Spectral Analysis of Laplacians on Certain Fractals

by

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A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Doctor of Philosophy
in
Pure Mathematics

Waterloo, Ontario, Canada, 2007

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Surprisingly, Fourier series on certain fractals can have better convergence properties than classical Fourier series. This is a result of the existence of gaps in the spectrum of the Laplacian. In this work we prove a general criterion for the existence of gaps. Most of the known examples on which the Laplacians admit spectral decimation satisfy the criterion. Then we analyze the infinite family of Vicsek sets, finding an explicit formula for the spectral decimation functions in terms of Chebyshev polynomials. The Laplacians on this infinite family of fractals are also shown to satisfy our criterion and thus have gaps in their spectrum.

Acknowledgements

This work would not have been possible without the support of many people.

I feel a deep sense of gratitude to my supervisor, Kathryn Hare, who read my numerous revisions and helped make some sense of the confusion, both mathematically and grammatically in English. Her help, stimulating suggestions and encouragement helped me in all the time of my research and the writing of my thesis. Many thanks to my committee members, Kenneth Davidson, Alexandru Nica, who offered guidance and support. I am grateful to my Master's thesis adviser, Zhongkai Li, who guided me to the world of harmonic analysis.

I am lucky to work in a friendly and cheerful group of fellow students in Waterloo. I would like to thank Keon Choi, Sean Cianflone, Wefeng Ge, Shengli Wu, Dilian Yang, Yanqiao Zhang, who has always been supportive. I am especially obligated to Lemmus Chan, my officemate for almost three years, who supported me in my research work. I want to thank him for all his help, support, interest and valuable suggestions.

Finally, I would like to give my special thanks to my wife, my son and my parents whose love enabled me to complete this work.

Contents

Chapter 1

Introduction

In harmonic analysis on the circle $\mathbb{T}$, we have the following classical results about the convergence of Fourier series ([15]).

Theorem 1.1 *(M. Riesz) For $1 < p < \infty$ and $f \in L^p(\mathbb{T})$, the partial sums $S_n(f)$ of the Fourier series converge to f in L^p.*

Note that this theorem is generally not true for $p = 1$, and for a continuous function, f, the convergence need not be uniform, both due to the L^1 unboundedness of the Dirichlet kernel.

On the other hand, if we let k_n be a summability kernel, such as Fejér's kernel, and let $\sigma_n(f) = k_n * f$, then $\sigma_n(f)$ converges to f in L^p for $1 \leq p < \infty$ and converges uniformly to f in $C(\mathbb{T})$.

Another classical result is the Littlewood-Paley theorem (see [9] and references therein).

Theorem 1.2 *(Littlewood-Paley) For each $1 < p < \infty$, there exist constants A_p and B_p such that*

$$A_p \|f\|_p \leq \|(\sum_{N \in \mathbb{Z}} |\tilde{S}_N f|^2)^{1/2}\|_p \leq B_p \|f\|_p,$$

where $\tilde{S}_N f$ is the N-th dyadic partial sum of the Fourier series of f, defined by

$$\tilde{S}_N f = \begin{cases} \displaystyle\sum_{2^{N-1} \le |j| < 2^N} \hat{f}(j) e^{ijx} & \text{if } N > 0, \\ \hat{f}(0) & \text{if } N = 0, \\ \displaystyle\sum_{-2^{N-1} \le |j| < -2^N} \hat{f}(j) e^{ijx} & \text{if } N < 0. \end{cases}$$

The Littlewood-Paley theorem can be viewed as a generalization of Parseval's formula to L^p as we can rewrite this formula as

$$\|f\|_2 = \|(\sum_{N \in \mathbb{Z}} |\hat{f}(N)|^2)^{1/2} = \|(\sum_{N \in \mathbb{Z}} |\hat{f}(N) e^{iNx}|^2)^{1/2}\|_2.$$

One might ask how far we can generalize the above results when the underlying space is a fractal? To answer this question, we first need to find an orthonormal basis for the L^2 space of functions on the fractal to form Fourier series. A natural choice would be eigenfunctions of the Laplacian, like the exponentials, cosines and sines in the classical case. This leads to the question of how to define a natural Laplacian on fractals.

Generally speaking, there are two ways to define a Laplacian on fractals. One is from a probabilistic point of view and the other is from an analytic point of view, and they are complementary to each other.

The first example of a Laplacian on fractal was defined on the Sierpinski gasket $\mathcal{SG}$ as a diffusion process by S. Kusuoka and S. Goldstein ([20] and [11]). J. Kigami constructed the Laplacian analytically, both as a renormalized limit of difference operators and through a weak formulation using the theory of Dirichlet forms [16]. Later, the theory of Laplacians was extended to other fractals, including nested fractals and p.c.f. self-similar sets by T. Lindsrøm [22] and J. Kigami [17].

Analysis on fractals has bloomed since then and many classical results of smooth analysis have found their analogues on the "rough" objects (see [18] and [33] and references therein). In particular, the spectrum and eigenfunctions of the Laplacian on the Sierpinski gasket were studied by M. Fukushima and T. Shima in [10] and [30] using the so-called spectral decimation method, which originated in physics literature ([1], [27] and [29]) and was generalized by T. Shima [30] and A. Teplyaev [26]. The spectra of Laplacians on a number of other fractals, including the the level-3 Sierpinski gasket $\mathcal{SG}_3$ [8], the infinite Sierpinski gasket [34], and the Pentagasket [2] has been analyzed either numerically or using the spectral decimation method.

There are startling difference from the smooth analysis. For example, there exist localized eigenfunctions [10]; there are eigenvalues with multiplicities higher than any given integer [33]; and the Weyl ratio, $\frac{\pi(x)}{x^{d/2}}$ with $\pi(x)$ being the eigenvalue counting function, does not have a limit for any choice of d ([10] and [19]).

One of the most striking results is that there can be gaps in the spectrum of the Laplacian. (For a given infinite sequence $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_k \leq \cdots$, we say that there exist **gaps** in the sequence if $\limsup_{k\geq 1} \dfrac{\alpha_{k+1}}{\alpha_k} > 1$.) This result was proved for the standard Laplacian on $\mathcal{SG}$ by M. Gibbons, A. Raj and R. Strichartz in [12] using results obtained by M. Fukushima and T. Shima [10].

The existence of gaps is an interesting phenomenon in itself, but it also has important applications to harmonic analysis on fractals. As R. Strichartz said in [32], "$\cdots$ the fractal world resembles the smooth world to some degree, but everything is worse." However, "when it comes to the convergence of Fourier series, things might be better $\cdots$." More precisely, he proved that the Fourier series (along a subsequence of the eigenvalues where the gaps occur) on the Sierpinski gasket converge for $1 \leq p < \infty$ and the Littlewood-Paley theorem holds for $1 < p < \infty$. One explanation of the convergence result is that the Dirichlet kernel behaves more like a summability kernel because of the gaps in the spectrum [32].

In Chapter 3 of this work, we prove general theorems giving criteria for the existence of gaps that is related to properties of the spectral decimation function. The known examples, including $\mathcal{SG}$ and $\mathcal{SG}_3$, are shown to satisfy our criterion, as does the infinite family of fractals, called the n-branch tree-like fractals.

In Chapters 4 and 6 we analyze in detail the infinite (symmetric) family of Vicsek sets. These are an interesting family of fractals whose Laplacians have been studied in a number of papers [14], [25], [26] and [30]. We verify that our criterion for gaps applies to this family as well. We also determine the ordering of the Dirichlet eigenvalues. This is the first infinite family to be analyzed whose spectral decimation functions have unbounded degrees. In Chapter 5, we use hypergraph theory to establish an upper bound for the Dirichlet and Neumann eigenvalues of the normalized (graph) Laplacian on a special class of self-similar graphs. In Chapter 2, we review basic techniques in the spectral decimation method.

Chapter 2

Spectral decimation method

2.1 P.C.F. self-similar sets and self-similar measures

Let $\{F_s\}_{s\in S}$ be a set of contraction maps with contraction ratios c_s, where S is a finite set with cardinality $|S|$. The unique non-empty compact set satisfying

$$\mathbf{K} = \bigcup_{1\leq s\leq |S|} F_s\mathbf{K}$$

is called a **self-similar set**. Its Hausdorff dimension d is the exponent such that $\sum_{s=1}^{|S|} c_i^d = 1$. We denote $W_n(S) = S^n$, the collection of words of length n. Further we let $W_*(S) = \cup_{n\geq 0} W_n(S)$.

For $w = w_1w_2\cdots w_n \in W_n(S)$, let

$$F_w = F_{w_1} \circ F_{w_2} \circ \cdots \circ F_{w_n},$$

and $\mathbf{K}_w = F_w\mathbf{K}$. $F_\varnothing$ is the identity map on $\mathbf{K}$.

We assume that there exists a continuous surjection $\pi :\ S^{\mathbb{N}} \to \mathbf{K}$ satisfying $\pi\circ s = F_s\circ\pi$ for every $s \in S$, where s denotes the map from $S^{\mathbb{N}}$ to $S^{\mathbb{N}}$ defined by $s(w_1w_2\cdots) =$

$sw_1w_2\cdots$. (π is usually the map $\pi(w) = \bigcap_{n=1}^{\infty} F_{w|n}(\mathbf{K})$, where $w|n = w_1\cdots w_n \in W_n$ [18].) The **critical set** $\mathcal{C}$ and the **post critical set** $\mathcal{P}$ are defined respectively by

$$\mathcal{C} = \pi^{-1}(\bigcup_{s,t\in S, s\neq t} (\mathbf{K}_s \cap \mathbf{K}_t)), \quad \mathcal{P} = \bigcup_{n\geq 1} \sigma^n(\mathcal{C}),$$

where $\sigma : \ S^{\mathbb{N}} \to S^{\mathbb{N}}$ is the one-sided shift map which removes the first letter of a given word. A connected self-similar set is called **post critically finite** (abbreviation p.c.f.) if the post critical set is finite [18]. Roughly speaking, those fractals are connected, but not too much. We can control the connectedness by a finite set.

We take $G_0 = (V_0,\ E_0)$ to be the complete graph on V_0, where V_0 is defined as

$$V_0 = \pi(\mathcal{P}).$$

Then define the **set of vertices at step** m, V_m, recursively by

$$V_m = \bigcup_s F_s V_{m-1}$$

and define the edge relation $(x,\ y) \in E_m$ (or $x \sim_m y$) to hold if there exist a word w of length $|w| = m$ such that $x,\ y \in F_w V_0$. It is not hard to see that the V_m's build up an ascending chain:

$$V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots .$$

Set

$$V_* = \bigcup_m V_m$$

and call the elements in V_* **vertices** and the elements in V_0 **boundary points**. It is clear that V_* is dense in $\mathbf{K}$ under the usual Euclidean metric and so we can use the sequence of finite sets V_m to approximate $\mathbf{K}$. Moreover, for continuous functions defined on $\mathbf{K}$ it is sufficient to understand their values on V_*.

Examples of p.c.f. self-similar sets include the Sierpinski gasket $\mathcal{SG}$, the level-3 Sierpinski gasket $\mathcal{SG}_3$ [8], the Vicsek set $\mathcal{VS}$ [25], and the tree-like fractal [30]. We describe the constructions of those sets below.

Example 1. (The Sierpinski Gasket $\mathcal{SG}$) Let $q_1 = (0,\ 0),\ q_2 = (1,\ 0)$, and $q_3 = (1/2, \sqrt{3}/2)$. Let $F_1,\ F_2,\ F_3$ be three contraction maps from $\mathbb{R}^2$ to $\mathbb{R}^2$ defined by

$$F_s(x) = q_s + 1/2(x - q_s),\ 1 \leq s \leq 3.$$

The Sierpinski gasket $\mathcal{SG}$ is defined as the unique compact $\mathbf{K}$ such that $\mathbf{K} = \bigcup_{1 \leq s \leq 3} F_s(\mathbf{K})$. It has Haudorff dimension $\frac{\log 3}{\log 2}$. Write $\dot{s} = (s, s, \cdots)$. Then the critical set

$$\mathcal{C} = \{(1\dot{3}), (3\dot{1}), (1\dot{2}), (2\dot{1}), (2\dot{3}), (3\dot{2})\}$$

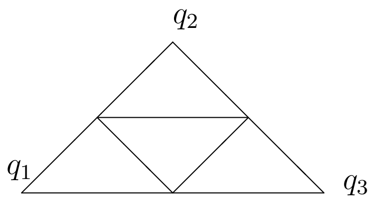
and the post critical set

$$\mathcal{P} = \{(\dot{1}), (\dot{2}), (\dot{3})\}.$$

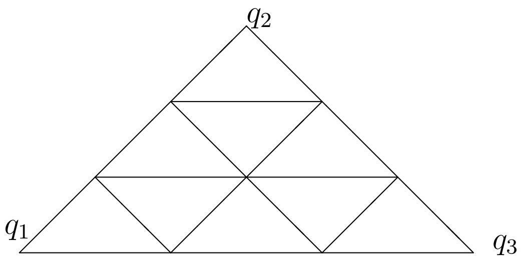
(See Examples 5.15 in [3].)

We can also think of this fractal as being constructed in the following fashion. We start with the equilateral triangle with vertices $q_1,\ q_2$ and q_3, then remove the open middle inscribed equilateral triangle of 1/4 the area. Remove the corresponding open middle inscribed triangles from each of the remaining 3 equilateral triangles and continue this way.

Clearly $V_0 = \{q_1,\ q_2,\ q_3\}$. The first step graph $G_1 = (V_1,\ E_1)$ is shown in the following figure.

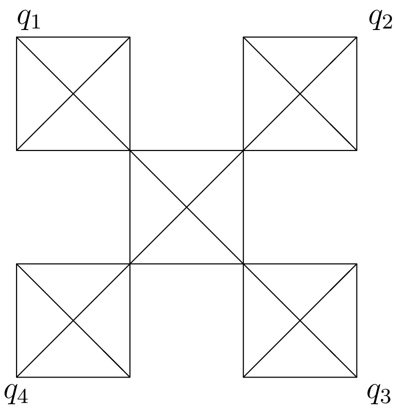


Example 2. (The level-3 Sierpinski Gasket $\mathcal{SG}_3$) To get the level-3 Sierpinski gasket, instead of bisecting the sides of a triangle and keeping the three of the four smaller triangles, we trisect the sides and keep the six of the nine smaller triangles. Then we repeat this process. See the following figure for the first step graph.

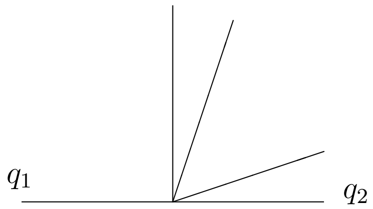


For this fractal, V_0 is the same as in the case of $\mathcal{SG}$.

Example 3. (The Vicsek Set $\mathcal{VS}$) For the Vicsek set, we start with a square with four corners $q_1,\ q_2,\ q_3$ and q_4, cut it into 3×3 equal pieces, and keep the four corner squares and the one in the center. Then we repeat the process. In this case, $V_0 = \{q_1,\ q_2,\ q_3,\ q_4\}$. See Section 4.1 for more details. The first step graph is shown in the following figure.



Example 4. (The Tree-Like Fractal) For the tree-like fractal, we stick 3 branches of length 1/2 to the middle of a unit interval with end points q_1 and q_2. We then repeated this process to each of the five intervals. V_0 consists of the two end points of the unit interval, q_1 and q_2. See Example 3 in Section 3.2 for more details. Below is the first step graph of this fractal.



For any given self-similar set $\mathbf{K}$, we choose a set of probability weights $\{p_i\}_{i=1,\cdots,|S|}$ so that $p_i > 0$ for all $i = 1, \cdots, |S|$ and $\sum_{i=1}^{|S|} p_i = 1$. Then we define a measure μ on $\mathbf{K}$ so that

$$\mu(\mathbf{K}_w) = p_w = p_{w_1} \cdots p_{w_m}$$

for any $w = w_1 \cdots w_m \in W_*$. This μ is called a **self-similar measure** on $\mathbf{K}$. It is the unique probability measure satisfying

$$\mu = \sum_{i=1}^{|S|} p_i \mu \circ F_i^{-1}.$$

2.2 Laplacian on finite graphs

For any set S, we use $\ell(S)$ to denote the set of real valued functions on S and

$$\ell_0(V_m) = \{f \in \ell(V_m) : \ f(p) = 0 \text{ for } p \in V_0\}.$$

For two sets U and V, we define

$$L(U,\ V) = \{A : \ \ell(U) \to \ell(V) \text{ and A is linear}\}.$$

In particular, $L(V)$ means $L(V,\ V)$.

Let G be a simple, finite graph with the set of vertices $V(G)$ and the set of edges $E(G)$. We say that two vertices $x,\ y$ are **neighbors** if they are connected by exactly one edge, denoted by $e(x,\ y)$, in the graph. For any vertex x, $\deg x$ is called the **vertex degree** of x in the graph and

$$\deg x = \sum_{e(x,y)\in E(G)} 1$$

We will assume that the graph G does not have any isolated points and so $\deg x$ is always positive.

First we give the notions of standard Laplacian and normalized Laplacian defined on the graph G.

Definition 2.1 *Given a function $f \in \ell(V(G))$, the* ***graph (non-normalized) Laplacian*** *of f at a vertex x is defined as*

$$\Delta f(x) = \sum_{e(x,\ y)\in E(G)} (f(y) - f(x)),$$

and the ***normalized (probability) Laplacian*** *is defined as*

$$\widehat{\Delta} f(x) \ = \ \frac{1}{\deg x} \sum_{e(x,\ y)\in E(G)} (f(y) - f(x)).$$

The symmetric matrix D corresponding to Δ is called the ***Laplacian matrix*** *and it has the expression*

$$D_{i,j} = \begin{cases} -\deg x_i, & \text{if } i = j, \\ 1, & \text{if } i \neq j \text{ and } e(x_i, x_j) \in E(G), \\ 0, & \text{otherwise,} \end{cases}$$

for $x_i,\ x_j \in V(G)$.

Suppose we are given a self-similar p.c.f. set $\mathbf{K}$. Assume that $\mathbf{K}$ can be approximated by a sequence of graphs $G_m = (V_m,\ E_m)$, where G_0 is the complete graph with vertices V_0 in the post-critical set and G_m is obtained from G_{m-1} by applying the contraction maps F_j to G_{m-1} and identifying points that are identical in $\mathbf{K}$. More precisely,

$$V_m = \bigcup_{w\in W_m} F_w(V_0) \quad (m \geq 1),$$

and $\mathbf{K} = \overline{\cup V_m}$. Furthermore, we denote $V_m^0 = V_m \setminus V_0$ and $V_*^0 = \cup_{m\geq 0} V_m \setminus V_0$.

Let D be the Laplacian matrix (a difference operator) on $G_0 = (V_0,\ E_0)$. By using the bijection $F_w|_{V_0} :\ V_0 \to B_w = F_w(V_0)$, we can identify $\ell(B_w)$ with $\ell(V_0)$ and hence regard D as an element of $L(B_w)$. Let $R_w \in L(V_m,\ B_w)$ be the restriction map to B_w, that is $R_w f = f|_{B_w}$. We then construct a matrix $H_m \in L(V_m)$ such that

$$H_m = \sum_{w\in W_m} R_w^t D R_w.$$

It is clear that $H_m = \Delta_m$, where Δ_m is the graph Laplacian on V_m as in Definition 2.1. Furthermore, if we define a measure $\hat{\mu}_m$ on V_m by

$$\hat{\mu}_m(x) = (\sum_{w\in W_m} R_w^t(-D)R_w)_{x,x},$$

then $\hat{\mu}_m(x) = \deg x$ and it is $(-H_m)_{x,x}$. The normalized Laplacian on V_m satisfies

$$\widehat{\Delta}_m f(x) = \frac{H_m f(x)}{\hat{\mu}_m(x)},$$

for $f \in \ell(V_m)$.

More generally, we can take D to be a symmetric matrix with row sums zero, and entries that are positive off the diagonal and negative on the diagonal. Choose $\mathbf{r} = (r_1^{-1},\ r_2^{-1},\ \cdots,\ r_{|S|}^{-1}) \in \ell(S)$ and call the number $r_0^{-1} := \sum_{s\in S} r_s^{-1}$ the **measure factor**. The reason of this terminology will be apparent later.

We define H_m by

$$H_m = \sum_{w\in W_m} r_w^{-1} R_w^t D R_w \tag{2.2.1}$$

and call $(H_m,\ \mathbf{r})$ the **generalized/combinatorial Laplacian with weight r** on the graph G_m. Decompose H_m into

$$H_m = \begin{bmatrix} T_m & J_m^t \\ J_m & X_m \end{bmatrix}, \tag{2.2.2}$$

where $T_m \in L(V_0),\ J_m \in L(V_0,\ V_m^0)$ and $X_m \in L(V_m^0)$. In particular, write $T = T_1,\ J = J_1$ and $X = X_1$.

If we define a more general measure $\hat{\mu}_m$ on V_m as

$$\hat{\mu}_m(x) = (\sum_{w \in W_m} r_w^{-1} R_w^t(-T)R_w)_{x,x},$$

then we can also generalize our definition of the normalized Laplacian on G_m to

$$\widehat{\Delta}_m f(x) \ := \frac{H_m f(x)}{\hat{\mu}_m(x)},$$

for $f \in \ell(V_m)$.

As a convention, if we do not specify what the weight vector $\mathbf{r}$ is, we will always assume that H_m and $\widehat{\Delta}_m$ are the generalized non-normalized Laplacian and the normalized Laplacian respectively on the graph G_m.

2.3 The spectral decimation function and forbidden eigenvalues

From now on, we shall assume that our p.c.f. fractal $\mathbf{K}$, generated by the graphs G_m, is connected and

$$\#(B_s \cap V_0) \leq 1 \text{ for every } s \in S, \tag{2.3.1}$$

where $B_s = F_s(V_0)$.The latter assumption implies that all vertices in V_0 are no longer neighbors in $G_1 = (V_1,\ E_1)$, so T is a diagonal matrix.

We define the diagonal matrices M and W such that $M_{i,i} = -X_{i,i}$ and $W = \begin{bmatrix} -T & 0 \\ 0 & M \end{bmatrix}$, where X and T are as in (2.2.2). It is clear that

$$\widehat{\Delta}_0 = -T^{-1}D,$$
$$\widehat{\Delta}_1 = W^{-1}H_1.$$

We also denote $G(\lambda) = (X + \lambda M)^{-1}$ if the inverse matrix exists.

Definition 2.2 *The generalized Laplacian* $(H_m, \mathbf{r})$ *is said to have a* ***strong harmonic structure*** *if there exist rational functions* $K_D(\lambda)$ *and* $K_T(\lambda)$ *such that when* $X + \lambda M$ *is invertible, then*

$$T - J^t(X + \lambda M)^{-1} J = K_D(\lambda) D + K_T(\lambda) T. \tag{2.3.2}$$

We call $K_D(0)^{-1}$ *the* ***energy renormalization constant****.*

We denote

$$\mathfrak{F} := \{\lambda \in \mathbf{R} : \ K_D(\lambda) = 0 \ \textit{or} \ \det(X + \lambda M) = 0\}$$

and call elements in $\mathfrak{F}$ *the* ***forbidden eigenvalues****. Moreover, we let*

$$\mathfrak{F}_k := \{\lambda \in \mathfrak{F} : \ \lambda \textit{ is an eigenvalue of } -\widehat{\Delta}_k\}$$

and call the elements in $\mathfrak{F}_k$ *the* ***forbidden eigenvalues at step*** k *or* ***initial eigenvalues at step*** k*. We call the function*

$$R(\lambda) := \frac{\lambda - K_T(\lambda)}{K_D(\lambda)}$$

the ***spectral decimation function****.*

Example 5. (The Sierpinski gasket) In this case, the set of boundary points $V_0 = \{(0,0),\ (1,0),\ (1/2, \sqrt{3}/4)\}$. We take D to be the Laplacian matrix on V_0, i.e.

$$D = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix},$$

and $\mathbf{r} = (1,\ 1,\ 1)$. The matrix corresponding to the standard Laplacian on V_1 is

$$H_1 = \begin{bmatrix} -2 & 0 & 0 & 0 & 1 & 1 \\ 0 & -2 & 0 & 1 & 0 & 1 \\ 0 & 0 & -2 & 1 & 1 & 0 \\ 0 & 1 & 1 & -4 & 1 & 1 \\ 1 & 0 & 1 & 1 & -4 & 1 \\ 1 & 1 & 0 & 1 & 1 & -4 \end{bmatrix}.$$

Hence

$$T = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix},$$

$$J = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix},$$

$$X = \begin{bmatrix} -4 & 1 & 1 \\ 1 & -4 & 1 \\ 1 & 1 & -4 \end{bmatrix}.$$

It follows then

$$(X + \lambda M)^{-1} = \frac{1}{2(4\lambda - 5)(2\lambda - 1)} \begin{bmatrix} 4\lambda - 3 & -1 & -1 \\ -1 & 4\lambda - 3 & -1 \\ -1 & -1 & 4\lambda - 3 \end{bmatrix}$$

and

$$\begin{aligned} T - J^t(X + \lambda M)^{-1}J \quad &= \quad \frac{1}{(4\lambda - 5)(2\lambda - 1)} \times \\ & \begin{bmatrix} -2(8\lambda^2 - 12\lambda + 3) & 3 - 2\lambda & 3 - 2\lambda \\ 3 - 2\lambda & -2(8\lambda^2 - 12\lambda + 3) & 3 - 2\lambda \\ 3 - 2\lambda & 3 - 2\lambda & -2(8\lambda^2 - 12\lambda + 3) \end{bmatrix}. \end{aligned}$$

From this, we can easily see that H_1 has a strong harmonic structure and we can read off the functions K_D and K_T:

$$K_D(\lambda) = \frac{3 - 2\lambda}{(4\lambda - 5)(2\lambda - 1)}, \; K_T(\lambda) = \frac{2\lambda}{2\lambda - 1}.$$

Hence the spectral decimation function is

$$R(\lambda) = \lambda(5 - 4\lambda),$$

and the forbidden eigenvalues are $1/2,\ 5/4,$ and $3/2$.

Suppose we are given a p.c.f. self-similar set (also satisfying our assumption (2.3.1)) and the generalized Laplacian has a strong harmonic structure. We then have the following spectral decimation property for the normalized Laplacian.

Proposition 2.3 *(Shima [30]) Suppose the generalized Laplacian has a strong harmonic structure. We have the following collective results:*
(1) If f is an eigenfunction of $-\widehat{\Delta}_{m+1}$ with eigenvalue λ, i.e. $-\widehat{\Delta}_{m+1}f = \lambda f$, and $\lambda \notin \mathfrak{F}$, then $-\widehat{\Delta}_m f|_{V_m} = R(\lambda) f|_{V_m}$,
(2) Conversely, if $-\widehat{\Delta}_m f = R(\lambda) f$, and $\lambda \notin \mathfrak{F}$, then there exists a unique extension $\overline{f}$ of f such that $-\widehat{\Delta}_{m+1}\overline{f} = \lambda\overline{f}$.

Proof. The full proof of this proposition can be found in [30]. Here we only show the case when $m = 1$. In other words, we shall prove the following claim:

For $\lambda \notin \mathfrak{F}$, λ is an eigenvalue of $-\widehat{\Delta}_1$ iff $R(\lambda)$ is an eigenvalue of $-\widehat{\Delta}_0$.

Before we prove the claim, let us first recall the notion of Schur complement from linear algebra. For a block structured matrix

$$I = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

with D invertible, the **Schur complement** of D in I is defined as $A - BD^{-1}C$. One important property of the Schur complement is that there exists an invertible matrix, say P, such that

$$P \begin{bmatrix} A & B \\ C & D \end{bmatrix} P^t = \begin{bmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{bmatrix}.$$

We are going to use this property to prove our claim.

Note that

$$H_1 + \lambda W = \begin{bmatrix} T - \lambda T & J^t \\ J & X + \lambda M \end{bmatrix},$$

so the Schur complement of $X+\lambda M$ in $H_1+\lambda W$ is $(T-\lambda T)-J^t(X+\lambda M)^{-1}J$ and hence there exists an invertible matrix P such that

$$P(H_1+\lambda W)P^t = \begin{bmatrix} (T-\lambda T)-J^tG(\lambda)J & 0 \\ 0 & X+\lambda M \end{bmatrix}.$$

If the Laplacian has a strong harmonic structure, it follows that

$$P(H_1+\lambda W)P^t = \begin{bmatrix} K_D(\lambda)D+(K_T(\lambda)-\lambda)T & 0 \\ 0 & X+\lambda M \end{bmatrix}.$$

Hence for those λ's which are not forbidden eigenvalues (i.e. $K_D(\lambda)\neq 0$ and $X+\lambda M$ is invertible), we have

$$H_1+\lambda W \ \text{ is singular iff } K_D(\lambda)D+(K_T(\lambda)-\lambda)T \text{ is singular.}$$

or equivalently,

$$W(W^{-1}H_1+\lambda) \text{ is singular iff } K_D(\lambda)T(T^{-1}D+\frac{K_T(\lambda)-\lambda}{K_D(\lambda)}) \text{ is singular.}$$

As W and $K_D(\lambda)T$ are invertible, the theorem follows by noting that $-W^{-1}H_1=-\widehat{\Delta}_1$ and $T^{-1}D=-\widehat{\Delta}_0$.

□

For the spectral decimation function, we have the following property.

Proposition 2.4 *(Shima [30]) The spectral decimation function R satisfies*

$$R(0)=0, \ \textit{and } R'(0)=\frac{1}{K_D(0)r_0}>1, \tag{2.3.3}$$

where we recall that $r_0^{-1}=\sum_{s\in S}r_s^{-1}$ is the measure factor.

Proof. For any set U, we denote the constant function 1 on U by 1_U. Since

$$0=H_1 1_{V_1}=\begin{bmatrix} T & J^t \\ J & X \end{bmatrix}\begin{bmatrix} 1_{V_0} \\ 1_{V_1\setminus V_0} \end{bmatrix},$$

we have the equations

$$\begin{cases} T1_{V_0} = -J^t 1_{V_1 \setminus V_0} \\ J1_{V_0} = -X1_{V_1 \setminus V_0} \end{cases}$$

We let $\lambda = 0$ in (2.3.2), and rewrite (2.3.2) as

$$J^t X^{-1} J = T - K_D(\lambda)D - K_T(\lambda)T.$$

Then we multiply both sides by $1^t_{V_0}$ on the left and 1_{V_0} on the right. Applying the above equations to the left-hand side of the resulting equality, we obtain

$$\begin{aligned} 1^t_{V_0}(J^t X^{-1} J)1_{V_0} &= 1^t_{V_0} J^t X^{-1}(-X1_{V_1 \setminus V_0}) \\ &= -1^t_{V_0} J^t 1_{V_1 \setminus V_0} \\ &= 1^t_{V_0} T 1_{V_0}. \end{aligned}$$

Since $1^t_{V_0} T 1_{V_0} = Tr(T)$, we have

$$1^t_{V_0}(J^t X^{-1} J)1_{V_0} = Tr(T). \tag{2.3.4}$$

On the other hand, notice that the sum of each column of the Laplacian matrix D is 0. Therefore,

$$1^t_{V_0} D 1_{V_0} = 0.$$

It follows that

$$1^t_{V_0}(T - K_T(0)T - K_D(0)D)1_{V_0} = (1 - K_T(0))Tr(T) \tag{2.3.5}$$

Comparing (2.3.4) and (2.3.5), we must have

$$Tr(T) = (1 - K_T(0))Tr(T).$$

Since $Tr(T) \neq 0$, $K_T(0)$ must be zero. It was proved in [30] that $K_D(0) > 0$, hence

$$R(0) = \frac{0 - K_T(0)}{K_D(0)} = 0.$$

Then we note that

$$\frac{d}{d\lambda}(X+\lambda M)^{-1} = -(X+\lambda M)^{-1}M(X+\lambda M)^{-1}$$

Hence differentiating both sides of (2.3.2) with respect to λ at 0 gives us

$$J^t X^{-1} M X^{-1} J = K'_D(0)D + K'_T(0)T. \tag{2.3.6}$$

Again, we multiply both sides of (2.3.6) by $1^t_{V_0}$ on the left and 1_{V_0} on the right. The left-hand side of (2.3.6) then becomes

$$\begin{aligned}
&1^t_{V_0}(J^t X^{-1} M X^{-1} J)1_{V_0} = -1^t_{V_1\setminus V_0} X^t (X^{-1} M X^{-1})(-X 1_{V_1\setminus V_0}) \\
&= 1^t_{V_1\setminus V_0} M 1_{V_1\setminus V_0} = Tr(M).
\end{aligned}$$

The second equality above holds because X is a symmetric matrix.

On the other hand, after multiplying on the left by $1^t_{V_0}$ and 1_{V_0} on the right, the right-hand side of (2.3.6) becomes

$$\begin{aligned}
1^t_{V_0}(K'_D(0)D + K'_T(0)T)1_{V_0} &= 1^t_{V_0} K'_T(0) T 1_{V_0} \\
&= Tr(T)K'_T(0).
\end{aligned}$$

Therefore,

$$K'_T(0) = \frac{Tr(M)}{Tr(T)}.$$

Note that

$$\begin{aligned}
R'(0) &= \frac{1-K'_T(0)}{K_D(0)} \\
&= [K_D(0)]^{-1}(1 - \frac{Tr(M)}{Tr(T)}) \\
&= [K_D(0)]^{-1}\frac{Tr(H_1)}{Tr(T)}
\end{aligned}$$

Note that by Definition (2.2.1),

$$H_1 = \sum_{w\in W_1} r_w^{-1} R^t_w D R_w.$$

It follows that

$$\frac{Tr(H_1)}{Tr(T)} = r_0^{-1},$$

the measure factor. Hence we obtain

$$R'(0) = \frac{1}{K_D(0) r_0}.$$

By Theorem 4.10 in [17], we know that for some $s \in S$, $r_s < K_D(0)^{-1}$. We thus have $R'(0) > 1$.

□

Let

$$\rho = R'(0) = \frac{1}{K_D(0) r_0}$$

and call it the **Laplacian renormalization constant**.

2.4 Laplacians on fractals and spectral decimation

We can define a (normalized) Laplacian Δ on $\mathbf{K}$ as a limit of the normalized discrete Laplacians $\widehat{\Delta}_m$.

Definition 2.5 *Let*

$$\mathcal{D} = \{u \in C(\mathbf{K}) : \text{ there exists a function } f \in C(\mathbf{K}) \text{ and } \lim_{m\to\infty} \rho^m \widehat{\Delta}_m u(x) = f(x) \text{ uniformly for } x \in V_* \setminus V_0\}.$$

We then define the ***(normalized) Laplacian on the fractal*** $\mathbf{K}$ *by*

$$\Delta u = f,$$

where f is the function appearing above.

If we choose D to be the Laplacian matrix on the complete graph G_0 and all r_i to be 1, then the corresponding Δ is called the **standard (normalized) Laplacian** on $\mathbf{K}$.

This Laplacian can also be obtained through a weak formulation. We use the notation $(\cdot\ ,\ \cdot)_\mu$ to denote the inner product on $L^2(\mathbf{K};\ \mu)$ with self-similar measure μ whose probabilities are given by $p_i = \frac{r_i}{r_0}$. For a finite set U, we denote the ordinary inner product on $\ell(U)$ by $< \cdot,\ \cdot >$.

Let $\mu_m = r_0^m \hat{\mu}_m$. Then it can be shown that $\{\mu_m\}_{m \geq 0}$ is weakly convergent to the self-similar measure μ. The weak formulation of the Laplacian on $\mathbf{K}$ is defined as follows.

Definition 2.6 *The Laplacian on the p.c.f. self-similar set* $\mathbf{K}$ *is defined as the self-adjoint operator* Δ_μ *on* $L^2(\mathbf{K};\ \mu)$*, which is characterized by the following relation:*

$$-(\Delta_\mu f,\ f)_\mu = \lim_{m\to\infty} -\rho^m (f,\ \widehat{\Delta}_m f)_m,$$

where $(\cdot,\ \cdot)_m$ *means the inner product on* $\ell(V_m)$ *weighted by* μ_m*.*

The inner product on the right-hand side of the equality in the definition can indeed be described in terms of Dirichlet forms.

Definition 2.7 *Let* $(H_m, \mathbf{r})$ *be a strong harmonic structure and let* ρ *be the Laplacian renormalization constant. Define the quadratic form* $\mathcal{E}_m$ *on* $\ell(V_m)$ *by*

$$\mathcal{E}_m(f,\ f) = -\rho^m < f,\ H_m f > .$$

Put

$$\mathcal{G}_0 = \{f \in \ell_0(V_*) :\ \lim_{m\to\infty} \mathcal{E}_m(f|_{V_m},\ f|_{V_m}) < \infty\}.$$

Define $(\mathcal{E},\ \mathcal{G}_0)$ *by*

$$\mathcal{E}(f,\ f) = \lim_{m\to\infty} \mathcal{E}_m(f|_{V_m},\ f|_{V_m}),\ \textit{for}\ f \in \mathcal{G}_0$$

The associated bilinear form is defined by the usual polarization identity:

$$\mathcal{E}(f,\ g) = 1/4[\mathcal{E}(f+g, f+g) - \mathcal{E}(f-g, f-g)].$$

Then $(\mathcal{E},\ \mathcal{G}_0)$ is a local Dirichlet form on $L^2(\mathbf{K};\mu)$ and Δ_μ as defined in Definition 2.6 is the operator associated with it. The Dirichlet form is an analog of $\int |\nabla f|^2 dx$ in the classical case (see [20] for definitions and properties of Dirichlet forms.) Recall that the classical Laplacian can be defined through the identity

$$\int |\nabla f|^2 dx = -\int f\Delta(f)dx.$$

for f with zero boundary values.

These two formulations of Laplacian described above are indeed equivalent ([18] and [33]).

Remark 2.8 *To define a Laplacian on a p.c.f fractal, we do not need a strong harmonic structure. Instead, we only require (2.3.2) to hold for* $\lambda = 0$*; i.e., there exists a number* $K_D(0)$*, such that*

$$T - J^t(X + \lambda M)^{-1}J = K_D(0)D.$$

We then define a Laplacian as the renormalized limit of the discrete Laplacian in the same way as in Definition 2.5 and Definition 2.6, where the renormalization constant is still $\rho = \frac{1}{K_D(0)r_0}$.

In some cases, spectra of Laplacians, with either Dirichlet or Neumann boundary conditions, can be obtained through a process called spectral decimation. Here by Dirichlet boundary condition, we mean the eigenfunction is zero on the boundary V_0, both for the discrete Laplacian and the Laplacian on the fractal, defined as a renormalized limit of the discrete Laplacians. The Neumann boundary condition needs a few words to explain. For discrete Laplacians, we imagine the graph G_m being imbedded in a lager graph by reflecting in each boundary vertex and we impose the step m eigenvalue equation on the even extension of the eigenfunction. (For the normalized Laplacian, the Neumann boundary condition simply means that we treat all boundary points the same as other (interior) points.) The Neumann boundary condition on the fractal means that the normal derivative of the function f on V_0 is zero, where the normal derivative of f is defined as follows.

Definition 2.9 *Let $x \in V_0$ and f be a continuous function of* $\mathbf{K}$*. We say the* ***normal derivative*** *of f, $\partial_n f(x)$, exists if the limit*

$$\lim_{m\to\infty} \rho^m \widehat{\Delta}_m f(x)$$

exists and we then define

$$\partial_n f(x) = \lim_{m\to\infty} \rho^m \widehat{\Delta}_m f(x).$$

The definition of normal derivatives corresponding to the non-normalized Laplacian is similar.

We continue to let ρ be the Laplacian renormalization constant and $\mathfrak{F}_k$ be the set of forbidden eigenvalues appearing at step k.

We start with an element in the set of forbidden eigenvalues $\mathfrak{F}$, say α, and suppose it is an eigenvalue of the discrete Laplacian $-\widehat{\Delta}_i$ for some i and associated with an eigenfunction f_i. By Proposition 2.3, we can extend $\alpha := \lambda_i$ to λ_{i+1} by applying inverses of the function R which appeared in that proposition. We will also obtain an eigenfunction f_{i+1} corresponding to $-\widehat{\Delta}_{i+1}$ whose restriction back to V_i is f_i. This iterative process can be repeated over and over again as long as we do not encounter any of the forbidden eigenvalues. Thus we will have a sequence $\{\lambda_k\}_{k\geq i}^{\infty}$ and a function f defined on $V_* = \bigcup_{m\geq 0} V_m$ whose restriction is an eigenfunction of the discrete Laplacian at any step. We call this function defined on V_* a **pre-eigenfunction**. Shima proved (Proposition 3.1 in [30]) that if the limit

$$\rho^i \lim_{m} \rho^m \lambda_m \tag{2.4.1}$$

converges to some positive number κ, then it must be an eigenvalue of the Laplacian $-\Delta$ on the fractal and the associated pre-eigenfunction is extended to an eigenfunction on the fractal belonging to κ.

This method certainly gives us a lot of eigenvalues of the Laplacian, but not necessarily all of them. However, if it does happen that all eigenvalues are constructed this way, then we say that the Laplacian admits **spectral decimation**.

Definition 2.10 *For a p.c.f. self-similar set* $\mathbf{K}$*, we say that the Laplacian,* $-\Delta$*, with Dirichlet boundary conditions, admits* ***spectral decimation with spectral decimation function*** R *if all eigenvalues of* $-\Delta$ *are of the form*

$$\rho^i \lim_{m\to\infty} \rho^m \phi_w(x),\ \ x \in \mathfrak{F}_{i+1} \text{ and } i \in \mathbb{N}\cup\{0\},$$

where $w = w_m \cdots w_1$ *is a word of length* m *with each* $w_j \in \{0, \cdots, \#(\text{inverse functions of } R)-1\}$*, and* $\phi_w = \phi_{w_m}\cdots\phi_{w_1}$ *with* ϕ_k *being the* $k+1$*-th inverse functions of* R *from bottom to top. In particular,* ϕ_0 *is the bottom inverse function of* R*.*

Remark 2.11 *(1) Note that in the above definition,* ϕ_w *has to be chosen such that the limit exists. As Shima proved [30], if* $\phi_0(z) < z$ *for all positive real numbers* z *on its domain, then the existence of the limit is equivalent to the condition that after finite steps, we only apply the bottom inverse function* ϕ_0*.*

In fact, if $\phi_0(z) < z$*, then* $\phi_0^{(n)}(z)$*, the* n *iterations of* ϕ_0*, is decreasing in* n *and so converges. But the limit is a fixed point of* ϕ_0 *and so it is 0. Hence after applying* ϕ_0 *certain times, the resulting value will be close to 0. Proposition 2.4 then tells us*

$$\phi_0(z) = \frac{1}{\rho}z + O(z^2), \text{ as } z \to 0.$$

Hence the limit (2.4.1) exists if we only apply ϕ_0 *after finite steps to do the extensions. Conversely, if we do not apply* ϕ_0 *after finite steps, then* $\phi_w(z)$ *with* $|w| = m$ *does not converge to 0 as* $m \to \infty$*. Since* $\rho > 1$*,* $\lim_m \rho^m\phi_w(z)$ *does not exist.*

Therefore, we conclude that if $\phi_0(z) < z$ *for all positive* z *on its domain, all eigenvalues of* $-\Delta$ *must be of the form*

$$\rho^i \lim_{m\to\infty} \rho^m \phi_0^{(m-j)}\phi_{w'}(z), \tag{2.4.2}$$

where $z \in \mathfrak{F}_{i+1}$*,* $|w'| = j$*, and* $i \in \mathbb{N}\cup\{0\}$*.*

(2) This definition can be applied to Laplacians with Neumann boundary conditions with $\mathfrak{F}$ *and* $\mathfrak{F}_i$ *replaced by* $\mathfrak{F}\cup\{0\}$ *and* $\mathfrak{F}_i\cup\{0\}$ *since constant functions are always Neumann eigenfunctions corresponding to eigenvalue zero.*

We denote the functions

$$y_0(z) = \lim_{m\to\infty} \rho^m \phi_0^{(m)}(z), \text{ if } w = 0, \tag{2.4.3}$$

$$y_w(z) = \lim_{m\to\infty} \rho^m \phi_0^{(m-j)} \phi_w(z), \text{ if } |w| = j, \ w_j \neq 0, \tag{2.4.4}$$

for all z such that the limits on the right-hand side exist. We are particularly interested about properties of the function y_0, which will be used in the proof of our criteria for finding spectral gaps.

Lemma 2.12 *If ϕ_0 is strictly convex on $[0,\ b]$, where b is the largest forbidden eigenvalue, and $\phi_0(b) < b$, then y_0 exists, is convex on its domain, strictly increasing and continuous.*

Proof. We first note that $y_0(z)$ is well-defined for all $0 \leq z \leq b$ because the strict convexity of ϕ_0 and $\phi_0(0) = 0$ gives

$$\frac{\phi_0(z)}{z} < \frac{\phi_0(b)}{b} < 1,$$

for all z in $[0,\ b]$ and hence by the comments in the previous remark this is enough to prove y_0 exists.

We note that continuity follows from convexity. We now prove convexity. We choose z_1 and z_2 from the domain of ϕ_0 with $0 \leq z_1 < z_2$ and let $0 < t < 1$. Since ϕ_0 is (strictly) convex, so is $\phi_0^{(m)}$ for all m. Hence,

$$\frac{t\, y_0(z_1) + (1-t)\, y_0(z_2)}{y_0\big(t\, z_1 + (1-t)\, z_2\big)} = \lim_m \frac{t\, \phi_0^{(m)}(z_1) + (1-t)\, \phi_0^{(m)}(z_2)}{\phi_0^{(m)}\big(t\, z_1 + (1-t)\, z_2\big)} \geq 1.$$

Next we prove that y_0 is strictly increasing. Since $y_0(0) = 0$, there is no loss of generality in taking $0 < z_1 < z_2$. By the strict convexity of ϕ_0,

$$\frac{\phi_0(z_1)}{z_1} < \frac{\phi_0(z_2)}{z_2},$$

which we rewrite as

$$\frac{\phi_0(z_2)}{\phi_0(z_1)} > \frac{z_2}{z_1}.$$

Repeated applications of this inequality gives

$$\frac{y_0(z_2)}{y_0(z_1)} = \lim_m \frac{\phi_0^{(m)}(z_2)}{\phi_0^{(m)}(z_1)} \geq \frac{z_2}{z_1} > 1.$$

□

Shima has proved the following theorem about the relationship between strong harmonic structure and spectral decimation.

Theorem 2.13 *(Shima [30]) Suppose the Laplacian $-\Delta$ has a strong harmonic structure. If*

$$|S| < \frac{1}{K_D(0)r_0}, \tag{2.4.5}$$

then $-\Delta$ admits spectral decimation with a rational function R.

If we take all r_i to be 1, then $1/r_0 = |S|$. By Theorem 4.10 in [17], we know that for some $s \in S$, $r_s < K_D(0)^{-1}$. Therefore $K_D(0) < 1$ and $|S| < \frac{1}{K_D(0)r_0}$. Hence we have the following corollary.

Corollary 2.14 *If a Laplacian has the strong harmonic structure and all $r_i = 1$, then it admits spectral decimation.*

The eigenfunctions of the Laplacian on the fractal $\mathbf{K}$, with either Dirichlet or Neumann boundary conditions, form an orthonormal basis for $L^2(\mathbf{K},\ \mu)$ and thus we can decompose any function $f \in L^2$ as

$$f = \sum_{j=1}^{\infty} c_j u_j \text{ with } c_j = \int_{\mathbf{K}} f u_j d\mu,$$

where $\{c_j\}$ are the the eigenvalues of the Laplacian and $\{u_j\}$ are the corresponding eigenfunctions. We call this sum the **Fourier series** of f and its partial sums converge to f in L^2 norm.

Chapter 3

Criteria for spectral gaps

First let us define what we mean when we say there exist gaps for a given sequence.

Definition 3.1 *Let $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_k \leq \cdots$ be an infinite sequence. We say that there exist* ***gaps*** *in the sequence if* $\limsup \dfrac{\alpha_{k+1}}{\alpha_k} > 1$.

M. Gibbons, A. Raj and R. Strichartz proved there were gaps in the spectrum of the standard Laplacian on the Sierpinski gasket $\mathcal{SG}$ in [12]. Together with a suitable heat kernel estimate (see [13]) and using a generic argument developed by X. Duong, E. Ouhabaz and A. Sikora ([7], Theorem 3.1 and 6.2), R. Strichartz proved the following theorems concerning convergence of the Fourier series on $\mathcal{SG}$ and the Littlewood-Paley theorem.

With either Dirichlet or Neumann boundary condition, there is a complete orthonormal basis of eigenfunctions, say $-\Delta u_j = \lambda_j u_j$, $j = 1, 2, \cdots$, and every L^2 function f has a Fourier series

$$f = \sum_{j=1}^{\infty} c_j u_j, \text{ with } c_j = \int_{\mathcal{SG}} f u_j d\mu,$$

where μ is the Hausdorff measure on $\mathcal{SG}$.

Theorem 3.2 *(Theorem 1 in [32]) Let $\{N_m\}$ be a sequence of integers such that $\dfrac{\lambda_{N_m+1}}{\lambda_{N_m}} - 1$ is bounded away from zero. Then the partial sums of the Fourier series $S_{N_m}f$ converge to f as $m \to \infty$ in L^p for $f \in L^p$ $(1 \le p < \infty)$ and uniformly if f is continuous.*

Theorem 3.3 *(Theorem 2 in [32]) Let $1 < p < \infty$. Let*

$$Sf(x) = (\sum_{m=1}^{\infty} |S_m f(x)|^2)^{1/2},$$

for

$$S_m f(x) = \sum_{j=N_{m-1}+1}^{N_m} c_j u_j(x),$$

where $\{N_m\}$ is the same sequence as in the above theorem. Then there exist constants A_p and B_p such that

$$A_p \parallel f \parallel_p \le \parallel Sf \parallel_p \le B_p \parallel f \parallel_p .$$

The argument used to prove the above two theorems can be applied to any other fractals where there exist gaps in the spectrum of the Laplacian and the heat kernel estimates specified in [7] are satisfied.

In this chapter we shall give three criteria for the existence of gaps in the spectrum of Laplacians on fractals which admit spectral decimation. We continue to denote by R the spectral decimation function, $\mathfrak{F}$ the set of forbidden eigenvales and ρ the Laplacian renormalization constant.

The proofs of the theorems we give are similar. In all cases, we show there are gaps in the spectrum between numbers A_k and B_k, where A_k, B_k are of the form

$$\begin{aligned} A_k &= \rho^k \lim_{m\to\infty} \rho^m \phi_0^{(m-1)}(x), \\ B_k &= \rho^k \lim_{m\to\infty} \rho^m \phi_0^{(m-1)}(y) \end{aligned}$$

with x, y consecutive elements of the set $R^{-1}(\mathfrak{F})$.

Following the proofs of the theorems we will give examples of fractals to which the theorems can be applied. The first theorem applies to the Sierpinski gaskets, $\mathcal{SG}$ and $\mathcal{SG}_3$. It is shown in [12] that there are two sequences where we can find gaps for $\mathcal{SG}$ and our first theorem detects one of them. Slightly modifying the conditions of the first theorem, we obtain a second theorem which can be used to find the other sequence for $\mathcal{SG}$ where gaps are already known to occur. The third theorem can be used to prove the existence of gaps for the tree-like fractals (see [30] or Section 2.1, Example 4). In Chapter 4 we will show the infinite Vicsek sets satisfy the criterion for the first theorem, so there are gaps in the spectrum for the standard Laplacian on these fractals, as well.

3.1 Gap Theorems

In our gap theorems, we assume the Laplacian on the fractal admits spectral decimation. Note Shima's approach makes it clear that the spectral decimation function, R, associated with a given Laplacian, is a rational function. As usual, we let $\mathfrak{F}_k = \{\lambda \in \mathfrak{F} : \lambda \text{ is an eigenvalue of } -\widehat{\Delta}_k\}$, be the set of forbidden/initial eigenvalues appearing at step k. We restrict the domain of R to the nonnegative real line. Suppose R has $n+1$ inverse functions and we denote the $n+1$ branches of the inverse function of $R(\lambda)$ from bottom to top by $\phi_0, \cdots, \phi_n$ as usual.

Theorem 3.4 *Let b be the largest forbidden eigenvalue. There exist gaps in the spectrum of the generalized Laplacian on the fractal if the following conditions are satisfied:*

(1) $R^{-1}([0, b]) \subseteq [0, b]$;
(2) $\phi_1(x)$ *is defined and decreasing on* $[0, b]$;
(3) $\phi_0(x)$ *is strictly convex and* $\phi_0(b) < \phi_1(b)$;
(4) there exists k_0 *such that for all* $k \geq k_0$ *and all* $x \in \mathfrak{F}_k$, $\phi_1(b) \leq x$.

Proof. Let

$$A_k = \rho^k \lim_{m\to\infty} \rho^m \phi_0^{(m)}(b),$$
$$B_k = \rho^k \lim_{m\to\infty} \rho^m \phi_0^{(m-1)}\phi_1(b).$$

The first condition tells us that we can further iterate $\phi_0(b)$ and $\phi_1(b)$ by inverse functions of R. By Remark 2.11, Lemma 2.12 and the assumptions on ϕ_0 and ϕ_1, the two limits defining A_k and B_k exist and so A_k and B_k are well-defined for any k.

Since ϕ_0 is strictly convex, by Lemma 2.12, we know that A_0 and B_0 are different and $B_0 > A_0$. Since $\frac{B_k}{A_k} = \frac{B_0}{A_0} > 1$, it is sufficient to show that there is no eigenvalue between A_k and B_k for all k greater than some k_0 as given in condition (4).

As the Laplacian admits spectral decimation, by Definition 2.10 and Remark 2.11, all eigenvalues must be of the form

$$\rho^i \lim_{m\to\infty} \rho^m \phi_0^{(m-j)}\phi_w(x),$$

where $i,\ j \in \mathbb{N} \cup \{0\},\ |w| = j$, and $x \in \mathfrak{F}_{i+1}$. Hence it suffices to prove the following two claims:

(i) For $i \geq 0$ and $x \in \mathfrak{F}_{i+1}$,

$$B_k \leq \rho^i \lim_{m\to\infty} \rho^m \phi_0^{(m-j)}\phi_w(x) \leq A_{k+1},$$

where $\phi_w = \phi_{w_j} \circ \cdots \circ \phi_{w_1}$ for $w = w_j \cdots w_1$ with $w_j \neq 0,\ |w| = j$, and $i + j = k + 1$;

(ii) For all $k \geq k_0 - 1$ and $x \in \mathfrak{F}_{k+2} \subseteq \mathfrak{F}$, we have

$$B_k \leq \rho^{k+1} \lim_{m\to\infty} \rho^m \phi_0^{(m)}(x) \leq A_{k+1}.$$

Once our claims are proved, only the eigenvalues $\lim_{m\to\infty} \rho^m \phi_0^{(m)}(x)$ with $x \in \mathfrak{F}_1$, and the eigenvalues $\rho^{k+1} \lim\limits_{m\to\infty} \rho^m \phi_0^{(m)}(x)$ with $x \in \mathfrak{F}_{k+2}$ and $k < k_0 - 1$ could lie in $\cup[A_j,\ B_j]$. As there are only finitely many such eigenvalues, this will not affect the existence of gaps in the sequence.

It is easy to see that the second inequality of (ii) follows directly from the monotonicity of ϕ_0. The the left-hand side inequality of (ii) reads

$$\rho^k \lim_{m\to\infty} \rho^m \phi_0^{(m-1)} \phi_1(b) \leq \rho^{k+1} \lim_{m\to\infty} \rho^m \phi_0^{(m)}(x),$$

which is equivalent to

$$\lim_{m\to\infty} \rho^m \phi_0^{(m-1)} \phi_1(b) \leq \rho \lim_{m\to\infty} \rho^m \phi_0^{(m)}(x),$$

If we replace m by $m' = m - 1$ on the right-hand side of the last inequality, we have

$$\lim_{m\to\infty} \rho^m \phi_0^{(m-1)} \phi_1(b) \leq \lim_{m\to\infty} \rho^m \phi_0^{(m-1)}(x).$$

The last inequality is true because condition (4) implies that $\phi_1(b) \leq x$ for all those forbidden eigenvalues which can appear at step k_0 or later.

To show (i), note that $\mathfrak{F}_{i+1} \subseteq \mathfrak{F}$ and

$$\rho^i \lim_{m\to\infty} \rho^m \phi_0^{(m-j)} \phi_w(x) = \lim_{m\to\infty} \rho^m \phi_0^{(m-i-j)} \phi_w(x).$$

It is sufficient to prove the following stronger inequalities:
(i') For $x \in \mathfrak{F}$,

$$B_k \leq \lim_{m\to\infty} \rho^m \phi_0^{(m-k-1)} \phi_w(x) \leq A_{k+1},$$

where $\phi_w = \phi_{w_j} \circ \cdots \circ \phi_{w_1}$ for $w = w_j \cdots w_1$ with $w_j \neq 0$.

To show the right-hand side inequality of (i'), note that $\phi_w(x) \leq b$ for any w and x by condition (1). Hence

$$\begin{aligned} \lim_{m\to\infty} \rho^m \phi_0^{(m-k-1)} \phi_w(x) &\leq \lim_{m\to\infty} \rho^m \phi_0^{(m-k-1)}(b) \\ &= \lim_{m\to\infty} \rho^{m+k+1} \phi_0^{(m)}(b) = A_{k+1}. \end{aligned}$$

Now we are only left with the left-hand side inequality of (i'), which is

$$\rho^k \lim_{m\to\infty} \rho^m \phi_0^{(m-1)} \phi_1(b) \leq \lim_{m\to\infty} \rho^m \phi_0^{(m-k-1)} \phi_w(x),$$

where $|w| = j$, $w_j \neq 0$, $x \in \mathfrak{F}$. Because of the monotonicity of ϕ_0, it is sufficient to show $\phi_1(b) \leq \phi_w(x) = \phi_{w_j} \circ \phi_{w'}(x)$, where $|w'| = |w| - 1$, $w_j \neq 0$, and $x \in \mathfrak{F}$. Note that to make $\phi_w(x)$ the smallest, w_j has to be 1. Since $x \leq b$ and ϕ_1 is decreasing, we have $\phi_1(b) \leq \phi_w(x)$.

□

Remark 3.5 *(1) It is possible that A_k and B_k are not true eigenvalues in the spectrum, but this will not affect the existence of gaps. What is important is that there is no eigenvalue between A_k and B_k for sufficiently large k. Indeed, if A_k and B_k are not true eigenvalues, then there are even larger gaps between the greatest (true) eigenvalue less than A_k and the least eigenvalues greater than B_k.*

(2) We can check the strict convexity of ϕ_0 by verifying if R is increasing and strictly concave on the image of ϕ_0. This fact can be seen either from the formula

$$\phi_0''(\lambda) = -\frac{R''(\phi_0(\lambda))}{[R'(\phi_0(\lambda))]^3}$$

or by direct proof.

We further remark that the strict convexity of ϕ_0 in our proof of the theorem is used to verify that A_0 and B_0 exist and are distinct.

(3) Recall that we have shown $\phi_0(0) = 0$ and $\phi_0'(0) = \frac{1}{\rho} > 0$ in Chapter 2, where ρ is the Laplacian renormalization constant, so ϕ_0 is always increasing on its domain. Because we divide the branches of the inverse function according to where the function turns, as long as R is continuous on $[0,\ a]$, where a is the least positive root of R, ϕ_1 will be decreasing.

Next we use the above theorem to prove the existence of gaps for the Sierpinski gasket, $\mathcal{SG}$, and the level-3 Sierpinski gasket, $\mathcal{SG}_3$.

Example 1. (The Sierpinski Gasket $\mathcal{SG}$) For the Sierpinski gasket, the spectral decimation function is

$$R(\lambda) = \lambda(5 - 4\lambda).$$

(See [34], [30] or Section 2.3.) Hence the inverse functions are

$$\phi_0(\lambda) = \frac{5 - \sqrt{25 - 16\lambda}}{8},$$
$$\phi_1(\lambda) = \frac{5 + \sqrt{25 - 16\lambda}}{8}.$$

The set of forbidden eigenvalues, $\mathfrak{F}$, is known to be $\{1/2,\ 5/4,\ 3/2\}$. (In [12], the spectral decimation function is $\lambda(5 - \lambda)$ and the forbidden eigenvalues are 2, 5, and 6 because of the normalization factor 4.) 1/2 does not appear as a forbidden eigenvalue at any step $k \geq 2$ (see Chapter 3 in [33]), so $\mathfrak{F}_k \subseteq \{5/4,\ 3/2\}$ for $k \geq 2$.

It is clear that 3/2 is in the domain of ϕ_0 and ϕ_1, $R^{-1}([0,\ 3/2]) = [0,\ 5/4] \subseteq [0,\ 3/2]$, and ϕ_1 is decreasing. The strict convexity of ϕ_0 can be easily obtained either by checking $\phi_0'' > 0$ or verifying $R'' < 0$ and ϕ_0 is increasing on its domain, as mentioned in Remark 3.5. The last condition is also satisfied since $\phi_1(3/2) = 3/4 < 5/4$, the smallest forbidden eigenvalue in $\mathfrak{F}_k$ for $k \geq 2$. Hence all conditions of Theorem 3.4 are satisfied and so there are gaps in the spectrum at two different places.

Note that $\phi_0(3/2) = 1/2$ and 1/2 does not appear as a forbidden eigenvalue at any step $k \geq 2$, so $A_k = 5^k \lim_{m\to\infty} \rho^m \phi_0^{(m)}(3/2)$ for $k \geq 2$ is not a true eigenvalue in the spectrum. But this does not affect the existence of gaps, as we pointed out in Remark 3.5. Indeed, we can replace A_k by the greatest eigenvalue less than A_k, say A_k', and obtain larger gaps. We claim

$$A_k' = 5^k \lim_{m\to\infty} 5^m \phi_0^{(m)}(5/4) \text{ for } k \geq 2.$$

To see this, we first note that since $\phi_l(x) \leq 5/4$ for all $x \in [0,\ 3/2]$ and $l = 0, 1$, it follows that if $i + j = k + 1$ and $x \in \mathfrak{F}_{i+1}$,

$$5^i \lim_{m\to\infty} 5^m \phi_0^{(m-j)} \phi_w(x) \leq A_{k+1}',\ |w| = j, \quad \text{and } w_j \neq 0.$$

Furthermore, as 5/4 is the second largest forbidden eigenvalue, we have

$$5^{k+1} \lim_{m\to\infty} 5^m \phi_0^{(m)}(x) \leq A_{k+1}'$$

for $x \in \mathfrak{F}_{k+2} \setminus \{3/2\}$. These observations establish the claim and show there are gaps in $(A'_k,\ B_k)$ for $B_k = 5^k \lim_{m\to\infty} 5^m \phi_0^{(m-1)}\phi_1(3/2)$, as previously proven in [12].

Example 2. (Level-3 Sierpinski Gasket $\mathcal{SG}_3$) The spectrum of the standard Laplacian on this fractal has been studied by S. Drenning in [8]. The spectral decimation function is

$$R(\lambda) = \frac{6\lambda(\lambda-1)(4\lambda-3)(4\lambda-5)}{6\lambda-7}$$

and the forbidden eigenvalues are $\frac{3\pm\sqrt{5}}{4}$, $3/4$, $7/6$, $5/4$, and $3/2$. The set of forbidden eigenvalues at step k, $\mathfrak{F}_k$, is equal to $\{3/4,\ 5/4,\ 3/2\}$ for $k \geq 2$.

In his paper, Drenning claimed that the numerical data suggested that there are gaps in the spectrum, but there was no proof to support his claim. Now we can apply Theorem 3.4 to this example and prove that his claim is correct.

As R is continuous on $[0,\ 7/6)$, ϕ_1 is decreasing. Note that $R(\lambda) = 0$ has four real roots 0, 3/4, 1, and 5/4. Furthermore,

$$R(\lambda) - 3/2 = \frac{3(4\lambda^2 - 6\lambda + 1)(16\lambda^2 - 24\lambda + 7)}{12\lambda - 14},$$

and it has four real roots $\frac{3\pm\sqrt{2}}{4}$, $\frac{3\pm\sqrt{5}}{4}$. Since

$$\begin{aligned} R^{-1}([0,\ 3/2]) &= [0, \max\{\text{largest root of } R(\lambda),\ \text{largest root of } R(\lambda) - 3/2\}] \\ &= [0,\ \frac{3+\sqrt{5}}{4}], \end{aligned}$$

which is contained in $[0,\ 3/2]$, the first condition of Theorem 3.4 is satisfied.

To check the third condition of the theorem, we note that R is differentiable on $(-\infty,\ 7/6)$ and

$$R'(x) = \frac{6(288x^4 - 1024x^3 + 1290x^2 - 658x + 105)}{(6x-7)^2}.$$

Using Maple, we find that R' are has only two real roots, 0.2880979998 and 0.8900943083, correct to six decimal places. The first turning point of R is 0.2880979998 and so the range of ϕ_0 is contained in $[0,\ 0.3]$. Using Maple again, we find

$$R''(x) = \frac{12(1728x^4 - 7104x^3 + 10752x^2 - 7056 + 1673)}{(6x-7)^3}$$

and the only two real roots of R'' are 0.5988200688 and 1.314052020. Particularly, this tells us that the sign of R'' remains unchanged on $[0,\ 0.3]$. We can easily check that it is negative, so R is strictly concave from 0 to 0.3. Therefore condition (3) is satisfied.

Note that $\phi_1(3/2)$ is equal to the second root of $R(\lambda) - 3/2 = 0$. Thus $\phi_1(3/2) = \frac{3-\sqrt{2}}{4}$, which is less than 3/4, the smallest element in $\mathfrak{F}_k$ for $k \geq 2$, so the last condition of the theorem is satisfied. Therefore, by Theorem 3.4, there exist gaps in the spectrum.

In [12], it is shown that there exists a second sequence of gaps for $\mathcal{SG}$. We can slightly modify our conditions in Theorem 3.4 and identify those gaps.

Theorem 3.6 *Let a and b be the two largest forbidden eigenvalues in $\mathfrak{F}$. There exist gaps in the spectrum of the generalized Laplacian if the following conditions are satisfied:*

(1) $R^{-1}([0,\ b]) \subseteq [0,\ a]$;
(2) $\phi_1(x)$ *is defined and decreasing on* $[0,\ b]$;
(3) $\phi_0(x)$ *is strictly convex;*
(4) there exists k_0 *such that for all* $k \geq k_0$ *and all* $x \in \mathfrak{F}_k$, $\phi_1(a) \leq x$.

Proof. The proof is quite similar to the above theorem. Let

$$A_k = \rho^k \lim_{m\to\infty} \rho^m \phi_0^{(m-1)} \phi_1(b),$$
$$B_k = \rho^k \lim_{m\to\infty} \rho^m \phi_0^{(m-1)} \phi_1(a).$$

We can show that all eigenvalues of the form

$$\rho^i \lim_{m\to\infty} \rho^m \phi_0^{(m-j)} \phi_w(x), \quad \text{with } |w| = j,\ w_j \neq 0,\ i + j = k+1,$$

are between B_k and A_{k+1}, except the case when $w = 1$ and $x = b$ when we obtain A_k. The rest of the proof is almost identical to that of Theorem 3.4.

□

Example 3. (Sierpinski Gasket $\mathcal{SG}$) In Example 1 we actually verified that the conditions of Theorem 3.6 were satisfied with $b = 3/2,\ a = 5/4$, and $k_0 = 2$. Therefore there are gaps between

$$A_k = 5^k \lim_{m\to\infty} \rho^m \phi_0^{(m-1)} \phi_1(3/2)$$

and

$$B_k = 5^k \lim_{m\to\infty} \rho^m \phi_0^{(m-1)} \phi_1(5/4)$$

as discovered in [12].

In Example 2 ($\mathcal{SG}_3$), condition (1) of the second theorem is not satisfied, so it does not apply in this setting.

We have the following alternative criterion to determine whether there are gaps in the spectrum. We shall use it to prove the existence of gaps for the tree-like fractals which will be introduced after the proof of the theorem.

Theorem 3.7 *Suppose $\alpha \le \beta$ are two consecutive forbidden eigenvalues in $\mathfrak{F}$. Let $c \ge b$ be such that $R^{-1}([0,\ b]) \subseteq [0,\ c]$, where b is the largest forbidden eigenvalue. If the following conditions are satisfied, then there must be gaps in the spectrum.*
(1) $\phi_1(x) \ge \beta$, for all $x \in [0,\ c]$;
(2) $\phi_0(c) \le \alpha$;
(3) $\phi_0(x)$ is strictly convex.

Proof. The proof of this theorem is also very similar to previous theorems.

For $k \ge 0$, we let

$$\begin{aligned} A_k &= \rho^k \lim_{m\to\infty} \rho^m \phi_0^{(m)}(\alpha), \\ B_k &= \rho^k \lim_{m\to\infty} \rho^m \phi_0^{(m)}(\beta). \end{aligned}$$

By the same reasoning as in the previous theorems, we can prove that both A_k and B_k exist and $\frac{B_k}{A_k} = \frac{B_0}{A_0} > 1$ for any k. We claim there is no eigenvalue between A_k and B_k for any k and hence there are gaps in the spectrum.

Let $k > 1$, $i + j = k$, $|w| = j$, and $w_j \neq 0$. We claim that for all $x \in \mathfrak{F}_{i+1} \subseteq \mathfrak{F}$, for which $\phi_0^{(m-j)}\phi_w(x)$ is defined,

$$B_k \leq \rho^i \lim_{m\to\infty} \rho^m \phi_0^{(m-j)}\phi_w(x) \leq A_{k+1}.$$

Note that the left-hand side inequality above reads

$$\rho^k \lim_{m\to\infty} \rho^m \phi_0^{(m)}(\beta) \leq \rho^i \lim_{m\to\infty} \rho^m \phi_0^{(m-j)}\phi_w(x),$$

which is equivalent to

$$\lim_{m\to\infty} \rho^m \phi_0^{(m)}(\beta) \leq \lim_{m\to\infty} \rho^{m+i-k} \phi_0^{(m-j)}\phi_w(x).$$

Since $i - k = -j$, if we let $m' = m - j$ on the right-hand side, then it would follow that this is equivalent to

$$\lim_{m\to\infty} \rho^m \phi_0^{(m)}(\beta) \leq \lim_{m\to\infty} \rho^{(m)} \phi_0^m \phi_w(x).$$

As ϕ_0 is increasing on its domain, it suffices to show that

$$\phi_w(x) \geq \beta,\ w_j \neq 0,\ x \in \mathfrak{F}_{i+1}.$$

By (1), $\phi_i(x) \geq \beta$ for all $x \in [0,\ c]$, so this is clearly true.

The right-hand side inequality reads

$$\rho^{k+1} \lim_{m\to\infty} \rho^m \phi_0^{(m)}(\alpha) \geq \rho^i \lim_{m\to\infty} \rho^m \phi_0^{(m-j)}\phi_w(x),$$

which is equivalent to

$$\lim_{m\to\infty} \rho^{m+k+1-i} \phi_0^{(m)}(\alpha) \geq \lim_{m\to\infty} \rho^m \phi_0^{(m-j)}\phi_w(x).$$

Again, notice that $k - i = j$ and let $m' = m + j + 1$ on the left-hand side. We would have

$$\lim_{m\to\infty} \rho^m \phi_0^{(m-j-1)}(\alpha) \geq \lim_{m\to\infty} \rho^m \phi_0^{(m-j)}\phi_w(x).$$

Hence it suffices to show that

$$\phi_0(\phi_w(x)) \leq \alpha$$

for all w. Since $R^{-1}([0,\ b]) \subseteq [0,\ c]$ and ϕ_0 is increasing, the largest possible value on the left-hand side is $\phi_0(c)$. Hence the last inequality is true by (2) and we have the desired result.

$\square$

Example 4. (The n-branch tree-like fractal) This infinite family of fractals is obtained by sticking $n-2$ intervals of length $1/2$ to the middle of a unit interval and then repeating this process (sticking $n-2$ branches to each interval obtained in the previous step with half of its length). We let $V_0 = \{p_1, p_2\}$ be the set of boundary points and $G_0 = (V_0,\ E_0)$ be the complete graph on V_0. Let $V_1 \setminus V_0 = \{q_1, q_2, \cdots, q_{n-1}\}$, the set of vertices in $V_1 \setminus V_0$, where q_1 is the only point on the line segment p_1p_2; in other words, q_1 is the only point in $V_1 \setminus V_0$ connected with p_1 and p_2. See the graph in Section 2.1, Example 4 for the case when $n = 5$.

For this infinite family of fractals there are more general Laplacians which admit spectral decimation (see [30] when $n = 5$). Indeed, we can choose a weighted vector $\mathbf{r} = (1,\ 1,\ \underbrace{r^{-1},\ \cdots,\ r^{-1}}_{n-2})$ with $r > 0$ so that the weights on the two outer branches are 1 and the weights on all $(n-2)$ inner branches are r. (If we take r to be 1, then we would have the standard Laplacian.)

Let D be the Laplacian matrix on the complete graph G_0 and H_1 be the matrices representing the graph Laplacians on V_0 and V_1 respectively (see Chapter 2 for the definition of H_m). We can see that $D = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ and

$$H_1 = \begin{bmatrix} T & J^t \\ J & X \end{bmatrix}.$$

Here T is a diagonal matrix with $T_{i,i} = D_{i,i}$ and J is the incidence matrix of V_0 and $V_1 \setminus V_0$; i.e., if q_i is a neighboring point of p_j, then $J_{i,j} = 1$; otherwise $J_{i,j} = 0$. Hence

$$J = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}_{(n-1)\times 2}$$

The matrix X is a square matrix with

$$X = \begin{bmatrix} -(2+(n-2)r) & r & r & \cdots & r \\ r & -r & 0 & \cdots & 0 \\ & \vdots & \vdots & & \\ r & 0 & 0 & \cdots & -r \end{bmatrix}_{(n-1)\times(n-1)}$$

and M is a diagonal matrix with $M_{i,i} = -X_{i,i}$, so

$$X + \lambda M = \begin{bmatrix} (2+(n-2)r)(\lambda-1) & r & r & \cdots & r \\ r & (\lambda-1)r & 0 & \cdots & 0 \\ & \vdots & \vdots & & \\ r & 0 & 0 & \cdots & (\lambda-1)r \end{bmatrix}_{(n-1)\times(n-1)}$$

Therefore the normalized Laplacian for any function f on $V_1 \setminus V_0$ is

$$\widehat{\Delta} f(q_1) = \frac{1}{2+(n-2)r}[f(p_1) + f(p_2) + r\sum_{i=2}^{n-1} f(q_i)] - f(q_1)$$

and for $2 \le i \le n-1$,

$$\widehat{\Delta} f(q_i) = f(q_1) - f(q_i).$$

Recall that the spectral decimation function R is given by

$$R(\lambda) = \frac{\lambda - K_T(\lambda)}{K_D(\lambda)},$$

where K_D and K_T are defined by

$$T - J^t(X+\lambda M)^{-1}J = K_D(\lambda)D + K_T(\lambda)T$$

Denote $G(\lambda) = (X + \lambda M)^{-1}$ if the inverse matrix exists. We can easily see that

$$K_D(\lambda) = (-J^t G(\lambda) J)_{1,2}.$$

Since

$$(J^t G(\lambda) J)_{1,2} = \sum_{k,\ j} J^t_{1,k} G_{k,j} J_{j,2} = G_{1,1},$$

we have that

$$K_D(\lambda) = -G(\lambda)_{1,1} = \frac{-1}{\det(X + \lambda M)} [(\lambda - 1)r]^{n-2}.$$

So now the question is to find $\det(X + \lambda M)$. Expanding by the last row of $\det(X + \lambda M)$, we have that

$$\begin{aligned}
\det(X + \lambda M) &= \begin{vmatrix} (2 + (n-2)r)(\lambda - 1) & r & r & \cdots & r \\ r & (\lambda - 1)r & 0 & \cdots & 0 \\ & \vdots & \vdots & & \\ r & 0 & 0 & \cdots & (\lambda - 1)r \end{vmatrix}_{(n-1)\times(n-1)} \\
&= (\lambda - 1)r \begin{vmatrix} (2 + (n-2)r)(\lambda - 1) & r & r & \cdots & r \\ r & (\lambda - 1)r & 0 & \cdots & 0 \\ & \vdots & \vdots & & \\ r & 0 & 0 & \cdots & (\lambda - 1)r \end{vmatrix}_{(n-2)\times(n-2)} \\
&+ (-1)^n r \begin{vmatrix} r & r & r & \cdots & r \\ (\lambda - 1)r & 0 & 0 & \cdots & 0 \\ & \vdots & \vdots & & \\ 0 & 0 & \cdots & (\lambda - 1)r & 0 \end{vmatrix}_{(n-2)\times(n-2)} \\
&= (\lambda - 1)r \begin{vmatrix} (2 + (n-2)r)(\lambda - 1) & r & \cdots & r \\ r & (\lambda - 1)r & \cdots & 0 \\ & \vdots & & \\ r & 0 & \cdots & (\lambda - 1)r \end{vmatrix} \\
&+ (-1)^{2n-3} r^2 ((\lambda - 1)r)^{n-3}
\end{aligned}$$

We expand by the last row again and we see that the last equation is equal to

$$
\begin{aligned}
& (\lambda-1)r(\lambda-1)r \begin{vmatrix} (2+(n-2)r)(\lambda-1) & r & \cdots & r \\ r & (\lambda-1)r & \cdots & 0 \\ & \vdots & & \\ r & 0 & \cdots & (\lambda-1)r \end{vmatrix}_{(n-3)\times(n-3)} \\
+ \quad & (\lambda-1)r\cdot r \begin{vmatrix} r & r & \cdots & r \\ (\lambda-1)r & 0 & \cdots & 0 \\ & \vdots & & \\ 0 & \cdots & (\lambda-1)r & 0 \end{vmatrix}_{(n-3)\times(n-3)} - r^2((\lambda-1)r)^{n-3} \\
= \quad & ((\lambda-1)r)^2 \begin{vmatrix} (2+(n-2)r)(\lambda-1) & r & \cdots & r \\ r & (\lambda-1)r & \cdots & 0 \\ & \vdots & & \\ r & 0 & \cdots & (\lambda-1)r \end{vmatrix} - 2r^2((\lambda-1)r)^{n-3}
\end{aligned}
$$

Repeatedly expanding by the last column, eventually we will have

$$
\begin{aligned}
\det(X+\lambda M) &= ((\lambda-1)r)^{n-3} \begin{vmatrix} (2+(n-2)r)(\lambda-1) & r \\ r & (\lambda-1)r \end{vmatrix} - (n-3)r^{n-1}(\lambda-1)^{n-3} \\
&= (\lambda-1)^{n-3}r^{n-2}[(2+(n-2)r)\lambda^2 - 2(2+(n-2)r)\lambda + 2].
\end{aligned}
$$

Therefore,

$$K_D(\lambda) = -\frac{\lambda-1}{(2+(n-2)r)\lambda^2 - 2(2+(n-2)r)\lambda + 2}.$$

Since

$$T - J^tG(\lambda)J = K_D(\lambda)D + K_T(\lambda)T,$$

we have

$$
\begin{aligned}
K_T(\lambda) &= \frac{T_{1,1} - (J^tG(\lambda)J)_{1,1} - K_D(\lambda)D_{1,1}}{T_{1,1}} \\
&= 1 - K_D(\lambda) + G_{1,1} \\
&= 1 - 2K_D(\lambda).
\end{aligned}
$$

It follows then

$$R(\lambda) = \frac{\lambda - K_T(\lambda)}{K_D(\lambda)} = -(2 + (n-2)r)\lambda(\lambda - 2).$$

Moreover, the forbidden eigenvalues, by definition, are zeros of K_D and $\det(X + \lambda M)$. If we let $\alpha_1 = 1 - \sqrt{\frac{(n-2)r}{2+(n-2)r}}$ and $\alpha_2 = 1 + \sqrt{\frac{(n-2)r}{2+(n-2)r}}$, then the set of forbidden eigenvalues $\mathfrak{F}$ is $\{1,\ \alpha_1,\ \alpha_2\}$.

We will use our second theorem to show that there are gaps in the spectrum of the Laplacian on this infinite family of fractals. Let $\phi_0,\ \phi_1$ be the 2 branches of the inverse function of $R(\lambda)$ from bottom to top

$$\begin{aligned}\phi_0(x) &= 1 - \sqrt{\frac{2 + (n-2)r - x}{2 + (n-2)r}},\\ \phi_1(x) &= 1 + \sqrt{\frac{2 + (n-2)r - x}{2 + (n-2)r}}.\end{aligned}$$

Note that $\phi_i(x) \leq 2$ for all $x \geq 0$ and $i = 0,\ 1$ and the domain of ϕ_0 and ϕ_1 (restricted to $[0,\ \infty]$) contains $[0,\ 2]$ for any n and r. Hence we can take $b = \alpha_2 < 2$ and $c = 2$ in Theorem 3.7 and the condition $R^{-1}([0,\ b]) \subseteq [0,\ c]$ is satisfied. Let $\alpha = \alpha_1$ and $\beta = 1$. Clearly $\phi_1(x) \geq 1 = \beta$ for all $x \in [0,\ 2]$ and so (1) is satisfied. As $\phi_0(2) = \alpha_1$, condition (2) is satisfied. As before, condition (3) of the strict convexity of ϕ_0 can be obtained by checking $R' > 0$ and $R'' < 0$ on $\mathrm{Im}(\phi_0)$.

Note that $\phi_1(x)$ is always greater than 1, hence the last condition in Theorem 3.4 cannot be met.

Theorem 3.7 says there are gaps between A_k and B_k, with

$$\begin{aligned}A_k &= \rho^k \lim_{m\to\infty} \rho^m \phi_0^{(m)}(\alpha),\\ B_k &= \rho^k \lim_{m\to\infty} \rho^m \phi_0^{(m)}(\beta).\end{aligned}$$

We can calculate ρ, the Laplacian renormalization constant by recalling that it is defined as the product of the energy renormalization constant and the measure factor. The energy renormalization constant is $K_D(0)^{-1}$, which one can easily see is 2. The measure renormalization factor is the sum of all the weights and equals $2+(n-2)r$. Thus $\rho = 2(2+(n-2)r)$.

In Chapter 4, we shall show that our first criterion for gaps, Theorem 3.4, applies to an infinite family of fractals, the n-branch Viesek set $\mathcal{VS}_n$, as well.

Chapter 4

Spectral decimation function for the standard Laplacian on $\mathcal{VS}_n$

In this chapter, we shall obtain the spectral decimation function for the standard Laplacian on an infinite family of fractals, the n-branch Vicsek sets $\mathcal{VS}_n$, in terms of the Chebyshev polynomials. We will then apply Theorem 3.4 in Chapter 3 to prove that there exist gaps in the spectrum of the standard Laplacian.

4.1 Spectral decimation function

We begin by recalling the definition of the Vicsek set, $\mathcal{VS}$. It is a p.c.f. self-similar set, which is constructed from the 1/3-similitudes, $F_1, \cdots, F_5$: $\mathbb{R}^2 \to \mathbb{R}^2$ with

$$F_1(x) = \frac{x}{3},\ F_2(x) = \frac{x}{3} + \frac{2}{3}(1,\ 0),\ F_3(x) = \frac{x}{3} + \frac{2}{3}(1,\ 1),$$
$$F_4(x) = \frac{x}{3} + \frac{2}{3}(0,\ 1),\ F_5(x) = \frac{x}{3} + \frac{2}{3}(1/2,\ 1/2).$$

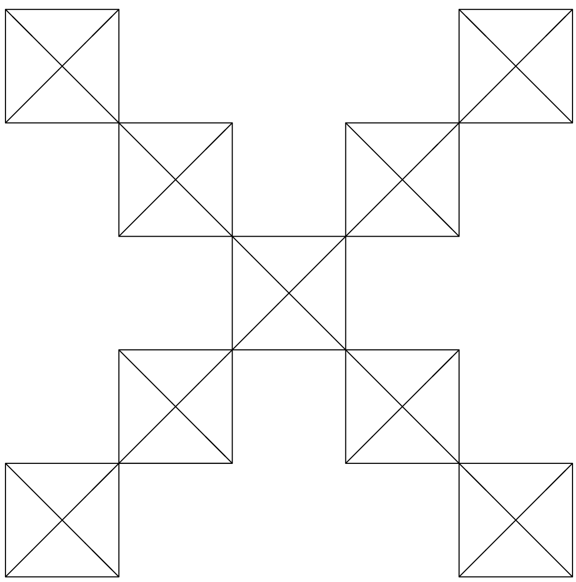
The Vicsek set $\mathcal{VS}$ is the unique compact set satisfying

$$\mathcal{VS} = \bigcup_{s=1}^{5} F_s(\mathcal{VS}).$$

It is easy to see that both the critical set $\mathcal{C}$ and the post critical set $\mathcal{P}$ are finite (see Example 5.15 in [3]).

The Hausdorff dimension of the Vicsek set is $\frac{\log 5}{\log 3}$. Each similitude, F_i, has a unique fixed point p_i, namely, $p_1 = (0,\ 0),\ p_2 = (1,\ 0),\ p_3 = (1, 1),\ p_4 = (0,\ 1),\ p_5 = (1/2,\ 1/2)$. Moreover, the set of boundary points, V_0, can be proved to be $\{p_1, p_2, p_3, p_4\}$. The set of m step vertices, V_m, is defined as $\bigcup_{w \in W_m} F_w(V_0)$ for all $m \in \mathbb{N}$. Each V_m is a subset of $\mathcal{VS}$ and $V_* := \bigcup_{m=1}^{\infty} V_m$ is dense in $\mathcal{VS}$, so we can use the sequence $(V_m)_{m\in\mathbb{N}}$ as a set of increasingly refined "grids" to approximate $\mathcal{VS}$.

We now define the n-branch Vicsek set, $\mathcal{VS}_n$, with the same four boundary points $p_1,\ p_2,\ p_3$, and p_4, but with n squares in each of the four directions. See the following figure for the first step graph of $\mathcal{VS}_3$.



With this notation, $\mathcal{VS} = \mathcal{VS}_2$. Let $N = 4n-3$ and $\lambda = \frac{1}{2n-1}$. We have N λ-similitudes $F_1, \cdots, F_N : \mathbb{R}^2 \to \mathbb{R}^2$. We define V_0, $V_m = V_m(n)$, and $V_* = V_*(n)$ as in the Vicsek set with 5 replaced by N. In particular, we let

$$V_1 \setminus V_0 = \{q_1, q_2, \cdots, q_{12(n-1)}\}$$

denote the set of vertices in $V_1 \setminus V_0$. The n-branch Vicsek set $\mathcal{VS}_n$ is the unique compact fixed point of $\bigcup_{s=1}^N F_s$. It is also a p.c.f. self-similar fractal (see [26]) with Hausdorff dimension $\frac{\log N}{\log \lambda}$.

In this chapter we will derive a formula for the spectral decimation function for the standard Laplacian with Dirichlet boundary condition on the n-branch Vicsek set $\mathcal{VS}_n$. To be precise, we will prove the following theorem.

Theorem 4.1 *Define*

$$\begin{aligned} f_n(\lambda) &:= T_n(3\lambda - 1) - 3T_{n-1}(3\lambda - 1), && (4.1.1)\\ g_n(\lambda) &:= U_{n-1}(3\lambda - 1) - U_{n-2}(3\lambda - 1), && (4.1.2)\\ h_n(\lambda) &:= U_{n-1}(3\lambda - 1) - 3U_{n-2}(3\lambda - 1), && (4.1.3)\\ l_n(\lambda) &:= U_{n-1}(3\lambda - 1) + U_{n-2}(3\lambda - 1). && (4.1.4) \end{aligned}$$

where T_n and U_n are Chebyshev polynomials of the first and the second kind defined by the same recurrence relation

$$P_{n+1}(x) = 2xP_n(x) - P_{n-1}(x), \tag{4.1.5}$$

with initial conditions $T_0(x) = 1$, $T_1(x) = x$ and $U_0(x) = 1$, $U_1(x) = 2x$. Then the spectral decimation function R is

$$R(\lambda) = \frac{\lambda - K_T(\lambda)}{K_D(\lambda)} = \lambda g_n(\lambda) h_n(\lambda) \tag{4.1.6}$$

and

$$3R(\lambda) - 4 = f_n(\lambda) l_n(\lambda), \tag{4.1.7}$$

Moreover, the forbidden eigenvalues are 4/3, and zeros of f_n and g_n. (See the Appendix for background about Chebyshev polynomials.)

The proof of this theorem will require a number of preliminary results.

We first establish some useful properties of the functions related to the spectral decimation function of the general Vicsek set which will be used in the proof of Theorem 4.1 and in applying the gap theorem.

We first fix n. Denote $y = 3\lambda - 1$. Then the functions $f_n,\ g_n,\ h_n$, and l_n can be rewritten as

$$\begin{aligned} f_n(\lambda) &= T_n(y) - 3T_{n-1}(y) && (4.1.8)\\ g_n(\lambda) &= U_{n-1}(y) - U_{n-2}(y) && (4.1.9)\\ h_n(\lambda) &= U_{n-1}(y) - 3U_{n-2}(y) && (4.1.10)\\ l_n(\lambda) &= U_{n-1}(y) + U_{n-2}(y) && (4.1.11) \end{aligned}$$

We will use $\alpha_i,\ \beta_i,\ \gamma_i$ and ξ_i to represent roots of $f_n,\ g_n,\ l_n$ and h_n respectively. First, let us find the explicit expressions for β_i and γ_i. If we let $y = 3\lambda - 1 = \cos\theta$, then using the definition for g_n and the subtraction formula for sine functions, we have

$$\begin{aligned} g_n(\lambda) &= U_{n-1}(\cos\theta) - U_{n-2}(\cos\theta)\\ &= \frac{\sin n\theta - \sin(n-1)\theta}{\sin\theta}\\ &= \frac{\cos\frac{2n-1}{2}\theta}{\cos\frac{\theta}{2}} \end{aligned}$$

Hence $g_n(\lambda) = 0$ would imply that $\frac{2n-1}{2}\theta = k\pi - \frac{\pi}{2}$ or $\theta = \frac{2k-1}{2n-1}\pi$, where $k = 1,\ 2,\ \cdots,\ n-1$. Since $\deg(g_n) = n-1$, these are all the roots of g_n. If we order the roots of g_n such that $\beta_1 < \beta_2 < \cdots < \beta_{n-1}$, then we would have

$$\beta_i = \frac{1 + \cos\frac{2(n-i)-1}{2n-1}\pi}{3}$$

for $i = 1,\ 2,\ \cdots,\ n-1$. Similarly, we can prove that γ_i, the roots of l_n, has the expression

$$\gamma_i = \frac{1 + \cos\frac{2(n-i)\pi}{2n-1}}{3}$$

with $i = 1,\ 2,\ \cdots,\ n-1$ and these are all the roots of l_n. Clearly, β_i and γ_i are positive and less than $2/3$. Moreover, we have the following three facts about the distributions of $\alpha_i,\ \beta_i,\ \gamma_i$ and ξ_i.

Proposition 4.2 *(a) The function f_n has exactly n real roots $\alpha_1,\ \alpha_2, \cdots, \alpha_n$ and if we naturally order them, then α_i and β_i are alternating and*

$$0 < \alpha_1 < \beta_1 < \alpha_2 < \cdots < \alpha_{n-1} < \beta_{n-1} < 2/3 < \alpha_n < 1 \tag{4.1.12}$$

(b) If we naturally order all the $n-1$ roots $\gamma_1,\ \cdots,\ \gamma_{n-1}$ of the function l_n, then $\alpha_1,\ \cdots,\ \alpha_n$ and $\gamma_1,\ \cdots,\ \gamma_{n-1}$ are alternating and

$$0 < \gamma_1 < \alpha_1 < \gamma_2 < \cdots < \gamma_{n-1} < \alpha_{n-1} < 2/3 < \alpha_n < 1. \tag{4.1.13}$$

(c) The function h_n has exactly $n-1$ roots $\xi_1,\ \cdots,\ \xi_{n-1}$ and if we naturally order them, then $\beta_1,\ \cdots,\ \beta_{n-1}$ and $\xi_1,\ \cdots,\ \xi_{n-1}$ are alternating and

$$0 < \beta_1 < \xi_1 < \beta_2 < \cdots < \xi_{n-2} < \beta_{n-1} < 2/3 < \xi_{n-1} < 1. \tag{4.1.14}$$

Proof. Note that to show (a), it suffices to show the following:

(i) For all $1 \leq i \leq n-2$, $f_n(\beta_{n-i})$ is positive if i is even and it is negative when i is odd, so there exists at least one root of f_n between β_i and β_{i+1}. This will give us at least $n-2$ roots of f_n.

(ii) $f_n(0)f_n(\beta_1) < 0$, so there exists at least one root between 0 and β_1.

(iii) $f_n(2/3)f_n(1) < 0$, so there exists at least one root between $2/3$ and 1.

In total, (i), (ii) and (iii) account for n roots of f_n, which is exactly the degree of f_n. Hence we have found all the roots of f_n. Moreover, if we can show that (i), (ii) and (iii) are all true, then clearly they are distributed as we described in (a).

To show (i), we note that for $1 \le i \le n-2$,

$$\begin{aligned} f_n(\beta_{n-i}) &= \cos n\frac{(2i-1)\pi}{2n-1} - 3\cos(n-1)\frac{(2i-1)\pi}{2n-1} \\ &= \cos[(i-1/2)\pi + \frac{(i-1/2)\pi}{2n-1}] - 3\cos[(i-1/2)\pi - \frac{(i-1/2)\pi}{2n-1}] \\ &= \begin{cases} 4\sin\frac{(i-1/2)\pi}{2n-1}, & \text{if } i \text{ is even,} \\ -4\sin\frac{(i-1/2)\pi}{2n-1}, & \text{if } i \text{ is odd.} \end{cases} \end{aligned}$$

Hence (i) is proved.

For (ii), we note that

$$\begin{aligned} f_n(0) &= T_n(-1) - 3T_{n-1}(-1) \\ &= \cos n\pi - 3\cos(n-1)\pi \\ &= 4(-1)^n \end{aligned}$$

and by the same argument as in the proof of (i), we can prove that

$$f_n(\beta_1) = \begin{cases} 4\sin\frac{(n-3/2)\pi}{2n-1}, & \text{if } i \text{ is even,} \\ -4\sin\frac{(n-3/2)\pi}{2n-1}, & \text{if } i \text{ is odd.} \end{cases}$$

It follows that the product $f_n(0)f_n(\beta_1)$ is always negative as desired.

By the equality $f_n(1) = T_n(2) - 3T_{n-1}(2)$ and the recursive formula for the Chebyshev polynomials, we know that

$$f_n(1) = T_{n-1}(2) - T_{n-2}(2).$$

Hence $f_n(1)$ is positive as $T_n(x)$ is increasing with n when $x > 1$. Direct evaluation gives $f_n(2/3) = T_n(1) - 3T_{n-1}(1) = -2$ and (iii) is proved.

Both (b) and (c) can be proved in a similar fashion. To show (b), it suffices to show that

$$f_n(\gamma_{n-i}) = \begin{cases} -2\cos\frac{i\pi}{2n-1}, & \text{if } i \text{ is even,} \\ 2\cos\frac{i\pi}{2n-1}, & \text{if } i \text{ is odd.} \end{cases}$$

To show (c), we note that

$$h_n(\beta_{n-i}) = \begin{cases} 2\cot\frac{(i-1/2)\pi}{2n-1}, & \text{if } i \text{ is even,} \\ -2\cot\frac{(i-1/2)\pi}{2n-1}, & \text{if } i \text{ is odd.} \end{cases}$$

Hence $h_n(\beta_{n-i})h_n(\beta_{n-i-1})$ is always negative. Since $U_{n-1}(1) = n$, we know that $h_n(2/3) = U_{n-1}(1) - 3U_{n-2}(1) = -2n + 3 < 0$ for all $n \geq 2$. On the other hand, using the fact that the Chebyshev polynomials $U_n(x)$ are increasing in n for $x \geq 1$ and their recursive formula, we know that

$$\begin{aligned} U_{n-1}(2) - 3U_{n-2}(2) &= 2 \cdot 2U_{n-2}(2) - U_{n-3}(2) - 3U_{n-2}(2) \\ &= U_{n-2}(2) - U_{n-3}(2) \end{aligned}$$

is always positive and so the product $h_n(2/3)h_n(1)$ is always negative as we expected.

□

We continue to use D to denote the Laplacian matrix on the complete graph $G_0 = (V_0,\ E_0)$, where E_0 denotes the set of edges, and let $H_1 = H_1(n)$ be the matrices representing the standard graph Laplacians on $V_1 = V_1(n)$. Hence

$$D = \begin{pmatrix} -3 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{pmatrix}.$$

We decompose H_1 as usual

$$H_1 = \begin{pmatrix} T & J^t \\ J & X \end{pmatrix},$$

where T is a diagonal matrix with

$$T_{i,i} = -(\text{ the number of neighboring points of } p_i)$$

and J is the incidence matrix of V_0 and $V_1 \setminus V_0$; i.e., if q_i is a neighboring point of p_j, then $J_{i,j} = 1$, otherwise $J_{i,j} = 0$. Hence there are 3 entries equal to 1 on each column of J and the rest are 0. The matrix X is a square matrix with $X_{i,i}$= -(the number of neighboring points of q_i). Moreover, if q_i is a neighboring point of q_j, then $X_{i,j} = 1$; otherwise, $X_{i,j} = 0$. Define M to be the diagonal matrix with $M_{i,i} = -X_{i,i}$.

It is easy to see that all diagonal entries of $T - J^t(X + \lambda M)J$ are identical and all the off diagonal entries are the same. Consequently, the standard Laplacian on the n-branch Vicsek set has a strong harmonic structure and so by Theorem 2.13, it admits spectral decimation.

Recall that the spectral decimation function is given by

$$R_n(\lambda) = R(\lambda) = \frac{\lambda - K_T(\lambda)}{K_D(\lambda)},$$

where K_D and K_T are defined to be

$$T - J^t(X + \lambda M)^{-1}J = K_D(\lambda)D + K_T(\lambda)T.$$

We let $\nu^{(i)}$ represent the i-th column of J. We can also think $\nu^{(i)}$ as a function defined on $V_1 \setminus V_0$, whose values are only 0 and 1.

It follows that

$$\begin{aligned} K_D(\lambda) &= (K_D \cdot D + K_T \cdot T)_{2,1} \\ &= (T - J^t(X + \lambda M)^{-1}J)_{2,1} \\ &= - < \nu^{(2)}, u >, \end{aligned}$$

where $u = (X + \lambda M)^{-1}\nu^{(1)}$. Here we remark that we can choose any non-diagonal entries to get the same result.

So now the question is how to find $u = (X + \lambda M)^{-1}\nu^{(1)}$. We shall solve the equation $(X + \lambda M)u = \nu^{(1)}$ to determine u, but first let us solve the homogeneous equation $(X + \lambda M)u = 0$ to find the expression of $\det(X + \lambda M)$ and hence a subset of the forbidden eigenvalues. (Recall that forbidden eigenvalues are those λ's such that $\det(X + \lambda M) = 0$ and $K_D(\lambda) = 0$. We shall later show that in the case of $\mathcal{VS}_n$, the roots of $\det(X+\lambda M) = 0$ are actually all of the forbidden eigenvalues.)

Notations: For any set V, we continue to denote all linear functions on V as $\ell(V)$ and all linear functions on V with zero boundary conditions $\ell_0(V)$. In the case of $\mathcal{VS}_n$, the values of $u \in \ell_0(V_1)$ on the j-th branch of $\mathcal{VS}_n$ (where $j = 1,\ 2,\ 3$ or 4) are written as

$$\begin{aligned} u(x_i, j) &:= u_{i,j}, \\ u(y_i', j) &:= u_{i,j}', \\ u(y_i'', j) &:= u_{i,j}'', \end{aligned}$$

where $u(x_i, j)$ is the function value of u on the i-th vertex on the diagonal (counting from the outside to the inside) of the j-th branch $(1 \le i \le n)$, $u(y_i', j)$ is the value of u on the i-th vertex below the diagonal of the j-th branch $(1 \le i \le n-1)$, and $u(y_i'', j)$ is the value of u on the i-th vertex above the diagonal of the j-th branch $(1 \le i \le n-1)$.

Recall that the normalized discrete Laplacian $\widehat{\Delta}_m$ on the m-step graph is defined as

$$\widehat{\Delta}_m u(x) = \frac{1}{\deg x} \sum_{y \sim_m x} (u(y) - u(x)),$$

for $u \in \ell(V_m \setminus V_0)$, where $y \sim_m x$ means that x and y are neighboring points in the m-step graph. Therefore, the Dirichlet eigenvalue problem $\widehat{\Delta}_1 u = -\lambda u$ (i.e. $(X + \lambda M)u = 0$) is given explicitly by the following system:

$$\begin{cases} u_{1,j} = 0, \\ u_{i,j} + (3\lambda - 3)u_{i,j}' + u_{i,j}'' + u_{i+1,j} = 0, \\ u_{i,j} + u_{i,j}' + (3\lambda - 3)u_{i,j}'' + u_{i+1,j} = 0, \\ u_{i-1,j} + u_{i-1,j}' + u_{i-1,j}'' + (6\lambda - 6)u_{i,j} + u_{i,j}' \\ +u_{i,j}'' + u_{i+1,j} = 0, \\ u_{n-1,j} + u_{n-1,j}' + u_{n-1,j}'' + (6\lambda - 6)u_{n,j} + \displaystyle\sum_{k=1, k\neq j}^{4} u_{n,k} = 0, \end{cases} \tag{4.1.15}$$

where $1 \le j \le 4$, the second and the third equations hold for $1 \le i \le n-1$, and the fourth equation holds for $2 \le i \le n-1$.

To simplify system (4.1.15), we introduce new variables

$$u_{i,j}^{+} := \frac{u'_{i,j} + u''_{i,j}}{2},$$
$$u_{i,j}^{-} := \frac{u'_{i,j} - u''_{i,j}}{2}.$$

Then we add and subtract the second and the third equations in (4.1.15) and rewrite the last two equations to get the following system.

$$\begin{cases} u_{1,j} = 0, \\ (3\lambda - 4)u_{i,j}^{-} = 0, \\ u_{i,j} + (3\lambda - 2)u_{i,j}^{+} + u_{i+1,j} = 0, \\ u_{i-1,j} + 2u_{i-1,j}^{+} + (6\lambda - 6)u_{i,j} + 2u_{i,j}^{+} + u_{i+1,j} = 0, \\ u_{n-1,j} + 2u_{n-1,j}^{+} + (6\lambda - 7)u_{n,j} + \displaystyle\sum_{k=1}^{4} u_{n,k} = 0, \end{cases} \tag{4.1.16}$$

where $1 \leq i \leq n-1$ for the second and the third equations, and the fourth equation holds for $2 \leq i \leq n-1$. Take i to be $i-1$ and i respectively in the third equation above and add them together, then subtract the fourth equation. We obtain a new equation

$$(3\lambda - 4)[u_{i-1,j}^{+} - 2u_{i,j} + u_{i,j}^{+}] = 0.$$

Similarly, we can use the last equation in (4.1.16) to subtract the third equation for $i = n-1$ to obtain another equation

$$(3\lambda - 4)[-u_{n-1,j}^{+} + 2u_{n,j}] + \sum_{k-1}^{4} u_{n,k} = 0.$$

Thus we have the following system equivalent to system (4.1.15)

$$\begin{cases} u_{1,j} = 0, \\ (3\lambda - 4)u_{i,j}^{-} = 0, & (1 \leq i \leq n-1), \\ u_{i,j} + (3\lambda - 2)u_{i,j}^{+} + u_{i+1,j} = 0, & (1 \leq i \leq n-1), \\ (3\lambda - 4)[u_{i-1,j}^{+} - 2u_{i,j} + u_{i,j}^{+}] = 0, & (2 \leq i \leq n-1), \\ (3\lambda - 4)[-u_{n-1,j}^{+} + 2u_{n,j}] + \sum_{k-1}^{4} u_{n,k} = 0, \end{cases} \tag{4.1.17}$$

where $1 \leq j \leq 4$.

We shall solve system (4.1.17) under the assumption $\lambda \neq 4/3$. We will soon see that this condition forces the four branches of the eigenfunction u to be mutually proportional. Moreover, there is a recursive formula for computing each non-zero branch of u; such a formula is valid for all eigenvalues $\lambda \neq 4/3$. These facts will follow from the first (boundary condition), the third and the fourth equations in (4.1.17).

We make an easy remark before we move on: by the second equation in (4.1.17) and the assumption $\lambda \neq 4/3$, we see that $u^-_{i,j} = 0$, or equivalently: $u'_{i,j} = u''_{i,j} (= u^+_{i,j})$. Due again to $\lambda \neq 4/3$, we have that the first four equations are equivalent to the following system:

$$\begin{cases} u_{1,j} = 0, \\ u^-_{i,j} = 0, \\ u_{i,j} + (3\lambda - 2)u^+_{i,j} + u_{i+1,j} = 0 \quad (1 \leq i \leq n-1), \\ u^+_{i-1,j} - 2u_{i,j} + u^+_{i,j} = 0. \end{cases} \tag{4.1.18}$$

Let us fix j and write out the above when $n = 4$:

$$\begin{cases} u_{1,j} = 0, \\ u_{1,j} + (3\lambda - 2)u^+_{1,j} + u_{2,j} = 0, \\ u^+_{1,j} - 2u_{2,j} + u^+_{2,j} = 0, \\ u_{2,j} + (3\lambda - 2)u^+_{2,j} + u_{3,j} = 0, \\ u^+_{2,j} - 2u_{3,j} + u^+_{3,j} = 0, \\ u_{3,j} + (3\lambda - 2)u^+_{3,j} + u_{4,j} = 0. \end{cases} \tag{4.1.19}$$

Reading from top to bottom, we see that all occurring variables can eventually be expressed as a product of some polynomial in λ and $u^+_{1,j}$. This is the same as saying that the space of solutions to the above system (4.1.19) is 1-dimensional. So as j varies from 1 to 4, it follows that the four branches of u are indeed mutually proportional.

We shall ultimately obtain the polynomials $f_n(\lambda)$ and $g_n(\lambda)$ from (4.1.18) and the last equation in (4.1.17). However, it seems to be helpful to solve (4.1.19) first, even though we may not be interested in the solution itself.

Recall that all variables are of the form $u_{1,j}^+$ times a polynomial in λ. Hence we may write:

$$\begin{cases} u_{i,j} := (-1)^{i-1} p_{i-1}(\lambda) \cdot u_{1,j}^+ \ 1 \le i \le n, \\ u_{i,j}^+ := (-1)^{i-1} q_{i-1}(\lambda) \cdot u_{1,j}^+ \ 1 \le i \le n-1 \end{cases} \tag{4.1.20}$$

for some polynomials p_i and q_i, which do not depend on j. (Note that we index p_i and q_i so that $\deg p_i = \deg q_i = i$.) We begin with $p_0(\lambda) = 0$ and $q_0(\lambda) = 1$, which follow from $u_1 = 0 \cdot u_1^+$ and $u_1^+ = 1 \cdot u_1^+$. From (4.1.18), we have

$$\begin{cases} u_{i+1} = -u_i - (3\lambda - 2)u_i^+, \\ u_{i+1}^+ = 2u_{i+1} - u_i^+. \end{cases} \tag{4.1.21}$$

We derive the linear recurrence relations

$$\begin{cases} p_i(\lambda) = p_{i-1}(\lambda) + (3\lambda - 2)q_{i-1}(\lambda) \\ q_i(\lambda) = 2p_i(\lambda) + q_{i-1}(\lambda) \end{cases} \tag{4.1.22}$$

with initial conditions $p_0 = 0,\ q_0 = 1$.

Lemma 4.3 *p_i and q_i have the expressions:*

$$\begin{cases} p_i(\lambda) = (3\lambda - 2)U_{i-1}(3\lambda - 1), \\ q_i(\lambda) = U_i(3\lambda - 1) - U_{i-1}(3\lambda - 1) = g_{i+1}(\lambda). \end{cases}$$

Proof. It is a straightforward induction.

□

Finally we combine the last equation in (4.1.17) with the solutions on the four branches (4.1.20) to determine exactly how the values of u on the four branches are related. Put (4.1.20) into (4.1.17) to get

$$(3\lambda - 4)[-(-1)^{n-2} q_{n-2}(\lambda) u_{1,j}^+ + 2(-1)^{n-1} p_{n-1}(\lambda) u_{1,j}^+] + \sum_{k=1}^{4} (-1)^{n-1} p_{n-1}(\lambda) u_{1,k}^+ = 0.$$

Hence

$$(3\lambda - 4)(2p_{n-1}(\lambda) + q_{n-2}(\lambda))u^+_{1,j} + p_{n-1}(\lambda)\sum_{k=1}^{4} u^+_{1,k} = 0.$$

By our recurrence relation (4.1.22), we have the four equations

$$(3\lambda - 4)q_{n-1}(\lambda)u^+_{1,j} + p_{n-1}(\lambda)\sum_{k=1}^{4} u^+_{1,k} = 0,\ (1 \leq j \leq 4). \tag{4.1.23}$$

Those equations are equivalent to the following linear combinations:

- For each $j = 2,\ 3,$ or 4, it is easy to see that

$$(3\lambda - 4)q_{n-1}(\lambda)(u^+_{1,1} - u^+_{1,j}) = 0.$$

As $\lambda \neq 4/3$, this simplifies to

$$q_{n-1}(\lambda)(u^+_{1,1} - u^+_{1,j}) = 0,\ \ j = 2,\ 3,\ 4. \tag{4.1.24}$$

- On the other hand, summing all four equations in (4.1.23), we have

$$[(3\lambda - 4)q_{n-1}(\lambda) + 4p_{n-1}(\lambda)]\sum_{k=4}^{4} u^+_{1,k} = 0. \tag{4.1.25}$$

Recall that

$$q_{n-1} = g_n(\lambda) = U_{n-1}(3\lambda - 1) - U_{n-2}(3\lambda - 1)$$

and we define

$$f^*_n(\lambda) := (3\lambda - 4)q_{n-1}(\lambda) + 4p_{n-1}(\lambda).$$

(We shall prove that indeed $f^*_n = f_n$ in the next lemma). Now (4.1.24) and (4.1.25) are equivalent to

$$\begin{cases} g_n(\lambda)(u^+_{1,1} - u^+_{1,2}) = 0, \\ g_n(\lambda)(u^+_{1,1} - u^+_{1,3}) = 0, \\ g_n(\lambda)(u^+_{1,1} - u^+_{1,4}) = 0, \\ f^*_n(\lambda)(\sum_{k=1}^{4} u^+_{1,k}) = 0. \end{cases} \tag{4.1.26}$$

Note that if $u_{1,j}^{+}$=0 for $j = 1, \cdots, 4$, then substituting this back into the second equation in (4.1.19) (the case when $j = 4$, or (4.1.18) in general), we can see that $u_{2,j} = 0$. Then the third equation gives $u_{2,j}^{+} = 0$ and so on. Hence we obtain that

$$u = 0 \text{ if } \; u_{1,1}^{+} = u_{1,2}^{+} = u_{1,3}^{+} = u_{1,4}^{+} = 0,$$

(4.1.26) implies that if $\lambda \neq 4/3$ is an eigenvalue, then $f_n^*(\lambda) = 0$ or $g_n(\lambda) = 0$.

Lemma 4.4 *f_n^* has the expression:*

$$f_n^*(\lambda) = T_n(3\lambda - 1) - 3T_{n-1}(3\lambda - 1) = f_n(\lambda). \tag{4.1.27}$$

Proof. By the definitions of p_n and q_n and Lemma 4.3, we have

$$\begin{aligned} f_n^*(\lambda) &= (3\lambda - 4)q_{n-1}(\lambda) + 4p_{n-1}(\lambda) \\ &= (y - 3)U_{n-1}(y) + (3y - 1)U_{n-2}(y), \end{aligned}$$

where $y = 3\lambda - 1$. It is enough to show the above equality holds for $y = \cos\theta$. Let $y = \cos\theta$ and then the right-hand side of the last equality becomes

$$\begin{aligned} & (\cos\theta - 3)\frac{\sin n\theta}{\sin\theta} + (3\cos\theta - 1)\frac{\sin(n-1)\theta}{\sin\theta} \\ = \; & \cos n\theta - 3\cos(n-1)\theta \\ = \; & T_n(y) - 3T_{n-1}(y) \end{aligned}$$

□

Using this Lemma, we can change f_n^* back to f_n in (4.1.26). Hence we have the following two cases about the λ-eigenspace of $-\widehat{\Delta}_1$, or equivalently, solutions to $(X+\lambda M) = 0$.

<u>Case 1</u>: Assume $\lambda \neq 4/3$ is a root of $f_n(\lambda)$. Then by (4.1.12), $g_n(\lambda) \neq 0$. By (4.1.26), we have $u_{1,1}^{+} = u_{1,2}^{+} = u_{1,3}^{+} = u_{1,4}^{+}$. Hence the values of u are completely symmetric and so the λ-eigenspace is 1-dimensional. As the first three equations in (4.1.26) suggest, we can construct an eigenfunction in the following way. We first determine it values in one of the

four directions and then let the values in the other three directions be the same as in that direction.

Case 2: Assume $\lambda \neq 4/3$ is a root of $g_n(\lambda)$. Then $f_n(\lambda) \neq 0$ and (4.1.26) says that $\sum_{k=1}^{4} u_{1,k}^{+} = 0$. This implies that the λ-eigenspace is 3-dimensional. From the last equation in (4.1.26), we know that a basis of three eigenfunctions in the eigenspace can be chosen as follows. We let $u_{1,1}^{+} = 1$, $u_{1,3}^{+} = -1$, and $u_{1,2}^{+} = u_{1,4}^{+} = 0$. Then substituting those values of $u_{1,j}^{+}$ back into (4.1.20), we obtain an eigenfunction for $-\widehat{\Delta}_1$ corresponding λ, a root of g_n. It is easy to see that that this eigenfunction has opposite values on the main diagonal (we call it antisymmetric) and zeros on the sub-diagonal. Later we will use this kind of eigenfunction to build three eigenfunctions for each discrete Laplacian $-\widehat{\Delta}_m$. Two other eigenfunctions of $-\widehat{\Delta}_1$ can be obtained similarly by letting $u_{1,1}^{+} = 1$, $u_{1,2}^{+} = -1$ (or $u_{1,4}^{+} = -1$ resp.), and $u_{1,3}^{+} = u_{1,4}^{+}($ or $u_{1,2}^{+}$ resp.$) = 0$.

The above result also shows that roots of f_n and g_n^3 are zeros of $\det(X + \lambda M)$. Since from (4.1.18) on, we were solving the Dirichlet eigenvalue problem $(X + \lambda M)u = 0$ under the assumption $\lambda \neq 4/3$, the only other zero of $\det(X + \lambda)$ must be $4/3$.

Note that $\det(X + \lambda M)$, f_n and g_n are all polynomials. It follows that

$$\det(X + \lambda M) = c f_n(\lambda) g_n^3(\lambda)(3\lambda - 4)^{8n-9}, \tag{4.1.28}$$

for some non-zero constant c.

Next we shall solve $(X + \lambda M)u = v$ by choosing a special v. Let $u \in \ell(V_1 \setminus V_0)$. Recall our change of coordinates

$$\begin{aligned} u_{i,j}^{+} &= \frac{u_{i,j}' + u_{i,j}''}{2}, \\ u_{i,j}^{-} &= \frac{u_{i,j}' - u_{i,j}''}{2}. \end{aligned}$$

We perform a kind of "averaging" through the four branches by introducing another change of coordinates:

$$\begin{cases} s_{i,1} = u_{i,1} + u_{i,2} + u_{i,3} + u_{i,4}, \\ s_{i,2} = u_{i,1} - u_{i,2}, \\ s_{i,3} = u_{i,1} - u_{i,3}, \\ s_{i,4} = u_{i,1} - u_{i,4}. \end{cases}$$

We can also do the same operations for $s^+_{i,j}$. Then the inverse coordinate change is

$$\begin{cases} u_{i,1} = 1/4(s_{i,1} + s_{i,2} + s_{i,3} + s_{i,4}), \\ u_{i,2} = 1/4(s_{i,1} - 3s_{i,2} + s_{i,3} + s_{i,4}), \\ u_{i,3} = 1/4(s_{i,1} + s_{i,2} - 3s_{i,3} + s_{i,4}), \\ u_{i,4} = 1/4(s_{i,1} + s_{i,2} + s_{i,3} - 3s_{i,4}), \end{cases}$$

and similarly for $u^+_{i,j}$ $(1 \le j \le 4)$. For any function $v \in \ell(V_1 \setminus V_0)$, we can write

$$\begin{array}{lll} j = 1: & t_{i,1} = v_{i,1} + v_{i,2} + v_{i,3} + v_{i,4}, & t^+_{i,1} = v^+_{i,1} + v^+_{i,2} + v^+_{i,3} + v^+_{i,4}, \\ j = 2,3,4: & t_{i,j} = v_{i,1} - v_{i,j}, & t^+_{i,j} = v^+_{i,1} - v^+_{1,j}. \end{array}$$

We shall choose a special v and $t_{i,j}$ and $t^+_{i,j}$ will then be the new changed variables. For example, if $v = v^{(1)}$ is the function on $V_1 \setminus V_0$ corresponding to the first column of J, i.e. $v'_{1,1} = v''_{1,1} = v_{2,1} = 1$ and the remaining entries are 0, then for all $1 \le j \le 4$, $t_{1,j} = 0,\ t^+_{1,j} = 1,\ t_{2,j} = 1,\ t^+_{i,j} = 0\ (i \ge 2),\ t_{i,j} = 0\ (i > 2),$ and $v^-_{i,j} = 0\ (1 \le i \le n-1)$.

Next we shall solve the equation $(X + \lambda M)u = v$ for $v = v^{(1)}$ and $u \in \ell(V_1 \setminus V_0)$. Performing the same actions as we did in (4.1.16), we can see that this is equivalent to the following system:

$$\begin{cases} u_{i,j} + (3\lambda - 2)u^+_{i,j} + u_{i+1,j} = v^+_{i,j} \quad (1 \le i \le n-1), \\ u_{i-1,j} + 2u^+_{i-1,j} + (6\lambda - 6)u_{i,j} + 2u^+_{i,j} + u_{i+1,j} = v_{i,j} \\ \quad (2 \le i \le n-1), \\ u_{n-1,j} + 2u^+_{n-1,j} + (6\lambda - 7)u_{n,j} + \sum_{k=1}^4 u_{n,k} = v_{n,j}, \\ (3\lambda - 4)u^-_{i,j} = v^-_{i,j} \quad (1 \le i \le n-1), \end{cases}$$

By summing together all four equations when $j = 1$ and subtracting two equations when

$j \neq 1$, and the change of coordinates, we see that it is also equivalent to

$$\begin{cases} s_{i,j} + (3\lambda - 2)s^+_{i,j} + s_{i+1,j} = t^+_{i,j} \quad (1 \leq i \leq n-1), \\ s_{i-1,j} + 2s^+_{i-1,j} + (6\lambda - 6)s_{i,j} + 2s^+_{i,j} + s_{i+1,j} = t_{i,j} \\ \ (2 \leq i \leq n-1), \\ s_{n-1,j} + 2s^+_{n-1,j} + (6\lambda - 7 + 4\delta_{1,j})s_{n,j} = t_{n,j}, \\ (3\lambda - 4)u^-_{i,j} = v^-_{i,j} \quad (1 \leq i \leq n-1) \end{cases}$$

Under the new system of coordinates, the matrix A representing $X + \lambda M$ is the direct sum of the following five blocks:

$$A_0 = \begin{bmatrix} 3\lambda - 4 & & & \\ & 3\lambda - 4 & & \\ & & \ddots & \\ & & & 3\lambda - 4 \end{bmatrix}_{4(n-1)\times 4(n-1)},$$

a diagonal matrix of size $4(n-1) \times 4(n-1)$ which corresponds to the last equation in the above system, and for $j = 1,\ 2,\ 3,$ and 4,

$$A_j = \begin{bmatrix} 3\lambda-2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 6\lambda-6 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3\lambda-2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 6\lambda-6 & 2 & 1 & 0 & 0 & 0 & 0 \\ & & & & & & \ddots & \ddots & \ddots & \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 3\lambda-2 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 2 & 6\lambda-7+4\delta_{1,j} \end{bmatrix}_{(2n-2)\times(2n-2)}.$$

The augmented matrix for the equation $(X + \lambda M)u = v$ is

$$[A_j|v] = \left[\begin{array}{cccccccccc|c} 3\lambda-2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t^+_{1,j} \\ 2 & 6\lambda-6 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & t_{2,j} \\ 0 & 1 & 3\lambda-2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & t^+_{2,j} \\ 0 & 1 & 2 & 6\lambda-6 & 2 & 1 & 0 & 0 & 0 & 0 & t_{3,j} \\ & & & & & & \ddots & \ddots & \ddots & & \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 3\lambda-2 & 1 & t^+_{n-1,j} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 2 & 6\lambda-7+4\delta_{1,j} & t_{n,j} \end{array}\right],$$

which can be row-reduced into the form

$$\left[\begin{array}{ccccccc|c} 3\lambda-2 & 1 & 0 & \cdots & 0 & 0 & 0 & t^+_{1,j} \\ 3\lambda-4 & -2(3\lambda-4) & 3\lambda-4 & \cdots & 0 & 0 & 0 & t^+_{1,j}-t_{2,j}+t^+_{2,j} \\ 0 & 1 & 3\lambda-2 & \cdots & 0 & 0 & 0 & t^+_{2,j} \\ 0 & 0 & 3\lambda-4 & \cdots & 0 & 0 & 0 & t^+_{2,j}-t_{3,j}+t^+_{3,j} \\ & & & & \ddots & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 3\lambda-2 & 1 & t^+_{n-1,j} \\ 0 & 0 & 0 & \cdots & 0 & 3\lambda-4 & -2(3\lambda-4)-4\delta_{1,j} & t^+_{n-1,j}-t_{n,j} \end{array}\right],$$

and

$$[A_0|v] = \left[\begin{array}{cccc|c} 3\lambda-4 & & & & v^-_{1,j} \\ & 3\lambda-4 & & & v^-_{2,j} \\ & & \ddots & & \vdots \\ & & & 3\lambda-4 & v^-_{n-1,j} \end{array}\right].$$

In the special case when v is the function corresponding to the first column of J, our linear system becomes

$$\begin{bmatrix} 3\lambda-2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 3\lambda-4 & -2(3\lambda-4) & 3\lambda-4 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 3\lambda-2 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 3\lambda-4 & \cdots & 0 & 0 & 0 \\ & & & & \ddots & \ddots & \\ 0 & 0 & 0 & \cdots & 1 & 3\lambda-2 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 3\lambda-4 & -2(3\lambda-4)-4\delta_{1,j} \end{bmatrix} \begin{bmatrix} s^+_{1,j} \\ s_{2,j} \\ s^+_{2,j} \\ s_{3,j} \\ \vdots \\ s^+_{n-1,j} \\ s_{n,j} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 3\lambda-4 & & & \\ & 3\lambda-4 & & \\ & & \ddots & \\ & & & 3\lambda-4 \end{bmatrix} \begin{bmatrix} u^-_{1,j} \\ u^-_{2,j} \\ \vdots \\ u^-_{n-1,j} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

which implies that when $\lambda \neq 4/3$, $u^-_{i,j} = 0$ for all $i,\ j$. By clearing all unnecessary factors $3\lambda - 4$, whenever possible, we have the following system

$$\begin{cases} (3\lambda-2)s^+_{1,j} + s_{2,j} = 1 \\ s^+_{1,j} - 2s_{2,j} + s^+_{2,j} = 0 \\ s_{2,j} + (3\lambda-2)s^+_{2,j} + s_{3,j} = 0 \\ \dots \quad \dots \quad \dots \\ s_{n-1,j} + (3\lambda-2)s^+_{n-1,j} + s_{n,j} = 0 \\ (3\lambda-4)s^+_{n-1,j} - [2(3\lambda-4) + 4\delta_{1,j}]s_{n,j} = 0 \end{cases} \tag{4.1.29}$$

The first $2n-3$ equations allow us to write all unknowns $s_{i,j}$ and $s^+_{i,j}$ ($i > 1$ and $1 \leq j \leq 4$) in terms of $s^+_{1,j}$ as

$$s^+_{i,j} = (-1)^{i-1}[a_i(\lambda)s^+_{1,j} + b_i(\lambda)],$$
$$s_{i,j} = (-1)^{i-1}[c_i(\lambda)s^+_{1,j} + d_i(\lambda)],$$

for some polynomials $a_i,\ b_i,\ c_i$ and d_i.

Lemma 4.5 *If $(X + \lambda M)u = v^{(1)}$ and $s_{i,j}$ and $s_{i,j}$ are defined as above, then for $i > 1$ and $1 \leq j \leq 4$,*

$$s^+_{i,j} = (-1)^{i-1}[(U_{i-1}(y) - U_{i-2}(y))s^+_{1,j} - 2U_{i-2}(y)], \tag{4.1.30}$$
$$s_{i,j} = (-1)^{i-1}[(y-1)U_{i-2}(y)s^+_{1,j} - (U_{i-2}(y) - U_{i-3}(y))], \tag{4.1.31}$$

where $y = 3\lambda - 1$. For $i = 1$, we have

$$s^+_{1,1} = \frac{(2y-2)U_{n-2}(y) - 4U_{n-3}(y)}{(2y^2-3y-1)U_{n-2}(y) - (y-3)U_{n-3}(y)} \tag{4.1.32}$$
$$s^+_{1,j} = \frac{2U_{n-2}}{U_{n-1} - U_{n-2}} \qquad (2 \leq j \leq 4). \tag{4.1.33}$$

Proof. We rewrite the first equation in 4.1.29 as

$$(-1) + (3\lambda - 2)s^+_{1,j} + s_{2,j} = 0$$

and we use the fictitious unknown $\hat{s}_{1,j} = -1$, so that we achieve a more symmetric equation

$$\hat{s}_{1,j} + (3\lambda - 2)s^+_{1,j} + s_{2,j} = 0.$$

This, together with the other equations

$$\begin{cases} s_{i,j} + (3\lambda - 2)s^+_{i,j} + s_{i+1,j} = 0 \\ s^+_{i,j} - 2s_{i+1,j} + s^+_{i+1,j} = 0 \end{cases}$$

imply that $a_i,\ b_i,\ c_i,$ and d_i satisfy the recurrence relations

$$\begin{cases} a_{i+1} = (6\lambda - 3)a_i + 2c_i, \\ b_{i+1} = (6\lambda - 3)b_i + 2d_i, \\ c_{i+1} = (3\lambda - 2)a_i + c_i, \\ d_{i+1} = (3\lambda - 2)b_i + d_i, \end{cases} \tag{4.1.34}$$

with the initial conditions $a_1 = 1,\ b_1 = 0,\ c_1 = 0,\ d_1 = -1$. In terms of the matrices

$$A = \begin{bmatrix} 6\lambda - 3 & 2 \\ 3\lambda - 2 & 1 \end{bmatrix} = \begin{bmatrix} 2y - 1 & 2 \\ y - 1 & 1 \end{bmatrix}$$

and

$$X_i = \begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix},$$

(4.1.34) can be written as $X_{i+1} = AX_i$ with

$$X_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Then the unique solution to (4.1.34) is clearly given by $X_i = A^{i-1}X_1$. Hence our proof will be finished once we have proved that

$$A^k = \begin{bmatrix} U_k(y) - U_{k-1}(y) & 2U_{k-1}(y) \\ (y-1)U_{k-1}(y) & U_{k-1}(y) - U_{k-2}(y) \end{bmatrix}$$

To prove this claim, we use induction on $k \geq 1$. When $k = 1$, note that $U_0 = 1$ and $U_1 = 2y$, so recursive formulas for Chebyshev polynomials gives us $U_{-1} = 0$. Hence

$$\begin{bmatrix} U_1 - U_0 & 2U_0 \\ (y-1)U_0 & U_0 - U_{-1} \end{bmatrix} = \begin{bmatrix} 2y-1 & 2 \\ y-1 & 1 \end{bmatrix} = A.$$

Next we assume that our claim is true for k. Hence by the induction assumption, we have

$$\begin{aligned} A^{k+1} &= \begin{bmatrix} 2y-1 & 2 \\ y-1 & 1 \end{bmatrix} \begin{bmatrix} U_k - U_{k-1} & 2U_{k-1} \\ (y-1)U_{k-1} & U_{k-1} - U_{k-2} \end{bmatrix} \\ &= \begin{bmatrix} (2yU_k - U_{k-1}) - U_k & 2(2yU_{k-1} - U_{k-2}) \\ (y-1)U_k & (2yU_{k-1} - U_{k-2}) - U_{k-1} \end{bmatrix} \\ &= \begin{bmatrix} U_{k+1} - U_k & 2U_k \\ (y-1)U_k & U_k - U_{k-1} \end{bmatrix} \end{aligned}$$

Here we have used the recursive identity $U_{k+1} = 2yU_k - U_{k-1}$ for Chebyshev polynomials in the last equality. Hence the claim is proved.

Having expressed all our unknowns in terms of the single variable $s^+_{1,j}$, we can solve for $s^+_{1,j}$ by making use of the very last equation:

$$(3\lambda - 4)s^+_{n-1,j} - [2(3\lambda - 4) + 4\delta_{1,j}]s_{n,j} = 0.$$

We first solve for $s^+_{1,j}$ when $j = 2,\ 3,\ 4$. Since $\delta_{1,j} = 0$, the above equation simplifies to $s^+_{n-1,j} = 2s_{n,j}$. Substituting the expressions for $s^+_{n-1,j}$ and $s_{n,j}$ from (4.1.30) and (4.1.31), and using the identity $U_{k+1} = 2yU_k - U_{k-1}$, we have that for $j = 2,\ 3$, and 4,

$$s^+_{1,j} = \frac{2U_{n-2}}{U_{n-1} - U_{n-2}}.$$

Next we solve for $s^+_{1,1}$, starting from $(6\lambda - 4)s_{n,1} - (3\lambda - 4)s^+_{n-1,1} = 0$ or $(2y-2)s_{n,1} - (y-3)s^+_{n-1,1} = 0$ as $y = 3\lambda - 1$. Again, using what we have shown in the first part of the lemma, we can find that

$$s^+_{1,1} = \frac{(2y-2)U_{n-2} - 4U_{n-3}}{(2y^2 - 3y - 1)U_{n-2} - (y-3)U_{n-3}}.$$

Moreover, we can further simplify the denominator by the identity $T_n = yU_{n-1} - U_{n-2}$ so that

$$s_{1,1}^{+} = \frac{(2y-2)U_{n-2} - 4U_{n-3}}{T_n - 3T_{n-1}} = \frac{2(T_{n-1} - U_{n-2} - U_{n-3})}{T_n - 3T_{n-1}}$$

□

Remark 4.6 *Notice that our formula does not hold for the boundary variables $s_{1,j} = 0$ (the formula would give $s_{1,1} = U_{-2}(y)$, which is -1 if we use the recursive formulas for Chebyshev polynomials). But this is not a major issue as we only consider functions on $V_1 \setminus V_0$ and so boundary values are irrelevant.*

Before we compute K_D and $\lambda - K_T$, we derive some formulas for future use.

Lemma 4.7 *The following three formulas can be obtained from properties of Chebyshev polynomials.*

$$(i) \qquad 1 + s_{1,1}^{+} = (y+1)\frac{U_{n-1}(y) - 3U_{n-2}(y)}{T_n(y) - 3T_{n-1}(y)}. \tag{4.1.35}$$

$$(ii) \qquad \begin{vmatrix} T_{n-1} - U_{n-2} - U_{n-3} & U_{n-2} \\ T_n - 3T_{n-1} & U_{n-1} - U_{n-2} \end{vmatrix} = 2, \ \textit{for any } n \geq 1. \tag{4.1.36}$$

$$(iii) \qquad s_{1,1}^{+} - s_{1,2}^{+} = \frac{4}{f_n(\lambda) g_n(\lambda)}. \tag{4.1.37}$$

Proof. To prove (4.1.35), we first substitute the expression for $s_{1,1}^{+}$ and regroup the numerator to obtain

$$\begin{aligned} LHS &= \frac{(T_n - 3T_{n-1}) + 2(T_{n-1} - U_{n-2} - U_{n-3})}{T_n - 3T_{n-1}} \\ &= \frac{(T_n + U_{n-2}) - (T_{n-1} + U_{n-3}) - 3U_{n-2} - U_{n-3}}{T_n - 3T_{n-1}}. \end{aligned}$$

Then use the formula

$$T_n = yU_{n-1} - U_{n-2} \tag{4.1.38}$$

and the recursive formula to simplify the right-hand side of the last equation to

$$\frac{yU_{n-1} - yU_{n-2} - 3U_{n-2} - (2yU_{n-2} - U_{n-1})}{T_n - 3T_{n-1}}$$

Simplifying this gives us

$$\frac{(y+1)U_{n-1} - 3(y+1)U_{n-2}}{T_n - 3T_{n-1}}$$

and the desired result follows.

To show (4.1.36), we first do the column operation $C_1 + y \times C_2 \to C_1$ to obtain

$$LHS = \begin{vmatrix} T_{n-1} - U_{n-2} + (yU_{n-2} - U_{n-3}) & U_{n-2} \\ (T_n + yU_{n-1}) - (T_{n-1} + yU_{n-2}) - 2T_{n-1} & U_{n-1} - U_{n-2} \end{vmatrix}.$$

Then by formula (4.1.38) and the formula

$$T_n = yU_n - yU_{n-1}, \tag{4.1.39}$$

we can simplify this determinant to

$$\begin{vmatrix} 2T_{n-1} - U_{n-2} & U_{n-2} \\ (U_n - U_{n-1}) - 2T_{n-1} & U_{n-1} - U_{n-2} \end{vmatrix}.$$

A row operation and column operation and the identity $2T_n = U_n - U_{n-2}$ give

$$\begin{aligned} \begin{vmatrix} 2T_{n-1} - U_{n-2} & U_{n-2} \\ (U_n - U_{n-1}) - U_{n-2} & U_{n-1} \end{vmatrix} &= \begin{vmatrix} 2T_{n-1} & U_{n-2} \\ U_n - U_{n-2} & U_{n-1} \end{vmatrix} \\ &= 2\begin{vmatrix} T_{n-1} & U_{n-2} \\ T_n & U_{n-1} \end{vmatrix}. \end{aligned}$$

Furthermore, the recursive formula gives

$$\begin{aligned} \begin{vmatrix} T_{n-1} & U_{n-2} \\ T_n & U_{n-1} \end{vmatrix} &= \begin{vmatrix} T_{n-1} & U_{n-2} \\ 2yT_{n-1} - T_{n-2} & 2yU_{n-2} - U_{n-3} \end{vmatrix} \\ &= 2y\begin{vmatrix} T_{n-1} & U_{n-2} \\ T_{n-1} & U_{n-2} \end{vmatrix} - \begin{vmatrix} T_{n-1} & U_{n-2} \\ T_{n-2} & U_{n-3} \end{vmatrix} \\ &= \begin{vmatrix} T_{n-2} & U_{n-3} \\ T_{n-1} & U_{n-2} \end{vmatrix}, \end{aligned}$$

Since this is true for all n, we know that this determinant will eventually be

$$\begin{vmatrix} T_0 & U_{-1} \\ T_1 & U_0 \end{vmatrix}$$

Substituting the expressions $U_{-1} = 0,\ U_0 = 1,\ T_0 = 1$ and $T_1(y) = y$, we see that the above determinant is equal to 1.

Formula (4.1.37) follows directly from the expressions for $s^+_{1,j}$, f_n, and g_n, and the result of (4.1.36).

$\square$

Now we are ready to prove Theorem 4.1 about the expressions of the spectral decimation function $R(\lambda)$ and $3R(\lambda) - 4$ at the beginning of this chapter.

Proof of Theorem 4.1. First, recall that at the beginning of this chapter we mentioned that

$$\begin{aligned} K_D &= (T - J^t(X + \lambda M)^{-1}J)_{2,1} \\ &= -< \nu^{(2)}, u >, \end{aligned}$$

where $\nu^{(j)}$ is the j-th column of J and $u = (X + \lambda M)^{-1}\nu^{(1)}$. By the definition of $\nu^{(2)}$, we know that

$$\begin{aligned} < \nu^{(2)},\ u > &= u^{'}_{1,2} + u^{''}_{1,2} + u_{2,2} \\ &= 2u^+_{1,2} + u_{2,2}. \end{aligned}$$

By the change of coordinates, we have that

$$\begin{aligned} u^+_{1,2} &= 1/4(s^+_{1,1} - 3s^+_{1,2} + s^+_{1,3} + s^+_{1,4}) \\ &= 1/4(s^+_{1,1} - s^+_{1,2}), \end{aligned}$$

where the last equality holds because Lemma 4.5 shows $s^+_{1,j}$ are the same for $j = 2,\ 3$, and

4. The same reasoning shows that

$$\begin{aligned} u_{2,2} &= \frac{1}{4}(s_{2,1} - 3s_{2,2} + s_{2,3} + s_{2,4}) \\ &= \frac{1}{4}(s_{2,1} - s_{2,2}) \\ &= \frac{1}{4}([1 - (y-1)s^+_{1,1}] - [1 - (y-1)s^+_{1,2}]) \\ &= -\frac{1}{4}(y-1)(s^+_{1,1} - s^+_{1,2}) \end{aligned}$$

Hence (4.1.37) and (iii) of Lemma 4.7 implies

$$\begin{aligned} K_D &= - < \nu^{(2)},\ u > \\ &= -(2u^+_{1,2} + u_{2,2}) \\ &= \frac{y-3}{4}(s^+_{1,1} - s^+_{1,2}) \\ &= \frac{3\lambda - 4}{f_n(\lambda) g_n(\lambda)}. \end{aligned}$$

As for $\lambda - K_T(\lambda)$, we first note that since the diagonal entries of D and T are -3,

$$\begin{aligned} -3K_D(\lambda) - 3K_T(\lambda) &= (K_D \cdot D + K_T \cdot T)_{1,1} \\ &= (T - J^t G(\lambda) J)_{1,1} \\ &= -3- < \nu^{(1)},\ u >, \end{aligned}$$

where we recall that $G(\lambda) = (X + \lambda M)^{-1}$, $\nu^{(1)}$ is the first column of J and $u = G(\lambda)\nu^{(1)}$. It follows that

$$\begin{aligned} 3(\lambda - K_T(\lambda)) &= 3\lambda - 3 + 3K_D - < \nu^{(1)},\ u > \qquad (4.1.40) \\ &= 3\lambda - 3 - 2(u^+_{1,1} + 3u^+_{1,2}) - (u_{2,1} + 3u_{2,2}). \end{aligned}$$

Our change of variables gives that

$$\begin{aligned} u^+_{1,1} + 3u^+_{1,2} &= s^+_{1,1} \\ u_{2,1} + 3u_{2,2} &= s_{2,1} \\ s_{2,1} &= 1 - (3\lambda - 2)s^+_{1,1} \end{aligned}$$

Substitute these back into (4.1.40) and use (4.1.35) to get

$$\begin{aligned} 3(\lambda - K_T) &= 3\lambda - 3 - 2s^+_{1,1} - 1 + (3\lambda - 2)s^+_{1,1} \\ &= (3\lambda - 4) + (3\lambda - 4)s^+_{1,1} \\ &= (3\lambda - 4)[(y+1)\frac{U_{n-1}(y) - 3U_{n-2}(y)}{T_n(y) - 3T_{n-1}(y)}] \\ &= (3\lambda - 4) \cdot 3\lambda \cdot \frac{h_n(\lambda)}{f_n(\lambda)}. \end{aligned}$$

Thus we have shown

$$\lambda - K_T = (3\lambda - 4) \cdot \lambda \cdot \frac{h_n(\lambda)}{f_n(\lambda)}.$$

By the definition of the spectral decimation function R, we have

$$R(\lambda) = \frac{\lambda - K_T(\lambda)}{K_D(\lambda)} = \lambda \cdot g_n(\lambda) \cdot h_n(\lambda).$$

Lastly, we compute $3R(\lambda) - 4$. Equivalently, we show

$$3(\lambda - K_T(\lambda)) - 4K_D(\lambda) = (3\lambda - 4)\frac{l_n(\lambda)}{g_n(\lambda)}.$$

Note that

$$\begin{aligned} 3(\lambda - K_T(\lambda)) - 4K_D(\lambda) &= 3\lambda - 3 - K_D - < \nu^{(1)},\ u > \\ &= 3\lambda - 3 + < \nu^{(2)},\ u > - < \nu^{(1)},\ u > \\ &= 3\lambda - 3 + (2u^+_{1,2} + u_{2,2}) - (2u^+_{1,1} + u_{2,1}) \end{aligned}$$

We first regroup the terms on the right-hand side to obtain

$$3\lambda - 3 - 2(u^+_{1,1} - u^+_{1,2}) - (u_{2,1} - u_{2,2}).$$

Then using our change of variables and the first equation in (4.1.29), we see that the last expression above is equal to

$$\begin{aligned} & 3\lambda - 3 - 2s^+_{1,2} - s_{2,2} \\ = \ & 3\lambda - 3 - 2s^+_{1,2} - (1 - (3\lambda - 2)s^+_{1,2}) \end{aligned}$$

Simplifying and applying Lemma 4.5 gives us

$$\begin{aligned} & (3\lambda-4)(1+s^{+}_{1,2}) \\ = \; & (3\lambda-4)\frac{U_{n-1}(y)+U_{n-2}(y)}{U_{n-1}(y)-U_{n-2}(y)} \\ = \; & (3\lambda-4)\frac{l_n(\lambda)}{g_n(\lambda)}. \end{aligned}$$

Thus

$$\begin{aligned} 3R(\lambda)-4 &= \frac{3(\lambda-K_T)-4K_D}{K_D} \\ &= \frac{\frac{(3\lambda-4)l_n}{g_n}}{\frac{3\lambda-4}{f_n g_n}} \\ &= f_n(\lambda)l_n(\lambda). \end{aligned}$$

By definition, the forbidden eigenvalues are the zeros of K_D, namely $4/3$, and the zeros of $\det(X+\lambda M)$, which are $4/3$, and the zeros of f_n and g_n. This completes the proof of the theorem.

□

Remark 4.8 *As far as we are aware, this is the first infinite family to be analyzed whose spectral decimation function grows unboundedly in degree.*

As a byproduct, since eigenfunctions of $-\widehat{\Delta}_1$ corresponding to different eigenvalues must be orthogonal, we can get an interesting corollary about a new property of the Chebyshev polynomials.

Corollary 4.9 *Suppose λ and μ are either different roots of f_n or different roots of g_n. Then*

$$\sum_{i=1}^{n-1}(p_i(\lambda)p_i(\mu)+q_{i-1}(\lambda)q_{i-1}(\mu))=0,$$

where p_i and q_i are defined in Lemma 4.3.

4.2 Gaps in the spectrum of the Laplacian on $\mathcal{VS}_n$

In this section, we shall prove that our gap theorem, Theorem 3.4 in Chapter 3, applies to the infinite family of fractals $\mathcal{VS}_n$ and so there exist gaps in the spectrum of the standard Laplacian.

Proposition 4.10 *There exist gaps in the spectrum of the standard Laplacian on $\mathcal{VS}_n$.*

Proof. Let α_i, β_i, γ_i and ξ_i represent roots of f_n, g_n, l_n and h_n respectively as in the previous section. By Proposition 4.2, we know that α_i and β_i, α_i and γ_i, β_i and ξ_i are all alternating. It follows that the set of forbidden eigenvalues

$$\mathfrak{F} = \{\alpha_1,\ \cdots,\ \alpha_n,\ \beta_1,\ \cdots,\ \beta_{n-1},\ 4/3\}$$

with α_1 the smallest forbidden eigenvalue and $4/3$ the largest. Also $R(\lambda) = 0$ has exactly $2n-1$ real roots $0,\ \beta_1,\ \cdots,\ \beta_{n-1},\ \xi_1,\ \cdots,\ \xi_{n-1}$ and $R(\lambda) - 4/3 = 0$ has exactly $2n-1$ real roots $\alpha_1,\ \cdots,\ \alpha_n,\ \gamma_1,\ \cdots,\ \gamma_{n-1}$. Hence $[0,\ 4/3]$ is in the domain of every branch of the inverse function.

As both the last root of R and $R-4/3$ occur before $x = 1$, it is clear that $R^{-1}[0,\ 4/3] \subseteq [0,\ 1]$, so the first condition is also satisfied for this infinite family of fractals.

Since the spectral decimation function R is a polynomial and its bottom inverse function ϕ_0 is increasing, the second inverse function must be decreasing and so the second condition of the theorem is satisfied.

To check the strict convexity of ϕ_0, we first note that the polynomial R has odd degree $d := 2n-1 \geq 3$, has positive leading coefficient, and has precisely d real zeros $\delta_1 < \cdots < \delta_d$. By Rolle's Theorem, we see that R' has $2n-2$ real zeros μ_i such that $\delta_1 < \mu_1 < \delta_2 < \mu_2 < \cdots < \mu_{d-1} < \delta_d$. The same reasoning shows that R'' has $2n-3$ real zeros ν_j such that $\mu_1 < \nu_1 < \mu_2 < \cdots < \nu_{2n-3} < \mu_{2n-2}$. In particular, $\mu_1 < \nu_1$. Since $\deg(R')$ is even and $\deg(R'')$ is odd, we see that $R' \geq 0$ and $R'' < 0$ throughout $\mathrm{Im}(\phi_0) = (-\infty, \mu_1]$. But the inverse of an increasing and strictly concave function is strictly convex. Hence the third condition is also satisfied.

From (4.1.13) we know that α_1 is the second smallest root of $R(\lambda) - 4/3 = 0$, so $\phi_1(4/3) = \alpha_1$. Because all other forbidden eigenvalues are greater than α_1 by (4.1.12), we have that $\phi_1(4/3)$ is less than or equal to all forbidden eigenvalues and so the last condition of the theorem holds. Therefore the gap theorem tells us that there exist gaps in the spectrum of the Laplacian on $\mathcal{VS}_n$ for $n = 2,\ 3, \cdots$.

$\square$

B. Hambly and T. Kumagai have proved in [13] that the necessary heat kernel estimate holds for the standard Laplacian on $\mathcal{VS}_n$. Hence we can obtain the following immediate corollaries using the same argument by R. Strichartz.

Corollary 4.11 *Let $\{N_m\}$ be a sequence of integers such that $\dfrac{\lambda_{N_m+1}}{\lambda_{N_m}} - 1$ is bounded away from zero. Then the partial sums of the Fourier series $S_{N_m}f$ converge to f as $m \to \infty$ in L^p for $f \in L^p$ $(1 \le p < \infty)$ and uniformly if f is continuous.*

Corollary 4.12 *Let $1 < p < \infty$. Let*

$$Sf(x) = (\sum_{m=1}^{\infty} |S_m f(x)|^2)^{1/2},$$

for

$$S_m f(x) = \sum_{j=N_{m-1}+1}^{N_m} c_j u_j(x),$$

where u_j are either Dirichlet or Neumann eigenfunctions of the Laplacian and $\{N_m\}$ is the same sequence as in the above theorem. Then there exist constants A_p and B_p such that

$$A_p \parallel f \parallel_p \le \parallel Sf \parallel_p \le B_p \parallel f \parallel_p .$$

Chapter 5

Hypergraphs and Laplacians

M. Fukushima and T. Shima [10], and R. Strichartz [33] proved that when $\mathbf{K}$ is the Sierpinski gasket, $3/2$ is a Dirichlet (and Neumann) eigenvalue of the discrete Laplacian, $-\widehat{\Delta}_m$, at any step $m > 1$, of multiplicity $\frac{3^m-3}{2}$. Similarly, when $\mathbf{K} = \mathcal{VS}_n$, the n-branch Vicsek set, we can prove that $4/3$ is always a Dirichlet (and Neumann) eigenvalue of $-\widehat{\Delta}_m$ ($m \geq 1$) by constructing independent eigenfunctions. Moreover, using the same argument as in [33], it can be shown that the multiplicity of $4/3$ is of order N^m, where N is the number of contraction maps.

Existence of such eigenvalues with large multiplicity are of special interest. Let

$$\rho(\lambda) = \#\{\text{eigenvalue } \alpha : \ \alpha \leq \lambda\}$$

be the eigenvalue counting function (by multiplicity). If there exist such eigenvalues, then $\lim_{\lambda\to\infty} \frac{\rho(\lambda)}{\lambda^d}$ does not exist for any d ([19] and [30]). This is quite different from the classical smooth analysis.

Note that the forbidden eigenvalue $3/2$ has the form $\frac{|V_0|}{|V_0|-1}$ for V_0 being the set of boundary points of the fractal $\mathcal{SG}$ (resp. $4/3$ in $\mathcal{VS}_n$). In this chapter, we shall study the nature of the eigenvalue $\frac{|V_0|}{|V_0|-1}$ of the Laplacian on finite graphs, built up from complete graphs of $|V_0|$ vertices. In doing so, it appears natural to borrow some concepts from the theory of hypergraphs. A standard reference on hypergraph theory is C. Berge [4]. However, this chapter is self-contained. All the necessary definitions are given in the next section.

5.1 Definitions and Notations

Here we restrict ourselves to finite hypergraphs. First we give a set-theoretic definition.

Definition 5.1 *A* ***hypergraph*** *is a pair $\mathcal{H} = (V, E)$, where V is a finite non-empty set and E is a set of subsets of V such that*

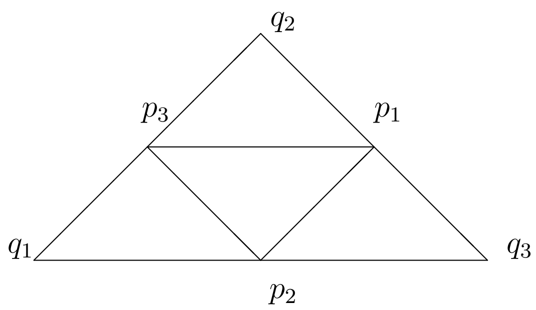
(1) $e \neq \varnothing$ for every $e \in E$; and

(2) $\bigcup E = V$.

Members of V are called **vertices** of $\mathcal{H}$. Members of E are called **hyperedges** (or simply **edges**) of $\mathcal{H}$.

Remark 5.2 *Whereas condition (1) is a natural one to make, condition (2) is related to the notion of duality, which we will not discuss here.*

Examples. The step-1 graph of the Sierpinski gasket, $\mathcal{SG}$, can be viewed as a hypergraph with 6 vertices, as in the traditional graph theory, and 3 hyperedges. Here we think each of the three sets of vertices, $\{q_1, p_2, p_3\}$, $\{q_2, p_1, p_3\}$ and $\{q_3, p_1, p_2\}$, (or the three corresponding small triangles) as a hyperedge. In contrast, as an ordinary graph, it has 9 edges.



Similarly, the finite graphs of $\mathcal{SG}_3$, $\mathcal{VS}_n$, the pentagasket (see [2] for the definition of pentagasket and the spectral analysis of the Laplacian on it), and many other fractals are hypergraphs, where we can think of each of the triangles, squares, or pentagons as a hyperedge.

Given a hypergraph $\mathcal{H} = (V, E)$, let us write $V = (x_1, \cdots, x_n)$ and $E = (e_1, \cdots, e_m)$ as ordered lists. The space $L(V, E)$ of real functions $f : E \times V \to \mathbb{R}$ can be identified with the space of $m \times n$ real matrices $f = \big(f_{e,x}\big)$. The **incidence matrix** of $\mathcal{H}$ is the matrix $A \in L(V, E)$ defined by

$$A_{e,x} = \begin{cases} 1 & \text{if } x \in e \\ 0 & \text{if } x \notin e. \end{cases}$$

Let $\ell(V)$ (or $\ell(E)$) be the space of real functions on V (resp. on E). Then the incidence matrix A becomes the linear transformation $A : \ell(V) \to \ell(E)$, defined by

$$Au(e) = \sum_{x \in e} u(x),$$

for $u \in \ell(V)$.

Definition 5.3 *The* ***hypergraph degree*** *of a vertex $x \in V$ is defined by*

$$\deg_{\mathcal{H}}(x) = \#\{e \in E : x \in e\}.$$

This definition generalizes the notion of degree in traditional graph theory, however, hyperdegree and degree need not be the same. For example, if x is any of the three interior points of the step-1 graph of $\mathcal{SG}$, then $\deg_{\mathcal{H}}(x) = 2$, while $\deg_G(x) = 4$.

Next we introduce a subclasses of hypergraphs which contain the finite graphs of many known examples of fractals.

Definition 5.4 *A hypergraph $\mathcal{H}$ is k-**uniform** for every $e \in E$, $|e| = k$, where $k \geq 0$ is an integer.*

The finite graphs of $\mathcal{SG}$, $\mathcal{VS}_n$, or the pentagasket are uniform hypergraphs, with each hyperedge containing 3, 4 or 5 vertices. In graph theory, a *simple graph* (a graph with no loops and no repeated edges) is just a 2-uniform simple hypergraph.

5.2 The Laplacian on hypergraphs

We fix a hypergraph $\mathcal{H} = (V, E)$ and a set of real positive weights $\mathcal{W} = \{r_e : e \in E\}$. Define diagonal matrices $S \in L(E)$ and $\tilde{M}$, N_1, $N \in L(V)$ by

$$\begin{aligned} S_{e,e} &= r_e, \\ \tilde{M}_{x,x} &= \sum_{e \ni x} r_e, \\ (N_1)_{x,x} &= \sum_{e \ni x} |e|\, r_e, \\ N_{x,x} &= \sum_{e \ni x} \big(|e| - 1\big)\, r_e. \end{aligned}$$

Thus, $N_1 = \tilde{M} + N$. We may think of the entries of $\tilde{M}$ as weighted hypergraph degrees. Indeed, setting $r_e = 1$ gives $\tilde{M}_{x,x} = \deg_{\mathcal{H}}(x)$.

The **Laplacian** with weights $\mathcal{W}$ is the operator $\Delta \in L(V)$ given by

$$\Delta u(x) = \sum_{e \ni x} r_e \sum_{y \in e} [u(y) - u(x)].$$

The associated **normalized Laplacian** is defined by $\hat{\Delta} = N^{-1}\Delta$.

Lemma 5.5 *$-\Delta$ is a positive operator, whose kernel contains at least the constant function $\mathbf{1} \in \ell(V)$.*

Proof. For $u, v \in \ell(V)$, we have the formula

$$(-\Delta u, v)_V = \sum_{e \in E} r_e \sum_{\{x,y\} \subset e} [u(x) - u(y)]\,[v(x) - v(y)].$$

Hence $-\Delta$ is positive. It is clear that $\Delta \mathbf{1} = 0$.

□

Our true starting point is:

Lemma 5.6 *For a general hypergraph $\mathcal{H}$,*

$$A^t\, S\, A = N_1 + \Delta, \tag{5.2.1}$$

where A is the incidence matrix of the hypergraph. If $\mathcal{H}$ is k-uniform, then

$$A^t\, S\, A = \tilde{M}\big(k + (k-1)\hat{\Delta}\big). \tag{5.2.2}$$

Proof. We first observe that when $\mathcal{H}$ is k-uniform, then $N_1 = k\tilde{M}$ and $N = (k-1)\tilde{M}$, and (5.2.2) follows from (5.2.1).

To prove (5.2.1), let $u \in \ell(V)$ and $x \in V$.

$$\begin{aligned}
\Delta u(x) &= \sum_{e \ni x} r_e \sum_{y \in e} [u(y) - u(x)] \\
&= \sum_{e \ni x} r_e \sum_{y \in e} u(y) - \sum_{e \ni x} r_e\, |e|\, u(x) \\
&= \Big(\sum_{e \in E} \sum_{y \in V} (A^t)_{x,e} S_{e,e} A_{e,y}\, u(y)\Big) - (N_1)_{x,x}\, u(x) \\
&= \sum_{y \in V} (A^t S\, A)_{x,y}\, u(y) - \sum_{y \in V} (N_1)_{x,y}\, u(y) \\
&= (A^t S\, A - N_1) u(x).
\end{aligned}$$

Hence $\Delta = A^t S\, A - N_1$, as desired.

$\square$

For the rest of this chapter, we maintain the assumption that $\mathcal{H}$ is a k-uniform hypergraph. We rewrite (5.2.2) as

$$\tilde{M}^{-1} A^t S A = k + (k-1)\hat{\Delta}. \tag{5.2.3}$$

From this, we draw several conclusions about the spectrum of $-\hat{\Delta}$.

Proposition 5.7 *For a k-uniform hypergraph $\mathcal{H}$ and the normalized Laplacian $-\hat{\Delta}$,*

(a) $\sigma(-\hat{\Delta}) \subset [0, \frac{k}{k-1}]$.

(b) $\ker\big(\frac{k}{k-1} + \hat{\Delta}\big) = \ker(A)$.

Proof. We begin with the observation that the spectra of $-\hat{\Delta} = -N^{-1}\Delta$ and $\tilde{M}^{-1}(A^t S A)$ lie within $[0, \infty)$. These follow from the simple fact that a product $M^{-1}X$, where M and X are positive operators, is positive with respect to the shifted inner product $\langle u, v\rangle = (u, Mv)_V$.

By the Spectral Mapping Theorem, there is a bijection $f : \sigma(-\hat{\Delta}) \to \sigma(\tilde{M}^{-1}A^t S A)$ given by $f(\lambda) = k - (k-1)\lambda$. Since $f^{-1}(0) = \frac{k}{k-1}$, it follows that $\sigma(-\hat{\Delta}) \subset [0, \frac{k}{k-1}]$. This proves (a).

For (b), note that since S is positive, we can write $A^t S A = (S^{1/2}A)^t(S^{1/2}A)$. Then for $u \in \ell(V)$,

$$\begin{aligned} u \in \ker\big(k + (k-1)\hat{\Delta}\big) \quad &\Leftrightarrow \quad u \in \ker\big(A^t S A\big) \\ &\Leftrightarrow \quad u \in \ker\big(S^{1/2} A\big) \\ &\Leftrightarrow \quad u \in \ker(A). \end{aligned}$$

The last equivalence holds, since S is invertible.

$\square$

Remark 5.8 *For $k \geq 2$, this improves upon L. Malozemov and A. Teplyaev who proved in [26] that 2 is an upper bound of $\sigma(-\hat{\Delta})$. Since we know that 3/2, 4/3 are eigenvalues of the discrete Laplacian on $\mathcal{SG}$ and $\mathcal{VS}_n$ respectively, our estimate of $\frac{k}{k-1}$ is sharp.*

Corollary 5.9 *The multiplicity of $\frac{k}{k-1}$ as an eigenvalue of $-\hat{\Delta}$ satisfies*

$$\mathrm{mult}\big(\frac{k}{k-1}\big) = \mathrm{nullity}(A) \geq \max\{|V| - |E|, 0\}.$$

Proof. The equality follows from (b) of the preceding proposition. For the inequality, note that $\mathrm{rank}(A) \leq |E|$ (since A has $|E|$ rows) and also that $\mathrm{nullity}(A) = |V| - \mathrm{rank}(A)$.

□

5.3 Neumann and Dirichlet

In this final section, we consider applications of the previous results to finite graphs of fractals which can be viewed as hypergraphs, such are $\mathcal{SG}$, $\mathcal{VS}_n$, the pentagasket and the tree-like fractal. Our main question is the multiplicity of $\frac{k}{k-1}$ as a Neumann eigenvalue or as a Dirichlet eigenvalue of the normalized Laplacian $-\hat{\Delta}$.

The Neumann problem is the easy one here: as the Laplacian equation is obeyed by boundary points, as well as any others, Corollary 5.9 directly applies. The equality in Corollary 5.9 hold for $\mathcal{SG}$, $\mathcal{SG}_3$ and $\mathcal{VS}_n$. It would be interesting to find general criteria for ensuring this.

We now turn to the Dirichlet problem. Since the set V_0 of boundary points is generally non-empty, we must redefine the Laplacian as an operator on the space $\ell_0(V) = \{u \in \ell(V) : u|V_0 = 0\}$ to conform with the fractal setting. Fortunately, the results in §5.2 apply to the Dirichlet case as well. The approach we propose below reduces the previous section to nothing but a specialization of the Dirichlet case with $V_0 = \varnothing$.

We begin with a hypergraph $\mathcal{H} = (V, E)$. A subset $V_0 \subset V$ fixed in advance is referred to as the set of **boundary points**. For convenience, denote the set of **interior points** by $V^\circ = V \backslash V_0$. Define the diagonal matrices $\tilde{M}^\circ$, N_1°, $N^\circ \in L(V^\circ)$ in the same way as their relatives in $L(V)$, except for the necessary restriction to V°. Define $A^\circ \in L(V^\circ, E)$, also by restricting A to $V^\circ \times E$.

The **Dirichlet Laplacian** $\Delta^\circ \in L(V^\circ)$ can be written as

$$\Delta^\circ u(x) = \sum_{e \ni x} r_e \sum_{y \in e} [u(y) - u(x)],$$

for $u \in \ell_0(V)$, $x \in V^\circ$. The associated **normalized Dirichlet Laplacian** is $\hat{\Delta}^\circ = (N^\circ)^{-1} \Delta^\circ$. We remark that in the definition of Δ°, the dummy variable y is intended to cover all elements of V, including those of V_0. As a result, the coefficient of $u(x)$ in this expression is the same regardless of whether V_0 is empty.

The positivity of $-\Delta^\circ$ holds, generalizing Lemma 5.5. The "energy form" $\mathcal{E}^\circ(u, v) = (-\Delta^\circ u, v)_{V^\circ}$ is modified to

$$\begin{aligned}\mathcal{E}^\circ(u, v) = &\sum_{e \in E} r_e \sum_{\{x,y\} \subset e \cap V^\circ} [u(x) - u(y)]\,[v(x) - v(y)] \\ &+ \sum_{e \in E} r_e \, |e \cap V_0| \sum_{x \in e} u(x)\, v(x).\end{aligned}$$

However, in general Δ° may be invertible (and hence $\Delta^\circ \mathbf{1} \neq 0$).

Lemma 5.6 extends to the current situation.

Lemma 5.10 *For a general hypergraph $\mathcal{H}$ with boundary points,*

$$(A^\circ)^t \, S \, A^\circ = N_1^\circ + \Delta^\circ. \tag{5.3.1}$$

If $\mathcal{H}$ is k-uniform, then

$$(A^\circ)^t \, S \, A^\circ = \tilde{M}^\circ \big(k + (k-1)\hat{\Delta}^\circ\big). \tag{5.3.2}$$

Proof. The argument is virtually unchanged.

□

Proposition 5.11 *For a k-uniform hypergraph $\mathcal{H}$ with boundary points V_0,*

(a) $\sigma(-\hat{\Delta}^\circ) \subset [0, \frac{k}{k-1}]$.

(b) $\ker\big(\frac{k}{k-1} + \hat{\Delta}^\circ\big) = \ker(A^\circ)$.

Proof. Same as that of Proposition 5.7.

□

Corollary 5.12 *The multiplicity of $\frac{k}{k-1}$ as an eigenvalue of $-\hat{\Delta}^\circ$ satisfies*

$$\mathrm{mul}\Big(\frac{k}{k-1}\Big) = \mathrm{nullity}(A^\circ) \geq \max\{|V^\circ| - |E|, 0\}.$$

Example 1. (Sierpinski Gasket) M. Fukushima and T. Shima [10] and R. Strichartz [33] proved that

$$M_m^{(N)}(3/2) = \frac{3^m + 3}{2} \tag{5.3.3}$$

$$M_m^{(D)}(3/2) = \frac{3^m - 3}{2}, \tag{5.3.4}$$

where $M_m^{(N)}(\lambda)$ and $M_m^{(D)}(\lambda)$ are multiplicities of λ as an Neumann and Dirichlet eigenvalues respectively. Notice that if we let V_m and E'_m be the set of vertices and hyperedges, then

$$|V_m| = \frac{3^{m+1} + 3}{2},$$
$$|E'_m| = 3^m.$$

We get the lower bounds on the multiplicity of 3/2 by Corollary 5.9 and Corollary 5.12. Thus these inequalities are sharp.

Example 2. (n-branch Vicsek Set) For the m-step graphs of the n-branch Vicsek set $\mathcal{VS}_n$, the corollaries imply

$$M_m^{(N)}(4/3) \geq 2(4n-3)^m + 1$$
$$M_m^{(D)}(4/3) \geq 2(4n-3)^m - 3.$$

Indeed, if we let V_m and E'_m be the set of vertices and hyperedges, then

$$|V_m| = 3(4n-3)^m + 1,$$
$$|E'_m| = (4n-3)^m.$$

The inequalities above follow from Corollary 5.9 and Corollary 5.12.

Chapter 6

Ordering the Dirichlet eigenvalues on $\mathcal{VS}_n$

In this chapter, we prove the ordering of the Dirichlet eigenvalues in Theorems 6.3 and 6.5.

6.1 Notation

We shall fix n from now on and always write $N = 2\,n - 2$, due to the frequent occurrence of this symbol. Let $R(\lambda)$ be the spectral decimation function, with its $2\,n-1$ inverses

$$\phi_0, \phi_1, \cdots, \phi_N$$

listed in increasing order. Let

$$\rho = R'(0) = (2\,n-1)\,(4\,n-3)$$

be the Laplacian renormalization constant (recall that it is the product of the energy renormalization constant, which is $K_D(0)^{-1} = 2n-1$, and the measure factor, which is $4n-3$, the number of contraction maps for the standard Laplacian). The list of forbidden eigenvalues is

$$\mathfrak{F} = \{\, \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_{n-1} < \beta_{n-1} < \alpha_n < 4/3 \,\},$$

where $\alpha_i,\, \beta_j$ $(i = 1, \cdots, n,\ j = 1, \cdots, n-1)$ are roots of $T_n(3x-1) - 3T_{n-1}(3x-1) = 0$ and $U_{n-1}(3x-1) - U_{n-2}(3x-1) = 0$ respectively, with T_n and U_n being Chebyshev polynomials of the first kind and the second kind.

Define the set of $2n-1$ symbols

$$\Sigma = \{0, 1, 2, \cdots, N\},$$

and let $W = \Sigma^*$ be the set of finite words on Σ (including the empty word). For any word $w \in W$ of length $j \geq 0$, where $w = w_j \ldots w_1$ with $w_1, \ldots, w_j \in \Sigma$, we set $\phi_w = \phi_{w_j} \circ \ldots \circ \phi_{w_1}$. For $\mu \in [0,\ 4/3]$, we define

$$\lambda_w(\mu) = \lim_{m \to \infty} \rho^m\, \phi_0^{(m-j)} \circ \phi_w(\mu).$$

Then the spectral decimation tells us that the entire set of Dirichlet eigenvalues of $-\Delta$ is a subset of

$$\Lambda^* = \{\rho^k\, \lambda_w(\mu) : k \geq 0,\ w \in W,\ \mu \in \mathfrak{F}\}.$$

Clearly, for any word w we have $\lambda_w = \lambda_{0\cdots 0w}$. Define an equivalence relation $\sim$ on W as follows to reduce this redundancy: $v \sim w$ iff there exists $u \in W$ such that $v = 0 \cdots 0u$ and $w = 0 \cdots 0u$ (the number of leading 0's need not be equal). Note that $v \sim w$ implies $\lambda_v = \lambda_w$.

Let $W_\sim$ denote the set of equivalence classes of W under $\sim$, whose members we shall call **reduced words**. Each member $[u]$ of $W_\sim$ contains a unique word of shortest length. As a rule, we shall generally denote the class $[u]_\sim$ by this shortest word. Perhaps an occasional exception is the class $[0]_\sim$, whose shortest word is the empty word, but we prefer to let 0 denote $[0]_\sim$. We now define the **length** $|w|$ of a class $w \in W_\sim$ to be the length of the shortest word in w. For example, the class $0 = 00 = 000 = \cdots$ has length 0, and the class $0011 = 011 = 11$ has length 2.

From now on, we shall work entirely on $W_\sim$. For $w \in W_\sim$, there is no ambiguity in writing

$$\lambda_w(\mu) = \lim_{m\to\infty} \rho^m \, \phi_0^{(m-|w|)} \circ \phi_w(\mu).$$

6.2 The refinement of Λ^*

In this section, we shall show that all α_i's cannot be an eigenvalue (neither Dirichlet nor Neumann) of the discrete Laplacian $-\widehat{\Delta}_m$ for all $m > 1$. Therefore numbers of the form $\rho^k \, \lambda_w(\alpha_i)$ with $k \geq 1$ cannot be in Λ^* and the set of Dirichlet eigenvalues of $-\Delta$ must be a subset of

$$\Lambda \quad := \quad \{\rho^k \, \lambda_w(\mu) : w \in W, \ \mu \in \mathfrak{F} \text{ if } k = 0 \text{ and } \mu \in \mathfrak{F} \setminus \{\alpha_i\}_{i=1}^n \text{ if } k \geq 1\}.$$

Theorem 6.1 *For each $1 \leq i \leq n$, α_i is not an eigenvalue of the discrete Laplacian $-\widehat{\Delta}_m$ for all $m > 1$. In other words, $\alpha_i \notin \mathfrak{F}_m$ for all $m > 1$. Therefore, $\rho^l \lambda_v(\alpha_i) \notin \Lambda$ for all $l > 0$.*

Proof. We shall use a dimension count argument to prove this theorem. We do the Neumann case first.

Since $\deg R = 2n - 1$, it has $2n - 1$ inverse functions. Hence when we extend one eigenvalue to the next step, we will normally have $2n - 1$ continued eigenvalues for the discrete Laplacian in the next step. But there are two exceptional cases. Recall that $R(\lambda) = \lambda g_n(\lambda) h_n(\lambda)$, with $\deg g_n = \deg h_n = n - 1$. Hence when we extend the eigenvalue 0 from one step to the next step, the resulting values will include roots of g_n, which are forbidden eigenvalues, so we can get only $(2n - 1) - (n - 1) = n$ ways to extend 0. Similarly, since $R(\lambda) - 3/4 = 1/4 f_n(\lambda) l_n(\lambda)$, with $\deg f_n = n$, and roots of f_n are forbidden eigenvalues, there are only $(2n - 1) - n = n - 1$ ways to extend $4/3$.

Notice that the extensions of α_i's and β_i's cannot be another forbidden eigenvalue. For instance, if we assume $\phi_k(\alpha_i) = \alpha_j$, then by applying R on both sides, we would have $\alpha_i = R(\alpha_j) = 4/3$, and this is a contradiction. The proof is similar for the other cases. Moreover, the extensions using different ϕ_w are also distinct because the ranges of $\phi_w([0,\ 4/3])$ are different. So there are always $2n-1$ ways to extend the eigenvalues except for the cases mentioned in the above paragraph. Next, we find all the eigenvalues for the discrete Laplacians at each step.

At step 0, we know by inspection that there are two eigenvalues, 0 and 4/3, of $-\widehat{\Delta}_0$, with multiplicity of 1 and 3. At step 1, as pointed above, there are n continued eigenvalues from 0 and $n-1$ ($3(n-1)$ counted by multiplicity) from 4/3. By Example 2, Section 5.3, the multiplicity of 4/3 at step m is $M_m^{(N)}(4/3) \geq 2(4n-3)^m + 1$. Hence $M_1^{(N)}(4/3) \geq 2(4n-3)+1$, we have found $n + 3(n-1) + 2(4n-3) + 1 = 3(4n-3) + 1$ eigenvalues and these must be all of the Neumann eigenvalues as $\#(V_1) = 3(4n-3)+1$.

We now investigate the general case. We first consider eigenvalues in the 0-series (continued eigenvalues from 0). Let a_m be the number of eigenvalues in the 0-series at step m. Since there are n ways to extend 0 and $2n-1$ ways for all other eigenvalues, a_m should satisfy

$$a_0 = 1,\ a_{m+1} = (2n-1)a_m - (n-1).$$

Therefore,

$$a_m = \frac{(2n-1)^m + 1}{2}.$$

We then look at eigenvalues in the first 4/3-series (eigenvalues extended from 4/3 as an initial eigenvalue of $-\widehat{\Delta}_0$). We have showed that when we extend 4/3 from step 0 to step 1, there are $3(n-1)$ continued eigenvalues, counted by multiplicity. When those $3(n-1)$ extended eigenvalues are extended to eigenvalues of $-\widehat{\Delta}_2$, the total number of those eigenvalues will be $3(n-1)(2n-1)$ since the degree of R is $2n-1$ and we will not encounter any forbidden eigenvalues this time and from then on. In general, the total number of those extended eigenvalues at step m is $3(n-1)(2n-1)^{m-1}$. That is the whole story of the eigenvalues in first 4/3-series.

For eigenvalues in the second 4/3-series (eigenvalues extended from 4/3 as an initial eigenvalue of $-\widehat{\Delta}_1$), note that 4/3 has multiplicity at least $2(4n-3)+1$ for $-\widehat{\Delta}_1$. Hence when we extend 4/3 to eigenvalues of $-\widehat{\Delta}_2$, it will give us at least $(n-1)[2(4n-3)+1]$ eigenvalues and when we extend it to eigenvalues of $-\widehat{\Delta}_3$, the number is at least $(n-1)(2n-1)[2(4n-3)+1]$. In general, when we extend 4/3 to eigenvalues of $-\widehat{\Delta}_m$, it will give us at least $(n-1)(2n-1)^{m-2}[2(4n-3)+1]$ eigenvalues ($n-1$ ways to extend 4/3 to $-\widehat{\Delta}_2$ and $2n-1$ for the rest). That completes the story for the second 4/3-series.

In general, there are at least $(n-1)(2n-1)^{m-1-k}[2(4n-3)^k+1]$ eigenvalues for the k-th 4/3-series, where $0 \leq k \leq m-1$.

Now if we can prove that the number of all those extended eigenvalues plus the multiplicity of 4/3 as an initial eigenvalue of $-\widehat{\Delta}_m$ is equal to the number of vertices of V_m, then we can say that we have found all of the Neumann eigenvalues since $\#(V_m) = 3(4n-3)^m+1$. In other words, it is sufficient to show the following:

$$\begin{aligned}
&3(4n-3)^m+1 \\
&= \frac{(2n-1)^m+1}{2} + 3(n-1)(2n-1)^{m-1} \\
&+(n-1)(2n-1)^{m-2}[2(4n-3)+1] \\
&+(n-1)(2n-1)^{m-3}[2(4n-3)^2+1] + \cdots \\
&+(n-1)[2(4n-3)^{m-1}+1] + 2(4n-3)^m + 1
\end{aligned} \tag{6.2.1}$$

Note that if we break the square bracket, then we will obtain a geometric series from all the constant terms 1:

$$\begin{aligned}
(n-1)(2n-1)^{m-2} + (n-1)(2n-1)^{m-3} + \cdots + (n-1) &= (n-1)\frac{1-(2n-1)^{m-1}}{1-(2n-1)} \\
&= \frac{(2n-1)^{m-1}-1}{2}.
\end{aligned}$$

Together with the first two terms on the right-hand side of (6.2.1), the sum is $(4n-3)(2n-1)^{m-1}$. Therefore, we need to show the following identity

$$(4n-3)(2n-1)^{m-1}+2(n-1)[(2n-1)^{m-2}(4n-3)+\cdots+(4n-3)^{m-1}]+2(4n-3)^m+1 = 3(4n-3)^m+1,$$

or equivalently,

$$\begin{aligned}
(4n-3)(2n-1)^{m-1} &+ 2(n-1)[(2n-1)^{m-2}(4n-3) \\
&+ \cdots + (4n-3)^{m-1}] = (4n-3)^m.
\end{aligned} \tag{6.2.2}$$

We shall prove this identity by induction. It is clear that this is true for $m = 2$. Now suppose it is true for $m > 2$. When m becomes $m + 1$, the left-hand side changes to

$$(4n-3)(2n-1)^m + 2(n-1)[(2n-1)^{m-1}(4n-3) + \cdots + (2n-1)(4n-3)^{m-1} + (4n-3)^m]$$

Taking out the common term $(2n - 1)$ except for the last term and using the induction hypothesis, we have

$$\begin{aligned}
&(2n-1)\{(4n-3)(2n-1)^{m-1} + 2(n-1)[(2n-1)^{m-2}(4n-3) \\
&+ \cdots + (4n-3)^{m-1}]\} + 2(n-1)(4n-3)^m \\
&= (2n-1)(4n-3)^m + 2(n-1)(4n-3)^m \\
&= (4n-3)^{m+1}
\end{aligned}$$

and we are done.

For the Dirichlet case, we first note that we can only start with step 1 as the Dirichlet boundary condition will give us a zero function on V_0. At step 1, recall that $|X + \lambda M| = Cf_n(\lambda)g_n(\lambda)(\lambda - 4/3)^{8n-9}$. Hence the eigenvalues of $-\widehat{\Delta}_1$ are $\alpha_1, \cdots, \alpha_n$ of multiplicity 1, $\beta_1, \cdots, \beta_{n-1}$ of multiplicity 3 and 4/3 of multiplicity $8n - 9$.

We then consider possible initial eigenvalues at step m for $m \geq 2$. Again, from Example 2 in Section 5.3, we know that the multiplicity of 4/3 as a Dirichlet eigenvalue of $-\widehat{\Delta}_k$ is at least $2(4n - 3)^k - 3$. Hence 4/3 is an initial eigenvalue of $-\widehat{\Delta}_k$ for any k. Other initial eigenvalues are $\beta_1, \cdots, \beta_{n-1}$ with multiplicity (at least) 3 since we can construct three eigenfunctions for each β_i $(1 \leq i \leq n - 1)$ as follows. Indeed, for each β_i, we can use one of the eigenfunctions corresponding to $-\widehat{\Delta}_1$, the one which is antisymmetric on the main diagonal (see Case 2 on page 56), as our building block. We can think the values of this eigenfunction on the upper main diagonal as the positive side of a battery and the lower main diagonal as the negative side. Then at any step $m \geq 2$, we can connect a chain of those batteries up to the center square to get values of an eigenfunction on the upper main diagonal. Then for each of the other directions, we can take minus values of the upper main diagonal to get three independent eigenfunctions.

Therefore at step $m \geq 2$, the total number of initial eigenvalues of $-\widehat{\Delta}_m$ is at least

$$2(4n-3)^m - 3 + 3(n-1).$$

Next we investigate the continued eigenvalues at step $m \geq 2$. Clearly for each $1 \leq i \leq n$, any continued eigenvalue in the α_i-series will have multiplicity 1. Hence the total number of all eigenvalues in the α_i-series for all i is $n(2n-1)^{m-1}$. (For each i, there are $2n-1$ ways to extend at each step and there are n such series).

For the first 4/3-series (eigenvalues extended from 4/3 which appear as an initial eigenvalue corresponding to $-\widehat{\Delta}_1$ with multiplicity $2(4n-3)-3 = 8n-9$), there are $n-1$ ways to extend 4/3 at step 2 and $2n-1$ ways to extend in the following $m-2$ steps, so the total number of eigenvalues in this series is $(n-1)(2n-1)^{m-2}[2(4n-3)-3]$. Similarly, the second 4/3-series (eigenvalues extended from 4/3 which appears as an initial eigenvalue corresponding to $-\widehat{\Delta}_2$ with multiplicity $2(4n-3)^2-3)$) has $(n-1)(2n-1)^{m-3}[2(4n-3)^2-3]$ eigenvalues. In general, for the k-th 4/3-series ($1 \leq k \leq m-1$), the total number of eigenvalues in that series is $(n-1)(2n-1)^{m-1-k}[2(4n-3)^k-3]$. The total number of eigenvalues of this type at step m is

$$\begin{aligned} & (n-1)(2n-1)^{m-2}[2(4n-3)-3] + (n-1)(2n-1)^{m-3}[2(4n-3)^2-3] \\ + & \cdots + (n-1)(2n-1)[2(4n-3)^{m-2}-3] + (n-1)[2(4n-3)^{m-1}-3] \end{aligned}$$

Breaking the square bracket and using (6.2.2), it can simplified to

$$\begin{aligned} & (4n-3)^m - (4n-3)(2n-1)^{m-1} \\ - & [3(n-1)(2n-1)^{m-2} + 3(n-1)(2n-1)^{m-3} + \cdots + 3(n-1)] \end{aligned}$$

and this is the (least) total number of the continued eigenvalues in all the 4/3-series.

Notice that each β_i can appear as an initial eigenvalue at any step with multiplicity (at least) 3. We fix i and consider the first β_i-series (eigenvalues extended from β_i corresponding to $-\widehat{\Delta}_1$ with multiplicity 3). There are $2n-1$ ways to extend at each step, so the total number of eigenvalues in that series is $3(2n-1)^{m-1}$. Similarly, the second β_i series (eigenvalues extended from β_i which appears as an initial eigenvalue corresponding to $-\widehat{\Delta}_2$ with multiplicity 3) has $3(2n-1)^{m-2}$ eigenvalues. In general, for the k-th β_i-series ($1 \leq k \leq m-1$) there are $3(2n-1)^{m-k}$ eigenvalues in that series. Summing for i from 1 to $m-1$, the total number of continued eigenvalues corresponding to each β_i is

$$3(2n-1)^{m-1} + 3(2n-1)^{m-2} + \cdots + 3(2n-1).$$

So the number of continued eigenvalues for all β_i-series is

$$(n-1)[3(2n-1)^{m-1}+3(2n-1)^{m-2}+\cdots+3(2n-1)].$$

Combining all results we have had above, we obtain that the total number of eigenvalues of $-\widehat{\Delta}_m$ ($m \geq 2$) is at least

$$\begin{aligned}
&\underbrace{2(4n-3)^m-3+3(n-1)}_{\text{ini e-val of } 4/3 \text{ and } \beta_i}+\underbrace{n(2n-1)^{m-1}}_{\text{ctd e-val from all } \alpha_i \text{ at step 1}}\\
+\;&\underbrace{(4n-3)^m-(4n-3)(2n-1)^{m-1}-[3(n-1)(2n-1)^{m-2}+\cdots+3(n-1)]}_{\text{ctd e-val from } 4/3}\\
+\;&\underbrace{3(n-1)(2n-1)^{m-1}+3(n-1)(2n-1)^{m-2}+\cdots+3(n-1)(2n-1)}_{\text{ctd e-val from all } \beta_i}
\end{aligned}$$

Note that the last $m-2$ terms cancel with the first $m-2$ terms in the square bracket and an easy calculation shows that the above expression is equal to $3(4n-3)^m-3$, which is equal to $\#(V_m \setminus V_0)$. Therefore we have found all Dirichlet eigenvalues for $-\widehat{\Delta}_m$.

Therefore we have proved that $\alpha_1, \cdots, \alpha_n$ can only be initial eigenvalues at step 1 with multiplicity 1.

$\square$

In the proof of the above theorem, we actually found the multiplicities of the Dirichlet eigenvalues of the Laplacian.

Corollary 6.2 *The multiplicities of the Dirichlet eigenvalues are as follows:*
$M_m^{(D)}\big(\lambda_v(\alpha_i)\big)=1$ *for all* $1 \leq i \leq n$,
$M_m^{(D)}\big(\rho^l\lambda_v(\beta_j)\big)=3$ *for all* $l,\ v$ *and* $1 \leq j \leq n-1$,
$M_m^{(D)}\big(\rho^l\lambda_v(4/3)\big)=2(4n-3)^{l+1}-3$ *if* $v_1=1,\ 3,\cdots,$ *or* $2n-3$ *and zero otherwise.*

6.3 Ordering of the eigenvalues

In this section we shall prove a proposition about the ordering on Λ. To do this, we first define several operations on reduced words.

We need the notion of **parity** of (reduced) words. A word $w = w_j \dots w_1 \in W_\sim$ is said to be **odd** (resp. **even**) if w contains an odd (resp. even) number of the odd symbols 1, 3, $\dots, N-1$. For example, 1 and 203 are odd, while 0 and 1032 are even. The **sign** of w is

$$\mathsf{sgn}(w) := (-1)^{w_1+\cdots+w_j} = (-1)^{(\#\text{ of odd digits in } w)}.$$

Clearly, w is even if and only if $sgn(w) = +1$.

We make the simple remark that ϕ_w is a strictly increasing (resp. strictly decreasing) function on the interval $[0, 4/3]$ if w is even (resp. odd) as the spectral decimation function R is a polynomial.

Fix $w = w_j \dots w_1 \in W_\sim$, where we choose $w_j > 0$. The **right shift** of w is the word w' (or $\sigma(w)$) obtained by deleting w_1:

$$w' := w_j \cdots w_2 \in W_\sim.$$

The most important operation on $W_\sim$ is the **successor** operator $w \to w^+$. For $w = w_j \cdots w_1$, consider

$$s = w_1 + \mathsf{sgn}(w') \in \Sigma \cup \{-1, N+1\}.$$

Then we define $w^+ \in W_\sim$ recursively by

$$w^+ := \begin{cases} w' \cdot s & \text{if } s \in \Sigma, \\ (w')^+ \cdot w_1 & \text{if } s \notin \Sigma. \end{cases}$$

We proceed with some examples using $n = 3$ (and $N = 4$).

Example 1. Let $w = 1230$, so that $w' = 123$ and $w_1 = 0$. Then $\mathsf{sgn}(w') = +1$, which tells us to increase w_1 by 1, provided that the resulting digit s still lies in Σ. Since

$s = 1 \in \Sigma$, we end up with $w^+ = 1231$. The next several successors are 1232, 1233 and 1234. We shall determine $(1234)^+$ in Example 3.

Example 2. Let $w = 1224$, so that $w' = 122$ and $w_1 = 4$. This time $\mathsf{sgn}(w') = -1$, so we shall *decrease* w_1 by 1 if we can. The result is $w^+ = 1223$. The next several successors are 1222, 1221 and 1220.

Example 3. What if $s = w_1 + \mathsf{sgn}(w') \notin \Sigma$? For instance, $(1234)^+ = (123)^+\,4 = 1224$, whereas $(1220)^+ = (122)^+\,0 = 1210$.

Example 4. Here we take $n = 2$. It should be clear that iterating $^+$ gives the following list of immediate successors starting from 0:

$$
\begin{aligned}
&0 \to 1 \to 2 \to 12 \to 11 \to 10 \to 20 \to 21 \to 22 \to 122 \to 121 \to 120 \to \\
&110 \to 111 \to 112 \to 102 \to 101 \to 100 \to 200 \to 201 \to 202 \to 212 \to \\
&211 \to 210 \to 220 \to 221 \to 222 \to 1222
\end{aligned}
$$

It is easy to see by induction that $w \mapsto w^+$ changes just one digit w_i, say, of w into $w_i \pm 1$. It follows that

$$\mathsf{sgn}(w^+) = -\mathsf{sgn}(w).$$

Regarding length, we see that $|w| \leq |w^+|$. Also, inequality holds only when $w = N^k = N \cdots N$ for some $k \geq 0$; in that case, $w^+ = 1w$ and $|w^+| = |w| + 1$.

For v, $w \in W_\sim$, we write $v <_+ w$ if $w = (v^+)^{\cdots +}$. For instance, in Example 4 above, we have

$$0 <_+ 1 <_+ 2 <_+ 12 <_+ \cdots <_+ 1222.$$

For two eigenvalues λ, $\mu \in \Lambda$, we write

$$\lambda \prec \mu$$

if $\lambda < \mu$ and if there does not exist any $\nu \in \Lambda$ such that $\lambda < \nu < \mu$ (i.e. λ and μ are consecutive eigenvalues.) The meaning of the converse $\succ$ should be clear.

Recall that if $i \in \Sigma$ and $k \geq 0$, we write $i^k = i \cdots i$ (repeated k times) and if $w \in W_\sim$, w_1 will mean the last digit of w (except if $w = 0$, when we say that $w_1 = 0$.)

The ordering of the eigenvalues is stated in the following theorem.

Theorem 6.3 *(1) For any $w \in W_\sim$ and $\mu, \nu \in [0, 4/3]$,*

$$\lambda_w(\mu) < \lambda_{w^+}(\nu).$$

(2) Let w be even. Then

$$\lambda_w(\alpha_1) \prec \cdots \prec \lambda_w(\alpha_i) \prec \lambda_w(\beta_i) \prec \lambda_w(\alpha_{i+1}) \prec \cdots \prec \lambda_w(\alpha_n) \prec \lambda_w(4/3).$$

If w is odd, then all occurrences of $\prec$ in the above are replaced with $\succ$.

(3) Let w be even. Then for any integers $0 \leq i < k$,

$$\rho^i \, \lambda_{w \cdot N^{k-i}}(4/3) \prec \rho^{i+1} \, \lambda_{w \cdot N^{k-i-1}}(4/3).$$

If w is odd, then $\prec$ is replaced with $\succ$. In particular, for any even w where $w_1 \neq N$, for any $k \neq 0$, we have $\rho^k \, \lambda_w(4/3) \prec \rho^k \, \lambda_{w^+}(4/3)$.

(4) For any odd w, let us write $w = v \cdot 0^l$, where $v_1 \neq 0$ and $l \geq 0$. Define the integer

$$p = \lceil v_1/2 \rceil \in \{1, \cdots, n-1\}.$$

Then

$$\lambda_w(\alpha_1) \prec \rho^{l+1} \, \lambda_{v'}(\beta_p) \prec \lambda_{w^+}(\alpha_1).$$

Before proving the theorem we mention some simple facts.

Lemma 6.4 *For any $w \in W_\sim$ and $v \in W$, λ_w is continuous and strictly monotone on $[0, 4/3]$ and $\lambda_{wv} = \rho^{|v|} \, \lambda_w \circ \phi_v$.*

Proof. In Section 3.2 we proved ϕ_0 is strictly convex and thus by Lemma 2.12, λ_0 is convex, strictly increasing and continuous on $[0, 4/3]$. Then $\lambda_w = \rho^{|w|}\,\lambda_0 \circ \phi_w$ is strictly monotone being a composite of strictly monotone functions.

The fact that $\lambda_{wv} = \rho^{|v|}\,\lambda_w \circ \phi_v$ is obvious.

□

Now we are ready to prove Theorem 6.3.

Proof. (1) We prove the claim by induction on the length of w.

If $|w| = 0$, then $w = 0$ and $w^+ = 1$. Since $\phi_0(\mu) < \phi_1(\nu)$, it follows that

$$\lambda_0(\mu) = \rho\,\lambda_0\big(\phi_0(\mu)\big) < \rho\,\lambda_0\big(\phi_1(\nu)\big) = \lambda_1(\nu).$$

For $|w| > 0$ we shall treat two cases.

Case 1: $s \in \Sigma$. If w' is even, then $s > w_1$ and hence $\phi_{w_1}(\mu) < \phi_s(\nu)$. As $\lambda_{w'}$ is strictly increasing,

$$\lambda_w(\mu) = \rho\,\lambda_{w'}\big(\phi_{w_1}(\mu)\big) < \rho\,\lambda_{w'}\big(\phi_s(\nu)\big) = \lambda_{w^+}(\nu).$$

Likewise, if w' is odd, then $s < w_1$ and $\phi_{w_1}(\mu) > \phi_s(\nu)$. Since $\lambda_{w'}$ is strictly decreasing, the inequality shown above remains unchanged.

Case 2: $s \notin \Sigma$. As $|w'| < |w|$, by induction we have $\lambda_{w'}\big(\phi_{w_1}(\mu)\big) < \lambda_{(w')^+}\big(\phi_{w_1}(\nu)\big)$. Therefore,

$$\lambda_w(\mu) = \rho\,\lambda_{w'}\big(\phi_{w_1}(\mu)\big) < \rho\,\lambda_{(w')^+}\big(\phi_{w_1}(\nu)\big) = \lambda_{w^+}(\nu).$$

(2) This follows trivially, by the monotonicity of λ_w.

(3) By induction, it is enough to prove that

$$\lambda_{wN^k}\big(4/3\big) < \rho\,\lambda_{wN^{k-1}}\big(4/3\big).$$

The proof of this statement is easy:

$$\lambda_{wN^k}\big(4/3\big) = \rho\,\lambda_{wN^{k-1}}\big(\phi_N(4/3)\big) = \rho\,\lambda_{wN^{k-1}}(\alpha_n) < \rho\,\lambda_{wN^{k-1}}\big(4/3\big).$$

The case where w is odd is just as obvious.

(4) Given $p = \lceil v_1/2 \rceil$, as in the hypothesis, we write $t = 2\,p - 1$.

Claim: $v \leq_+ v't \leq_+ v^+$. (In fact, if v' is even, then $v't = v$, while if v' is odd, then $v't = v^+$.)

To see this, note that since v is odd, it follows that v' is even iff v_1 is odd. If v' is even then $v_1 = 2p - 1$ and $t = v_1$, so $v't = v$. Alternatively, if v' is odd, then $t = v_1 - 1$ and $v't = v^+$.

Together with (1) this implies

$$\lambda_v\big(\phi_0^{(k)}(\alpha_1)\big) < \lambda_v(0) \leq \lambda_{v't}(0) \leq \lambda_{v^+}(0) < \lambda_{v^+}\big(\phi_0^{(k)}(\alpha_1)\big).$$

Multiplying the terms above by ρ^k gives

$$\begin{aligned} \rho^k\,\lambda_v\big(\phi_0^{(k)}(\alpha_1)\big) &< \rho^k\,\lambda_{v't}(0) < \rho^k\,\lambda_{v^+}\big(\phi_0^{(k)}(\alpha_1)\big) \\ \Leftrightarrow\ \lambda_{v0^k}(\alpha_1) &< \rho^{k+1}\,\lambda_{v'}\big(\phi_t(0)\big) < \lambda_{v^+0^k}(\alpha_1) \\ \Leftrightarrow\ \lambda_w(\alpha_1) &< \rho^{k+1}\,\lambda_{v'}(\beta_p) < \lambda_{w^+}(\alpha_1) \end{aligned}$$

where the last statement is due to the fact that $\beta_p = \phi_{2\,p-1}(0)$.

$\square$

For each even word w, write

$$w = uN^k \text{ and } w^+ = v0^l,$$

where $k,\, l \geq 0$ and $u_1 \neq N$, $v_1 \neq 0$. Set $p = \lceil v_1/2 \rceil$. Define sets $\Lambda_w^{(r)}$ as follows. Let

$$\begin{aligned} \Lambda_w^{(1)} &= \{\lambda_w(\alpha_i),\ \lambda_w(\beta_j) :\ i = 1, \cdots, n; j = 1, \cdots, n-1\}, \\ \Lambda_w^{(3)} &= \{\lambda_{w^+}(\alpha_i),\ \lambda_{w^+}(\beta_j) :\ i = 1, \cdots, n; j = 1, \cdots, n-1\}. \end{aligned}$$

By Theorem 6.3 (2), the order of the elements in $\Lambda_w^{(1)}$ is

$$\lambda_w(\alpha_i) < \lambda_w(\beta_i) < \lambda_w(\alpha_{i+1}),$$

and in $\Lambda_w^{(3)}$ is

$$\lambda_{w^+}(\alpha_i) > \lambda_{w^+}(\beta_i) > \lambda_{w^+}(\alpha_{i+1}),$$

for $i = 1, \cdots, n-1$. We also define

$$\begin{aligned}
\Lambda_w^{(2)} &= \{\rho^i \lambda_{uN^{k-i}}(4/3),\ \rho^j \lambda_{u^+N^{k-j}}(4/3) :\ i, j = 1, \cdots, k\} \\
\Lambda_w^{(4)} &= \{\rho^{l+1} \lambda_{v'}(\beta_p)\}.
\end{aligned}$$

Since u is even, by Theorem 6.3 (3), the order of elements in $\Lambda_w^{(2)}$ is

$$\begin{aligned}
&\rho^i \lambda_{uN^{k-i}}(4/3) < \rho^j \lambda_{uN^{k-j}}(4/3) \\
&\rho^i \lambda_{u^+N^{k-i}}(4/3) > \rho^j \lambda_{u^+N^{k-j}}(4/3)
\end{aligned}$$

for $0 \le i < j \le k$, and

$$\rho^k \lambda_u(4/3) < \rho^k \lambda_{u^+}(4/3).$$

Finally, we define the "w-subsequence", Λ_w, for even words w as

$$\Lambda_w = \bigcup_{r=1}^{4} \Lambda_w^{(r)}.$$

For two sets S and T, we write $S \precsim T$ if the largest element in S is less than the smallest element in T.

Theorem 6.3 implies that if $i < j$, then $\Lambda_w^{(i)} \precsim \Lambda_w^{(j)}$ and if u and v are even words with $u <_+ v$, then $\Lambda_u^{(i)} \precsim \Lambda_v^{(j)}$ for all i and j.

Theorem 6.5 *The set of Dirichlet eigenvalues of Laplacian on $\mathcal{VS}_n$ is given by*

$$\Lambda = \bigcup_{w\ even} \Lambda_w.$$

Proof. Let $w \to w^-$ denote the inverse of $w \to w^+$.
If $\mu = \lambda_v(\alpha_i)$ or $\mu = \lambda_v(\beta_i)$, then $\mu \in \Lambda_w^{(1)}$ or $\Lambda_w^{(3)}$ for some even word w.
If $\mu = \rho^k \lambda_v\big(4/3\big)$ and v is even, set $w = vN^k$. If v is odd, take $w = (vN^k)^-$. Then $\mu \in \Lambda_w^{(2)}$ in either case.
If $\mu = \rho^{l+1} \lambda_v(\beta_p)$ and v is even, choose $t = 2p - 1$. If v is odd, choose $t = 2p$. Set $u = vt$. In both cases, u is odd and $p = \lceil t/2 \rceil$. Taking $w = (u0^p)^-$, we see that $\mu \in \Lambda_w$.

As these are all the possibilities for elements of Λ, the desired result follows.

□

Remark 6.6 *We remind the reader that the multiplicities of the eigenvalues were calculated in Corollary 6.2.*

6.4 Weyl's Theorem

In this section, we describe the asymptotic behavior of the Dirichlet spectrum. We first consider a bottom part in the spectrum. There are $3(4n-3)^m - 3$ eigenvalues corresponding to $-\Delta_m$ for any m. If we extend those eigenvalues by using ϕ_0, then we will have the smallest eigenvalues for $-\Delta_{m+1}$ because the largest of those continued eigenvalues is $\phi_0(4/3) = \gamma_1 < \beta_1$, the smallest initial eigenvalue. Therefore, if we extend those $3(4n-3)^m - 3$ eigenvalues by using ϕ_0 for each $m' > m$ and pass to the limit, we will obtain the smallest $3(4n-3)^m - 3$ eigenvalues for the Laplacian on $\mathcal{VS}_n$. Note that the largest of those eigenvalues on VS_n is $x_m := \rho^{m-1}\lambda_0(4/3)$.

Define the Dirichlet eigenvalue counting function

$$\pi(x) = \{\lambda : \ \lambda \text{ is a Dirichlet eigenvalue and } \lambda \leq x\}.$$

Recall that in the classical case, when D is a bounded domain in $\mathbb{R}^d$, then $\pi(x)$ has a remarkable property shown by Weyl [21]:

$$\pi(x) = Cx^{d/2} + o(x^{d/2}).$$

In contrast, Shima proved the following theorem in [30].

Theorem 6.7 *Let* $\deg R$ *denote the degree of the spectral decimation function* R*. If* $\deg R < |S| < \rho$*, then*

$$0 < \liminf_{\lambda\to\infty} \frac{\pi(\lambda)}{\lambda^{d_s/2}} < \limsup_{\lambda\to\infty} \frac{\pi(\lambda)}{\lambda^{d_s/2}} < \infty, \tag{6.4.1}$$

where $d_s = 2\frac{\log|S|}{\log\rho}$ *and* ρ *is the Laplacian renormalization constant.*

The number d_s is called the spectral dimension and it is not necessarily the same as the Hausdorff dimension. Indeed, in our problem, $d_s = 2\frac{\log(4n-3)}{\log(2n-1)(4n-3)}$ while the Hausdorff dimension is $\frac{\log(4n-3)}{\log(2n-1)}$.

Since $\pi(x_m) = 3(4n-3)^m - 3$,

$$\frac{3(4n-3)}{(\lambda_0(4/3))^{d_s/2}} = \lim_{m\to\infty} \frac{\pi(x_m)}{x_m^{d_s/2}} \le \limsup_{x\to\infty} \frac{\pi(x)}{x^{d_s/2}}$$

On the other hand, since the multiplicity of x_m is $2(4n-3)^m - 3$,

$$\lim_{x\to x_m^-} \pi(x) = (4n-3)^m.$$

Hence

$$\liminf_{x\to\infty} \frac{\pi(x)}{x^{d_s/2}} \le \frac{(4n-3)}{(\lambda_0(4/3))^{d_s/2}}$$

and therefore $\lim_{x\to\infty} \frac{\pi(x)}{x^{d_s/2}}$ does not exist. Furthermore, given any x, choose m such that $x \in [x_{m-1},\ x_m]$. As $\frac{x_m}{x_{m-1}} = \rho$,

$$\frac{\pi(x_{m-1})}{(4n-3)x_{m-1}^{d_s/2}} \le \frac{\pi(x)}{x^{d_s/2}} \le \frac{(4n-3)\pi(x_m)}{x_m^{d_s/2}}.$$

Letting $x \to \infty$, we obtain an alternative proof of the inequalities (6.4.1) for $\mathcal{VS}_n$.

Appendix A

Chebyshev polynomials

In this section, we will give definitions and some basic properties of the special functions, the Chebyshev polynomials. A good reference is *Chebyshev polynomials* by Theodore J. Rivlin [28].

Definition A.1 *The Chebyshev polynomials of the first kind, T_n, are defined by the trigonometric identity:*

$$T_n(x) = \cos(n\theta)$$

for n a nonnegative integer, $x = \cos\theta$, and $0 \le \theta \le \pi$.

As

$$\cos n\theta = \sum_{k=0}^{\lceil n/2 \rceil} \left[(-1)^k \sum_{j=k}^{\lceil n/2 \rceil} \binom{n}{2j} \binom{j}{k} \right] \cos^{n-2k}\theta, \tag{A.0.1}$$

$T_n(x)$, defined above by its value in $[-1,\ 1]$, is a polynomial of degree n and hence, by analytic continuation, it is defined for all x (indeed for all complex numbers x). Let us list the first few of them:

$$
\begin{aligned}
&T_0(x) = 1,\\
&T_1(x) = x,\\
&T_2(x) = 2x^2 - 1,\\
&T_3(x) = 4x^3 - 3x,\\
&T_4(x) = 8x^4 - 8x^2 + 1,\\
&T_5(x) = 16x^5 - 20x^3 + 5x,
\end{aligned}
$$

We can also develop a closed-form generating formula for Chebyshev polynomials of the first kind. Since

$$
\begin{aligned}
\cos n\theta &= \frac{e^{in\theta} + e^{-in\theta}}{2}\\
&= \frac{(\cos\theta + \sqrt{\cos^2\theta - 1})^n + (\cos\theta + \sqrt{\cos^2\theta - 1})^{-n}}{2},
\end{aligned}
$$

if we replace $\cos\theta$ with x, we can alternatively write:

$$T_n(x) = \frac{(x + \sqrt{x^2 - 1})^n + (x + \sqrt{x^2 - 1})^{-n}}{2}.$$

For any $a > 1$, the map $y \to \frac{a^y + a^{-y}}{2}$ is an increasing function, thus $T_n(x)$ is increasing in n.

Chebyshev polynomials of the second kind are defined by

$$U_n(x) = \frac{1}{n} T_n'(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad (x = \cos\theta)$$

and are structurally quite similar to the Dirichlet kernel. The first few Chebyshev polynomials of the second kind are

$$
\begin{aligned}
&U_0(x) = 1\\
&U_1(x) = 2x\\
&U_2(x) = 4x^2 - 1\\
&U_3(x) = 8x^3 - 4x\\
&U_4(x) = 16x^4 - 12x^2 + 1\\
&U_5(x) = 32x^5 - 32x^3 + 6x
\end{aligned}
$$

Chebyshev polynomials of both kinds satisfy the same recurrence relation

$$P_{i+1}(x) = 2xP_i(x) - P_{i-1}(x), \tag{A.0.2}$$

but with different initial conditions $T_0(x) = 1,\ T_1(x) = x$ and $U_0(x) = 1,\ U_1(x) = 2x$. They are closely related by the following equations, which will be used often in Chapter 4.

$$T_n'(x) = nU_{n-1}(x) \tag{A.0.3}$$

$$T_n(x) = \frac{1}{2}(U_n(x) - U_{n-2}(x)) \tag{A.0.4}$$

$$T_{n+1}(x) = xT_n(x) - (1 - x^2)U_{n-1}(x) \tag{A.0.5}$$

$$T_n(x) = U_n(x) - xU_{n-1}(x) \tag{A.0.6}$$

$$T_n(x) = xU_{n-1}(x) - U_{n-2}(x) \tag{A.0.7}$$

Moreover, the Chebyshev polynomials are solutions to the Pell equation:

$$T_n^2 - (x^2 - 1)U_{n-1}^2 = 1.$$

Hence for fixed $x > 1$, the Chebyshev polynomials of the second kind are also increasing in n.

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