Posets of Non-Crossing Partitions of Type B and Applications

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

The thesis is devoted to the study of certain combinatorial objects called *non-crossing partitions.* The enumeration properties of the lattice $\mathsf{NC}^{\mathsf{A}}(\mathsf{n})$ of *non-crossing partitions* were studied since the work of G. Kreweras in 1972. An important feature of $\mathsf{NC}^{\mathsf{A}}(\mathsf{n})$, observed by P. Biane in 1997, is that it embeds into the symmetric group $\mathfrak{S}_n$; via this embedding, $\mathsf{NC}^{\mathsf{A}}(\mathsf{n})$ is canonically identified to the interval $[\varepsilon, \gamma_o] \subseteq \mathfrak{S}_n$ (considered with respect to a natural partial order on $\mathfrak{S}_n$), where ε is the unit of $\mathfrak{S}_n$ and γ_o is the forward cycle.

There are two extensions of the concept of non-crossing partitions that were considered in the recent research literature. On the one hand, V. Reiner introduced in 1997 the analogue of *type B* for $\mathsf{NC}^{\mathsf{A}}(\mathsf{n})$. This poset is denoted $\mathsf{NC}^{\mathsf{B}}(\mathsf{n})$ and it is isomorphic to the interval $[\varepsilon, \gamma_o]$ of the hyperoctahedral group B_n, where now γ_o stands for the natural forward cycle of B_n. On the other hand, J. Mingo and A. Nica studied in 2004 a set of *annular* non-crossing partitions (diagrams drawn inside an annulus – unlike the partitions from $\mathsf{NC}^{\mathsf{A}}(\mathsf{n})$ or from $\mathsf{NC}^{\mathsf{B}}(\mathsf{n})$, which are drawn inside a disc).

In this thesis the type B and annular objects are considered in a unified framework. The forward cycle of B_n is replaced by a permutation which has two cycles, $\gamma = [1, 2, \ldots, p][p+1, \ldots, p+q]$, where $p + q = n$. Two equivalent characterizations of the interval $[\varepsilon, \gamma] \subseteq B_n$ are found – one of them is in terms of a *genus inequality*, while the other is in terms of *annular crossing patterns.* A corresponding poset $\mathsf{NC}^{\mathsf{B}}(\mathsf{p},\,\mathsf{q})$ of *annular non-crossing partitions of type B* is introduced, and it is proved that $[\varepsilon, \gamma] \simeq \mathsf{NC}^{\mathsf{B}}(\mathsf{p},\,\mathsf{q})$, where the partial order on $\mathsf{NC}^{\mathsf{B}}(\mathsf{p},\,\mathsf{q})$ is the usual reverse refinement order for partitions.

The posets $\mathsf{NC}^{\mathsf{B}}(\mathsf{p},\,\mathsf{q})$ are not lattices in general, but a remarkable exception is found to occur in the case when $q = 1$. Moreover, it is shown that the meet operation in the lattice $\mathsf{NC}^{\mathsf{B}}(\mathsf{n\text{-}1},\,\mathsf{1})$ is the usual "intersection meet" for partitions. Some results concerning the enumeration properties of this lattice are obtained, specifically concerning its rank generating function and its Möbius function.

The results described above in type B are found to also hold in connection to the Weyl groups of *type D*. The poset $\mathsf{NC}^{\mathsf{D}}(\mathsf{n\text{-}1},\,\mathsf{1})$ turns out to be equal to the poset $\mathsf{NC}^{\mathsf{D}}(\mathsf{n})$ constructed by C. Athanasiadis and V. Reiner in a paper in 2004. The non-crossing partitions of type D of Athanasiadis and Reiner are thus identified as annular objects.

Non-crossing partitions of type A are central objects in the combinatorics of free probability. A parallel concept of *free independence of type B*, based on non-crossing partitions of type B, was proposed by P. Biane, F. Goodman and A. Nica in a paper in 2003. This thesis introduces a concept of *scarce $\mathbb{G}$-valued probability spaces*, where $\mathbb{G}$ is the algebra of Graßman numbers, and recognizes free independence of type B as free independence in the "scarce $\mathbb{G}$-valued" sense.

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Chapter 1

Introduction and Thesis Outline

The lattice $\mathsf{NC}^{\mathsf{A}}(\mathsf{n})$ of non-crossing partitions of type A has been studied since the early 70's, starting with the work of Kreweras. In 1997, Biane observed that $\mathsf{NC}^{\mathsf{A}}(\mathsf{n})$ embeds into the symmetric group $\mathfrak{S}_n$, considered with a natural partial order; via this embedding $\mathsf{NC}^{\mathsf{A}}(\mathsf{n})$ is identified to the interval $[\varepsilon, \gamma_o]$ of $\mathfrak{S}_n$, where ε denotes the identity permutation on the set $\{1, 2, \ldots, n\} \stackrel{\text{def}}{=} [n]$, and γ_o stands for the forward cycle.

A similar result was obtained by Reiner, also in 1997, this time in connection to the hyperoctahedral group B_n. Reiner defined a type B analogue for $\mathsf{NC}^{\mathsf{A}}(\mathsf{n})$, denoted $\mathsf{NC}^{\mathsf{B}}(\mathsf{n})$, which is also identified to the interval $[\varepsilon, \gamma_o]$, where now $\gamma_o \in B_n$ denotes the natural forward cycle on the set $\{1, 2, \ldots, n, -1, -2, \ldots, -n\} \stackrel{\text{def}}{=} [\pm n]$.

In 2004, moving into a different direction, Mingo and Nica studied a set of annular non-crossing partitions, denoted $\mathsf{NC}^{\mathsf{A}}(\mathsf{p}, \mathsf{q})$. These partitions still connect to the symmetric groups, where now the natural forward cycle is replaced by a permutation which has two cycles

$$\gamma = (1, 2 \ldots, p)(p + 1, p + 2, \ldots, p + q).$$

The partitions from $\mathsf{NC}^{\mathsf{A}}(\mathsf{p}, \mathsf{q})$ are drawn inside an annulus, unlike the partitions from $\mathsf{NC}^{\mathsf{A}}(\mathsf{n})$ and $\mathsf{NC}^{\mathsf{B}}(\mathsf{n})$, which are drawn inside a disc. However, the results in the annular framework are not as nice as in the disc case; this is due to a good extent to the fact that $\mathsf{NC}^{\mathsf{A}}(\mathsf{p}, \mathsf{q})$ *cannot* be identified as an interval of $\mathfrak{S}_{p+q}$.

Our main point in this thesis is that by imposing *symmetry* conditions of type B, the problems encountered in the annular framework disappear.

We introduce a poset of *annular non-crossing partitions of type B*, which we denote $\mathsf{NC}^{\mathsf{B}}(\mathsf{p},\ \mathsf{q})$ and we prove that this poset *is* isomorphic to the interval $[\varepsilon, \gamma]$ of the hyperoctahedral group B_{p+q}, where

$$\gamma = (1, 2, \ldots, p, -1, -2, \ldots, -p)(p+1, p+2, \ldots, p+q, -(p+1), -(p+2), \ldots, -(p+q)).$$

The posets $\mathsf{NC}^{\mathsf{B}}(\mathsf{p},\ \mathsf{q})$ are not lattices in general, but a remarkable exception is found to occur in the case when $q = 1$. Moreover, we show that the meet operation in the lattice $\mathsf{NC}^{\mathsf{B}}(\mathsf{n\text{-}1},\ 1)$ is the usual "intersection meet" for partitions. Some results concerning the enumeration properties of this lattice are obtained, specifically concerning its rank generating function and its Möbius function.

The results described above in type B are found to also hold in connection to the Weyl groups of *type D*. The poset $\mathsf{NC}^{\mathsf{D}}(\mathsf{n\text{-}1},\ 1)$ turns out to be equal to the poset $\mathsf{NC}^{\mathsf{D}}(\mathsf{n})$ constructed by C. Athanasiadis and V. Reiner in a paper in 2004. Therefore, the non-crossing partitions of type D of Athanasiadis and Reiner are identified as annular objects, thus giving a first application of the above results.

Non-crossing partitions of type A are central objects in the combinatorics of free probability. A parallel concept of *free independence of type B*, based on non-crossing partitions of type B, was proposed by P. Biane, F. Goodman and A. Nica in a paper in 2003. This thesis introduces a concept of *scarce $\mathbb{G}$-valued probability spaces*, where $\mathbb{G}$ is the algebra of Graßman numbers, and recognizes free independence of type B as free independence in the "scarce $\mathbb{G}$-valued" sense.

In the remaining part of the introduction we give a more detailed description of the results in the thesis and we also indicate how the thesis is organized.

Non-crossing partitions

After a short review, given in Section 2.1, of some general facts and terminology from poset theory that are used in this thesis, the rest of Chapter 2 is devoted to *non-crossing partitions*. Thus section 2.2 deals with non-crossing partitions of "type A", originally introduced and studied by G. Kreweras in [Kre72]. Block containment (or, equivalently, reversed refinement) defines a partial order relation on the set $\Pi(\mathsf{n})$ of *all* partitions of the set $X = \{1, 2, \ldots, n\} = [n]$. However, we will focus on the subposet, denoted $\mathsf{NC}^{\mathsf{A}}(\mathsf{n})$, of partitions which satisfy a certain *non-crossing* condition. We note that $\mathsf{NC}^{\mathsf{A}}(\mathsf{n})$ itself is a lattice, but it is *not* a sublattice of $\Pi(\mathsf{n})$ (the meet operation on $\mathsf{NC}^{\mathsf{A}}(\mathsf{n})$ coincides with the one on $\Pi(\mathsf{n})$, but the join operation is not the same anymore). Partitions are usually drawn in the following way: the points of the set $[n]$ are spread on a circle, in clockwise sense and then joined into some *convex hulls* according to their block structure. By looking at such pictures the choice of the name "non-crossing" in the name of $\mathsf{NC}^{\mathsf{A}}(\mathsf{n})$ becomes self-explanatory – a partition π of $[n]$ is non-crossing when the convex hulls drawn for the blocks of π are pairwise disjoint. Such a drawing is shown in Figure 2.1 on page 16. We also mention that $\mathsf{NC}^{\mathsf{A}}(\mathsf{n})$ plays a fundamental rôle in the combinatorics of *free probability* (see e.g. [NS06]).

Section 2.3 concerns the case of non-crossing partitions of type B which were introduced by V. Reiner in [Rei97]. They are partitions of the ordered set $\{1, 2, \ldots, n, -1, -2, \ldots, -n\} \stackrel{\text{def}}{=} [\pm n]$. The extra condition which is imposed on a partition π from $\mathsf{NC}^{\mathsf{B}}(\mathsf{n})$ is a *symmetry* condition which must be satisfied by the blocks of π (condition (2.24) specifically). $\mathsf{NC}^{\mathsf{B}}(\mathsf{n})$ is a sublattice of $\mathsf{NC}^{\mathsf{A}}(\pm\,\mathsf{n})$. The symmetric arrangement of the blocks of a type B non-crossing partition appears very clearly on their pictures (see Figure 2.2 on page 19).

The marked group framework

Starting with the paper [Bia97] it became apparent that there is an intimate connection between partitions and *permutations*. This connection is best understood in the *marked group framework*, which is presented in detail in Section 3.1. The main idea is the following: consider a group G with the group operation written multiplicatively and the set of generators T and look at the minimal factorizations of an arbitrary element x of G as products of the generators. The number of generators appearing in any of the minimal factorizations gives the *length* of x, denoted $\ell_G(x)$. Under some extra assumptions on the set T, the length induces a partial order on G by declaring

$$x \leqslant_G y \Longleftrightarrow \ell_G(y) = \ell_G(x) + \ell_G(x^{-1}y). \tag{1.1}$$

Most of the facts from Sections 3.1 are presented with their proofs as, although they are well-known, we were not able to find a suitable reference in the existing literature.

In Section 3.2 we apply these notions to concrete examples of groups, most notably to the *symmetric group* $\mathfrak{S}_n$, the *hyperoctahedral* group

$$B_n = \{\tau \in \mathfrak{S}_{\pm n} \mid \tau(-i) = -\tau(i), i \in X = [\pm n]\},$$

and the *Weyl group*

$$D_n = \{\tau \in B_n \mid \tau \quad \text{is an even permutation}\}.$$

Three ways of looking at $\mathfrak{S}_{\mathsf{nc}}^{\mathsf{A}}(X,\gamma_o)$

In Section 3.3 we start looking at the set of non-crossing *permutations*, denoted $\mathfrak{S}_{\mathsf{nc}}^{\mathsf{A}}(X,\gamma_o)$, where $X=[n]$ and γ_o denotes the forward cycle of $[n]$. There are three points of view that one can take when looking at $\mathfrak{S}_{\mathsf{nc}}^{\mathsf{A}}(X,\gamma_o)$; the following table gives a summary of how this goes.

I	A permutation τ of X is in $\mathfrak{S}_{\mathsf{nc}}^{\mathsf{A}}(X,\gamma_o)$ if and only if every cycle of τ is compatible with γ_o, and if τ avoids a suitably defined *crossing pattern* with respect to γ_o.
II	A permutation τ of X is in $\mathfrak{S}_{\mathsf{nc}}^{\mathsf{A}}(X,\gamma_o)$ if and only if it achieves equality in a certain *genus inequality for τ and γ_o.*
III	$\mathfrak{S}_{\mathsf{nc}}^{\mathsf{A}}(X,\gamma_o)$ is the *interval* $[\varepsilon,\gamma_o]$ in the symmetric group $\mathfrak{S}_n$, where ε is the identity permutation of X

Table 1. Three ways of looking at $\mathfrak{S}_{\mathsf{nc}}^{\mathsf{A}}(X,\gamma_o)$.

The "genus inequality for τ and γ_o" simply says that

$$\#(\tau)+\#(\tau^{-1}\gamma_o)\leqslant n+1, \tag{1.2}$$

where $\#(\tau)$ denotes the number of cycles of τ, counting the fixed points.

The equivalence between II and III is thus immediate by comparing (1.1) and (1.2).

Let us also point out that non-crossing permutations are related to non-crossing partitions via the poset isomorphism

$$[\varepsilon,\gamma_o]\ \ni\tau\longmapsto\Omega(\tau)\in\mathsf{NC}^{\mathsf{A}}(\mathsf{n})\qquad\text{(see [Bia97])}, \tag{1.3}$$

where $\Omega(\tau)$ denotes the partition of X into the orbits of $\tau\in\mathfrak{S}_n$.

Type B analogue of $\mathfrak{S}_{\mathsf{nc}}^{\mathsf{A}}(X, \gamma_o)$

In Section 3.4 we look at the type B analogues of the facts from Section 3.3. If we denote

$$\mathfrak{S}_{\mathsf{nc}}^{\mathsf{B}}(X, \gamma_o) = \mathfrak{S}_{\mathsf{nc}}^{\mathsf{A}}(X, \gamma_o) \cap B_n \tag{1.4}$$

it turns out that the approach III from Table 1 has a very nice type B counterpart; that is, one has

$$\mathfrak{S}_{\mathsf{nc}}^{\mathsf{B}}(X, \gamma_o) = [\varepsilon, \gamma_o]_{B_n} = \{\tau \in B_n \mid \varepsilon \leqslant_{B_n} \tau \leqslant_{B_n} \gamma_o\}$$

where $\gamma_o = (1, 2, \dots, n, -1, -2, \dots, -n)$ denotes the forward cycle on $[\pm n]$. Moreover, one has the type B analogue for the poset isomorphism from (1.3), also implemented by the same orbit map Ω:

$$[\varepsilon, \gamma_o]_{B_n} \ni \tau \longmapsto \Omega(\tau) \in \mathsf{NC}^{\mathsf{B}}(\mathsf{n}) \ . \tag{1.5}$$

The proofs of these facts are much more recent than for the corresponding facts in type A – they appeared independently in [BW02], [Bes03], [BGN03], not long after the lattice $\mathsf{NC}^{\mathsf{B}}(\mathsf{n})$ was introduced by Reiner.

Annular objects

In Section 4.1 we take on another variation of the situation presented in Table 1, recently considered in [MN04], where the forward cycle γ_o is now replaced by a permutation γ which has *two orbits*. We denote the orbits of γ by Y and Z (disjoint non-empty subsets of X, such that $Y \cup Z = X$). Both the approaches II and III in Table 1 turn out to have straightforward analogues in this situation. However, the analogue of (III) does not look very good, as it turns out that the interval $[\varepsilon, \gamma]_{\mathfrak{S}_X}$ is canonically identified to the direct product $[\varepsilon, \alpha] \times [\varepsilon, \beta] \subseteq \mathfrak{S}_Y \times \mathfrak{S}_Z$, where $\alpha \in \mathfrak{S}_X$ and $\beta \in \mathfrak{S}_Z$) are the cyclic permutations induced by γ. (In other words, we have $[\varepsilon, \gamma] \simeq \mathfrak{S}_{\mathsf{nc}}^{\mathsf{A}}(X, \alpha) \times \mathfrak{S}_{\mathsf{nc}}^{\mathsf{A}}(X, \beta)$, and this takes us back to the case when we consider "reference" permutations with only one orbit.) It is thus more promising to define $\mathfrak{S}_{\mathsf{nc}}^{\mathsf{A}}(X, \gamma)$ via the analogue of (II) from Table 1:

$$\mathfrak{S}_{\mathsf{nc}}^{\mathsf{A}}(X, \gamma) = \left\{ \tau \in \mathfrak{S}_X \;\middle|\; \begin{array}{c} \text{the genus inequality for } \tau \text{ and } \gamma \\ \text{holds with equality} \end{array} \right\}. \tag{1.6}$$

The permutations in $\mathfrak{S}_{\mathsf{nc}}^{\mathsf{A}}(X, \gamma)$ are said to be *annular non-crossing*; the word "annular" is used here in connection to the fact that such permutations are usually drawn in an annulus, where the elements of Y are represented as points on the outer circle of the annulus, while the elements of Z are represented as points on the inner circle of the annulus, in the cyclic orders indicated by γ. Examples of such drawings can be seen in Figure 4.1 on page 51.

It is easily seen that the set $\mathfrak{S}_{\mathsf{nc}}^{\mathsf{A}}(X,\gamma)$ defined by Equation (1.6) contains the interval $[\varepsilon,\gamma]_{\mathfrak{S}_X}$, and the inclusion is strict. To be precise, $[\varepsilon,\gamma]_{\mathfrak{S}_X}$ turns out to consist of exactly those permutations in $\mathfrak{S}_{\mathsf{nc}}^{\mathsf{A}}(X,\gamma)$ which are "γ – disconnected" (in the sense of 4.1.1).

Referring again to Table 1, one can then ask what is the annular analogue, relevant to $\mathfrak{S}_{\mathsf{nc}}^{\mathsf{A}}(X,\gamma)$, for the approach I in that table. This question was answered in [MN04]. In that paper it is explained what is the suitable definition for the fact that "every cycle of τ is compatible with γ", and what are the crossing patterns that have to be avoided by τ, in order for τ to belong to $\mathfrak{S}_{\mathsf{nc}}^{\mathsf{A}}(X,\gamma)$. The description of $\mathfrak{S}_{\mathsf{nc}}^{\mathsf{A}}(X,\gamma)$ via annular crossing patterns looks a bit ad-hoc, but is nevertheless very useful, due to the fact that it is done in terms of *localized* conditions: if a permutation $\tau\in\mathfrak{S}_X$ does not belong to $\mathfrak{S}_{\mathsf{nc}}^{\mathsf{A}}(X,\gamma)$, then this is detectable by inspecting a subset of not more than six elements of X, and by finding that they fail to satisfy either an annular crossing pattern, or a condition of compatibility with γ.

Finally, let us record here a troubling issue which appears when we move to the annular framework: the orbit map $\tau\mapsto\Omega(\tau)$ is *not one-to-one* on $\mathfrak{S}_{\mathsf{nc}}^{\mathsf{A}}(X,\gamma)$. In Section 4 of [MN04], the pathology causing the non-injectivity of the orbit map is identified precisely. Because of this (and because of the fact that $[\varepsilon,\gamma]$ is here only a proper subset of $\mathfrak{S}_{\mathsf{nc}}^{\mathsf{A}}(X,\gamma)$), the nice poset isomorphism mentioned in (1.3) above is quite far from having an annular counterpart.

***Symmetric* annular objects**

Starting with Section 4.2 the new things which are brought in by this thesis are presented. Our main point is to show that

the problems encountered in the annular framework disappear
when symmetry conditions of type B are added.

We will denote

$$\gamma=(1,2,\ldots,p,-1,-2,\ldots-p)(p+1,\ldots,n,-(p+1),\ldots,-n)$$

and, by analogy with (1.4) let us define

$$\mathfrak{S}_{\mathsf{nc}}^{\mathsf{B}}(X,\gamma)\,=\mathfrak{S}_{\mathsf{nc}}^{\mathsf{A}}(X,\gamma)\,\cap B_n.$$

The first issue which did not hold in the type A annular framework was that $\mathfrak{S}_{\mathsf{nc}}^{\mathsf{A}}(X,\gamma)\neq[\varepsilon,\gamma_o]_{\mathfrak{S}_n}$. But in the type B annular framework we obtain that

$$\mathfrak{S}_{\mathsf{nc}}^{\mathsf{B}}(X,\gamma)\,=[\varepsilon,\gamma]_{B_n}\qquad\text{(see Theorem 4.2.3).}$$

The second fact which did not hold in the general annular framework, but does hold for $\mathfrak{S}^{\mathsf{B}}_{\mathsf{nc}}(X,\gamma)$, is that the orbit map Ω *is one-to-one* on $\mathfrak{S}^{\mathsf{B}}_{\mathsf{nc}}(X,\gamma)$. We will work with a small adjustment of the orbit map, which in addition to being one-to-one is also order-preserving between the partial order on $\mathfrak{S}^{\mathsf{B}}_{\mathsf{nc}}(X,\gamma)$ (viewed as an interval in B_n) and the reversed refinement order on partitions of X. The adjusted orbit partition of a permutation $\tau \in \mathfrak{S}^{\mathsf{B}}_{\mathsf{nc}}(X,\gamma)$ will be denoted by $\widetilde{\Omega}$; $\widetilde{\Omega}(\tau)$ is obtained by bundling together all the non-zero blocks of $\Omega(\tau)$ into a single block of $\widetilde{\Omega}(\tau)$. By letting

$$\mathsf{NC}^{\mathsf{B}}(\mathsf{p},\,\mathsf{q}) = \{\widetilde{\Omega}(\tau) \mid \tau \in \mathfrak{S}^{\mathsf{B}}_{\mathsf{nc}}(X,\gamma)\}$$

we prove that the following analogue for the poset isomorphism from (1.5):

$$\mathfrak{S}^{\mathsf{B}}_{\mathsf{nc}}(X,\gamma) \ni \tau \longmapsto \widetilde{\Omega}(\tau) \in \mathsf{NC}^{\mathsf{B}}(\mathsf{p},\,\mathsf{q}) \qquad \text{(see Theorem 4.2.18)}$$

is a poset isomorphism, where $\mathfrak{S}^{\mathsf{B}}_{\mathsf{nc}}(X,\gamma)$ is partially ordered as an interval of B_n, while $\mathsf{NC}^{\mathsf{B}}(\mathsf{p},\,\mathsf{q})$ is partially ordered by reversed refinement.

$\mathsf{NC}^{\mathsf{B}}(\mathsf{n\text{-}1},\,\mathsf{1})$ is a lattice

One of the reasons for the interest shown to the posets of non-crossing partitions is that many times they are lattices. The poset $\mathsf{NC}^{\mathsf{B}}(\mathsf{p},\,\mathsf{q})$ is not a lattice if $p, q \geqslant 2$, but it is interesting to see that that $\mathsf{NC}^{\mathsf{B}}(\mathsf{n\text{-}1},\,\mathsf{1})$ *is* a lattice, for every $n \geqslant 2$. The meet operation on $\mathsf{NC}^{\mathsf{B}}(\mathsf{n\text{-}1},\,\mathsf{1})$ is the restriction of the meet operation on the lattice $\Pi(\mathsf{n})$ of all partitions of $[n]$.

The rank cardinalities for $\mathsf{NC}^{\mathsf{B}}(\mathsf{n\text{-}1},\,\mathsf{1})$ are found in Section 5.3, specifically in formula (5.16). When comparing formula (5.16) to Proposition 6 of [Rei97], one notices the somewhat surprising fact that $\mathsf{NC}^{\mathsf{B}}(\mathsf{n\text{-}1},\,\mathsf{1})$ has exactly the same rank generating function as the lattice $\mathsf{NC}^{\mathsf{B}}(\mathsf{n})$. An immediate thought related to this is that perhaps $\mathsf{NC}^{\mathsf{B}}(\mathsf{n\text{-}1},\,\mathsf{1})$ might just be equal to (or at least isomorphic to) $\mathsf{NC}^{\mathsf{B}}(\mathsf{n})$. The particular case $n = 2$ does not contradict this thought, as it is indeed true that $\mathsf{NC}^{\mathsf{B}}(\mathsf{1},\,\mathsf{1}) = \mathsf{NC}^{\mathsf{B}}(\mathsf{2})$. But, already for $n = 3$ it is no longer true that $\mathsf{NC}^{\mathsf{B}}(\mathsf{2},\,\mathsf{1}) = \mathsf{NC}^{\mathsf{B}}(\mathsf{3})$ (a direct way to see that $\mathsf{NC}^{\mathsf{B}}(\mathsf{2},\,\mathsf{1}) \not\simeq \mathsf{NC}^{\mathsf{B}}(\mathsf{3})$ is by examining the Hasse diagrams of these two lattices) and in fact, the Möbius formula for $\mathsf{NC}^{\mathsf{B}}(\mathsf{n\text{-}1},\,\mathsf{1})$, which is found in Section 5.4, shows that $\mathsf{NC}^{\mathsf{B}}(\mathsf{n\text{-}1},\,\mathsf{1}) \not\simeq \mathsf{NC}^{\mathsf{B}}(\mathsf{n})$, for all $n \geqslant 3$.

A first application

A first application of our results is found in Chapter 6. In section 6.2 it is observed how type D analogues can be obtained for the above Theorems 4.2.3 and 4.2.18. In fact, for each of these two theorems, the analogue of type D is an easy consequence of the corresponding fact in type

B. The key observation is to recognize the non-crossing partitions from $\mathsf{NC}^{\mathsf{D}}(\mathsf{n})$ appearing in the type D construction proposed in the paper [AR04] (which is explained in Section 6.1) as *annular* objects coming from $\mathsf{NC}^{\mathsf{D}}(\mathsf{n\text{-}1,\ 1})$, where

$$\mathsf{NC}^{\mathsf{D}}(\mathsf{p,\ q}) = \{\widetilde{\Omega}(\tau) \mid \tau \in \mathfrak{S}^{\mathsf{D}}_{\mathsf{nc}}(X,\gamma) = \mathfrak{S}^{\mathsf{A}}_{\mathsf{nc}}(X,\gamma) \cap D_n\}.$$

The fact that $\ell_{B_n} = \ell_{D_n}$ finishes the argument.

In [AR04] it is also shown that $\mathsf{NC}^{\mathsf{D}}(\mathsf{n})$ is a lattice. However this does not follow from our result since the meet operation considered in that paper does not coincide with the usual *non-empty block intersection* meet operation from $\Pi(\pm\mathsf{n})$.

A second application

In Chapter 7 a second application is found. So far [BGN03] is the only paper in the literature which replaces $\mathsf{NC}^{\mathsf{A}}(\mathsf{n})$ with $\mathsf{NC}^{\mathsf{B}}(\mathsf{n})$ in the context of combinatorics of free probability and proposes a *type B freeness* theory. We introduce the concept of *scarce* $\mathbb{A}$ *-valued probability space*, where $\mathbb{A}$ is a commutative algebra over the algebra $\mathbb{C}$ of complex numbers. When $\mathbb{A}$ is taken to be the commutative algebra $\mathbb{G}$ of Graßman numbers, we show how the concept of "type B freeness" from that paper can be seen in the framework of scarce $\mathbb{G}$ -valued probability space.

Chapter 2

Non-Crossing Partitions

2.1 Background on Posets and Möbius Inversion

2.1.1 Definition. We will use the abbreviation *poset* for "partially ordered set". We will only consider finite posets. The partial order on a poset will be usually denoted by "$\leqslant$", or, when necessary, by "$\leqslant_P$".

i. • For $a, b \in P$ we denote

$$a < b \quad \overset{\text{def}}{\Longleftrightarrow} \quad [a \leqslant b \quad \& \quad a \neq b].$$

• b is called a *cover* for a, or b is said to cover a if $a < b$ and if there is no other element c in the poset P strictly between a and b, that is

$$b \quad \text{covers} \quad a \quad \overset{\text{def}}{\Longleftrightarrow} \quad [a < b \quad \& \quad (a \leqslant c \leqslant b \Longrightarrow c \in \{a, b\})]. \tag{2.1}$$

• The set $\{c \in P \mid a \leqslant c \leqslant b\}$ is called the *interval* $[a, b]$.

• A totally ordered subset of P is called a *chain.* The *length* of a finite chain is defined to be the number of elements in that chain less one. A poset P is called *ranked* if all maximal chains are finite and if they all have the same length. In a ranked poset, the length of any of the maximal chains is called the *rank* of the poset.

ii. • For a subset $S \subseteq P$ the sets of the *minorants* (or *lower bounds*) of S, $\mathcal{L}(S)$, and that of the *majorants* (or *upper bounds*) of S, $\mathcal{U}(S)$, are defined as follows

$$\begin{aligned}\mathcal{L}(S) &\stackrel{\text{def}}{=} \{b \in P \mid b \leqslant a, \quad \forall a \in S\} \quad \text{and, respectively,}\\ \mathcal{U}(S) &\stackrel{\text{def}}{=} \{b \in P \mid b \geqslant a, \quad \forall a \in S\}.\end{aligned}$$

• Given $P \supseteq S$ a subposet of P, the smallest majorant of S, if it exists, is called the *least upper bound* of S and it is denoted l.u.b.(S); the greatest minorant is called the *greatest lower bound* of S, and it is denoted g.l.b.(S). If l.u.b.(S), g.l.b.(S) exist then they are the unique elements of P which satisfy

$$\begin{aligned}&\text{g.l.b.}(S) \in \mathcal{U}(S) \quad \& \quad \text{g.l.b.}(S) \leqslant b, \quad \forall b \in \mathcal{U}(S) \quad \text{and,}\\ &\text{l.u.b.}(S) \in \mathcal{L}(S) \quad \& \quad \text{l.u.b.}(S) \geqslant b, \quad \forall b \in \mathcal{L}(S).\end{aligned}$$

• For a finite subset $A =: \{a_1, a_1, \ldots, a_n\} \subset P$, we will use the "meet" and "join" notations for g.l.b.(A), l.u.b.(A). Thus,

$$\begin{aligned}&a_1 \wedge a_2 \wedge \cdots \wedge a_n = \text{l.u.b.}\{a_1, a_2, \ldots, a_n\} \quad \text{and}\\ &a_1 \vee a_2 \vee \cdots \vee a_n = \text{g.l.b.}\{a_1, a_2, \ldots, a_n\}.\end{aligned}$$

iii. The poset P is called a *lattice* if any two elements of P have both a meet and a join.

2.1.2 Remark. Let P be a finite lattice. An induction argument shows that any family $\{a_1, \ldots, a_n\} \subseteq P$ must have meet and join. In particular, by listing $\{a_1, \ldots, a_n\} = P$ it follows that the finite lattice P must have minimal and maximal elements, denoted 0_P and 1_P, respectively: $0_P \leqslant x \leqslant 1_P, \quad \forall x \in P$.

The following lemma gives a criterion which is useful in verifying that concrete posets are lattices.

2.1.3 Lemma. *Let P be a finite poset. If P has a maximum element 1_P and if every two elements $a, b \in P$ have a meet $a \wedge b$ then P is a lattice.*

Proof. Let $a, b \in P$. The goal is to show that they have a join. The set of upper bounds for a, b, must be finite since P itself is finite. Also, the set $\mathcal{U}(\{a, b\})$ is not empty, since it contains 1_P. List $\mathcal{U}(\{a, b\}) = \{u_1, u_2, \ldots u_n\}$. Since we know that every two elements have a meet then any finite family must have a meet, in particular $\{u_1, u_2, \ldots u_n\}$. The meet of the family $\{u_1, u_2, \ldots u_n\}$ is the join of a and b, i.e.

$$a \vee b = u_1 \wedge u_2 \wedge \cdots \wedge u_n.$$

■

2.1.4 Remark. Let $P_1, P_2, \ldots, P_n$ be posets. The direct product $P_1 \times \cdots \times P_n$ becomes a poset w.r.t. the partial order defined by

$$(\sigma_1, \ldots, \sigma_n) \leqslant (\pi_1, \ldots, \pi_n) \Longleftrightarrow \sigma_i \leqslant \pi_i, \quad \forall 1 \leqslant i \leqslant n.$$

The meet and join operations are given by

$$\begin{aligned}(\sigma_1, \ldots, \sigma_n) \wedge (\pi_1, \ldots, \pi_n) &= (\sigma_1 \wedge \pi_1, \ldots, \sigma_n \wedge \pi_n)\\ (\sigma_1, \ldots, \sigma_n) \vee (\pi_1, \ldots, \pi_n) &= (\sigma_1 \vee \pi_1, \ldots, \sigma_n \vee \pi_n).\end{aligned}$$

2.1.5 The Möbius Inversion Formula in Finite Posets. Let P be a finite poset and let

$$P^{(2)} = \{(\pi, \sigma) \in P \times P \mid \pi \leqslant \sigma\}. \tag{2.2}$$

For $F, G : P^{(2)} \longrightarrow \mathbb{C}$ we define their *convolution* $F * G$ to be the function

$$F * G : P^{(2)} \longrightarrow \mathbb{C}\ , \quad (F * G)(\pi, \sigma) \stackrel{\text{def}}{=} \sum_{\substack{\rho \in P \\ \rho \in [\pi, \sigma]}} F(\pi, \rho) G(\rho, \sigma). \tag{2.3}$$

Also, for functions $f : P \longrightarrow \mathbb{C}$ and $G : P^{(2)} \longrightarrow \mathbb{C}$ their convolution is defined by

$$f * G : P \longrightarrow \mathbb{C}\ , \quad (f * G)(\sigma) \stackrel{\text{def}}{=} \sum_{\substack{\rho \in P \\ \rho \leqslant \sigma}} f(\rho) G(\rho, \sigma). \tag{2.4}$$

The convolution operations defined above are associative and distributive w.r.t. the operation of taking linear combinations of functions on $P^{(2)}$ or on P. The *unit* for the convolution operations is the function

$$\delta : P^{(2)} \longrightarrow \mathbb{C}\ , \quad \delta(\pi, \sigma) = \begin{cases} 1, & \text{if} \quad \pi = \sigma \\ 0, & \text{if} \quad \pi < \sigma. \end{cases} \tag{2.5}$$

The following lemma describes the invertible functions w.r.t. convolution.

2.1.6 Lemma. *Let P be a finite poset and consider the convolution operation for functions on $P^{(2)}$ as in (2.3). A function $F : P^{(2)} \longrightarrow \mathbb{C}$ is invertible w.r.t. convolution iff $F(\pi, \pi) \neq 0$ for every $\pi \in P$.*

2.1.7 Definition. Let P be a finite poset. The *zeta function* of P is

$$\zeta : P^{(2)} \longmapsto \mathbb{C}\ , \qquad \zeta(\pi, \sigma) = 1, \qquad \forall\, (\pi, \sigma) \in P^{(2)}. \tag{2.6}$$

The inverse of ζ under convolution is called the *Möbius function* of P, and it is denoted by μ or, when necessary, by μ_P.

The Möbius function $\mu : P^{(2)} \longrightarrow \mathbb{C}$ satisfies the *Möbius Inversion Formula*, that is, for every pair of functions $f, g : P \longrightarrow \mathbb{C}$, we have that

$$f(\pi) = \sum_{\sigma \leqslant \pi} g(\sigma) \qquad \Longleftrightarrow \qquad g(\sigma) = \sum_{\pi \leqslant \sigma} f(\pi)\mu(\pi, \sigma), \qquad \forall\, \pi, \sigma \in P. \tag{2.7}$$

The above equivalence is immediate, being equivalent to

$$f = g * \zeta \Longleftrightarrow g = f * \mu.$$

Moreover, the function μ satisfies

$$\mu(\pi, \pi) = 1, \qquad \forall\, \pi \in P, \tag{2.8}$$

and

$$\sum_{\rho \in [\pi, \sigma]} \mu(\pi, \rho) = 0 = \sum_{\rho \in [\pi, \sigma]} \mu(\rho, \sigma), \qquad \forall\, \pi < \sigma. \tag{2.9}$$

Equation (2.9) above can be used to compute the Möbius function recursively, by induction on the length of interval $[\pi, \rho]$.

2.1.8 Lemma. *Let $P, Q, P_1, P_2, \ldots, P_n$ be finite posets.*

i. *If $\Phi : P \longrightarrow Q$ is a poset isomorphism, then*

$$\mu_Q(\Phi(\pi), \Phi(\sigma)) = \mu_P(\pi, \sigma), \qquad \forall\, \pi \leqslant_P \sigma.$$

ii. *The Möbius function is* multiplicative*. That is,*

$$(\pi_1, \ldots, \pi_k) \leqslant (\sigma_1, \ldots, \sigma_k) \Longrightarrow \mu_P((\pi_1, \ldots, \pi_k), (\sigma_1, \ldots, \sigma_k)) = \mu_{P_1}(\pi_1, \sigma_1) \cdots \mu_{P_k}(\pi_k, \sigma_k). \tag{2.10}$$

2.2 Non-Crossing Partitions of Type A

Let X be a finite non-empty set. The cardinality of X is denoted by $|X|$. The following special notations are used without reference throughout the thesis

$$\{1, 2, \ldots, n\} = [n], \qquad \text{and} \qquad \{\pm 1, \pm 2, \ldots, \pm n\} = [\pm n]. \tag{2.11}$$

2.2.1 Partitions. Let $B_1, \ldots, B_k$ be mutually disjoint non-empty subsets of X which cover X. Then $\pi = \{B_1, \ldots, B_k\}$ is called a *partition* of X. That is, π is a partition of X if the sets B_i are such that

$$\begin{cases} B_i \neq \emptyset, \quad 1 \leqslant i \leqslant k \\ B_i \cap B_j = \emptyset, \quad i \neq j, \quad 1 \leqslant i, j \leqslant k \\ B_1 \cup \ldots \cup B_k = X. \end{cases}$$

The sets $B_1, \ldots, B_k$ are called the *blocks* of π. The set of all partitions of X is denoted by $\Pi(X)$. If n is a positive integer, we write $\Pi(\mathsf{n})$ instead of $\Pi([n])$ and $\Pi(\pm\mathsf{n})$ instead of $\Pi([\pm n]\,)$.

• If $\pi \in \Pi(X)$, then $\#(\pi)$ denotes the number of blocks of π.

• We denote by $\overset{\pi}{\sim}$ the equivalence relation on X determined by the partition π, and which is defined by

$$a \overset{\pi}{\sim} b \overset{\text{def}}{=} \quad a \text{ and } b \text{ belong to the same block of } \pi. \tag{2.12}$$

• If Y is a non-empty subset of X and $\pi \in \Pi(X)$, then the *restriction* of π to Y is the partition in $\Pi(X)$ denoted $\pi|_Y$ and defined by

$$a \overset{\pi|_Y}{\sim} b \overset{\text{def}}{\Longleftrightarrow} a \overset{\pi}{\sim} b, \qquad a, b \in Y.$$

2.2.2 The Poset $\Pi(X)$. The set $\Pi(X)$ is partially ordered by *reverse refinement*:

$$\pi \leqslant \rho \overset{\text{def}}{\Longleftrightarrow} \quad \text{every block of} \quad \rho \quad \text{is a union of blocks of} \quad \pi. \tag{2.13}$$

If $\pi \leqslant \rho$ then π is also called a refinement of ρ.

The smallest partition (w.r.t. the reverse refinement order) is the one made of singletons only and it is denoted by 0_X while the largest is the one with the single block $\{X\}$ and it is denoted by 1_X. For the special cases when $X = [n]$ and $X = [\pm n]$, $1_X, 0_X$ are denoted by $1_n, 0_n$ and $1_{\pm n}, 0_{\pm n}$, respectively.

An immediate exercise shows that reverse refinement is a partial order which induces a lattice structure on $\Pi(X)$. The meet and join operations are denoted by "$\wedge$" and "$\vee$".

• The meet of two partitions is simply given by non-empty block intersection, i.e. if $\pi = \{P_1, \cdots, P_m\}$ and $\rho = \{R_1, \cdots, R_n\}$ are two partitions in $\Pi(X)$ then

$$\pi \wedge \rho = \{P_i \cap R_j \quad | \quad 1 \leqslant i \leqslant m, \quad 1 \leqslant j \leqslant n, \quad P_i \cap R_j \neq \emptyset\}. \tag{2.14}$$

• The join in $\Pi(X)$ is described in the following way. Given two partitions $\pi, \rho \in \Pi(X)$ look at the equivalence relations on the set X which they determine, denoted $\overset{\pi}{\sim}$, and respectively $\overset{\rho}{\sim}$. Define now a new equivalence relation on the set X, as follows.

$$x \sim y \overset{\text{def}}{\Longleftrightarrow} \begin{cases} \exists\ m \geqslant 1, \exists\ x_o, x_1, \ldots, x_{2m} \in X \quad \text{such that} \\ x = x_o \overset{\pi}{\sim}\ x_1 \overset{\rho}{\sim}\ x_2 \overset{\pi}{\sim}\ x_3 \cdots \overset{\pi}{\sim}\ x_{2m-1} \overset{\rho}{\sim}\ x_{2m} = y. \end{cases} \tag{2.15}$$

It is easily verified that "$\sim$" is indeed an equivalence relation. The equivalence classes of "$\sim$" form a partition in $\Pi(X)$. This partition is exactly the join "$\pi \vee \rho$" of π and ρ.

2.2.3 Remark and Notation. The property of a partition of having crossings makes sense on any finite linearly ordered ground set X. On the other hand any such set X is identified as an ordered set with the set $\{1, 2, \ldots, n\} = [n]$. For this reason we will only present the non-crossing partitions of $[n]$. We also use the convention that elements in each block are listed in increasing order, i.e.

$$\Pi(\mathsf{n}) \ni \pi = \{B_1, \ldots, B_k\} \ni B_i = \{b_{i_1}, \ldots, b_{i_m}\} \quad \Longrightarrow \quad b_{i_1} < \cdots < b_{i_m}.$$

2.2.4 Non-Crossing Partitions. ***i.*** A partition $\pi \in \Pi(\mathsf{n})$ is called *non-crossing* if

$$[i_1 < i_2 < j_1 < j_2 \quad \& \quad i_1 \overset{\pi}{\sim}\ j_1 \quad \& \quad i_2 \overset{\pi}{\sim} j_2] \Longrightarrow i_1 \overset{\pi}{\sim}\ i_2 \overset{\pi}{\sim}\ j_1 \overset{\pi}{\sim}\ j_2. \tag{2.16}$$

ii. The subset of $\Pi(\mathsf{n})$ consisting of all non-crossing partitions is denoted $\mathsf{NC}^{\mathsf{A}}(\mathsf{n})$. The superscript A in the notation will become meaningful after reading Section 2.3 which deals with type B non-crossing partitions.

There are two standard ways of graphically representing partitions, linear and circular. We choose to draw partitions from $\Pi(\mathsf{n})$ on a disc. Figure 2.1 illustrates how this is done.

2.2.5 Remark. *i.* It is clear that every non-crossing partition of $[n]$ has at least one block which is an interval (which may be a singleton). If

$$\pi = \{B_1, \ldots, B_k\}, B_i = \{b_{i_1}, \ldots, b_{i_m}\}, 1 \leqslant i \leqslant k$$

is a partition in $\Pi(\mathsf{n})$, then

$$\pi \in \mathsf{NC}^{\mathsf{A}}(\mathsf{n}) \iff \text{the interval } [b_{i_1}, b_{i_m}] \text{ consists of a union of blocks of } \pi.$$

ii. It is immediate that $\mathsf{NC}^{\mathsf{A}}(\mathsf{n})$ contains 1_n and 0_n and it is a sublattice of $\Pi(\mathsf{n})$ (one needs to verify here that if $\mathsf{NC}^{\mathsf{A}}(\mathsf{n}) \supseteq \{\pi, \rho\}$ then $\pi \wedge \rho \in \mathsf{NC}^{\mathsf{A}}(\mathsf{n})$ as well. This is immediate from (2.14) and best seen via the graphical description of non-crossing partitions).

iii. However, the situation with the join operation is different. In $\mathsf{NC}^{\mathsf{A}}(\mathsf{n})$ the join operation is not given by the procedure described in 2.2.2 (formula (2.15)). For instance,

$$\big\{\{1,3\},\{2\},\{4\}\big\} \quad \vee \quad \big\{\{1\},\{2,4\},\{3\}\big\} = \begin{cases} \big\{\{1,3\},\{2,4\}\big\} \notin \mathsf{NC}^{\mathsf{A}}(4), & \text{in } \Pi(4) \\ \big\{\{1,2,3,4\}\big\} = 1_4, & \text{in } \mathsf{NC}^{\mathsf{A}}(4). \end{cases}$$

iii. The total number of non-crossing partitions is the n^{th} Catalan number

$$card\, \mathsf{NC}^{\mathsf{A}}(\mathsf{n}) = \frac{1}{n+1}\binom{2n}{n}. \tag{2.17}$$

Given $1 \leqslant k \leqslant n$ the number of non-crossing partitions in $\mathsf{NC}^{\mathsf{A}}(\mathsf{n})$ having $n-k$ blocks is called the (type A) Narayana number, and is given by

$$card\{\pi \in \mathsf{NC}^{\mathsf{A}}(\mathsf{n}) \mid rank(\pi) = k\} = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}. \tag{2.18}$$

One fundamental difference between the lattice $(\Pi(\mathsf{n}), \leqslant)$ and $(\mathsf{NC}^{\mathsf{A}}(\mathsf{n}), \leqslant)$ is that the one of non-crossing partitions is *self-dual* while the lattice of all partitions is not. A counterexample for the last statement is provided by looking at the Hasse diagram of $\Pi(4)$, 4 being the first number for which the notion of crossing makes sense. An order-reversing lattice isomorphism on $\mathsf{NC}^{\mathsf{A}}(\mathsf{n})$ is implemented by the *Kreweras complementation map* which is introduced below.

2.2.6 The Kreweras Complement. Aside from the finite linearly ordered set $[n]$ consider the "dual" finite linearly ordered set $[n'] \stackrel{\text{def}}{=} \{1' \leqslant 2' \leqslant \ldots \leqslant n'\}$ and the linearly ordered set $[n : n']$ obtained by "interlacing *to the right*" $[n]$ and $[n']$, i.e.

$$[n : n'] \stackrel{\text{def}}{=} \{1 \leqslant 1' \leqslant 2 \leqslant 2' \leqslant \ldots \leqslant n \leqslant' n\}$$

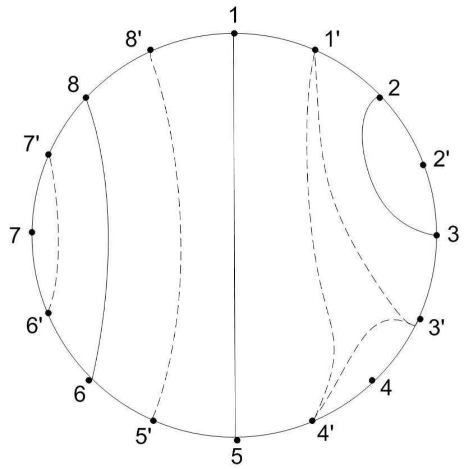


Figure 2.1: The non-crossing partition $\pi = \{\{1,5\},\{2,3\},\{4\},\{6,8\},\{7\}\}$ and its Kreweras complement $K(\pi) = \{\{1,3,4\},\{2\},\{5,8\},\{6,7\}\}$.

Via the natural poset isomorphism $[n] \longrightarrow [n'] : k \longmapsto k'$, elements of $\Pi(\mathsf{n}')$ will be identified as partitions in $\Pi(\mathsf{n})$. Given $\pi \in \Pi(\mathsf{n})$ and $\sigma' \in \Pi(\mathsf{n}')$, then by taking the union of all the blocks of both π and σ' (a union of disjoint subsets of two disjoint sets) one obtains a partition in $\Pi(\mathsf{n} : \mathsf{n}')$ which, being obtained as a set union, will be denoted by $\pi \cup \sigma'$. Even if π and σ' are chosen to be non-crossing it is not true in general that their union $\pi \cup \sigma'$ is non-crossing on $[n : n']$ (immediate counterexamples can be found).

Given $\pi \in \mathsf{NC}^{\mathsf{A}}(\mathsf{n})$, the *Kreweras complement* of π is defined to be the largest non-crossing partition of the set $[n']$ for which $\pi \cup \sigma$ is non-crossing on the set $[n : n']$. The Kreweras complement of the partition π is denoted $K(\pi)$.

• By considering the "left-hand" version of the set $[n : n']$, i.e. by considering the set

$$[n' : n] \stackrel{\text{def}}{=} \{1' \leqslant 1 \leqslant 2' \leqslant 2 \leqslant \cdots \leqslant n' \leqslant n\},$$

and by repeating the definition given for K above, one obtains a "left-hand" version of the Kreweras complement, denoted K'. The left and right Kreweras complements are related by the formula

$$K' \circ K = id, \quad \text{where} \quad id \quad \text{denotes the identity function on} \quad \mathsf{NC}^{\mathsf{A}}(\mathsf{n})\,.$$

• The map

$$K : \mathsf{NC}^{\mathsf{A}}(\mathsf{n}) \longrightarrow \mathsf{NC}^{\mathsf{A}}(\mathsf{n}) \ : \pi \longmapsto K(\pi)$$

is a lattice anti-isomorphism, i.e.

$$\pi \leqslant \sigma \Longleftrightarrow K(\pi) \geqslant K(\sigma). \tag{2.19}$$

• Another useful fact regarding the Kreweras complement is that

$$\#(\pi) + \#(K(\pi)) = n + 1, \quad \forall \pi \in \mathsf{NC}^{\mathsf{A}}(\mathsf{n}) \ . \tag{2.20}$$

2.2.7 The Möbius Function of $\mathsf{NC}^{\mathsf{A}}(\mathsf{n})$. If we denote by $\hat{0}, \hat{1}$ the minimal and, respectively, maximal element of $\mathsf{NC}^{\mathsf{A}}(\mathsf{n})$ and if we put $M_n^A \stackrel{\text{def}}{=} \mu_{\mathsf{NC}^{\mathsf{A}}(\mathsf{n})}(\hat{0}, \hat{1})$ then

$$M_n^A = (-1)^n \frac{1}{n}\binom{2n-2}{n-1}, \quad n \geqslant 1 \qquad \text{(see [Kre72])}. \tag{2.21}$$

The numbers M_n^A determine completely the Möbius functions for the lattices $\mathsf{NC}^{\mathsf{A}}(\mathsf{n})$, due to a phenomenon of "canonical factorization of intervals". Indeed, it turns out that for every $n \geqslant 1$ and for every $\pi \leqslant \rho$ in $\mathsf{NC}^{\mathsf{A}}(\mathsf{n})$ one has a canonical factorization of the subposet $[\pi, \rho] \subseteq \mathsf{NC}^{\mathsf{A}}(\mathsf{n})$ as

$$[\pi, \rho] \simeq \left(\mathsf{NC}^{\mathsf{A}}(1)\right)^{p_1} \times \left(\mathsf{NC}^{\mathsf{A}}(2)\right)^{p_2} \times \cdots \left(\mathsf{NC}^{\mathsf{A}}(\mathsf{n})\right)^{p_n}, \tag{2.22}$$

for some integers $p_1, p_2, \ldots, p_n \geqslant 0$ (see e.g. Lecture 10 of [NS06]). Equation (2.22) immediately implies, via the multiplicativity property of the Möbius functions, that we have

$$\mu_{\mathsf{NC}^{\mathsf{A}}(\mathsf{n})}(\pi, \rho) = \left(M_1^A\right)^{p_1}\left(M_1^A\right)^{p_1}\left(M_2^A\right)^{p_2}\cdots\left(M_n^A\right)^{p_n}. \tag{2.23}$$

2.3 Non-Crossing Partitions of Type B

In this section we fix the finite linearly ordered ground set

$$[\pm n] \stackrel{\text{def}}{=} \{1 \leqslant 2 \leqslant \cdots \leqslant n \leqslant -1 \leqslant -2 \leqslant \cdots \leqslant -n\},$$

which, under the obvious map, is isomorphic to

$$[2n] = \{1 \leqslant 2 \leqslant \cdots \leqslant n \leqslant n+1 \leqslant \cdots \leqslant 2n\}.$$

$\Pi(2\mathsf{n})$ and $\Pi(\pm\mathsf{n})$, and respectively, $\mathsf{NC}^{\mathsf{A}}(2\mathsf{n})$ and $\mathsf{NC}^{\mathsf{A}}(\pm\ \mathsf{n})$ are thus isomorphic as posets. The set $[\pm n]$ comes with its natural "inversion map" $x \longmapsto -x$. The image of a subset $B \subseteq [\pm n]$ under the inversion map is denoted $-B$.

2.3.1 Definitions, Notations and Remarks. ***i.*** A partition $\Pi(\pm \mathsf{n}) \ni \pi$ is called invariant under the inversion map if

$$B \quad \text{is a block of} \quad \pi \Longleftrightarrow -B \quad \text{is a block of} \quad \pi, \tag{2.24}$$

and the notation for the set of all invariant partitions is

$$\Pi(\pm \mathsf{n}) \supsetneq \Pi^{\mathsf{B}}(\mathsf{n}) \stackrel{\text{def}}{=} \{ \text{all invariant partitions of} \quad [\pm n] \ \}.$$

Given $\Pi^{\mathsf{B}}(\mathsf{n}) \ni \pi$, one distinguishes two different kinds of blocks of π: those which are, and those which are not inversion invariant.

- An invariant block $B = -B$ of a partition is called *zero-block*.
- By (2.24) it is clear that the nonzero-blocks must come in pairs.
- An invariant partition $\pi \in \Pi^{\mathsf{B}}(\mathsf{n})$ will be thus listed as

$$\pi = \{ \underbrace{B_1, B_2, \ldots, B_k, -B_1, -B_2, \ldots, -B_k,}_{\text{nonzero-blocks}} \underbrace{Z_1, Z_2, \ldots, Z_p}_{\text{zero-blocks}} \}. \tag{2.25}$$

- Given $\pi \in \mathsf{NC}^{\mathsf{B}}(\mathsf{n})$, the number of pairs of non-zero blocks of π will be denoted by $\#_{\mathsf{paired}}(\pi)$.

ii. A partition which is *invariant* and *non-crossing* is called *type B* non-crossing partition. The set of all type B non-crossing partitions is

$$\boxed{\mathsf{NC}^{\mathsf{B}}(\mathsf{n}) \stackrel{\text{def}}{=} \mathsf{NC}^{\mathsf{A}}(\pm\, \mathsf{n}) \cap \Pi^{\mathsf{B}}(\mathsf{n})}. \tag{2.26}$$

- Because of the non-crossing condition which appears in the definition of $\mathsf{NC}^{\mathsf{B}}(\mathsf{n})$ it is immediate that a partition $\pi \in \mathsf{NC}^{\mathsf{B}}(\mathsf{n})$ can have *at most one zero-block.*

- It is also immediate from the definition that $\mathsf{NC}^{\mathsf{B}}(\mathsf{n})$ is a sublattice of $\mathsf{NC}^{\mathsf{A}}(\pm\, \mathsf{n})$. $\mathsf{NC}^{\mathsf{B}}(\mathsf{n})$ has the same minimal and maximal elements $0_{\pm n}$ and $1_{\pm n}$ respectively, and the meet operation is the same as in $\mathsf{NC}^{\mathsf{A}}(\pm\, \mathsf{n})$, given by non-empty block intersection, as in (2.14).

- It is easily seen that $\mathsf{NC}^{\mathsf{B}}(\mathsf{n})$ is invariant for both right and left Kreweras complementation maps $K, K' : \mathsf{NC}^{\mathsf{A}}(\pm\, \mathsf{n}) \longrightarrow \mathsf{NC}^{\mathsf{A}}(\pm\, \mathsf{n})$,i.e.

$$K(\mathsf{NC}^{\mathsf{B}}(\mathsf{n})\ ) \subseteq \mathsf{NC}^{\mathsf{B}}(\mathsf{n}) \quad \& \quad K'(\mathsf{NC}^{\mathsf{B}}(\mathsf{n})\ ) \subseteq \mathsf{NC}^{\mathsf{B}}(\mathsf{n})\ , \tag{2.27}$$

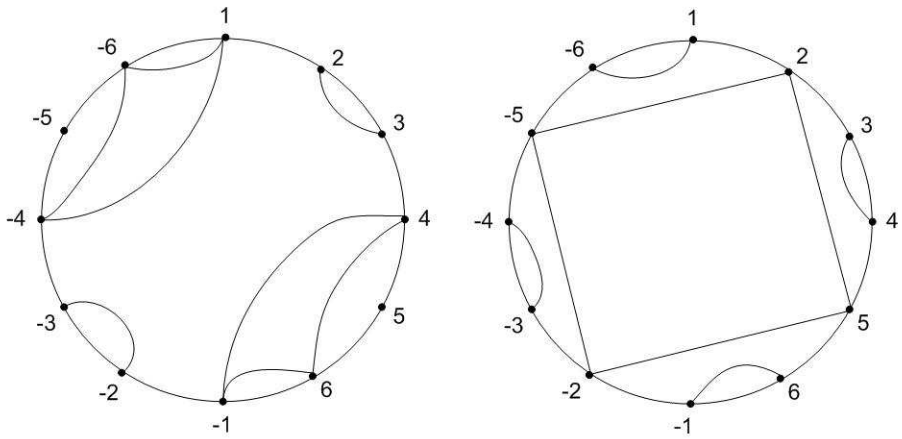


Figure 2.2: Two type B non-crossing partitions: without zero-block $\{\{1,-4,-6\},\{-1,4,6\},\{2,3\},\{-2,-3\},\{5\},\{-5\}\}$, and with zero-block $\{\{1,-6\},\{-1,6\},\{3,4\},\{-3,-4\},\{2,5,-2,-5\}\}$

therefore the notations $K(\pi), K'(\pi)$ are unambiguous regardless of whether the partition π is viewed in $\mathsf{NC}^{\mathsf{B}}(\mathsf{n})$ or in $\mathsf{NC}^{\mathsf{A}}(\pm\,\mathsf{n})$.

- Observe that formula (2.20) gives that

$$\#(\pi) + \#(K(\pi)) = 2n + 1, \quad \text{for any} \quad \pi \in \Pi(\pm\mathsf{n}), \tag{2.28}$$

therefore, since $2n+1$ is an odd number and since both π and $K(\pi)$ have at most one zero-block (and an even number (possibly 0) of nonzero-blocks), it must be that *exactly* one of π and $K(\pi)$ has a zero-block.

The convention that we use throughout this thesis is to draw partitions of a finite linearly ordered set X inside a circle: put the elements of X on the circle, going clockwise in their increasing order, at equal distance from each other and connect them in the disc according to the block structure of the partition. The very special block-structure for the type B non-crossing partitions will reflect very clearly in their pictures. Some of them will have a zero-block which contains the center of the circle in its interior, and all the other blocks will be symmetrically arranged, as they come in pairs. The partitions without zero-block will just have pairs of non-crossing blocks inside the disc. Figure 2.2 illustrates how the diagrams of such partitions may look.

2.3.2 Absolute Value in $\mathsf{NC}^{\mathsf{B}}(\mathsf{n})$. The relationship between $\mathsf{NC}^{\mathsf{A}}(\mathsf{n})$ and $\mathsf{NC}^{\mathsf{B}}(\mathsf{n})$ is best shown by using the idea of "absolute value" of a type B partition. The absolute value map is defined as follows.

$$\text{Abs} : [\pm n] \longrightarrow [n] : \pm i \longmapsto i, \quad \text{for every} \quad 1 \leqslant i \leqslant n,$$

and it is then extended to $\Pi(\pm\mathsf{n})$

$$\text{Abs} : \Pi(\pm\mathsf{n}) \longrightarrow \Pi(\mathsf{n}) : \pi \longmapsto \text{Abs}(\pi) \stackrel{\text{def}}{=} \big\{ \text{Abs}(B) \mid B \quad \text{block of} \quad \pi \big\}.$$

The following results about the absolute value map appear in [BGN03].

2.3.3 Lemma. $\pi \in \mathit{NC}^{B}(n) \Longrightarrow \mathit{Abs}(\pi) \in \mathit{NC}^{A}(n)$ *.*

2.3.4 Lemma. $\pi \in \mathit{NC}^{B}(n) \Longrightarrow \mathit{Abs}(K(\pi)) = K(\mathit{Abs}(\pi))$*, i.e. the following diagram is commutative.*

$$\begin{tikzcd} \mathit{NC}^{B}(n) \arrow[r, "Abs"] \arrow[d, "K"'] & \mathit{NC}^{A}(n) \arrow[d, "K"] \\ \mathit{NC}^{B}(n) \arrow[r, "Abs"] & \mathit{NC}^{A}(n) \end{tikzcd}$$

2.3.5 Proposition. $\mathit{NC}^{B}(n) \ni \pi \stackrel{Abs}{\longmapsto} \mathit{Abs}(\pi) \in \mathit{NC}^{A}(n)$ *is an $(n+1)$–to–1 map.*

Given $p \in \mathsf{NC}^{\mathsf{A}}(\mathsf{n})$, the set $\text{Abs}^{-1}(p)$ is given a detailed description. There are $(n+1)$ partitions from $\mathsf{NC}^{\mathsf{B}}(\mathsf{n})$ mapped by "Abs" to p.

The number of non-crossing partitions of type B (as counted in [Rei97]) is obtained as a consequence of the above Proposition 2.3.5.

$$|\ \mathsf{NC}^{\mathsf{A}}(\mathsf{n})\ | = \frac{1}{n+1}\binom{2n}{n} \quad \Longrightarrow \quad |\ \mathsf{NC}^{\mathsf{B}}(\mathsf{n})\ | = \binom{2n}{n}. \tag{2.29}$$

The Narayana numbers for $\mathsf{NC}^{\mathsf{B}}(\mathsf{n})$ are also counted in [Rei97]. They are given by

$$card\{\pi \in \mathsf{NC}^{\mathsf{B}}(\mathsf{n}) \mid rank(\pi) = k\} = \binom{n}{k}^2. \tag{2.30}$$

2.3.6 The Möbius Function of $\mathsf{NC}^{\mathsf{B}}(\mathsf{n})$. Similarly to the situation from the lattice $\mathsf{NC}^{\mathsf{A}}(\mathsf{n})$, the minimal and maximal elements of $\mathsf{NC}^{\mathsf{B}}(\mathsf{n})$ (which in fact are the same as in $\mathsf{NC}^{\mathsf{A}}(\mathsf{n})$) are denoted by $\hat{0}$ and, respectively, $\hat{1}$. Further, if we let $M_n^{B} \stackrel{\text{def}}{=} \mu_{\mathsf{NC}^{\mathsf{B}}(\mathsf{n})}(\hat{0}, \hat{1})$, then

$$M_n^{B} = (-1)^n \binom{2n-1}{n}, \quad n \geqslant 1 \qquad \text{(see [Rei97])}. \tag{2.31}$$

The numbers from M_n^B determine completely the Möbius functions for the lattices $\mathsf{NC}^{\mathsf{B}}(\mathsf{n})$. The "canonical factorization of intervals" also holds in the case of $\mathsf{NC}^{\mathsf{B}}(\mathsf{n})$. Indeed, it turns out that for every $n \geqslant 1$ and for every $\pi \leqslant \rho$ in $\mathsf{NC}^{\mathsf{B}}(\mathsf{n})$ one has a canonical factorization of the subposet $[\pi, \rho] \subseteq \mathsf{NC}^{\mathsf{B}}(\mathsf{n})$ as

$$[\pi, \rho] \simeq \prod_{i=1}^{n} \left(\mathsf{NC}^{\mathsf{A}}(\mathsf{i})\right)^{p_i} \times \prod_{j=1}^{n} \left(\mathsf{NC}^{\mathsf{B}}(\mathsf{j})\right)^{q_j}, \tag{2.32}$$

with $p_1, \ldots, p_n, q_1, \ldots, q_n \geqslant 0$. The factorization (2.32) gives, of course, a corresponding formula for $\mu_{\mathsf{NC}^{\mathsf{B}}(\mathsf{n})}(\pi, \rho)$, namely

$$\mu_{\mathsf{NC}^{\mathsf{B}}(\mathsf{n})}(\pi, \rho) = \prod_{i=1}^{n} \left(M_i^A \right)^{p_i} \cdot \prod_{j=1}^{n} \left(M_j^B \right)^{q_j}. \tag{2.33}$$

Chapter 3

Non-Crossing Permutations

3.1 Marked Groups

Let us consider a multiplicative group G with a finite set of generators T. Assume also that T does not contain the unit e of G.

3.1.1 Definition. The pair (G, T) is called a *marked group* if the set T is invariant under taking the inverse and if the conjugate of any of the generators is also a generator, i.e. if the following two conditions are met.

MG1. $x \in T \Longrightarrow x^{-1} \in T$.

MG2. $x \in T,\, c \in G \Longrightarrow c^{-1}xc \in T$.

3.1.2 Definition and Remark. Let (G, T) be a marked group and $e \neq a \in G$. The *length* of the element a is defined to be the smallest positive integer n such that there exists a factorization $a = x_1 \cdots x_n$, $x_i \in T$, $1 \leqslant i \leqslant n$ and is denoted by $\mid a \mid$, or by $\ell_G(a)$. By convention the unit e has length 0 (and it is the only element with this property). The only elements of length 1 are the generators.

If the generators of the group are thought of as "letters", then arbitrary products of them would produce "words". With this analogy in mind, a product of generators $x_1 \cdots x_n = a$ can be thought of as a "reduced word", whenever $\mid a \mid = n$. The product $x_1 \cdots x_n$ would then be *a shortest factorization* of a as product of generators.

3.1.3 Properties of the Length Function. The length function $|\cdot|: G \longrightarrow \mathbb{N}$ has the following properties.

L1. $a, b \in G \Longrightarrow | ab | \leqslant | a | + | b |$ (the "triangle inequality" of the length function).

L2. $a \in G \Longrightarrow | a^{-1} | = | a |$.

L3. $a, c \in G \Longrightarrow | c^{-1}ac | = | a |$.

Proof. Suppose that a, b are reduced words of lengths n, m, respectively.

$$\begin{cases} a &= x_1 \cdots x_n, \quad | a | = n, \quad x_i \in T, \quad 1 \leqslant i \leqslant n \\ b &= z_1 \cdots z_m, \quad | b | = m, \quad z_i \in T, \quad 1 \leqslant i \leqslant m. \end{cases}$$

L1. ab must be $x_1 \cdots x_n \cdot z_1 \cdots z_m$ hence its length must be at most $m + n$.

L2. follows immediately from *MG1*.

L3. Follows immediately from *MG2*.

■

3.1.4 Distance on a marked group. The length function induces a *distance* function denoted d, defined by

$$\mathsf{d} : G \times G \longrightarrow \mathbb{N} : (a, b) \longmapsto \mathsf{d}(a, b) = | a^{-1}b | \, . \tag{3.1}$$

From now on, the distance between two elements a, b in a marked group will be, as needed, any element in the set

$$\{ \, | a^{-1}b |, \quad | b^{-1}a |, \quad | ba^{-1} |, \quad | ab^{-1} | \, \} = \{ \, \mathsf{d}(a, b) \, \}.$$

Also, d is symmetric, satisfies the triangle inequality (hence it is indeed a distance on G). Moreover d is invariant under left and right translation by elements of G . Thus for any $a, b, c, f \in G$, d satisfies the following.

D1. $\mathsf{d}(a, a) = 0 \quad \& \quad \mathsf{d}(a, b) = \mathsf{d}(b, a)$ since $| a^{-1}b | = | b^{-1}a |$. [by *L2.*]

D2. $\mathsf{d}(a, b) \leqslant \mathsf{d}(a, c) + \mathsf{d}(c, b)$ since $| a^{-1}b | \leqslant | a^{-1}c | + | c^{-1}b |$. [by *L1.*]

D3. $\mathsf{d}(caf, cbf) = \mid f^{-1}a^{-1}c^{-1}cbf \mid = \mid f^{-1}a^{-1}bf \mid = \mid a^{-1}b \mid = \mathsf{d}(a,b)$. [by *L3.*]

D4. $\mathsf{d}(a,b) = \mathsf{d}(a^{-1}, b^{-1})$. [by *D3.* with $c = b^{-1}$ & $f = a^{-1}$, and then apply *D2.*]

3.1.5 Partial order on a marked group. The distance function on a marked group induces an order relation "$\leqslant$". Let a, b be elements of the marked group G with unit e. If $a, b \in G$ then

$$a \leqslant b \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad \mathsf{d}(e,a) + \mathsf{d}(a,b) = \mathsf{d}(e,b). \tag{3.2}$$

Taking into account (3.1) the partial order can be described by

$$\boxed{a \leqslant b \Longleftrightarrow \mid a \mid + \mid a^{-1}b \mid = \mid b \mid}. \tag{3.3}$$

3.1.6 Remark. Any subset H of a marked group (G, T) becomes thus a poset w.r.t. the inherited partial order. Of course, it may happen that a certain subset H is a subgroup which may be generated by a different set of generators T' which still satisfy conditions ***MG1, MG2*** and hence make (H, T') into a marked group too, having a (possibly) different length function, distance and hence partial order. In order to avoid any possible confusion, at certain occasions the partial order on (G, T) might appear as "$\leqslant_G$" rather then just "$\leqslant$".

3.1.7 Lemma. *"$\leqslant$" is a partial order on (G, T).*

Proof. The proof is immediate. E.g., let us check the transitivity:

$$a \leqslant b \quad \& \quad b \leqslant c \Longrightarrow a \leqslant c.$$

We need to prove that $\mid a \mid + \mid a^{-1}c \mid = \mid c \mid$.

$\boxed{\leqslant}$ $\mid c \mid \leqslant \mid a \mid + \mid a^{-1}c \mid$. [by the triangle inequality ***L1*** applied to $\mid a \mid$ & $\mid a^{-1}c \mid$]

$\boxed{\geqslant}$

$$\begin{aligned} \mid a^{-1}c \mid &\leqslant \mid a^{-1}b \mid + \mid b^{-1}c \mid && \text{[by the triangle inequality for } a^{-1}b \;\; \& \;\; b^{-1}c\text{]} \\ \Longleftrightarrow \mid a \mid + \mid a^{-1}c \mid &\leqslant \mid a \mid + \mid a^{-1}b \mid + \mid b^{-1}c \mid && \text{[by adding } \mid a \mid \text{ on both sides]} \\ &= \mid b \mid + \mid b^{-1}c \mid && \text{[since } a \leqslant b\text{]} \\ &= \mid c \mid . && \text{[since } b \leqslant c\text{]} \end{aligned}$$

■

Continuing for a moment with the "generators" as "letters" analogy, the order relation can thus be formulated as follows: $a \leqslant b$ if and only if there exists a shortest factorization of a as a product of generators which is a "prefix" of some shortest factorization of b.

3.1.8 Remark. While the minimal element in any marked group is always unique, it is possible for some marked group $(G, \leqslant)$ to have more than one maximal element.

3.2 Intervals in Marked Groups.

3.2.1 Remark. ***i.*** Let a, b be elements of the marked group G. Then

$$a \quad \text{covers} \quad b \Longleftrightarrow \begin{cases} a \leqslant b \quad \& \quad a^{-1}b \quad \text{is a generator,} \\ \text{and} \quad \mid a \mid +1 = \mid b \mid . \end{cases}$$

We note the following immediate properties.

$$\begin{cases} a \leqslant b & \Longrightarrow \quad \mid a \mid \leqslant \mid b \mid \\ a \leqslant b & \Longleftrightarrow \quad a^{-1} \leqslant b^{-1} \\ b \quad \text{covers} \quad a & \Longleftrightarrow \quad b^{-1} \quad \text{covers} \quad a^{-1}. \end{cases}$$

ii. It is easily seen that any interval in a marked group is a ranked poset. If x is any element of the marked group G then the rank of the interval $[e, x]$ will be simply called the rank of the element x. A quick argument shows that in fact the rank of x is in fact equal to the length of x, i.e.

$$rank(x) = \ell_G(x). \tag{3.4}$$

It is advantageous to reduce the study of arbitrary intervals $[a, b]$ to the particular intervals having the identity e as minimal element, i.e. intervals of the form $[e, b]$.

3.2.2 Definition. For a, b elements in a marked group G the element $a^{-1}b$ will be denoted by $\mathsf{C}_b(a)$.

3.2.3 Proposition. *Let (G, T) be a marked group and $b \in G$. Then*

i. $a \leqslant b \Longleftrightarrow \mathsf{C}_b(a) \leqslant b$.

ii. $[e,b] \ni a \longmapsto \mathsf{C}_b(a) \in [e,b]$ *is well-defined (by i.), decreasing (ii.1.) and bijective (ii.2.).*

Proof. *i.* A direct computation shows that

$$\begin{aligned} a \leqslant b &\Longleftrightarrow \mid a \mid + \mid a^{-1}b \mid = \mid b \mid \\ &\Longleftrightarrow \mid b^{-1}ab \mid + \mid a^{-1}b \mid = \mid b \mid && \text{[by } L3\text{]} \\ &\Longleftrightarrow \mid (a^{-1}b)^{-1}b \mid + \mid a^{-1}b \mid = \mid b \mid \\ &\Longleftrightarrow a^{-1}b = \mathsf{C}_b(a) \leqslant b. \end{aligned}$$

ii.1. We must prove that $[x \leqslant y \Longrightarrow \mathsf{C}_b(x) \geqslant \mathsf{C}_b(y)]$. We know that $x \leqslant y \leqslant b$, so

$$\begin{cases} \mid b \mid - \mid y \mid = \mid y^{-1}b \mid \\ \mid y \mid - \mid x \mid = \mid x^{-1}y \mid \\ \mid b \mid - \mid x \mid = \mid x^{-1}b \mid . \end{cases}$$

$$\begin{aligned} \text{Then} \qquad & \mid b \mid - \mid x \mid = \mid (\mid b \mid - \mid y \mid) + (\mid y \mid - \mid x \mid) \\ &\Longleftrightarrow \mid x^{-1}b \mid = \mid y^{-1}b \mid + \mid x^{-1}y \mid \\ &\Longleftrightarrow \mid x^{-1}b \mid = \mid y^{-1}b \mid + \mid yx^{-1} \mid && \text{[by } L2\text{]} \\ &\Longleftrightarrow \mid x^{-1}b \mid = \mid y^{-1}b \mid + \mid b^{-1}yx^{-1}b \mid && \text{[by } L3\text{]} \\ &\Longleftrightarrow \mid x^{-1}b \mid = \mid y^{-1}b \mid + \mid (y^{-1}b)^{-1}(x^{-1}b) \mid \\ &\Longleftrightarrow \mathsf{C}_b(x) \geqslant \mathsf{C}_b(y). \end{aligned}$$

ii.2. That the above mapping is one-to-one is obvious: $[x^{-1}b = y^{-1}b \Longrightarrow x = y]$. Now for the surjectivity part let $y \in [e,b]$. Since $\mathsf{C}_b(by^{-1}) = y$ it suffices to show that $by^{-1} \in [e,b]$. Indeed

$$\begin{aligned} \mid by^{-1} \mid + \mid (by^{-1})^{-1}b \mid &= \mid y^{-1}b \mid + \mid y \mid && \text{[by property } L2\text{]} \\ &= \mid b \mid . && \text{[since } y \leqslant b\text{]} \end{aligned}$$

■

3.2.4 Proposition. *Let (G,T) be a marked group. Then*

i. $a \leqslant b \leqslant c \Longrightarrow \mathsf{C}_b(a) \leqslant \mathsf{C}_c(a)$.

ii. *Suppose that $a \leqslant b$ in G. Then $[a,b] \ni t \longmapsto \Phi(t) = \mathcal{C}_t(a) \in [e, \mathcal{C}_b(a)]$ is a poset isomorphism.*

Proof. ***i.*** A direct computation gives that

$$\begin{aligned} a^{-1}b \leqslant a^{-1}c &\Longleftrightarrow \mid a^{-1}b \mid + \mid (a^{-1}b)^{-1}a^{-1}c \mid = \mid a^{-1}c \mid && \text{[by (3.3)]} \\ &\Longleftrightarrow \mid a^{-1}b \mid + \mid b^{-1}c \mid = \mid a^{-1}c \mid \\ &\Longleftrightarrow (\mid b \mid - \mid a \mid) + (\mid c \mid - \mid b \mid) = \mid c \mid - \mid a \mid . && \text{[because } a \leqslant b \leqslant c] \end{aligned}$$

ii. Φ is well-defined and increasing by ***i.*** and also one-to-one, since

$$a^{-1}b = a^{-1}c \Longrightarrow b = c.$$

To prove that it is also surjective let $y \in [e, a^{-1}b]$. It suffices to prove that $ay \in [a,b]$ since $\Phi(ay) = a^{-1}(ay) = y$, i.e. to prove that $[y \leqslant a^{-1}b \Longrightarrow a \leqslant ay \leqslant b]$. Observe that

$$\begin{cases} a \ \leqslant ay \\ ay \leqslant b \end{cases} \Longleftrightarrow \begin{cases} A =: \mid a \mid + \mid a^{-1}(ay) \mid = \mid ay \mid =: B \\ C =: \mid ay \mid + \mid y^{-1}a^{-1}b \mid = \mid b \mid =: D, \end{cases}$$

so the goal is to prove that $A = B \quad \& \quad C = D$. Note first that $A + C = B + D$. Indeed,

$$\begin{aligned} A + C = B + D &\Longleftrightarrow \mid a \mid + \mid y \mid + \mid y^{-1}a^{-1}b \mid = \mid b \mid \\ &\Longleftrightarrow \mid y \mid + \mid y^{-1}a^{-1}b \mid = \mid b \mid - \mid a \mid \\ &\Longleftrightarrow \mid a^{-1}b \mid = \mid a^{-1}b \mid . && \text{[since } y \leqslant a^{-1}b \text{ and } a \leqslant b]. \end{aligned}$$

It is also true that

$$\begin{cases} A \geqslant B, & \text{by the triangle inequality for} \quad a \quad \& \quad y \\ C \geqslant D, & \text{by the triangle inequality for} \quad ay \quad \& \quad (ay)^{-1}b. \end{cases}$$

Now it becomes clear that $A = B \quad \& \quad C = D$ must be the case.

Hence Φ is an increasing bijection from from $[a,b]$ to $[e, a^{-1}b]$, but this is not enough yet to ensure that Φ is indeed a poset isomorphism. Φ^{-1} maps $[e, a^{-1}b]$ back to $[a,b]$, is of course bijective too, is defined by the formula $\Phi^{-1}(z) = az$ for every $z \in [e, a^{-1}b]$, but we still need to prove that Φ^{-1} is also increasing, i.e. that $[x \leqslant y \leqslant a^{-1}b \Longrightarrow ax \leqslant ay]$.

Now, $ax \leqslant ay \Longleftrightarrow \mid ax \mid + \mid (ax)^{-1}ay \mid = \mid ay \mid \Longleftrightarrow \mid ax \mid + \mid x^{-1}y \mid = \mid ay \mid$.
To prove the last equality write down individually each of the terms involved:

$$\mid ax \mid = \mid b \mid - \mid (ax)^{-1}b \mid \qquad \text{[since } \Phi^{-1}(x) = ax \leqslant b]$$

$$\begin{aligned} &= \mid b \mid - \mid a^{-1}b \mid - \mid x \mid && [\text{since } x \leqslant a^{-1}b] \\ \mid ay \mid &= \mid b \mid - \mid (ay)^{-1}b \mid && [\text{since } \Phi^{-1}(y) = ay \leqslant b] \\ &= \mid b \mid - \mid a^{-1}b \mid - \mid y \mid . && [\text{since } y \leqslant a^{-1}b] \end{aligned}$$

Also $\mid x^{-1}y \mid = \mid y \mid - \mid x \mid$, so everything adds up as it should and the proof is complete.

■

3.2.5 Remark. Let G be a marked group and x be any element in G. If y is any element in the conjugacy class of x then the interval $[e, x]$ is isomorphic to the interval $[e, y]$.

Proof. Let $y = c^{-1}xc$ for some $c \in G$ and define

$$\begin{aligned} &\Phi : [e, x] \longrightarrow [e, y], \Phi(a) = c^{-1}ac, \qquad \text{and} \\ &\Psi : [e, y] \longrightarrow [e, x], \Phi(b) = cbc^{-1}. \end{aligned}$$

It is immediately verified that Φ and Ψ are poset isomorphisms inverse to each other and thus $[e, x] \simeq [e, y]$. ■

3.3 Examples of Marked Groups

The symmetric group $\mathfrak{S}_n$, the hyperoctahedral group B_n and the group D_n are most relevant to this thesis. We will apply the general notions of length, induced partial order, intervals and covers to these examples. The first example, $(\mathbb{Z}_2^n, \cdot)$, is included to illustrate the marked group framework in a different context than that of permutations.

3.3.1 Example. The Group $(\mathbb{Z}_\mathbf{2}^\mathbf{n}, \cdot)$.

$\mathbb{Z}_\mathbf{2}^\mathbf{n}$ is the group of cardinality 2^n whose elements are n-tuples with entries either 1 or -1. Two such n-tuples are multiplied coordinatewise :

$$\mathbb{Z}_2^n = \{(\varepsilon_1, \cdots, \varepsilon_n) \mid \varepsilon_i \in \{-1, 1\}, 1 \leqslant i \leqslant n\}, \quad (\varepsilon_1, \cdots, \varepsilon_n)(\sigma_1, \cdots, \sigma_n) = (\varepsilon_1\sigma_1, \cdots, \varepsilon_n\sigma_n).$$

Of course, this is a commutative group and the unit is the n-tuple $\varepsilon = (1, \cdots, 1)$. We let our set of generators be

$$T = \{t_i = (\varepsilon_1, \cdots, \varepsilon_i, \cdots, \varepsilon_n) \mid \varepsilon_i = -1 \quad \text{and} \quad \varepsilon_j = 1 \quad \text{for} \quad i \neq j\}.$$

A generator is thus an n-tuple with exactly one "-1" entry. The position at which the "-1" is to be found in the n-tuple gives the name of the generator.

The axioms of a marked group are trivially satisfied.

MG1: it is clear that $t_i = t_i^{-1}, 1 \leqslant i \leqslant n$ (and in fact that every element in this group equals its own inverse).

MG2: T is also invariant under conjugation since the conjugacy classes are singletons:

$$g \in \mathbb{Z}_2^n, t_i \in T \Longrightarrow g^{-1}t_ig = g^{-1}gt_i = t_i.$$

Length formula on $(\mathbb{Z}_2^n, T)$. The length of an element equals the number of its "-1" entries. Indeed, the length of e is 0 and also the length of any generator is 1. Suppose now that g is an n-tuple with $k \geqslant 2$ "-1" entries, say

$$g = (\varepsilon_1, \cdots, \varepsilon_n), \quad \varepsilon_{i_1} = \varepsilon_{i_2} = \cdots = \varepsilon_{i_k} = -1 \quad \& \quad \varepsilon_j = 1 \quad \text{for} \quad j \notin \{i_1, \cdots, i_k\}.$$

It it clear that g can be written as $g = t_{i_1}t_{i_2}\cdots t_{i_k}$ and consequently that $\mid g \mid= k$, since at least k generators are necessary to produce k occurrences of"-1" among the entries of g. This is in fact just another way of saying that (in the particular case of $(\mathbb{Z}_\mathbf{2}^\mathbf{n}, T)$, but not in general) the length of the product of k generators is k precisely when they are *pairwise distinct*. In this group a generator can appear in a shortest factorization at most once and in fact shortest factorizations are unique, up to order: if g has "-1" in the positions $i_1, \ldots, i_k$ then $g = t_{i_1}\cdots t_{i_k}$.

Partial order on $(\mathbb{Z}_2^n, T)$. The partial order on a marked group (G, T) was defined according to formula (3.3), which read

$$a \leqslant b \Longleftrightarrow \mid a \mid + \mid a^{-1}b \mid = \mid b \mid .$$

Now pick a, b reduced words in $\mathbb{Z}_2^n$, say $a = t_{i_1}\cdots t_{i_k}$ and $b = t_{j_1}\cdots t_{j_p}$. Then

$$a \leqslant b \Longleftrightarrow k + \mid a^{-1}b \mid = p \Longleftrightarrow \mid a^{-1}b \mid = \mid t_{i_k}^{-1}\cdots t_{i_1}^{-1}t_{j_1}\cdots t_{j_p} \mid = p - k.$$

Obviously this cannot happen unless $k \leqslant p \quad \& \quad (i_1, \ldots, i_k) = (j_1, \ldots, j_k)$, i.e. unless a shortest factorization of a is a prefix of a shortest factorization of b. It is now easy to see that a is less than b if and only if, whenever an entry of a is "-1" then the corresponding entry of b must be "-1" too (but if "1" is the entry of a in a certain position then b is allowed to have a "-1" in that particular position; in fact the more often this occurs the "further away" b is from a). Note that $(-1,-1,\cdots,-1)$ is greater than any other element in the group.

Covering relations in $(\mathbb{Z}_2^n, T)$. Let $a = t_{i_1}\cdots t_{i_k}$ and $b = t_{j_1}\cdots t_{j_p}$. Then b covers a when $a \leqslant b \quad \& \quad p = k+1$, i.e. b is a cover of a if they are different in *exactly one* position: a position where a has a "1" and b has a "-1".

Intervals in $(\mathbb{Z}_2^n, T)$. Let $a = t_{i_1}\cdots t_{i_k}$ and $b = t_{j_1}\cdots t_{j_p}$, as before. The positions where a has "-1" are "locked in" throughout the interval (i.e. every element in this interval has "-1" in those positions where a does). What Proposition 3.2.4 says in this case is that $[e, a^{-1}b]$ (isomorphic to $[a,b]$) is the interval obtained by first writing down $[a,b]$ and then switching all the "-1" entries of a to "1" in every element of the interval.

In view of Proposition 3.2.4 it suffices to consider intervals of the form $[\varepsilon, c]$. $[\varepsilon, c]$, where $c = a^{-1}b, c = t_{j_{k+1}}\cdots t_{j_p}, m = p-k$. This interval has 2^m elements (by deleting the "1" entries of c it becomes isomorphic to $\mathbb{Z}_{\mathbf{2}}^{\mathbf{m}}$ and there are exactly $\binom{m}{r}$ elements of length r. For instance, if $\mathbb{Z}_2^4 \ni b = t_1t_3t_4 = (-1,1,-1,-1)$ then

$$[e,b] = \{e, t_1, t_3, t_4, t_1t_3, t_1t_4, t_3t_4, b\}.$$

Remark. It is quite easy to notice that in fact $(\mathbb{Z}_2^n, \leqslant)$ is a isomorphic to the lattice of all subsets of the set $\{1,2,\ldots,n\}$, ordered by inclusion. The identification is the following: write a subset (listed in increasing order) $\{i_1, i_2, \ldots, i_k\}$ of $\{1,2,\ldots,n\}$ as an n-tuple with "-1" in the positions $i_1, i_2, \ldots, i_k$ and "1" everywhere else.

Next in line are three more examples of marked groups, each one of them being a group of permutations: the symmetric group, the hyperoctahedral group B_n and the Weyl group D_n . The purpose of this section is to describe the length formulas and the covering relations for each of these groups.

3.3.2 Definitions, Notations and Remarks. The conventions and notations regarding partitions of a finite set used in Chapter 2 are being kept here. Some of them are recalled for convenience.

Let X be a finite set. The number of elements of X is denoted by $|X|$. The sets $\{1,2,\cdots,n\}$ and $\{1,2,\cdots,n\}\cup\{-1,-2,\cdots,-n\}$ are denoted by $[n]$ and $[\pm n]$, respectively.

i. • The set of all permutations of X is denoted by $\mathfrak{S}_X$. Cycle notation will be employed for permutations in $\mathfrak{S}_X$.

• A permutation with a single cycle is called a *long cycle.*

• Regardless of what the ground set X is, the identity permutation will be always denoted by ε .

• If the set X is linearly ordered then the *forward cycle* (that is, the unique permutation τ of X with the property that $x \leqslant \tau(x), \forall x \in X \setminus \{\max X\}$ and $\tau(\max X) = \min X$) will always be denoted by γ_o. The ground set will always be clear from the context in concrete situations.

• In the two particular cases when $X = [n]$ and $X = [\pm n]$, $\mathfrak{S}_X$ is denoted $\mathfrak{S}_n$ and $\mathfrak{S}_{\pm n}$, respectively, and, according to the previous convention, the special symbol γ_o denotes both the *forward cycles* $(1, 2, \cdots, n)$ and $(1, 2, \cdots, n, -1, -2, \cdots, -n)$ on $\mathfrak{S}_n$ and $\mathfrak{S}_{\pm n}$, respectively.

• The number of cycles of a permutation $\tau \in \mathfrak{S}_X$, is denoted by $\#(\tau)$. Note that fixed points are counted in $\#(\tau)$. However, when writing down permutations the fixed points are usually omitted (for instance "$\tau = (a_1, a_2, a_3)(b_1, b_2) \in \mathfrak{S}_X$" means that a_1, a_2, a_3, b_1, b_2 are five distinct elements of X, that we have $\tau(a_1) = a_2, \tau(a_2) = a_3, \tau(a_3) = a_1$ and $\tau(b_1) = b_2, \tau(b_2) = b_1$, and that $\tau(x) = x$ for every $x \in X \setminus \{a_1, a_2, a_3, b_1, b_2\}$). We will strive to avoid any ambiguities that might arise in this way.

ii. Let τ be a permutation of X. The action of τ on X splits X into a collection of subsets called *orbits* of τ (where $x, y \in X$ are in the same orbit of τ iff there exists $m \in \mathbb{Z}$ such that $\tau^m(x) = y$). The orbits of a given permutation τ form a partition of X. An *orbit-map* Ω is thus obtained from permutations to partitions

$$\mathfrak{S}_X \ni \tau \longmapsto \Omega(\tau) \in \Pi(X). \tag{3.5}$$

The number of blocks of $\Omega(\tau)$ is clearly equal to the number $\#(\tau)$ of cycles of τ.

3.3.3 Example. The Symmetric Group $\mathfrak{S}_n$

The set of generators T_n is taken to be the set of *all* transpositions $(i, j), i, j \in [n]$. The cardinality of T_n is $\frac{n(n-1)}{2}$. The axioms of a marked group are easily checked.

MG1: T_n is obviously closed under taking the inverse since every transposition is its own inverse.

`MG2`: T_n is also closed under conjugation: for every $\tau \in S_n$ and for every transposition $(i,j) \in T_n$ the permutation $\tau^{-1}(i,j)\tau$ is also a transposition, namely $(\tau^{-1}(i), \tau^{-1}(j))$, as a direct verification shows.

Therefore $(\mathfrak{S}_n, T_n)$ is a marked group. It should be mentioned here that the symmetric group can be generated by fewer elements, e.g. only by the *consecutive* transpositions but then condition `MG2` would not be fulfilled since one can easily find a permutation $\tau \in \mathfrak{S}_n$ such that $(\tau^{-1}(i), \tau^{-1}(j))$ is not a consecutive transposition even if (i,j) is.

Length formula on $\mathfrak{S}_n$. It is well known that the length formula in $\mathfrak{S}_n$ is given by

$$\ell_{\mathfrak{S}_n}(\tau) = |\tau| = n - \#(\tau). \tag{3.6}$$

Thus, the maximum length a permutation in $\mathfrak{S}_n$ can achieve is $n-1$, and it is attained by any long cycle.

• For example, if $\mathfrak{S}_5 \ni \tau = (1)(2,5,3)(4,6)$ then

$$\begin{cases} \Omega(\tau) = \{1\} \cup \{2,3,5\} \cup \{4,6\} \in \Pi(6) \\ |\,\tau\,| = 6 - 3 = 3. \end{cases} \tag{3.7}$$

In particular this implies that $(2,5)(3,5)(4,6)$ is a minimal factorization of τ.

• Any finite linearly ordered set X is poset isomorphic to $[n]$. For this reason the length formula in $\mathfrak{S}_X$ is the same as in $\mathfrak{S}_n$,

$$\ell_{\mathfrak{S}_X}(\tau) = |\,\tau\,| \quad = \quad |X| - \#(\tau). \tag{3.8}$$

The covering relations in $(\mathfrak{S}_n, T_n)$ are explained by the following observation: assume $\tau \in \mathfrak{S}_n$ has p orbits. When multiplied by the transposition $t = (i,j)$ the result $\tau \cdot t$ has $p+1$ or $p-1$ orbits, depending on whether the set $\{i,j\}$ is contained in the same orbit of τ or in two different orbits, respectively. In the first case the orbit which contains both i,j is split into two orbits whereas in the second case the orbit which contains i is joined to the one containing j.

This fact is recorded in following lemma.

Lemma (Covering Relations in $\mathfrak{S}_n$). *Let $\sigma, \tau \in \mathfrak{S}_n$, $|\,\tau\,| \notin \{0, n-1\}$. Then*

$$\sigma \quad \textit{covers} \quad \tau \quad \Longleftrightarrow \quad \exists\, t \text{ such that } \begin{cases} \sigma = \tau \cdot t \quad \textit{with} \quad t = (i,j) \\ \textit{and} \quad i,j \quad \textit{belong to different orbits of} \quad \tau \end{cases} \tag{3.9}$$

Proof. Assume that the unique decomposition of τ into disjoint cycles (counting fixed points) is $\tau = c_1 \cdots c_p$. The length of τ is thus $n - p$.

$\boxed{\Longrightarrow}$

We know that σ covers τ hence (from the description $\boldsymbol{i}$. in 3.2.1 of covers in a marked group) that means that we have that

$$\begin{cases} \mid \tau \mid & + \mid \tau^{-1}\sigma \mid = \mid \sigma \mid, \quad \text{and} \\ \mid \tau \mid & +1 = \mid \sigma \mid, \end{cases}$$

hence $\mid \tau^{-1}\sigma \mid = 1$ so $\tau^{-1}\sigma$ must be a generator, say $\tau^{-1}\sigma = (i, j) = t \in T_n$. It follows that $\sigma = \tau \cdot t$.

We must now prove that i, j cannot lie in the same orbit of τ.

Assume the contrary is true, i.e. that i, j belong to the same orbit, w.l.o.g. say c_1 (since the cycles are disjoint they commute) of $\tau = c_1 \cdots c_p$. To fix the notations assume also that

$$c_1 = (i_1, \cdots, \boxed{i_s = i}, i_{s+1}, \cdots, \boxed{i_l = j}, i_{l+1}, \cdots, i_m), \quad m \geqslant 2.$$

A direct computation shows that

$$c_1 t = c_1(i_s, i_l) = (i_1, \cdots, \boxed{i_s = i}, i_{l+1}, \cdots, i_m)(i_{s+1}, \cdots, \boxed{i_l = j}).$$

This means that $\mid \sigma \mid = n - p - 1 < n - p = \mid \tau \mid$, contrary to our assumption that $\mid \sigma \mid = \mid \tau \mid +1$.

$\boxed{\Longleftarrow}$

It is clear that $\mid \tau^{-1}\sigma \mid = |(i, j)| = 1$ so we only need to prove that $\mid \sigma \mid = \mid \tau \mid +1$ or equivalently that $\#(\tau) - \#(\sigma) = 1$, whenever $\sigma = \tau \cdot t, t = (i, j)$ with i, j coming from different orbits of τ.

W.l.o.g., as in proving the other direction, assume that i comes from c_1, j comes from c_2 and let

$$c_1 = \big(i_1, \cdots, \boxed{i_s = i}, i_{s+1}, \cdots, i_k\big), \quad c_2 = \big(j_1, \cdots, \boxed{j_r = j}, j_{r+1}, \cdots, j_m\big), \quad k, m \geqslant 1.$$

A direct computation shows that

$$c_1 c_2 t = c_1 c_2 (i, j) = \big(i_1, \cdots, \boxed{i_s = i}, j_{r+1}, \cdots, j_m, j_1, \cdots, j_{r-1}, \boxed{j_r = j}, i_{s+1}, \cdots, i_k\big),$$

which implies that $\mid \tau \mid - \mid \sigma \mid = (n - p) - (n - p - 1) = 1$, as desired. ∎

Intervals in $\mathfrak{S}_n$. In view of Proposition 3.2.4 we know that it suffices to look at intervals which start at the identity permutation ε. Furthermore, Remark 3.2.5 combined with the fact that the conjugacy class of any permutation $\tau \in \mathfrak{S}_n$ is determined by the cycle structure of τ imply that the cycles of the permutation τ completely determine the interval $[\varepsilon, \sigma^{-1}\tau\sigma]$, for any $\sigma \in \mathfrak{S}_n$.

In his paper [Bia97], P. Biane shows that if the orbits of τ are $X_1, X_2, \ldots, X_m$ then

$$[\varepsilon, \tau] \simeq \mathsf{NC}^{\mathsf{A}}(|X_1|) \times \mathsf{NC}^{\mathsf{A}}(|X_2|) \times \cdots \times \mathsf{NC}^{\mathsf{A}}(|X_m|), \tag{3.10}$$

where $|X_i|$ denotes the cardinality of the orbit X_i, for $1 \leqslant i \leqslant m$.

3.3.4 Example. The Hyperoctahedral Group B_n

• For $A \subseteq [\pm n]$, denote $-A = \{-i \mid i \in A\}$.

• A subset A of $[\pm n]$ is called *inversion-invariant* if $-A = A$.

• The cycles of a permutation $\tau \in \mathfrak{S}_{\pm n}$ are called inversion-invariant if the orbits which they determine are such.

• A permutation $\tau \in \mathfrak{S}_{\pm n}$ is called *signed* if it has the property that

$$\tau(-i) = -\tau(i), i \in [\pm n]. \tag{3.11}$$

The following special notations are consistently and without reference used throughout the thesis.

- $(i_1, \cdots, i_k)(-i_1, \cdots, -i_k) =: ((i_1, \cdots, i_k))$, for any $\{i_1, \cdots, i_k\} \subseteq [\pm n], |i_j| \neq |i_l|$, for $j \neq l$.
- $(i_1, \cdots, i_k, -i_1, \cdots, -i_k) =: [i_1, \cdots, i_k]$ for any $\{i_1, \cdots, i_k\} \subseteq [\pm n], i_j \neq i_l$, for $j \neq l$.

Direct verifications show the following immediate formulas

$$\begin{cases} [i][j] = ((i,j))((i,-j)) = ((i,-j))((i,j)) \\ [i_1, \cdots, i_k] = [i_1]((i_1, i_2)) \cdots ((i_{k-1}, i_k)) \\ ((i_1, \cdots, i_k)) = ((i_1, i_2)) \cdots ((i_{k-1}, i_k). \end{cases} \tag{3.12}$$

Definition. The subset of $\mathfrak{S}_{\pm n}$ which consists of *signed permutations* is clearly a subgroup. This subgroup is called the hyperoctahedral group, and it is denoted B_n. Thus

$$B_n = \{\tau \in \mathfrak{S}_{\pm n} \mid \tau(-i) = -\tau(i), i \in [\pm n]\}. \tag{3.13}$$

The set of generators of B_n is taken to be

$$R_n = \underbrace{\{[i] \mid i \in [\pm n]\,]\}}_{\text{denoted } R_n^i} \quad \cup \quad \underbrace{\{((i,j)) \mid i,j \in [\pm n]\,, \mid i \mid \neq \mid j \mid\}}_{\text{denoted } R_n^p} = R_n^i \cup R_n^p. \tag{3.14}$$

The superscripts i and p stand for *invariant* and *paired*, respectively.

• For a cycle c of a permutation $\tau \in B_n$ there are only two possibilities.

1^o c is inversion-invariant, i.e. of the form $[i_1, \cdots, i_k]$; the orbit corresponding to such a cycle will be called *invariant orbit.*

2^o c is not inversion-invariant; in this case "$-c$" must also appear (because of (3.13)) in the cycle decomposition of τ and the orbit corresponding to such a cycle "c" will be called *paired (non-invariant)* orbit.

• The number of *pairs* of non-invariant orbits of a (signed) permutation τ will be denoted by $\#_{\mathsf{paired}}(\tau)$. For instance if τ has no invariant orbits then $\frac{1}{2}\,\#(\tau) = \#_{\mathsf{paired}}(\tau)$.

Therefore in the cycle decomposition of a permutation from B_n the non-inversion-invariant cycles must come in pairs (and so do the corresponding paired orbits). One such pair will be a cycle of the form $((i_1, \cdots, i_k))$. Naturally, the number of paired orbits of a permutation $\tau \in B_n$ is always even. The disjoint cycle decomposition of a permutation τ in B_n will thus look like:

$$\tau = \underbrace{((\ldots))}_{-c_1,c_1}\underbrace{((\ldots))}_{-c_2,c_2}\cdots\underbrace{((\ldots))}_{-c_k,c_k}\underbrace{[\ldots]}_{z_1}\underbrace{[\ldots]}_{z_2}\cdots\underbrace{[\ldots]}_{z_p}. \tag{3.15}$$

Note the similarity with the way a non-crossing partition of type B is listed in (2.25).

Let us now verify the axioms of a marked group for the set T of generators.

MG1: R_n^p and R_n^i are both invariant under taking the inverse since every element of R_n is its own inverse.

MG2: R_n is also closed under conjugation by elements of B_n (but not by elements of $\mathfrak{S}_{\pm n}$ in general). In fact even more is true: each of the sets R_n^p and R_n^i is invariant under conjugation.

Verification for R_n^p: for every $\tau \in B_n$ and for every transposition $((i,j)) \in R_n^p$ the permutation $\tau^{-1}((i,j))\tau$ is also a transposition in R_n^p, namely $((\tau^{-1}(i), \tau^{-1}(j)))$, as a direct verification shows (and which uses the fact that τ is signed).

Verification for R_n^i: for every $\tau \in B_n$ and for every transposition $[i] \in R_n^i$ the permutation $\tau^{-1}[i]\tau$ is also a transposition in R_n^i, namely $[\tau^{-1}(i)]$. The verification of this also uses the fact that τ is signed.

Again, like in the case of the symmetric group $\mathfrak{S}_n$, it should be mentioned here that the hyperoctahedral group B_n can be generated by fewer signed transpositions, for instance one could take only the consecutive ones $\{((i,i+1)), 1 \leqslant i \leqslant n-1\} \subsetneq R_n^p$ and the transposition $[1] \in R_n^i$. However, this set of generators is not closed under conjugation so *MG2* fails.

Length Formula on B_n. The symbols ℓ_B, ℓ_{B_n}, $|\cdot|$ will denote the length function on B_n, and will be used interchangeably depending on the level of specificity required in concrete situations.

The minimal number of signed transpositions whose product is a signed permutation τ is obtained by subtracting the number of pairs of non-invariant orbits of τ from n. The length on the hyperoctahedral group B_n is thus given by the formula:

$$\ell_{B_n}(\tau) = \ell_B(\tau) = |\tau| = n - \#_{\mathsf{paired}}(\tau). \tag{3.16}$$

Covering Relations in B_n. Let us now look at the covering relations in B_n. Suppose $\tau, \sigma \in B_n$ are such that $\tau \neq \sigma$ and $\tau \leqslant_{B_n} \sigma$. By the description of covers from *i.* in 3.2.1 we know that

$$\sigma \quad \text{covers} \quad \tau \quad \Longleftrightarrow \quad \begin{cases} \sigma = \tau \cdot t, & \text{and } t \text{ is a generator,} \\ \text{and } |\sigma| = |\tau| + 1 \end{cases} \tag{3.17}$$

Now, any generator t is either balanced, $t = [i]$, or paired, $t = ((i,j))$. The condition that $|\sigma| = |\tau| + 1$ from equation (3.17) is equivalent to the fact that $\sigma = \tau \cdot t$ falls into either one of the following four possible cases. These cases are presented below according to the relevant cycles of τ and σ.

$$\begin{array}{rcll}
\overbrace{[i_1, i_2, \ldots, i_k]}^{\text{cycle(s) of } \sigma} & = & \overbrace{((i_1, i_2, \ldots, i_k))}^{\text{cycles(s) of } \tau} \cdot \overbrace{[i_k]}^{t} & (Bc\ 1) \\
[i_1, i_2, \ldots, i_k] & = & [i_1, i_2, \ldots, i_j]((i_{j+1}, i_{j+2}, \ldots, i_k)) \cdot ((i_j, i_k)) & (Bc\ 2) \\
((i_1, i_2, \ldots, i_k)) & = & ((i_1, i_2, \ldots, i_j))((i_{j+1}, i_{j+2}, \ldots, i_k)) \cdot ((i_j, i_k)) & (Bc\ 3) \\
\underbrace{[i_1, i_2, \ldots, i_j][i_{j+1}, i_{j+2}, \ldots, i_k]}_{\text{cycle(s) of } \sigma} & = & \underbrace{((i_1, i_2, \ldots, i_k))}_{\text{cycle(s) of } \tau} \cdot \underbrace{((-i_j, i_k))}_{t}. & (Bc\ 4)
\end{array}$$

(see e.g. Example 2.6 in [BGN03]).

Intervals in B_n. Paralleling the situation described in in the case of $\mathfrak{S}_n$, the paper [BGN03] gives a factorization similar to the one from (3.10), and which works for any interval $[\varepsilon, \tau]$ with $\tau \leqslant_{B_n} \gamma_o$.

In that paper it is shown that if $\tau \in B_n$ has an inversion-invariant orbit $Z = -Z$ and if the other orbits which are not singletons are denoted $X_1, -X_1, \ldots, X_m, -X_m$ then

$$[\varepsilon, \tau] \simeq \mathsf{NC}^{\mathsf{A}}(|X_1|) \times \mathsf{NC}^{\mathsf{A}}(|X_2|) \times \cdots \times \mathsf{NC}^{\mathsf{A}}(|X_m|) \times \mathsf{NC}^{\mathsf{B}}(|Z|/2)\ . \tag{3.18}$$

In the case when the permutation $\tau \leqslant_{B_n} \gamma_o$ has no invariant orbits and the other orbits with more than one element are $X_1, -X_1, \ldots, X_m, -X_m$, it is shown that

$$[\varepsilon, \tau] \simeq \mathsf{NC}^{\mathsf{A}}(|X_1|) \times \mathsf{NC}^{\mathsf{A}}(|X_2|) \times \cdots \times \mathsf{NC}^{\mathsf{A}}(|X_m|)\,. \tag{3.19}$$

3.3.5 Example. The Group D_n

As with the hyperoctahedral group, the symbols $\ell_{D_n}, \ell_D, |\cdot|$ will denote the length function on the marked group (D_n, R_n^p).

Definition. D_n denotes the subgroup of index 2 of B_n consisting of even permutations. That is,

$$D_n \quad \overset{\text{def}}{=} \quad \left\{\tau \in \mathfrak{S}_{\pm n} \;\middle|\; \begin{array}{c} \tau(-i) = -\tau(i),\ \ i \in [\pm n]\,,\ \text{and} \\ \tau \text{ is an even permutation} \end{array} \right\}. \tag{3.20}$$

It is easily seen that D_n is generated by $R_n^p = R_n \setminus R_n^i = \{((i,j)) \mid i,j \in [\pm n]\,, \mid i \mid \neq \mid j \mid\}$. Indeed, by writing τ as a product of generators of B_n and then by successively using formulas (3.12) τ can be written as

$$\tau = \underbrace{((i_1,j_1))((i_2,j_2))\ldots((i_k,j_k))}_{\text{even number of these: } 2k}\ \underbrace{[c_1][c_2]\ldots[c_m]}_{m \text{ even since } \tau \text{ is even}}\ .$$

Since $[i][j] = ((i,j))((i,-j))$ we are done.

The marked group axioms *MG1*, *MG2* for D_n have already been verified at the same time with the group B_n when we noted that both of the sets R_n^p and R_n^i are invariant under taking the inverse and under conjugation.

Remark. Let us point out that in fact the length formula on the group D_n is the same as the one on B_n. That is,

$$\ell_{B_n}(\tau) = n - \#_{\mathsf{paired}}(\tau) = |\,\tau\,| = \ell_{D_n}(\tau) = \ell_D(\tau), \quad \forall \tau \in D_n. \tag{3.21}$$

Proof. We want to prove that the following implication holds

$$x \in D_n \Longrightarrow \ell_{B_n}(x) = \ell_{D_n}(x). \tag{3.22}$$

We will prove this by double inequality. Recall that the length of an element x in a marked group is by definition the minimal number of generators whose product is x.

$\boxed{\leqslant}$. This is immediate since D_n has fewer generators than B_n: $R_n^p \subseteq R_n$.

$\boxed{\geqslant}$. Let $g_1 \cdots g_n$ be a shortest factorization of x, $g_1, \ldots, g_n \in R_n$.

Case 1. If all of $g_1, \ldots, g_n$ happen to be from the set R_n^p of generators of D_n then we are done.

Case 2. Suppose now that some of the g_i's come from the set R_n^i, i.e. they are transpositions of the form $[i] = (i, -i)$. The number of such transpositions must be even, as x itself is an even permutation ($[i]$ is odd, while $((i,j))$ is even, $\forall i, j$). The element x will thus look like

$$x = \ldots ((\, , \,))[\;]((\, , \,)) \ldots [\;] \ldots ((\, , \,)) \ldots \;.$$

If we can prove that we find a shortest factorization of x where all the odd generators $[\;]$ are grouped together,

$$x = ((\, , \,))((\, , \,)) \ldots ((\, , \,))[\;][\;] \ldots [\;]\,, \tag{3.23}$$

then we are done because by using the formula $[i][j] = ((i,j))((i,-j))$ the total number of generators in the factorization of x does not change.

The grouping as in Equation (3.23) can be done due to the following equality, which is checked directly,

$$[i]((u,v)) = ((u,-v))[i], \quad \forall\, i, u, v \in [\pm n].$$

■

If $\jmath$ denotes the inclusion function on $\mathfrak{S}_{\pm n}$ then it is clear that

$$\begin{aligned} &(D_n, \leqslant_{D_n}) \overset{\jmath}{\hookrightarrow} (B_n, \leqslant_{B_n}) \quad \text{is order preserving} \\ \text{i.e.} \qquad &\tau \leqslant_{D_n} \sigma \Longleftrightarrow \tau \leqslant_{B_n} \sigma, \quad \forall \quad \tau, \sigma \in D_n \end{aligned} \tag{3.24}$$

The covering relations in D_n are not used in this thesis. They are obtained in a similar manner as in the case of B_n (see e.g. [BW02]).

As for intervals in D_n, the above Equation (3.24) can be used to infer that for any $x, y \in D_n$ we have in fact that

$$[x, y]_{D_n} = [x, y]_{B_n} \cap D_n.$$

The above equality will be used again in Chapter 6 when we will derive type D results from their type B counterparts.

3.4 Permutations and Partitions: Type A

Starting with the paper [Bia97] it became apparent that there is an important relationship between non-crossing partitions and permutations. In that paper the lattice $\mathsf{NC}^{\mathsf{A}}(\mathsf{n})$ was embedded in $\mathfrak{S}_n$ and shown to be poset isomorphic to the interval $[\varepsilon, \gamma_o]_{\mathfrak{S}_n}$.

We will embed $\mathsf{NC}^{\mathsf{A}}(\mathsf{n})$ into $\mathfrak{S}_n$, following the approach from [Bia97]. The idea is to associate to each block of a partition a cycle in a unique way, determined by a compatibility condition with the forward cycle γ_o. Let us begin by introducing the necessary tools.

3.4.1 Definition and Remark. ***i.*** Let X be a finite set, let $Y \subseteq X$ be a subset of X and fix the permutation $\tau \in \mathfrak{S}_X$. The permutation *induced* by τ on the set Y is denoted by $\tau \downarrow Y$ and it is the permutation in $\mathfrak{S}_Y$ defined by letting $(\tau \downarrow Y)(y)$ be the first element in the list $\{\tau(y), \tau^2(y), \ldots, \tau^k(y), \ldots\}$ which belongs to Y. For example consider

$$\mathfrak{S}_4 \ni \tau = (1,2)(3,4) \quad \text{and} \quad Y = \{1,2,3\} \subseteq [4].$$

Then

$$\begin{cases} \tau(1) = 2 \in Y \Longrightarrow (\tau \downarrow Y)(1) = 2 \\ (\tau)(2) = 1 \in Y \Longrightarrow (\tau \downarrow Y)(2) = 1 \\ \tau(3) = 4 \notin Y, \tau^2(3) = \tau(4) = 3 \in Y \Longrightarrow (\tau \downarrow Y)(3) = 3, \end{cases}$$

and thus $\mathfrak{S}_3 \ni \tau \downarrow Y = (1,2)(3)$.

ii. Let σ denote a fixed permutation in $\mathfrak{S}_X$. The permutation $\tau \in \mathfrak{S}_X$ is called *compatible* with σ if

$$\tau \downarrow V = \sigma \downarrow V, \qquad \text{for every orbit } V \text{ of } \quad \tau. \tag{3.25}$$

iii. In the case when $\tau \in \mathfrak{S}_n$ is compatible with the forward cycle γ_o on $[n], \tau$ is said to be *standard in the disc sense.*

A quick look at the above example in *i.* makes it clear that the relation between $\Omega(\tau)$ and $\Omega(\tau \downarrow Y)$ is the following: the orbits of $\tau \downarrow Y$ are obtained by intersecting Y with the orbits of τ (when the intersection is non-empty, of course). $\tau \downarrow Y$ is a cyclic permutation of Y if and only if one of the orbits of τ completely contains Y. It is also immediate that if $\emptyset \neq Z \subseteq Y \subseteq X$ then $(\tau \downarrow Y) \downarrow Z = \tau \downarrow Z$.

We now move on to another significant fact which holds for permutations of an arbitrary set X and which plays an important rôle in this thesis.

3.4.2 The Genus Inequality. Let τ and β be permutations in $\mathfrak{S}_X$. The action of the subgroup of $\mathfrak{S}_X$ generated by $\{\tau, \beta\}$ splits the ground set X into orbits. Their number is denoted $\#(\tau, \beta)$. The following inequality, known as "the genus inequality", is well-known and appears in various forms in the literature.

3.4.3 Proposition.

$$\#(\tau) + \#(\tau^{-1}\beta) + \#(\beta) \quad \leqslant \quad |\, X \,| + 2\,\#(\tau, \beta) \qquad (\mathit{GI})$$

For a proof see e.g. Section 2 in [GJ97].

We will be especially interested in those permutations which manage to *achieve equality* in *GI*, for a fixed β. Because of their importance to this thesis, we assign them a special notation.

$$\mathfrak{S}_{\mathsf{genus}}(X, \beta) \quad \stackrel{\text{def}}{=} \quad \big\{\tau \in \mathfrak{S}_X \;\mid\; \#(\tau) + \#(\tau^{-1}\beta) + \#(\beta) \quad = \quad |\, X \,| + 2\,\#(\tau, \beta)\big\}. \quad (\mathit{GE})$$

3.4.4 Definitions, Notations and Remarks. *i.* Define the map

$$\Pi(n\Pi(\mathsf{n}) \ni \pi \longmapsto \mathfrak{perm}_{\gamma_o}(\pi) \in \mathfrak{S}_n\,,$$

where $\mathfrak{perm}_{\gamma_o}(\pi)$ denotes the unique permutation in $\mathfrak{S}_n$ which is compatible with γ_o and whose orbits are exactly the blocks of the partition π. $\mathfrak{perm}_{\gamma_o}$ does the following: given a partition π, $\mathfrak{perm}_{\gamma_o}$ turns each block of π into a cycle, *by keeping the increasing order* into which the elements of that block were listed. After transforming each block into a cycle, it multiplies the cycles together. The result is $\mathfrak{perm}_{\gamma_o}(\pi)$.

We will denote

$$\boxed{\mathfrak{S}_{\mathsf{compatible}}(n, \gamma_o) \stackrel{\text{def}}{=} \{\tau \in \mathfrak{S}_n \mid \tau \quad \text{is standard in the disc sense}\}}$$

Observe that the set $\mathfrak{S}_{\mathsf{compatible}}(n,\gamma_o)$ is precisely the image of $\Pi(\mathsf{n})$ through the map $\mathfrak{perm}_{\gamma_o}$:

$$\mathfrak{S}_{\mathsf{compatible}}(n,\gamma_o) \quad = \quad \mathfrak{perm}_{\gamma_o}\left(\Pi(\mathsf{n})\right) \subsetneq \mathfrak{S}_n\,, \tag{3.26}$$

and also that $\mathfrak{perm}_{\gamma_o}$ is one-to-one onto its image and that its inverse is precisely the orbit map Ω

$$\mathfrak{S}_{\mathsf{compatible}}(n,\gamma_o) \quad \underset{\longleftarrow\ \Omega}{\overset{\mathfrak{perm}_{\gamma_o}\ \longrightarrow}{\longleftrightarrow}} \quad \Pi(\mathsf{n})\,. \tag{3.27}$$

Let also note that

ii. Let $\mathfrak{S}_{\mathsf{DNC}}(n,\gamma_o)$ denote the subset of permutations in $\mathfrak{S}_n$ which *do not* satisfy a "crossing condition" in the disc, denoted *DC*, that is,

$$\boxed{\mathfrak{S}_{\mathsf{DNC}}(n,\gamma_o) \quad \stackrel{\text{def}}{=} \quad \{\tau\in\mathfrak{S}_n \mid \tau \quad \textit{does not satisfy} \text{ condition } \quad \textit{DC}\,\}} \subsetneq \mathfrak{S}_n\,, \tag{3.28}$$

where τ satisfies property *DC* if

$$\exists \quad \text{distinct elements} \quad [n]\ni a,b,c,d \quad \text{and} \quad \begin{cases}\gamma_o\downarrow\{a,b,c,d\} & =(a,b,c,d)\\ \tau\downarrow\{a,b,c,d\} & =(a,c)(b,d).\end{cases} \tag{\textit{DC}}$$

iii. The subset of $\mathfrak{S}_n$ of permutations which are both *standard in the disc sense* and *have no crossings* is also assigned a special notation,

$$\boxed{\mathfrak{S}_{\mathsf{disc\text{-}nc}}(n,\gamma_o) \quad \stackrel{\text{def}}{=} \quad \mathfrak{S}_{\mathsf{compatible}}(n,\gamma_o)\cap\mathfrak{S}_{\mathsf{DNC}}(n,\gamma_o)} \subsetneq \mathfrak{S}_n\,. \tag{3.29}$$

iv. In the particular case when $X=[n]$ and $\beta=$ the forward cycle γ_o, a permutation τ satisfies the genus inequality *GI* iff

$$\begin{aligned}\#(\tau)+\#(\tau^{-1}\beta)+\#(\beta) &\leqslant \ \mid X\mid+2\,\#(\tau,\beta)\Longleftrightarrow\\ \#(\tau)+\#(\tau^{-1}\gamma_o)+\#(\gamma_o) &\leqslant \ n+\,2\,\#(\tau,\gamma_o)\Longleftrightarrow\\ \#(\tau)+\#(\tau^{-1}\gamma_o) &\leqslant n+2-1 \qquad\qquad [\text{since } \#(\gamma_o)=\#(\tau,\gamma_o)=1]\\ &= n+1.\end{aligned}$$

The last inequality is nothing but the triangle inequality *L3* applied to τ and $\tau^{-1}\gamma_o$. Indeed,

$$\ell_{\mathfrak{S}_n}(\tau)+\ell_{\mathfrak{S}_n}(\tau^{-1}\gamma_o)\leqslant\ell_{\mathfrak{S}_n}(\gamma_o)\Longleftrightarrow \qquad [\text{by the length formula in } \mathfrak{S}_n]$$

$$
\begin{aligned}
[n - \#(\tau)] + [n - \#(\tau^{-1}\gamma_o)] \leqslant n - \#(\gamma_o) &\iff && \text{[since } \gamma_o \text{ has one orbit]} \\
2n - [\#(\tau) + \#(\tau^{-1}\gamma_o)] \leqslant n + 1 &\iff && \\
\#(\tau) + \#(\tau^{-1}\gamma_o) \leqslant n + 1. & &&
\end{aligned}
$$

We define the set of permutations which achieve equality in the genus inequality to be

$$\boxed{\mathfrak{S}_{\mathsf{genus}}(n, \gamma_o) \quad \stackrel{\text{def}}{=} \quad \{\tau \in \mathfrak{S}_n \mid \#(\tau) + \#(\tau^{-1}\gamma_o) = n + 1\}} \subsetneq \mathfrak{S}_n\,. \tag{3.30}$$

As a consequence, we obtain the following lemma.

3.4.5 Lemma. $\mathfrak{S}_{\mathit{genus}}(n, \gamma_o) = [\varepsilon, \gamma_o]_{\mathfrak{S}_n}$.

Proof.

$$
\begin{aligned}
\tau \in [\varepsilon, \gamma_o] &\stackrel{\text{def}}{\iff} \ell_{\mathfrak{S}_n}(\tau) + \ell_{\mathfrak{S}_n}(\tau^{-1}\gamma_o) = \ell_{\mathfrak{S}_n}(\gamma_o) \\
&\iff \#(\tau) + \#(\tau^{-1}\gamma_o) = n + 1. && \text{[by (3.4.4, } \boldsymbol{iv}\text{)]}
\end{aligned}
$$

∎

Even though the proof is very simple, we should however note that $\mathfrak{S}_{\mathsf{genus}}(n, \gamma_o)$ provides us with a very "compact" characterization of the interval $[\varepsilon, \gamma_o]$, which only uses the number of cycles of permutations.

3.4.6 Lemma. *$\mathfrak{S}_{\mathit{disc\text{-}nc}}(n, \gamma_o)$ and $\mathit{NC}^A(n)$ are mapped into one another by the poset isomorphisms $\mathfrak{perm}_{\gamma_o}$ and Ω.*

$$\mathfrak{S}_n \supsetneq \mathfrak{S}_{\mathit{disc\text{-}nc}}(n, \gamma_o) \quad \underset{\Omega \ \rightarrow}{\overset{\leftarrow \mathfrak{perm}_{\gamma_o}}{\longleftrightarrow}} \quad \mathit{NC}^A(n) \subsetneq \Pi(n)\,).$$

Proof. Condition DC is in a sense the "$\mathfrak{S}_n$" version of the crossing condition (2.16) for partitions in $\Pi(\mathsf{n})$. One might naturally expect then to be able to identify $\mathsf{NC}^{\mathsf{A}}(\mathsf{n})$ with $\mathfrak{S}_{\mathsf{DNC}}(n, \gamma_o)$. Modulo adjusting $\mathfrak{S}_{\mathsf{DNC}}(n, \gamma_o)$, by keeping only its permutations which are standard in the disc sense (i.e. by intersecting it with $\mathfrak{S}_{\mathsf{compatible}}(n, \gamma_o)$), this is in fact exactly what happens. That Ω and $\mathfrak{perm}_{\gamma_o}$ are bijective and inverse to each other is obvious from their definition. ∎

$\mathfrak{S}_{\mathsf{disc\text{-}nc}}(n, \gamma_o)$ is thus the set of "non-crossing permutations" (coming from $\mathfrak{S}_{\mathsf{DNC}}(n, \gamma_o)$) which are also "standard in the disc sense" (compatible with the forward cycle γ_o, i.e. coming from $\mathfrak{S}_{\mathsf{compatible}}(n, \gamma_o)$). The compatibility condition is chosen to provide us with a "clockwise sense" of winding on the cycles. This also makes the orbit map Ω injective. However, since we will only look at permutations which are standard in the disc sense, the elements of $\mathfrak{S}_{\mathsf{disc\text{-}nc}}(n, \gamma_o)$ will be simply called "non-crossing permutations" (rather than "non-crossing permutations standard in the disc sense"). Moreover, the word "disc" is being kept in the notation because starting with Chapter 4 we will move to an "annular" framework.

The theorem below shows that $\mathsf{NC}^{\mathsf{A}}(\mathsf{n})$ is in fact isomorphic to an interval in $\mathfrak{S}_n$.

3.4.7 Theorem. $\mathfrak{S}_{\textsf{\textit{disc-nc}}}(n, \gamma_o) = [\varepsilon, \gamma_o]_{\mathfrak{S}_n}$.

The proof of this result is found in [Bia97] (see Theorem 1 from Section 1).

3.4.8 Remark. Regarding the isomorphism between the lattices $\mathsf{NC}^{\mathsf{A}}(\mathsf{n})$ and $\mathfrak{S}^{\mathsf{A}}_{\mathsf{nc}}(X, \gamma_o)$, we mention here that the Kreweras complementation map $K : \mathsf{NC}^{\mathsf{A}}(\mathsf{n}) \longrightarrow \mathsf{NC}^{\mathsf{A}}(\mathsf{n})$ corresponds to the order reversing map $[\varepsilon, \gamma_o]_{\mathfrak{S}_n} \ni \tau \longmapsto C_{\gamma_o}(\tau) = \tau^{-1}\gamma_o$, i.e.

$$\Omega(\tau) = \pi \Longrightarrow \Omega(\tau^{-1}\gamma_o) = K(\pi). \tag{3.31}$$

3.4.9 Type A Summarizing Diagram. The three equivalent descriptions of the interval $[\varepsilon, \gamma_o]_{\mathfrak{S}_n}$ are summarized in the following "type A diagram".

$$\boxed{\begin{array}{c} \mathfrak{S}_{\mathsf{genus}}(n, \gamma_o) \ = \mathfrak{S}_{\mathsf{disc\text{-}nc}}(n, \gamma_o) \ = [\varepsilon, \gamma_o]_{\mathfrak{S}_n} \\ \Omega \downarrow \Big\updownarrow \uparrow \mathfrak{perm}_{\gamma_o} \\ \mathsf{NC}^{\mathsf{A}}(\mathsf{n}) \end{array}}$$

3.4.10 Definition. An element of any of the three sets in the upper row of the above diagram is called *type A non-crossing permutation.* Thus, the set of all type A non-crossing permutations is defined to be

$$\boxed{\mathfrak{S}^{\mathsf{A}}_{\mathsf{nc}}(X, \gamma_o) \quad \stackrel{\text{def}}{=} \quad \mathfrak{S}_{\mathsf{genus}}(n, \gamma_o) \ = \mathfrak{S}_{\mathsf{disc\text{-}nc}}(n, \gamma_o) \ = [\varepsilon, \gamma_o]_{\mathfrak{S}_n}} \tag{3.32}$$

3.5 Permutations and Partitions: Type B

The notation from Sections 2.2 and 2.3 are used without reference in the present section. We recall here that the symbol γ_o is used for *both* forward long cycles

$$(1, 2, \ldots, n) \quad \text{on} \quad \mathfrak{S}_n\,, \quad \text{and, respectively}$$

$$(1,2,\ldots,n,-1,-2,\ldots,-n) = [1,2,\ldots,n] \quad \text{on} \quad \mathfrak{S}_{\pm n}\,.$$

We are now looking at partitions in the set $\Pi(\pm \mathsf{n})$, and at permutations in $\mathfrak{S}_{\pm n}$. Obviously, the diagram which summarized the preceding section carries through to the present setting, where the ground set $[n]$ is replaced by $[\pm n]$. To fix the notations, we reproduce it with the updated symbols

$$\begin{array}{c} \mathfrak{S}_{\mathsf{genus}}(\pm n,\gamma_o) \;=\mathfrak{S}_{\mathsf{disc\text{-}nc}}(\pm n,\gamma_o) \;= [\varepsilon,\gamma_o]_{\mathfrak{S}_{\pm n}} \\ \Omega\downarrow \Big\updownarrow \uparrow \mathfrak{perm}_{\gamma_o} \\ \mathsf{NC}^{\mathsf{A}}(\pm\ \mathsf{n}) \end{array}$$

The element of the the upper row which we focus upon, as the image of $\mathsf{NC}^{\mathsf{A}}(\pm\ \mathsf{n})$, is the interval $[\varepsilon,\gamma_o]_{\mathfrak{S}_{\pm n}}$.

The first thing we note here is that the forward cycle $\mathfrak{S}_{\pm n}\ni\gamma_o = (1,2,\ldots,n,-1,-2,\ldots,-n) = [1,2,\ldots,n]$ happens to be a signed permutation, and thus it belongs to the hyperoctahedral group B_n. Referring to the formulas (3.12), we have that

$$\gamma_o = [1,2,\ldots,n] = [1]((1,2))((2,3))\cdots((n-1,n)). \tag{3.33}$$

Let us also record the length of γ_o, as an element of $\mathfrak{S}_{\pm n}$ and B_n, respectively. Since γ_o is made of a single cycle, which is invariant, the length formulas in $\mathfrak{S}_{\pm n}$ and B_n give that $\ell_{\mathfrak{S}_{\pm n}}(\gamma_o) = 2n-1$ while $\ell_{B_n}(\gamma_o) = n$.

We are therefore exactly in the situation described in Remark 3.1.6, with the group $G=\mathfrak{S}_{\pm n}$ generated by all transpositions and its subgroup $H = B_n$ generated by all *signed* transpositions. Hence we are dealing with two different partial orders, which are denoted by $\leqslant_{\mathfrak{S}_{\pm n}}$ and $\leqslant_{B_n}$. This is the reason for having the subscript in $[\varepsilon,\gamma_o]_{\mathfrak{S}_{\pm n}}$. Since $\gamma_o\in B_n$, it makes sense to consider the set $[\varepsilon,\gamma_o]_{B_n}$ of all signed permutations which sit under γ_o, with respect to the partial order defined by the set of signed transpositions.

3.5.1 Remark. In the paper [BGN03] it is shown that, if $\jmath$ denotes the inclusion function on $\mathfrak{S}_{\pm n}$, then

$$(\,[\varepsilon,\gamma_o]_{B_n}\,,\leqslant_{B_n})\overset{\jmath}{\hookrightarrow}(\mathfrak{S}_{\pm n}\,,\leqslant_{\mathfrak{S}_{\pm n}}) \quad \text{is order preserving.}$$

$$\text{That is,}\quad \tau_1\ \leqslant_{B_n}\ \tau_2\ \leqslant_{B_n}\ \gamma_o \quad\Longrightarrow\quad \tau_1\ \leqslant_{\mathfrak{S}_{\pm n}}\ \tau_2. \tag{3.34}$$

Let us observe that the fact that both τ_1, τ_2 are sitting under γ_o w.r.t. $\leqslant_{B_n}$, really matters. One can easily find $\tau_1 \leqslant_{B_n} \tau_2$ but $\tau_1 \not\leqslant_{\mathfrak{S}_{\pm n}} \tau_2$. For instance, pick

$$\begin{cases} \tau_1 = ((1,2)), & \ell_{B_n}(\tau_1) = 1, \ell_{\mathfrak{S}_{\pm n}}(\tau_1) = 2 \\ \tau_2 = [1][2], & \ell_{B_n}(\tau_2) = 2, \ell_{\mathfrak{S}_{\pm n}}(\tau_2) = 2, \end{cases}$$

and then use the formula $[1][2] = ((1,2))((1,-2))$.

In the same paper [BGN03] it is also shown that the poset isomorphism $\mathsf{NC}^{\mathsf{A}}(\mathsf{n}) \rightleftarrows [\varepsilon, \gamma_o]_{\mathfrak{S}_n}$ (and, implicitly $\mathsf{NC}^{\mathsf{A}}(\pm\, \mathsf{n}) \rightleftarrows [\varepsilon, \gamma_o]_{\mathfrak{S}_{\pm n}}$) has the perfect "B" analogue:

3.5.2 Theorem. *$\mathsf{NC}^B(n)$ is poset isomorphic to $[\varepsilon, \gamma_o]_{B_n}$.*

3.5.3 Remark. ***i.*** By spelling out the formula for the length function in B_n we get that

$$\begin{aligned} \tau \leqslant_{B_n} \gamma_o &\overset{\text{def}}{\iff} \ell_{B_n}(\tau) + \ell_{B_n}(\tau^{-1}\gamma_o) = \ell_{B_n}(\gamma_o) \\ &\iff [n - \#_{\mathsf{paired}}(\tau)] + [n - \#_{\mathsf{paired}}(\tau^{-1}\gamma_o) = n \\ &\iff \#_{\mathsf{paired}}(\tau) + \#_{\mathsf{paired}}(\tau^{-1}\gamma_o) = n \end{aligned}$$

and therefore $[\varepsilon, \gamma_o]_{B_n} = \{\tau \in B_n \mid \#_{\mathsf{paired}}(\tau) + \#_{\mathsf{paired}}(\tau^{-1}\gamma_o) = n\}$.

ii. Let us now look at the "type B" versions of the equivalent descriptions of $\mathfrak{S}^{\mathsf{A}}_{\mathsf{nc}}(X, \gamma_o)$ from the type A diagram of non-crossing permutations. It is clear from Remark 3.5.1 that the following equality (this is simply an equality of sets) holds.

$$\{\tau \in \mathfrak{S}_{\pm n} \mid \tau \leqslant_{B_n} \gamma_o\} = \{\tau \in \mathfrak{S}_{\pm n} \mid \tau \leqslant_{\mathfrak{S}_{\pm n}} \gamma_o\} \cap B_n$$

that is, $$[\varepsilon, \gamma_o]_{B_n} = [\varepsilon, \gamma_o]_{\mathfrak{S}_{\pm n}} \cap B_n. \tag{3.35}$$

Now, since $\mathfrak{S}_{\mathsf{genus}}(\pm n, \gamma_o) = [\varepsilon, \gamma_o]_{\mathfrak{S}_{\pm n}}$ [by Lemma 3.4.5]

and $[\varepsilon, \gamma_o]_{B_n} = [\varepsilon, \gamma_o]_{\mathfrak{S}_{\pm n}} \cap B_n$, [by (3.35)]

it follows that $[\varepsilon, \gamma_o]_{B_n} = \mathfrak{S}_{\mathsf{genus}}(\pm n, \gamma_o) \cap B_n$.

Similarly, $\mathfrak{S}_{\mathsf{disc\text{-}nc}}(\pm n, \gamma_o) = [\varepsilon, \gamma_o]_{\mathfrak{S}_{\pm n}}$ [by Lemma 3.4.6]

and $[\varepsilon, \gamma_o]_{B_n} = [\varepsilon, \gamma_o]_{\mathfrak{S}_{\pm n}} \cap B_n$ [by (3.35)]

imply that $[\varepsilon, \gamma_o]_{B_n} = \mathfrak{S}_{\mathsf{disc\text{-}nc}}(\pm n, \gamma_o) \cap B_n$.

3.5.4 Type B Summarizing Diagram. The facts presented in this section are summarized in the following "type B diagram"

$$\boxed{\begin{array}{ccccc} \mathfrak{S}_{\mathsf{genus}}(\pm n,\gamma_o)\cap B_n & = & [\varepsilon,\gamma_o]_{B_n} & = & \mathfrak{S}_{\mathsf{disc\text{-}nc}}(\pm n,\gamma_o)\cap B_n \\ & & \Omega\downarrow \Big\updownarrow \uparrow \mathfrak{perm}_{\gamma_o} & & \\ & & \mathsf{NC}^{\mathsf{B}}(\mathsf{n}) & & \end{array}}$$

3.5.5 Definition. Any element of any of the three sets in the upper row of the type B diagram is called a *type B non-crossing permutation.* Thus, the set of type B non-crossing permutations is defined to be

$$\boxed{\mathfrak{S}^{\mathsf{B}}_{\mathsf{nc}}(X,\gamma_o) \quad \stackrel{\text{def}}{=} \quad \mathfrak{S}_{\mathsf{genus}}(\pm n,\gamma_o)\cap B_n = \mathfrak{S}_{\mathsf{disc\text{-}nc}}(\pm n,\gamma_o)\cap B_n = [\varepsilon,\gamma_o]_{B_n}} \tag{3.36}$$

3.5.6 Type A - Type B Summarizing Diagram. Type B non-crossing permutations (partitions) are intrinsically depending on the type A non-crossing permutations (partitions) as they are defined as subsets of $\mathfrak{S}_{\pm n}$ ($\mathsf{NC}^{\mathsf{A}}(\pm\ \mathsf{n})$), respectively. The way how "type B sits inside type A" is also compactly presented below in a "type A–type B" diagram

$$\boxed{\begin{array}{ccccc} \mathfrak{S}_{\pm n} \quad \supsetneq & & \mathfrak{S}^{\mathsf{A}}_{\mathsf{nc}}(X,\gamma_o) & \overset{3.5.1}{\underset{''\supsetneq''}{\longleftrightarrow}} & \mathfrak{S}^{\mathsf{B}}_{\mathsf{nc}}(X,\gamma_o) \\ & & \Omega\downarrow \Big\updownarrow \uparrow \mathfrak{perm}_{\gamma_o} & & \Omega\downarrow \Big\updownarrow \uparrow \mathfrak{perm}_{\gamma_o} \\ \Pi(\pm\mathsf{n}) & \supsetneq & \mathsf{NC}^{\mathsf{A}}(\pm\ \mathsf{n}) & \supsetneq & \mathsf{NC}^{\mathsf{B}}(\mathsf{n}) \end{array}}$$

Chapter 4

Type B Annular Posets

4.1 Permutations and Partitions: The Annular Case

The set $\mathsf{NC}^{\mathsf{B}}(\mathsf{n})$ of type B non-crossing partitions, is obtained from the set of non-crossing partitions of type A by changing the ground set from $[n]$ to $[\pm n]$ and asking that the condition (2.24) is satisfied:

$$B \quad \text{is a block of} \quad \pi \Longleftrightarrow -B \quad \text{is a block of} \quad \pi. \tag{2.24}$$

In section 3.4 and section 3.5 the posets $\mathsf{NC}^{\mathsf{A}}(\mathsf{n})$ and $\mathsf{NC}^{\mathsf{B}}(\mathsf{n})$ were embedded inside $\mathfrak{S}_n$ and B_n, respectively, and their images were seen to be in fact the intervals consisting of all the permutations which sit under the corresponding forward cycle γ_o, with respect to the appropriate partial orders $\leqslant_{\mathfrak{S}_n}$ and $\leqslant_{B_n}$. The images of $\mathsf{NC}^{\mathsf{A}}(\mathsf{n})$ and $\mathsf{NC}^{\mathsf{B}}(\mathsf{n})$ under the map $\mathfrak{perm}_{\gamma_o}$ were denoted by $\mathfrak{S}^{\mathsf{A}}_{\mathsf{nc}}(X, \gamma_o)$, and $\mathfrak{S}^{\mathsf{B}}_{\mathsf{nc}}(X, \gamma_o)$, respectively.

Non-crossing partitions are drawn inside a disc. The points of the ground set are spread on the boundary of the disc, in clockwise sense, as indicated by the linear order of the ground set and they are connected according to the block structure of the partition which is being drawn. A non-crossing partition will produce a figure whose contours will not intersect.

The exact same drawing which is associated to a non-crossing partition also represents a non-crossing permutation, with the only difference that we must put some arrows on the boundary of the contour associated to each block. These new "arrowed contours" represent the cycles, and the arrows indicate the clockwise sense of winding inside each cycle. The clockwise orientation is given algebraically by a compatibility condition with the forward cycle

γ_o. It is clear that the very same drawing (the one with arrows) can be used to represent both partitions and permutations.

The importance of drawings should not be diminished by the fact that all the formal arguments were carried through without explicitly using them. They are important as they really help with the intuition. We view these graphical representations as some sort of intermediate objects between the world of permutations and the world of partitions, as objects which we do not rigorously define but which can help the intuitive understanding of what is going on.

In this section we look at *annular* objects. They were introduced in [MN04]. Before making the concrete definitions let us make a short heuristic presentation, modeled on the "non-rigorously defined" drawings.

Suppose that instead of one circle we now have two circles, an *inner* circle drawn inside an *outer* circle and suppose that we put some points (say p points) on the outer circle and some other points (say q of them) on the inner circle. If we start connecting these $p + q$ points between them we would get some contours which lie inside the annulus determined by the two circles. We say that one such contour can be *connected* if it is obtained by using points from both circles, and that it is *disconnected* in the opposite case. The regions enclosed by the contours may, or may not intersect. Of course only drawings with non-intersecting contours are to be of any relevance. We also might consider placing some arrows on a contour if at some point we are to make it into a cycle of a permutation. Figure 4.1 exemplifies the kind of drawings that we have in mind.

It turns out that the type of annular drawings just described works better with permutations rather than partitions. Three issues must be addressed before making the definition of *non-crossing annular permutations*.

• What is the ground set where the points from the two circles arise.

• The "annular" version of the map $\mathfrak{perm}_{\gamma_o}$ has a chance to work only if we can fix a unique sense of winding inside each cycle (thus on each contour of a drawing); what is the analogue of γ_o (say γ) which would determine a "standard in the annular sense" condition.

• What are the possible crossing patterns and what are their algebraic descriptions (the annular versions of the crossing condition DC).

These conditions lead to the following working hypotheses and tentative answers.

•• If we have p points on the outer circle and q points on the inner circle then the ground set

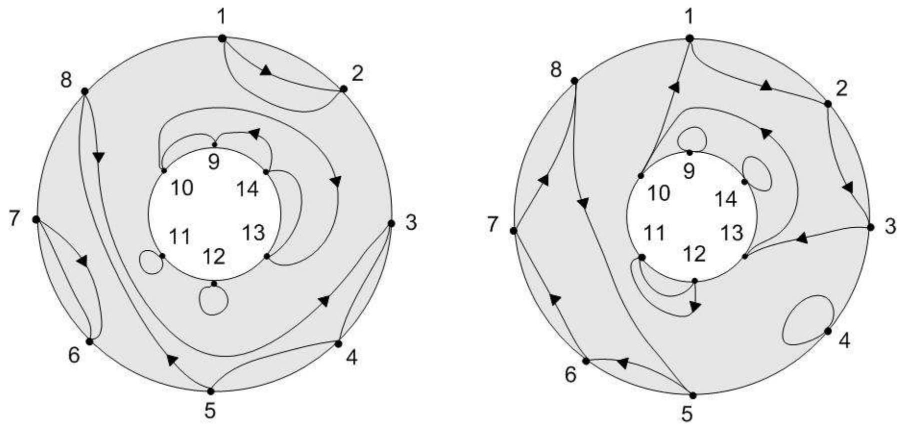


Figure 4.1: Examples of annular non-crossing permutations.
disconnected:(1,2)(3,4,-1,-4)(5,6,-6,-7)(-5)(7), and
connected: (1,2,3,-6,6)(-5,7)(-1,2,-3,-4)(4)(5)(-7).

is $[p+q]$, the points on the outer circle are $Y = \{1, 2, \ldots, p\}$ and the points on the inner circle are $Z = \{p+1, p+2, \ldots, p+q\}$. The annular non-crossing permutations will live in the set $\mathfrak{S}_{p+q}$; also, to be consistent with the clockwise orientation for connected contours, the points on the inner circle must be placed in *counterclockwise sense.*

•• If some contour of a drawing is obtained only with the p points from the outer circle then the orientation inside that contour must be "standard in the disc sense", thus it will be obtained from a compatibility condition with the " outer forward cycle", denoted γ_{ext} , on the set $[p]$ (γ_{ext} is a permutation of $\mathfrak{S}_n$, we call it "outer forward cycle" because it really permutes forward only the outer points keeping the inner ones fixed). Similarly the "inner forward cycle", denoted γ_{int} , on the set $[p+q] \setminus [p] = \{p+1, p+2, \ldots, p+q\}$ will implement the orientation on the hulls made only with the q points from the inner circle. The orientation for contours made with points from both the outer and the inner circle will be implemented by $\gamma \overset{\text{def}}{=} \gamma_{\mathsf{ext}}\,\gamma_{\mathsf{int}}$.

•• The crossing patterns in the annulus turn out to be more complicated than in the disc. The annular version of the crossing condition ***DC*** is in fact a set of three conditions which will be denoted ***AC1*** , ***AC2*** and ***AC3*** .

Let us now formalize the tentative answers and elaborate on the above ideas.

4.1.1 Definitions, Notations and Remarks. *i.* Let us fix two positive integers p, q with $p+q=n$ and make the following notations.

$$\bullet \begin{cases} X & \stackrel{\text{def}}{=} & [p+q]=[n] & \rightsquigarrow & \text{the ground set} \\ Y & \stackrel{\text{def}}{=} & [p] & \rightsquigarrow & \text{the points on the outer circle} \\ Z & \stackrel{\text{def}}{=} & [p+q]\setminus[p]=\{p+1,\dots,p+q\} & \rightsquigarrow & \text{the points on the “inner” circle} \\ \gamma_{\mathsf{ext}} & \stackrel{\text{def}}{=} & (1,2,\dots,p) & \rightsquigarrow & \text{the “outer” forward cycle} \\ \gamma_{\mathsf{int}} & \stackrel{\text{def}}{=} & (p+1,p+2,\dots,p+q) & \rightsquigarrow & \text{the “inner” forward cycle} \\ \gamma & \stackrel{\text{def}}{=} & \gamma_{\mathsf{ext}}\,\gamma_{\mathsf{int}}=\gamma_{\mathsf{int}}\,\gamma_{\mathsf{ext}}=(1,2,\dots,p)(p+1,p+2,\dots,p+q). \end{cases}$$

It is clear that

$$\gamma\downarrow Y=\gamma_{\mathsf{ext}}\downarrow Y=\gamma_o \quad \text{the forward cycle on } Y,$$
$$\text{and also,}\qquad \gamma\downarrow Z=\gamma_{\mathsf{int}}\downarrow Z=\gamma_o \quad \text{the forward cycle on } Z.$$

• A subset $A\subseteq X$ with the property that $A\cap Y\neq\emptyset\neq A\cap Z$ will be called $\gamma-$ *connected*.

• A partition $\pi\in\Pi(X)$ is called $\gamma-$connected if it has at least one $\gamma-$connected block, and it will be said $\gamma-$ *disconnected* in the opposite case.

• $\mathfrak{S}_X\ni\tau$ is called $\gamma-$ *connected* $\stackrel{\text{def}}{\Longleftrightarrow}$ $\Omega(\tau)\in\Pi(X)$ is $\gamma-$ *connected*.

• $\mathfrak{S}_X\ni\tau$ is called $\gamma-$ *disconnected* $\stackrel{\text{def}}{\Longleftrightarrow}$ $\Omega(\tau)\in\Pi(X)$ is $\gamma-$ *disconnected*.

• Viewing $\mathfrak{S}_X$ (and $\Pi(X)$) as being made of the disjoint union of the $\gamma-$connected and $\gamma-$disconnected permutations (and, respectively, partitions) proves useful to our purposes; for this reason we assign these sets the following special notations

$$\begin{cases} \mathfrak{S}_X^{\prime} \stackrel{\text{def}}{=} \{\tau\in\mathfrak{S}_X \mid \tau \text{ is } \gamma-\text{connected}\} \\ \mathfrak{S}_X^{\prime\prime} \stackrel{\text{def}}{=} \{\tau\in\mathfrak{S}_X \mid \tau \text{ is } \gamma-\text{disconnected}\}, \end{cases}$$

$$\text{and}\qquad \begin{cases} \Pi^{\prime}(X) \stackrel{\text{def}}{=} \{\pi\in\Pi(X) \mid \pi \text{ is } \gamma-\text{connected}\} \\ \Pi^{\prime\prime}(X) \stackrel{\text{def}}{=} \{\pi\in\Pi(X) \mid \pi \text{ is } \gamma-\text{disconnected}\}. \end{cases}$$

It is clear from the definition that the orbit map Ω sends $\mathfrak{S}_X^{\prime}$ to $\Pi^{\prime}(X)$ and $\mathfrak{S}_X^{\prime\prime}$ to $\Pi^{\prime\prime}(X)$, respectively.

ii. A permutation is standard in the disc sense if the winding direction inside each of its cycles is the clockwise direction. Algebraically speaking, this was seen to be equivalent with that permutation being compatible (as in 3.4.1) with the forward cycle γ_o; in the annular framework, compatibility (as in 3.4.1) with γ is not sufficient to reflect pictorially the "clockwise winding" condition on the cycles. A permutation $\tau \in \mathfrak{S}_X$ will be called *standard in the annular sense* if for every orbit A of τ the following two conditions are met:

$$\tau \downarrow (A \cap Y) = \gamma \downarrow (A \cap Y) \quad \& \quad \tau \downarrow (A \cap Z) = \gamma \downarrow (A \cap Z) \qquad (\texttt{AS 1})$$

$$\begin{cases} \exists \quad \text{at most one element} \quad a_{Y} \in A \cap Y \quad \text{such that} \quad \tau(a_{Y}) \in Z, \quad \text{and} \\ \exists \quad \text{at most one element} \quad a_{Z} \in A \cap Z \quad \text{such that} \quad \tau(a_{Z}) \in Y. \end{cases} \qquad (\texttt{AS 2})$$

We denote

$$\boxed{\mathfrak{S}_{\mathsf{compatible}}(X, \gamma) \quad \stackrel{\text{def}}{=} \quad \{\tau \in \mathfrak{S}_X \mid \tau \quad \text{is standard in the annular sense}\}} \qquad (4.1)$$

Let us note that if $\tau \in \mathfrak{S}_X$ is standard in the annular sense then both of $\tau \downarrow Y \in \mathfrak{S}_Y$ and $\tau \downarrow Z \in \mathfrak{S}_Z$ are standard in the disc sense.

iii. Condition *DC* was used to explain when two hulls cross, in the disc case. In the annular case it turns out that there are three possible crossing patterns, denoted *AC1* , *AC2* and *AC3* . The crossing pattern *AC1* is

$$\exists \quad 4 \quad \text{distinct elements} \quad a, b, c, d \in X \quad \text{such that}$$
$$\gamma \downarrow \{a, b, c, d\} = (a, b, c, d) \quad \& \quad \tau \downarrow \{a, b, c, d\} = (a, c)(b, d). \qquad (\texttt{AC1})$$

For the remaining two crossing patterns we introduce the following *"AC-test permutation"*: for every $y \in Y$ and $z \in Z$ let

$$\lambda_{y,z} \quad \stackrel{\text{def}}{=} \quad \big(\gamma(y), \gamma^2(y), \ldots, \gamma^{p-1}(y), \gamma(z), \gamma^2(z), \ldots, \gamma^{q-1}(z)\big). \qquad (4.2)$$

It is obvious that if we switch y and z in the above definition then $\lambda_{z,y} = \lambda_{y,z}$. They will be used interchangeably as needed. $\lambda_{y,z}$ is thus the permutation in $\mathfrak{S}_X$ which fixes y and z and organizes $X \setminus \{y, z\}$ in a cycle in the following way

$$\begin{tikzcd}
\gamma_{\mathsf{ext}}(y) \arrow[r] \arrow[d] & \gamma^2_{\mathsf{ext}}(y) \arrow[r] & \cdots \arrow[r] & \gamma^{p-1}_{\mathsf{ext}}(y) \arrow[d] \\
\gamma^{q-1}_{\mathsf{int}}(z) & \cdots \arrow[l] & \gamma^2_{\mathsf{int}}(z) \arrow[l] & \gamma_{\mathsf{int}}(z) \arrow[l]
\end{tikzcd}$$

If we imagine the annulus "cut" along the curve which connects y and z then a topological disc is obtained and the points of $X \setminus \{y, z\}$ lie on the boundary of this disc in the order prescribed by $\lambda_{y,z}$. The crossing patterns *AC2* and *AC3* are defined below.

$$\begin{gathered}\exists \quad 5 \quad \text{distinct elements} \quad a, b, c \in X, y \in Y, z \in Z \quad \text{such that} \\ \lambda_{y,z} \downarrow \{a, b, c\} = (a, b, c) \quad \& \quad \tau \downarrow \{a, b, c, y, z\} = (a, c, b)(y, z)\end{gathered} \tag{AC2}$$

and,

$$\begin{gathered}\exists \quad 6 \quad \text{distinct elements} \quad a, b, c, d \in X, y \in Y, z \in Z \quad \text{such that} \\ \lambda_{y,z} \downarrow \{a, b, c, d\} = (a, b, c, d) \quad \& \quad \tau \downarrow \{a, b, c, d, y, z\} = (a, c)(b, d)(y, z).\end{gathered} \tag{AC3}$$

iv. We define the set of permutations of X which have no crossings to be

$$\boxed{\mathfrak{S}_{\mathsf{ANC}}(X, \gamma) \quad \stackrel{\text{def}}{=} \quad \{\tau \in \mathfrak{S}_X \mid \tau \quad \textit{does not satisfy} \text{ conditions } \quad AC1,\ AC2,\ AC3\}} \tag{4.3}$$

In analogy with the disc case, the *annular non-crossing permutations* are defined to be those permutations which have no crossings and are standard in the annular sense. The notation is

$$\boxed{\mathfrak{S}_{\mathsf{ann\text{-}nc}}(X, \gamma) \quad \stackrel{\text{def}}{=} \quad \mathfrak{S}_{\mathsf{compatible}}(X, \gamma) \cap \mathfrak{S}_{\mathsf{ANC}}(X, \gamma)} \subsetneq \mathfrak{S}_X \tag{4.4}$$

Since any permutation in $\mathfrak{S}_X$ is either $\gamma-$connected or $\gamma-$disconnected , so is any annular non-crossing permutation. The set $\mathfrak{S}_{\mathsf{ann\text{-}nc}}(X, \gamma)$ is thus split into the disjoint union of connected and disconnected permutations, denoted $\mathfrak{S}'_{\mathsf{ann\text{-}nc}}(X, \gamma)$ and $\mathfrak{S}''_{\mathsf{ann\text{-}nc}}(X, \gamma)$, respectively.

4.1.2 Remark. Suppose that τ is a $\gamma-$ disconnected permutation. Then

$$\tau \in \mathfrak{S}''_{\mathsf{ann\text{-}nc}}(X, \gamma) \iff \begin{cases} \tau \downarrow Y \in \mathfrak{S}^{\mathsf{A}}_{\mathsf{nc}}(Y, \gamma_o) & \text{and} \\ \tau \downarrow Z \in \mathfrak{S}^{\mathsf{A}}_{\mathsf{nc}}(Z, \gamma_o) . \end{cases} \tag{4.5}$$

For a proof see Remark 3.8 in [MN04]. The proof is carried out by observing that for permutations which are $\gamma-$ disconnected conditions *AC2* and *AC3* do not even apply, while *AC1* leads directly to (4.5).

4.1.3 Corollary. Remark 4.1.2 has the following immediate consequence.

$$\mathfrak{S}''_{\mathsf{ann\text{-}nc}}(X, \gamma) = [\varepsilon, \gamma_o]_{\mathfrak{S}_Y} \times [\varepsilon, \gamma_o]_{\mathfrak{S}_Z} = [\varepsilon, \gamma]_{\mathfrak{S}_X} .$$

The last equality in the equation above follows from Remark 3.10. It is thus clear that, unlike in the disc case, the subset of $\mathfrak{S}_X$ of annular non-crossing permutations *cannot* be identified with the interval $[\varepsilon,\gamma]_{\mathfrak{S}_X}$. That is,

$$\mathfrak{S}_{\mathsf{ann\text{-}nc}}(X,\gamma) \supsetneq \mathfrak{S}''_{\mathsf{ann\text{-}nc}}(X,\gamma) = [\varepsilon,\gamma]_{\mathfrak{S}_X} .$$

4.1.4 Annular Non-crossing Permutations and The Genus Inequality. Let us now look at the subset of permutations of $\mathfrak{S}_X = \mathfrak{S}_{p+q}$ which satisfy the genus equality *GE* (where now we take $\beta = \gamma$ in Equation (*GE*)). Thus

$$\mathfrak{S}_{\mathsf{genus}}(X,\gamma) \overset{\text{def}}{=} \{\tau \in \mathfrak{S}_X \mid \#(\tau) + \#(\tau^{-1}\gamma) + \#(\gamma) = \mid X \mid +2 \cdot \#(\tau,\gamma)\}. \tag{4.6}$$

Now, since $\mathfrak{S}_X = \mathfrak{S}'_X \cup \mathfrak{S}''_X$, the set $\mathfrak{S}_{\mathsf{genus}}(X,\gamma)$ is split accordingly into $\mathfrak{S}'_{\mathsf{genus}}(X,\gamma) \cup \mathfrak{S}''_{\mathsf{genus}}(X,\gamma)$. These two sets are described in the following remark.

4.1.5 Remark. Obviously $|X| = n = p+q$, $\#(\gamma) = 2$ and $\#(\tau,\gamma) = 2$ if τ is $\gamma-$disconnected or $\#(\tau,\gamma) = 1$ if τ is $\gamma-$connected.

i. Suppose τ is a $\gamma-$connected permutation. Since the genus inequality *GI* is satisfied by all permutations we have that

$$\begin{aligned}\tau \in \mathfrak{S}'_X \implies \quad & [\ \#(\tau) + \#(\tau^{-1}\gamma) + \#(\gamma) \leqslant \mid X \mid +2 \cdot \#(\tau,\beta) \iff \\ & \#(\tau) + \#(\tau^{-1}\gamma) + 2 \leqslant \mid X \mid +2 \cdot 1 \iff \\ & \#(\tau) + \#(\tau^{-1}\gamma) \leqslant |X| \quad].\end{aligned}$$

Therefore,

$$\mathfrak{S}'_{\mathsf{genus}}(X,\gamma) = \{\tau \in \mathfrak{S}_X \mid \#(\tau) + \#(\tau^{-1}\gamma) = |X|\}.$$

ii. Similarly, if τ is a $\gamma-$disconnected permutation we get that

$$\begin{aligned}\tau \in \mathfrak{S}''_X \implies \quad & [\ \#(\tau) + \#(\tau^{-1}\gamma) + \#(\gamma) \leqslant \mid X \mid +2 \cdot \#(\tau,\beta) \iff \\ & \#(\tau) + \#(\tau^{-1}\gamma) + 2 \leqslant \mid X \mid +2 \cdot 2 \iff \\ & \#(\tau) + \#(\tau^{-1}\gamma) \leqslant |X| + 2 \quad].\end{aligned}$$

Therefore,

$$\mathfrak{S}''_{\mathsf{genus}}(X,\gamma) = \{\tau \in \mathfrak{S}_X \mid \#(\tau) + \#(\tau^{-1}\gamma) = |X| + 2\}.$$

4.1.6 Theorem.

$$\mathfrak{S}_{genus}(X,\gamma) = \mathfrak{S}_{ann\text{-}nc}(X,\gamma) . \tag{4.7}$$

For a proof see [MN04] (Theorem 6.1). This theorem is the very nice annular analogue of the equality

$$\mathfrak{S}_{\mathsf{genus}}(n, \gamma_o) = \mathfrak{S}_{\mathsf{disc\text{-}nc}}(n, \gamma_o)\,,$$

from the disc case. The proof is carried out by showing separately that

$$\begin{aligned} \mathfrak{S}'_{\mathsf{genus}}(X, \gamma) &= \mathfrak{S}'_{\mathsf{ann\text{-}nc}}(X, \gamma) \quad \text{and} \\ \mathfrak{S}''_{\mathsf{genus}}(X, \gamma) &= \mathfrak{S}''_{\mathsf{ann\text{-}nc}}(X, \gamma)\,. \end{aligned}$$

4.1.7 Corollary.

$$\tau \in \mathfrak{S}_{\textsf{\textit{ann-nc}}}(X, \gamma) \iff \tau^{-1}\gamma \in \mathfrak{S}_{\textsf{\textit{ann-nc}}}(X, \gamma)\,. \tag{4.8}$$

Proof. Observe first that both of the sets $\mathfrak{S}'_X$ and $\mathfrak{S}''_X$ are invariant under the complementation map $\tau \longmapsto \mathsf{C}_\gamma(\tau) = \tau^{-1}\gamma$. Indeed

$$\tau \quad \text{is} \quad \gamma - \text{disconnected} \iff \tau^{-1} \quad \text{is} \quad \gamma - \text{disconnected} \iff \tau^{-1}\gamma \quad \text{is} \quad \gamma - \text{disconnected}\,.$$

Also, a direct verification shows that τ and $\tau^{-1}\gamma$ satisfy the genus equality exactly at the same time, thus

$$\tau \in \mathfrak{S}_{\mathsf{genus}}(X, \gamma) \iff \tau^{-1}\gamma \in \mathfrak{S}_{\mathsf{genus}}(X, \gamma)\,.$$

In fact, by Remark 4.1.5, we have that

$$\begin{aligned} & \tau \in \mathfrak{S}'_{\mathsf{genus}}(X, \gamma) \iff \tau^{-1}\gamma \in \mathfrak{S}'_{\mathsf{genus}}(X, \gamma) \\ \text{and implicitly that} \quad & \tau \in \mathfrak{S}''_{\mathsf{genus}}(X, \gamma) \iff \tau^{-1}\gamma \in \mathfrak{S}''_{\mathsf{genus}}(X, \gamma)\,. \end{aligned} \tag{4.9}$$

The result follows by the above Theorem 4.1.6. ■

4.1.8 Annular Non-crossing Partitions. By analogy with the disc case, the set of *non-crossing annular permutations* is defined to be the image in $\Pi(X)$ of the set of non-crossing annular partitions through Ω.

$$\boxed{\Omega(\mathfrak{S}_{\mathsf{ann\text{-}nc}}(X, \gamma)) \quad \stackrel{\text{def}}{=} \quad \mathsf{NC}^{\mathsf{A}}(\mathsf{p},\, \mathsf{q}) \;\subsetneq \Pi(X)} \tag{4.10}$$

Again, just like in the disc case, the superscript "A" is due to the existence of an analogue of type B which will be discussed in the next section.

4.1.9 Proposition. *Suppose π is a γ − disconnected partition. Then*

i. $\exists \pi_{ext} \in \Pi(Y)$ *and* $\pi_{int} \in \Pi(Z)$ *such that* $\pi = \{\pi_{ext}\,, \pi_{int}\}$.

$$\textit{ii.}\ \ \pi \in NC^A(p,\, q) \iff \begin{cases} \pi_{int} \in NC^A(Z)\ (= NC^A(p)) \\ \pi_{ext} \in NC^A(Y)\ (\simeq NC^A(q)). \end{cases}$$

$$\textit{iii.}\ \ \pi \in NC^A(p,\, q) \implies \exists!\, \tau \in \mathfrak{S}''_{ann\text{-}nc}(X,\gamma) \quad \textit{such that} \quad \Omega(\tau) = \pi.$$

Proof. *i.* and *ii.* follow directly from Remark 4.1.2. For *iii.*, denote $\mathfrak{perm}_{\gamma_o}(\pi_{\text{ext}}) = \tau_{\text{ext}}$ and $\mathfrak{perm}_{\gamma_o}(\pi_{\text{int}}) = \tau_{\text{int}}$. Then, if we pick $\tau = \tau_{\text{int}} \cdot \tau_{\text{ext}}$ it follows that $\Omega(\tau) = \pi$. This τ is denoted by $\mathfrak{perm}_{\gamma}(\pi)$. ∎

4.1.10 Proposition. *Suppose π is a $\gamma -$ connected partition and suppose also that π has at least two $\gamma -$ connected blocks. Then*

$$\pi \in NC^A(p,\, q) \implies \exists!\ \tau \in \mathfrak{S}'_{ann\text{-}nc}(X,\gamma) \quad \textit{such that} \quad \Omega(\tau) = \pi.$$

4.1.11 Remark. We have seen in the disc case that the "orbit map" Ω and the "cycle map" $\mathfrak{perm}_{\gamma_o}$ were poset isomorphisms inverse to each other between $\mathsf{NC}^{\mathsf{A}}(\mathsf{n})$ and $[\varepsilon,\gamma_o]_{\mathfrak{S}_n}$, and also between $\mathsf{NC}^{\mathsf{B}}(\mathsf{n})$ and $[\varepsilon,\gamma_o]_{B_n}$.

However, in the annular framework, the orbit map Ω is *not* injective anymore. Indeed, suppose that C is the only $\gamma -$ connected block of the fixed partition $\pi \in \Pi(\mathsf{p{+}q})$. If we denote $C_Y = C \cap Y$ and $C_Z = C \cap Z$, then there are exactly $\mid C_Y \mid \cdot \mid C_Z \mid$ permutations in $\mathfrak{S}_{\mathsf{ann\text{-}nc}}(X,\gamma)$ which are sent to π by the orbit map Ω. That is

$$\text{card}\,\{\tau \in \mathfrak{S}_{\mathsf{ann\text{-}nc}}(X,\gamma)\ \mid \Omega(\tau) = \pi\} = \mid C_Y \mid \cdot \mid C_Z \mid \tag{4.11}$$

For further details see Proposition 4.6 in [MN04].

4.1.12 Annular Type A Summarizing Diagram. The facts presented in this section are summarized in the following "type A annular diagram"

$$\boxed{\begin{array}{c} \mathfrak{S}_{\mathsf{genus}}(X,\gamma) = \mathfrak{S}_{\mathsf{ann\text{-}nc}}(X,\gamma) \supsetneq [\varepsilon,\gamma]_{\mathfrak{S}_X} \\ \Big\downarrow \text{bad correspondence} \\ \mathsf{NC}^{\mathsf{A}}(\mathsf{p},\, \mathsf{q}) \end{array}}$$

4.1.13 Definition. Any element in either of the sets $\mathfrak{S}_{\mathsf{ann\text{-}nc}}(X,\gamma)$ or $\mathfrak{S}_{\mathsf{genus}}(X,\gamma)$ is called a *type A annular non-crossing permutation.* Thus, the set of all type A annular non-crossing permutations is

$$\boxed{\mathfrak{S}^{\mathsf{A}}_{\mathsf{nc}}(X,\gamma) \quad \stackrel{\text{def}}{=} \quad \mathfrak{S}_{\mathsf{ann\text{-}nc}}(X,\gamma) = \mathfrak{S}_{\mathsf{genus}}(X,\gamma)} \tag{4.12}$$

4.2 Posets of Type B Annular Non-Crossing Partitions

In this section we start presenting some of the new things brought in by this thesis.

We have seen in the previous section that the properties of annular non-crossing permutations are not as nice as in the disc case. First, we have seen that $\mathfrak{S}_{\textsf{ann-nc}}(X,\gamma)$ is not equal to $[\varepsilon,\gamma]_{\mathfrak{S}_X}$. Secondly, $\textsf{NC}^{\textsf{A}}(\textsf{p, q})$, the set of non-crossing annular partitions, (defined as the image of $\mathfrak{S}_{\textsf{ann-nc}}(X,\gamma)$ through the orbit map Ω) is not poset isomorphic to $\mathfrak{S}_{\textsf{ann-nc}}(X,\gamma)$. And finally, the orbit map Ω fails to be injective in the annular case.

By simply modifying the ground set from $X=[n]=[p+q]$ to $X=[\pm n]$, it is clear that everything that was done in Section 4.1 still holds. Thus $X=Y\cup Z$, where

$$\begin{cases} Y & \stackrel{\text{def}}{=} & \{1,\dots,p\}\cup\{-1,\dots,-p\} \\ \\ Z & \stackrel{\text{def}}{=} & \{p+1,\dots,n\}\cup\{-(p+1),\dots,-n\}. \end{cases} \tag{4.13}$$

The order on X is $\{1\leqslant 2\leqslant\cdots\leqslant n\leqslant -1\leqslant -2\leqslant\cdots\leqslant -n\}$. We are now looking at annular non-crossing permutations inside $\mathfrak{S}_{\pm n}$.

However, something new makes sense now: we can restrict our attention only to the *signed* annular permutations and hope that maybe some of the things which failed in the type A annular case will get fixed in the new type B annular framework. This is indeed the case but it requires a certain amount of work before becoming apparent.

Let us briefly recapitulate: we have two circles and an annulus between them, we also have a total of $2n$ points. $2p$ of them (the ones in the set Y) are to be found on the outer circle in increasing order and clockwise sense. The remaining $2q$ points (the ones from Z) are spread in increasing order and counterclockwise sense on the inner circle.

The outer and inner forward cycles on the sets Y and Z, denoted $\gamma_{\textsf{ext}}$ and $\gamma_{\textsf{int}}$, respectively, and their commutative product $\gamma\ \stackrel{\text{def}}{=}\ \gamma_{\textsf{ext}}\cdot\gamma_{\textsf{int}}$ are listed below.

$$\begin{cases} \gamma_{\textsf{ext}} = \big(1,2,\dots,p,-1,-2,\dots,-p) = [1,2,\dots,p] \quad \& \quad \gamma_{\textsf{ext}}\downarrow Y=\gamma\downarrow Y=\gamma_o. \\ \\ \gamma_{\textsf{int}} = (p+1,\dots,n,-(p+1),\dots,-n\big)= [p+1,p+2,\dots,n],\ \gamma_{\textsf{int}}\downarrow Z=\gamma\downarrow Z=\gamma_o. \\ \\ \gamma \quad \stackrel{\text{def}}{=} \quad \gamma_{\textsf{ext}}\cdot\gamma_{\textsf{int}} = [1,2,\dots,p][p+1,p+2,\dots,n]. \end{cases}$$

The permutation $\gamma_{\mathsf{ext}} \in \mathfrak{S}_{\pm n}$ permutes forward the elements of Y and keeps the elements of Z fixed. Similarly $\gamma_{\mathsf{int}} \in \mathfrak{S}_{\pm n}$ permutes forward the elements of Z and keeps the elements of Y fixed. They are called "forward cycles" with this convention in mind.

Let us now take another look at the "type A annular diagram" with the symbols updated and ready for a type B approach

$$\begin{array}{c} \mathfrak{S}_{\mathsf{genus}}(X,\gamma) \; = \mathfrak{S}_{\mathsf{ann\text{-}nc}}(X,\gamma) \; \supsetneq [\varepsilon,\gamma]_{\mathfrak{S}_{\pm n}} \\ \text{bad properties of} \Big\updownarrow \Omega \text{ and } \mathfrak{perm}_{\gamma} \\ \mathsf{NC}^{\mathsf{A}}(\pm\mathsf{p},\pm\mathsf{q}) \end{array}$$

The upper row in the diagram consists of subsets of $\mathfrak{S}_{\pm n}$. It is clear that

$$\mathfrak{S}_{\mathsf{genus}}(X,\gamma) \cap B_n = \mathfrak{S}_{\mathsf{ann\text{-}nc}}(X,\gamma) \cap B_n. \tag{4.14}$$

where B_n denotes the hyperoctahedral group, as in Section 3.2.

4.2.1 Definition. Any element in either of the sets $\mathfrak{S}_{\mathsf{ann\text{-}nc}}(X,\gamma) \cap B_n$ or $\mathfrak{S}_{\mathsf{genus}}(X,\gamma) \cap B_n$ is called a *type B annular non-crossing permutation.* Thus, the set of all type B annular non-crossing permutations is defined to be

$$\boxed{\mathfrak{S}^{\mathsf{B}}_{\mathsf{nc}}(X,\gamma) \quad \overset{\text{def}}{=} \quad \mathfrak{S}_{\mathsf{ann\text{-}nc}}(X,\gamma) \cap B_n = \mathfrak{S}_{\mathsf{genus}}(X,\gamma) \cap B_n} \tag{4.15}$$

4.2.2 Remark. It is obvious that $\gamma = [1, 2, \ldots, p][p+1, p+2, \ldots, n] \in B_n$. Note that γ has only two cycles, both of them inversion-invariant. Thus $\#_{\mathsf{paired}}(\gamma) = 0$ and so $\ell_{B_n}(\gamma) = n$; the above decomposition is a shortest factorization.

The first fact which failed in the type A annular setting but is recovered in the type B annular framework is shown in the following theorem.

4.2.3 Theorem. *The set of type B annular non-crossing permutations forms an interval in the hyperoctahedral group B_n. That is*

$$\mathfrak{S}^{B}_{nc}(X,\gamma) \; = [\varepsilon,\gamma]_{B_n}. \tag{4.16}$$

The statement of the theorem will follow by analyzing separately the cases of $\gamma-$disconnected permutations (in Proposition 4.2.4 below) and $\gamma-$connected permutations (in Proposition 4.2.7 below).

4.2.4 Proposition. *Suppose that τ is a $\gamma-$disconnected permutation in $\mathfrak{S}_X$. The following are equivalent.*

(1) $\tau \in \mathfrak{S}^{B}_{nc}(X,\gamma)$ *(i.e. τ is a type B annular non-crossing permutation).*

(2) $\tau \downarrow Y \in \mathfrak{S}^{B}_{nc}(Y,\gamma_o)$ *and* $\tau \downarrow Z \in \mathfrak{S}^{B}_{nc}(Z,\gamma_o)$.

(3) $\tau \in [\varepsilon,\gamma]_{B_n}$.

Proof. $\boxed{(1) \Longleftrightarrow (2)}$. This is immediate from Remark 4.1.2, which reads

$$\mathfrak{S}_{\textsf{ann-nc}}(X,\gamma) \ni \tau \text{ is } \gamma-\text{disconnected} \Longleftrightarrow [\tau \downarrow Y \in \mathfrak{S}^{\textsf{A}}_{\textsf{nc}}(Y,\gamma_o) \quad \text{and} \quad \tau \downarrow Z \in \mathfrak{S}^{\textsf{A}}_{\textsf{nc}}(Z,\gamma_o)\,],$$

combined with the simple observation that

$$\tau \in B_n \Longleftrightarrow [\tau \downarrow Y \in B_Y \qquad \text{and} \qquad \tau \downarrow Z \in B_Z], \tag{4.17}$$

where B_Y and B_Z denote the type B Weyl groups on Y and Z, respectively.

$\boxed{(2) \Longrightarrow (3)}$. From the hypotheses in (2) it is clear that $\tau \in B_n$. On the other hand, by using the type B length formula, it is immediately verified that

$$\ell_{B_n}(\tau) = \ell_{B_Y}(\tau \downarrow Y) + \ell_{B_Z}(\tau \downarrow Z), \tag{4.18}$$

where $\ell_{B_Y}(\cdot)$ and $\ell_{B_Z}(\cdot)$ are the lengths of type B defined (in the obvious way) on the groups B_Y and B_Z, respectively. We also have an equation analogous to (4.18), which refers to $\tau^{-1}\gamma$ instead of τ (this is because $\tau^{-1}\gamma$ also is a γ-disconnected permutation in B_n, same as τ); in other words, we have that

$$\begin{aligned} \ell_{B_n}(\tau^{-1}\gamma) &= \ell_{B_Y}\left((\tau^{-1}\gamma) \downarrow Y \right) + \ell_{B_Z}[(\tau^{-1}\gamma) \downarrow Z] \\ &= \ell_{B_Y}[(\tau \downarrow Y)^{-1}\gamma_o] + \ell_{B_Z}[(\tau \downarrow Z)^{-1}\gamma_o]. \end{aligned} \tag{4.19}$$

By adding together Equations (4.18) and (4.19) we obtain that

$$\begin{array}{ccccc} \ell_{B_n}(\tau) & + & \ell_{B_n}(\tau^{-1}\gamma) & & = \\ \underbrace{\ell_{B_Y}(\tau \downarrow Y) + \ell_{B_Y}[(\tau \downarrow Y)^{-1}\gamma_o]} & + & \underbrace{\ell_{B_Z}(\tau \downarrow Z) + \ell_{B_Z}[(\tau \downarrow Z)^{-1}\gamma_o]} & & = \\ \ell_{B_Y}(\gamma_o) & + & \ell_{B_Z}(\gamma_o) & & = \end{array}$$

$$= p \quad + \quad q.$$

At the second equality sign in the above calculation we used the fact that $(\tau \downarrow Y) \leqslant_B \gamma_o$ in B_Y and $(\tau \downarrow Z) \leqslant_B \gamma_o$ in B_Z; this fact comes from the hypotheses that $\tau \downarrow Y \in \mathfrak{S}^{\mathsf{B}}_{\mathsf{nc}}(Y, \gamma_o) = [\varepsilon, \gamma_o]_{B_Y}$ and $\tau \downarrow Z \in \mathfrak{S}^{\mathsf{B}}_{\mathsf{nc}}(Z, \gamma_o) = [\varepsilon, \gamma_o]_{B_Z}$.

We have thus obtained that $\ell_{B_n}(\tau) + \ell_{B_n}(\tau^{-1}\gamma) = n$, which is exactly the desired conclusion that $\tau \leqslant_{B_n} \gamma$ in B_n.

$\boxed{(2) \Longleftarrow (3)}$. From the hypotheses of *(3)* it is clear that $\tau \downarrow Y \in B_Y$ and $\tau \downarrow Z \in B_Z$. Note also that the formulas found in (4.18) and (4.19) above hold here as well (indeed, these two formulas only depended on the fact that τ is a $\gamma -$ disconnected permutation in B_n).

As in the proof of *(2)* $\Longrightarrow$ *(3)*, we will rely on the known fact that

$$\mathfrak{S}^{\mathsf{B}}_{\mathsf{nc}}(Y, \gamma_o) = [\varepsilon, \gamma_o]_{B_Y} \quad \text{and} \quad \mathfrak{S}^{\mathsf{B}}_{\mathsf{nc}}(Z, \gamma_o) = [\varepsilon, \gamma_o]_{B_Z}.$$

Thus our goal in this part of the proof is to check that

$$(\tau \downarrow Y) \leqslant_B \gamma_o \in B_Y \quad \text{and} \quad (\tau \downarrow Z) \leqslant_B \gamma_o \in B_Z,$$

or in other words, we have to check that the inequalities

$$\begin{cases} \ell_{B_Y}(\tau \downarrow Y) + \ell_{B_Y}[(\tau \downarrow Y)^{-1}\gamma_o)] & \geqslant \quad \ell_{B_Y}(\gamma_o) \qquad \text{and} \\ \ell_{B_Z}(\tau \downarrow Z) + \ell_{B_Z}[(\tau \downarrow Z)^{-1}\gamma_o)] & \geqslant \quad \ell_{B_Z}(\gamma_o) \end{cases}$$

both hold with equality. Clearly, it suffices to verify that the sum of the two inequalities holds with equality. And indeed, we have:

$$\begin{aligned} &\Big[\ell_{B_Y}(\tau \downarrow Y) + \ell_{B_Y}[(\tau \downarrow Y)^{-1}\gamma_o]\Big] \quad + \quad \Big[\ell_{B_Z}(\tau \downarrow Z) + \ell_{B_Z}[(\tau \downarrow Z)^{-1}\gamma_o]\Big] = \\ =&\Big[\ell_{B_Y}(\tau \downarrow Y) + \ell_{B_Z}(\tau \downarrow Z)\Big] \quad + \quad \Big[\ell_{B_Y}[(\tau \downarrow Y)^{-1}\gamma_o] + \ell_{B_Z}[(\tau \downarrow Z)^{-1}\gamma_o]\Big] \\ =&\ell_B(\tau) + \ell_B(\tau^{-1}\gamma) && \text{[by Equations (4.18) and (4.19)]} \\ =&\ell_B(\gamma) && [\text{due to the hypothesis that } \tau \leqslant_{B_n} \gamma] \\ =&\ell_{B_Y}(\gamma_o) + \ell_{B_Z}(\gamma_o). && [\text{since } n = p + q] \end{aligned}$$

$\blacksquare$

We now move towards proving the $\gamma-$connected case of the equality appearing in Theorem 4.2.3 by first observing that $\gamma-$connected permutations in $\mathfrak{S}^{\mathsf{B}}_{\mathsf{nc}}(X,\gamma)$ cannot have inversion-invariant orbits. This is a useful fact since the partial order $\leqslant_{B_n}$ on $[\varepsilon,\gamma]_{B_n}$ is defined in terms of the length formula ℓ_{B_n} , which in turn involves only the non-invariant cycles. It thus seems like a good idea to count the number of pairs $\#_{\mathsf{paired}}(\tau)$ of a $\gamma-$connected permutation $\tau \in \mathfrak{S}^{\mathsf{B}}_{\mathsf{nc}}(X,\gamma_o)$.

4.2.5 Lemma. *Let τ be any permutation in $\mathfrak{S}^{B}_{nc}(X,\gamma_o)$. Then τ cannot have a $\gamma-$connected orbit which is inversion invariant.*

Proof. By contradiction: assume that τ has an orbit A which is inversion-invariant and $\gamma-$connected. Then there exist i,j such that

$$A\cap Y \supseteq \{i\} \quad \& \quad \{j\}\subseteq A\cap Z \quad \text{such that} \quad \tau(i)=j, \qquad \text{[since } A \text{ is } \gamma-\text{connected]}$$
$$A\cap Y \supseteq \{-i\} \quad \& \quad \{-j\}\subseteq A\cap Z \quad \text{such that} \quad \tau(-i)=-j. \quad \text{[since } A \text{ is inversion-invariant]}$$

Condition ***AS 2*** (from 4.1.1) is not fulfilled since both of $\tau(i)$ and $\tau(-i)$ belong to Z. Thus τ cannot be standard in the annular sense, and implicitly $\tau \notin \mathfrak{S}^{\mathsf{B}}_{\mathsf{nc}}(X,\gamma_o)$. ■

4.2.6 Proposition. *Let τ be a $\gamma-$connected permutation in $\mathfrak{S}^{B}_{nc}(X,\gamma_o)$. Then τ has no inversion-invariant orbits.*

Proof. By assumption τ has at least one $\gamma-$connected orbit, C, which cannot be inversion-invariant (as the preceding lemma shows) hence $-C\cap C=\emptyset$. Let us fix the (unique, by condition ***AS 2***) elements i,j with the properties

$$C\cap Y \supseteq \{i\} \quad \& \quad \{j\}\subseteq C\cap Z \quad \text{such that} \quad \tau(i)=j, \qquad \text{[since } C \text{ is } \gamma-\text{connected]}$$
$$-C\cap Y \supseteq \{-i\} \quad \& \quad \{-j\}\subseteq -C\cap Z \quad \text{such that} \quad \tau(-i)=-j. \qquad \text{[since } \tau \text{ is signed]}$$

We are left with the possibility that τ has a $\gamma-$disconnected orbit A which is inversion-invariant. Suppose, by contradiction, that this is indeed the case and let $\{k,-k\}$ be any pair of elements of A. Then the 6 distinct elements $-i,-j,k,-k\in X, i\in Y, j\in Z$ satisfy the annular crossing pattern

$$\textit{AC3}: \qquad \begin{cases} \lambda_{i,j}\downarrow\{-i,-j,k,-k\} & =(-i,-j,k,-k) \quad \& \\ \tau\downarrow\{-i,-j,k,-k,i,j\} & =(-i,k)(-j,-k)(i,j). \end{cases}$$

■

4.2.7 Proposition. *Suppose that τ is a $\gamma-$connected permutation in $\mathfrak{S}_X$. Then*

$$\tau \in \mathfrak{S}^{B}_{nc}(X,\gamma) \quad \Longleftrightarrow \quad \tau \leqslant_{B_n} \gamma.$$

Proof. $\boxed{\Longrightarrow}$ Corollary 4.1.7 and Lemma 4.2.2 imply that the following equivalence holds

$$\mathfrak{S}^{\mathsf{B}}_{\mathsf{nc}}(X,\gamma_o) \ni \tau \quad \text{is} \quad \gamma-\text{connected} \quad \Longleftrightarrow \quad \mathfrak{S}^{\mathsf{B}}_{\mathsf{nc}}(X,\gamma_o) \ni \tau^{-1}\gamma \quad \text{is} \quad \gamma-\text{connected}\,, \tag{4.20}$$

and therefore, by Proposition 4.2.6, both τ and $\tau^{-1}\gamma$ have no inversion-invariant orbits. On the other hand, the description of $\mathfrak{S}^{'}_{\mathsf{genus}}(X,\gamma)$ from Remark 4.1.5 implies that

$$\#(\tau) + \#(\tau^{-1}\gamma) = 2n.$$

Now let us use the length formula in B_n,

$$\begin{aligned}
\ell_{B_n}(\tau) = n - \#_{\mathsf{paired}}(\tau) \quad & \& \quad \ell_{B_n}(\tau^{-1}\gamma) = n - \#_{\mathsf{paired}}(\tau^{-1}\gamma) && \Longleftrightarrow \\
\ell_{B_n}(\tau) = n - \frac{1}{2}\#(\tau) \quad & \& \quad \ell_{B_n}(\tau^{-1}\gamma) = n - \frac{1}{2}\#(\tau^{-1}\gamma) && \Longrightarrow \\
\ell_{B_n}(\tau) + \ell_{B_n}(\tau^{-1}\gamma) \quad & = \quad 2n - \frac{1}{2}\big[\#(\tau) + \#(\tau^{-1}\gamma)\big] = 2n - n = n = \ell_{B_n}(\gamma) && \Longleftrightarrow \\
& \tau \leqslant_{B_n} \gamma. && \text{[by definition (3.3) of partial order]}
\end{aligned}$$

$\boxed{\Longleftarrow}$ We use again the description of $\mathfrak{S}^{'}_{\mathsf{genus}}(X,\gamma)$ from Remark 4.1.5, so we need to prove that

$$\#(\tau) + \#(\tau^{-1}\gamma) = 2n.$$

As observed there, it is true in general for any $\gamma-$connected permutation $\tau \in \mathfrak{S}^{'}_X$ that

$$\#(\tau) + \#(\tau^{-1}\gamma) \leqslant 2n,$$

hence in order to prove the equality we just need to prove that

$$\#(\tau) + \#(\tau^{-1}\gamma) \geqslant 2n. \tag{4.21}$$

Let us denote the number of invariant orbits of τ and $\tau^{-1}\gamma$ by k and l, respectively. Now

$$\begin{aligned}
\ell_{B_n}(\tau) = n - \#_{\mathsf{paired}}(\tau) \quad & \& \quad \ell_{B_n}(\tau^{-1}\gamma) = n - \#_{\mathsf{paired}}(\tau^{-1}\gamma) && \Longleftrightarrow \\
\ell_{B_n}(\tau) = n - \frac{1}{2}\big[\#(\tau) - k\big] \quad & \& \quad \ell_{B_n}(\tau^{-1}\gamma) = n - \frac{1}{2}\big[\#(\tau^{-1}\gamma) - l\big] && \Longrightarrow
\end{aligned}$$

$$\begin{aligned} \ell_{B_n}(\gamma) = n = \ell_{B_n}(\tau) + \ell_{B_n}(\tau^{-1}\gamma) &= 2n - \frac{1}{2}\big[\#(\tau) + \#(\tau^{-1}\gamma) - k - l\big] \qquad \Longrightarrow \\ \#(\tau) + \#(\tau^{-1}\gamma) - k - l &= 2n + k + l > 2n \end{aligned}$$

and thus inequality (4.21) is proved. ■

At this point we have thus "fixed" the strict inequality $\mathfrak{S}^{\mathsf{A}}_{\mathsf{nc}}(X,\gamma) \supsetneq [\varepsilon,\gamma]_{\mathfrak{S}_{\pm n}}$ which appeared in the "type A annular" diagram, in the sense that, when moving to type B annular we now have that

$$\mathfrak{S}^{\mathsf{B}}_{\mathsf{nc}}(X,\gamma) = [\varepsilon,\gamma]_{B_n}.$$

In view of the same type A diagram, let us now move on towards fixing the "bad" correspondence $\mathfrak{S}^{\mathsf{A}}_{\mathsf{nc}}(X,\gamma) \longleftrightarrow \mathsf{NC}^{\mathsf{A}}(\pm\mathsf{p},\pm\mathsf{q})$, where $\mathsf{NC}^{\mathsf{A}}(\pm\mathsf{p},\pm\mathsf{q})$ was simply defined as the image of $\mathfrak{S}^{\mathsf{A}}_{\mathsf{nc}}(X,\gamma)$ through the orbit map Ω. The first problem in type A annular was that Ω was not injective. It looks encouraging that in type B this is not possible since there are no type B annular permutations with a single $\gamma-$connected block, thus the annoying non-injectivity phenomenon cannot occur in this framework.
However, it is not a good idea to define "type B annular non-crossing permutations" as the image of $\mathfrak{S}^{\mathsf{B}}_{\mathsf{nc}}(X,\gamma)$ through Ω because, even if the orbit map is now injective, *it is not order preserving* (the partial orders being $\leqslant_{B_n}$ and the reversed refinement $\leqslant$, respectively). Indeed, if $\tau \in \mathfrak{S}^{\mathsf{B}}_{\mathsf{nc}}(X,\gamma)$ is $\gamma-$connected then

$$\tau \leqslant_{B_n} \gamma \qquad \text{but} \qquad \Omega(\tau) \nleqslant \Omega(\gamma) = \{Y,Z\}. \tag{4.22}$$

This shortcoming is remedied without any damage by replacing Ω with an adjusted orbit map denoted $\widetilde{\Omega}$ and which is introduced in (4.2.13) below. Before that, some work must be done.

4.2.8 Orbits of permutations from $\mathfrak{S}^{\mathsf{B}}_{\mathsf{nc}}(X,\gamma)$. We will denote

$$\mathcal{O}^{\mathsf{B}}_{\mathsf{nc}}(p,q) \quad \overset{\text{def}}{=} \quad \left\{ A \subseteq X \;\middle|\; \begin{array}{l} \exists\, \tau \in \mathfrak{S}^{\mathsf{B}}_{\mathsf{nc}}(X,\gamma) \text{ such that} \\ A \text{ is an orbit of } \tau \end{array} \right\}. \tag{4.23}$$

4.2.9 Remark. Let A be a set in $\mathcal{O}^{\mathsf{B}}_{\mathsf{nc}}(p,q)$. A permutation in $\mathfrak{S}^{\mathsf{B}}_{\mathsf{nc}}(X,\gamma)$ which has A as an orbit must also have $-A$ as an orbit, and this implies that either $A=-A$, or $A\cap(-A)=\emptyset$. In the case when $A=-A$, we must have that $A\subseteq Y$ or $A\subseteq Z$, because a permutation in $\mathfrak{S}^{\mathsf{B}}_{\mathsf{nc}}(X,\gamma)$ which has an inversion-invariant orbit must be $\gamma-$disconnected (see Proposition 4.2.6).

4.2.10 Lemma. ***i.*** *Let $A \in \mathcal{O}^{B}_{nc}(p,q)$ be such that A is $\gamma-$ disconnected (that is, we have $A \subseteq Y$ or $A \subseteq Z$). Let $\tau \in \mathfrak{S}^{B}_{nc}(X,\gamma)$ be such that A is an orbit of τ. Then*

$$\tau \downarrow A \;=\; \gamma \downarrow A. \tag{4.24}$$

ii. *Let $A \in \mathcal{O}^{B}_{nc}(p,q)$ be such that A is $\gamma-$ connected (that is, $A \cap Y \neq \emptyset \neq A \cap Z$). Let $\tau \in \mathfrak{S}^{B}_{nc}(X,\gamma)$ be such that A is an orbit of τ. On the other hand consider two elements $y \in A \cap Y$ and $z \in A \cap Z$, and look at the* AC*-test permutation $\lambda_{y,z} \in \mathfrak{S}_X$ (defined as in Equation (4.2)). Then*

$$\tau \downarrow A \;=\; \lambda_{-y,-z} \downarrow A. \tag{4.25}$$

Proof. ***i.*** If $A \subseteq Y$, then $\tau \downarrow A = (\tau \downarrow Y) \downarrow A = (\gamma \downarrow Y) \downarrow A = \gamma \downarrow A$ (we used the equality $\tau \downarrow Y = \gamma \downarrow Y$, which is part of the requirements of compatibility between τ and γ). The case when $A \subseteq Z$ is analogous.

ii. As observed in Remark 4.2.9, we have $A \cap (-A) = \emptyset$. So $-y, -z \notin A$, which in turn implies that $\lambda_{-y,-z} \downarrow A$ is a cyclic permutation of A.
If $|A| \leq 2$, then the equality (4.25) follows just from the fact that both $\lambda_{-y,-z} \downarrow A$ and $\tau \downarrow A$ are cyclic permutations of A.
Suppose then that $|A| \geqslant 3$. If the equality (4.25) would not hold, then there would exist three distinct elements $a, b, c \in A$ such that

$$\lambda_{-y,-z} \downarrow \{a,b,c\} = (a,b,c), \quad \tau \downarrow \{a,b,c\} = (a,c,b).$$

But then the five elements $a, b, c, -y, -z$ would produce an occurrence of the crossing pattern *AC2* in τ – contradiction.

■

4.2.11 Definition. Let A be a set in $\mathcal{O}^{\mathsf{B}}_{\mathsf{nc}}(p,q)$. From the preceding lemma it is immediate that if τ_1, τ_2 are permutations in $\mathfrak{S}^{\mathsf{B}}_{\mathsf{nc}}(X,\gamma)$ which have A as an orbit, then we must have $\tau_1 \downarrow A = \tau_2 \downarrow A$. It thus makes sense to define a permutation $\mu_A \in \mathfrak{S}_A$ by stipulating that

$$\mu_A = \tau \downarrow A, \tag{4.26}$$

where τ is an arbitrary permutation in $\mathfrak{S}^{\mathsf{B}}_{\mathsf{nc}}(X,\gamma)$ having A as an orbit. We will refer to μ_A as the *canonical permutation* of A.

4.2.12 Remark. Let A be a set in $\mathcal{O}^{\mathsf{B}}_{\mathsf{nc}}(p,q)$, and consider the canonical permutation $\mu_A \in \mathfrak{S}_A$ defined above.

i. Equations (4.24) and (4.25) from Lemma 4.2.10 give us "explicit" formulas for μ_A: if A is γ − disconnected then

$$\mu_A = \gamma \downarrow A, \tag{4.27}$$

while if A is γ − connected (which implies that $A \cap (-A) = \emptyset$) then

$$\mu_A = \lambda_{-y,-z} \downarrow A, \tag{4.28}$$

for an arbitrary choice of $y \in A \cap Y$ and $z \in A \cap Z$.

ii. Note that in the case when A is γ − connected we still have that

$$\mu_A \downarrow (A \cap Y) = \gamma \downarrow (A \cap Y), \quad \mu_A \downarrow (A \cap Z) = \gamma \downarrow (A \cap Z). \tag{4.29}$$

The first of these two equalities follows from the immediate observation that

$$\lambda_{-y,-z} \downarrow (Y \setminus \{-y\}) = \gamma \downarrow (Y \setminus \{-y\}),$$

combined with the fact that $A \cap Y \subseteq Y \setminus \{-y\}$. We use a similar argument for the second equality, this time in reference to $A \cap Z$.

iii. Let us record here a fact that will be used later: suppose that A is γ-connected, and that we are given four distinct elements $a, b, c, d \in A$, such that $\mu_A \downarrow \{a, b, c, d\} = (a, b, c, d)$. Then it is not possible to have $a, c \in Y$ and $b, d \in Z$. Indeed, let us pick some elements $y \in A \cap Y$ and $z \in A \cap Z$. From part *i.* of this remark it follows that

$$\lambda_{-y,-z} \downarrow \{a, b, c, d\} = \mu_A \downarrow \{a, b, c, d\} = (a, b, c, d);$$

and it is clear, directly from the definition of $\lambda_{-y,-z}$, that $\lambda_{-y,-z} \downarrow \{a, b, c, d\}$ could not be (a, b, c, d) if we were to have $a, c \in Y$ and $b, d \in Z$.

4.2.13 The Partitions $\Omega(\tau)$ and $\widetilde{\Omega}(\tau)$. We move on from individual orbits to *orbit partitions* for permutations in $\mathfrak{S}^{\mathsf{B}}_{\mathsf{nc}}(X,\gamma)$, that is, we look at the orbit map

$$\mathfrak{S}^{\mathsf{B}}_{\mathsf{nc}}(X,\gamma) \ni \tau \longmapsto \Omega(\tau) \in \Pi(X).$$

It is worth noting that this map is *one-to-one*. Indeed, if we know that a permutation $\tau \in \mathfrak{S}^{\mathsf{B}}_{\mathsf{nc}}(X,\gamma)$ has orbit partition $\pi \in \Pi(X)$, then we know how to determine τ – we just have to

"multiply" (in the sense of viewing $\mathfrak{S}_A \subseteq \mathfrak{S}_X$ with the elements from $X \setminus A$ fixed) together the canonical permutations $\mu_A \in \mathfrak{S}_A$, where A runs in the set of blocks of π – by analogy with the disc case, τ obtained in this fashion from π will be denoted $\mathfrak{perm}_\gamma(\pi)$.
As noted in Equation (4.22), the orbit map Ω is not order-preserving. We introduce an adjusted orbit map denoted $\widetilde{\Omega}$ as follows.

4.2.14 Definition. Let τ be a permutation in B_n. We define $\widetilde{\Omega}(\tau)$ to be the partition of X which is obtained from $\Omega(\tau)$ by grouping together all the inversion-invariant blocks of $\Omega(\tau)$ (if such blocks exist) into only one block of $\widetilde{\Omega}(\tau)$. That is: if

$$\Omega(\tau) = \{A_1, \dots, A_k, B_1, -B_1, \dots, B_l, -B_l\},$$

with $A_i = -A_i$ for $1 \leqslant i \leqslant k$, then

$$\widetilde{\Omega}(\tau) \quad \stackrel{\text{def}}{=} \quad \{A_1 \cup \cdots \cup A_k, B_1, -B_1, \dots, B_l, -B_l\}. \tag{4.30}$$

Before starting to prove that $\widetilde{\Omega}$ and $\mathfrak{perm}_\gamma$ are indeed the right B-annular counterparts of Ω and $\mathfrak{perm}_{\gamma_o}$ from the disc case, let us first make the following definition.

4.2.15 Definition. The image through $\widetilde{\Omega}$ of $\mathfrak{S}^{\mathsf{B}}_{\mathsf{nc}}(X,\gamma)$ is defined to be the set of type B annular non-crossing partitions, and it is denoted as follows

$$\boxed{\widetilde{\Omega}(\mathfrak{S}^{\mathsf{B}}_{\mathsf{nc}}(X,\gamma)) \quad \stackrel{\text{def}}{=} \quad \mathsf{NC}^{\mathsf{B}}(\mathsf{p},\mathsf{q}) \subsetneq \Pi(X)} \tag{4.31}$$

If $\tau \in \mathfrak{S}^{\mathsf{B}}_{\mathsf{nc}}(X,\gamma)$ then moving back and forth from $\Omega(\tau)$ to $\widetilde{\Omega}(\tau)$ is a fairly minor adjustment, as explained in the next lemma.

4.2.16 Lemma. *Let τ be a permutation in $\mathfrak{S}^{B}_{nc}(X,\gamma)$, and consider the following condition on the partition $\widetilde{\Omega}(\tau)$:*

$$\text{"There exists a block } A \text{ of } \widetilde{\Omega}(\tau) \text{ which is inversion-invariant and } \gamma-\text{connected."} \tag{4.32}$$

If this condition is fulfilled, then the block A with the deemed properties ($A = -A$ and $A \cap Y \neq \emptyset \neq A \cap Z$) is uniquely determined, and $\Omega(\tau)$ is obtained from $\widetilde{\Omega}(\tau)$ by splitting A into $A \cap Y$ and $A \cap Z$. In the opposite case, when the above condition is not fulfilled, we have that $\Omega(\tau) = \widetilde{\Omega}(\tau)$.

Proof. We discuss separately the cases when τ is $\gamma-$connected and when it is $\gamma-$disconnected.

$\underline{\tau \text{ is } \gamma-\text{connected}}$: then we know that τ has no inversion-invariant orbits (by Proposition 4.2.6). In this case we observe that $\widetilde{\Omega}(\tau) = \Omega(\tau)$, and that, on the other hand, $\widetilde{\Omega}(\tau)$ does not satisfy the condition (4.32). The conclusion of the lemma checks out.

$\underline{\tau \text{ is } \gamma-\text{disconnected}}$: then $\Omega(\tau)$ has at most two inversion-invariant orbits; and moreover, if $\Omega(\tau)$ has exactly two inversion-invariant orbits, then one of them is contained in Y and the other is contained in Z. This follows immediately from Proposition 4.2.4, and the fact that a permutation in $\mathsf{NC}^{\mathsf{B}}(\mathsf{p})$ or in $\mathsf{NC}^{\mathsf{B}}(\mathsf{q})$ has at most one inversion-invariant orbit. It is thus clear that the only possibility for $\widetilde{\Omega}(\tau) \neq \Omega(\tau)$ is when both $\tau \downarrow Y$ and $\tau \downarrow Z$ have inversion-invariant orbits. This also is the only possibility for having $\widetilde{\Omega}(\tau)$ satisfy the condition (4.32) – hence the conclusion of the lemma checks out in this case as well. ■

4.2.17 Corollary. *The adjusted orbit map* $(B_n, \leqslant_{B_n}) \ni \tau \longmapsto \widetilde{\Omega}(\tau) \in (\Pi(X), \leqslant)$ *is one-to-one and order preserving.*

Proof. For the "order preserving" part, let us first note that it suffices to prove that $\widetilde{\Omega}(\tau_1) \leqslant \widetilde{\Omega}(\tau_2)$ whenever τ_2 covers τ_1. This follows from a case-by-case analysis of the covering relations $\mathit{Bc}\ \ i$, $1 \leqslant i \leqslant 4$, in the hyperoctahedral group B_n, which were reviewed in Section 3.3.

The "one-to-one" part is also immediate: if $\sigma, \tau \in \mathfrak{S}^{\mathsf{B}}_{\mathsf{nc}}(X,\gamma)$ are such that $\widetilde{\Omega}(\sigma) = \widetilde{\Omega}(\tau)$, then Lemma 4.2.16 implies that $\Omega(\sigma) = \Omega(\tau)$, and then the injectivity of the orbit map Ω implies that $\sigma = \tau$. ■

Let us recall at this point that our goal in this section is to establish the isomorphism between the posets $(\mathfrak{S}^{\mathsf{B}}_{\mathsf{nc}}(X,\gamma), \leqslant_{B_n})$ and $(\mathsf{NC}^{\mathsf{B}}(\mathsf{p}, \mathsf{q}), \leqslant)$. Up to now we have seen that the adjusted orbit map $\widetilde{\Omega}$ sends permutations to partitions in a one-to-one and order-preserving fashion. Of course, it remains to prove that whenever $\widetilde{\Omega}(\sigma) \leqslant \widetilde{\Omega}(\tau)$ it follows that $\sigma \leqslant_{B_n} \tau$. The theorem which we want to prove is thus:

4.2.18 Theorem. *Let us denote*

$$NC^B(p,\, q) \stackrel{\text{def}}{=} \{\widetilde{\Omega}(\tau) \mid \tau \in \mathfrak{S}^{B}_{nc}(X,\gamma)\}. \tag{4.33}$$

Then the map

$$\mathfrak{S}^{B}_{nc}(X,\gamma) \ni \tau \longmapsto \widetilde{\Omega}(\tau) \in NC^B(p,\, q) \tag{4.34}$$

is a poset isomorphism, where $\mathfrak{S}^{B}_{nc}(X,\gamma)$ *is partially ordered as an interval of* B_n*, while* $NC^B(p,\, q)$ *is partially ordered by reversed refinement.*

The proof will follow from Corollary 4.2.17 and Proposition 4.2.22 below.

We need to prove first several technical lemmas concerning the canonical permutations μ_A with A coming from $\mathcal{O}^{\mathsf{B}}_{\mathsf{nc}}(p,q)$.

4.2.19 Lemma. *Let $A, B \in \mathcal{O}^{B}_{nc}(p,q)$ be such that $A \subseteq B$. Then $\mu_B \downarrow A = \mu_A$.*

Proof. If A is $\gamma-$disconnected, then both μ_A and $\mu_B \downarrow A$ are equal to $\gamma \downarrow A$. Let us then assume that A is $\gamma-$connected, and let us pick two elements $y \in A \cap Y$ and $z \in A \cap Z$. We have in particular that $y \in B \cap Y$ and $z \in B \cap Z$, and it follows that both μ_A and $\mu_B \downarrow A$ are equal to $\lambda_{-y,-z} \downarrow A$. ■

4.2.20 Lemma. *Let $A \in \mathcal{O}^{B}_{nc}(p,q)$, and suppose that σ is a permutation in $\mathfrak{S}^{B}_{nc}(X, \gamma_o)$ such that A is a union of orbits of σ. Then $\sigma \downarrow A \in \mathfrak{S}^{A}_{nc}(A, \mu_A)$.*

Proof. We will use the description of $\mathfrak{S}^{\mathsf{A}}_{\mathsf{nc}}(A, \mu_A)$ in terms of crossing pattern DC, as in Definition 3.4.4.

We first check that $\sigma \downarrow A$ is compatible with μ_A. This amounts to checking that for every orbit B of $\sigma \downarrow A$ we have

$$(\sigma \downarrow A) \downarrow B = \mu_A \downarrow B. \tag{4.35}$$

But every orbit B of $\sigma \downarrow A$ is in fact an orbit of σ (since it is given that A is a union of orbits of σ); thus $B \in \mathcal{O}^{\mathsf{B}}_{\mathsf{nc}}(p,q)$, and both sides of Equation (4.35) are equal to the canonical permutation μ_B (where on the right-hand side we invoke the preceding lemma).

We now prove that $\sigma \downarrow A$ cannot display the crossing pattern DC with respect to μ_A. Assume by contradiction that there exist four distinct points $a, b, c, d \in A$ such that

$$\mu_A \downarrow \{a,b,c,d\} = (a,b,c,d), \quad (\sigma \downarrow A) \downarrow \{a,b,c,d\} = (a,c)(b,d). \tag{4.36}$$

We distinguish two cases.

<u>Case 1.</u> $\{a,b,c,d\}$ is a $\gamma-$disconnected subset of X; that is, we have that either $\{a,b,c,d\} \subseteq Y$ or $\{a,b,c,d\} \subseteq Z$.

In this case, Equation (4.29) from Remark 4.2.12 implies that $\mu_A \downarrow \{a,b,c,d\} = \gamma \downarrow \{a,b,c,d\}$. Thus the conditions in (4.36) amount to

$$\gamma \downarrow \{a,b,c,d\} = (a,b,c,d), \quad \sigma \downarrow \{a,b,c,d\} = (a,c)(b,d),$$

and this implies that σ displays the crossing pattern *AC1* with respect to γ. But this is in contradiction with the hypothesis that $\sigma \in \mathfrak{S}^{\mathsf{B}}_{\mathsf{nc}}(X, \gamma) \subseteq \mathfrak{S}^{\mathsf{A}}_{\mathsf{nc}}(X, \gamma)$ (and where we invoke the description of $\mathfrak{S}^{\mathsf{A}}_{\mathsf{nc}}(X, \gamma)$ in terms of annular crossing patterns *AC1* , *AC2* , *AC3*).

<u>*Case 2.*</u> $\{a, b, c, d\}$ is a $\gamma -$ connected subset of X; that is, $\{a, b, c, d\} \cap Y \neq \emptyset \neq \{a, b, c, d\} \cap Z$.

In this case we must have that at least one of the two sets $\{a, c\}$ and $\{b, d\}$ is $\gamma -$ connected. Indeed, if both $\{a, c\}$ and $\{b, d\}$ were $\gamma -$ disconnected, then it would follow that either we have $a, c \in Y$ and $b, d \in Z$, or vice-versa, we have $a, c \in Z$ and $b, d \in Y$; but this comes in contradiction with Remark 4.2.12 *iii.* In the remaining part of the proof we will assume that $\{b, d\}$ is $\gamma -$ connected (the discussion based on the assumption "$\{a, c\}$ is $\gamma -$ connected" is similar and omitted here).

Let us next record that the six elements $a, b, c, d, -b, -d$ of X are distinct from each other. Indeed, we have that a, b, c, d are distinct elements of A, while $-b, -d$ are distinct elements of $-A$, and Remark 4.2.9 implies that $A \cap (-A) = \emptyset$ (we use here the fact that A is $\gamma -$connected, which holds because $A \supseteq \{b, d\}$).

¿From the second equality stated in (4.36) and the fact that $\sigma \in B_n$ it is immediate that

$$\sigma \downarrow \{a, b, c, d, -b, -d\} = (a, c)(b, d)(-b, -d),$$

while on the other hand we see that

$$\begin{aligned} \lambda_{-b,-d} \downarrow \{a, b, c, d\} &= (\lambda_{-b,-d} \downarrow A) \downarrow \{a, b, c, d\} \\ &= \mu_A \downarrow \{a, b, c, d\} \qquad \text{[by Equation (4.28) in Remark 4.2.12]} \\ &= (a, b, c, d). \end{aligned}$$

Hence σ displays the crossing pattern *AC3* with respect to γ – contradiction. ■

4.2.21 Lemma. *Let B and C be sets in $\mathcal{O}^{B}_{nc}(p, q)$ such that $B = -B \subseteq Y$ and $C = -C \subseteq Z$. We denote $B \cup C = A$. Suppose that σ is a permutation in $\mathfrak{S}^{B}_{nc}(X, \gamma)$ such that A is a union of orbits of σ. Then $\sigma \downarrow A \in \mathfrak{S}^{A}_{nc}(A, \gamma_A)$.*

Proof. The permutation $\gamma \downarrow A$ has exactly two orbits, namely B and C. We will prove that $\sigma \downarrow A \in \mathfrak{S}^{\mathsf{A}}_{\mathsf{nc}}(A, \gamma_A)$ by using the description of $\mathfrak{S}^{\mathsf{A}}_{\mathsf{nc}}(A, \gamma_A)$ in terms of the annular crossing patterns.

Let us first look at the verification that $\sigma \downarrow A$ is compatible with $\gamma \downarrow A$. Here we have to check that every orbit U of $\sigma \downarrow A$ satisfies the conditions *AS1* and *AS2* of Definition 4.1.1 *ii.*, in the appropriate reformulation where Y and Z are replaced by B and C. And indeed, these reformulated conditions *AS-i* $(1 \leqslant i \leqslant 2)$ are immediate consequences of the corresponding conditions *AS-i* satisfied by $\sigma \in \mathfrak{S}^{\mathsf{A}}_{\mathsf{nc}}(X, \gamma)$, and where we use the same U. In order to illustrate what happens, let us work out for instance the condition *AS1*. In the reformulation for $\sigma \downarrow A$, this condition has the form

$$\text{“}(\sigma \downarrow A) \downarrow (U \cap B) = (\gamma \downarrow A) \downarrow (U \cap B)\text{”},$$

where U is an orbit of σ such that $U \subseteq A$. So we are required to check that σ and γ induce the same permutation on $U \cap B$. But the corresponding condition which we know to be satisfied by σ is that $\sigma \downarrow (U \cap Y) = \gamma \downarrow (U \cap Y)$, and this does indeed imply that $\sigma \downarrow (U \cap B) = \gamma \downarrow (U \cap B)$, since $U \cap Y \supseteq U \cap B$.

The verification that $\sigma \downarrow A$ does not display any of the annular crossing patterns *AC1*, *AC2*, *AC3* with respect to $\gamma \downarrow A$ goes on the same lines as in the preceding paragraph. That is, if $\sigma \downarrow A$ would display a crossing pattern *AC-i* with respect to $\gamma \downarrow A$ $(1 \leqslant i \leqslant 3)$, then the same set of 4, 5 or 6 points of A could be used to infer that σ displays the crossing pattern *AC-i* with respect to γ. The verification of this fact is straightforward and it is skipped. We only note here that when treating the crossing patterns *AC2* and *AC3* one has to take into account the following simple observation: if $b \in B$, $c \in C$, and $\lambda_{b,c} \in \mathfrak{S}_X$ is the *AC*-test permutation defined as in Equation (4.2), then the counterpart of $\lambda_{b,c}$ with respect to to $\mathfrak{S}^{\mathsf{A}}_{\mathsf{nc}}(A, \gamma_A)$ coincides with $\lambda_{b,c} \downarrow A$. ■

4.2.22 Proposition. *Let $\sigma, \tau \in \mathfrak{S}^{B}_{nc}(X, \gamma_o)$ be such that $\widetilde{\Omega}(\sigma) \leqslant \widetilde{\Omega}(\tau)$. Then $\sigma \leqslant_{B_n} \tau$.*

Proof. <u>*Case 1.*</u> Both σ and τ are $\gamma-$disconnected.

In this case each of σ and τ is completely determined by its restriction to Y and to Z. Let B_Y and B_Z be the Weyl groups of type B corresponding to the sets Y and Z, respectively. It is immediate that the required inequality $\sigma \leqslant_{B_n} \tau$ in B_n will follow if we can prove that $\sigma \downarrow Y \leqslant_{B_Y} \tau \downarrow Y$ in B_Y and $\sigma \downarrow Z \leqslant_{B_Z} \tau \downarrow Z$ in B_Z.

Now, from the hypothesis that $\widetilde{\Omega}(\sigma) \leq \widetilde{\Omega}(\tau)$ it follows that $\Omega(\sigma \downarrow Y) \leq \Omega(\tau \downarrow Y)$, since the blocks of $\sigma \downarrow Y$ (respectively $\tau \downarrow Y$) are obtained by intersecting the blocks of $\widetilde{\Omega}(\sigma)$ (respectively the blocks of $\Omega(\tau)$) with Y. But Proposition 4.2.4 gives us that $\sigma \downarrow Y, \tau \downarrow Y \in B_Y \simeq B_p$. Thus if we know that $\Omega(\sigma \downarrow Y) \leqslant \Omega(\tau \downarrow Y)$, then we can invoke the poset isomorphism between $\mathsf{NC}^{\mathsf{B}}(\mathsf{n})$ and $\mathfrak{S}^{\mathsf{B}}_{\mathsf{nc}}(X, \gamma_o)$ (as in the type B summarizing diagram) to conclude that $\sigma \downarrow Y \leqslant_{B_Y} \tau \downarrow Y$. The inequality $\sigma \downarrow Z \leqslant_{B_Z} \tau \downarrow Z$ is obtained in a similar manner.

Case 2. τ has no inversion-invariant orbits.

In this case σ cannot have inversion-invariant orbits either. We have $\widetilde{\Omega}\,(\sigma) = \Omega(\sigma)$ and $\widetilde{\Omega}\,(\tau) = \Omega(\tau)$, thus our hypothesis is that $\Omega(\sigma) \leqslant \Omega(\tau)$.

Let A be an orbit of τ. Then A is a union of orbits of σ, and Lemma 4.2.20 gives us that $\sigma \downarrow A \in \mathfrak{S}^{\mathsf{A}}_{\mathsf{nc}}(A, \tau_A)$. Observe that

$$(\sigma \downarrow A)^{-1}(\tau \downarrow A) = (\sigma^{-1}\tau) \downarrow A,$$

thus the genus equality gives us that

$$\#(\sigma \downarrow A) + \#(\sigma^{-1}\tau) \downarrow A) = 1 + |A|. \tag{4.37}$$

In Equation (4.37) let us sum over all orbits A of τ, where we take into account that every orbit of σ is contained in precisely one orbit of τ, and that (consequently) the same is true for every orbit of $\sigma^{-1}\tau$. We get that

$$\#(\sigma) \ + \ \#(\sigma^{-1}\tau) \ = \ \#(\tau) + 2n. \tag{4.38}$$

Finally, we convert Equation (4.38) into a formula which involves lengths in B_n. If a permutation $\varphi \in B_n$ has no inversion-invariant orbits, then the relation between the length $\ell_B\,(\varphi)$ and the number of cycles $\#(\varphi)$ is

$$\#(\varphi) = 2(n - \ell_B\,(\varphi)). \tag{4.39}$$

This formula applies to each of $\sigma, \sigma^{-1}\tau$ and τ (where in the case of $\sigma^{-1}\tau$, the absence of inversion-invariant orbits follows from the inequality $\Omega(\sigma^{-1}\tau) \leqslant \Omega(\tau)$). By substituting this into (4.38) we get precisely that $\ell_B\,(\sigma) + \ell_B\,(\sigma^{-1}\tau) = \ell_B\,(\tau)$, and the required inequality $\sigma \leqslant_{B_n} \tau$ follows.

Case 3. σ and τ are neither as in _Case 1_ nor as in _Case 2_.

In this case τ must have inversion-invariant orbits (otherwise _Case 2_ would apply). Proposition 4.2.6 thus implies that τ is $\gamma-$disconnected. But then σ has to be $\gamma-$connected, otherwise _Case 1_ would apply. From the given inequality $\widetilde{\Omega}\,(\sigma) \leqslant \widetilde{\Omega}\,(\tau)$ and the fact that σ is $\gamma-$connected we next infer that the partition $\widetilde{\Omega}\,(\tau)$ is $\gamma-$connected.

In the preceding paragraph we saw that τ is $\gamma-$ disconnected, but the partition $\widetilde{\Omega}(\tau)$ is $\gamma-$connected. The only way this can happen is if τ has exactly two inversion-invariant orbits, an orbit $B=-B\subseteq Y$ and an orbit $C=-C\subseteq Z$. Then, denoting $B\cup C=A_o$, we have that A_o is the unique $\gamma-$ connected block of $\widetilde{\Omega}(\tau)$ (while all the other blocks of $\widetilde{\Omega}(\tau)$ are actual orbits of τ, and each of them is either contained in Y or contained in Z). In the preceding paragraph we also saw that σ is $\gamma-$connected; note that, due to the inequality $\Omega(\sigma)\leqslant\widetilde{\Omega}(\tau)$, all the $\gamma-$ connected orbits of σ must be contained in A_o.

We now start to count orbits of σ and of $\sigma^{-1}\tau$, in the same way as we did in the *Case 2*. For every orbit A of τ such that $A\neq B,C$ we have that A is a union of orbits of σ and we can do exactly the same calculation as shown in *Case 2*. We obtain, analogously to Equation (4.37) from Case 2, that

$$\#(\sigma\downarrow A)+\#(\sigma^{-1}\tau)\downarrow A)=1+|A|,\quad\forall\, A \text{ orbit of } \tau,\ A\neq B,C. \tag{4.40}$$

On the other hand, $A_o=B\cup C$ also is a union of orbits of σ. Lemma 4.2.21 applies to this situation, and gives us that $\sigma\downarrow A_o\in\mathfrak{S}_{\mathsf{nc}}^{\mathsf{A}}(A,\tau\downarrow A_o)$. It is convenient to replace here $\gamma\downarrow A_o$ by $\tau\downarrow A_o$ (the equality $\gamma\downarrow A_o=\tau\downarrow A_o$ is the combination of the two equalities $\gamma\downarrow B=\tau\downarrow B$ and $\gamma\downarrow C=\tau\downarrow C$, which hold because τ is compatible with γ. So we obtain that $\sigma\downarrow A_o\in\mathfrak{S}_{\mathsf{nc}}^{\mathsf{A}}(A,\tau\downarrow A_o)$, and the genus inequality for $\sigma\downarrow A_o$ and $\tau\downarrow A_o$ must thus hold with equality. Out of the two cases discussed in Remark 4.1.5 we have to pick the first one, case ***i.*** (indeed, σ has $\gamma-$ connected blocks which are contained in A_o, and this implies that $\sigma\downarrow A_o$ is $(\tau\downarrow A_o)$-connected). Hence the equality which we add to those recorded in Equation (4.40) is

$$\#(\sigma\downarrow A_o)+\#((\sigma^{-1}\tau)\downarrow A_o)=|A_o|. \tag{4.41}$$

Let us now sum in Equation (4.40) over all the orbits $A\neq B,C$ of τ, and let us also add up Equation (4.41) to the result of that summation. We get (analogously to Equation (4.38) from *Case 2*), that

$$\#(\sigma)\;+\;\#((\sigma^{-1}\tau)\;=\;\Big(\#(\tau)-2\Big)+2n. \tag{4.42}$$

Finally, we convert Equation (4.42) into a formula which involves lengths in B_n. We omit here the verification of the fact that the permutations σ and $\sigma^{-1}\tau$ do not have inversion-invariant orbits (the verification has only one non-trivial point, namely the absence of inversion-invariant orbits of $(\sigma^{-1}\tau)\downarrow A_o$, which is obtained by applying a "re-denoted" version of Proposition 4.2.6 to the permutation $(\sigma^{-1}\tau)\downarrow A_o\in\mathfrak{S}_{\mathsf{nc}}^{\mathsf{A}}(A,\tau\downarrow A_o)$). Hence the conversion from $\#(\sigma)$

and $\#(\sigma^{-1}\tau)$ to the lengths $\ell_B(\sigma)$ and $\ell_B(\sigma^{-1}\tau)$ is done via the same formula (4.39) as we used in *Case 2*. The permutation τ has two inversion-invariant orbits, hence the formula used for τ has to be

$$\#(\tau) \;=\; 2(n - \ell_B(\tau) + 1).$$

When we use these formulas in order to rewrite Equation (4.42) in terms of lengths in B_n, we get that $\ell_B(\sigma) + \ell_B(\sigma^{-1}\tau) = \ell_B(\tau)$, and the required inequality $\sigma \leqslant_{B_n} \tau$ is obtained in this case as well. ■

Finally, it is clear that Theorem 4.2.18 now follows, when we combine the statements of Corollary 4.2.17 and of Proposition 4.2.22.

Chapter 5

The Lattice NC^{B} (n-1, 1)

5.1 Intersection Meets for Partitions in NC^{B} (p, q)

In this section we continue to use the framework from Chapter 4. The two positive integers p and q, and all the notations introduced in connection to them, specifically in Section 4.2. are fixed for the present chapter.

$$\begin{cases} X = \{1, \ldots, n\} \cup \{-1, \ldots, -n\}, \\ Y = \{1, \ldots, p\} \cup \{-1, \ldots, -p\}, \\ Z = \{p+1, \ldots, n\} \cup \{-(p+1), \ldots, -n\}, \end{cases} \qquad \text{for} \quad n = p + q.$$

We are thus looking at $\mathsf{NC}^{\mathsf{B}}(\mathsf{p}, \mathsf{q})$, which is a set of partitions of X. For any partitions π, ρ of X we will use the notation $\pi \wedge \rho$ to refer to the **intersection meet** of π and ρ. That is, $\pi \wedge \rho$ is the partition of X into blocks of the form $A \cap B$ where A is a block of π, B is a block of ρ, and $A \cap B \neq \emptyset$. It is immediate that $\pi \wedge \rho$ is the meet for π and ρ in the lattice $\Pi(X)$ of all partitions of X.

In connection to the notation $\pi \wedge \rho$, we emphasize that the implication

$$\pi, \rho \in \mathsf{NC}^{\mathsf{B}}(\mathsf{p}, \mathsf{q}) \implies \pi \wedge \rho \in \mathsf{NC}^{\mathsf{B}}(\mathsf{p}, \mathsf{q})$$

is not true in general. And in fact, while $\mathsf{NC}^{\mathsf{B}}(\mathsf{p}, \mathsf{q})$ is always a ranked poset with partial order given by reverse refinement, it is not true in general that $\mathsf{NC}^{\mathsf{B}}(\mathsf{p}, \mathsf{q})$ would be a lattice with respect to this partial order. The goal of the present section can be summarized by saying that we look at the following question: if $\pi, \rho \in \mathsf{NC}^{\mathsf{B}}(\mathsf{p}, \mathsf{q})$ and if it is to be that $\pi \wedge \rho \notin \mathsf{NC}^{\mathsf{B}}(\mathsf{p}, \mathsf{q})$), then why exactly does this happen?

5.1.1 The case when $\pi \wedge \rho$ is $\gamma-$ disconnected

5.1.1 Definition. Let θ be a partition in $\mathsf{NC}^{\mathsf{B}}(\mathsf{p})$, and let ω be a partition in $\mathsf{NC}^{\mathsf{B}}(\mathsf{q})$. We define a partition π of X which will be denoted by "$\Phi(\theta, \omega)$", and is described as follows.

i. Whenever A is a block of θ such that $A \neq -A$, we take A to be a block of π.

ii. Whenever B is a block of ω such that $B \neq -B$, we take B' to be a block of π, where

$$B' := \{b+p \mid b \in B,\ b > 0\} \cup \{b-p \mid b \in B,\ b < 0\} \subseteq \{p+1, \ldots, n\} \cup \{-(p+1), \ldots, -n\} \subseteq X.$$

iii. Let $U \subseteq X$ be the union of all the blocks of π considered in (i) and (ii) above. If $U \neq X$, then we take $X \setminus U$ to be a block of π.

5.1.2 Remark. Let θ, ω and $\pi = \Phi(\theta, \omega)$ be as above.

1^o It is clear that if M is a block of π, then $-M$ is a block of π as well. It is also clear that π can have at most one inversion-invariant block M, namely the block $X \setminus U$ from *iii.* of Definition 5.1.1 (if it is the case that $U \neq X$). A moment's thought shows that the construction of π can be succinctly described as follows: "Every block of θ and every block of ω is identified to a subset of X, in the natural way; this gives a partition π_o of X. Then π is obtained from π_o by joining together all the inversion-invariant blocks of π_o (if such blocks exist) into one block of π".

2^o By using the canonical identifications

$$\mathfrak{S}_{\mathsf{nc}}^{\mathsf{A}}(Y, \gamma_o) \cap B_Y = \mathfrak{S}_{\mathsf{nc}}^{\mathsf{B}}(Y, \gamma_o) \simeq \mathsf{NC}^{\mathsf{B}}(\mathsf{p}) \text{ and } \mathfrak{S}_{\mathsf{nc}}^{\mathsf{A}}(Z, \gamma_o) \cap B_Z = \mathfrak{S}_{\mathsf{nc}}^{\mathsf{B}}(Z, \gamma_o) \simeq \mathsf{NC}^{\mathsf{B}}(\mathsf{q}) \tag{5.1}$$

the construction of the partition $\pi = \Phi(\theta, \omega)$ can also be described as follows: we identify θ and ω with permutations from $\mathfrak{S}_{\mathsf{nc}}^{\mathsf{A}}(Y, \gamma_o) \cap B_Y$ and respectively from $\mathfrak{S}_{\mathsf{nc}}^{\mathsf{A}}(Z, \gamma_o) \cap B_Z$, in the canonical way from (5.1). The two permutations so obtained (one of Y and one of Z) can be combined together into one permutation τ of X. Note that, by Proposition 4.2.4, τ is a $\gamma-$ disconnected permutation in $\mathfrak{S}_{\mathsf{nc}}^{\mathsf{B}}(X, \gamma)$. Then π can be defined as being the partition $\widetilde{\Omega}(\tau)$ for this particular $\tau \in \mathfrak{S}_{\mathsf{nc}}^{\mathsf{B}}(X, \gamma)$.

5.1.3 Proposition. 1^o *For every* $\theta \in NC^B(p)$ *and* $\omega \in NC^B(q)$*, the partition* $\Phi(\theta, \omega)$ *defined above belongs to* $NC^B(p, q)$.

2^o *The map* $\Phi : NC^B(p) \times NC^B(q) \to NC^B(p, q)$ *is injective, and its range-set can be described as* $\{\widetilde{\Omega}(\tau) \mid \tau \in \mathfrak{S}_{nc}^B(X, \gamma),\ \tau \text{ is } \gamma - \text{disconnected}\}$.

Proof. Part 1^o and the description of the range-set of Φ in part 2^o follow from the description of $\Phi(\theta, \omega)$ observed in Remark 5.1.2.2. The injectivity of Φ is immediate from the description of $\Phi(\theta, \omega)$ given in Definition 5.1.1. ■

5.1.4 Corollary. *The subset* $\{\widetilde{\Omega}(\tau) \mid \tau \in \mathfrak{S}_{nc}^B(X, \gamma),\ \tau \text{ is } \gamma - \text{disconnected}\}$ *of* $NC^B(p, q)$ *is closed under the operation "$\wedge$" of intersection meet.*

Proof. This is immediate from Proposition 5.1.3 and the straightforward verification (made directly from Definition 5.1.1) that we have $\Phi(\theta, \omega) \wedge \Phi(\theta', \omega') = \Phi(\theta \wedge \theta', \omega \wedge \omega')$, for every $\theta, \theta' \in \mathsf{NC}^{\mathsf{B}}(\mathsf{p})$ and every $\omega, \omega' \in \mathsf{NC}^{\mathsf{B}}(\mathsf{q})$. ■

5.1.5 Corollary. *Let* π, ρ *be in* $NC^B(p, q)$*, and let us denote* $\pi \wedge \rho =: \nu$*. If* ν *has inversion-invariant blocks, then* $\nu \in NC^B(p, q)$.

Proof. Let N be an inversion-invariant block of ν, and let us write $N = M \cap M'$ where M is a block of π and M' is a block of ρ. Then $M \cap (-M) \supseteq N$, hence $M \cap (-M) \neq \emptyset$, and M must be an inversion-invariant block of π. Similarly, M' must be an inversion-invariant block of ρ. From Proposition 4.2.6 it follows that we must have $\pi = \widetilde{\Omega}(\tau)$ and $\rho = \widetilde{\Omega}(\tau')$ for some $\gamma -$disconnected permutations $\tau, \tau' \in \mathfrak{S}_{\mathsf{nc}}^{\mathsf{B}}(X, \gamma)$. But then Corollary 5.1.4 gives us that ν also is of the form $\widetilde{\Omega}(\sigma)$ for some $\gamma -$ disconnected permutation $\sigma \in \mathfrak{S}_{\mathsf{nc}}^{\mathsf{B}}(X, \gamma)$, and in particular we find that $\nu \in \mathsf{NC}^{\mathsf{B}}(\mathsf{p}, \mathsf{q})$. ■

In the remaining part of this subsection we will prove another statement going on the same lines as the above corollary, but where the hypothesis on ν will be that it is $\gamma -$disconnected. When doing that, it will come in handy to use the following notation.

5.1.6 Notations. Let π be a partition of X.

***i*.** We will denote by $\Psi_1(\pi)$ the partition of $\{1, \ldots, p\} \cup \{-1, \ldots, -p\}$ into blocks of the form $A = M \cap Y$, with M a block of π such that $M \cap Y \neq \emptyset$.

ii. We will denote by $\Psi_2(\pi)$ the partition of $\{1,\dots,q\} \cup \{-1,\dots,-q\}$ into blocks of the form

$$B = \{b - p \mid b \in M \cap Z,\ b > 0\} \cup \{b + p \mid b \in M \cap Z,\ b < 0\}$$

with M a block of π such that $M \cap Z \neq \emptyset$.

5.1.7 Lemma. *Let π be a partition in $NC^B(p, q)$, and consider the partitions $\theta := \Psi_1(\pi)$ and $\omega := \Psi_2(\pi)$ from the preceding notation. Then $\theta \in NC^B(p)$ and $\omega \in NC^B(q)$.*

Proof. We denote by τ the unique permutation in $\mathfrak{S}^{\mathsf{B}}_{\mathsf{nc}}(X,\gamma)$ which has $\widetilde{\Omega}(\tau) = \pi$.

Assume by contradiction that $\theta \notin \mathsf{NC}^{\mathsf{B}}(\mathsf{p})$. Then there exist two distinct blocks A and A' of θ and elements $a, c \in A$, $b, d \in A'$ such that $\alpha \downarrow \{a, b, c, d\} = (a, b, c, d)$, where

$$\alpha := \gamma \downarrow Y = (1, \dots, p, -1, \dots, -p) \in \mathfrak{S}_Y .$$

The blocks A and A' can be written as $M \cap Y$ and respectively $M' \cap Y$, where M and M' are two distinct blocks of π. By using the fact that $\pi = \widetilde{\Omega}(\tau)$, it is easily seen that $\tau \downarrow \{a, b, c, d\} = (a, c)(b, d)$. On the other hand it is clear that

$$\gamma \downarrow \{a, b, c, d\} = \alpha \downarrow \{a, b, c, d\} = (a, b, c, d),$$

and it follows that τ satisfies the crossing pattern *AC1* – contradiction.

The verification that $\omega \in \mathsf{NC}^{\mathsf{B}}(\mathsf{q})$ is made on the same lines as shown for θ in the preceding paragraph. ∎

5.1.8 Corollary. *Let π, ρ be in $NC^B(p, q)$, and let us denote $\pi \wedge \rho =: \nu$. If ν is $\gamma-$disconnected, then $\nu \in NC^B(p, q)$.*

Proof. We will assume that ν has no inversion-invariant blocks (if it has such blocks, then we just invoke Corollary 5.1.5).

Consider the partitions $\Psi_1(\nu)$ and $\Psi_2(\nu)$; we claim that $\Psi_1(\nu) \in \mathsf{NC}^{\mathsf{B}}(\mathsf{p})$ and $\Psi_2(\nu) \in \mathsf{NC}^{\mathsf{B}}(\mathsf{q})$. Indeed, directly from how the maps $\Psi_1(\cdot)$ and $\Psi_2(\cdot)$ are defined (see Notation 5.1.6) it is immediately checked that

$$\Psi_1(\nu) = \Psi_1(\pi \wedge \rho) = \Psi_1(\pi) \wedge \Psi_1(\rho), \text{ and } \Psi_2(\nu) = \Psi_2(\pi \wedge \rho) = \Psi_2(\pi) \wedge \Psi_2(\rho).$$

But $\Psi_1(\pi), \Psi_1(\rho) \in \mathsf{NC}^{\mathsf{B}}(\mathsf{p})$ (by Lemma 5.1.7), and $\mathsf{NC}^{\mathsf{B}}(\mathsf{p})$ is closed under intersection meets, hence $\Psi_1(\nu) \in \mathsf{NC}^{\mathsf{B}}(\mathsf{p})$. A similar argument shows that $\Psi_2(\nu) \in \mathsf{NC}^{\mathsf{B}}(\mathsf{q})$.

Now let us look at the partition $\Phi(\,\Psi_1(\nu), \Psi_2(\nu)\,)$. Note that $\Psi_1(\nu)$ and $\Psi_2(\nu)$ have no inversion-invariant blocks (due to the hypothesis that ν has no such blocks). The description of $\Phi(\,\Psi_1(\nu), \Psi_2(\nu)\,)$ given in Remark 5.1.2.1 thus says that $\Phi(\,\Psi_1(\nu), \Psi_2(\nu)\,)$ is simply obtained by identifying the blocks of $\Psi_1(\nu)$ and of $\Psi_2(\nu)$ to subsets of X, in the natural way. But then it becomes clear that $\Phi(\,\Psi_1(\nu), \Psi_2(\nu)\,)$ is ν itself, and Proposition 5.1.3.1 implies that $\nu \in \mathsf{NC}^{\mathsf{B}}\,(\mathsf{p},\,\mathsf{q})\,)$, as required. ■

5.1.2 The case when $\pi \wedge \rho$ is $\gamma -$ connected

5.1.9 Lemma. *Consider the collection of sets $\mathcal{O}^{B}_{nc}(p,q)$ introduced in 4.2.8 . Let B be a set in $\mathcal{O}^{B}_{nc}(p,q)$ such that $B \cap (-B) = \emptyset$, and let A be a non-empty subset of B. Then $A \in \mathcal{O}^{B}_{nc}(p,q)$.*

Proof. By the definition of $\mathcal{O}^{\mathsf{B}}_{\mathsf{nc}}(p,q)$, we can find a permutation $\tau \in \mathfrak{S}^{\mathsf{B}}_{\mathsf{nc}}(X,\gamma)$ such that B is an orbit of τ. Then $-B$ is an orbit of τ as well. Let σ be the permutation of X defined as follows:

1^o The sets A and $-A$ are orbits of σ, and we have $\sigma \downarrow \pm A = \tau \downarrow \pm A$.

2^o Every element of $B \setminus A$ and every element of $(-B) \setminus (-A)$ is a fixed point for σ.

3^o On the set $X \setminus (B \cup (-B))$ (which is a union of orbits of τ) the permutation σ acts exactly as τ does.

We claim that $\sigma \in \mathfrak{S}^{\mathsf{B}}_{\mathsf{nc}}(X,\gamma)$. Indeed, on the one hand it is clear that $\sigma \in B_n$. On the other hand, the fact that $\sigma \in \mathfrak{S}^{\mathsf{A}}_{\mathsf{nc}}(X,\gamma)$ is easily verified by using the description of $\mathfrak{S}^{\mathsf{A}}_{\mathsf{nc}}(X,\gamma)$ in terms of annular crossing patterns: the compatibility of σ with γ follows immediately from the compatibility of τ with γ, and it is also immediate that if σ would satisfy the crossing pattern *AC-i* for some $1 \leqslant i \leqslant 3$ then τ would satisfy the same crossing pattern, for the same set of points of X. (In the verification of the latter fact one uses the obvious remark that fixed points of permutations of X can not be involved in any of the crossing patterns *AC1* , *AC2* , *AC3* .)

So $\sigma \in \mathfrak{S}^{\mathsf{B}}_{\mathsf{nc}}(X,\gamma)$ and A is an orbit of σ, which implies that $A \in \mathcal{O}^{\mathsf{B}}_{\mathsf{nc}}(p,q)$. ■

5.1.10 Proposition. *Let π, ρ be in $NC^{B}\,(p,\,q)$, and let us denote $\pi \wedge \rho =: \nu$. Suppose that ν is $\gamma -$ connected, and has no inversion-invariant block. Then*

1^o *Every block A of ν belongs to the collection of sets $\mathcal{O}^{B}_{nc}(p,q)$, and we can thus talk about the canonical permutation μ_A, defined as in 4.2.11.*

2^o *Let τ be the permutation of X which is uniquely determined by the requirements that $\Omega(\tau) = \nu$ and that $\tau \downarrow A = \mu_A$, for every block A of ν. Then τ belongs to the group B_n, it is compatible with γ, and does not display the crossing patterns* ***AC1*** *and* ***AC2****.*

Proof. 1^o Let A be a block of ν, and let us write $A = B \cap C$ where B is a block of π and C is a block of ρ. It cannot happen that B and C are both inversion-invariant (if $B = -B$ and $C = -C$ then it would follow that $A = -A$, in contradiction to the hypotheses given on ν). Assume for instance that B is not inversion-invariant.

Observe that $B \in \mathcal{O}^{B}_{nc}(p,q)$. Indeed, B is a block of π, and π is of the form $\widetilde{\Omega}(\varphi)$ for some $\varphi \in \mathfrak{S}^{B}_{nc}(X,\gamma)$. ¿From the definition of $\widetilde{\Omega}(\varphi)$ it follows that either B is an orbit of φ or a union of orbits of φ. But the latter possibility is ruled out by the fact that $B \cap (-B) = \emptyset$ (by the definition of $\widetilde{\Omega}$ as in (4.30)). Hence B is an orbit of φ, and hence $B \in \mathcal{O}^{B}_{nc}(p,q)$.

But then Lemma 5.1.9 applies to B and A, and gives us that $A \in \mathcal{O}^{B}_{nc}(p,q)$ as well.

2^o The fact that $\tau \in B_n$ is immediate. It is also immediate that τ satisfies the conditions of compatibility with γ. Indeed, these conditions are actually defined for the individual cycles of τ, hence they have to be fulfilled since (by the definition of the canonical permutations μ_A) every cycle of τ comes from some permutation in $\mathfrak{S}^{B}_{nc}(X,\gamma)$.

The proof that τ cannot satisfy ***AC1*** and ***AC2*** relies essentially on the fact that the definition for each of these crossing patterns involves elements from *only two* orbits of τ. We will only show the proof for ***AC2***, the argument for ***AC1*** being analogous.

So let us assume by contradiction that τ displays the crossing pattern ***AC2***, hence that there exist five distinct points $a, b, c, y, z \in X$, with $y \in Y$ and $z \in Z$, such that

$$\lambda_{y,z} \downarrow \{a,b,c\} = (a,b,c) \text{ and } \tau \downarrow \{a,b,c,y,z\} = (a,c,b)(y,z). \tag{5.2}$$

We claim that $\{a,b,c\}$ must be a $\gamma -$ connected subset of X. Indeed, let A be the orbit of τ which contains $\{a,b,c\}$. If it happened that $\{a,b,c\} \subseteq Y$ or $\{a,b,c\} \subseteq Z$ then we would deduce that

$$\begin{aligned} \tau \downarrow \{a,b,c\} &= \mu_A \downarrow \{a,b,c\} && \text{[by definition of } \tau] \\ &= \gamma \downarrow \{a,b,c\} && \text{[by Equation (4.29) in Remark 4.2.12} \, \textit{ii.}] \end{aligned}$$

$$= \lambda_{y,z} \downarrow \{a,b,c\} \qquad \text{[directly from the definition of } \lambda_{y,z}\text{]}$$

in contradiction to what was assumed in (5.2).

Now let A be as above and let A' denote the orbit of τ which contains $\{y,z\}$. Then A, A' are blocks of ν, so we can write $A = B \cap C$ and $A' = B' \cap C'$ where B, B' are blocks of π and C, C' are blocks of ρ. We have that either $B \neq B'$ or $C \neq C'$ (in the opposite case it would follow that $A = A'$, in contradiction to how τ acts on $\{a,b,c,y,z\}$). By swapping the roles of π and ρ if necessary, we will assume that $B \neq B'$. Note that each of the two blocks B and B' of π is $\gamma-$connected (since $B \supseteq \{a,b,c\}$ and $B' \supseteq \{y,z\}$).

Let φ be the unique permutation in $\mathfrak{S}^{\mathsf{B}}_{\mathsf{nc}}(X,\gamma)$ with the property that $\widetilde{\Omega}\,(\varphi) = \pi$. Observe that φ is $\gamma-$connected: indeed, if φ was to be $\gamma-$disconnected then (as seen directly from the definition of $\widetilde{\Omega}$) the partition $\widetilde{\Omega}\,(\varphi)$ would have at most one $\gamma-$connected block, while we know that π has at least two such blocks, namely B and B'. From the fact that φ is $\gamma-$connected we further infer that φ has no inversion-invariant orbits (Proposition 4.2.6). This implies that $\Omega(\varphi) = \widetilde{\Omega}\,(\varphi) = \pi$, and we can therefore be certain that B and B' are orbits of φ.

We next prove that $\varphi \downarrow \{a,b,c\} = (a,c,b)$. To this end we consider the canonical permutation μ_B associated to the set $B \in \mathcal{O}^{\mathsf{B}}_{\mathsf{nc}}(p,q)$ and we write:

$$\begin{aligned}
\varphi \downarrow \{a,b,c\} &= \mu_B \downarrow \{a,b,c\} && \text{[by definition of } \mu_B\text{]}\\
&= (\mu_B \downarrow A) \downarrow \{a,b,c\} && \text{[because } B \supseteq A \supseteq \{a,b,c\}\text{]}\\
&= \mu_A \downarrow \{a,b,c\} && \text{[by Lemma 4.2.19])}\\
&= \tau \downarrow \{a,b,c\} && \text{[by definition of } \tau\text{]}\\
&= (a,c,b) && \text{[by (5.2)].}
\end{aligned}$$

We have thus found that φ has two distinct orbits B and B' such that $B \supseteq \{a,b,c\}$, $B' \supseteq \{y,z\}$, and such that $\varphi \downarrow \{a,b,c\} = (a,c,b)$. It is then clear that $\varphi \downarrow \{a,b,c,y,z\} = (a,c,b)(y,z)$; in conjunction with the equality $\lambda_{y,z} \downarrow \{a,b,c\} = (a,b,c)$ from (5.2) this shows that φ satisfies the crossing pattern *AC2* – contradiction. ■

5.1.11 Remark. At this moment we have narrowed down quite a bit the possibilities for how it can happen that $\pi, \rho \in \mathsf{NC}^{\mathsf{B}}\,(\mathsf{p},\ \mathsf{q})$, but $\nu := \pi \wedge \rho \notin \mathsf{NC}^{\mathsf{B}}\,(\mathsf{p},\ \mathsf{q})$: we must have that ν is $\gamma-$connected and without inversion-invariant blocks (because of Corollaries 5.1.5 and 5.1.8), and the permutation τ constructed in Proposition 5.1.10 must display the crossing pattern *AC3* .

It is somewhat disappointing to see that if $p, q \geqslant 2$, this one possibility that was left (with τ displaying the crossing pattern *AC3*) *can in fact occur.* This is immediately seen by looking at the example where $\pi = \Omega(\sigma)$ and $\rho = \Omega(\tau)$ for

$$\begin{cases} \sigma &= ((1, 2, p+1, p+2)), \\ \tau &= ((1, -(p+2), p+1, -2)). \end{cases} \tag{5.3}$$

In fact, if $p, q \geq 2$ then one can argue directly that $\mathsf{NC}^{\mathsf{B}}(\mathsf{p},\ \mathsf{q})$ is not a lattice, in the following way: let σ, τ be as in (5.3), and consider on the other hand the permutations

$$\sigma_o = ((1, p+1)), \quad \tau_o = ((2, p+2)) \in B_n. \tag{5.4}$$

We denote $\Omega(\sigma) = \pi$, $\Omega(\tau) = \rho$, $\Omega(\sigma_o) = \pi_o$, $\Omega(\tau_o) = \rho_o$. It is straightforward to check that π, ρ, π_o, ρ_o all belong to $\mathsf{NC}^{\mathsf{B}}(\mathsf{p},\ \mathsf{q})$, satisfy the inequalities $\pi_o \leqslant \pi$, $\pi_o \leqslant \rho$, $\rho_o \leqslant \pi$, $\rho_o \leqslant \rho$, and yet there is no partition $\nu \in \mathsf{NC}^{\mathsf{B}}(\mathsf{p},\ \mathsf{q})$ such that $\pi_o, \rho_o \leqslant \nu \leqslant \pi, \rho$.

Figure 5.1 shows how the partitions and permutations of this example look, in the particular case when $p = q = 2$.

On the other hand, note that the above example takes advantage of the existence of at least 4 points on each of the two circles of the annulus. This detail really turns out to be essential – in the next section we will see that it is possible to "finish the argument" for the fact that $\pi \wedge \rho \in \mathsf{NC}^{\mathsf{B}}(\mathsf{p},\ \mathsf{q})$, if we place ourselves in the particular situation when $p = n - 1$ and $q = 1$.

5.2 NC^{B} (n-1, 1) is a lattice

This section inherits all the notation from Section 5.1 with the specification that the positive integers p, q are now set to be

$$p = n - 1,\ q = 1, \quad \text{for some } n \geq 2. \tag{5.5}$$

So the set X continues to be $\{1, 2, \ldots, n\} \cup \{-1, -2, \ldots, -n\}$, but Y and Z have now become

$$Y = \{1, 2, \ldots, n-1\} \cup \{-1, -2, \ldots, -(n-1)\}, \quad Z = \{n, -n\},$$

γ is the permutation

$$\gamma = ((1, \ldots, n-1)) \cdot [n] \in B_n,$$

etc. Our goal for the section is to show that we have the following.

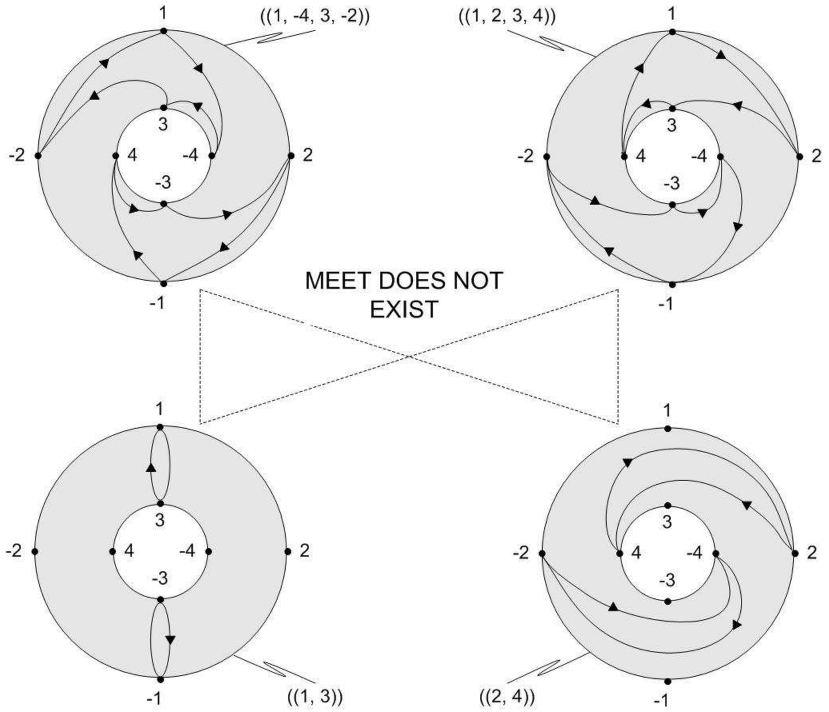


Figure 5.1: Type B annular permutations with no meet

5.2.1 Theorem. *NC^B (n-1, 1) is a lattice.*

5.2.2 Remark. In order to prove that NC^B (n-1, 1) is a lattice, all we need to do is prove that NC^B (n-1, 1) is closed under the operation "$\wedge$" of intersection meet (which is reviewed in Section 2.2). Indeed, once this is established, it becomes clear that every $\pi, \rho \in$ NC^B (n-1, 1) have a greatest common lower bound in NC^B (n-1, 1), which is precisely their intersection meet, hence "$\wedge$" really gives a meet operation on NC^B (n-1, 1). On the other hand it is obvious that NC^B (n-1, 1) has a largest element, the partition of X into only one block. Therefore Lemma 2.1.3 can be used here.

5.2.3 Remark. Let π, ρ be two partitions in NC^B (n-1, 1), and consider their intersection meet $\nu := \pi \wedge \rho$. Let us suppose that ν is γ – connected and has no inversion-invariant blocks, and let τ be the permutation of X defined as in Proposition 5.1.10 above: the orbit partition of τ is equal to ν, and for every block A of ν we have that $\tau \downarrow A = \mu_A$ (the canonical permutation of A as in Definition 4.2.11). We will spend most of the present section in examining whether τ can display the crossing pattern *AC3*, in order to eventually conclude that this cannot happen.

So let us assume that τ does satisfy *AC3* , i.e. that there exist six distinct elements $a, b, c, d, y, z \in X$ such that $y \in Y$, $z \in Z$, and where we have

$$\lambda_{y,z} \downarrow \{a, b, c, d\} = (a, b, c, d), \quad \tau \downarrow \{a, b, c, d, y, z\} = (a, c)(b, d)(y, z). \tag{5.6}$$

In the current remark we make some observations about what this entails, and we set some notations.

The main observation we want to record here is that *exactly one of the sets* $\{a, c\}$ *and* $\{b, d\}$ *is* $\gamma -$ *connected.* Indeed, it is clear that $\{a, c\}$ and $\{b, d\}$ can't both be $\gamma -$ connected, as this would imply that among a, b, c, d, y, z there are three distinct elements of Z (namely z, one element from $\{a, c\} \cap Z$ and one from $\{b, d\} \cap Z$). This is not possible, since $Z = \{n, -n\}$ has only two elements.

Suppose on the other hand that none of $\{a, c\}$ and $\{b, d\}$ are $\gamma -$ connected, i.e. that each of them is either contained in Y or contained in Z. Note that it is not possible to have $\{a, b, c, d\} \subseteq Y$ or $\{a, b, c, d\} \subseteq Z$. Indeed, if we had for instance that $\{a, b, c, d\} \subseteq Y$ then it would follow that

$$\begin{aligned} \lambda_{y,z} \downarrow \{a, b, c, d\} &= \Big(\lambda_{y,z} \downarrow (Y \setminus \{y\})\Big) \downarrow \{a, b, c, d\} \\ &= \gamma \downarrow \{a, b, c, d\}. \end{aligned}$$

This would lead to

$$\gamma \downarrow \{a, b, c, d\} = (a, b, c, d), \quad \tau \downarrow \{a, b, c, d\} = (a, c)(b, d),$$

and would imply that τ satisfies the crossing pattern *AC1* , in contradiction to Proposition 5.1.10. So if we assume that $\{a, c\}$ and $\{b, d\}$ are both $\gamma -$ disconnected then it must follow that $\{a, c\} \subseteq Y$ and $\{b, d\} \subseteq Z$ or vice-versa ($\{a, c\} \subseteq Z$ and $\{b, d\} \subseteq Y$). But this situation can't occur either, because, as explained in Remark 4.2.12*iii.*, it is not compatible with the assumption that $\lambda_{y,z} \downarrow \{a, b, c, d\} = (a, b, c, d)$.

Hence we know that exactly one of $\{a, c\}$ and $\{b, d\}$ is $\gamma -$connected. By considering a circular permutation of a, b, c, d (which does not affect the two equalities from (5.6)) we may assume that $\{a, c\}$ is $\gamma -$ connected, and moreover, that $a \in Z$ and $c \in Y$.

Now, a and z are distinct elements of Z; since $|Z| = 2$, we deduce that

$$a = -z, \quad Z = \{a, z\}, \tag{5.7}$$

and the remaining four elements b, c, d, y that play a role in (5.6) all belong to Y. It is useful to also record here that the cyclic order of b, c, d, y on Y is given by the formula

$$\gamma \downarrow \{b, c, d, y\} = (b, c, d, y); \tag{5.8}$$

this follows immediately by using the assumption (5.6) that $\lambda_{y,z} \downarrow \{a, b, c, d\} = (a, b, c, d)$, and by checking how the long cycle of $\lambda_{y,z}$ goes, when one starts at the point $a \in Z$.

In what follows we will denote by A, A' and A'' the three distinct orbits of τ (equivalently, blocks of ν) which contain $\{a, c\}$, $\{b, d\}$ and $\{y, z\}$, respectively. Since $\nu = \pi \wedge \rho$, we can write

$$A = B \cap C, \; A' = B' \cap C', \; A'' = B'' \cap C'', \tag{5.9}$$

where B, B', B'' are blocks of π and C, C', C'' are blocks of ρ. Note that we have the relations

$$B'' = -B, \; C'' = -C, \tag{5.10}$$

which hold because $B'' \ni z = -a \in -B$ and $C'' \ni z = -a \in -C$.

5.2.4 Lemma. *Consider the setting of the Remark 5.2.3.*

1^o *It is not possible that any two of the three blocks B, B', B'' of π are distinct from each other. Similarly, it is not possible that any two of the three blocks C, C', C'' of ρ are distinct from each other.*

2^o *It is not possible that $B = B' = B''$, and similarly, it is not possible that $C = C' = C''$.*

Proof. 1^o Assume for a contradiction that B, B' and B'' are three distinct blocks of π. Let φ be the unique permutation in NC^{B} (n-1, 1) with the property that $\widetilde{\Omega}(\varphi) = \pi$. Since π has at least two distinct γ-connecting blocks (namely B and B''), we can use Lemma 4.2.16 to infer that $\widetilde{\Omega}(\varphi)$ coincides in this case with the orbit partition $\Omega(\varphi)$. Hence B, B', B'' are three distinct orbits of φ, where $B \supseteq A \supseteq \{a, c\}$, $B' \supseteq A' \supseteq \{b, d\}$, and $B'' \supseteq A'' \supseteq \{y, z\}$. It is then clear that

$$\varphi \downarrow \{a, b, c, d, y, z\} = (a, c)(b, d)(y, z),$$

and in conjunction with our standing assumption that $\lambda_{y,z} \downarrow \{a, b, c, d\} = (a, b, c, d)$ (made in Equation (5.6)), this implies that φ satisfies the crossing pattern *AC3* – contradiction.

The argument that C, C', C'' cannot be three distinct blocks of ρ is identical to the one shown above for B, B', B''.

2^o If we had that $B = B' = B''$ then it would follow that C, C', C'' are three distinct blocks of ρ (since the intersections $A = B \cap C$, $A' = B' \cap C'$ and $A'' = B'' \cap C''$ give three distinct orbits of τ); but this is not possible, by part 1^o of the lemma. A similar argument rules out the possibility that $C = C' = C''$. ■

5.2.5 Lemma. *Consider the setting of the Remark 5.2.3. Then $B \neq B''$ and $C \neq C''$.*

Proof. Assume for a contradiction that $B = B''$. We observed above (see (5.10)) that we also have $B'' = -B$; hence B is an inversion-invariant block of π. It is moreover clear that B is γ-connected, since $B \cap Y \ni c, y$ and $B \cap Z \ni a, z$.

Let φ be the unique permutation in $\mathfrak{S}^{\mathsf{B}}_{\mathsf{nc}}(X, \gamma_o)\,(n-1, 1)$ with the property that $\widetilde{\Omega}\,(\varphi) = \pi$. By Lemma 4.2.16, B is the unique block of π which is both inversion-invariant and $\gamma-$connected. The same lemma tells us that the partition $\Omega(\varphi)$ of X into orbits of φ consists of $B \cap Y$, $B \cap Z$, and all the blocks of π which are different from B. Note in particular that B' has to be an orbit of φ (indeed, B' is a block of π, and cannot be equal to $B = B''$, by part 2^o of the preceding lemma).

But then let us look at the distinct orbits $B \cap Y$ and B' of φ, and at the elements $c, y \in B \cap Y$ and $b, d \in B'$. All these four elements belong to Y, and we have $\gamma \downarrow \{b, c, d, y\} = (b, c, d, y)$ (see Equation (5.8) above). This leads us to the conclusion that φ satisfies the crossing pattern $\mathtt{AC\,1}$ – contradiction.

So the assumption that $B = B''$ leads to contradiction, hence $B \neq B''$. The fact that $C \neq C''$ follows in the same way. ■

5.2.6 Remark. Consider the setting of the Remark 5.2.3. Due to the facts proved in this setting in Lemmas 5.2.4 and 5.2.5, we now know that the blocks B, B', B'' of π are such that either $B' = B$ or $B' = B''$ (indeed, Lemma 5.2.5 states that $B \neq B''$, so having $B' \neq B$ and $B' \neq B''$ would contradict Lemma 5.2.4.1). Similarly, the blocks C, C', C'' of ρ are such that either $C' = C$ or $C' = C''$.

Observe that it is not possible to have $B' = B$ and $C' = C$, because $A = B \cap C$ and $A' = B' \cap C'$ are distinct orbits of the permutation τ. Similarly, it is not possible to have that $B' = B''$ and $C' = C''$. So we are either in the case when $B' = B$, $C' = C''$, or we are in the case when

$B' = B''$, $C' = C$. By swapping, if necessary, the roles of π and of ρ in the above discussion, we can (and will) assume in what follows that it is the first of these two cases which takes place.

So from now on we can continue our discussion by writing everything in terms of the blocks B and C. Indeed, the blocks B', B'' and C', C'' that were introduced in (5.6) can now be replaced in terms of B and C:

$$B' = B,\ B'' = -B,\ \ C' = C'' = -C. \tag{5.11}$$

In terms of B and C alone, the statement of Lemma 5.2.5 becomes that B and C are not inversion-invariant; hence we know that

$$B \cap (-B) = \emptyset,\ \text{and}\ C \cap (-C) = \emptyset. \tag{5.12}$$

It is useful to also record here that (as an immediate consequence of (5.11) and of how B, B', B'' and C, C', C'' were defined in Remark 5.2.3) we have

$$a, b, c, d, -y \in B,\ \ a, -b, c, -d, -y \in C. \tag{5.13}$$

5.2.7 Proposition. *Let π, ρ be two partitions in NC^B (n-1, 1), and consider their intersection meet $\nu := \pi \wedge \rho$. Suppose that ν is γ-connected and has no inversion-invariant blocks, and let τ be the permutation of X defined as in Proposition 5.1.10 above: the orbit partition of τ is equal to ν, and for every block A of ν we have that $\tau \downarrow A = \mu_A$ (the canonical permutation of A). Then $\tau \in \mathfrak{S}^B_{nc}(X, \gamma)$, where $\gamma = [1, 2, \ldots, n-1] \cdot [n]$.*

Proof. The only thing to be proved about τ which was left out in Proposition 5.1.10 is that it does not satisfy the crossing pattern *AC3*. Assume for a contradiction that τ does satisfy *AC3*, and consider six distinct points $a, b, c, d, y, z \in X$ with $y \in Y$ and $z \in Z$, such that the relations (5.6) from Remark 5.2.3 hold. The arguments presented in Remark 5.2.3, in Lemmas 5.2.4 and 5.2.5, and in Remark 5.2.6 then tell us the following: at the cost of doing a cyclic permutation of a, b, c, d and of swapping if necessary the roles of π and ρ, we may assume that there exist a block B of π and a block C of ρ such that (5.12) and (5.13) hold. Moreover, the cyclic permutation we performed on a, b, c, d ensures that

$$a = -z,\ \{a, z\} = Z,\ \ \text{and}\ \ b, c, d, y \in Y,\ \gamma \downarrow \{b, c, d, y\} = (b, c, d, y)$$

(see Equations (5.7) and (5.8) in Remark 5.2.3).

Let φ and ψ be the permutations in $\mathfrak{S}^{\mathsf{B}}_{\mathsf{nc}}(X, \gamma)$ which have $\widetilde{\Omega}\,(\varphi) = \pi$ and $\widetilde{\Omega}\,(\psi) = \rho$. Observe that B is an orbit of φ. Indeed, the only way B could be a block of $\widetilde{\Omega}\,(\varphi)$ but not an orbit of φ would be if B was the union of two inversion-invariant orbits of φ; but this would imply that $B = -B$, and we know from (5.12) that $B \neq -B$. A similar argument shows that C is an orbit of ψ.

Let us next look at the elements $b, -b, c, d, y \in Y$. We claim that these are five distinct elements of Y. Indeed, b, c, d, y have to be distinct because they are part of the set of six distinct elements $a, b, c, d, y, z \in X$ that we started with. We next observe that $-b$ is distinct from b, c, d because $b, c, d \in B$, $-b \in -B$ (by (5.13)), and $B \cap (-B) = \emptyset$ (by (5.12)); a similar argument shows that $-b \neq y$ (we have $-b \in C$, $y \in -C$, and $C \cap (-C) = \emptyset$).

We consider the cyclic permutation induced by γ on the set $\{b, -b, c, d, y\}$. Since we know that $\gamma \downarrow \{b, c, d, y\} = (b, c, d, y)$, there are in fact only four possibilities for what $\gamma \downarrow \{b, -b, c, d, y\}$ can be. We group these four possibilities into two cases, and we argue that each of the two cases leads to contradiction.

Case 1. $\gamma \downarrow \{b, -b, c, d, y\} = (b, -b, c, d, y)$, or $\gamma \downarrow \{b, -b, c, d, y\} = (b, c, -b, d, y)$.

In this case we have that $\gamma \downarrow \{b, -b, d, y\} = (b, -b, d, y)$, with $b, d \in B$ and $-b, y \in -B$. Since B and $-B$ are two distinct orbits of φ, it follows that $\tau \downarrow \{b, -b, d, y\} = (b, d)(-b, y)$, and we find that φ satisfies the crossing pattern *AC1* – contradiction.

Case 2. $\gamma \downarrow \{b, -b, c, d, y\} = (b, c, d, -b, y)$, or $\gamma \downarrow \{b, -b, c, d, y\} = (b, c, d, y, -b)$.

In this case we have that $\gamma \downarrow \{b, c, d, -b\} = (b, c, d, -b)$, with $b, d \in -C$ and $c, -b \in C$. Since C and $-C$ are two distinct orbits of ψ, it follows that $\tau \downarrow \{b, c, d, -b\} = (b, d)(c, -b)$, and we find that ψ satisfies the crossing pattern *AC1* – contradiction. ■

5.2.8 Corollary. *If π, ρ are two partitions in NC^B (n-1, 1), then the intersection meet $\pi \wedge \rho$ also belongs to NC^B (n-1, 1).*

Proof. This follows immediately when the statement of Proposition 5.2.7 is combined with the discussion given in Remark 5.1.11 at the end of the preceding section. ■

Finally, Theorem 5.2.1 follows from Corollary 5.2.8, in the way observed in the above Remark 5.2.2.

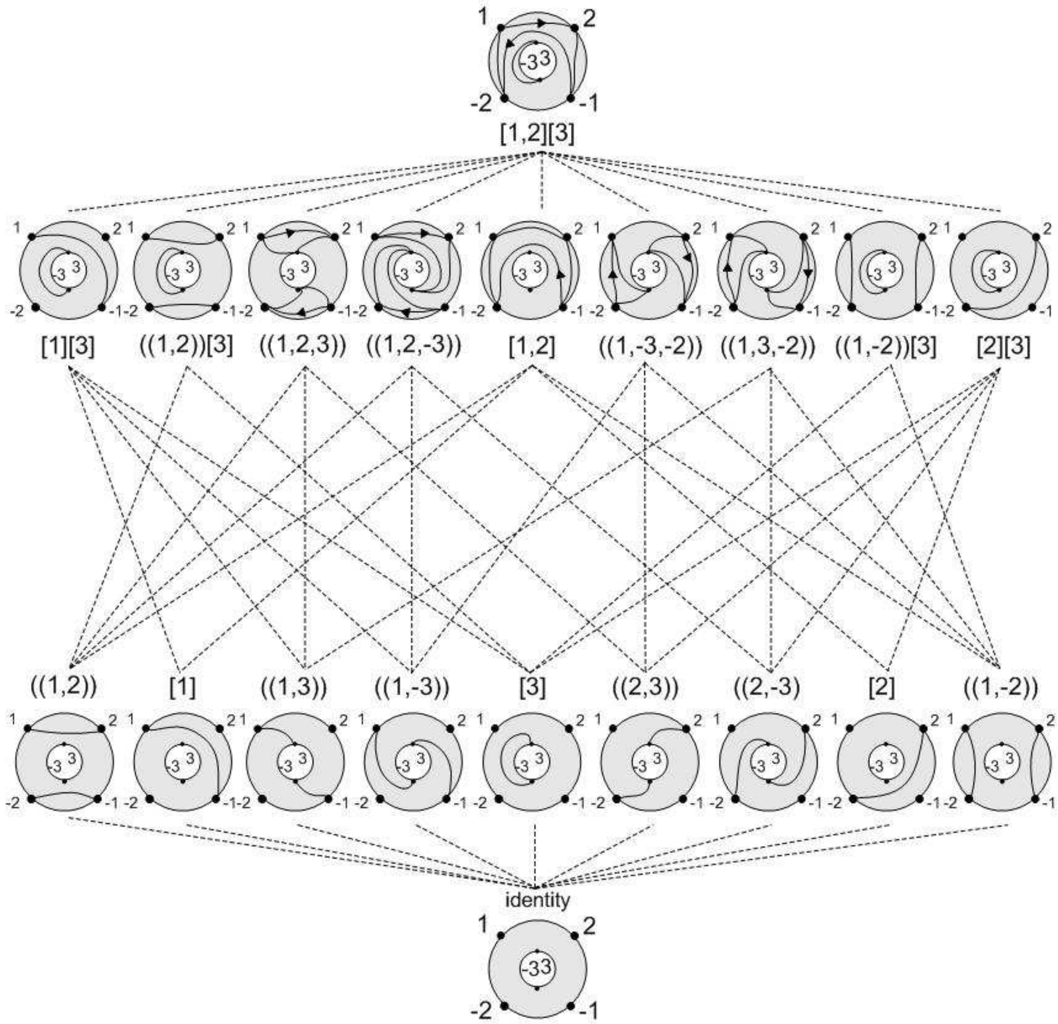


Figure 5.2: The lattice NC^B (2, 1)

5.3 Rank Cardinalities in NC^{B} (n-1, 1)

5.3.1 Remark. In any marked group an interval $[e, x]$ is a ranked poset and the rank of an element $y \in [e, x]$ is simply given by its length. In particular, $\mathsf{NC}^{\mathsf{B}}(\mathsf{n})$ and $\mathsf{NC}^{\mathsf{B}}(\mathsf{p},\ \mathsf{q})$, being isomorphic as posets with intervals coming from the hyperoctahedral group B_n are also ranked posets.

The goal in this section is to find the number of elements having rank k in the lattice $\mathsf{NC}^{\mathsf{B}}(\mathsf{n\text{-}1},\ 1)$. We start by pointing out in Remark 5.3.2 below that this was answered in the case of $\mathsf{NC}^{\mathsf{B}}(\mathsf{n})$ in [Rei97].

5.3.2 Remark. The number of elements in $\mathsf{NC}^{\mathsf{B}}(\mathsf{n})$ of fixed rank k is

$$card\,\{\pi \in \mathsf{NC}^{\mathsf{B}}(\mathsf{n})\ \mid rank(\pi) = k\} = \binom{n}{k}^2. \tag{5.14}$$

This matches the fact that the total number of elements of $\mathsf{NC}^{\mathsf{B}}(\mathsf{n})$ is $\binom{2n}{n}$, since, as it is well-known, we have

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}. \tag{5.15}$$

It is interesting to see that $\mathsf{NC}^{\mathsf{B}}(\mathsf{n\text{-}1},\ 1)$ has in fact the same generating function as $\mathsf{NC}^{\mathsf{B}}(\mathsf{n})$ (see Proposition 5.3.3 below) and thus a natural question one may ask is: couldn't it be that $\mathsf{NC}^{\mathsf{B}}(\mathsf{n\text{-}1},\ 1)$ is isomorphic to (or maybe even the same as) $\mathsf{NC}^{\mathsf{B}}(\mathsf{n})$? An immediate argument shows that for $n = 2$ we have in fact that $\mathsf{NC}^{\mathsf{B}}(1,\ 1) = \mathsf{NC}^{\mathsf{B}}(2)$. However we shall see in the next section that, already for $n = 3$, $\mathsf{NC}^{\mathsf{B}}(2,\ 1)$ is not isomorphic to $\mathsf{NC}^{\mathsf{B}}(3)$, and in general $\mathsf{NC}^{\mathsf{B}}(\mathsf{n\text{-}1},\ 1)$ is not isomorphic to $\mathsf{NC}^{\mathsf{B}}(\mathsf{n})$.

Let us now state and prove the main result of the section.

5.3.3 Proposition. *Given $n \geqslant 2$ and $0 \leqslant k \leqslant n$ we have that*

$$card\,\{\pi \in \mathit{NC}^{B}\,\textit{(n-1, 1)} \mid rank(\pi) = k\} = \binom{n}{k}^2. \tag{5.16}$$

Proof. Let us fix $n \geqslant 2$ and $0 \leqslant k \leqslant n$. We will show equivalently that

$$card\,\{\tau \in \mathfrak{S}^{\mathsf{B}}_{\mathsf{nc}}(X, \gamma)\ \mid rank(\tau) = k\} = \binom{n}{k}^2, \quad \text{where} \quad X = [\pm n], \gamma = [1, 2, \ldots, n-1][n]. \tag{5.17}$$

We will do the count separately for the $\gamma-$connected and respectively $\gamma-$disconnected permutations of $\mathfrak{S}^{\mathsf{B}}_{\mathsf{nc}}(X,\gamma)$. Let us denote

$$\begin{aligned} \{\tau \in \mathfrak{S}^{\mathsf{B}}_{\mathsf{nc}}(X,\gamma) \mid rank(\tau) = k \quad \text{and} \quad \tau \text{ is } \gamma-\text{connected}\} &= C \\ \{\tau \in \mathfrak{S}^{\mathsf{B}}_{\mathsf{nc}}(X,\gamma) \mid rank(\tau) = k \quad \text{and} \quad \tau \text{ is } \gamma-\text{disconnected}\} &= D \\ \{\tau \in D \mid \tau(n) = n \quad \text{and} \quad \tau(-n) = -n\} &= D_1 \\ \{\tau \in D \mid \tau(n) = -n \quad \text{and} \quad \tau(-n) = n\} &= D_2 \end{aligned}$$

Claim 1.

$$card\, D = card\, D_1 + card\, D_2 = \binom{n-1}{k}^2 + \binom{n-1}{k-1}^2. \tag{5.18}$$

Verification of Claim 1. Since $\gamma = [1,2,\dots,n-1][n]$ and since τ is $\gamma-$disconnected it follows that there are only two ways of permuting n and $-n$ (the only two points on the inner circle), thus $D = D_1 \cup D_2$. The remaining points $[\pm(n-1)]$ are all sitting on the outer circle and are permuted by τ in a non-crossing and symmetric fashion. It becomes now clear that

$$card\, D_1 = card\,\{\rho \in \mathsf{NC}^{\mathsf{B}}(\text{n-1}) \mid rank(\rho) = k\} = \binom{n-1}{k}^2. \qquad \text{[by (5.14)]}$$

As for D_2, since the points on the inner circle are not fixed it must be that exactly two points on the outer circle are fixed, hence

$$card\, D_2 = card\,\{\rho \in \mathsf{NC}^{\mathsf{B}}(\text{n-1}) \mid rank(\rho) = k-1\} = \binom{n-1}{k-1}^2$$

and thus (5.18) follows.

Claim 2.

$$card\, C = 2\binom{n-1}{k}\binom{n-1}{k-1}. \tag{5.19}$$

Let us first note that if the above Equation (5.19) is assumed to be true then it implies, together with Equation (5.18) that Equation (5.16) follows:

$$\begin{aligned} \binom{n-1}{k}^2 + 2\binom{n-1}{k}\binom{n-1}{k-1} + \binom{n-1}{k-1}^2 &= \left[\binom{n-1}{k} + \binom{n-1}{k-1}\right]^2 \\ &= \binom{n}{k}^2. \qquad \text{[by Pascal's Triangle]} \end{aligned}$$

Verification of Claim 2. The main point is to observe that the maps α and β in the diagram below are bijections and thus $\gamma = \beta \circ \alpha$ is also bijective.

$$\begin{tikzcd}
\{(\sigma, z) \mid \sigma \in \mathsf{NC}^{\mathsf{A}}(\mathsf{n\text{-}1}),\, z \in [\pm(n-1)]\} & \{\tau \in \mathsf{NC}^{\mathsf{B}}(\mathsf{n\text{-}1,\ 1}) \mid \tau \,\text{is}\, \gamma - \text{connected}\} \arrow[l, "\gamma=\beta\circ\alpha"'] \arrow[dl, "\alpha"] \\
\{(\rho, z) \mid \rho \in \mathsf{NC}^{\mathsf{B}}(\mathsf{n\text{-}1}) \text{ with zero-block } Z \,\text{and}\, z \in Z\} \arrow[u, "\beta=(\rho,z)\mapsto(Abs(\rho),z)"] &
\end{tikzcd}$$

Let us now explain how α is defined. Suppose that $\tau \in \mathsf{NC}^{\mathsf{B}}(\mathsf{n\text{-}1,\ 1})$ is $\gamma -$ connected and the pair of blocks connecting to $n, -n$ are denoted $B_1, -B_1$ and the rest of the blocks of τ are $C_1, -C_1, \ldots, C_l, -C_l$. Now if we let $Z = \{B_1 \cup -B_1\} \setminus \{n, -n\}$ then it is clear that $\rho = \{Z, C_1, -C_1, \ldots, C_l, -C_l\} \in \mathsf{NC}^{\mathsf{B}}(\mathsf{n\text{-}1})$. Finally, z is always chosen to be the point on the outer circle which is the image of n τ.

Let us illustrate how the diagram above works by looking at a concrete example. Let $n = 5$ and $\tau = ((1, 5, -4))((2, 3))$. Then $\rho = [1, 4]((2, 3)), z = -4$ and $\sigma = (1, 4)(2, 3)$ (permutations are identified with partitions (via the poset isomorphisms $\widetilde{\Omega}$) for reasons of brevity).

Conversely, σ and $z = -4$ recover ρ: we are choosing the block $\{1, 4\} \ni 4 = |z|$ of σ to be lifted into the zero-block Z of ρ. A moment's thought shows that (ρ, z) together with the non-crossing conditions in an annulus and the clockwise winding sense inside cycles completely reconstruct τ. Thus γ is indeed a bijection.

Let us pick now an element $\tau \in \mathfrak{S}^{\mathsf{B}}_{\mathsf{nc}}(X, \gamma) \simeq \mathsf{NC}^{\mathsf{B}}(\mathsf{n\text{-}1,\ 1})$ of rank k. The length formula in the hyperoctahedral group gives that $k = \ell_{B}(\tau) = n - \frac{1}{2}\#_{\mathsf{paired}}(\tau) = n - \frac{1}{2}\#(\tau)$ and therefore the number of blocks of τ is $\#(\tau) = 2(n-k)$.

By following the arrows in the diagram above it is clear that the number of blocks of $\alpha(\tau)$ is $2(n-k) - 1$ and also the number of blocks of $\sigma = \gamma(\tau)$ is $n - k$. The rank of σ is thus $n - 1 - \#(\sigma) = n - 1 - (n-k) = k - 1$.

Since σ and z are chosen independently, we have thus obtained that

$$\begin{aligned}
&card\{\tau \in \mathfrak{S}^{\mathsf{B}}_{\mathsf{nc}}(X,\gamma)\ |rank(\tau) = k \,\text{and}\, \tau\,\text{is}\,\gamma-\text{connected}\,\} = \\
&\qquad = card\{(\sigma, z) \mid \sigma \in \mathsf{NC}^{\mathsf{A}}(\mathsf{n\text{-}1}),\, \#(\sigma) = n-k\,\text{and}\, z \in [\pm(n-1)]\} \\
&\qquad = card\{(\sigma, z) \mid \sigma \in \mathsf{NC}^{\mathsf{A}}(\mathsf{n\text{-}1}),\, \#(\sigma) = n-k\} \cdot 2(n-1) \\
&\qquad = card\{(\sigma, z) \mid \sigma \in \mathsf{NC}^{\mathsf{A}}(\mathsf{n\text{-}1}),\, rank(\sigma) = k-1\} \cdot 2(n-1) \\
&\qquad = \frac{1}{n-1}\binom{n-1}{k-1}\binom{n-1}{k} \cdot 2(n-1) \qquad \text{[by using equation (2.18)]} \\
&\qquad = 2\binom{n-1}{k-1}\binom{n-1}{k}.
\end{aligned}$$

■

5.4 The Möbius Function of NC^{B} (n-1, 1)

In this section we look at the Möbius function for the lattices $\mathsf{NC}^{\mathsf{B}}\,(\mathsf{n\text{-}1,\ 1})$, $n \geqslant 2$. We will in fact concentrate on calculating explicitly the numbers

$$M^{B}_{n-1,1} \stackrel{\text{def}}{=} \mu_{\mathsf{NC}^{\mathsf{B}}\,(\mathsf{n\text{-}1,\ 1})}(\hat{0}, \hat{1}), n \geqslant 2 \tag{5.20}$$

where in (5.20) the symbols $\hat{0}, \hat{1}$ denote the minimal and respectively the maximal element of $\mathsf{NC}^{\mathsf{B}}\,(\mathsf{n\text{-}1,\ 1})$.

In order to explain why the numbers $M^{B}_{n-1,1}$ are important, let us start by recalling that the main point in the Möbius function calculations for $\mathsf{NC}^{\mathsf{A}}(\mathsf{n})$ and $\mathsf{NC}^{\mathsf{B}}(\mathsf{n})$ (as in Section 2.2 and Section 2.3) is to determine explicitly what are the numbers

$$M^{A}_{n} \stackrel{\text{def}}{=} \mu_{\mathsf{NC}^{\mathsf{A}}(\mathsf{n})}(\hat{0}, \hat{1}) \qquad \text{and} \qquad M^{B}_{n} \stackrel{\text{def}}{=} \mu_{\mathsf{NC}^{\mathsf{B}}(\mathsf{n})}(\hat{0}, \hat{1}), \quad n \geqslant 1 \tag{5.21}$$

More precisely, we have that

$$M^{A}_{n} = (-1)^n \frac{1}{n}\binom{2n-2}{n-1}, \quad n \geqslant 1$$

and that

$$M^{B}_{n} = (-1)^n \binom{2n-1}{n}, n \geqslant 1.$$

Returning to the lattices $\mathsf{NC}^{\mathsf{B}}(\mathsf{n\text{-}1,\ 1})$, it can be shown that one has canonical factorizations analogous to those from (2.22) and (2.32), which now look as follows

$$[\pi, \rho] \simeq \prod_{i=1}^{n} \big(\mathsf{NC}^{\mathsf{A}}(\mathsf{i})\big)^{p_i} \times \prod_{j=1}^{n} \big(\mathsf{NC}^{\mathsf{B}}(\mathsf{j})\big)^{q_j} \times \prod_{k=1}^{n} \big(\mathsf{NC}^{\mathsf{B}}(\mathsf{n\text{-}1,\ 1})\big)^{r_k}, \tag{5.22}$$

for an interval $[\pi, \rho] \subseteq \mathsf{NC}^{\mathsf{B}}(\mathsf{n\text{-}1,\ 1})$, and where $p_1, \ldots, p_n, q_1, \ldots, q_n, r_1, \ldots, r_n$ are non-negative integers. Similarly as in (2.23) and (2.33), the factorization (5.22) then provides us with a formula for what $\mu_{\mathsf{NC}^{\mathsf{B}}(\mathsf{n\text{-}1,\ 1})}(\pi, \rho)$ is, in terms of the numbers from the sequences appearing in (5.20) and (5.21).

The canonical factorization from (5.22) is not hard to obtain, but it is somewhat inconvenient (in terms of notation) to describe explicitly. Here we will not pursue the explicit description of the factorization, but we will rather focus on determining concretely the numbers $M^{B}_{n-1,1}$ from (5.20). In order to do so, we will rely on a general fact concerning the Möbius function of a lattice, which is reviewed in the next proposition.

5.4.1 Proposition. *Let P be a finite lattice, let $\hat{0}$ and $\hat{1}$ denote the minimal and the maximal element of P, respectively, and let ω be a fixed element of P, where $\omega \neq \hat{0}$. Then*

$$\sum_{\substack{\pi \in P \\ \pi \vee \omega = \hat{1}}} \mu_P(\hat{0}, \pi) = 0. \tag{5.23}$$

Proof. For a proof of Proposition 5.4.1, we refer to Section 3.9 of [Sta97] or (in identical formulation and notations as here) to Corollary 10.13 of [NS06]. ∎

The main result of the present section is then stated as follows.

5.4.2 Theorem. *The number $M^{B}_{n-1,1}$ introduced in the above Equation (5.20) is*

$$M^{B}_{n-1,1} = (-1)^n \cdot \frac{5n-4}{2n-2} \cdot \binom{2n-2}{n}. \tag{5.24}$$

Proof. The idea is to apply Proposition 5.4.1 above to the particular case when $P = \mathsf{NC}^{\mathsf{B}}(\mathsf{n\text{-}1,\ 1})$ considered with reverse refinement order, while ω is the orbit partition of $\tau = [n] = (n, -n) \in \mathfrak{S}^{\mathsf{B}}_{\mathsf{nc}}(X, \gamma)$ with $X = [\pm n], \gamma = [1, 2, \ldots, n-1][n]$.

The first step when applying (5.23) to this specific situation is to determine exactly what are the partitions $\pi \in \mathsf{NC}^{\mathsf{B}}(\mathsf{n\text{-}1,\ 1})$ which satisfy "$\pi \vee \omega = \hat{1}$". We note that

$$\pi \vee \omega = \hat{1} \iff K(\pi) \wedge K(\omega) = \hat{0}, \tag{5.25}$$

where K is the Kreweras complementation map on $\mathsf{NC}^{\mathsf{B}}(\mathsf{n\text{-}1,\ 1})$. We recall here that the map K is the order-reversing isomorphism on $\mathsf{NC}^{\mathsf{B}}(\mathsf{n\text{-}1,\ 1})$ obtained, via the isomorphism $\mathsf{NC}^{\mathsf{B}}(\mathsf{n\text{-}1,\ 1}) \simeq [\varepsilon, \gamma]_{B_n}$, from the order-reversing isomorphism

$$\mathsf{C}_\gamma : [\varepsilon, \gamma]_{B_n} \longrightarrow [\varepsilon, \gamma]_{B_n} : \mathsf{C}_\gamma(\tau) \stackrel{\text{def}}{=} \tau^{-1}\gamma,$$

as the following diagram shows.

$$\begin{tikzcd}
{[\varepsilon, \gamma]_{B_n}} \arrow[r, "\mathsf{C}_\gamma"] & {[\varepsilon, \gamma]_{B_n} = \mathfrak{S}^{\mathsf{B}}_{\mathsf{nc}}(X, \gamma)} \arrow[d, "\widetilde{\Omega}"] \\
\mathsf{NC}^{\mathsf{B}}(\mathsf{n\text{-}1,\ 1}) \arrow[u, "\simeq"] & \mathsf{NC}^{\mathsf{B}}(\mathsf{n\text{-}1,\ 1}) \arrow[l, "K"']
\end{tikzcd}$$

Thus, we calculate $K(\omega)$ by moving from $\mathsf{NC}^{\mathsf{B}}(\mathsf{n\text{-}1,\ 1})$ to $\mathfrak{S}^{\mathsf{B}}_{\mathsf{nc}}(X, \gamma)$,

$$K(\omega) = \widetilde{\Omega}\,(\tau^{-1}\gamma) = \widetilde{\Omega}\,([1, 2, \ldots, n-1]) = \widetilde{\Omega}\,(\gamma_{\mathsf{ext}}) = [\pm(n-1)]. \tag{5.26}$$

We need to describe explicitly the set

$$\{\rho \in \mathsf{NC}^{\mathsf{B}}(\mathsf{n\text{-}1,\ 1}) \mid \rho \wedge K(\omega) = \hat{0}\}. \tag{5.27}$$

When looking at (5.27), we prefer to use partition language, since we know that "$\wedge$" for $\mathsf{NC}^{\mathsf{B}}(\mathsf{n\text{-}1,\ 1})$ is just the usual intersection meet. We get that the set in (5.27) actually equals

$$\{\hat{0}, \omega\} \cup \{\rho_i \mid i \in [\pm(n-1)]\}, \tag{5.28}$$

where $\rho_i \stackrel{\text{def}}{=} ((i, n))$, for $|i| \leqslant n-1$.

By taking K^{-1} in (5.28) we get that

$$\{\pi \in \mathsf{NC}^{\mathsf{B}}(\mathsf{n\text{-}1,\ 1}) \mid \pi \vee \omega = \hat{1}\} = \{\hat{1}, K^{-1}(\omega)\} \cup \{\pi_i \mid i \in [\pm(n-1)]\}, \tag{5.29}$$

where $\pi_i = K^{-1}(\rho_i)$, for $|i| \leqslant n-1$.

For $K^{-1}(\omega)$ we get the same partition as for $K(\omega)$ in (5.26), due to the fact that $\tau^{-1}\gamma = \gamma\tau^{-1}$:

$$K^{-1}(\omega) = \widetilde{\Omega}\,(\gamma\tau^{-1}) = \widetilde{\Omega}\,([1, 2, \ldots, n-1]) = \widetilde{\Omega}\,(\gamma_{\mathsf{ext}}) = [\pm(n-1)]. \tag{5.30}$$

We now compute π_i:

$$\pi_i = K^{-1}(\rho_i) = \widetilde{\Omega}\left(\gamma \cdot ((i, n))\right) = ((i+1, i+2, \ldots, -i, n)). \tag{5.31}$$

Hence π_i is the partition made of exactly one pair of non-invariant blocks which are obtained from the cycles of the permutation which sends n to $i+1$. It is then clear that $[\hat{0}, \pi_i]$ is poset isomorphic with $\mathsf{NC}^{\mathsf{A}}(\mathsf{n})$ and therefore we have

$$\mu(\hat{0}, \pi_i) = (-1)^{n+1} \frac{(2n-2)!}{(n-1)!\; n!}. \tag{5.32}$$

On the other hand we have that the interval $[\hat{0}, K^{-1}(\omega)]$ is poset isomorphic with $\mathsf{NC}^{\mathsf{B}}(\mathsf{n\text{-}1})$ and thus

$$\mu\big(\hat{0}, K^{-1}(\omega)\big) = (-1)^{n-1} \binom{2n-3}{n-1}, \tag{5.33}$$

by following the paper [Rei97] which gives that $\mu\big(\hat{0}_{\mathsf{NC}^{\mathsf{B}}(\mathsf{n})}, \hat{1}_{\mathsf{NC}^{\mathsf{B}}(\mathsf{n})}\big) = (-1)^n \binom{2n-1}{n}$.

We can now see that Equation (5.21) becomes, in the this particular case

$$0 = \mu(\hat{0}, \hat{1}) + \mu\big(\hat{0}, K^{-1}(\omega)\big) + \sum_{|i| \leqslant n-1} \mu(\hat{0}, \pi_i) \tag{5.34}$$

which implies that

$$\begin{aligned}
-\mu(\hat{0}, \hat{1}) &= (-1)^{n-1} \binom{2n-3}{n-1} + (2n-2) \cdot (-1)^{n-1} \cdot \frac{(2n-2)!}{(n-1)!\; n!} \\
&= (-1)^{n-1} \cdot \frac{(2n-3)!}{(n-1)!\; (n-2)!} \cdot \left(1 + \frac{2(2n-2)}{n}\right) \\
&= (-1)^{n-1} \cdot \frac{(2n-3)!}{(n-1)!\; (n-2)!} \cdot \frac{5n-4}{n}.
\end{aligned}$$

We have thus obtained that

$$\mu(\hat{0}, \hat{1}) = \mu\left(\hat{0}_{\mathsf{NC}^{\mathsf{B}}(\mathsf{n\text{-}1})}, \hat{1}_{\mathsf{NC}^{\mathsf{B}}(\mathsf{n\text{-}1})}\right) = (-1)^n \cdot \frac{(2n-3)!}{n!\; (n-2)!} \cdot (5n-4). \tag{5.35}$$

■

5.4.3 Remark. By comparing Equations (2.31) and (5.24) it is easy to check that

$$M_n^{B} \neq M_{n-1,1}^{B}, n \geqslant 3,$$

which implies that

$$\mathsf{NC}^{\mathsf{B}}(\mathsf{n\text{-}1,\ 1}) \not\simeq \mathsf{NC}^{\mathsf{B}}(\mathsf{n}), n \geqslant 3.$$

For $n = 2$ we have $M_{1,1}^{B} = M_2^{B}$ $(= 3)$; this equality had to be true since $\mathsf{NC}^{\mathsf{B}}(\mathsf{1,\ 1}) = \mathsf{NC}^{\mathsf{B}}(\mathsf{2})$. But for larger values of n we find that

$$M_{2,1}^{B} = -11 \neq -10 = M_3^{B}, \quad M_{3,1}^{B} = 40 \neq 35 = M_4^{B} \quad \text{etc.}$$

Chapter 6

Type D Annular Posets

6.1 Non-Crossing Partitions of Type D

Type D non-crossing partitions were introduced fairly recently in [AR04]. The present section is intended to provide the reader with the definition "type D non-crossing partitions" as it appeared in the cited paper.

The notations which we employ to define type D non-crossing partitions are consistent with the ones used in the thesis (specifically with the notations from Chapter 2) but some of them will differ from the ones used in [AR04].

6.1.1 Type D Partitions. A partition $\pi \in \Pi(\mathsf{n})$ is called a *type D partition* if it is of type B and if it satisfies the following property: the zero-block, if present, does not consist of a single pair $\{i, -i\}$. The set of all type D partitions, denoted $\Pi^{\mathsf{D}}(\mathsf{n})$ is the subposet of the poset of non-crossing partitions of type B defined below

$$\Pi^{\mathsf{D}}(\mathsf{n}) \overset{\text{def}}{=} \{\pi \in \Pi^{\mathsf{B}}(\mathsf{n}) \mid B \quad \text{zero-block of} \quad \pi \quad \Longrightarrow B \neq \{i, -i\}, 1 \leqslant i \leqslant n\}. \tag{6.1}$$

We have seen in the previous section that the non-crossing condition in type B is inherited from the "original" non-crossing condition from type A. However, in type D this is not the case. The non-crossing condition, as it appeared in [AR04] goes as follows.

6.1.2 Type D Non-Crossing Partitions. Let us label the vertices of a regular $(2n-2)$-gon, going in clockwise order, by $1, 2, \ldots, n-1, -1, -2, \ldots, -(n-1)$. The centroid of the same $(2n-2)$-gon is labeled by n and $-n$, simultaneously.

Now let $\pi \in \Pi^{\mathsf{D}}(\mathsf{n})$ and B a block of π. Denote the convex hull of the points coming from the block B by $\rho(B)$.

i. The blocks A and B of the partition $\pi \in \Pi^{\mathsf{D}}(\mathsf{n})$ are said to *cross* if $\rho(A) \neq \rho(B)$ and if the intersection of the relative interior of $\rho(A)$ and $\rho(B)$ is nonempty.

ii. A partition $\pi \in \Pi^{\mathsf{D}}(\mathsf{n})$ is said to be *non-crossing* if no two of its blocks cross. The poset of type D non-crossing partitions is thus defined as

$$\mathsf{NC}^{\mathsf{D}}(\mathsf{n}) \overset{\text{def}}{=} \{\pi \in \Pi^{\mathsf{D}}(\mathsf{n}) \mid \pi \quad \text{is non-crossing}\}. \tag{6.2}$$

One of the main results in [AR04] is given in the proposition below (see Proposition 3.1 in [AR04]).

6.1.3 Proposition. *The poset $\mathsf{NC}^{\mathsf{D}}(\mathsf{n})$ is a lattice.*

Figure 6.1 shows the Hasse diagram of $\mathsf{NC}^{\mathsf{D}}(3)$. The partitions are drawn exactly as in [AR04].

The Narayana numbers and the Möbius function of $\mathsf{NC}^{\mathsf{D}}(\mathsf{n})$ are also given in the same paper (see Theorem 1.2 in [AR04]). We include them for completeness.

6.1.4 Theorem. *i.* $\mathit{card}\{\pi \in \mathsf{NC}^{\mathsf{D}}(\mathsf{n}) \mid \mathit{rank}\,\pi = k\} = \binom{n}{k}^2 - \frac{n}{n-1}\binom{n-1}{k}\binom{n-1}{k-1}$.

ii. $\mathit{card}\,\mathsf{NC}^{\mathsf{D}}(\mathsf{n}) = \binom{2n}{n} - \binom{2n-2}{n-1}$.

iii. $\mu_{\mathsf{NC}^{\mathsf{D}}(\mathsf{n})}(\hat{0}, \hat{1}) = (-1)^n \left(2\binom{2n-2}{n} - \binom{2n-3}{n-1}\right)$.

6.2 $\mathsf{NC}^{\mathsf{D}}(\mathsf{n}) = \mathsf{NC}^{\mathsf{D}}(\mathsf{n}\text{-}1,\ 1)$

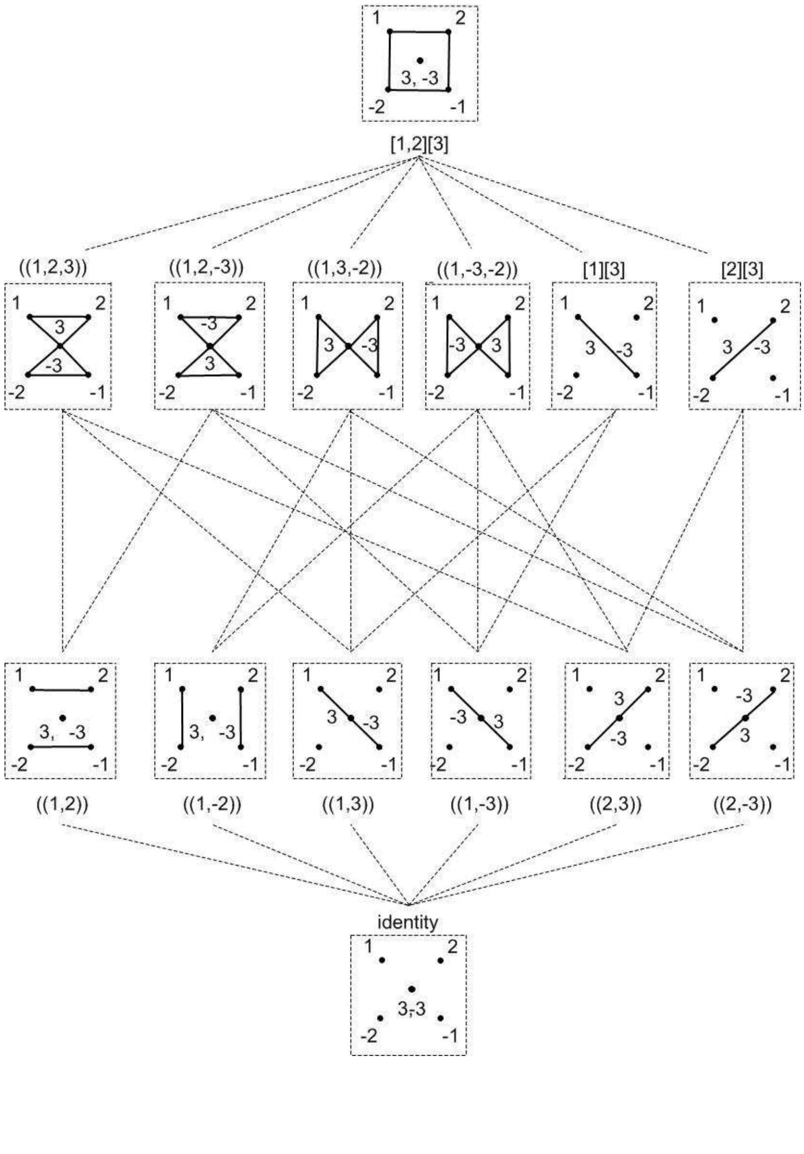
In Section 3.3 we have seen that the Weyl group D_n is a subgroup of index 2 in B_n and, more importantly, that the length function on D_n coincides with the restriction of the length function from B_n. Equation (3.24) read:

$$\tau \leqslant_{D_n} \sigma \Longleftrightarrow \tau \leqslant_{B_n} \sigma, \quad \forall \quad \tau, \sigma \in D_n. \tag{3.24}$$

Considering *type D annular objects* comes thus naturally.

• In the disc case the forward cycle γ_o which is not an *even* permutation (it has only one inversion) and therefore γ_o does not belong to D_n (but γ_o does belong to B_n).

Figure 6.1: The lattice $\mathsf{NC}^{\mathsf{D}}(3)$



• Since we have seen that in the type B annular case things turn out nicely, at the level of permutations as well as at the level of partitions, we make the following definition.

6.2.1 Definition. Paralleling the B case, we define the set of *type D annular non-crossing permutations* to be

$$\mathfrak{S}_{\mathsf{nc}}^{\mathsf{D}}(X,\gamma) \quad \stackrel{\text{def}}{=} \quad \mathfrak{S}_{\mathsf{ann\text{-}nc}}(X,\gamma) \cap D_n. \tag{6.3}$$

An immediate corollary of the corresponding type B result (Theorem 4.2.3) is that the set of type D annular permutations is an interval of the group D_n.

6.2.2 Corollary. $\mathfrak{S}_{nc}^{D}(X,\gamma) = [\varepsilon,\gamma]_{D_n}$.

Proof.

$$\begin{aligned}
[\varepsilon,\gamma]_{D_n} &= [\varepsilon,\gamma]_{B_n} \cap D_n && \text{[because of equation 3.24]}\\
&= \mathfrak{S}_{\mathsf{nc}}^{\mathsf{B}}(X,\gamma) \cap D_n && \text{[by Theorem 4.2.3]}\\
&= [\mathfrak{S}_{\mathsf{ann\text{-}nc}}(X,\gamma) \cap B_n] \cap D_n && \text{[by definition 4.2.1 of } \mathfrak{S}_{\mathsf{nc}}^{\mathsf{B}}(X,\gamma)\,]\\
&= \mathfrak{S}_{\mathsf{ann\text{-}nc}}(X,\gamma) \cap D_n && \text{[since } D_n \subseteq B_n]\\
&= \mathfrak{S}_{\mathsf{nc}}^{\mathsf{D}}(X,\gamma)\,. && \text{[by definition 6.2.1 of } \mathfrak{S}_{\mathsf{nc}}^{\mathsf{D}}(X,\gamma)\,]
\end{aligned}$$

■

Similarly, the counterpart of type D for Theorem 4.2.18 is a corollary of Theorem 4.2.18.

6.2.3 Corollary. *Let us denote*

$$NC^{D}(p,\, q) := \{\widetilde{\Omega}(\tau) \mid \tau \in \mathfrak{S}_{nc}^{D}(X,\gamma)\,\}. \tag{6.4}$$

Then the map

$$\mathfrak{S}_{nc}^{D}(X,\gamma) \ni \tau \mapsto \widetilde{\Omega}(\tau) \in NC^{D}(p,\, q) \tag{6.5}$$

is a poset isomorphism, where $\mathfrak{S}_{nc}^{D}(X,\gamma)$ is partially ordered by "$\leqslant_{D_n}$" as an interval of D_n, while $NC^{D}(p,\, q)$ is partially ordered by reversed refinement "$\leqslant$".

Proof. ¿From the equivalence in (3.24) it follows that the partial order considered on $\mathfrak{S}_{\mathsf{nc}}^{\mathsf{D}}(X,\gamma)$ is the one induced from $\mathfrak{S}_{\mathsf{nc}}^{\mathsf{B}}(X,\gamma)$. On the other hand it is clear that the partial order on $\mathsf{NC}^{\mathsf{D}}(\mathsf{p},\,\mathsf{q})$ is the one induced from $\mathsf{NC}^{\mathsf{B}}(\mathsf{p},\,\mathsf{q})$ (since for $\pi,\rho \in \mathsf{NC}^{\mathsf{D}}(\mathsf{p},\,\mathsf{q})$ the inequality "$\pi \leqslant \rho$" means that every block of ρ is a union of blocks of π, and this is independent of whether π, ρ are viewed as elements of $\mathsf{NC}^{\mathsf{D}}(\mathsf{p},\ \mathsf{q})$ or as elements of $\mathsf{NC}^{\mathsf{B}}(\mathsf{p},\ \mathsf{q})$). But then the fact that in (6.5) we have a poset isomorphism follows by appropriately restricting the poset isomorphism from Theorem 4.2.18. ■

Finally, let us discuss the counterpart of type D for Theorem 5.2.1. This does hold, that is, $\mathsf{NC}^{\mathsf{D}}(\mathsf{n}\text{-}1,\ 1)$ *is* a lattice with respect to the partial order given by reversed refinement "$\leqslant$". But this is not an immediate corollary of Theorem 5.2.1. Indeed, $\mathsf{NC}^{\mathsf{D}}(\mathsf{n}\text{-}1,\ 1)$ is a *subposet* of $\mathsf{NC}^{\mathsf{B}}(\mathsf{n}\text{-}1,\ 1)$, but is not a *sublattice* of $\mathsf{NC}^{\mathsf{B}}(\mathsf{n}\text{-}1,\ 1)$ – for $\pi, \rho \in \mathsf{NC}^{\mathsf{D}}(\mathsf{n}\text{-}1,\ 1)$), the meet of π and ρ in $\mathsf{NC}^{\mathsf{D}}(\mathsf{n}\text{-}1,\ 1)$ doesn't generally coincide with the "intersection meet" $\pi \wedge \rho$ from the lattice $\Pi(X)$ of all partitions of the set X ! The result is however true by the work of Athanasiadis and Reiner in [AR04].

We conclude this section by pointing out a couple of clues that have to be followed in order to make the connection between the poset $\mathsf{NC}^{\mathsf{D}}(\mathsf{n})$ from [AR04] and the poset $\mathsf{NC}^{\mathsf{D}}(\mathsf{n}\text{-}1,\ 1)$ of this paper. The construction made in [AR04] goes by drawing the points $1, 2, \ldots, n-1, -1, -2, \ldots, -(n-1)$ around a circle, and by placing both n and $-n$ at the center of the circle. But if instead of putting n and $-n$ right at the center we put them on a small circle concentric with the one containing $\pm 1, \pm 2, \ldots, \pm(n-1)$, then the partitions considered in the definition of $\mathsf{NC}^{\mathsf{D}}(\mathsf{n})$ (see Section 3 in [AR04]) become *annular* non-crossing. Another point in [AR04] which looks puzzling at first sight is that if a partition $\pi \in \mathsf{NC}^{\mathsf{D}}(\mathsf{n})$ has an "inversion-invariant" block (also called in [AR04] a "zero-block" – a block B such that $B = -B$), then $\pm n$ are forced to belong to that block. But this corresponds exactly to the passage from $\Omega(\tau)$ to $\widetilde{\Omega}(\tau)$ discussed in the paragraph which precedes Theorem 4.2.18. Indeed, if a permutation $\tau \in \mathfrak{S}^{\mathsf{D}}_{\mathsf{nc}}(X, \gamma)$) has inversion-invariant orbits, then it turns out that τ must have exactly two such orbits, M and N, where $M \subseteq \{1, \ldots, n-1\} \cup \{-1, \ldots, -(n-1)\}$ and N is forced to be $\{n, -n\}$; so the partition $\widetilde{\Omega}(\tau)$ has exactly one inversion-invariant block, $M \cup N$, which is forced to contain $\pm n$.

Figure 6.2 shows the Hasse diagram for the lattice $\mathsf{NC}^{\mathsf{D}}(2,\ 1)$ (which is, of course, the same lattice as $\mathsf{NC}^{\mathsf{D}}(3)$). We have included this diagram to illustrate more clearly how type D non-crossing partitions can be viewed as annular objects.

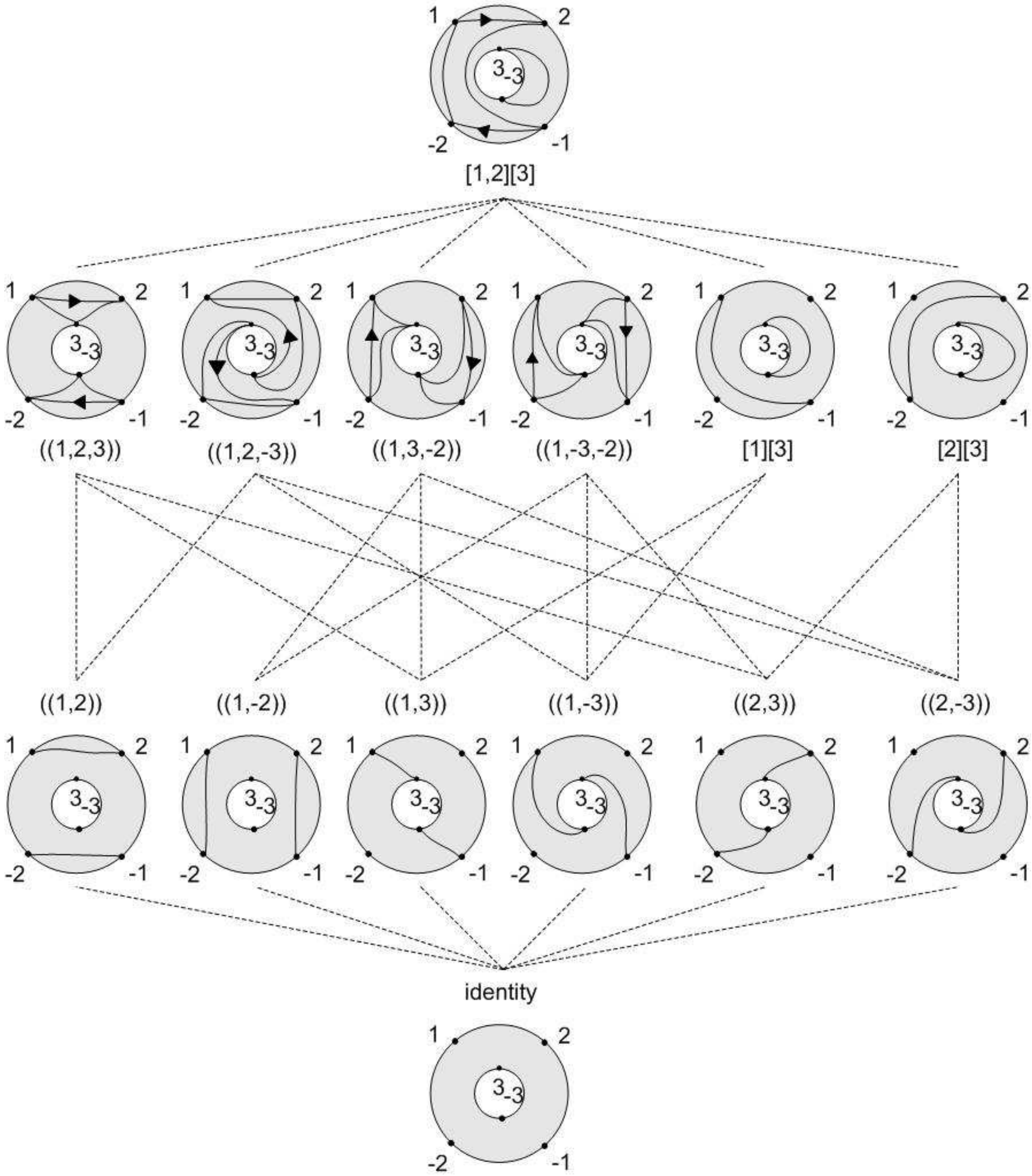


Figure 6.2: The lattice $\mathsf{NC}^{\mathsf{D}}(2, 1)$

Chapter 7

$\mathbb{G}$ -valued Non-Crossing Cumulants

7.1 Overview of Free Independence.

The definition of the concept of $\mathbb{B}$ - valued probability space, as originally introduced by D. Voiculescu goes as follows.

7.1.1 Definition. A $\mathbb{B}$ - valued non-commutative probability space is a pair $(\mathcal{M}, E)$ where $\mathcal{M}$ is a unital algebra over the algebra $\mathbb{C}$ of complex numbers, such that

$$\mathbb{B} \subseteq \mathcal{M}, \qquad \text{as a unital subalgebra,} \tag{7.1}$$

and $E : \mathcal{M} \longrightarrow \mathbb{B}$ is complex-linear and satisfies

$$\begin{cases} E(b) = b, \forall b \in \mathbb{B}, \quad \text{and} \\ E(b_1 x b_2) = b_1 E(x) b_2, \quad \forall x \in \mathcal{M}, b_1, b_2 \in \mathbb{B}. \end{cases} \tag{7.2}$$

A linear functional satisfying conditions (7.2) above is called *conditional expectation.*

7.1.2 Non-commutative probability spaces. A prominent particular case is obtained when we specialize to the case $\mathbb{B} = \mathbb{C}$. In this situation we will re-denote E by φ and the pair $(\mathcal{M}, \varphi)$ will be simply called a *non-commutative probability space.* An element of $a \in \mathcal{M}$ is called a *random variable* and the values $\varphi(a^n)$ are called the *moments* of a.

A non-commutative probability space is thus a pair $(\mathcal{M}, \varphi)$ with the properties that $\mathcal{M}$ is a unital algebra over $\mathbb{C}$ (with unit denoted $1_{\mathcal{M}}$) and φ is a $\mathbb{C}$ - linear functional on $\mathcal{M}$ satisfying

$$\varphi(1_{\mathcal{M}}) = 1. \tag{7.3}$$

Equation (7.3) is the particular case of Equation (7.2) when $\mathbb{B} = \mathbb{C}$.

Non-commutative probability spaces provide the natural framework for the theory of *free probability.*

7.1.3 Freeness. Let $(\mathcal{M}\ , \varphi)$ be a non-commutative probability space. The unital subalgebras $\mathcal{A}_1\ , \mathcal{A}_2\ , \ldots, \mathcal{A}_k$ of $\mathcal{M}$ are said to be *freely independent*, with respect to φ, if the following implication holds.

$$\begin{cases} n \geqslant 1 \quad \text{and} \quad 1 \leqslant i_1, \ldots, i_n \leqslant k, i_1 \neq i_2 \neq \ldots \neq i_n \\ a_1 \in \mathcal{A}_{i_1}\ , a_2 \in \mathcal{A}_{i_2}\ , \ldots, a_n \in \mathcal{A}_{i_n} \\ \varphi(a_1) = \varphi(a_2) = \ldots = \varphi(a_n) = 0 \end{cases} \implies \varphi(a_1 a_2 \cdots a_n) = 0. \tag{7.4}$$

The random variables $a_1, \ldots, a_n$ are called freely independent if the algebras they generate are.

7.1.4 Examples of computations with free random variables. Let a be a random variable in $\mathcal{M}$. It is clear that

$$\varphi(a - \varphi(a) \cdot 1_{\mathcal{M}}\ ) = \varphi(a) - \varphi(a) \cdot \varphi(1_{\mathcal{M}}\ ) = \varphi(a) - \varphi(a) = 0. \tag{7.5}$$

The assignment $\mathcal{M}\ \ni a \longmapsto a - \varphi(a) \cdot 1_{\mathcal{M}}$ is called *centering* of the random variable a.

The following examples illustrate how *mixed moments* of free random variables can be computed from the individual moments of the variables, using the method of centering.

i. Let a and b be random variables freely independent in $(\mathcal{M}\ , \varphi)$. We have that

$$\begin{aligned} 0 &= \varphi\big((a - \varphi(a) \cdot 1_{\mathcal{M}}\ ) \cdot (b - \varphi(b) \cdot 1_{\mathcal{M}}\ )\big) \\ &= \varphi(ab) - \varphi(a \cdot 1_{\mathcal{M}}\ )\varphi(b) - \varphi(a)\varphi(1_{\mathcal{M}}\ \cdot b) + \varphi(a)\varphi(b)\varphi(1_{\mathcal{M}}\ ) \\ &= \varphi(ab) - \varphi(a)\varphi(b). \end{aligned}$$

We have thus obtained that

$$\varphi(ab) = \varphi(a)\varphi(b).$$

More generally, one finds that

$$\varphi(a^n \cdot b^m \cdot a^r) = \varphi(a^{n+r})\varphi(b^m), \qquad \forall n, m, r \geqslant 0.$$

ii. By a similar computation, starting from

$$\varphi\big((a-\varphi(a)\cdot 1_{\mathcal{M}})(b-\varphi(b)\cdot 1_{\mathcal{M}})(a-\varphi(a)\cdot 1_{\mathcal{M}})(b-\varphi(b)\cdot 1_{\mathcal{M}})\big)=0$$

one obtains that

$$\varphi(abab)=\varphi(aa)\varphi(b)\varphi(b)+\varphi(a)\varphi(a)\varphi(bb)-\varphi(a)\varphi(b)\varphi(a)\varphi(b). \tag{7.6}$$

7.2 Cumulant Functionals on a Scarce $\mathbb{A}$ -Valued Probability Space

We now consider another variation around the concept of $\mathbb{B}$ -valued non-commutative probability space, where we no longer ask that $\mathbb{B} \subseteq \mathcal{M}$, as in Equation (7.1). We fix a unital commutative algebra $\mathbb{A}$ over $\mathbb{C}$. The unit of $\mathbb{A}$ is denoted by $1_{\mathbb{A}}$.

7.2.1 Definition. The pair $(\mathcal{M}, \Psi)$ is called a *scarce* $\mathbb{A}$ *-valued probability space* if

$$\Psi:\mathcal{M}\longrightarrow\mathbb{A}\ :1_{\mathcal{M}}\overset{\Psi}{\longmapsto}1_{\mathbb{A}}\quad\text{is }\mathbb{C}\text{-linear}, \tag{7.7}$$

where $\mathcal{M}$ is a unital algebra over $\mathbb{C}$ and $1_{\mathcal{M}}$ is the unit of $\mathcal{M}$.

7.2.2 Remark. As already pointed out above, we no longer ask that $\mathbb{A}$ is canonically embedded as a unital subalgebra inside $\mathcal{M}$ and that Ψ is a conditional expectation. This explains the choice of the adjective "scarce" in Definition 7.2.1. "Scarce" is an ad-hoc term and it is used in this thesis only.

7.2.3 Definitions, Notations and Remarks. Let $(\mathcal{M}, \Psi)$ be a scarce $\mathbb{A}$ -valued probability space.

i. For $n \geqslant 1$ define $\lambda_n^{\psi}:\mathcal{M}^n\longrightarrow\mathbb{A}$ by

$$\lambda_n^{\psi}(a_1,a_2\ldots,a_n)\overset{\text{def}}{=}\Psi(a_1a_2\cdots a_n). \tag{7.8}$$

ii. More generally, for $n \geqslant 1$ and $\pi\in\mathsf{NC}^{\mathsf{A}}(\mathsf{n})$ consider the $\mathbb{C}$ -multilinear map

$$\lambda_\pi^{\psi}:\mathcal{M}^n\longrightarrow\mathbb{A},$$

defined by

$$\lambda_\pi^{\psi}(a_1,a_2,\ldots,a_n)\overset{\text{def}}{=}\prod_{A\text{ block of }\pi}\lambda_{|A|}^{\psi}\big((a_1,\ldots,a_n)\,|_A\big), \tag{7.9}$$

where we have used the following notation: if the block $A = \{i_1, \ldots, i_s\}$ then

$$(a_1, \ldots, a_n)\,|_A \;\stackrel{\text{def}}{=} a_{i_1} \cdot a_{i_2} \cdots a_{i_s}.$$

iii. Let us note that (7.9) is indeed more general than (7.8) as $\lambda^{\psi}_{n} = \lambda^{\psi}_{1_n}$, where 1_n is the partition of $\mathsf{NC}^{\mathsf{A}}(\mathsf{n})$ made with a single block. For every $n \geqslant 1$ and for every $\pi \in \mathsf{NC}^{\mathsf{A}}(\mathsf{n})$, λ^{ψ}_{n} and λ^{ψ}_{π} are called the *moment functionals* of $(\mathcal{M}\ , \Psi)$. Let us also note the product on the right hand-side of (7.9) is well-defined due to the commutativity of $\mathbb{A}$.

7.2.4 Definition. Let $(\mathcal{M}\ , \Psi)$ be a scarce $\mathbb{A}$ -valued probability space. For $n \geqslant 1$ define $\kappa^{\psi}_{n} : \mathcal{M}^{\ n} \longrightarrow \mathbb{A}$ by

$$\kappa^{\psi}_{n}(a_1, \ldots, a_n) \;\stackrel{\text{def}}{=} \sum_{\pi \in \mathsf{NC}^{\mathsf{A}}(\mathsf{n})} \lambda^{\psi}_{\pi}(a_1, \ldots, a_n)\mu(\pi, 1_n), \tag{7.10}$$

and, for every $\rho \in \mathsf{NC}^{\mathsf{A}}(\mathsf{n})$ define $\kappa^{\psi}_{\rho} : \mathcal{M}^{\ n} \longrightarrow \mathbb{A}$ by

$$\kappa^{\psi}_{\rho}(a_1, \ldots, a_n) \;\stackrel{\text{def}}{=} \sum_{\substack{\pi \in \mathsf{NC}^{\mathsf{A}}(\mathsf{n}) \\ \pi \leqslant \rho}} \lambda^{\psi}_{\pi}(a_1, \ldots, a_n)\mu(\pi, 1_n), \tag{7.11}$$

where 1_n denotes the maximal element of $\mathsf{NC}^{\mathsf{A}}(\mathsf{n})$.

Note that $\kappa^{\psi}_{n} = \kappa^{\psi}_{1_n}$ and that every κ^{ψ}_{ρ} is $\mathbb{C}$ -multilinear, as a linear combination of the $\mathbb{C}$ -multilinear functionals λ^{ψ}_{π}. The maps defined in (7.11) and (7.10) are called *non-crossing cumulant functionals* of $(\mathcal{M}\ , \Psi)$.

7.2.5 Proposition. *Consider the scarce $\mathbb{A}$ -valued probability space. The formula which defines the cumulants κ^{ψ}_{ρ} in terms of λ^{ψ}_{π} can be reversed in the following way: for $n \geqslant 1$ and $\sigma \in NC^A(n)$ we have*

$$\lambda^{\psi}_{\sigma}(a_1, \ldots, a_n) = \sum_{\substack{\rho \in NC^A(n) \\ \rho \leqslant \sigma}} \kappa^{\psi}_{\rho}(a_1, \ldots, a_n). \tag{7.12}$$

Thus, in particular, when $\sigma = 1_n$ we get that

$$\lambda^{\psi}_{n}(a_1, \ldots, a_n) = \sum_{\substack{\rho \in NC^A(n) \\ \rho \leqslant 1_n}} \kappa^{\psi}_{\rho}(a_1, \ldots, a_n) = \sum_{\rho \in NC^A(n)} \kappa^{\psi}_{\rho}(a_1, \ldots, a_n). \tag{7.13}$$

Proof. For the proof of the equality (7.12) we start with the right hand-side of the stated equality and get that

$$\begin{aligned}
\sum_{\substack{\rho\in \mathsf{NC}^{\mathsf{A}}(\mathsf{n})\\ \rho\leqslant\sigma}} \kappa^{\psi}_{\rho}(a_1,\dots,a_n) &= \sum_{\rho\leqslant\sigma}\Big(\sum_{\pi\leqslant\rho}\lambda^{\psi}_{\pi}(a_1,\dots,a_n)\mu(\pi,\rho)\Big)\\
&= \sum_{\pi\leqslant\sigma}\Big(\sum_{\rho\in[\pi,\sigma]}\lambda^{\psi}_{\pi}(a_1,\dots,a_n)\mu(\pi,\rho)\Big)\\
&= \sum_{\pi\leqslant\sigma}\lambda^{\psi}_{\pi}(a_1,\dots,a_n)\Big(\sum_{\rho\in[\pi,\sigma]}\mu(\pi,\rho)\Big)\\
&\qquad\qquad\text{[by distributivity and associativity in } \mathbb{A}\text{]}\\
&= \lambda^{\psi}_{\sigma}(a_1,\dots,a_n)\cdot 1. \qquad\text{[by Equations (2.8) and (2.9)]}
\end{aligned}$$

■

7.2.6 Proposition. *For $n \geqslant 1$ and $\rho\in NC^{A}(n)$ we have that*

$$\kappa^{\psi}_{\rho}(a_1,\dots,a_n) = \prod_{B\ \text{block of}\ \rho}\kappa^{\psi}_{|B|}\big((a_1,\dots,a_n)\mid_B\big). \tag{7.14}$$

Proof. The proof relies essentially on two facts. The first is that the order relation "$\leqslant$" on $\mathsf{NC}^{\mathsf{A}}(\mathsf{n})$ is given by block containment and secondly, the fact that the Möbius function is multiplicative. Indeed,

$$\begin{aligned}
\kappa^{\psi}_{\rho}(a_1,\dots,a_n) &= \sum_{\pi\leqslant\rho}\lambda^{\psi}_{\pi}(a_1,\dots,a_n)\mu(\pi,\rho) &&\text{[by definition of } \kappa^{\psi}_{\rho}\text{]}\\
&= \sum_{\pi\leqslant\rho}\Big(\prod_{A\,\text{block of}\,\pi}\lambda^{\psi}_{|A|}\big((a_1,\dots,a_n)|_A\big)\Big)\mu(\pi,\rho) &&\text{[by definition of } \lambda^{\psi}_{\pi}\text{]}\\
&= \sum_{\pi\leqslant\rho}\Big(\prod_{B\,\text{block of}\,\rho}\Big(\prod_{\substack{A\,\text{block of}\,\pi\\ \text{such that}\, A\subseteq B}}\lambda^{\psi}_{|A|}(a_1,\dots,a_n)|_A\Big)\Big)\mu(\pi,\rho)\\
&&&\text{[by definition of } \leqslant\text{]}\\
&= \sum_{\pi\leqslant\rho}\Big(\prod_{B\,\text{block of}\,\pi}\lambda^{\psi}_{\pi_B}\big((a_1,\dots,a_n)\mid_B\big)\Big)\mu(\pi,\rho) &&\text{[where } \pi_B=\pi\mid_B\in \mathsf{NC}^{\mathsf{A}}(\mathsf{B})\text{]}\\
&= \sum_{\pi\leqslant\rho}\Big(\prod_{B\,\text{block of}\,\pi}\lambda^{\psi}_{\pi_B}\big((a_1,\dots,a_n)\mid_B\big)\Big)\prod_{B\,\text{block of}\,\rho}\mu(\pi_B,1_B)\\
&&&\text{[since } \mu \text{ is multiplicative]}\\
&= \sum_{\pi\leqslant\rho}\prod_{B\,\text{block of}\,\pi}\Big(\lambda^{\psi}_{\pi_B}\big((a_1,\dots,a_n)\mid_B\big)\mu(\pi_B,1_B)\Big)
\end{aligned}$$

$$= \prod_{B \text{ block of } \rho} \underbrace{\Big(\sum_{\pi_B \in \mathsf{NC}^{\mathsf{A}}(\mathsf{B})} \lambda^{\psi}_{\pi_B}\big((a_1, \ldots, a_n) \mid_B\big) \mu(\pi_B, 1_B) \Big)}$$

$$= \prod_{B \text{ block of } \rho} \overbrace{\kappa^{\psi}_{|B|}\big((a_1, \ldots, a_n) \mid_B\big)}.$$

[the underbraced summation and the overbraced formula are equal by the definition of $\kappa^{\psi}_{|B|}$]

■

7.3 Free Independence in a Scarce $\mathbb{A}$ -valued Probability Space

In this section we continue to use of the same fixed commutative unital algebra $\mathbb{A}$ over $\mathbb{C}$ from the preceding section.

7.3.1 Definition. Let $(\mathcal{M}, \Psi)$ be a scarce $\mathbb{A}$ -valued probability space and let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k$ be unital subalgebras of $\mathcal{M}$. The subalgebras $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k$ are said to be "freely independent" with respect to Ψ if the following implication holds.

$$\begin{cases} n \geqslant 1 \quad \text{and} \quad 1 \leqslant i_1, \ldots, i_n \leqslant k \\ a_1 \in \mathcal{A}_{i_1}, a_2 \in \mathcal{A}_{i_2}, \ldots, a_n \in \mathcal{A}_{i_n} \\ \exists \quad 1 \leqslant p < q \leqslant n \quad \text{such that} \quad i_p \neq i_q \end{cases} \implies \kappa^{\psi}_n(a_1, a_2, \ldots, a_n) = 0 \tag{7.15}$$

7.3.2 Remark. Condition (7.15) states that "mixed cumulants" vanish. Thus we have defined free independence via the cumulant functionals, specifically by the requirement that mixed cumulants vanish. Condition (7.15) can be used as a recipe for computing joint moments of elements from $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k$, as shown in the following example.

7.3.3 Example. Let $(\mathcal{M}, \Psi)$ be a scarce $\mathbb{A}$ -valued probability space and take $\mathcal{A}$, $\mathcal{B}$ unital subalgebras of $\mathcal{M}$ which are freely independent with respect to Ψ.

i. Let $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Then (7.15) gives that $\kappa^{\psi}_n(a, b) = 0$. But

$$\kappa^{\psi}_n(a, b) = \Psi(ab) - \Psi(a)\Psi(b), \qquad \text{[by Equation (7.10)]}$$

hence by equating with 0 we get that

$$\Psi(ab) = \Psi(a)\Psi(b). \tag{7.16}$$

Equation (7.16) provides in some sense the justification for using the term "independence" in Definition 7.15.

ii. Let $a, a' \in \mathcal{A}$ and $b \in \mathcal{B}$. We want to compute $\Psi(aba')$ in terms of $\Psi|_{\mathcal{A}}$ and of $\Psi|_{\mathcal{B}}$. We are using the moment-cumulant formula (7.13).

$$\begin{aligned}
\underline{\Psi(aba')} &= \overset{\psi}{\lambda}_3(a, b, a') \\
&= \underbrace{\overset{\psi}{\kappa}_n(a, b, a') + \overset{\psi}{\kappa}_1(a) + \overset{\psi}{\kappa}_n(b, a') + \overset{\psi}{\kappa}_n(a, b)\overset{\psi}{\kappa}_1(a')} \\
&\quad + \overset{\psi}{\kappa}_n(a, a')\overset{\psi}{\kappa}_1(b) + \overset{\psi}{\kappa}_1(a)\overset{\psi}{\kappa}_1(b)\overset{\psi}{\kappa}_1(a') \\
&= \Big(\overset{\psi}{\kappa}_n(a, a') + \overset{\psi}{\kappa}_1(a)\overset{\psi}{\kappa}_1(a')\Big)\overset{\psi}{\kappa}_1(b) \qquad \text{[the underbraced part is 0 by (7.15)]} \\
&= \underline{\Psi(aa')\Psi(b)}. \qquad (7.17)
\end{aligned}$$

iii. Let $a, a' \in \mathcal{A}$ and $b, b' \in \mathcal{B}$. A similar calculation as the one above gives that

$$\begin{aligned}
\underline{\Psi(aba'b')} &= \overset{\psi}{\kappa}_n(a, a')\overset{\psi}{\kappa}_1(b)\overset{\psi}{\kappa}_1(b') + \overset{\psi}{\kappa}_1(a)\overset{\psi}{\kappa}_1(a')\overset{\psi}{\kappa}_n(b, b') + \overset{\psi}{\kappa}_1(a)\overset{\psi}{\kappa}_1(b)\overset{\psi}{\kappa}_1(a')\overset{\psi}{\kappa}_1(b') \\
&= \Big(\Psi(aa') - \Psi(a)\Psi(a')\Big)\Psi(b)\Psi(b') + \Psi(a)\Psi(a')\Big(\Psi(bb') - \Psi(b)\Psi(b')\Big) + \\
&\quad + \Psi(a)\Psi(b)\Psi(a')\Psi(b') \\
&= \underline{\Psi(aa')\Psi(b)\Psi(b') + \Psi(a)\Psi(a')\Psi(bb') - \Psi(a)\Psi(b)\Psi(a')\Psi(b')}. \qquad (7.18)
\end{aligned}$$

7.3.4 Proposition. *Let $(\mathcal{M}, \Psi)$ be a scarce $\mathbb{A}$ -valued probability space and suppose that $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k$ are freely independent with respect to Ψ. Then the following implication holds.*

$$\begin{cases} n \geqslant 1 \quad \textit{and} \quad 1 \leqslant i_1, \ldots, i_n \leqslant k, i_1 \neq i_2 \neq \ldots \neq i_n \\ a_1 \in \mathcal{A}_{i_1}, a_2 \in \mathcal{A}_{i_2}, \ldots, a_n \in \mathcal{A}_{i_n} \\ \Psi(a_1) = \Psi(a_2) = \ldots = \Psi(a_n) = 0 \end{cases} \implies \Psi(a_1 a_2 \cdots a_n) = 0. \quad (7.19)$$

Proof. We want to prove that implication (7.19) holds. To this end, let

$$\begin{cases} n \geqslant 1 \quad \text{and} \quad 1 \leqslant i_1, \ldots, i_n \leqslant k, i_1 \neq i_2 \neq \ldots \neq i_n \\ a_1 \in \mathcal{A}_{i_1}, a_2 \in \mathcal{A}_{i_2}, \ldots, a_n \in \mathcal{A}_{i_n} \\ \Psi(a_1) = \Psi(a_2) = \ldots = \Psi(a_n) = 0. \end{cases}$$

The goal is to prove that $\Psi(a_1 a_2 \cdots a_n) = 0$.

<u>*Case $n = 1$*</u> : is clear, since $\overset{\psi}{\kappa}_1(a) = \Psi(a)$.

Case $n \geqslant 2$: we know that

$$\begin{aligned}\Psi(a_1 a_2 \cdots a_n) &= \lambda^{\psi}_{n}(a_1, \ldots, a_n) && \text{[by definition of } \lambda^{\psi}_{n}\text{]}\\ &= \boxed{\sum_{\rho \in \mathsf{NC}^{\mathsf{A}}(\mathsf{n})} \kappa^{\psi}_{\rho}(a_1, \ldots, a_n)}. && \text{[by formula (7.12)]}\end{aligned}$$

We will show that every term in the boxed summation above is 0. In order to do so let us fix $\rho \in \mathsf{NC}^{\mathsf{A}}(\mathsf{n})$ and look at

$$\kappa^{\psi}_{\rho}(a_1, \ldots, a_n) = \underbrace{\prod_{B \text{ block of } \rho} \kappa^{\psi}_{|B|}\big((a_1, \ldots, a_n) \mid_B\big)}. \qquad \text{[which is Equation (7.14)]}$$

Case 1 : one of the blocks of ρ is a singleton, $B = \{m\}$. In this case one of the factors in the underbraced product must be $\kappa^{\psi}_{1}(a_m) = \Psi(a_m) = 0$.

Case 2 : none of the blocks of $\rho \in \mathsf{NC}^{\mathsf{A}}(\mathsf{n})$ is a singleton. If that is the case then there must exist $1 \leqslant p < q \leqslant n$ such that $B = [p, q] \cap \mathbb{Z}$ is a block of ρ and so one of the factors in the underbraced product must be $\kappa^{\psi}_{q-p+1}(a_p, a_{p+1}, \ldots, a_q) = 0$, by the vanishing of $\mathbb{A}$ -cumulants assumption.

■

7.3.5 Remark. The implication (7.19) is the usual Voiculescu condition defining free independence. In the particular case when $\mathbb{A} = \mathbb{C}$ it can be shown that, conversely, the Voiculescu condition (7.19) implies condition (7.15) of vanishing cumulants. Thus when $\mathbb{A} = \mathbb{C}$ condition (7.15) really is equivalent to the usual definition of free independence.

For an arbitrary commutative unital algebra $\mathbb{A}$ it isn't true that (7.19) $\Longrightarrow$ (7.15). Indeed, for general $\mathbb{A}$, the hypothesis that (7.15) holds isn't for instance sufficient to yield the independence formula (7.19) (when $\mathbb{A} = \mathbb{C}$ one can obtain (7.15) from (7.19) by using the method of *centering*, which is no longer available for arbitrary $\mathbb{A}$).

7.4 Scarce $\mathbb{G}$ -valued Probability Spaces and Their Connection to $\mathsf{NC}^{\mathsf{B}}(\mathsf{n})$

In this section we consider the particular case of a scarce $\mathbb{A}$ -valued probability space where the commutative unital algebra $\mathbb{A}$ is the Graßmann algebra $\mathbb{G}$.

7.4.1 Review of the Graßman algebra $\mathbb{G}$. ***i.*** Consider the vector space over $\mathbb{C}$

$$\mathbb{G} \stackrel{\text{def}}{=} \{z = z_b + \epsilon \cdot z_s \mid z_b, z_s \in \mathbb{C}\ , \quad \epsilon^2 = 0\} = \mathbb{C} \oplus \epsilon \cdot \mathbb{C}\ ,$$

with the natural addition and scalar multiplication operations. On $\mathbb{G}$ we define the operation of multiplication in the following way

$$z_1 \cdot z_2 = (z_{b_1} + \epsilon \cdot z_{s_1})(z_{b_1} + \epsilon \cdot z_{s_1}) \quad \stackrel{\text{def}}{=} \quad z_{b_1} z_{b_2} + \epsilon \cdot (z_{b_1} z_{s_2} + z_{s_1} z_{b_2}). \tag{7.20}$$

The multiplication is easily seen to be associative, distributive and commutative. $\mathbb{G}$ thus becomes a commutative algebra with unit $1_{\mathbb{G}} = 1 + \epsilon \cdot 0 = 1 \in \mathbb{C}$.

Alternatively, $\mathbb{G}$ can be viewed as an algebra of upper triangular 2×2 matrices considered with the usual matrix multiplication

$$\mathbb{G} = \left\{ z = \begin{bmatrix} z_b & z_s \\ 0 & z_b \end{bmatrix} \quad \text{with} \quad z_b, z_s \in \mathbb{C} \right\}. \tag{7.21}$$

ii. We will refer to $\mathbb{G}$ as *the Graßmann algebra*. An element $\mathbb{G} \ni z = z_b + \epsilon \cdot z_s$ will be called *a Graßmann number*, z_b will be called the *body* of the Graßmann number z and z_s will be called the *soul* of the Graßmann number z.

iii. Let us consider now the "body" and "soul" operations on $\mathbb{G}$, defined as follows

$$\mathsf{Bo} : \mathbb{G} \longrightarrow \mathbb{C} \ : z = z_b + z_s \longmapsto \mathsf{Bo}\,(z) = z_b \tag{7.22}$$

$$\mathsf{So} : \mathbb{G} \longrightarrow \mathbb{C} \ : z = z_b + z_s \longmapsto \mathsf{So}\,(z) = z_s. \tag{7.23}$$

Both body and soul operations are linear. While the body operation "Bo" is also multiplicative, "So" is not. More precisely

$$\mathsf{Bo}\,(pq) = \mathsf{Bo}\,(p)\mathsf{Bo}\,(q), \qquad p, q \in \mathbb{G}$$

$$\mathsf{So}\,(pq) = \mathsf{Bo}\,(p)\mathsf{So}\,(q) + \mathsf{So}\,(p)\mathsf{Bo}\,(q), \qquad p, q \in \mathbb{G}\ . \tag{7.24}$$

iv. Note that the equation for $\mathsf{So}\,(pq)$ extends to a product of n factors in the form

$$\mathsf{So}\,(p_1 p_2 \cdots p_n) = \sum_{m=1}^{n} \Big(\mathsf{So}\,(p_m) \cdot \prod_{\substack{1 \leqslant i \leqslant n \\ i \neq m}} \mathsf{Bo}\,(p_i) \Big) \tag{7.25}$$

If X is a set and $f : X \longrightarrow \mathbb{G}$ is a function, we will denote by $\mathsf{Bo}\,(f), \mathsf{So}\,(f)$ the functions from X to $\mathbb{C}$ defined by $\begin{cases} \big(\mathsf{Bo}\,(f)\big)(x) = \mathsf{Bo}\,\big(f(x)\big) \\ \big(\mathsf{So}\,(f)\big)(x) = \mathsf{So}\,\big(f(x)\big) \end{cases}$, $x \in X$.

7.4.2 Remark. The type B non-commutative probability spaces considered in [BGN03] can be treated as scarce $\mathbb{G}$ -valued probability spaces. Indeed, the structure considered in [BGN03] is of the form

$$(\mathcal{A}, \varphi, \mathcal{V}, f, \Phi), \tag{7.26}$$

where $(\mathcal{A}, \varphi)$ is a non-commutative probability space over $\mathbb{C}$, Φ is a two-sided action of $\mathcal{A}$ on the vector space $\mathcal{V}$, and $f : \mathcal{V} \longrightarrow \mathbb{C}$ is a linear functional. It is pointed out in [BGN03] that from the structure (7.26) one builds an algebra structure on $\mathcal{M} = \mathcal{A} \times \mathcal{V}$, with multiplication defined by

$$(a, \xi) \cdot (b, \eta) = (ab, a\eta + \xi b), \qquad a, b \in \mathcal{A}, \xi, \eta \in \mathcal{V} . \tag{7.27}$$

By considering the natural $\mathbb{C}$ -linear map

$$\Psi : \mathcal{M} \longrightarrow \mathbb{G} : (a, v) \longmapsto \Psi\big((a, v)\big) = \varphi(a) + \epsilon \cdot f(v), \qquad a \in \mathcal{A}, v \in \mathcal{V} \tag{7.28}$$

a scarce $\mathbb{G}$ -valued probability space $(\mathcal{M}, \Psi)$ is thus obtained. It is easily seen that the "cumulant functionals of type B" defined in section 6.2 of [BGN03] for $(\mathcal{A}, \varphi, \mathcal{V}, f, \Phi)$ are exactly the $\mathbb{G}$ -valued cumulant functionals for $(\mathcal{M}, \Psi)$ as defined in (7.11). Moreover, the concept of "freeness of type B" defined in Section 7 of the same paper is obtained via a condition of vanishing of mixed cumulants, thus the statement from [BGN03] that

"$(\mathcal{A}_1, \mathcal{V}_1), (\mathcal{A}_2, \mathcal{V}_2), \ldots, (\mathcal{A}_k, \mathcal{V}_k)$ are freely independent in $(\mathcal{A}, \varphi, \mathcal{V}, f, \Phi)$"

is equivalent to the fact that the unital subalgebras $\mathcal{A}_1 \times \mathcal{V}_1, \mathcal{A}_2 \times \mathcal{V}_2, \ldots, \mathcal{A}_k \times \mathcal{V}_k$ of $\mathcal{M}$ are freely independent with respect to Ψ, in the sense of the above Definition 7.15.

In conclusion, the "freeness of type B" studied in [BGN03] can be seen in the framework of the preceding sections, where we make $\mathbb{A}$ to be the Graßmann algebra $\mathbb{G}$.

One can thus hope that further study of scarce $\mathbb{G}$ -valued probability spaces (not necessarily of the kind appearing in [BGN03]) may lead to a better understanding of what "free probability of type B" should be about.

7.4.3 Definition. Let $(\mathcal{M}, \Psi)$ be a scarce $\mathbb{A}$ -valued probability space over the Graßmann algebra $\mathbb{G}$ and consider a partition with zero-block

$$\mathsf{NC}^{\mathsf{B}}(\mathsf{n}) \ni \rho = \{Z, C_1, -C_1, \ldots, C_m, -C_m\}, \quad \text{where } Z \text{ is the zero-block of } \rho.$$

For $a_1, \ldots, a_n \in \mathcal{M}$ define

$$\kappa\kappa^{\psi}_{\rho}(a_1, \ldots, a_n) \overset{\text{def}}{=} \mathsf{So}\left[\kappa^{\psi}_{|Abs(Z)|}\big((a_1, \ldots, a_n)\,|_{Z\cap[n]}\big) \cdot \prod_{j=1}^{m} \mathsf{Bo}\left(\kappa^{\psi}_{|C_j|}\big((a_1, \ldots, a_n)\,|_{Abs(C_j)}\big)\right)\right] \tag{7.29}$$

where *Abs* denotes the absolute value map, as reviewed in Section 2.3.

We then obtain the following.

7.4.4 Proposition.

$$So\,(\Psi(a_1,\ldots,a_n)) = \sum_{\substack{\rho\in NC^B(n)\\ \rho \text{ has zero-block}}} \kappa\kappa^{\psi}_{\rho}\,(a_1,\ldots,a_n). \tag{7.30}$$

Proof. We know that $\Psi : \mathcal{M} \longrightarrow \mathbb{G}$, $\Psi(a_1,\ldots,a_n) = \sum_{\pi\in\mathsf{NC}^{\mathsf{A}}(\mathsf{n})} \kappa^{\psi}_{\pi}\,(a_1,\ldots,a_n)$. Thus

$$\begin{aligned}
\mathsf{So}\,(\Psi(a_1,\ldots,a_n)) &= \sum_{\pi\in\mathsf{NC}^{\mathsf{A}}(\mathsf{n})} \mathsf{So}\left(\kappa^{\psi}_{\pi}\,(a_1,\ldots,a_n)\right) && \text{[since So is linear]}\\
&= \sum_{\pi\in\mathsf{NC}^{\mathsf{A}}(\mathsf{n})} \mathsf{So}\Big(\prod_{B\text{ block of }\pi} \kappa^{\psi}_{|B|}\left((a_1,\ldots,a_n)\,|_B\right)\Big) && \text{[by definition (7.11) of } \kappa^{\psi}_{\pi}]\\
&= \underbrace{\sum_{\pi\in\mathsf{NC}^{\mathsf{A}}(\mathsf{n})} \sum_{B\text{ block of }\pi}} \mathsf{So}\Big[\kappa^{\psi}_{|B|}\left((a_1,\ldots,a_n)\,|_B\right)\cdot \prod_{\substack{B'\text{ block of }\pi\\ B'\neq B}} \mathsf{Bo}\Big(\kappa^{\psi}_{|B'|}\left((a_1,\ldots,a_n)\,|_{B'}\right)\Big)\Big]\\
& && \text{[by (7.25)]}\\
&= \overbrace{\sum_{\substack{\rho\in\mathsf{NC}^{\mathsf{B}}(\mathsf{n})\\ \rho\text{ has zero-block}}}} \mathsf{So}\Big[\kappa^{\psi}_{|B|}\left((a_1,\ldots,a_n)\,|_B\right)\cdot \prod_{\substack{B'\text{ block of }\pi\\ B'\neq B}} \mathsf{Bo}\Big(\kappa^{\psi}_{|B'|}\left((a_1,\ldots,a_n)\,|_{B'}\right)\Big)\Big]\\
&= \sum_{\substack{\rho\in\mathsf{NC}^{\mathsf{B}}(\mathsf{n})\\ \rho\text{ has zero-block}}} \kappa\kappa^{\psi}_{\rho}\,(a_1,\ldots,a_n). && \text{[by Definition 7.4.3]}
\end{aligned}$$

For the equality between the underbraced double sum and the overbraced sum above, note that the double sum can be in fact written as the single sum indexed by the set $\{\rho \in \mathsf{NC}^{\mathsf{B}}(\mathsf{n}) \mid \rho \ \text{ has zero-block}\}$. This is possible by writing $\mathsf{NC}^{\mathsf{A}}(\mathsf{n}) \ni \pi = Abs(\rho)$ and $B = Abs(Z)$. ■

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