On a Question of Wintner Concerning the Sequence of Integers Composed of Primes from a Given Set

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Jeongsoo Kim

Abstract

Wintner asked the following question :

Does there exist an infinite set S of prime numbers such that if $n_0 < n_1 < \ldots < n_i < \ldots$ is the sequence of all positive integers composed of the primes in S then

$$\lim_{i \to \infty} (n_{i+1} - n_i) \quad = \quad \infty \quad ?$$

In 1973 Tijdeman [38] proved that the answer to the question is yes. In this thesis, we shall investigate Wintner's question in more detail.

Tijdeman [38] proved that for each real number θ with $0 < \theta < 1$ there exists an infinite set of primes S such that if $n_0 < n_1 < \ldots < n_i < \ldots$ is the sequence of all positive integers composed of the primes in S then $n_{i+1} - n_i > n_i^{1-\theta}$ for $i = 0, 1, \ldots$.

Given such a θ, we shall show that we can find an infinite set $S = \{p_1, p_2, \ldots\}$ of primes with $p_1 < p_2 < \ldots$ so that the n-th term p_n does not grow too quickly. In particular, we shall show that

$$p_n \quad < \quad \exp\left(\frac{c_1 n^2}{\theta} \; \log\left(\frac{c_2 n}{\theta}\right)\right)$$

where c_1, c_2 are explicit numbers.

We shall also investigate the following question. We shall look for a function $\mathcal{L}(x)$ which grows quickly and yet for which there is an infinite set of primes S such that the associated sequence of power products $n_0 < n_1 < n_2 < \ldots$ satisfies

$$n_{i+1} - n_i \quad > \quad \mathcal{L}(n_i)$$

for $i = 0, 1, \ldots$.

We define a family of functions

$$F_{k,\theta}(x) \quad = \quad \exp^k\left((\log_k(x))^\theta\right)$$

where k is an non-negative integer and θ is a real number and $\log_k$ is k-iterated logarithms and $\exp^k$ is k-iterated exponentiations. And we prove that for given non-negative integer k and a real number θ with $0 < \theta < 1$ there is an infinite set $S(k, \theta)$ of prime numbers such that if $n_0 < n_1 < \ldots < n_i < \ldots$ is the sequence of

all positive integers composed of the primes in $S(k,\theta)$ then

$$n_{i+1} - n_i \quad > \quad \frac{n_i}{F_{k,\theta}(n_i)}$$

for $i = 0, 1, \ldots$.

Finally, we shall consider prime pairs (p,q) such that if $n_0 < n_1 < \ldots$ is the sequence of all positive integers composed of the primes p, q then

$$n_{i+1} - n_i \quad > \quad \sqrt{n_i}.$$

We find all such prime pairs (p,q) with $2 \leq p < q < e^8$ by computational work. Given two such primes p, q we can find an infinite set of primes $\{p, q, p_3, p_4, \ldots\}$ such that if $n_0 < n_1 < n_2 < \ldots$ is the sequence of all positive integers composed of the primes then

$$n_{i+1} - n_i \quad > \quad \sqrt{n_i}.$$

for $i = 0, 1, \ldots$.

These results generalize and develop the answer to Wintner's question due to Tijdeman.

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Give Thanks...

Jeongsoo Kim

Dedication

Dedicated to my parents : D.-H. Kim and J.-J. Cho Kim

and

to my grandmother : K.-S. Bae Cho

Contents

Chapter 1

Introduction

Let us define the following set:

Definition Let S be a set of prime numbers. Define

$$\mathcal{N}(S) \quad = \quad \left\{ x \in \mathbf{N} \;\middle|\; x = \prod_{p \in S} p^a, \;\; a \in \mathbf{N} \cup \{0\} \right\}.$$

We see that $\mathcal{N}(S) = \{n_0, n_1, \ldots\}$ is the set of all positive integers composed of the primes in S. That means for given set S of prime numbers, we see that for a positive integer x, x is in $\mathcal{N}(S)$ if and only if for any prime number p with $p|x$, p is in S.

We have some examples.

Examples

1. If S is the set of all prime numbers then $\mathcal{N}(S)$ is the set of all positive integers.
2. If $S = \emptyset$ then $\mathcal{N}(S) = \{1\}$ and this is the only case $\mathcal{N}(S)$ is a finite set.
3. If $S = \{2\}$ then $\mathcal{N}(S) = \{2^i | i \in \mathbf{N} \cup \{0\}\}$
4. If S is the set of all odd prime numbers then $\mathcal{N}(S)$ is the set of all positive odd integers.

We want to see $\mathcal{N}(S)$ from an additive point of view. First, without loss of generality, we can order the elements of the set

$$S = \{p_1 < p_2 < \ldots\},$$
$$\mathcal{N}(S) = \{n_0 < n_1 < n_2 < \ldots\}.$$

We see that $n_0 = 1$ and $n_1 = p_1$ the smallest prime in S. We denote the cardinality of a set A by $|A|$.

1.1 When $|S|$ is Finite

In 1898, Størmer [36] proved the following theorem.

Theorem 1.1 (Størmer)**.** *Let S be a finite subset of odd primes. Then for $n_i \in \mathcal{N}(S)$*

$$\liminf_{i\to\infty} (n_{i+1} - n_i) > 2.$$

This result was improved by Thue [41] in 1908.

Theorem 1.2 (Thue)**.** *Let S be a finite set of primes and $n_i \in \mathcal{N}(S)$. Then*

$$\lim_{i\to\infty} (n_{i+1} - n_i) = \infty.$$

Thue derived this theorem from his result on the approximation of algebraic numbers by rational numbers.

Størmer proposed the question of determining for a given finite set of prime numbers, the pairs $(a, a+1)$ of consecutive integers such that both a and $a+1$ belong to $\mathcal{N}(S)$. He proved [35] that given a finite set S of t primes, there are only finitely many pairs $(a, a+1)$ such that both a and $a+1$ belong $\mathcal{N}(S)$. He used explicit methods involving Pell's equations and showed that the number of such pairs is at most $3^t - 2^t$.

Lehmer [21] generalized this question to that of finding, for a given finite set S of primes, all pairs $(a, a+k)$ such that a and $a+k$ are in $\mathcal{N}(S)$ for $k = 1, 2, 4$. He was interested in an efficient way to determine the number of these pairs. Using a result of Gelfond [15], Cassels [12] gave an explicit upper bound for the size of the numbers. And he gave necessary and sufficient conditions to determine when both a and $a+k$ are in $\mathcal{N}(S)$. Recently, Jones [18] extended Lehmer's results to the case when k is an arbitrary positive integer.

In 1918, Pólya [29] proved the same result as Theorem 1.2 with a different approach. His proof uses an estimate for the sum of the divisors of $p-1$ for primes p up to x. Pólya proved that

Theorem 1.3 (Pólya). *If S is any finite subset of primes and $n_i \in \mathcal{N}(S)$ then $n_{i+1} - n_i$ tends to infinity. Moreover, if $|S| \geq 2$ then*

$$\lim_{i \to \infty} \frac{n_{i+1}}{n_i} = 1.$$

From Pólya's proof, we have information of upper bounds of the sequence of the quotients $\frac{n_{i+1}}{n_i}$.

Erdös [13] observed this using the results of Siegel [32] and Mahler [23].

Theorem 1.4 (Erdös). *Let S be a finite subset of primes. Let $0 < \theta < 1$. Then there is $N(\theta, S) > 0$ such that*

$$n_{i+1} - n_i > n_i^{\theta}$$

for all $n_i \in \mathcal{N}(S)$ with $n_i > N(\theta, S)$.

But in both Siegel's and Mahler's methods $N(\theta, S)$ is not effectively computable.

In 1973 and 1974, Tijdeman [38, 39] resolved these problems. He uses Fel'dman's estimates [14] for linear forms in the logarithms of algebraic numbers.

Theorem 1.5 (Tijdeman [38]). *Let $n_1 < n_2 < \ldots$ be the sequence of integers composed of primes not greater than p. Then there exists an effectively computable positive number $C_1 = C(p)$ such that*

$$n_{i+1} - n_i > \frac{n_i}{(\log n_i)^{C_1}}$$

for $n_i \geq 3$.

In 1974, Tijdeman proved the following theorem by applying estimates for linear forms in logarithms and using some elementary properties of continued fraction expressions.

Theorem 1.6 (Tijdeman [39]). *Let $S = \{p_1 < p_2\}$ and $n_i \in \mathcal{N}(S)$. Then there exist effectively computable numbers $C_2 = C(p_1, p_2)$ and $N = N(p_1, p_2)$ such that*

$$n_{i+1} - n_i < \frac{n_i}{(\log n_i)^{C_2}}$$

for $n_i \geq N$.

Tijdeman proved the following theorem without estimates for linear forms in the logarithms of algebraic numbers.

Theorem 1.7 (Tijdeman [38]). *Let* $S = \{p_1 < \ldots < p_t\}$ *be a given set of* t *prime numbers and* $t > 1$. *Then there are infinitely many pairs* x, y *in* $\mathcal{N}(S)$ *such that*

$$0 \quad < \quad x - y \quad < \quad \frac{(t \log p_t)^t \cdot y}{(\log y)^{t-1}}. \tag{1.1}$$

Remark We shall include the proof of this theorem for completeness and rewrite the proof in terms of our notation. The proof may be found in [38, Theorem 2].

Proof. Let $S = \{p_1 < \ldots < p_t\}$ be given. Let M be a positive integer and consider a set $\mathcal{N}(S, M)$ such that

$$\mathcal{N}(S, M) \quad = \quad \{x \in \mathcal{N}(S) \mid x = p_1^{a_1} \cdots p_t^{a_t},\ 0 \leq a_i \leq M \text{ for } i = 1, \ldots, t\}.$$

Then,

$$|\mathcal{N}(S, M)| \quad = \quad (M+1)^t. \tag{1.2}$$

For any $x \in \mathcal{N}(S, M)$ with $x = p_1^{a_1} \cdots p_t^{a_t}$

$$0 \quad \leq \quad \log x \quad = \quad a_1 \log p_1 + \cdots + a_t \log p_t \quad \leq \quad M \cdot t \cdot \log p_t. \tag{1.3}$$

By (1.2) and (1.3) there are $x, y \in \mathcal{N}(S, M)$ such that $y < x$ and

$$0 \quad < \quad \log x - \log y \quad \leq \quad \frac{M \cdot t \cdot \log p_t}{(M+1)^t - 1}. \tag{1.4}$$

Let $x = p_1^{a_1} \cdots p_t^{a_t}$ and $y = p_1^{b_1} \cdots p_t^{b_t}$. We may assume without loss of generality $a_i \cdot b_i = 0$ for $i = 1, \ldots, t$. We have $y^2 < xy < p_t^{Mt}$. Therefore, $M > \frac{2 \cdot \log y}{t \cdot \log p_t}$. Substituting this estimate in (1.4) then we see that

$$\log \frac{x}{y} \quad = \quad \log x - \log y \quad < \quad \frac{(t \log p_t)^t}{(2 \log y)^{t-1}}. \tag{1.5}$$

We see that from (1.4) and since $t > 1$, if M goes to infinity then $\log \frac{x}{y}$ goes to 0.

Since $\frac{x}{y} > 1$ we have

$$\log \frac{x}{y} \quad \geq \quad \frac{1}{2}\left(\frac{x}{y} - 1\right) \tag{1.6}$$

for sufficiently large M. We note that $\mathcal{N}(S) = \cup_{M=1}^{\infty}\mathcal{N}(S, M)$. Therefore, by (1.5),(1.6) and $t > 1$, if M goes to infinity then we have infinitely many $x, y \in \mathcal{N}(S)$ that satisfy (1.1) as required. □

By the above Theorem, we note that the constant $C_1 = C(p)$ in Theorem 1.5 cannot be replaced by a constant smaller than $\pi(p) - 1$ where $\pi(x)$ denotes the number of primes less than or equal to x.

Theorem 1.8 (Tijdeman [38, 39]). *Let S be a finite subset of t prime numbers and $t \geq 2$. Then there are effectively computable numbers C_3, C_4 and N that only depend on S such that*

$$\frac{n_i}{(\log n_i)^{C_3}} \quad < \quad n_{i+1} - n_i \quad < \quad \frac{n_i}{(\log n_i)^{C_4}}$$

for $n_i \in \mathcal{N}(S)$ with $n_i \geq N$.

By Theorem 1.5 and Theorem 1.7 we see that the numbers in Theorem 1.8 satisfy $C_3 \geq t - 1$ and $C_4 \leq t - 1$.

This result is very satisfactory not only because we can deduce all the previous theorems from Theorem 1.8 but also for finite S we see the difference $n_{i+1} - n_i$ behaves like $\frac{n_i}{(\log n_i)^C}$ on average.

1.2 Wintner's Question

What can be said if S is an infinite subset of primes ?

In the review paper [13] Erdös mentioned the following question introduced by Wintner.

Question (Wintner) Does there exist an infinite sequence of primes $p_1 < p_2 < \ldots$ such that if $n_0 < n_1 < \ldots$ is the sequence of all positive integers composed of p's in the sequence of primes then

$$\lim_{i \to \infty} (n_{i+1} - n_i) \quad = \quad \infty \text{ ?}$$

And Erdös mentioned that it seems certain that such a sequence exists.

This looks like a natural question after what we know about the sequence of gaps $n_{i+1} - n_i$ in $\mathcal{N}(S)$ when S is finite. But, we meditate on what this question means.

In the additive point of view, we can construct the set of all positive integers on the Peano axioms. In the multiplicative point of view, we can construct the set of all positive integers by the set of all prime numbers P.

These two aspects of construction for the same set, we ask whether there is some relation between the successor function in Peano Axioms and the set of all prime numbers P and in this case we can write $\mathcal{N}(P) = \mathbf{N}$.

In this point, we can say Wintner's question is a question of finding some relation between the additive structure and the multiplicative structure of the set of integers composed of primes from a given set. In other words, we can ask whether there is an infinite subset S of prime numbers such that for $n_i \in \mathcal{N}(S)$ we can define the sequence of gaps $n_{i+1} - n_i$ as a successor like function $\mathcal{L}(n_i)$ and the behavior of $\mathcal{L}(n_i)$ is similar to the case $|S|$ is finite.

Just after Baker [8] proved a sharpening of the bounds for linear forms in logarithms, Tijdeman applied the theorem and proved such an infinite set S of primes exists.

Theorem 1.9 (Tijdeman)**.** *Let $0 < \theta < 1$. Then there is an infinite set S of prime numbers with*

$$n_{i+1} - n_i \quad > \quad n_i^{1-\theta} \tag{1.7}$$

for all $n_i \in \mathcal{N}(S)$.

Proof. See [38, Theorem 7]. □

1.3 Motivation

Tijdeman proved that for given $0 < \theta < 1$, if we have a set of t prime numbers $S_t = \{p_1 < \ldots < p_t\}$ such that $n_{i+1} - n_i > n_i^{1-\theta}$ for all $n_i \in \mathcal{N}(S_t)$ then there is a prime number p_{t+1} with $p_{t+1} > p_t$ such that $m_{j+1} - m_j > m_j^{1-\theta}$ for all $m_j \in \mathcal{N}(S_t \cup \{p_{t+1}\})$. And in the last part of his paper [38] he remarked the following two things. (Here, we have relabeled Theorem and Equation numbers so that they correspond to the numbering in this thesis.)

Remarks ([38, Remarks])

1. It follows from the proof of Theorem 1.9 that for every θ with $0 < \theta < 1$ it is possible effectively to give a sequence $T_1, T_2, \ldots$ such that there exists a sequence $p_1 < p_2 < \ldots$ with required property and with $\frac{T_n}{2} \leq p_n \leq T_n$ for all $n = 1, 2, \ldots$.
2. It follows from Theorem 1.7, there does not exist a constant C such that Theorem 1.9 is valid if (1.7) is replaced by the inequality

$$n_{i+1} - n_i \quad > \quad \frac{n_i}{(\log n_i)^C} \,.$$

3. Remark 1 is discussed in more detail in Section 1.3.2. Remark 2 is discussed in section 1.3.3.

So, we want to know what a sequence of such T_n could be like. Moreover, for given θ we want to find a formula $T(n, \theta)$ such that there is $p_n < T(n, \theta)$ with required property.

And then, we are interested in the lower bounds of the sequence of gaps $n_{i+1} - n_i$. We want to find a function $\mathcal{L}(x)$ such that there is an infinite set S of primes with

$$n_{i+1} - n_i \quad > \quad \mathcal{L}(n_i) \,.$$

That means we want to know the behavior of the sequence of gaps $n_{i+1} - n_i$ that makes it possible for there to exist an infinite subset S of prime numbers that satisfies Wintner's question.

1.3.1 Lower bounds for p_n

Let $S = \{p_1 < p_2 < \ldots\}$ be an infinite set of prime numbers such that for all $n_i \in \mathcal{N}(S)$, $n_{i+1} - n_i > \sqrt{n_i}$ hold. Then it is not difficult to find a lower bound for p_n in S. Since the p_n's are primes we know that $n \log n < p_n$ for n sufficiently large by the prime number theorem. Now we have a non-trivial lower bound for $p_n \in S$.

Proposition 1.1. *Let $S = \{p_1 < p_2 < \ldots\}$ be an infinite set of prime numbers such that $n_{i+1} - n_i > \sqrt{n_i}$ hold for all $n_i \in \mathcal{N}(S)$. Then there is a positive number C such that $p_n \geq Cn^2$ for sufficiently large n.*

Proof. Consider a set $\mathcal{X}(a) = \{x_0 < x_1 < \ldots < x_i < \ldots\}$ that is generated by the following recursive relation :

$$x_0 = a, \quad x_i = x_{i-1} + \sqrt{x_{i-1}}$$

for some $a \geq 4$ and $i = 1, 2, \ldots$.

For any set W of real numbers and a real number u we denote

$$f\,(W, u) \quad = \quad \Big|\, \{\, w \in W \;\big|\; w \,\leq\, u \,\} \,\Big|.$$

First, it is clear that

$$\log x \quad < \quad f\,(\mathcal{X}(a), x) \quad < \quad x$$

for sufficiently large x. Since $\lim_{i\to\infty}(x_i - x_{i-1}) = \infty$, we have $f\,(\mathcal{X}(a), x) < x$. Since $\sqrt{x_i} < x_i$ for all $x_i \in \mathcal{X}(a)$ we see that $x_{i+1} < 2x_i$. Hence, for such given $x > 0$ there are at least k members in $\mathcal{X}(a)$ where k is the largest number satisfying $2^k\, a < x$. Therefore $\log x \,<\, f\,(\mathcal{X}(a), x)$ for sufficiently large x.

Now we are interested in a non-trivial upper bound for $f(\mathcal{X}(a), x)$.

The idea for this proof as following:

Step 1 For any set S of prime numbers with required property $n_{i+1} - n_i > \sqrt{n_i}$ for all $n_i \in \mathcal{N}(S)$, we observe

$$f(\mathcal{N}(S), x) \quad < \quad f\,(\mathcal{X}(a), x) \tag{1.8}$$

for sufficiently large x. For any set S of prime numbers, we know $S \subsetneq \mathcal{N}(S)$ and

$$f(S, x) \quad < \quad f(\mathcal{N}(S), x). \tag{1.9}$$

Step 2 For given positive real number x, we will claim that that

$$f\,(\mathcal{X}(4), x) \quad \leq \quad 3\sqrt{x}. \tag{1.10}$$

Step 3 On the other hand, we suppose a set S of prime numbers such that $f(S, x) \geq 3\sqrt{x} \geq f\,(\mathcal{X}(4), x)$ for some $x > 0$. Then by definition of $\mathcal{X}(4)$ there are many such integers $n_i \in \mathcal{N}(S)$ composed of these primes in S that the integers will be close and cannot satisfy the relation $n_{i+1} - n_i > \sqrt{n_i}$.

Therefore, if we produce a non-trivial upper bound for $f(\mathcal{X}(4), x)$ then we get a non-trivial lower bound for $p_n \in S$ for any set S of prime numbers with $n_{i+1} - n_i > \sqrt{n_i}$ for all $n_i \in \mathcal{N}(S)$.

This is a brief idea of this proposition.

Before proving (1.10), we observe the following relation.

For any non-negative integer i,

$$x_i \geq \begin{cases} \left(\frac{i}{3}+2\right)^2 & \text{if } i \equiv 0 \pmod 3 \\ \left(\frac{i-1}{3}+2\right)^2 + \left(\frac{i-1}{3}+2\right) & \text{if } i \equiv 1 \pmod 3 \\ \left(\frac{i-2}{3}+2\right)^2 + 2\left(\frac{i-2}{3}+2\right) & \text{if } i \equiv 2 \pmod 3 \end{cases} \tag{1.11}$$

We shall use induction on i. When $i = 0$ then $i = 0 \equiv 0 \pmod 3$, and as we assumed that $x_0 = a = 4$ so we have $x_0 = \left(\frac{0}{3}+2\right)^2$ as required. Now we suppose that for all $i \leq k-1$ (1.11) hold. When $i = k$ we consider 3 cases.

(Case 1) $k \equiv 0 \pmod 3$ and $k > 0$: Then $k - 1 \equiv 2 \pmod 3$ and by the inductive hypothesis x_{k-3} satisfies the inequality (1.11) so that

$$x_{k-1} \geq \left(\frac{k-3}{3}+2\right)^2 + 2\left(\frac{k-3}{3}+2\right).$$

Obviously for positive integer k, $x_{k-1} \geq 1$ and so $\sqrt{x_{k-1}} \geq 1$. Therefore, we have

$$x_k = x_{k-1} + \sqrt{x_{k-1}} \geq \left(\frac{k-3}{3}+2\right)^2 + 2\left(\frac{k-3}{3}+2\right) + 1.$$

And we know that $(a+1)^2 = a^2 + 2a + 1$, so

$$x_k \geq \left(\left(\frac{k-3}{3}+2\right)+1\right)^2. \tag{1.12}$$

And since $k \equiv 0 \pmod 3$, (1.12) is equivalent to

$$x_k \geq \left(\frac{k}{3}+2\right)^2$$

as required.

(Case 2) $k \equiv 1 \pmod 3$: Then $k-1 \equiv 0 \pmod 3$ so we have by inductive hypothesis $x_{k-1} \geq \left(\frac{k-1}{3}+2\right)^2$. Therefore,

$$x_k \quad = \quad x_{k-1} + \sqrt{x_{k-1}} \quad \geq \quad \left(\frac{k-1}{3}+2\right)^2 + \left(\frac{k-1}{3}+2\right)$$

as required.

(Case 3) $k \equiv 2 \pmod 3$: Then $k-1 \equiv 1 \pmod 3$ so we have by inductive hypothesis $x_{k-1} \geq \left(\frac{k-1}{3}+2\right)^2 + \left(\frac{k-1}{3}+2\right)$. We note that $\sqrt{x_{k-1}} \geq \left(\frac{k-1}{3}+2\right)$. Therefore,

$$x_k \quad = \quad x_{k-1} + \sqrt{x_{k-1}} \quad \geq \quad \left(\frac{k-1}{3}+2\right)^2 + 2\left(\frac{k-1}{3}+2\right)$$

as required.

Therefore, for all 3 cases, we have (1.11).

Now we can show (1.10) that, for given $x > 0$

$$f\left(\mathcal{X}(4), x\right) \quad \leq \quad 3\sqrt{x}.$$

By (1.11), if k is the greatest number satisfying

$$\left(\frac{k}{3}\right)^2 \quad \leq \quad x$$

then $f\left(\mathcal{X}(4), x\right) \leq k$ and $k \leq 3\sqrt{x}$.

Suppose that there is a set S of prime numbers such that $f(S,x) \geq 3\sqrt{x}$ and $n_{i+1} - n_i > \sqrt{n_i}$ for all $n_i \in \mathcal{N}(S)$. Then we get

$$f(\mathcal{X}(4), x) \quad \leq \quad 3\sqrt{x} \quad \leq \quad f(S,x) \quad \leq \quad f(\mathcal{N}(S), x).$$

This contradict to the relations (1.8) and (1.9).

Hence, for all S with required property $f(S,x) < 3\sqrt{x}$. Therefore, for any set S of prime numbers with required property, we have $p_n > Cn^2$. □

We have a non-trivial lower bound Cn^2 for p_n in S where S satisfies Wintner's condition with $n_{i+1} - n_i > \sqrt{n_i}$ for $n_i \in \mathcal{N}(S)$.

1.3.2 Upper bounds for p_n

Recall from the Remark on page 7, it is possible to find a sequence T_n such that $\frac{T_n}{2} \leq p_n \leq T_n$ and p_n have the desired property. In this section, we discuss this sequence T_n in more detail.

Theorem A There are effectively computable positive numbers c_1 and c_2 such that for any real number θ with $0 < \theta < 1$ there exists an infinite set S of prime numbers $p_1 < p_2 < \ldots$ for which the integers composed of the primes satisfy (1.7) with $\frac{1}{2}T\left(n\right) \leq p_n \leq T\left(n\right)$, where

$$T\left(n\right) \quad = \quad \exp\left(\frac{c_1 n^2}{\theta}\log\left(\frac{c_2 n}{\theta}\right)\right).$$

In Theorem A, we give an effectively computable upper number $T\left(n\right)$ such that there is a prime p_n in the interval $[\frac{T(n)}{2}, T(n)]$ for each $n = 1, 2, \ldots$ and when the n_i's are composed of the primes in the sequence $p_1 < p_2 < \ldots$ then the inequality (1.7) holds. In the proof of our Main Theorem A we apply Waldschmidt's estimate for linear forms in the logarithms of algebraic numbers and use nested recursive induction to construct $T(n)$ as required.

Our objective for Theorem A is the following :

For given $0 < \theta < 1$, we want to construct a set $S = \{p_1 < p_2 < \ldots\}$ of prime numbers such that

1. $n_{i+1} - n_i \quad > \quad n_i^{1-\theta}$ for all $n_i \in \mathcal{N}\left(S\right)$.
2. For given initial t primes $p_1 < \ldots < p_t \in S$ we can find the next prime p_{t+1} such that $p_{t+1} \leq T\left(t+1\right)$.
3. The sequence $T\left(t\right)$ is effectively computable in terms of t and grows slowly.

1.3.3 The Sequence of Gaps $n_{i+1} - n_i$

Now we go back to Tijdeman's answer to Wintner's conjecture with an additive point of view. We ask what make it possible for there to exist a set of primes with the required property.

We investigate the inequality

$$n_{i+1} - n_i \quad > \quad \frac{n_i}{F\left(n_i\right)}$$

where the n_i's are composed of a given set of primes and $F\left(x\right) < x^{\theta}$ for any θ with $0 < \theta < 1$. By Theorem 1.7, we can prove the following Proposition.

Proposition 1.2. *Let* $F\left(x\right) = \left(\log x\right)^C$ *for any real number* C*. Then, we cannot find infinitely many primes* $S = \{p_1 < p_2 < \ldots\}$ *such that if* $n_i \in \mathcal{N}\left(S\right)$ *then*

$$n_{i+1} - n_i \quad > \quad \frac{n_i}{F\left(n_i\right)}.$$

Proof. Suppose that there is a real number C and there is an infinite set S of primes such that

$$n_{i+1} - n_i \quad > \quad \frac{n_i}{\left(\log n_i\right)^C} \tag{1.13}$$

for all n_i in $\mathcal{N}\left(S\right)$. By Theorem 1.3, such a C is a positive number.

For the set $S = \{p_1 < p_2 < \ldots\}$, we consider a subset of S with initial r primes $S_r = \{p_1 < \ldots < p_r\}$ with $r > 2\left(C+1\right)$. By Theorem 1.7, there are infinitely many y, x in $\mathcal{N}\left(S_r\right)$ that satisfy (1.1). Moreover we can choose such x and y with $x > y > \exp\left(r^r p_r^r\right)$. We see that these x, y are in $\mathcal{N}\left(S\right)$ also. So there is a positive integer i such that $n_i = y$ and $n_i \in \mathcal{N}\left(S\right)$. Since, S satisfies Wintner's condition with respect to $F\left(a\right) = \left(\log a\right)^C$, and S_r is a subset of S we have

$$x - y \quad \geq \quad n_{i+1} - y \quad > \quad \frac{y}{\left(\log y\right)^C}.$$

Since, x and y satisfy (1.1) we have

$$\frac{y}{\left(\log y\right)^C} \quad < \quad x - y \quad < \quad \frac{\left(r \log p_r\right)^r \cdot y}{\left(\log y\right)^{r-1}}.$$

By our choices of r and x, y we see the above inequality gives us a contradiction as follows

$$1 \quad < \quad \frac{\left(r \log p_r\right)^r}{\left(\log y\right)^{r-1-C}} \quad < \quad \frac{r^r p_r^r}{\left(\log y\right)^{C+1}} \quad < \quad \frac{r^r p_r^r}{\left(r^r p_r^r\right)^{C+1}}.$$

$\square$

So, we want to figure out what conditions on the gaps $n_{i+1} - n_i$ with

$$\lim_{i \to \infty} (n_{i+1} - n_i) \quad = \quad \infty$$

allow us the possibility of finding an infinite set of primes satisfying Wintner's condition.

So, our objective is as follows:

To find a function $F(x)$ which grows as slowly as possible and yet for which there is an infinite set S of prime numbers such that $n_{i+1} - n_i > \frac{n_i}{F(n_i)}$ for $n_i \in \mathcal{N}(S)$.

In Chapter 4, we construct an infinite set S of primes for Wintner's question with respect to a family of functions which grow quite slowly. In particular, we prove the following result.

Theorem B Let θ be a real number with $0 < \theta < 1$ and k be a positive integer. For $a \geq \exp^k(1)$ we define

$$F(a) \quad = \quad \exp^k\left((\log_k a)^\theta\right).$$

There is an infinite set S of primes such that if $n_i, n_{i+1} \in \mathcal{N}(S)$ then

$$n_{i+1} - n_i \quad > \quad \frac{n_i}{F(n_i)}$$

where $\exp^k$ denotes the k-th iterated exponentiation and $\log_k$ denotes the k-th iterated logarithm.

In order to prove Theorem A and B we shall build on the argument given by Tijdeman [38] in his solution of Wintner's problem.

1.3.4 Computation

After finding theoretical upper bounds for p_n in Wintner's question, we want to find the initial few primes in the question practically. We shall review some related problems.

In 1974, Tijdeman and Meijer [40] found a relation between the convergents of the continued fraction of $\xi = \frac{\log p_1}{\log p_2}$ and the exponents in the sequence $\frac{n_{i+1}}{n_i}$ with $n_i, n_{i+1} \in \mathcal{N}(\{p_1, p_2\})$. They considered one-sided convergents to $\xi = \frac{\log p_1}{\log p_2}$ as defined below :

Let ξ be an irrational number with the continued fraction expansion $[a_0, a_1, \ldots]$. The n-th convergent $[a_0, \ldots, a_n]$ to ξ is denoted by $\frac{A_n}{B_n}$. We recall that for $n = 0, 1, \ldots$

$$A_{n+1} = a_{n+1}A_n + A_{n-1}$$

$$B_{n+1} = a_{n+1}B_n + B_{n-1}$$

where we define $A_{-1} = 1,\ A_0 = a_0,\ B_{-1} = 0$ and $B_0 = 1$.

Definition. A rational number $\frac{A}{B}$ is said to be *one-sided convergent* to $\xi = [a_0, a_1, \ldots]$ if there is a non-negative integer n such that for the n-th convergent $\frac{A_n}{B_n}$ to ξ and for some j with $1 \leq j \leq a_{n+1}$ we have

$$\frac{A}{B} = \frac{jA_n + A_{n-1}}{jB_n + B_{n-1}}.$$

They showed that

Theorem 1.10 (Tijdeman, Meijer [40])**.** *Let α, β be real numbers with $\alpha > \beta > 1$ and such that $\xi = \frac{\log \beta}{\log \alpha}$ is irrational. Let $n_1 < n_2 < \ldots$ be the sequence composed of α and β i.e., for all i we can express $n_i = \alpha^{a_i}\beta^{b_i}$ for some non-negative integers a_i, b_i. Let*

$$W = \left\{ \frac{n_{i+1}}{n_i} \,\middle|\, i = 1, 2, \ldots \right\}.$$

Then W is the set of all products $\alpha^{-k}\beta^{l}$ and $\alpha^{k}\beta^{-l}$ which are greater than 1 and such that $\frac{k}{\ell}$ is a one-sided convergent to ξ.

We note that if β and α are different primes then $\xi = \frac{\log \beta}{\log \alpha}$ is irrational.

In 1982, Stroeker and Tijdeman [37] found all the positive integer solutions a, b of the inequality

$$\left| p^a - q^b \right| < p^{\frac{a}{2}} \tag{1.14}$$

for all primes p, q with $p < q < 20$.

They first proved that the linear form

$$\Lambda = a \log p - b \log q$$

has a value close to zero when a, b is a solution of (1.14). And then they split the exponents a in (1.14) into three cases : a is "very large", a is "medium large" and

a is “small”. These cases correspond approximately to $a \geq 2^{43}$, $10 < a < 2^{43}$ and $a \leq 10$.

When they applied the estimate of linear forms in logarithms of p, q, they got a “very large” bound of $M_1 = \max\{a, b\}$ such that if $a \geq M_1$ then there is no solution of (1.14). This is because Baker’s theory implies that the linear forms cannot be close to zero so that there is no solution with “very large” a. In order to solve (1.14), in the “medium large” of a, they can avoid checking all the a in the range. For this they investigated the size of the linear forms $a \log p - b \log q$. If

$$\left| \frac{\log p}{\log q} - \frac{b}{a} \right| \quad < \quad \frac{1}{2a^2}$$

then $\frac{b}{a}$ is a convergent of the continued fraction of $\frac{\log p}{\log q}$. Hence it is suffice to check only the a’s which are denominators of the convergents of the continued fraction of $\frac{\log p}{\log q}$. Finally, for “small” values of a they calculate directly. They found that all solutions of (1.14) have “small” a.

In the 1980’s, de Weger gave computational methods to reduce the upper bounds for the solution of Diophantine equations. He studied a linear form Λ that is close to 0 together with a large but explicitly known upper bound for the absolute values of the coefficients of Λ. And then he showed that there is no solution between the known bound and the reduced bound he computed. In 1987, de Weger [45] gave a table with numerical data for the following inequalities :

$$\left| p^a - q^b \right| \quad < \quad \left(\min\{p^a, q^b\} \right)^{\delta} \tag{1.15}$$

for p, q primes such that $p < q < 200$ and a, b positive integers with $a \geq 2, b \geq 2$ and either $\delta = \frac{1}{2}$ or $\delta = 0.9$, $\min\{p^a, q^b\} > 10^{15}$.

In Chapter 5, we shall investigate the inequality (1.15) where $\delta = \frac{1}{2}$ by computational methods because much sharper estimates have been established on linear forms in logarithms. In addition supercomputers and computer technology have improved greatly since the 1980’s, and there are computer packages for performing various number-theoretic calculations. We shall apply the estimates for linear forms in 2 logarithms by Laurent, Mignotte and Nesterenko [22], follow the ideas that have been applied in computation by Stroeker and Tijdeman [37] and de Weger [45], and then use MAPLE for number-theoretic calculation, specifically continued fraction expansion for a given real number to a certain precision. In this way we prove the follow result.

Theorem C There are 2086 pairs of prime numbers (p_1, p_2) with $2 \leq p_1 < p_2 <$

e^8 such that

$$x - y \quad < \quad \sqrt{y}$$

where $x, y \in \mathcal{N}(\{p_1, p_2\})$ with $y < x$, $\gcd(x, y) = 1$. And they are listed in the Table I (page 71).

It follows from the proof of Theorem A that if p_1 and p_2 are prime numbers with $p_1 < p_2$ and for which $n_{i+1} - n_i \geq \sqrt{n_i}$ where n_i is the i-th term in $\mathcal{N}(\{p_1, p_2\})$ then we may extend $\{p_1, p_2\}$ to an infinite set $S = \{p_1, p_2, \ldots\}$ of prime numbers for which $n_{i+1} - n_i \geq \sqrt{n_i}$ but with now n_i the i-th term in $\mathcal{N}(S)$.

Chapter 2

Preliminaries

2.1 Definitions

We define some terminology we will use in our thesis

Definition Let α be an algebraic number of degree d over $\mathbf{Q}$ with conjugates $\sigma_1\alpha, \ldots, \sigma_d\alpha$ and minimal polynomial

$$c_0X^d + \cdots + c_d \quad = \quad c_0 \cdot \prod_{i=1}^{d} (X - \sigma_i\alpha)$$

where c_i's are integers with $c_0 > 0$.

1. Height (or classical height) $H(\alpha)$ is defined by

$$H(\alpha) \quad = \quad \max\{c_0, |c_1|, \ldots, |c_d|\}.$$

2. Weil's absolute logarithmic height $h(\alpha)$ of α is defined by

$$h(\alpha) \quad = \quad \frac{1}{d} \cdot \left(\log c_0 + \sum_{i=0}^{d} \log \max\{1, |\sigma_i\alpha|\} \right).$$

2.2 Linear Forms in Logarithms

2.2.1 Baker's Theorems

In the 1960's, Baker made a major breakthrough in transcendental number theory in his celebrated series of papers [1, 2, 3, 4].

Theorem 2.1 (Baker). *If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are non-zero algebraic numbers such that*

$\log \alpha_1, \ldots, \log \alpha_n$ are linearly independent over the field of rational numbers

then $1, \log \alpha_1, \ldots, \log \alpha_n$ are linearly independent over the field of all algebraic numbers where log *denotes the principal branch of the logarithm functions.*

Theorem 2.2 (Baker). *If $\alpha_1, \alpha_2, \ldots, \alpha_n$, $\beta_1, \beta_2, \ldots, \beta_n$ are non-zero algebraic numbers and*

$$\Lambda = \beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n,$$

where log *denotes the principal branch of the logarithm functions then $\Lambda = 0$ or Λ is transcendental.*

2.2.2 Trivial Estimate

In the special case that all α_i and β_i are rational integers we have the following trivial estimates.

Proposition 2.1. *Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are rational integers with the a_i greater than 1. We assume that*

$$a_1^{b_1} \cdots a_n^{b_n} \neq 1.$$

Then

$$\left| a_1^{b_1} \cdots a_n^{b_n} - 1 \right| \geq \exp\left(-nB \log A\right)$$

where $B = \max\{|b_1|, \ldots, |b_n|\}$ and $A = \max\{a_1, \ldots, a_n\}$.

Proof. We know that the absolute value of a non-zero rational number is at least as large as the inverse of a denominator so

$$\begin{aligned}\left|a_1^{b_1}\cdots a_n^{b_n}-1\right| &\geq \prod_{b_i<0} a_i^{b_i} \\ &\geq \exp\left(-\sum_{i=1}^{n}|b_i|\log a_i\right) \\ &\geq \exp\left(-nB\log A\right). \qquad (2.1)\end{aligned}$$

□

We shall call (2.1) *Liouville's inequality.* The dependence in n and A in Liouville's inequality is sharp, but the main interest for applications is with the dependence in B.

2.2.3 Estimates on Linear Forms in Logarithms

Baker gives effective lower bounds for $|\Lambda|$ in the case $\Lambda \neq 0$.

Baker's work affected a wide range of research, directed both towards improving his estimates and to applying them to specific arithmetic problems. Many problems of Diophantine analysis reduce to lower estimates for the absolute values of the Λ. The bounds are given as functions of the degrees and the heights of these numbers $\alpha_1, \alpha_2, \ldots, \alpha_n,\ \beta_1, \beta_2, \ldots, \beta_n$. Baker's general effective estimates led to significant applications in number theory and opened a new era in the theory of Diophantine equations. In the 1970's and 1980's Baker, Fel'dman, Stark, Waldschmidt, Wüstholz and many others gave quantitative estimates for the bounds. The bounds have been improved in terms of heights and other parameters over the years.

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be non-zero algebraic numbers with $\alpha_i \neq 1$ for $i = 1, \ldots, n$. Let $\mathbf{Q}(\alpha_1, \ldots, \alpha_n)$ have degree at most d over $\mathbf{Q}$. Let the heights of α_i be $H(\alpha_i) \leq A_i$ where $A_i \geq 4$ for $i = 1, \ldots n$. Put $\Omega = (\log A_1)\cdots(\log A_n)$, $\Omega' = (\log A_1)\cdots(\log A_{n-1})$. Let $\beta_1, \beta_2, \ldots, \beta_n$ be algebraic numbers with the classical heights $H(\beta_i) \leq B$ where $B \geq 4$ for $i = 1, \ldots n$. Let

$$\Lambda = \beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n,$$

where log denotes the principal branch of the logarithm functions.

In 1977, Baker proved that

Theorem 2.3 (Baker)**.** *If* $\Lambda \neq 0$ *then*

$$|\Lambda| \quad > \quad (B\Omega)^{-C\Omega\log\Omega'}$$

where $C = (16nd)^{200n}$*. In the special case that if* $\beta_1, \ldots \beta_n$ *are rational integers then the bracketed factor* Ω *has been eliminated to yield*

$$|\Lambda| \quad > \quad B^{-C\Omega\log\Omega'}.$$

This bound has been improved in terms of the constants and the factor $\Omega \cdot \log\Omega'$.

In 1993, Baker and Wüstholtz [11] proved the following Theorem.

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be non-zero algebraic numbers with $\alpha_i \neq 1$ for $i = 1, \ldots, n$ and $\mathbf{Q}(\alpha_1, \ldots, \alpha_n)$ have degree at most d over $\mathbf{Q}$. Let $b_1, \ldots, b_n$ be rational integers, not all 0 with $B = \max\{|b_1|, \ldots, |b_n|, e^{\frac{1}{d}}\}$ and $A_i = \max\{H(\alpha_i), e\}$ for $i = 1, \ldots, n$. Let $\Omega = \log A_1 \cdots \log A_n$.

Theorem 2.4 (Baker-Wüstholtz)**.** *If* $\Lambda \neq 0$ *then*

$$|\Lambda| \quad \geq \quad \exp\left(-C\left(n,d\right)\,\Omega\,\log B\right)$$

where $C(n,d) = (16nd)^{2n+4}$.

We see that this estimation is fully explicit with respect to all parameters. Moreover, we note that the factor $\Omega' = \log A_1 \cdots \log A_{n-1}$ in Theorem 2.3 has been removed.

It is conjectured that the product of the logarithms in $\Omega = \log A_1 \cdots \log A_n$ may be replaced by the sum of logarithms.

Conjecture (Lang-Waldschmidt). Let $a_1, \ldots, a_n$ be positive rational numbers and $b_1, \ldots, b_n$ be integers. For $j = 1, \ldots, n$ let $B_j = \max\{H(b_j), 1\}$, $A_j = H(a_j)$, $B = \max\{B_1, \ldots, B_n\}$, $A = \max\{A_1, \ldots, A_n\}$ and $\Lambda = b_1 \log a_1 + \cdots + b_n \log a_n$.

Let $\epsilon > 0$. There exists $C(\epsilon) > 0$ depending only on ϵ such that if $|\Lambda| \neq 0$ then

$$|\Lambda| \quad > \quad \frac{C(\epsilon)^n\, B}{\left(B_1 \cdots B_n \cdot A_1^2 \cdots A_n^2\right)^{1+\epsilon}}.$$

Remark ([20, p.213]). This conjecture is motivated from the uniform distribution. Suppose that $B_1, \ldots, B_n, A_1, \ldots, A_n$ are sufficiently large. Let $\mathcal{S}$ be the set of numbers

$$b_1 \log a_1 + \cdots + b_n \log a_n$$

with $H(b_j) \leq B_j$ and $H(a_j) \leq A_j$ for $j = 1, \ldots, n$. Since b_j are integers and a_j are rational numbers for $j = 1, \ldots, n$, $\mathcal{S}$ has cardinality at most

$$(2B_1 + 1) \cdots (2B_n + 1) \cdot (2A_1 + 1)^2 \cdots (2A_n + 1)^2.$$

This set $\mathcal{S}$ is contained in the interval

$$[-nB \log A, nB \log A].$$

If this set is uniformly distributed in this interval, then the distance from 0 to the closest non-zero element of $\mathcal{S}$ in absolute vale would be

$$\frac{2nB \log A}{(2B_1 + 1) \cdots (2B_n + 1) \cdot (2A_1 + 1)^2 \cdots (2A_n + 1)^2}.$$

This motivates their conjecture.

For Diophantine equations the first application of Baker's estimates were given by Baker himself and by Baker and Davenport [6]. In the last forty years very extensive Diophantine investigations were made by using Baker's theory on linear forms in logarithms. For various general classes of equations, theorems regarding upper bounds for the solutions of the equation have been established. These provide explicit upper bounds on the solutions.

In many applications only two or three logarithms occur. In these cases bounds with better constants are available. In 1995, Laurent, Mignotte, and Nesterenko [22] gave the following estimates for linear forms in two logarithms of algebraic numbers.

Let α_1, α_2 be non-zero algebraic numbers and suppose they are multiplicatively independent. Let $\mathbf{Q}(\alpha_1, \alpha_2)$ have degree at most D over $\mathbf{Q}$. Let $A_i > 1$ be a real number satisfying

$$\log A_i \quad \geq \quad \max\left\{ h(\alpha_i), \frac{|\log \alpha_i|}{D}, \frac{1}{D} \right\},$$

where log denotes the principal branch of logarithm. Further, let b_1 and b_2 be two positive integers. Define

$$b' \quad = \quad \frac{b_1}{D \log A_2} + \frac{b_2}{D \log A_1}$$

and

$$\log B \quad = \quad \max \left\{ \log b', \frac{21}{D}, \frac{1}{2} \right\}.$$

Lemma 2.1 (Laurent, Mignotte, Nesterenko [22]). *Let $\Lambda = b_2 \log \alpha_2 - b_1 \log \alpha_1$. Then*

$$|\Lambda| \quad \geq \quad \exp\left(-30.9 D^4 (\log B)^2 \log A_1 \log A_2\right).$$

In the proof of this Lemma, they applied Laurent's interpolation determinants and a refined zero estimate due to Nesterenko. And the constant 30.9 is much smaller than 270 from the previous estimates due to Mignotte-Waldschmidt [27].

2.2.4 Sharpening Estimates

Baker refined his estimates from [1, 2, 3] and [4] in a new series of papers [7, 8, 9] generalized and deepened them. His estimate is best possible for both fixed A and variable B and for fixed B and variable A. He later generalized this result, obtaining the following theorem.

Theorem 2.5 (Baker [7]). *Let $a_1, a_2, \ldots, a_{n+1}$ be non-zero algebraic numbers with degrees at most d. Suppose that the heights of $a_1, a_2, \ldots, a_n$ and a_{n+1} are at most A_n and $A \geq 2$ respectively. There is an effectively computable number $C > 0$ depending on n, d and A_n such that*

$$0 \quad < \quad |b_1 \log a_1 + \cdots + b_{n+1} \log a_{n+1}| \quad < \quad C^{-\log A \log B}$$

have no solution in rational integers $b_1, b_2, \ldots, b_{n+1}$ with absolute values at most $B \geq 2$.

And he established the following generalization.

Theorem 2.6 (Baker [8]). *Let $a_1, a_2, \ldots, a_{n+1}$ be non-zero algebraic numbers with degrees at most d. Suppose that the heights of $a_1, a_2, \ldots, a_n$ and a_{n+1} are at most A_n and $A \geq 2$ respectively. There is an effectively computable number C, depending only on n, d and A_n such that, for any θ with $0 < \theta < \frac{1}{2}$, the inequalities*

$$0 < |b_1 \log a_1 + \cdots + b_{n+1} \log a_{n+1}| < \left(\frac{\theta}{B_{n+1}}\right)^{C \log A} \exp(-\theta B_n) \quad (2.2)$$

have no solution in rational integers $b_1, \ldots, b_n$ *and* b_{n+1} $(\neq 0)$ *with absolute values at most* B_n *and* B_{n+1}*, respectively.*

Note that, on taking $\theta = \frac{1}{B_n}$ and assuming that $B_n \leq B_{n+1}$ we obtain the result of Theorem 2.5.

His generalized sharpening of the bounds for linear forms in logarithms (2.2) has a particular significance in connection with applications. Specifically, Tijdeman [38] applied Theorem 2.6 in order to prove Wintner's conjecture.

Baker's works generalized Gelfond's method. In [42], Waldschmidt gave estimates for linear forms in logarithms based on Schneider's method. He also gives a lower bound for linear forms in logarithms of algebraic numbers in integer coefficients with an explicit constant. Finally, Waldschmidt [43] stated, using an extended method of Schneider, a completely explicit lower bound when $\beta_1, \ldots, \beta_n$ are rational integers.

In our proofs of Theorem A and Theorem B, we applied a Theorem of Waldschmidt, which is the subject of the next section.

2.2.5 Waldschmidt's Theorem

For any rational number x we may write $x = \frac{b}{a}$ with a and b co-prime integers. We see the height of x to be the maximum of $|a|$ and $|b|$.

Let $a_1, \ldots, a_n$ and a_{n+1} be rational numbers with heights at most $A_1, \ldots, A_n$ and A_{n+1} respectively. We shall suppose that $A_i \geq 4$ for $i = 1, \ldots, n+1$. Next let $b_1, \ldots, b_n$ and b_{n+1} be rational integers. Suppose that B and B_{n+1} are positive real numbers with $B \geq |b_j|$ for $j = 1, \ldots, n$ and $B_{n+1} \geq \max(3, |b_{n+1}|)$. Put

$$\Lambda = b_1 \log a_1 + \cdots + b_n \log a_n + b_{n+1} \log a_{n+1},$$

$$\Omega_n = \log A_1 \log A_2 \cdots \log A_n,$$

where log denotes the principal branch of the logarithm functions.

Lemma 2.2 (Waldschmidt [43]). *There exists an effectively computable positive number C such that if $\Lambda \neq 0$ then*

$$|\Lambda| \quad > \quad \exp\left(-C\,(n+1)^{4(n+1)}\ \Omega_n\ \log A_{n+1}\ \log\left(B_{n+1} + \frac{B}{\log A_{n+1}}\right)\right).$$

Remark We shall include in the thesis the proof of Lemma 2.2 given by Stewart and Tijdeman for completeness. The proof may be found in [34, Lemma 1].

Proof of Lemma 2.2 This follows from the estimates by Waldschmidt [43, Corollaire 10.1]. He proved this result under the assumption that $b_{n+1} \neq 0$. If $b_{n+1} = 0$ then we apply the same theorem with b_{n+1} replaced by b_j where j is the largest integer for which $b_j \neq 0$. Notice that $j \geq 1$ since $\Lambda \neq 0$. Since $\log A_{n+1} \log\left(3 + \frac{B}{\log A_{n+1}}\right)$ is larger than $\frac{1}{2}\log B$, the result follows. □

Remark In Lemmas 2.1 and 2.2, the logarithms are supposed to have their principal values, but this is not a restriction, since we shall be concerned exclusively with positive real numbers.

2.3 Explicit Determination

Baker [5] showed in 1968 how his estimates for linear forms of logarithms of algebraic numbers can be used to give effective upper bounds for the solutions of the Thue equation. Then Baker and Davenport [6] introduced a simple but powerful lemma, the so-called Davenport's Lemma, that is related to Diophantine approximation. Applying this lemma they found much smaller upper bounds for the solutions. They then combined the reduction algorithms and computational techniques to find all the solutions of certain types of equations practically.

Győry [17] reviewed some classical strategy for solving some classes of Diophantine equations or inequalities while applying Baker's theory. The main steps are as follows.

1. Transform the equation into a purely exponential equation i.e., a Diophantine equation where the unknowns are all in the exponents. Each type of equation needs a particular kind of transformation. It uses some arguments from algebraic number theory, theory of recurrence sequences, and geometry of numbers. This transformation makes it possible to apply Baker's theory.

2. Apply Baker's theory to derive an explicit upper bound for the solutions. In general, the upper bounds are so large that they cannot be used to determine all solutions in practice.

3. Reduce the explicit upper bound to a much smaller bound. In this step we apply theory from Diophantine approximations.

4. Determine all the solutions under the smaller bound from above step, using some search techniques with computation and specific properties of the initial equation.

In Chapter 5, we shall apply the above strategy and procedure used by Stroeker, Tijdeman [37] and de Weger [45] for finding the first two primes p_1, p_2 so that $n_0 < n_1 < \ldots$ is the sequence of integers composed of the two primes then $n_{i+1} - n_i > \sqrt{n_i}$ for all $i = 0, 1, \ldots$.

Chapter 3

First Main Theorem

In this chapter, we shall show that for a given real number θ with $0 < \theta < 1$, we can find an infinite set $S = S(\theta) = \{p_1, p_2, \ldots\}$ of primes with $p_1 < p_2 < \ldots$ such that n-th term p_n in S does not grow too quickly and if $n_i \in \mathcal{N}(S)$ then ${n_{i+1} - n_i > n_i}^{1-\theta}$ for $i = 0, 1, 2, \ldots$.

In particular, for a given $0 < \theta < 1$, we shall find sequence $T(n) = T(n, \theta)$ such that

1. $T(n)$ is effectively computable and grows slowly.
2. We can find the n-th prime p_n in S with $\frac{1}{2}T(n) \leq p_n \leq T(n)$.
3. If $n_0 < n_1 < \ldots$ is the sequence of all integers composed of the primes in S then

$$n_{i+1} - n_i \quad > \quad n_i^{1-\theta} \tag{3.1}$$

   for $i = 0, 1, 2, \ldots$.

3.1 Lemma

We give an auxiliary lemma due to Pethö and de Weger [28]. This one enables us to find an upper bound in closed form for some real number $x > 1$ that is bounded by a polynomial in $\log x$.

Lemma 3.1 (Pethö, de Weger [28]). *Let $u \geq 0$, $v > 0$, $h \geq 1$ and let x be a real number with $x > 1$ satisfying*

$$x \quad \leq \quad u + v (\log x)^h .$$

If $v > \left(\frac{e^2}{h}\right)^h$ then

$$x \quad < \quad 2^h \left(u^{\frac{1}{h}} + v^{\frac{1}{h}} \log\left(h^h v\right)\right)^h$$

and if $v \leq \left(\frac{e^2}{h}\right)^h$ then

$$x \quad \leq \quad 2^h \left(u^{\frac{1}{h}} + 2e^2\right)^h .$$

Remark We shall include in this thesis the proof of this Lemma for completeness. The proof may be found in [45, Lemma 2.1]. We can see also [28, Lemma 2.2].

Proof of Lemma 3.1 Because x is bounded above, we may assume that x is the largest solution of

$$x \quad = \quad u + v (\log x)^h .$$

Since, $x^{\frac{1}{h}}$ is concave when $h \geq 1$ if z_1 and z_2 are positive real numbers then

$$(z_1 + z_2)^{\frac{1}{h}} \quad \leq \quad z_1^{\frac{1}{h}} + z_2^{\frac{1}{h}},$$

hence we have

$$x^{\frac{1}{h}} \quad \leq \quad u^{\frac{1}{h}} + c \cdot \log\left(x^{\frac{1}{h}}\right)$$

where $c = hv^{\frac{1}{h}}$. Define y by

$$x^{\frac{1}{h}} \quad = \quad (1 + y)\, c \log c.$$

If $c \geq e^2$ then from

$$\log c \quad < \quad \log (c \log c)$$

it follows that

$$c^h (\log c)^h \quad < \quad v \left(\log\left(c^h (\log c)^h\right)\right)^h ,$$

which implies

$$x \quad > \quad c^h (\log c)^h .$$

Hence $y > 0$. Now, we see that

$$\begin{array}{rclcl} (1+y)\, c \log c & = & x^{\frac{1}{h}} & \leq & u^{\frac{1}{h}} + c \log (1+y) + c \log c + c \log \log c \\ & & & < & u^{\frac{1}{h}} + cy + c \log c + c \log \log c. \end{array}$$

Therefore,

$$yc\,(\log c - 1) \quad < \quad u^{\frac{1}{h}} + c \log \log c.$$

Since, $c \geq e^2$

$$\begin{array}{rcl} x^{\frac{1}{h}} & = & c \log c + yc \log c \\ & < & c \log c + \dfrac{\log c}{\log c - 1} \left(u^{\frac{1}{h}} + c \log \log c \right) \\ & < & 2 \left(u^{\frac{1}{h}} + c \log c \right) \end{array}$$

as required. If $c \leq e^2$ then note that $x \leq u + \left(\frac{e^2}{h}\right)^h (\log x)^h$. So, we may assume $c = e^2$ in this case. The result follows. □

3.2 Terminology

Definition 3.1. *Let $F(x)$ be a function with $\lim_{x \to \infty} F(x) = \infty$. A set S of prime numbers is said to satisfy Wintner's condition with respect to F if*

(1) *S is infinite.*

(2) *For any $n_i, n_{i+1} \in \mathcal{N}(S)$, $\quad n_{i+1} - n_i > F(n_i)$.*

In this thesis, we reserve p for a prime number, S as a subset of prime numbers with ordering

$$S \quad = \quad \{p_1 < p_2 < \ldots\},$$

and $\mathcal{N}(S)$ as the set of all positive integers composed of primes in S with ordering

$$\mathcal{N}(S) \quad = \quad \{n_0 < n_1 < n_2 < \ldots\}.$$

3.3 First Main Theorem

Theorem 3.1. *There are effectively computable positive numbers c_1 and c_2 such that for any real number θ with $0 < \theta < 1$ there exists an infinite set S of prime numbers $p_1 < p_2 < \ldots$ which satisfies Wintner's question with respect to x^θ and all p_n in S we have $\frac{1}{2}T(n) \leq p_n \leq T(n)$ where*

$$T(n) \quad = \quad \exp\left(\frac{c_1 n^2}{\theta}\log\left(\frac{c_2 n}{\theta}\right)\right).$$

Proof. Given $0 < \theta < 1$. Let $c_1 = 2^7$ and $c_2 = 2C$ where C is the effectively computable positive number in Lemma 2.2. Put

$$T(n) \quad = \quad \exp\left(\frac{c_1 n^2}{\theta}\log\left(\frac{c_2 n}{\theta}\right)\right) \tag{3.2}$$

for $n = 1, 2, \ldots$. Note that for the given $0 < \theta < 1$

$$T(1) \quad < \quad T(2) \quad < \quad \ldots$$

and

$$2T(n) \quad < \quad T(n+1). \tag{3.3}$$

We will use induction on n to prove our result.

When $n = 1$. We can take a prime p_1 with $\frac{1}{2}T(1) \leq p_1 \leq T(1)$ since Rosser and Schoenfeld [31] proved that for an integer T with $T \geq 41$, the number of primes in the interval $[\frac{T}{2}, T]$ is greater than $\frac{3T}{10\log T}$ and we see that $T(1) \geq 41$. Let $S_1 = \{p_1\}$. We know that the $n_i \in \mathcal{N}(S_1)$ can be expressed by $n_i = p^i$ for $i = 0, 1, 2, \ldots$ so that

$$n_{i+1} - n_i \quad = \quad {p_1}^{i+1} - {p_1}^{i} \quad = \quad {p_1}^{i}(p_1 - 1) \quad > \quad p_1^i \quad \geq \quad \left(p_1^i\right)^{1-\theta}$$

for $i = 0, 1, 2, \ldots$. Therefore, (3.1) holds for powers of p_1.

Now suppose that we have $S_n = \{p_1 < p_2 < \ldots < p_n\}$ satisfying $\frac{1}{2}T(j) \leq p_j \leq T(j)$ for $j = 1, 2, \ldots, n$ and if $n_0 < n_1 < \ldots < n_i < \ldots$ is the sequence of all positive integers composed of the primes in S_n then $n_{i+1} - n_i > n_i^{1-\theta}$ for $i = 0, 1, 2, \ldots$.

We claim that we can find the next prime $p_{n+1} > p_n$ satisfying $\frac{1}{2}T(n+1) \leq p_{n+1} \leq T(n+1)$ such that if $n_0 < \ldots < n_i < \ldots$ is the sequence of all positive integers composed of the primes in $S_n \cup \{p_{n+1}\}$ then $n_{i+1} - n_i > n_i^{1-\theta}$ for $i = 0, 1, 2, \ldots$.

Put $T = T(n+1)$ for brevity. Consider any prime p with $p \in [T, \frac{T}{2}]$. Then by (3.3), $p_n < p$. Suppose that there are $y, x \in \mathcal{N}(S_n \cup \{p\})$ such that

$$0 \quad < \quad x - y \quad < \quad y^{1-\theta}. \tag{3.4}$$

Suppose $y = 1$ and $x \geq p_1$ then $x - y \geq p_1 - 1 > 1$ holds for any $p_1 > 2$. Therefore, $y > 1$. In particular we note that since y is less than x and $0 < \theta < 1$, we get

$$y \quad < \quad x \quad < \quad y + y^{1-\theta} \quad < \quad 2y. \tag{3.5}$$

Let $y = {p_1}^{a_1}{p_2}^{a_2} \cdots {p_n}^{a_n} p^a$ and $x = {p_1}^{b_1}{p_2}^{b_2} \cdots {p_n}^{b_n} p^b$ be the prime factorizations of y and x, respectively. Then we can see that $a \neq b$, since if $a = b$ then we can consider $y' = \frac{y}{p^a}, x' = \frac{x}{p^b} = \frac{x}{p^a}$ and by (3.4) we get

$$0 \quad < \quad x - y \quad = \quad p^a (x' - y') \quad < \quad p^a (y')^{1-\theta}.$$

But this contradicts our inductive hypothesis since $y', x' \in \mathcal{N}(S_n)$, hence

$$x' - y' \quad > \quad y'^{(1-\theta)}.$$

Therefore, $a \neq b$.

Put $\Lambda = \log \frac{x}{y}$ where log denotes the principal branch of logarithm. Then,

$$\Lambda \quad = \quad \sum_{j=1}^{n} (b_j - a_j) \log p_j + (b - a) \log p \quad > \quad 0.$$

Further, by (3.4)

$$0 \quad < \quad \log \frac{x}{y} \quad < \quad \frac{x}{y} - 1 \quad < \quad y^{-\theta}. \tag{3.6}$$

Furthermore, since $a_j, b_j \geq 0$ and $3 \leq p_j$ for $j = 1, 2, \ldots, n$ so by (3.5) we have

$$|b_j - a_j| \quad \leq \quad \max_{j=1,\ldots,n} (b_j, a_j) \quad \leq \quad \max(\log x, \log y) \quad < \quad \log 2y \tag{3.7}$$

and since $a, b \geq 0$, we have

$$|b - a| \quad \leq \quad \max(b, a) \quad \leq \quad \max\left(\frac{\log x}{\log p}, \frac{\log y}{\log p}\right) \quad < \quad \frac{\log 2y}{\log p}. \tag{3.8}$$

Now we suppose that $y \geq p^8$. Then $\frac{\log 2y}{\log p} > 3$. Applying Lemma 2.2 with $A_i = T(i)$ for $i = 1, \ldots, n$, $A_{n+1} = p$, $B = \log 2y$ and $B_{n+1} = \frac{\log 2y}{\log p}$ to our $\Lambda \neq 0$, we see that there exists an effectively computable constant C such that

$$\Lambda \quad > \quad \exp\left(-C\,(n+1)^{4(n+1)} \log T\,(1) \cdots \log T\,(n) \log p \log\left(4\frac{\log 2y}{\log p}\right)\right).$$

Then by (3.3), (3.5) and (3.6),

$$y^{\theta} \quad < \quad \exp\left(C\,(n+1)^{4(n+1)} \left(\log T\,(n)\right)^n \log p \log\left(4\frac{\log 2y}{\log p}\right)\right).$$

Now, we take logarithms of both sides and divide by $\theta \log p$ to get

$$\frac{\log y}{\log p} \quad < \quad \frac{1}{\theta} C\,(n+1)^{4(n+1)} \left(\log T\,(n)\right)^n \log\left(4\frac{\log 2y}{\log p}\right). \tag{3.9}$$

Let

$$X \quad = \quad \frac{\log y}{\log p}\,.$$

Then since $y \geq p^8$ we obtain from (3.9) that

$$X \quad < \quad C_1 \log(8X) \quad \leq \quad 2C_1 \log X \tag{3.10}$$

where $C_1 = \frac{1}{\theta} C\,(n+1)^{4(n+1)} \left(\log T\,(n)\right)^n$.

We apply Lemma 3.1 with $u = 0$, $v = 2C_1$, $h = 1$ and $x = X > 8$ to (3.10) then

$$X \quad \leq \quad 2\left(2C_1 \log\left(2C_1\right)\right).$$

Therefore, we can have

$$X \quad < \quad \frac{1}{2} U\,(n) \tag{3.11}$$

where

$$U\,(n) \quad = 16C_1^2.$$

If $y < p^8$ and so $X < 8$ then we also have that (3.11) holds.

Further, by (3.7) and (3.8), we see

$$a_j, b_j \quad \leq \quad U(n) \log p \tag{3.12}$$

for $j = 1, 2 \ldots, n$ and

$$a, b \quad \leq \quad \frac{\log 2y}{\log p} \quad \leq \quad 2X \quad \leq \quad U(n). \tag{3.13}$$

Hence, for each prime $p \in [\frac{T}{2}, T]$, the number of possible pairs (y, x) for which $0 < x - y < y^{1-\theta}$ with $y = {p_1}^{a_1}{p_2}^{a_2} \cdots {p_n}^{a_n} p^a$ and $x = {p_1}^{b_1}{p_2}^{b_2} \cdots {p_n}^{b_n} p^b$ is at most the number of possible choices of the exponents $a_1, \ldots, a_n, a, b_1, \ldots, b_n$ and b. Moreover, from (3.12) and (3.13) it is at most

$$(U(n) \log T + 1)^{2n} (U(n) + 1)^2 \quad < \quad (U(n) + 1)^{2n+2} (\log T)^{2n}. \tag{3.14}$$

Now, we assume that these exponents $a_1, \ldots, a_n, a,\ \ b_1, \ldots, b_n,\ b$ are fixed. Then by (3.5)

$$1 \quad < \quad \frac{x}{y} \quad = \quad {p_1}^{b_1 - a_1}{p_2}^{b_2 - a_2} \cdots {p_n}^{b_n - a_n} p^{b-a} \quad < \quad 1 + y^{-\theta}.$$

Put $K = {p_1}^{a_1 - b_1}{p_2}^{a_2 - b_2} \cdots {p_n}^{a_n - b_n}$. Then,

$$K \quad < \quad p^{b-a} \quad < \quad K\left(1 + y^{-\theta}\right) \tag{3.15}$$

Since, $a \neq b$, we have 2 cases.

(Case 1) $b > a$. Then,

$$\begin{aligned} K^{\frac{1}{b-a}} \quad < \quad p \quad &< \quad K^{\frac{1}{b-a}}\left(1 + y^{-\theta}\right)^{\frac{1}{b-a}} \\ &< \quad K^{\frac{1}{b-a}}\left(1 + y^{-\theta}\right) \end{aligned}$$

Hence, p is contained in a fixed interval of length

$$K^{\frac{1}{b-a}} \cdot y^{-\theta}$$

and by (3.15) and $p \in [\frac{T}{2}, T]$

$$K^{\frac{1}{b-a}}\left(y^{-\theta}\right) \quad < \quad py^{-\theta} \quad \leq \quad Ty^{-\theta}.$$

(Case 2) $b < a$. Then, by (3.15)

$$K^{\frac{1}{b-a}}\left(1+y^{-\theta}\right)^{\frac{1}{b-a}} \quad < \quad p \quad < \quad K^{\frac{1}{b-a}}. \tag{3.16}$$

So, p is contained in an interval of length $K^{\frac{1}{b-a}}\left(1-\left(1+y^{-\theta}\right)^{\frac{1}{b-a}}\right)$. Moreover, by (3.16) we see that $K^{\frac{1}{b-a}} < p\left(1+y^{-\theta}\right)^{\frac{1}{a-b}}$. Hence, we have

$$\begin{aligned} K^{\frac{1}{b-a}}\left(1-\left(1+y^{-\theta}\right)^{\frac{1}{b-a}}\right) \quad &< \quad p\left(1+y^{-\theta}\right)^{\frac{1}{a-b}}\left(\left(1-\left(1+y^{-\theta}\right)\right)^{\frac{1}{b-a}}\right) \\ &= \quad p\left(\left(1+y^{-\theta}\right)^{\frac{1}{a-b}}-1\right). \end{aligned} \tag{3.17}$$

Since $b < a$ we have $\left(1+y^{-\theta}\right)^{\frac{1}{a-b}} \leq \left(1+y^{-\theta}\right)$ and then from (3.17)

$$p\left(\left(1+y^{-\theta}\right)^{\frac{1}{a-b}}-1\right) \quad \leq \quad py^{-\theta}. \tag{3.18}$$

Since we take $p \in [\frac{T}{2}, T]$

$$py^{-\theta} \quad \leq \quad Ty^{-\theta}. \tag{3.19}$$

That means by (3.17), (3.18) and (3.19), the length of the interval that contains p is bounded by

$$K^{\frac{1}{b-a}}\left(1-\left(1+y^{-\theta}\right)^{\frac{1}{b-a}}\right) \quad < \quad Ty^{-\theta}.$$

In both cases, the number of primes p with fixed exponents $a_1, \ldots, a_n, a$ and $b_1, \ldots, b_n, b$, for which $y, x \in \mathcal{N}\left(S_n \cup \{p\}\right)$ with $y = {p_1}^{a_1}{p_2}^{a_2}\cdots{p_n}^{a_n}p^a$ and $x = {p_1}^{b_1}{p_2}^{b_2}\cdots{p_n}^{b_n}p^b$ such that $0 < x - y < y^{1-\theta}$ does not exceed $Ty^{-\theta}$. Since, we have $\frac{T}{2} \leq p \leq x < 2y$, we see that

$$Ty^{-\theta} \quad \leq \quad T\left(\frac{T}{4}\right)^{-\theta} \quad = \quad 4^{\theta}T^{1-\theta} \quad < \quad 4T^{1-\theta}. \tag{3.20}$$

That means by (3.14) and (3.20) the total number of possible primes $p \in [\frac{T}{2}, T]$ for which there exist $y, x \in \mathcal{N}(S_n \cup \{p\})$ with $0 < x - y < y^{1-\theta}$, is at most,

$$4T^{1-\theta}\left((U(n)+1)^{2n+2}(\log T)^{2n}\right).$$

We want to exclude these primes in the interval $[\frac{T}{2}, T]$. Now we claim that for $T = T(n+1)$ we can find the next prime p for which $\mathcal{N}(S_n \cup \{p\})$ satisfying (3.1). For this it is sufficient to show that the number of primes in $[\frac{T}{2}, T]$ is larger than the number of the excluded primes

$$4T^{1-\theta}\left((U(n)+1)^{2n+2}(\log T)^{2n}\right).$$

Recall that the number of primes in $[\frac{T}{2}, T]$ at least $\frac{3T}{10\log T}$. Thus we want to show that for our $T = T(n+1)$,

$$4T^{1-\theta}\left((U(n)+1)^{2n+2}(\log T)^{2n}\right) < \frac{3T}{10\log T}.$$

Since $4\frac{10}{3} < 2^4$, $2T(n) < T(n+1)$ and $U(n)+1 \leq 2U(n)$, it suffices to show that

$$2^4 (2U(n))^{2n+2}(\log T)^{2n+1} < T^{\theta}. \tag{3.21}$$

In the right hand of (3.21), we see that

$$\begin{aligned} T(n+1)^{\theta} &= \left(\frac{c_2(n+1)}{\theta}\right)^{c_1(n+1)^2} \\ &= \mathcal{R}_c \mathcal{R}_\theta \mathcal{R}_n \end{aligned} \tag{3.22}$$

where

$$\begin{aligned} \mathcal{R}_c &= c_2^{c_1(n+1)^2} \\ \mathcal{R}_\theta &= \left(\frac{1}{\theta}\right)^{c_1(n+1)^2} \\ \mathcal{R}_n &= (n+1)^{c_1(n+1)^2}. \end{aligned}$$

We recall that $U(n) = 16C_1^2 = 16\left(\frac{1}{\theta}C(n+1)^{4(n+1)}(\log T(n))^n\right)^2$.

For the left side of (3.21), we note that

$$\log T(n) \quad = \quad \frac{c_1 n^2}{\theta} \log\left(\frac{c_2 n}{\theta}\right) \quad < \quad \frac{c_1 c_2 n^3}{\theta^2}. \tag{3.23}$$

When we put $c_1 = 2^7$ and $c_2 = 2C$ where C is the constant in Lemma 2.2 then by (3.23),

$$\begin{aligned} 2U(n) \quad &\leq \quad 2^5 \cdot \frac{1}{\theta^2} \cdot C^2 \cdot (n+1)^{8(n+1)} \left(\frac{c_1 c_2 n^3}{\theta^2}\right)^{2n} \\ &\leq \quad 2^{16n+5} \cdot \left(\frac{1}{\theta}\right)^{4n+2} \cdot C^{2n+2} \cdot (n+1)^{8(n+1)+6n}. \end{aligned} \tag{3.24}$$

And we know that

$$\log T(n+1) \quad = \quad \frac{c_1 (n+1)^2}{\theta} \log\left(\frac{c_2 (n+1)}{\theta}\right) \quad < \quad \frac{c_1 c_2 (n+1)^3}{\theta^2}. \tag{3.25}$$

Hence the left side of (3.21) satisfies by (3.23), (3.24) and (3.25),

$$2^4 (2U(n))^{2n+2} (\log T)^{2n+1} \quad < \quad \mathcal{L}_c \mathcal{L}_\theta \mathcal{L}_n \tag{3.26}$$

where

$$\begin{aligned} \mathcal{L}_c \quad &= \quad (2C)^{96(n+1)^2} \\ \mathcal{L}_\theta \quad &= \quad \left(\frac{1}{\theta}\right)^{8(n+1)^2} \\ \mathcal{L}_n \quad &= \quad (n+1)^{32(n+1)^2}. \end{aligned}$$

We compare (3.22) and (3.26). Since $\mathcal{L}_c \leq \mathcal{R}_c$, $\mathcal{L}_\theta \leq \mathcal{R}_\theta$ and $\mathcal{L}_n \leq \mathcal{R}_n$, hence the inequality (3.21) holds. Also, we observe that (3.3) holds. Therefore we can find a prime $p > p_n$ in the interval $[\frac{T(n+1)}{2}, T(n+1)]$ with the required property and we put $p = p_{n+1}$. □

Chapter 4

Second Main Theorem

In this chapter, we will consider Wintner's question with respect to lower bounds for the sequence of gaps $n_{i+1} - n_i$. We shall look for a function $\mathcal{L}(x)$ which grows quickly and yet, for which we can still prove that there is an infinite set of primes S such that the associated sequence of power products $n_0 < n_1 < n_2 < \ldots$ satisfies

$$n_{i+1} - n_i \quad > \quad \mathcal{L}(n_i) \tag{4.1}$$

for $i = 0, 1, 2, \ldots$.

Because we already know Theorem 1.7 and Theorem 1.9, we are interested in $\mathcal{L}(x)$ which for any real number $C > 0$ and any real number θ with $0 < \theta < 1$ satisfy

$$n_i^{1-\theta} \quad < \quad \mathcal{L}(n_i) \quad < \quad \frac{n_i}{(\log n_i)^C} \tag{4.2}$$

for n_i sufficiently large.

4.1 Basic Properties

Remark First, we observe some basic properties of $\mathcal{N}(S)$ for a given set S of prime numbers.

1. $S_1 = S_2$ if and only if $\mathcal{N}(S_1) = \mathcal{N}(S_2)$.
2. $S_1 \subseteq S_2$ if and only if $\mathcal{N}(S_1) \subseteq \mathcal{N}(S_2)$.

3. If $S_1 \subseteq S_2$ and $a \in \mathcal{N}(S_1) \cap \mathcal{N}(S_2)$ then there are non-negative integers i, j with $i \leq j$ such that $a = n_i = m_j$ with $n_i \in \mathcal{N}(S_1)$, $m_j \in \mathcal{N}(S_2)$. Further we see that $n_{i+1} - n_i \geq m_{j+1} - m_j$.

4.2 Nice Functions

In this section, we define a family of functions and investigate some properties of the functions.

We use the following notation for iterated logarithms and iterated exponentials.

Notation. For any non-negative integer n, we denote n-iterated exponentiation by

$$\exp^0(x) = x, \quad \exp^1(x) = \exp(x) = e^x, \quad \exp^{n+1}(x) = \exp(\exp^n(x)),$$

and n-iterated logarithms by

$$\log_0(x) = x, \quad \log_1(x) = \log(x), \quad \log_{n+1}(x) = \log(\log_n(x)).$$

We note that for any non-negative integer k and for any real number x, $\exp^k(x)$ is a well defined positive continuous function. And for $x \geq \exp^k(1)$, $\log_k(x)$ is a well defined non-negative continuous function.

Further $\exp^k(\log_k(x)) = \log_k(\exp^k(x)) = x$ as expected.

We shall investigate the derivatives of the above functions. For convenience we denote for a non-negative integer k,

$$E^k(x) \quad = \quad \exp^k(x)$$

for a real number x and

$$L_k(x) \quad = \quad \log_k(x)$$

for a real number with $x \geq \exp^k(1)$.

The following two propositions are simple applications of the chain rule.

Proposition 4.1. *For a given positive integer k we have the following :*

For any real number x

$$\left(E^k(x)\right)' \quad = \quad E^k(x) \cdot E^{k-1}(x) \cdots E^1(x) \ , \tag{4.3}$$

and for any real number x with $x \geq \exp^k(1)$ we see that

$$(L_k(x))' = \frac{1}{L_0(x) \cdot L_1(x) \cdots L_{k-1}(x)} . \tag{4.4}$$

Proof. We first show (4.3), the derivative of the iterated exponential function. We use induction on k. When $k = 1$, $E^1(x) = e(x)$ so $\left(E^1(x)\right)' = (e(x))' = e(x) = E^1(x)$ as required. Suppose for any $k \leq n-1$ (4.3) hold. For $k = n$, by definition of $E^n(x)$ we see $E^n(x) = \exp\left(E^{n-1}(x)\right)$ and by property of exponential function and the inductive hypothesis we have

$$\begin{aligned} (E^n(x))' &= \left(\exp\left(E^{n-1}(x)\right)\right)' \\ &= \left(\exp\left(E^{n-1}(x)\right)\right) \cdot \left(E^{n-1}(x)\right)' \\ &= E^n(x) \cdot E^{n-1}(x) \cdots E^1(x) \end{aligned}$$

as required.

Now, we show (4.4), the derivative of the iterated logarithm function using induction on k. When $k = 1$ and $x \geq \exp^1(1)$, $L_1(x) = \log(x)$ so $(L_1(x))' = (\log(x))' = \frac{1}{x} = \frac{1}{L_0(x)}$ as required. Suppose for any $k \leq n-1$ (4.4) hold. Then for $k = n$ and $x \geq \exp^n(1)$, by definition of $L_n(x)$ we see $L_n(x) = \log\left(L_{n-1}(x)\right)$ and by property of logarithm function and the inductive hypothesis we have

$$\begin{aligned} (L_n(x))' &= \left(\log\left(L_{n-1}(x)\right)\right)' \\ &= \frac{\left(L_{n-1}(x)\right)'}{L_{n-1}(x)} \\ &= \left(L_{n-1}(x)\right)' \cdot \frac{1}{L_{n-1}(x)} \\ &= \frac{1}{L_0(x) \cdot L_1(x) \cdots L_{n-2}(x)} \cdot \frac{1}{L_{n-1}(x)} \end{aligned}$$

as required.

□

Definition 4.1. *Let k be a non-negative integer and θ be a real number such that $0 < \theta < 1$. Let a be a real with $a \geq \exp^k(1)$. Define*

$$F_{k,\theta}(a) = \exp^k\left(\left(\log_k(a)\right)^\theta\right).$$

For convenience, we define $F_{k,\theta}(1) = 1$.

Remark

1. If $k = 0$ then for $a \geq \exp^0(1) = 1$ we have $F_{k,\theta}(a) = a^\theta$ for any real θ with $0 < \theta < 1$.
2. If $\theta = 0$ then $F_{k,\theta}(a) = \exp^k(1) = C$ for any non-negative integer k and for a with $a \geq \exp^k(1)$.
3. For given non-negative integer k and a real number θ with $0 < \theta < 1$ we see that $F_{k,\theta}(x)$ is an increasing function on $x \geq \exp^k(1)$ since $F_{k,\theta}(x)$ is a composition function of the increasing functions x^θ, $\log_k x$ and $\exp^k(x)$ on $x \geq \exp^k(1)$.

Note. We will consider $\mathcal{L}(x)$ in (4.1) and (4.2) as $\mathcal{L}(x) = \frac{x}{F_{k,\theta}(x)}$ for proper ranges of k and θ.

Now we discuss Wintner's question regarding the function $F_{k,\theta}(x)$.

Proposition 4.2. *Let k be a non-negative integer. If $\theta = 0$ then we cannot find an infinite set S of primes satisfying*

$$n_{i+1} - n_i \quad > \quad \frac{n_i}{F_{k,\theta}(n_i)}$$

for $n_i \in \mathcal{N}(S)$.

Proof. When $\theta = 0$ for any real number x and for any non-negative integer k, we have $F_{k,\theta}(x) = \exp^k(1)$. Let $C = \exp^k(1) > 0$. Suppose we can find an infinite set S of primes satisfying $n_{i+1} - n_i > \frac{n_i}{C}$. So $n_{i+1} > \left(1 + \frac{1}{C}\right) n_i$. We apply the third part of the Remark of Section 4.1 then this contradicts Theorem 1.3 which asserts that, $\lim_{i\to\infty} \frac{n_{i+1}}{n_i} = 1$. □

Proposition 4.3. *Let k be a positive integer. For given $0 < \theta < 1$,*

$$F_{0,\theta}(a) \quad > \quad F_{1,\theta}(a) \quad > \quad \cdots \quad > \quad F_{k,\theta}(a)$$

for sufficiently large a.

Proof. Consider $a \geq \exp^k(1)$. Let t be a non-negative integer with $0 \leq t \leq k-1$. We compare $F_{t,\theta}(a)$ and $F_{t+1,\theta}(a)$. Up to taking t times logarithms on $F_{t,\theta}(a)$ and $F_{t+1,\theta}(a)$, we get $(\log_t(a))^\theta$ and $\exp^1\left(\left(\log_{t+1}(a)\right)^\theta\right)$ respectively. After taking the logarithms one more time of both sides we have $\theta \cdot \log_{t+1}(a)$, $\left(\log_{t+1}(a)\right)^\theta$ and since $0 < \theta < 1$, there is a positive real number a_t such that

$$\theta \cdot \log_{t+1}(a) \quad > \quad \left(\log_{t+1}(a)\right)^\theta$$

for $a > a_t$. Hence for $a > \max\{a_0, a_1, \ldots, a_{t-1}\}$ we have

$$F_{0,\theta}(a) \quad > \quad F_{1,\theta}(a) \quad > \quad \cdots \quad > \quad F_{k,\theta}(a)$$

as required. □

Proposition 4.4. *Let k be a non-negative integer and let θ_1 and θ_2 be real numbers with $0 < \theta_1 < \theta_2$. Then for all real numbers a with $a \geq \exp^k(1)$*

$$F_{k,\theta_1}(a) \quad < \quad F_{k,\theta_2}(a).$$

Proof. When we take logarithms k times we see that $(\log_k(a))^{\theta_1} < (\log_k(a))^{\theta_2}$, since $a \geq \exp^k(1)$ and $\theta_1 < \theta_2$. □

Remark For any non-negative integer k we have restricted our attention in Definition 4.1 to $F_{k,\theta}(a)$ with $0 < \theta < 1$. If $\theta \geq 1$ then by Proposition 4.4, we see that $F_{k,\theta}(a) > F_{k,1}(a) = a$. Hence, if we consider $n_i \in \mathcal{N}(S)$ for a given infinite set S of prime numbers we see that

$$n_{i+1} - n_i \quad \geq \quad 1 \quad = \quad \frac{n_i}{n_i} \quad > \quad \frac{n_i}{F_{k,\theta}(n_i)}$$

for any integers $n_{i+1} > n_i \geq \exp^k(1)$. And, it is obvious for any integers with $n_{i+1} > n_i$.

Therefore, we will consider $\mathcal{L}(x)$ in (4.2) as $\mathcal{L}(x) = \frac{x}{F_{k,\theta}(x)}$ for any non-negative integer k and a real number θ with $0 < \theta < 1$.

Proposition 4.5. *For any real θ_1, θ_2 with $0 < \theta_1 < 1$ and $0 < \theta_2 < 1$ we see that*

$$F_{k,\theta_1}(a) \quad > \quad F_{k+1,\theta_2}(a) \tag{4.5}$$

for sufficiently large a.

Proof. After taking logarithms $k+1$ times on $F_{k,\theta_1}(a)$ and $F_{k+1,\theta_2}(a)$ we see that

$$\theta_1 \cdot \log_{k+1}(a) \quad > \quad \left(\log_{k+1}(a)\right)^{\theta_2}.$$

Since $0 < \theta_2 < 1$, we have the inequality (4.5) for sufficiently large a. □

Remark By Proposition 4.5, for given non-negative integer k and for any $0 < \theta_i < 1$ where $i = 1, 2, \ldots, k$, we have that

$$F_{1,\theta_1}(a) \quad > \quad F_{2,\theta_2}(a) \quad > \quad \cdots \quad > \quad F_{k,\theta_k}(a)$$

for sufficiently large a.

Proposition 4.6. *For any positive integers k_1, k_2 and any positive real numbers θ_1 and θ_2 with $0 < \theta_i < 1$ for $i = 1, 2$, we see that*

$$\exp^{k_1}\left(\left(\log_{k_1}(a)\right)^{\theta_1}\right) \quad > \quad \exp^{k_2}\left(\left(\log_{k_2+1} a\right)^{\theta_2}\right) \tag{4.6}$$

for sufficiently large a.

Proof. If $k_1 = k_2$ then it is clear by Proposition 4.5. If $k_1 > k_2$ then we take logarithms k_2 times on both sides of (4.6). Then we want to show

$$\exp^{k_1-k_2}\left(\left(\log_{k_1}(a)\right)^{\theta_1}\right) \quad > \quad \left(\log_{k_2+1}(a)\right)^{\theta_2}.$$

When we take logarithms $k_1 - k_2$ times of both sides of the above inequality we have

$$\left(\log_{k_1}(a)\right)^{\theta_1} \quad > \quad C + \left(\log_{k_1+1}(a)\right),$$

which is then for sufficiently large a as required. If $k_1 < k_2$ then we take logarithms k_1 times on both sides. Then we want to show

$$\left(\log_{k_1}(a)\right)^{\theta_1} \quad > \quad \exp^{k_2-k_1}\left(\left(\log_{k_2+1}(a)\right)^{\theta_2}\right).$$

When we take logarithms $k_2 - k_1$ times on both sides of the above inequality again, since $0 < \theta < 1$ we have

$$C + \log_{k_2}(a) \quad > \quad \left(\log_{k_2+1}(a)\right)^{\theta_2},$$

which holds for sufficiently large a as required. □

4.3 Lemmas

We see the following relation. Indeed, it is related to the fact that for any real numbers a, b if $2 \leq a \leq b$ then $a + b \leq 2b \leq ab$.

Lemma 4.1. *Let n be a non-negative integer and let A and B be real numbers with $A, B \geq \exp^{n+1}(2)$. Then*

$$\log_n (\log A \cdot \log B) \quad \leq \quad \big(\log_{n+1}(A)\big) \cdot \big(\log_{n+1}(B)\big). \tag{4.7}$$

Proof. When $n = 0$ then both sides of (4.7) are equal to $\log A \log B$. We use induction on $n \geq 1$. If $n = 1$, then $\log(\log A \cdot \log B) = \log_2 A + \log_2 B \leq \log_2 A \cdot \log_2 B$ since by $A, B \geq \exp^k(2)$ both $\log_2 A$ and $\log_2 B$ are greater than or equal to 2. Suppose that the statement is true for all $n \leq k - 1$. Now we want to show that it is true for $n = k$. By the inductive hypothesis and properties of the log function we see that

$$\begin{aligned}
\log_k (\log A \cdot \log B) &= \log\big(\log_{k-1}(\log A \cdot \log B)\big) \\
&\leq \log(\log_k A \cdot \log_k B) \\
&= \log(\log_k A) + \log(\log_k B) \\
&= \log_{k+1} A + \log_{k+1} B \\
&\leq \log_{k+1} A \cdot \log_{k+1} B
\end{aligned}$$

since $A, B \geq \exp^{k+1}(2)$, so we have both $\log_{k+1} A$ and $\log_{k+1} B$ are greater than or equal to 2. □

Lemma 4.2. *Let k be a positive integer and θ be a real number with $0 < \theta < 1$. For any real number $x \geq \exp^k(2)$ we define*

$$f(x) \quad = \quad f_{k,\theta}(x) \quad = \quad \frac{\exp^{k-1}\Big((\log_k(x))^{\theta}\Big)}{\log x}. \tag{4.8}$$

Then $f(x)$ has the following properties.

1. *$0 < f(x) < 1$ for any $x \geq \exp^k(2)$.*
2. *$f(x)$ is a decreasing function on $x \geq \exp^k(2)$.*
3. *$G(x) = x^{-f(x)}$ is a decreasing function on $x \geq \exp^k(2)$.*

4. *Let* $F(x) = \frac{1}{G(x)}$ *where* $G(x)$ *is defined above. Then* $\frac{x}{F(x)}$ *is an increasing function on* $x \geq \exp^k(2)$ *and* $\lim_{x\to\infty} \frac{x}{F(x)} = \infty$.

Remark. We will claim that $\frac{x}{F(x)}$ is the function $\mathcal{L}(x)$ we consider in Wintner's question.

Proof. Let a positive integer k and a real number θ with $0 < \theta < 1$ be given.

1. Let x be a real number with $x \geq \exp^k(2)$. Then, $\log(x) > 0$ and $\exp(y)$ is positive for any real number y. Therefore $f(x) > 0$. We can show that

$$\exp^{k-1}\left((\log_k(x))^\theta\right) \quad < \quad \log(x) \tag{4.9}$$

since after taking logarithms $k-1$ times of both sides of (4.9) we have, since $0 < \theta < 1$

$$(\log_k(x))^\theta \quad < \quad \log_k(x).$$

Therefore $f(x) < 1$. □

2. We shall show that the derivative of $f(x)$ is negative on $x \geq \exp^k(2)$. Recall the notation $E^k(x)$ and $L_k(x)$, then

$$f(x) \quad = \quad f_{k,\theta}(x) \quad = \frac{E^{k-1}\left((L_k(x))^\theta\right)}{L_1(x)}. \tag{4.10}$$

And, the derivative of $f(x)$ is

$$(f(x))' \quad = \quad \frac{\left(E^{k-1}\left((L_k(x))^\theta\right)\right)' L_1(x) - E^{k-1}\left((L_k(x))^\theta\right)\frac{1}{L_0(x)}}{(L_1(x))^2}. \tag{4.11}$$

But, the denominator of (4.11) is square and so positive, we need to determine the sign of numerator of (4.11). By the property of exponential function

$$\left(E^{k-1}\left(L_k(x)\right)^\theta\right)' \quad = \quad \left(E^{k-1}\left((L_k(x))^\theta\right)\right)\cdot\left(E^{k-2}\left(L_k(x)^\theta\right)\right)'. \tag{4.12}$$

Moreover,

$$\begin{aligned}
\left(E^{k-2}\left((L_k(x))^\theta\right)\right)' &= \left(E^{k-2}\left((L_k(x))^\theta\right)\right)\cdot\left(E^{k-3}\left((L_k(x))^\theta\right)\right)' \\
\left(E^{k-3}\left((L_k(x))^\theta\right)\right)' &= \left(E^{k-3}\left((L_k(x))^\theta\right)\right)\cdot\left(E^{k-4}\left((L_k(x))^\theta\right)\right)' \\
&\vdots \\
\left(E^{1}\left((L_k(x))^\theta\right)\right)' &= E^{1}\left((L_k(x))^\theta\right)\cdot\left((L_k(x))^\theta\right)'. \qquad (4.13)
\end{aligned}$$

And, by Proposition 4.1, we have

$$\begin{aligned}
\left((L_k(x))^\theta\right)' &= \theta\cdot(L_k(x))^{\theta-1}\,(L_k(x))' \\
&= \theta\cdot(L_k(x))^{\theta-1}\,\frac{1}{L_0(x)L_1(x)\cdots L_{k-1}(x)}. \qquad (4.14)
\end{aligned}$$

Therefore, by (4.12), (4.13) and (4.14)

$$\begin{aligned}
&\left(E^{k-1}\,(L_k(x))^\theta\right)' \\
= \quad &\left(\prod_{j=1}^{k-1}E^j\left((L_k(x)^\theta)\right)\right)\cdot\theta\cdot(L_k(x))^{\theta-1}\cdot\frac{1}{\prod_{i=0}^{k-1}L_i(x)}. \qquad (4.15)
\end{aligned}$$

Therefore, the numerator of (4.11) is by (4.15)

$$\left(\frac{E^{k-1}\left((L_k(x))^\theta\right)}{L_0(x)}\right)\cdot\left(\frac{\mathcal{E}}{\mathcal{L}}\cdot\theta\cdot\frac{(L_k(x))^\theta}{L_k(x)}-1\right) \qquad (4.16)$$

where

$$\begin{aligned}
\mathcal{E} &= L_1(x)\cdot\prod_{j=1}^{k-2}E^j\left((L_k(x))^\theta\right), \qquad (4.17)\\
\mathcal{L} &= L_1(x)\cdot\prod_{i=2}^{k-1}L_i(x). \qquad (4.18)
\end{aligned}$$

We claim that for any real number x with $x \geq \exp^k(2)$, (4.16) is negative.

First we note that $\frac{E^{k-1}\left((L_k(x))^\theta\right)}{L_0(x)}$ is positive for a real x with $x \geq \exp^k(2)$. Hence it is sufficient to show that for given real number $x \geq \exp^k(2)$, $\left(\frac{\mathcal{E}}{\mathcal{L}} \cdot \theta \cdot \frac{(L_k(x))^\theta}{L_k(x)} - 1\right)$ is negative or equivalently $\frac{\mathcal{E}}{\mathcal{L}} \cdot \theta \cdot \frac{(L_k(x))^\theta}{L_k(x)}$ is less than 1.

Since $\theta < 1$ and $\frac{(L_k(x))^\theta}{L_k(x)} < 1$ we shall show that $\frac{\mathcal{E}}{\mathcal{L}} < 1$.

We recall that

$$\frac{\mathcal{E}}{\mathcal{L}} = \frac{E^{k-2}\left((L_k(x))^\theta\right)}{L_2(x)} \cdots \frac{E^1\left((L_k(x))^\theta\right)}{L_{k-1}(x)}.$$

But for each i for $i = 2, \ldots, k-1$ we can show that

$$\frac{E^{k-i}\left((L_k(x))^\theta\right)}{L_i(x)} < 1.$$

Or equivalently we shall show that,

$$E^{k-i}\left((L_k(x))^\theta\right) < L_i(x). \tag{4.19}$$

The above inequality holds because after taking $k-i$ logarithms on both sides of (4.19) we have since $0 < \theta < 1$

$$(L_k(x))^\theta < L_{i+k-i}(x) = L_k(x)$$

as required.

So, we proved our claim that for any $x \geq \exp^k(2)$ (4.16) is negative so $f(x)'$ is negative.

Therefore, $f(x)$ is a decreasing function on $x \geq \exp^k(2)$. □

3. Now we want show that $G(x) = x^{-f(x)}$ is a decreasing function on $x \geq \exp^k(2)$ or equivalently, $G(x)'$ is negative. But we know that $G(x) = \exp(-f(x)\log(x))$ and the derivative of $G(x)$ is

$$\begin{aligned} (\exp(-f(x)\log(x)))' &= \exp(-f(x)\log(x)) \cdot (-f(x)\log(x))' \\ &= \exp(-f(x)\log(x)) \cdot \left((-f(x))' \cdot \log x + (-f(x))\frac{1}{x}\right). \end{aligned} \tag{4.20}$$

For any real number x, $\exp(x)$ is positive. Hence the sign of (4.20) is determined by the sign of

$$\left((-f(x))' \cdot \log x + (-f(x)) \frac{1}{x} \right). \tag{4.21}$$

To show (4.21) is negative we claim that

$$f(x) \quad > \quad (-f(x))' \cdot \log x \cdot x. \tag{4.22}$$

By (4.10) and (4.16) we see that

$$\begin{aligned}
(-\,f(x))' \cdot \log x \cdot x \quad &= \quad (-\,f(x))' \cdot L_1(x) \cdot L_0(x) \\
= \quad (-\,L_0(x)) \cdot &\frac{\left(E^{k-1}\left((L_k(x))^{\theta}\right)\right)' L_1(x) - E^{k-1}\left((L_k(x))^{\theta}\right) \frac{1}{L_0(x)}}{(L_1(x))} \\
= \quad -\left(E^{k-1}\left((L_k(x))^{\theta}\right)\right)' &\cdot L_0(x) - f(x).
\end{aligned} \tag{4.23}$$

By (4.15) and (4.23), to show (4.22) is equivalent to show the following inequality

$$2\ f(x) \qquad > \qquad \theta \cdot \frac{(L_k(x))^{\theta}}{L_k(x)} \cdot \frac{E^1\left((L_k(x))^{\theta}\right)}{L_{k-1}(x)} \cdots \frac{E^{k-1}\left(L_k(x)^{\theta}\right)}{L_1(x)}.$$

By (4.10), we can divide both sides of the above inequality by $f(x)$ we get

$$2 \quad > \quad \theta \cdot \frac{(L_k(x))^{\theta}}{L_k(x)} \cdot \frac{E^1\left((L_k(x))^{\theta}\right)}{L_{k-1}(x)} \cdots \frac{E^{k-2}\left(L_k(x)^{\theta}\right)}{L_2(x)}. \tag{4.24}$$

By (4.19) for $i = 1, \ldots, k-1$ and $0 < \theta < 1$, the right hand side of inequality (4.24) is less than 1. Hence (4.24) holds on $x \geq \exp^k(2)$. Therefore, $x^{-f(x)}$ is a decreasing function on $x \geq \exp^k(2)$. □

4. Finally, we want to show that $x^{1-f(x)}$ is increasing on $x \geq \exp^k(2)$. Note that $x^{1-f(x)} = \exp\left((1 - f\,(x))\log x\right)$, so we see

$$\left(x^{1-f(x)}\right)' = \exp\left((1-f(x))\log x\right) \cdot \left((1-f(x))\log x\right)'. \tag{4.25}$$

Since for any real x, $\exp(x)$ is positive, for given x with $x \geq \exp^k(2)$, the sign of (4.25) is determined by the sign of

$$\begin{aligned}(\log x - f(x)\log x)' &= \frac{1}{x} - (f(x))'\log x - \frac{f(x)}{x} \\ &= \left(\frac{1-f(x)}{x} - (f(x))'\log x\right). \end{aligned} \tag{4.26}$$

But, the sign of (4.26) is positive on $x \geq \exp^k(2)$ by the first and second part of Lemma 4.2 that $0 < f(x) < 1$ and $f(x)' < 0$.

Finally, we shall show that $\lim_{x\to\infty} \frac{x}{F(x)} = \infty$.

For any positive number N, we shall show that for sufficiently large x

$$\frac{x}{F(x)} > N$$

or equivalently that

$$x > N \cdot F(x) = N \cdot \exp\left(f(x)\log x\right). \tag{4.27}$$

But the above inequality holds for sufficiently large x because after taking the logarithm on both sides of (4.27) we have

$$\log x > \log N + f(x)\log x = \log N + \exp^{k-1}\left((\log_k(x))^\theta\right). \tag{4.28}$$

By taking the logarithms $k-1$ time on both sides of (4.28), we see the inequality holds for sufficiently large x since $0 < \theta < 1$.

Hence $\frac{x}{F(x)}$ is an unbounded increasing function and so $\lim_{x\to\infty} \frac{x}{F(x)} = \infty$. $\square$

We are ready to prove the main theorem.

4.4 Second Main Theorem

Theorem 4.1. *Let k be a non-negative integer and let θ be a real number with $0 < \theta < 1$. For $a \geq \exp^k(2)$, we define*

$$F(a) \quad = \quad \exp^k\left((\log_k(a))^{\theta} \right).$$

Also, we define $F(1) = 1$. Then we can find a set S of infinitely many primes such that if $n_i, n_{i+1} \in \mathcal{N}(S)$ then

$$n_{i+1} - n_i \quad > \quad \frac{n_i}{F(n_i)} \tag{4.29}$$

for $i = 0, 1, \ldots$.

Remark $n_0 = 1 \in \mathcal{N}(S)$ for all S so $F(1)$ needs to be defined. We can choose S such that $p_1 \geq \exp^k(2)$ for any non-negative integer k, so $F(x)$ is well defined for all $n_i \in \mathcal{N}(S)$.

Proof. For $k = 0$, this is Theorem 1.9. So we shall suppose that $k \geq 1$. For a given positive integer k and a real θ with $0 < \theta < 1$, we want to construct a sequence of primes $p_1 < p_2 < \ldots < p_n < \ldots$ satisfying (4.29) inductively.

We can take p_1 to be the least prime greater than $\exp^k(2)$. Let $S_1 = \{p_1\}$. Then $n_0 = 1$ and $n_1 = p_1 > \exp^k(2)$. So $n_1 - n_0 > 1$ as required. For all $n_i \in \mathcal{N}(S_1)$ with $n_i \geq n_1$, $F(n_i) > 1$ and so (4.29) holds since

$$n_{i+1} - n_i \quad = \quad {p_1}^{i+1} - {p_1}^{i} \quad = \quad {p_1}^{i}(p_1 - 1) \quad > \quad p_1^i \quad > \quad \frac{p_1^i}{F(p_1^i)}.$$

Now suppose that we have $S_n = \{p_1 < p_2 < \ldots < p_n\}$ satisfying (4.29).

First we note that for $x \geq \exp^k(2)$,

$$\frac{x}{F(x)} \quad = \quad x^{1-f(x)}$$

where

$$f(x) \quad = \quad \frac{\exp^{k-1}\left((\log_k x)^{\theta} \right)}{\log x}. \tag{4.30}$$

Consider any prime $p > p_n$. Suppose that there are $x, y \in \mathcal{N}(S_n \cup \{p\})$ such that

$$0 \quad < \quad x - y \quad < \quad y^{1-f(y)}, \tag{4.31}$$

where

$$f(y) = \frac{\exp^{k-1}\left((\log_k y)^{\theta}\right)}{\log y}.$$

If $y < x < p$ then $x, y \in \mathcal{N}(S_n)$ and so by the inductive hypothesis x, y satisfies (4.29). We note that $p \leq x$. Moreover, $y^{1-f(y)} \leq y$ by the first part of Lemma 4.2. Hence we observe that $p < x < y + y^{1-f(y)} < 2y < 2x$. Let $y = {p_1}^{a_1}{p_2}^{a_2} \cdots {p_n}^{a_n} p^a$ and $x = {p_1}^{b_1}{p_2}^{b_2} \cdots {p_n}^{b_n} p^b$ be the prime factorizations of y and x respectively. Then we can see that $a \neq b$, since if $a = b$ then we can consider $y' = \frac{y}{p^a}, x' = \frac{x}{p^b} = \frac{x}{p^a}$ so,

$$\begin{aligned} 0 \quad < \quad x - y \quad &= \quad p^a (x' - y') \\ &< \quad (p^a y')^{1-f(p^a y')} \\ &= \quad (p^a)^{1-f(p^a y')} \cdot (y')^{1-f(p^a y')}. \end{aligned} \tag{4.32}$$

Moreover, we note that by the third part of Lemma 4.2

$$(p^a y')^{-f(p^a y')} \quad < \quad (y')^{-f(y')}.$$

Hence, in (4.32)

$$\begin{aligned} p^a (x' - y') \quad &< \quad p^a \cdot (y')^{-f(y')} \cdot (y') \\ &= \quad p^a \cdot (y')^{1-f(y')}. \end{aligned}$$

But this contradicts the inductive hypothesis since $y', x' \in \mathcal{N}(S_n)$ so $x' - y' > y'^{(1-f(y'))}$.

Put $\Lambda = \log \frac{x}{y}$. Then

$$\Lambda \quad = \quad \sum_{j=1}^{n} (b_j - a_j) \log p_j + (b - a) \log p \quad > \quad 0,$$

and by (4.31)

$$\log\frac{x}{y} \quad < \quad \frac{x}{y}-1 \quad < \quad y^{-f(y)} \tag{4.33}$$

where log denotes the principal branch of logarithm.

Furthermore, since $a_j, b_j \geq 0$ for $j = 1, \ldots, n$ and $y < x < 2y$ we have,

$$|b_j - a_j| \quad \leq \quad \max_{j=1,\ldots,n}(b_j, a_j) \quad \leq \quad \max(\log x, \log y) \quad < \quad \log 2y, \tag{4.34}$$

for $j = 1, 2, ..., n$, and since $a, b \geq 0$

$$|b-a| \quad \leq \quad \max(b,a) \quad \leq \quad \max\left(\frac{\log x}{\log p}, \frac{\log y}{\log p}\right) \quad < \quad \frac{\log 2y}{\log p}. \tag{4.35}$$

Put $X = \frac{\log y}{\log p}$ and assume that $X \geq \exp^k(2)$.

Then by Lemma 2.2 with $A_j = p_j$ for $j = 1, \ldots, n$, $A_{n+1} = p$, $B = \log 2y$ and $B_{n+1} = \frac{\log 2y}{\log p}$, there exists an effectively computable constant C such that by (4.33),

$$y^{-f(y)} \quad > \quad \Lambda \quad > \quad \exp\left(-C(n+1)^{4(n+1)}\log p_1 \cdots \log p_n \cdot \log p \log\left(4\frac{\log 2y}{\log p}\right)\right).$$

We will denote by $C_1, C_2, C_3, ...$ positive numbers which depend on $n, p_1, \ldots p_n$ but do not depend on p. Let $C_1 = C(n+1)^{4(n+1)}\log p_1 \cdots \log p_n$. Then, we see that

$$y^{f(y)} \quad < \quad \exp\left(C_1 \cdot \log p \cdot \log\left(4\frac{\log 2y}{\log p}\right)\right).$$

Now, we take logarithms of both sides and divide by $\log p$.

Then, there is a real number C_2, C_3 with $C_2 > 1$ and $C_3 > 1$ such that

$$f(y)\frac{\log y}{\log p} \quad < \quad C_2 \cdot \log\left(8\frac{\log y}{\log p}\right) \quad < \quad C_3 \cdot \log\left(\frac{\log y}{\log p}\right). \tag{4.36}$$

Then by (4.30) and (4.36),

$$f(y)X \quad = \quad \frac{\exp^{k-1}\left((\log_k y)^{\theta}\right)}{X\log p}X \quad < \quad C_3 \log X$$

We multiply each side by $\log p$ and recall the definition of X, then we have

$$\exp^{k-1}\left(\left(\log_{k-1}(X\log p)\right)^{\theta}\right) \;<\; C_3\log p\log X. \tag{4.37}$$

We note that $X \geq \exp^{k}(2)$. We can take $k-1$ times logarithms of both sides again then, since $p^{C_3} > p_n^{C_3} \geq \exp^{k}(2)$, by (4.37) and Lemma 4.1

$$\begin{aligned}\left(\log_{k-1} X\right)^{\theta} &< \left(\log_{k-1}(X\log p)\right)^{\theta}\\ &< \log_{k-1}(C_3\log p\log X)\\ &\leq \log_k\left(p^{C_3}\right)\log_k X.\end{aligned}$$

Let $Z = \left(\log_{k-1} X\right)^{\theta}$. Then,

$$\begin{aligned}Z &< \log_k(p^{C_3})\cdot\log\left(Z^{\frac{1}{\theta}}\right)\\ &= \log_k(p^{C_3})\cdot\frac{1}{\theta}\cdot(\log Z).\end{aligned}$$

When we apply Lemma 3.1, with $u = 0, h = 1$ and $v = \log_k(p^{C_3})\cdot\frac{1}{\theta}$ we have

$$Z \;\leq\; \left(\frac{1}{\theta}\cdot\log_k\left(p^{C_3}\right)\right)^{2}$$

or equivalently

$$\log_{k-1} X \;<\; \left(\frac{1}{\theta}\cdot\log_k\left(p^{C_3}\right)\right)^{\frac{2}{\theta}}. \tag{4.38}$$

Let us define

$$U_1(p) \;=\; \exp^{k-1}\left(\left(\frac{1}{\theta}\cdot\log_k\left(p^{C_3}\right)\right)^{\frac{2}{\theta}}\right) \tag{4.39}$$

and define

$$\mathcal{U}(p) \;=\; \max\left(2\exp^{k}(2), 2U_1(p)\right). \tag{4.40}$$

Then, if $X \geq \exp^k(2)$ then by (4.38) and (4.39) for any $p > p_n$ we have

$$X \leq \frac{1}{2}\,\mathcal{U}(p).$$

And, if $X < \exp^k(2)$ then by (4.40) we have

$$X \leq \frac{1}{2}\,\mathcal{U}(p).$$

Therefore, for any $X > 0$ we have $X \leq \frac{1}{2}\,\mathcal{U}(p)$.

Let T be an integer with $\frac{T}{2} > p_n$. We recall the result of Rosser and Schoenfeld [31] that the number of primes in the interval $[\frac{T}{2}, T]$ is larger than $\frac{3T}{10\log T}$ for $T \geq 41$. For each prime $p \in [\frac{T}{2}, T]$, we first count the number of integers $y, x \in \mathcal{N}(S_n \cup \{p\})$ such that $0 < x - y < y^{1-f(y)}$. We observed in (4.34)

$$a_j, b_j \leq \mathcal{U}(T)\log T$$

for $j = 1, 2, \ldots, n$, and in (4.35)

$$a, b \leq \frac{\log 2y}{\log p} \leq 2X \leq \mathcal{U}(T).$$

We note that $p < x < 2y < y^2$ and $p < T$. Therefore, the number of possible choices of the exponents $a_1, \ldots, a_n, a, b_1, \ldots, b_n$ and b is at most

$$(2\,\mathcal{U}(T)\log T)^{2n} \cdot (2\,\mathcal{U}(T))^2. \tag{4.41}$$

Now, we assume that all these exponents are fixed. Then, we have

$$1 < \frac{x}{y} = {p_1}^{b_1-a_1}{p_2}^{b_2-a_2}\cdots{p_n}^{b_n-a_n}p^{b-a} < 1 + y^{-f(y)}.$$

Put $K = {p_1}^{a_1-b_1}{p_2}^{a_2-b_2}\cdots{p_n}^{a_n-b_n}$. Then,

$$K < p^{b-a} < K\left(1 + y^{-f(y)}\right). \tag{4.42}$$

Since, $a \neq b$, we have two cases.

(Case 1) $b > a$. Then,

$$K^{\frac{1}{b-a}} < p < K^{\frac{1}{b-a}}\left(1 + y^{-f(y)}\right)^{\frac{1}{b-a}} < K^{\frac{1}{b-a}}\left(1 + y^{-f(y)}\right).$$

Hence p is contained in an interval of the length $K^{\frac{1}{b-a}}\left(y^{-f(y)}\right)$ and by (4.42)

$$K^{\frac{1}{b-a}}\left(y^{-f(y)}\right) \quad < \quad p\cdot y^{-f(y)} \quad \leq \quad T\cdot y^{-f(y)}.$$

(Case 2) $b<a$. Then, by (4.42)

$$\left(1+y^{-f(y)}\right)^{\frac{1}{b-a}} K^{\frac{1}{b-a}} \quad < \quad p \quad < \quad K^{\frac{1}{b-a}}. \tag{4.43}$$

Hence p is contained in an interval of the length

$$K^{\frac{1}{b-a}}\left(1-\left(1+y^{-f(y)}\right)^{\frac{1}{b-a}}\right). \tag{4.44}$$

But, $\left(1+y^{-f(y)}\right)>0$ so by (4.43)

$$K^{\frac{1}{b-a}} \quad < \quad p\left(1+y^{-f(y)}\right)^{\frac{1}{a-b}} \tag{4.45}$$

and so we get by (4.44) and (4.45)

$$\begin{aligned}
& K^{\frac{1}{b-a}}\left(1-\left(1+y^{-f(y)}\right)^{\frac{1}{b-a}}\right) \\
< \;& p\cdot\left(1+y^{-f(y)}\right)^{\frac{1}{a-b}}\left(1-\left(1+y^{-f(y)}\right)^{\frac{1}{b-a}}\right) \\
= \;& p\cdot\left(\left(1+y^{-f(y)}\right)^{\frac{1}{a-b}}-1\right) \\
< \;& p\cdot y^{-f(y)} \\
\leq \;& T\cdot y^{-f(y)}.
\end{aligned}$$

Fix the exponents $a_1,\dots,a_n,a$ and $b_1,\dots b_n,b$. In both cases, the number of primes p for which $y,x\in\mathcal{N}\left(S_n\cup\{p\}\right)$ have prime factorizations $y={p_1}^{a_1}{p_2}^{a_2}\cdots{p_n}^{a_n}p^a$ and $x={p_1}^{b_1}{p_2}^{b_2}\cdots{p_n}^{b_n}p^b$ and satisfy $0<x-y<y^{1-f(y)}$ does not exceed $Ty^{-f(y)}$. We replace this bound with a bound that does not depend on y. Indeed we claim that

$$Ty^{-f(y)} \quad \leq \quad 4T^{1-f(T)}. \tag{4.46}$$

Note that $\frac{T}{2}\leq p<x<2y$ and so $y>\frac{T}{4}$. Moreover, by Lemma 4.2 we note that, $x^{-f(x)}$ is a decreasing function and $f(x)$ is a decreasing function and $0<f(x)<1$.

Therefore,

$$\begin{aligned} T \cdot y^{-f(y)} &\leq T\left(\frac{T}{4}\right)^{-f\left(\frac{T}{4}\right)} \\ &\leq T\left(\frac{T}{4}\right)^{-f(T)} \\ &\leq T^{1-f(T)} \cdot 4^{f(T)} \\ &\leq 4T^{1-f(T)} \end{aligned}$$

We know that the number of primes in the interval $[\frac{T}{2}, T]$ is larger than $\frac{3T}{10\log T}$. On the other hand we know the number of possible exponents by (4.41) and for each set of exponents the number of primes p in the interval is at most $1 + 4T^{1-f(T)}$. Hence it suffices to check the following inequality :

$$\left(4T^{1-f(T)}\right)\left(2\,\mathcal{U}\,(T)\right)^{2n+2}(\log T)^{2n} \quad \leq \quad \frac{3T}{10\log T} \tag{4.47}$$

That means for sufficiently large T, we can find a prime that does not have a pair y, x with $x - y \leq y^{1-f(y)}$. We see that (4.47) is equivalent to

$$C_4\left(2\,\mathcal{U}\,(T)\right)^{2n+2}(\log T)^{2n+1} \quad \leq \quad T^{f(T)}.$$

By (4.30) we want to show, equivalently, that

$$C_5 + C_6 \cdot \log \mathcal{U}\,(T) + C_7 \log_2 T \quad < \quad \exp^{k-1}\left((\log_k T)^{\theta}\right). \tag{4.48}$$

By (4.39) and (4.40), we observe that for sufficiently large T,

$$\log \mathcal{U}\,(T) \quad < \quad \exp^{k-1}\left((\log_k T)^{\theta}\right).$$

And,

$$\log_2 T \quad < \quad \exp^{k-1}\left((\log_k T)^{\theta}\right).$$

Therefore, (4.48) holds for sufficiently large T or equivalently (4.47) holds.

Therefore we can find a prime p in the interval $[\frac{T}{2}, T]$ with the required property and we put $p = p_{n+1}$. $\square$

4.5 Further Research

In this section, we report on a question related to Theorem 4.1 which we are not able to answer. We introduce a family of functions.

Definition 4.2. *For a given non-negative integer k and a real number δ with $\delta > 1$, we put*

$$G_{k,\delta}(a) \quad = \quad \exp^k\left(\left(\log_{k+1}(a)\right)^{\delta}\right)$$

for $a \geq \exp^{k+1}(1)$.

Proposition 4.7. *For given $\delta > 1$,*

$$G_{1,\delta}(a) \quad < \quad G_{2,\delta}(a) \quad < \quad \ldots \quad < \quad G_{k,\delta}(a)$$

for $a \geq \exp^{k+1}(1)$.

Proposition 4.8. *For any non-negative integers k_1, k_2, and for any real $\theta > 0$ and $\delta > 1$ we have*

$$F_{k_1,\theta}(a) \quad > \quad G_{k_2,\delta}(a)$$

for sufficiently large a.

We have no idea whether we can find an infinite set S of primes such that if $n_0 < n_1 < \ldots < n_i < \ldots$ is the set of all positive integers composed of the primes in S then

$$n_{i+1} - n_i \quad > \quad \mathcal{L}(n_i)$$

when we replace our functions $\mathcal{L}(n_i) = \frac{n_i}{F_{k,\theta}(n_i)}$ with functions $\mathcal{L}(n_i) = \frac{n_i}{G_{k,\delta}(n_i)}$ where

$$G_{k,\delta}(a) \quad = \quad \exp^k\left(\left(\log_{k+1}(a)\right)^{\delta}\right)$$

where $\delta > 1$.

Moreover, we want to find relations between the sequences of functions G_k and F_k. We have the following question :

Question What functions $H(a)$ have the following properties :

$$\lim_{a\to\infty} \frac{F_{k,\theta}(a)}{H(a)} = \infty$$

and

$$\lim_{a\to\infty} \frac{H(a)}{G_{k,\delta}(a)} = \infty$$

for $k = 1, 2, \cdots$?

Remark. The existence of the function $H(a)$ may be related to tetration that occurs in the fourth place in the logical progression : addition, multiplication, exponentiation, tetration. [30]

Chapter 5

Computation

In this chapter, we shall determine all prime pairs (p_1, p_2) with $2 \le p_1 < p_2 < e^8$ such that for all $n_i \in \mathcal{N}(\{p_1, p_2\})$

$$n_{i+1} - n_i \quad > \quad \sqrt{n_i}. \tag{5.1}$$

These prime pairs (p_1, p_2) can be extended to an infinite sequence of prime numbers in Wintner's question with respect to $\sqrt{x}$ when we appeal to the proof of Theorem 4.1 with $k = 0$ and $\theta = \frac{1}{2}$.

Remark The largest prime less than e^8 is 2971 which is the 429-th prime number.

First we have the following Theorem.

Theorem 5.1. *There are 2086 prime pairs (p_1, p_2) with $2 \le p_1 < p_2 < e^8$ for which*

$$0 \quad < \quad x - y \quad < \quad \sqrt{y} \tag{5.2}$$

has a solution x, y with $x, y \in \mathcal{N}(\{p_1, p_2\})$ and $\gcd(x, y) = 1$. We list p_1, p_2, x, y in Table I for all the prime pairs (p_1, p_2) as above except for those with $x = p_2, y = p_1$.

Before showing Theorem 5.1, we remark on some assumptions we may make and about the feasibility of our computation.

Remark A

1. In Theorem 5.1, we consider $x, y \in \mathcal{N}(\{p_1, p_2\})$ such that $\gcd(x, y) = 1$ without loss of generality. If we have $x > y$ in $\mathcal{N}(\{p_1, p_2\})$ satisfying (5.2)

with $\gcd(x, y) = d > 1$ then $x' = \frac{x}{d}$, $y' = \frac{y}{d}$ are also in $\mathcal{N}(\{p_1, p_2\})$ and these x', y' satisfy (5.2) since

$$0 \quad < \quad x' - y' \quad = \quad \frac{x-y}{d} \quad < \quad \frac{\sqrt{y}}{d} \quad < \quad \frac{\sqrt{y}}{\sqrt{d}} \quad = \quad \sqrt{y'}.$$

2. Therefore, we can write $x, y \in \mathcal{N}(\{p_1, p_2\})$ with $\gcd(x, y) = 1$ that satisfy (5.2) as

$$\begin{aligned} & x = p^a, \quad y = q^b \\ & \text{with } p, q \in \{p_1, p_2\} \quad \text{and } p \neq q \\ & \text{for some non-negative integers } a, b. \end{aligned} \tag{5.3}$$

From now on we reserve the expression (5.3) for $x, y \in \mathcal{N}(\{p_1, p_2\})$ with $\gcd(x, y) = 1$.

Remark B We review that the computation for Theorem 5.1 is feasible for some given range of primes.

1. In computation, we are given a range of primes $p_1 < p_2 < \mathcal{U}$. This $\mathcal{U}$ depends on computational power : for finding primes and for accurate calculations for each step in the proof of the theorem, etc.
2. By Theorem 1.4, for given $p_1 < p_2$ there are only finitely many $x, y \in \mathcal{N}(\{p_1, p_2\})$ satisfying (5.2).
3. Moreover, by (3.11) in Theorem 3.1, there is an effectively computable positive number $C(p_1, p_2)$ such that if $x, y \in \mathcal{N}(\{p_1, p_2\})$ satisfy (5.2) then $x < C(p_1, p_2)$.
4. The inequality (5.2) suffices to check (5.1). If we find $x, y \in \mathcal{N}(\{p_1, p_2\})$ satisfying (5.2) then there is non-negative integer i such that $n_i = y$. Moreover, $n_{i+1} \leq x$ since $x > y$. So,

$$0 \quad < \quad n_{i+1} - n_i \quad \leq \quad x - y \quad < \quad \sqrt{y} \quad = \quad \sqrt{n_i}.$$

Note that, $x - y \neq \sqrt{y}$ since x, y are integers with $\gcd(x, y) = 1$.

Remark C By Remark B, if for given p_1, p_2 there is no $x, y \in \mathcal{N}(\{p_1, p_2\})$ with $\gcd(x, y) = 1$ that satisfies (5.2) then all $n_i \in \mathcal{N}(p_1, p_2)$ satisfy (5.1).

Strategy for the Proof of Theorem 5.1. We consider separately each prime pair (p_1, p_2) with $2 \leq p_1 < p_2 < e^8$. Let a and b be positive integers and put $M = \max\{a, b\}$. We wish to determine when

$$|p_1^a - p_2^b| \quad < \quad \left(\min\left\{p_1^a, p_2^b\right\}\right)^{\frac{1}{2}}. \tag{5.4}$$

We split the search for examples into three ranges for M. The first is for $M < 20$ and we check this range by a direct search through all possible exponent pairs. The second range is for $20 \leq M < 2^{18}$ and in this range we use properties of the continued fraction of $\frac{\log p_1}{\log p_2}$ to determine if these are any solutions of (5.4). The third range is for $M \geq 2^{18}$ and we prove, see Proposition 5.2, that there are no solutions in this range. But we should mention that these 3 ranges for M are dependent on the upper and lower bounds for the primes p_1 and p_2. In our computation, $2 \leq p_1 < p_2 < e^8$.

We have already used some of the properties we shall discuss in this chapter but at this time we recall and show them clearly. We can see some similar propositions in the following sections to those in the work of Stroeker and Tijdeman [37] and the work of de Weger [45].

5.1 General Upper Bound

Proposition 5.1. *Let $x, y \in \mathcal{N}(\{p_1, p_2\})$ with $\gcd(x, y) = 1$ and expressed by (5.3). Suppose that x, y satisfy the inequality (5.2). Define $\Lambda_1 = \log x - \log y$. Then*

$$\Lambda_1 \quad < \quad \sqrt{\frac{2}{p_1^M}} \tag{5.5}$$

where $M = \max\{a, b\}$.

Proof. First note that $y > 1$ since if $y = 1$ then $0 < x - y < 1$ has no solution. We also have $x < 2y$ since if $x \geq 2y$ then $x - y \geq 2y - y = y > \sqrt{y}$. Let $M = \max\{a, b\}$. Since x, y satisfy the inequality (5.2) and $1 < y < x < 2y$ so we have

$$0 \quad < \quad \Lambda_1 \quad = \quad \log\frac{x}{y} \quad < \quad \frac{x}{y} - 1 \quad < \quad \frac{1}{\sqrt{y}} \quad < \quad \sqrt{\frac{2}{x}}. \tag{5.6}$$

We observe that $x \geq p_1^M$. For this we let $m = \min\{a, b\}$. Recall that we assume that $\gcd(x, y) = 1$.

1. $p_1|x$ then $x = p_1^M$ since $x > y$ and $p_1 < p_2$.
2. If $p_2|x$ then we consider 2 cases.
   (a) If $y = p_1^M$ it is clear since $x > y = p_1^M$.
   (b) If $y = p_1^m$ then $x = p_2^M > p_1^M$. Therefore $x \geq p_1^M$.

Therefore, by (5.6) we obtain

$$0 \quad < \quad \Lambda_1 \quad < \quad \sqrt{\frac{2}{p_1^M}}.$$

□

Proposition 5.2. *Let p_1, p_2 be given with $2 \leq p_1 < p_2 < e^8$. Let $\mathcal{B} = 2^{18}$. Then there are no $x, y \in \mathcal{N}(\{p_1, p_2\})$ satisfying (5.2) expressed by (5.3) such that $M = \max\{a, b\} \geq \mathcal{B}$.*

Proof. Let $\Lambda_1 = \log x - \log y = a \log p - b \log q > 0$. Let $b' = \frac{b}{\log p} + \frac{a}{\log q}$. Since $M = \max\{a, b\}$ and $2 \leq p_1 < p_2 < e^8$, we have

$$\frac{M}{8} \quad \leq \quad b' \quad \leq \quad 3M. \tag{5.7}$$

Then by Lemma 2.1 with $D = 1, A_1 = p_1,\ A_2 = p_2$ and $\log B = \max\left\{\log b', \frac{21}{D}, \frac{1}{2}\right\}$

$$\Lambda_1 \quad > \quad \exp\left(-31\,(\log B)^2 \log p_1 \log p_2\right).$$

Suppose that x, y satisfy (5.2). Then by (5.5),

$$\exp\left(-31\,(\log B)^2 \log p_1 \log p_2\right) \quad < \quad \Lambda_1 \quad < \quad \sqrt{\frac{2}{p_1^M}}. \tag{5.8}$$

By taking logarithms of both sides of (5.8) and multiplying by $-\frac{2}{\log p_1}$, we have

$$M - \frac{\log 2}{\log p_1} \quad < \quad 62\,(\log B)^2 \log p_2. \tag{5.9}$$

We divide into 2 cases.

(Case 1) If $\log B = 21$ then we have that by (5.9)

$$M \quad < \quad 62 \cdot (21)^2 \cdot 8 \quad < \quad 2^{18}.$$

(Case 2) If $21 < \log B = \log b'$ then by (5.7) and (5.9)

$$3M \quad < \quad 3 \cdot 62 \cdot (\log B)^2 \cdot 8 + 1 \quad < \quad 3 \cdot 496 \cdot (\log 3M\,)^2 + 1.$$

Hence

$$3M \quad < \quad C \cdot (\log 3M)^2$$

where $C = 1489$. We apply Lemma 3.1 with $u = 0, v = C, \quad h = 2$ and $x = 3M$ to get

$$M \quad < \quad 2^{18}.$$

But this contradicts the fact (5.7) of that $e^{21} < 3M$.

Therefore, we always have the case 1 and $M < 2^{18}$.

That means there is no solution $x, y \in \mathcal{N}(\{p_1, p_2\})$ satisfying (5.2) expressed by (5.3) such that $M = \max\{a, b\} \ \geq 2^{18} = \mathcal{B}$.

□

Therefore, to find x, y satisfying (5.2) we may suppose that both exponents a and b are less than $\mathcal{B} = 2^{18}$.

5.2 Reduced Number of Calculations

There remains the problem of covering this range $M = \max\{a, b\} < \mathcal{B}$ without a prohibitive amount of computation. We resolve this question by applying some results from Diophantine approximation.

Proposition 5.3. *Let $x, y \in \mathcal{N}(\{p_1, p_2\})$ be expressed by (5.3) which satisfy (5.2) such that $M = \max\{a, b\}$. Then M is the exponent of the smaller prime.*

Proof. We recall that $x = p^a$, $y = q^b$ and $p, q \in \{p_1, p_2\}$ with $p \neq q$. If $p = p_1 < p_2 = q$ then since $x > y$ so $M = a \geq b$ is the exponent of the smaller prime.

Let $q = p_1 < p_2 = p$ and suppose that $M = a > b$. Then $x = p_2^a$ and $y = p_1^b$. But in this case $y < 2y \leq p_1 \cdot p_1^b \leq p_1^a < p_2^a = x$ this contradicts our choice of x, y as $x < 2y$. Therefore, $M = b$ is the exponent of the smaller prime. □

Now, we want to apply some properties of continued fractions. For this we need to restrict our search range as follows.

Proposition 5.4. *Let x, y be expressed by (5.3) and let $M = \max\{a, b\}$. Suppose that $p_1^M \geq 2^{18}$. Then,*

$$p_1^{\frac{M}{2}} > 23M. \tag{5.10}$$

Proof. Suppose not, then

$$2^{\frac{M}{2}} \leq p_1^{\frac{M}{2}} \leq 23M.$$

Hence $M \leq 18$. This is a contradiction since we then have

$$414 \geq 23M \geq p_1^{\frac{M}{2}} \geq 2^9.$$

□

Remark D We will see in Proposition 5.7 that for a given pair of primes (p_1, p_2) with $2 \leq p_1 < p_2 < e^8$, if we want to find $x, y \in \mathcal{N}(\{p_1, p_2\})$ satisfying (5.2) and expressed by (5.3) by using properties of the continued fraction of $\frac{\log p_1}{\log p_2}$ then it suffices to restrict y to $y \geq 2^{18}$ by Proposition 5.3 and Proposition 5.4.

But we note that if x and y satisfy (5.2) then $x < 2y$. Therefore we compute whether $x - y < \sqrt{y}$ directly for all $x, y \in \mathcal{N}(\{p_1, p_2\})$ with $y < x < 2^{19}$. Note the associated powers a, b satisfy $0 \leq a, b < 20$ and this is the first range for $M = \max\{a, b\}$.

If we can not find any $x, y \in \mathcal{N}(\{p_1, p_2\})$ with $1 < y < x < 2^{19}$ that satisfy (5.2) by a direct search through the first range for M then we apply properties of the continued fraction of $\frac{\log p_1}{\log p_2}$ for finding x, y satisfying (5.2).

We recall some definitions and facts about continued fraction expressions.

Definitions Let α be a real number. We denote by $[\alpha]$ the greatest integer n less than or equal to α. Let $a_0 = [\alpha]$. If $\alpha \neq a_0$ there is a real number $\alpha_1 > 1$ such that

$\alpha = a_0 + \frac{1}{\alpha_1}$. Put $a_1 = [\alpha_1]$. Next, if $\alpha_1 \neq a_1$ then $\alpha_1 = a_1 + \frac{1}{\alpha_2}$ and let $a_2 = [\alpha_2]$. By repeated application of the above, we produce a sequence of integers $a_0, a_1, \ldots$, and a sequence of real numbers $\alpha_1, \alpha_2, \ldots$. The sequences may be finite.

1. The integers $a_0, a_1, \ldots$ are called the partial quotients of α.
2. If the sequence of integers is finite, say $a_0, a_1, \ldots, a_n$

   then α is a rational number and

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}.$$

   This expression is known as a finite continued fraction expansion of α

   and we denote $\alpha = [a_0, a_1, \ldots, a_n]$.

3. If the algorithm does not terminate we write

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots \cfrac{1}{a_{n-1} + \cfrac{1}{a_n + \ddots}}}}.$$

   and we call this the infinite continued fraction of α and denote

   $\alpha = [a_0, a_1, a_2, \ldots]$.

4. Let $\alpha = [a_0, a_1, \ldots, a_k]$. We write $\frac{p_n}{q_n} = [a_0, a_1, \ldots, a_n]$ for $n = 0, 1, 2, \ldots, k$ with $\gcd(p_n, q_n) = 1$ and $q_n > 0$.

   The rational number $\frac{p_n}{q_n}$ is called the n-th convergent of α.

   Let $\alpha = [a_0, a_1, \ldots]$. We write $\frac{p_n}{q_n} = [a_0, a_1, \ldots, a_n]$ for $n = 0, 1, 2, \ldots$

   with $\gcd(p_n, q_n) = 1, q_n > 0$. The rational number $\frac{p_n}{q_n}$ is called the n-th convergent of α.

We recall some important properties of continued fractions. The proofs of the next two Propositions may be found in [19].

Proposition 5.5. *Let* $\theta = [a_0, a_1, \ldots, a_n, \ldots]$ *and* $\frac{p_n}{q_n}$ *be the n-th convergent of* θ. *Then*

$$\frac{1}{(a_{n+1}+2)\, q_n^2} \quad < \quad \left|\theta - \frac{p_n}{q_n}\right| \quad < \quad \frac{1}{a_{n+1} q_n^2}. \tag{5.11}$$

Proposition 5.6. *Let* θ *be a real number and* $\frac{p}{q}$ *be a rational number* $\frac{p}{q}$ *satisfying the following inequality*

$$\left|\theta - \frac{p}{q}\right| \quad < \quad \frac{1}{2q^2}.$$

Then there is a non-negative integer k *such that* $\frac{p}{q}$ *is the k-th convergent of* θ.

Now we are ready to prove the following Proposition.

Proposition 5.7. *Let* x, y *satisfy (5.2) and be expressed by (5.3). Let* $M = \max\{a, b\}$ *and* $m = \min\{a, b\}$.

If $p_1^{\frac{M}{2}} > 23M$ *then* $\frac{m}{M}$ *is a convergent of* $\frac{\log p_1}{\log p_2}$.

Proof. By Proposition 5.3, M is the exponent of the smaller prime p_1. We recall that $2 \le p_1 < p_2$. Dividing $|\Lambda_1| = |a \log p - b \log q| = |M \log p_1 - m \log p_2|$ by $M \log p_2$, then by (5.5) we have

$$\begin{aligned} \left|\frac{m}{M} - \frac{\log p_1}{\log p_2}\right| \quad &\le \quad \sqrt{\frac{2}{p_1^M}} \cdot \frac{1}{M \log p_2} \\ &\le \quad \frac{\sqrt{2}}{23M} \cdot \frac{1}{M \log p_2} \\ &\le \quad \frac{1}{17M^2} \\ &< \quad \frac{1}{2M^2}. \end{aligned} \tag{5.12}$$

By Proposition 5.6 we can find a non-negative integer k such that $\frac{m}{M}$ is the k-th convergent of $\frac{\log p_1}{\log p_2}$. □

Remark E Recall that $M = \max\{a, b\}$ and $m = \min\{a, b\}$.

1. It clearly suffices to make a check of all numbers a, b in the relevant range $20 \le M < 2^{18}$ which are denominators of the convergent of $\frac{\log p_1}{\log p_2}$. But the following Propositions ensure us that we only need to compute and check m, M when $\frac{m}{M}$ is one of the convergent of $\frac{\log p_1}{\log p_2}$ up to $M < 2^{18} = \mathcal{B}$ with special partial quotients. We shall discuss this in Proposition 5.8.

2. In Proposition 5.4, for given range of primes $p_1 < p_2 < \mathcal{U}$ we have found some relation between 2^{18}, the lower bound of p_1^M in the hypothesis and 23 the coefficient of M in (5.12). This represents a trade off between direct computation for the small range for M and the probability distribution of the "large enough" n-th partial quotients of the continued fraction expansion to check the approximation property. (See Proposition 5.8.) In our case, "large enough" means greater than or equal to 15.

5.3 Find a Proper Expression

Now, our interest is to find a proper expression of $\frac{\log p_1}{\log p_2}$. In computation there are some restrictions to represent irrational numbers. Hence we need to check the required accuracy before computing. The following remarks and proposition tell us the relation between the convergents of an irrational number and the convergents of a close rational number.

Remark F For given 2 primes p_1, p_2 with $2 \le p_1 < p_2 < e^8$, let $\xi = \frac{\log p_1}{\log p_2}$. Then ξ is irrational number. Let the continued fraction expansion of ξ be given by

$$\xi \quad = \quad [a_0, a_1, a_2, \cdots],$$

and let $\frac{m_k}{M_k}$ be the k-th convergent of ξ for $k = 0, 1, 2 \ldots$.

Let x, y satisfy (5.2) and be expressed by (5.3).

1. By Proposition 5.2 and Proposition 5.7, M_k is bounded. So, we need to find all the convergents $\frac{m_k}{M_k}$ of ξ up to $M_k < \mathcal{B} = 2^{18}$. In this case we only need to consider the continued fraction expansion of ξ up to the k-th step where

$$k \quad \le \quad -1 + \frac{\log\left(\sqrt{5}\ \mathcal{B} + 1\right)}{\log\left(\frac{1+\sqrt{5}}{2}\right)}. \tag{5.13}$$

de Weger [45] showed this using the fact that if $\frac{p_n}{q_n}$ is the n-th convergent of a real number, then q_n is at least the $(n+1)$-th Fibonacci number.

2. Therefore, for finding k-th convergents $\frac{m_k}{M_k}$ up to $M_k < \mathcal{B} = 2^{18}$ in (5.13), it suffices to compute the continued fraction expansion up to the 26-th partial quotient.

3. For computing, it suffices to find a rational number θ such that for every n up to 26 the n-th convergent of ξ is exactly the same as that of θ.

The following theorem tells us the required accuracy in order to apply the continued fraction algorithm while the denominator of the k-th convergent is less than $\mathcal{B} = 2^{18}$.

Theorem 5.2. *Let $\xi = \frac{\log p_1}{\log p_2}$ and θ be a rational number with $|\xi - \theta| < \epsilon$ where $\epsilon = 2^{-39}$. Then every convergent $\frac{p_n}{q_n}$ of ξ with $20 \le q_n < 2^{18}$ is a convergent of θ.*

Proof. Let $\frac{p_n}{q_n}$ be a convergent of ξ and $20 \le q_n < 2^{18}$. Since $p_1^M \ge 2^{18}$, we see by Proposition 5.4 and Proposition 5.7

$$\left|\theta - \frac{p_n}{q_n}\right| \quad \le \quad |\theta - \xi| + \left|\xi - \frac{p_n}{q_n}\right| \quad \le \quad \epsilon + \frac{1}{8q_n^2} \quad < \quad \frac{1}{2q_n^2} \tag{5.14}$$

Therefore, $\frac{p_n}{q_n}$ is also a convergent of θ. □

Remark G

1. Therefore, for given $\xi = \frac{\log p_1}{\log p_2}$, if we represent ξ by a rational number θ with $|\xi - \theta| < 2^{-39}$ then any convergent $\frac{m_k}{M_k}$ of ξ with $k \le 26$ and $M_k \ge 20$ is also a convergent of θ.

2. Note that if we have two rational numbers θ_1, θ_2 such that

$$\theta_1 \quad < \quad \xi \quad < \quad \theta_2$$

and the continued fraction expansions of θ_1 and θ_2 are the same up to the k-th partial quotient then ξ also has the same continued fraction expansion up to k-th partial quotient.

3. When we use **Maple**, we shall find the continued fraction expansion of ξ up to the 30-th partial quotient that satisfies the above two conditions we mentioned in this Remark in order to guarantee the accuracy of our computations.

5.4 Criteria for Solution

We have already reduced the number of calculations in Section 5.2. For a given prime pair (p_1, p_2), we only need to check 26 candidates pairs (M, m) in the whole "medium" range of $20 \leq \max\{a, b\} < \mathcal{B} = 2^{18}$. But, in computation the most time and memory consuming part is exponentiation. So, reducing the number of exponentiations is one of the critical parts for feasibility.

In our question, for given two prime numbers p, q, we need to compute exponentiations p^a, q^b where a, b are huge numbers almost up to $\mathcal{B} = 2^{18}$. In this section, we shall discuss the nice criteria for deciding to calculate exponentiation. And this is the answer to the question we mentioned in Remark E.

Proposition 5.8. *Let x, y be expressed by (5.3) and $M = \max\{a, b\}$ and $m = \min\{a, b\}$. And suppose that $p_1^M \geq 2^{18}$. If (x, y) is a solution of (5.2) then there is a positive integer r with $0 \leq r \leq 26$ such that*

1. *$\frac{m}{M}$ is the r-th convergent of $\frac{\log p_1}{\log p_2}$.*
2. *The $(r+1)$-th partial quotient a_{r+1} of the continued fraction expansion of $\frac{\log p_1}{\log p_2}$ is greater than or equal to 15.*

Proof. By (5.12) of Proposition 5.7 we see

$$\left|\frac{\log p_1}{\log p_2} - \frac{m}{M}\right| \quad < \quad \frac{1}{17M^2}. \tag{5.15}$$

And, by Proposition 5.5,

$$\frac{1}{(a_{r+1}+2)\, M_r^2} \quad < \quad \left|\frac{\log p_1}{\log p_2} - \frac{m_r}{M_r}\right| \quad < \quad \frac{1}{a_{r+1} M_r^2}. \tag{5.16}$$

By (5.15) and (5.16) we see that $a_{r+1} \geq 15$. □

5.5 Some Remarks on the Program

5.5.1 The Small Range

In the execution of our program we treat all prime pairs (p_1, p_2) with $2 \leq p_1 < p_2 < e^8$. For given p_1, p_2, the aim is to find $x, y \in \mathcal{N}(\{p_1, p_2\})$ such that

1. $x = p^a, y = p^b$ where $p, q \in \{p_1, p_2\}$, $p \neq q$ and

2. $0 < x - y < \sqrt{y}$.

We note that $0 \leq a, b < 20$ cover the "small" range for $1 \leq \max\{a, b\} < 20$. For each pair (p_1, p_2) the solutions x, y with a, b in this range are detected by direct checking for all possible pairs x, y with $1 < y < x < 2y$. We generate all positive integers p^a, q^b where $p, q \in \{p_1, p_2\}$, $p \neq q$ with $a, b < 20$ and check whether

$$|p^a - q^b| \quad < \quad \left(\min\left\{p^a, q^b\right\}\right)^{\frac{1}{2}}.$$

If we find p^a, q^b that satisfy the above inequality and $a > 1$ or $b > 1$ then write $p_1, p_2, a, b, \max\{p^a, q^b\}, \min\{p^a, q^b\}$ in Table 1. Hence in Table 1 we have all solutions $x, y \in \mathcal{N}(\{p_1, p_2\})$ with $\gcd(x, y) = 1$ that satisfy $0 < x - y < \sqrt{y}$ except $\{x, y\} = \{p_1, p_2\}$.

After checking all the range for small M we go "medium" range for M.

5.5.2 The Medium Range

We call the range of exponents $20 \leq \max\{a, b\} < 2^{18}$ "medium". We searched x, y that satisfy (5.2) and the associated the maximum exponents M are in "medium" by the strategy from Diophantine Approximations.

We review the algorithm that has been used for finding solutions x, y for given p_1, p_2.

Given two prime numbers p_1, p_2 with $2 \leq p_1 < e^8$.

Let $\xi := \frac{\log p_1}{\log p_2}$.

Call continued fraction expansion of ξ up to the 30-th partial quotient and we get

$$\theta \quad = \quad [a_0, a_1, \ldots, a_{30}].$$

We note that for required accuracy it is enough to find the continued fraction up to the 30-th partial quotient. The error between $\frac{\log p_1}{\log p_2}$ and computational expressed number so that

$$[a_0, a_1, \ldots, a_{28}] \quad < \quad \xi \quad < \quad [a_0, a_1, \ldots, a_{29}].$$

And so θ is a proper expression of ξ. Let

$$\frac{m_k}{M_k} = [a_0, a_1, \ldots, a_k] \tag{5.17}$$

for $k = 1, \ldots, 30$.

Apply Proposition 5.2, we are interested in the case the denominator of the k-th convergent (5.17) is less than $\mathcal{B} = 2^{18}$. We have reduced level with

$$reducedLevel = \max_{0 \le k \le 26} \{ k \,|\, \text{the k-th denominator } M_k < 2^{18} \}.$$

For i from 0 to $reducedLevel$ do

If the i-th partial quotient is such that $a_{i+1} \ge 15$.

Then, we set $\frac{m_i}{M_i} = [a_0, a_1, \ldots, a_i]$.

Check the inequality (5.2) whether

$$\left| p_1^{M_i} - p_2^{m_i} \right| < \left(\min\{p_1^{M_i}, p_2^{m_i}\} \right)^{\frac{1}{2}}. \tag{5.18}$$

If (5.18) is true

then we add $p_1, p_2, p_1^{M_i}, p_2^{m_i}$ to the Table I.

else $i := i + 1$.

Else $i := i + 1$.

5.5.3 Computing Environment

We use the package **Maple 10** on grayling server (SunFire V20/40z systems with 2 CPUs and 4GB memory) in University of Waterloo. The total time for computing for the code based on the Appendix B is around 9 days.

5.6 Further Research

We may try to find an initial 3 or more primes in Wintner's question with respect to $\sqrt{x}$. In this case we should apply the LLL algorithm [45]. For computational feasibility, we need sharp estimates of linear forms in n logarithms if we are dealing with more than 2 primes.

Finally, we mention a consequence of the abc conjecture. The abc conjecture links the additive and multiplicative structure of the integers.

Conjecture (Oesterlé-Masser)
Let a, b, and c be non-zero integers and define

$$G \quad = \quad G(a,b,c) \quad = \quad \prod_{\substack{p\ \mid abc \\ p\ a\, prime}} p.$$

Suppose that a, b, and c are co-prime and that

$$a+b+c \quad = \quad 0.$$

For each $\epsilon > 0$ there is a $C(\epsilon) > 0$ such that

$$\max\{|a|,|b|,|c|\} \quad < \quad C(\epsilon)\cdot G^{1+\epsilon}.$$

This conjecture is known as the abc conjecture.

Remark For S and for any $n_i \in \mathcal{N}(S)$ we note

$$(n_{i+1} - n_i) + n_i \quad = \quad n_{i+1}.$$

We observe that Theorem 1.4 an immediate consequence of abc conjecture when we take $\theta = \frac{1}{1+\epsilon}$.

Appendix A

Table I

In the following table we list the prime pairs (p_1, p_2) with $2 \leq p_1 < p_2 < e^8$ for which there is a co-prime pair of integers x, y from $\mathcal{N}(\{p_1, p_2\})$ with $0 < x - y < \sqrt{y}$. Note that $x = \max\{p_1^a, p_2^b\}$ and $y = \min\{p_1^a, p_2^b\}$. We also list **ALL** such integers x and y and the associated powers a and b of p_1 and p_2 respectively for which $\{x, y\} \neq \{p_1, p_2\}$ and $0 < x - y < \sqrt{y}$ **EXCEPT** $x = p_2, y = p_1$.

We should mention that all the solutions are found in the "small" range for M.

p_1	p_2	a	b	x	y
2	3	2	1	4	3
2	3	3	2	9	8
2	3	5	3	32	27
2	3	8	5	256	243
2	5	2	1	5	4
2	5	7	3	128	125
2	7	3	1	8	7
2	11	7	2	128	121
2	13	4	1	16	13
2	17	4	1	17	16
2	19	4	1	19	16
2	23	9	2	529	512
2	29	5	1	32	29
2	31	5	1	32	31
2	37	5	1	37	32
2	59	6	1	64	59
2	61	6	1	64	61
2	67	6	1	67	64
2	71	6	1	71	64
2	127	7	1	128	127
2	131	7	1	131	128
2	137	7	1	137	128
2	139	7	1	139	128
2	181	15	2	32768	32761
2	241	8	1	256	241

2	251	8	1	256	251
2	257	8	1	257	256
2	263	8	1	263	256
2	269	8	1	269	256
2	271	8	1	271	256
2	491	9	1	512	491
2	499	9	1	512	499
2	503	9	1	512	503
2	509	9	1	512	509
2	521	9	1	521	512
2	523	9	1	523	512
2	997	10	1	1024	997
2	1009	10	1	1024	1009
2	1013	10	1	1024	1013
2	1019	10	1	1024	1019
2	1021	10	1	1024	1021
2	1031	10	1	1031	1024
2	1033	10	1	1033	1024
2	1039	10	1	1039	1024
2	1049	10	1	1049	1024
2	1051	10	1	1051	1024
2	2011	11	1	2048	2011
2	2017	11	1	2048	2017
2	2027	11	1	2048	2027
2	2029	11	1	2048	2029
2	2039	11	1	2048	2039
2	2053	11	1	2053	2048
2	2063	11	1	2063	2048
2	2069	11	1	2069	2048
2	2081	11	1	2081	2048
2	2083	11	1	2083	2048
2	2087	11	1	2087	2048
2	2089	11	1	2089	2048
3	5	3	2	27	25
3	7	2	1	9	7
3	11	2	1	11	9
3	13	7	3	2197	2187
3	23	3	1	27	23
3	29	3	1	29	27

3	31	3	1	31	27
3	47	7	2	2209	2187
3	73	4	1	81	73
3	79	4	1	81	79
3	83	4	1	83	81
3	89	4	1	89	81
3	229	5	1	243	229
3	233	5	1	243	233
3	239	5	1	243	239
3	241	5	1	243	241
3	251	5	1	251	243
3	257	5	1	257	243
3	421	11	2	177241	177147
3	709	6	1	729	709
3	719	6	1	729	719
3	727	6	1	729	727
3	733	6	1	733	729
3	739	6	1	739	729
3	743	6	1	743	729
3	751	6	1	751	729
3	2141	7	1	2187	2141
3	2143	7	1	2187	2143
3	2153	7	1	2187	2153
3	2161	7	1	2187	2161
3	2179	7	1	2187	2179
3	2203	7	1	2203	2187
3	2207	7	1	2207	2187
3	2213	7	1	2213	2187
3	2221	7	1	2221	2187
5	11	3	2	125	121
5	23	2	1	25	23
5	29	2	1	29	25
5	127	3	1	127	125
5	131	3	1	131	125
5	601	4	1	625	601
5	607	4	1	625	607
5	613	4	1	625	613
5	617	4	1	625	617
5	619	4	1	625	619

5	631	4	1	631	625
5	641	4	1	641	625
5	643	4	1	643	625
5	647	4	1	647	625
7	19	3	2	361	343
7	43	2	1	49	43
7	47	2	1	49	47
7	53	2	1	53	49
7	331	3	1	343	331
7	337	3	1	343	337
7	347	3	1	347	343
7	349	3	1	349	343
7	353	3	1	353	343
7	359	3	1	359	343
7	907	7	2	823543	822649
7	2357	4	1	2401	2357
7	2371	4	1	2401	2371
7	2377	4	1	2401	2377
7	2381	4	1	2401	2381
7	2383	4	1	2401	2383
7	2389	4	1	2401	2389
7	2393	4	1	2401	2393
7	2399	4	1	2401	2399
7	2411	4	1	2411	2401
7	2417	4	1	2417	2401
7	2423	4	1	2423	2401
7	2437	4	1	2437	2401
7	2441	4	1	2441	2401
7	2447	4	1	2447	2401
11	113	2	1	121	113
11	127	2	1	127	121
11	131	2	1	131	121
11	401	5	2	161051	160801
11	1297	3	1	1331	1297
11	1301	3	1	1331	1301
11	1303	3	1	1331	1303
11	1307	3	1	1331	1307
11	1319	3	1	1331	1319
11	1321	3	1	1331	1321

11	1327	3	1	1331	1327
11	1361	3	1	1361	1331
11	1367	3	1	1367	1331
13	47	3	2	2209	2197
13	89	7	4	62748517	62742241
13	157	2	1	169	157
13	163	2	1	169	163
13	167	2	1	169	167
13	173	2	1	173	169
13	179	2	1	179	169
13	181	2	1	181	169
13	2153	3	1	2197	2153
13	2161	3	1	2197	2161
13	2179	3	1	2197	2179
13	2203	3	1	2203	2197
13	2207	3	1	2207	2197
13	2213	3	1	2213	2197
13	2221	3	1	2221	2197
13	2237	3	1	2237	2197
13	2239	3	1	2239	2197
13	2243	3	1	2243	2197
17	277	2	1	289	277
17	281	2	1	289	281
17	283	2	1	289	283
17	293	2	1	293	289
19	83	3	2	6889	6859
19	347	2	1	361	347
19	349	2	1	361	349
19	353	2	1	361	353
19	359	2	1	361	359
19	367	2	1	367	361
19	373	2	1	373	361
19	379	2	1	379	361
23	509	2	1	529	509
23	521	2	1	529	521
23	523	2	1	529	523
23	541	2	1	541	529
23	547	2	1	547	529
29	821	2	1	841	821

29	823	2	1	841	823
29	827	2	1	841	827
29	829	2	1	841	829
29	839	2	1	841	839
29	853	2	1	853	841
29	857	2	1	857	841
29	859	2	1	859	841
29	863	2	1	863	841
31	173	3	2	29929	29791
31	937	2	1	961	937
31	941	2	1	961	941
31	947	2	1	961	947
31	953	2	1	961	953
31	967	2	1	967	961
31	971	2	1	971	961
31	977	2	1	977	961
31	983	2	1	983	961
31	991	2	1	991	961
37	1361	2	1	1369	1361
37	1367	2	1	1369	1367
37	1373	2	1	1373	1369
37	1381	2	1	1381	1369
37	1399	2	1	1399	1369
41	263	3	2	69169	68921
41	1657	2	1	1681	1657
41	1663	2	1	1681	1663
41	1667	2	1	1681	1667
41	1669	2	1	1681	1669
41	1693	2	1	1693	1681
41	1697	2	1	1697	1681
41	1699	2	1	1699	1681
41	1709	2	1	1709	1681
41	1721	2	1	1721	1681
43	1811	2	1	1849	1811
43	1823	2	1	1849	1823
43	1831	2	1	1849	1831
43	1847	2	1	1849	1847
43	1861	2	1	1861	1849
43	1867	2	1	1867	1849

43	1871	2	1	1871	1849
43	1873	2	1	1873	1849
43	1877	2	1	1877	1849
43	1879	2	1	1879	1849
43	1889	2	1	1889	1849
47	2179	2	1	2209	2179
47	2203	2	1	2209	2203
47	2207	2	1	2209	2207
47	2213	2	1	2213	2209
47	2221	2	1	2221	2209
47	2237	2	1	2237	2209
47	2239	2	1	2239	2209
47	2243	2	1	2243	2209
47	2251	2	1	2251	2209
53	2767	2	1	2809	2767
53	2777	2	1	2809	2777
53	2789	2	1	2809	2789
53	2791	2	1	2809	2791
53	2797	2	1	2809	2797
53	2801	2	1	2809	2801
53	2803	2	1	2809	2803
53	2819	2	1	2819	2809
53	2833	2	1	2833	2809
53	2837	2	1	2837	2809
53	2843	2	1	2843	2809
53	2851	2	1	2851	2809
53	2857	2	1	2857	2809
53	2861	2	1	2861	2809
113	1201	3	2	1442897	1442401
131	1499	3	2	2248091	2247001
163	2081	3	2	4330747	4330561

Appendix B1

Maple Code

The following code is based on the Maple Code we used in our computation specially for the "Medium" range for M.

```
> PRIME_1 := given;
> PRIME_2 := given;

> with(numtheory);

> generalLevel := 30;
> veryLarge := 2^(18);

> enoughLarge := 15;

> outputFilefp:= fopen("tabel1.txt", WRITE, TEXT);
> fclose(outputFilefp);

> runFilefp:= fopen("runningReport.txt", WRITE, TEXT);
> fclose(runFilefp);


> p1 := PRIME_1;
> p2 := PRIME_2;

> x := log (p1)/ log (p2);

> BOUND := 0;
> reducedLevel := 0;
```

```
> cf := cfrac(x, generalLevel, 'quotients');
> print(cf);

> for BB from 1 to generalLevel do # BB

> BOUND := nthdenom(cf, BB);

> if (BOUND > veryLarge ) #BOUND

> then
> printf("This BOUND is too BIG = %d > veryLarge = %d \n",
> BOUND, 2^(18) );
> break; #then
> else reducedLevel := reducedLevel + 1;
> end if; #BOUND

> end do; # BB

> printf("Reduced Level is %d \n", reducedLevel );

> for i from 1 to reducedLevel by 1 do #from nthLevel to reduced
> step by very Large #

> #### NEW PART CRITERIA
> Criteria := cf[i+1] - enoughLarge;

> if (signum(Criteria) > -1)
> then
> printf("%d + 1 the partial quotient %d is
> bigger than 14 ", i, cf[i+1] );

> kk:=p1^(nthdenom(cf, i));
> printf("\n\n p1^nthdemum = %d^(%d) \n = kk = %d \n",p1,
```

```
> nthdenom(cf, i), kk);
> tt:=p2^(nthnumer(cf, i));
> printf("\n\n p2^nthnumer = %d^(%d) \n = tt = %d \n", p2,
> nthnumer(cf, i), tt);
> LL := abs ( kk - tt );
> printf("\n\n LL = abs(kk - tt) \n = %d \n", LL);

> RR := sqrt( min(kk, tt ) );
> printf("\n\n RR = sqrt ( min(kk, tt ) ) \n = %g \n", RR);


> printf("\n\n LL - RR %g \n", LL-RR);
> if ( signum(LL-RR) < 0)
> then
> ouputFilefp:= fopen("tabel1.txt", APPEND, TEXT);
> fprintf(outputFilefp, "p1 = %d a = %d p2 = %d b = %d ",
> p1,nthdenom(cf, i), p2, nthnumer(cf, i));
> fprintf(outputFilefp, " abs(%d - %d) = abs(%d) = %d ", kk, tt,
> abs(kk-tt), LL);
> fprintf(outputFilefp, " sqrt(%d) = %g \n ", min(kk,tt), RR);
> fclose(ouputFilefp);


> end if; # signum(LL - RR) < 0


> end if; #criteria

> end do; #i reducedLevel for n-th denum nth numer
```

```
> runFilefp:= fopen("runningReport.txt", APPEND, TEXT); #betterthan
> binary
> fprintf(runFilefp, "We've done check prime pair (%d %d) \n", p1,
> p2);
> fclose(runFilefp);
```

Appendix B2

Running Sample

We attach a sample running result for the Maple Code in Appendix B with special the case $p_1 = 43$ and $p_2 = 1013$.

$$PRIME\_1 := 43$$

$$PRIME\_2 := 1013$$

[GIgcd, bigomega, cfrac, cfracpol, cyclotomic, divisors, factorEQ, factorset, fermat,
imagunit, index, integral_basis, invcfrac, invphi, issqrfree, jacobi, kronecker, λ,
legendre, *mcombine*, *mersenne*, *migcdex*, *minkowski*, *mipolys*, *mlog*, *mobius*,
mroot, *msqrt*, *nearestp*, *nthconver*, *nthdenom*, *nthnumer*, *nthpow*, *order*, *pdexpand*,
ϕ, π, *pprimroot*, *primroot*, *quadres*, *rootsunity*, *safeprime*, σ, *sq2factor*, *sum2sqr*,
τ, *thue*]

$$generalLevel := 30$$

$$veryLarge := 262144$$

$$enoughLarge := 15$$

$$outputFilefp := 0$$

$$runFilefp := 0$$

$$p1 := 43$$

$$p2 := 1013$$

$$x := \frac{\ln(43)}{\ln(1013)}$$

$$BOUND := 0$$

$$reducedLevel := 0$$

cf := [0, 1, 1, 5, 3, 1, 94, 3, 10, 4 , 1, 10, 5, 3, 3, 2, 2, 83, 1, 8, 19, 1, 1,
3, 3, 2, 1, 24, 2, 4, 4,
...]

[0, 1, 1, 5, 3, 1, 94, 3, 10, 4, 1, 10, 5, 3, 3, 2, 2, 83, 1, 8, 19, 1, 1, 3, 3, 2, 1, 24, 2, 4, 4, ...]

$$BOUND := 1$$

$$BOUND := 2$$

$$BOUND := 11$$

$$BOUND := 35$$

$$BOUND := 46$$

$$BOUND := 4359$$

$$BOUND := 13123$$

$$BOUND := 135589$$

$$BOUND := 555479$$

```
This BOUND is too BIG = 555479  > veryLarge = 262144
```

```
Reduced Level is 8
```

$$Criteria := -14$$

$$Criteria := -14$$

$$Criteria := -10$$

$$Criteria := -12$$

$$Criteria := -14$$

$$Criteria := 79$$

```
6 + 1 the partial quotient  94 is bigger than 14
```

p1^nthdemum = 43^(4359)

= kk =
1945350546965597461267744209398273250646028240711581661547099348267840
3609808039190805551699238710442124964225107349991785325168154812514416
6295184488546311653873678583979723756599484431149855457616853602364356
6403002633859621041054448265850800949938813711695288000239724941836812
3660459924116117217586380417184745114439582656571416390079931319321085
2118946940700411796928518914321838267987668566375970936959079563825117
2036096342280287183897799809832551747339889090000811108566929704171583
4639545168814200326125370732480615392486228938632406346105795996100592
7171190414679741657515720110103808669882451956620909154369578324665962
0161721367674182154620435227153673702590682680404454503247205341770685
6222295355667980560565418462351020128795752152078126376627555826924426
1222708517738999367683339867381054501851579367864435403528227057806683
8362732223783436159089966551118629512830185880967969999339854067749419
2742095149872550723501668396982787920711589778017485964133738237954276
9770850305734105675555292343346352343366622603512967756748291911918936
1288523964273933744584645809981563221983802067811884579598652362534765
2725092366080614750897778764071878704141828556694878309775008755763907
1190241194883720350664524331251814588670008469145377848952743374682873
1850239979273812374548605097611667549119155255879203019673675620384252
1125405706238020180983318695134031075151558541449805632695220022725749
5365972446683819981099759379675312993750813100795683111978025924480171
3095000179386949706874770089279713924123179774681212504940286170951507
3899744524817140453249121001979422088992751741278481578005652116718728
0450663412516080773405456628884898119953798909153621727818696051097348
6979047298488501764041947002853753530163544004200014854903858077378597
5516699034808932407328533525409864257866513236552317904559302751622744
0037920659057050227565069325524128068692231744705835447349980457004226
3827193986014915603544394308481536591333697871442696957151611615772212
9530765770826862555526447316191270074296179960812385303472828051884308
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4197611034827577116631842010592019636008021717674255065383253643612315
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2349773686142277108337769199299177153694596339793867349054635376504764
9841542263255359672739549739428841880817574977059693809020513999091661
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8776299804688101277101598569897851246400917457770691076826110872046895
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3641259616229688242720495831618826808445657233723035736623853836651597
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0350272119354762738075418052956640382229011451340577525153141055761439
7799940334520434434401399769011701416027726523531223657977704639065209
0103222855691168579636236792110855488129174922809419356069936973408999

8574768412663188853335677626105609260169496113385299594665215352707494
6215877701597275068340974298559267714708108850350014404036124420168552
7600141990314174141702307710503577065505045144379232366982577492784441
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5942918087482140719304745711268694305711895327634808830921655413470207
3815271837335838958248025395509618074629817015759704819883169828434362
7353840692342543236556855847224659840769681840327943630845886549999332
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0239517693829899650006076844660058004393720886619945165643330907756189
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2218615995114863062685413214020248153997357027425826182075219557017677
7462794120278161250755154269503104394672446705599535532886035479735454
5898626999538506567304957933403121191128056826041685654587514672166002
9263811291600466131689936947576851626828688573895677299587019352723113
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9707541474646032338744605503910500176498273479950871117411091598370055
9412080825080705589692602872190846919177988740398153084698528448706680
3594063949681590402010712476631535104320944324541830860404899682846935
5361243330224402106609727388329433837736752956988424025033617574189913
3282884687754874001279857730103377025012611479728083925826453169314709
9774018989418318014821201386015759998337060341031888494229919931755060
1624101015223423355188893578067594843703863249438300472612941869499722
5898403258639786443267658197147969585587350775433863808233108320905320
6634744460435482007960137725126886948018516884631155312258223406564806
3019139762021274611030451009792656914540873358908882155433072884373511
3389183915371046135793080072359449332766342654493604846963542682893654
2495169863658631400546122421451250200728502387473821128124113793698379
0490284169807180517150822331575759933804124362130910537707078749873154
7638501545415058731278221101716719994484909060636706994867947602943852
3458485318449112296910798134873766124826646953178998187462633922133974
1529651067580120571866580132196143511103127687199443927105583758804087
0415519634660886006965014640502936868521856069941646550786785444573540
4765029442257980424031526772891867316817995305338748436611233135232368
9049915000655607437873031608388287038730143783004663111805227498901088
0379672035191663271841978173892862427108466732933385669710312975252518
3411873970805273536953534259222300297150361178573360499010917127865459
2982133975430159019587660470646143530137965732633655176411215887386215
4530934424608313983194782646059119597290793762218950641607637390114990
6717007695944704311570208989833474006603113559570363052773459889412436
2351775255265471704565586489918530304621175551900989432538332860820929
8325560622929645050607214385733509202298777505741365661296016461601222
9670109426389920021469620046507795331517297667799528074972919765477539
3050960374585486815945176111184468485249894927811860210747133251585671
1542795343088854850947156648906591074124566134939711651532689033768502
6836418330880564915819233828973882889352320351584019471720138857741362

4792122414871453512612753601389964495104109800459114214336990524462823
0930381778764375116824431400481630924279320168025707852553686643388631
2376147010746933648539813786285013535228641255592181707845698711189247
6565488976655243602134586018342456528813767530258448047354595587303145
1336638056463289622897355191785385391614096155138868256636787064601700
0141503812941842197070800142499710271179777329390984915409145844787275
1545964345010097255477925366134818746798144302260080833284787458198444
5421334061259934014790119256105458838866195510119000239735474017828214
922944552360218939261896240708905066587188974271907

```
 p2^nthnumer = 1013^(2369)

  = tt =
```

1944357226269164679897750981689865790619836834498690045927205930194194
8008369761927153225625067338855917425826409456777525480466840878615794
8234744647925005092520043177122150321513620563436758817162335905092500
3988855692076535654125535357772383253567342237449917330821642626030821
4668211369813031644840803369299466306918594105271579462501176075663174
9756623568933641683761241070059505577274473440454569960042879032674585
9097315961658599308056581714390933969386969788253919289750560553692662
9860731552286200507465919533310946523043712754264253400156170982356211
3754957452457497229240784777453192891715230474506336459974532488100688
3869992013727866848478994351608072823564344906854665062991219426202750
4655870281589947922661402286327987295412576264603649353301608180157125
5110400518200694804287010419771486440836474972057074772649058149030910
6842812382159839682197152071143573729223262794735939322755726805490618
4206032620863604699056855440106987835713557047110538339955384402669812
9471663338242152838045049053651369901731227386098740594170139418934719
4973333222183511194472508622718590915878016208327299518861184619590155
0847570865177762679306648894304425986244040147305895890693593766278477
3659500590806334175729362315585326393463435873615105792395756007702193
9498132039570698601814394971849812055051877972547283356285159664807436
4289329540420410715831292946122537090811701372357781925313248455345494
5577473971589152004761058493671475616048724979282208763253871078059698
0280257082573601602172302089233769167181060696406822271602037549025982
6588422223446349800940005712222258145315923300019469792691220519001147
5646593009463058504342724441662742330160271180912877101677975556996776
4785905836051109152786859015479427247422412453362534180837625091533177
1565776474250193909901268661860570330283218139226604155532247700046709
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7772659399667763037378356771984319921079864117037786622830015569536926
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9166760960467841837481359776423949038662259482407976340722610792872891
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4480710424532678731482857026976034508126458907121338838397680188894583
4302926087315953927239598299603956686028530884138658913790410878079660
2500949778429480979877554474542161536063865051091661162104530398256578
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8524174617576617273368286498700211049263337697869039290074335481222559
3588912858320405855871005477455372636917515285367659179585766444500472
5238002192338835108860544931816130827398211953375423227443229546969098
3611176012372648028923776491077409560789701844900616900833990252651586
7345463693538736947798484397481919413310752927670901218465259004631981
5447395290512160371380286421700923892617462632360698566285026004376421
5980685769356148334253143998637776278059901989594014896717367601643873
7106428912064314679780270840522829992015709553854973750586195805745076
6836336729113452866711064258302805719741992992555542169489342935946571
6658286646381459507739143101658497108728112745665616334547783571472012
6436970515895937321410522141534656864557827278957511777461773332889151
8444503272383683621825897563272308424120816477460838900412613671623271
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6641790484325926959686257441625143914862597828097946032987746282429181
0239804508483326735163887688047616992455523929716743383867678048622416
1479035179618648465402045331327291222057488553413968832001698374962825
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1076798067369207837209103881794065392452369922943237623928552835167304
6202102667172170482844441820542523643958186814477005672205578355201307
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4457868345449482789860276638165777928288353441189491636023786288094283
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9836127584845558610878820480860040584362236299480689428228457079713280
4842492504294905471298768126132994579297350472341050689440285006893122
6476055803859029767481186885000036645877722474536680607895808233486001
4821336017693610336415347256074952418351448655543544313350932649422885
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9222627807429553156314870731486723528484281884523320389974046410259105
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5210908311493773527366374815006887221448162936863569012760483637968875
2095806730836134096766698789705832014984130112368123758183381749494555

4070517955869696847238116783951994070369962686507236099494160139595621
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2433277514636358678797851697331201949407933944894023134195740802301178
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7192608250294354078562006657788036397969967774661277932266622031115515
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9220135435008196299593869832659493682412681077160643544588708746379305
6362953340593755713824447559295345677924552787131192458970242621765629
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LL = abs(kk - tt)
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RR = sqrt ( min(kk, tt )  )

= 1.39440e+3560



LL  - RR  9.93321e+7116
```

$$Criteria := -12$$

$$Criteria := -5$$

$$runFilefp := 1$$

41

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