, then we know that there exists some $B_1 \leq H$ with $A_1 \underbrace{\cong}_{\phi_1} B_1$.

Let $B_2 = \langle B_1 \cup \{h_1, h_2\} \rangle \leq H$. Then since H is locally finite, we have that B_2 is finite. It is also clear that we have enlarged: $B_1 \leq B_2$.

Since G and H have identical skeletons, we know that there exists some $A_2^* \leq G$ with $B_2 \underbrace{\cong}_{\phi_2} A_2^*$. We now wish to obtain $A_2 \leq G$ such that $A_1 \leq A_2$. Therefore we must construct A_2 using A_2^*. There must exist a subgroup $A_1^* \leq A_2^*$ such that $A_1 \cong A_1^*$. To see it, recall the following facts demonstrated above:

$$\begin{aligned} A_1 &\cong B_1 \\ B_1 &\leq B_2 \\ B_2 &\cong A_2^* \end{aligned}$$

Then since G and H satisfy property 2, we have that $A_1 = g^{-1}A_1^*g$, for some $g \in G$.

Now,

$$\begin{aligned} A_1^* &\leq A_2^* \\ g^{-1}A_1^*g &\leq g^{-1}A_2^*g \end{aligned}$$

Let $A_2 = g^{-1}A_2^*g$. Then we obtain $A_1 \leq A_2$, as required.

Now let $A_3 = \langle A_2 \cup \{g_1, g_2, g_3\}\rangle \leq G$. Then $A_2 \leq A_3$. Since G and H have identical skeletons, obtain: $A_3 \underbrace{\cong}_{\phi_3} B_3^*$. The pattern of the construction is now clear. Then inductively, we obtain the following subgroups and isomorphisms ϕ_i:

$$\begin{array}{ccccccccc} A_1 & \leq & A_2 & \leq & A_3 & \leq & \cdots & \leq & G \\ \phi_1 \downarrow & & \phi_2 \uparrow & & \phi_3 \downarrow & & & & \\ B_1 & \leq & B_2 & \leq & B_3 & \leq & \cdots & \leq & H \end{array}$$

Take infinite unions to obtain the groups G and H. From the construction above, it is clear that every element of G and H will get included in the respective infinite union. G and H will be equal to the infinite unions as required. To complete the proof, we must exhibit an isomorphism from G to H.

Exhibiting the Isomorphism from G to H: Since all the ϕ_is are isomorphisms, we are free to write them all in the same direction:

$$\begin{array}{ccccccccc} A_1 & \leq & A_2 & \leq & A_3 & \leq & \cdots & \leq & G \\ \phi_1 \downarrow & & \phi_2 \downarrow & & \phi_3 \downarrow & & & & \\ B_1 & \leq & B_2 & \leq & B_3 & \leq & \cdots & \leq & H \end{array}$$

Fix any $g_1, g_2 \in G$. Then by the construction above, there exists $A_i \leq G$ such that $g_1, g_2 \in A_i$, for some i. Then take the map $\phi_i : G \to H$ and restrict: $\phi_i|_{A_i}$. This restriction is an isomorphism, having g_1, g_2 in its domain. Further, the same holds for $\phi_j|_{A_j}, \forall j \geq i$. Therefore ϕ_j has the isomorphism behaviour we require with respect to g_1, g_2. Using this method, we can find an isomorphism from G to H that works for any $g_1, g_2 \in G$. Therefore, we get the desired result that $G \cong H$. ■

The following theorem may help to narrow our search for a group which answers our question positively.

Theorem (Lawrence): 2.5.4 *Suppose that a group G is infinite, with finitely may k-conjugacy classes, for all $k = 1, 2, 3, \ldots$. Then there exists a countably infinite subgroup $H \leq G$, with finitely may k-conjugacy classes, for all $k = 1, 2, 3, \ldots$.*

Outline of Proof: Start with any countable subgroup. Fix a positive integer k. Inductively enlarge the group by adding elements of G which conjugate one k-tuple onto another. Then construct a countable group in which the set of k-conjugacy classes is a subset of the k-conjugacy classes of G. Last, notice that repeating this construction for all $k = 1, 2, 3, \ldots$ still yields a countable group which answers our question positively.

Proof: Let $H_1 < G$ be any countable subgroup. Fix a positive integer k. Consider the countably many k-tuples $(a_1, \ldots, a_k) \in H_1^k$. If $(a_1, \ldots, a_k)$ and $(b_1, \ldots, b_k)$ lie in the same k-conjugacy class of G, then there exists $g_1 \in G$ such that $(a_1, \ldots, a_k) = (g_1^{-1} b_1 g_1, \ldots, g_1^{-1} b_k g_1)$. Take all such g_is and form a larger subgroup:

$$H_2 = \langle H_1, g_1, g_2, \ldots \rangle$$

Then H_2 is also countable. Repeat the above construction starting from H_2 to create a countable:

$$H_3 = \langle H_2, g_1, g_2, \ldots \rangle$$

In this way, construct a chain of countable groups:

$$H_1 \leq H_2 \leq H_3 \leq \cdots$$

Let:

$$H_k^* = \bigcup_{j=1,2,3,\ldots} H_j$$

Then H_k^* is a countable group. Perform the above construction for all $k = 1, 2, 3, \ldots$. Take:

$$H = \bigcup_{k=1,2,3,\ldots} H_k^*$$

Then $H < G$ is a countable group. Also, by the above construction, the k-conjugacy classes of H are a subset of the k-conjugacy classes of G. Therefore, we have that H has only finitely many k-conjugacy classes for all $k = 1, 2, 3, \ldots$, since G does. ■

This corollary may help to further focus our search for a positive answer, or a proof that no such positive answer exists.

Corollary (Lawrence): 2.5.5 *The group H which is constructed in Theorem (2.5.4) embeds into the countable universal locally finite group U.*

Remark: The existence and uniqueness of the countable universal locally finite group U is proved in Theorem (3.3.4).

Proof: Since H has finitely many k-conjugacy classes, for all $k = 1, 2, 3, \ldots$, we have by Corollary (2.4.2) that H is locally finite. Then since H is countable, it embeds into U by Theorem (3.3.1 3). ■

Later we shall show that H must be a proper subgroup of U.

2.6 Other Remarks

If a group G has finitely many k-conjugacy classes, for all $k = 1, 2, 3, \ldots$, then (by Theorem (2.4.1)) G has a uniform bound on the size of its k-generated subgroups, for all $k = 1, 2, 3, \ldots$. This implies (by Corollary (2.4.2)) that G is locally finite. Therefore we wish to explore locally finite groups further. In particular, we want to look at Phillip Hall's universal locally finite group, since this group also satisfies Property 2, which was one of the hypotheses of Theorem (2.5.2).

If we do not seek a group G which is a positive answer for all $k = 1, 2, 3, \ldots$, but for the moment fix a particular k, then we can use a construction called an HNN extension to make isomorphic k-generated subgroups conjugate in the HNN extension of G. If in addition G is existentially closed, then we can make the isomorphic k-generated subgroups conjugate in G itself. Through these means we can get some control over the number of k-conjugacy classes of G. Therefore we investigate existentially closed groups and HNN extensions further.

The identity on extended commutators proved in Theorem (2.4.3) also suggests that we should further investigate Engel groups.

Chapter 3

Phillip Hall's Universal Locally Finite Group

3.1 Introduction

Universal locally finite groups are relevant to our k-conjugacy class problem because they satisfy Property 1 (2.1.6 1) and Property 2 (2.1.6 2) from our Preliminary Results chapter. If we could find a universal locally finite group that also satisfied Property 3 (2.1.6 3), we would have a positive answer to our question, by Theorem (2.5.2).

Thus we will take a closer look at universal locally finite groups.

3.2 Definition of Universal Locally Finite Group

Definition (Universal): A locally finite group U is *universal* if:

1. Every finite group can be embedded into U.
2. Any two isomorphic finite isomorphic subgroups of U are conjugate in U.

Remark: The name "universal" is used because a universal locally finite group provides a "universe" for doing finite group theory.

3.3 Theorems

Theorem (P. Hall): 3.3.1 *Let U be a universal locally finite group. Then:*

1. *For any two finite subgroups A, B of U, every isomorphism of A onto B is induced by an inner automorphism of U.*

2. *If A is a subgroup of the finite group B, then every embedding of A into U can be extended to an embedding of B into U.*

3. *U contains a copy of every countable, locally finite group.*

4. *Let C_m denote the set of all elements of order $m > 1$ in U. Then C_m is a single class of conjugate elements and $U = C_m C_m$. In particular, U is simple.*

Outline of Proof (1): First, note that this result is indeed stronger than the definition of universal. The definition of universal gives us that there is *some* inner automorphism of U sending A to B. This result says that *any* automorphism of U sending A to B must be inner.

Form the holomorph of A, $Hol\ A$. Notice that since $Hol\ A$ is finite, it also embeds into U. Obtain a subgroup C of the holomorph isomorphic to A and therefore also isomorphic to B. Construct an automorphism of C using an arbitrary isomorphism from A to B, and two elements of U which conjugate A and B onto C respectively. Use this automorphism of C to obtain an element of U which conjugates C onto itself. Use this element of U to obtain a new element of U which acts by conjugation on A in precisely the same way as the arbitrary isomorphism onto B.

Proof (1): Let A, B be finite isomorphic subgroups of U. Let:

$$\alpha : A \underbrace{\longrightarrow}_{\cong} B$$

be an isomorphism. Let $H = Hol\ A$ denote the holomorph of A. Since A is finite, H is also finite. Therefore H can also be embedded into U. Thus by properties of the holomorph, U contains finite subgroups C and G where $C \cong A$, G normalizes C, and G acts by conjugation on C as its full group of automorphisms.

Here, think of:

$$Hol\ A = \underbrace{A^\rho}_{C} \rtimes \underbrace{(Aut\ A)}_{G}$$

where A^ρ denotes the right regular representation of A. G normalizes C because A^ρ is a normal subgroup of $Hol\ A$.

Since U is universal, there exist $a, b \in U$ such that $a^{-1}Aa = b^{-1}Bb = C$. The mapping $c \mapsto b^{-1}((aca^{-1})\alpha)b$ defines an automorphism of C. We can see that this map defines an automorphism of C by following the compositions of isomorphisms:

$$C \xrightarrow[\text{conjugate by } a^{-1}]{\cong} A \xrightarrow[\alpha]{\cong} B \xrightarrow[\text{conjugate by } b]{\cong} C$$

By the construction of the holomorph, every automorphism of C is induced by conjugation by some $g \in G$. In other words, we then have that there exists $g \in G$ such that:

$$b^{-1}((aca^{-1})\alpha)b = g^{-1}cg, \forall c \in C$$

Let $y \in A$ be arbitrary. Then, by the above argument:

$$\begin{aligned}
(y)\alpha &= (a(a^{-1}ya)a^{-1})\alpha \\
&= b(b^{-1}((a(a^{-1}ya)a^{-1})\alpha)b)b^{-1} \\
&= b(g^{-1}(a^{-1}ya)g)b^{-1} \\
&= (agb^{-1})^{-1}y(agb^{-1})
\end{aligned}$$

In other words, conjugation by $agb^{-1} \in U$ induces the automorphism α. Since α was chosen arbitrarily, the result must hold for any isomorphism from A to B.

Outline of Proof (2): Use the fact that $A \leq B$ can both be embedded into U to produce an isomorphism between the 2 embeddings of A (one from embedding A and the other from embedding the copy of A contained in B). Then by part (1) this isomorphism must be induced by conjugation by some element of U. Use this element to produce an embedding of B into U which restricts to the original embedding of A into U.

Proof (2): By hypothesis, there exist embeddings $\phi : A \to U$ and $\psi : B \to U$. Then $\psi^{-1}\phi$ induces an isomorphism of A^{ψ} onto A^{ϕ}. Then by part (1), there exists $g \in U$ which induces this isomorphism. In other words, $a^{\psi g} = a^{\phi}, \forall a \in A$. But then the map $b \mapsto b^{\psi g}$ is an embedding of B into U. Moreover, its restriction to A is ϕ. This shows that the embedding of A into U extends to an embedding of B into U, as required.

Outline of Proof (3): Consider the local system of the countable locally finite group G. One such system must exist by Lemma (2.1.5.1). Show that each group of the local system can be embedded into U. Then by induction, produce an embedding of G into U.

Proof (3): Let G be any countable, locally finite group. Then by Lemma (2.1.5.1), G contains a local system of finite subgroups G_i, linearly ordered with respect to inclusion. The proof is by induction on the local system. Let $n \in \mathbb{N}$ be such that for all $i \in \mathbb{N}$, $i \leq n$, embeddings $\phi_i : G_i \to U$ have been determined such that, if $i + 1 \leq n$, the embedding ϕ_i is the restriction to G_i of the embedding ϕ_{i+1}.

Then, by part (2), there is an embedding ϕ_{n+1} of G_{n+1} into U extending ϕ_n. So inductively, one may choose a sequence $\{\phi_i\}_{i\in\mathbb{N}}$, and this sequence determines an embedding of G into U. In other words, U contains a copy of G.

Outline of Proof (4):

1. Demonstrate that any group which is generated by each of its non-trivial conjugacy classes is a simple group.

2. The cyclic groups generated by each element of order m are all finite and all isomorphic to one another. Therefore, all these groups embed into U and moreover they are all conjugate in U, since U is universal. The generators of each cyclic group of order m are therefore all conjugate in U. Therefore all elements of order m are conjugate in U.

3. Lemma (3.3.2) demonstrates that an arbitrary element of the group U can always be written as a product of 2 elements of order m. Then it is clear that two copies of the conjugacy class of elements of order m generate the whole of U.

Proof (4):

Claim:

$$U \textit{ is generated by each non}-\textit{trivial conjugacy class} \implies U \textit{ is simple}$$

Proof of Claim: We prove the contrapositive. Suppose that U is not simple. Then let $N \lhd U$ be a non-trivial normal subgroup properly contained in U. Then let $1 \neq n \in N$. Such an n must exist since N is non-trivial. Then the conjugacy class of n is contained in N, since N is fixed setwise by conjugation by all elements of U. Thus the conjugacy class of N does not generate U, since it is properly contained in U. The conjugacy class of n is non-trivial, since $n \neq 1$. Thus we have a non-trivial conjugacy class which does not generate U. This completes the proof of the claim. $\dashv$ (Claim)

Let $u, v \in U$ have order m. Then $\langle u\rangle, \langle v\rangle$ are both cyclic of order m, and therefore isomorphic to one another. Apply (1) to $\langle u\rangle, \langle v\rangle$ and obtain that these groups are conjugate in U.

Now let $x \in U$ have order n. Suppose that there exists a finite 2-generator group $\langle a, b\rangle$ where a, b have order m, and the element ab has order n. Since this group is finite, there exists an embedding ϕ of $\langle a, b\rangle$ into U.

We shall show that such a group always exists in Lemma (3.3.2).

Observe that $a^\phi b^\phi$ and x both have order n. Therefore there exists $g \in U$ such that $(a^\phi b^\phi)^g = a^{\phi g} b^{\phi g} = x$. This shows that the arbitrary element $x \in U$ can be written as a product of two elements or order m. The result now follows by Lemma (3.3.2). ■

Part (2) tells us that if such a group exists, then it has a local system consisting of finite symmetric groups. This must be true since every finite group may be embedded into some finite symmetric group, by Cayley's Theorem.

Additional Points Regarding This Theorem from [5]:

1. Since $C_m = C_m^{-1}$, part (4) may also be expressed by saying that $U = JJ^{-1}$ for every class of conjugate elements J in U, other than the unit class. It would be interesting to know whether there exist any finite simple groups with this property. On the whole, it seems unlikely.

   Examples of Finite Simple Groups that Do Not Have the Desired Property: It is well known that the alternating groups A_n are simple for $n \geq 5$. We show here that in all cases except possibly $n = 6$, A_n is a finite simple group not having the desired property.

   When $n \geq 7$: Take J to be the conjugacy class of A_n containing all the 3-cycles. For a proof that these 3-cycles all lie in the same conjugacy class of A_n, refer to [18], Theorem 3.8 i). The inverse of any 3-cycle is another 3-cycle, so $J = J^{-1}$. It is also clear that J is non-trivial.

   Since $n \geq 7$, A_n contains a 7-cycle. However, it is clear that we cannot write a 7-cycle as a product of 2 3-cycles. Therefore, $JJ^{-1} = JJ$ does not generate the whole of A_n for $n \geq 7$.

   When $n = 6$: We believe that A_6 also does not have this property, although we have not rigorously proved it yet.

   When $n = 5$: Take J to be the conjugacy class of (12345) in A_5. Then J is clearly non-trivial. Recall that A_5 has 2 conjugacy classes containing 5-cycles, each of size 12. For a proof, refer to [1], p128.

   Claim 1: $J = J^{-1}$.

Proof of Claim 1: Let $j \in J$ be arbitrary. Then we can write $j = \sigma^{-1}(12345)\sigma$, for some $\sigma \in A_5$. Then $j^{-1} = \sigma^{-1}(54321)\sigma$. It is enough to show that $j^{-1} \in J$.

Let $\tau = (12)(35) \in A_5$. Then $\tau^{-1} = \tau$. Also:

$$\begin{aligned} \tau^{-1}(12345)\tau &= ((12)(35))(12345)((12)(35)) \\ &= (54321) \end{aligned}$$

Therefore:

$$\begin{aligned} j^{-1} &= \sigma^{-1}(54321)\sigma \\ &= \sigma^{-1}(\tau^{-1}(12345)\tau)\sigma \\ &= (\tau\sigma)^{-1}(12345)(\tau\sigma) \end{aligned}$$

showing that $j^{-1} \in J$, as required. $\dashv$ (Claim 1)

Claim 2: $(12)(34) \notin JJ = JJ^{-1}$.

Proof of Claim 2: We shall show that any product of 2 5-cycles which equals (12)(34) requires one 5-cycle from J and one 5-cycle from outside of J. Without loss of generality, we may start with:

$$(12)(34) = (5 * * * *)(5 * * * *)$$

We must send 5 somewhere in cycle 1. Without loss of generality because of symmetry, send it to 1:

$$(12)(34) = (51 * **)(5 * * * *)$$

Since 5 is fixed by the LHS, we must then send 1 to 5 in cycle 2:

$$(12)(34) = (51 * **)(5 * * * 1)$$

For the next position of cycle 1, note that 2 is not possible. If we write a 2 here, then we have nowhere to put our 2 in cycle 2 which preserves the (12) portion of the LHS. Therefore the next position of cycle 1 must be 3 or 4. Without loss of generality because of symmetry, let it be 4:

$$(12)(34) = (514 * *)(5 * * * 1)$$

For the next position of cycle 1, note that 3 is not possible. If we write a 3 here, then we have nowhere to put our 3 in cycle 2 which preserves the (34) portion of the LHS. Therefore the next position of cycle 1 can only be 2. Cycle 1 is now completely known:

$$(12)(34) = (51423)(5 * * * 1)$$

Since we have $2 \mapsto 3$ in cycle 1, we require $3 \mapsto 1$ in cycle 2:

$$(12)(34) = (51423)(5 * *31)$$

Since we have $4 \mapsto 2$ in cycle 1, we require $2 \mapsto 3$ in cycle 2. Cycle 2 is now completely known:

$$(12)(34) = (51423)(54231)$$

Now since $(243)^{-1}(12345)(243) = (51423)$, we have that $(51423) \in J$. Also since $(2534)^{-1}(12345)(2534) = (54231)$, we have that an odd permutation is required to conjugate (12345) onto (54231), in other words $(54231) \notin J$. $\dashv$ (Claim 2)

We have explicitly exhibited an element of A_5, namely $(12)(34)$, which is not in JJ^{-1}. Therefore JJ^{-1} does not generate the whole of A_5, and A_5 does not have the required property, as claimed.

2. We claim that the fact that every finite group can be embedded into U implies that U contains infinitely many copies of each non-trivial finite group.

   Proof of Claim: Let G denote a non-trivial finite group. By contradiction, suppose that U contains only n copies of G, where n is finite. Since G is finite, the following is also a finite group:

$$H = \underbrace{G \times \cdots \times G}_{n+1}$$

   Therefore H also embeds into U, since U is universal. But H clearly contains $n+1$ subgroups, each isomorphic to G. Therefore U also contains $n+1$ subgroups, each isomorphic to G. This contradicts the fact that G contains only n copies of G. Therefore the number of copies of G in U is infinite, as claimed. $\dashv$ (Claim)

3. Part (4) of the theorem also implies that given any $u \in U$, we can solve:

$$x^m = y^m = 1; \; xy = u$$

   for $x, \; y$ in U, for any $m > 1$.

Lemma: 3.3.2 *For any two integers $m > 1, n \geq 1$, there exists a finite 2-generator group $\langle a, b \rangle$ such that a, b have order m and ab has order n.*

Outline of Proof: Let $\langle a\rangle$ be a cyclic group of order m. Let $\langle c\rangle$ be a cyclic group of order n. Let G be the standard wreath product $\langle c\rangle \wr \langle a\rangle$ of $\langle c\rangle$ by $\langle a\rangle$. Now take this element of the base group:

$$d = (\underbrace{c, c^{-1}, 1, \ldots, 1}_{m})$$

Notice that the order of d in the base group is n, since the order of c is n. Let $b = d^a$. Show that the order of b in the wreath product is m. Show that the subgroup $\langle a, b\rangle$ of G is a finite group with $|a| = |b| = m$ and $|b^{a^{-1}}| = |d| = n$.

Remark: Refer to [4] for a different proof of this result, using free products rather than wreath products.

Proof: Let $\langle a\rangle$ be a cyclic group of order m. Let $\langle c\rangle$ be a cyclic group of order n. Let G be the standard wreath product $\langle c\rangle \wr \langle a\rangle$ of $\langle c\rangle$ by $\langle a\rangle$.

Mnemonic Diagram For This Wreath Product:

$$\begin{gathered} A = C_m = \langle a\rangle \\ \downarrow \ \textit{acts on} \\ \underbrace{\langle c_1\rangle \times \cdots \times \langle c_m\rangle}_{m} \end{gathered}$$

The action of a on $\underbrace{\langle c_1\rangle \times \cdots \times \langle c_m\rangle}_{m}$ is to rotate all co-ordinates one position to the right.

$$\begin{array}{ccccc} a & : & \underbrace{\langle c_1\rangle \times \cdots \times \langle c_m\rangle}_{m} & \underbrace{\rightarrow}_{\textit{acts on}} & \underbrace{\langle c_1\rangle \times \cdots \times \langle c_m\rangle}_{m} \\ & & (C_1, \ldots, C_m)^a & = & (C_m, C_1, \ldots, C_{m-1}) \end{array}$$

The base group B of G is the set of m-tuples:

$$\{(C_1, \ldots, C_m) : C_i \in \langle c\rangle, \ \forall i\}$$

Now let $d = (\underbrace{c, c^{-1}, 1, \ldots, 1}_{m})$. Notice that $d \in B$. Also notice that the order of d in B is n, since the order of c is n. Let $b = d^a$.

Claim: The order of b in the wreath product $\langle c\rangle \wr \langle a\rangle$ is m.

Proof of Claim: The wreath product $\langle c\rangle \wr \langle a\rangle$ is the semi-direct product:

$$(\langle c_1\rangle \times \cdots \times \langle c_m\rangle) \rtimes \langle a\rangle$$

where a acts on the normal subgroup $(\langle c_1\rangle \times \cdots \times \langle c_m\rangle)$ by rotating the co-ordinates one position to the right, i.e.

$$(C_1, \ldots, C_m)^a = (C_m, C_1, \ldots, C_{m-1})$$

So we may understand $b = d^a$ as the ordered pair:

$$(d, a) = ((c, c^{-1}, 1 \ldots, 1), a)$$

Denote the operation in the semi-direct product by $*$, then we have:

$$\begin{aligned}
(d^a)^2 &= (d^a) * (d^a) \\
&= ((c, c^{-1}, 1 \ldots, 1), a) * ((c, c^{-1}, 1 \ldots, 1), a) \\
&= ((c, c^{-1}, 1 \ldots, 1)(1, c, c^{-1}, 1 \ldots, 1), a^2) \\
&= ((c, 1, c^{-1}, 1 \ldots, 1), a^2)
\end{aligned}$$

Now that we can see the pattern for taking powers of $b = d^a$ in the semi-direct product, it is clear that $(d^a)^m = 1_{\langle c \rangle \wr \langle a \rangle}$, and no lesser power than m of $b = d^a$ can equal $1_{\langle c \rangle \wr \langle a \rangle}$. Thus b has order m in $\langle c \rangle \wr \langle a \rangle$ as claimed. $\dashv$ (Claim)

Therefore the subgroup $\langle a, b \rangle$ of G is a finite group with $|a| = |b| = m$ and $|b^{a^{-1}}| = |(d^a)^{a^{-1}}| = |d| = n$. $\blacksquare$

Foundation for Defining the Constricted Symmetric Group: Let G be any locally finite group. Let $\overline{S}$ be the full symmetric group on the set G. Let ρ denote the right regular representation of G in $\overline{S}$.

Recall: The right regular representation ρ identifies elements of G as follows:

$$\begin{aligned}
g^\rho \;:\; G &\rightarrow \overline{S} \\
g &\mapsto (x \mapsto xg)
\end{aligned}$$

Then, for all $x, y \in G$:

$$\begin{aligned}
(x\langle y \rangle)^{y^\rho} &= x\langle y \rangle y \text{ (\textit{i.e. act by y^ρ means multiply on right by y})} \\
&= x\langle y \rangle
\end{aligned}$$

Here we can think of $x\langle y \rangle$ in a more set-theoretic way:

$$x\langle y \rangle = \{x, xy, xy^2, \ldots, xy^t\}$$

where $|y| = t + 1$. Notice that since G is locally finite, any $y \in G$ must have finite order by definition. Let:

$$S = \{\sigma \in \overline{S} : \exists \textit{ finite subgroup } F_\sigma < G \textit{ such that } (xF_\sigma)^\sigma = xF_\sigma, \forall x \in G\}$$

Notes on the Above Definition:

1. In words, "There must be a finite subgroup F_σ such that every left coset of F_σ is fixed setwise by σ."
2. The existence of F_σ depends on the choice of σ. There is not necessarily one choice of F_σ that works for all choices of σ. As long as for the chosen σ, at least one F_σ exists which satisfies the definition, then σ passes the test to lie in S.
3. $xF_\sigma \subseteq G$, since $F_\sigma < G$ and $x \in G$
4. $(xF_\sigma)^\sigma$ denotes the action of σ on the left coset xF_σ.
5. $(xF_\sigma)^\sigma = xF_\sigma$ indicates that σ fixes the left coset xF_σ setwise when it acts.

Let $y \in G$ be arbitrary. Then $y^\rho \in \overline{S}$ by the definition of $\overline{S}$.

Claim 1: Moreover, $y^\rho \in S$, where S is defined as above.

Proof of Claim 1: Recall we showed above that $(x\langle y\rangle)^{y^\rho} = x\langle y\rangle$ for all $x,\ y \in G$. Let $\sigma = y^\rho$. Take $F_\sigma = \langle y\rangle$. Then F_σ must be finite since it is finitely generated and G is locally finite. Also, xF_σ is clearly fixed setwise by $\sigma = y^\rho$, by the above argument. Then $y^\rho \in S$, and the claim is proved. $\dashv$ (Claim 1)

So we see that when we embed a locally finite group G into $\overline{S}$ via its right regular representation, we are really embedding G into the constricted symmetric group S, which is properly contained in $\overline{S}$. This distinction will turn out to be important later, when we seek to apply Lemma (3.3.3) to show that the group we construct during the proof of Theorem (3.3.4) is in fact universal.

Claim 2: $S \subseteq \overline{S}$ as defined above is a locally finite group.

Proof of Claim 2: Let T be any finitely generated subgroup of S. Then $T = \langle \sigma_1, \dots, \sigma_r\rangle$ where $\sigma_i \in S$. Let $F = \langle F_{\sigma_1}, \dots, F_{\sigma_r}\rangle$. Notice that the F_{σ_i}s must exist by the definition of S. By the definition of S, the F_{σ_i}s are finite. The F_{σ_i}s are subgroups of a locally finite group G. Therefore we can see that F, viewed as a set, must be finite.

Then $(xF)^\sigma = xF, \forall \sigma \in T$ and $\forall x \in G$. This embeds T into a finite (r-fold) Cartesian product of copies of the symmetric group on the finite set F. Therefore T is finite.

Therefore S is locally finite, as claimed. $\dashv$ (Claim 2)

Definition (Constricted Symmetric Group): We call the group S defined as above the *constricted symmetric group on G*. Notice that if G is finite then $S = \overline{S}$. Since S depends on the structure of G, it is not a canonical subgroup of $\overline{S}$.

Note that any σ which fixes all but finitely many $x \in G$ lies in S (take the finite group F_σ to be the the group generated by the finite number of elements which σ moves). This shows that S_∞ embeds into S in a natural way. However, S is larger than S_∞, as we will now demonstrate.

Example of Constricted Symmetric Group: Let:

$$G = \bigotimes_{i=1}^{\infty} \mathbb{Z}_2$$

G is the direct sum of infinitely many copies of $\mathbb{Z}_2$. Let us think of building this group up from right to left. This makes the most sense since we will shortly define a bijection between elements of G and the natural numbers, via the binary representation using the "digits" from G. Let us require that all elements of G can have only finitely many 1s, and the rest of the digits 0.

The group operation is component-wise addition, performed in $\mathbb{Z}_2$. The identity of G is: $(\ldots, 0, 0, 0)$.

It is easy to see that G is locally finite, since any finite subset of G has an upper bound on how many 1 bits its elements can have. Since our addition is performed in $\mathbb{Z}_2$, this upper bound must be preserved by any subgroup generated by the finite subset.

As mentioned earlier, define a bijection between elements of G and $\mathbb{N}$ using the binary representation from the digits of G.

Let H be the subgroup of G generated by: $(\ldots, 0, 0, 1)$. Then $H \cong C_2$, in particular, H is finite.

Define a permutation σ on G by:

$$\begin{array}{rccc} \sigma & : & G & \rightarrow & G \\ & & g & \mapsto & g + (\ldots, 0, 0, 1) \end{array}$$

In words, σ changes the last bit in the binary representation. When we view σ as a bijection on $\mathbb{N}$, it is equivalent to the following permutation:

$$(0\ 1)(2\ 3)(4\ 5)(6\ 7)\cdots$$

In particular, it is easy to see that σ moves infinitely many natural numbers, equivalently it moves infinitely many elements of G.

We need to show that σ fixes each coset of H setwise. Let $g \in G$ be arbitrary. Then we can write:

$$g = (\ldots, d_4, d_3, d_2, d_1)$$

Then the coset $g + H$ in G has the form:

$$\{(\ldots, d_4, d_3, d_2, 0), (\ldots, d_4, d_3, d_2, 1)\}$$

Now it is clear that since σ affects only the last digit, σ must fix every coset of H in G.

Therefore $\sigma \in S$. S contains a permutation which moves infinitely many elements of G. So we can see that the constricted symmetric group is larger than S_∞.

Lemma: 3.3.3 *Let G be a locally finite group. Let ρ denote the regular representation of G in the constricted symmetric group S on G. Then any two finite isomorphic subgroups of G^ρ are conjugate in S.*

Outline of Proof: Start with finite isomorphic subgroups of G. Argue that any finite isomorphic subgroups of G^ρ must arise from finite isomorphic subgroups of G. Then show that any 2 such subgroups of G^ρ are conjugate.

Proof: Let K_1, K_2 be finite isomorphic subgroups of G. Let $H = \langle K_1, K_2 \rangle$. Note that since $K_1,\ K_2$ are finite, and G is locally finite, therefore H is finite.

Let an isomorphism from K_1 to K_2 be denoted as follows:

$$\begin{array}{cccc} \alpha & : & K_1 & \rightarrow & K_2 \\ & & x & \mapsto & x^* \end{array}$$

1. Let $\{x_i : i \in I\}$ be a complete set of left coset representatives of H in G. The left cosets of H in G are then: $\{x_i H : i \in I\}$.
2. Let $\{y_1, \ldots, y_r\}$ be a complete set of left coset representatives of K_1 in H. The left cosets of K_1 in H are then: $\{y_j K_1 : 1 \leq j \leq r\}$. The left cosets of K_1 in G are then: $\{x_i(y_j K_1) : i \in I, 1 \leq j \leq r\}$.
3. Let $\{z_1, \ldots, z_r\}$ be a complete set of left coset representatives of K_2 in H. The left cosets of K_2 in H and G are defined analogously to those for K_1.

Note that since $K_1 \cong K_2$, K_1 and K_2 have the same number of left cosets in H and G, by a quick Lagrange argument. This is why we use the same r for both sets of left cosets.

Define the elements $\sigma \in S$ by picking out all the permutations $\sigma \in \overline{S}$ such that:

$$(x_i y_j x)^\sigma = x_i y_j^* x^*, \ i \in I, \ 1 \le j \le r \ and \ x \in K_1$$

Explanatory Notes on the Above:

1. $(x_i y_j x)^\sigma$ denotes the action of σ on $(x_i y_j x)$.
2. We select the σs that fix x_i and move y_j and x.
3. $x_i y_j x$ is a representative of a left coset of K_1 in G (since $x \in K_1$), as mentioned above.

Then $\sigma \in S$ by taking F_σ to be H as we have defined it here. Then for any $k \in K_1$:

$$\begin{aligned}(x_i z_j x^*)^{\sigma^{-1} k^\rho \sigma} &= (x_i y_j x)^{k^\rho \sigma} \ applying \ \sigma^{-1} \ first, \ since \ \sigma, \sigma^{-1} \ fix \ x_i \\ &= (x_i y_j x k)^\sigma \ right \ multiply \ by \ k \ (under \ representation \ \rho) \\ &= x_i z_j (xk)^* \ since \ \sigma, \sigma^{-1} \ fix \ x_i \\ &= (x_i z_j x^*)^{(k^*)^\rho}\end{aligned}$$

since α is an isomorphism, in particular it is a homomorphism. Hence, for all $k \in K_1$:

$$\sigma^{-1} k^\rho \sigma = (k^*)^\rho$$

Therefore K_1^ρ and K_2^ρ are conjugate in S, as required. ■

Theorem: 3.3.4 *There exist countable universal locally finite groups and any two such groups are isomorphic.*

Outline of Proof:

1. Demonstrate a construction for a countable locally finite group. (At each step, embed the group constructed so far into its full symmetric group via the right regular representation.) Then prove that the construction yields a universal group.
2. Given a second countable universal locally finite group, demonstrate a construction of an isomorphism between the second group and the first group.

Proof (Existence): Inductively define a direct system of finite groups (and embeddings) as follows:

1. Let U_1 be any finite group such that $|U_1| \geq 3$.
2. If $n \geq 1$ and the group U_n is already chosen, let U_{n+1} be the full symmetric group on the set U_n. Embed U_n into U_{n+1} via its right regular representation.
3. This family of groups and embeddings forms a direct system consisting of finite groups, linearly ordered with respect to inclusion. Take U to be the direct limit:

$$U = \lim_{\rightarrow} U_n$$

   Then by Lemma (2.1.5.1), U is a countable, locally finite group.

It just remains to prove that U is universal. For more convenient notation throughout the rest of the proof, we shall identify U_n with its image under embedding in U.

Embedding Any Finite Group Into U: The order $|U_n|$ tends to infinity with n. Thus, if G is any finite group, then there exists an integer n such that $|G| \leq |U_n|$. Recall that U_{n+1} was taken to be the full symmetric group on the underlying set of U_n. Therefore since G can always be embedded into the symmetric group on $|G|$ letters (by Cayley's Theorem), G can be embedded into U_{n+1}. Then G is isomorphic to a subgroup of U_{n+1}, and hence to a subgroup of U.

Finite subgroups of U are conjugate: Let $G,\ H$ be any two isomorphic finite subgroups of U. Then there exists an integer j such that $\langle G,\ H\rangle \subseteq U_j$. By construction, U_j is embedded into U_{j+1} via its right regular representation.

Therefore, G and H are finite isomorphic subgroups in the right regular representation of U in the the constricted symmetric group of U. Therefore the hypothesis of Lemma (3.3.3) is satisfied. Hence the subgroups $G,\ H$ are conjugate in $U_{j+1} \leq U$. Therefore G and H are conjugate in U.

Thus U is a universal locally finite group. Existence is now proved. $\dashv$ (Existence)

Proof (Uniqueness up to Isomorphism): We use a "back-and-forth" approach to construct an isomorphism between U and any other countable universal locally finite group V.

Let V be any other countable universal locally finite group. Then by Lemma (2.1.5.1), there is a local system $\{V_n\}_{n \in \mathbb{N}}$ of finite subgroups of V, linearly ordered by inclusion.

Since each U_r is finite, it can be embedded into the universal group V. Let ϕ be any embedding of U_r into V.

Then one has $U_r^{\phi} \subsetneq V_s$, for some integer s. Then by Theorem (3.3.1 2), there exists an embedding ψ of V_s into U such that the composite map $\phi\psi$ restricts to the identity on U_r. N.B. Here we are composing left to right, i.e. $\phi\psi$ means apply ϕ first, then ψ.

There is an integer r' with $V_s^{\psi} \subsetneq U_{r'}$. Again by Theorem (3.3.1 2), there exists an embedding ϕ' of $U_{r'}$ into V such that $\psi\phi'$ restricts to the identity map on V_s.

By choosing an arbitrary embedding ϕ_1 of U_1 into V, one may in this way choose inductively two strictly ascending sequences of integers:

$$\begin{array}{ccccccc} 1 & = & r_1 & < & r_2 & < & \cdots \\ 1 & = & s_1 & < & s_2 & < & \cdots \end{array}$$

and two sequences of proper embeddings:

$$\left.\begin{array}{ccccc} \phi_i & : & U_{r_i} & \to & V_{s_i} \\ \psi_i & : & V_{s_i} & \to & U_{r_{i+1}} \end{array}\right\} i \in \mathbb{P}$$

such that:

1. $\phi_i\psi_i$ is the identity on U_{r_i}, and
2. $\psi_i\phi_{i+1}$ is the identity on V_{s_i}

The following diagram shows the construction:

$$\begin{tikzcd}
U_{r_1} \arrow[d, "\phi_1"'] & \leq & U_{r_2} \arrow[d, "\phi_2"'] & \leq & U_{r_3} \arrow[d, "\phi_3"'] & \leq & \cdots \\
V_{s_1} \arrow[urr, "\psi_1"] & \leq & V_{s_2} \arrow[urr, "\psi_2"] & \leq & V_{s_3} \arrow[urr, "\psi_3"] & \leq & \cdots
\end{tikzcd}$$

It follows that for each index i the embeddings ϕ_{i+1} and ψ_{i+1} are, respectively, extensions of ϕ_i and ψ_i.

Thus the sequences $\{\phi_i\}_{i\in\mathbb{N}}$ and $\{\psi_i\}_{i\in\mathbb{N}}$ determine injective homomorphisms:

$$\begin{array}{ccccc} \phi & : & U & \to & V \\ \psi & : & V & \to & U \end{array}$$

such that:

1. $\phi\psi$ is the identity on U, and
2. $\psi\phi$ is the identity on V

Therefore $\phi,\ \psi$ are isomorphisms. In other words, $U \cong V$, for any other countable universal locally finite group V. $\dashv$ (Uniqueness up to isomorphism) ■

We now briefly turn our attention to universal locally finite groups with larger cardinalities.

Theorem: 3.3.5 *Every infinite locally finite group G can be embedded into a universal locally finite group of cardinal $|G|$. In particular, there exist universal locally finite groups of arbitrary infinite cardinal.*

Outline of Proof: Use the restricted symmetric group on a countable set to construct a new universal locally finite group containing G as a subgroup, with the same cardinality as G.

Proof: Let S denote the restricted symmetric group on some countable set. Then S is countable, locally finite, and contains an isomorphic copy of every finite group. Let $U_0 = G \times S$. For $i = 0, 1, 2, \ldots$ define U_{i+1} and $\rho_i : U_i \to U_{i+1}$ inductively, as follows:

1. ρ_i is the regular representation of U_i into the symmetric group on U_i.
2. U_{i+1} is the subgroup of the constricted symmetric group on U_i generated by $U_i^{\rho_i}$.
3. For each pair of isomorphic finite subgroups of $U_i^{\rho_i}$, an element of this constricted subgroup conjugates one onto the other.

So every pair of finite isomorphic subgroups of $U_i^{\rho_i}$ are conjugate in U_{i+1}. Furthermore, U_i and U_{i+1} have the same cardinal. By induction, $|G|$ will also be the same cardinal.

Then $U = \lim_{\to} U_i$ is a universal locally finite group of cardinality $|G|$, containing a subgroup isomorphic to G. ■

Lemma: 3.3.6 *Let K be a finite group. Let M be a subgroup of index $d > 1$ in K. Let L be the symmetric group of order $(cd)!$, for some c. Let H be a semi-regular subgroup of L of order c. Let θ be an embedding of M into H such that $[H : M^\theta] > 2$. Then θ extends to embeddings θ_1, θ_2 of K into L such that:*

$$H \cap K^{\theta_1} = M^\theta = H \cap K^{\theta_2} \text{ and } K^{\theta_1} \neq K^{\theta_2}$$

Outline of Proof: Partition the set permuted by L in 2 different ways. Use the distinct partitions to explicitly construct distinct $\theta_1,\ \theta_2$ extending θ with the desired properties.

Proof: Let $\{1\} \cup S$ and $\{1\} \cup T$ be the left traversals respectively of M in K and M^θ in H, where $1 \notin S \cup T$. Since H occurs as a subgroup of L in its regular representation repeated d times, we may identify the permutand of L (i.e. the set cd of elements permuted by L) with the set $P = H \cup SH$, where SH denotes the set of formal products of $s \in S$ and $h \in H$, and H permutes P by right multiplication. Since $H = M^\theta \cup TM^\theta$, we may write P as the disjoint union of four sets $M^\theta,\ TM^\theta,\ SM^\theta,\ STM^\theta$, the products being written formally.

Since θ is an embedding, it is an injective map. Therefore, the following maps are bijections:

$$\begin{array}{ccccc} \theta & : & M & \to & M^\theta \\ \theta & : & TM & \to & TM^\theta \\ \theta & : & SM & \to & SM^\theta \\ \theta & : & TSM & \to & TSM^\theta \end{array}$$

Since $M^\theta,\ TM^\theta,\ SM^\theta,\ STM^\theta$ are disjoint, so are $M,\ TM,\ SM,\ STM$. Define Q to be the disjoint union of the four sets of formal products $M,\ TM,\ SM,\ TSM$. Let the elements of K permute Q by right multiplication.

Let ψ be any bijection from the set TS onto the set ST. Denote by $\overline{\theta}$ the bijection of Q onto P given by:

$$\begin{array}{rcl} x^{\overline{\theta}} & = & x^\theta \\ (tx)^{\overline{\theta}} & = & tx^\theta \\ (sx)^{\overline{\theta}} & = & sx^\theta \\ (tsx)^{\overline{\theta}} & = & (ts)^\psi x^\theta \end{array}$$

where $x \in M,\ s \in S$ and $t \in T$. We may use the mapping $\overline{\theta}$ to transfer the representation of K on Q to a representation $\rho = \rho(\psi)$ of K on P by defining:

$$pk^\rho = (p^{\overline{\theta}^{-1}} k)^{\overline{\theta}},\ \textit{where } p \in P \textit{ and } k \in K$$

We claim that ρ is an extension of θ. We show it for each of the four sets.

1. Suppose that $x, y \in M$. Then:

$$\begin{aligned}(x^\theta)y^\rho &= ((x^{\overline{\theta}})^{\overline{\theta}^{-1}} y)^{\overline{\theta}} \\ &= (xy)^{\overline{\theta}} \\ &= (x^\theta)y^\theta\end{aligned}$$

2. Suppose that $t \in T$ and $x, y \in M$. Then:

$$\begin{aligned}(tx^\theta)y^\rho &= ((tx^\theta)^{\overline{\theta}^{-1}} y)^{\overline{\theta}} \\ &= (((tx)^{\overline{\theta}})^{\overline{\theta}^{-1}} y)^{\overline{\theta}} \\ &= (txy)^{\overline{\theta}} \\ &= (tx^\theta)y^\theta\end{aligned}$$

3. Suppose that $s \in S$ and $x, y \in M$. Then:

$$\begin{aligned}(sx^\theta)y^\rho &= ((sx^\theta)^{\overline{\theta}^{-1}} y)^{\overline{\theta}} \\ &= (((sx)^{\overline{\theta}})^{\overline{\theta}^{-1}} y)^{\overline{\theta}} \\ &= (sxy)^{\overline{\theta}} \\ &= (sx^\theta)y^\theta\end{aligned}$$

4. Suppose that $s \in S$, $t \in T$ and $x, y \in M$. Then:

$$\begin{aligned}(stx^\theta)y^\rho &= ((st)^{\psi^{-1}} xy)^{\overline{\theta}} \\ &= st(xy)^\theta \\ &= (stx^\theta)y^\theta\end{aligned}$$

Claim: $H \cap K^\rho = M^\theta$.

Proof of Claim: $M^\theta \subseteq H \cap K^\rho$ is clear. For a contradiction, suppose that $M^\theta \subsetneq H \cap K^\rho$. Then we may select $h \in H \cap K^\rho \setminus M^\theta$ and $k \in K \setminus M$, such that $k^\rho = h$.

If $h \in H \setminus M^\theta$, then h maps M^θ onto TM^θ. If $k \in K \setminus M$, then k maps M onto SM and thus k^ρ maps M^θ into SM^θ. But SM^θ and TM^θ are disjoint, so we have a contradiction. Therefore the only possibility is that $H \cap K^\rho = M^\theta$, as claimed. $\dashv$ (Claim)

Since the index $[H : M^\theta] = 1 + |T| > 2$ (and the order of S is at least 1), there exist bijections ψ_1 and ψ_2 of TS onto ST such that for some $t \in T$, we have:

$$(tS)^{\psi_1} \neq (tS)^{\psi_2}$$

Let $\theta_1 = \rho(\psi_1)$ and $\theta_2 = \rho(\psi_2)$. Then $\theta_1,\ \theta_2$ are embeddings of K into L extending θ such that $\underbrace{H \cap K^{\theta_1} = M^\theta}_{by\ the\ claim}$, $\underbrace{H \cap K^{\theta_2} = M^\theta}_{by\ the\ claim}$. We have only to show that $K^{\theta_1} \neq K^{\theta_2}$.

The orbit $tK = tM \cup tSM$ of K containing t corresponds to the orbits:

$$\begin{aligned} tK^{\theta_1} &= tM^\theta \cup (tS)^{\psi_1} M^\theta \\ tK^{\theta_2} &= tM^\theta \cup (tS)^{\psi_2} M^\theta \end{aligned}$$

of $K^{\theta_1},\ K^{\theta_2}$ containing t. By contradiction, assume that $K^{\theta_1} = K^{\theta_2}$. Then:

$$\begin{aligned} (tS)^{\psi_1} M^\theta &= tK^{\theta_1} \cap STM^\theta \\ &= tK^{\theta_2} \cap STM^\theta \\ &= (tS)^{\psi_2} M^\theta \end{aligned}$$

But then $(tS)^{\psi_1} = (tS)^{\psi_2}$, contradicting the choice of t. Therefore $K^{\theta_1} \neq K^{\theta_2}$ as required. ■

The following theorem is a strengthening of Theorem (3.3.1 3).

Theorem: 3.3.7 *Let V be any universal locally finite group. Let G be any countably infinite, locally finite group. Then there exist at least $2^{\aleph_0}$ distinct subgroups of V isomorphic to G.*

Outline of Proof: Construct a countable universal locally finite group U using the finite subgroups of G. Use the previous result to construct $2^{\aleph_0}$ distinct embeddings of G into U. Notice that V contains a subgroup isomorphic to U, therefore we have $2^{\aleph_0}$ distinct embeddings of G into V.

Proof: Since G is countably infinite, and locally finite, G contains a tower:

$$\{1\} = G_0 < G_1 < \cdots < G_i < \cdots < G$$

of distinct finite subgroups such that $G = \bigcup_{i=1}^{\infty} G_i$. For each natural number $i \geq 1$, let $d_i = [G_i : G_{i-1}]$. Clearly from the construction in Theorem (3.3.4) we may assume that $d_1 \geq 3$.

Define:
$$c_i = \begin{cases} d_i & if \quad i = 1 \\ (c_{i-1})!d_i & if \quad i > 1 \end{cases}$$
Let U_i be the symmetric group of order $c_i!$, and for each i embed U_i into U_{i+1} by a semi-regular representation (i.e. its regular representation repeated d_{i+1} times). Let $U = \lim_{\rightarrow} U_i$. It follows from Lemma (3.3.3) and the proof of Theorem (3.3.4) that U is a countable universal locally finite group. Again, identify U_i with its image in U.

Suppose that θ is an embedding of G_i into U_i (an embedding must exist, by Cayley's Theorem). Notice that applying our counting formula inductively gives:
$$|G_i| = d_1 d_2 \cdots d_i$$
while:
$$|U_i| = (\cdots(d_1!d_2)!d_3 \cdots d_{i-1})!d_i)! > 2(d_1 d_2 \cdots d_i)$$
since $d_1 \geq 3$.

In Lemma (3.3.6), take:

1. $M = G_i$
2. $K = G_{i+1}$
3. $H = U_i$
4. $L = U_{i+1}$

Then we have that there exist embeddings $\theta_1,\ \theta_2$ of G_{i+1} into U_{i+1} extending θ such that:
$$\begin{aligned} U_i \cap G_{i+1}^{\theta_1} &= G_i^{\theta} \\ &= U_i \cap G_{i+1}^{\theta_2} \end{aligned}$$
and $G_{i+1}^{\theta_1} \neq G_{i+1}^{\theta_2}$.

It follows that there exists $2^{\aleph_0}$ distinct sequences $\{\phi_i\}_{i \in \mathbb{N}}$, where ϕ_i is an embedding of G_i into U_i extending ϕ_{i-1} and satisfying:
$$U_{i-1} \cap G_i^{\phi_i} = G_{i-1}^{\phi_{i-1}}$$
such that for any two distinct sequences $\{\phi_i\}$ and $\{\psi_i\}$ there exists some j such that $G_j^{\phi_j} \neq G_j^{\psi_j}$.

The sequences $\{\phi_i\}$ and $\{\psi_i\}$ determine embeddings ϕ and ψ, respectively, of G into U and:

$$\begin{aligned} U_j \cap G^{\phi} &= G_j^{\phi_j} \\ &\neq G_j^{\psi_j} \\ &= U_j \cap G^{\psi} \end{aligned}$$

Hence $\phi \neq \psi$, and there exist $2^{\aleph_0}$ distinct subgroups of U isomorphic to G. But finally by Theorem (3.3.1 3), the universal locally finite group V contains a subgroup isomorphic to the countable group U. ■

How Embeddings of Subgroups are Linked in General: Let $A \cong B$ be isomorphic subgroups of the universal locally finite group U.

Note that here we do not require A and B to be finite. If A and B are finite, then we immediately have by the definition of a universal locally finite group that A and B are conjugate in U. Therefore if $B = g^{-1}Ag$ for some $g \in U$, then we have an automorphism of U which sends to A to B, namely the inner automorphism of U induced by g.

We would like to have some information about how the embeddings of A, B into U are linked in general. In general, we claim that no automorphism of U transforms A into B.

To see a counterexample, take A to be any infinite maximal elementary abelian p-subgroup of U for some prime p, and take B to be any infinite proper subgroup of A such that $|A| = |B|$. Then A is a maximal elementary abelian p-subgroup but B is properly contained in an elementary abelian p-subgroup (and thus is not maximal). Any automorphism of U sending A to B must preserve the structure of A, in particular the property of being maximal in U. Thus in this situation there cannot be any automorphism of U which transforms A into B.

A and B may be regarded as isomorphic vector spaces over $\mathbb{Z}_p$, where $B \subsetneq A$. Again we can see that no automorphism of U could send A to B.

Theorem: 3.3.8 *The automorphism group A of the countable universal locally finite group U has cardinal $2^{\aleph_0}$.*

Outline of Proof: Explicitly construct $2^{\aleph_0}$ distinct automorphisms of the countable universal locally finite group.

Proof: It is clear that $|A| \leq 2^{\aleph_0}$. We need to show that equality holds. By the proof of Theorem (3.3.4), the countable universal locally finite group is the union of an ascending sequence $\{U_i\}_{i\in\mathbb{N}}$ of subgroups U_i which are isomorphic to symmetric groups of order $n_i!$, where $n_1 \geq 3$, $n_{i+1} = n_i!$, and U_i is embedded into U_{i+1} via its right regular representation.

Put $C_{i+1} = C_{U_{i+1}}(U_i)$ (the centralizer of U_i in U_{i+1}). By Theorem 6.3.1 on p 86 of [4], the left and right regular representations are centralizers for each other. So the subgroup C_{i+1} is the left regular representation of U_i in U_{i+1}. In particular, $|C_{i+1}| \geq 2$ for each natural number i.

Choose $c_1 \in U_1$ and for every natural number $i > 1$ choose any element $c_i \in C_i$. If ϕ_i denotes the inner automorphism of U_i defined by:

$$x \mapsto x^{c_1 \cdots c_i}, x \in U_i$$

then for every natural number i the automorphism ϕ_i of U_i is equal to the restriction to U_i of the automorphism ϕ_{i+1} of U_{i+1}. Thus the sequence $\{\phi_i\}_{i\in\mathbb{N}}$ determines an automorphism ϕ of U.

Let $\{d_i\}_{i\in\mathbb{N}}$ be a second sequence of elements of U with $d_1 \in U_1$ and $d_i \in C_i$ for $i > 1$, and $\{\psi_i\}_{i\in\mathbb{N}}$ the corresponding sequence of inner automorphisms of the groups U_i determining the automorphism ψ of U.

We shall suppose that for some natural number j, one has $c_j \neq d_j$, and we shall prove that this implies $\phi \neq \psi$.

If $j = 1$, then $\phi_1 \neq \psi_1$, since the group U_1 has trivial centre, and thus $\phi \neq \psi$.

If $j > 1$, then for all elements $x \in C_j$, one has:

$$x^{\phi} = x^{\phi_j} = x^{c_1 \cdots c_j} = x^{c_j}$$

since $c_1 \cdots c_{j-1} \in U_{j-1} \subseteq C_U C_j$. Similarly $x^{\psi} = x^{d_j}$. Since the subgroup C_j is isomorphic to the symmetric group of order $n_{j-1}!$, its centre is trivial and thus the inner automorphisms of C_j induced by two distinct elements are distinct. Thus, for at least one element $x \in C_j$:

$$x^{\phi} = x^{c_j} \neq x^{d_j} = x^{\psi}$$

Therefore $\phi \neq \psi$.

The number of distinct sequences of the sort described, and therefore the number of distinct automorphisms is clearly $2^{\aleph_0}$. ■

3.4 Questions to Be Answered Later

1. Are any two universal locally finite groups of the same cardinality isomorphic?

   The answer to this is no. See Theorem (4.2.3).

2. If for the cardinal $\aleph > \aleph_0$, there is more than one isomorphism type of universal locally finite groups, do there exists $2^{\aleph_0}$ pairwise non-isomorphic universal locally finite groups of cardinal $\aleph$?

   The answer is yes to a slightly modified version of this question. See Corollary (4.3.4).

3. Does every universal locally finite group U contain an isomorphic copy of every locally finite group G satisfying $|G| \leq |U|$?

   The answer is no. See Theorem (4.2.2).

3.5 Phillip Hall's Universal Locally Finite Group Cannot Answer Our Question Positively

If we could find a universal locally finite group that also satisfied property 3, we would have a positive answer to our original problem (as shown in Theorem (2.5.3)).

However, we note here that one property of the countable universal locally finite group U prevents it from being a positive answer to our original question.

The universal locally finite group U has to contain copies of every finite group. In particular, U contains copies of $C_2,\ C_3,\ C_4, \ldots$. From this it is clear that U contains elements of arbitrarily high order. Then for all $k = 1, 2, 3, \ldots$, there is no uniform bound on the size of k-generated subgroups of U.

By Theorem (2.4.1), any group with finitely many $(k+1)$-conjugacy classes has a uniform bound on the size of its k-generated subgroups. Therefore there is no way that a universal locally finite group U can answer our question positively for any k.

Recall that in Theorem (2.5.4), we proved that if a group G exists which answers our question positively, then there must be a countable subgroup H which also answers the question positively. Moreover, this H must be a proper subgroup of the countable universal locally finite group U. We proved in Corollary (2.5.5) that H must be a subgroup of U. The fact that H must be a proper subgroup of U is now clear from the fact that U itself cannot answer the question positively.

If possible, we want to bound the number if isomorphism classes of k-generated subgroups of our candidate group. We also wish to find a weakened condition on our candidate group which might permit it to answer our original question positively. This leads us naturally to the topics of existentially closed groups and HNN extensions, and the related property of ω-homogeneity.

Before investigating these topics, we pause briefly to include some further results which Shelah has obtained on uncountable universal local finite groups.

Chapter 4

Uncountable Universal Locally Finite Groups

4.1 Two Papers by Shelah

Here, we summarize some further results on universal locally finite groups of bigger cardinality than the one discovered by Phillip Hall. Whereas Phillip Hall was primarily interested in the countable case, Shelah obtains some results in uncountable cardinalities.

In particular, we want to capture the relevant open questions from Phillip Hall which are answered here.

We state the results without proof. Including the proofs would take us far off the group theory track, and deep into model theory.

4.2 Theorems from [15]

Lemma: 4.2.1 *For any $\kappa \geq \aleph_0$, there are universal locally finite groups of cardinal κ.*

Proof: Refer to [15], Lemma 2. ■

Theorem: 4.2.2 *There is a locally finite group H of cardinal $\aleph_1$ such that for each $\kappa \geq \aleph_1$, there is a locally finite group G of cardinal κ, such that H is not embeddable in G.*

Proof: Refer to [15], Theorem 3. ■

Theorem: 4.2.3 *For each cardinal $\kappa \geq \aleph_1$ there are several non-isomorphic universal locally finite groups of cardinal κ.*

Proof: Refer to [15], Theorem 4. ■

Theorem: 4.2.4 *Let $\mu(\kappa)$ be the number of isomorphism types of universal locally finite groups of cardinal κ. Then $\mu(\kappa) = 2^\kappa$ if $\kappa \geq \aleph_1$.*

Proof: Refer to [15], Theorem 5. ■

Theorem: 4.2.5 *For each regular uncountable κ there are 2^κ pairwise non-embeddable universal locally finite groups of cardinal κ.*

Proof: Refer to [15], Theorem 8. ■

4.3 Theorems from [3]

Nice Facts

1. By Theorem (3.3.1), every countable universal locally finite group is $\aleph_0$-universal. So in the category $LF_{\aleph_0}$, a universal object exists.

2. This can be understood as a generalization of the fact that S_n is universal for the category of finite groups of cardinality $\leq n$.

Theorem: 4.3.1 *For every uncountable cardinal λ which satisfies:*

1. *$\lambda = \lambda^{\aleph_0}$ or*

2. *There exists a cardinal μ such that $\lambda < \mu \leq \lambda^{\aleph_0}$ and $2^\mu < 2^\lambda$*

there is no universal object in ULF_λ.

Proof: Refer to [3], Theorem 3. ■

Corollary: 4.3.2 *Theorem (4.3.1) implies that:*

1. *There is no universal object in $ULF_{2^{\aleph_0}}$.*

2. *If $2^{\aleph_0} < 2^{\aleph_1}$, then there is no universal object in $ULF_{\aleph_1}$.*

Proof: Refer to [3], Corollary 4. ■

Theorem: 4.3.3 *Assume that λ satisfies: there exists an infinite cardinal μ such that $\lambda < \mu \leq \lambda^{\aleph_0}$ and $2^\lambda < 2^\mu$. Then in ULF_λ, there are 2^μ non-isomorphic groups.*

Proof: Refer to [3], Theorem 10. ■

Corollary: 4.3.4 *If $2^{\aleph_0} < 2^{\aleph_1}$, then there are $2^{\aleph_1}$ isomorphism types of groups from $ULF_{\aleph_1}$.*

Proof: Refer to [3], Corollary 11. ■

4.4 Considerable Interaction between Group Theory and Model Theory

It is obvious from the results stated here that model theory can be used to prove results which on the surface are purely group theoretic. Since the focus of this thesis is on group theory, we have refrained from venturing into the realm of pure model theory.

4.5 An Uncountable Universal Locally Finite Group Cannot Answer Our Question Positively

The same explanation for the countable universal locally finite group applies in the uncountable case. We cannot have such a group answering our original problem since we require some uniform bound on the order of the elements of a group which answers our question. In an uncountable universal locally finite group, there can be no such uniform bound. Therefore, an uncountable universal locally finite group also could not answer our original question positively.

Chapter 5

Existentially Closed Groups

5.1 Introduction

For this chapter, we are no longer trying to answer the question for all $k = 1, 2, 3, \ldots$, but are trying to get some information for a particular fixed k only. We investigate existentially closed groups because they, along with the related construction of HNN extensions (to be defined shortly) allow us to get some control over the number of k-conjugacy classes for a particular fixed k.

We observe at the end of the chapter that the property of being *existentially closed* is too strong for our purposes here, i.e. an existentially closed group could not answer our question positively. We shall soon see in Theorem (5.3.3) that:

$$\textit{existentially closed} \Longrightarrow \omega - \textit{homogeneous}$$

It is therefore natural to ask whether there is a way that we can use the weaker property of ω-homogeneity to construct a group which answers our question positively.

A Partial Answer: It is not entirely clear whether ω-homogeneous groups could be used to answer the original question. The condition of being ω-homogeneous is both weaker and stronger than what we need, as follows:

1. Professor Park's requirement is that the isomorphism between k-generated subgroups be induced by conjugation. This is stronger than what the definition of ω-homogeneous requires.

2. Professor Park does not require the lifting of isomorphisms to automorphisms of the group, as the definition of ω-homogeneous does. This is weaker than what the definition of ω-homogeneous requires.

5.2 Definitions and Notation

5.2.1 Free Product

Definition (Free Product): Let $A,\ B$ be groups with presentations:

$$A = \langle a_1, \ldots \mid r_1, \ldots \rangle$$

$$B = \langle b_1, \ldots \mid s_1, \ldots \rangle$$

where the sets of generators $\{a_1, \ldots\}$ and $\{b_1, \ldots\}$ are disjoint. Then the *free product*, $A \ast B$ of the groups $A,\ B$ is the group:

$$A \ast B = \langle a_1, \ldots,\ b_1, \ldots \mid r_1, \ldots,\ s_1, \ldots \rangle$$

The groups $A,\ B$ are the *factors* of $A \ast B$. The free product $A \ast B$ is independent of the presentations chosen for $A,\ B$.

Definition (Normal Form / Reduced Sequence for Free Product): A *normal form* or *reduced sequence* is a sequence $g_1, \ldots, g_n$ of elements of $A \ast B$ such that:

1. Each $g_i \neq 1$
2. Each g_i is in one of the factors
3. No successive $g_i,\ g_{i+1}$ are in the same factor

Examples: Let $A \ast B = \langle a, b | a^7, b^5 \rangle$

1. Reduced:
   (a) a^5, b^3, a^2, b
2. Not Reduced:
   (a) a, b^5, a
   (b) a^2, a^3, b^3

Normal Form Theorem for Free Products: 5.2.1 *Consider the free product $A \ast B$. The following equivalent statements hold.*

1. *If $w = g_1 \cdots g_n,\ n > 0$, where $g_1, \ldots, g_n$ is a reduced sequence, then $w \neq 1$ in $A \ast B$.*
2. *Each element of $A \ast B$ can be uniquely expressed as a product $w = g_1 \cdots g_n$, where $g_1, \ldots, g_n$ is a reduced sequence.*

Proof: Refer to [14], Chapter IV, Theorem 1.2. ■

5.2.2 Free Product with Amalgamated Subgroup

Definition (Free Product with Amalgamated Subgroup): Let G and H be groups with subgroups $A \leq G$ and $B \leq H$, and with $\phi : A \to B$ an isomorphism. Form the group:

$$P = (G * H) \ / \ \langle a = \phi(a), \ \forall a \in A \rangle$$

P is the quotient of the free product $G * H$ by the normal subgroup generated by $\{a\phi(a)^{-1} : a \in A\}$.

Note that in the *free product with amalgamated subgroup* (also known as *amalgamated free product*), the two isomorphic groups A and B inside G and H are made equal.

Definition (Normal Form / Reduced Sequence for Free Product with Amalgamated Subgroup): A sequence of elements $c_1, \ldots, c_n, \ n \geq 1$ of elements of $G * H$ is called *reduced* if:

1. Each c_i is in one of the factors G or H.
2. Successive $c_i, \ c_{i+1}$ come from different factors.
3. If $n > 1$ then no c_i is in A or B.
4. If $n = 1$ then no $c_1 \neq 1$.

Normal Form Theorem for Free Products with Amalgamation: 5.2.2 *If $c_1, \ldots, c_n$ is a reduced sequence, $n \geq 1$, then the product $c_1 \cdots c_n \neq 1$ in P. In particular, G and H are embedded in P by the maps $g \mapsto g$ and $h \mapsto h$.*

Proof: Refer to [14], Chapter IV, Theorem 2.6. ■

5.2.3 HNN Extension

Definition (HNN Extension): Let G be a group. Let $A, \ B$ be isomorphic subgroups of G. Let $\phi : A \to B$ be an isomorphism. Then the HNN *extension of G relative to $A, \ B, \ \phi$* is the group:

$$G^* = \langle G, \ t \mid t^{-1}at = \phi(a), \ \forall a \in A \rangle$$

Notice that in the HNN extension, the isomorphic subgroups A and B are made conjugate.

Definition (Reduced Sequence for HNN Extension): A sequence $g_0, t^{\epsilon_1}, g_1, \ldots, t^{\epsilon_n}, g_n$ ($\epsilon_i = \pm 1,\ n \geq 0$) is said to be *reduced* if there is no consecutive subsequence t^{-1}, g_i, t with $g_i \in A$ or t, g_j, t^{-1} with $g_j \in B$.

Britton's Lemma: 5.2.3 *In the HNN extension of G:*

$$G^* = \langle G, t \mid t^{-1}at = \phi(a), \forall a \in A\rangle$$

if the sequence $g_0, t^{\epsilon_1}, g_1, \ldots, t^{\epsilon_n}, g_n$ is reduced and $n \geq 1$, then $g_0 t^{\epsilon_1} g_1 \cdots t^{\epsilon_n} g_n \neq 1$ in G^*.

Proof: Refer to [18], Theorem 11.81. ■

Definition (Normal Form for HNN Extensions): A *normal form* is a sequence

$$g_0, t^{\epsilon_1}, g_1, \ldots, t^{\epsilon_n}, g_n \ (n \geq 0)$$

where:

1. g_0 is an arbitrary element of G.
2. If $\epsilon_i = -1$, then g_i is a representative of a left coset of A in G.
3. If $\epsilon_i = 1$, then g_i is a representative of a left coset of B in G.
4. There is no consecutive subsequence $t^{\epsilon}, 1, t^{-\epsilon}$.

Normal Form Theorem for HNN Extensions: 5.2.4 *Let:*

$$G^* = \langle G,\ t \mid t^{-1}at = \phi(a),\ \forall a \in A\rangle$$

be an HNN extension. Then:

1. *The group G embeds into G^* by the map $g \mapsto g$. If $g_0 t^{\epsilon_1} g_1 \cdots t^{\epsilon_n} g_n = 1$ in G^*, then $g_0, t^{\epsilon_1}, g_1, \ldots, t^{\epsilon_n}, g_n$ is not reduced.*
2. *Every element $w \in G^*$ has a unique representation as $w = g_0 t^{\epsilon_1} g_1 \cdots t^{\epsilon_n} g_n$, where $g_0, t^{\epsilon_1}, g_1, \ldots, t^{\epsilon_n}, g_n$ is a normal form.*

Proof: Refer to [14], Chapter IV, Theorem 2.1. ■

Bounded HNN Extensions: Since our original question would require a locally finite group to provide a positive answer, it is natural to ask whether it is possible to construct *bounded HNN extensions*. In other words, when can 2 isomorphic subgroups be made conjugate by adjoining an element t of finite order, instead of the t of infinite order in the basic HNN extension construction?

Exploring this idea fully would require introducing far too much new material. The interested reader is referred to [8] for full details.

5.2.4 Equations and Inequalities

Definition (Equations and Inequalities): For a group G and variables $x, y, \ldots$, we use the expression *equation over* G and *inequality over* G in the natural way.

For example, if $g, h \in G$, then:

$$x^3 = g$$

is an equation over G, and:

$$xgx^{-1} \neq h$$

is an inequality over G.

Definition (Soluble): A set of equations and inequalities over G is said to be *soluble in* G if we can replace each variable by an element of G so as to make every member of the set true simultaneously.

A set of equations and inequalities over G is said to be *soluble over* G if it is soluble in some group containing G (i.e. G can be embedded into the larger group where the solution exists).

More formally, we can form the free product $G * F$ where F is the free group generated by the list of variables $x, y, \ldots$. Then an equation over G is a statement of the form $w = 1$, and an inequality over G is a statement of the form $w \neq 1$, where in each case $w \in G * F$ is a word.

The elements of $G * F$ are called the *terms* over G.

Examples of Terms:

1. x
2. $x^5 y^{-1}$
3. $g_1 x^2 g_2 y^{-2}$

A set of equations and inequalities over G:

$$\{w_i = 1, w_j \neq 1 : i \in I, j \in J\}$$

is *soluble over* G if there is a group $H \supseteq G$ and a homomorphism:

$$\theta : G * F \to H$$

such that, for all $g \in G$, $i \in I$, $j \in J$:

1. $\theta(g) = g$
2. $w_i \in ker\theta$
3. $w_j \notin ker\theta$

The set is *soluble in* G if we can take $H = G$.

Theorem: 5.2.5 *The above set of equations and inequalities is soluble over G if and only if N, the least normal subgroup of $G*F$ containing $\{w_i : i \in I\}$:*

1. *intersects G trivially, i.e. $N \cap G = \{1\}$, and*
2. *contains no w_j, for $j \in J$*

Outline of Proof ($\Longrightarrow$): Assuming that either condition does not hold quickly contradicts the definition of being soluble over G.

Outline of Proof ($\Longleftarrow$): Form the normal subgroup N of $G * F$. Define a quotient map into $H = (G * F)/N$ and show that the definition of soluble over G holds for this H.

Proof ($\Longrightarrow$): By the definition of being soluble over G, we have $H \supseteq G$ and $\theta : G * F \to H$ such that:

$$\{w_i : i \in I\} \subseteq ker\ \theta$$

Notice that $ker\ \theta$ is then a normal subgroup of $G * F$, which contains $\{w_i : i \in I\}$. Therefore, by the definition of N, we have that $N \subseteq ker\ \theta$.

If we have a $g \in N \cap G$, $g \neq 1$, then $g \in N \subseteq ker\ \theta$. This implies that $\theta(g) = 1 \neq g$. This violates the definition of being soluble over G. So $N \cap G = \{1\}$.

If there is some $j \in J$ such that $w_j \in N$, then since $N \subseteq ker\ \theta$, we have that $w_j \in ker\ \theta$. This violates the definition of being soluble over G. So N contains no w_j, for $j \in J$.

Proof ($\Longleftarrow$)**:** Let the normal subgroup N of $G * F$ be defined by taking the closure of $\langle w_i : i \in I\rangle$ under products, inverses, and conjugates. Then it is clear that N is the least normal subgroup of $G * F$ which contains $\{w_i : i \in I\}$. By hypothesis, N intersects G trivially. Define the map θ as the quotient map:

$$\begin{array}{rcl} \theta \ : \ G * F & \to & \underbrace{(G * F)/N}_{=H} \\ g & \mapsto & g, \ \forall\ g \in G \\ w_j & \mapsto & w_j, \ \forall\ j \in J \end{array}$$

It is clear that θ is a homomorphism. Since $G \cap N = \{1\}$, we have that θ embeds G into H. Also, all the w_i lie in $N = ker\ \theta$. Last, we have that $ker\ \theta$ contains no w_j, as required. We can now see that our set of equalities and inequalities is soluble over G. ■

Examples:

1. The set:
$$\mathcal{S} = \{x^{-1}gx = g, x^{-1}hx = h, x^{-1}kx \neq k\}$$
is not soluble over G if $g, h, k \in G$ and $k = gh$.

   Proof: By contradiction, suppose there exists $H \supseteq G$ where a solution exists. Let $a \in H$ be a solution. Then:
$$\begin{aligned} k \neq a^{-1}ka &= a^{-1}(gh)a \\ &= a^{-1}g(aa^{-1})ha \\ &= (a^{-1}ga)(a^{-1}ha) \\ &= (g)(h) \\ &= k \end{aligned}$$

   This contradiction tells us that no solution can exist in any extension of G.

2. If $C_G(g) \cap C_G(h) \subseteq C_G(k)$, but $k \notin \langle g, h\rangle$, then $\mathcal{S}$ is soluble over G, but not in G.

   Proof (not soluble in G): By contradiction, suppose it is soluble in G. Let $a \in G$ be a solution. Then:
$$\begin{aligned} a^{-1}ga &= g \\ a^{-1}ha &= h \\ \Longrightarrow a &\in C_G(g) \cap C_G(h) \\ \Longrightarrow a &\in C_G(k) \\ \Longrightarrow a^{-1}ka &= k \end{aligned}$$

This contradiction tells us that no solution can exist in G.

Proof (soluble over G): Take $A = B = \langle g, h \rangle < G$. Then take $\phi : A \to B$ to be the identity map. Then ϕ is trivially an isomorphism. Form the HNN extension:

$$G^* = \langle G, t \mid t^{-1}at = a, \ a \in \langle g, h \rangle \rangle$$

It is clear that in G^*, we have:

$$\begin{array}{rcl} t^{-1}gt & = & g \\ t^{-1}ht & = & h \end{array}$$

Now notice that since $k \notin \langle g, h \rangle$, we also have $k^{-1} \notin \langle g, h \rangle$ and hence the sequence:

$$\underbrace{1}_{g_0}, \underbrace{t^{-1}}_{t^{-1}}, \underbrace{k}_{g_1}, \underbrace{t}_{t}, \underbrace{k^{-1}}_{g_2}$$

is reduced. Therefore, by Britton's Lemma (5.2.3), we have that in G^*:

$$\begin{array}{rcl} t^{-1}ktk^{-1} & \neq & 1 \\ t^{-1}kt & \neq & k \end{array}$$

Thus in the HNN extension G^* of G, we can find a solution t for the system as claimed.

3. The set $\mathcal{S} = \{g = x^{-1}gx\}$ is soluble in G for any group G, and any $g \in G$.

   Proof: Take $x = 1$.

Definition (Algebraically Closed): A group M is said to be *algebraically closed* if every finite set of equations defined over M that is soluble over M is soluble in M.

Definition (Existentially Closed): A group M is said to be *existentially closed* if every finite set of equations and inequalities defined over M that is soluble over M is soluble in M.

Remark: In the definition of existentially closed, inequalities are allowed, whereas in the definition of algebraically closed inequalities are not allowed.

From this, it is clear that existentially closed implies algebraically closed.

Equivalent (Model Theoretic) Definition: A group M is said to be *existentially closed* if every finite formula in $\mathcal{B}$ that is satisfiable in some group containing M is satisfiable in M.

5.2.5 ω-Homogeneity

Definition (ω-Homogeneous): A group K is said to be ω*-homogeneous* if, given any finite set $\{g_1, \ldots, g_r, h\} \subseteq K$ and any injective homomorphism:

$$\theta : \langle g_1, \ldots, g_r \rangle \to K$$

we can extend θ to an injective homomorphism:

$$\phi : \langle g_1, \ldots, g_r, h \rangle \to K$$

If the group is countable, this is equivalent to saying that given any 2 finite subsets of the same type, there exists an automorphism of K which sends one to the other.

5.3 Theorems

Theorem: 5.3.1 *A group is existentially closed if and only if it is non-trivial and algebraically closed.*

Outline of Proof ($\Longrightarrow$): This is a straightforward application of the definitions.

Outline of Proof ($\Longleftarrow$): Transform the given set of equations / inequalities into an equivalent set of equations. Then use the fact that our group is algebraically closed.

Proof ($\Longrightarrow$): An existentially closed group is clearly algebraically closed. The trivial group is not existentially closed, for the inequality $x \neq 1$ is soluble in some group, therefore over, and not in, the trivial group.

Proof ($\Longleftarrow$): Let M be a non-trivial algebraically closed group. Consider the equations and inequalities:

$$\mathcal{S} = \{u_1 = 1, \ldots, u_n = 1, v_1 \neq 1, \ldots, v_m \neq 1\}$$

Suppose that $\mathcal{S}$ is soluble in $K \supseteq M$. Since M is non-trivial, we may choose a non-trivial element $h \in M$. The statement $v_i \neq 1$ is equivalent to $(v_i = 1) \Rightarrow (h = 1)$. By Lemma 1.5 of [9], this is equivalent to the solubility of the equation:

$$s_i^{-1} v_i s_i t_i^{-1} v_i t_i = h$$

over any group containing both v_i and h. The elements s_i and t_i ($1 \leq i \leq m$) are new variables, not involved in the u_j or v_i. Thus the new set of equations:

$$\mathcal{S}^* = \{u_j = 1, h = s_i^{-1} v_i s_i t_i^{-1} v_i t_i : 1 \leq j \leq n,\ 1 \leq i \leq m\}$$

is soluble over K and hence over M. Since M is algebraically closed, this set has a solution in M. This solution must satisfy $u_j = 1$, $v_i \neq 1$, for $1 \leq j \leq n,\ 1 \leq i \leq m$. Therefore M is existentially closed. ■

Theorem: 5.3.2 *Let M be an existentially closed group. Then:*

1. *M cannot be finitely generated.*
2. *M contains every finitely presented simple group (and hence every finite group).*
3. *M is simple.*

Outline of Proof:

1. For a contradiction, assume that M is finitely generated.
2. Given any finitely presented group G, construct a system of equations which demonstrates that M contains an isomorphic image of G.
3. Show that for any $h, g \in M$, h lies in the normal subgroup generated by g.

Proof:

1. Let $\{g_1, \ldots, g_k\} \subseteq M$ be any finite subset. We shall show that this always implies $\langle g_1, \ldots, g_k \rangle \subsetneq M$.

   Consider the set of equations and inequalities:

   $$\mathcal{S} = \{x^{-1} g_1 x = g_1,\ \ldots,\ x^{-1} g_k x = g_k,\ x^{-1} y x \neq y\}$$

   We can solve $\mathcal{S}$ over M (e.g. in the direct product $M \times G$, where G is any nonabelian group). Since M is existentially closed, therefore we can solve $\mathcal{S}$ in M. Therefore there exists some $y \in M$ such that $y \notin \langle g_1, \ldots, g_k \rangle$. Therefore $\langle g_1, \ldots, g_k \rangle \subsetneq M$.

2. Let $G = \langle g_1, \ldots, g_k \mid w_1(g), \ldots, w_r(g) \rangle$ be a finitely presented group containing a non-trivial element $u(g)$. Let $\{x_1, \ldots, x_k\}$ be distinct variables. Then we can solve the set:

   $$\mathcal{S} = \{w_1(x) = \cdots = w_r(x) = 1, u(x) \neq 1\}$$

   over M (in $M \times G$) and hence in M. So M contains a non-trivial homomorphic image of G. If G is simple, then this image can only be the whole of G. Then G embeds into M.

3. If $g, h \in M$ and $g \neq 1$, then $(g = 1) \Rightarrow (h = 1)$ holds in M. So by Lemma 1.5 of [9], we can solve this equation over M, and hence in M:

$$x^{-1}gxy^{-1}gy = h$$

   Thus every $h \in M$ lies in the normal subgroup generated by g, for any $g \neq 1$. So M is simple. ■

Theorem: 5.3.3 *An existentially closed group M is ω-homogeneous.*

Outline of Proof: Construct a useful HNN extension.

Proof: There exists an HNN-extension of M in which θ is equivalent to conjugation. Since M is existentially closed, this implies that θ is equivalent to conjugation in M. Thus θ can be extended to an inner automorphism of M. ■

Theorem: 5.3.4 *If A is an abelian group and contains an element of infinite order, then A is ω-homogeneous if and only if A is divisible.*

Outline of Proof ($\Longrightarrow$): Depending on whether the element we wish to divide has finite or infinite order, we can always use our element of infinite order to define an injective map which allows us to divide, once we extend using ω-homogeneity.

Outline of Proof ($\Longleftarrow$): Assuming that the element we wish to adjoin to our finitely generated group is not already in the finitely generated subgroup, we use our structure theorem for divisible groups to demonstrate that we can always extend our embedding as required.

Proof ($\Longrightarrow$): We have that A is ω-homogeneous. Let $a \in A$ have infinite order. Let $b \in A$ be arbitrary. Let $n \in \mathbb{Z},\ n > 0$ be arbitrary.

If b has infinite order: Define:

$$\begin{array}{rcccl} \theta & : & \langle na \rangle & \rightarrow & A \\ & & na & \mapsto & b \end{array}$$

It is clear that θ is a homomorphism.

Claim: θ is injective.

Proof of Claim: Suppose that, for some $k,\ l \in \mathbb{Z}$:

$$\begin{aligned} (kna)\theta &= (lna)\theta \\ k(na)\theta &= l(na)\theta \\ kb &= lb \\ (k-l)b &= 0 \\ \Longrightarrow k-l &= 0,\ \textit{since b has infinite order} \\ \Longrightarrow k &= l \end{aligned}$$

⊣ (Claim)

Now, since A is ω-homogeneous, we can extend θ to ϕ:

$$\begin{array}{cccc} \phi & : & \langle na, a\rangle & \to & A \\ & & na & \mapsto & b \\ & & a & \mapsto & c \end{array}$$

for some $c \in A$. But then:

$$\begin{aligned} b &= (na)\phi \\ &= n(a)\phi \\ &= nc \end{aligned}$$

showing that A is divisible, as claimed.

If b has finite order: Say $|b| = m$ for some $m > 0$.

Claim 1: $a = na^*$ for some $a^* \in A$.

Proof of Claim 1: Define:

$$\begin{array}{ccccc} \theta_1 & : & \langle na\rangle & \to & A \\ & & na & \mapsto & a \end{array}$$

θ_1 is clearly a homomorphism.

Sub-Claim 1: θ_1 is injective.

Proof of Sub-Claim 1: Suppose:

$$\begin{aligned} 0 &= (kna)\theta_1,\ \textit{for some } k \in \mathbb{Z} \\ 0 &= k(na)\theta_1 \\ &= ka \\ \Longrightarrow 0 &= k,\ \textit{since a has infinite order} \end{aligned}$$

⊣ (Sub-Claim 1)

Now since A is ω-homogeneous, we can extend θ_1:

$$\begin{array}{rcccc} \phi_1 & : & \langle na, a\rangle & \rightarrow & A \\ & & na & \mapsto & a \\ & & a & \mapsto & a^*, \; for \; some \; a^* \in A \end{array}$$

Then:

$$\begin{aligned} a &= (na)\phi_1 \\ &= n(a)\phi_1 \\ &= na^* \end{aligned}$$

$\dashv$ (Claim 1)

Now define:

$$\begin{array}{rcccc} \theta_2 & : & \langle na\rangle & \rightarrow & A \\ & & na & \mapsto & a+b \end{array}$$

θ_2 is clearly a homomorphism.

Claim 2: θ_2 is injective.

Proof of Claim 2: Suppose:

$$\begin{aligned} 0 &= (kna)\theta_2, \; for \; some \; k \in \mathbb{Z} \\ 0 &= k(na)\theta_2 \\ 0 &= k(a+b) \end{aligned} \tag{5.1}$$

Sub-Claim 2: The element $(a+b)$ has infinite order.

Proof of Sub-Claim 2: Suppose that:

$$\begin{aligned} 0 &= l(a+b), \; for \; some \; l \in \mathbb{Z} \\ \Longrightarrow 0m &= lm(a+b) \\ \Longrightarrow 0 &= l(ma+mb) \\ 0 &= l(ma), \; since \; mb = 0 \\ \Longrightarrow 0 &= lm, \; since \; a \; has \; infinite \; order \\ \Longrightarrow 0 &= l, \; since \; m > 0 \end{aligned}$$

$\dashv$ (Sub-Claim 2)

So by Sub-Claim 1, equation (5.1) implies that $k = 0$ as required. $\dashv$ (Claim 2)

Now since A is ω-homogeneous, we can extend θ_2:

$$\begin{array}{rcll}\phi_2 & : & \langle na, a\rangle & \rightarrow & A \\ & & na & \mapsto & a+b \\ & & a & \mapsto & c,\ for\ some\ c \in A\end{array}$$

Then:

$$\begin{array}{rcl} a+b & = & (na)\phi_2 \\ & = & n(a)\phi_2 \\ & = & nc \\ na^* + b & = & nc \\ b & = & n(c-a^*) \end{array}$$

completing the proof that A is divisible in this case also.

Proof ($\Longleftarrow$): We have that A is divisible. Let $\{g_1, \ldots, g_r, h\} \subseteq A$ be any finite subset. We adopt the notation that $D\langle g_1, \ldots, g_r\rangle$ denotes the divisible subgroup of A generated by $\{g_1, \ldots, g_r\}$, i.e. the smallest divisible subgroup of A which contains $\{g_1, \ldots, g_r\}$.

Since A is divisible, we have by Theorem 4.1.5 of [17] that A is isomorphic to some number (possibly infinite) of copies of $(\mathbb{Q}, +)$ and $C_{p_i^\infty}$, for some prime p_i. Let $D_1 = D\langle g_1, \ldots, g_r\rangle$. Since D_1 is finitely generated, it is isomorphic to:

$$\bigoplus_{finite} \{(\mathbb{Q}, +)\ or\ C_{p_i^\infty}\}$$

Let $\theta : \langle g_1, \ldots, g_r\rangle \rightarrow A$ be any injective homomorphism. We have the following 2 cases to handle.

1. $\underline{h \in D_1}$: Then we can find $a_1, \ldots, a_r \in \mathbb{Z}$ and $n_1, \ldots, n_r \in \mathbb{Z},\ n_i > 0$ such that:
$$h = \frac{a_1}{n_1}g_1 + \cdots + \frac{a_r}{n_r}g_r$$

   Then define ϕ extending θ by setting:

$$h\phi = \frac{a_1}{n_1}(g_1)\theta + \cdots + \frac{a_r}{n_r}(g_r)\theta$$

   Then ϕ is a homomorphism, since θ is. We need to check that ϕ is injective. Suppose that:

$$\begin{array}{rcl} 0 & = & (b_1g_1 + \cdots + b_rg_r + b_{r+1}h)\phi \\ & = & b_1(g_1)\phi + \cdots + b_r(g_r)\phi + b_{r+1}(h)\phi \end{array}$$

$$
\begin{aligned}
&= b_1(g_1)\theta + \cdots + b_r(g_r)\theta + b_{r+1}(\frac{a_1}{n_1}(g_1)\theta + \cdots + \frac{a_r}{n_r}(g_r)\theta) \\
&= (b_1 + b_{r+1}\frac{a_1}{n_1})(g_1)\theta + \cdots + (b_r + b_{r+1}\frac{a_r}{n_r})(g_r)\theta \\
&= ((b_1 + b_{r+1}\frac{a_1}{n_1})g_1 + \cdots + (b_r + b_{r+1}\frac{a_r}{n_r})g_r)\theta \\
&= (b_1g_1 + \cdots + b_rg_r + b_{r+1}(\frac{a_1}{n_1}g_1 + \cdots + \frac{a_r}{n_r})g_r)\theta \\
&= (b_1g_1 + \cdots + b_rg_r + b_{r+1}h)\theta \\
\implies 0 &= b_1g_1 + \cdots + b_rg_r + b_{r+1}h, \ \textit{since } \theta \textit{ is injective}
\end{aligned}
$$

Therefore ϕ is injective as required.

2. $\underline{h \notin D_1}$: Let $D_2 = D\langle h\rangle$. Then D_2 is indecomposable, since it is generated by one element. Therefore D_2 is isomorphic to one copy of $(\mathbb{Q}, +)$ or $C_{p_i^\infty}$. Let $D_3 = \langle g_1, \ldots, g_r, h\rangle$. Then it is clear that $D_3 = D_1 + D_2$. Also, since $h \notin D_1$, we have that $D_1 \cap D_2 = \emptyset$. Therefore we have that $D_3 = D_1 \oplus D_2$.

   Let $D_4 = D\langle g_1\theta, \ldots, g_r\theta\rangle$. Then $\theta : D_1 \to D_4$ is an isomorphism.

   Now since $D_3 \leq A$, $D_3 \cong D_1 \oplus D_2$ and $D_1 \cong D_4 \leq A$, we can find $D_5 \leq A$ such that $D_5 \cong D_2$ and $D_1 \oplus D_2 \cong D_4 \oplus D_5$. Let $\beta : \underbrace{D_1 \oplus D_2}_{=D_3} \to D_4 \oplus D_5$ be an isomorphism that restricts to θ on $\langle g_1, \ldots, g_r\rangle$. It is possible to find such a β because of the direct sum construction. Then in particular, β is injective.

   Now since $\langle g_1, \ldots, g_r, h\rangle \subseteq D\langle g_1, \ldots, g_r, h\rangle = D_3$, we may take ϕ to be the restriction of β to $\langle g_1, \ldots, g_r, h\rangle$. Then from construction it is clear that ϕ is injective and restricts to θ on $\langle g_1, \ldots, g_r\rangle$.

In either case, we have shown that A is ω-homogeneous, as required. ■

Theorem: 5.3.5 *If A is an abelian torsion group, write:*

$$A = A_1 \oplus A_2 \oplus \cdots$$

where A_i is a p_i-group, for some prime p_i, and $p_i \neq p_j$ when $i \neq j$. Then:

1. *A is ω-homogeneous if and only if each A_i is ω-homogeneous.*
2. *A_i is ω-homogeneous if and only if it is divisible or a direct sum of cyclic groups of the same order.*

Outline of Proof:

1. This is straightforward from the definition of ω-homogeneity.

2. (a) For ($\Longrightarrow$), we use a result from [11] to obtain that A_i is a direct sum of cyclic groups. Either there is a uniform bound on the order of elements of A_i or there is no uniform bound. We shall show that if there is a uniform bound, say p_i^k, then $A_i = C_{p_i^k} \oplus C_{p_i^k} \oplus \cdots$. If there is no uniform bound, then we show that A_i is divisible.

   (b) For ($\Longleftarrow$), we show that it suffices to prove that $C_{p_i^k}$ (for a finite k) and $C_{p_i^\infty}$ are both ω-homogeneous. We then apply the definition of ω-homogeneity for a countable group which requires that for any 2 isomorphic finitely generated subgroups, the isomorphism can be extended to an automorphism of the whole group.

Proof:

1. (a) $\underline{\Longrightarrow}$ We have that A is ω-homogeneous. Let A_i be arbitrary. We want to show that A_i is ω-homogeneous. Let $\{g_1, \ldots, g_r, h\} \subseteq A_i$ be any finite set. Without loss of generality, assume $h \neq 0$. Let $\theta : \langle g_1, \ldots, g_r \rangle \to A_i$ be any injective homomorphism.

   Since $A_i \hookrightarrow A$ in a natural way, we may regard θ as an embedding of $\langle g_1, \ldots, g_r \rangle$ into A. Since A is ω-homogeneous, we can obtain an injective homomorphism $\phi : \langle g_1, \ldots, g_r, h \rangle \to A$ extending θ. Since ϕ extends θ, we have that:

   $$\langle g_1\phi, \ldots, g_r\phi \rangle \subseteq A_i$$

   So we need to make sure that $h\phi \in A_i$ also.

   Recall that $h \in A_i$, a p_i-group. Also recall that $p_i \neq p_j$ when $i \neq j$. Now since ϕ is injective, $h\phi$ must have the same order as h, i.e. some positive power of p_i. Thus the only possibility is that $h\phi \in A_i$. Therefore $\langle g_1\phi, \ldots, g_r\phi, h\phi \rangle \subseteq A_i$. Thus ϕ is an embedding of $\langle g_1, \ldots, g_r, h \rangle$ into A_i, and A_i is ω-homogeneous as required.

   (b) $\underline{\Longleftarrow}$ Let $\{g_1, \ldots, g_r, h\} \subseteq A$ be any finite set. Let $\theta : \langle g_1, \ldots, g_r \rangle \to A$ be any injective homomorphism. We need to show that we can extend θ to an injective homomorphism $\phi : \langle g_1, \ldots, g_r, h \rangle \to A$.

   Since $A = A_1 \oplus A_2 \oplus \cdots$, we can write uniquely:

   $$\begin{array}{ccccccc} g_1 & = & g_{11} & + & g_{12} & + & \cdots \\ g_2 & = & g_{21} & + & g_{22} & + & \cdots \\ & \vdots & & & & & \cdots \\ h & = & h_1 & + & h_2 & + & \cdots \end{array}$$

where for all j, $g_{ij} \in A_j$, $h_j \in A_j$. We regard θ as a combination of $\theta_j = \theta|_{A_j}$, for each j.

Claim: Each θ_j injective.

Proof of Claim: Suppose $(b_1 g_{1j} + b_2 g_{2j} + \cdots + b_r g_{rj} + b_{r+1} h_j)\theta_j = 0$. Then $b_1 g_{1j} + b_2 g_{2j} + \cdots + b_r g_{rj} + b_{r+1} h_j \in A_j \subseteq A$. And since $\theta|_{A_j} = \theta_j$, we then have that $(b_1 g_{1j} + b_2 g_{2j} + \cdots + b_r g_{rj} + b_{r+1} h_j)\theta = 0$. Since θ is injective, this implies that $b_1 g_{1j} + b_2 g_{2j} + \cdots + b_r g_{rj} + b_{r+1} h_j = 0$, and θ_j is therefore injective, as claimed. $\dashv$ (Claim)

Recall that each A_j is ω-homogeneous. Therefore we can extend each $\theta_j : \langle g_{1j}, \ldots, g_{rj} \rangle \to A_j$ to an injective map $\phi_j : \langle g_{1j}, \ldots, g_{rj}, h_j \rangle \to A_j$. Then extend θ to ϕ by defining $\phi(h) = \phi_1(h_1) + \phi_2(h_2) + \cdots$. Since A is a direct sum, this clearly gives us an injective map into A and we are done.

2. (a) $\underline{\Longrightarrow}$

If p_i^k is a uniform bound for the order of elements of A: By Theorem 6 on p. 17 of [11], we have that A_i is a direct sum of cyclic groups. Since A_i is a p_i-group, we may write:

$$A_i = \underbrace{C_{p_i^{e_1}} \oplus C_{p_i^{e_2}} \oplus \cdots}_{\textit{possibly finite, the same argument works}}$$

We want to show that $k = e_1 = e_2 = \cdots$. For a contradiction, suppose that this does not hold. Without loss of generality, suppose that we have:

$$A_i = C_{p_i^l} \oplus C_{p_i^k} \oplus C_{p_i^k} \oplus \cdots$$

where $l < k$.

Let b_1 generate $C_{p_i^l}$. Let b_2 generate $C_{p_i^k}$. Define:

$$\begin{array}{cccc} \theta & : & \langle p_i^{(k-l)} b_2 \rangle & \to & A_i \\ & & p_i^{(k-l)} b_2 & \mapsto & b_1 \end{array}$$

Notice that $p_i^{(k-l)} b_2$ generates a copy of $C_{p_i^l}$ inside $C_{p_i^k}$. Since b_1 is a generator of $C_{p_i^l}$, we have that θ is an isomorphism. In particular, θ is an injective homomorphism.

Since A_i is ω-homogeneous, we can extend θ to an injective homomorphism:

$$\begin{array}{rccc} \phi & : & \langle p_i^{(k-l)} b_2, b_2 \rangle & \rightarrow & A_i \\ & & p_i^{(k-l)} b_2 & \mapsto & b_1 \\ & & b_2 & \mapsto & \sum_j x_j b_j \end{array}$$

for some $\sum_j x_j b_j \in A_i = C_{p_i^l} \oplus C_{p_i^k} \oplus C_{p_i^k} \oplus \cdots$ (i.e. b_j generates the jth summand, $x_j \in \mathbb{Z}$, $\forall j$). Then since ϕ is a homomorphism, we have that:

$$\begin{array}{rcl} (p_i^{(k-l)} b_2)\phi & = & p_i^{(k-l)} (b_2)\phi \\ & = & p_i^{(k-l)} (\sum_j x_j b_j) \end{array}$$

We look at the first co-ordinate of $(p_i^{(k-l)} b_2)\phi$. Since ϕ extends θ, the first co-ordinate must equal b_1. From the above equation, the first co-ordinate must equal $p_i^{(k-l)} x_1 b_1$. Since ϕ is well-defined, we must have:

$$\begin{array}{rcl} b_1 & = & p_i^{(k-l)} x_1 b_1 \\ p_i^l b_1 & = & p_i^k x_1 b_1 \\ 0 & = & p_i^k x_1 b_1, \textit{ since } |b_1| = p_i^l \\ \Longrightarrow p_i^k x_1 & | & p_i^l, \textit{ since } |b_1| = p_i^l \end{array}$$

This is a contradiction, since $x_1 \in \mathbb{Z}$ and $l < k$. Thus this part of the proof is completed, and the summands of A_i must all be of the same order.

If there is no uniform bound for the order of elements of A: A_i is a p_i-group. Since p_i is prime, we can always divide by anything coprime with p_i. So to show that A_i is divisible, it suffices to show that we can divide any element of A_i by p_i.

Let $b \in A_i$ be arbitrary. Since A_i is a p_i-group, $|b| = p_i^k$, for some non-negative $k \in \mathbb{Z}$. Since there is no uniform bound on the order of the elements of A_i, we can always find $a \in A_i$ such that $|a| = p_i^{k+1}$. Then the following map is an injective homomorphism:

$$\begin{array}{rcccc} \theta & : & \langle p_i a \rangle & \rightarrow & A_i \\ & & p_i a & \mapsto & b \end{array}$$

Now since A_i is ω-homogeneous, we can find an injective homomorphism ϕ extending θ:

$$\begin{array}{rclcll} \phi & : & \langle p_i a, a\rangle & \to & A_i & \\ & & p_i a & \mapsto & b & \\ & & a & \mapsto & c, & for\ some\ c \in A_i \\ & & p_i a & \mapsto & p_i c & \end{array}$$

But then $p_i a \mapsto b = p_i c$. This shows that we can always divide b by p_i. Thus A_i is divisible, as claimed.

(b) $\Longleftarrow$ If A_i is a divisible p_i-group, then by Theorem 4.1.5 of [17], we know that A_i is isomorphic to a direct sum of copies of $C_{p_i^\infty}$. Thus by part (1) of the proof, it suffices to prove the following claim.

If A_i is a direct sum of cyclic groups of the same order, then by part (1) of the proof, it again suffices to prove:

Claim: $C_{p_i^k}$ and $C_{p_i^\infty}$ are ω-homogeneous.

Proof of Claim: Let $A \cong B$ be any 2 finitely generated isomorphic subgroups of $C_{p_i^k}$ or $C_{p_i^\infty}$. Since $A,\ B$ are finitely generated p_i-groups, we can find some finite n such that $A,\ B \leq C_{p_i^n}$. In fact, this implies that $A = B$, since a finite cyclic group has only one subgroup of a given fixed order. Then the isomorphism $A \cong B$ is an automorphism of $C_{p_i^n}$. This automorphism then lifts to an automorphism of $C_{p_i^k}$ or $C_{p_i^\infty}$ in the natural way. We started with any 2 finitely generated isomorphic subgroups and we lifted to an automorphism of the whole group which sends one subgroup to the other. Then by the equivalent definition of ω-homogeneous in the case of a countable group, we are done. $\dashv$ (Claim)

■

Theorem: 5.3.6 *Phillip Hall's universal locally finite group is ω-homogeneous.*

Outline of Proof: This is a straightforward consequence of Theorem (3.3.1 2).

Proof: Let U denote Phillip Hall's universal locally finite group. Let $\{g_1, \ldots, g_r, h\} \subset U$ be any finite subset. Let $\theta : \langle g_1, \ldots, g_r\rangle \to U$ be any injective homomorphism.

Now since U is locally finite, we have that both of $\langle g_1, \ldots, g_r \rangle$ and $\langle g_1, \ldots, g_r, h \rangle$ are finite subgroups. It is clear that $\langle g_1, \ldots, g_r \rangle \leq \langle g_1, \ldots, g_r, h \rangle$. Then by Theorem (3.3.1 2), we have that θ can be extended to an embedding:

$$\phi : \langle g_1, \ldots, g_r, h \rangle \to U$$

But this embedding ϕ must be injective. Therefore we have that U satisfies the definition of ω-homogeneous, and we are done. ■

Recall that earlier we defined:

Definition (Skeleton): For any group G, the *skeleton* of G, denoted by $Sk\ G$, is the class of all finitely generated groups that can be embedded in G.

Theorem: 5.3.7

1. *If G is a countable group and if K is an ω-homogeneous group with $Sk\ G \subseteq Sk\ K$, then G can be embedded into K.*
2. *Two countable ω-homogeneous groups are isomorphic if and only if they have the same skeletons.*
3. *Any isomorphism between finitely generated subgroups of a countable ω-homogeneous group K can be extended to an automorphism of K.*

Outline of Proof:

1. Use the containment of the skeletons and the ω-homogeneity of K to construct an embedding of G.
2. This is a standard "back-and-forth" argument.
3. Modify the previous argument slightly to get the desired result.

Proof:

1. Let $G = \{g_1, g_2, \ldots\}$. Let $G_n = \langle g_1, \ldots, g_n \rangle$. We show that we can construct embeddings $\phi_n : G_n \hookrightarrow K$, such that ϕ_{n+1} extends ϕ_n, for all $n \geq 1$. Then the map:

$$\begin{array}{ccccc} \phi & : & G & \to & K \\ & & g_n & \mapsto & g_n\phi_n \end{array}$$

is an embedding of G into K.

Since G_n is finitely generated, and since $Sk\ G \subseteq Sk\ K$, there exists an embedding $\theta_n : G_n \hookrightarrow K$. Let $\phi_1 = \theta_1$.

We define ϕ_{n+1} inductively, as follows: Suppose that $\phi_n : G_n \to K$ extends $\phi_{n-1} : G_{n-1} \to K$. The map:

$$(\theta_{n+1}|_{G_n})^{-1} \circ \phi_n : \langle g_1\theta_{n+1}, \ldots, g_n\theta_{n+1}\rangle \to K$$

is an embedding of $\langle g_1\theta_{n+1}, \ldots, g_n\theta_{n+1}\rangle$ into K, which sends $g_i\theta_{n+1}$ to $g_i\phi_n$ $(1 \leq i \leq n)$. Since K is ω-homogeneous, this embedding can be extended to an embedding ψ_{n+1} of $Im\ \theta_{n+1}$ into K.

Let $\phi_{n+1} = \theta_{n+1} \circ \psi_{n+1}$. Then we have that, for $(1 \leq i \leq n)$:

$$\begin{aligned} g_i\phi_{n+1} &= g_i\theta_{n+1} \circ \psi_{n+1} \\ &= g_i\theta_{n+1} \circ (\theta_{n+1}^{-1} \circ \phi_n) \\ &= g_i\phi_n \end{aligned}$$

so ϕ_{n+1} extends ϕ_n, as required.

2. Suppose that $K_1 = \langle g_1, g_2, \ldots\rangle$ and $K_2 = \langle h_1, h_2, \ldots\rangle$ are 2 ω-homogeneous groups that have the same skeletons. Let $G_1 = \langle g_1\rangle$. Choose an embedding $\theta_1 : G_1 \to K_2$. We may do this, since G_1 is finitely generated and $Sk\ K_1 = Sk\ K_2$.

   Then the group $H_1 = \langle g_1\theta_1, h_1\rangle$ belongs to $Sk\ K_2$ and hence to $Sk\ K_1$. Therefore we may choose an embedding $\phi_1 : H_1 \hookrightarrow K_1$. Since K_1 is ω-homogeneous, we may choose ϕ_1 such that $g_1\theta_1\phi_1 = g_1$.

   Let $G_2 = \langle H_1\phi_1, g_2\rangle$. Then since K_2 is ω-homogeneous, we may choose an embedding $\theta_2 : G_2 \hookrightarrow K_2$ extending θ_1, so that $h_1\phi_1\theta_2 = h_1$. Let $H_2 = \langle G_2\theta_2, h_2\rangle$.

   Continuing in this way, we may choose ϕ_i extending ϕ_{i-1} and so that $g_i = g_i\theta_i\phi_i$, then choose θ_{i+1} extending θ_i so that $h_i = h_i\phi_i\theta_{i+1}$. Thus we can define embeddings:

$$\begin{array}{ccccccccccc} \theta & : & K_1 & \to & K_2 & and & \phi & : & K_2 & \to & K_1 \\ & & g_i & \mapsto & g_i\theta_i & & & & h_i & \mapsto & h_i\phi_i \end{array}$$

   for $i \in \mathbb{N}$. From this we can see that $\theta \circ \phi = \phi \circ \theta = 1$. Thus $K_1 \cong K_2$ as claimed.

3. This follows by an obvious modification to the argument for (2). Take $K_1 = K_2 = K$. Take θ_1 to be the given isomorphism with G its domain. Then extend θ_1 to an automorphism of K, as in Theorem (2). ■

Theorem: 5.3.8 *If M is a countable existentially closed group, and if G is a finitely generated subgroup of M with $Z(G) = 1$, then $C_M(G)$ is isomorphic to M.*

Outline of Proof: By Theorem (5.3.3), M is ω-homogeneous. $C_M(G)$ is countable, since M is. So show that $C_M(G)$ is ω-homogeneous and that $Sk\ C_M(G) = Sk\ M$. Then appeal to Theorem (5.3.7 2).

Proof ($C_M(G)$ is ω-homogeneous): Let H be a finitely generated subgroup of $C_M(G)$. Let $\theta : H \to C_M(G)$ be an injective homomorphism.

Claim 1:

$$H \cap G = H\theta \cap G = 1$$

Proof of Claim 1: For a contradiction, suppose that $1 \neq h \in H \cap G$. Then since $h \in H \subseteq C_M(G)$, we have that $hg = gh$, $\forall g \in G$. We also have $h \in G$, so this implies that $h \in Z(G)$. Since $h \neq 1$, we have a contradiction with $Z(G) = 1$. Therefore $H \cap G = 1$ as claimed.

An analogous argument works to show that $H\theta \cap G = 1$. $\dashv$ (Claim 1)

By the Claim 1, we have that:

$$\begin{aligned} \langle H, G \rangle &\cong H \times G \\ \langle H\theta, G \rangle &\cong H\theta \times G \end{aligned}$$

Thus θ extends to a monomorphism $\hat{\theta} : H \times G \to M$, which fixes G elementwise.

Claim 2: There exists $k \in M$ such that for all $x \in H \times G$, $k^{-1}xk = x\hat{\theta}$.

Proof of Claim 2: G and H are finitely generated subgroups of M. Since $H \cap G = \{1\}$, and both are subgroups of M, we have that $H \times G$ is also a subgroup of M. Therefore $H \times G$ and $(H \times G)\hat{\theta}$ are isomorphic subgroups of M. Form the HNN extension: $M^* = \langle M, t \mid t^{-1}(H \times G)t = (H \times G)\hat{\theta} \rangle$. Then $t \in M^*$ is a solution of:

$$\forall x \in (H \times G),\ y^{-1}xy = x\hat{\theta}$$

Since G, H are finitely generated, this can be written as a finite system of equations over M. Since M is existentially closed, we can find $k \in M$ which also solves the system. Thus the claim is proved. $\dashv$ (Claim 2)

Let $g \in G$ be arbitrary. Then notice that since $\hat{\theta}$ is the identity map for all $g \in G$, this implies that:

$$\begin{aligned} k^{-1}gk &= g\hat{\theta} \\ &= g \\ gk &= kg \\ \Longrightarrow k &\in C_M(G) \end{aligned}$$

Therefore θ is the restriction of an inner automorphism of $C_M(G)$. This shows that $C_M(G)$ is ω-homogeneous, as required.

Proof ($Sk\, C_M(G) = Sk\; M$): Since $C_M(G) \subseteq M$, it is clear that $Sk\, C_M(G) \subseteq Sk\; M$. Let F be a finitely generated subgroup of M. We want to show that $F \in Sk\; C_M(G)$, i.e. that F centralizes G. It suffices to exhibit an embedding of F into $C_M(G)$. This will show that $Sk\; M \subseteq Sk\; C_M(G)$ and complete the proof of equality.

Take any group $F_1 \cong F$ and form the direct product $M \times F_1$. Then $M \times F_1$ contains a copy of $F \leq M$ and a copy of F_1. Since these subgroups of $M \times F_1$ are isomorphic, we may form the HNN extension $M^* = \langle M \times F_1, t \mid t^{-1}Ft = F_1 \rangle$.

Claim 3: $t^{-1}Ft$ centralizes G in M^*.

Proof of Claim 3: Any $g \in G \leq M$ clearly commutes with everything in the copy of F_1 in $M \times F_1$. But in the HNN extension, $F_1 = t^{-1}Ft$. Therefore g commutes with everything in $t^{-1}Ft$, completing the proof. $\dashv$ (Claim 3)

Therefore we have that $t^{-1}Ft \subseteq C_{M^*}(G)$. We still need to show that $t^{-1}Ft \subseteq C_M(G)$. Notice that the $t \in M^*$ is a solution of $x^{-1}Fx \subseteq C_M(G)$. This can be expressed as a finite system of equations since both F and G are finitely generated. Also, the solution t lies in $M \times F_1$, which contains M. Since M is existentially closed, there must exist $m \in M$ such that $m^{-1}Fm \subseteq C_M(G)$. Conjugation by m is an automorphism of M, in particular it is injective. Therefore conjugation by m induces an embedding of F into $C_M(G)$.

This shows that $Sk\; M \subseteq Sk\; C_M(G)$, and therefore $Sk\; M = Sk\; C_M(G)$.

Thus by Theorem (5.3.7 2) we are done. ■

Notation: A class $\mathcal{H}$ of groups will from now on be isomorphism closed, i.e. any group isomorphic to a group in the class also lies in the class. Also, we call $\mathcal{H}$ *trivial* if it consists of precisely the groups with one element.

Definition: A class $\mathcal{H}$ of finitely generated groups is said to satisfy:

1. <u>SC (Subgroup Closure)</u>: if whenever $F \in \mathcal{H}$ and G is a finitely generated subgroup of F, then $G \in \mathcal{H}$

2. JEP (Joint Embedding Property): if, for any $F, G \in \mathcal{H}$, there exist a group $H \in \mathcal{H}$ and injective homomorphisms θ, ϕ such that $\theta : F \to H$ and $\phi : G \to H$.

3. AEP (Amalgamated Embedding Property): if, for any $F, G, H \in \mathcal{H}$, and for any injective homomorphisms $\alpha : F \to G$ and $\beta : F \to H$, there exist $K \in \mathcal{H}$ and injective homomorphisms $\gamma : G \to K$ and $\delta : H \to K$ such that $\alpha\gamma = \beta\delta$.

4. AC (Algebraic Closure): if whenever $F \in \mathcal{H}$ and $\mathcal{S}$ is a finite set of equations defined over F and soluble over F, then $\mathcal{S}$ is soluble in some group $G \in \mathcal{H}$ that contains F.

Theorem: 5.3.9 *Let $\mathcal{H}$ be a non-empty class of finitely generated groups, which contains at most a countable set of isomorphism types of groups. Then $\mathcal{H}$ is the skeleton of a countable group if and only if it satisfies SC and JEP.*

Outline of Proof: Just apply the definitions.

Proof ($\Longrightarrow$): Let K be a countable group. Let $\mathcal{H} = Sk\ K$. Then it is clear that $\mathcal{H}$ satisfies SC.

Let $F, G \in \mathcal{H}$. Then $F \cong F_1 \leq K$ and $G \cong G_1 \leq K$. Let $H = \langle F_1, G_1 \rangle \leq K$. Then H is finitely generated. Therefore $H \in \mathcal{H}$. Also, there exist monomorphisms $\theta : F \to F_1 \leq H$ and $\theta : G \to G_1 \leq H$. Therefore K satisfies JEP.

Proof ($\Longleftarrow$): Let $\mathcal{H}$ satisfy SC and JEP. Since $\mathcal{H}$ is countable, let $G_0, G_1, \ldots$ be representatives of the isomorphism classes of $\mathcal{H}$.

Let $H_0 = G_0$. Then inductively take H_{i+1} to be a group in $\mathcal{H}$ in which both H_i and G_{i+1} are embedded. We can always find such a group in $\mathcal{H}$ since $\mathcal{H}$ satisfies JEP and $H_i,\ G_{i+1} \in \mathcal{H}$. Identify H_i with its embedding in H_{i+1}. Then form $K = \bigcup_{i \in \mathbb{N}} H_i$.

Since each H_i is countable, K is countable. By construction, every G_i embeds into K, therefore $\mathcal{H} \subseteq Sk\ K$. Let F be a finitely generated subgroup of K. Then, for some i, $F \leq H_i \in \mathcal{H}$. In other words, $Sk\ K \subseteq \mathcal{H}$. So we have $Sk\ K = \mathcal{H}$, and K is the required group. ■

Theorem: 5.3.10 *Let $\mathcal{H}$ be a non-empty class of finitely generated groups, which contains at most a countable set of isomorphism types. Then $\mathcal{H}$ is the skeleton of a countable ω-homogeneous group if and only if it satisfies SC and AEP.*

Outline of Proof: For ($\Longrightarrow$), show that ω-homogeneous implies SC and AEP.

To start the ($\Longleftarrow$) direction, we show that if any non-empty class satisfies SC and AEP, it satisfies JEP. Then by Theorem (5.3.9), $\mathcal{H} = Sk\ K_0$, for some countable group K_0. Here is the plan of attack for the rest of the ($\Longleftarrow$) direction:

1. Fix any particular choice of a finitely generated group $G \leq K_0$, an element $h \in K_0$, and a monomorphism $\theta : G \hookrightarrow K_0$. Show that, for this choice of triple (G, h, θ), there exists a countable group $\hat{K}_0$, containing K_0, and a monomorphism $\hat{\theta} : \langle G, h\rangle \hookrightarrow \hat{K}_0$, such that $\hat{\theta}$ extends θ and $Sk\ \hat{K}_0 = \mathcal{H}$.

2. Show there exists a countable group X_0, containing K_0 such that, for every choice of a finitely generated group $G \leq K_0$, an element $h \in K_0$ and a monomorphism $\theta : G \hookrightarrow K_0$, there is a monomorphism $\hat{\theta} : \langle G, h\rangle \hookrightarrow X_0$ extending θ. In (1), we showed that a $\hat{K}_0$ exists for a particular choice of (G, h, θ). Here we show that X_0 works for all possible choices of (G, h, θ) simultaneously. We show further that we can choose X_0 such that $\mathcal{H} = Sk\ X_0$.

3. Show that there exists a countable group X, with $Sk\ X = \mathcal{H}$, such that for each finitely generated group $G \leq X$, element $h \in X$, and monomorphism $\theta : G \hookrightarrow X$, we can find a monomorphism $\hat{\theta} : \langle G, h\rangle \hookrightarrow X$, which extends θ. So this X is the required countable ω-homogeneous group.

Proof ($\Longrightarrow$): Let $\mathcal{H} = Sk\ X$, where X is a countable ω-homogeneous group. Then $\mathcal{H}$ satisfies SC. We now show that $\mathcal{H}$ satisfies AEP also.

Let $F,\ G,\ H \in \mathcal{H} = Sk\ X$ be arbitrary. Then since there exist isomorphisms:

$$\begin{array}{ccccccc} \theta & : & F & \rightarrow & F_1 & \leq & X \\ \phi & : & G & \rightarrow & G_1 & \leq & X \\ \psi & : & H & \rightarrow & H_1 & \leq & X \end{array}$$

Now suppose that we have embeddings:

$$\begin{array}{ccccc} \alpha & : & F & \hookrightarrow & G \\ \beta & : & F & \hookrightarrow & H \end{array}$$

Then we have an embedding:

$$\theta^{-1}\beta\psi \ : \ F_1 \ \rightarrow \ H_1 \ \leq \ X$$

Since X is ω-homogeneous, we can extend the inverse of the above embedding to:

$$\delta_1 \ : \ H_1 \ \hookrightarrow \ X$$

Similarly, we have an embedding:

$$\theta^{-1}\alpha\phi \ : \ F_1 \ \to \ G_1 \ \leq \ X$$

Since X is ω-homogeneous, we can extend the inverse of the above embedding to:

$$\gamma_1 \ : \ G_1 \ \hookrightarrow \ X$$

Let $K = \langle H_1\delta_1, G_1\gamma_1\rangle \leq X$. Let $\delta = \psi\delta_1$. Let $\gamma = \phi\gamma_1$. Then we have:

$$\begin{array}{rcccl} \delta & : & H & \hookrightarrow & K \\ \gamma & : & G & \hookrightarrow & K \end{array}$$

and

$$\begin{aligned} \alpha\gamma &= \theta\,\underbrace{(\theta^{-1}\alpha\phi\gamma_1)}_{=id} \\ &= \theta \\ &= \theta\,\underbrace{(\theta^{-1}\beta\psi\delta_1)}_{=id} \\ &= \beta\delta \end{aligned}$$

Therefore $\mathcal{H}$ satisfies AEP as required.

Proof ($\Longleftarrow$)**:** Suppose that $\mathcal{H}$ satisfies SC and AEP. Since $\mathcal{H}$ is non-empty and satisfies SC, we have $1 \in \mathcal{H}$. Let $H, G \in \mathcal{H}$ be arbitrary. Let $F = 1$. Then there exist monomorphisms $\alpha : F \hookrightarrow H$ and $\alpha : F \hookrightarrow G$. Since $\mathcal{H}$ satisfies AEP, there exists $K \in \mathcal{H}$ such that G and H can both be embedded in K. Thus $\mathcal{H}$ satisfies JEP.

Then by Theorem (5.3.9), we have that $\mathcal{H} = Sk\ K_0$, for some countable group K_0. Let $G \leq K_0$ be finitely generated. Let $h \in K_0$. Let $\theta : G \hookrightarrow K_0$ be a monomorphism.

1. Since K_0 is countable, we may write K_0 as $\bigcup_{i\in\mathbb{N}} G_i$ where each G_i is finitely generated and

   $$G, G\theta, \{h\} \subseteq G_0 \subseteq G_1 \subseteq G_2 \subseteq \cdots$$

   Thus we get a sequence of embeddings:

   $$\begin{tikzcd} G \arrow[r, "\theta"] \arrow[d, "1"'] & G_0 \arrow[r, "1"] & G_1 \arrow[r, "1"] & \cdots \arrow[r, "1"] & G_i \arrow[r, "1"] & G_{i+1} \arrow[r, "1"] & \cdots \\ G_0 \end{tikzcd}$$

Since $\mathcal{H}$ satisfies AEP, we can find a finitely generated group $\hat{G}_0 \in \mathcal{H}$ and embeddings $\alpha_0 : G_0 \hookrightarrow \hat{G}_0$ and $\beta_0 : G_0 \hookrightarrow \hat{G}_0$ such that $\theta\alpha_0 = \beta_0$. By induction, for each $i \in \mathbb{N}$, we can find a finitely generated group $\hat{G}_i \in \mathcal{H}$ and embeddings $\alpha_i : G_i \hookrightarrow \hat{G}_i$ and $\beta_i : \hat{G}_{i-1} \hookrightarrow \hat{G}_i$ such that $\alpha_{i-1}\beta_i = \alpha_i$. Thus using AEP we can extend the above diagram to a commuting ladder:

$$\begin{tikzcd}
G \arrow[r, "\theta"] \arrow[d, "1"'] & G_0 \arrow[r, "1"] \arrow[d, "\alpha_0"'] & G_1 \arrow[r, "1"] \arrow[d, "\alpha_1"'] & \cdots \arrow[r, "1"] & G_i \arrow[r, "1"] \arrow[d, "\alpha_i"'] & G_{i+1} \arrow[r, "1"] \arrow[d, "\alpha_{i+1}"'] & \cdots \\
G_0 \arrow[r, "\beta_0"'] & \hat{G}_0 \arrow[r, "\beta_1"'] & \hat{G}_1 \arrow[r, "\beta_2"'] & \cdots \arrow[r, "\beta_i"'] & \hat{G}_i \arrow[r, "\beta_{i+1}"'] & \hat{G}_{i+1} \arrow[r, "\beta_{i+2}"'] & \cdots
\end{tikzcd}$$

where $\hat{G}_i \in \mathcal{H}$ and $\alpha_i\beta_{i+1} = \alpha_{i+1}$. $\alpha_i : G_i \to \hat{G}_i$ is an embedding. We are free to replace $\hat{G}_i$ by an isomorphic $\hat{G}_i^*$ such that $\alpha_i : G_i \to \hat{G}_i^*$ is the identity map. Since $\mathcal{H}$ is isomorphism closed, and since $\hat{G}_i^* \cong \hat{G}_i$, we then have that $\hat{G}_i^* \in \mathcal{H}$. So without loss of generality, we may choose $\hat{G}_i$ so that $\alpha_i = 1$. Then $\beta_i = 1$, for $i \geq 1$, and θ is the restriction of β_0 to G.

Take $\hat{\theta}$ to be the restriction of β_0 to $\langle G, h\rangle \leq G_0$. Let $\hat{K}_0 = \bigcup_{i\in\mathbb{N}} \hat{G}_i$. Then $\hat{\theta} : \langle G, h\rangle \to \hat{K}_0$ is a monomorphism which extends θ. By construction, $\hat{K}_0$ is countable.

We claim that $Sk\ \hat{K}_0 = \mathcal{H}$. $\mathcal{H} = Sk\ K_0 \subseteq Sk\ \hat{K}_0$, since $K_0 \leq \hat{K}_0$. Also, each $\hat{G}_i \in \mathcal{H}$. Therefore $Sk\ \hat{K}_0 \subseteq \mathcal{H}$. So we have that $Sk\ \hat{K}_0 = \mathcal{H}$ as claimed. Thus the first stage is complete.

2. There are only countably many triples (G, h, θ) with $G \leq K_0$ finitely generated, $h \in K_0$, and $\theta : G \hookrightarrow K_0$ a monomorphism. We may list these triples:

$$(G_0, h_0, \theta_0),\ (G_1, h_1, \theta_1),\ (G_2, h_2, \theta_2), \ldots$$

Now construct $K_1 = \hat{K}_0$ as in Theorem (1), so that $\theta_0 : G_0 \hookrightarrow K_0$ extends to $\hat{\theta}_0 : \langle G_0, h_0\rangle \hookrightarrow K_1$. Then $\theta_1 : G_1 \hookrightarrow K_0 \leq K_1$. Thus we can construct $K_2 = \hat{K}_1$ as in Theorem (1), so that θ_1 extends to $\hat{\theta}_1 : \langle G_1, h_1\rangle \hookrightarrow K_2$.

By this process we construct a sequence of groups:

$$K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots$$

each of which is countable and such that $Sk\ K_i = \mathcal{H},\ \forall i$. Let $X_0 = \bigcup_{i\in\mathbb{N}} K_i$. Then X_0 is countable and $Sk\ X_0 = \mathcal{H}$. Further, for each finitely generated group $G \leq K_0$, and each $h \in K_0$, each monomorphism $\theta : G \hookrightarrow K_0$ extends to a monomorphism $\hat{\theta} : \langle G, h\rangle \hookrightarrow K_i \leq X_0$, since every finitely generated subgroup of X_0 lies in some K_i.

3. Now X_0 is countable and so there are only countably many triples (G, h, θ) such that $G \leq X_0$ is finitely generated, $h \in X_0$, and $\theta : G \hookrightarrow X_0$ is a monomorphism. So as in Theorem (2), we can construct a countable group X_1, with $Sk\ X_1 = \mathcal{H}$ and such that, for any triple (G, h, θ) as above, the monomorphism $\theta : G \hookrightarrow X_0$ extends to a monomorphism $\hat{\theta} : \langle G, h \rangle \hookrightarrow X_1$.

   We can do the same with X_1 as in (2), to get a countable group X_2, with $Sk\ X_2 = \mathcal{H}$ and such that, for each $G \leq X_1$ and each $h \in X_1$, each monomorphism $\theta : G \hookrightarrow X_1$ extends to a monomorphism $\hat{\theta} : \langle G, h \rangle \hookrightarrow X_2$. In this way we construct a sequence:

$$X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots$$

   of countable groups, such that $Sk\ X_i = \mathcal{H}$, for each i.

   Let $X = \bigcup_{i \in \mathbb{N}} X_i$. Then X is countable. Also, $Sk\ X = \mathcal{H}$. If $G \leq X$ is a finitely generated group, if $k \in X$, and if $\theta : G \hookrightarrow X$ is a monomorphism, then there exists i such that $G, G\theta, h$ all lie in X_i. By construction, there exists a monomorphism $\hat{\theta} : \langle G, h \rangle \hookrightarrow X_{i+1}$, which extends $\theta : G \hookrightarrow X_i$. Thus X is ω-homogeneous. ■

Theorem: 5.3.11 *If $\mathcal{H}$ is a non-trivial non-empty class of finitely generated groups that consists of at most a countable set of distinct isomorphism types, then $\mathcal{H}$ is the skeleton of a countable existentially closed group if and only if it satisfies SC, JEP and AC.*

Outline of Proof ($\Longrightarrow$): This is straightforward since existentially closed groups are algebraically closed.

Outline of Proof ($\Longleftarrow$): Show that $\mathcal{H}$ satisfies AEP. Use Theorem (5.3.10) to obtain a countable ω-homogeneous group K such that $\mathcal{H} = Sk\ K$. Then show that K is algebraically closed. Since K is non-trivial, this implies by Theorem (5.3.1) that K is existentially closed.

Proof ($\Longrightarrow$): Let M be a countable existentially closed group. Let $\mathcal{H} = Sk\ M$. The skeleton of any group must satisfy SC and JEP. So we just have to show that $\mathcal{H}$ satisfies AC also.

Let $F \in \mathcal{H}$. Then F is isomorphic to some subgroup $F_1 \leq M$. Let $\mathcal{S}$ be a finite set of equations defined over F. Let $\mathcal{S}_1$ be the corresponding set of equations defined over F_1. If $\mathcal{S}$ is soluble over F, then $\mathcal{S}_1$ is soluble over F_1, say in G_1. Then $\mathcal{S}_1$ is soluble over M (for example in $M *_{M \cap G_1} G_1$, the free product of M and G_1 with amalgamated subgroup $M \cap G_1$). Since M

is existentially closed, this implies that $\mathcal{S}_1$ is soluble in M. Therefore $\mathcal{S}_1$ is soluble in a finitely generated subgroup (say H_1) of M containing F_1. Let H be a group containing F that is isomorphic to H_1. Then $\mathcal{S}$ is soluble in H. Also, $H \in \mathcal{H}$. Therefore $\mathcal{H}$ satisfies AC as required.

Proof ($\Longleftarrow$): Let $\mathcal{H}$ satisfy SC, JEP and AC.

Claim: $\mathcal{H}$ satisfies AEP.

Proof of Claim: Let $F, G, H \in \mathcal{H}$ be arbitrary. Let $\alpha : F \hookrightarrow G$ and $\beta : F \hookrightarrow H$ be monomorphisms. Since $\mathcal{H}$ satisfies JEP, there exist some $K_1 \in \mathcal{H}$ and monomorphisms $\theta : G \hookrightarrow K_1$ and $\phi : H \hookrightarrow K_1$. Then $F\alpha\theta$ and $F\beta\phi$ are isomorphic subgroups of K_1.

Since F is finitely generated, write $F = \langle f_1, \ldots, f_n \rangle$. Then since $F\alpha\theta \cong F\beta\phi$, we may form the HNN extension:

$$K_1^* = \langle K_1, t \mid t^{-1}(F\alpha\theta)t = F\beta\phi \rangle$$

Then the finite set of equations:

$$\mathcal{T} = \{x^{-1}(f_i\alpha\theta)x = f_i\beta\phi : 1 \leq i \leq n\}$$

is soluble in K_1^*.

Since $\mathcal{H}$ satisfies AC, there exists some $K \in \mathcal{H}$ such that K contains K_1 and K contains an element k such that $k^{-1}(f_i\alpha\theta)k = f_i\beta\phi$, for $1 \leq i \leq n$.

Define the following map:

$$\begin{array}{rccc} \rho & : & G\theta & \rightarrow & K \\ & & g\theta & \mapsto & k^{-1}(g\theta)k \end{array}$$

Then ρ is a monomorphism which sends $F\alpha\theta$ to $F\beta\phi$. Define $\gamma = \theta\rho$ and $\delta = \phi$. Then γ and δ are monomorphisms and for each $f \in F$, we have:

$$\begin{aligned} f\alpha\gamma &= f\alpha(\theta\rho) \\ &= k^{-1}(f\alpha\theta)k \\ &= f\beta\phi \\ &= f\beta\delta \end{aligned}$$

Therefore, we have that $\alpha\gamma = \beta\delta$ and $\mathcal{H}$ satisfies AEP as claimed. $\dashv$ (Claim)

Then by Theorem (5.3.10), there exists a countable ω-homogeneous group K such that $Sk\ K = \mathcal{H}$.

Let $\mathcal{S}$ be a finite set of equations defined over K. Then $\mathcal{S}$ is also a set of equations over some finite subgroup $F \leq K$. Since $\mathcal{H}$ satisfies AC, there is then some subgroup $G \in \mathcal{H}$ that contains F, such that $\mathcal{S}$ is soluble in G.

Now G is isomorphic to some $G\theta \leq K$. Since K is ω-homogeneous, the monomorphism $(\theta|_F)^{-1} : F\theta \hookrightarrow K$ extends to a monomorphism $\hat{\theta} : G\theta \hookrightarrow K$, so that $\theta\hat{\theta}$ fixes F elementwise. Since $G \cong G\theta\hat{\theta}$, The set $\mathcal{S}$ is soluble in $G \cong G\theta\hat{\theta}$, and hence in K.

This shows that K is algebraically closed. But $K \neq 1$, since $\mathcal{H}$ contains some non-trivial group. Therefore, by Theorem (5.3.1), K is existentially closed, as required. ■

Theorem: 5.3.12 *There exists a locally finitely presented countable existentially closed group.*

Outline of Proof:

1. Let $\mathcal{H}$ be the class of finitely generated subgroups of finitely presented groups.
   (a) Show that $\mathcal{H}$ satisfies SC.
   (b) Show that $\mathcal{H}$ satisfies JEP.
   (c) Show that $\mathcal{H}$ satisfies AC.
   (d) Show that $\mathcal{H}$ contains at most countably many isomorphism types.

   Then appeal to Theorem (5.3.11) to obtain a countable existentially closed group M such that $Sk\ M = \mathcal{H}$.
2. Show that M is locally finitely presented.

Proof:

1. Let $\mathcal{H}$ be the class of finitely generated subgroups of finitely presented groups.
   (a) Then $\mathcal{H}$ satisfies SC.
   (b) Let $A, B \in \mathcal{H}$ be arbitrary. Let G, H be finitely presented groups containing A, B respectively. Then the direct product $X = A \times B$ is a subgroup of $G \times H$, which is finitely presented. Therefore $X \in \mathcal{H}$. There are natural embeddings $\theta : A \hookrightarrow X$ and $\phi : B \hookrightarrow X$. Therefore $\mathcal{H}$ satisfies JEP.
   (c) Let A be any finitely generated subgroup of a finitely presented group G. Let $\mathcal{S}$ be any finite set of equations defined over A which are soluble in $L \geq A$. Let $\{\overline{x_1}, \ldots, \overline{x_n}\} \subseteq L$ be a solution to $\mathcal{S}$ lying in L. Let:
   $$H = \langle A, h_1, \ldots, h_n \mid w(h_1, \ldots, h_n) = 1,\ \forall w(x_1, \ldots, x_n) \in \mathcal{S} \rangle$$

It is clear from construction that H contains a solution to $\mathcal{S}$. We are done if we can show that $H \in \mathcal{H}$. Since $\mathcal{S}$ is soluble in L, there is a homomorphism:

$$\begin{array}{rcccll} \theta & : & H & \to & L & \\ & & a & \mapsto & a, & \forall a \in A \\ & & h_i & \mapsto & \overline{x_i}, & 1 \leq i \leq n \end{array}$$

Since $A \leq L$, the map $\theta|_A$ is an embedding. Therefore we may regard A as a subgroup of H.

Form $F = H *_A G$, the free product of H and G amalgamating A. Then F is finitely generated.

Claim: F is finitely presented.

Proof of Claim: Since F is finitely generated, it suffices to show that the relations defining F are finite. Define:

i. $\Gamma = \{$defining relations for $G\}$ (finite by assumption)
ii. $\Omega = \{$defining relations for A as a subgroup of $H\}$ (finite since A is a subgroup of the finitely presented group H)
iii. $\Delta = \{$relations identifying generating elements of A in G with the corresponding elements of A in H when we amalgamate$\}$ (finite since A is finitely generated)

Notice that Ω is then a consequence of $\Gamma \cup \mathcal{S}$. So for defining relations of F we may take $\Gamma \cup \mathcal{S} \cup \Delta$. This is a finite set, and the claim is proved. $\dashv$ (Claim)

Therefore $F \geq H$ is finitely presented. Therefore H is also finitely presented. In other words, $H \in \mathcal{H}$. Therefore $\mathcal{H}$ satisfies AC.

(d) Up to isomorphism, there are at most countably many finitely presented groups. Each finitely presented group has at most countably many finitely generated subgroups. Therefore $\mathcal{H}$ contains at most countably many isomorphism types.

Therefore by Theorem (5.3.11), there is a countable existentially closed group M such that $Sk\ M = \mathcal{H}$.

2. Let A be any finitely generated subgroup of M. We have to show that A is finitely presented.

Since A is finitely generated, $A \in Sk\ M$. Therefore $A \in \mathcal{H}$. By the definition of $\mathcal{H}$, there exists a finitely presented group $G \geq A$. Since G is a finitely generated subgroup of itself, we have that $G \in \mathcal{H}$. Then G is isomorphic to some subgroup G_1 of M. Then A is isomorphic to some subgroup A_1 of G_1.

Since A and A_1 are isomorphic subgroups of M, we can find an HNN extension of M in which $y^{-1}A_1y = A$, for some y. Since A is finitely generated, we can view the element y as a solution to a finite set of equations defined over M. Since M is existentially closed, this implies that we have an $x \in M$ such that $x^{-1}A_1x = A$.

Then since $A_1 \leq G_1$, we have that:

$$\begin{array}{rcl} A & = & x^{-1}A_1x \\ \Longrightarrow A & \subseteq & \underbrace{x^{-1}G_1x}_{finitely\ presented} \end{array}$$

Therefore A is finitely presented as required. A was chosen arbitrarily, therefore all finitely generated subgroups of M are finitely presented. In other words, M is locally finitely presented.

Since M is countable, it is equal to the union of its finitely generated subgroups. By the above, M is equal to the union of its finitely presented subgroups. Thus the finitely presented subgroups of M form a local system. Therefore M is a countable locally finitely presented existentially closed group. ■

5.4 An Existentially Closed Group Cannot Answer our Question Positively

In this section we demonstrate why existentially closed groups cannot answer our original question positively.

By Theorem (5.3.2 2), an existentially closed group G contains a copy of every finite group. In particular, G contains copies of $C_2, C_3, C_4, \ldots$. From this it is clear that G contains elements of arbitrarily high order. Then for all $k = 1, 2, 3, \ldots$, there is no uniform bound on the size of k-generated subgroups of G.

By Theorem (2.4.1), any group with finitely many $(k+1)$-conjugacy classes has a uniform bound on the size of its k-generated subgroups. Therefore there is no way that an existentially closed group G can answer our question positively for any k.

The property of being ω-homogeneous is weaker than the property of being existentially closed. Therefore it might be possible to construct a positive answer by starting with an ω-homogeneous group which is not existentially closed. It remains unclear at this point how to continue the construction of such a positive answer.

Chapter 6

Engel Groups

6.1 Introduction

Engel groups are connected with this problem because of some partial results we have already obtained using the extended commutator notation defined in section 2.1.7.

6.2 Commutator Identities

Commutator Identities: 6.2.1 *Let G be a group. Let $x, y, z \in G$. Then:*

1. $[x, y] = [y, x]^{-1}$
2. $[xy, z] = [x, z]^y [y, z]$ *and* $[x, yz] = [x, z][x, y]^z$
3. $[x, y^{-1}] = ([x, y]^{y^{-1}})^{-1}$ *and* $[x^{-1}, y] = ([x, y]^{x^{-1}})^{-1}$
4. $[x, y^{-1}, z]^y [y, z^{-1}, x]^z [z, x^{-1}, y]^x = 1$ *(Hall-Witt identity)*

where x^y denotes x conjugated by y: $y^{-1}xy$.

Outline of Proofs: Apply the commutator definitions above to get the desired results.

Proof:

1.

$$\begin{aligned} [y, x]^{-1} &= (y^{-1}x^{-1}yx)^{-1} \\ &= x^{-1}y^{-1}xy \\ &= [x, y] \end{aligned}$$

2. (a)

$$\begin{aligned}
[xy, z] &= (xy)^{-1}(z)^{-1}(xy)(z) \\
&= y^{-1}x^{-1}z^{-1}xyz \\
&= y^{-1}x^{-1}z^{-1}x(zz^{-1})yz \\
&= y^{-1}x^{-1}z^{-1}xz(yy^{-1})z^{-1}yz \\
&= y^{-1}(x^{-1}z^{-1}xz)y(y^{-1}z^{-1}yz) \\
&= [x, z]^y[y, z]
\end{aligned}$$

(b)

$$\begin{aligned}
[x, yz] &= (x)^{-1}(yz)^{-1}(x)(yz) \\
&= x^{-1}z^{-1}y^{-1}xyz \\
&= x^{-1}z^{-1}(xz)(z^{-1}x^{-1})y^{-1}xyz \\
&= (x^{-1}z^{-1}xz)z^{-1}(x^{-1}y^{-1}xy)z \\
&= [x, z][x, y]^z
\end{aligned}$$

3. (a)

$$\begin{aligned}
([x, y]^{y^{-1}})^{-1} &= ((x^{-1}y^{-1}xy)^{y^{-1}})^{-1} \\
&= (y(x^{-1}y^{-1}xy)y^{-1})^{-1} \\
&= (yx^{-1}y^{-1}x)^{-1} \\
&= x^{-1}yxy^{-1} \\
&= [x, y^{-1}]
\end{aligned}$$

(b)

$$\begin{aligned}
([x, y]^{x^{-1}})^{-1} &= ((x^{-1}y^{-1}xy)^{x^{-1}})^{-1} \\
&= (x(x^{-1}y^{-1}xy)x^{-1})^{-1} \\
&= (y^{-1}xyx^{-1})^{-1} \\
&= xy^{-1}x^{-1}y \\
&= [x^{-1}, y]
\end{aligned}$$

4. Let:

   (a) $u = xzx^{-1}yx$.
   (b) $v = yxy^{-1}zy$.
   (c) $w = zyz^{-1}xz$.

   Then:

   (a) $[x, y^{-1}, z]^y = u^{-1}v$.

(b) $[y, z^{-1}, x]^z = v^{-1}w$.

(c) $[z, x^{-1}, y]^x = w^{-1}u$.

The identity is then obvious. ■

Commutator Automorphism Identity 1: 6.2.2 *Let G be a group. Let $a, b \in G$. Let $\alpha \in Aut(G)$. Then:*

$$\alpha([a, b]) = [\alpha(a), \alpha(b)]$$

Proof:

$$\begin{aligned} \alpha([a, b]) &= \alpha(a^{-1}b^{-1}ab) \\ &= \alpha(a^{-1})\alpha(b^{-1})\alpha(a)\alpha(b) \\ &= \alpha(a)^{-1}\alpha(b)^{-1}\alpha(a)\alpha(b) \\ &= [\alpha(a), \alpha(b)] \quad ■ \end{aligned}$$

Remark: Notice that this result tells us that in any group G the derived subgroup is always *fully invariant*, i.e. fixed by any automorphism of G.

Commutator Automorphism Identity 2: 6.2.3 *Let G be a group. Let $a, b \in G$. Let $\alpha \in Aut(G)$. Then, for any n:*

$$\alpha([a, \ _n b]) = [\alpha(a), \ _n \alpha(b)]$$

Proof: This is immediate from Commutator Automorphism Identity 1 (6.2.2), and the definition of the extended commutator. ■

6.3 Engel Elements

Definition (Engel Element): Let G denote a group, and $g \in G$ denote any element. Then g is a *right Engel element* $\Longleftrightarrow$ for each $x \in G$, there is a positive integer $n = n(g, x)$ such that $[g, \ _n x] = 1_G$. The element g is called right Engel element and the x appears on the right.

If n can be chosen independently of x, the g is a right n-Engel element of G (or less precisely a bounded right Engel element). The sets of right Engel and bounded right Engel elements of G are denoted respectively:

$$R(G) \quad and \quad \overline{R}(G)$$

Left Engel elements are defined similarly. The sets of left Engel and bounded left Engel elements of G are denoted respectively:

$$L(G) \quad and \quad \overline{L}(G)$$

Claim: The above 4 subsets are invariant under all automorphisms of G.

Outline of Proof of Claim: We show the details for the case of a right Engel element only, as the other cases are analogous. Show that for an arbitrary right Engel element and an arbitrary automorphism, the image of the element under the automorphism is again a right Engel element.

Proof of Claim: We show the details for the case of a right Engel element only, as the other cases are analogous.

Let $r \in R(G)$ be arbitrary. r is a right Engel element. Let $\alpha \in Aut(G)$ be arbitrary. We need to show that $\alpha(r) \in R(G)$, i.e. that $\alpha(r)$ is a right Engel element.

Let $x \in G$ be arbitrary. Consider $[r,\ {}_n\alpha^{-1}(x)]$. Notice that:

$$(\alpha \in Aut(G)) \Longrightarrow (\alpha^{-1} \in Aut(G))$$

Therefore:

$$(x \in G) \Longrightarrow (\alpha^{-1}(x) \in G)$$

Since r is a right Engel element, we know that there is some n such that:

$$[r,\ {}_n\alpha^{-1}(x)] = 1$$

Apply α to this equation to obtain:

$$\begin{aligned} \alpha([r,\ {}_n\alpha^{-1}(x)]) &= \alpha(1) \\ [\alpha(r),\ {}_n\alpha(\alpha^{-1}(x))] &= 1, \textit{ by Commutator Automorphism Identity } 2 \ (6.2.3) \\ [\alpha(r),\ {}_nx] &= 1 \end{aligned}$$

showing that $\alpha(r)$ is also a right Engel element, as required. ■

Remark: It is not yet known whether these 4 subsets are always subgroups of G.

The following is a straightforward consequence of Commutator Identity (6.2.1 2):

Lemma - Commutator Identity: 6.3.1 *Let G be any group. Let $a, b \in G$ be any elements. Then $[ab, a] = [b, a]$.*

Theorem (Heineken): 6.3.2 *In any group G the inverse of a right Engel element is a left Engel element and the inverse of a right n-Engel element is a left $(n+1)$-Engel element. Thus:*

$$R(G)^{-1} \subseteq L(G) \quad \textit{and} \quad \overline{R}(G)^{-1} \subseteq \overline{L}(G)$$

Outline of Proof:

1. Assume g^{-1} is a right n-Engel element.
2. Show that this implies g is a left $(n+1)$-Engel element.
3. Then both parts of the conclusion follow.

Proof: Let $x, g \in G$. Assume g^{-1} is a right n-Engel element. In other words, $[g^{-1},_n x] = 1, \forall x \in G$. Then using the commutator identities we obtain:

$$\begin{aligned}
[x,_{n+1} g] &= [[x,g],_n g] \\
[x,_{n+1} g] &= [[g^{-1},x]^g,_n g]
\end{aligned}$$

Since:

$$\begin{aligned}
[x,g] &= x^{-1}g^{-1}xg \\
&= (g^{-1}g)x^{-1}g^{-1}xg \\
&= g^{-1}(gx^{-1}g^{-1}x)g \\
&= g^{-1}[g^{-1},x]g \\
&= [g^{-1},x]^g
\end{aligned}$$

Then:

$$\begin{aligned}
[x,_{n+1} g] &= [[g^{-1},x]^g,_n g] \\
[x,_{n+1} g] &= [[g^{-1},x],_n g]^g \; \textit{since conjugation by } g \textit{ does not affect } {}_ng \\
[x,_{n+1} g] &= [gg^{-x},_n g]^g
\end{aligned}$$

Since:

$$\begin{aligned}
gg^{-x} &= g(g^{-1})^x \\
&= gx^{-1}g^{-1}x \\
&= [g^{-1},x]
\end{aligned}$$

Then:

$$\begin{aligned}
[x,_{n+1} g] &= [gg^{-x},_n g]^g \\
[x,_{n+1} g] &= [g^{-x},_n g]^g
\end{aligned}$$

Since (by 6.3.1):

$$\begin{aligned}
[gg^{-x},_n g] &= [\cdots[[gg^{-x},\underbrace{g],g]\cdots g]}_{n} \\
&= [\cdots[[[g^{-x},\underbrace{g],g],g]\cdots g]}_{n} \; \textit{by } (6.3.1) \\
&= [g^{-x},_n g]
\end{aligned}$$

So we finally obtain that:

$$[x,_{n+1} g] \quad = \quad [g^{-x},_n g]^g$$

Now, g^{-1} is right n-Engel, by hypothesis. So in particular,

$$\begin{array}{rrcl} & [g^{-1},_n g^{x^{-1}}] & = & 1 \\ \underbrace{\Longleftrightarrow}_{conjugate\ by\ x} & [g^{-x},_n g] & = & 1 \\ \underbrace{\Longleftrightarrow}_{conjugate\ by\ g} & [g^{-x},_n g]^g & = & 1 \\ \underbrace{\Longleftrightarrow}_{argument\ above} & [x,_{n+1} g] & = & 1 \end{array}$$

Then $[g^{-x},_n g]^g = 1 \Longrightarrow [x,_{n+1} g] = 1$. The result now follows. ■

Remark: It is still an open question whether every right Engel element is a left Engel element.

6.4 Engel Groups

Background: Engel groups are useful because they are a generalization of nilpotent groups which are not locally nilpotent.

The origins of Engel groups lie outside of group theory, in the theory of Lie rings.

Definition: (Engel Group) Let G denote a group. If $G = L(G) = R(G)$, then G is an *Engel group*.

Remark: Every locally nilpotent group is an Engel group. The converse is false, as shown by an example of Golod [2].

Definition (n-Engel Group): Let G denote a group, and n denote a positive integer. Then G is an *n-Engel group* $\Longleftrightarrow [a,\ _n b] = 1, \forall a, b \in G$. In other words, every element is both a left and right n-Engel element. Observe that a nilpotent group of class n is an n-Engel group. Also observe that n-Engel groups need not be nilpotent.

Definition (Bounded Engel Group): A group is a *bounded Engel group* if it is n-Engel for some n.

Other Nice Facts:

1. 0-Engel groups have order 1.
2. 1-Engel groups are precisely the abelian groups.
3. 2-Engel groups are structurally more complex. In particular, every group of exponent 3 is a 2-Engel group. For a proof refer to [17], Theorem 12.3.5.

Before proving that a finite Engel group is nilpotent, we need to recall one useful result.

Theorem: 6.4.1 *If all the proper subgroups of G are nilpotent, then G is solvable.*

Proof: Refer to Theorem 6.5.7 (iv) on p. 148 of [20]. ■

Theorem: 6.4.2 *A finite Engel group is nilpotent.*

Proof: Let G be a finite Engel group. Notice that every subgroup and every quotient group of G is therefore an Engel group. Let $|G| = n$. The proof is by induction on n.

Base $(n = 1)$**:** The trivial group is clearly nilpotent.

Induction $(n > 1)$**:** By Theorem (6.4.1), we have that G is solvable. In particular, G has some non-trivial abelian quotient. Since in any group G/G' is the largest abelian quotient, we then have that G/G' is a non-trivial finite abelian group. Therefore we can find some normal subgroup $H^* \lhd G/G'$, where $\frac{G/G'}{H^*}$ is cyclic of order p, for some prime p.

By the correspondence theorem, we may pull H^* back to $H \lhd G$. Then we have that $G' \leq H$ and G/H is cyclic of order p. Moreover:

$$\begin{array}{rcl} |G| & = & \underbrace{|G/H|}_{=p} \cdot |H| \\ \Longrightarrow \quad p & | & |G| \end{array}$$

If G is a p-group, then we are done since finite p-groups are nilpotent. So for the rest of the proof assume that G is not a p-group.

Since G is not a p-group, there exists another prime q such that $q|\ |G|$. Then q must divide $|H|$ since $|G| = \underbrace{|G/H|}_{=p} \cdot |H|$, and p and q are distinct primes.

H is a proper subgroup of G, so the induction hypothesis applies to H. Since H is nilpotent, H is a direct product of its Sylow subgroups. In particular, all the Sylow subgroups of H are normal, therefore unique in H. Let $Q \leq H$ be the unique Sylow q-subgroup of H. Notice that Q is then a characteristic subgroup of H.

$Z(Q)$, the centre of Q, is a characteristic subgroup of Q. Since Q is a characteristic subgroup of H, $Z(Q)$ is also a characteristic subgroup of H. Also, since H is a direct product of its Sylow subgroups, $Z(Q) \leq Z(H)$.

We are finished if we can show that $Z(G) \neq \{1\}$. If G has a non-trivial centre, then $|G/Z(G)| \leq |G|$, so by the induction hypothesis, $G/Z(G)$ is nilpotent. Then we can construct the normal series for $G/Z(G)$, pull back to G via the correspondence theorem, then add the group $Z(G)$ at the start of the series to complete the series showing that G is nilpotent.

Notice that since Q is a non-trivial q-group, it has a non-trivial centre: $Z(Q) \neq 1$. So there exists a non-trivial $h \in Z(Q)\setminus\{1\}$. Since $Z(Q) \leq Z(H)$, this $h \in Z(H)$, i.e. h commutes with everything in H.

If h satisfies $[h, g] = 1$, then h also commutes with g, therefore with $\overline{g}$. Then since h commutes with everything in H, and commutes with the generator $\overline{g}$ of G/H, we have that h commutes with everything in G, i.e. $h \in Z(G)$. If this happens then we are done.

For a contradiction, suppose that $\forall h \in Z(Q) \setminus \{1\},\ [h, g] \neq 1$. Now:

$$[h, g] = \underbrace{h^{-1}}_{\in Z(Q)} \underbrace{g^{-1}hg}_{\in Z(Q)}$$

with the second containment holding since conjugation by g induces an automorphism of H and $Z(Q)$ is characteristic in H. Then $[h, g] \in Z(Q) \setminus \{1\}$. Then by assumption:

$$\begin{aligned}
[\underbrace{[h, g]}_{\in Z(Q)\setminus\{1\}}, g] &\neq 1 \\
[h,\ {}_2g] &\neq 1 \\
&\ \vdots \\
[h,\ {}_mg] &\neq 1,\ \forall m
\end{aligned}$$

This contradicts the hypothesis that G is an Engel group, and we are done. ■

6.5 How Engel Identities Relate to Our Original Question

A group which answers our original question positively for $k = 2$ must satisfy the condition on extended commutators from Theorem (2.4.3). So there is an identity similar to that satisfied by Engel elements, which must hold in any group which answers our question positively.

It remains unclear how to use this fact to construct a group which has all the needed properties to answer our original question positively. In particular, the connection between Engel groups and the other properties which a solution must have is difficult to see.

Further investigation can be done into locally finite groups satisfying an Engel condition.

Chapter 7

Conclusion

7.1 Summary of Implications between Classes of Groups

Here we take the opportunity to record the implications that link all the important properties of groups we have investigated.

$$\begin{array}{c} \textit{algebraically closed} \\ \Uparrow \\ \textit{existentially closed} \\ \Downarrow \\ \omega - \textit{homogeneous} \end{array}$$

We also have:

Theorem: 7.1.1 *Restricting to the class of locally finite existentially closed groups,*

$$universal\ locally\ finite \Longleftrightarrow existentially\ closed\ and\ locally\ finite$$

Outline of Proof ($\Longrightarrow$)**:** Obtain a finitely generated locally finite extension of our base group in which a solution exists. Then, since this extension is a finite group, it can therefore be embedded into the universal locally finite group.

Outline of Proof ($\Longleftarrow$)**:** By Theorem (5.3.2), an existentially closed group contains every finite group. Form the HNN extension which makes a pair of finite isomorphic subgroups conjugate. Then since our group is existentially closed, an element which conjugates one subgroup onto the other must lie in the group itself.

Proof ($\Longrightarrow$): Let U denote a universal locally finite group. Let $\mathcal{S}$ denote a finite set of equations and inequalities defined over U. Let G be the subgroup of U generated by all the coefficients that appear in $\mathcal{S}$. Then since $\mathcal{S}$ is finite, G is finitely generated, therefore finite. We may regard $\mathcal{S}$ as being defined over G.

Suppose there is a locally finite group $H \geq U$ where a solution to $\mathcal{S}$ exists. Then $H \geq G$ also. Since $\mathcal{S}$ is finite, we may list the variables appearing in $\mathcal{S}$:

$$\{x_1, \ldots, x_m\}$$

Then there exist $h_1, \ldots, h_m \in H$ such that if we put $x_1 = h_1, \ldots, x_m = h_m$, then every equation / inequality in $\mathcal{S}$ is satisfied in H. Take:

$$G^* = \langle \underbrace{G}_{finite}, \underbrace{h_1, \ldots, h_m}_{finitely\ many} \rangle \leq H$$

Then it is clear that $\mathcal{S}$ is soluble in G^*. Also, G^* is a finitely generated subgroup of the locally finite group H. Therefore G^* is finite.

Since $G \leq U$, let $\theta : G \hookrightarrow U$ be an embedding. Since $G \leq G^*$, by Theorem (3.3.1 2) we have that θ extends to an embedding $\theta^* : G^* \hookrightarrow U$. Then $\mathcal{S}$ is soluble in U. This shows that U is existentially closed, as required.

Proof ($\Longleftarrow$): We have that a locally finite group U is existentially closed. We want to show that U is a universal locally finite group.

By (5.3.2), an existentially closed group contains every finite group. Let A, B be finite subgroups of U, with $\theta : A \to B$ an isomorphism. Since A, B are finite, we may write:

$$\begin{aligned} A &= \{a_1, \ldots, a_n\} \\ B &= \{b_1, \ldots, b_n\} \\ &= \{a_1\theta, \ldots, a_n\theta\} \end{aligned}$$

We seek a $t \in U$ such that:

$$\begin{aligned} b_1 &= t^{-1}a_1t \\ b_2 &= t^{-1}a_2t \\ &\vdots \\ b_n &= t^{-1}a_nt \end{aligned}$$

Form the HNN extension:

$$U^* = \langle U, t \mid \theta(A) = t^{-1}At \rangle$$

Then U^* is an extension of U containing a solution to our system of equations. Since U is existentially closed, we therefore have a solution $t \in U$. Thus U is universal as required. ■

An Example to Show the Restriction to Locally Finite Existentially Closed Groups is Required in Theorem (7.1.1): Let $\mathcal{S} = \{x^{-1}b^2x = b^3, [x^{-1}bx, b] \neq 1\}$. We shall demonstrate that $\mathcal{S}$ has a solution in a non-locally finite group, but in no locally finite group.

Proof that a Solution Exists in a non-Locally Finite Group: Let $G = \langle b \rangle \cong C_\infty$. Then since $\langle b \rangle$ is infinite cyclic, so are $\langle b^2 \rangle$ and $\langle b^3 \rangle$. Moreover, the following map is an isomorphism:

$$\begin{array}{rccc} \theta & : \langle b^2 \rangle & \to & \langle b^3 \rangle \\ & b^2 & \mapsto & b^3 \end{array}$$

So we may form the HNN extension:

$$H = \langle G, t \mid t^{-1}b^2t = b^3 \rangle$$

Then, in H we have $t^{-1}b^2t = b^3$. Therefore taking $x = t$, we have a solution to the equality of $\mathcal{S}$.

Notice that:

$$\begin{aligned} [t^{-1}bt, b] &= (t^{-1}bt)^{-1}(b)^{-1}(t^{-1}bt)(b) \\ &= (t^{-1}b^{-1}t)(b^{-1})(t^{-1}bt)(b) \\ &= t^{-1}b^{-1}tb^{-1}t^{-1}btb \\ &= \underbrace{1}_{g_0}\underbrace{t^{-1}}_{t^{-1}}\underbrace{b^{-1}}_{g_1}\underbrace{t}_{t}\underbrace{b^{-1}}_{g_2}\underbrace{t^{-1}}_{t^{-1}}\underbrace{b}_{g_3}\underbrace{t}_{t}\underbrace{b}_{g_4} \end{aligned}$$

The only g_i that lies in either of $\langle b^2 \rangle$ or $\langle b^3 \rangle$ is $g_0 = 1$. Therefore this sequence is reduced. Therefore by Britton's Lemma (5.2.3), we have that $[t^{-1}bt, b] \neq 1$ in H, as required. So $x = t$ is a solution of $\mathcal{S}$ lying in H. H is clearly not locally finite, so this part of the example is complete.

Proof that No Solution Exists in a Locally Finite Group: Suppose for a contradiction that a solution $x = a$ of $\mathcal{S}$ exists in some locally finite group G. Then since $b \in G$, $|b| = m < \infty$. Then:

$$|b^2| = \frac{m}{GCD(2, m)}$$

$$
\begin{array}{rcl}
|b^3| & = & \dfrac{m}{GCD(3,m)} \\
|a^{-1}b^2a| & = & |b^2| \\
So\ since\ a^{-1}b^2a & = & b^3 \\
|a^{-1}b^2a| & = & |b^3| \\
\dfrac{m}{GCD(2,m)} & = & \dfrac{m}{GCD(3,m)} \\
\underbrace{GCD(3,m)}_{1\ or\ 3} & = & \underbrace{GCD(2,m)}_{1\ or\ 2} \\
GCD(3,m) & = & GCD(2,m) = 1 \\
|b^2| & = & |b^3| = m
\end{array}
$$

So we have that $|b|$ is not a multiple of 2 or 3. From the above facts, we can show that $\langle b^2 \rangle = \langle b \rangle$. $\langle b^2 \rangle \subseteq \langle b \rangle$ is clear. Since we showed above that $|b^2| = m = |b|$, the subgroup $\langle b^2 \rangle$ cannot be properly contained, and we must have that $\langle b^2 \rangle = \langle b \rangle$. The same argument gives us that $\langle b^3 \rangle = \langle b \rangle$ also.

Since $\langle b^2 \rangle = \langle b \rangle$, we have that $b \in \langle b^2 \rangle$. Write $b = (b^2)^l$ for some l. Then we have:

$$
\begin{array}{rcl}
a^{-1}ba & = & a^{-1}(b^{2l})a \\
 & = & (a^{-1}b^2a)^l \\
 & = & b^{3l} \\
\Longrightarrow \langle a^{-1}ba \rangle & = & \langle b^{3l} \rangle
\end{array}
$$

Then since $a^{-1}ba = b^k$ for some k, we have that $a^{-1}ba$ commutes with b, in other words $[a^{-1}ba, b] = 1$. We have reached a contradiction, completing the proof.

The second part of this example has shown that no solution to $\mathcal{S}$ can exist in any locally finite group. In particular, any existentially closed group containing $G = \langle b \rangle \cong C_\infty$ is not locally finite. So to sum up, the restriction to the class of locally finite existentially closed groups is critical for the equivalence in Theorem (7.1.1) to hold.

7.2 Conclusion

Although we have not yet obtained a positive answer to our problem, or a proof that no positive answer could exist, we have explored some interesting classes of groups which may ultimately yield an answer in the future.

The investigation will continue, in particular in the following areas:

1. proper subgroups of U, the countable universal locally finite group
2. bounded HNN extensions
3. ω-homogeneous groups
4. locally finite groups satisfying an Engel condition

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