Probabilistic Robust Design for Dynamic Systems Using Metamodelling

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Designers use simulations to observe the behaviour of a system and to make design decisions to improve dynamic performance. However, for complex dynamic systems, these simulations are often time-consuming and, for robust design purposes, numerous simulations are required as a range of design variables is investigated. Furthermore, the optimum set is desired to meet specifications at particular instances in time. In this thesis, the dynamic response of a system is broken into discrete time instances and recorded into a matrix. Each column of this matrix corresponds to a discrete time instance and each row corresponds to the response at a particular design variable set. Singular Value Decomposition (SVD) is then used to separate this matrix into two matrices: one that consists of information in parameter-space and the other containing information in time-space. Metamodels are then used to efficiently and accurately calculate the response at some arbitrary set of design variables at any time. This efficiency is especially useful in Monte Carlo simulation where the responses are required at a very large sample of design variable sets. This work is then extended where the normalized sensitivities along with the first and second moments of the response are required at specific times. Later, the procedure of calculating the metamodel at specific times and how this metamodel is used in parameter design or integrated design for finding the optimum parameters given specifications at specific time steps is shown. In conclusion, this research shows that SVD and metamodelling can be used to apply probabilistic robust design tools where specifications at certain times are required for the optimum performance of a system.

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Table of Contents

List of Figures

List of Tables

Nomenclature

Common English Characters

$f(\mathbf{v},\boldsymbol{\beta})$ = function in terms of design variables and model parameters

$f_{t_n}(\mathbf{v})$= metamodel at the nth time step

$g_{t_n}(\mathbf{v})$ = limit-state function at the nth time step

p = number of design variables

s_{dom} = dominant singular values of **S**

t_n = nth time step

tol_{v_i} = tolerance of v_i

v_i = design variable i

$\bar{v}_i$ = mean of design variable i

v_i^{nom} = nominal value of ith design variable

w_i = weight of response i in Kriging interpolation.

$\left(x_i^j - x_i^k\right)$ = difference between training points of design variable i.

$y_m(\mathbf{v})$ = metamodel found for the mth column of **D**

$\hat{y}_m(\mathbf{v}_0)$= estimate for $\mathbf{v}_0$ at the mth column of **D**

$z(v_m,t_n)$ = response at the mth design variable combination at the nth time step.

Capital English Characters

C_{LQ} = loss of quality cost

C_p = production cost

C_S = scrap cost

C_T = total cost

$\Pr\left(S_{t_n}\right)$ = probability of success or conformance at n^{th} time step

$S_{t_n}^{v_i}$ = normalized sensitivity at n^{th} time step for v_i

Common Bold English Characters

$\mathbf{d}_m$ = m^{th} column of **D**

v – vector of design variables

$\mathbf{v}_0$ **=** vector containing the arbitrary design variable combination

$\mathbf{y}_{1\times m}$ **=** row vector of all metamodels found for the m columns of **D**

$\hat{\mathbf{y}}\left(\mathbf{v}_0\right)$ = row estimate of **D** corresponding to $\mathbf{v}_0$.

$\mathbf{z}(\mathbf{v}_0)$ = dynamic response at $\mathbf{v}_0$

Capital Bold English Characters

$\mathbf{C}_v$ = matrix of covariances

E = vector of experimental response residuals

S = matrix of singular values found from Singular Value Decomposition

$\mathbf{S}_t^{v_i}$ = normalized sensitivities over time for v_i

X = matrix used in the calculation of model parameters

$\mathbf{X}_{tr}$ = matrix of design variables used to generate initial experimental design otherwise known as the 'training points'.

$\mathbf{V}^T$ = matrix containing information from **Z** in time-space found from Singular Value Decomposition

Z = m x n matrix containing all the response time history information obtained from Simulations

Greek Characters

$\lfloor \mu_{z,t} \rfloor_{1\times n}$ = 1 x n matrix containing the mean of the response at each of the n time steps

$\left[\sigma^2_{z,t}\right]_{1\times n}$ = 1 x n matrix containing the variance of the response at each of the n time steps

μ_{z,t_n} = mean of the response at the n^{th} time step

μ_{v_i} = mean of the i^{th} design variable

$\sigma^2_{v_i}$ = variance of the i^{th} design variable

ζ_{L,t_n} = lower limit specification at t_n

ζ_{U,t_n} = upper limit specification at t_n

Bold Greek Characters

$\boldsymbol{\beta}_i$ = model parameters used in Response Surface Models

$(\boldsymbol{\gamma}^*(\mathbf{v}))$ = function in **v** of the correlation between some arbitrary set of design variables and the training points.

$(\boldsymbol{\gamma}^*(\mathbf{v_0}))$ = vector containing the covariance between $\mathbf{v_0}$ and the training designs $v_i^1 \ldots v_j^m$ for a particular design variable.

$\boldsymbol{\theta}_i$ = correlation function parameters used in Kriging model

Γ = correlation matrix, used in Kriging, representing the correlation between all design sites for all design variable found using a specified correlation function.

Other

$\frac{\partial \mathbf{y}}{\partial \mathbf{v}^T}$ = matrix of first derivatives of all metamodels of **D**.

$\frac{\partial f_t(\mathbf{v})}{\partial v_i}$ = first derivative of the metamodels over time with respect to the i^{th} design variable.

Chapter 1

Introduction

1.1. Problem Statement

Designers are often required to analyze a system in order to make design decisions. However, most real-world systems are quite complex and their use in experiments is financially undesirable. Due to these complications, the designer then develops a computer simulation model of the system. Although simulations are quite helpful and are cheaper than physical models of the real system, even these computer models may be too complex for analysis and for the implementation of robust design techniques. In order to reduce these complexities, analysts require less complicated but accurate models of the original system.

Furthermore, in search of the design variables that result in the optimum performance of a dynamic system, the response of the system over time is desired at different combinations of the design variables. Now suppose the designer developed a simulation of a real system, segregates the continuous response of the dynamic system into discrete time steps and records the initial results into a matrix where each column represents a time step and each row represents a particular design variable combination. Even if the analyst finds simple models to fit these initial results at each time, interpolation at each time step to find the response at some arbitrary design variable set can become a tedious process especially if a large number of time steps is involved. In addition, for the purpose of design where an optimum result is desired, finding the response at many design variable combinations in a Monte Carlo simulation becomes tedious. Therefore, some method must be found for fast and accurate interpolation.

Also, the responses at particular times are important when finding the optimum design of a dynamic system such as the 'settling' or 'rise' time. Consider Figure (1-1) where the dynamic response of some system is recorded at various design variable sets. An optimum system is required such that the responses at t_1 and t_2 meet some desired specifications. At t_1, the analyst may be interested in limiting the overshoot and at t_2 ensuring the response 'settles' at some specific target or within a specific range may be important. A method is therefore needed to easily pick various time steps and perform robust design such that specifications at those times are met.

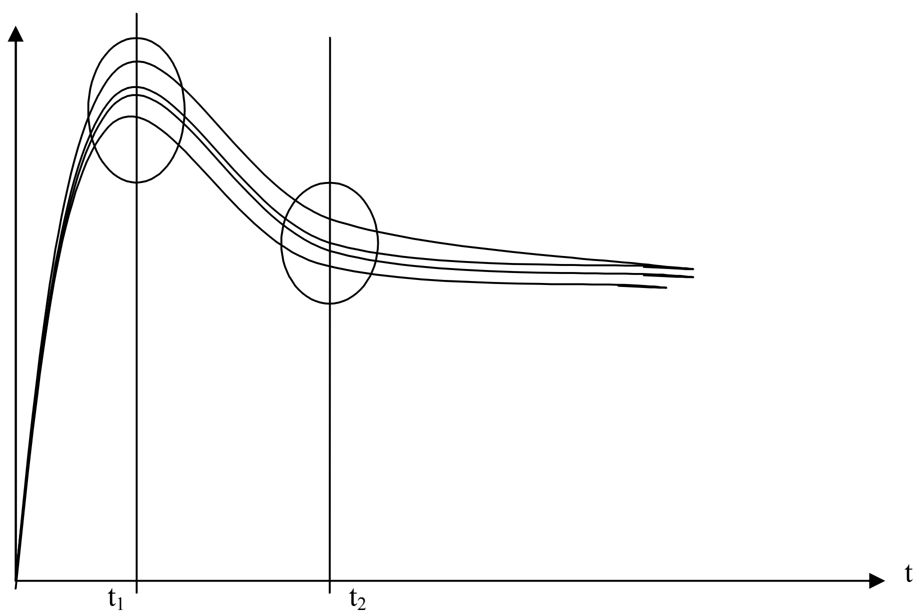

Figure 1-1 showing the dynamic response of some arbitrary systems at various design variable sets.

1.2. Objective

In order to address the problems mentioned, the objectives of this work are

- To develop models in order to simplify complex simulation models
- To find a technique in which interpolation can be quickly performed and can be extended for the purposes of design.
- To apply sensitivity analysis and probabilistic robust design to find optimum parameters of dynamic systems given specifications at certain discrete times.

1.3. Methodology

Before achieving the objectives stated, some research is done to find the mathematical principles and concepts that will be applicable. Among these, the concepts of 'metamodelling' and the characteristic of singular value decomposition (SVD) to separate a matrix of response time histories into parameter and time-space is first presented. Then, this feature of SVD is combined with metamodelling to allow quick and accurate interpolation of the original matrix to find the time history response at an arbitrary design variable set. This efficient method of interpolation ultimately becomes useful for generating the response at a large set of arbitrary design variable combinations as in the case of a Monte Carlo simulation. Furthermore, since parameter information and time information are separated, picking specific time steps for robust design methods to be applied is enabled.

In Chapter 2, a brief description of the concept of 'metamodelling' and the mathematical theory of two types of metamodels that are used during the course of this work is presented. The recent work in using SVD to partition a matrix, consisting of response time histories, into parameter and time space is introduced. Some research is also presented to show how other authors have used metamodelling in their work and the comparison of Kriging with Regression Models. Furthermore, the various tools of robust design, sensitivity analysis techniques and the use of the Taylor series expansion to find first and second moments of the response are presented.

Chapter 3 shows how SVD can be combined with metamodelling to reduce interpolation calculations. Three case studies are used to illustrate the theory. In each of these cases, the experimental results are arranged into a matrix of dynamic responses at various design variable sets. SVD is then applied to this matrix thus separating the matrix into three matrices, two of which are parameter-dependent and the last being time-dependent. The two parameter-dependent matrices are then multiplied and metamodels are developed for the columns of this matrix. Two metamodelling techniques, the Spatial Correlation Model (Kriging) and the Response Surface Model (RSM), are then used and the results obtained from each are recorded and compared. Along with predictions at arbitrary design variable sets, the statistical coefficient of determination is calculated as a measure of comparing the two metamodelling techniques in fitting the experimental data.

In Chapter 4, the procedure of using SVD to calculate the normalized sensitivities and first and second moments of the response at each time is shown. In addition, SVD and metamodelling is combined with robust design tools where specifications at certain times are given. Case studies are used to illustrate these ideas. For each case study, the coefficient of determination is calculated to choose the more suitable metamodel to be used for robust design. A Monte Carlo simulation is then used to find the probability of conformance given specifications at some important time and, if necessary, improvements to this probability will be made through parameter design or integrated design. For cases with multiple random design variables, normalized sensitivities of each design variable are calculated at the time of interest.

Chapter 5 then presents a discussion of the results obtained and the problems that arose. A brief discussion of the suitability of the Kriging Model and the Response Surface Model for use in robust design is also given.

Chapter 2

Literature Review and Theory

Although much research has been done in maximizing the performance or improving the quality of a system through robust design techniques, research in the design of a dynamic system where specifications at certain periods of time have to be met is quite young. This chapter will first present the mathematical theories of Singular Value Decomposition (SVD), metamodelling and robust design techniques for efficient design of a dynamic system. The two metamodels, Kriging and Response Surface Models, will be used for comparison later in this thesis; therefore, their respective mathematical theories of general model form and estimating model parameters will be presented. A brief description on the work of previous authors in the comparison of these two methods and their uses in various fields will be presented. Later, the theory of normalized sensitivity and probability calculations will be presented along with robust design tools of parameter and integrated design.

2.1 Singular Value Decomposition

Consider the m x n matrix, **Z**

$$\mathbf{Z} = \begin{bmatrix} z(v_1,t_1) & z(v_1,t_2) & \dots & z(v_1,t_n) \\ z(v_2,t_1) & z(v_2,t_2) & \dots & z(v_2,t_n) \\ \vdots & \vdots & \ddots & \vdots \\ z(v_m,t_1) & z(v_m,t_2) & \dots & z(v_m,t_n) \end{bmatrix}_{m\times n}$$

where $z(v_m, t_n)$ represents the response at the mth design variable combination and at the nth time step. The rows of **Z** represent the dynamic response at the various design variable combinations and each column corresponds to a particular time step. SVD factorizes **Z** into the product $\mathbf{USV}^T$ where **U** is a column-orthogonal m x m matrix with each column being the left eigenvector of **Z**. $\mathbf{V}^T$ is an orthogonal n x n matrix of the right eigenvectors of **Z** and form an orthonormal basis for the response time histories of the various design variable combinations**.**

$$\mathbf{Z} = \mathbf{U}_{m\times m}\mathbf{S}_{m\times n}\mathbf{V}^{T}_{n\times n} = \mathbf{D}\mathbf{V}^{T} \tag{2.1}$$

where **D** is a matrix obtained from the product of **U** and **S**. **S** is a diagonal matrix containing all singular values of **Z** where

$$s_1 \geq s_2 \geq \ldots \geq s_n \geq 0$$

$$\mathbf{S} = \begin{bmatrix} s_1 & & & \\ & s_2 & & \\ & & \ddots & \\ & & & s_n \end{bmatrix}$$

The magnitude of the singular values also provides a measure of how closely **Z** can be approximated by a matrix of smaller rank (Leon, 1998). The rank of a matrix is defined here as the number of linearly independent rows or columns. An interesting characteristic of SVD is its ability to factorize **Z** into parameter-space and time-space (Wehrwein and Mourelatos, 2006). Although very little research was found on the application of this characteristic in design application, the use of SVD in principal component analysis (PCA) is fairly well known (Berrar D.P., Dubitzky, W. and Granzow, M., 2003 and Leon, S.J., 1998). In order to gain an appreciation of how SVD separates the matrix into time and parameter-space, reference is made to work done in the use of SVD and PCA for gene expression analysis (Berrar, Dubitzky and Granzow, 2003). The theory presented in this work is applied to **Z** consisting of dynamic responses.

One way to calculate the SVD is to first calculate $\mathbf{V}^{\mathrm{T}}$ and **S** by diagonalizing $\mathbf{Z}^{\mathrm{T}}\mathbf{Z}$ (Berrar, D.P., Dubitzky, W. and Granzow, M., 2003)

$$\mathbf{Z}^{T}\mathbf{Z} = \mathbf{V}\mathbf{S}^{2}\mathbf{V}^{T} \tag{2.2}$$

and then to calculate **U** as follows

$$\mathbf{U} = \mathbf{Z}\mathbf{V}\mathbf{S}^{-1} \tag{2.3}$$

The elements of the i^{th} row of $\mathbf{Z}$ form the n-dimensional vector G_i referred to as the time history response of the i^{th} design variable set and the elements of the j^{th} column of $\mathbf{Z}$ form the m-dimensional vector a_j referred to as the response profile of the j^{th} time step. This response profile gives the responses at the various design variable combinations at the particular time step. If $\mathbf{Z}$ is conditioned by centering each column, then

$$\mathbf{Z}^T\mathbf{Z} = \sum_i G_i G_i^T \tag{2.4}$$

is proportional to the covariance matrix of the variables of G_i. A centered vector is one with zero mean value for the elements and the covariance matrix for a set of variables $\{z^k\}$ are given by $c_{ij} = C(z^i,z^j)$. By equation (2.2), diagonalization of $\mathbf{Z}^T\mathbf{Z}$ yields $\mathbf{V}^T$, which also yields the principal components of G_i. So, the right eigenvectors found in $\mathbf{V}^T$ are the same as the principal components of G_i. The eigenvalues of $\mathbf{Z}^T\mathbf{Z}$ are proportional to the variances of the principal components. The matrix **US** then contains the principal component scores, which are the parameter information in the space of principal components (Berrar, D.P., Dubitzky, W. and Granzow, M., 2003).

In other words, the matrix **US** contains all the parameter-dependent information of $\mathbf{Z}$ whilst $\mathbf{V}^T$ contains all its time-dependent information and each row in $\mathbf{D}$ corresponds to the information of $\mathbf{Z}$ in parameter-space with respect to each design variable combination. In order to then find the parameter information of $\mathbf{Z}$ at some other design variable set, $\mathbf{v}_0$, a row in $\mathbf{D}$ is needed that corresponds to $\mathbf{v}_0$. The entire time-dependent response at $\mathbf{v}_0$ is then found by multiplying this new row by $\mathbf{V}^T$. This new row is calculated using metamodels.

$$\mathbf{Z}(\mathbf{v}_0) = \mathbf{U}(\mathbf{v}_0)\mathbf{S}(\mathbf{v}_0)\mathbf{V}^T = \mathbf{D}(\mathbf{v}_0)\mathbf{V}^T \tag{2.5}$$

$\mathbf{D}$ is an m x m matrix and if n>>m, meaning the number of time steps involved is much larger than the number of design variable combinations, then fewer metamodels are derived. Metamodels can be developed for each time step and can be ultimately used in the calculation of the response at some arbitrary design variable combination. However, if a very large number of time steps are involved, developing a metamodel for each column of $\mathbf{Z}$ becomes extremely time-consuming especially if more complex metamodels are used. Hence, the above method can be employed.

2.2 Metamodelling

Computer simulation models are normally used in design to model a real-life system in order to make decisions. These models are used because it is expensive to either construct prototypes of the real system or to even use these real systems in experiments; however, they may also be too complex for use in analysis and design. Therefore, simpler models of these simulation models, also known as metamodels, are still required.

A metamodel is a model of the input/output function or a simple function that approximates the relationship between system performance and the controllable factors (Salvendy, and Kleijnen and Van Beers, 2004), and a simulation input–output model may be represented mathematically as (Barton, 1998)

$$y = f(\mathbf{v})$$

where $\mathbf{v}$ is a vector of design variables. The major issues in metamodelling include:

i) the choice of the function form for the metamodel

ii) the design of experiments

iii) the assessment of the adequacy of the fitted metamodel

Some popular examples of metamodels are splines, radial basis functions, neural networks, Kriging Models and Response Surface Models (Barton, 1998). Since this research will mainly focus on Response Surface and Kriging models, further details on the general functional form and calculation of model parameters will now be presented.

2.2.1 Response Surface Models (RSM)

Response Surface Models, sometimes known as regression models, are one of the most commonly used and simplest techniques used to generate metamodels. The construction of these models through regression techniques is well known from their use in fitting data from physical experiments in statistical applications. In RSM-construction, the response, y(**v**), is modelled as the realization of a stochastic variable (Montgomery, 2005 and Walpole, R.E., Myers, R.H., Myers, S.L. and Ye, K., 2002)

$$y = f(\mathbf{v}, \boldsymbol{\beta}) + \varepsilon \tag{2.6}$$

where f(**v**,**β**) is a function of the design variables and model parameters **β** with

$$\mathbf{v} = \begin{bmatrix} v_1 & v_2 & \dots & v_p \end{bmatrix} \quad \boldsymbol{\beta} = \begin{bmatrix} \beta_1 & \beta_2 & \dots & \beta_p \end{bmatrix}^T$$

and ε is the error term.

When developing a RSM, the form of the relationship between the response and the independent variables is unknown and must be approximated. Usually, a low-order polynomial in some region of the independent variables is employed. If the response can be modelled well by a linear function of the independent variables, then a first-order model is used to approximate the function

$$y = \beta_0 + \beta_1 v_1 + \beta_2 v_2 + \dots + \beta_p v_p + \varepsilon \tag{2.7}$$

However, if there is curvature in the system, then a polynomial model of higher degree may be used

$$y = \beta_0 + \sum_{i=1}^{p} \beta_i v_i + \sum_{i=1}^{p} \beta_{ii} v_i^2 + \sum_{i<j}\sum \beta_{ij} v_i v_j + \varepsilon \tag{2.8}$$

The model parameters, **β**, are calculated using the ordinary least squares equation

$$\boldsymbol{\beta} = \left(\mathbf{X}^T \mathbf{X}\right)^{-1} \mathbf{X}^T \mathbf{Y} \tag{2.9}$$

After calculating these model parameters, an estimate of the output at some arbitrary set of design variables, $\hat{y}(\mathbf{v}_0)$, is found from

$$\hat{y}(\mathbf{v}_0) = f(\mathbf{v}_0, \beta) \tag{2.10}$$

2.2.2 Spatial Correlation Models or Kriging Models

The spatial correlation metamodel, also known as the Kriging model, is another popular type of metamodel. Several recent researchers (Sacks, J., Welch, W.J., Mitchell, T.J. and Wynn, H.P., 1989) have developed a spatial correlation parametric regression modelling approach that shares some common features with spline smoothing and kernel metamodelling (Barton, 1998).

Kriging was named after a South African mining engineer D.G. Krige and began as an interpolation method (Kleijnen and Van Beers, 2004). In order to perform predictions, Kriging uses a weighted linear combination of all the output values already observed. The distance between the location to be predicted and the locations already observed determines the weights used in Kriging. The use of Kriging in deterministic simulation became popular from some previous research (Sacks, J., Welch, W.J., Mitchell, T.J. and Wynn, H.P. 1989). Many authors after have since used Kriging as both a metamodel and an interpolation technique in a wide variety of applications. Authors have also compared Kriging to Response Surface Models in a variety of applications, however, very little research was found on the use of Kriging in probabilistic design of dynamic systems.

2.2.3 Kriging in Interpolation

The Kriging predictor for some unobserved input $\mathbf{v}_0$ given initial data, is a weighted linear combination of the entire m observed responses (Kleijnen and Van Beers, 2004 and Sakata, S., Ashida, F. and Zako, M., 2003)

$$\hat{y}(\mathbf{v}_0) = \sum_{i=1}^{m} w_i Y_i \tag{2.11}$$

where Y_i represents the observed ouput or experimental response and w_i is the weight given to Y_i. This method of estimating $\hat{y}(\mathbf{v}_0)$ is used in spatial estimation and for simple interpolation. However, for design purposes, a metamodel is needed to describe the simulation output.

2.2.4 Kriging as a Metamodel

Some authors have used a Kriging metamodel that fits the response as the realization of a (Gaussian) stochastic process (Simpson and Martin, 2005 and Simpson, T.W., Peplinski, J.D., Koch, P.N. and Allen, J.K., 2001)

$$y(\mathbf{v}) = f(\mathbf{v}, \boldsymbol{\beta}) + \mathrm{E}(\mathbf{v}) \tag{2.12}$$

This model is the combination of a 'global' regression model $f(\mathbf{v},\boldsymbol{\beta})$ and a random process $\mathrm{E}(\mathbf{v})$ that allows for 'local' corrections to the 'global' model. Research in this type of metamodel found that previous authors used various ‘global’ models such as a linear regression model (Martin and Simpson, 2003) or even a constant term (Simpson, T.W., Peplinski, J.D., Koch, P.N. and Allen, J.K., 2001). Others suggested the use of a polynomial regression model (Rijpkema, J.J.M, Etman, L.F.P. and Schoofs, A.J.G., 2001).

The random process, E(**v**), is assumed to have zero mean as well as a spatial covariance for design sites x_i^j and x_i^k of design variable i which is the product of a process variance σ^2 and a correlation function $\gamma\left(x_i^j - x_i^k\right)$ (Rijpkema, J.J.M, Etman, L.F.P. and Schoofs, A.J.G., 2001)

$$\mathrm{cov}\left(z\left(x_i^j\right), z\left(x_i^k\right)\right) = \sigma^2 \gamma\left(x_i^j, x_i^k\right) \tag{2.13}$$

There are different types of correlation functions that can be used and the particular correlation function chosen depends on the preference of the user. The random process of the Kriging model uses a correlation function to “pull” the model through the observed locations in the domain (Martin and Simpson, 2003). This function also affects the smoothness of the model and the impact or weight of nearby points on the prediction. Some correlation functions and their properties are shown in Table (2-1)

Table 2-1 showing various forms of Correlation Functions and their Properties (Martin and Simpson, 2003).

Name	**Γ(d)**	**Parameter Restriction**
Gaussian	$e^{-\theta d^2}$	$\theta > 0$
Exponential	$e^{-\theta \lvert d \rvert^p}$	$\theta > 0$ $1 \leq p < 2$
Cubic Spline	$1-6\left(\frac{d}{\theta}\right)^2+6\left(\frac{\lvert d \rvert}{\theta}\right)^3, \quad \lvert d \rvert < \frac{\theta}{2}$ $2\left(1-\frac{\lvert d \rvert}{\theta}\right)^3, \quad \frac{\theta}{2} \leq \lvert d \rvert < \theta$ $0, \quad \lvert d \rvert \geq \theta$	$\theta > 0$
Matern	$\frac{(\theta \lvert d \rvert)^v}{\Gamma[v]2^{v-1}} K_v[\theta \lvert d \rvert]$	$\theta > 0, v > 0$

where $d = \left(x_i^j - x_i^k\right)$, **Γ**(d) is the function used to calculated the correlation matrices and θ is the model parameter to be estimated. Although a wide variety of correlation functions are available, the most popular function is the Gaussian correlation function. Kriging requires a lot of iterative calculations to estimate model parameters, therefore, a correlation function with the least number of parameters to estimate is most desirable. Therefore, the Gaussian correlation function is chosen

$$\mathbf{\Gamma} = \gamma\left(x_i^j - x_i^k\right) = \prod_{i=1}^{p} \exp\left(-\theta_i \left(x_i^j - x_i^k\right)^2\right) \tag{2.14}$$

Similar to RSM, in order to estimate the parameters used in the Kriging model an experimental design has to be selected containing m design variable combinations for which simulations have to be carried out. These initial design variable combinations are called the training points.

$$y = f(\mathbf{v}, \boldsymbol{\beta}) + \mathbf{E} \tag{2.15}$$

Equation (2.15) above is similar to the general funtional form of the RSM shown in equation (2.6); however, the residuals, **E**, are now correlated according to a correlation function specified by the user

$$\mathrm{cov}(\mathbf{E}) = \sigma^2 * \begin{pmatrix} \gamma(x_i^1, x_i^1) & \dots & \gamma(x_i^1, x_i^m) \\ \vdots & \ddots & \vdots \\ \gamma(x_i^m, x_i^1) & \dots & \gamma(x_i^m, x_i^m) \end{pmatrix} \equiv \sigma^2 \boldsymbol{\Gamma} \tag{2.16}$$

The model parameters $\boldsymbol{\beta}$ are best estimated using the following equation:

$$\boldsymbol{\beta} = \left(\mathbf{X}^T \boldsymbol{\Gamma} \mathbf{X}\right)^{-1} \mathbf{X}^T \boldsymbol{\Gamma}^{-1} \mathbf{Y} \tag{2.17}$$

and an estimate, s^2, for σ^2 can be derived from

$$s^2 = \frac{1}{m} \mathbf{E}^T \boldsymbol{\Gamma}^{-1} \mathbf{E} \tag{2.18}$$

However, these estimates depend on the correlation function parameters θ through **Γ**. Therefore, these parameters are usually estimated first from the experimental data using a Maximum Likelihood approach resulting in the maximization of the log-likelihood function

$$L(\boldsymbol{\theta}) = -\left(m\left(\ln\left(s^2\right)\right) + \ln\left(\det(\boldsymbol{\Gamma})\right)\right) \tag{2.19}$$

The iterative process of maximizing $L(\boldsymbol{\theta})$ can become computationally expensive, since for every evaluation of $L(\boldsymbol{\theta})$ estimates of s^2, b as well as det(**Γ**) must be calculated. Hence, most authors have strived to reduce this computational burden by choosing a correlation function with very few parameters.

The general form of the Kriging model is

$$y = \beta_0 + \sum_{i=1}^{p} \beta_i v_i + \left(\boldsymbol{\gamma}^*(\mathbf{v})\right)^T \boldsymbol{\Gamma}^{-1} \mathbf{E} \tag{2.20}$$

where $\gamma^*(\mathbf{v})$ is a function in **v** that calculates the correlation between $\mathbf{v}_0$ and the training points and $\mathbf{E}$ is a vector of the residuals of the experimental results calculated using equation (2.21)

$$\mathbf{E} = \mathbf{Y} - f(\mathbf{v}, \boldsymbol{\beta}) \tag{2.21}$$

where **Y** is a vector of the experimental responses and the training points are substituted into the linear regression model, f(**v**,**β**) to obtain an estimate of the responses at the training points. Once model parameters and correlation function parameters are estimated, the best linear unbiased prediction of the output at some arbitrary design $\mathbf{v}_0$, can be generated from:

$$\hat{y}(\mathbf{v}_0) = f(\mathbf{v}_0, \boldsymbol{\beta}) + \left(\boldsymbol{\gamma}^*(\mathbf{v}_0)\right)^T \boldsymbol{\Gamma}^{-1}\mathbf{E} \tag{2.22}$$

and $\boldsymbol{\gamma}^*(\mathbf{v}_0)$ is in the form of a Gaussian correlation function

$$\boldsymbol{\gamma}^*(\mathbf{v}_0) = \prod_{i=1}^{p} \exp\left(-\theta_i \left(v_i^0 - x_i^j\right)^2\right) \tag{2.23}$$

where v_i^0 represents some arbitrary value of design variable i. The second expression in equation (2.20) for the predicted response is in fact an interpolation of the residuals of the regression model $f(\mathbf{v}_0, \boldsymbol{\beta})$. Therefore, exact predictions are obtained at the initially observed experimental responses.

The general Kriging model is shown in equation (2.20) and the following summary of steps outlines the procedure used to estimate the parameters, **θ** used in the Kriging model.

1. Make an initial guess for $\boldsymbol{\theta}$
2. Use this initial guess to calculate $\boldsymbol{\Gamma}$ using equation (2.14).
3. Substitute $\boldsymbol{\Gamma}$ in equation (2.17) to calculate $\boldsymbol{\beta}$
4. Calculate the residuals for the training design using equation (2.21).
5. Use $\boldsymbol{\beta}$ to calculate s^2 using equation (2.18) and then use this estimate of s^2 to calculate $L(\boldsymbol{\theta})$ using equation (2.19).
6. Repeat steps 1 – 5 until $L(\boldsymbol{\theta})$ is maximized.

Several authors have done research on comparing the performance of Kriging with RSM in deterministic simulations. Among these are: Rijpkema, J.J.M, Etman, L.F.P. and Schoofs, A.J.G., 2001 who applied RSM and Kriging to a simple two-variable analytical test function and Simpson, T.W., Peplinski, J.D., Koch, P.N. and Allen, J.K., 2001, who investigated the use of Kriging models as alternatives to traditional second-order polynomial response surfaces for constructing global approximations in the design of an aerospike nozzle. Similar research was also done by Jin, R., Du, X. and Chen, W., 2003, Simpson, T.W., Peplinski, J.D., Koch, P.N. and Allen, J.K., 2001, Sakata, S., Ashida, F. and Zako, M. 2003, Martin and Simpson, 2005 and Kleijnen and Van Beers, 2003. All these authors have found that response surface models, although quite simple and even with the availability of second-order polynomial models for non-linear functions, did not perform as accurately as Kriging. However, Kriging models require an iterative procedure to estimate model parameters that can be quite time consuming. Although Kriging models show great promise in its ability to fit 'noisy' data, but, its limited use in engineering applications may be due to the lack of readily available software.

Later, this research will use the statistical coefficient of determination to calculate the adequacy of each model in fitting the experimental data. Since the RSM is simpler than the Kriging model, if the RSM fits the experimental data well, it will be used for robust design; otherwise, the Kriging model is used.

2.3 Model Adequacy

Now that the general functional forms of the Kriging and Response Surface models have been presented along with the method of calculating model parameters, a measure to compare the adequacy of each metamodel in fitting the experimental data is required. To determine if a model is a good fit for the experimental data, a Goodness-of-fit Test is performed in which the coefficient of determination, R^2, is calculated (Walpole, R.E., Myers, R.H., Myers, S.L. and Ye, K., 2002 and Montgomery, 2005). R^2 is a dimensionless quantity used in statistical applications to check how well the metamodel performs in fitting the experimental data. The "R-squared" value is calculated from

$$R^2 = 1 - \frac{SS_{error}}{SS_{total}} \tag{2.24}$$

From equation (2.24)

$$SS_{error} = \sum_{i=1}^{m} \left(z(x_i) - \hat{z}(x_i) \right)^2 \tag{2.25}$$

$$SS_{total} = \sum_{i=1}^{m} \left(z(x_i) - \bar{z} \right)^2 \tag{2.26}$$

where $z(x_i)$ represents the response at training point i, $\hat{\mathbf{z}}(x_i)$ represents the estimate of the response at i^{th} training point obtained from the metamodel and $\bar{z}$ represents the mean of the observed responses. If the calculated R^2 is close to 1, then the model is a good fit.

2.4 Robust Design

"Robust Design is an engineering methodology for optimizing the product and process conditions which are minimally sensitive to the various causes of variation, and which produce high-quality products with low development and manufacturing costs" (Sung, 1996). Two of the most important tools in robust design are Taguchi's parameter design and integrated design. In parameter design, the design variables are chosen to minimize the effect of noise factors that can affect the quality of the product. In integrated design, means and tolerances of the design variables are chosen in order to minimize system failure. Before any of these tools are used, some analysis should be done to determine the effect of the design variables on the response. All the robust design tools and their equations presented here were obtained from Savage, 2007.

2.4.1 Normalized Sensitivities

In the analysis of dynamic systems, and also for the purpose of robust design, the impact of each design variable on the response is desired. This information is found by calculating the first-order sensitivity factors and then the normalized sensitivities. The normalized sensitivities calculate the percentage change of the response for a 1% change in the design variable. For the function

$$y = f(\mathbf{v})$$

the first-order sensitivity factors, (FOS) factors, with respect to v_i are found from

$$FOS_{v_i} = \frac{\partial f(\mathbf{v})}{\partial v_i} \tag{2.27}$$

which is just the first-order derivative of the function with respect to v_i. Using the FOS factor, the sensitivity function, or normalized sensitivity, of a particular design variable is calculated using

$$S_{v_i} = \frac{\partial f(\mathbf{v})}{\partial v_i} \cdot \frac{v_i^{nom}}{f(\mathbf{v})|_{\mathbf{v}^{nom}}} \tag{2.28}$$

where v_i^{nom} is the nominal value of design variable i and $f(\mathbf{v})|_{\mathbf{v}^{nom}}$ is the value of the function evaluated at the nominal values of all the design variables

Calculation of the FOS factors using the Response Surface Model is very easy; however, this is not so obvious with the Kriging model. Consider equation (2.20) written in the form

$$y(\mathbf{v}) = f(\mathbf{v}, \boldsymbol{\beta}) + \left(\boldsymbol{\gamma}^*(\mathbf{v})\right)^T \boldsymbol{\Gamma}^{-1}\mathbf{E}$$

where the first part of the model is the ‘global’ estimation and the second part is an interpolation of the residuals. Taking first-order derivatives of the ‘global’ part of the model is quite simple even if a polynomial approximation is used. The second part of the model consists of the two correlation matrices and a vector of the residuals of the experimental design. Now, $\mathbf{\Gamma}^{-1}$ is a constant matrix and does not change as $\mathbf{v}_0$ changes and neither does $\mathbf{E}$ change since these depend only on the experimental design. Therefore, this correlation matrix and vector can be considered to be constants. Now, $\boldsymbol{\gamma}^*(\mathbf{v})$ is not a constant matrix but is a function in $\mathbf{v}$

$$\boldsymbol{\gamma}^*(\mathbf{v}) = \prod_{i=1}^{p} \exp\left(-\theta_i (v_i - \mathbf{x}_i)^2\right)$$

$$\frac{\partial \boldsymbol{\gamma}^*(v)}{\partial v_i} = -2\theta_i (v_i - \mathbf{x}_i) \prod_{k=1}^{p} \exp\left(-\theta_i (v_i - \mathbf{x}_i)^2\right) \tag{2.29}$$

and the first-order derivative of equation (2.20) becomes

$$\frac{\partial y(\mathbf{v})}{\partial v_i} = \frac{\partial f(\mathbf{v}, \boldsymbol{\beta})}{\partial v_i} + \left(\frac{\partial \boldsymbol{\gamma}^*(\mathbf{v})}{\partial v_i}\right)^T \mathbf{\Gamma}^{-1}\mathbf{E} \tag{2.30}$$

Therefore, for a Kriging model consisting of one design variable

$$y = \beta_0 + \beta_1 v_1 + \left[\exp\left(-\theta_1 (\mathrm{v}_1 - \mathbf{x}_1)^2\right)\right]^T \mathbf{\Gamma}^{-1}\mathbf{E}$$

where $\mathbf{x}_1$ is a vector of the training points of v_1. The first derivative of y with respect to v_1 is therefore

$$\frac{dy}{dv_1} = \beta_1 + \left[-2\theta_1 (\mathrm{v}_1 - \mathbf{x}_1) \exp\left(-\theta_1 (\mathrm{v}_1 - \mathbf{x}_1)^2\right)\right]^T \mathbf{\Gamma}^{-1}\mathbf{E}$$

and the second derivative is

$$\frac{d^2 y}{dv_1^2} = \left[-2\theta_1 \exp\left(-\theta_1 (v_1 - \mathbf{x}_1)^2\right) + 4\theta_1^2 (v_1 - \mathbf{x}_1)^2 \exp\left(-\theta_1 (v_1 - \mathbf{x}_1)\right)\right]^T \mathbf{\Gamma}^{-1}\mathbf{E}$$

This method can be easily extended to multiple design variables and an example of how these equations are used will be shown later.

2.4.2 Probability Calculations

Given the mean and variance of each random design variable and the design specifications, the probability of conformance can be easily calculated using a variety of methods; Monte Carlo Simulation and the Second Moment Method.

Transmission of Moments

In order to eventually determine the probability of conformance of the response at a particular time step, it is necessary to calculate the mean and variance of the response given the mean and variance of the design variables. Given the function

$$y = f(\mathbf{v})$$

and the information

$$\mu_{\mathbf{v}} = \begin{bmatrix} \mu_{v_1} \\ \mu_{v_2} \\ \vdots \\ \mu_{v_p} \end{bmatrix} \qquad \mathbf{C}_v = \begin{bmatrix} \sigma_{v_1}^2 & \operatorname{cov}(v_1, v_2) & \dots & \operatorname{cov}(v_1, v_p) \\ \operatorname{cov}(v_2, v_1) & \sigma_{v_2}^2 & \dots & \operatorname{cov}(v_2, v_p) \\ \vdots & \vdots & \ddots & \vdots \\ \operatorname{cov}(v_p, v_1) & \operatorname{cov}(v_p, v_2) & \dots & \sigma_{v_p}^2 \end{bmatrix}$$

where $\mathbf{C}_{\mathbf{v}}$ is a matrix of covariances and $\mu_{\mathbf{v}}$ is a vector of the means of the design variables. Taylor series expansion is now performed on y = f(**v**) to get

$$y = f(\mathbf{v})\Big|_{\mu_{\mathbf{v}}} + \frac{\partial f(\mathbf{v})}{\partial v_1}\bigg|_{\mu_{\mathbf{v}}} (v_1 - \mu_{v_1}) + \frac{\partial f(\mathbf{v})}{\partial v_2}\bigg|_{\mu_{\mathbf{v}}} (v_2 - \mu_{v_2}) + \dots + \frac{1}{2}\frac{\partial^2 f(\mathbf{v})}{\partial v_1^2}\bigg|_{\mu_{\mathbf{v}}} (v_1 - \mu_{v_1})^2 + \frac{1}{2}\frac{\partial^2 f(\mathbf{v})}{\partial v_2^2}\bigg|_{\mu_{\mathbf{v}}} (v_2 - \mu_{v_2})^2 + \dots + \frac{\partial^2 f(\mathbf{v})}{\partial v_1 \partial v_2 \dots}\bigg|_{\mu_{\mathbf{v}}} \mathbf{C}_{\mathbf{v}} + H.O.T. \tag{2.31}$$

where H.O.T. represents higher order terms and are neglected. Therefore, the mean and variance of the response is calculated as

$$\mu_Z \approx f(\mathbf{v})\Big|_{\mu_{\mathbf{v}}} + \frac{1}{2}\frac{\partial^2 f(\mathbf{v})}{\partial (\mathbf{v})^2}\bigg|_{\mu_{\mathbf{v}}} vec\mathbf{C}_{\mathbf{v}} \tag{2.32}$$

$$\sigma_Z^2 \approx \begin{bmatrix} \frac{\partial f}{\partial v_1} & \frac{\partial f}{\partial v_2} & \cdots & \frac{\partial f}{\partial v_p} \end{bmatrix} \begin{bmatrix} \sigma_{v_1}^2 & \operatorname{cov}(v_1, v_2) & \cdots & \operatorname{cov}(v_1, v_p) \\ \operatorname{cov}(v_2, v_1) & \sigma_{v_2}^2 & \cdots & \operatorname{cov}(v_2, v_p) \\ \vdots & \vdots & \ddots & \vdots \\ \operatorname{cov}(v_p, v_1) & \operatorname{cov}(v_p, v_2) & \cdots & \sigma_{v_p}^2 \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial v_1} \\ \frac{\partial f}{\partial v_2} \\ \vdots \\ \frac{\partial f}{\partial v_p} \end{bmatrix} \quad (2.33)$$

where μ_z denotes the mean of the response and σ_z^2 denotes the variance of the response.

The specified lower and upper limits are denoted as ζ_L and ζ_U, where these limits may be related to the target and some known tolerances. The quality characteristic and the limits are connected through the use of "limit-state functions" denoted as g(**v**). Then, for the ith quality characteristic, and say some upper limit ζ_i, the limit-state function is written as

$$g_i(\mathbf{v}) = \zeta_i - z_i(\mathbf{v}) \quad (2.34)$$

where

$g_i(\mathbf{v}) = 0 \;\; \mathbf{v} \in$ Limit-state surface

$g_i(\mathbf{v}) > 0 \;\; \mathbf{v} \in$ Conformance region (S)

$g_i(\mathbf{v}) < 0 \;\; \mathbf{v} \in$ Non – conformance region (F)

More specifically,

$$\Pr(S) = \Pr\{g_i(\mathbf{v}) > 0\} \quad (2.35)$$

$$\Pr(F) = \Pr\{g_i(\mathbf{v}) \leq 0\} \quad (2.36)$$

where Pr(S) is the probability of success or conformance and Pr(F) is the probability of failure or nonconformance. The probability of conformance can also be calculated using the mean and variance of the response calculated using the transmission of moment theory. Therefore,

$$\Pr(S) \approx 1 - \Phi\left(\frac{-(\zeta_U - \mu_z)}{\sigma_z}\right) - \Phi\left(\frac{-(\mu_z - \zeta_L)}{\sigma_z}\right) \quad (2.37)$$

where Φ denotes the normal cumulative distribution function.

Sometimes it is necessary to convert the limit-state functions from the original **v**-space into the standard normal **u**-space. The Rosenblatt transformation is used for this conversion. Consider the case where p-design variables are normal and the parameters are in matrix form

$$\mu_{\mathbf{v}} = \begin{bmatrix} \mu_{v_1} \\ \mu_{v_2} \\ \vdots \\ \mu_{v_p} \end{bmatrix} \quad \text{and} \quad \mathbf{C}_{\mathbf{v}} = \begin{bmatrix} \sigma_{v_1}^2 & \mathrm{cov}(v_1, v_2) & \dots & \mathrm{cov}(v_1, v_p) \\ \mathrm{cov}(v_2, v_1) & \sigma_{v_2}^2 & \dots & \mathrm{cov}(v_2, v_p) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{cov}(v_p, v_1) & \mathrm{cov}(v_p, v_2) & \dots & \sigma_{v_p}^2 \end{bmatrix}$$

In standard normal space, by definition

$$\mu_{\mathbf{v}} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{C}_{\mathbf{v}} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

The transformation is a generalization of the one-dimensional form $\mathbf{U} = \frac{1}{\sigma}(\mathbf{V} - \mu)$ and the linear form is written as

$$\mathbf{U} = \mathbf{A}(\mathbf{V} - \mathrm{E}(\mathbf{V})) \tag{2.38}$$

where $\mathbf{A}^{-1}$ = Cholesky($\mathbf{C}_{\mathbf{v}}$) and $\mathbf{A}^{-1}$ also gives the reverse transformation

$$\mathbf{V} = \mathbf{A}^{-1}\mathbf{U} + \mathrm{E}(\mathbf{V}) \tag{2.39}$$

and E(**V**) is the expected value operator of the design variables.

Monte Carlo Simulation

Another method of calculating probabilities is to generate a very large sample of data and count the instances when the specifications are met. In a Monte Carlo simulation, a sample of design variables is generated from their probability distributions and the corresponding responses are found by substituting the sample of design variables into the derived metamodel. Given the mean and tolerance of a particular design variable used in the generation of data from the simulation, the variance is found from

$$\sigma_{v_p}^2 = \frac{tol\%}{300}\mu_{v_p} \tag{2.40}$$

where $\sigma_{v_p}^2$ and μ_{v_p} denotes the variance and mean of v_p (Savage, 2007) and tol% is the percentage tolerance.

2.4.3 Parameter Design

Parameter design is used to calculate the mean value of the design variables, given constant tolerances that will result in the response having an acceptable probability of conformance. There are various methods available for parameter design.

One method is by balancing conformance indices. In this method, probability is associated with β

$$\text{min. } Q[\underline{\mu}] = \sum_{i=1}^{k} e^{-\alpha_i \beta_i} \tag{2.41}$$

where

$$\alpha = sign(g(\mathbf{u}) = 0) = \pm 1 \tag{2.42}$$

and g(**u**) is the limit-state function in **u**-space and for bi-linear models

$$\beta_k = \frac{|-h_0|}{\sqrt{h_1^2 + h_2^2 + \ldots}} \tag{2.43}$$

where

$$h_0 = a_0 + \sum_{i=1}^{p} a_i \mu_i - \zeta\left(b_0 - \sum_{i=1}^{p} b_i \mu_i\right) \tag{2.44}$$

$$\begin{aligned} h_1 &= \sigma_1(a_1 - \zeta b_1) \\ h_2 &= \sigma_2(a_2 - \zeta b_2) \\ &\vdots \\ h_p &= \sigma_p(a_p - \zeta b_p) \end{aligned} \tag{2.45}$$

and ζ can be an upper or lower specification or both. For two specifications, two equations are generated for each design variable. A bi-linear model assumes the form

$$y = \frac{a_0 + a_1 v_1 + a_2 v_2 + \ldots a_p v_p}{b_0 + b_1 v_1 + b_2 v_2 + \ldots b_p v_p} \tag{2.46}$$

The above method of finding the optimal design parameters will be applied to response surface models and will be shown later. However, for non-linear models such as the Kriging model, this method is inappropriate.

Another method that can be attempted for Kriging models is using a probability objective function where the probability of conformance is calculated from the second moment method. The objective function becomes either minimizing or maximizing the probability of failure or conformance. Expressions are developed for the mean and variance of the response in terms of the means of the design parameters. Eventually, the objective function becomes an expression in terms of the means of the design variables. Recall equation (2.37)

$$\max \Pr(S) \approx 1 - \Phi\left(\frac{-(\zeta_U - \mu_z)}{\sigma_z}\right) - \Phi\left(\frac{-(\mu_z - \zeta_L)}{\sigma_z}\right)$$

where the mean and variance of the response is a function of the design variables using equations (2.31) and (2.32). For bi-linear models, the probability objective function can also be described in terms of conformance indices

$$\min \ \mathrm{Q} = \sum_i \Phi(-a_i \beta_i) \tag{2.47}$$

where Q above represents the probability of non-conformance or failure.

2.4.4 Integrated Design

In some cases, the optimum values of both the tolerance and the mean of the design variable are required. For these cases, integrated design is performed. Since choosing tolerances affects the production costs as tighter tolerances usually results in higher costs, the objective function in performing this method minimizes the production cost while minimizing the probability of failure. This problem is stated as

$$\min \; C_T(\mu_v, tol_v) \tag{2.48}$$

subject to

Pr(F) ≤ some specified limit

where C_T is a function of the means and tolerances of the design variables.

Robust design enables the cost of quality placed onto the manufacturer to be reduced and provides a way of delivering a product to the customer that meets specifications at the lowest cost to the manufacturer. Overall, robust design provides a way to make the product insensitive to variation in the raw material, manufacturing and in the operating environment. Therefore, the total cost, C_T, is a function that is made up of the production cost that comprises material and component tolerances and the 'loss of quality' costs is made up of the factors stated above. Some examples of equations that are used to calculate the production cost are shown in Table (2-2)

Table 2-2 showing production cost models (Savage, 2007).

Model Name	**Cost Model (C_p)**
Reciprocal	$a + \frac{b}{tol}$
Reciprocal Power	$a + \frac{b}{tol^k}$
Exponential	be^{-mtol}
Piecewise Linear	$a_i + \frac{b_i}{tol_i}$

where *a* and *b* are cost parameters set by a particular manufacturing process.

The 'loss of quality' cost is calculated using the equation

$$C_{LQ} = \frac{C_s}{\Delta^2}\left((\mu_z - T)^2 + \sigma_z^2\right) \tag{2.49}$$

where C_s is the scrap cost and T is the target value. The mean and variance of the response is calculated using equations (2.32) and (2.33). Now, all this theory will be applied to dynamic systems where specifications at different time steps are required for optimum system performance.

Chapter 3

Singular Value Decomposition Combined With Metamodels

The combination of Singular Value Decomposition (SVD) with metamodelling provides a way of reducing the computational burden required to calculate the response at some arbitrary design variable set (Wehrwein and Mourelatos, 2006). All the responses obtained from an initial set of simulation runs are arranged into a matrix, **Z**

$$\mathbf{Z} = \begin{bmatrix} z(v_1, t_1) & z(v_1, t_2) & \dots & z(v_1, t_n) \\ z(v_2, t_1) & z(v_1, t_2) & \dots & z(v_2, t_n) \\ \vdots & \vdots & \ddots & \vdots \\ z(v_m, t_1) & z(v_m, t_2) & \dots & z(v_m, t_n) \end{bmatrix}_{m \times n} \tag{3.1}$$

where $z(v_m, t_n)$ represents the response at design variable combination v_m and time step t_n. SVD is then applied to **Z** according to the relation

$$\mathbf{Z} = \mathbf{U}_{m \times m} \mathbf{S}_{m \times n} \mathbf{V}_{n \times n}^{T} = \mathbf{D}\mathbf{V}^{T} \tag{3.2}$$

SVD partitions **Z** into matrices of parameter and time-dependent information. **D** is a matrix of the information of **Z** in parameter-space and **V** comprises the time-dependent information of **Z.** In order to obtain responses and perform robust design according to the parameters, metamodels are developed for the columns of **D**. A description of how SVD partitions this matrix is given Chapter 2 and for a complete description of this procedure, reference is made to Berrar, D.P., Dubitzky, W. and Granzow, M., 2003. There are other applications of the use of SVD in principal component analysis that is similar to this work (Leon, S.J., 1998, Berrar, D.P., Dubitzky, W. and Granzow, M., 2003).

If the experimental design set is much less than the number of time steps used, (m<<n), then finding metamodels for each of the m columns of **D** is an advantage. However, if m is still very large, this task may also be tedious. Previous research on SVD states that singular values gives an indication of how closely the original matrix can be approximated; therefore, if only the dominant singular values in **S** are kept, equation (3.2) can be partitioned as

$$\mathbf{Z} = \begin{bmatrix} \mathrm{U} & \mathrm{U_n} \end{bmatrix} \begin{bmatrix} \mathrm{s_{dom}} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{V} \\ \mathrm{V_n} \end{bmatrix} \tag{3.3}$$

where

$$\mathbf{U} = \begin{bmatrix} \mathrm{U} & \mathrm{U_n} \end{bmatrix} \quad \mathbf{S} = \begin{bmatrix} s_{\mathrm{dom}} & 0 \\ 0 & 0 \end{bmatrix} \quad \mathbf{V}^T = \begin{bmatrix} \mathrm{V} \\ \mathrm{V_n} \end{bmatrix}$$

and s_{dom} denotes the dominant singular values. In determining whether a singular value is dominant or not, depends on the degree of accuracy desired by the analyst. In this research, a dominant singular value was in the order of 10^{-2} or greater.

For determining whether a particular singular value is dominant or not depends on the analyst and their preference of how accurate an approximation is desired. However, a look at the relation between the coefficient of determination, R^2, and the singular value can show how the magnitude of the singular value can affect the adequacy of the metamodel in fitting the experimental responses. The non-dominant singular values of **S** have been truncated to zero and equation (3.2) becomes

$$\mathbf{Z = USV^T = DV^T} \tag{3.4}$$

The time dependent response of an arbitrary design variable set, $\mathbf{v}_0$, which is different from all m sample points, is calculated using a nonlinear interpolation of each column of matrix **D** in equation (3.2)

$$\hat{z}(\mathbf{v}_0) = \mathbf{U}(\mathbf{v}_0)\mathbf{S}(\mathbf{v}_0)\mathbf{V}^T = \mathbf{D}(\mathbf{v}_0)\mathbf{V}^T \tag{3.5}$$

Therefore, metamodels are used to estimate the row in **D** corresponding to $\mathbf{v}_0$ and the time-dependent information in **V** stays the same. Let each column of **D** be $\mathbf{d}_m$ and the metamodels found for each column of **D** are

$$\mathbf{y}_{1\times m} = \begin{bmatrix} y_1 & y_2 & \dots & y_m \end{bmatrix} \tag{3.6}$$

where y_m represents the metamodel for the m^{th} column of **D.** After substituting $\mathbf{v}_0$ into each of the metamodels, an estimate of the row in **D** corresponding to $\mathbf{v}_0$ is obtained. Then multiplying this new row by $\mathbf{V}^T$, the estimate of the response at $\mathbf{v}_0$ is obtained

$$\hat{z}(\mathbf{v}_0) = \begin{bmatrix} \hat{y}_1(\mathbf{v}_0) & \hat{y}_2(\mathbf{v}_0) & \dots & \hat{y}_m(\mathbf{v}_0) \end{bmatrix} \mathbf{V}^T \tag{3.7}$$

where $\hat{y}_m(\mathbf{v}_0)$ is the estimate of $\mathbf{v}_0$ in the m^{th} column of **D**. Later, it will be shown how this combination of SVD and metamodelling can be useful in robust design applications.

This theory is very useful especially for cases where a very large number of time steps are involved since only the "significant" columns of **D** are used. In order to show how the number of columns of **D** affects the calculated response, the coefficient of determination (R^2) at each time step is calculated using equations

$$R_{t_n}^2 = 1 - \frac{SS_{error}^{t_n}}{SS_{total}^{t_n}} \tag{3.8}$$

$$SS_{error}^{t_n} = \sum_{i=1}^{m} \left(z(v_i, t_n) - \hat{z}(v_i, t_n) \right)^2 \tag{3.9}$$

$$SS_{total}^{t_n} = \sum_{i=1}^{m} \left(z(v_i, t_n) - \bar{z}_{t_n} \right)^2 \tag{3.10}$$

where $\bar{z}_{t_n}$ represents the mean of the responses at the n^{th} time step and $\hat{z}(v_i, t_n)$ is the estimate of the response at v_i and the n^{th} time step. Two case studies will now be presented to show how the combination of SVD and metamodelling is used and how the number of columns of **D** used affects the R^2 values and the calculated response.

3.1. Case Study 1 – Servo with one random design variable

Consider the servo-system (Chandrashekar, M. and Savage, G.J., 1997) shown in Figure (3-1)

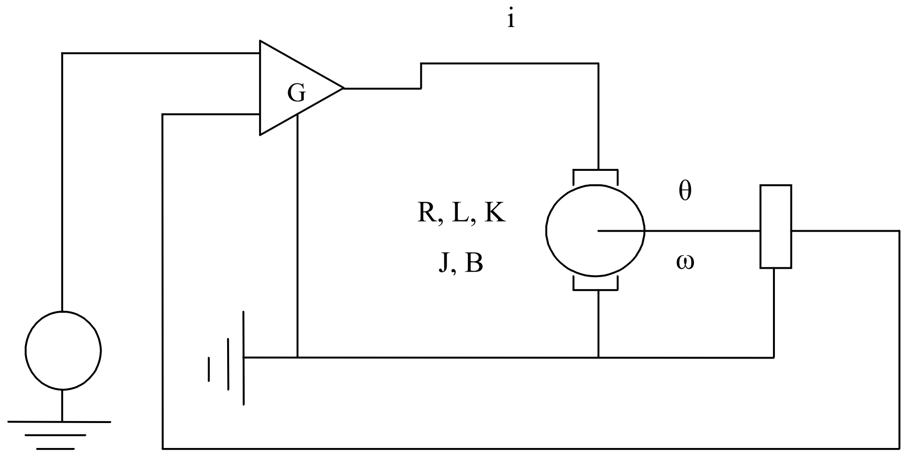

Figure 3-1 showing a schematic of a position control Servo-System.

The motor equations are

$$v = Ri + L\frac{di}{dt} + K\omega \tag{3.11}$$

$$T = Ki - B\omega - J\frac{d\omega}{dt} \tag{3.12}$$

where R is the winding resistance - Ω, L is the winding inductance - H, K is the torque constant – Nm/A, B is friction – Nms/rad, J is shaft inertia – kgm^2 and G is the gain of the amplifier. These variables make up the design variables of the servo. The servo consists of three responses, angular position, angular speed and inductor current, the state equation of each is

$$\begin{bmatrix} \dot{\theta} \\ \dot{\omega} \\ \dot{i} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -B/J & K/J \\ -G/L & -K/L & -R/L \end{bmatrix} \begin{bmatrix} \theta \\ \omega \\ i \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ G/L \end{bmatrix} \theta_{in} \tag{3.13}$$

MAPLE was used to run the simulation and achieve the response time history at the specified design variable combinations.

3.1.1 Angular Speed Response – One Design Variable

Consider the case where winding inductance, L, is a random design variable and is denoted as v_1 with all other design variables held constant; the response of interest is the angular speed of the servo. Response time histories at three levels of v_1 were generated using MAPLE. The simulation was run from t = 0.005s to t = 0.100s and the dynamic response was recorded at intervals of 0.005s resulting in a total of twenty discrete time steps. The matrix **Z** shows an extract of the response obtained, at the low, medium and high values of the design variable

$$\mathbf{v} = \begin{bmatrix} v_1^{low} = 4.00 \\ v_1^{med} = 4.40 \\ v_1^{high} = 4.84 \end{bmatrix}$$

$$\mathbf{Z} = \begin{bmatrix} 35.4135 & 50.2836 & 49.2782 & \ldots & 0.3880 \\ 32.4767 & 46.6146 & 46.6894 & \ldots & 0.4199 \\ 29.7636 & 43.1454 & 44.0685 & \ldots & 0.2897 \end{bmatrix}_{3\times 20}$$

The rows of **Z** correspond to the dynamic angular speed at the three levels of v_1 and each column represents a particular time step. Therefore, the first row of **Z** corresponds to the angular velocity time history at v_1^{low} and the third row is the response time history at v_1^{high}.

Singular Value Decomposition is now performed on **Z** to yield the matrices **U, S** and **V**. These values are

$$\mathbf{S} = \begin{bmatrix} 155.6462 & 0 & 0 \\ 0 & 8.9566 & 0 \\ 0 & 0 & 0.4325 \end{bmatrix}_{3\times 3}$$

From **S**, the singular values decrease rapidly and from the theory of SVD, using the first column alone of **S** and equation (3.4) would result in a matrix that very closely approximates the original matrix. From the product of **U** and **S**, **D** was found to be

$$\mathbf{D} = \begin{bmatrix} -93.6644 & 6.2518 & 0.1679 \\ -98.9039 & -0.4008 & -0.3525 \\ -85.8489 & -6.4012 & 0.1860 \end{bmatrix}_{3\times 3}$$

Each row of **D** corresponds to the information of **Z** in parameter-space at each level of v_1; therefore, the first row of **D** corresponds to the low value of the design variable. The design variables used to generate these initial results are called the "training points" and will eventually be used to estimate model parameters in the metamodels. In this example, **D** consists of only three columns; therefore, metamodels are developed for all columns. Later, examples will be presented where using all columns of **D** will be too time-consuming.

3.1.2 Response Surface Model

Since only one design variable is assumed to be random, a linear RSM is assumed

$$y_m = \beta_0 + \beta_1 v_1 + \varepsilon \tag{3.14}$$

where y_m represents the metamodel of the m^{th} column of **D** and β_0 and β_1 are the model parameters. These model parameters are calculated using equation (2.9) restated here for convenience

$$\boldsymbol{\beta} = \left(\mathbf{X}^T\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{Y}$$

To show how the model parameters are calculated, consider the first column of **D**

$$\mathbf{Y} = \mathbf{d}_1 = \begin{bmatrix} -93.6644 \\ -89.9039 \\ -85.8489 \end{bmatrix}$$

and the matrix **X** as

$$\mathbf{X} = \begin{bmatrix} 1 & 4.00 \\ 1 & 4.40 \\ 1 & 4.84 \end{bmatrix}$$

and after substituting these matrices into equation (2.9), the model parameters are found to be

$$\boldsymbol{\beta} = \begin{bmatrix} -130.8617 \\ 9.3027 \end{bmatrix}$$

Therefore, on substituting these parameters into equation (3.14) as β_0 and β_1, the RSM to represent the first column of **D** is

$$y_1 = -130.8617 + 9.3027 v_1 \quad (3.15)$$

This procedure of estimating model parameters is then repeated to the other two columns of **D** and the metamodels found are

$$y_2 = 66.1902 - 15.0393 v_1 \quad (3.16)$$

$$y_3 = -0.1831 + 0.0416 v_1 \quad (3.17)$$

3.1.3 Kriging Metamodel

Recall the general form of the Kriging model

$$y = f(\mathbf{v}, \boldsymbol{\beta}) + \mathrm{E}(\mathbf{v}) \quad (3.18)$$

where f(**v, β**) is assumed to take the form of a linear RSM model similar to equation (3.14). Therefore, the Kriging model to fit the experimental results of the m^{th} column of **D**, for one design variable is

$$y_m = \beta_0 + \beta_1 v_1 + \left(\boldsymbol{\gamma}^*(\mathbf{v})\right)_m^T \boldsymbol{\Gamma}_m^{-1} \mathbf{E}_m \quad (3.19)$$

where $\gamma^*(\mathbf{v}) = \exp\left(-\theta_1 (v_1 - x_1)^2\right)$. Consider $\mathbf{d}_1$

$$\mathbf{Y} = \mathbf{d}_1 = \begin{bmatrix} -93.6644 \\ -89.9039 \\ -85.8489 \end{bmatrix}$$

with training points of v_1 being

$$\mathbf{x}_1 = \begin{bmatrix} 4.00 \\ 4.40 \\ 4.84 \end{bmatrix}$$

To begin the iterative procedure outlined in Chapter 2 to calculate the Kriging model parameters **β**, an initial estimate of **θ** is made. The correlation function that will be used is the Gaussian correlation function and according to Simpson & Martin, 2003, **θ** must be greater than zero. Consider an initial estimate of **θ** = 20. This value is then used to calculate **Γ** using equation (2.14), which for one design variable becomes equation (3.20)

$$\mathbf{\Gamma} = \exp\left(-\theta\left(x_1^j - x_1^k\right)^2\right) \tag{3.20}$$

where $\left(x_1^j - x_1^k\right)$ is the difference between the training points. These values are found in Table (3-1)

Table 3-1 showing the calculation of $\left(x_1^j - x_1^k\right)$

	$x^{j=1} = 4$	$x^2 = 4.4$	$x^3 = 4.84$
$x^{i=1} = 4$	4 – 4 = 0	-0.4	-0.84
$x^2 = 4.4$	0.44	0	-0.44
$x^3 = 4.84$	0.84	0.44	0

and after substituting these values into equation (3.20) above, the matrix of the correlation between each of the training points becomes

$$\mathbf{\Gamma} = \begin{bmatrix} 1.0000 & 0.0408 & 0 \\ 0.0408 & 1.0000 & 0.0208 \\ 0 & 0.0208 & 1.0000 \end{bmatrix}_{3\times 3}$$

In order to clarify how this matrix is obtained, consider the value found in $\mathbf{\Gamma}_{23}$; that is, the 2nd row and 3rd column of $\mathbf{\Gamma}$. This number represents the correlation between the second and third training points of the design variable, therefore,

$$\mathbf{\Gamma}_{23} = \exp\left(-20\left(x_1^2 - x_1^3\right)^2\right) = \exp\left(-20(-0.44)^2\right) = 0.0208$$

This is then repeated to the other numbers found in $\left(x_1^j - x_1^k\right)$ to find the entire correlation matrix.

Following the procedure to calculate the optimum **θ,** the correlation matrix is then substituted into equation (2.17) to find an initial estimate of **β**.

$$\boldsymbol{\beta} = \begin{bmatrix} -130.8645 \\ 9.3033 \end{bmatrix}$$

This estimate of **β** is then substituted into equation (2.18) to obtain an estimate of s^2

$$s^2 = \frac{1}{m}\left(\mathbf{Y} - f(\mathbf{v},\boldsymbol{\beta})\right)^T \boldsymbol{\Gamma}^{-1}\left(\mathbf{Y} - f(\mathbf{v},\boldsymbol{\beta})\right) = 3.5007 \times 10^{-4}$$

and s^2 is now substituted into equation (2.19) to obtain an estimate of L(**θ**).

$$L(\boldsymbol{\theta}) = 23.8742$$

Now, the first estimate of **θ** is then changed and the entire procedure is repeated until a value of **θ** is achieved that maximizes L(**θ**). Eventually, an optimum **θ** of 65 is reached to yield the following metamodel to fit the data in $\mathbf{D}_1$

$$y_1 = -130.8617 + 9.3027 v_1 + \left(\boldsymbol{\gamma}^*(\mathbf{v})\right)_1^T \boldsymbol{\Gamma}_1^{-1}\mathbf{E}_1 \tag{3.21}$$

This entire procedure of estimating an optimum value of **θ** to maximize L(**θ**) is then repeated to the other two columns of **D** and the Kriging metamodels obtained for $\mathbf{d_2}$ and $\mathbf{d_3}$ are

$$y_2 = 66.1902 - 15.0393 v_1 + \left(\boldsymbol{\gamma}^*(\mathbf{v})\right)_2^T \boldsymbol{\Gamma}_2^{-1}\mathbf{E}_2 \tag{3.22}$$

$$y_3 = -0.1831 + 0.0416 v_1 + \left(\boldsymbol{\gamma}^*(\mathbf{v})\right)_3^T \boldsymbol{\Gamma}_3^{-1}\mathrm{E}_3 \tag{3.23}$$

The final estimate of **θ** that maximizes L(**θ**) was found using the MATLAB function ‘fminsearch’.

On comparing the RSMs with the Kriging metamodels, it is clear that the model parameters found for the ‘global’ part of the Kriging model are identical to those found for the RSMs. The only difference between these two metamodels is the interpolation of the residuals present in the second part of the Kriging metamodel. Now that the two different metamodels have been developed for each of the three columns of **D**, suppose the angular velocity time history at $\mathbf{v}_0$ = [4.2] is desired. Since each row of **D** corresponds to a particular design variable then to calculate the entire response time history, a new row in **D** has to be calculated that corresponds to the parameter-dependent information of $\mathbf{v}_0$ and this new row is then multiplied by $\mathbf{V}^T$.

3.1.4 Response Calculation at $\mathbf{v}_0$ - Response Surface Model

Recall the three equations (3.15), (3.16) and (3.17), found to describe the experimental data found in the three columns of **D**. Predicting the response time history at $\mathbf{v}_0$ is easy using the RSM. Here, $\mathbf{v}_0$ is just substituted into each metamodel to obtain the estimate at each column of **D**. Therefore, substituting $\mathbf{v}_0 = [4.2]$ into the metamodel representing the first column of **D**

$$\hat{y}_1(\mathbf{v}_0) = -130.8617 + 9.3027(4.2) = -91.7904$$

and repeating this procedure for the other two equations, the row estimate becomes

$$\hat{\mathbf{y}}(\mathbf{v}_0) = [-91.7894 \quad 3.0129 \quad -0.0186]$$

This row in then multiplied by $\mathbf{V}^T$ as described in equation (3.7) to obtain the response time history at $\mathbf{v}_0$; the results of which are shown graphically in Figure (3-2).

3.1.5 Response Calculation at $\mathbf{v}_0$ - Kriging Metamodel

When using the Kriging metamodel, the correlation between $\mathbf{v}_0$ and the training points, $(\boldsymbol{\gamma}^*(\mathbf{v}))^T$, is needed and in order to calculate this matrix, the difference between $\mathbf{v}_0$ and all the training points are first calculated

Table 3-2 showing the calculation of $(v_1^0 - x_1^j)$

	$x_1^1 = 4$	$x_1^2 = 4.4$	$x_1^3 = 4.84$
$v_0 = 4.2$	0	-0.4	-0.84

Consider the first column of **D**. The correlation between $\mathbf{v}_0$ and the training points was found to be

$$\boldsymbol{\gamma}^*(\mathbf{v}_0) = \begin{bmatrix} 0.0743 \\ 0.0743 \\ 0 \end{bmatrix}$$

using the equation

$$\boldsymbol{\gamma}^*(\mathbf{v}_0) = \exp\left(-\theta_1 \left(v_1^0 - x_1^j\right)\right)$$

and the values found in Table (3-2). The number in the first row of matrix $\gamma^*(\mathbf{v}_0)$ represents the correlation between $\mathbf{v}_0$ and the first training point. The details of this calculation are

$$\boldsymbol{\gamma}^*(\mathbf{v}_0) = \exp\left(-65\left(v_1^0 - x_1^1\right)\right) = \exp\left(-65(4.2-4)^2\right) = 0.0743$$

Eventually, the estimate, $\hat{y}_1$ is -91.7894 and the entire estimate of the row corresponding to $\mathbf{v}_0$ is

$$\hat{\mathbf{y}}(\mathbf{v}_0) = \begin{bmatrix} -91.7894 & 3.0129 & -0.0186 \end{bmatrix}$$

After multiplying $\hat{\mathbf{y}}(\mathbf{v}_0)$ by $\mathbf{V}^\mathbf{T}$, the angular velocity time history at $\mathbf{v}_0$ = [4.2] is obtained

$$\hat{z}(\mathbf{v}_0) = \begin{bmatrix} -91.7894 & 3.0129 & -0.0186 \end{bmatrix} \begin{bmatrix} -0.3629 & -0.5203 & \dots & -0.0041 \\ 0.2226 & 0.2430 & \dots & 0.0050 \\ 0.1750 & 0.1836 & \dots & -0.1550 \end{bmatrix}_{3\times 20}$$

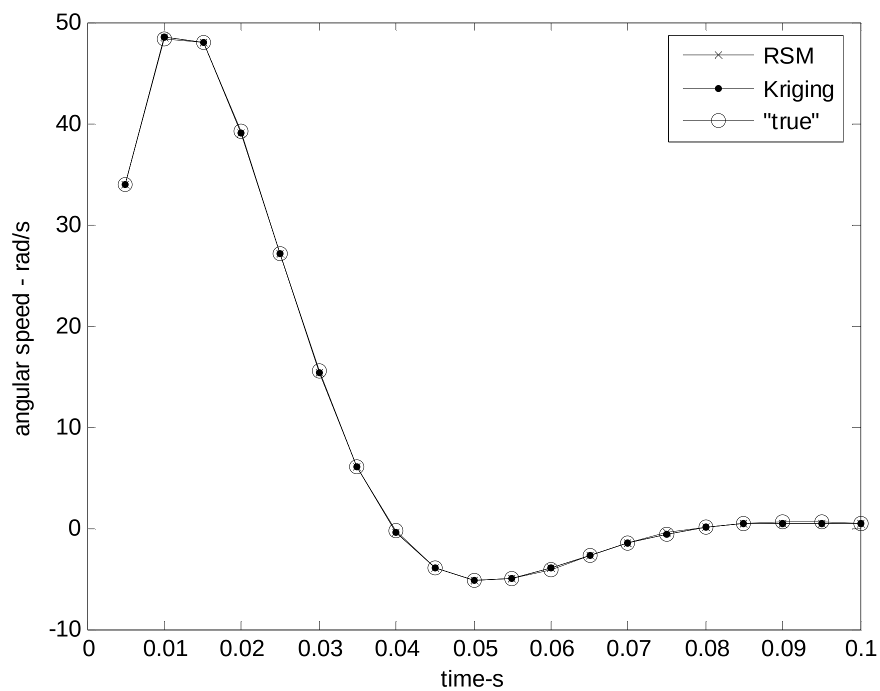

Figure 3-2 showing the angular speed response at v_0 calculated using RSM and Kriging.

From Figure 3-2, it is clear that both metamodels seem to produce accurate results after comparison with those obtained from the simulation. On looking at these results more closely, as in Table 3-3, the Kriging model seems to give a closer prediction than the RSM.

One of the attractive features of the Kriging metamodel is that the derived model exactly predicts response at the training points. To show this, consider $\mathbf{v}_0 = [4]$ and after substituting this number into each of the metamodels, the row estimate is

$$\hat{y}(\mathbf{v}_0) = [-93.6644 \quad 6.2518 \quad 0.1679]$$

which is exactly equal to the first row of **D** and after multiplication with $\mathbf{V}^T$ yields the exact angular speed response at $v_1 = 4$. This feature is especially useful for very 'noisy' responses and leads to more accurate predictions. Later, an example with a 'noisy' response will be presented and the difference in RSM and Kriging is clearly seen.

Table 3-3 showing the angular speed response obtained from the RSM, Kriging models and the simulation.

Time	RSM	Error (%)	Kriging	Error (%)	Actual
0.005	33.9842	0.3037	33.9794	0.290	33.8813
0.010	48.4919	0.2301	48.4866	0.219	48.3806
0.015	48.0009	0.0861	47.9988	0.0817	47.9596
0.020	39.0973	-0.1280	39.0993	-0.123	39.1474
0.025	27.03	-0.4383	27.0353	-0.419	27.1490
0.030	15.3436	-0.9291	15.3502	-0.887	15.4875
0.035	5.9766	-2.049	5.9825	-1.95	6.1016
0.040	-0.3917	24.35	-0.3879	23.1	-0.3150
0.045	-3.9178	0.4616	-3.9165	0.428	-3.8998
0.050	-5.2028	-0.6493	-5.2040	-0.626	-5.2368
0.055	-4.9907	-1.358	-4.9936	-1.301	-5.0594
0.060	-3.9721	-2.027	-3.9758	-1.93	-4.0543
0.065	-2.6823	-2.787	-2.6858	-2.66	-2.7592
0.070	-1.4723	-3.840	-1.4751	-3.658	-1.5311
0.075	-0.5257	-6.209	-0.5276	-5.87	-0.5605
0.080	0.1016	12.27	0.1008	11.4	0.0905
0.085	0.4359	-1.758	0.4360	-1.74	0.4437
0.090	0.5209	-7.773	0.5246	-7.12	0.5648
0.095	0.4955	-6.948	0.4980	-6.48	0.5325
0.100	0.3915	-6.586	0.3930	-6.23	0.4191

3.2 Case Study 2 - Servo with Three Random Design Variables

Now, consider the same servo with multiple design variables L (v_1), K (v_2) and R (v_3). As before, these variables are assumed to be random and the low, medium and high values of each variable used to generate the training design are

$$\begin{pmatrix} v_1^{low} \\ v_1^{med} \\ v_1^{high} \end{pmatrix} = \begin{pmatrix} 0.0010 \\ 0.0011 \\ 0.0012 \end{pmatrix} \qquad \begin{pmatrix} v_2^{low} \\ v_2^{med.} \\ v_3^{high} \end{pmatrix} = \begin{pmatrix} 8.00\times10^{-3} \\ 8.80\times10^{-3} \\ 9.68\times10^{-3} \end{pmatrix} \begin{pmatrix} v_3^{low} \\ v_3^{med.} \\ v_3^{high} \end{pmatrix} = \begin{pmatrix} 4.00 \\ 4.40 \\ 4.84 \end{pmatrix}$$

The angular speed response time histories at each of the twenty-seven design variable combinations are

$$\mathbf{Z} = \begin{bmatrix} 35.4135 & 50.2836 & \dots & 0.3880 \\ 32.4767 & 46.6146 & \cdots & 0.4199 \\ \vdots & \vdots & \ddots & \vdots \\ 35.1859 & 49.6381 & \dots & 0.3436 \end{bmatrix}_{27x20}$$

The first row of **Z** corresponds to all the low values of v_1, v_2 and v_3 and the last row represents the response at all the high values of the design variables. The rows in-between represent the dynamic responses at the various design variable combinations. After SVD, metamodels were developed for the first 11 columns of **D**.

3.2.1 Response Surface Model

Normally, for multiple design variables, a quadratic RSM is assumed. However, if a linear RSM fits the experimental design well, then there is no need for a more complex model. In this research, the choice of metamodel is between the RSM or Kriging model. Therefore, if the linear RSM does not provide a good fit, the Kriging model is chosen.

The linear RSM has the form

$$y = \beta_0 + \beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3 \tag{3.24}$$

In estimating the RSM model parameters, consider the matrices **X** and $\mathbf{d_1}$

$$\mathbf{X} = \begin{bmatrix} 1 & 0.001 & 0.008 & 4 \\ 1 & 0.001 & 0.008 & 4.4 \\ 1 & 0.001 & 0.008 & 4.84 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 0.0012 & 0.00968 & 4.84 \end{bmatrix} \qquad \mathbf{Y} = \mathbf{d}_1 = \begin{bmatrix} -93.8644 \\ -89.6539 \\ -85.2081 \\ \vdots \\ -92.4988 \end{bmatrix}$$

After using the least squares theory $\boldsymbol{\beta}$ is found to be

$$\boldsymbol{\beta} = \begin{bmatrix} -99.5914 \\ -953.6270 \\ -3790.1 \\ 9.3335 \end{bmatrix}$$

and eventually the following equations are obtained for columns 1 and 2 respectively

$$y_1 = -99.5914 - 953.6270 v_1 - 3790.1 v_2 + 9.3335 v_3 \tag{3.25}$$

$$y_2 = 3.4160 - 563.1305 v_1 + 7407.9 v_2 - 15.5230 v_3 \tag{3.26}$$

3.2.2 Kriging

The general form of the Kriging model for the mth column of **D** for three random design variables is

$$y_m = \beta_0 + \beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3 + \left(\boldsymbol{\gamma}^*(\mathbf{v})\right)_m^T \boldsymbol{\Gamma}_m^{-1} \mathbf{E}_m \tag{3.27}$$

where $\gamma^*(\mathbf{v}) = \exp\left(-\theta_1 (v_1 - \mathbf{x}_1)^2\right) \cdot \exp\left(-\theta_2 (v_2 - \mathbf{x}_2)^2\right) \cdot \exp\left(-\theta_3 (v_3 - \mathbf{x}_3)^2\right)$. In order to show how Kriging model parameters are estimated using multiple design variables, consider $\mathbf{d}_1$ and the initial estimate of $\boldsymbol{\theta}$

$$\boldsymbol{\theta}_1 = [10000 \quad 10000 \quad 1]$$

For three design variables, the Gaussian correlation function becomes

$$\boldsymbol{\Gamma} = \exp\left(-\theta_1 \left(x_1^j - x_1^k\right)^2\right) \cdot \exp\left(-\theta_2 \left(x_2^j - x_2^k\right)^2\right) \cdot \exp\left(-\theta_3 \left(x_3^j - x_3^k\right)^2\right) \tag{3.28}$$

where $\left(x_1^j - x_1^k\right)$ is the difference between all training points for the first design variable and so forth. An extract of the matrix of the experimental design variable combinations is

$$\mathbf{X}_{tr} = \begin{bmatrix} 0.001 & 0.008 & 4 \\ 0.001 & 0.008 & 4.4 \\ 0.001 & 0.008 & 4.84 \\ \vdots & \vdots & \vdots \\ 0.0012 & 0.00968 & 4.84 \end{bmatrix}_{27\times 3}$$

where the first row of $\mathbf{X}_{tr}$ corresponds to all the low values of the design variables and the last row represents all the high values of the design variables. The first column of $\mathbf{X}_{tr}$**,** $\mathbf{x}_1$ is the training design for the first design variable. The entire matrix is shown in the appendix. In order to estimate the model parameters, the difference between training points for each design variable is required. Extracts of these matrices are

$$\left[x_1^j - x_1^k\right] = \begin{bmatrix} 0 & 0 & \dots & -0.0002 & -0.0002 \\ 0 & 0 & \dots & -0.0002 & -0.0002 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0.0002 & 0.0002 & \dots & 0 & 0 \\ 0.0002 & 0.0002 & \dots & 0 & 0 \end{bmatrix}_{27\times 27}$$

$$\left[x_2^j - x_2^k\right] = \begin{bmatrix} 0 & 0 & \dots & -0.0017 & -0.0017 \\ 0 & 0 & \dots & -0.0017 & -0.0017 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0.0017 & 0.0017 & \dots & 0 & 0 \\ 0.0017 & 0.0017 & \dots & 0 & 0 \end{bmatrix}_{27\times 27}$$

$$\left[x_3^j - x_3^k\right] = \begin{bmatrix} 0 & -0.40 & \dots & -0.40 & -0.84 \\ 0.40 & 0 & \dots & 0 & -0.44 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -0.44 & 0.40 & \dots & 0 & -0.44 \\ 0.84 & 0.44 & \dots & 0.44 & 0 \end{bmatrix}$$

These matrices are then used, along with $\boldsymbol{\theta}_1$**,** in the calculation of $\boldsymbol{\Gamma}$ and eventually, $\boldsymbol{\Gamma}$ is found to be

$$\mathbf{\Gamma} = \begin{bmatrix} 1 & 0.8521 & 0.4938 & \dots & 0.4799 \\ 0.8521 & 1 & 0.8240 & \dots & 0.8007 \\ 0.4938 & 0.8240 & 1 & \dots & 0.9718 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0.4799 & 0.8007 & 0.9718 & \dots & 1 \end{bmatrix}_{27x27}$$

In order to show how the calculations are performed, consider $\mathbf{\Gamma}_{23.}$ This value is obtained by using the value found in the 2nd row and 3rd column of each of the three matrices shown above. Then using equation (3.28) as follows

$$\mathbf{\Gamma}_{23} = e^{-10000(0)^2} \times e^{-10000(0)^2} \times e^{-1(-0.44)^2} = 1 \times 1 \times 0.8240 = 0.8240$$

The above correlation matrix, $\mathbf{\Gamma}$, and $\mathbf{X}$ is then substituted into equation (2.17) to obtain an initial estimate of $\boldsymbol{\beta}$

$$\mathbf{X} = \begin{bmatrix} 1 & 0.001 & 0.008 & 4 \\ 1 & 0.001 & 0.008 & 4.4 \\ 1 & 0.001 & 0.008 & 4.84 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 0.0012 & 0.00968 & 4.84 \end{bmatrix} \qquad \boldsymbol{\beta}_1 = \begin{bmatrix} 860340 \\ -690 \\ -106990 \\ -159700 \end{bmatrix}$$

This initial $\boldsymbol{\beta}_1$ is then used to obtain estimates of $\mathbf{E}$ using the initial simulation runs, the linear RSM model used to represent f($\mathbf{v}, \boldsymbol{\beta}$) and equation (2.21).

$$\mathbf{E} = \begin{bmatrix} -220760 \\ -156990 \\ -86600 \\ \vdots \\ -86430 \end{bmatrix}_{27x1}$$

These preliminary estimates of $\mathbf{E}$ and $\boldsymbol{\beta}$ are then substituted into equation (2.18) to get

$$s^2 = 8.8952 \times 10^{10}$$

which is then substituted into equation (2.19) to get

$$L(\theta) = -319.8718$$

Now, this entire procedure is then repeated for another estimate of **θ** until L(**θ**) is maximized. Eventually, an optimal estimate of $\boldsymbol{\theta}_1$ is found to be

$$\boldsymbol{\theta}_1 = [90000000 \quad 20000000 \quad 10]$$

and

$$\boldsymbol{\beta}_1 = \begin{bmatrix} -99.8 \\ -950.7960 \\ -3794.6 \\ 9.3927 \end{bmatrix}$$

The Kriging model to fit the data in $\mathbf{D}_1$ is then

$$y_1 = -99.8330 - 950.7960 v_1 - 3794.6 v_2 + 9.3927 v_3 + \left(\boldsymbol{\gamma}^*(\mathbf{v})\right)_1^T \boldsymbol{\Gamma}_1^{-1}\mathbf{E}_1 \tag{3.29}$$

Now the metamodels will be used along with SVD to estimate the response at some arbitrary design variable set.

3.2.3 Estimation of Response at $\mathbf{v}_0$.

An estimate of the angular velocity at $\mathbf{v}_0 = [0.00115 \quad 0.008 \quad 4.6]$ is desired. In using the linear RSM, $\mathbf{v}_0$ is just substituted into the metamodel to obtain the row estimate for **D** that is then multiplied by $\mathbf{V}^T$. These results are shown in Table (3-4). For the Kriging model, as with one random design variable, **γ*(v)** has to be estimated. For three design variables, this equation becomes

$$\boldsymbol{\gamma}^*(\mathbf{v}_0) = \exp\left(-\theta_1 \left(v_1^0 - x_1^j\right)^2\right) \cdot \exp\left(-\theta_2 \left(v_2^0 - x_2^j\right)^2\right) \cdot \exp\left(-\theta_3 \left(v_3^0 - x_3^j\right)^2\right) \tag{3.30}$$

Extracts of the matrices containing the differences between $\mathbf{v}_0$ and the training points are shown below

$$\left[v_1^0 - x_1^j\right] = \begin{bmatrix} 1.5\times10^{-4} \\ 1.5\times10^{-4} \\ \vdots \\ -5\times10^{-5} \\ -5\times10^{-5} \end{bmatrix}_{27\times1}$$

$$\left[v_2^0 - x_2^j\right] = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ -1.68\times10^{-3} \\ -1.68\times10^{-3} \end{bmatrix}_{27\times1}$$

$$\left[v_3^0 - x_3^j\right] = \begin{bmatrix} 0.6 \\ 0.2 \\ \vdots \\ 0.2 \\ -0.24 \end{bmatrix}_{27\times1}$$

After substituting these values into equation (3.30), the correlation between $\mathbf{v}_0$ and the training points was found to be

$$\boldsymbol{\gamma}^*(\mathbf{v}_0) = \begin{bmatrix} 0.0036 \\ 0.0036 \\ \vdots \\ 0 \end{bmatrix}_{27x1}$$

and substituting all this information in equation (3.29), the following estimate is obtained for $\mathbf{v}_0$ for $\mathbf{d_1}$

$$\hat{y}_1(\mathbf{v}_0) = -93.6432$$

These calculations are then repeated to the other columns of **D** to get the estimate of the row corresponding to $\mathbf{v}_0$. Eventually, the response at $\mathbf{v}_0$ is calculated from equation (3.7) the results of which, using both the RSM and Kriging models are shown in Table 3-4.

3.2.4 Model Adequacy

In order to further determine the adequacy of the Kriging and Response Surface Models, the coefficient of determination (R^2) is calculated at each time step. R^2 is also used to show how the model adequacy changes when the number of columns of **D** used changes. Table (3-5) shows these results.

From Table (3-5), 12 columns of **D** exactly predict the experimental response time histories using the Kriging model. Also, the Response Surface models obtained are quite good for most of the time steps but there are some exceptions. Reducing the number of columns of **D** to 5 also gives fairly acceptable results for both Kriging and RSM. Now, calculating R^2 just determines how well the metamodel models the experimental results. The metamodels can also be tested by calculating the response at some $\mathbf{v}_0$ and comparing these results with those obtained from running the simulation at $\mathbf{v}_0$. From the predictions found in Table (3-4), the predictions obtained from both models were quite close as can also be seen in Figure (3-3). Now, a system with a noisier response will be used.

Table 3-4 showing response at v_0 for Servo Example with three random design variables calculated using the RSM, Kriging and from the simulation.

Time	**RSM**	**Error (%)**	**Kriging**	**Error (%)**	**Actual**
0.005	31.0771	0.0634	31.0743	0.054	31.0574
0.010	45.1115	0.2832	44.9835	-0.0013	44.9841
0.015	45.7757	0.4695	45.5277	-0.0748	45.5618
0.020	38.6222	0.5200	38.3476	-0.1947	38.4224
0.025	28.2108	0.5844	27.9775	-0.2474	28.0469
0.030	17.5700	0.2602	17.4679	-0.3224	17.5244
0.035	8.5586	-0.9777	8.6154	-0.3205	8.6431
0.040	1.9668	-8.729	2.1626	0.3573	2.1549
0.045	-2.1447	12.69	-1.8625	-2.139	-1.9032
0.050	-4.1484	6.141	-3.8441	-1.645	-3.9084
0.055	-4.6040	4.338	-4.3355	-1.747	-4.4126
0.060	-4.0984	3.047	-3.9039	-1.843	-3.9772
0.065	-3.1201	1.417	-3.0151	-1.996	-3.0765
0.070	-2.0354	-1.093	-2.0145	-2.109	-2.0579
0.075	-1.0717	-5.900	-1.1154	-2.063	-1.1389
0.080	-0.3400	-20.45	-0.4220	-1.264	-0.4274
0.085	0.1353	176.7	0.0403	-17.59	0.0489
0.090	0.3550	13.46	0.2506	-19.91	0.3129
0.095	0.4446	7.834	0.3683	-10.67	0.4123
0.100	0.4194	4.458	0.3741	-6.824	0.4015

Table 3-5 showing the coefficient of determination at each time Step using different numbers of columns of D for the Servo with three random Design Variables.

Time	R^2 (12 Columns of D)		R^2 (5 Columns of D)		R^2 (2 Columns of D)	
	RSM	Kriging	RSM	Kriging	RSM	Kriging
0.005	0.9968	1.0000	0.9968	1.0000	0.9928	0.9935
0.010	0.9983	1.0000	0.9983	1.0000	0.9980	0.9983
0.015	0.9962	1.0000	0.9962	1.0000	0.9955	0.9983
0.020	0.3360	1.0000	0.3360	1.0000	0.3209	0.7161
0.025	0.9521	1.0000	0.9521	1.0000	0.9487	0.9886
0.030	0.9874	1.0000	0.9874	1.0000	0.9873	0.9990
0.035	0.9963	1.0000	0.9963	1.0000	0.9958	0.9990
0.040	0.9980	1.0000	0.9980	1.0000	0.9955	0.9927
0.045	0.9879	1.0000	0.9879	1.0000	0.9807	0.9712
0.050	0.9257	1.0000	0.9257	0.9999	0.9006	0.8877
0.055	0.4164	1.0000	0.4163	0.9992	0.2946	0.3551
0.060	0.6828	1.0000	0.6828	0.9993	0.6404	0.7359
0.065	0.9342	1.0000	0.9342	0.9993	0.9296	0.9638
0.070	0.9851	1.0000	0.9850	0.9992	0.9849	0.9973
0.075	0.9957	1.0000	0.9956	0.9989	0.9945	0.9930
0.080	0.9786	1.0000	0.9784	0.9986	0.9731	0.9591
0.085	0.9031	1.0000	0.9027	0.9982	0.8865	0.8567
0.090	0.5585	1.0000	0.5460	0.9387	0.5159	0.4846
0.095	0.0797	1.0000	0.0653	0.9399	0.0061	0.0522
0.100	0.6220	1.0000	0.6198	0.9888	0.5987	0.6749

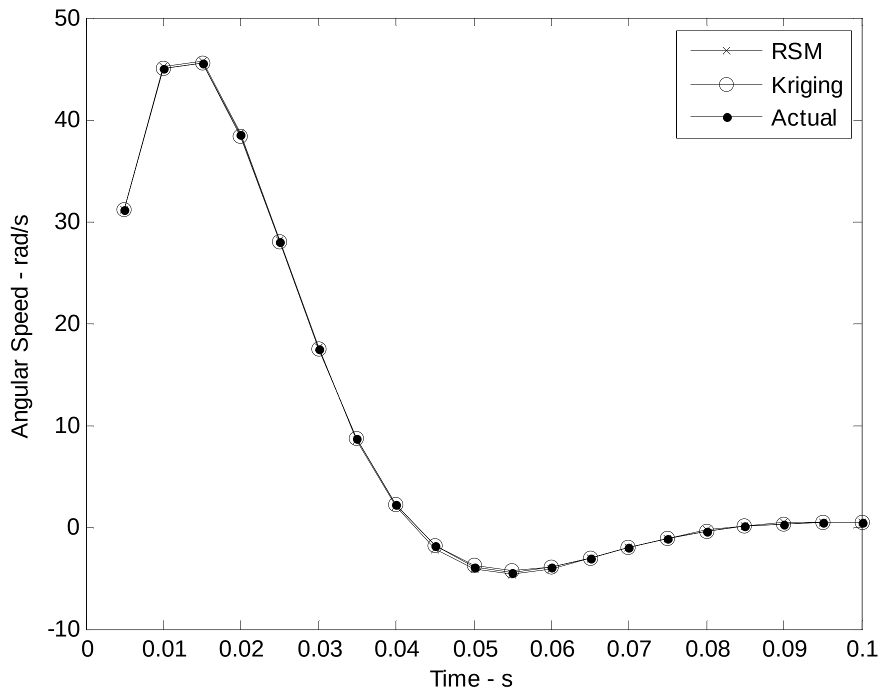

Figure 3-3 showing the response at v_0 calculated using both the RSM and Kriging Model for the servo with three random design variables.

3.3 Case Study 3 – Simulation of a Piano String

Another example of a dynamic system is the action of a hammer hitting the string in a piano. The response of this system is more erratic than that of the servo. The string velocities were obtained from simulations performed by Motion Research Group of the University of Waterloo. A diagram of the experimental setup

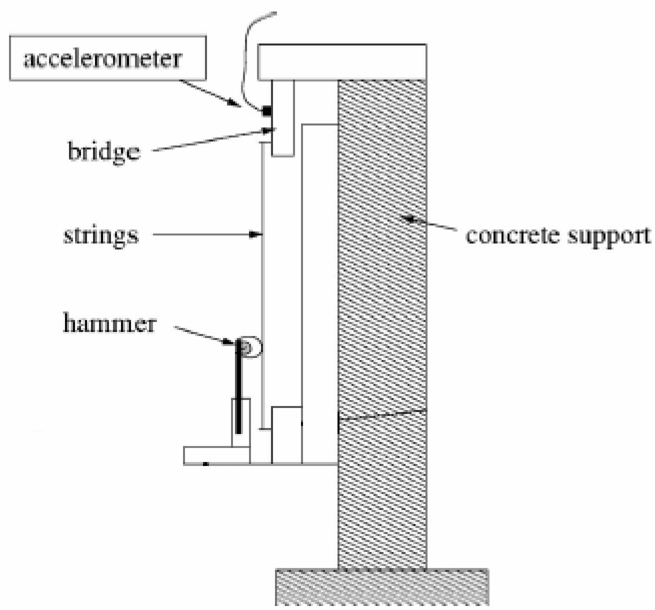

Figure 3-4 showing the experimental setup to obtain sample piano string velocities. (Bensa, J., Gipouloux, O. and Kronland-Martinet, R., 2005).

3.3.1 Simulation Results

The most important parameters in piano design are relative striking position of the hammer, hammer-string mass ratio and string stiffness (Askenfelt, A. and Chaigne, A., 1994). For this case study, the three random variables were initial hammer velocity (v_1), string stiffness (v_2) and striking position of hammer (v_3). The three levels of each design variable used in experimental design are

$$\begin{bmatrix} v_1^{low} \\ v_1^{med} \\ v_1^{high} \end{bmatrix} = \begin{bmatrix} 3.50 \\ 4.00 \\ 4.50 \end{bmatrix} \qquad \begin{bmatrix} v_2^{low} \\ v_2^{med} \\ v_2^{high} \end{bmatrix} = \begin{bmatrix} 3.60\times10^{-5} \\ 3.90\times10^{-5} \\ 4.20\times10^{-5} \end{bmatrix} \qquad \begin{bmatrix} v_3^{low} \\ v_3^{med} \\ v_3^{high} \end{bmatrix} = \begin{bmatrix} 7.44\times10^{-2} \\ 8.06\times10^{-2} \\ 8.68\times10^{-2} \end{bmatrix}$$

The 27 combinations of these three design variables yield 27 string velocity time histories from 0s to 0.005s using 252 time steps. An extract of these results are shown in **Z**

$$\mathbf{Z} = \begin{bmatrix} 0 & -0.001361 & -0.005883 & \dots & 0.8626 \\ 0 & -0.0007399 & -0.002602 & \dots & 0.5833 \\ 0 & -0.001331 & -0.005742 & \dots & 1.2870 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -0.002564 & -0.01144 & \dots & 1.5444 \end{bmatrix}_{27x252}$$

This simulation is quite expensive and can deter the analyst from observing the response at various design variable combinations. Metamodelling and SVD are quite useful since interpolation is performed on a much smaller scale. **Z** gives some initial simulation runs at the different combinations of the three design variables and SVD was then performed on this matrix. For accurate predictions and modelling of the experimental design, the most dominant values of **S** are used. A graphical representation of the dynamic string velocity is shown in Figure (3-5).

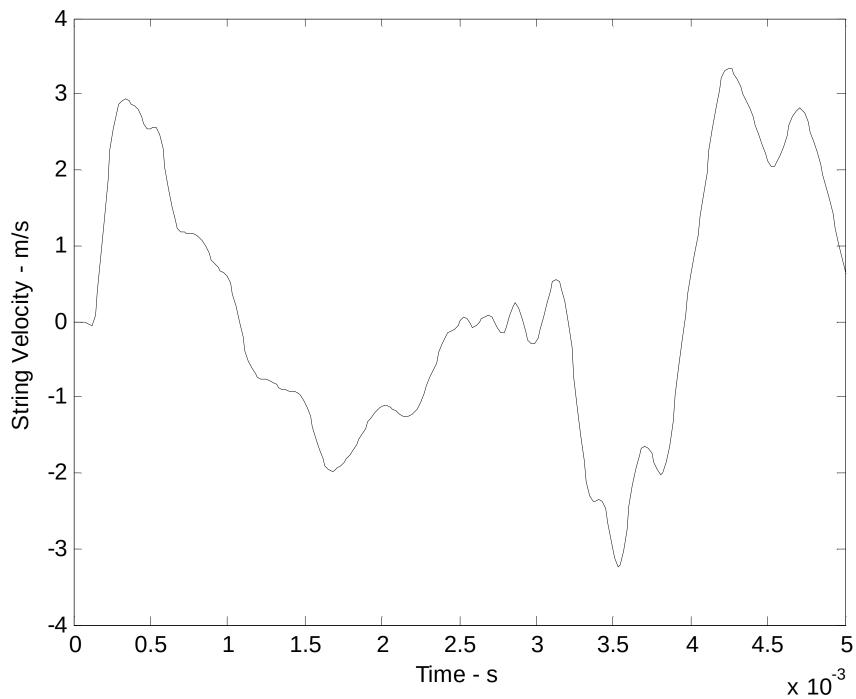

Figure 3-5 showing the string velocity output when v_1 = 3.50, v_2 = 3.60×10^{-5} and v_3 = 7.44×10^{-2}

3.3.2 Response Surface and Kriging Models

Metamodels were generated for the first ten columns of **D**, however, the first four RSM and Kriging models are shown here in equations (3-31) – (3-38)

$$y_1 = -7.78 - 7.99v_1 - 1548.45v_2 - 84.34v_3 \tag{3-31}$$

$$y_2 = 54.24 + 0.28v_1 + 1226.80v_2 - 686.92v_3 \tag{3-32}$$

$$y_3 = 0.30 - 0.020v_1 + 6832.90v_2 - 6.08v_3 \tag{3-33}$$

$$y_4 = -13.52 + 2.00v_1 + 1.30\times10^5 v_2 + 4.72v_3 \tag{3-34}$$

$$y_1 = -8.70 - 7.94v_1 - 1.77\times10^3 v_2 - 95.08v_3 + \left(\gamma^*(\mathbf{v})\right)_1^T \mathbf{\Gamma}_1^{-1}\mathbf{E}_1 \tag{3-35}$$

$$y_2 = 41.10 + 0.19v_1 - 4.03\times10^3 v_2 - 518.57v_3 + \left(\gamma^*(\mathbf{v})\right)_2^T \mathbf{\Gamma}_2^{-1}\mathbf{E}_2 \tag{3-36}$$

$$y_3 = -4.54 - 0.18v_1 - 338.33v_2 + 59.11v_3 + \left(\gamma^*(\mathbf{v})\right)_3^T \mathbf{\Gamma}_3^{-1}\mathbf{E}_3 \tag{3-37}$$

$$y_4 = -12.30 + 1.78v_1 + 1.25\times10^5 v_2 + 2.43v_3 + \left(\gamma^*(\mathbf{v})\right)_4^T \mathbf{\Gamma}_4^{-1}\mathbf{E}_4 \tag{3-38}$$

Now, the response at $\mathbf{v}_0 = \begin{bmatrix}3.50 & 3.80\times10^{-5} & 8.10\times10^{-2}\end{bmatrix}$ is desired. Therefore, after substituting these numbers into the RSMs and using equation (3-7) as in the previous case study, the response was found and is shown graphically in Figure (3-6). For the Kriging models, the correlation between $\mathbf{v}_0$ and the training points was calculated and a row in **D** was estimated. The estimated response is also shown in Figure (3-6).

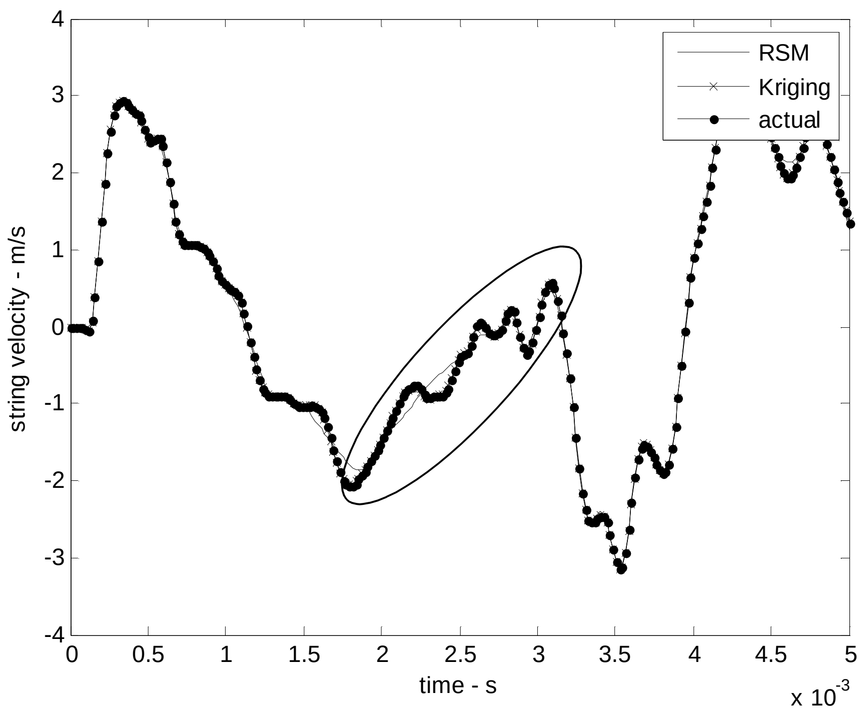

Figure 3-6 showing the response at v_0 calculated using Kriging and RSM.

From Figure 3-6 there is a noticeable difference between the response found using the Kriging model and the RSM. The RSM does not capture some of the behaviour of the string velocity between 2 x 10^{-3} and 3 x 10^{-3}s; the RSM just “smoothes” the response. Table 3-6 shows the R^2 value between 2 x 10^{-3}s and 3 x 10^{-3}s and from this table, it is clear that the RSM does not model the experimental results as well as the Kriging model. Therefore, if design analysis is to be made at any of these time steps, the Kriging model is preferred. The entire table of results is shown in appendix A.

Table 3-6 showing R^2 calculated for times $2\times10^{-3}s$ to $3\times10^{-3}s$

Time x10^{-3}	**2.0120**	**2.032**	**2.0518**	**2.0717**	**2.0916**	**2.1116**	**2.1315**
RSM	0.9596	0.883	0.7461	0.5416	0.3252	0.1901	0.1393
Kriging	0.9994	0.999	0.9983	0.9986	0.9988	0.9993	0.9986
Time x10^{-3}	**2.1514**	**2.1713**	**2.1912**	**2.2112**	**2.2311**	**2.2510**	**2.2709**
RSM	0.1523	0.2522	0.4414	0.6551	0.8385	0.7689	0.4759
Kriging	0.9984	0.9974	0.9917	0.9886	0.9934	0.9948	0.9952
Time x10^{-3}	**2.2908**	**2.311**	**2.3307**	**2.3506**	**2.3705**	**2.3904**	**2.4104**
RSM	0.2963	0.170	0.0529	0.0837	0.2419	0.3624	0.4552
Kriging	0.9935	0.992	0.9954	0.9980	0.9963	0.9980	0.9991
Time x10^{-3}	**2.4303**	**2.4502**	**2.470**	**2.4900**	**2.5100**	**2.5299**	**2.550**
RSM	0.6053	0.8151	0.919	0.9392	0.9535	0.9589	0.932
Kriging	0.9985	0.9965	0.997	0.9956	0.9964	0.9983	0.992
Time x10^{-3}	**2.5697**	**2.5896**	**2.6096**	**2.630**	**2.6494**	**2.6693**	**2.6892**
RSM	0.8504	0.6646	0.5296	0.620	0.7714	0.9076	0.9456
Kriging	0.9911	0.9847	0.9813	0.997	0.9942	0.9920	0.9945
Time x10^{-3}	**2.7092**	**2.7291**	**2.7490**	**2.7689**	**2.7888**	**2.8088**	**2.8287**
RSM	0.5869	0.2358	0.8016	0.9096	0.8373	0.4610	0.4635
Kriging	0.9667	0.9784	0.9864	0.9801	0.9926	0.9852	0.9647
Time x10^{-3}	**2.8486**	**2.8685**	**2.8884**	**2.908**	**2.9283**	**2.9482**	**2.9681**
RSM	0.8460	0.9533	0.9548	0.824	0.2735	0.6074	0.8714
Kriging	0.9909	0.9978	0.9940	0.988	0.9860	0.9943	0.9979

Chapter 4

Sensitivity Analysis and Robust Design

In dynamic systems, the response at certain time steps may determine whether or not a specific system is optimum. An acceptable design may require that the overshoot of the response at some time lies between specific limits or that the response stabilizes at a particular settling time. Therefore, it is important that design calculations are done such that specifications at different times are met. Before parameter design is performed, however, some analysis of the system must be first performed. First, a suitable metamodel must be chosen; that is, one that fits the experimental results well. For this, the statistical coefficient of determination is used. When a suitable model is chosen, sensitivity analysis is then performed to determine the effect of each design variable on the response. This can give the analyst an idea of how sensitive is the response to each design variable. Then, robust design calculations are performed to find the optimum system.

This chapter shows how normalized sensitivities are calculated over time for each design variable and how robust design through parameter design or integrated design is done to find the optimum system given specifications at certain time steps. The theory is then illustrated through the use of several case studies: the first of which is the design of a position control servo where only one design variable is random, the second considers the same servo but now three design variables are random. The third case study is the design of a mobile sign used in shop windows and the fourth looks at choosing variables to allow the velocity of a piano string to meet specifications at specific time steps.

4.1 Normalized Sensitivities

Normalized sensitivities are used to determine the effect of the design variables on the response of the system and are calculated using equation (2.28). SVD can, again, be used to quickly calculate normalized sensitivities over time especially for systems with very large time steps. The First-Order Sensitivity factor (FOS) of equation (2.28) is just the first-order derivative of a function of the design variables. Therefore, to obtain the FOS factors over time, this first order derivative can be applied to the metamodels obtained for the columns of **D** and then multiplied by the matrix **V**

$$\frac{\partial f_t(\mathbf{v})}{\partial v_i} = \left[\frac{\partial y_1(\mathbf{v})}{\partial v_i} \quad \frac{\partial y_2(\mathbf{v})}{\partial v_i} \quad \cdots \quad \frac{\partial y_m(\mathbf{v})}{\partial v_i} \right]_{v_i^{nom}} \cdot \mathbf{V}^T \tag{4.1}$$

where $\frac{\partial y_1(\mathbf{v})}{\partial v_i}$ is the first-order derivative of the metamodel of the first column of **D** with respect to v_i and v_i^{nom} is just the nominal value of design variable i. Therefore, the time history of normalized sensitivities is obtained from

$$\mathbf{S}_t^{v_i} = \left. \frac{\partial f_t(\mathbf{v})}{\partial v_i} \right|_{\mathbf{v}_i^{nom}} \cdot \frac{v_i^{nom}}{f_t(\mathbf{v})|_{\mathbf{v}_i^{nom}}} \tag{4.2}$$

4.2 Robust Design

A metamodel has been developed for each time step along with the coefficients of determination and normalized sensitivities. Taking all this information into account, robust design can now be performed. Although Kriging tends to always fit the experimental results quite well, it is non-linear and finding sensitivities and first and second moment information of the response is more difficult than using the RSM. Therefore, if the RSM is acceptable in fitting the experimental data it should be used as a first choice.

Before robust design is performed, the probability of conformance is calculated using the initial means of the design variables to determine how well the initial system meets specifications. A very high probability of conformance would indicate an acceptable system.

If the probability of conformance is too low, robust design is then performed and the probability of conformance re-calculated.

Normalized sensitivities are also calculated to give the analyst an idea of the important variables for design. A variable with a very small or negligible effect on the response can be ignored since changing this variable will not very likely change the response by very much. Also, including this variable will add complexities to the calculations that are not necessary. The application of robust design techniques to both the RSM and Kriging models will be presented to show how an optimum system may be calculated.

4.2.1 Probability Calculations

The probability of conformance at specific times can be calculated using either the Monte Carlo simulation with limit state functions or the Second Moment Method. Limit-state functions are easily derived at each time using equation (2.34) and for limit-state functions in **u**-space, the transformation of equation (2.39) is used.

At the n^{th} time, t_n, suppose ζ_{t_n} denotes the upper limit with metamodel $f_{t_n}(\mathbf{v})$. Therefore, the limit-state function, $g_{t_n}(\mathbf{v})$, becomes

$$g_{t_n}(\mathbf{v}) = \zeta_{t_n} - f_{t_n}(\mathbf{v}) \tag{4.3}$$

where the probability of success is found from

$$\Pr(S_{t_n}) = \Pr(g_{t_n}(\mathbf{v}) > 0) \tag{4.4}$$

For the case where ζ_{t_n} is a lower limit, then

$$g_{t_n}(\mathbf{v}) = f_{t_n}(\mathbf{v}) - \zeta_{t_n} \tag{4.5}$$

and the probability of success is calculated using equation (4.4) again. Now that the limit-state function has been derived, Monte Carlo simulation is used to generate a large sample of results given the mean and variance of each design variable and the number of instances when the specifications are met is counted.

The other method of probability calculation, the Second Moment method, utilizes the transmission of moment methodology to calculate the probability of conformance. The theories presented in the literature review in equations (2.32 and 2.33) can be applied to situations where the response of the system varies over time. The first-order derivative of each design variable for each metamodel is arranged into the matrix shown in equation (4.6) below

$$\left[\frac{\partial \mathbf{y}^T}{\partial \mathbf{v}}\right]_{m\times p} = \begin{bmatrix} \frac{\partial y_1}{\partial v_1} & \frac{\partial y_1}{\partial v_2} & \cdots & \frac{\partial y_1}{\partial v_p} \\ \frac{\partial y_2}{\partial v_1} & \frac{\partial y_2}{\partial v_2} & \cdots & \frac{\partial y_2}{\partial v_p} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_m}{\partial v_1} & \frac{\partial y_m}{\partial v_2} & \cdots & \frac{\partial y_m}{\partial v_p} \end{bmatrix}_{m\times p} \tag{4.6}$$

where $\frac{\partial y_m}{\partial v_p}$ represents the derivative of the metamodel of the m[th] column of **D** with respect to design variable p. A matrix of these derivatives over time is found using equation (4.7)

$$\frac{\partial f_t(\mathbf{v})}{\partial \mathbf{v}^T} = \left(\frac{\partial \mathbf{y}^T}{\partial \mathbf{v}}\right)^T \mathbf{V}^T \tag{4.7}$$

and can then be used to find the mean and variance of the response at each time step using equations (4.8) and (4.9) given the means and variances of the design variables

$$\left[\mu_{z,t}\right]_{1\times n} \approx f_t(\mathbf{v})\big|_{\bar{\mathbf{v}}} + \frac{1}{2}\left(\frac{\partial^2 f_t(\mathbf{v})}{\partial\left(\mathbf{v}^T\right)^2}\bigg|_{\bar{v}}\right)^T vec\mathbf{C}_{\mathbf{v}} \tag{4.8}$$

$$\left[\sigma^2_{z,t}\right]_{1\times n} \approx \left(\frac{\partial f_t(\mathbf{v})}{\partial \mathbf{v}^T}\bigg|_{\bar{v}}\right)^T \mathbf{C}_v \left(\frac{\partial f_t(\mathbf{v})}{\partial \mathbf{v}^T}\bigg|\right)_{\bar{v}} \tag{4.9}$$

4.2.2 Parameter Design using the Response Surface Model

Parameter design is easily done using the response surface model. For a linear response surface model the minimization of conformance indices method is used. Consider the general linear RSM

$$f(\mathbf{v},\boldsymbol{\beta}) = \beta_0 + \beta_1 v_1 + \beta_2 v_2 + \ldots + \beta_p v_p$$

that can be considered to be a bi-linear model of the form shown in equation (4.10)

$$f(\mathbf{v},\boldsymbol{\beta}) = \frac{\beta_0 + \beta_1 v_1 + \beta_2 v_2 + \ldots + \beta_p v_p}{1} \tag{4.10}$$

where $a_0 = \beta_0$, $a_1 = \beta_1$, $a_2 = \beta_2 \ldots a_p = \beta_p$ and $b_0 = 1$ according to equation (2.46). These numbers are then substituted into equations (2.43), (2.44) and (2.45) to achieve the objective function in equation (2.41) repeated below

$$\text{min. } Q = \sum_{i=1}^{k} e^{-a_i \beta_i}$$

The RSM at a specific time step is found by multiplying the row of metamodels found for each column of **D** by the row in **V** that corresponds to the time of interest. This method is shown in equation (4.11)

$$f_{t_n}(\mathbf{v}) = [y_1(\mathbf{v}) \quad y_2(\mathbf{v}) \quad \ldots \quad y_m(\mathbf{v})] \cdot \mathbf{V}_{t_n}^T \tag{4.11}$$

where $f_{t_n}(\mathbf{v})$ refers to the metamodel at the n^{th} time step and $\mathbf{V}_{t_n}^T$ refers to the column in $\mathbf{V}^T$ that represents the n^{th} time step.

In many instances, an optimum system is desired where the response at several time steps is specified. For such a case, constraints can be introduced where each constraint specifies the minimum acceptable probability of success using the limits at the specific time steps

$$\text{min } Q = \sum_{i=1}^{k} e^{\alpha_i \beta_i}$$

subject to

$$\Pr(S_{t_2}) \geq x_{t_2}$$
$$\Pr(S_{t_3}) \geq x_{t_3}$$
$$\vdots$$
$$\Pr(S_{t_n}) \geq x_{t_n}$$

where x_{t_n} is the desired probability of conformance at the n^{th} time step. This problem was solved using 'fmincon' in MATLAB.

4.2.3 Parameter Design using the Kriging Model

Parameter design using the Kriging model cannot be done using the above procedure. Instead, a probability objective function is used where the probability is calculated using equation (2.37) for the time of importance

$$\max\ \Pr\left(S_{t_n}\right) \approx 1 - \Phi\left(\frac{-\left(\zeta_{U,t_n} - \mu_{z,t_n}\right)}{\sigma_{z,t_n}}\right) - \Phi\left(\frac{-\left(\mu_{z,t_n} - \zeta_{L,t_n}\right)}{\sigma_{z,t_n}}\right) \tag{4.12}$$

or

$$\min\ \Pr\left(F_{t_n}\right) = 1 - \Pr\left(S_{t_n}\right)$$

where $\Pr\left(S_{t_n}\right)$ denotes the probability of success or conformance at the specific time and, likewise, $\Pr\left(F_{t_n}\right)$ denotes the probability of failure or non-conformance. The expressions μ_t and σ_t are calculated from equations (4.8) and (4.9). Similar to parameter design using the Response Surface model, for multiple time steps

$$\begin{aligned} &\min\ \Pr\left(F_{t_1}\right) \\ &\text{subject to} \\ &\Pr\left(S_{t_2}\right) \geq x_{t_2} \\ &\quad\vdots \\ &\Pr\left(S_{t_n}\right) \geq x_{t_n} \end{aligned} \tag{4.13}$$

4.2.4 Integrated Design

Like parameter design, integrated design for dynamic systems is done to meet certain specifications at particular times by finding the mean and tolerance of each design variable to give a theoretical optimum system and this is done by minimizing the total cost while ensuring that the probability of failure is within acceptable limits.

Now, the total cost is made up of the production and loss of quality costs. The production cost is simply calculated using the Reciprocal cost model shown in Table (2-2) where the assumptions a = 0 and b = 1 are made. However, the loss of quality cost requires the mean and variance of the response at the time of interest are calculated using equations (4.14) and (4.15) below

$$\mu_{z,t_n} \approx f_{t_n}(\mathbf{v})\Big|_{\bar{\mathbf{v}}} + \frac{1}{2}\left(\frac{\partial^2 f_{t_n}(\mathbf{v})}{\partial\left(\mathbf{v}^T\right)^2}\Bigg|_{\bar{\mathbf{v}}}\right)^T vec\mathbf{C}_{\mathbf{v}} \tag{4.14}$$

$$\sigma_{z,t_n}^2 \approx \left(\left[\frac{\partial f_{t_n}(\mathbf{v})}{\partial \mathbf{v}^T}\right]\Bigg|_{\bar{\mathbf{v}}}\right)^T \left[\mathbf{C}_v\right]\left[\frac{\partial f_{t_n}(\mathbf{v})}{d\mathbf{v}^T}\right]\Bigg|_{\bar{\mathbf{v}}} \tag{4.15}$$

A particular time step is selected by picking the row in **V** that corresponds to this point in time. Therefore,

$$f_{t_n}(\mathbf{v}) = \left[y_1(\mathbf{v}) \quad y_2(\mathbf{v}) \quad \ldots \quad y_m(\mathbf{v})\right] \cdot \mathbf{V}_{t_n}^T \tag{4.16}$$

$$\frac{\partial y_{t_n}}{\partial v_i} = \left[\frac{\partial y}{\partial \mathbf{v}}\right]^T \cdot \mathbf{V}_{t_n}^T \tag{4.17}$$

and the loss of quality cost at the nth time step becomes

$$C_{LQ} = \frac{C_S}{\Delta^2}\left(\left(\mu_{z,t_n} - T\right)^2 + \sigma_{z,t_n}^2\right) \tag{4.18}$$

4.3 Calculating a specific response given specifications at every time step.

Sometimes, the analyst may require the dynamic response of a system to follow a specific pattern (Yue, H. and Jiang, W., 2002). For this case, specifications at each time step have to be met. Normally, if the system has to meet specifications at only two times, metamodels can be developed only for these two times instead of performing SVD. However, for this case of meeting specifications at each time, developing metamodels at each time is impractical and in this case, applying SVD becomes useful. When metamodels are developed for the columns of the reduced matrix, **D**, the metamodel for any other time is found using equation (4.11). Given specifications at each time, the optimum system can be found from

$$\min \ \Pr\left(F_{system}\right) \tag{4.19}$$

where

$$\Pr\left(F_{system}\right) = \left(\Pr\left(F_{t_1}\right) + \Pr\left(F_{t_2}\right) + \ldots + \Pr\left(F_{t_n}\right)\right) - \Pr\left(F_{t_1} \cap F_{t_2} \cap \ldots \cap F_{t_n}\right)$$

4.4 Case Study 1 – Servo with One Random Design Variable

Suppose the servo presented in Chapter 3 contains one random design variable, winding resistance denoted as v_1. However, for an optimum system, the performance of the angular position of the servo at specific times is important. Like the previous examples in chapter 3, three levels of v_1

$$\begin{bmatrix} v_1^{low} \\ v_1^{med} \\ v_1^{high} \end{bmatrix} = \begin{bmatrix} 8.00\times 10^{-3} \\ 8.80\times 10^{-3} \\ 9.68\times 10^{-3} \end{bmatrix}$$

are used to generate a sample of angular position time histories

$$\mathbf{Z} = \begin{bmatrix} 0.0937 & 0.3161 & 0.5701 & \dots & 0.9990 \\ 0.1025 & 0.3425 & 0.6105 & \dots & 1.0009 \\ 0.1120 & 0.3702 & 0.6513 & \dots & 1.0002 \end{bmatrix}_{3\times 20}$$

The graphical plot of the experimental responses is shown in Figure 4-1 where 'low', 'medium' and 'high' refers to the three levels of v_1. After SVD of **Z**, **D** becomes

$$\mathbf{D} = \begin{bmatrix} -4.2313 & 0.0832 & 0.0022 \\ -4.2558 & -0.0009 & -0.0045 \\ -4.2750 & -0.0815 & 0.0023 \end{bmatrix}_{3\times 3}$$

and since **D** is so small, metamodels are developed for all three columns.

The response surface models are

$$y_1 = -4.0248 - 25.9654 v_1 \tag{4.20}$$

$$y_2 = 0.8649 - 97.9526 v_1 \tag{4.21}$$

$$y_3 = -0.0016 + 0.1861 v_1 \tag{4.22}$$

and the Kriging models are

$$y_1 = -4.0248 - 25.9654 v_1 + \left(\boldsymbol{\gamma}^*(\mathbf{v})\right)_1^T \boldsymbol{\Gamma}_1^{-1} \mathbf{E}_1 \tag{4.23}$$

$$y_2 = 0.8649 - 97.9526 v_1 + \left(\boldsymbol{\gamma}^*(\mathbf{v})\right)_2^T \boldsymbol{\Gamma}_2^{-1} \mathbf{E}_2 \tag{4.24}$$

$$y_3 = -0.0016 + 0.1861 v_1 + \left(\boldsymbol{\gamma}^*(\mathbf{v})\right)_3^T \boldsymbol{\Gamma}_3^{-1} \mathbf{E}_3 \tag{4.25}$$

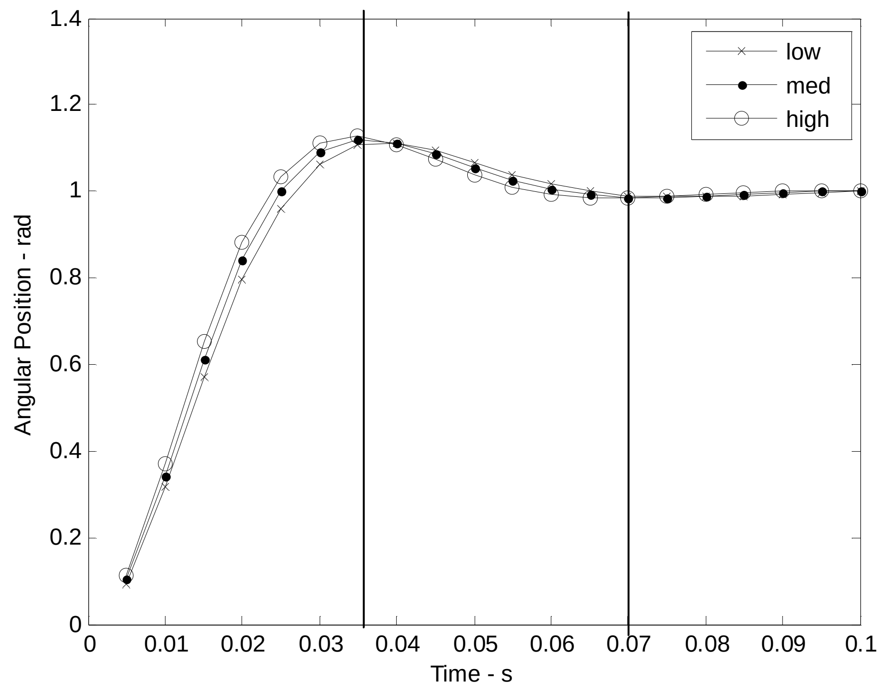

Figure 4-1 showing the Angular Position Response of the Experimental Design.

R^2 was then calculated at each time step to determine how well each model fits the experimental data. These results are shown in Table (4-1).

For an optimum system, we want to control the overshoot of the response at t = 0.035s and ensure that the response "settles" at 0.070s as seen from Figure (4-1). First, robust design is done using only one time and then later, both times are used to find optimum parameters. Before robust design calculations are performed, a suitable model at this time has to be found; therefore, reference is made to Table (4-1). At t = 0.035s, the response surface model has a high R^2 of 0.9478; however, the Kriging model exactly predicts the experimental data. The Kriging model would be better to use in robust design calculations but the response surface model is easier. Since R^2 for the RSM is fairly large, the RSM would be acceptable for use in these calculations. For this case study, the two models are now used to show how the design calculations are performed using the different metamodels to find the optimum system.

Table 4-1 showing R^2 over time for each model.

	Time	0.005	0.010	0.015	0.020	0.025	0.030	0.035	0.040
R^2	**RSM**	1.0000	0.9998	0.9994	0.9984	0.9961	0.9889	0.9478	0.6566
R^2	**Kriging**	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	Time	**0.045**	**0.050**	**0.055**	**0.060**	**0.065**	**0.070**	**0.075**	**0.080**
R^2	**RSM**	0.9920	0.9999	0.9984	0.9929	0.9771	0.9102	0.0017	0.9561
R^2	**Kriging**	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	Time	**0.085**	**0.090**	**0.095**	**0.100**				
R^2	**RSM**	0.9970	0.9992	0.9888	0.9679				
R^2	**Kriging**	1.0000	1.0000	1.0000	1.0000				

4.4.1 Moments of the Response using the RSM

A suitable system is desired such that the angular position response at t = 0.035s falls between $\zeta_U = 1.10$ and $\zeta_L = 1.05$. However, before optimum parameters are found, the mean and variance of the response at each time should be determined in order to have an idea of the distribution of the data. The mean and variance of v_1 are initially $\mu_1 = 0.0088$ and $\sigma_1^2 = \left(2.93 \times 10^{-4}\right)^2$ where the variance is calculated using

$$\sigma_1^2 = \left(\frac{tol\%}{300}\mu_1\right)^2 \tag{4.26}$$

with a 10% tolerance.

In order to calculate the mean and variance of the response at a specific time, equations (4.14) and (4.15) are used. To illustrate this procedure, consider t = 0.005s, the RSM model and equation (4.17) to find the first order derivatives at this single time step time

$$\frac{\partial f_{t_n}(\mathbf{v})}{\partial \mathbf{v}} = [-25.9654 \quad -97.9526 \quad 0.1861][-0.0242 \quad -0.1046 \quad 0.0885]^T \tag{4.27}$$

$$\frac{\partial f_{t_n}(\mathbf{v})}{\partial \mathbf{v}} = 10.8907$$

From equation (4.14), the mean of the response at t_1 is

$$\mu_{z,t_1} = 0.1024 + 0 = 0.1024$$

and from equation (4.15), the variance of the response at t_1 is

$$\sigma_{z,t_1}^2 = 10.8907 \times \left(2.93 \times 10^{-4}\right)^2 \times 10.8907 = 0.0102 \times 10^{-3}$$

4.4.2 Moments of the Response using the Kriging model

Now, using the Kriging model, the first and second order derivatives are

$$\frac{\partial y_1}{\partial v_1} = -25.9654 + \left(\frac{\partial \gamma^*(\mathbf{v})}{\partial v_1}\right)_1^T \mathbf{\Gamma}_1^{-1}\mathbf{E}_1 \tag{4.28}$$

$$\frac{\partial^2 y_1}{\partial v_1^2} = \left(\frac{\partial^2 \gamma^*(\mathbf{v})}{\partial v_1^2}\right)_1^T \mathbf{\Gamma}_1^{-1}\mathbf{E}_1 \tag{4.29}$$

$$\frac{\partial y_2}{\partial v_1} = -97.9526 + \left(\frac{\partial \gamma^*(\mathbf{v})}{\partial v_1}\right)_2^T \mathbf{\Gamma}_2^{-1}\mathbf{E}_2 \tag{4.30}$$

$$\frac{\partial^2 y_2}{\partial v_1^2} = \left(\frac{\partial^2 \gamma^*(\mathbf{v})}{\partial v_1^2}\right)_2^T \mathbf{\Gamma}_2^{-1}\mathbf{E}_2 \tag{4.31}$$

$$\frac{\partial y_3}{\partial v_1} = 0.1861 + \left(\frac{\partial \gamma^*(\mathbf{v})}{\partial v_1}\right)_3^T \mathbf{\Gamma}_3^{-1}\mathbf{E}_3 \tag{4.32}$$

$$\frac{\partial^2 y_3}{\partial v_1^2} = \left(\frac{\partial^2 \gamma^*(\mathbf{v})}{\partial v_1^2}\right)_3^T \mathbf{\Gamma}_3^{-1}\mathbf{E}_3 \tag{4.33}$$

where $\gamma^*(\mathbf{v})$ is a function of the design variables of the correlation between $\mathbf{v}_0$ and the training points

$$\gamma^*(\mathbf{v}) = \exp\left(-\theta_1(\mathbf{v}_1 - \mathbf{x}_1)^2\right) \tag{4.34}$$

and $\mathbf{x}_1$ is a matrix representing the training points of $\mathbf{v}_1$. Also, the first and second derivatives at this first time step, (0.005s), are found using equations (4.35) and (4.36)

$$\frac{\partial f_{t_1}(\mathbf{v})}{\partial v_1} = \begin{bmatrix} \frac{\partial y_1}{\partial v_1} & \frac{\partial y_2}{\partial v_1} & \frac{\partial y_3}{\partial v_1} \end{bmatrix} \mathbf{V}_{t_1}^T \tag{4.35}$$

$$\frac{\partial^2 f_{t_1}(\mathbf{v})}{\partial v_1^2} = \begin{bmatrix} \frac{\partial^2 y_1}{\partial v_1^2} & \frac{\partial^2 y_2}{\partial v_1^2} & \frac{\partial^2 y_3}{\partial v_1^2} \end{bmatrix} \mathbf{V}_{t_1}^T \tag{4.36}$$

After substituting the mean of v_1 into equations (4.28) to (4.36) and then the results obtained from these equations into equations (4.14) and (4.15), the mean and variance at 0.005s using the Kriging model was found to be

$$\mu_{z,t_1} = 0.1025 + \frac{1}{2}\left(-5.9540 \times 10^{3}\right)\left(8.6044 \times 10^{-8}\right) = 0.1022$$

$$\sigma_{z,t_1}^{2} = 10.8912 \times \left(2.93 \times 10^{-4}\right)^{2} \times 10.8912 = 0.0102 \times 10^{-3}$$

The first and second moments of the response at the remaining time steps are shown in table 4-2. These first and second moments can be used to calculate the probability of conformance at each time step using equation (2.37). However, a simpler approach is to use a Monte Carlo simulation where a large sample of responses is generated from the mean and variance of v_1.

From table (4-2), at 0.035s, the mean and variance of the response would indicate that the probability of conformance given the limits specified previously would be 0. Using a Monte Carlo simulation to check this assumption and a sample of 10000 values of $v_{1,}$ the probability of conformance was found to be 0 using both the Kriging and RSM models. In order to improve this probability, robust design is done.

Table 4-2 showing first and second moments at each time step calculated using Kriging and RSM.

Time	RSM		Kriging	
	$\mu_{Z,t}$	$\sigma_{Z,t}^2 \times 10^{-3}$	$\mu_{Z,t}$	$\sigma_{Z,t}^2 \times 10^{-3}$
0.005	0.1024	0.0102	0.1029	0.0102
0.010	0.3421	0.0892	0.3425	0.0892
0.015	0.6093	0.2007	0.6080	0.2007
0.020	0.8373	0.2361	0.8329	0.2361
0.025	0.9968	0.1709	0.9893	0.1709
0.030	1.0859	0.0737	1.0763	0.0737
0.035	1.1177	0.0123	1.1081	0.0123
0.040	1.1113	0.0008	1.1031	0.0008
0.045	1.0852	0.0143	1.0799	0.0143
0.050	1.0534	0.0255	1.0514	0.0255
0.055	1.0250	0.0242	1.0261	0.0242
0.060	1.0043	0.0152	1.0072	0.0152
0.065	0.9920	0.0062	0.9958	0.0062
0.070	0.9868	0.0012	0.9905	0.0012
0.075	0.9866	0	0.9895	0
0.080	0.9893	0.0004	0.9909	0.0004
0.085	0.9929	0.0010	0.9935	0.0010
0.090	0.9963	0.0010	0.9959	0.0010
0.095	0.9989	0.0006	0.9978	0.0006
0.100	1.0006	0.0003	0.9993	0.0003

4.4.3 Parameter Design Using RSM – Balancing Conformance Indices

Now, the optimum mean of v_1 will now be calculated by first using the RSM. For robust design, the optimization function becomes

$$\min \ Q = e^{-\alpha_1 \beta_1} + e^{-\alpha_2 \beta_2} \tag{4.37}$$

where the constants α_1, β_1, α_2 and β_2 are found using equations (2.42) and (2.43).The RSM at t = 0.035s is

$$y_{t=0.035s} = 1.0126 + 11.9468 v_1 \tag{4.38}$$

with the limit state functions being (in **u**-space)

$$g_1 = 1.10 - \left(1.0126 + 11.9468\left(\mu_1 + \sigma_1 u_1\right)\right) \tag{4.39}$$

$$g_2 = \left(1.0126 + 11.9468\left(\mu_1 + \sigma_1 u_1\right)\right) - 1.05 \tag{4.40}$$

then, from equations (2.42), (2.43), (2.44) and (2.45)

$$a_0 = 1.0126 \qquad a_1 = 11.9468 \qquad b_0 = 1$$

$$h_{0U} = a_0 + a_1\mu_1 - 1.10 \qquad h_{0L} = a_0 + a_1\mu_1 - 1.05 \qquad h_1 = a_1\sigma_1$$

$$\beta_1 = \frac{\left|-h_{0U}\right|}{\sqrt{h_1^2}} = \frac{\left|-\left(1.0126 + 11.9468\mu_1 - 1.10\right)\right|}{\sqrt{\left(11.9468\sigma_1\right)^2}} \tag{4.41}$$

$$\beta_2 = \frac{\left|-h_{0L}\right|}{\sqrt{h_1^2}} = \frac{\left|-\left(1.0126 + 11.9468\mu_1 - 1.05\right)\right|}{\sqrt{\left(11.9468\sigma_1\right)^2}} \tag{4.42}$$

with

$$\alpha_1 = sign\left(g_1\left(u = 0\right)\right)$$

$$\alpha_2 = sign\left(g_2\left(u = 0\right)\right)$$

After minimizing Q using 'fsolve' in MATLAB, the optimum design was found to be μ = 0.0064.

Using a Monte Carlo simulation, the probability of conformance was found to be

$$\Pr\left(S_{t=0.035s}\right) = \frac{10000}{10000} = 1$$

The angular displacement estimates from the Monte Carlo simulation are shown in Figure (4-2). For the upper and lower specifications of 1.10 and 1.05, it can be seen from Figure (4-2) that all the responses using the theoretical optimum of v_1 lies within this range with a mean value of about 1.09.

4.4.4 Parameter Design using Kriging Model – Probability Objective Function

Using the above set of equations is easy for linear or bi-linear models. However, the non-linear Kriging model requires a different approach. The objective becomes maximizing the probability of success or minimizing the probability of failure where the probability of success at 0.035s (7^{th} time step) is calculated using

$$\max \ \Pr\left(S_{t_7}\right) = 1 - \Phi\left(-\frac{\left(\zeta_{U,t_7} - \mu_{z,t_7}\right)}{\sigma_{z,t_7}}\right) - \Phi\left(-\frac{\left(\mu_{z,t_7} - \zeta_{L,t_7}\right)}{\sigma_{z,t_7}}\right) \tag{4.43}$$

where ζ_{U,t_7} and ζ_{L,t_7} are the upper and lower limits respectively at 0.035s and μ_{z,t_7} and σ_{z,t_7} are the mean and standard deviation of the response at 0.035s. Using this equation and the function 'fsolve' in MATLAB, the optimum v_1 was found to be 0.0064.

From the Monte Carlo Simulation, the probability of conformance was found to be

$$\Pr\left(S_{t_7}\right) = \frac{9996}{10000} = 0.9996$$

Figure (4-3) shows the data from the Monte Carlo simulation using as a histogram and it can be seen that the data follows a similar distribution as that of Figure (4-2).

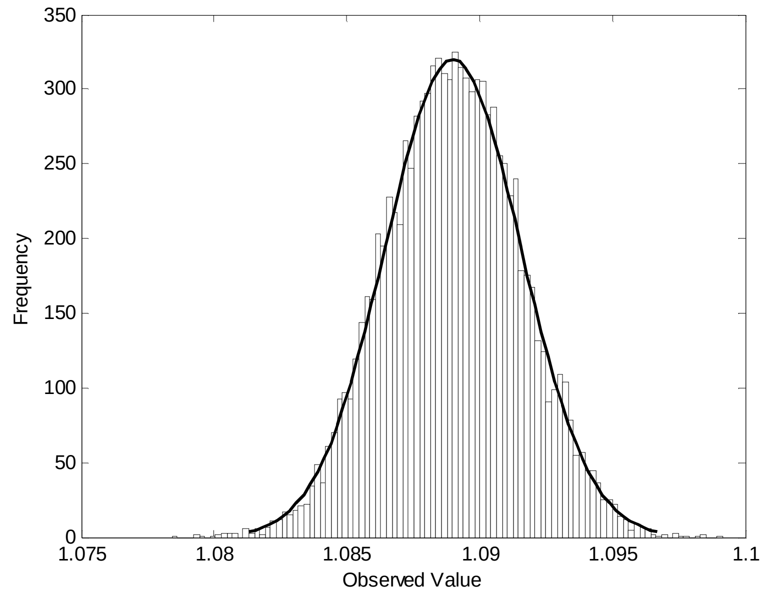

Figure 4-2 showing a histogram of the results from Monte Carlo at the theoretical optimum mean of v_1 and RSM

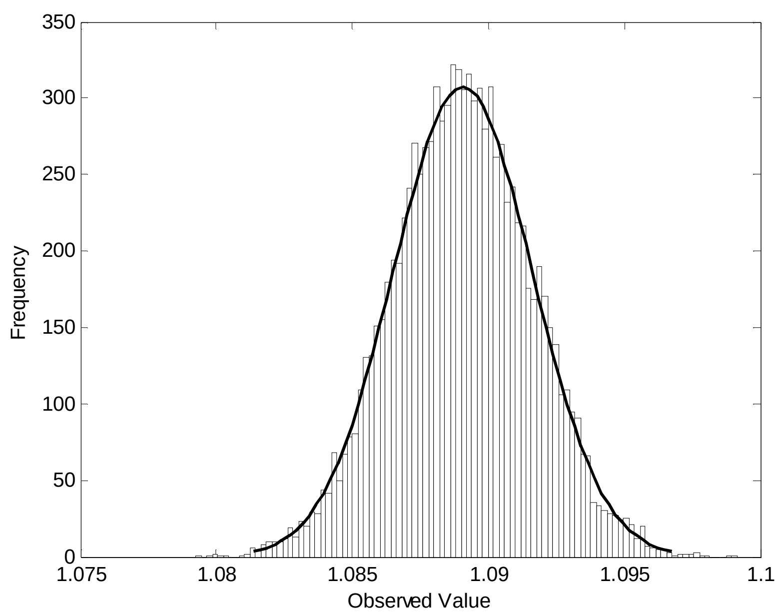

Figure 4-3 showing a histogram of the results from Monte Carlo using the theoretical optimum and Kriging model.

4.4.5 Robust Design Using Multiple Time Steps

The angular position response of the servo contains two important time steps; the 'rise' time and the 'settling' time. The 'rise' time is where the response overshoots and we want to limit the amount of overshoot that takes place. This design was done previously; however, at the 'settling' time, we also want to ensure that the response stabilizes at t = 0.070s and at this time the specifications are $0.995 \leq \theta_{t=0.070s} \leq 1.005$. Since the RSM provides a good fit of the experimental design, it will be used to determine the optimum design variables. The RSM at this time is

$$y_{t=0.07s} = 1.0196 - 3.7179v_1 \tag{4.44}$$

In order to find the best design variables such that both specifications are met, constraints can be introduced to equation (4.43) and these constraints specify that the probability of failure at other times does not exceed some stated amount. The optimization problem then becomes

$$\begin{gathered} \min \ \Pr\left(F_{t=0.035s}\right) \\ \text{subject to} \\ \Pr\left(F_{t=0.07s}\right) \leq 0.10 \end{gathered} \tag{4.45}$$

where the probability of nonconformance is calculated using either equation (2.37) or (2.47). The optimum mean of v_1 was found to be 0.0063. When using this result to generate a sample of 10000 results using the Monte Carlo simulation, the probabilities of conformance are

$$\Pr\left(1.05 \leq \theta_{t=0.035s} \leq 1.10\right) = 1$$

$$\Pr\left(0.995 \leq \theta_{t=0.070s} \leq 1.005\right) = 0.927$$

The results from the Monte Carlo simulation are shown in Figures (4-4) and (4-5). From the figures, the mean at 0.035s and 0.070s are 1.0875 and 0.996. It can also be seen from Figure (4-5) that some of the observed values are less than 0.995. However, the overall probability of conformance is acceptable.

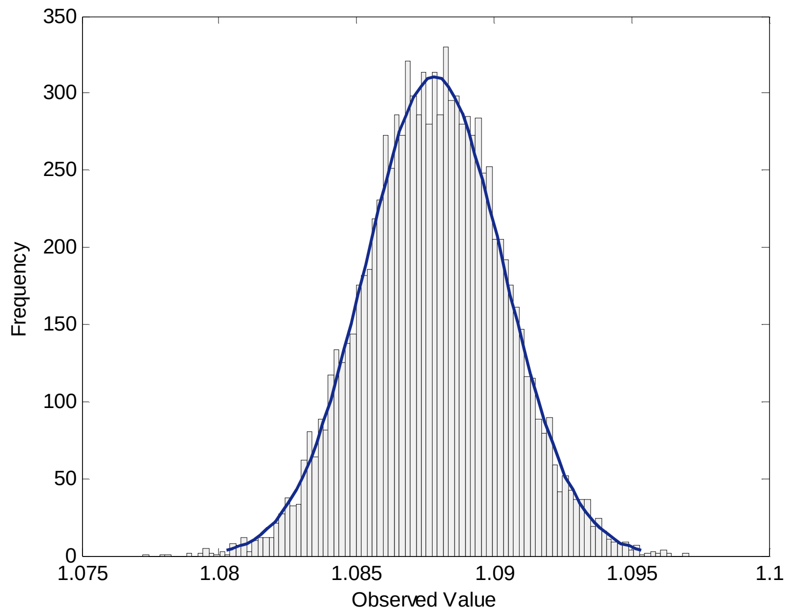

Figure 4-4 showing a histogram of the distribution of the response at 0.035s from Monte Carlo using the theoretical optimum mean

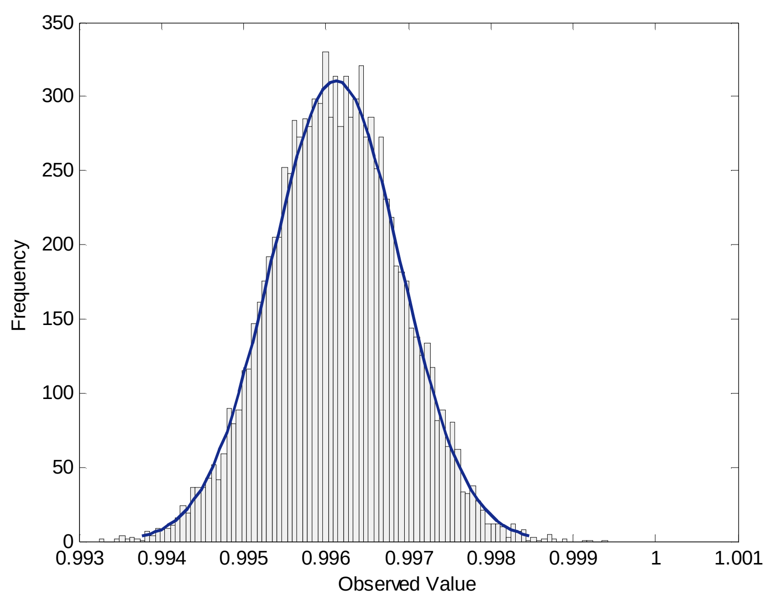

Figure 4-5 showing a histogram of the distribution of the response at 0.070s from Monte Carlo using the theoretical optimum mean.

4.5 Case Study 2- Servo with Three Random Design Variables

The variables winding inductance, v_1, winding resistance, v_2, and torque constant, v_3 are now considered to be random and the response time histories at the various combinations of the three levels of each design variable

$$\begin{pmatrix} v_1^{low} \\ v_1^{med.} \\ v_1^{high} \end{pmatrix} = \begin{pmatrix} 1.00\times10^{-3} \\ 1.10\times10^{-3} \\ 1.20\times10^{-3} \end{pmatrix} \qquad \begin{pmatrix} v_2^{low} \\ v_2^{med.} \\ v_3^{high} \end{pmatrix} = \begin{pmatrix} 8.00\times10^{-3} \\ 8.80\times10^{-3} \\ 9.68\times10^{-3} \end{pmatrix} \qquad \begin{pmatrix} v_3^{low} \\ v_3^{med.} \\ v_3^{high} \end{pmatrix} = \begin{pmatrix} 4.00 \\ 4.40 \\ 4.84 \end{pmatrix}$$

are recorded into **Z**

$$\mathbf{Z} = \begin{bmatrix} 0.0937 & 0.3161 & 0.5701 & \dots & 0.9990 \\ 0.0861 & 0.2911 & 0.5289 & \dots & 0.9968 \\ 0.0791 & 0.2677 & 0.4898 & \dots & 0.9954 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0.0934 & 0.3136 & 0.5637 & \dots & 0.9987 \end{bmatrix}_{27x20}$$

Metamodels were found for the first eight columns of **D;** however, the first four are shown below. The response surface models obtained are

$$y_1 = -4.2973 - 4.8982v_1 - 29.6885v_2 + 0.0777v_3 \tag{4.46}$$

$$y_2 = 0.1068 - 2.6635v_1 + 98.4941v_2 - 0.2207v_3 \tag{4.47}$$

$$y_3 = 0.0806 + 15.4473v_1 - 5.9340v_2 - 0.0103v_3 \tag{4.48}$$

$$y_4 = 0.0073 + 7.7310v_1 - 0.9482v_2 - 0.0017v_3 \tag{4.49}$$

and the Kriging models are

$$y_1 = -4.2847 - 4.9833v_1 - 29.8292v_2 + 0.0751v_3 + \left(\boldsymbol{\gamma}^*(\mathbf{v})\right)_1^T \boldsymbol{\Gamma}_1^{-1}\mathbf{E}_1 \tag{4.50}$$

$$y_2 = 0.1135 - 2.7075v_1 + 98.7854v_2 - 0.2229v_3 + \left(\boldsymbol{\gamma}^*(\mathbf{v})\right)_2^T \boldsymbol{\Gamma}_2^{-1}\mathbf{E}_2 \tag{4.51}$$

$$y_3 = 0.0516 + 15.5365v_1 - 5.9147v_2 - 0.0033v_3 + \left(\boldsymbol{\gamma}^*(\mathbf{v})\right)_3^T \boldsymbol{\Gamma}_3^{-1}\mathbf{E}_3 \tag{4.52}$$

$$y_4 = 0.0085 + 7.7285v_1 - 0.9177v_2 - 0.0020v_3 + \left(\boldsymbol{\gamma}^*(\mathbf{v})\right)_4^T \boldsymbol{\Gamma}_4^{-1}\mathbf{E}_4 \tag{4.53}$$

The coefficient of determination at each time step calculated using both metamodels are shown in Table (4-3). Later, these values will be used to determine the metamodel more desirable for use in robust design.

Table 4-3 showing R^2 at each time step calculated using both metamodels for Servo with multiple random design variables.

Time	**0.005**	**0.010**	**0.015**	**0.020**	**0.025**	**0.030**	**0.035**	**0.040**
RSM	0.997	0.997	0.998	0.999	0.998	0.991	0.961	0.755
Kriging	1	1	1	1	1	1	1	1
Time	**0.045**	**0.050**	**0.055**	**0.060**	**0.065**	**0.070**	**0.075**	**0.080**
RSM	0.525	0.925	0.986	0.997	0.988	0.950	0.816	0.310
Kriging	1	1	1	1	1	1	1	1
Time	**0.085**	**0.090**	**0.095**	**0.100**				
RSM	0.436	0.873	0.970	0.989				
Kriging	1	1	1	1				

The FOS factors at each time step are then found using equation (4.1) and using these FOS factors and equation (4.2), the normalized sensitivities can then be calculated for each variable at each time step. These results are shown in Table (4-4).

In order to show how these sensitivities are calculated for multiple random design variables, consider t = 0.005s and the response surface models. The FOS factors are calculated as shown in equation (4.54)

$$\left[\frac{\partial f_{t_1}(\mathbf{v})}{\partial \mathbf{v}^T}\right]_{\bar{\mathbf{v}}} = \begin{bmatrix} \frac{\partial y_1}{\partial v_1} & \frac{\partial y_2}{\partial v_1} & \cdots & \frac{\partial y_8}{\partial v_1} \\ \frac{\partial y_1}{\partial v_2} & \frac{\partial y_2}{\partial v_2} & \cdots & \frac{\partial y_8}{\partial v_2} \\ \frac{\partial y_1}{\partial v_3} & \frac{\partial y_1}{\partial v_3} & \cdots & \frac{\partial y_8}{\partial v_3} \end{bmatrix}_{\bar{\mathbf{v}}} \begin{bmatrix} -0.0223 & 0.0840 & \ldots & 0.0997 \end{bmatrix}^T \qquad (4.54)$$

where the first matrix is evaluated at the mean of the design variables and the second matrix is the row in **V** corresponding to 0.005s. The first row of the first matrix of equation (4.54) corresponds to the first-order derivatives with respect to v_1. All this information is then substituted into equation (4.2) to get the normalized sensitivities of each design variable at each time step.

Consider now the same time and the Kriging models. The first derivative with respect to v_1 of the metamodel of $\mathbf{D}_1$ is

$$\frac{\partial y_1}{\partial v_1} = -4.9833 + \frac{\partial \gamma^*(\mathbf{v})}{\partial v_1}\mathbf{\Gamma}_1^{-1}\mathbf{E}$$

After finding the required derivatives and again using equation (4.7) and the first row of $\mathbf{V}^T$ for 0.005s, the FOS factors can be calculated. The normalized sensitivities are then ultimately calculated using equation (4.2).

From Table (4-4), v_1 has the smallest effect on the angular position response of the servo, whereas the magnitudes of the other two variables are about the same. Also, comparison of the sensitivities calculated using Kriging and RSM with the analytical results showed that in some cases the Kriging model seemed to perform better but in others, the RSM performed better. Therefore, for this example, no model always gave better results. These results will just give the analyst an idea as to how the response will behave when trying to find optimal results. Since the effect of v_1 is so small, v_1 will be considered to be a constant at its mean value and v_2 and v_3 will be used in parameter design.

Table 4-4 showing the normalized sensitivities calculated at each time step for angular position response of the servo.

	v_1			v_2			v_3		
Time	**RSM**	**Krig.**	**Actual**	**RSM**	**Krig.**	**Actual**	**RSM**	**Krig.**	**Actual**
0.005	-0.085	-0.088	-0.085	0.937	0.943	0.939	-0.873	-0.878	-0.811
0.010	-0.031	-0.032	-0.032	0.841	0.845	0.835	-0.852	-0.865	-0.791
0.015	-0.011	-0.012	-0.012	0.721	0.724	0.709	-0.769	-0.782	-0.723
0.020	-0.002	-0.002	-0.001	0.585	0.587	0.567	-0.653	-0.664	-0.624
0.025	0.004	0.004	0.004	0.440	0.441	0.415	-0.516	-0.525	-0.506
0.030	0.006	0.006	0.007	0.294	0.294	0.264	-0.370	-0.376	-0.376
0.035	0.007	0.007	0.007	0.156	0.156	0.124	-0.224	-0.228	-0.241
0.040	0.006	0.006	0.007	0.038	0.039	0.007	-0.092	-0.094	-0.115
0.045	0.005	0.004	0.005	-0.051	-0.051	-0.079	0.015	0.015	-0.008
0.050	0.006	0.003	0.003	-0.106	-0.106	-0.127	0.091	0.094	0.071
0.055	0.0008	0.0004	0.001	-0.115	-0.126	-0.139	0.124	0.127	0.117
0.060	-0.0007	-0.001	-0.001	-0.126	-0.119	-0.121	0.128	0.131	0.130
0.065	-0.001	-0.002	-0.001	-0.118	-0.092	-0.087	0.108	0.111	0.117
0.070	-0.0018	-0.002	-0.002	-0.092	-0.059	-0.048	0.076	0.079	0.089
0.075	-0.0016	-0.002	-0.001	-0.059	-0.027	-0.013	0.043	0.045	0.056
0.080	-0.0012	-0.001	-0.002	-0.003	-0.003	0.011	0.014	0.016	0.024
0.085	-0.0007	-0.0005	-0.001	0.013	0.013	0.024	-0.006	-0.004	0.001
0.090	-0.0004	-0.0003	0	0.020	0.021	0.028	-0.017	-0.016	-0.014
0.095	-0.0003	-0.0002	0	0.021	0.022	0.025	-0.021	-0.020	-0.022
0.100	0.0002	0.0002	0	0.018	0.018	0.019	-0.020	-0.018	-0.022

4.5.1 Moments of the Response

Now, to find the mean and variance of the response, the first-order derivatives have already been obtained and since the RSMs are linear, the second-order derivative is 0. All the required information is then substituted into equations (4.8) and (4.9) to obtain the first and second moments of the response.

When using the Kriging model to calculate the moments, the first and second derivatives of the Kriging model representing $\mathbf{D}_1$ are

$$\frac{\partial y_1}{\partial v_1} = -4.9833 + \frac{\partial \gamma^*(\mathbf{v})}{\partial v_1}\mathbf{\Gamma}_1^{-1}\mathbf{E}$$

$$\frac{\partial^2 y_1}{\partial v_1^2} = \frac{\partial^2 \gamma^*(\mathbf{v})}{\partial v_1^2}\mathbf{\Gamma}_1^{-1}\mathbf{E}$$

where $\gamma^*(\mathbf{v}) = \exp\left(-\theta_1(v_1 - \mathbf{x}_1)^2\right)\exp\left(-\theta_2(v_2 - \mathbf{x}_2)^2\right)\exp\left(-\theta_3(v_3 - \mathbf{x}_3)^2\right)$. After substituting the required information into equations (4.8) and (4.9), the moments of the response are calculated at each time step. The results obtained using both metamodels were calculated and are found in Table (4-5).

Table 4-5 showing the First and Second Moments of Angular Position Response at Each Time Step.

	RSM		**Kriging**	
Time	**μ**	**σ^2**	**μ**	**σ^2**
0.005	0.0949	0.0161	0.0942	0.0166
0.010	0.3195	0.1590	0.3179	0.1682
0.015	0.5742	0.3981	0.5723	0.4282
0.020	0.7972	0.5322	0.7959	0.5806
0.025	0.9599	0.4636	0.9599	0.5134
0.030	1.0580	0.2738	1.0595	0.3087
0.035	1.1016	0.0997	1.1042	0.1156
0.040	1.1065	0.0135	1.1096	0.0162
0.045	1.0890	0.0037	1.0921	0.0020
0.050	1.0623	0.0245	1.0644	0.0245
0.055	1.0360	0.0373	1.0376	0.0396
0.060	1.0145	0.0347	1.0153	0.0382
0.065	1.0000	0.0224	1.0000	0.0254
0.070	0.9920	0.0102	0.9915	0.0120
0.075	0.9891	0.0028	0.9885	0.0034
0.080	0.9896	0.0002	0.9890	0.0003
0.085	0.9919	0.0002	0.9915	0.0002
0.090	0.9946	0.0008	0.9946	0.0008
0.095	0.9972	0.0010	0.9973	0.0011
0.100	0.9991	0.0008	0.9995	0.0009

4.5.2 Probability of Conformance Calculations

For an optimum system, the upper and lower specifications at t = 0.035s are $\zeta_U = 1.10$ and $\zeta_L = 1.05$. The means and variances of each design variable are

$$\begin{bmatrix} \mu_{v_1} \\ \mu_{v_2} \\ \mu_{v_3} \end{bmatrix} = \begin{bmatrix} 0.0011 \\ 0.0088 \\ 4.4 \end{bmatrix} \qquad \begin{bmatrix} \sigma_{v_1}^2 \\ \sigma_{v_2}^2 \\ \sigma_{v_3}^2 \end{bmatrix} = \begin{bmatrix} 3.67\times10^{-5} \\ 2.93\times10^{-4} \\ 0.15 \end{bmatrix}$$

where the variance is calculated using

$$\sigma_{v_i}^2 = \frac{10\mu_{v_i}}{300}$$

and the probability of conformance at 0.035s is

$$\Pr\left(S_{t_7}^{RSM}\right) = 1 - \left(\Phi\left(\frac{-(1.10-1.0966)}{\sqrt{0.0997\times10^{-3}}}\right) + \Phi\left(-\frac{(1.0966-1.05)}{\sqrt{0.0997\times10^{-3}}}\right)\right) = 1\text{- } (0.3667 + 0) = 0.6333$$

A sample of 10000 design variable combinations was generated using the means and variances of each variable. Among the 10000 responses of angular position at t = 0.035s, the total number of responses that conformed to specifications is 6288 and, from the Monte Carlo simulation, the probability of conformance was found to be

$$\Pr\left(S_{t_7}^{MC}\right) = \frac{6288}{10000} = 0.6288$$

Now, using the second moment method and the Kriging model,

$$\Pr\left(S_{t_7}^{kriging}\right) = 1 - \left(\Phi\left(\frac{-(1.10-1.0942)}{\sqrt{0.0839\times10^{-3}}}\right) + \Phi\left(-\frac{(1.0942-1.05)}{\sqrt{0.0839\times10^{-3}}}\right)\right) = 1\text{-}(0.2634+0) = 0.7366$$

and from Monte Carlo using the Kriging metamodels to estimate responses, the probability of conformance was found to be

$$\Pr\left(S_{t_7}^{MC}\right) = 0.6386$$

4.5.3 Parameter Design using the RSM – Balancing of Conformance Indices

In order to increase the probability of conformance, parameter design is now performed to select the means of the design variables that will yield satisfactory results. From the sensitivity calculations, v_1 has a very small effect on angular position; therefore this variable is considered to be constant. The parameter design method that will be used is balancing conformance indices. Although the RSM is simpler than the Kriging model and is preferred for robust design calculations, the coefficient of determination is needed to determine how well it models the experimental data. From Table (4-3), the coefficient of determination calculated at 0.035s shows that the RSM is quite acceptable and will now be used for parameter design.

Using equation (4.12), the RSM model for the response at 0.035s is

$$y = 1.1635 + 6.7901 v_1 + 19.4638 v_2 - 0.0588 v_3$$

and the objective function is

$$\min Q = e^{-\alpha_1 \beta_1} + e^{-\alpha_2 \beta_2}$$

with

$$\alpha_1 = sign(g_1(u=0))$$

$$\alpha_2 = sign(g_2(u=0))$$

$$\beta_1 = \frac{\left|-(1.16 + 6.79\mu_1 + 19.46\mu_2 + (-0.059\mu_3) - 1.10)\right|}{\sqrt{(6.798\sigma_1)^2 + (19.46\sigma_2)^2 + (-0.059\sigma_3)^2}}$$

$$\beta_2 = \frac{\left|-(1.16 + 6.79\mu_1 + 19.46\mu_2 + (-0.059\mu_3) - 1.05)\right|}{\sqrt{(6.79\sigma_1)^2 + (19.46\sigma_2)^2 + (-0.059\sigma_3)^2}}$$

The optimum design was then found to be

$$\begin{bmatrix} \mu_{v_1} \\ \mu_{v_2} \\ \mu_{v_3} \end{bmatrix} = \begin{bmatrix} 0.0011 \\ 0.0063 \\ 3.9998 \end{bmatrix}$$

Using the Monte Carlo simulation, from a sample of 10000 responses at 0.035s, the probability of success was found to be

$$\Pr(S) = \frac{9844}{10000} = 0.9844$$

and a histogram of the data from the Monte Carlo simulation is shown in Figure (4-6). Given upper and lower limits of 1.10 and 1.05 respectively, the results Monte Carlo simulation seems to conform to these specifications. Although from Figure (4-6) some observed values fall below the lower specification, the overall probability of conformance is still acceptable. From the figure, the mean is approximately 1.07.

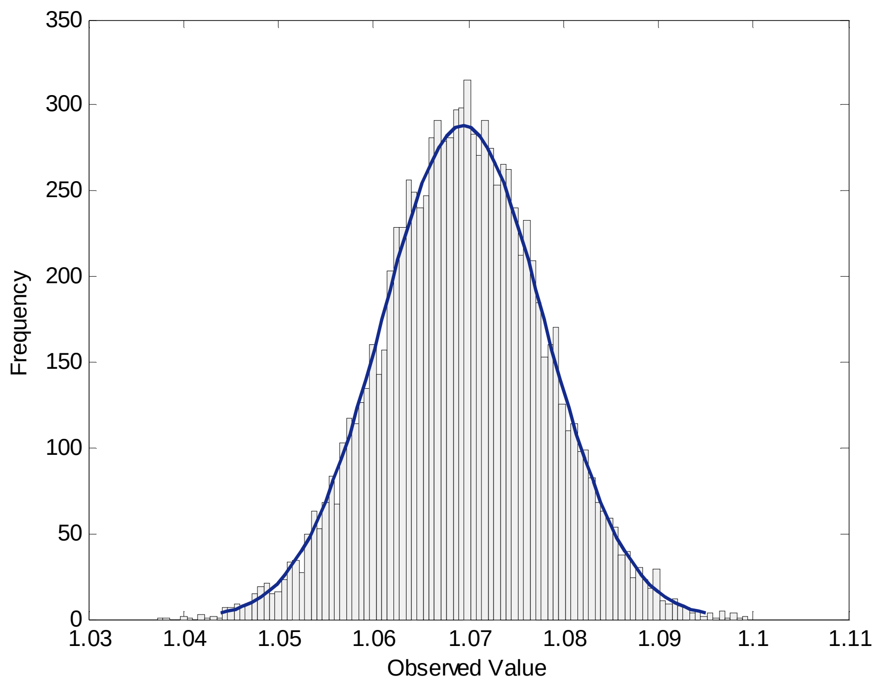

Figure 4-6 showing a histogram of the data obtained from Monte Carlo for the Servo with three random design variables at t = 0.035s.

4.5.4 Multiple Responses

We want to design the system, now considering all three random design variables with v_1 held constant, to meet the specifications previously stated at times t = 0.035s and t = 0.070s. As before, the optimization problem becomes

$$\begin{aligned} &\min \ \Pr\left(F_{t=0.035s}\right) \\ &\text{s.t.} \\ &\quad \Pr\left(F_{t=0.070s}\right) \le 0.100 \end{aligned}$$

and using 'fsolve', the optimum design was found to be $v_2 = 0.0077$ $v_3 = 4.3799$ where the probabilities of conformance at the two time steps are

$$\Pr\left(1.05 \le \theta_{t=0.035s} \le 1.10\right) = 0.9939$$

$$\Pr\left(0.995 \le \theta_{t=0.070s} \le 1.005\right) = 0.8979$$

The results from the Monte Carlo simulation show that an infeasible solution has been reached since the constraint was not met. These results are clearly seen in Figures (4-7) and (4-8). Although the responses at 0.035s fall within the specifications, many of the responses at 0.070s are greater than 1.005. On looking at Table (4-4), the problem lies in the normalized sensitivities of the variables. At 0.035s, a 1% change in v_2 would increase the response by 0.156%. However, this same percentage change at 0.070s decreases the response by -0.059%. A similar situation occurs for v_3. Therefore, it is difficult to find a set of design variables that can achieve the desired results. The designer must, therefore, be willing to compromise. Perhaps the probability of conformance at 0.035s can be reduced in order to increase the conformance at 0.070s. If this is unacceptable, then no feasible solution is achieved.

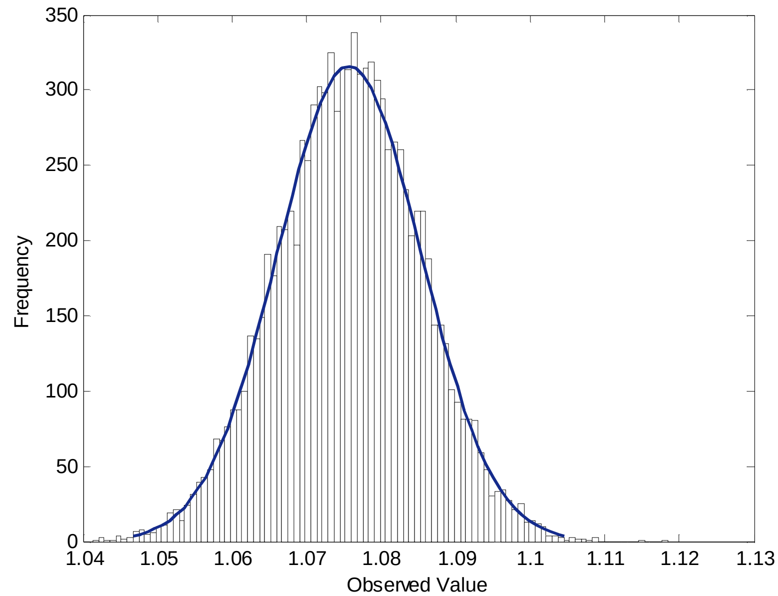

Figure 4-7 showing a histogram of the responses at 0.035s from the Monte Carlo simulation using the optimum means.

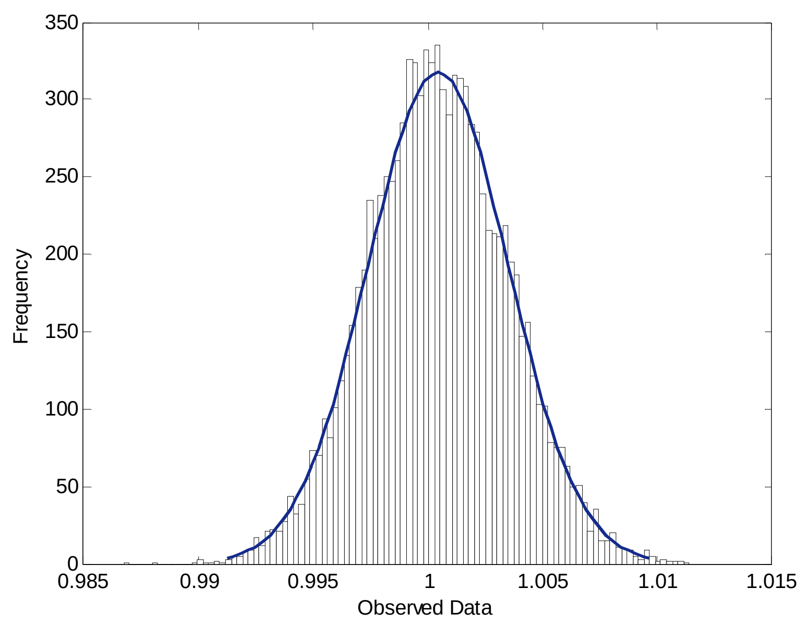

Figure 4-8 showing a histogram of the response at 0.070s from the Monte Carlo simulation using the optimum means.

4.6 Case Study 3 –Window Sign

Consider the case of finding the design variables to give an optimum performance of a mobile display (Cochin, 1980). The sign uses a windup oscillating device to attract shoppers to a shop window. The mechanism is made up of two steel spheres on either end of a rod and is hung on a thin wire that can be twisted many times without breaking. At the start of the business day, the device is wound up 4000˚ which is approximately 11 revolutions. A design is desired such that the motion of the display, at the end of the business day, decays to approximately 10˚. A diagram of the device is shown below

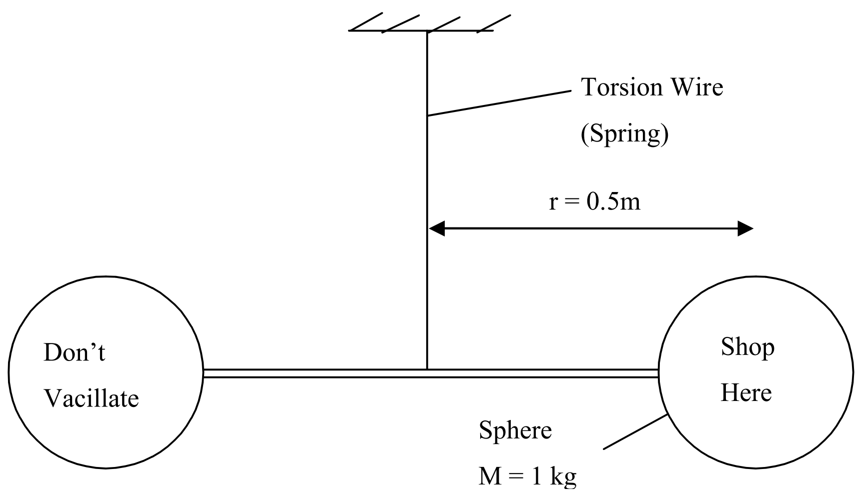

Figure 4-9 showing the mechanism of a windup oscillating display sign.

The system is torsional with a torsion spring, K, being the thin wire. This wire connects the rigid rod to ground. The system also has an initial displacement of 4000˚ and the equation of the system is

$$\theta = \theta_0 e^{-at}$$

where $a = \zeta\omega_n$, $\omega_n = \sqrt{\frac{K}{J}}$, $\zeta = \frac{D}{2J\omega_n}$ and $J = 2Mr^2$.

4.6.1 Simulation Results

The two design variables of interest are K – Nm/rad and r - m with constants M = 1kg, and θ_0 = 4000˚. The angular displacement response of the system was observed from t = 0s to t = 30000s (8.3hr) using the following low, medium and high values of each design variable

$$\begin{bmatrix} r^{low} \\ r^{med} \\ r^{high} \end{bmatrix} = \begin{bmatrix} 0.45 \\ 0.495 \\ 0.54 \end{bmatrix} \qquad \begin{bmatrix} K^{low} \\ K^{med} \\ K^{high} \end{bmatrix} = \begin{bmatrix} 1.82\times10^{-4} \\ 2.00\times10^{-4} \\ 2.20\times10^{-4} \end{bmatrix}$$

The time history responses at the nine design variable combinations were recorded and an extract of these results are show in **Z**

$$\mathbf{Z} = \begin{bmatrix} 4000 & 3656.20 & 3341.90 & \dots & 4.7271 \\ 4000 & 3623.8 & 3283 & \dots & 2.427 \\ 4000 & 3588.2 & 3218.8 & \dots & 1.571 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 4000 & 3709.3 & 3439.8 & \dots & 13.952 \end{bmatrix}_{9\times76}$$

A plot of the experimental responses are shown in Figure (4-10)

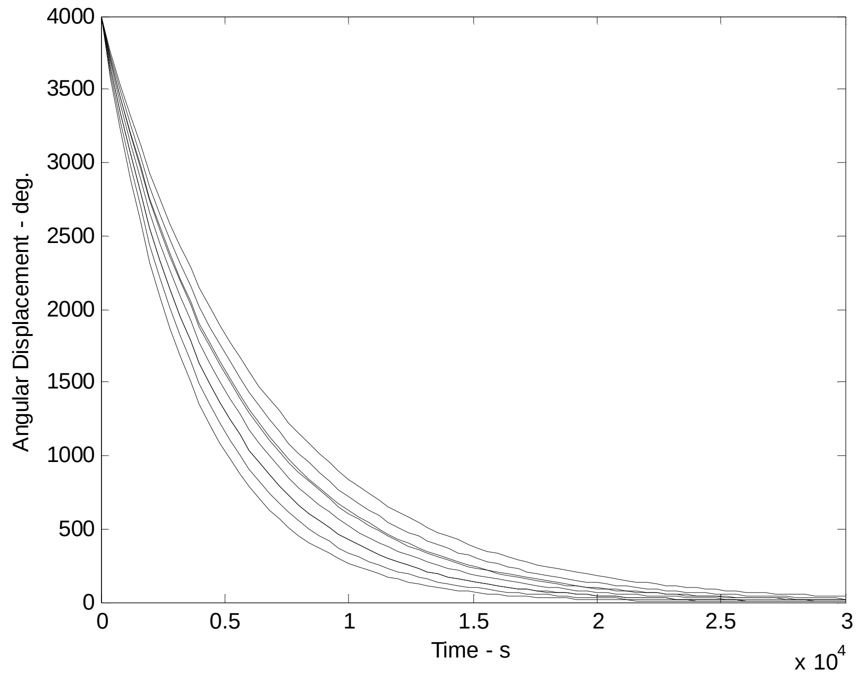

Figure 4-10 showing the experimental responses for the Display Sign

Now, a business day for this problem is considered to be 30000s; therefore, an optimal design is required such that the angular displacement at this time is between 11˚ and 9˚ with a target of 10˚. The problem with the original design variables and tolerances is that too much variability exists at the time of interest. Therefore, a set of design variables and tolerances has to be found.

Like the servo example, SVD is applied to **Z** and metamodels are developed for the significant columns of **D**. Initially, response surface models will be derived for the first four columns of **D** and the R^2 value will be calculated at t = 30000s. If the calculated R^2 is found to be acceptable, an RSM will be used for Robust Design. However, if R^2 indicated an inadequate model, then a Kriging model will be used for design. R^2 at 30000s was found to be 0.88. This value indicates that the RSM does not provide a very good model of the experimental design at t = 30000s. Using the Kriging models, R^2 was found to be 1 meaning that an exact model of the experimental design is achieved.

4.6.2 Robust Design

At the time of interest, the upper and lower specifications are 11˚ and 9˚ respectively with a target of 10˚. The Kriging model will be used for design calculations since it provides a better 'fit' of the experimental data than the response surface model. The normalized sensitivities of each design variable at this time are $v_1 = 13.0827$ and $v_2 = -7.5125$ indicating that the response is very sensitive to a change in either one of these design variables, therefore, optimum means as well as tolerances are required.

$$\min C_T = C_{LQ} + C_P$$

subject to

$$1 - \Phi\left(-\frac{\left(\zeta_{U,t_{76}} - \mu_{z,t_{76}}\right)}{\sigma_{z,t_{76}}}\right) - \Phi\left(-\frac{\left(\mu_{z,t_{76}} - \zeta_{L,t_{76}}\right)}{\sigma_{z,t_{76}}}\right) \geq 0.97$$

$$0.1 \leq tol_1 \leq 10$$

$$0.1 \leq tol_2 \leq 10$$

The optimum set of means and tolerances was then found to be

$\mu_{v_1} = 0.5003$, $\mu_{v_2} = 2 \times 10^{-4}$, $tol_{v_1} = 0.7807$ and $tol_{v_2} = 1.1338$

and from a Monte Carlo simulation, the probability of conformance is 0.9794. A histogram of the results from the Monte Carlo simulation is shown in Figure (4-11) and it is clear that the responses fall within specifications with a mean of 10.

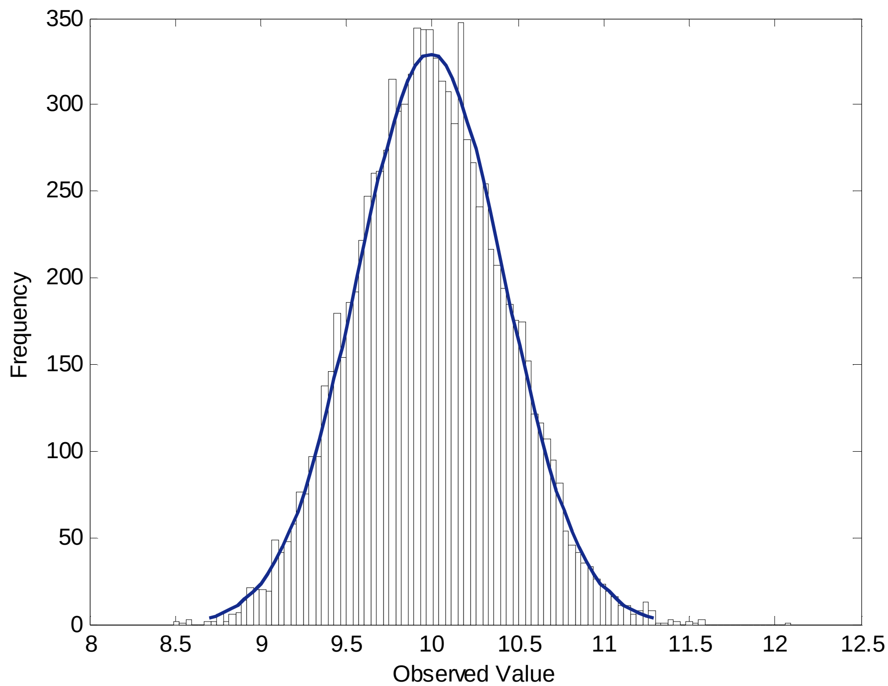

Figure 4-11 showing a histogram and normal distribution fit of the observed data from Monte Carlo simulation using the theoretical optimum means.

4.7 Case Study 4 – Design of a Piano String

From Chapter 3, the velocity of a piano string is observed while varying the initial hammer velocity, spring stiffness and the striking position of the hammer. For illustrative purposes, suppose an optimum set of design variables is desired such that the response at particular time steps meets some stated specifications. The responses obtained from the initial set of design variable combinations are shown graphically in Figure 4-12.

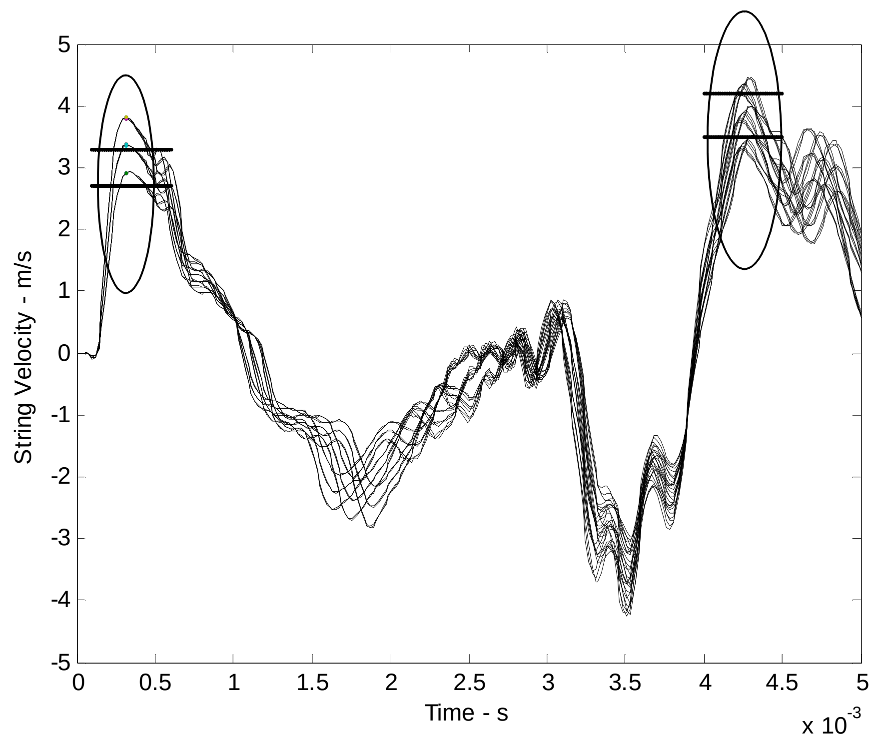

Figure 4-12 showing the responses at the training points from the simulation of the piano string.

The initial means and variances of each design variable are

$$\begin{bmatrix} \mu_{v_1} \\ \mu_{v_2} \\ \mu_{v_3} \end{bmatrix} = \begin{bmatrix} 4.00 \\ 3.90 \times 10^{-5} \\ 8.06 \times 10^{-2} \end{bmatrix} \quad \begin{bmatrix} \sigma_{v_1}^2 \\ \sigma_{v_2}^2 \\ \sigma_{v_3}^2 \end{bmatrix} = \begin{bmatrix} 1.33 \times 10^{-1} \\ 1.69 \times 10^{-12} \\ 7.22 \times 10^{-6} \end{bmatrix}$$

where the variance is calculated from

$$\sigma_{v_i}^2 = \left(\frac{10}{300}\mu_{v_i}\right)^2$$

4.7.1 Design Specifications

Consider the times t = 0.0003187s and t = 0.004243s which correspond to the 17th and 214th time steps respectively. The specifications at these times, stated below, are shown in Figure (4-12)

$$\zeta_{L,t_{17}} = 2.7 \qquad \zeta_{U,t_{17}} = 3.3$$

$$\zeta_{L,t_{214}} = 3.5 \qquad \zeta_{U,t_{214}} = 4.2$$

The optimum means and tolerances of the three design variables are required to meet these specifications. Consider, first, the specifications at t = 0.0003187s. To find the desired means and tolerances, integrated design is used. The optimization problem is then

$$\min\ C_T^{t_{17}}\left(\mu_1, \mu_2, \mu_3, tol_1, tol_2, tol_3\right)$$

subject to

$$\Pr\left(F_{t_{17}}\right) \le 0.020$$

where $C_T^{t_{17}}\left(\mu_1, \mu_2, \mu_3, tol_1, tol_2, tol_3\right)$ denotes the total cost at 0.0003187s and is a function of the means and tolerances of the design variables. This cost is made up of the production and loss of quality costs which for t = 0.003187s are

$$C_p = \frac{1}{tol_{v_1}} + \frac{1}{tol_{v_2}} + \frac{1}{tol_{v_3}}$$

$$C_{LQ} = \frac{C_s}{(3.3-2.7)^2}\left(\left(\mu_{Z,t_{17}} - 3\right)^2 + \sigma_{Z,t_{17}}^2\right)$$

where the scrap cost, C_s, is assumed to be \$1 and the mean and variance of the response at t_{17} is calculated using equations (4.14) and (4.15). The production cost takes the reciprocal model form with the assumptions of a = 0 and b = 1.

The normalized sensitivities of each design variable at this time are

v_1 = 1.0616

v_2 = 0.0054

v_3 = 0.0217

indicating that v_2 has a very small effect on the string velocity. If v_2 is kept constant, the problem becomes

$$\min \; C_T^{t_{17}}\left(\mu_1, \mu_3, tol_1, tol_2, tol_3\right)$$
$$\text{subject to}$$
$$\Pr\left(F_{t_{17}}\right) \leq 0.020$$

where $C_T^{t_{17}}\left(\mu_1, \mu_3, tol_1, tol_2, tol_3\right)$ is the total cost as a function of the means and tolerances of v_1 and v_3.

4.7.2 Multiple Responses

The optimum design is now found to meet specifications at the 17th and 214th time steps when the means and tolerances of v_1 and v_3 are unknown.

$$\text{Total Cost, } C_T = C_T^{t_{17}} + C_T^{t_{214}}$$
$$\min \; C_T = C_T^{t_{17}} + C_T^{t_{214}}$$
$$\text{subject to}$$
$$P\left(F_{t_{17}}\right) \leq 0.02$$
$$P\left(F_{t_{214}}\right) \leq 0.02$$

Optimum design at $v_1 = 3.8586$, $v_2 = 3.90 \times 10^{-5}$, $v_3 = 0.1146$, $tol_1 = 1.6086$, $tol_2 = 10$ and $tol_3 = 9.4375$. Using a Monte Carlo simulation and 5000 runs, at t = 0.003187s, the probability of conformance is

$$\Pr\left(S_{t_{17}}\right) = 0.98$$

and at 0.004243s,

$$\Pr\left(S_{t_{214}}\right) = 0.99$$

Histograms of the data obtained from the Monte Carlo simulation at these times using the optimum means and tolerances of the design variables are shown in Figures (4-13) and (4-14). From these figures the spread of the observed responses fall within the specifications with means of approximately 3.26 and 3.65 and the 17th and 214th time steps respectively.

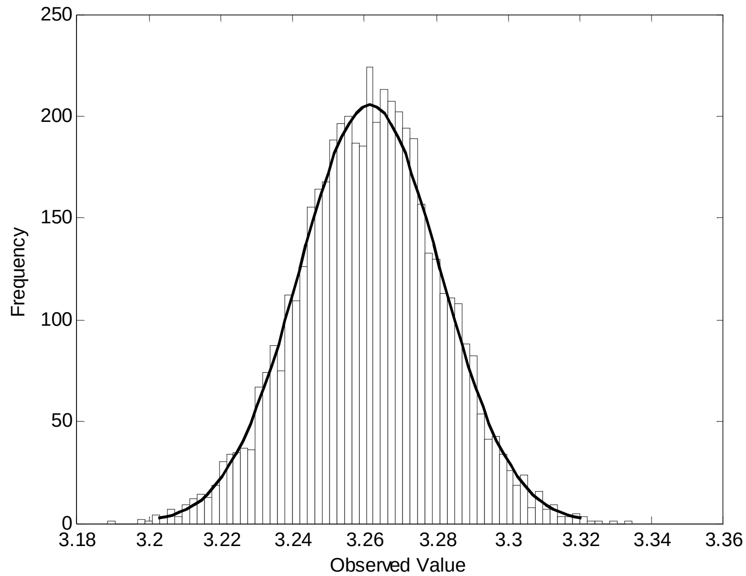

Figure 4-13 showing a histogram of the data obtained from Monte Carlo at the 17th time step using the optimum values of the means and tolerances of the design variables.

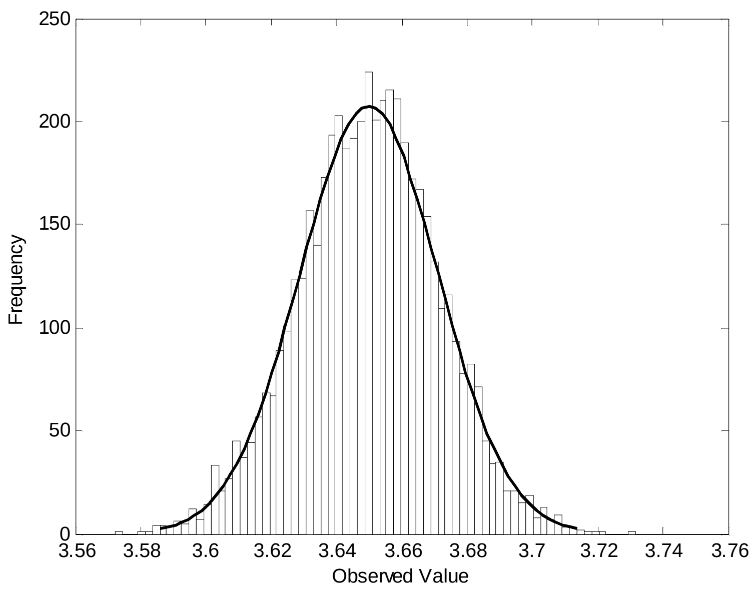

Figure 4-14 showing a histogram of the data obtained from Monte Carlo at the 214th time step using the optimum values of the means and tolerances of the design variables.

Chapter 5

Discussion and Conclusion

5.1 Discussion

Previous research focussed on using metamodelling for fitting experimental results and making predictions at some arbitrary set of design variables. However, very little research has been found on the use of metamodelling in the design of dynamic systems where specifications at certain times are desired. This research has introduced a method of combining Singular Value Decomposition (SVD) and Metamodelling in order to find the set of design variables that resulted in an optimum dynamic system given specifications at individual time steps. The practicality of two popular metamodels, Response Surface Models (RSM) and Kriging models, has been compared by way of several case studies. The findings of this research will now be presented.

The dynamic response of some system has been broken into discrete time steps and recorded in a matrix. Normally, to find responses at multiple design variable sets, metamodels are developed for each time step. However, for cases where the number of time steps is of the order of one hundred and above, finding a metamodel for each step will take a very long time. To address this problem, SVD has been applied to factorize this matrix into matrices containing information in parameter- and time-space. Metamodels are then found only for the columns of the matrix in parameter-space, **D**. The response at any specific time is then found from the product of the new row in **D**, corresponding to $\mathbf{v}_0$, with the matrix of time-dependent information. Furthermore, the numbers found in the columns of **D** are decreasing in magnitude. The coefficient of determination has been used to show how a few columns of **D** can also achieve acceptable results in fitting the experimental responses.

This feature of SVD has been used for designing a system where the optimum performance depends on specifications at certain times. Several case studies have been used to illustrate the theory of combining SVD with metamodelling for robust design applications. In order to determine which metamodel fit the experimental results more accurately the statistical coefficient of determination, R^2, was calculated. If R^2 indicated that the RSM provided a suitable fit of the experimental response, the RSM was used for robust design. For systems with multiple design variables, normalized sensitivities were calculated for each design variable and the design variable with a negligible effect on the response (of order 10^{-3}) was considered constant when applying robust design calculations.

Also, the comparison of the dynamic response at some arbitrary set of design variables calculated using both metamodels with the results obtained from the actual simulation served as a method of evaluating the metamodel performance. For a system containing a simple response, although the Kriging model fit the experimental results exactly according to the R^2 value, the RSM is also quite acceptable. For the same system, predictions at v_0 using the Kriging model were not much better than those obtained from the RSM. However, for a system with a 'noisy' response, the advantage of the Kriging model in better fitting the data is more clearly seen. A plot of the response showed that in some areas, the RSM smoothes the data rather than modelling it exactly.

5.2 Conclusion

Overall, SVD with metamodelling helps to greatly reduce the number of calculations required by reducing the number of columns over which interpolation is needed. This method is very helpful when using a Monte Carlo simulation to generate responses at a large sample of design variable combinations. Also, for robust design at specific time steps, for a system with a simple dynamic response, as in the case of the servo, a simple RSM is suitable for design since calculations required to estimate Kriging model parameters are quite demanding. However, for a "noisy" response like that of the velocity of the piano string, the iterative process in estimating Kriging model parameters is offset by very accurate results in fitting the experimental data and, thus, the Kriging model is preferred for robust design calculations.

More importantly, this method of using SVD to separate a matrix of dynamic response into parameter- and time-space was found to be very helpful for robust design since the metamodel, normalized sensitivities, first and second moments and probability of conformance can be easily found.

5.3 Future Work

Future work includes

- comparing the predictions obtained using different correlation functions in the Kriging metamodel
- applying this methodology for the case of multiple responses and for situations where there are specifications at every time step
- using first-order reliability method (FORM) as opposed to a Monte Carlo simulation
- applying this methodology to cases where specifications at every time step have to be met

Appendix A

R^2 calculated at each time step using the RSM and Kriging model.

Time $x10^{-3}$	0.000	0.020	0.0398	0.0598	0.0797	0.0996	0.1195
RSM	0.000	0.429	0.3190	0.0230	0.5482	0.9800	0.9947
Kriging	1.000	0.996	0.9979	0.9981	0.9933	0.9964	0.9994
Time $x10^{-3}$	**0.1394**	**0.159**	**0.179**	**0.1992**	**0.2191**	**0.2390**	**0.2590**
RSM	0.9891	0.996	0.999	0.9996	1.0000	1.0000	1.0000
Kriging	0.9924	0.997	0.999	0.9997	0.9999	0.9999	0.9999
Time $x10^{-3}$	**0.2789**	**0.2988**	**0.319**	**0.339**	**0.3586**	**0.3785**	**0.3984**
RSM	0.9999	0.9999	1.000	0.999	0.9995	0.9994	0.9975
Kriging	0.9999	0.9999	1.000	1.000	1.0000	0.9998	1.0000
Time $x10^{-3}$	**0.4183**	**0.4383**	**0.4582**	**0.478**	**0.498**	**0.5180**	**0.5379**
RSM	0.9989	0.9879	0.9926	0.990	0.953	0.9679	0.9909
Kriging	0.9999	1.0000	0.9995	1.000	1.000	0.9998	0.9998
Time $x10^{-3}$	**0.5578**	**0.5777**	**0.5976**	**0.6175**	**0.638**	**0.657**	**0.6773**
RSM	0.9390	0.8660	0.9152	0.9740	0.988	0.965	0.9055
Kriging	0.9999	0.9995	0.9996	0.9999	1.000	0.999	0.9996
Time $x10^{-3}$	**0.6972**	**0.7171**	**0.7371**	**0.7570**	**0.7769**	**0.797**	**0.817**
RSM	0.8460	0.9128	0.9839	0.9952	0.9912	0.992	0.986
Kriging	0.9994	0.9994	0.9996	0.9995	0.9991	0.999	0.999

Time x10^{-3}	**0.8367**	**0.8566**	**0.8765**	**0.8964**	**0.9163**	**0.9363**	**0.956**
RSM	0.9677	0.9152	0.8513	0.8752	0.9601	0.8991	0.608
Kriging	0.9994	0.9995	0.9992	0.9983	0.9974	0.9976	0.999
Time x10^{-3}	**0.976**	**0.9960**	**1.0159**	**1.0359**	**1.0558**	**1.0757**	**1.0956**
RSM	0.340	0.3248	0.4630	0.3619	0.4918	0.6549	0.8169
Kriging	0.999	0.9975	0.9917	0.9984	0.9994	0.9998	1.0000
Time x10^{-3}	**1.116**	**1.136**	**1.1554**	**1.1753**	**1.1952**	**1.2151**	**1.2351**
RSM	0.935	0.982	0.9691	0.9195	0.8462	0.7528	0.6696
Kriging	1.000	0.999	0.9997	0.9999	1.0000	0.9998	0.9990
Time x10^{-3}	**1.2550**	**1.275**	**1.294**	**1.3147**	**1.3347**	**1.3546**	**1.3745**
RSM	0.7094	0.885	0.976	0.9355	0.8738	0.8688	0.9236
Kriging	0.9974	0.997	0.997	0.9982	0.9989	0.9992	0.9994
Time x10^{-3}	**1.3944**	**1.4143**	**1.434**	**1.4542**	**1.4741**	**1.4940**	**1.5139**
RSM	0.9774	0.9666	0.924	0.9221	0.9695	0.9695	0.8563
Kriging	0.9995	0.9996	0.999	0.9994	0.9991	0.9987	0.9986
Time x10^{-3}	**1.5339**	**1.5538**	**1.5737**	**1.5936**	**1.6135**	**1.6335**	**1.6534**
RSM	0.7394	0.6899	0.7030	0.7601	0.8440	0.9307	0.9787
Kriging	0.9990	0.9995	0.9998	1.0000	0.9997	0.9993	0.9990
Time x10^{-3}	**1.6733**	**1.6932**	**1.713**	**1.7331**	**1.753**	**1.7729**	**1.7928**
RSM	0.9577	0.8784	0.775	0.6795	0.603	0.5591	0.5899
Kriging	0.9989	0.9991	0.999	0.9999	0.999	0.9995	0.9987
Time x10^{-3}	**1.8127**	**1.8327**	**1.8526**	**1.873**	**1.8924**	**1.9124**	**1.9323**
RSM	0.7185	0.8521	0.9253	0.959	0.9751	0.9842	0.9895
Kriging	0.9986	0.9993	0.9996	0.999	0.9998	0.9997	0.9997

Time x10^{-3}	**1.9522**	**1.9721**	**1.9920**	**2.0120**	**2.032**	**2.0518**	**2.0717**
RSM	0.9931	0.9955	0.9902	0.9596	0.883	0.7461	0.5416
Kriging	0.9998	0.9999	0.9994	0.9994	0.999	0.9983	0.9986
Time x10^{-3}	**2.0916**	**2.1116**	**2.1315**	**2.1514**	**2.1713**	**2.1912**	**2.2112**
RSM	0.3252	0.1901	0.1393	0.1523	0.2522	0.4414	0.6551
Kriging	0.9988	0.9993	0.9986	0.9984	0.9974	0.9917	0.9886
Time x10^{-3}	**2.2311**	**2.2510**	**2.2709**	**2.2908**	**2.311**	**2.3307**	**2.3506**
RSM	0.8385	0.7689	0.4759	0.2963	0.170	0.0529	0.0837
Kriging	0.9934	0.9948	0.9952	0.9935	0.992	0.9954	0.9980
Time x10^{-3}	**2.3705**	**2.3904**	**2.4104**	**2.4303**	**2.4502**	**2.470**	**2.4900**
RSM	0.2419	0.3624	0.4552	0.6053	0.8151	0.919	0.9392
Kriging	0.9963	0.9980	0.9991	0.9985	0.9965	0.997	0.9956
Time x10^{-3}	**2.5100**	**2.5299**	**2.550**	**2.5697**	**2.5896**	**2.6096**	**2.630**
RSM	0.9535	0.9589	0.932	0.8504	0.6646	0.5296	0.620
Kriging	0.9964	0.9983	0.992	0.9911	0.9847	0.9813	0.997
Time x10^{-3}	**2.6494**	**2.6693**	**2.6892**	**2.7092**	**2.7291**	**2.7490**	**2.7689**
RSM	0.7714	0.9076	0.9456	0.5869	0.2358	0.8016	0.9096
Kriging	0.9942	0.9920	0.9945	0.9667	0.9784	0.9864	0.9801
Time x10^{-3}	**2.7888**	**2.8088**	**2.8287**	**2.8486**	**2.8685**	**2.8884**	**2.908**
RSM	0.8373	0.4610	0.4635	0.8460	0.9533	0.9548	0.824
Kriging	0.9926	0.9852	0.9647	0.9909	0.9978	0.9940	0.988
Time x10^{-3}	**2.9283**	**2.9482**	**2.9681**	**2.9880**	**3.0080**	**3.0279**	**3.0478**
RSM	0.2735	0.6074	0.8714	0.9535	0.9772	0.9666	0.8989
Kriging	0.9860	0.9943	0.9979	0.9980	0.9985	0.9994	0.9977

Time $x10^{-3}$	**3.068**	**3.0876**	**3.1076**	**3.1275**	**3.1474**	**3.1673**	**3.1873**
RSM	0.606	0.5079	0.8229	0.9207	0.9567	0.9726	0.9811
Kriging	0.992	0.9915	0.9973	0.9977	0.9970	0.9978	0.9990
Time $x10^{-3}$	**3.2072**	**3.227**	**3.2470**	**3.2669**	**3.2869**	**3.3068**	**3.3267**
RSM	0.9868	0.989	0.9878	0.9891	0.9908	0.9888	0.9857
Kriging	0.9992	0.999	0.9998	0.9994	0.9985	0.9990	0.9986
Time $x10^{-3}$	**3.347**	**3.3665**	**3.3865**	**3.4064**	**3.4263**	**3.4462**	**3.4661**
RSM	0.983	0.9690	0.9560	0.9713	0.9856	0.9796	0.9821
Kriging	0.994	0.9930	0.9970	0.9949	0.9925	0.9970	0.9994
Time $x10^{-3}$	**3.4861**	**3.506**	**3.5259**	**3.5458**	**3.5657**	**3.5857**	**3.6056**
RSM	0.9908	0.989	0.9768	0.9711	0.9780	0.9726	0.9402
Kriging	0.9976	0.995	0.9953	0.9963	0.9954	0.9929	0.9927
Time $x10^{-3}$	**3.6255**	**3.6454**	**3.665**	**3.6853**	**3.7052**	**3.7251**	**3.7450**
RSM	0.9316	0.9575	0.968	0.9693	0.9784	0.9903	0.9920
Kriging	0.9906	0.9860	0.989	0.9966	0.9982	0.9959	0.9962
Time $x10^{-3}$	**3.7649**	**3.7849**	**3.8048**	**3.825**	**3.8446**	**3.8645**	**3.8845**
RSM	0.9890	0.9905	0.9926	0.983	0.9478	0.8908	0.8382
Kriging	0.9986	0.9987	0.9959	0.994	0.9947	0.9924	0.9600
Time $x10^{-3}$	**3.9044**	**3.9243**	**3.9442**	**3.9641**	**3.9841**	**4.0040**	**4.0239**
RSM	0.8177	0.8396	0.8852	0.9385	0.9758	0.9881	0.9842
Kriging	0.9636	0.9928	0.9983	0.9983	0.9985	0.9990	0.9993
Time $x10^{-3}$	**4.0438**	**4.0637**	**4.0837**	**4.104**	**4.1235**	**4.143**	**4.1633**
RSM	0.9704	0.9575	0.9630	0.979	0.9911	0.997	0.9988
Kriging	0.9985	0.9982	0.9988	0.999	0.9992	0.999	0.9997

Time x10^{-3}	**4.1833**	**4.2032**	**4.2231**	**4.2430**	**4.2629**	**4.2829**	**4.303**
RSM	0.9946	0.9847	0.9750	0.9730	0.9798	0.9896	0.992
Kriging	0.9997	0.9994	0.9993	0.9996	0.9994	0.9994	0.999
Time x10^{-3}	**4.3227**	**4.3426**	**4.3625**	**4.3825**	**4.4024**	**4.4223**	**4.4422**
RSM	0.9845	0.9802	0.9846	0.9852	0.9659	0.9343	0.9184
Kriging	0.9990	0.9993	0.9995	0.9986	0.9984	0.9989	0.9991
Time x10^{-3}	**4.4622**	**4.4821**	**4.502**	**4.5219**	**4.5418**	**4.5618**	**4.5817**
RSM	0.9313	0.9614	0.981	0.9578	0.8716	0.7467	0.6791
Kriging	0.9994	0.9997	0.999	0.9988	0.9988	0.9992	0.9994
Time x10^{-3}	**4.6016**	**4.6215**	**4.6414**	**4.6614**	**4.6813**	**4.701**	**4.7211**
RSM	0.7403	0.8500	0.9328	0.9769	0.9819	0.949	0.8875
Kriging	0.9995	0.9996	0.9996	0.9995	0.9994	0.999	0.9995
Time x10^{-3}	**4.7410**	**4.7610**	**4.7809**	**4.8008**	**4.8207**	**4.8406**	**4.8606**
RSM	0.8116	0.7404	0.7249	0.8033	0.9059	0.9597	0.9750
Kriging	0.9991	0.9982	0.9971	0.9981	0.9981	0.9984	0.9991
Time x10^{-3}	**4.8805**	**4.900**	**4.9203**	**4.940**	**4.9602**	**4.9801**	**5.0000**
RSM	0.9805	0.984	0.9776	0.959	0.9407	0.9374	0.9487
Kriging	0.9982	0.998	0.9986	0.9989	0.9988	0.9987	0.9975

Appendix B

Training points used for the position control servo

$$
\mathbf{X}_{tr} = \begin{bmatrix}
0.001 & 0.008 & 4 \\
0.001 & 0.008 & 4.4 \\
0.001 & 0.008 & 4.84 \\
0.001 & 0.0088 & 4 \\
0.001 & 0.0088 & 4.4 \\
0.001 & 0.0088 & 4.84 \\
0.001 & 0.00968 & 4 \\
0.001 & 0.00968 & 4.4 \\
0.001 & 0.00968 & 4.84 \\
0.0011 & 0.008 & 4 \\
0.0011 & 0.008 & 4.4 \\
0.0011 & 0.008 & 4.84 \\
0.0011 & 0.0088 & 4 \\
0.0011 & 0.0088 & 4.4 \\
0.0011 & 0.0088 & 4.84 \\
0.0011 & 0.00968 & 4 \\
0.0011 & 0.00968 & 4.4 \\
0.0011 & 0.00968 & 4.84 \\
0.0012 & 0.008 & 4 \\
0.0012 & 0.008 & 4.4 \\
0.0012 & 0.008 & 4.84 \\
0.0012 & 0.0088 & 4 \\
0.0012 & 0.0088 & 4.4 \\
0.0012 & 0.0088 & 4.84 \\
0.0012 & 0.00968 & 4 \\
0.0012 & 0.00968 & 4.4 \\
0.0012 & 0.00968 & 4.84
\end{bmatrix}_{27\times 3}
$$

Appendix C

R^2 calculated at each time step, for the Servo Example (three random design variables and Angular Position Response), using both metamodels.

Time	**0.005**	**0.010**	**0.015**	**0.020**	**0.025**	**0.030**	**0.035**
RSM	0.997	0.997	0.998	0.999	0.998	0.991	0.961
Kriging	1	1	1	1	1	1	1
Time	**0.040**	**0.045**	**0.050**	**0.055**	**0.060**	**0.065**	**0.070**
RSM	0.755	0.525	0.925	0.986	0.997	0.988	0.950
Kriging	1	1	1	1	1	1	1
Time	**0.075**	**0.080**	**0.085**	**0.090**	**0.095**	**0.100**	
RSM	0.816	0.310	0.436	0.873	0.970	0.989	
Kriging	1	1	0.9999	0.9999	1	0.9998	

Appendix D

The coefficient of determination calculated at each time step, using both metamodels, for the Grocery Sign example.

Time	**0**	**400**	**800**	**1200**	**1600**	**2000**	**2400**	**2800**
RSM	1	0.992	0.993	0.994	0.995	0.996	0.997	0.998
Krig	1	1	1	1	1	1	1	1
Time	**3200**	**3600**	**4000**	**4400**	**4800**	**5200**	**5600**	**6000**
RSM	0.998	0.998	0.999	0.999	0.999	0.999	0.999	0.999
Krig	1	1	1	1	1	1	1	1
Time	**6400**	**6800**	**7200**	**7600**	**8000**	**8400**	**8800**	**9200**
RSM	0.999	0.999	0.999	0.998	0.998	0.997	0.996	0.995
Krig	1	1	1	1	1	1	1	1
Time	**9600**	**10000**	**10400**	**10800**	**11200**	**11600**	**12000**	**12400**
RSM	0.995	0.994	0.993	0.991	0.990	0.989	0.988	0.986
Krig	1	1	1	1	1	1	1	1
Time	**12800**	**13200**	**13600**	**14000**	**14400**	**14800**	**15200**	**15600**
RSM	0.985	0.983	0.981	0.980	0.978	0.976	0.974	0.972
Krig	1	1	1	1	1	1	1	1
Time	**16000**	**16400**	**16800**	**17200**	**17600**	**18000**	**18400**	**18800**
RSM	0.970	0.968	0.966	0.964	0.961	0.959	0.957	0.954
Krig	1	1	1	1	1	1	1	1
Time	**19200**	**19600**	**20000**	**20400**	**20800**	**21200**	**21600**	**22000**
RSM	0.952	0.949	0.947	0.944	0.941	0.939	0.936	0.933
Krig	1	1	1	1	1	1	1	1
Time	**22400**	**22800**	**23200**	**23600**	**24000**	**24400**	**24800**	**25200**
RSM	0.930	0.928	0.925	0.922	0.919	0.916	0.913	0.910

Krig	1	1	1	1	1	1	1	1
Time	**25600**	**26000**	**26400**	**26800**	**27200**	**27600**	**28000**	**28400**
RSM	0.907	0.904	0.901	0.898	0.895	0.891	0.888	0.885
Krig	1	1	1	1	1	1	1	1
Time	**28800**	**29200**	**29600**	**30000**				
RSM	0.882	0.879	0.876	0.872				
Krig	1	1	1	1				

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Glossary of Terms

Design Variable – input variable that affects the output of a system

Design Variable Combination or Design Variable Set – set of two or more design variables used to generate one response. E.g. Angular response when $v_1 = 0.0011$ and $v_2 = 0.0088$ is a design variable combination

Dynamic System – a system in which the response changes over time

Experimental design – design consisting of training points and experimental responses.

Experimental responses – set of responses obtained from the simulation at the training points.

FOS Factors or first-order sensitivity factors – first derivative of a function with respect to a particular design variable

Integrated Design – robust design technique used to find the means and tolerances of the design variables to maximize the probability of conformance

Kriging – type of metamodel initially used as an interpolation method in spatial estimation
Metamodelling – process of developing metamodels to fit experimental data

Metamodel – a simple model used to fit the experimental data.

Normalized Sensitivities or Sensitivities – gives the percentage change of the response for a 1% change of a particular design variable

Parameter Design – robust design technique used to find the means of the design variables at constant tolerances to maximize the probability of conformance

PCA or Principal Component Analysis – a statistical process that groups individual variables into separate components of factors and uses these rather than individual variables as a basis for measuring similarities between areas. (Badiru, 2006).

Response – output of a system

Response Time History – response of a system recorded at consecutive time steps

RSM or Response Surface Model – otherwise known as regression model; a type of metamodel common in statistical applications for fitting experimental data

SVD or Singular Value Decomposition – matrix decomposition technique

Time Step – a discrete instance in time

Training points – set of initial design variables or design variable combinations used for generating initial response data from the simulation