Cantor sets and numbers with restricted partial quotients

by

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Abstract

For $j = 1, \ldots, k$ let C_j be a Cantor set constructed from the interval I_j, and let $\epsilon_j = \pm 1$. We derive conditions under which

$$\epsilon_1 C_1 + \cdots + \epsilon_k C_k = \epsilon_1 I_1 + \cdots + \epsilon_k I_k \quad \text{and} \quad C_1^{\epsilon_1} \cdots C_k^{\epsilon_k} = I_1^{\epsilon_1} \cdots I_k^{\epsilon_k}.$$

When these conditions do not hold, we derive a lower bound for the Hausdorff dimension of the above sum and product. We use these results to make corresponding statements about the sum and product of sets $F(B_j)$, where B_j is a set of positive integers and $F(B_j)$ is the set of real numbers x such that all partial quotients of x, except possibly the first, are members of B_j. We also examine cases where our conditions do not hold, but in which it is still the case that $C_1 + C_2$ contains an interval.

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Chapter 1

Introduction

Let x be a real number. We say that x is *badly approximable* if there exists a positive integer n such that for every rational number p/q,

$$\left| x - \frac{p}{q} \right| > \frac{1}{nq^2}.$$

It can be shown that this set is of Lebesgue measure zero; however, it is still quite large. In 1947 Marshall Hall [8] showed that every real number can be expressed as the sum of two badly approximable numbers. In particular, for a positive integer m let $F(m)$ denote the set of numbers

$$F(m) = \{[t, a_1, a_2, \ldots]\, ;\, t \in \mathbb{Z}, 1 \le a_i \le m \text{ for } i \ge 1\}$$

where by $[a_0, a_1, a_2, \ldots]$ we denote the *infinite continued fraction*

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots}}}$$

with *partial quotients* a_0, a_1, a_2 and so on. It can be shown that for every $x \in F(4)$ and every $p/q \in \mathbb{Q}$,

$$\left| x - \frac{p}{q} \right| > \frac{1}{6q^2}$$

so that $F(4)$ is a set of badly approximable numbers. Hall proved that

$$F(4) + F(4) = \mathbb{R}$$

where we define the sum of two sets of real numbers A and B by

$$A + B = \{a + b; a \in A \text{ and } b \in B\}.$$

In 1973 Bohuslav Diviš [6] and Tom Cusick [3] showed independantly that one could not do much better than Hall's result, namely that

$$F(3) + F(3) \neq \mathbb{R}.$$

In 1975 James Hlavka [10] generalized Hall's results to the case of different sets $F(m)$ and $F(n)$. He proved that

$$F(m) + F(n) = \mathbb{R} \tag{1.1}$$

holds for (m, n) equal to (2,7) or (3,4), but does not hold for (m, n) equal to (2,4). Now, if (1.1) holds, then the same equation holds with m and n replaced by m' and n' respectively, where $m' \geq m$ and $n' \geq n$. Further, if either m or n is equal to one then trivially (1.1) does not hold, since $F(1)$ consists of the points $\{[t, 1, 1, 1 \ldots]; t \in \mathbb{Z}\}$. Hence the only cases of interest left are $(m, n) = (2, 5)$ and $(m, n) = (2, 6)$. Hlavka conjectured that in these two cases (1.1) would not hold. We will show that in both cases Hlavka's conjecture is false.

We can also examine the difference of two sets $F(m)$ and $F(n)$. If A is a set of real numbers we define $-A$ by

$$-A = \{-a; a \in A\}$$

and denote $A + (-B)$ by $A - B$. We have the following result.

Theorem 1.0.1 *Let m and n be integers. The equations*

$$F(m) + F(n) = \mathbb{R} \quad \text{and} \quad F(m) - F(n) = \mathbb{R}$$

hold if (m, n) equals $(2, 5)$ or $(3, 4)$. Neither of the above equations hold if (m, n) equals $(2, 4)$. Additionally,

$$F(3) + F(3) \neq \mathbb{R} \quad \text{and} \quad F(3) - F(3) = \mathbb{R}.$$

In 1971 Thomas Cusick [2] examined the complementary case of sums of real numbers whose continued fraction expansion contains only large partial quotients. For each positive integer l we define the set $G(l)$ by

$$G(l) = \{[t, a_1, a_2, \dots]\,;\, t \in \mathbb{Z} \text{ and } a_i \geq l \text{ for } i \geq 1\}$$
$$\bigcup \{[t, a_1, a_2, \dots, a_k]\,;\, t, k \in \mathbb{Z},\, k \geq 0 \text{ and } a_i \geq l \text{ for } 1 \leq i \leq k\}.$$

Cusick proved that

$$G(2) + G(2) = \mathbb{R}.$$

The above results are special cases of the following general problem. Let B be a set of positive integers. If B is a finite set, we let $F(B)$ denote the set of real numbers which have an infinite continued fraction expansion with all partial quotients, except possibly the first, members of B. For B infinite, we define $F(B)$ similarly, but also allow numbers with finite continued fraction expansions. Thus if we define

$$L_m = \{1, 2, \dots, m\} \quad \text{and} \quad U_l = \{l, l + 1, \dots\}$$

for positive integers m and l, then $F(m) = F(L_m)$ and $G(l) = F(U_l)$. For sets of positive integers B_1 and B_2, we wish to know when

$$F(B_1) + F(B_2) = \mathbb{R} \tag{1.2}$$

and when

$$F(B_1) - F(B_2) = \mathbb{R}. \tag{1.3}$$

We shall derive conditions on the sets B_1 and B_2 such that (1.2) and (1.3) follow. Let $B = \{b_1, b_2, \ldots\}$ be a non-empty set of positive integers with $b_1 < b_2 < \ldots$. If $|B| = 1$ then we put $\tau(B) = 0$. Otherwise, we set

$$l = l(B) = \min B \quad \text{and} \quad \Delta_i = \Delta_i(B) = b_{i+1} - b_i$$

for $1 \leq i < |B|$. If B is a finite set with $|B| > 1$ then we put

$$m = m(B) = \max B, \quad \delta = \delta(B) = \frac{-lm + \sqrt{l^2m^2 + 4lm}}{2},$$

and

$$\tau(B) = \min_{1 \leq i < |B|} \min \left\{ \frac{\delta(m-l)}{\Delta_i lm - \delta(m-l)} \cdot \frac{b_{i+1}lm + m + \delta l}{b_i lm + m + \delta l}, \right. \left. \frac{(m - b_{i+1})lm + \delta(m-l)}{\Delta_i lm - \delta(m-l)} \cdot \frac{b_i l + \delta}{lm + \delta} \right\}.$$

If B is an infinite set then we put

$$\tau(B) = \inf_{1 \leq i} \min \left\{ \frac{1}{\Delta_i l - 1} \cdot \frac{b_{i+1} l + 1}{b_i l + 1}, \frac{b_i l + 1}{\Delta_i l - 1} \right\}. \tag{1.4}$$

If we let $\Delta = \max_i \Delta_i$ then if $1 < |B| < \infty$ we have

$$\tau(B) \geq \frac{\delta(m-l)}{\Delta lm - \delta(m-l)} \cdot \frac{lm + \delta - l\Delta}{lm + \delta}$$

and if B is infinite then

$$\tau(B) \geq \frac{1}{\Delta l - 1}.$$

It is a simple matter to calculate $\tau(B)$ for various sets B (see Table 1.1).

We denote the Hausdorff dimension of a set S by $\dim_H S$. To help determine whether (1.2) and (1.3) hold we will prove the following theorem.

Table 1.1: Values of $\tau(B)$ for certain B

B	$\tau(B)$
L_2	$(-1+\sqrt{3})/2 = 0.366\ldots$
L_3	$(-6+4\sqrt{21})/15 = 0.822\ldots$
L_4	$(-3+15\sqrt{2})/14 = 1.300\ldots$
L_5	$4\sqrt{5}/5 = 1.788\ldots$
L_6	$(15+35\sqrt{15})/66 = 2.281\ldots$
L_7	$(42+24\sqrt{77})/91 = 2.775\ldots$
U_l	$1/(l-1)$

Theorem 1.0.2 *Let B_1 and B_2 be sets of positive integers, and take $\epsilon_1, \epsilon_2 \in \{1, -1\}$.*

1. *If $\tau(B_1)\tau(B_2) \geq 1$ then $\epsilon_1 F(B_1) + \epsilon_2 F(B_2) = \mathbf{R}$.*

2. *If $\tau(B_1)\tau(B_2) < 1$ then*

$$\dim_H(\epsilon_1 F(B_1) + \epsilon_2 F(B_2)) \geq \frac{\log 2}{\log\left(2 + \frac{1-\tau(B_1)\tau(B_2)}{\tau(B_1)+\tau(B_2)+2\tau(B_1)\tau(B_2)}\right)}.$$

For example, we have

$$\dim_H(F(2) + F(2)) \geq 0.658\ldots .$$

The positive results in Theorem 1.0.1 follow in part from Theorem 1.0.2. In addition we mention the following corollaries.

Corollary 1.0.3 *If l and m are positive integers with $m \geq 2l$, then*

$$F(m) + G(l) = \mathbf{R}.$$

Corollary 1.0.4 *Let B_o denote the set of positive odd integers. Then*

$$F(B_o) + F(B_o) = \mathbb{R}.$$

Furthermore, if B is any finite set of odd positive integers then we have

$$F(B) + F(B) \neq \mathbb{R}.$$

Note that if $B(m)$ is the set of positive odd integers less than m, then $\tau(B(m))$ approaches one as m tends to infinity. Thus part 1 of Theorem 1.0.2 is tight in the sense that we cannot replace 1 by any smaller number.

Diviš [6], Cusick [3], and Hlavka [10] also developed techniques that allowed them to examine the sum of more than two $F(m)$'s. Diviš and Cusick showed independantly that

$$F(3) + F(3) + F(3) = \mathbb{R} \quad \text{and} \quad F(2) + F(2) + F(2) + F(2) = \mathbb{R}$$

while

$$F(2) + F(2) + F(2) \neq \mathbb{R}.$$

Hlavka proved that

$$F(l) + F(m) + F(n) = \mathbb{R}$$

holds if (l, m, n) equals (2,2,4) or (2,3,3) but does not hold for (l, m, n) equal to $(2, 2, 3)$. Together with the work on sums of two $F(m)$'s, these results allow us to determine those finite sets of positive integers $\{m_1, \ldots, m_k\}$ for which

$$\sum_{j=1}^{k} F(m_j) = \mathbb{R}.$$

In the case of sums of integers with large partial quotients, Thomas Cusick and Robert Lee [5] showed in 1971 that

$$lG(l) = \mathbb{R} \tag{1.5}$$

for every positive integer l, where $lG(l)$ denotes the sum of l copies of $G(l)$. We shall extend this to the case where the summands are unequal.

Theorem 1.0.5 *If k and $l_1, l_2, \ldots, l_k$ are positive integers with*

$$\sum_{j=1}^{k} \frac{1}{l_j} \geq 1$$

then

$$G(l_1) + \cdots + G(l_k) = \mathbb{R}.$$

Note that if l is a positive integer and we set $k = l$ and $l_1 = l_2 = \cdots = l_k = l$ then we recover (1.5).

For a non-empty set of positive integers B we define $\gamma(B)$ by

$$\gamma(B) = \frac{\tau(B)}{\tau(B) + 1}.$$

Theorem 1.0.5 is a consequence of the following general theorem.

Theorem 1.0.6 *Let k be a positive integer and $B_1, B_2, \ldots, B_k$ be non-empty sets of positive integers. Let $\epsilon_j \in \{1, -1\}$ for $j = 1, \ldots, k$. If*

$$\sum_{j=1}^{k} \gamma(B_j) \geq 1$$

then

$$\epsilon_1 F(B_1) + \cdots + \epsilon_k F(B_k) = \mathbb{R}. \tag{1.6}$$

Otherwise

$$\dim_H(\epsilon_1 F(B_1) + \cdots + \epsilon_k F(B_k)) \geq \frac{\log 2}{\log\left(1 + \frac{1}{\gamma(B_1) + \cdots + \gamma(B_k)}\right)}. \tag{1.7}$$

Hall and Cusick also examined products of numbers with bounded partial quotients. For sets A and B of real numbers, we define the product of A and B by

$$AB = A \cdot B = \{ab;\ a \in A \text{ and } b \in B\}$$

and A^{-1} by

$$A^{-1} = \{1/a;\ a \in A \text{ and } a \neq 0\}.$$

We also denote by A/B the set $A \cdot (B^{-1})$. Hall [8] proved that

$$[1, \infty) \subseteq F(4) \cdot F(4) \tag{1.8}$$

while Cusick [2] established that

$$[1, \infty) \subseteq G(2) \cdot G(2). \tag{1.9}$$

We shall derive the following multiplicative analogue of Theorem 1.0.6.

Theorem 1.0.7 *Let k be a positive integer. For $j = 1, \ldots, k$ let B_j be a set of positive integers and let $\epsilon_j \in \{1, -1\}$. Set*

$$S_\gamma = \gamma(B_1) + \cdots + \gamma(B_k),$$
$$S_\epsilon = \epsilon_1 + \cdots + \epsilon_k$$

and

$$F = F(B_1)^{\epsilon_1} F(B_2)^{\epsilon_2} \cdots F(B_k)^{\epsilon_k}.$$

1. *If $S_\gamma > 1$ and $S_\epsilon = k$ then there exists a positive real number c_1 such that*

$$F \supseteq (-\infty, -c_1] \cup [c_1, \infty).$$

2. *If $S_\gamma > 1$ and $|S_\epsilon| < k$ then*

$$F \supseteq (-\infty, 0) \cup (0, \infty).$$

3. *If $S_\gamma > 1$ and there exists r such that $|B_r| = \infty$ and $\epsilon_r = 1$ then*

$$F = \mathbb{R}.$$

4. *If $S_\gamma = 1$ and $S_\epsilon = k$ then there exists a positive real number c_2 such that*

$$F \supseteq [c_2, \infty).$$

5. *If $S_\gamma = 1$ and $|S_\epsilon| < k$ then $(0, \infty) \subseteq F$.*

6. *If $S_\gamma = 1$ and there exists r such that $|B_r| = \infty$, $\epsilon_r = 1$ and $\Delta_i(B_r)$ is constant, then $F = \mathbb{R}$.*

7. *If $S_\gamma < 1$ then*

$$\dim_H F \geq \frac{\log 2}{\log\left(1 + \frac{1}{S_\gamma}\right)}.$$

For particular choices of B_j we can calculate c_1 and c_2 in the above theorem explicitly. If we denote by $\langle a_1, a_2, \ldots \rangle$ the continued fraction $[0, a_1, a_2, \ldots]$ and let $[a_0, \ldots, a_k, \overline{b_1, b_2}] = [a_0, \ldots, a_k, b_1, b_2, b_1, b_2, \ldots]$, then we have the following improvements to (1.8).

Theorem 1.0.8 *Let m and n be integers with $n \geq m$, $m \geq 3$ and $n \geq 4$, or $(m, n) = (2, 7)$. Then*

$$F(m) \cdot F(n) = \left(-\infty, -(1 - \langle \overline{1, m} \rangle)\langle \overline{n, 1} \rangle\right] \bigcup \left[(1 - \langle \overline{1, m} \rangle)(1 - \langle \overline{1, n} \rangle), \infty\right).$$

Theorem 1.0.9 *Let $k \geq 3$ and for $1 \leq t \leq k$ let $m_t \in \mathbb{Z}$ with $m_t \geq 2$. Assume that $m_1 \geq m_t$ for $1 \leq t \leq k$ and that*

$$\sum_{t=1}^{k} m_t \geq 8.$$

Put

$$\alpha = \langle\overline{m_1, 1}\rangle \prod_{t=2}^{k}(1 - \langle\overline{1, m_t}\rangle) \quad \text{and} \quad \beta = \prod_{t=1}^{k}(1 - \langle\overline{1, m_t}\rangle).$$

Then

$$\prod_{t=1}^{k} F(m_t) = \begin{cases} (-\infty, -\alpha] \cup [\beta, \infty), & \text{if } k \text{ is even;} \\ (-\infty, -\beta] \cup [\alpha, \infty), & \text{if } k \text{ is odd.} \end{cases}$$

We may also strengthen and generalize (1.9).

Theorem 1.0.10 *If k and $l_1, l_2, \ldots, l_k$ are positive integers with*

$$\sum_{j=1}^{k} \frac{1}{l_j} \geq 1$$

then

$$G(l_1) \cdots G(l_k) = \mathbb{R}.$$

If $\tau(L_m)\tau(L_n) < 1$, then it still might be the case that $F(m) + F(n)$ and $F(m)F(n)$ contain intervals; in fact in 1977 Hanno Schecker [14] proved that $F(3)+F(3)$ contains an interval. Inspired by his approach we will show that

$$F(5) \pm F(2) = \mathbb{R} \quad \text{and} \quad F(3) - F(3) = \mathbb{R}$$

which will partially prove Theorem 1.0.1. Additionally, we shall provide the following characterization of $F(3) + F(3)$. If $w = d_1 d_2 \cdots d_t$ is a finite word with $d_i \in \mathbb{Z}^+$ for $1 \leq i \leq t$ then we will abuse notation by putting

$$[a_0, \ldots, a_k, w, a_{k+1}, \ldots] = [a_0, \ldots, a_k, d_1, d_2, \ldots, d_t, a_{k+1}, \ldots]$$

and

$$\langle a_1, \ldots, a_k, w, a_{k+1}, \ldots \rangle = \langle a_1, \ldots, a_k, d_1, d_2, \ldots, d_t, a_{k+1}, \ldots \rangle.$$

Theorem 1.0.11 *Put*

$$R = \bigcup_{v \in \mathcal{M}} [[\langle v, \overline{3,1} \rangle + \langle v, 2, \overline{1,3} \rangle, \langle v, \overline{1,3} \rangle + \langle v, 1, 2, \overline{1,3} \rangle]]$$

and

$$S = \{2\langle d_1, d_2, \ldots \rangle \,; d_i \in \{1,3\} \text{ and if } d_i = 1 \text{ then } d_{i+1} = 3, \text{ for } i \geq 1\}$$

where $\mathcal{M}$ is the set of finite words $v = d_1 \cdots d_t$ such that $t \geq 0$, $d_i \in \{1,3\}$ for $1 \leq i \leq t$, and if $d_k = 1$ for some k then $t \geq k+1$ and $d_{k+1} = 3$. Then

$$F(3) + F(3) = (\mathbb{Z} + R) \cup (\mathbb{Z} + S).$$

Furthermore, $\mathbb{Z} + S$ contains an uncountable set of points not contained in $\mathbb{Z} + R$.

It should be noted that Theorem 1.0.11 contradicts work of Gregori Freiman. In [7] Freiman claims to have proven that $C(L_3) + C(L_3) = R$ where R is defined as in Theorem 1.0.11. However, in his proof that $x + y \in R$ for every $x, y \in C(L_3)$ he assumes that $x \neq y$ and hence misses the set S described in Theorem 1.0.11.

Finally, we shall examine the products $F(5) \cdot F(2)$ and $F(3) \cdot F(3)$, achieving the following result.

Theorem 1.0.12 *There exists a real number c_1 such that*

$$(-\infty, c_1] \cup [c_1, \infty) \subseteq F(5) \cdot F(2) \quad \text{and} \quad (-\infty, c_1] \cup [c_1, \infty) \subseteq F(3) \cdot F(3).$$

As in the works of Hall, Cusick and Lee, Diviš and Hlavka, our results hinge on the study of certain Cantor sets. For any set of positive integers B we define the set $C(B)$ by

$$C(B) = \{\langle a_1, a_2, \ldots \rangle \; ; \; a_i \in B \text{ for every } i\}$$

where numbers with a finite continued fraction expansion are included in $C(B)$ if and only if B is an infinite set. We shall show that the sets $C(B)$ may be viewed as Cantor sets. We then derive results on sums and products of Cantor sets to prove our results.

The majority of the results contained in Chapters 3 through 5 and Sections 6.1 and 6.2 will appear shortly [1]. For a good general survey of work related to numbers with bounded partial quotients, see [15].

Chapter 2

Background Information

2.1 Continued Fractions

Let $a_1, \dots, a_n$ be a finite sequence of positive integers, and let a_0 be an integer. We denote by $[a_0, a_1, \dots, a_n]$ the number

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\cdots + \cfrac{1}{a_n}}}.$$

By an abuse of notation we also call $[a_0, a_1, \dots, a_n]$ a *continued fraction representation* of α. The integers $a_0, \dots, a_n$ are called *partial quotients* of α. If $a_0 \in \mathbb{Z}$ and $a_1, a_2, \dots$ is an infinite sequence of positive integers then we denote by $[a_0, a_1, \dots]$ the limit

$$\lim_{n \to \infty} [a_0, a_1, \dots, a_n]$$

provided the limit exists. The following lemma guarantees that it does.

Lemma 2.1.1 *Let $a_0 \in \mathbb{Z}$ and let $a_1, a_2, \ldots$ be an infinite sequence of positive integers. Then*

$$\lim_{n\to\infty} [a_0, a_1, \ldots, a_n]$$

exists. Furthermore, if α is a positive real number then there exists a sequence $a_0, a_1, \ldots$ such that

$$\alpha = [a_0, a_1, \ldots].$$

Proof. See [9, Theorems 162, 165 and 170] .

□

Let $a_0 \in \mathbb{Z}$ and $a_1, a_2, \ldots$ be a sequence of positive integers, and put $\alpha = [a_0, a_1, \ldots]$. For $n \in \mathbb{Z}$ such that a_n exists we define p_n and q_n to be the coprime integers with $q_n > 0$ and

$$\frac{p_n}{q_n} = [a_0, a_1, \ldots, a_n].$$

We call p_n/q_n then n^{th} *convergent* to α.

We also define p_n and q_n for $n = -2$ or $n = -1$ by

$$p_{-2} = q_{-1} = 0 \quad \text{and} \quad p_{-1} = q_{-2} = 1.$$

By elementary properties of continued fractions we have the following lemma.

Lemma 2.1.2 *Let p_n/q_n be the n^{th} convergent to $[a_0, a_1, \ldots]$. Then*

$$p_n = a_n p_{n-1} + p_{n-2} \quad \text{and} \quad q_n = a_n q_{n-1} + q_{n-2} \tag{2.1}$$

for $n \geq 0$.

Proof. See [9], Theorem 149.

□

We can extend the definition of $[a_0, a_1, \ldots]$ to allow any real numbers not less than one to be partial quotients. Further, we can take a_0 to be any real number. The previous lemmas hold with this extended definition, and additionally we have the following result.

Lemma 2.1.3 *For a fixed $r \geq 0$ and $1 \leq i \leq 4$ assume that $G_i = [a_0, a_1, \ldots, a_r, g_i]$ for some real $g_i > 0$. For $0 \leq n \leq r$ let p_n/q_n be the n^{th} convergent to $[a_0, \ldots, a_r]$, and put $Q = q_{r-1}/q_r$. Then*

1.

$$|G_1 - G_2| = \frac{|g_1 - g_2|}{q_r^2(g_1 + Q)(g_2 + Q)}$$

and

2.

$$\left|\frac{G_1 - G_2}{G_3 - G_4}\right| = \left|\frac{g_1 - g_2}{g_3 - g_4}\right| \cdot \frac{(g_3 + Q)(g_4 + Q)}{(g_1 + Q)(g_2 + Q)}.$$

Proof. See [10], Lemmas 4 and 5.

□

2.2 Hausdorff Dimension

For any subset U of $\mathbb{R}$ we define the *diameter* of U, denoted $d(U)$, to be

$$d(U) = \sup_{x,y \in U} \{|x - y|\}.$$

Let $E \subseteq \mathbb{R}$ and take ϵ to be any positive real number. We define a *countable ϵ-cover* of E to be any countable collection $\{U_i\}$ of subsets of $\mathbb{R}$ such that

$$E \subseteq \bigcup_{i=1}^{\infty} U_i \quad \text{and} \quad d(U_i) \leq \epsilon$$

for $i \geq 1$. We also define, for positive real numbers s,

$$\mathcal{H}_\epsilon^s(E) = \inf \sum_{i=1}^{\infty} d(U_i)^s$$

where the infimum is taken over all countable ϵ-covers $\{U_i\}$ of E. For $s > 0$ we define the *Hausdorff s-dimensional outer measure* of E, denoted $\mathcal{H}^s(E)$, to be

$$\mathcal{H}^s(E) = \lim_{\epsilon \to \infty} \mathcal{H}_\epsilon^s(E) = \sup_{\epsilon > 0} \mathcal{H}_\epsilon^s(E).$$

The *Hausdorff dimension* of E, denoted by $\dim_H(E)$, is defined to be the unique number s such that

$$\mathcal{H}^t(E) = \begin{cases} \infty & \text{for } t < s, \\ 0 & \text{for } t > s. \end{cases}$$

If $E \subseteq \mathbb{R}$ and $f : E \to \mathbb{R}$ is a function then we define the *image* of f to be

$$f(E) = \{f(x);\ x \in E\}.$$

We have the following results.

Lemma 2.2.1 *Let E be a set of real numbers with $f : E \to \mathbb{R}$ such that for some positive constant c,*

$$|f(x) - f(y)| \leq c|x - y|$$

for all $x, y \in E$. Then

$$\dim_H(f(E)) \leq \dim_H(E).$$

Proof. See [11, p. 44] .

□

Lemma 2.2.2 *Let $S \subseteq [a, b]$ be a set of real numbers with $a > 0$. Then*

$$\dim_H(S) = \dim_H(\log S).$$

Proof. For every $x, y \in S$,

$$\frac{1}{b}|x - y| \leq |\log x - \log y| \leq \frac{1}{a}|x - y|$$

and the lemma follows from Lemma 2.2.1.

□

2.3 Cantor Sets

Let T be a connected directed graph. We say that T is a *tree* if every vertex V of T has at most one edge terminating at V, and one vertex V_R has no edges terminating at V_R. We call V_R the *root* of T. If there is an edge connecting V_1 to V_2, then we say that V_2 is a *subvertex* of V_1. A vertex with no subvertices is called a *leaf.* A tree where each vertex has at most t subvertices is called a tree of *valence* t. A tree of valence 2 is also called a *binary* tree.

We define a *generalized Cantor set* (henceforth known as a *Cantor set*) to be any set C of real numbers of the form

$$C = I \setminus \bigcup_{i \geq 1} O_i$$

where I is a finite closed interval and $\{O_i \,;\, i \geq 1\}$ is a countable (finite or infinite) collection of disjoint open intervals contained in I. We may inductively define a binary tree $\mathcal{D}$ that will represent C. Let the root of the tree be the interval I. We say that $\{I\}$ is the *zeroth level* of the tree. Now say that we have defined our tree up to the n^{th} level. We define the $(n+1)^{th}$ level of the tree as follows. Let I^w be an n^{th} level vertex of our tree. Assume first that

$$I^w \bigcap \left(\bigcup_{i \geq 1} O_i \right) \neq \emptyset.$$

Let O_{I^w} be the interval in the set $\{O_i \,;\, i \geq 1\}$ of least index which is contained in I^w, and let I^{w0} and I^{w1} be closed intervals with

$$I^w = I^{w0} \cup O_{I^w} \cup I^{w1}$$

where the union is disjoint. We let I^{w0} and I^{w1} be subvertices of I^w in $\mathcal{D}$. If

$$I^w \bigcap \left(\bigcup_{i \geq 1} O_i \right) = \emptyset$$

then we set $I^{w0} = I^w$ and let I^{w0} be a subvertex of I^w in $\mathcal{D}$. We repeat this process for every vertex I^w in the n^{th} level of $\mathcal{D}$. The $(n+1)^{th}$ level of the tree is the set of vertices I^v in $\mathcal{D}$ with $|v| = n+1$, where $|v|$ denotes the length of the word v. We continue this process inductively, creating the infinite tree $\mathcal{D}$. Note that

$$\{O_{I^w} \,;\, I^w \text{ is a bridge of } \mathcal{D}\} = \{O_i \,;\, i \geq 1\}$$

hence

$$C = \bigcap_{n=0}^{\infty} \left(\bigcup_{|w|=n} I^w \right).$$

Any tree with this property is said to be a *derivation* of the Cantor set C from I. The intervals $I, I^0, \ldots$ are called *bridges* of the derivation, while the open intervals $O_I, O_{I^0}, \ldots$ are called *gaps* of C. If I^w is a bridge of $\mathcal{D}$ then we say that I^w *splits* as

$$I^w = I^{w0} \cup O_{I^w} \cup I^{w1}.$$

For example, if C is the usual middle–third Cantor set we may take a derivation of C to be the tree $\mathcal{D}$ with root $I = [0,1]$ and bridges

$$I^{d_1 d_2 \cdots d_t} = \left[2\sum_{i=1}^{t} \frac{d_i}{3^i}, \frac{1}{3^t} + 2\sum_{i=1}^{t} \frac{d_i}{3^i} \right]$$

for all finite binary words $d_1 d_2 \ldots d_t$.

Note that the derivation $\mathcal{D}$ of a Cantor set C is not uniquely determined by C; for example, if we change the order in which the open intervals are removed then we get a different derivation but the same Cantor set.

We denote the length of an interval I by $|I|$. We say that a derivation $\mathcal{D}$ is *ordered* if for any bridges A and B of $\mathcal{D}$ with $A = A^0 \cup O_A \cup A^1$, $B = B^0 \cup O_B \cup B^1$ and $B \subseteq A$ we have $|O_A| \geq |O_B|$.

Cantor sets arise in the study of real numbers whose partial quotients are members of a given set. Let $B = \{b_1, b_2, \ldots, b_t\}$ be a finite set of positive integers with $t \geq 2$ and $b_1 < \cdots < b_t$. We set $l = l(B) = \min B = b_1$, $m = m(B) = \max B = b_t$,

$$C(B) = F(B) \bigcap [0,1] = \{\langle a_1, a_2, \ldots \rangle \,;\, a_i \in B \text{ for } i = 1, 2, \ldots\},$$

and let $I(B)$ be the closed interval

$$I(B) = [\langle \overline{m, l} \rangle, \langle \overline{l, m} \rangle].$$

We have $C(B) \subseteq I(B)$. We now inductively construct a derivation $\mathcal{D}(B)$ of $C(B)$ from $I(B)$. For any real a and b, we denote by $[[a,b]]$ and $((a,b))$ the intervals

$$[[a,b]] = [\min\{a,b\}, \max\{a,b\}]$$

and

$$((a,b)) = (\min\{a,b\}, \max\{a,b\}).$$

If, for $i < t$,

$$A = [[\langle a_1, \ldots, a_r, b_i, \overline{m,l}\rangle, \langle a_1, \ldots, a_r, m, \overline{l,m}\rangle]] \tag{2.2}$$

is a bridge of $\mathcal{D}(B)$ of level n, then we form the subvertices of A by setting

$$A^0 = [[\langle a_1, \ldots, a_r, b_i, \overline{m,l}\rangle, \langle a_1, \ldots, a_r, b_i, \overline{l,m}\rangle]],$$
$$O_A = ((\langle a_1, \ldots, a_r, b_i, \overline{l,m}\rangle, \langle a_1, \ldots, a_r, b_{i+1}, \overline{m,l}\rangle))$$

and

$$A^1 = [[\langle a_1, \ldots, a_r, b_{i+1}, \overline{m,l}\rangle, \langle a_1, \ldots, a_r, m, \overline{l,m}\rangle]].$$

In this manner we construct the $(n+1)^{th}$ level of the derivation from the n^{th} level. Note that A^0 is of the form (2.2) with $a_{r+1} = b_i$ and b_i replaced by l. Similarly A^1 is also of the form (2.2). Since $I(B)$ is of the form (2.2) with $r = 0$ and $i = 1$, by induction we obtain the *canonical* derivation $\mathcal{D}(B)$ of $C(B)$ from $I(B)$.

If B is an infinite set then we may construct a similar derivation. Assume that $B = \{b_1, \ldots\}$ with $b_i < b_{i+1}$ for $i \geq 1$. If we set $l = l(B) = \min B = b_1$ then we have $C(B) \subseteq I(B)$ where $I(B) = [0, 1/l]$ and

$$C(B) = \{\langle a_1, a_2, \ldots\rangle\,;\ a_i \in B \text{ for } i \geq 1\}$$
$$\bigcup \{\langle a_1, a_2, \ldots, a_k\rangle\,;\ k \in \mathbf{Z},\ k \geq 0 \text{ and } a_i \in B \text{ for } 1 \leq i \leq k\}.$$

If

$$A = [[\langle a_1, \ldots, a_r, b_i \rangle, \langle a_1, \ldots, a_r \rangle]] \tag{2.3}$$

is a bridge, then we split A by setting

$$A^0 = [[\langle a_1, \ldots, a_r, b_i \rangle, \langle a_1, \ldots, a_r, b_i, l \rangle]],$$
$$O_A = ((\langle a_1, \ldots, a_r, b_i, l \rangle, \langle a_1, \ldots, a_r, b_{i+1} \rangle))$$

and

$$A^1 = [[\langle a_1, \ldots, a_r, b_{i+1} \rangle, \langle a_1, \ldots, a_r \rangle]]$$

where by convention we set $\langle a_1, \ldots, a_r \rangle = 0$ if $r = 0$. As above, we construct the canonical derivation $\mathcal{D}(B)$ of $C(B)$ from $I(B)$ using this process.

For given sets of integers B_j, $j = 1, \ldots, k$, we would like to be able to determine if

$$\sum_{j=1}^{k} C(B_j) = \sum_{j=1}^{k} I(B_j). \tag{2.4}$$

To do this we shall derive criteria on general Cantor sets that guarantee (2.4) holds. Our conditions will be less stringent than those derived previously. Let C be a Cantor set with derivation $\mathcal{D}$, and let A be a bridge of $\mathcal{D}$. We define the *thickness of A with respect to $\mathcal{D}$*, denoted by $\tau_{\mathcal{D}}(A)$, to be positive infinity if A does not split. Otherwise we set

$$\tau(A) = \tau_{\mathcal{D}}(A) = \min\left\{\frac{|A^0|}{|O_A|}, \frac{|A^1|}{|O_A|}\right\}$$

where throughout this paper we adopt the convention that $x/0 = \infty$ for any $x > 0$. We define the *thickness* $\tau(\mathcal{D})$ of the derivation $\mathcal{D}$ by

$$\tau(\mathcal{D}) = \inf_A \tau_{\mathcal{D}}(A)$$

where the infimum is taken over all bridges A of $\mathcal{D}$. We also define $\tau(C)$, the *thickness* of the Cantor set C, by

$$\tau(C) = \sup_{\mathcal{D}} \tau(\mathcal{D})$$

where the supremum is taken over all derivations $\mathcal{D}$ of C. For example, if C is the middle-third Cantor set then $\tau(C) = 1$. An equivalent definition of $\tau(C)$ may be found in [13, p. 61] . It follows from Lemma 3.2.1 that $\tau(C) = \tau(\mathcal{D}_o)$, where $\mathcal{D}_o$ is any ordered derivation of C. The following observation is trivial yet crucial in our use of thickness.

Lemma 2.3.1 *Let C be a Cantor set. Then C is an interval if and only if $\tau(C)$ equals infinity.*

Proof. Let C be derived from I and take $\mathcal{D}$ to be any ordered derivation of C from I. By Lemma 3.2.1 we have $\tau(C) = \tau(\mathcal{D})$. If $C \neq I$ then I, the root of $\mathcal{D}$, must split in $\mathcal{D}$ with a nontrivial gap, so

$$\tau(\mathcal{D}) \leq \tau_{\mathcal{D}}(I) < \infty.$$

If, on the other hand, $C = I$, then no gaps are removed from C in the derivation, so $\tau(\mathcal{D}) = \infty$ as required.

□

To relate thickness to Hausdorff dimension we use the following result.

Lemma 2.3.2 *If C is a Cantor set then*

$$\dim_H(C) \geq \frac{\log 2}{\log\left(2 + \frac{1}{\tau(C)}\right)}.$$

Proof. See [13, p. 77].

□

Thus, for example, if C is the middle-third Cantor set then by Lemma 2.3.2 we have

$$\dim_H C \geq \frac{\log 2}{\log 3}. \tag{2.5}$$

In fact, it can be shown that in this case equality holds in (2.5), hence Lemma 2.3.2 is best possible.

As in [8] and [2] we employ the logarithm function to treat products and quotients of Cantor sets. Given a set S of positive numbers, we form the set S^* by putting

$$S^* = \{\log x\,;\, x \in S\}.$$

If C is Cantor set of positive numbers, then C^* will also be a Cantor set. We can construct a derivation of C^* by taking our bridges to be of the form $[\log a, \log b]$, where $[a, b]$ is a bridge of our derivation $\mathcal{D}$ of C. By Lemma 2.2.2 we have

$$\dim_H(C^*) = \dim_H(C).$$

Chapter 3

Sums of Cantor Sets

3.1 Main Results

Let C be the middle-third Cantor set. Hugo Steinhaus proved in 1917 [16] that $C + C = [0, 2]$. Here we shall consider the more general problem of determining the sum of a finite number of arbitrary Cantor sets. For sums of two Cantor sets we shall prove the following result.

Theorem 3.1.1 *For $j = 1, 2$ let C_j be a Cantor set derived from I_j, with O_j a gap of maximal size in C_j. Assume that*

$$|O_1| \leq |I_2| \quad \textit{and} \quad |O_2| \leq |I_1|.$$

1. *If $\tau(C_1)\tau(C_2) \geq 1$ then $C_1 + C_2 = I_1 + I_2$.*

2. *If $\tau(C_1)\tau(C_2) < 1$ then*

$$\tau(C_1 + C_2) \geq \frac{\tau(C_1) + \tau(C_2) + 2\tau(C_1)\tau(C_2)}{1 - \tau(C_1)\tau(C_2)}.$$

Part 1 of Theorem 3.1.1 may be derived from work of Sheldon Newhouse; our approach shall give an alternative proof. Newhouse [12] established the following result.

Theorem 3.1.2 *Let K_1 and K_2 be Cantor sets derived from I_1 and I_2 respectively, with $\tau(K_1)\tau(K_2) > 1$. Then either $I_1 \cap I_2 = \emptyset$, K_1 is contained in a gap of K_2, K_2 is contained in a gap of K_1 or $K_1 \cap K_2 \neq \emptyset$.*

In fact, if Newhouse's proof is slightly altered then we may replace the condition "$\tau(K_1)\tau(K_2) > 1$" in Theorem 3.1.2 with the weaker condition "$\tau(K_1)\tau(K_2) \geq 1$".

To see that Part 1 of Theorem 3.1.1 follow from this modified version of Theorem 3.1.2, we assume that $\tau(C_1)\tau(C_2) \geq 1$ and let k be any number in $I_1 + I_2$. Upon applying Theorem 3.1.2 (modified) with $K_1 = k - C_1$ and $K_2 = C_2$ we find that $(k - C_1) \cap C_2 \neq \emptyset$ and hence $k \in C_1 + C_2$.

If C is a Cantor set then we define the *normalized thickness* of C, denoted by $\gamma(C)$, by

$$\gamma(C) = \frac{\tau(C)}{\tau(C) + 1}.$$

Theorem 3.1.1 is a special case of the following theorem.

Theorem 3.1.3 *Let k be a positive integer and for $j = 1, 2, \ldots, k$ let C_j be a Cantor set derived from I_j, with O_j a gap of maximal size in C_j. Let $S_\gamma = \gamma(C_1) + \cdots + \gamma(C_k)$.*

1.

$$\dim_H(C_1 + \cdots + C_k) \geq \frac{\log 2}{\log\left(1 + \frac{1}{\min\{S_\gamma, 1\}}\right)}.$$

2. *If* $S_\gamma \geq 1$ *then* $C_1 + \cdots + C_k$ *contains an interval. Otherwise* $C_1 + \cdots + C_k$ *contains a Cantor set of thickness at least*

$$\frac{S_\gamma}{1 - S_\gamma}.$$

3. *If*

$$|I_{r+1}| \geq |O_j| \qquad \text{for } r = 1, \ldots, k-1 \text{ and } j = 1, \ldots, r, \tag{3.1}$$

$$|I_1| + \cdots + |I_r| \geq |O_{r+1}| \qquad \text{for } r = 1, \ldots, k-1 \tag{3.2}$$

and $S_\gamma \geq 1$ *then*

$$C_1 + \cdots + C_k = I_1 + \cdots + I_k.$$

4. *If (3.1) and (3.2) hold and* $S_\gamma < 1$ *then*

$$\tau(C_1 + \cdots + C_k) \geq \frac{S_\gamma}{1 - S_\gamma}.$$

Theorem 3.1.3 is best possible in the sense that the condition $S_\gamma \geq 1$ in part 2 or part 3 cannot be replaced by $S_\gamma \geq \eta$ for any $\eta < 1$. Similarly if we multiply the bound for the thickness or the Hausdorff dimension of the sum by $1 + \delta$ for any $\delta > 0$ then the results do not hold in general.

3.2 A Few Preliminary Lemmas

To prove Theorem 3.1.3 we require several lemmas.

Lemma 3.2.1 *Let* $\mathcal{D}$ *be any derivation of* C *from* I. *Then there exists an ordered derivation* $\mathcal{D}_o$ *of* C *from* I *with*

$$\tau(\mathcal{D}) \leq \tau(\mathcal{D}_o).$$

Furthermore, if $\mathcal{D}_1$ and $\mathcal{D}_2$ are two ordered derivations of C from I then

$$\tau(\mathcal{D}_1) = \tau(\mathcal{D}_2).$$

Proof. Let $\mathcal{D}$ be a derivation of C from I, and assume that $\mathcal{D}$ is not ordered. Then there exists a bridge A of $\mathcal{D}$ which splits as $A = A^0 \cup O_A \cup A^1$, with A^0 and A^1 splitting as $A^0 = A^{00} \cup O_{A^0} \cup A^{01}$ and $A^1 = A^{10} \cup O_{A^1} \cup A^{11}$ respectively, such that either $|O_{A^0}| > |O_A|$ or $|O_{A^1}| > |O_A|$. Assume without loss of generality that $|O_{A^0}| > |O_A|$. Consider the derivation $\mathcal{D}_s$ which is identical to $\mathcal{D}$ except that the positions of O_{A^0} and O_A in the tree have been switched, that is, O_{A^0} is removed before O_A. If we set $A_s = A$,

$$\begin{aligned} A_s^0 &= A^{00}, \quad O_{A_s} = O_{A^0}, \qquad & A_s^1 &= A^{01} \cup O_A \cup A^1 \\ A_s^{10} &= A^{01}, \quad O_{A_s^1} = O_A \quad \text{and} & A_s^{11} &= A^1 \end{aligned}$$

then in $\mathcal{D}_s$, $A = A_s$ splits as $A_s = A_s^0 \cup O_{A_s} \cup A_s^1$ and A_s^1 splits as $A_s^1 = A_s^{10} \cup O_{A_s^1} \cup A_s^{11}$. We claim that

$$\tau(\mathcal{D}) \leq \tau(\mathcal{D}_s). \tag{3.3}$$

To prove (3.3) it suffices to show that

$$\min\left\{\frac{|A^0|}{|O_A|}, \frac{|A^1|}{|O_A|}, \frac{|A^{00}|}{|O_{A^0}|}, \frac{|A^{01}|}{|O_{A^0}|}\right\} \leq \min\left\{\frac{|A_s^0|}{|O_{A_s}|}, \frac{|A_s^1|}{|O_{A_s}|}, \frac{|A_s^{10}|}{|O_{A_s^1}|}, \frac{|A_s^{11}|}{|O_{A_s^1}|}\right\}. \tag{3.4}$$

Now,

$$\begin{aligned} \frac{|A_s^0|}{|O_{A_s}|} &= \frac{|A^{00}|}{|O_{A^0}|}, & \frac{|A_s^1|}{|O_{A_s}|} &= \frac{|A^{01} \cup O_A \cup A^1|}{|O_{A^0}|} > \frac{|A^{01}|}{|O_{A^0}|}, \\ \frac{|A_s^{10}|}{|O_{A_s^1}|} &= \frac{|A^{01}|}{|O_A|} > \frac{|A^{01}|}{|O_{A^0}|} \quad \text{and} & \frac{|A_s^{11}|}{|O_{A_s^1}|} &= \frac{|A^1|}{|O_A|} \end{aligned}$$

since $|O_A| < |O_{A^0}|$, and so (3.4) holds.

We construct our ordered derivation $\mathcal{D}_o$ as follows. First we modify $\mathcal{D}$ to form a new tree $\mathcal{D}^1$ with the property that the first open interval removed is of maximal size. We form this tree by switching (a finite number of times) the order in which open intervals are removed in $\mathcal{D}$, as outlined above. Next we perform the same process on the bridges of level 1 in $\mathcal{D}^1$, forming a new derivation $\mathcal{D}^2$ which has its first two levels ordered. We continue this procedure inductively, forming $\mathcal{D}^{n+1}$ from $\mathcal{D}^n$ by switching the order open intervals are removed until for every bridge A of level n in $\mathcal{D}^{n+1}$, the next open interval removed from A is of maximal size. Our ordered derivation $\mathcal{D}_o$ is the derivation with the same root as $\mathcal{D}$ and for which the n^{th} level of $\mathcal{D}_o$ consists of the same bridges as the n^{th} level of $\mathcal{D}^n$.

We will use (3.3) to prove the first part of our lemma. For $k \in \mathbb{Z}^+$ let $\mathcal{O}_o^k$ be the set of all gaps between intervals in the k^{th} level of the derivation $\mathcal{D}_o$. Let n_k be the minimal number of levels of $\mathcal{D}$ we must descend before all intervals in $\mathcal{O}_o^k$ have been removed. Further, let D_o^k consist of all bridges occurring in the first k levels of $\mathcal{D}_o$, and let D^{n_k} denote the set of all bridges occurring in the first n_k levels of the derivation $\mathcal{D}$. Then for every $k \in \mathbb{Z}^+$ we have

$$\tau(\mathcal{D}) \leq \min_{A \in D^{n_k}} \tau_{\mathcal{D}}(A) \leq \min_{A \in D_o^k} \tau_{\mathcal{D}_o}(A)$$

by a finite number of applications of (3.3). Thus

$$\tau(\mathcal{D}_o) = \inf_k \min_{A \in D_o^k} \tau_{\mathcal{D}_o}(A) \geq \tau(\mathcal{D})$$

as required.

Now assume that $\mathcal{D}_1$ and $\mathcal{D}_2$ are two ordered derivations of C from I. Let $(t_j)_j$ be the sequence of different lengths of open intervals removed in the derivations, in decreasing order (note that both derivations remove the same set of intervals). If no intervals are removed then $C = I$ and

$$\tau(\mathcal{D}_1) = \infty = \tau(\mathcal{D}_2).$$

Otherwise for every j let B_j be a bridge of minimal width in $\mathcal{D}_1$ such that, in the notation of section 2.3, $B_j = A^{wd}$ for some binary word w and $d \in \{0, 1\}$, with $|O^w| = t_j$. Then

$$\tau(\mathcal{D}_1) = \inf_j \frac{|B_j|}{t_j}.$$

However, B_j satisfies the same condition with $\mathcal{D}_1$ replaced by $\mathcal{D}_2$, whence

$$\tau(\mathcal{D}_2) = \inf_j \frac{|B_j|}{t_j}$$

and the lemma follows.

□

Lemma 3.2.2 *Let C_1 and C_2 be Cantor sets derived by derivations $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively. Put*

$$\tau_1 = \tau(\mathcal{D}_1) \quad \textit{and} \quad \tau_2 = \tau(\mathcal{D}_2).$$

If both τ_1 and τ_2 are greater than zero and neither C_1 nor C_2 contains an interval, then there exist bridges A and B of $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively which split as

$$A = A^0 \cup O_A \cup A^1 \quad \textit{and} \quad B = B^0 \cup O_B \cup B^1$$

such that

$$|A| \geq \frac{\tau_1}{\tau_1 + 1}(\tau_2 + 1)|O_B| \quad \textit{and} \quad |B| \geq \frac{\tau_2}{\tau_2 + 1}(\tau_1 + 1)|O_A|.$$

Proof. Let $S = (A_i)_{i=1}^{\infty}$ be a sequence of bridges of $\mathcal{D}_1$, where if $\mathcal{D}_1$ contains a bridge of width t then $|A_i| = t$ for some i, and $|A_i| > |A_{i+1}|$ for $i \geq 1$. Since C_1 does not contain an interval, all A_i split, and $|A_i|$ tends to zero as i increases. We define the sequence $(B_j)_{j=1}^{\infty}$ from $\mathcal{D}_2$ in a similar manner. If O_{A_i} and O_{B_j} are the open intervals removed when A_i and B_j split, then

$$|O_{A_i}| \leq \frac{|A_i|}{2\tau_1 + 1} \quad \text{and} \quad |O_{B_j}| \leq \frac{|B_j|}{2\tau_2 + 1} \tag{3.5}$$

for $i, j \geq 1$. Therefore to prove the lemma it suffices to exhibit A_r and B_s with

$$\frac{\tau_2 + 1}{2\tau_2 + 1} \cdot \frac{\tau_1}{\tau_1 + 1} \leq \frac{|A_r|}{|B_s|} \leq \frac{2\tau_1 + 1}{\tau_1 + 1} \cdot \frac{\tau_2 + 1}{\tau_2}$$

or equivalently

$$\frac{\tau_2}{2\tau_2 + 1}|B_s| \leq \frac{\tau_1 + 1}{\tau_1} \cdot \frac{\tau_2}{\tau_2 + 1}|A_r| \leq \frac{2\tau_1 + 1}{\tau_1}|B_s|. \tag{3.6}$$

By (3.5), for $j \geq 1$ we have

$$\begin{aligned} \max\{|B_j^0|, |B_j^1|\} &\geq \frac{1}{2}\left(|B_j| - |O_{B_j}|\right) \\ &\geq \frac{1}{2}\left(|B_j| - \frac{|B_j|}{2\tau_2 + 1}\right) \\ &= \frac{\tau_2}{2\tau_2 + 1}|B_j|, \end{aligned}$$

hence

$$\frac{|B_{j+1}|}{|B_j|} \geq \frac{\tau_2}{2\tau_2 + 1} \quad \text{for } j \geq 1.$$

Therefore

$$\frac{2\tau_1 + 1}{\tau_1}|B_j| \geq |B_j| \quad \text{and} \quad \frac{\tau_2}{2\tau_2 + 1}|B_j| \leq |B_{j+1}|$$

so to establish (3.6) it is enough to find r and s such that

$$|B_{s+1}| \leq \frac{\tau_1 + 1}{\tau_1} \cdot \frac{\tau_2}{\tau_2 + 1}|A_r| \leq |B_s|. \tag{3.7}$$

Since $\{|A_i|\}_i$ and $\{|B_j|\}_j$ are both sequences which are monotonically decreasing to zero and $\tau_2 \neq 0$, (3.7) must have a solution (r, s) and the lemma follows.

□

3.3 The Crucial Proposition

The next proposition is the key result in this thesis. In its proof we shall make use of the concept of compatibility of bridges, which is similar to an approach used by Hlavka ([10, Theorem 3]).

Proposition 3.3.1 *For $j = 1, 2$ let C_j be a Cantor set derived from I_j with O_j the largest gap in C_j. Let $S_\gamma = \gamma(C_1) + \gamma(C_2)$.*

1. *Let α' and β' be any positive real numbers for which $\alpha'\beta' = \tau(C_1)\tau(C_2)$, and put $\alpha = \min\{1, \alpha'\}$ and $\beta = \min\{1, \beta'\}$. If*

$$\beta|O_1| \le |I_2| \quad \textit{and} \quad \alpha|O_2| \le |I_1| \tag{3.8}$$

   then

$$\tau(C_1 + C_2) \ge \min\left\{\frac{\tau(C_1) + \beta}{1 - \beta}, \frac{\tau(C_2) + \alpha}{1 - \alpha}\right\}.$$

2. *If $|O_1| \le |I_2|$, $|O_2| \le |I_1|$ and $S_\gamma \ge 1$ then*

$$C_1 + C_2 = I_1 + I_2.$$

3. *If (3.8) holds with*

$$\alpha' = \gamma(C_1)(\tau(C_2) + 1), \quad \beta' = \gamma(C_2)(\tau(C_1) + 1)$$

   and $S_\gamma < 1$, then

$$\tau(C_1 + C_2) \ge \frac{S_\gamma}{1 - S_\gamma}.$$

4. *If $S_\gamma \ge 1$ then $C_1 + C_2$ contains an interval. Otherwise $C_1 + C_2$ contains a Cantor set of thickness at least*

$$\frac{S_\gamma}{1 - S_\gamma}.$$

Proof. We first prove part 1. Assume that (3.8) holds and set

$$\tau = \min\left\{\frac{\tau(C_1)+\beta}{1-\beta}, \frac{\tau(C_2)+\alpha}{1-\alpha}\right\}.$$

We will show that $\tau(C_1 + C_2) \geq \tau$. To do so we will construct a tree of valence 2 to represent $C_1 + C_2$. This tree might not be a derivation, since bridges of the tree may overlap. However, we will use this tree to construct a derivation of $C_1 + C_2$ with the required thickness.

We will construct our first tree inductively, by setting the root to be $I_1 + I_2$ and showing how each bridge in the tree splits. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be ordered derivations of C_1 and C_2 respectively. If A and B are bridges of $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively, we say that A and B are *compatible*, denoted $A \sim B$, if

$$|A| \geq \alpha|O_B| \quad \text{and} \quad |B| \geq \beta|O_A|$$

where A and B split as

$$A = A^0 \cup O_A \cup A^1$$

and

$$B = B^0 \cup O_B \cup B^1.$$

If A does not split but B does, then we say $A \sim B$ if $|A| \geq \alpha|O_B|$, and similarly if A splits but B does not, then $A \sim B$ if $|B| \geq \beta|O_A|$. Finally, for all bridges A and B, neither of which split, we put $A \sim B$.

We shall construct a derivation for $C_1 + C_2$ using the derivations of C_1 and C_2. Let A and B be bridges of $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively with $A \sim B$, and set $D = A + B$. Assume first that both A and B split. Then

$$\frac{\min\{|A^0|, |A^1|\}}{|O_A|} \cdot \frac{\min\{|B^0|, |B^1|\}}{|O_B|} \geq \tau(C_1)\tau(C_2) \geq \alpha\beta$$

so

$$\min\{|A^0|,|A^1|\}\min\{|B^0|,|B^1|\} \geq \alpha\beta|O_A||O_B|,$$

thus either

$$\min\{|A^0|,|A^1|\} \geq \alpha|O_B| \tag{3.9}$$

or

$$\min\{|B^0|,|B^1|\} \geq \beta|O_A|. \tag{3.10}$$

Assume that (3.9) holds, and let O_{A^0} and O_{A^1} be the open intervals removed in the splitting of A^0 and A^1 respectively. Since the derivations are ordered,

$$\beta|O_{A^0}| \leq \beta|O_A| \leq |B| \quad \text{and} \quad \beta|O_{A^1}| \leq \beta|O_A| \leq |B|$$

as $A \sim B$. By (3.9) we have

$$\alpha|O_B| \leq |A^0| \quad \text{and} \quad \alpha|O_B| \leq |A^1|$$

whence

$$A^0 \sim B \quad \text{and} \quad A^1 \sim B.$$

We put

$$D^0 = A^0 + B, \quad D^1 = A^1 + B \quad \text{and} \quad O_D = D \setminus (D^0 \cup D^1). \tag{3.11}$$

We have

$$|D^0| = |A^0| + |B|, \quad |D^1| = |A^1| + |B|$$

and if O_D is non-empty,

$$|O_D| = |O_A| - |B|.$$

Note that if $\beta = 1$ then O_D is necessarily empty. Thus either there is no gap between D^0 and D^1 or

$$\frac{\min\{|D^0|,|D^1|\}}{|O_D|} = \frac{\min\{|A^0|,|A^1|\}+|B|}{|O_A|-|B|} \geq \frac{\tau(C_1)+\beta}{1-\beta} \geq \tau \tag{3.12}$$

For $d = 0$ or $d = 1$, to determine the splitting of D^d we repeat the above process with A replaced with A^d.

If we find that (3.10) holds instead of (3.9), we perform the same process, except we split B instead of A. Again we may bound $\min\{|D^0|,|D^1|\}/|O_D|$ by τ.

If A splits but B does not then we define D^0, D^1 and O_D as in (3.11), and find that either O_D is empty or (3.12) holds. If B splits but A does not then we proceed in an analogous manner. Finally, if neither A nor B split then we let D be the vertex $A+B$, and place under D an infinite stalk composed of vertices D^w where w is a binary word composed of zeros, and $D^w = D$ as intervals.

Since $I_1 \sim I_2$ we find by induction that we may construct a tree T_S of closed intervals $\{D^w\}$ such that

$$C_1 + C_2 = \bigcap_{m\geq 0} \bigcup_{w} D^w$$

where the union is taken over all binary words w of length m such that D^w is a vertex of T_S. We further have that if V is a vertex of T_S then either V does not split or

$$\min\{|V^0|,|V^1|\} \geq \tau|O_V|. \tag{3.13}$$

Now T_S might not be a derivation of $C_1 + C_2$ since we may have some overlap of intervals associated with vertices. We will however use T_S to construct a derivation for $C_1 + C_2$ with the required thickness. Let $H^0 = I_1 + I_2$, $H^1 = D^0 \cup D^1$ and in general

$$H^m = \bigcup_{w} D^w$$

where the union is over all binary words w of length m with D^w in T_S. For each m, H^m will be the union of a finite number of disjoint closed intervals $\{H_i^m\}$. We next define a tree T_H by taking as vertices all intervals $\{H_i^m\}$ and as edges all lines joining vertices H_i^m to H_j^{m+1}, where $H_j^{m+1} \subseteq H_i^m$ as sets. We will convert T_H into a tree where every vertex has at most two subvertices. Let N be a vertex of T_H. We will construct a finite tree T_N with root N and having as leaves the subvertices of N in T_H, such that T_N is of valence 2 and T_N satisfies a condition similar to (3.13).

Let N have subvertices $N_1, N_2, \ldots, N_t$ in T_H. If $t \leq 2$ then we let T_N be the tree with root N and leaves $N_1, \ldots, N_t$. Otherwise, we have that as intervals,

$$N = N_1 \cup G_1 \cup N_2 \cup G_2 \cup \cdots \cup G_{t-1} \cup N_t \tag{3.14}$$

where $G_1, \ldots, G_{t-1}$ are open intervals. For intervals J_1 and J_2 we write $J_1 \rightarrow J_2$ if $|J_1| \geq \tau |J_2|$. We start by making the following claim.

Claim 1 *Let G_r and G_s be two open intervals in (3.14) with $r < s$. Let J denote the entire closed interval between G_r and G_s. Then $J \rightarrow G_r$ or $J \rightarrow G_s$. Further, if G_r is any open interval in (3.14) then*

$$\left(N_r \cup \bigcup_{1 \leq n < r} (N_n \cup G_n) \right) \rightarrow G_r \tag{3.15}$$

and

$$\left(N_t \cup \bigcup_{r < n < t} (N_n \cup G_n) \right) \rightarrow G_r \tag{3.16}$$

Proof of Claim 1. Since J contains points of $C_1 + C_2$ and $(C_1 + C_2) \cap (G_r \cup G_s) = \emptyset$, there exists a vertex $V = V^0 \cup O_V \cup V^1$ of T_S with $V \cap J \neq \emptyset$ and either $G_r \subseteq O_V$ or $G_s \subseteq O_V$. Assume without loss of generality that $G_r \subseteq O_V$. If $V^1 \subseteq J$ then

$$\frac{|J|}{|G_r|} \geq \frac{|V^1|}{|O_V|} \geq \tau$$

so $J \to G_r$. Otherwise $G_s \subseteq V^1$, and since $(C_1 + C_2) \cap G_s = \emptyset$ there exists a vertex $W = W^0 \cup O_W \cup W^1$ in T_S with $W \subseteq V^1$ and $G_s \subseteq O_W$. In this case $W^0 \subseteq J$ so

$$\frac{|J|}{|G_s|} \geq \frac{|W^0|}{|O_W|} \geq \tau$$

so $J \to G_s$ and the first part of the claim follows.

To prove the second part of the claim we denote by J_r^0 and J_r^1 the left sides of (3.15) and (3.16) respectively. As above, we have a vertex V of T_S with $V^0 \subseteq J_r^0$, $V^1 \subseteq J_r^1$ and $G_r \subseteq O_V$, and the claim follows.

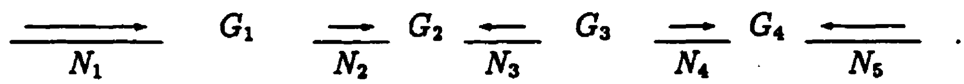
By the claim we have

$$N_1 \to G_1, \qquad\qquad N_t \to G_{t-1}$$

and

$$N_j \to G_{j-1} \quad \text{or} \quad N_j \to G_j$$

for $j = 2, \ldots, t-1$. For example,



Thus there must be some G_{r_1} with

$$N_{r_1} \to G_{r_1} \quad \text{and} \quad N_{r_1+1} \to G_{r_1}. \tag{3.17}$$

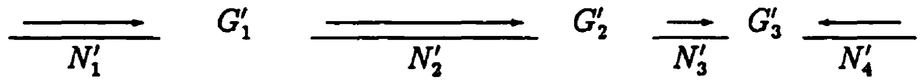
We set $t' = t - 1$,

$$N'_j = \begin{cases} N_j & \text{if } 1 \leq j < r_1 \\ N_{r_1} \cup G_{r_1} \cup N_{r_1+1} & \text{if } j = r_1 \\ N_{j+1} & \text{if } r_1 < j \leq t' \end{cases}$$

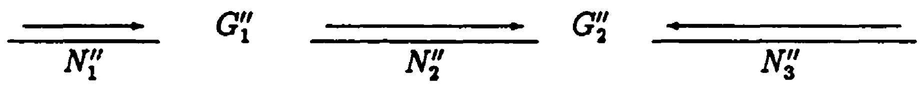
and

$$G_j' = \begin{cases} G_j & \text{if } 1 \le j < r_1 \\ G_{j+1} & \text{if } r_1 \le j \le t'. \end{cases}$$

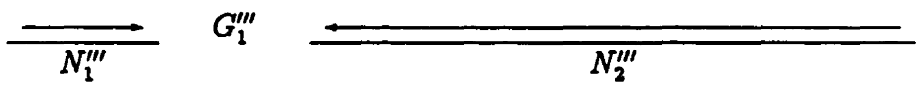
By the claim, $N_{r_1}' \to G_{r_1-1}'$ or $N_{r_1}' \to G_{r_1}'$, i.e.



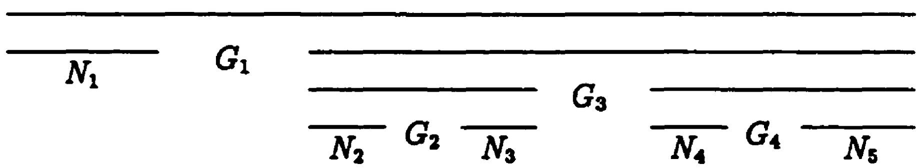
We continue this process until we have only two closed intervals left. In our example, the next step results in



while the last step yields



We are now ready to construct our finite tree T_N. Let G_{r_i} be the open interval satisfying (3.17) at the i^{th} step, for $i = 1, \dots, t-2$. Further, let $G_{r_{t-1}}$ be the open interval remaining when our process terminates. We form T_N by removing, in order, the open intervals $G_{r_{t-1}}, G_{r_{t-2}}, \dots, G_{r_1}$.



By our construction, if N^w is a vertex in T_N which splits as

$$N^w = N^{w0} \cup O_{N^w} \cup N^{w1}$$

then

$$|N^{w0}| \geq \tau |O_{N^w}| \quad \text{and} \quad |N^{w1}| \geq \tau |O_{N^w}|.$$

To construct our derivation $\mathcal{D}$ of $C_1 + C_2$, we take as vertices and edges of $\mathcal{D}$ the sets

$$V = \bigcup_{N \in T_S} V(T_N) \quad \text{and} \quad E = \bigcup_{N \in T_S} E(T_N)$$

respectively, where for a tree T we denote the set of vertices of T by $V(T)$ and the set of edges by $E(T)$. We have $\tau(\mathcal{D}) \geq \tau$ and the first part of the lemma follows.

We will use part 1 of the lemma to prove parts 2 and 3. Let

$$\alpha' = \gamma(C_1)(\tau(C_2) + 1) \quad \text{and} \quad \beta' = \gamma(C_2)(\tau(C_1) + 1) \tag{3.18}$$

and define α and β by

$$\alpha = \min\{1, \alpha'\} \quad \text{and} \quad \beta = \min\{1, \beta'\}. \tag{3.19}$$

Assume that $|O_1| \leq |I_2|$, $|O_2| \leq |I_1|$ and $S_\gamma \geq 1$. Then $\tau(C_1)\tau(C_2) \geq 1$, which implies that $\alpha = \beta = 1$. Therefore, by part 1, $\tau(C_1 + C_2) = \infty$ and part 2 follows.

To prove part 3 we first define α', β', α and β by (3.18) and (3.19). Note that if $S_\gamma < 1$ then $\alpha = \alpha'$ and $\beta = \beta'$, hence

$$\frac{\tau(C_1) + \beta}{1 - \beta} = \frac{\tau(C_1) + \beta'}{1 - \beta'} = \frac{S_\gamma}{1 - S_\gamma}$$

and

$$\frac{\tau(C_2) + \alpha}{1 - \alpha} = \frac{\tau(C_2) + \alpha'}{1 - \alpha'} = \frac{S_\gamma}{1 - S_\gamma}$$

so by part 1 of the lemma

$$\tau(C_1 + C_2) \geq \frac{S_\gamma}{1 - S_\gamma}$$

and part 3 follows.

To prove part 4 we first note that if $\tau(C_1) = 0$ or $\tau(C_2) = 0$ then the result follows trivially, whence we may assume $\tau(C_1)$ and $\tau(C_2)$ are both greater than zero. If either C_1 or C_2 contains a bridge that does not split then $C_1 + C_2$ will contain an interval, hence a set of infinite thickness. Otherwise, by Lemma 3.2.2 there exist bridges A and B of $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively, with

$$A = A^0 \cup O_A \cup A^1 \quad \text{and} \quad B = B^0 \cup O_B \cup B^1$$

such that

$$|A| \geq \alpha |O_B| \quad \text{and} \quad |B| \geq \beta |O_A|$$

where α and β are as defined in (3.19). By parts 2 and 3 of Proposition 3.3.1 applied to the Cantor sets

$$C_A = C_1 \cap A \quad \text{and} \quad C_B = C_2 \cap B$$

we have

$$C_A + C_B = A + B$$

if $S_\gamma \geq 1$ and

$$\tau(C_A + C_B) \geq \frac{S_\gamma}{1 - S_\gamma}$$

otherwise, and part 4 of the lemma follows.

□

3.4 Proofs of our Main Results

Proof of Theorem 3.1.3. For real numbers γ_1, γ_2 and γ_3 in $[0,1]$ with $\gamma_1+\gamma_2 < 1$ we put

$$\tau_{12} = \frac{\gamma_1+\gamma_2}{1-\gamma_1-\gamma_2}.$$

Note that

$$\frac{\tau_{12}}{1+\tau_{12}} = \gamma_1+\gamma_2$$

so

$$\frac{\tau_{12}}{1+\tau_{12}} + \gamma_3 = \gamma_1+\gamma_2+\gamma_3. \tag{3.20}$$

We first prove part 2. Assume $S_\gamma \geq 1$ and let t be the smallest integer with $\gamma(C_1) + \cdots + \gamma(C_t) \geq 1$. Using Proposition 3.3.1 part 4 and (3.20) we find by induction that $C_1 + \cdots + C_t$ contains an interval, whence $C_1 + \cdots + C_k$ contains an interval.

If $S_\gamma < 1$ then we find by Proposition 3.3.1 part 4, (3.20) and induction that $C_1 + \cdots + C_k$ contains a Cantor set of thickness at least $S_\gamma/(1-S_\gamma)$, so by Lemma 2.3.2,

$$\dim_H(C_1+\cdots+C_k) \geq \frac{\log 2}{\log(1+\frac{1}{S_\gamma})}$$

and part 1 of the theorem follows.

To prove parts 3 and 4 we first note that by (3.1) and (3.2) the sets $I_1 + \cdots + I_r$ and I_{r+1} satisfy (3.8) with $\alpha = \beta = 1$, for $r = 1,\ldots,k-1$. We find by induction, Proposition 3.3.1 part 2 and (3.20) that if $S_\gamma \geq 1$ then

$$C_1+\cdots+C_k = I_1+\cdots+I_k$$

and part 3 of the theorem follows. Similarly if $S_\gamma < 1$ then by induction, Proposition 3.3.1 part 3 and (3.20) we have

$$\tau(C_1 + \cdots + C_k) \geq \frac{S_\gamma}{1 - S_\gamma}$$

and the theorem follows.

□

Proof of Theorem 3.1.1. Theorem 3.1.1 follows from Theorem 3.1.3 with $k = 2$, since in that case

$$\tau(C_1)\tau(C_2) \geq 1 \qquad \text{if and only if} \qquad S_\gamma \geq 1$$

and

$$\frac{\tau(C_1) + \tau(C_2) + 2\tau(C_1)\tau(C_2)}{1 - \tau(C_1)\tau(C_2)} = \frac{S_\gamma}{1 - S_\gamma}.$$

□

Chapter 4

Numbers with Restricted Partial Quotients

4.1 Bounds on the Thickness of $C(B)$

To apply Theorems 3.1.1 and 3.1.3 to the cases where the Cantor sets are of the form $C(B_j)$ for some $B_j \subseteq \mathbb{Z}^+$, we need only calculate the thicknesses of the Cantor sets in question.

Lemma 4.1.1 *Let $t \geq 2$ be an integer and $B = \{b_1, b_2, \ldots, b_t\}$ a finite set of positive integers with $b_i < b_{i+1}$ for $i = 1, 2, \ldots, t-1$. Let $l = b_1$ and $m = b_t$, and set $\Delta_i = b_{i+1} - b_i$ for $i = 1, 2, \ldots, t-1$. Put*

$$\delta = \frac{-lm + \sqrt{l^2m^2 + 4lm}}{2}.$$

Then

$$\tau(\mathcal{D}(B)) = \min_{1\le i<t} \min \left\{ \frac{\delta(m-l)}{\Delta_i lm - \delta(m-l)} \cdot \frac{b_{i+1}lm + m + \delta l}{b_i lm + m + \delta l}, \right.$$
$$\left. \frac{(m - b_{i+1})lm + \delta(m-l)}{\Delta_i lm - \delta(m-l)} \cdot \frac{b_i l + \delta}{lm + \delta} \right\}.$$

Proof. Assume that our bridge is of the form (2.2). To compute a lower bound for $|A^0|/|O_A|$ we use part 2 of Lemma 2.1.3 with

$$g_1 = [b_i, \overline{l,m}], \quad g_2 = [b_i, \overline{m,l}], \quad g_3 = [b_{i+1}, \overline{m,l}] \quad \text{and} \quad g_4 = [b_i, \overline{l,m}]$$

to find that

$$\begin{aligned}
\frac{|A^0|}{|O_A|} &= \frac{(b_i + \langle\overline{l,m}\rangle) - (b_i + \langle\overline{m,l}\rangle)}{(b_{i+1} + \langle\overline{m,l}\rangle) - (b_i + \langle\overline{l,m}\rangle)} \cdot \frac{b_{i+1} + \langle\overline{m,l}\rangle + Q}{b_i + \langle\overline{m,l}\rangle + Q} \\
&= \frac{\langle\overline{l,m}\rangle - \langle\overline{m,l}\rangle}{\Delta_i + \langle\overline{m,l}\rangle - \langle\overline{l,m}\rangle} \cdot \frac{b_{i+1} + \langle\overline{m,l}\rangle + Q}{b_i + \langle\overline{m,l}\rangle + Q} \\
&= \frac{\delta/l - \delta/m}{\Delta_i + \delta/m - \delta/l} \cdot \frac{b_{i+1} + \delta/m + Q}{b_i + \delta/m + Q} \\
&= \frac{\delta(m-l)}{\Delta_i lm - \delta(m-l)} \cdot \frac{(b_{i+1} + Q)m + \delta}{(b_i + Q)m + \delta}.
\end{aligned} \tag{4.1}$$

Similarly we use part 2 of Lemma 2.1.3 with

$$g_1 = [m, \overline{l,m}], \quad g_2 = [b_{i+1}, \overline{m,l}], \quad g_3 = [b_{i+1}, \overline{m,l}] \quad \text{and} \quad g_4 = [b_i, \overline{l,m}]$$

and find that

$$\frac{|A^1|}{|O_A|} = \frac{(m - b_{i+1})lm + \delta(m-l)}{\Delta_i lm - \delta(m-l)} \cdot \frac{(b_i + Q)l + \delta}{(m + Q)l + \delta}.$$

Thus

$$\begin{aligned}\tau(\mathcal{D}(B)) &= \inf_{Q} \min_{1\le i<t} \min\left\{\frac{\delta(m-l)}{\Delta_i lm-\delta(m-l)}\cdot\frac{(b_{i+1}+Q)m+\delta}{(b_i+Q)m+\delta},\right.\\ &\qquad\left.\frac{(m-b_{i+1})lm+\delta(m-l)}{\Delta_i lm-\delta(m-l)}\cdot\frac{(b_i+Q)l+\delta}{(m+Q)l+\delta}\right\}\\ &= \min_{1\le i<t} \min\left\{\frac{\delta(m-l)}{\Delta_i lm-\delta(m-l)}\cdot\frac{b_{i+1}lm+m+\delta l}{b_i lm+m+\delta l},\right.\\ &\qquad\left.\frac{(m-b_{i+1})lm+\delta(m-l)}{\Delta_i lm-\delta(m-l)}\cdot\frac{b_i l+\delta}{lm+\delta}\right\}\end{aligned}$$

since $0 \le Q \le 1/l$, and

$$\frac{q_{-1}}{q_0} = 0 \quad\text{and}\quad \frac{q_0}{q_1} = \frac{1}{l}$$

if $a_1 = l$.

□

A similar but simpler result holds in the infinite case.

Lemma 4.1.2 *Let $B = \{b_1, b_2, \dots\}$ be an infinite set of integers with $b_i < b_{i+1}$ for $i \ge 1$. Let $l = b_1$ and set $\Delta_i = b_{i+1} - b_i$ for $i \ge 1$. Then*

$$\tau(\mathcal{D}(B)) = \inf_{i\ge 1} \min\left\{\frac{1}{\Delta_i l-1}\cdot\frac{b_{i+1}l+1}{b_i l+1}, \frac{b_i l+1}{\Delta_i l-1}\right\}.$$

Proof. We use the same strategy as in the proof of Lemma 4.1.1. If A is a bridge of the form (2.3) then by part 2 of Lemma 2.1.3 with

$$g_1 = [b_i, l], \quad g_2 = b_i, \quad g_3 = b_{i+1} \quad\text{and}\quad g_4 = [b_i, l]$$

we find

$$\frac{|A^0|}{|O_A|} = \frac{1}{\Delta_i l-1}\cdot\frac{b_{i+1}+Q}{b_i+Q}. \tag{4.2}$$

Now if $r = 0$ then

$$\frac{|A^1|}{|O_A|} = \frac{b_i l + 1}{\Delta_i l - 1},$$

while if $r > 0$ we apply part 2 of Lemma 2.1.3 with

$$g_1 = [a_r, b_{i+1}], \quad g_2 = a_r, \quad g_3 = [a_r, b_i, l] \quad \text{and} \quad g_4 = [a_r, b_{i+1}]$$

and conclude that

$$\begin{aligned}
\frac{|A^1|}{|O_A|} &= \frac{\langle b_{i+1}\rangle}{\langle b_i, l\rangle - \langle b_{i+1}\rangle} \cdot \frac{[a_r, b_i, l] + \frac{q_{r-2}}{q_{r-1}}}{a_r + \frac{q_{r-2}}{q_{r-1}}} \\
&= \frac{b_i l + 1}{\Delta_i l - 1} \cdot \frac{a_r q_{r-1} + q_{r-2} + q_{r-1}\langle b_i, l\rangle}{a_r q_{r-1} + q_{r-2}} \\
&= \frac{b_i l + 1}{\Delta_i l - 1} \cdot \frac{q_r + q_{r-1}\langle b_i, l\rangle}{q_r} \\
&= \frac{(b_i + Q)l + 1}{\Delta_i l - 1} \geq \frac{b_i l + 1}{\Delta_i l - 1}
\end{aligned}$$

by (2.1) and since $Q \geq 0$. Therefore

$$\tau(\mathcal{D}(B)) = \inf_Q \inf_i \min \left\{ \frac{1}{\Delta_i l - 1} \cdot \frac{b_{i+1} + Q}{b_i + Q}, \frac{b_i l + 1}{\Delta_i l - 1}, \right\} \tag{4.3}$$

and the lemma follows upon minimizing (4.3), since as in the proof of Lemma 4.1.1 we have $0 \leq Q \leq 1/l$ with $Q = 0$ if $r = 0$ and $Q = 1/l$ if $r = 1$ and $a_1 = l$.

□

Note that $\tau(\mathcal{D}(B))$ equals $\tau(B)$ (as defined in the first chapter).

Lemma 4.1.3 *Let B be a set of positive integers with $|B| > 1$. If $\Delta_i(B) = \Delta$ is constant then $\mathcal{D}(B)$ is ordered and so*

$$\tau(C(B)) = \tau(\mathcal{D}(B)) = \tau(B).$$

Proof. Assume first that B is finite. In the notation of Lemma 4.1.1 put

$$O\left((a_1,\ldots,a_r),b_i\right) = \left(\left(\langle a_1,\ldots,a_r,b_i,\overline{l,m}\rangle,\langle a_1,\ldots,a_r,b_{i+1},\overline{m,l}\rangle\right)\right)$$

for $b_i < m$. Then by part 1 of Lemma 2.1.3 with

$$g_1 = [b_{i+1},\overline{m,l}] \quad \text{and} \quad g_2 = [b_i,\overline{l,m}]$$

we have

$$|O\left((a_1,\ldots,a_r),b_i\right)| = \frac{\Delta + \delta/m - \delta/l}{q_r^2(b_{i+1} + \delta/m + Q)(b_i + \delta/l + Q)}.$$

Thus

$$|O\left((a_1,\ldots,a_r),b_i\right)| > |O\left((a_1,\ldots,a_r),b_j\right)|$$

for $j > i$, and

$$|O\left((a_1,\ldots,a_r),b_i\right)| > |O\left((a_1,\ldots,a_r,b_i),b_j\right)|$$

for $b_j \in B$, so $\mathcal{D}(B)$ is an ordered derivation. By Lemma 3.2.1 we have $\tau(C(B)) = \tau(\mathcal{D}(B))$ and Lemma 4.1.3 follows for B finite.

If B is infinite then we use an analogous approach, where in this case we define

$$O\left((a_1,\ldots,a_r),b_i\right) = \left(\left(\langle a_1,\ldots,a_r,b_i,l\rangle,\langle a_1,\ldots,a_r,b_{i+1}\rangle\right)\right).$$

□

4.2 Sums of Continued Fractions

For n an integer and B a set of positive integers with $|B| > 1$ we define $C(n;B)$ by

$$C(n;B) = n + C(B).$$

Using the derivation $\mathcal{D}(B)$ of $C(B)$ we may construct the canonical derivation $n+\mathcal{D}(B)$ of $C(n;B)$ from $n+I(B)$ by translating every interval in $\mathcal{D}$ by n. Similarly we may construct the canonical derivation $n-\mathcal{D}(B)$ of $n-C(B)$ from $n-I(B)$.

Proof of Theorem 1.0.6. Put

$$S_\gamma = \sum_{j=1}^{k} \gamma(B_j).$$

Assume first that $S_\gamma \geq 1$, and for $N \geq 1$ and $j = 1, \dots, k$ set

$$C_j^N = \bigcup_{n=-N}^{N} C(n; B_j)$$

and

$$I_j^N = [-N + \min C(B_j), N + \max C(B_j)].$$

For $j = 1, \dots, k$ we construct a derivation $\mathcal{D}_j^N$ of C_j^N from I_j^N as follows. Assume that $|I(B_j)| < 1$. Remove from I_j^N the interval $(n + \max C(B_j), n + 1 + \min C(B_j))$ for $n = -N, \dots, N-1$, so that if $A_j = I_j^N$ then

$$A_j^0 = -N + I(B_j), \qquad A_j^1 = [-N + 1 + \min C(B_j), N + \max C(B_j)],$$
$$A_j^{10} = -N + 1 + I(B_j), \qquad A_j^{11} = [-N + 2 + \min C(B_j), N + \max C(B_j)]$$

and ultimately,

$$A_j^{1\cdots 10} = N - 1 + I(B_j) \quad \text{and} \quad A_j^{1\cdots 11} = N + I(B_j). \tag{4.4}$$

We complete $\mathcal{D}_j^N$ by using the derivations $n + \mathcal{D}(B_j)$, $n = -N, \dots, N$, to split A_j^0, $A_j^{10}, \dots, A_j^{1\cdots 10}$ and $A_j^{1\cdots 11}$. Note that if B_j is finite then

$$\frac{|n + I_j|}{|(n + \max C(B_j), n + 1 + \min C(B_j))|} = \frac{\langle \overline{l,m} \rangle - \langle \overline{m,l} \rangle}{1 + \langle \overline{m,l} \rangle - \langle \overline{l,m} \rangle}$$
$$= \frac{\delta(m-l)}{ml - \delta(m-l)} > \tau(B_j)$$

and that if B_j is infinite then

$$\frac{|n+I_j|}{|(n+\max C(B_j), n+1+\min C(B_j))|} = \frac{1}{l-1} \geq \tau(B_j)$$

so in either case $\tau(\mathcal{D}_j^N) \geq \tau(B_j)$.

If $|I_j| = 1$ then we form $\mathcal{D}_j^N$ by removing the gaps in C_j^N in order of descending width, so that again we have $\tau(\mathcal{D}_j^N) \geq \tau(B_j)$.

For $j = 1, \ldots, k$, I_j^N has width greater than 2, and all gaps in C_j^N are of width less than 1, whence (3.1) and (3.2) hold. Since $S_\gamma \geq 1$ and for a Cantor set C, $\tau(-C) = \tau(C)$, by part 3 of Theorem 3.1.3 we have

$$\begin{aligned}\epsilon_1 C_1^N + \cdots + \epsilon_k C_k^N &= \epsilon_1 I_1^N + \cdots + \epsilon_k I_k^N \\ &\supseteq [-k(N-1), k(N-1)]\end{aligned}$$

and (1.6) follows upon letting N tend to infinity.

If $S_\gamma < 1$ then (1.7) follows from part 1 of Theorem 3.1.3 with $C_j = C(B_j)$ for $j = 1, \ldots, k$.

□

Proof of Theorem 1.0.2. Theorem 1.0.2 is a special case of Theorem 1.0.6, since

$$\gamma(B_1) + \gamma(B_2) \geq 1 \qquad \text{if and only if} \qquad \tau(B_1)\tau(B_2) \geq 1$$

and further,

$$-1 + \frac{1}{\gamma(B_1) + \gamma(B_2)} = \frac{1 - \tau(B_1)\tau(B_2)}{\tau(B_1) + \tau(B_2) + 2\tau(B_1)\tau(B_2)}.$$

□

Proof of Corollary 1.0.3. Note that $\delta(L_m) > m/(m+1)$, whence

$$\tau(L_m) > \frac{m(m-1)}{m(m+1)-m(m-1)} \cdot \frac{(m-1)(m+1)+m}{m(m+1)+m} > \frac{(m-1)^2}{2m}.$$

Since

$$\tau(U_l) = \frac{1}{l-1} \geq \frac{2}{m-2}$$

we have

$$\tau(L_m)\tau(U_l) > \frac{(m-1)^2}{m(m-2)} > 1$$

and the result follows from part 1 of Theorem 1.0.2.

□

Before proving Corollary 1.0.4 we need a preliminary lemma.

Lemma 4.2.1 *If B is a finite set of odd positive integers then $1 \notin 2C(B)$.*

Proof. Let $m = \max B$ and assume that $1 \in 2C(B)$. Then $1 \in S$, where

$$S = \left[\langle a_1, a_2, \overline{m,1}\rangle, \langle a_1, a_2, \overline{1,m}\rangle\right] + \left[\langle b_1, b_2, \overline{m,1}\rangle, \langle b_1, b_2, \overline{1,m}\rangle\right]$$

for some odd a_1, a_2, b_1 and b_2 between 1 and m inclusive. Now if both a_1 and b_1 are greater than 1 then $1 \notin S$, so we may assume without loss of generality that $a_1 = 1$. Thus

$$S = \left[\frac{a_2+\theta}{a_2+\theta+1} + \frac{b_2+\theta}{b_1(b_2+\theta)+1}, \frac{a_2+\rho}{a_2+\rho+1} + \frac{b_2+\rho}{b_1(b_2+\rho)+1}\right]$$

where $\theta = \langle\overline{m,1}\rangle$ and $\rho = \langle\overline{1,m}\rangle$. Therefore we have

$$1 \geq \frac{a_2+\theta}{a_2+\theta+1} + \frac{b_2+\theta}{b_1(b_2+\theta)+1} \tag{4.5}$$

and

$$1 \leq \frac{a_2 + \rho}{a_2 + \rho + 1} + \frac{b_2 + \rho}{b_1(b_2 + \rho) + 1}. \tag{4.6}$$

It can be shown that (4.5) and (4.6) are equivalent to

$$a_2 + 1 - b_1 \leq \frac{1}{b_2 + \theta} - \theta \tag{4.7}$$

and

$$a_2 + 1 - b_1 \geq \frac{1}{b_2 + \rho} - \rho \tag{4.8}$$

respectively. But for any integer $n \geq 1$ and real $x \in (0, 1)$ we have

$$-1 < \frac{1}{n + x} - x < 1$$

and so by (4.7) and (4.8) we must have $a_2 + 1 - b_1 = 0$. But this is not possible, since both a_2 and b_1 are odd, and the lemma follows.

□

Proof of Corollary 1.0.4. We find that $\tau(B_o) = 1$, and so by part 1 of Theorem 1.0.2,

$$F(B_o) + F(B_o) = \mathbb{R}.$$

Now if B is a finite set of positive odd integers then $0 \notin 2C(B)$ and $2 \notin 2C(B)$. By Lemma 4.2.1 we have $1 \notin 2C(B)$, whence

$$1 \notin \mathbb{Z} + 2C(B) = F(B) + F(B)$$

and the result follows.

□

Chapter 5

Products of Cantor Sets

5.1 General Result

Recall from Section 2.3 that for $E \subseteq \mathbb{R}^+$ we define E^* by

$$E^* = \{\log x;\ x \in E\}.$$

We have the following multiplicative analogue of Theorem 3.1.3.

Theorem 5.1.1 *Let k be a positive integer and for $j = 1, 2, \ldots, k$ let C_j be a Cantor set derived from $I_j \subseteq (0, \infty)$, with O_j a gap in C_j chosen so that $|O_j^*|$ is maximal. Put $S_\gamma = \gamma(C_1^*) + \cdots + \gamma(C_k^*)$.*

1. *If $S_\gamma \geq 1$ then $C_1 \cdots C_k$ contains an interval.*

2. *If $S_\gamma < 1$ then*
$$\dim_H(C_1 \cdots C_k) \geq \frac{\log 2}{\log\left(1 + \frac{1}{S_\gamma}\right)}.$$

3. If

$$|I^*_{r+1}| \geq |O^*_j| \qquad \text{for } r = 1, \ldots, k-1 \text{ and } j = 1, \ldots, r,$$
$$|I^*_1| + \cdots + |I^*_r| \geq |O^*_{r+1}| \qquad \text{for } r = 1, \ldots, k-1$$

and $S_\gamma \geq 1$ then

$$C_1 \cdots C_k = I_1 \cdots I_k.$$

Proof. Note that by Lemma 2.2.2,

$$\dim_H(C_1 \cdot\dots\cdot C_k) = \dim_H(C^*_1 + \dots + C^*_k).$$

We apply Theorem 3.1.3 to the Cantor sets $C^*_1, \ldots, C^*_k$ and the theorem follows.

□

5.2 Bounding $\tau(C^*)$

Before we can use Theorem 5.1.1 we must find a bound for $\tau(C^*)$. We start by generalizing a lemma of Cusick ([2, Lemma 2]). For real x and positive integers n we put

$$\binom{x}{n} = \frac{x(x-1)\cdots(x-n+1)}{n!}.$$

Lemma 5.2.1 *Let $E = [a, b] \subseteq (0, \infty)$ be an interval of real numbers. Suppose that $E = E_1 \cup O \cup E_2$ where*

$$E_1 = [a, a+r], \quad O = (a+r, a+r+s) \quad \text{and} \quad E_2 = [a+r+s, a+r+s+t].$$

If $s < a + r$ and $\tau > 0$ is a real number such that

$$\frac{t - \tau s}{s^2} \geq \frac{1}{a + r}\left(1 + \sum_{2 \leq n < \tau + 1} \binom{\tau + 1}{n}\right)$$

then

$$\frac{|E_2^*|}{|O^*|} \geq \tau.$$

Proof. We have $|E_2^*| \geq \tau |O^*|$ if and only if

$$\log(a + r + s + t) - \log(a + r + s) \geq \tau(\log(a + r + s) - \log(a + r))$$

which is equivalent to

$$(a + r + s + t)(a + r)^{\tau} \geq (a + r + s)^{\tau + 1}$$

or alternatively,

$$1 + \frac{s + t}{a + r} \geq \left(1 + \frac{s}{a + r}\right)^{\tau + 1}. \tag{5.1}$$

Using the power series expansion for $(1 + x)^y$ for real y and $|x| < 1$ we find that (5.1) is equivalent to

$$\frac{t - \tau s}{a + r} \geq \sum_{n=2}^{\infty} \binom{\tau + 1}{n} \left(\frac{s}{a + r}\right)^n. \tag{5.2}$$

Let R be the unique positive integer such that $R \leq \tau + 1 < R + 1$, and for $n \geq 2$ let C_n denote the binomial coefficient in (5.2). If $n > \tau + 1$ then

$$C_n = \binom{\tau + 1}{n} = \frac{\tau + 1}{n} \cdot \frac{\tau}{n - 1} \cdots \frac{\tau + 1 - R}{n - R} \cdot \frac{\tau - R}{n - R - 1} \cdots \frac{\tau + 2 - n}{1}.$$

Observe that $(|C_n|)_{n \geq \tau + 1}$ is a non-increasing sequence. Further, $C_n C_{n+1} \leq 0$ for $n \geq \tau + 1$. Therefore

$$\left| \sum_{\substack{\tau + 1 \leq n \\ 2 \leq n}} \binom{\tau + 1}{n} \left(\frac{s}{a + r}\right)^n \right| \leq \left(\frac{s}{a + r}\right)^2. \tag{5.3}$$

The lemma follows from (5.2) and (5.3).

□

As a corollary we may find a bound for $\tau(C^*)$.

Corollary 5.2.2 *Let $C \subseteq \mathbb{R}^+$ be a Cantor set derived by $\mathcal{D}$ and let τ be a real number which is at most $\tau(C)$. Assume that for all bridges $A = [a,b]$ of $\mathcal{D}$ with A^e to the right of A^d $(d, e \in \{0,1\})$ for which*

$$\frac{|A^e| - \tau|O_A|}{|O_A|^2} < \frac{1}{a}\left(1 + \sum_{2 \leq n < \tau+1} \binom{\tau+1}{n}\right) \tag{5.4}$$

we have

$$\frac{|A^{e*}|}{|O_A^*|} \geq \tau. \tag{5.5}$$

Then

$$\tau(C^*) \geq \tau.$$

Proof. Let $A = A^0 \cup O_A \cup A^1$ be a bridge of $\mathcal{D}$ with A^d to the left of A^e, for $(d,e) = (1,0)$ or $(d,e) = (0,1)$. Since the logarithm function has decreasing slope it follows that

$$\frac{|A^{d*}|}{|O_A^*|} \geq \frac{|A^d|}{|O_A|} \geq \tau.$$

If (5.4) does not hold then (5.5) holds by Lemma 5.2.1, so in any case

$$\min\left\{\frac{|A^{0*}|}{|O_A^*|}, \frac{|A^{1*}|}{|O_A^*|}\right\} \geq \tau$$

as required.

□

Chapter 6

Products of Continued Fractions

6.1 Bounding the Thickness of $(n \pm C(B))^*$

We may use Corollary 5.2.2 to find a bound for $\tau((n \pm C(B))^*)$ for n sufficiently large.

Lemma 6.1.1 *Let B be a set of positive integers with $|B| > 1$.*

1. *There exists $M_1 \in \mathbb{Z}^+$ such that*

$$\tau(C(n; B)^*) \geq \tau(B)$$

for all $n \geq M_1$.

2. *If τ is a real number with $\tau < \tau(B)$ then there exists $M_2 \in \mathbb{Z}^+$ such that*

$$\tau((n - C(B))^*) \geq \tau$$

for all $n \geq M_2$.

3. If $|B| = \infty$ and $\Delta_i(B) = \Delta$ is constant then there exists $M_3 \in \mathbb{Z}^+$ such that

$$\tau((n - C(B))^*) \geq \tau(B)$$

for all $n \geq M_3$.

Proof. For positive real numbers x we define

$$h(x) = 1 + \sum_{2 \leq t < x+1} \binom{x+1}{t}. \tag{6.1}$$

Since the lemma holds trivially if $\tau(B) = 0$ or $\tau(B) = \infty$ we may assume that $0 < \tau(B) < \infty$. We first prove part 2. Assume that $\tau < \tau(B)$, say $\tau = \tau(B) - \eta$ where $\eta > 0$. Choose an integer M_2 such that

$$M_2 > \frac{h(\tau)}{\eta} + 1.$$

Let $n \geq M_2$ be an integer and let $\mathcal{D}$ be the canonical derivation of $n - C(B)$. If A is any bridge of $\mathcal{D}$ then $A \subseteq [n-1, \infty)$ and

$$\frac{|A^e|}{|O_A|} - \tau \geq \tau(B) - \tau = \eta$$

for $e = 0, 1$. Thus

$$\frac{|A^e| - \tau|O_A|}{|O_A|^2} \geq \frac{\eta}{|O_A|} > \eta > \frac{h(\tau)}{M_2 - 1} \geq \frac{h(\tau)}{n-1}$$

and so (5.4) never holds. Therefore, by Corollary 5.2.2

$$\tau_{\mathcal{D}^*}(A^*) \geq \tau$$

and part 2 of the lemma follows.

We next prove part 3. Let M_3 and n be integers with $M_3 \geq \Delta h(\tau(B)) + 1$ and $n \geq M_3$. Let $\mathcal{D}$ be the canonical derivation of $n - C(B)$. To bound the quantity

$$\frac{|A^e| - \tau|O_A|}{|O_A|^2}$$

for all bridges A of $\mathcal{D}$ we need only compute the bound for all bridges of $\mathcal{D}(B)$. Let A be a bridge of type (2.3). From (4.2) we have

$$\frac{|A^0|}{|O_A|} - \tau(B) = \frac{1}{\Delta l - 1}\left(\frac{b_{i+1}+Q}{b_i+Q} - 1\right) = \frac{\Delta}{(\Delta l - 1)(b_i + Q)}$$

By Lemma 2.1.3 part 1 with $g_1 = [b_{i+1}]$ and $g_2 = [b_i, l]$ we have

$$|O_A| = \langle a_1 \ldots, a_r, g_1\rangle - \langle a_1 \ldots, a_r, g_2\rangle = \frac{\Delta - 1/l}{q_r^2(b_{i+1}+Q)(b_i + 1/l + Q)}$$

Therefore

$$\begin{aligned}\frac{|A^0| - \tau|O_A|}{|O_A|^2} &= \frac{\Delta}{(\Delta l - 1)(b_i+Q)} \cdot \frac{q_r^2(b_{i+1}+Q)(b_i+1/l+Q)}{\Delta - 1/l}\\ &> \frac{q_r^2 b_{i+1}}{\Delta l} \geq \frac{1}{\Delta} > \frac{h(\tau(B))}{n-1}.\end{aligned}$$

Similarly we find that

$$\frac{|A^1| - \tau|O_A|}{|O_A|^2} > \frac{q_r^2 b_i^2 b_{i+1}}{\Delta^2} > \frac{1}{\Delta} > \frac{h(\tau(B))}{n-1}$$

and part 3 of the lemma follows from Corollary 5.2.2.

We now prove part 1. Assume first that B is finite and put

$$T = \frac{\delta(m-l)}{\max\{\Delta_i\}lm - \delta(m-l)}.$$

Let M_1 and n be integers with $M_1 \geq (m+2)^2 h(\tau(B))/T$ and $n \geq M_1$. If A is a bridge of $\mathcal{D}(B)$ of the form (2.2) then from (4.1) we have

$$\begin{aligned}\frac{|A^0|}{|O_A|} - \tau &\geq \frac{\delta(m-l)}{\Delta_i lm - \delta(m-l)}\left(\frac{(b_{i+1}+Q)m+\delta}{(b_i+Q)m+\delta} - \frac{(b_{i+1}+1/l)m+\delta}{(b_i+1/l)m+\delta}\right)\\ &\geq T \cdot \frac{m^2\Delta_i(1/l - Q)}{((b_i+Q)m+\delta)((b_i+1/l)m+\delta)}\\ &\geq T \cdot \frac{m^2\Delta_i q_{r-2}}{lq_r((b_i+Q)m+\delta)((b_i+1/l)m+\delta)}\end{aligned}$$

since $q_r \geq lq_{r-1} + q_{r-2}$. From Lemma 2.1.3 part 1 with $g_1 = [b_{i+1}, \overline{m, l}]$ and $g_2 = [b_i, \overline{l, m}]$ we have

$$|O_A| = \frac{\Delta_i + \delta/m - \delta/l}{q_r^2(b_{i+1} + \delta/m + Q)(b_i + \delta/l + Q)}$$

whence if $r \neq 1$ then

$$\frac{|A^0| - \tau|O_A|}{|O_A|^2} > T\frac{q_{r-2}q_r}{l} > \frac{h(\tau(B))}{n}. \tag{6.2}$$

Similarly we have

$$\frac{|A^1| - \tau|O_A|}{|O_A|^2} > T\frac{q_{r-1}q_r b_i b_{i+1}}{\Delta_i(m+2)^2} > T\frac{q_{r-1}q_r b_i}{(m+2)^2} \geq \frac{h(\tau(B))}{n} \tag{6.3}$$

if $r \neq 0$.

Now if $r = 1$ then in $C(n; B)$ we have A^1 on the right side of A^0, and if $r = 0$ then A^0 is to the right of A^1. Thus by Corollary 5.2.2, part 1 of the lemma follows for B finite.

If B is infinite then we take M_1 greater than $(\max \Delta_i)l^2 h(\tau(B))$ and use an analogous argument to the above to establish our result.

□

6.2 Improving Cusick's Product Result

Proof of Theorem 1.0.10. Let S_γ denote the number

$$S_\gamma = \sum_{j=1}^{k} \frac{1}{l_j}.$$

If $l_j = 1$ for any j then the theorem follows trivially, so we may assume that $l_j \geq 2$ for $j = 1, \ldots, k$.

First assume that $k = 2$, then $l_1 = l_2 = 2$. For $M \geq 2$ sufficiently large we have by Lemma 6.1.1 part 1 that

$$\tau(C(n; U_2)^*) \geq \tau(U_2) = 1$$

for $n \geq M$. For positive integers $N > M$ we define C^N and I^N by

$$C^N = \left[N + \frac{1}{2}, N + 1\right] \cup \bigcup_{n=M}^{N} C(n; U_2)$$

and

$$I^N = [M, N + 1].$$

We may construct a derivation $\mathcal{D}^N$ of C^N from I^N in a similar manner to that used in the proof of Theorem 1.0.6, the only difference being that the rightmost interval in every level of the tree contains the closed interval $[N + 1/2, N + 1]$, so that

$$\tau((C^N)^*) \geq \tau(U_2) = 1.$$

Thus $(C^N)^{-1}$ and $C(M; U_2)$ satisfy the requirements of Theorem 5.1.1 part 3, so

$$\frac{C(M; U_2)}{C^N} = \frac{[M, M + 1/2]}{[M, N + 1]} = \left[\frac{M}{N + 1}, 1 + \frac{1}{2M}\right]$$

therefore

$$\frac{C(M; U_2)}{\bigcup_{n=M}^{N} C(n; U_2)} \supseteq \left[\frac{2M + 1}{2N + 1}, 1\right].$$

But

$$C(0; U_2) = \frac{1}{\bigcup_{n=2}^{\infty} C(n; U_2)}$$

whence

$$C(M; U_2) \cdot C(0; U_2) \supseteq \left[\frac{2M + 1}{2N + 1}, 1\right].$$

This holds for every $N > M$, thus

$$C(M; U_2) \cdot C(0; U_2) \supseteq (0, 1]. \tag{6.4}$$

By taking reciprocals we have

$$C(M; U_2)^{-1} \cdot C(0; U_2)^{-1} \supseteq [1, \infty).$$

However

$$C(M; U_2)^{-1} \subseteq C(0; U_2) \quad \text{and} \quad C(0; U_2)^{-1} \subseteq G(2)$$

so

$$C(0; U_2) \cdot G(2) \supseteq [1, \infty). \tag{6.5}$$

By (6.4) and (6.5) we have

$$(0, \infty) \subseteq G(2) \cdot G(2). \tag{6.6}$$

By part 3 of Lemma 6.1.1 we may extend our results to the negative real axis by replacing the set $C(M; U_2)$ by $C(-M; U_2)$, so that

$$(-\infty, 0) \subseteq G(2) \cdot G(2). \tag{6.7}$$

Since $0 \in G(2)$, we have by (6.6) and (6.7) that

$$G(2) \cdot G(2) = \mathbb{R} \tag{6.8}$$

as required.

Now assume that $k > 2$. To prove the theorem we will use an approach similar to that used to establish (6.8). Without loss of generality we may assume that

$$l_1 \geq l_2 \geq \cdots \geq l_k.$$

Now, for M sufficiently large, for all $n \geq M$ and all $j = 1, \ldots, k$ the largest gap in $C(n; U_{l_j})^*$ is $(O_j^n)^*$, where O_j^n is the largest gap in $C(n; U_{l_j})$, namely

$$O_j^n = ([n, l_j + 1], [n, l_j, l_j]) .$$

By Lemma 6.1.1 part 1 there exists a positive integer $M_1 \geq M$ such that

$$M_1 > \max\left\{l_1, \frac{l_{k-1}^2}{l_k}\right\}, \qquad (l_2 l_3 \cdots l_k) \text{ divides } M_1$$

and

$$\tau(C(n; U_{l_j})^*) \geq \tau(U_{l_j}) \tag{6.9}$$

for all $n \geq M_1$ and all $j = 1, \ldots, k$. For $j = 2, \ldots, k$ we set

$$M_j = \frac{l_{j-1}^2}{l_j} M_{j-1}. \tag{6.10}$$

By our choice of $M_1, \ldots, M_k$ we have, by calculation,

$$|(O_j^{M_j})^*| \leq |I(M_{j+1}; U_{j+1})^*| \leq |I(M_j; U_{l_j})^*| \tag{6.11}$$

for $j = 1, \ldots, k-1$. For integers $N > 2M_k$ we define C_k^N and I_k^N by

$$C_k^N = \left[N + \frac{1}{l_k}, N + 1\right] \cup \bigcup_{n=M_k}^{N} C(n; U_{l_k}) \quad \text{and} \quad I_k^N = [M_k, N+1]. \tag{6.12}$$

Now the largest gap in $(C_k^N)^*$ is either $(O_k^{M_k})^*$ or $(M_k + 1/l_k, M_k + 1)^*$. Since $l_k \leq l_{k-1}$ and $k \geq 2$ we have

$$\frac{l_k}{l_1 M_1 (l_1 \cdots l_{k-1})} \leq \frac{1}{l_1 M_1}. \tag{6.13}$$

Also, by (6.10) it follows that

$$M_j = \frac{l_1 M_1}{l_j^2} (l_1 \cdots l_j) \tag{6.14}$$

for $j = 1, \ldots, k$, whence with (6.13) we find

$$\frac{1}{M_k} \leq \frac{1}{l_1 M_1}$$

thus

$$\log\left(\frac{M_k + 1}{M_k + 1/l_k}\right) \leq \log\left(\frac{M_1 + 1/l_1}{M_1}\right).$$

Equivalently,

$$|(M_k + 1/l_k, M_k + 1)^*| \leq |I(M_1; U_{l_1})^*|. \tag{6.15}$$

As in the proof of the $k = 2$ case we have

$$\tau((C_k^N)^*) \geq \tau(U_{l_k}). \tag{6.16}$$

By (6.9), (6.11), (6.15), (6.16) and Theorem 5.1.1 part 3,

$$\frac{C(M_1; U_{l_1}) \cdots C(M_{k-1}; U_{l_{k-1}})}{C_k^N} = \left[\frac{M_1 \cdots M_{k-1}}{N+1}, \frac{(M_1 + 1/l_1) \cdots (M_{k-1} + 1/l_{k-1})}{M_k}\right].$$

Thus

$$\frac{C(M_1; U_{l_1}) \cdots C(M_{k-1}; U_{l_{k-1}})}{\bigcup_{n=M_k}^{N} C(n; U_{l_k})} \supseteq \left[\frac{(M_1 + 1/l_1) \cdots (M_{k-1} + 1/l_{k-1})}{N + 1/l_k}, \frac{M_1 \cdots M_{k-1}}{M_k}\right]. \tag{6.17}$$

Since $M_1 > l_{k-1}^2 / l_k$ and $k > 2$ we have by (6.14) that

$$\frac{M_1 \cdots M_{k-1}}{M_k} > 1. \tag{6.18}$$

Also, since $M_k \geq l_k$,

$$\left(\bigcup_{n=M_k}^{N} C(n; U_{l_k})\right)^{-1} \subseteq C(0; U_{l_k}) \tag{6.19}$$

so by (6.17), (6.18) and (6.19), upon taking the limit as N approaches infinity we have

$$(0,1] \subseteq G(l_1)\cdots G(l_k). \tag{6.20}$$

Since $M_j \geq l_j$ for $j = 1,\ldots,k-1$ we may take reciprocals in (6.17) and let N tend to infinity, so that

$$[1,\infty) \subseteq G(l_1)\cdots G(l_k).$$

With (6.20) we have

$$(0,\infty) \subseteq G(l_1)\cdots G(l_k).$$

As before we may extend our results to the negative reals, finding

$$(-\infty,0) \subseteq G(l_1)\cdots G(l_k).$$

Since $0 \in G(l_1)$ our result follows.

□

6.3 Products of Continued Fractions

Proof of Theorem 1.0.7. We may assume without loss of generality that $0 < \tau(B_j) < \infty$ for $1 \leq j \leq k$. Assume first that $S_\tau > 1$. Let η be the positive real number

$$\eta = \frac{S_\tau - 1}{k}.$$

We will follow a similar approach to that used in the proof of Theorem 1.0.6. For $j = 1,\ldots,k$, by Lemma 6.1.1 part 1 there exists a positive integer M_j such that for $n \geq M_j$,

$$\tau(C(n;B_j)^*) \geq \tau(B_j).$$

Let $M = \max M_j$. We define $\tilde{C}_j^N$ and $\tilde{I}_j^N$ for $N \geq M$ by

$$\tilde{C}_j^N = \bigcup_{n=M}^{N} C(n; B_j)$$

and

$$\tilde{I}_j^N = [M + \min C(B_j), N + \max C(B_j)].$$

Our definition of $\tilde{\mathcal{D}}_j^N$ is analogous to that for $\mathcal{D}_j^N$ in the proof of Theorem 1.0.6. Note that since the logarithm function is not linear, in the notation of (4.4) we may have

$$\frac{|(A^{1\cdots 11})^*|}{|(O_{A^{1\cdots 1}})^*|} < \tau(B_j). \tag{6.21}$$

However, if we set

$$\eta' = \frac{\eta(\tau(B_j) + 1)^2}{1 + \eta(\tau(B_j) + 1)}$$

and take N sufficiently large, then

$$\frac{|(A^{1\cdots 11})^*|}{|(O_{A^{1\cdots 1}})^*|} > \tau(B_j) - \eta'.$$

Thus

$$\tau((\tilde{C}_j^N)^*) > \tau(B_j) - \eta'$$

so

$$\gamma((\tilde{C}_j^N)^*) > \gamma(B_j) - \eta,$$

whence it follows that

$$\gamma((\tilde{C}_1^N)^*) + \cdots + \gamma((\tilde{C}_k^N)^*) > S_\gamma - k\eta = 1.$$

Now

$$\gamma(((\tilde{C}_j^N)^{-1})^*) = \gamma((\tilde{C}_j^N)^*)$$

so if $N > M + 1$ then all the conditions of Theorem 5.1.1 part 3 are satisfied and we find that

$$\begin{aligned}(\tilde{C}_1^N)^{\epsilon_1} \cdots (\tilde{C}_k^N)^{\epsilon_k} &= (\tilde{I}_1^N)^{\epsilon_1} \cdots (\tilde{I}_k^N)^{\epsilon_k} \\ &\supseteq \left[\frac{(M+1)^{S_\epsilon^+}}{N^{S_\epsilon^-}}, \frac{N^{S_\epsilon^+}}{(M+1)^{S_\epsilon^-}}\right] \end{aligned} \tag{6.22}$$

where $S_\epsilon^+ = |\{j;\ \epsilon_j = 1\}|$ and $S_\epsilon^- = k - S_\epsilon^+$. We let N tend to infinity in (6.22) and find that if $S_\epsilon = k$ then

$$\left[(M+1)^k, \infty\right) \subseteq F \tag{6.23}$$

and if $|S_\epsilon| < k$ then

$$(0, \infty) \subseteq F. \tag{6.24}$$

To extend these results to the negative axis we consider the set

$$\tilde{C}_1^{N-} = \bigcup_{n=M}^{N} (n - C(B_1))$$

for $N > M + 1$. By Lemma 6.1.1 part 2 we find that for M and N sufficiently large,

$$\tau((\tilde{C}_1^{N-})^*) > \tau(B_1) - \eta'.$$

As before we have by part 3 of Theorem 5.1.1 that

$$\left[\frac{(M+1)^{S_\epsilon^+}}{(N-1)^{S_\epsilon^-}}, \frac{(N-1)^{S_\epsilon^+}}{(M+1)^{S_\epsilon^-}}\right] \subseteq (\tilde{C}_1^{N-})^{\epsilon_1} (\tilde{C}_2^N)^{\epsilon_2} \cdots (\tilde{C}_k^N)^{\epsilon_k}.$$

However,

$$n - C(B_1) = -(-n + C(B_1)) \subseteq -F(B_1)$$

for every n, whence

$$\left[\frac{(M+1)^{S_\epsilon^+}}{(N-1)^{S_\epsilon^-}}, \frac{(N-1)^{S_\epsilon^+}}{(M+1)^{S_\epsilon^-}}\right] \subseteq -F.$$

Taking the limit as N approaches infinity we find that

$$(-\infty, -(M+1)^k] \subseteq F \tag{6.25}$$

if $S_\epsilon = k$, and

$$(-\infty, 0) \subseteq F \tag{6.26}$$

if $|S_\epsilon| < k$. Part 1 of the theorem follows from (6.23) and (6.25), while part 2 is a consequence of (6.24) and (6.26).

We now assume that $S_\gamma > 1$ and $|B_r| = \infty$ for some r with $\epsilon_r = 1$. Now,

$$C(B_r) = \bigcup_{b_i \in B_r} \frac{1}{C(b_i; B_r)} = \frac{1}{\bigcup_{b_i \in B_r} C(b_i; B_r)}.$$

This is similar to the case $S_\gamma > 1$ and $|S_\epsilon| < k$, where instead of dividing by the set

$$\bigcup_{n=M}^{N} C(n; B_r)$$

we are dividing by

$$C' = \bigcup_{\substack{M \leq n \leq N \\ n \in B_r}} C(n; B_r).$$

Notice that

$$\frac{|[b_{i+1}, b_{i+1} + 1/l_r]|}{|(b_i + 1/l_r, b_{i+1})|} = \frac{1}{\Delta_i l_r - 1}. \tag{6.27}$$

Choose M sufficiently large so that $\Delta(M) = \max_{j \geq s} \Delta_j(B_r)$ occurs infinitely often in the sequence $(\Delta_1(B_r), \Delta_2(B_r), \dots)$, where $s = s(M)$ is the unique positive integer with $b_{s-1} < M \leq b_s$. Then by (1.4),

$$\tau(B_r) \leq \frac{1}{\Delta(M) l_r - 1}$$

so by (6.27) $\tau((C')^*) \geq \tau(B_r) - \eta'$ for N sufficiently large. We proceed in a similar manner as in the proof of part 2 of the theorem and find that

$$(-\infty, 0) \cup (0, \infty) \subseteq F.$$

However, $0 \in F(B_r)$ and part 3 of the theorem follows.

Now assume that $S_\gamma = 1$. We will first examine the sets $\tilde{C}_j^N$ in more detail. Assume that for some j, B_j is not equal to U_l for any $l \geq 1$. We claim that for M and N sufficiently large,

$$\tau((\tilde{C}_j^N)^*) \geq \tau(B_j).$$

To see this we first take M so that by Lemma 6.1.1 part 1,

$$\tau(C(n; B_j)^*) \geq \tau(B_j)$$

for $n \geq M$. Assume that $|B_j| < \infty$, say $B_j = \{b_1, \ldots, b_t\}$ with $l = b_1 < b_2 < \cdots < b_t = m$. Then, in the notation of (6.21),

$$\begin{aligned} \frac{|(A^{1\cdots 11})|}{|(O^{1\cdots 1})|} &= \frac{\langle \overline{l,m} \rangle - \langle \overline{m,l} \rangle}{1 - \langle \overline{l,m} \rangle + \langle \overline{m,l} \rangle} \\ &= \frac{\delta(m-l)}{lm - \delta(m-l)} \\ &> \frac{\delta(m-l)}{lm - \delta(m-l)} \cdot \frac{b_{t-1}l + \delta}{ml + \delta} \geq \tau(B_j) \end{aligned}$$

by Lemma 4.1.1. Hence for N sufficiently large

$$\tau((\tilde{C}_j^N)^*) \geq \tau(B_j).$$

Now assume that $|B_j| = \infty$ but $\Delta_i > 0$ for some i. Then by Lemma 4.1.2

$$\begin{aligned} \tau(B_j) &\leq \frac{1}{\Delta_i l - 1} \cdot \frac{(b_i + \Delta_i)l + 1}{b_i l + 1} < \frac{1}{\Delta_i l - 1} \cdot \frac{(b_i + \Delta_i)}{b_i} \\ &\leq \frac{1}{\Delta_i l - 1} + \frac{\Delta_i}{l(\Delta_i l - 1)} \leq \frac{1}{2l-1} + \frac{2}{l(2l-1)} \\ &= \frac{l+2}{l(2l-1)} < \frac{1}{l-1} = \frac{|(A^{1\cdots 11})|}{|(O^{1\cdots 1})|} \end{aligned}$$

since $(l - 1)(l + 2) < l(2l - 1)$ for $l \geq 1$, and our claim follows.

If $B_l \neq U_l$ for all $l \geq 1$ and $1 \leq j \leq k$ then by part 3 of Lemma 5.1.1, if $S_\epsilon = k$ then

$$[M^k, N^k] \subseteq \tilde{C}_1^N \dots \tilde{C}_k^N$$

while if $|S_\epsilon| < k$ then

$$\left[\frac{M^{k-1}}{N}, \frac{N}{M^{k-1}}\right] \subseteq (\tilde{C}_1^N)^{\epsilon_1} \dots (\tilde{C}_k^N)^{\epsilon_k}$$

and so letting N approach infinity we find that

$$[M^k, \infty) \subseteq F \quad \text{or} \quad (0, \infty) \subseteq F$$

as required.

If for all j we have $B_j = U_{l_j}$ for some l_j then, since

$$C(N; U_l)^{-1} \subseteq C(0; U_l)$$

for $N \geq l$ we may assume that $S_\epsilon = k$, and our results follow from Theorem 1.0.10.

We now assume that for some t in the range $1 \leq t < k$ we have $B_j = U_{l_j}$ for $j = 1, \dots, t$, $l_1 \geq \dots \geq l_t$ and $B_j \neq U_l$ for any $j > t$ and $l \geq 1$. As in the proof of Theorem 1.0.10 we may choose M_j, $j = 1, \dots, t$ such that, in the notation of Theorem 1.0.10,

$$|(O_j^{M_j})^*| \leq |I(M_{j+1}; U_{j+1})^*| \leq |I(M_j; U_{l_j})^*|$$

For N large define C_t^N and I_t^N by (6.12), substituting t for k. Without loss of generality we may assume that $\epsilon_k = 1$ (if not, consider the set F^{-1}). If we choose M sufficiently large so that $|I(M_j; U_{l_j})^*| > |[M, M+1]^*|$ for $j = 1, \dots, t$ then by Lemma 5.1.1 we have

$$\frac{C(M_1; U_{l_1})^{\epsilon_1} \dots C(M_{t-1}; U_{l_{t-1}})^{\epsilon_1} \cdot (\tilde{C}_{t+1}^N)^{\epsilon_{t+1}} \dots (\tilde{C}_k^N)^{\epsilon_k}}{C_t^{N^k}} \supseteq \left[\frac{M^{k-1}}{N^k + 1}, \frac{N}{M^{k-1}}\right]$$

since $M > M_j$ for $j = 1, \ldots, t$. Thus

$$\frac{C(M_1; U_{l_1})^{\epsilon_1} \ldots C(M_{t-1}; U_{l_{t-1}})^{\epsilon_1} \cdot (\tilde{C}_{t+1}^N)^{\epsilon_{t+1}} \ldots (\tilde{C}_k^N)^{\epsilon_k}}{\bigcup_{n=M_k}^{N^k} C(n; U_{l_t})} \supseteq \left[\frac{N^{k-1}}{N^k}, \frac{N}{M^{k-1}}\right] = \left[\frac{1}{N}, \frac{N}{M^{k-1}}\right]$$

and parts 4 and 5 of the theorem follow upon letting N tend to infinity, since

$$\bigcup_{n=M_k}^{N^k} C(n; U_{l_t}) \subseteq C(0; U_{l_t}).$$

To prove part 6 of the theorem we extend our results to the negative reals by using part 3 of Lemma 6.1.1, as was done in the proof of Theorem 1.0.10.

Finally, if $S_\gamma < 1$ then by Lemma 6.1.1 part 1 we have

$$\tau(C(M; B_j)^*) \geq \tau(B_j)$$

for $j = 1, \ldots, k$ and M sufficiently large, whence by Theorem 5.1.1 part 2,

$$\dim_H(C(M; B_1)^{\epsilon_1} \cdots C(M; B_k)^{\epsilon_k}) \geq \frac{\log 2}{\log\left(1 + \frac{1}{S_\gamma}\right)}$$

and the theorem follows.

□

Our methods of proving Theorems 1.0.10, 1.0.7, 1.0.8 and 1.0.9 differ from that employed by Hall in [8]. He covers part of the real line by intervals of the form

$$I(n; L_4) \cdot I(n; L_4) \qquad \text{or} \qquad I(n; L_4) \cdot I(n+1; L_4)$$

and then shows that

$$C(n; L_4) \cdot C(n; L_4) = I(n; L_4) \cdot I(n, L_4)$$

and

$$C(n; L_4) \cdot C(n+1; L_4) = I(n; L_4) \cdot I(n+1, L_4).$$

6.4 Products of Sets of the Form $F(m)$

Before we prove Theorems 1.0.8 and 1.0.9 we will prove several preliminary lemmas.

Lemma 6.4.1 *If* $0 < m \leq n$ *then*

$$(1 - \langle\overline{1,m}\rangle)\langle\overline{n,1}\rangle \leq (1 - \langle\overline{1,n}\rangle)\langle\overline{m,1}\rangle.$$

Proof. For positive real x put

$$f(x) = \frac{x(1 - \langle\overline{1,x}\rangle)}{\langle\overline{1,x}\rangle}$$

then, from Maple we have

$$f'(x) = \frac{x(2 + x - \sqrt{x(x+4)})^2}{(x - \sqrt{x(x+4)})^2\sqrt{x(x+4)}} > 0$$

for $x > 0$. Therefore

$$\frac{m(1 - \langle\overline{1,m}\rangle)}{\langle\overline{1,m}\rangle} \leq \frac{n(1 - \langle\overline{1,n}\rangle)}{\langle\overline{1,n}\rangle}$$

for $m \leq n$, and the lemma follows, since $\langle\overline{1,m}\rangle = m\langle\overline{m,1}\rangle$.

□

Lemma 6.4.2 *If* $m > 0$ *then* $1 - \langle\overline{1,m}\rangle < \langle\overline{m,1}\rangle$.

Proof. We have

$$\langle\overline{1,m}\rangle > \frac{m}{m+1} \quad \text{and} \quad \langle\overline{m,1}\rangle > \frac{1}{m+1}$$

hence

$$1-\langle\overline{1,m}\rangle < 1-\frac{m}{m+1} = \frac{1}{m+1} < \langle\overline{m,1}\rangle$$

as required.

□

For positive integers m and N we put

$$F^+(m) = F(m)\bigcap[0,\infty), \qquad C_N^+(m) = \bigcup_{n=0}^{N}(n+C(m)),$$

$$F^-(m) = F(m)\bigcap(-\infty,0) \quad \text{and} \quad C_N^-(m) = \bigcup_{n=1}^{N}(n-C(m)).$$

We have the following results.

Lemma 6.4.3 *If N is sufficiently large then*

$$\tau(C_N^+(2)^*) > 0.3660, \qquad \tau(C_N^-(2)^*) > 0.2999,$$

$$\tau(C_N^+(3)^*) > 0.8220, \qquad \tau(C_N^-(3)^*) > 0.7675,$$

$$\tau(C_N^+(4)^*) > 1.3009, \qquad \tau(C_N^-(4)^*) > 1.2557,$$

$$\tau(C_N^+(5)^*) > 1.7888, \qquad \tau(C_N^-(5)^*) > 1.7504,$$

$$\tau(C_N^+(6)^*) > 2.2811, \qquad \tau(C_N^-(6)^*) > 2.2477,$$

$$\tau(C_N^+(7)^*) > 2.7758, \qquad \tau(C_N^-(7)^*) > 2.7463,$$

$$\tau(C_N^+(8)^*) > 3.2719, \qquad \tau(C_N^-(8)^*) > 3.2456,$$

$$\tau(C_N^+(9)^*) > 3.7690 \quad \textit{and} \quad \tau(C_N^-(9)^*) > 3.7452.$$

Proof. We first note that since the second derivative of the logarithm function is decreasing, if N is large enough then (as in the proof of Theorem 1.0.7)

$$\frac{\log(N+\langle\overline{1,m}\rangle)-\log(N+\langle\overline{m,1}\rangle)}{\log(N+\langle\overline{m,1}\rangle)-\log(N-1+\langle\overline{1,m}\rangle)} > \tau(L_m)$$

and

$$\frac{\log(N-\langle\overline{m,1}\rangle)-\log(N-\langle\overline{1,m}\rangle)}{\log(N-\langle\overline{1,m}\rangle)-\log(N-1-\langle\overline{m,1}\rangle)} > \tau(L_m)$$

Thus it suffices to determine $\tau(C(L_m)^*)$ and $\tau((1-C(L_m))^*)$ for $2 \le m \le 9$. Choose m in the range $2 \le m \le 9$ and assume that $r > 1$, $d \in \{0,1\}$ and A is a bridge of $C(L_m)$ of the form (2.2). Then by (6.2) and (6.3) we have

$$\frac{|A^d|-\tau|O_A|}{|O_A|^2} > T\frac{q_{r-1}q_r}{(m+2)^2}$$

where

$$T = \frac{\langle\overline{1,m}\rangle(m-1)}{m-\langle\overline{1,m}\rangle(m-1)}.$$

We use a Maple program to determine

$$\min_A \tau(A^*) \quad \text{and} \quad \min_A \tau((1-A)^*)$$

where the minima are taken over all bridges A of $C(L_m)$ with

$$q_r q_{r-1} < \frac{(m+2)^2}{T}\cdot\frac{h(\tau(L_m))}{1-\langle\overline{1,m}\rangle} \quad \text{or} \quad r \le 1$$

where $h(x)$ is defined as in (6.1) (note that $1-\langle\overline{1,m}\rangle < \langle\overline{m,1}\rangle$ by Lemma 6.4.2). Our results follow from Corollary 5.2.2.

□

Lemma 6.4.4 *For $m \ge 9$ and N sufficiently large,*

$$\tau(C_N^+(m)^*) > 3.4 \quad \textit{and} \quad \tau(C_N^-(m)^*) > 3.4\,. \tag{6.28}$$

Proof. We will use induction on m. For $m = 9$, (6.28) follows from Lemma 6.4.3. Assume that (6.28) holds for $m < t$, for some integer $t \geq 10$. We must bound $\tau(C_N^+(t)^*)$ and $\tau(C_N^-(t)^*)$. As in the proof of Lemma 6.4.3 it suffices to bound $\tau(C(L_t)^*)$ and $\tau((1 - C(L_t))^*)$. Let

$$A = [[\langle a_1, \dots, a_r, k, \overline{t, 1}\rangle, \langle a_1, \dots, a_r, t, \overline{1, t}\rangle]]$$

be a bridge of $\mathcal{D}(L_t)$. If all a_i's are less than t and $k < t - 1$ then A contains some bridge B of $\mathcal{D}(L_{t-1})$ with $A^0 \supset B^0$ and $A^1 \supset B^1$, so (6.28) follows from the induction hypothesis. Therefore we may assume that either some $a_i = t$ or $k = t - 1$.

Now let D be either A or $1 - A$. Assume that $D^0 < D^1$ and put

$$D = [a, d] \qquad O_D = (b, c)$$
$$D^0 = [a, b] \qquad D^1 = [c, d]$$

for some a, b, c and d with $a < b < c < d$. We claim that

$$a > |D| = d - a. \tag{6.29}$$

To see this assume first that $k = t - 1$. Then the largest possible intervals for D are I_1, I_2, $1 - I_1$ and $1 - I_2$ where

$$I_1 = [\langle \overline{t, 1}\rangle, \langle t - 1, \overline{t, 1}\rangle] \quad \text{and} \quad I_2 = [\langle 1, t - 1, \overline{t, 1}\rangle, \langle 1, \overline{t, 1}\rangle].$$

Now

$$\frac{1}{t+1} < \langle \overline{t, 1}\rangle < \frac{1}{t} \quad \text{and} \quad \frac{t}{t+1} < \langle \overline{1, t}\rangle < \frac{t+1}{t+2}$$

hence

$$\begin{aligned} |I_1| = \langle t - 1, \overline{t, 1}\rangle - \langle \overline{t, 1}\rangle &< \frac{1}{t - 1 + \frac{1}{t+1}} - \frac{1}{t+1} \\ &= \frac{2t+1}{t^2(t+1)} < \frac{1}{t+1} < \langle \overline{t, 1}\rangle < 1 - \langle t - 1, \overline{t, 1}\rangle \end{aligned}$$

and

$$|I_2| = \langle 1, \overline{t, 1}\rangle - \langle 1, t-1, \overline{t, 1}\rangle < \frac{1}{1+\frac{1}{t+1}} - \frac{1}{1+\frac{1}{t-1+\frac{1}{t+1}}}$$
$$= \frac{2t+1}{(t+2)(t^2+t+1)} < \frac{1}{t+2} < 1 - \langle 1, \overline{t, 1,}\rangle < \langle 1, t-1, \overline{t, 1}\rangle.$$

Therefore if $k = t-1$ then (6.29) holds. Otherwise $a_i = t$ and so D is contained in a bridge which has $k = t-1$, and so again (6.29) holds.

Now we know that

$$\frac{|D^{0*}|}{|O_D^*|} > \tau(L_t) > 3.4$$

by the decreasing slope of the logarithm function and since $\tau(L_t) \geq \tau(L_{10}) = 4.266\ldots$. Also

$$\frac{|D^{1*}|}{|O_D^*|} = \frac{\log d - \log c}{\log c - \log b} = \frac{\log\left(1 + \frac{|D^1|}{c}\right)}{\log\left(1 + \frac{|O_D|}{b}\right)} > \frac{\frac{|D^1|}{c} - \frac{1}{2}\left(\frac{|D^1|}{c}\right)^2}{\frac{|O_D|}{b}}$$

by the power series expansion of the logarithm function, as $c > a > |D^1|$ and $b > a > |O_D|$. Therefore

$$\frac{|D^{1*}|}{|O_D^*|} > \frac{|D^1|}{|O_D|} \cdot \frac{b}{c}\left(1 - \frac{1}{2}\frac{|D^1|}{c}\right)$$
$$\geq \tau(L_t)\frac{a + \tau(L_t)|O_D|}{a + (\tau(L_t)+1)|O_D|}\left(1 - \frac{1}{2}\frac{\tau(L_t)|O_D|}{a + (\tau(L_t)+1)|O_D|}\right)$$

since

$$\frac{|D^1|}{|O_D|} \geq \tau(L_t).$$

But $a > |D| \geq (2\tau(L_t)+1)|O_D|$, hence

$$\frac{|D^{1*}|}{|O_D^*|} > \tau(L_t)\frac{3\tau(L_t)+1}{3\tau(L_t)+2}\left(1 - \frac{\tau(L_t)}{6\tau(L_t)+4}\right)$$
$$> 3.4$$

as $\tau(L_t) \geq 4.266\ldots$. Therefore

$$\tau(C_N^+(L_t)^*) > 3.4 \quad \text{and} \quad \tau(C_N^-(L_t)^*) > 3.4$$

for N sufficiently large, and the lemma follows by induction.

□

Lemma 6.4.5

$$F(3)F(4) = (-\infty, (1 - \langle\overline{1,3}\rangle)\langle\overline{4,1}\rangle] \cup [\langle\overline{3,1}\rangle\langle\overline{4,1}\rangle, \infty) \tag{6.30}$$

and

$$F(2)F(7) = (-\infty, (1 - \langle\overline{1,2}\rangle)\langle\overline{7,1}\rangle] \cup [\langle\overline{2,1}\rangle\langle\overline{7,1}\rangle, \infty). \tag{6.31}$$

Proof. By Lemma 6.4.3 we have

$$\tau(C_N^+(3)^*) \geq 0.8220\ , \qquad \tau(C_N^+(4)^*) \geq 1.3009$$

and

$$\tau(C_N^-(4)^*) \geq 1.2557.$$

Since $0.822 \times 1.255 > 1$ we find by an approach analogous to that used in the proof of part 1 of Theorem 1.0.7 that

$$F^+(3) \cdot F^+(4) = [\langle\overline{3,1}\rangle\langle\overline{4,1}\rangle,\ \infty) \tag{6.32}$$

and

$$F^+(3) \cdot F^-(4) = (-\infty,\ \langle\overline{3,1}\rangle(-1 + \langle\overline{1,4}\rangle)]. \tag{6.33}$$

Now put

$$I_3^- = [1-\langle 1,\overline{3,1}\rangle, 1-\langle 1,\overline{1,3}\rangle] = [0.2087\ldots, 0.4417\ldots],$$
$$I_4^+ = [\langle 4,\overline{1,4}\rangle, \langle 3,\overline{4,1}\rangle] = [0.2071\ldots, 0.3118\ldots],$$
$$I_4^- = [1-\langle 1,\overline{4,1}\rangle, 1-\langle 1,\overline{1,4}\rangle] = [0.1715\ldots, 0.4530\ldots],$$

$$C_3^- = C(L_3)\cap I_3^-, \quad C_4^+ = C(L_4)\cap I_4^+ \quad \text{and} \quad C_4^- = (1-C(L_4))\cap I_4^-.$$

Since $1-I_3^-$ is a bridge of $\mathcal{D}(L_3)$, we may use $\mathcal{D}(L_3)$ to construct a derivation of C_3^- from I_3^-. To bound $\tau((C_3^-)^*)$ we use the same process that was used to establish the results contained in Lemma 6.4.3. Specifically we find

$$\tau((C_3^-)^*) \geq \tau(L_3). \tag{6.34}$$

Similarly we have

$$\tau((C_4^-)^*) \geq \tau(L_4) \quad \text{and} \quad \tau((C_4^+)^*) \geq \tau(L_4). \tag{6.35}$$

The largest gap in $(C_3^-)^*$, $(C_4^+)^*$ and $(C_4^-)^*$ has width $0.1563\ldots$, $0.0943\ldots$ and $0.1256\ldots$ respectively. Further,

$$|(I_3^-)^*| = 0.7497\ldots, \quad |(I_4^+)^*| = 0.4091\ldots \quad \text{and} \quad |(I_4^-)^*| = 0.9710\ldots$$

so by (6.34), (6.35) and part 3 of Theorem 5.1.1,

$$C_3^-\cdot C_4^+ = I_3^-\cdot I_4^+ = [(1-\langle\overline{1,3}\rangle)\langle\overline{4,1}\rangle, (1-\langle 1,\overline{1,3}\rangle)\langle 3,\overline{4,1}\rangle] \tag{6.36}$$

and

$$C_3^-\cdot C_4^- = I_3^-\cdot I_4^- = [(1-\langle\overline{1,3}\rangle)(1-\langle\overline{1,4}\rangle), (1-\langle 1,\overline{1,3}\rangle)(1-\langle 1,\overline{1,4}\rangle)]. \tag{6.37}$$

Since

$$F(3)\cdot F(4) \subseteq \left(-\infty, (-1+\langle\overline{1,3}\rangle)\langle\overline{4,1}\rangle\right] \cup \left[(-1+\langle\overline{1,3}\rangle)(-1+\langle\overline{1,4}\rangle), \infty\right)$$

by Lemma 6.4.1 and Lemma 6.4.2, (6.30) follows from (6.32), (6.33), (6.36) and (6.37).

We now prove (6.31). By Lemma 6.4.3 we have

$$\tau(C_N^+(2)^*)\tau(C_N^-(7)^*) > 1$$

and

$$\tau(C_N^+(2)^*)\tau(C_N^+(7)^*) > 1$$

for N sufficiently large. Therefore by Theorem 5.1.1

$$\begin{aligned} F(2)F(7) &\supseteq (-\infty, -\langle\overline{2,1}\rangle(1-\langle\overline{1,7}\rangle)] \cup [\langle\overline{2,1}\rangle\langle\overline{7,1}\rangle, \infty) \\ &= (-\infty, -0.0411\ldots] \cup [0.0464\ldots, \infty). \end{aligned} \tag{6.38}$$

Let

$$I^1 = [\langle 1, \overline{1,2}\rangle, \langle\overline{1,2}\rangle] \quad \text{and} \quad C^1 = C(L_2) \cap I^1,$$

then using Maple and Corollary 5.2.2 (as in the proof of Lemma 6.4.3) we find that

$$\tau((1-C^1)^*) \geq 0.366.$$

Let O^- and O^+ denote the largest gap in $(1-C(L_7))^*$ and $C(L_7)^*$ respectively. Then

$$O^- = (1-\langle 1,\overline{1,7}\rangle, 1-\langle 2,\overline{7,1}\rangle)^* \quad \text{and} \quad O^+ = (\langle 2,\overline{7,1}\rangle, \langle 1,\overline{1,7}\rangle)^*.$$

Further,

$$|O^-| = 0.119\ldots, \qquad |O^+| = 0.119\ldots,$$
$$|(1-I^1)^*| = 0.4557\ldots, \qquad |I(L_7)^*| = 1.94\ldots$$

and

$$|(1 - I(L_7))^*| = 2.04\ldots .$$

Therefore by Theorem 5.1.1 we have

$$\begin{aligned}(1 - C^1)(1 - C(L_7)) &= [(1 - \langle\overline{1,2}\rangle)(1 - \langle\overline{1,7}\rangle), (1 - \langle 1,\overline{1,2}\rangle)(1 - \langle\overline{7,1}\rangle)] \\ &= [0.0301\ldots, 0.369\ldots]\end{aligned} \tag{6.39}$$

and

$$\begin{aligned}-(1 - C^1)C(L_7) &= [-(1 - \langle 1,\overline{1,2}\rangle)\langle\overline{1,7}\rangle, -(1 - \langle\overline{1,2}\rangle)\langle\overline{7,1}\rangle] \\ &= [-0.375\ldots, -0.0339\ldots].\end{aligned} \tag{6.40}$$

Since

$$F(2) \cdot F(7) \subseteq \big(-\infty, (-1 + \langle\overline{1,2}\rangle)\langle\overline{7,1}\rangle\big] \cup \big[(-1 + \langle\overline{1,2}\rangle)(-1 + \langle\overline{1,7}\rangle), \infty\big)$$

by Lemma 6.4.1 and Lemma 6.4.2, (6.31) follows from (6.38), (6.39) and (6.40).

□

Lemma 6.4.6

$$F(2)F(2)F(2)F(2) = (-\infty, -(1 - \langle\overline{1,2}\rangle)^3\langle\overline{2,1}\rangle] \cup [(1 - \langle\overline{1,2}\rangle)^4, \infty).$$

Proof. By Lemma 6.4.3 we have, for N sufficiently large,

$$\gamma(C_N^-(L_2)^*) > 0.2307 \quad \text{and} \quad \gamma(C_N^+(L_2)^*) > 0.2679.$$

Therefore by Theorem 5.1.1 and letting N tend to infinity we have

$$\begin{aligned}F(2)^4 &\supseteq (-\infty, -\langle\overline{2,1}\rangle^3(1 - \langle\overline{1,2}\rangle)] \cup [\langle\overline{2,1}\rangle^4, \infty) \\ &= (-\infty, -0.0131\ldots] \cup [0.0179\ldots, \infty).\end{aligned} \tag{6.41}$$

Similar to the approach used in the proof on Lemma 6.4.5 we put

$$I^+ = [\langle\overline{2,1}\rangle, \langle 2,\overline{2,1}\rangle], \qquad I^- = [1-\langle 1,\overline{1,2}\rangle, 1-\langle\overline{1,2}\rangle],$$
$$C^+ = C(L_2)\cap I^+ \qquad \text{and} \quad C^- = (1-C(L_2))\cap I^-.$$

Using Maple and Corollary 5.2.2 we find that

$$\tau(C^{+*}), \tau(C^{-*}) > 0.366$$

therefore

$$\gamma(C^{+*}), \gamma(C^{-*}) > 0.2679. \tag{6.42}$$

Now I^- splits as $I^- = I_1^- \cup O \cup I_2^-$ where

$$\begin{aligned} I_1^- &= [1-\langle\overline{1,2}\rangle, 1-\langle 1,2,\langle\overline{2,1}\rangle\rangle], \\ O &= (1-\langle 1,2,\langle\overline{2,1}\rangle\rangle, 1-\langle 1,2,\overline{1,2}\rangle) \\ \text{and} \quad I_2^- &= [1-\langle 1,2,\overline{1,2}\rangle, 1-\langle 1,\overline{1,2}\rangle]. \end{aligned}$$

Further,

$$|I_1^{-*}| = 0.103\ldots, \qquad |I_2^{-*}| = 0.1437\ldots\ ,$$

$$\begin{aligned} 3(I_1^{-*}\cup I_2^{-*}) &= [-3.95\ldots, -3.641\ldots]\cup[-3.638\ldots, -3.288\ldots] \\ &\cup[-3.327, -2.936]\cup[-3.015, -2.583] \end{aligned}$$

and

$$|I^{+*}| = 0.1438\ldots$$

hence $|I^{+*}|$ is greater than the size of the largest gap in $3C^{-*}$, so by (6.42) and Theorem 5.1.1

$$\begin{aligned} C^+(C^-)^3 = I^+(I^-)^3 &= [(1-\langle\overline{1,2}\rangle)^3\langle\overline{2,1}\rangle, (1-\langle 1,\overline{1,2}\rangle)^3\langle 2,\overline{2,1}\rangle] \\ &= [0.00704\ldots, 0.0319\ldots]. \end{aligned} \tag{6.43}$$

Also we have, by (6.42) and Theorem 5.1.1,

$$\begin{aligned}(C^-)^4 = (I^-)^4 &= [(1-\langle\overline{1,2}\rangle)^4, (1-\langle 1,\overline{1,2}\rangle)^4] \\ &= [0.00514\ldots, 0.0319\ldots].\end{aligned} \tag{6.44}$$

The lemma follows from (6.41), (6.43), (6.44), Lemma 6.4.1 and Lemma 6.4.2.

□.

Proof of Theorem 1.0.8. If $(m,n) = (3,4)$ or $(m,n) = (2,7)$ then the theorem follows from Lemma 6.4.5. Otherwise by Lemmas 6.4.1 and 6.4.2 we have

$$F(m)\cdot F(n) \subseteq \left(-\infty, -(1-\langle\overline{1,m}\rangle)\langle\overline{n,1}\rangle\right] \bigcup \left[(1-\langle\overline{1,m}\rangle)(1-\langle\overline{1,n}\rangle), \infty\right).$$

By Lemmas 6.4.3 and 6.4.4

$$\tau(C_N^-(m)^*)\tau(C_N^+(n)^*) > 1 \quad \text{and} \quad \tau(C_N^-(m)^*)\tau(C_N^-(n)^*) > 1$$

so by Theorem 5.1.1

$$\left(-(N-\langle\overline{m,1}\rangle)(N+\langle\overline{1,n}\rangle), -(1-\langle\overline{1,m}\rangle)\langle\overline{n,1}\rangle\right] \subseteq F(m)\cdot F(n)$$

and

$$\left[(1-\langle\overline{1,m}\rangle)(1-\langle\overline{1,n}\rangle), (N-\langle\overline{m,1}\rangle)(N-\langle\overline{n,1}\rangle)\right) \subseteq F(m)\cdot F(n)$$

and the theorem follows upon letting N tend to infinity.

□

Proof of Theorem 1.0.9. Using an approach similar to that used to prove Theorem 1.0.8, Theorem 1.0.9 follows from Lemmas 6.4.1, 6.4.2, 6.4.3, 6.4.4 and 6.4.6, and Theorem 5.1.1.

□

Chapter 7

Cantor Sets with Small Thickness

7.1 Asymptotic and Maximal Thicknesses

Let C be a Cantor set with ordered derivation $\mathcal{D}$. We define the *asymptotic thickness* of C by

$$\tau_A(C) = \sup_A \tau(A \cap C)$$

where the supremum is taken over all bridges A of $\mathcal{D}$. We also define the *normalized asymptotic thickness* of C by

$$\gamma_A(C) = \frac{\tau_A(C)}{1 + \tau_A(C)}.$$

Using the results of Chapter 3 we may obtain the following theorem.

Theorem 7.1.1 *Let k be a positive integer with $C_1, \ldots, C_k$ Cantor sets. If $\gamma_A(C_1) + \cdots + \gamma_A(C_k) > 1$ then $C_1 + \cdots + C_k$ contains an interval. Otherwise*

$$\dim_H(C_1 + \cdots + C_k) \geq \frac{\log 2}{\log\left(1 + \frac{1}{\gamma_A(C_1) + \cdots + \gamma_A(C_k)}\right)}.$$

Proof. Let $S = \gamma_A(C_1) + \cdots + \gamma_A(C_k)$ and assume first that $S > 1$. Put

$$\delta = \frac{S-1}{k}.$$

Then for $j = 1, \ldots, k$ there exists a Cantor set C_j' contained in C_j with

$$\gamma(C_j') \geq \gamma_A(C_j) - \delta.$$

Note that

$$\gamma(C_1') + \cdots + \gamma(C_k') \geq S - k\delta = 1$$

hence by Theorem 3.1.3 $C_1' + \cdots + C_k'$ contains an interval and the first part of Theorem 7.1.1 follows.

Assume next that $S \leq 1$. For any $\epsilon > 0$ put

$$\delta = \frac{S(1-2^{-\epsilon})}{k}.$$

Notice that

$$\begin{aligned}\frac{\log 2}{\log\left(1+\frac{1}{S}\right)} - \frac{\log 2}{\log\left(1+\frac{1}{S-k\delta}\right)} &= \frac{\log 2\left(\log\left(1+\frac{1}{S-k\delta}\right) - \log\left(1+\frac{1}{S}\right)\right)}{\log\left(1+\frac{1}{S}\right)\log\left(1+\frac{1}{S-k\delta}\right)} \\ &\leq \frac{\log\left(\frac{S-k\delta+1}{S-k\delta+1-k\delta/S}\right)}{\log 2} \\ &< \frac{\log\left(\frac{1}{1-k\delta/S}\right)}{\log 2} = \epsilon \qquad (7.1)\end{aligned}$$

since $S - k\delta > 0$ and

$$\log\left(1+\frac{1}{S-k\delta}\right) > \log\left(1+\frac{1}{S}\right) \geq \log 2.$$

For $j = 1, \ldots, k$ there exists a Cantor set C_j' with

$$\gamma(C_j') \geq \gamma_A(C_k) - \delta.$$

Thus by Theorem 3.1.3

$$\dim_H(C_1' + \cdots + C_k') \geq \frac{\log 2}{\log\left(1 + \frac{1}{S - k\delta}\right)}.$$

Using (7.1) and the fact that $C_j' \subseteq C_j$ for $1 \leq j \leq k$ we have

$$\dim_H(C_1 + \cdots + C_k) \geq \frac{\log 2}{\log\left(1 + \frac{1}{S}\right)} - \epsilon,$$

and upon letting ϵ tend to zero we establish the required result.

$\square$

While use of the asympotic thickness will often give a better result, there are many cases where there will not be much improvement over use of ordinary thickness. Consider the Cantor set $C = C(\{1, 2, 3, 4, 10\})$. We have

$$\tau(C) = 0.0717 \quad \text{and} \quad \tau_A(C) = 0.07247$$

However $C(L_4) = C(\{1, 2, 3, 4\}) \subseteq C$ and $\tau(C(L_4)) = 1.300\ldots$, so C contains a set of large thickness. We define the *maximal thickness* and *normalized maximal thickness* of a Cantor set C to be

$$\tau_M(C) = \sup_{C' \subseteq C} \tau(C') \quad \text{and} \quad \gamma_M(C) = \frac{\tau_M(C)}{\tau_M(C) + 1}$$

respectively, where the supremum is taken over all Cantor sets C' contained in C. Note that for any Cantor set C we have

$$\tau(C) \leq \tau_A(C) \leq \tau_M(C).$$

We have the following results.

Lemma 7.1.2 *If C is a Cantor set then*

$$\dim_H(C) \geq \frac{\log 2}{\log\left(2 + \frac{1}{\tau_M(C)}\right)}.$$

Proof. For any ϵ in the range $0 < \epsilon < \tau_M(C)$ there exists a Cantor set $C' \subseteq C$ such that $\tau(C') \geq \tau_M(C) - \epsilon$. By Lemma 2.3.2 we have

$$\dim_H(C) \geq \dim_H(C') \geq \frac{\log 2}{\log\left(2 + \frac{1}{\tau_M(C)-\epsilon}\right)}.$$

and the lemma follows upon letting ϵ tend to infinity.

□

Theorem 7.1.3 *Let $C_1, \ldots, C_k$ be Cantor sets. If*

$$\gamma_M(C_1) + \cdots + \gamma_M(C_k) > 1$$

then $C_1 + \cdots + C_k$ contains an interval. Otherwise

$$\dim_H(C_1 + \cdots + C_n) \geq \frac{\log 2}{\log\left(1 + \frac{1}{\gamma_M(C_1) + \cdots + \gamma_M(C_k)}\right)}.$$

Proof. The proof of Theorem 7.1.3 is similar to that of Theorem 7.1.1, with maximal thickness and normalized maximal thickness replacing asymptotic thickness and normalized asymptotic thickness respectively.

□

7.2 Another Technique

If C_1 and C_2 have small maximal thicknesses (i.e., $\tau_M(C_1)\tau_M(C_2) < 1$) then it may still be possible to prove that $C_1 + C_2$ contains an interval. We will examine three such cases, motivated by the examples of $C(L_6) + C(L_2)$, $C(L_5) + C(L_2)$ and $C(L_3) + C(L_3)$ respectively. The techniques used were inspired in part by work of Hanno Schecker [14] who first proved that $C(L_3) + C(L_3)$ contains an interval. Schecker made the following definitions. Put

$$\theta = \langle \overline{1,2} \rangle, \qquad \eta = \langle \overline{1,3} \rangle$$

and for any positive integer n define

$$\theta_n = \begin{cases} \theta/2 & \text{if } n \text{ is even,} \\ \theta & \text{otherwise,} \end{cases} \quad \text{and} \quad \eta_n = \begin{cases} \eta/3 & \text{if } n \text{ is even,} \\ \eta & \text{otherwise.} \end{cases}$$

Note that $\theta/2 = \langle \overline{2,1} \rangle$ and $\eta/3 = \langle \overline{3,1} \rangle$. We also define $\mathcal{B}$ by

$$\mathcal{B} = \{(a_i)_{i=-h}^{k};\ h, k \in \mathbb{Z} \text{ and } 1 \le a_i \le 3 \text{ for } -h \le i \le k\}.$$

For any tuple $A = (a_{-h}, \ldots, a_k) \in \mathcal{B}$ we define the *T-interval* $I(A)$ to be the closed interval with lower endpoint

$$\min\{\langle a_{-1}, \ldots, a_{-h} + \eta_h \rangle + \langle a_1, \ldots, a_k + \theta_k \rangle, \langle a_{-1}, \ldots, a_{-h} + \theta_h \rangle + \langle a_1, \ldots, a_k + \eta_k \rangle\}$$

and upper endpoint

$$\max\{\langle a_{-1}, \ldots, a_{-h} + \eta_{h+1} \rangle + \langle a_1, \ldots, a_k + \theta_{k+1} \rangle, \langle a_{-1}, \ldots, a_{-h} + \theta_{h+1} \rangle + \langle a_1, \ldots, a_k + \eta_{k+1} \rangle\}.$$

We also define the *T-ratio* of A, $v(A)$, to be

$$v(A) = \left| \frac{\langle a_1, \ldots, a_k + \eta_1 \rangle - \langle a_1, \ldots, a_k + \eta_2 \rangle}{\langle a_{-1}, \ldots, a_{-h} + \eta_1 \rangle - \langle a_{-1}, \ldots, a_{-h} + \eta_2 \rangle} \right|.$$

We say that the T-interval $I(A)$ is *valid* if

$$h \geq 2 \quad \text{or} \quad a_{-1} \geq 2, \qquad k \geq 2 \quad \text{or} \quad a_1 \geq 2$$

and

$$\frac{5}{17} \leq v(A) \leq \frac{17}{5}.$$

Let $B = (b_{-h'}, \ldots, b_{k'}) \in \mathcal{B}$. Then we say that $I(B)$ is a *successor* of $I(A)$ if

$$k' \geq k, \qquad h' \geq h, \qquad h' + k' > h + k$$

and

$$b_i = a_i \qquad \text{for } i = -h, \ldots, k.$$

Schecker established the following result.

Theorem 7.2.1 *Every valid T-interval is covered by its valid successors.*

Proof. See [14, Lemma 1].

□

By induction we have the following corollary.

Theorem 7.2.2 *Every valid T-interval is contained in $C(3) + C(3)$.*

Proof. See [14, Theorem 1].

□

Before we proceed to our results we must introduce additional notation. Let C be a Cantor set derived from I and let n be an integer greater than one. We may define an n-ary tree $\mathcal{D}^n$ as follows. Let the root of $\mathcal{D}^n$ be the closed interval I. Suppose that A is a vertex of $\mathcal{D}^n$ already defined but without sub-vertices. Let $O_A^1, \ldots, O_A^{n-1}$ be $n-1$ gaps occurring in $C \cap A$ and write

$$A = A^1 \cup O_A^1 \cup A^2 \cup \cdots \cup O_A^{n-1} \cup A^n$$

where $A^1, \ldots, A^n$ are closed intervals. We let the sub-vertices of A be the intervals $A^1, \ldots, A^n$ and continue defining $\mathcal{D}^n$ recursively. Let the length of a word w be denoted by $|w|$. If

$$C = \lim_{t \to \infty} \bigcup_{|w|=t} I^w$$

then we say that $\mathcal{D}^n$ is an *n-ary derivation* of C from I, and that C is *derived* from I by $\mathcal{D}^n$. The vertices of $\mathcal{D}^n$ are also called *bridges* of the derivation. We say that $\mathcal{D}^n$ is *ordered* if for any bridge A of $\mathcal{D}^n$ and any j with $1 \le j \le n$, O_A^j is a gap of maximal size in $(A^j \cup O_A^j \cup \cdots \cup A^n) \cap C$.

For example, if $C = C(L_n)$ then one ordered n-ary derivation of C is the n-ary tree $\mathcal{D}^n(L_n)$ defined by setting

$$I = I(L_n) = [\langle \overline{n,1} \rangle, \langle \overline{1,n} \rangle]$$
$$\text{and} \quad I^{d_1 \cdots d_t} = [[\langle d_1, \ldots, d_t, \overline{n,1} \rangle, \langle d_1, \ldots, d_t, \overline{1,n} \rangle]]$$

for any positive integer t and integers $d_1, \ldots, d_t$ between 1 and n. Note that the gaps in the derivation are of the form

$$O^j_{I^{d_1 \cdots d_t}} = (((\langle d_1, \ldots, d_t, j, \overline{1,n} \rangle, \langle d_1, \ldots, d_t, j+1, \overline{n,1} \rangle))).$$

For sets of real numbers X and Y we write $X < Y$ ($X > Y$) if for every $x \in X$ and $y \in Y$, $x < y$ ($x > y$). Assume henceforth that in the derivation $\mathcal{D}^n$ we have, for all bridges A of $\mathcal{D}^n$,

$$A^1 < A^2 < \cdots < A^n \quad \text{or} \quad A^1 > A^2 > \cdots > A^n.$$

Then we may construct a normal binary derivation $\mathcal{D}^{n\prime}$ of C by using $\mathcal{D}^n$. Let the root of $\mathcal{D}^{n\prime}$ be $B = I$, where I is the root of $\mathcal{D}^n$. We define $\mathcal{D}^{n\prime}$ recursively as follows. If, for a binary word v we have that the interval B^v is a bridge of $\mathcal{D}^n$, say $B^v = A^w$, then we put

$$\begin{array}{ll} B^{v0} = A^{w1}, & B^{v1} = A^{w2} \cup O^2_{A^w} \cup \cdots \cup A^{wn}, \\ B^{v10} = A^{w2}, & B^{v11} = A^{w3} \cup \cdots \cup A^{wn}, \\ \vdots & \vdots \\ B^{v11\cdots10} = A^{w(n-1)} & B^{v11\cdots11} = A^{wn}. \end{array}$$

It is easy to see that the tree $\mathcal{D}^{n\prime}$ so constructed will be a derivation for C in the traditional sense. For example, if $\mathcal{D}^n(L_n)$ is as defined above then $\mathcal{D}^n(L_n)'$ will be the derivation $\mathcal{D}(L_n)$ defined in (2.2).

Note that if $n = 2$ then the trees $\mathcal{D}^n$ and $\mathcal{D}^{n\prime}$ will be identical. If $n = 2$ and A is a bridge of $\mathcal{D}^n$ then we often denote O^1_A simply by O_A.

It may be the case that in the derivation $\mathcal{D}^{n\prime}$ all local thicknesses are large except for those where the ratio is of the form

$$\frac{|B^{v11\cdots11}|}{|O_B^{v11\cdots1}|}.$$

In such a situation we would hope that if we could in some sense "ignore" the bridges of the form $B^{v11\cdots11}$ then we could use the fact that the other local thicknesses are large to show that $C + C$ contains an interval. This concept was used by Schecker

[14] and motivates our next definition. If A is a bridge of $\mathcal{D}^n$ then we define $\overline{A}$ to be

$$\overline{A} = \bigcap_{D \in \mathcal{M}} D$$

where $\mathcal{M}$ is the set of closed intervals D such that

$$A = \bigcup_{t=1}^{k_1} (A^{w_t n} \cup O^{n-1}_{A^{w_t}} \cup D \cup \bigcup_{t=1}^{k_2} (O^{n-1}_{A^{u_t}} \cup A^{u_t n})$$

for some $k_1, k_2 \in \mathbb{Z}^+$ and words w_i and u_j with $A^{w_i n} < D < A^{u_j n}$ for $1 \le i \le k_1$ and $1 \le j \le k_2$. Equivalently, we form $\overline{A}$ from A by stripping off from both sides of A as many intervals of the form $O^{n-1}_{A^w} A^{wn}$ as possible. For example, if A is a bridge of $\mathcal{D}^n(L_n)$, say

$$A = [[\langle a_1, \ldots, a_t, \overline{n, 1}\rangle, \langle a_1, \ldots, a_t, \overline{1, n}\rangle]]$$

then

$$\overline{A} = [[\langle a_1, \ldots, a_t, \overline{n-1, 1}\rangle, \langle a_1, \ldots, a_t, \overline{1, n-1}\rangle]].$$

We define A^L and A^R to be the two half-open intervals with

$$A = A^L \cup \overline{A} \cup A^R \quad \text{and} \quad A^n \subseteq A^R$$

and for $t = 1, \ldots, n-1$ put

$$\overline{O^t_A} = X \cup O^t_A \cup Y$$

where

$$X = \begin{cases} A^{tR} & \text{if } A^{tL} < A^{tR} < O^t_A \text{ or } A^{tL} > A^{tR} > O^t_A, \\ A^{tL} & \text{otherwise.} \end{cases}$$

and

$$Y = \begin{cases} A^{(t+1)R} & \text{if } A^{(t+1)L} < A^{(t+1)R} < O_A^t \text{ or } A^{(t+1)L} > A^{(t+1)R} > O_A^t, \\ A^{(t+1)L} & \text{otherwise.} \end{cases}$$

Additionally, for $1 \leq s, t \leq n$ with $s \neq t$ we put

$$A^s A^t = A^s \cup O_A^s \cup \cdots \cup A^t \quad \text{and} \quad \overline{A^s A^t} = \overline{A^s} \cup \overline{O_A^s} \cup \cdots \cup \overline{A^t}.$$

Hence, for example,

$$A = A^L \cup \overline{A^1} \cup \overline{O_A^1} \cup \overline{A^2} \cup \cdots \cup \overline{O_A^{n-2}} \cup \overline{A^{n-1}} \cup A^R.$$

If O_A^k is a gap in $C(L_n)$, say

$$O_A^k = ((\langle a_1, \ldots, a_t, k, \overline{1, n}\rangle, \langle a_1, \ldots, a_t, k, \overline{n, 1}\rangle))$$

then

$$\overline{O_A^k} = ((\langle a_1, \ldots, a_t, k, \overline{1, n-1}\rangle, \langle a_1, \ldots, a_t, k, \overline{n-1, 1}\rangle)).$$

Let C_1 and C_2 be Cantor sets derived by $\mathcal{D}_1^n$ and $\mathcal{D}_2^m$ from I_1 and I_2 respectively. Assume that for every interval $J \subseteq I_1$, if O_1 is the largest gap in $C_1 \cap J$ then $|\overline{O_1}| = \sup |\overline{O}|$, where the supremum is taken over all gaps O contained in $C_1 \cap J$. If $\mathcal{D}_1^n$ is ordered and has this property then we say that $\mathcal{D}_1^n$ is *B-ordered.* Let J_1 and J_2 be closed intervals of the form $A^s A^t$ contained in I_1 and I_2 respectively. We say that $\overline{J_1}$ and J_2 are *B-compatible*, denoted $\overline{J_1} \approx J_2$, if

$$|\overline{J_1}| > |O_2| \quad \text{and} \quad |J_2| > |\overline{O_1}|$$

where O_1 and O_2 are the largest gaps in $C_1 \cap J_1$ and $C_2 \cap J_2$ respectively.

If $I = I(L_3) = [\langle\overline{3,1}\rangle, \langle\overline{1,3}\rangle]$ then in Shecker's notation we have

$$I(A) = (\overline{I^{a_{-1} \cdots a_{-h}}} + I^{a_1 \cdots a_k}) \bigcup (\overline{I^{a_1 \cdots a_k}} + I^{a_{-1} \cdots a_{-h}}).$$

7.3 $F(6) + F(2)$

Although it follows from Theorem 7.4.3 that $F(6) \pm F(2) = \mathbb{R}$, it is instructive to examine an independent proof that $F(6) \pm F(2)$ contains an interval, since the proof is much simpler than the proof of Theorem 7.4.3. Thus the main ideas behind the proof will be more obvious. We have the following general theorem.

Theorem 7.3.1 *Let C_1 and C_2 be Cantor sets with derivations $\mathcal{D}_1^n$ and $\mathcal{D}_2^2$ respectively. Assume that $\mathcal{D}_1^n$ is B-ordered and $\mathcal{D}_2^2$ is ordered. Let I_1 and I_2 be the roots of $\mathcal{D}_1^n$ and $\mathcal{D}_2^2$ respectively, and assume that $\overline{I_1} \approx I_2$ and for every bridge A of $\mathcal{D}_1^n$ and B of $\mathcal{D}_2^2$*

$$|\overline{A^1}| \geq |\overline{A^2}| \geq \cdots \geq |\overline{A^n}|, \qquad |\overline{O_A^1}| \geq |\overline{O_A^2}| \geq \cdots \geq |\overline{O_A^{n-1}}|, \tag{7.2}$$

$$|B^1| \geq |B^2| \quad \textit{and} \quad |O_{B^1}| \geq |O_{B^2}|. \tag{7.3}$$

Assume further that

$$\frac{|\overline{A^{n-1}A^n}|}{|\overline{A^{n-2}}|} \geq 1, \tag{7.4}$$

$$\frac{|\overline{A^n}|}{|\overline{O_A^{n-2}}|} \cdot \frac{|B^2|}{|O_{B^1}|} \geq 1, \tag{7.5}$$

$$\frac{|\overline{A^{n-1}}|}{|\overline{O_A^{n-2}}|} \cdot \frac{|B^1|}{|O_B|} \geq 1, \tag{7.6}$$

$$\frac{|\overline{A^k}|}{|\overline{O_A^k}|} \cdot \tau(C_2) \geq 1 \tag{7.7}$$

for $k = 1, \ldots, n-2$ and

$$\frac{|\overline{A^{k+1}A^{n-1}}|}{|\overline{O_A^k}|} \cdot \tau(C_2) \geq 1 \tag{7.8}$$

for $k = 1, \ldots, n-3$. Then

$$\overline{I_1} + I_2 \subseteq (I_1 \cap C_1) + (I_2 \cap C_2).$$

Proof. Let A and B be bridges of $\mathcal{D}_1^n$ and $\mathcal{D}_2^2$ respectively. Let k be an integer in the range $1 \leq k \leq n-2$, and assume that $\overline{A^k A^{n-1}} \approx B$. We will show that $\overline{A^k A^{n-1}} + B$ can be covered by a union of intervals of the form

$$\overline{A^{wt} A^{w(n-1)}} + B^v \tag{7.9}$$

where either $t > k$ or $w \neq \emptyset$ or $v \neq \emptyset$, and $\overline{A^{wt} A^{w(n-1)}} \approx B^v$. By repeating this process we can cover $\overline{A^k A^{n-1}} + B$ by intervals of the form (7.9) where either $v \neq \emptyset$ or $|\overline{O^t_{A^w}}| < |B^2| < |B^1|$. In this case if $v = \emptyset$ then it follows that

$$\overline{A^{wt} A^{w(n-1)}} \approx B^1, \qquad \overline{A^{wt} A^{w(n-1)}} \approx B^2$$

and

$$\overline{A^{wt} A^{w(n-1)}} + B = (\overline{A^{wt} A^{w(n-1)}} + B^1) \cup (\overline{A^{wt} A^{w(n-1)}} + B^2).$$

Hence it is possible to cover $\overline{A^k A^{n-1}} + B$ by intervals of the form (7.9) where $v \neq \emptyset$. Similar to the above argument we may repeat this process and cover $\overline{A^k A^{n-1}} + B$ by a union of intervals of the form (7.9) where $v \neq \emptyset$ and either $w \neq \emptyset$ or $|O_{B^v}| < |\overline{A^{n-1}}|$. If $w = \emptyset$ then

$$\overline{A^r} \approx B^v$$

for $t \leq r \leq n-1$, and

$$\overline{A^t A^{n-1}} + B^v = (\overline{A^t} + B^v) \cup \cdots \cup (\overline{A^{n-1}} + B^v).$$

Therefore it is possible to cover $\overline{A^k A^{n-1}} + B$ by a union of intervals of the form (7.9) where $w \neq \emptyset$ and $v \neq \emptyset$. Therefore by induction on $\min\{|w|, |v|\}$ we may cover $\overline{A^k A^{n-1}} + B$ by intervals of the form (7.9) where w and v are as long as desired. By letting $\min\{|w|, |v|\}$ tend to infinity we have

$$\overline{A^k A^{n-1}} + B \subseteq C_1 + C_2.$$

Since $\overline{I_1} = \overline{I_1^1 I_1^{n-1}}$ and $\overline{I_1} \approx I_2$ it will then follow that

$$\overline{I_1} + I_2 \subseteq C_1 + C_2.$$

Let A and B be bridges of $\mathcal{D}_1^n$ and $\mathcal{D}_2^2$ and assume that k is an integer with $1 \leq k \leq n-2$ and $\overline{A^k A^{n-1}} \approx B$. We must cover $\overline{A^k A^{n-1}} + B$ by a union of intervals of the form (7.9), as described above. Assume without loss of generality that $A^1 < A^2$. We identify two different situations.

Situation 1: $1 \leq k \leq n-3$

Case 1: $|B^2| \geq |\overline{O_A^k}|$

Since $|B^1| \geq |B^2|$ we have

$$\overline{A^k A^{n-1}} \approx B^1 \quad \text{and} \quad \overline{A^k A^{n-1}} \approx B^2$$

and since $\overline{A^k A^{n-1}} \approx B$ we know that $\overline{A^k A^{n-1}} + B^1$ overlaps $\overline{A^k A^{n-1}} + B^2$. Therefore

$$\overline{A^k A^{n-1}} + B = (\overline{A^k A^{n-1}} + B^1) \cup (\overline{A^k A^{n-1}} + B^2).$$

Case 2: $|B^2| < |\overline{O_A^k}|$

By (7.7) and (7.8) we have

$$|\overline{A^k}| > |O_B| \quad \text{and} \quad |\overline{A^{k+1} A^{n-1}}| > |O_B|$$

hence

$$\overline{A^k A^{n-1}} + B = (\overline{A^k} + B) \cup (\overline{A^{k+1} A^{n-1}} + B)$$

with

$$\overline{A^k} \approx B \quad \text{and} \quad \overline{A^{k+1} A^{n-1}} \approx B.$$

Situation 2: $k = n-2$

Case 1: $|B^2| \geq |\overline{O_A^{n-2}}|$

As before we find that

$$\overline{A^{n-2}A^{n-1}} + B = (\overline{A^{n-2}A^{n-1}} + B^1) \cup (\overline{A^{n-2}A^{n-1}} + B^2)$$

with $\overline{A^{n-2}A^{n-1}} \approx B^1$ and $\overline{A^{n-2}A^{n-1}} \approx B^2$.

Case 2: $|\overline{A^{n-1}}| \geq |O_B|$

In this case it follows that

$$\overline{A^{n-2}A^{n-1}} + B = (\overline{A^{n-2}} + B) \cup (\overline{A^{n-1}} + B)$$

where $\overline{A^{n-2}} \approx B$ and $\overline{A^{n-1}} \approx B$.

Case 3: $|B^2| < |\overline{O_A^{n-2}}|$ and $|\overline{A^{n-1}}| < |O_B|$

By (7.4), (7.7) and (7.5) we have

$$|\overline{A^{n-1}A^n}| \geq |\overline{A^{n-2}}| > |O_B| \quad \text{and} \quad |\overline{A^{n-1}}| > |\overline{A^n}| > |O_{B^1}| > |O_{B^2}| \tag{7.10}$$

By (7.6) and (7.7) we find that

$$|B^1| > |\overline{O_A^{n-2}}| \geq |\overline{O_A^{n-1}}| \quad \text{and} \quad |B^2| > |\overline{O^1_{A^{n-1}}}|.$$

Therefore with (7.10) we have

$$\overline{A^{n-2}} \approx B, \qquad \overline{A^{n-2}A^{n-1}} \approx B^1, \qquad \overline{A^n} \approx B^1 \quad \text{and} \quad \overline{A^{n-1}} \approx B^2.$$

Now if $B^2 < B^1$ then

$$\overline{A^{n-2}A^{n-1}} + B = (\overline{A^{n-2}} + B) \cup (\overline{A^{n-2}A^{n-1}} + B^1)$$

while if $B^1 < B^2$ then

$$\overline{A^{n-2}A^{n-1}} + B \subseteq (\overline{A^{n-2}} + B) \cup (\overline{A^{n-2}A^{n-1}} + B^1) \cup (\overline{A^n} + B^1) \cup (\overline{A^{n-1}} + B^2).$$

Notice that if $B^1 < B^2$ then $\overline{A^{n-2}A^{n-1}} + B^1$ intersects $\overline{A^n} + B^1$, since $|B^1| > |\overline{O_A^{n-1}}|$, while $\overline{A^n} + B^1$ intersects $\overline{A^{n-1}} + B^2$, since

$$|\overline{A^{n-1}A^n}| > |O_B| \quad \text{and} \quad |B| > |\overline{O_A^{n-1}}| + |\overline{A^n}|.$$

Theorem 7.3.1 follows by induction, as described at the beginning of the proof.

□

Corollary 7.3.2 *Let I_1 and I_2 be bridges of $\mathcal{D}^6(L_6)$ and $\mathcal{D}^2(L_2)$ respectively, with*

$$I_1 \notin \{I(L_6), I(L_6)^1\}, \qquad I_2 \notin \{I(L_2), I(L_2)^1\}$$

If $\overline{I}_1 \approx I_2$ then

$$\overline{I_1} + I_2 \subseteq (I_1 \cap F(6)) + (I_2 \cap F(2)) \quad \textit{and} \quad \overline{I_1} - I_2 \subseteq (I_1 \cap F(6)) - (I_2 \cap F(2)).$$

Proof. Let $A = I(L_6)^w$ be a bridge of $\mathcal{D}^6(L_6)$ and let $B = \pm I(L_2)^v$ be a bridge of either $\mathcal{D}^2(L_2)$ or $-\mathcal{D}^2(L_2)$. Assume that

$$w \notin \{\emptyset, 1\} \quad \text{and} \quad v \notin \{\emptyset, 1\}.$$

Let $w = a_1 \cdots a_r$ and $v = b_1 \cdots b_s$. Then

$$\frac{1}{7} \leq \langle a_r, \ldots, a_1 \rangle \leq \frac{7}{8} \quad \text{and} \quad \frac{1}{3} \leq \langle b_s, \ldots, b_1 \rangle \leq \frac{3}{4}$$

Note that by Lemma 2.1.3 we have

$$\frac{|\overline{A^k}|}{|\overline{A^{k+1}}|} \geq \frac{([k+1, \overline{1,4}] + 7/8)([k+1, \overline{4,1}] + 7/8)}{([k, \overline{1,4}] + 7/8)([k, \overline{4,1}] + 7/8)} > 1 \tag{7.11}$$

for $1 \leq k \leq n-1$. Also

$$\frac{|\overline{O_A^k}|}{|\overline{O_A^{k+1}}|} \geq \frac{([k+2,\overline{4,1}]+7/8)([k+1,\overline{1,4}]+7/8)}{([k+1,\overline{4,1}]+7/8)([k,\overline{1,4}]+7/8)} > 1 \tag{7.12}$$

for $1 \leq k \leq n-2$, and (7.2) follows. Also

$$\frac{|B^1|}{|B^2|} \geq \frac{([2,\overline{1,2}]+3/4)([2,\overline{2,1}]+3/4)}{([1,\overline{1,2}]+3/4)([1,\overline{2,1}]+3/4)} > 1 \tag{7.13}$$

and

$$\frac{|O_{B^1}|}{|O_{B^2}|} \geq \frac{([2,2,\overline{2,1}]+3/4)([2,1,\overline{1,2}]+3/4)}{([1,2,\overline{2,1}]+3/4)([1,1,\overline{1,2}]+3/4)} > 1 \tag{7.14}$$

so (7.3) holds.

By an approach similar to that used in Section 4.1 we must determine quantities of the form

$$\inf_{x \leq Q \leq y} \operatorname{fr}(g_1, g_2, g_3, g_4, Q) \tag{7.15}$$

where

$$\operatorname{fr}(g_1, g_2, g_3, g_4, Q) = \frac{(g_1 - g_2)}{(g_3 - g_4)} \cdot \frac{(g_3 + Q)(g_4 + Q)}{(g_1 + Q)(g_2 + Q)}$$

and x, y, g_1, g_2, g_3 and g_4 are certain positive real numbers. Now if g_3/g_1 and g_4/g_2 are both greater than one or both less than one then the infimum in (7.15) clearly occurs at either x or y. Otherwise, from calculus we see that $\operatorname{fr}(g_1, g_2, g_3, g_4, Q)$, considered as a function of Q, will have a critical point at Q if

$$0 = aQ^2 + bQ + c$$

where

$$a = g_1 + g_2 - g_3 - g_4, \qquad b = 2g_1g_2 - 2g_3g_4 \quad \text{and} \quad c = (g_3 + g_4)g_1g_2 - (g_1 + g_2)g_3g_4.$$

Thus if $a = 0$ the critical point is $Q = -c/b$, while if $a \neq 0$ then the critical points are

$$\frac{-b+\sqrt{b^2-4ac}}{2a} \quad \text{and} \quad \frac{-b-\sqrt{b^2-4ac}}{2a}$$

if $b^2 - 4ac \geq 0$. Therefore to determine (7.15) we calculate $\mathrm{fr}(g_1, g_2, g_3, g_4, Q)$ at the above critical points and at x and y.

We have

$$\frac{|\overline{A^5A^6}|}{|\overline{A^4}|} \geq \mathrm{fr}([6,\overline{1,5}],[5,\overline{5,1}],[4,\overline{1,5}],[4,\overline{5,1}],1/7) > 1.428, \tag{7.16}$$

$$\frac{|\overline{A^5}|}{|\overline{O_A^4}|} \geq \frac{|\overline{A^6}|}{|\overline{O_A^4}|} \geq \mathrm{fr}([6,\overline{1,5}],[6,\overline{5,1}],[5,\overline{5,1}],[4,\overline{1,5}],1/7) > 1.296,$$

$$\frac{|\overline{A}|}{|\overline{O_A^1}|} \geq \mathrm{fr}([5,\overline{1,5}],[1,\overline{5,1}],[2,\overline{5,1}],[1,\overline{1,5}],1/7) > 8.672, \tag{7.17}$$

$$\frac{|\overline{A^{k+1}A^5}|}{|\overline{O_A^k}|} \geq \frac{|\overline{A^4A^5}|}{|\overline{O_A^3}|} \geq \mathrm{fr}([5,\overline{1,5}],[4,\overline{5,1}],[4,\overline{5,1}],[3,\overline{1,5}],1/7) > 3.542 \tag{7.18}$$

for $k = 1, 2, 3$, and

$$\frac{|\overline{A^k}|}{|\overline{O_A^k}|} \geq \frac{|\overline{A^4}|}{|\overline{O_A^4}|} \geq \mathrm{fr}([4,\overline{1,5}],[4,\overline{5,1}],[5,\overline{5,1}],[4,\overline{1,5}],7/8) > 2.584 \tag{7.19}$$

for $1 \leq k \leq 4$. Also

$$\frac{|B^2|}{|O_{B^1}|} \geq \mathrm{fr}([2,\overline{1,2}],[2,\overline{2,1}],[1,1,\overline{1,2}],[1,2,\overline{2,1}],1/3) > 0.9593$$

and

$$\frac{|B^1|}{|O_B|} \geq \mathrm{fr}([1,\overline{1,2}],[1,\overline{2,1}],[2,\overline{2,1}],[1,\overline{1,2}],3/4) > 0.8501.$$

Therefore

$$\frac{|\overline{A^6}|}{|\overline{O_A^4}|}\cdot\frac{|B^2|}{|O_{B^1}|} > 1.24 \quad \text{and} \quad \frac{|\overline{A^5}|}{|\overline{O_A^4}|}\cdot\frac{|B^1|}{|O_B|} > 1.10\,. \tag{7.20}$$

Further

$$\tau(B \cap C(L_2)) = \text{fr}([2,\overline{1,2}],[2,\overline{2,1}],[2,\overline{2,1}],[1,\overline{1,2}],1/3) > 0.3890$$

so that

$$\tau(B \cap C(L_2))^{-1} < 2.571\ . \tag{7.21}$$

Corollary 7.3.2 follows from (7.11), (7.12), (7.13), (7.14), (7.16), (7.20), (7.21), (7.17), (7.18), (7.19) and Theorem 7.3.1.

□

7.4 $F(5) + F(2)$

There will be three main differences between the proof we will use to establish that $F(5) \pm F(2) = \mathbb{R}$ and the proof of Theorem 7.3.1. Let A and B be bridges of $\mathcal{D}(L_5)$ and $\mathcal{D}(L_2)$ respectively, with $\overline{A} \approx B$. In our proof we will differentiate between the case where $A^1 < A^2$ and $B^1 < B^2$ (or $A^1 > A^2$ and $B^1 > B^2$) and that where $A^1 < A^2$ and $B^1 > B^2$ (or $A^1 > A^2$ and $B^1 < B^2$). We will also have to occasionally descend more than one level at a time in our derivations, using sums of the form $\overline{A^w} + B^u$ where either $|w| > 1$ or $|u| > 1$. Finally, we will occasionally "backtrack", as in (7.24).

Let $J_1 = [x_1, y_1]$ and $J_2 = [x_2, y_2]$ be intervals. We say that J_1 is *almost linked to* J_2 if $x_2 \leq y_1$. Notice that if J_1 and J_2 overlap then J_1 is almost linked to J_2.

Lemma 7.4.1 *Let $n \in \mathbb{Z}^+$, and for $1 \le t \le n$ let $J_t = [x_t, y_t]$ be a non-trivial interval. Assume that $x_1 < y_n$ and that J_t is almost linked to J_{t+1} for $1 \le t \le n-1$. then*

$$[x_1, y_n] \subseteq \bigcup_{t=1}^{n} J_t.$$

Proof. See [14, *Hilfssatz* 1].

□

Lemma 7.4.2 *Let I_1 and I_2 be two bridges of $\mathcal{D}^5(L_5)$ and $\mathcal{D}^2(L_2)$ respectively, with*

$$I_1 \notin \{I(L_5), I(L_5)^1\}, \qquad I_2 \notin \{I(L_2), I(L_2)^1\}$$

and $\overline{I_1} \approx I_2$. Then

$$\overline{I_1} + I_2 \subseteq (I_1 \cap F(5)) + (I_2 \cap F(2)) \quad \textit{and} \quad \overline{I_1} - I_2 \subseteq (I_1 \cap F(5)) - (I_2 \cap F(2)).$$

Proof. Let A and B be bridges of $\mathcal{D}(L_5)^5$ and $\pm\mathcal{D}(L_2)^2$ respectively, with

$$A \notin \{I(L_5), I(L_5)^1\} \quad \text{and} \quad B \notin \{\pm I(L_2), \pm I(L_2)^1\}.$$

As in the proof of Theorem 7.3.1 we will use induction over $\min\{|w|, |v|\}$ to prove that for $1 \le k \le 3$, if $\overline{A^k A^4} \approx B$ then for any $n \ge 0$, $\overline{A^k A^4} + B$ can be covered by a finite union of intervals of the form

$$\overline{A^{wt} A^{w}4} + B^v$$

where $\min\{|w|, |v|\} > n$. Thus by letting n tend to infinity we have

$$\overline{A^k A^4} + B \subseteq (\overline{A^k A^4} \cap F(5)) + (B \cap F(2))$$

or

$$\overline{A^kA^4} + B \subseteq (\overline{A^kA^4} \cap F(5)) + (B \cap (-F(2)))$$

as required.

Let A and B be bridges of $\mathcal{D}^5(L_5)$ and $\pm\mathcal{D}^2(L_2)$ respectively with

$$A \notin \{I(L_5), I(L_5)^1\} \quad \text{and} \quad B \notin \{\pm I(L_2), \pm I(L_2)^1\}.$$

and let k be an integer with $1 \leq k \leq 3$ and $\overline{A^kA^4} \approx B$. Note first that

$$|\overline{A^1}| > |\overline{A^2}| > \cdots > |\overline{A^5}|, \qquad |\overline{O_A^1}| > |\overline{O_A^2}| > \cdots > |\overline{O_A^4}|$$

and

$$|B^1| > |B^2|.$$

We must show that $\overline{A^kA^4} + B$ can be covered by a union of intervals of the form $\overline{A^{wt}A^{w4}} + B^v$, where either $t > k$ or $w \neq \emptyset$ or $v \neq \emptyset$, and $\overline{A^{wt}A^{w4}} \approx B^v$.

The rationale followed when determining the covering intervals is simple. In many cases it was found that one could cover $\overline{A^kA^4} + B$ if intervals of the form $A^w + B^v$ were used, but not if they were replaced with those of the form $\overline{A^w} + B^v$. To make up the difference we add to the union intervals of the form $\overline{A^{w3}} + B^{vu}$. To maintain B-compatibility it is usually necessary to have $u \neq \emptyset$. This of course makes it difficult to ensure that $\overline{A^w} + B^v$ is almost linked to $\overline{A^{w3}} + B^{vu}$.

We may assume without loss of generality that $A^1 < A^2$. We first examine the case when $B^1 < B^2$. We identify two situations.

Situation 1: $k = 1$ or $k = 2$

Case 1: $|B^2| \geq |\overline{O_A^k}|$

Then $\overline{A^kA^4} \approx B^2$ and $\overline{A^kA^4} \approx B^1$ since $|B^1| > |B^2|$. Also,

$$\overline{A^kA^4} + B = (\overline{A^kA^4} + B^1) \cup (\overline{A^kA^4} + B^2).$$

Case 2: $|B^2| < |\overline{O_A^k}|$

If $k = 1$ then $|\overline{A^{k+1}A^4}| > |O_B|$ by (A.6) so $\overline{A^{k+1}A^4} \approx B$. If $k = 2$ then define w and v by letting $A = I(L_5)^w$ and $B = \pm I(L_2)^v$. If $v \neq (1\ 2)$ then by (A.7) we have $|\overline{A^{k+1}A^4}| > |O_B|$. Otherwise $v = (1\ 2)$. We find by calculation that if $w \neq (2)$ and $w \neq (1\ 1)$ then $|\overline{O_A^2}| < |B^2|$, a contradiction, whence either $w = (2)$ or $w = (1\ 1)$. In both of these cases we have

$$|\overline{A^3A^4}| > 0.02 > |O_B|$$

and thus if $k = 2$ we have $|\overline{A^{k+1}A^4}| > |O_B|$, and so $\overline{A^{k+1}A^4} \approx B$.

Case 2(a): $|\overline{A^k}| \geq |O_B|$

In this case $\overline{A^k} \approx B$ and

$$\overline{A^kA^4} + B = (\overline{A^k} + B) \cup (\overline{A^{k+1}A^4} + B).$$

Case 2(b): $|\overline{A^k}| < |O_B|$

We have that $|B^1| > |\overline{O_A^k}|$ by (A.5), thus $\overline{A^kA^4} \approx B^1$ and

$$\overline{A^kA^4} + B = (\overline{A^kA^4} + B^1) \cup (\overline{A^{k+1}A^4} + B).$$

Situation 2: $k = 3$

Case 1: $|B^2| \geq |\overline{O_A^3}|$

In this case $\overline{A^3A^4} \approx B^1$, $\overline{A^3A^4} \approx B^2$ and

$$\overline{A^3A^4} + B = (\overline{A^3A^4} + B^1) \cup (\overline{A^3A^4} + B^2).$$

Case 2: $|\overline{A^4}| \geq |O_B|$

Then $\overline{A^3} \approx B$, $\overline{A^4} \approx B$ and

$$\overline{A^3A^4} + B = (\overline{A^3} + B) \cup (\overline{A^4} + B).$$

Case 3: $|B^2| < |\overline{O^3_A}|$ and $|\overline{A^4}| < |O_B|$

We have

$$|\overline{A^4A^5}| > |O_B|, \qquad |\overline{A^4}| > |O_{B^2}| \quad \text{and} \quad |\overline{A^5}| > |O_{B^{11}}| \tag{7.22}$$

by (A.8), (A.12) and (A.11). Also,

$$|B^1| > |\overline{O^3_A}| \quad \text{and} \quad |B^2| > |\overline{O^1_{A^4}}| \tag{7.23}$$

by (A.16) and (A.17) respectively. By (7.22) and (7.23) it follows that

$$\overline{A^3A^4} \approx B^1 \quad \text{and} \quad \overline{A^4} \approx B^2.$$

Case 3(a): $|\overline{A^5}| \geq |O_{B^1}|$

Then since $|B^1| > |\overline{O^3_A}| > |\overline{O^1_{A^5}}|$ we have $\overline{A^5} \approx B^1$. Since $|B^1| > |\overline{O^3_A}| > |\overline{O^4_A}|$ we know that $\overline{A^3A^4} + B^1$ is almost linked to $\overline{A^5} + B^1$. Also $\overline{A^5} + B^1$ is almost linked to $\overline{A^4} + B^2$, as $|\overline{A^4A^5}| > |O_B|$. Thus

$$\overline{A^3A^4} + B \subseteq (\overline{A^3A^4} + B^1) \cup (\overline{A^5} + B^1) \cup (\overline{A^4} + B^2)$$

by Lemma 7.4.1.

Case 3(b): $|\overline{A^5}| < |O_{B^1}|$

In this case $|B^{11}| > |\overline{O^4_A}| > |\overline{O^1_{A^5}}|$ by (A.18). With (7.22) we have

$$\overline{A^5} \approx B^{11}.$$

Also, since $|B^{11}| > |\overline{O^4_A}|$ we know that $\overline{A^3A^4} + B^1$ is almost linked to $\overline{A^5} + B^{11}$. Similar to the above we find that $\overline{A^5} + B^{11}$ is almost linked to $\overline{A^4} + B^2$, since $|\overline{A^4A^5}| > |O_B|$. Therefore

$$\overline{A^3A^4} + B \subseteq (\overline{A^3A^4} + B^1) \cup (\overline{A^5} + B^{11}) \cup (\overline{A^4} + B^2)$$

by Lemma 7.4.1.

We now assume that $B^1 > B^2$. Cases 1 and 2(a) of Situation 1 may be handled in the same way as for $B^1 < B^2$. Also, the proofs for Case 1 and Case 2 of Situation 2 are the same as in the case where $B^1 < B^2$. Thus it suffices to examine Situation 1, Case 2(b) and Situation 2, Case 3.

Situation 1: $k = 1$ or $k = 2$
Case 2(b): $|B^2| < |\overline{O_A^k}|$ and $|\overline{A^k}| < |O_B|$
As in the case where $B^1 < B^2$ we have

$$\overline{A^{k+1}A^4} \approx B \quad \text{and} \quad \overline{A^kA^4} \approx B^1.$$

Also $|B^2| > |\overline{O^1_{A^k}}|$ and $|O_{B^2}| < |B^2| < |\overline{O_A^k}| < |\overline{A^k}|$ by (A.5) and (A.4), whence $\overline{A^k} \approx B^2$. Further, $|\overline{A^{k5}}| > |O_{B^{112}}|$ and $|B^{112}| > |\overline{O^4_{A^k}}|$ by (A.20) and (A.21), so $\overline{A^{k5}} \approx B^{112}$, and $\overline{A^{k5}} + B^{112}$ is almost linked to $\overline{A^kA^4} + B^1$.

Assume that $k = 1$. Then $|A^1| - |A^{1L}| - |A^{15L}| > |O_B|$ by (A.22), so $\overline{A^1} + B^2$ is almost linked to $\overline{A^{15}} + B^{112}$. Therefore by Lemma 7.4.1

$$\overline{A^1A^4} + B \subseteq (\overline{A^1} + B^2) \cup (\overline{A^{15}} + B^{112}) \cup (\overline{A^1A^4} + B^1).$$

Assume next that $k = 2$. By (A.24) we have

$$|A^2| - |A^{2L}| - |A^{25L}| > |O_B| \quad \text{or} \quad |B^1| > |O_A^1| + |A^{1L}| + |A^{25L}|.$$

If $|A^2| - |A^{2L}| - |A^{25L}| > |O_B|$ then $\overline{A^2} + B^2$ is almost linked to $\overline{A^{25}} + B^{112}$, so

$$\overline{A^2A^4} + B \subseteq (\overline{A^2} + B^2) \cup (\overline{A^{25}} + B^{112}) \cup (\overline{A^2A^4} + B^1)$$

by Lemma 7.4.1. Otherwise $|B^1| > |O_A^1| + |A^{1L}| + |A^{25L}|$ so $\overline{A^1} + B$ is almost linked to $\overline{A^{25}} + B^{112}$. Now $|\overline{A^1}| > |O_B|$ and $|B| > |\overline{O^1_{A^1}}|$ by (A.27), hence $\overline{A^1} \approx B$ and

$$\overline{A^2A^4} + B \subseteq (\overline{A^2} + B^2) \cup (\overline{A^1} + B) \cup (\overline{A^{25}} + B^{112}) \cup (\overline{A^2A^4} + B^1)$$

by Lemma 7.4.1.

Situation 2: $k = 3$

Case 3: $|B^2| < |\overline{O_A^3}|$ and $|\overline{A^4}| < |O_B|$

Then $|\overline{A^3}| > |O_{B^2}|$ and $|B^2| > |\overline{O_{A^3}^1}|$ by (A.19) and (A.31), whence $\overline{A^3} \approx B^2$. Also, $|B^1| > |\overline{O_A^3}|$ by (A.16), so $\overline{A^3A^4} \approx B^1$.

Case 3(a): $|\overline{A^3}| \geq |O_B|$

In this case

$$\overline{A^3A^4} + B \subseteq (\overline{A^3} + B) \cup (\overline{A^3A^4} + B^1)$$

where $\overline{A^3} \approx B$ and $\overline{A^3A^4} \approx B^1$.

Case 3(b): $|\overline{A^3}| < |O_B|$

Then $|B^1| > |\overline{O_A^2}|$ by (A.32), so $\overline{A^2A^4} \approx B^1$. Since $|O_B| \leq |\overline{A^3A^4}| < |\overline{A^2A^3}|$ we have

$$\overline{A^3A^4} + B \subseteq (\overline{A^3} + B^2) \cup (\overline{A^2A^4} + B^1). \tag{7.24}$$

The lemma follows by induction on the level of the bridges used in the sums, as described at the beginning of the proof.

□

Theorem 7.4.3

$$F(5) + F(2) = \mathbb{R} \quad \textit{and} \quad F(5) - F(2) = \mathbb{R}.$$

Proof. Let I_1 and I_2 be the roots of $\mathcal{D}^5(L_5)$ and $D^2(L_2)$ respectively. Note that

$$\begin{array}{cccc}
|\overline{I_1^{11}}| = 0.099\ldots, & |\overline{I_1^{12}}| = 0.050\ldots, & |\overline{I_1^{13}}| = 0.030\ldots, & |\overline{I_1^{14}}| = 0.020\ldots, \\
|\overline{O^1_{I_1^{11}}}| = 0.015\ldots, & |\overline{O^1_{I_1^{12}}}| = 0.007\ldots, & |\overline{O^1_{I_1^{13}}}| = 0.004\ldots, & |\overline{O^1_{I_1^{14}}}| = 0.003\ldots, \\
|\overline{I_1^{2}}| = 0.099\ldots, & |\overline{I_1^{3}}| = 0.050\ldots, & |\overline{I_1^{4}}| = 0.030\ldots, & |\overline{I_1^{5}}| = 0.020\ldots, \\
|\overline{O^1_{I_1^{2}}}| = 0.015\ldots, & |\overline{O^1_{I_1^{3}}}| = 0.007\ldots, & |\overline{O^1_{I_1^{4}}}| = 0.004\ldots, & |\overline{O^1_{I_1^{5}}}| = 0.003\ldots.
\end{array}$$

Also

$$\begin{array}{cccc}
|I_2^{2}| = 0.056\ldots, & |I_2^{11}| = 0.056\ldots, & |I_2^{12}| = 0.029\ldots, & |I_2^{21}| = 0.0219\ldots, \\
|O_{I_2^{2}}| = 0.024\ldots, & |O_{I_2^{11}}| = 0.024\ldots, & |O_{I_2^{12}}| = 0.012\ldots, & |O_{I_2^{21}}| = 0.009\ldots.
\end{array}$$

Therefore

$$\begin{array}{cccc}
\overline{I_1^{5}} \approx I_2^{21}, & \overline{I_1^{4}} \approx I_2^{2}, & \overline{I_1^{3}} \approx I_2^{2}, & \overline{I_1^{2}} \approx I_2^{2}, \\
\overline{I_1^{5}} \approx I_2^{12}, & \overline{I_1^{11}} \approx I_2^{2}, & \overline{I_1^{12}} \approx I_2^{2}, & \overline{I_1^{11}} \approx I_2^{11}, \\
\overline{I_1^{11}} \approx I_2^{12}, & \overline{I_1^{13}} \approx I_2^{11}, & \overline{I_1^{12}} \approx I_2^{12}, & \overline{I_1^{13}} \approx I_2^{12}, \\
\overline{I_1^{14}} \approx I_2^{12}, & \overline{I_1^{2}} \approx I_2^{11}, & \overline{I_1^{2}} \approx I_2^{12}, & \overline{I_1^{12}} \approx I_2^{11}, \\
\overline{I_1^{13}} \approx I_2^{2}, & \overline{I_1^{14}} \approx I_2^{21}, & \overline{I_1^{4}} \approx I_2^{11}, & \overline{I_1^{3}} \approx I_2^{12}
\end{array}$$

and

$$\overline{I_1^{4}} \approx I_2^{12}.$$

Now

$$\overline{I_1^5} + I_2^{21} = [0.537\ldots, 0.580\ldots], \qquad \overline{I_1^4} + I_2^2 = [0.573\ldots, 0.660\ldots],$$
$$\overline{I_1^3} + I_2^2 = [0.627\ldots, 0.734\ldots], \qquad \overline{I_1^2} + I_2^2 = [0.719\ldots, 0.875\ldots],$$
$$\overline{I_1^5} + I_2^{12} = [0.874\ldots, 0.924\ldots], \qquad \overline{I_1^{11}} + I_2^2 = [0.912\ldots, 1.069\ldots],$$
$$\overline{I_1^{12}} + I_2^2 = [1.054\ldots, 1.161\ldots], \qquad \overline{I_1^{11}} + I_2^{11} = [1.124\ldots, 1.280\ldots],$$
$$\overline{I_1^{11}} + I_2^{12} = [1.249\ldots, 1.378\ldots], \qquad \overline{I_1^{13}} + I_2^{11} = [1.339\ldots, 1.426\ldots],$$
$$\overline{I_1^{12}} + I_2^{12} = [1.391\ldots, 1.470\ldots], \qquad \overline{I_1^{13}} + I_2^{12} = [1.465\ldots, 1.524\ldots]$$

and

$$\overline{I_1^{14}} + I_2^{12} = [1.510\ldots, 1.560\ldots].$$

hence by Lemma 7.4.2 we have

$$[0.538, 1.560] \subseteq F(5) + F(2)$$

so $F(5) + F(2) = \mathbb{R}$. Further,

$$\overline{I_1^5} - I_2^{12} = [-0.560\ldots, -0.510\ldots], \qquad \overline{I_1^4} - I_2^{12} = [-0.525\ldots, -0.465\ldots],$$
$$\overline{I_1^3} - I_2^{12} = [-0.470\ldots, -0.391\ldots], \qquad \overline{I_1^4} - I_2^{11} = [-0.426\ldots, -0.339\ldots],$$
$$\overline{I_1^2} - I_2^{12} = [-0.378\ldots, -0.249\ldots], \qquad \overline{I_1^2} - I_2^{11} = [-0.280\ldots, -0.124\ldots],$$
$$\overline{I_1^{11}} - I_2^{12} = [-0.185\ldots, -0.056\ldots], \qquad \overline{I_1^{11}} - I_2^{11} = [-0.087\ldots, 0.069\ldots],$$
$$\overline{I_1^{12}} - I_2^{11} = [0.054\ldots, 0.161\ldots], \qquad \overline{I_1^{11}} - I_2^2 = [0.124\ldots, 0.280\ldots],$$
$$\overline{I_1^{12}} - I_2^2 = [0.265\ldots, 0.372\ldots], \qquad \overline{I_1^{13}} - I_2^2 = [0.339\ldots, 0.426\ldots]$$

and

$$\overline{I_1^{14}} - I_2^{21} = [0.419\ldots, 0.462\ldots].$$

Therefore by Lemma 7.4.2 we have

$$[-0.560, 0.462] \subseteq F(5) - F(2)$$

and the theorem follows.

□

7.5 $F(3) + F(3)$

The proof that $F(3) + F(3)$ contains an interval will be more complicated than the proofs in the previous two sections. Heuristically this is to be expected, since $\tau(L_3)\tau(L_3) < \tau(L_5)\tau(L_2)$. However, we will have an advantage we did not have before, that since we will be adding together two copies of the same Cantor set we can use sums of the form $\overline{B^w} + A^u$ as well as those of the form $\overline{A^u} + B^w$.

When comparing an interval of the form $\overline{B^v} + A^w$ with one of the form $\overline{A^s} + B^t$ the following notation will be useful. If m and n are integers with $1 \leq m, n \leq 3$ and A is a bridge of $\mathcal{D}^3(L_3)$ we put

$$\ulcorner\overline{A^m A^n} := \overline{A^n A^m}\urcorner = A^m \cup \overline{A^m A^n}.$$

We also define

$$\ulcorner\overline{A} = A^L \cup \overline{A}, \qquad \overline{A}\urcorner = \overline{A} \cup A^R$$

and for $k = 1$ or $k = 2$ put

$$\ulcorner\overline{O^k_A} = O^k_A \cup A^{(k+1)R} \quad \text{and} \quad \overline{O^k_A}\urcorner = A^{kL} \cup O^k_A.$$

Lemma 7.5.1 *Let I_1 and I_2 be two bridges of $\mathcal{D}^3(L_3)$, with*

$$I_1, I_2 \not\in \{I(L_3), I(L_3)^1\}$$

and $\overline{I_1} \approx I_2$. *Then*

$$\overline{I_1} + I_2 \subseteq (I_1 \cap F(3)) + (I_2 \cap F(3)) \quad \textit{and} \quad \overline{I_1} - I_2 \subseteq (I_1 \cap F(3)) - (I_2 \cap F(3)).$$

Proof. Our proof will be similar to those of Theorem 7.3.1 and Lemma 7.4.2. Let A be a bridge of $\mathcal{D}^3(L_3)$ and B be bridges of $\pm\mathcal{D}^3(L_3)$ with $A, B \neq \pm I(L_3)$ and $A, B \neq \pm I(L_3)^1$, and let $k = 1$ or $k = 2$. We will show that if $\overline{A} \approx B^k B^3$ then $\overline{A} + B^k B^3$ may be covered by a union of intervals, where each interval is either of the form

$$\overline{A^w} + B^{vt} B^{v3} \tag{7.25}$$

where $\overline{A^w} \approx B^{vt} B^{v3}$ and either $t > k$ or $w \neq \emptyset$ or $v \neq \emptyset$, or of the form

$$\overline{B^v} + A^{wr} A^{w3} \tag{7.26}$$

where $\overline{B^v} \approx A^{wr} A^{w3}$ and $r = 1$ or $r = 2$, and either $w \neq \emptyset$ or $v \neq \emptyset$. As in the proofs of Theorem 7.3.1 and Lemma 7.4.2 it follows that we can cover $\overline{A} + B^k B^3$ by intervals of the form (7.25) or (7.26) where $\min\{|w|, |v|\}$ is arbitrarily large. Thus

$$\overline{A} + B^k B^3 \subseteq (A \cap F(3)) + (B^k B^3 \cap F(3))$$

if $B \subseteq \mathbb{R}^+$, while

$$\overline{A} + B^k B^3 \subseteq (A \cap F(3)) + (B^k B^3 \cap (-F(3)))$$

if $B \subseteq \mathbb{R}^-$, and the lemma follows.

Let A and B be bridges of $\pm\mathcal{D}^3(L_3)$ with $A, B \neq \pm I(L_3)$ and $A, B \neq I(L_3)^1$, and assume that $k = 1$ or $k = 2$, and that $\overline{A} \approx B^k B^3$. We must cover $\overline{A} + B^k B^3$ by a union of intervals of the form (7.25) or (7.26) where $t > k$ or $w \neq \emptyset$ or $v \neq \emptyset$. Without loss of generality we may assume that $A^1 < A^2$. Assume first that $B^1 < B^2$.

Situation 1: $k = 1$

Case 1: $\min\{|B^1|, |B^2B^3|\} \geq |\overline{O^1_A}|$

Then $\overline{A} \approx B^1$, $\overline{A} \approx B^2B^3$ and

$$\overline{A} + B = (\overline{A} + B^1) \cup (\overline{A} + B^2B^3).$$

Case 2: $|\overline{A^2}| \geq |O^1_B|$

Since $|\overline{A^1}| > |\overline{A^2}|$ we know that $\overline{A^1} \approx B$, $\overline{A^2} \approx B$ and

$$\overline{A} + B = (\overline{A^1} + B) \cup (\overline{A^2} + B).$$

Case 3: $\min\{|B^1|, |B^2B^3|\} < |\overline{O^1_A}|$ and $|\overline{A^2}| < |O^1_B|$

In this case $|\overline{A^1}| > |O^1_B|$, $\min\{|A^1|, |A^2A^3|\} > |\overline{O^1_B}|$, $|\overline{A^2}| > |O^2_B|$, and $|\overline{B}| > |O^1_A|$ by (B.12), (B.13), (B.14) and (B.15). Further,

$$|\overline{O^1_{A^2}}| < |\overline{A^2}| < |O^1_B| < |B^2B^3|.$$

Since $|\overline{B}| > |O^1_A|$ we know that $|\overline{B}| > |O^1_{A^1}|$ and $|\overline{B}| > |O^2_A|$. Therefore

$$\overline{A^1} \approx B, \qquad \overline{B} \approx A^1, \qquad \overline{B} \approx A^2A^3, \qquad \overline{A^2} \approx B^2B^3$$

and

$$\overline{A} + B \subseteq (\overline{A^1} + B) \cup (\overline{B} + A^1) \cup (\overline{B} + A^2A^3) \cup (\overline{A^2} + B^2B^3).$$

Note that $\overline{B} + A^1$ is almost linked to $\overline{B} + A^2A^3$ since $|\overline{B}| > |O^1_A|$.

Situation 2: $k = 2$

Case 1: $|B^2| \geq |\overline{O^1_A}|$

Then

$$\overline{A} \approx B^2 \tag{7.27}$$

and

$$|\overline{B^2}| > |O^2_A|, \qquad |B^3| > |\overline{O^1_{A^2}}|, \tag{7.28}$$

$$|B^{21}| > |\overline{O^1_{A^3}}| \quad \text{and} \quad |\overline{B^{213}}| > |O^2_{A^{31}}| \tag{7.29}$$

by (B.18), (B.33), (B.31) and (B.32).

Case 1(a): $|B^3| \geq |\overline{O^1_A}|$

Then $\overline{A} \approx B^2$, $\overline{A} \approx B^3$ and

$$\overline{A} + B^2B^3 = (\overline{A} + B^2) \cup (\overline{A} + B^3).$$

Case 1(b): $|\overline{A^2A^3}| \geq |\overline{O^2_B}|$ and $|B^3| < |\overline{O^1_A}|$

Then

$$|A^2A^3| > |\overline{O^1_{B^2}}| \tag{7.30}$$

by (B.18), and

$$|\overline{A^2}| > |O^1_{B^3}| \tag{7.31}$$

by (B.33). Thus

$$\overline{A} \approx B^2, \qquad \overline{B^2} \approx A^2A^3 \quad \text{and} \quad \overline{A^2} \approx B^3$$

by (7.27), (7.28), (7.30) and (7.31). Also $\overline{B^2} + A^2A^3$ is almost linked to $\overline{A^2} + B^3$ since $|\overline{A^2A^3}| \geq |\overline{O^2_B}|$. Thus

$$\overline{A} + B^2B^3 = (\overline{A} + B^2) \cup (\overline{B^2} + A^2A^3) \cup (\overline{A^2} + B^3).$$

Case 1(c): $|\overline{A^2A^3}| < |\overline{O^2_B}|$ and $|B^3| < |\overline{O^1_A}|$

In this case

$$|\overline{B^2}| > |O^1_A| \tag{7.32}$$

by (B.17). Also

$$|\overline{A^2}| > |O^1_{B^3}|, \qquad |\overline{A^3}| > |O^1_{B^{21}}| \quad \text{and} \quad |A^{312}A^{313}| > |\overline{O^1_{B^{213}}}| \tag{7.33}$$

by (B.33), (B.31) and (B.32). Thus

$$\overline{A} \approx B^2, \qquad \overline{B^2} \approx A, \qquad \overline{A^3} \approx B^{21}, \qquad \overline{B^{213}} \approx A^{312}A^{313} \quad \text{and} \quad \overline{A^2} \approx B^3$$

by (7.27), (7.32), (7.29), (7.33) and (7.28).

Case 1(c)(i): $|\overline{A^2A^3}'| \geq |O^2_B| + |B^{213L}|$
Then $\overline{B^{213}} + A^{312}A^{313}$ is almost linked to $\overline{A^2} + B^3$, so

$$\overline{A} + B^2B^3 \subseteq (\overline{A} + B^2) \cup (\overline{B^2} + A) \cup (\overline{A^3} + B^{21}) \cup (\overline{B^{213}} + A^{312}A^{313}) \cup (\overline{A^2} + B^3).$$

Case 1(c)(ii): $|\overline{A^2A^3}'| < |O^2_B| + |B^{213L}|$
Say $A = \pm I(L_3)^w$ and $\pm B = I(L_3)^v$. Assume first that $w = (1\ 3)$. If $v \neq (1\ 1)$ and $v \neq (2)$ then by calculation we have

$$|\overline{O^2_B}'| < |\overline{A^2A^3}'|.$$

Further, if $v = (1\ 1)$ or $v = (2)$ then

$$|B^3| > 0.008 > |\overline{O^1_A}|.$$

However we know that

$$|\overline{A^2A^3}'| < |\overline{O^2_B}'| \quad \text{and} \quad |B^3| < |\overline{O^1_A}|$$

hence we must have $w \neq (1\ 3)$. Therefore $|B^{33}| > |\overline{O^2_{A^2}}| > |\overline{O^1_{A^{23}}}|$ by (B.16), $|\overline{A^{23}}| > |O^1_{B^{33}}|$ by (B.42), $|\overline{B^3}| > |O^1_{A^1}|$ by (B.43) and $|A^1| > |\overline{O^1_{B^3}}|$ by (B.43). Therefore

$$\overline{B^3} \approx A^1 \quad \text{and} \quad \overline{A^{23}} \approx B^{33}.$$

Now $\overline{B^2}+A$ is almost linked to $\overline{B^3}+A^1$, since $|A| > |\overline{O^2_B}|$. Also $|\overline{B^3}| > |O^1_A|+|A^{23L}|$ by (B.44), so $\overline{B^3}+A^1$ is almost linked to $\overline{A^{23}}+B^{33}$. Finally, $\overline{A^{23}}+B^{33}$ is almost linked to $\overline{A^2}+B^3$, as $|B^{33}| > |\overline{O^2_{A^2}}|$. Therefore

$$\overline{A}+B^2B^3 = (\overline{A}+B^2) \cup (\overline{B^2}+A) \cup (\overline{B^3}+A^1) \cup (\overline{A^{23}}+B^{33}) \cup (\overline{A^2}+B^3).$$

Case 2: $|B^2| < |\overline{O^1_A}|$

We have $|\overline{A^1}| > |O^2_B|$ by (B.48), hence

$$\overline{A^1} \approx B^2B^3.$$

Case 2(a): $|\overline{A^2}| \geq |O^2_B|$

Then $\overline{A^2} \approx B^2B^3$ and

$$\overline{A}+B^2B^3 = (\overline{A^1}+B^2B^3) \cup (\overline{A^2}+B^2B^3).$$

Case 2(b): $|\overline{A^2}| < |O^2_B|$

Then

$$|\overline{A^2}| > |O^1_{B^3}| \quad \text{and} \quad |B^3| > |\overline{O^1_{A^2}}| \tag{7.34}$$

by (B.63),

$$|\overline{A^2}| > |O^1_{B^2}| \quad \text{and} \quad |B^2| > |\overline{O^1_{A^2}}|$$

by (B.62) and (7.34), since $|B^2| > |B^3|$, and

$$|A^3| > |\overline{O^1_{B^2}}| \quad \text{and} \quad |\overline{B^2}| > |O^1_{A^3}|$$

by (B.48) and (B.64). Therefore

$$\overline{A^2} \approx B^3, \qquad \overline{A^2} \approx B^2 \quad \text{and} \quad A^3 \approx \overline{B^2}.$$

Also, $|\overline{O_A^2}| < |\overline{B^2}|$ by (B.65), so $\overline{A^2} + B^2$ intersects $\overline{B^2} + A^3$. Similarly $|\overline{O_B^2}| < |\overline{A^2 A^3}|$ so $\overline{B^2} + A^3$ intersects $\overline{A^2} + B^3$. Hence we have

$$\overline{A} + B^2B^3 = (\overline{A^1} + B^2B^3) \cup (\overline{A^2} + B^2) \cup (\overline{B^2} + A^3) \cup (\overline{A^2} + B^3).$$

We now assume that $B^1 > B^2$. Cases 1 and 2 of Situation 1 may be dealt with in the same manner as those where $B^1 < B^2$, so it suffices to examine Situation 1, Case 3 and Situation 2.

Situation 1: $k = 1$

Case 3: $\min\{|B^1|, |B^2B^3|\} < |\overline{O_A^1}|$ and $|\overline{A^2}| < |O_B^1|$

As in Situation 1 we have

$$\overline{A^1} \approx B, \qquad \overline{B} \approx A^1 \quad \text{and} \quad \overline{B} \approx A^2A^3.$$

Also,

$$|\overline{A^2}| > |O_{B^1}^1| \quad \text{and} \quad |B^1| > |O_B^1| > |\overline{A^2}| > |\overline{O_{A^2}^1}|$$

by (B.81) and (B.2), so

$$\overline{A} + B \subseteq (\overline{A^1} + B) \cup (\overline{B} + A^1) \cup (\overline{B} + A^2A^3) \cup (\overline{A^2} + B^1).$$

As before, $\overline{B} + A^1$ is almost linked to $\overline{B} + A^2A^3$ since $|\overline{B}| > |O_A^1|$.

Situation 2: $k = 2$

Case 1: $|B^3| \geq |\overline{O_A^1}|$

Then $\overline{A} \approx B^2$, $\overline{A} \approx B^3$ and

$$\overline{A} + B^3B^2 = (\overline{A} + B^3) \cup (\overline{A} + B^2).$$

Case 2: $|\overline{A^1}| \geq |O_B^2|$

In this case $\overline{A^1} \approx B^3B^2$. Also we have

$$|\overline{A^2}| > |O^1_{B^3}|, \qquad |A^2| > |\overline{O^1_{B^2}}|, \qquad |A^3| > |\overline{O^1_{B^3}}| \quad \text{and} \quad |\overline{A^2}| > |O^1_{B^2}| \qquad (7.35)$$

by (B.77), (B.78), (B.89) and (B.62).

Case 2(a): $|\overline{A^2}| \geq |O_B^2|$

Then $\overline{A^2} \approx B^3B^2$ and

$$\overline{A} + B^3B^2 = (\overline{A^1} + B^3B^2) \cup (\overline{A^2} + B^3B^2).$$

Note that $B^3B^2 = B^2B^3$ hence $\overline{A^1} + B^3B^2$ and $\overline{A^2} + B^3B^2$ are both of the form (7.25). We have used the notation B^3B^2 rather than B^2B^3 to remind the reader that $B^3 < B^2$.

Case 2(b): $|B^2| \geq |\overline{O_A^1}|$

We have $\overline{A} \approx B^2$ and

$$\overline{A} + B^3B^2 = (\overline{A^1} + B^3B^2) \cup (\overline{A} + B^2).$$

Case 2(c): $|\overline{A^2}| < |O_B^2|$ and $|B^2| < |\overline{O_A^1}|$

Then $|B^3| > |\overline{O^1_{A^2}}|$, $|\overline{B^2}| > |O^1_{A^2}|$ and $|B^2| > |\overline{O^1_{A^2}}|$ by (B.77), (B.78) and (B.62), so with (7.35) we have

$$\overline{A^2} \approx B^3, \qquad \overline{B^2} \approx A^2 \quad \text{and} \quad \overline{A^2} \approx B^2.$$

Also

$$|\overline{B^{213}}| > |O^1_{A^{23}}| \quad \text{and} \quad |\overline{B^3}| > |O^1_{A^3}|$$

by (B.79) and (B.89). Further,

$$|\overline{A^{213}}| > |O^1_{B^{33}}| \quad \text{and} \quad |A^3| > |\overline{O^1_{B^3}}| \qquad (7.36)$$

by (B.90) and (B.89). Therefore

$$\overline{B^3} \approx A^3.$$

Case 2(c)(i): $|\ulcorner\overline{A^2}| \geq |O^2_B| + |(B^{213})^L|$

Then $\overline{A^2} + B^3$ is almost linked to $\overline{B^{213}} + A^{23}$. Also $|A^{23}| > |\overline{O^2_{B^{21}}}| > |\overline{O^1_{B^{213}}}|$ by (B.67) so $\overline{B^{213}} + A^{23}$ is almost linked to $\overline{B^2} + A^2$. Thus

$$\overline{A} + B^3B^2 \subseteq (\overline{A^1} + B^3B^2) \cup (\overline{A^2} + B^3) \cup (\overline{B^{213}} + A^{23}) \cup (\overline{B^2} + A^2) \cup (\overline{A^2} + B^2).$$

Case 2(c)(ii): $|\ulcorner\overline{A^2}| < |O^2_B| + |(B^{213})^L|$

Then $|B^{33}| > |\overline{O^2_{A^{21}}}| > |\overline{O^1_{A^{213}}}|$ by (B.69), so $\overline{A^2} + B^3$ is almost linked to $\overline{A^{213}} + B^{33}$, and with (7.36) we have $\overline{A^{213}} \approx B^{33}$.

Case 2(c)(ii) (A): $|\overline{B^3}^{\urcorner}| \geq |O^2_A| + |(A^{213})^L|$

In this case $\overline{A^{213}} + B^{33}$ is almost linked to $\overline{B^3} + A^3$. Also $|\overline{A^2A^3}^{\urcorner}| > |\overline{O^2_B}^{\urcorner}|$ by (B.90), so $\overline{B^3} + A^3$ is almost linked to $\overline{A^2} + B^2$. We have

$$\overline{A} + B^3B^2 = (\overline{A^1} + B^3B^2) \cup (\overline{A^2} + B^3) \cup (\overline{A^{213}} + B^{33}) \cup (\overline{B^3} + A^3) \cup (\overline{A^2} + B^2).$$

Case 2(c)(ii) (B): $|\overline{B^3}^{\urcorner}| < |O^2_A| + |(A^{213})^L|$

Then $|A^2| - |(A^{213})^L| > |O^2_B| + |(B^{213})^L|$ by (B.70) hence $\overline{A^{213}} + B^{33}$ is almost linked to $\overline{B^{213}} + A^{23}$. Additionally $|A^{23}| > |\overline{O^2_{B^{21}}}|$ by (B.71), so as in Case 2(c)(i) we have $\overline{B^{213}} \approx A^{23}$. Hence

$$\overline{A}+B^3B^2 = (\overline{A^1}+B^3B^2)\cup(\overline{A^2}+B^3)\cup(\overline{A^{213}}+B^{33})\cup(\overline{B^{213}}+A^{23})\cup(\overline{B^2}+A^2)\cup(\overline{A^2}+B^2).$$

Case 3: $|\overline{A^1}| < |O^2_B|$ and $|B^3| < |\overline{O^1_A}|$

Then $|B^2| > |\overline{O^1_A}|$ by (B.12), so $\overline{A} \approx B^2$. Also $|A| > |\overline{O^1_{B^2}}|$, $|\overline{B^2}| > |O^1_A|$, $|\overline{A^1}| > |O^1_{B^3}|$, $|B^3| > |\overline{O^1_{A^1}}|$, $|A^{13}| > |\overline{O^2_{B^{21}}}| > |\overline{O^1_{B^{213}}}|$ and $|\overline{B^{213}}| > |O^1_{A^{13}}|$ by (B.107), (B.104), (B.106) and (B.105), hence

$$\overline{B^2} \approx A, \qquad \overline{A^1} \approx B^3, \qquad \overline{B^{213}} \approx A^{13}$$

and $\overline{B^{213}} + A^{13}$ is almost linked to $\overline{B^2} + A$. Further, $|\overline{A^2}| > |O^1_{B^3}|$, $|B^3| > |\overline{O^1_{A^2}}|$, $|A^2| > |\overline{O^1_{B^3}}|$ and $|\overline{B^3}| > |O^1_{A^2}|$ by (B.33) and (B.114), so

$$\overline{A^2} \approx B^3 \quad \text{and} \quad \overline{B^3} \approx A^2.$$

Finally

$$|\overline{A^{113}}| > |O^1_{B^{33}}| \tag{7.37}$$

by (B.113).

Case 3(a): $|\ulcorner\overline{A^1}| \geq |O^2_B| + |(B^{213})^L|$

Then $\overline{A^1} + B^3$ is almost linked to $\overline{B^{213}} + A^{13}$, and so

$$\overline{A} + B^3B^2 \subseteq (\overline{A^1} + B^3) \cup (\overline{B^{213}} + A^{13}) \cup (\overline{B^2} + A) \cup (\overline{A} + B^2).$$

Case 3(b): $|\ulcorner\overline{A^1}| < |O^2_B| + |(B^{213})^L|$

We have $|B^{33}| > |\overline{O^2_{A^{11}}}| > |\overline{O^1_{A^{113}}}|$ by (B.95), so $\overline{A^1}+B^3$ is almost linked to $\overline{A^{113}}+B^{33}$. Also, with (7.37) we have $\overline{A^{113}} \approx B^{33}$. Finally $|\overline{B^3}^{\urcorner}| > |O^1_A| + |(A^{113})^R|$ by (B.96), so $\overline{A^{113}} + B^{33}$ is almost linked to $\overline{B^3} + A^2$. Thus

$$\overline{A} + B^3B^2 = (\overline{A^1} + B^3) \cup (\overline{A^{113}} + B^{33}) \cup (\overline{B^3} + A^2) \cup (\overline{A^2} + B^3) \cup (\overline{A} + B^2).$$

The lemma follows by induction as described at the beginning of the proof.

□

Theorem 7.5.2

$$F(3) - F(3) = \mathbb{R}.$$

Proof. Let $I = [\langle\overline{3,1}\rangle, \langle\overline{1,3}\rangle]$. Then

$$|\overline{I^{11}}| = 0.056\ldots, \qquad |\overline{I^{12}}| = 0.029\ldots, \qquad |\overline{I^{13}}| = 0.017\ldots, \tag{7.38}$$

$$|\overline{O^1_{I^{11}}}| = 0.024\ldots, \qquad |\overline{O^1_{I^{12}}}| = 0.012\ldots, \qquad |\overline{O^1_{I^{13}}}| = 0.007\ldots, \tag{7.39}$$

$$|\overline{I^{2}}| = 0.056\ldots, \qquad |\overline{I^{21}}| = 0.021\ldots, \qquad |\overline{I^{22}}| = 0.009\ldots, \tag{7.40}$$

$$|\overline{O^1_{I^{2}}}| = 0.024\ldots \qquad |\overline{O^1_{I^{21}}}| = 0.009\ldots, \qquad |\overline{O^1_{I^{22}}}| = 0.004\ldots, \tag{7.41}$$

$$|I^{11}| = 0.083\ldots, \qquad |I^{12}| = 0.042\ldots, \qquad |I^{13}| = 0.025\ldots, \tag{7.42}$$

$$|O^1_{I^{11}}| = 0.018\ldots, \qquad |O^1_{I^{12}}| = 0.009\ldots, \qquad |O^1_{I^{13}}| = 0.005\ldots, \tag{7.43}$$

$$|I^3| = 0.042\ldots \quad \text{and} \quad |O^1_{I^3}| = 0.009\ldots. \tag{7.44}$$

Thus

$$\overline{I^{11}} \approx I^{11}, \qquad \overline{I^{11}} \approx I^{12}, \qquad \overline{I^{2}} \approx I^{11}, \qquad \overline{I^{2}} \approx I^{12}, \tag{7.45}$$

$$\overline{I^{22}} \approx I^{13}, \qquad \overline{I^{21}} \approx I^{13}, \qquad \overline{I^{12}} \approx I^{3} \quad \text{and} \quad \overline{I^{13}} \approx I^{3}. \tag{7.46}$$

Also,

$$I^{11} - \overline{I^{11}} = [-0.075\ldots, 0.064\ldots] \qquad I^{12} - \overline{I^{11}} = [0.059\ldots, 0.158\ldots] \tag{7.47}$$

$$I^{11} - \overline{I^{2}} = [0.135\ldots, 0.275\ldots] \qquad I^{12} - \overline{I^{2}} = [0.270\ldots, 0.370\ldots] \tag{7.48}$$

$$I^{13} - \overline{I^{22}} = [0.342\ldots, 0.378\ldots] \qquad I^{13} - \overline{I^{21}} = [0.377\ldots, 0.425\ldots] \tag{7.49}$$

$$\overline{I^{12}} - I^{3} = [0.396\ldots, 0.468\ldots] \quad \text{and} \quad \overline{I^{13}} - I^{3} = [0.464\ldots, 0.524\ldots] \tag{7.50}$$

Since $F(3) - F(3)$ is symmetric about zero, the theorem follows from (7.45), (7.46), (7.47), (7.48), (7.49), (7.50) and Lemma 7.5.1.

□

Proof of Theorem 1.0.1. As shown by Diviš [6] and Cusick [3] we have

$$F(3) + F(3) \neq \mathbb{R}.$$

Also, Hlavka [10] established that

$$F(2) + F(4) \neq \mathbb{R} \quad \text{and} \quad F(3) + F(4) = \mathbb{R}.$$

Using Theorem 1.0.2 we find

$$F(3) - F(4) = \mathbb{R}$$

and from Theorems 7.4.3 and 7.5.2 we have that

$$F(2) + F(5) = F(2) - F(5) = F(3) - F(3) = \mathbb{R}.$$

Since

$$I(L_2) - I(L_4) \subseteq [-0.462\ldots, 0.524\ldots]$$

we know that

$$F(2) - F(4) \neq \mathbb{R}$$

and the theorem follows.

□

With slightly more work we can improve Lemma 7.5.1, describing $F(3) + F(3)$ completely. We have the following three lemmas.

Lemma 7.5.3 *Let A be a bridge of $\mathcal{D}^3(C(L_3))$ and let $n \geq 0$ be an integer. Put $w = (1\,3)^n\,1$. Then*

$$|A^{2w3}| \geq |\overline{O^2_{A^{3w}}}| \tag{7.51}$$

$$|\overline{A^{3w2}}| > |\overline{A^{3w3}}| \geq |O^1_{A^{2w3}}| \tag{7.52}$$

Proof. By Lemma 2.1.3 there exists Q in the range $0 \le Q \le 1$ such that

$$\frac{|A^{2w3}|}{|\overline{O^2_{A^{3w}}}|} = \mathrm{fr}([2,w,3,\overline{1,3}],[2,w,3,\overline{3,1}],[3,w,3,\overline{2,1}],[3,w,2,\overline{1,2}],Q)$$

$$= \frac{\langle w,3,\overline{1,3}\rangle - \langle w,3,\overline{3,1}\rangle}{\langle w,3,\overline{2,1}\rangle - \langle w,2,\overline{1,2}\rangle} \cdot \frac{([3,w,3,\overline{2,1}]+Q)([3,w,2,\overline{1,2}]+Q)}{([2,w,3,\overline{1,3}]+Q)([2,w,3,\overline{3,1}]+Q)}. \quad (7.53)$$

Now

$$\frac{\langle w,3,\overline{1,3}\rangle - \langle w,3,\overline{3,1}\rangle}{\langle w,3,\overline{2,1}\rangle - \langle w,2,\overline{1,2}\rangle} = \mathrm{fr}([3,\overline{1,3}],[3,\overline{3,1}],[3,\overline{2,1}],[2,\overline{1,2}],\langle 1,(3,1)^n\rangle)$$

$$\ge \mathrm{fr}([3,\overline{1,3}],[3,\overline{3,1}],[3,\overline{2,1}],[2,\overline{1,2}],\langle\overline{1,3}\rangle)$$

$$= 0.6558\ldots . \quad (7.54)$$

Further

$$\frac{([3,w,3,\overline{2,1}]+Q)([3,w,2,\overline{1,2}]+Q)}{([2,w,3,\overline{1,3}]+Q)([2,w,3,\overline{3,1}]+Q)} \ge \frac{([3,w,3,\overline{2,1}]+1)([3,w,2,\overline{1,2}]+1)}{([2,w,3,\overline{1,3}]+1)([2,w,3,\overline{3,1}]+1)}$$

$$\ge \frac{([3,1,3,\overline{2,1}]+1)([3,1,2,\overline{1,2}]+1)}{([2,1,3,\overline{1,3}]+1)([2,1,3,\overline{3,1}]+1)}$$

$$= 1.570\ldots . \quad (7.55)$$

Therefore

$$\frac{|A^{2w3}|}{|\overline{O^2_{A^{3w}}}|} > 1.02$$

by (7.53), (7.54) and (7.55), and (7.51) follows.

Note that

$$\frac{|\overline{A^{3w2}}|}{|\overline{A^{3w3}}|} \ge \mathrm{fr}([2,\overline{1,2}],[2,\overline{2,1}],[3,\overline{1,2}],[3,\overline{2,1}],1) = 1.644\ldots \quad (7.56)$$

so to establish (7.52) it suffices to prove that

$$\frac{|\overline{A^{3w3}}|}{|O^1_{A^{2w3}}|} \ge 1.$$

Now

$$\frac{|\overline{A^{3w3}}|}{|O^1_{A^{2w3}}|} = \mathrm{fr}([3,w,3,\overline{1,2}],[3,w,3,\overline{2,1}],[2,w,3,1,\overline{1,3}],[2,w,3,2,\overline{3,1}],Q)$$
$$= \frac{\langle w,3,\overline{1,2}\rangle - \langle w,3,\overline{2,1}\rangle}{\langle w,3,1,\overline{1,3}\rangle - \langle w,3,2,\overline{3,1}\rangle} \cdot \frac{([2,w,3,1,\overline{1,3}]+Q)([2,w,3,2,\overline{3,1}]+Q)}{([3,w,3,\overline{1,2}]+Q)([3,w,3,\overline{2,1}]+Q)}. \tag{7.57}$$

However

$$\frac{\langle w,3,\overline{1,2}\rangle - \langle w,3,\overline{2,1}\rangle}{\langle w,3,1,\overline{1,3}\rangle - \langle w,3,2,\overline{3,1}\rangle} = \mathrm{fr}([3,\overline{1,2}],[3,\overline{2,1}],[3,1,\overline{1,3}],[3,2,\overline{3,1}],\langle 1,(3,1)^n\rangle)$$
$$\geq \mathrm{fr}([3,\overline{1,2}],[3,\overline{2,1}],[3,1,\overline{1,3}],[3,2,\overline{3,1}],\langle\overline{1,3}\rangle)$$
$$= 3.075\ldots . \tag{7.58}$$

Also

$$\frac{([2,w,3,1,\overline{1,3}]+Q)([2,w,3,2,\overline{3,1}]+Q)}{([3,w,3,\overline{1,2}]+Q)([3,w,3,\overline{2,1}]+Q)} \geq \frac{([2,1,3,1,\overline{1,3}]+0)([2,1,3,2,\overline{3,1}]+0)}{([3,\overline{1,3}]+0)([3,\overline{1,3}]+0)}$$
$$= 0.5367. \tag{7.59}$$

Thus by (7.57), (7.58) and (7.59) we have

$$\frac{|\overline{A^{3w3}}|}{|O^1_{A^{2w3}}|} > 1.65. \tag{7.60}$$

By (7.56) and (7.60) we have (7.52), and the lemma follows.

□

Lemma 7.5.4 *Let $I = [\langle\overline{3,1}\rangle, \langle\overline{1,3}\rangle]$ and assume that $A = I^v$ is a bridge of $\mathcal{D}^3(L_3)$ for some word v. Put*

$$C = A \cap F(3), \qquad C^3 = A^3 \cap F(3) \quad \textit{and} \quad C^{13} = A^{13} \cap F(3).$$

Then

$$C + C = (C^3 + C^3) \bigcup [[\langle v, \overline{3,1}\rangle + \langle v, 2, \overline{1,3}\rangle, \langle v, \overline{1,3}\rangle + \langle v, 1, 2, \overline{1,3}\rangle]] \bigcup (C^{13} + C^{13})$$

where the union is disjoint.

Proof. We claim that

$$[[\langle v, \overline{2,1}\rangle + \langle v, \overline{3,1}\rangle, \langle v, \overline{1,2}\rangle + \langle v, \overline{1,3}\rangle]] = \overline{A} + A \subseteq C + C. \quad (7.61)$$

If $A \neq I^1$ and $A \neq I$ then (7.61) follows immediately from Lemma 7.5.1. Otherwise assume first that $A = I^1$. By (7.38), (7.39), (7.42) and (7.43) we have

$$\overline{I^{11}} \approx I^{11}, \quad \overline{I^{12}} \approx I^{11}, \quad \overline{I^{11}} \approx I^{13}, \quad \overline{I^{12}} \approx I^{12},$$
$$\overline{I^{13}} \approx I^{12} \quad \text{and} \quad \overline{I^{12}} \approx I^{13}.$$

Also

$$\overline{I^{11}} + I^{11} = [1.135, 1.275], \qquad \overline{I^{12}} + I^{11} = [1.261, 1.373],$$
$$\overline{I^{11}} + I^{13} = [1.342, 1.425], \qquad \overline{I^{12}} + I^{12} = [1.396, 1.468],$$
$$\overline{I^{13}} + I^{12} = [1.464, 1.524] \quad \text{and} \quad \overline{I^{12}} + I^{13} = [1.468, 1.523].$$

Therefore

$$\overline{I^1} + I^1 \subseteq (I^1 \cap F(3)) + (I^1 \cap F(3)) \quad (7.62)$$

and so our claim follows for $A = I^1$. If $A = I$ then by (7.38), (7.39), (7.40), (7.41), (7.42), (7.43) and (7.44) we have

$$\overline{I^2} \approx I^3, \quad \overline{I^2} \approx I^2, \quad \overline{I^{11}} \approx I^3, \quad (7.63)$$

$$\overline{I^2} \approx I^{11} \quad \text{and} \quad \overline{I^2} \approx I^{12}. \quad (7.64)$$

Note that $\overline{I^2} \approx I^2$ follows trivially. Further,

$$\overline{I^2} + I^3 = [0.629\ldots, 0.728\ldots], \qquad \overline{I^2} + I^2 = [0.724\ldots, 0.864\ldots], \tag{7.65}$$

$$\overline{I^{11}} + I^3 = [0.841\ldots, 0.940\ldots], \qquad \overline{I^2} + I^{11} = [0.924\ldots, 1.064\ldots], \tag{7.66}$$

$$\overline{I^2} + I^{12} = [1.059\ldots, 1.158\ldots] \quad \text{and} \quad \overline{I^1} + I^1 = [1.135\ldots, 1.523\ldots]. \tag{7.67}$$

Therefore (7.61) holds for $A = I$, by (7.63), (7.64), (7.65), (7.66), (7.67), (7.62) and Lemma (7.5.1), and our claim follows.

For $n \geq 0$ put $w_n = (1\ 3)^n\ 1$. Then by Lemma 7.5.3 we have

$$\overline{A^{3w_n3}} \approx A^{2w_n3}, \qquad \overline{A^{3w_n2}} \approx A^{2w_n3} \tag{7.68}$$

and

$$|A^{2w_n3}| \geq |\overline{O^2_{A^{3w_n}}}|. \tag{7.69}$$

By (7.68) and Lemma 7.5.1

$$(\overline{A^{3w_n3}} + A^{2w_n3}) \cup (\overline{A^{3w_n2}} + A^{2w_n3}) \subseteq C + C$$

and by (7.69) we know that $\overline{A^{3w_n3}} + A^{2w_n3}$ and $\overline{A^{3w_n2}} + A^{2w_n3}$ overlap, hence

$$\overline{A^{3w_n3}A^{3w_n2}} + A^{2w_n3} \subseteq C + C. \tag{7.70}$$

Now

$$\overline{A^{3w_n312}} \subseteq \overline{A^{3w_n3}} \quad \text{and} \quad A^{2w_n313} \subseteq A^{2w_n3}$$

so

$$\overline{A^{3w_{n+1}3}A^{3w_{n+1}2}} + A^{2w_{n+1}3} \quad \text{and} \quad \overline{A^{3w_n3}A^{3w_n2}} + A^{2w_n3} \qquad \text{overlap.} \tag{7.71}$$

Also

$$\frac{|\overline{A^{313}A^{312}}|}{|\overline{O^2_{A^{21}}}|} \geq \text{fr}([3,1,3,\overline{1,2}],[3,1,2,\overline{2,1}],[2,1,3,\overline{2,1}],[2,1,2,\overline{1,2}],0) = 1.18\ldots$$

hence $|\overline{A^{313}A^{312}}| > |\overline{O^2_{A^{21}}}|$ and so

$$\overline{A^{313}A^{312}} + A^{213} \quad \text{and} \quad \overline{A} + A \qquad \text{overlap.} \tag{7.72}$$

By (7.61), (7.70), (7.71) and (7.72) we have

$$[[\langle v, 3, (1,3)^n, 1, 3, \overline{1,2}\rangle + \langle v, 2, \overline{1,3}\rangle, \langle v, \overline{1,2}\rangle + \langle v, \overline{1,3}\rangle]] \subseteq C + C. \tag{7.73}$$

Since $\langle v, \overline{3,1}\rangle + \langle v, 2, \overline{1,3}\rangle \in C + C$, upon letting n tend to infinity in (7.73) we find that

$$[[\langle v, \overline{3,1}\rangle + \langle v, 2, \overline{1,3}\rangle, \langle v, \overline{1,2}\rangle + \langle v, \overline{1,3}\rangle]] \subseteq C + C. \tag{7.74}$$

We may perform the same process using A^1 instead of A, finding that

$$[[\langle v, 1, \overline{1,2}\rangle + \langle v, 1, \overline{1,3}\rangle, \langle v, 1, \overline{3,1}\rangle + \langle v, 1, 2, \overline{1,3}\rangle]] \subseteq C + C.$$

With (7.74) we have

$$[[\langle v, \overline{3,1}\rangle + \langle v, 2, \overline{1,3}\rangle, \langle v, \overline{1,3}\rangle + \langle v, 1, 2, \overline{1,3}\rangle]] \subseteq C + C.$$

Since

$$A^2A^1 + A^3A^2, \quad \text{and} \quad A^{11}A^{12} + A^1$$

are both contained in

$$[[\langle v, \overline{3,1}\rangle + \langle v, 2, \overline{1,3}\rangle, \langle v, \overline{1,3}\rangle + \langle v, 1, 2, \overline{1,3}\rangle]]$$

we know that

$$C + C = (C^3 + C^3) \bigcup [[\langle v, \overline{3,1}\rangle + \langle v, 2, \overline{1,3}\rangle, \langle v, \overline{1,3}\rangle + \langle v, 1, 2, \overline{1,3}\rangle]] \bigcup (C^{13} + C^{13}).$$

The above union is disjoint since for any bridge B of $\mathcal{D}^3(C(L_3))$ we have

$$\frac{|B^3|}{|O^2_B|} \leq \mathfrak{fr}([3, \overline{1,3}], [3, \overline{3,1}], [3, \overline{3,1}], [2, \overline{1,3}], 1) = 0.883$$

hence $|A^3| < |O^2_A|$ and $|A^{13}| < |O^2_{A^1}|$. The lemma follows.

□

Corollary 7.5.5 *Let I^u be a bridge of $\mathcal{D}^3(C(L_3))$ and put $C^u = I^u \cap F(3)$. Then*

$$C^u + C^u = \{2\langle u, d_1, d_2, \ldots\rangle\ ; d_i \in \{1,3\} \text{ and if } d_i = 1 \text{ then } d_{i+1} = 3, \text{ for } i \geq 1\}$$
$$\bigcup \left(\bigcup_{v \in \mathcal{M}} [[\langle u, v, \overline{3,1}\rangle + \langle u, v, 2, \overline{1,3}\rangle, \langle u, v, \overline{1,3}\rangle + \langle u, v, 1, 2, \overline{1,3}\rangle]] \right),$$

where $\mathcal{M}$ is the set of finite words $v = d_1 \cdots d_t$ such that $t \geq 0$, $d_i \in \{1,3\}$ for $1 \leq i \leq t$ and if $d_k = 1$ for some k then $t \geq k+1$ and $d_{k+1} = 3$. The union is disjoint.

Proof. Put

$$R = \bigcup_{v \in \mathcal{M}} [[\langle u, v, \overline{3,1}\rangle + \langle u, v, 2, \overline{1,3}\rangle, \langle u, v, \overline{1,3}\rangle + \langle u, v, 1, 2, \overline{1,3}\rangle]]$$

and

$$S = \{2\langle u, d_1, d_2, \ldots\rangle\ ; d_i \in \{1,3\} \text{ and if } d_i = 1 \text{ then } d_{i+1} = 3, \text{ for } i \geq 1\}.$$

Then by repeated application of Lemma 7.5.4 we have

$$R \subseteq C^u + C^u.$$

Also we have

$$S \subseteq C^u + C^u$$

by the definition of C^u. Let $x \in C^u + C^u$ and suppose that $x \notin R$. Then by Lemma 7.5.4 we have

$$x \in I^{u3} + I^{u3} \quad \text{or} \quad x \in I^{u13} + I^{u13}.$$

Similarly if $x \in I^{ud_1 \cdots d_t} + I^{ud_1 \cdots d_t}$ then

$$x \in I^{ud_1 \cdots d_t 3} + I^{ud_1 \cdots d_t 3} \quad \text{or} \quad I^{ud_1 \cdots d_t 13} + I^{ud_1 \cdots d_t 13}.$$

Thus by induction we find that $x \in S$. By the disjointness of the union in Lemma 7.5.4 we have $R \cap S = \emptyset$, and the corollary follows.

□

Proof of Theorem 1.0.11. By Corollary 7.5.5 we have

$$F(3) + F(3) = (\mathbb{Z} + R) \cup (\mathbb{Z} + S).$$

Now,

$$[\langle\overline{3,1}\rangle + \langle 2,\overline{1,3}\rangle, \langle\overline{1,3}\rangle + \langle 1,2,\overline{1,3}\rangle] = [0.622\ldots, 1.527\ldots].$$

Further, if $I = [\langle\overline{3,1}\rangle, \langle\overline{1,3}\rangle]$ then we have

$$I^{13} + I^{13} = [1.530\ldots, 1.582\ldots]$$

and

$$I^{33} + I^{33} = [0.604\ldots, 0.612\ldots].$$

Therefore if $x \in (1 + I^{33} + I^{33}) \cap (F(3) + F(3))$ and

$$x = [a_0, a_1, \ldots] + [0, b_1, \ldots]$$

where $a_0 \in \mathbb{Z}$ and $a_i, b_i \in \{1,2,3\}$ for $i \geq 1$ then we must have

$$a_0 = 1 \quad \text{and} \quad a_1 = a_2 = b_1 = b_2 = 3.$$

Put $C^{33} = I^{33} \cap F(3)$. Then by Corollary 7.5.5 there are an uncountable number of points in $C^{33} + C^{33}$ that are not in any interval contained in $C^{33} + C^{33}$, and the theorem follows.

□

7.6 $F(5)\cdot F(2)$ and $F(3)\cdot F(3)$

Given Lemmas 7.4.2 and 7.5.1 one would not be surprised to learn that $F(5)\cdot F(2)$ and $F(3)\cdot F(3)$ contain intervals; however it is possible to prove even more, namely that each contains two infinite rays.

To prove our result we will use the following two lemmas.

Lemma 7.6.1 *Let I_1 and I_2 be bridges of $\mathcal{D}^{l_1}(L_{l_1})$ and $\mathcal{D}^{l_2}(L_{l_2})$ respectively. Assume that $\overline{I_1} \approx I_2$ and $|I_2| > 2|\overline{O^1_{I_1}}|$. Then for n sufficiently large and $n \le m \le 2n$,*

$$(n+\overline{I_1})^* \approx (m+I_2)^*.$$

Proof. Choose N such that for $n \ge N$,

$$\frac{n-1}{n}|(n+I_2)^*| \ge 2|(n+\overline{O^1_{I_1}})^*| \quad \text{and} \quad |(n+\overline{I_1})^*| \ge |(n+O^1_{I_2})^*|. \tag{7.75}$$

Then if $n \ge N$ and $n \le m \le 2n$ we have

$$|(n+\overline{I_1})^*| \ge |(n+O^1_{I_2})^*| \ge |(m+O^1_{I_2})^*|. \tag{7.76}$$

Assume that $I_2 = [\alpha,\beta]$. Then

$$\begin{aligned}
\frac{|(m+I_2)^*|}{|(n+I_2)^*|} &= \frac{\log(m+\beta)-\log(m+\alpha)}{\log(n+\beta)-\log(n+\alpha)} \\
&= \frac{\log\left(1+\frac{\beta-\alpha}{m+\alpha}\right)}{\log\left(1+\frac{\beta-\alpha}{n+\alpha}\right)} \\
&\ge \frac{\frac{\beta-\alpha}{m+\alpha} - \frac{1}{2}\left(\frac{\beta-\alpha}{m+\alpha}\right)^2}{\left(\frac{\beta-\alpha}{n+\alpha}\right)} \\
&= \frac{n+\alpha}{m+\alpha}\left(1-\frac{1}{2}\frac{\beta-\alpha}{m+\alpha}\right) \\
&> \frac{n}{m}\cdot\frac{n-1}{n}.
\end{aligned}$$

Therefore, since $m \leq 2n$,

$$|(m+I_2)^*| > \frac{n-1}{2n}|(n+I_2)^*| \geq |(n+\overline{O^1_{I_1}})^*| \tag{7.77}$$

by (7.75). The lemma follows from (7.76) and (7.77).

□

Lemma 7.6.2 *Let α_1, α_2, β_1, β_2 be real numbers with $0 \leq \alpha_1 < \beta_1 \leq 1$ and $0 \leq \alpha_2 < \beta_2 \leq 1$. Let δ be any number in the range*

$$1-(\beta_1-\alpha_1)-(\beta_2-\alpha_2) < \delta < 1. \tag{7.78}$$

Then for n sufficiently large,

$$(n+k+1+\alpha_1)(2n-k-1+\alpha_2) < (n+k+\beta_1)(2n-k+\beta_2) \tag{7.79}$$

for k in the range

$$\left\lfloor \delta\frac{n}{2}\right\rfloor \leq k \leq \left\lfloor \frac{n}{2}\right\rfloor$$

Furthermore, for n sufficiently large

$$(n+1+\left\lfloor \delta\frac{n+1}{2}\right\rfloor+\alpha_1)(2n+2-\left\lfloor \delta\frac{n+1}{2}\right\rfloor+\alpha_2) < (n+\left\lfloor \frac{n}{2}\right\rfloor+\beta_1)(2n-\left\lfloor \frac{n}{2}\right\rfloor+\beta_2). \tag{7.80}$$

Proof. To prove the first part of the theorem we note that (7.79) is equivalent to

$$n < n(\beta_2-\alpha_2+2(\beta_1-\alpha_1)) + k(2+\beta_2-\alpha_2+\alpha_1-\beta_1) + c_1 \tag{7.81}$$

for some constant c_1. However

$$\frac{n}{2} \geq k \geq \frac{\delta n}{2} + c_2$$

for some constant c_2, whence there exists c_3 such that (7.81) holds if

$$n < n(\delta + \beta_2 - \alpha_2 + \frac{3}{2}(\beta_1 - \alpha_1)) + c_3 \tag{7.82}$$

However by (7.78) we know that (7.82) holds for n sufficiently large and our result follow.

To prove the second part of the lemma we note that

$$\left\lfloor \delta \frac{n+1}{2} \right\rfloor = \delta \frac{n+1}{2} + c_4 \quad \text{and} \quad \left\lfloor \frac{n}{2} \right\rfloor = \frac{n}{2} + c_5$$

for some constants c_4 and c_5. Hence (7.80) is equivalent to

$$n^2 \left(2 + \frac{\delta}{2} - \frac{\delta^2}{4}\right) + c_6 n + c_7 < \frac{9}{4} n^2 + c_8 n + c_9$$

for some constants c_6, c_7, c_8 and c_9, or alternatively

$$c_{10} n + c_{11} < n^2 \left(\frac{1}{4} - \frac{\delta}{2} + \frac{\delta^2}{4}\right) \tag{7.83}$$

for some constants c_{10} and c_{11}. But

$$\frac{1}{4} - \frac{\delta}{2} + \frac{\delta^2}{4} = \left(\frac{\delta}{2} - \frac{1}{2}\right)^2 > 0$$

since $\delta < 1$, and so (7.83) holds for n sufficiently large, and the lemma follows.

□

Proof of Theorem 1.0.12. Let

$$I_5 = [\langle 1, 1, \overline{1, 5} \rangle, \langle 1, 1, \overline{5, 1} \rangle], \qquad I_2 = [\langle 1, 1, \overline{1, 2} \rangle, \langle 1, 1, \overline{2, 1} \rangle]$$

and

$$I_3 = [\langle 1,1,\overline{1,3}\rangle, \langle 1,1,\overline{3,1}\rangle].$$

Then by Lemma 7.6.1 there exists N_0 such that

$$(n+\overline{I_5})^* \approx (m+I_2)^*$$

for $n \geq N_0$ and $n \leq m \leq 2n$. Let $\epsilon > 0$ be given. Then for n sufficiently large, the values of the ratios in Appendix A corresponding to bridges from $(n+\mathcal{D}^5(L_5))^*$ and $(n+\mathcal{D}^2(L_2))^*$ will be within ϵ of the values given in Appendix A for bridges of $\mathcal{D}^5(L_5)$ and $\mathcal{D}^2(L_2)$. Since the products in Appendix A are all strictly greater than one, we find by an approach analogous to the proof of Lemma 7.4.2 that

$$[(n+\alpha_1)(m+\alpha_2), (n+\beta_1)(m+\beta_2)] \subseteq F(5)\cdot F(2) \tag{7.84}$$

for n sufficiently large and $n \leq m \leq 2n$, where

$$\alpha_1 = \langle 1,1,\overline{1,4}\rangle, \qquad \beta_1 = \langle 1,1,\overline{4,1}\rangle,$$
$$\alpha_2 = \langle 1,1,\overline{1,2}\rangle \quad \text{and} \quad \beta_2 = \langle 1,1,\overline{2,1}\rangle.$$

Put

$$I(n,m) = [(n+\alpha_1)(m+\alpha_2), (n+\beta_1)(m+\beta_2)].$$

By Lemma 7.6.2 for N_1 sufficiently large

$$\bigcup_{n\geq N_1} \bigcup_{\lfloor \frac{4n}{7} \rfloor \leq k \leq \lfloor \frac{n}{2} \rfloor} I(n+k, 2n-k) \tag{7.85}$$

is an interval. Therefore by (7.84) and (7.85) there exists a constant c_3 such that

$$[c_3, \infty) \subseteq F(5)\cdot F(2).$$

By a similar process we find that there exists real numbers c_4, c_5 and c_6 such that

$$[c_4, \infty) \subseteq -F(5)\cdot F(2), \qquad [c_5, \infty) \subseteq F(3)\cdot F(3)$$

and

$$[c_6, \infty) \subseteq -F(3) \cdot F(3)$$

and the theorem follows.

□

7.7 Final Remarks

There are many related questions not answered in this thesis. For example, it is unknown whether $F(4) \pm F(2)$ or $F(2) \pm F(2) \pm F(3)$ contains an interval. Also, it should be possible to use the work of Section 7.5 to obtain an alternate proof of the lower bound for Hall's ray. Further, we have not examined the sets $F(5)/F(2)$ and $F(3)/F(3)$. Finally, the general work on sums of Cantor sets should have many applications beyond the field of number theory.

Appendix A

Calculations for $F(5) + F(2)$

Let $A = I(L_5)^w$ and $B = \pm I(L_2)^v$ be bridges of $\mathcal{D}^5(L_5)$ and $\pm\mathcal{D}^2(L_2)$ with $w \notin \{\emptyset, 1\}$ and $v \notin \{\emptyset, 1\}$. Say

$$w = a_1\, a_2 \cdots a_r \quad \text{and} \quad v = b_1\, b_2 \cdots b_s$$

and put

$$Q_A = \langle a_r, \ldots, a_1 \rangle \quad \text{and} \quad Q_B = \langle b_s, \ldots, b_1 \rangle.$$

Then

$$\frac{1}{6} = \langle 5, 1 \rangle \le Q_A \le \langle 1, 5, 1 \rangle = \frac{6}{7} \quad \text{and} \quad \frac{1}{3} = \langle 2, 1 \rangle \le Q_B \le \langle 1, 2, 1 \rangle = \frac{3}{4}.$$

We have

$$\frac{|B^1|}{|O_B|} \ge \mathrm{fr}([1, \overline{1, 2}], [1, \overline{2, 1}], [2, \overline{2, 1}], [1, \overline{1, 2}], 3/4) = 0.8501\ldots\,, \tag{A.1}$$

$$\frac{|B^2|}{|O_B|} \ge \mathrm{fr}([2, \overline{1, 2}], [2, \overline{2, 1}], [2, \overline{2, 1}], [1, \overline{1, 2}], 1/3) = 0.3890\ldots\,, \tag{A.2}$$

$$\frac{|\overline{A^2 A^4}|}{|\overline{O_A^1}|} \ge \mathrm{fr}([4, \overline{1, 4}], [2, \overline{4, 1}], [2, \overline{4, 1}], [1, \overline{1, 4}], 1/6) = 2.764\ldots\,,$$

$$\frac{|\overline{A^3A^4}|}{|\overline{O^2_A}|} \geq \mathrm{fr}([4,\overline{1,4}],[3,\overline{4,1}],[3,\overline{4,1}],[2,\overline{1,4}],1/6) = 2.567\ldots\,,$$

$$\frac{|\overline{A}|}{|\overline{O^1_A}|} \geq \mathrm{fr}([4,\overline{1,4}],[1,\overline{4,1}],[2,\overline{4,1}],[1,\overline{1,4}],0.2087\ldots) = 6.599\ldots \tag{A.3}$$

and for $k = 1, 2$ we have

$$\frac{|\overline{A^k}|}{|\overline{O^k_A}|} \geq \frac{|\overline{A^3}|}{|\overline{O^3_A}|} \geq \mathrm{fr}([3,\overline{1,4}],[3,\overline{4,1}],[4,\overline{4,1}],[3,\overline{1,4}],6/7) = 2.044\ldots\,. \tag{A.4}$$

Thus

$$\frac{|\overline{A^k}|}{|\overline{O^k_A}|}\cdot\frac{|B^1|}{|O_B|} > 1.73 \quad \text{and} \quad \frac{|\overline{A^k}|}{|\overline{O^1_{A^k}}|}\cdot\frac{|B^2|}{|O_B|} > 2.56 \tag{A.5}$$

for $k = 1$ or $k = 2$, and

$$\frac{|\overline{A^2A^4}|}{|\overline{O^1_A}|}\cdot\frac{|B^2|}{|O_B|} > 1.07\,. \tag{A.6}$$

Now if $v \neq (1\ 2)$ then

$$Q_B > \langle 2,1,2,1\rangle = \frac{4}{11}.$$

In this case we have

$$\frac{|B^2|}{|O_B|} \geq \mathrm{fr}([2,\overline{1,2}],[2,\overline{2,1}],[2,\overline{2,1}],[1,\overline{1,2}],4/11) = 0.3908\ldots$$

and so if $v \neq (1\ 2)$ then

$$\frac{|\overline{A^3A^4}|}{|\overline{O^2_A}|}\cdot\frac{|B^2|}{|O_B|} > 1.003\,. \tag{A.7}$$

More generally we have

$$\frac{|\overline{A^4A^5}|}{|\overline{O^3_A}|} \geq \mathrm{fr}([5,\overline{1,4}],[4,\overline{4,1}],[4,\overline{4,1}],[3,\overline{1,4}],1/6) = 2.853\ldots$$

hence

$$\frac{|\overline{A^4A^5}|}{|\overline{O^3_A}|}\cdot\frac{|B^2|}{|O_B|} > 1.11\,. \tag{A.8}$$

Now

$$\frac{|\overline{A^5}|}{|\overline{O_A^3}|} \geq \mathrm{fr}([5,\overline{1,4}],[5,\overline{4,1}],[4,\overline{4,1}],[3,\overline{1,4}],1/6) = 0.8899\ldots\ ,$$
$$\frac{|B^2|}{|O_{B^{11}}|} \geq \mathrm{fr}([2,\overline{1,2}],[2,\overline{2,1}],[1,1,2,\overline{2,1}],[1,1,1,\overline{1,2}],1/3) = 2.570\ldots$$

and

$$\frac{|B|}{|O_B|} \geq \mathrm{fr}([2,\overline{1,2}],[1,\overline{2,1}],[2,\overline{2,1}],[1,\overline{1,2}],3/4) = 2.261\ldots\ . \tag{A.9}$$

Therefore

$$\frac{|B^2|}{|O_{B^2}|} > 2.261\ , \tag{A.10}$$

$$\frac{|\overline{A^5}|}{|\overline{O_A^3}|}\cdot\frac{|B^2|}{|O_{B^{11}}|} > 2.28 \tag{A.11}$$

and

$$\frac{|\overline{A^4}|}{|\overline{O_A^3}|}\cdot\frac{|B^2|}{|O_{B^2}|} > 2.01 \tag{A.12}$$

since $|\overline{A^4}| \geq |\overline{A^5}|$. Further

$$\frac{|\overline{A^4}|}{|\overline{O_A^3}|} \geq \mathrm{fr}([4,\overline{1,4}],[4,\overline{4,1}],[4,\overline{4,1}],[3,\overline{1,4}],1/6) = 1.312\ldots \tag{A.13}$$

and

$$\frac{|\overline{A^5}|}{|\overline{O_A^4}|} \geq \mathrm{fr}([5,\overline{1,4}],[5,\overline{4,1}],[5,\overline{4,1}],[4,\overline{1,4}],1/6) = 1.367\ldots\ . \tag{A.14}$$

Also

$$\frac{|\overline{A^4}|}{|\overline{O_{A^4}^1}|} \geq 6.599 \quad \text{and} \quad \frac{|B^{11}|}{|O_{B^1}|} \geq 0.8502 \tag{A.15}$$

by (A.3) and (A.1). Therefore

$$\frac{|\overline{A^4}|}{|\overline{O^3_A}|} \cdot \frac{|B^1|}{|O_B|} > 1.11\ , \tag{A.16}$$

$$\frac{|\overline{A^4}|}{|\overline{O^1_{A^4}}|} \cdot \frac{|B^2|}{|O_B|} > 2.56\ , \tag{A.17}$$

$$\frac{|\overline{A^5}|}{|\overline{O^4_A}|} \cdot \frac{|B^{11}|}{|O_{B^1}|} > 1.16 \tag{A.18}$$

and

$$\frac{|\overline{A^3}|}{|\overline{O^3_A}|} \cdot \frac{|B^2|}{|O_{B^2}|} > 4.62 \tag{A.19}$$

by (A.13), (A.1), (A.15), (A.2), (A.14), (A.4) and (A.10). Also

$$\frac{|\overline{A^{15}}|}{|\overline{O^1_A}|} \geq \mathrm{fr}([1,5,\overline{4,1}],[1,5,\overline{1,4}],[2,\overline{4,1}],[1,\overline{1,4}],6/7) = 0.1070\dots\ ,$$

$$\frac{|\overline{A^{25}}|}{|\overline{O^2_A}|} \geq \mathrm{fr}([2,5,\overline{4,1}],[2,5,\overline{1,4}],[3,\overline{4,1}],[2,\overline{1,4}],6/7) = 0.08768\dots$$

and

$$\frac{|B^2|}{|O_{B^{112}}|} \geq \mathrm{fr}([2,\overline{1,2}],[2,\overline{2,1}],[1,1,2,1,\overline{1,2}],[1,1,2,2,\overline{2,1}],1/3) = 14.67\dots$$

thus

$$\frac{|\overline{A^{k5}}|}{|\overline{O^k_A}|} \cdot \frac{|B^2|}{|O_{B^{112}}|} > 1.28 \tag{A.20}$$

for $k = 1$ or $k = 2$. Now

$$\frac{|\overline{A}|}{|\overline{O^4_A}|} \geq \mathrm{fr}([4,\overline{1,4}],[1,\overline{4,1}],[5,\overline{4,1}],[4,\overline{1,4}],6/7) = 28.09\dots$$

and

$$\frac{|B^{112}|}{|O_B|} \geq \mathrm{fr}([1,1,2,\overline{1,2}],[1,1,2,\overline{2,1}],[2,\overline{2,1}],[1,\overline{1,2}],3/4) = 0.05838\dots$$

so

$$\frac{|\overline{A^k}|}{|\overline{O^4_{A^k}}|}\cdot\frac{|B^{112}|}{|O_B|} > 1.63 \tag{A.21}$$

for $k=1$ or $k=2$. Further

$$\frac{|A^1|-|A^{1L}|-|A^{15L}|}{|\overline{O^1_A}|} \geq \mathrm{fr}([1,\overline{1,4}],[1,5,\overline{1,4}],[2,\overline{4,1}],[1,\overline{1,4}],6/7) = 2.620\ldots$$

and so with (A.2) we have

$$\frac{|A^1|-|A^{1L}|-|A^{15L}|}{|\overline{O^1_A}|}\cdot\frac{|B^2|}{|O_B|} > 1.01\ . \tag{A.22}$$

Additionally,

$$\frac{|A^2|-|A^{2L}|-|A^{25L}|}{|O^1_A|+|A^{1L}|+|A^{25L}|} \geq \mathrm{fr}([2,\overline{1,4}],[2,5,\overline{1,4}],[2,5,\overline{1,4}],[1,\overline{1,4}],1/6) = 1.241\ldots \tag{A.23}$$

hence

$$\frac{|A^2|-|A^{2L}|-|A^{25L}|}{|O^1_A|+|A^{1L}|+|A^{25L}|}\cdot\frac{|B^1|}{|O_B|} > 1.05 \tag{A.24}$$

by (A.23) and (A.1). Further

$$\frac{|\overline{A^1}|}{|\overline{O^2_A}|} \geq \mathrm{fr}([1,\overline{1,4}],[1,\overline{4,1}],[3,\overline{4,1}],[2,\overline{1,4}],6/7) = 4.433\ldots \tag{A.25}$$

and

$$\frac{|\overline{A^2}|}{|\overline{O^1_{A^1}}|} \geq \mathrm{fr}([2,\overline{1,4}],[2,\overline{4,1}],[1,1,\overline{1,4}],[1,2,\overline{4,1}],1/6) = 2.584\ldots \tag{A.26}$$

so

$$\frac{|\overline{A^1}|}{|\overline{O^2_A}|}\cdot\frac{|B^2|}{|O_B|} > 1.73 \quad\text{and}\quad \frac{|\overline{A^2}|}{|\overline{O^1_{A^1}}|}\frac{|B|}{|O_B|} > 5.84 \tag{A.27}$$

by (A.25), (A.2), (A.26) and (A.9). We also have

$$\frac{|\overline{A^1 A^2}|}{|\overline{A^2 A^4}|} = \mathrm{fr}([2, \overline{1, 4}], [1, \overline{4, 1}], [4, \overline{1, 4}], [2, \overline{4, 1}], 6/7) = 1.416\ldots . \tag{A.28}$$

Finally,

$$\frac{|\overline{A^4}|}{|\overline{O^1_{A^3}}|} \geq \mathrm{fr}([4, \overline{1, 4}], [4, \overline{4, 1}], [3, 1, \overline{1, 4}], [3, 2, \overline{4, 1}], 1/6) = 4.074\ldots \tag{A.29}$$

and

$$\frac{|\overline{A^3}|}{|\overline{O^2_A}|} \geq \mathrm{fr}([3, \overline{1, 4}], [3, \overline{4, 1}], [3, \overline{4, 1}], [2, \overline{1, 4}], 1/6) = 1.230\ldots \tag{A.30}$$

so

$$\frac{|\overline{A^4}|}{|\overline{O^1_{A^3}}|} \cdot \frac{|B^2|}{|O_B|} > 1.59 \tag{A.31}$$

by (A.29) and (A.2), while

$$\frac{|\overline{A^3}|}{|\overline{O^2_A}|} \cdot \frac{|B^1|}{|O_B|} > 1.04 \tag{A.32}$$

by (A.30) and (A.1).

Appendix B

Calculations for $F(3) + F(3)$

Let $A = \pm I(L_3)^w$ and $B = \pm I(L_3)^v$ be bridges of $\pm\mathcal{D}^3(L_3)$ with $w, v \notin \{\emptyset, 1\}$. Assume that

$$w = a_1\, a_2 \cdots a_r \quad \text{and} \quad v = b_1\, b_2 \cdots b_s$$

and put

$$Q_A = \langle a_r, \ldots, a_1 \rangle \quad \text{and} \quad Q_B = \langle b_s, \ldots, b_1 \rangle.$$

Then

$$\frac{1}{4} = \langle 3, 1 \rangle \leq Q_A, Q_B \leq \langle 1, 3, 1 \rangle = \frac{4}{5}.$$

We have

$$\frac{|\overline{A^1}|}{|\overline{O_A^1}|} \geq \mathfrak{fr}([1, \overline{1, 2}], [1, \overline{2, 1}], [2, \overline{2, 1}], [1, \overline{1, 2}], 4/5) = 0.8438\ldots\,, \tag{B.1}$$

$$\frac{|B^1|}{|O_B^1|} \geq \mathfrak{fr}([1, \overline{1, 3}], [1, \overline{3, 1}], [2, \overline{3, 1}], [1, \overline{1, 3}], 4/5) = 1.657\ldots\,, \tag{B.2}$$

$$\frac{|B^2 B^3|}{|O_B^1|} \geq \mathfrak{fr}([3, \overline{1, 3}], [2, \overline{3, 1}], [2, \overline{3, 1}], [1, \overline{1, 3}], 1/4) = 1.633\ldots\,, \tag{B.3}$$

$$\frac{|A^1|}{|\overline{O_A^1}|} \geq \mathrm{fr}([1,\overline{1,3}],[1,\overline{3,1}],[2,\overline{2,1}],[1,\overline{1,2}],4/5) = 1.247\dots, \tag{B.4}$$

$$\frac{|A^2A^3|}{|\overline{O_A^1}|} \geq \mathrm{fr}([3,\overline{1,3}],[2,\overline{3,1}],[2,\overline{2,1}],[1,\overline{1,2}],1/4) = 1.229\dots, \tag{B.5}$$

$$\frac{|\overline{A^2}|}{|\overline{O_A^1}|} \geq \mathrm{fr}([2,\overline{1,2}],[2,\overline{2,1}],[2,\overline{2,1}],[1,\overline{1,2}],1/4) = 0.3837\dots, \tag{B.6}$$

$$\frac{|B^1|}{|O_B^2|} \geq \mathrm{fr}([1,\overline{1,3}],[1,\overline{3,1}],[3,\overline{3,1}],[2,\overline{1,3}],4/5) = 3.046\dots, \tag{B.7}$$

$$\frac{|B^2B^3|}{|O_B^2|} \geq \mathrm{fr}([3,\overline{1,3}],[2,\overline{3,1}],[3,\overline{3,1}],[2,\overline{1,3}],4/5) = 3.354\dots, \tag{B.8}$$

$$\frac{|\overline{A^2}|}{|O_A^1|} \geq \mathrm{fr}([2,\overline{1,2}],[2,\overline{2,1}],[2,\overline{3,1}],[1,\overline{1,3}],1/4) = 0.5095\dots, \tag{B.9}$$

$$\frac{|\overline{B}|}{|O_B^1|} \geq \mathrm{fr}([2,\overline{1,2}],[1,\overline{2,1}],[2,\overline{3,1}],[1,\overline{1,3}],4/5) = 3.000\dots \tag{B.10}$$

and

$$\frac{|\overline{A^1}|}{|O_A^1|} \geq \mathrm{fr}([1,\overline{1,2}],[1,\overline{2,1}],[2,\overline{3,1}],[1,\overline{1,3}],4/5) = 1.121\dots. \tag{B.11}$$

Thus

$$\frac{|\overline{A^1}|}{|\overline{O_A^1}|} \cdot \frac{\min\{|B^1|,|B^2B^3|\}}{|O_B^1|} > 1.37 \tag{B.12}$$

by (B.1), (B.2) and (B.3),

$$\frac{\min\{|A^1|,|A^2A^3|\}}{|\overline{O_A^1}|} \cdot \frac{\min\{|B^1|,|B^2B^3|\}}{|\overline{O_B^1}|} > 1.51 \tag{B.13}$$

by (B.4) and (B.5),

$$\frac{|\overline{A^2}|}{|\overline{O_A^1}|} \cdot \frac{\min\{|B^1|,|B^2B^3|\}}{|O_B^2|} > 1.16 \tag{B.14}$$

by (B.6), (B.7) and (B.8), and

$$\frac{|\overline{A^2}|}{|O_A^1|} \cdot \frac{|\overline{B}|}{|O_B^1|} > 1.52 \tag{B.15}$$

by (B.9) and (B.10). Further, if $w \neq (1\ 3)$ then

$$Q_A \geq \langle 3,1,3,1\rangle = \frac{5}{19}.$$

We have

$$\frac{|\overline{A^2A^3}|}{|\overline{O^2_{A^2}}|} \geq \mathrm{fr}([3,\overline{1,3}],[2,\overline{2,1}],[2,\overline{2,1}],[2,3,\overline{2,1}],5/19) = 13.05\ldots$$

and

$$\frac{|B^{33}|}{|O^2_B| + |B^{213L}|} \geq \mathrm{fr}([3,3,\overline{3,1}],[3,\overline{3,1}],[3,\overline{3,1}],[2,1,3,\overline{1,2}],1/4) = 0.07667\ldots\ .$$

Thus if $w \neq (1\ 3)$ then

$$\frac{|\overline{A^2A^3}|}{|\overline{O^2_{A^2}}|} \cdot \frac{|B^{33}|}{|O^2_B| + |B^{213L}|} > 1.0005\ . \tag{B.16}$$

Now

$$\frac{|\overline{A^2A^3}|}{|O^1_A|} \geq \mathrm{fr}([3,\overline{1,3}],[2,\overline{2,1}],[2,\overline{3,1}],[1,\overline{1,3}],1/4) = 1.464\ldots$$

and

$$\frac{|\overline{B^2}|}{|\overline{O^2_B}|} \geq \mathrm{fr}([2,\overline{1,2}],[2,\overline{2,1}],[3,\overline{3,1}],[2,\overline{1,2}],4/5) = 0.8835\ldots$$

so

$$\frac{|\overline{A^2A^3}|}{|O^1_A|}\frac{|\overline{B^2}|}{|\overline{O^2_B}|} > 1.29\ . \tag{B.17}$$

As well,

$$\frac{|A^2A^3|}{|\overline{O^1_A}|} \geq \mathrm{fr}([3,\overline{1,3}],[2,\overline{3,1}],[2,\overline{2,1}],[1,\overline{1,2}],1/4) = 1.229\ldots\ ,$$

$$\frac{|B^3|}{|\overline{O^1_{B^2}}|} \geq \mathrm{fr}([3,\overline{1,3}],[3,\overline{3,1}],[2,1,\overline{1,2}],[2,2,\overline{2,1}],1/4) = 1.814\ldots\ '$$

$$\frac{|\overline{O^1_A}|}{|O^2_A|} \geq \mathrm{fr}([2,\overline{2,1}],[1,\overline{1,2}],[3,\overline{3,1}],[2,\overline{1,3}],4/5) = 2.442\ldots$$

and

$$\frac{|\overline{B^2}|}{|B^2|} \geq \text{fr}([2,\overline{1,2}],[2,\overline{2,1}],[2,\overline{1,3}],[2,\overline{3,1}],1/4) = 0.6799\ldots$$

therefore

$$\frac{|A^2A^3|}{|\overline{O^1_A}|}\cdot\frac{|B^3|}{|\overline{O^1_{B^2}}|} > 2.22 \quad\text{and}\quad \frac{|\overline{O^1_A}|}{|O^2_A|}\cdot\frac{|\overline{B^2}|}{|B^2|} > 1.66\,. \tag{B.18}$$

Now,

$$\frac{|\overline{A^3}|}{|\overline{O^1_A}|} \geq \text{fr}([3,\overline{1,2}],[3,\overline{2,1}],[2,\overline{2,1}],[1,\overline{1,2}],1/4) = 0.2079\ldots\,, \tag{B.19}$$

$$\frac{|B^3|}{|O^1_{B^{21}}|} \geq \text{fr}([3,\overline{1,3}],[3,\overline{3,1}],[2,1,2,\overline{3,1}],[2,1,1,\overline{1,3}],1/4) = 6.097\ldots\,, \tag{B.20}$$

$$\frac{|\overline{O^1_A}|}{|\overline{O^1_{A^3}}|} \geq \text{fr}([2,\overline{2,1}],[1,\overline{1,2}],[3,1,\overline{1,2}],[3,2,\overline{2,1}],4/5) = 9.449\ldots\,, \tag{B.21}$$

$$\frac{|B^{21}|}{|B^2|} \geq \text{fr}([2,\overline{1,3}],[2,1,\overline{1,3}],[2,\overline{1,3}],[2,\overline{3,1}],1/4) = 0.3954\ldots\,, \tag{B.22}$$

$$\frac{|A^{312}A^{313}|}{|\overline{O^1_A}|} \geq \text{fr}([3,\overline{1,3}],[3,1,2,\overline{3,1}],[2,\overline{2,1}],[1,\overline{1,2}],1/4) = 0.05012\ldots\,, \tag{B.23}$$

$$\frac{|B^3|}{|\overline{O^1_{B^{213}}}|} \geq \text{fr}([3,\overline{1,3}],[3,\overline{3,1}],[2,1,3,1,\overline{1,2}],[2,1,3,2,\overline{2,1}],1/4) = 44.56\ldots\,, \tag{B.24}$$

$$\frac{|\overline{O^1_A}|}{|O^2_{A^{31}}|} \geq \text{fr}([2,\overline{2,1}],[1,\overline{1,2}],[3,1,3,\overline{3,1}],[3,1,2,\overline{1,3}],4/5) = 56.03\ldots\,, \tag{B.25}$$

$$\frac{|\overline{B^{213}}|}{|B^2|} \geq \text{fr}([2,1,3,\overline{1,2}],[2,1,3,\overline{2,1}],[2,\overline{1,3}],[2,\overline{3,1}],1/4) = 0.02796\ldots\,, \tag{B.26}$$

$$\frac{|\overline{A^2}|}{|\overline{O^1_A}|} \geq \text{fr}([2,\overline{1,2}],[2,\overline{2,1}],[2,\overline{2,1}],[1,\overline{1,2}],1/4) = 0.3837\ldots\,, \tag{B.27}$$

$$\frac{|B|}{|O^1_B|} \geq \text{fr}([3,\overline{1,3}],[1,\overline{3,1}],[2,\overline{3,1}],[1,\overline{1,3}],0.5332\ldots) = 4.475\ldots\,, \tag{B.28}$$

$$\frac{|\overline{O^1_A}|}{|\overline{O^1_{A^2}}|} \geq \text{fr}([2,\overline{2,1}],[1,\overline{1,2}],[2,1,\overline{1,2}],[2,2,\overline{2,1}],4/5) = 5.563\ldots \tag{B.29}$$

and

$$\frac{|B^3|}{|B^2|} \geq \mathrm{fr}([3,\overline{1,3}],[3,\overline{3,1}],[2,\overline{1,3}],[2,\overline{3,1}],1/4) = 0.5383\ldots \tag{B.30}$$

therefore

$$\frac{|\overline{A^3}|}{|\overline{O^1_A}|}\cdot\frac{|B^3|}{|O^1_{B^{21}}|} > 1.26 \quad \text{and} \quad \frac{|\overline{O^1_A}|}{|\overline{O^1_{A^3}}|}\cdot\frac{|B^{21}|}{|B^2|} > 3.73 \tag{B.31}$$

by (B.19), (B.20), (B.21) and (B.22),

$$\frac{|A^{312}A^{313}|}{|\overline{O^1_A}|}\cdot\frac{|B^3|}{|\overline{O^1_{B^{213}}}|} > 2.23 \quad \text{and} \quad \frac{|\overline{O^1_A}|}{|O^2_{A^{31}}|}\frac{|\overline{B^{213}}|}{|B^2|} > 1.56 \tag{B.32}$$

by (B.23), (B.24), (B.25) and (B.26), and

$$\frac{|\overline{A^2}|}{|\overline{O^1_A}|}\cdot\frac{|B^3|}{|O^1_{B^3}|} > 1.72 \quad \text{and} \quad \frac{|\overline{O^1_A}|}{|\overline{O^1_{A^2}}|}\cdot\frac{|B^3|}{|B^2|} > 2.99 \tag{B.33}$$

by (B.27), (B.28), (B.29) and (B.30). Further,

$$\frac{|\overline{A^{23}}|}{|\overline{O^1_A}|} \geq \mathrm{fr}([2,3,\overline{2,1}],[2,3,\overline{1,2}],[2,\overline{2,1}],[1,\overline{1,2}],1/4) = 0.03715\ldots\ , \tag{B.34}$$

$$\frac{|B^3|}{|O^1_{B^{33}}|} \geq \mathrm{fr}([3,\overline{1,3}],[3,\overline{3,1}],[3,3,2,\overline{3,1}],[3,3,1,\overline{1,3}],1/4) = 48.81\ldots\ , \tag{B.35}$$

$$\frac{|O^1_A|}{|\overline{O^1_{A^{23}}}|} \geq \mathrm{fr}([2,\overline{3,1}],[1,\overline{1,3}],[2,3,2,\overline{2,1}],[2,3,1,\overline{1,2}],4/5) = 44.85\ldots\ , \tag{B.36}$$

$$\frac{|B^{33}|}{|\overline{B^2}|} \geq \mathrm{fr}([3,3,\overline{3,1}],[3,\overline{3,1}],[2,\overline{1,2}],[2,\overline{2,1}],1/4) = 0.07271\ldots\ , \tag{B.37}$$

$$\frac{|O^1_A|}{|O^1_{A^1}|} \geq \mathrm{fr}([2,\overline{3,1}],[1,\overline{1,3}],[1,1,\overline{1,3}],[1,2,\overline{3,1}],1/4) = 2.417\ldots\ , \tag{B.38}$$

$$\frac{|\overline{B^3}|}{|\overline{B^2}|} \geq \mathrm{fr}([3,\overline{1,2}],[3,\overline{2,1}],[2,\overline{1,2}],[2,\overline{2,1}],1/4) = 0.5417\ldots\ , \tag{B.39}$$

$$\frac{|\overline{A^2A^3}|}{|O^1_A|+|A^{23L}|} \geq \mathrm{fr}([3,\overline{1,3}],[2,\overline{2,1}],[2,3,\overline{1,2}],[1,\overline{1,3}],1/4) = 1.453\ldots \tag{B.40}$$

and

$$\frac{|\ulcorner\overline{B^3}|}{|O^2_B|+|B^{213L}|} \geq \mathrm{fr}([3,\overline{1,2}],[3,\overline{3,1}],[3,\overline{3,1}],[2,1,3,\overline{1,2}],1/4) = 0.7521\ldots \quad \text{(B.41)}$$

thus we have

$$\frac{|\overline{A^{23}}|}{|\overline{O^1_A}|}\cdot\frac{|B^3|}{|O^1_{B^{33}}|} > 1.81\ , \qquad \frac{|O^1_A|}{|O^1_{A^{23}}|}\cdot\frac{|B^{33}|}{|\overline{B^2}|} > 3.26\ , \quad \text{(B.42)}$$

by (B.34), (B.35), (B.36) and (B.37),

$$\frac{|A^1|}{|\overline{O^1_A}|}\cdot\frac{|B^3|}{|O^1_{B^3}|} > 5.58\ , \qquad \frac{|O^1_A|}{|O^1_{A^1}|}\cdot\frac{|\overline{B^3}|}{|\overline{B^2}|} > 1.30 \quad \text{(B.43)}$$

by (B.4), (B.28), (B.38) and (B.39) and

$$\frac{|\overline{A^2A^3}\urcorner|}{|O^1_A|+|A^{23L}|}\cdot\frac{|\ulcorner\overline{B^3}|}{|O^2_B|+|B^{213L}|} > 1.09 \quad \text{(B.44)}$$

by (B.40) and (B.41). Now

$$\frac{|B^2|}{|O^2_B|} \geq \mathrm{fr}([2,\overline{1,3}],[2,\overline{3,1}],[3,\overline{3,1}],[2,\overline{1,3}],4/5) = 1.480\ldots\ , \quad \text{(B.45)}$$

$$\frac{|A^3|}{|\overline{O^1_A}|} \geq \mathrm{fr}([3,\overline{1,3}],[3,\overline{3,1}],[2,\overline{2,1}],[1,\overline{1,2}],1/4) = 0.3038\ldots \quad \text{(B.46)}$$

and

$$\frac{|B^2|}{|\overline{O^1_{B^2}}|} \geq \mathrm{fr}([2,\overline{1,3}],[2,\overline{3,1}],[2,1,\overline{1,2}],[2,2,\overline{2,1}],1/4) = 3.370\ldots \quad \text{(B.47)}$$

hence

$$\frac{|\overline{A^1}|}{|\overline{O^1_A}|}\cdot\frac{|B^2|}{|O^2_B|} > 1.24 \quad \text{and} \quad \frac{|A^3|}{|\overline{O^1_A}|}\cdot\frac{|B^2|}{|\overline{O^1_{B^2}}|} > 1.02 \quad \text{(B.48)}$$

by (B.1), (B.45), (B.46) and (B.47). Further,

$$\frac{|\overline{A^2}|}{|\overline{A^1}|} \geq \mathrm{fr}([2,\overline{1,2}],[2,\overline{2,1}],[1,\overline{1,2}],[1,\overline{2,1}],1/4) = 0.4105\ldots, \tag{B.49}$$

$$\frac{|O_B^2|}{|\overline{O_{B^2}^1}|} \geq \mathrm{fr}([3,\overline{3,1}],[2,\overline{1,3}],[2,1,\overline{1,3}],[2,2,\overline{3,1}],1/4) = 2.868\ldots, \tag{B.50}$$

$$\frac{|\overline{A}|}{|\overline{O_A^1}|} \geq \mathrm{fr}([2,\overline{1,2}],[1,\overline{2,1}],[2,\overline{2,1}],[1,\overline{1,2}],4/5) = 2.257\ldots, \tag{B.51}$$

$$\frac{|B^2|}{|O_B^2|} \geq \mathrm{fr}([2,\overline{1,3}],[2,\overline{3,1}],[3,\overline{3,1}],[2,\overline{1,3}],4/5) = 1.480\ldots, \tag{B.52}$$

$$\frac{|O_B^2|}{|\overline{O_{B^3}^1}|} \geq \mathrm{fr}([3,\overline{3,1}],[2,\overline{1,3}],[3,1,\overline{1,3}],[3,2,\overline{3,1}],4/5) = 5.136\ldots, \tag{B.53}$$

$$\frac{|B^3|}{|O_B^2|} \geq \mathrm{fr}([3,\overline{1,3}],[3,\overline{3,1}],[3,\overline{3,1}],[2,\overline{1,3}],1/4) = 0.8402\ldots, \tag{B.54}$$

$$\frac{|\overline{A^2}|}{|O_{A^3}^1|} \geq \mathrm{fr}([2,\overline{1,2}],[2,\overline{2,1}],[3,1,\overline{1,2}],[3,2,\overline{2,1}],4/5) = 3.910\ldots, \tag{B.55}$$

$$\frac{|\overline{B^2}|}{|O_B^2|} \geq \mathrm{fr}([2,\overline{1,2}],[2,\overline{2,1}],[3,\overline{3,1}],[2,\overline{1,3}],4/5) = 1.011\ldots, \tag{B.56}$$

$$\frac{|\overline{A^2}|}{|\overline{O_A^2}^{\urcorner}|} \geq \mathrm{fr}([2,\overline{1,2}],[2,\overline{2,1}],[3,\overline{3,1}],[2,\overline{1,2}],4/5) = 0.8835\ldots, \tag{B.57}$$

$$\frac{|\overline{B^2}^{\urcorner}|}{|O_B^2|} \geq \mathrm{fr}([2,\overline{1,3}],[2,\overline{2,1}],[3,\overline{3,1}],[2,\overline{1,3}],4/5) = 1.155\ldots, \tag{B.58}$$

$$\frac{|\overline{A^2A^3}^{\urcorner}|}{|\overline{A^1}|} \geq \mathrm{fr}([3,\overline{1,3}],[2,\overline{2,1}],[1,\overline{1,2}],[1,\overline{2,1}],1/4) = 1.179\ldots, \tag{B.59}$$

$$\frac{|O_B^2|}{|\overline{O_B^2}^{\urcorner}|} \geq \mathrm{fr}([3,\overline{3,1}],[2,\overline{1,3}],[3,\overline{3,1}],[2,\overline{1,2}],1/4) = 0.8712\ldots, \tag{B.60}$$

and

$$\frac{|O_B^2|}{|\overline{O_{B^2}^1}|} \geq \mathrm{fr}([3,\overline{3,1}],[2,\overline{1,3}],[2,1,\overline{1,2}],[2,2,\overline{2,1}],1/4) = 2.159\ldots. \tag{B.61}$$

Thus

$$\frac{|\overline{A^2}|}{|\overline{A^1}|}\cdot\frac{|O_B^2|}{|O_{B^2}^1|} > 1.17 \quad \text{and} \quad \frac{|\overline{A^2}|}{|\overline{O_{A^2}^1}|}\cdot\frac{|B^2|}{|O_B^2|} > 3.34 \tag{B.62}$$

by (B.49), (B.50), (B.51) and (B.52),

$$\frac{|\overline{A^2}|}{|\overline{A^1}|}\cdot\frac{|O^2_B|}{|O^1_{B^3}|} > 2.10 \quad \text{and} \quad \frac{|\overline{A^2}|}{|\overline{O^1_{A^2}}|}\cdot\frac{|B^3|}{|O^2_B|} > 1.89 \tag{B.63}$$

by (B.49), (B.53), (B.51) and (B.54),

$$\frac{|\overline{A^2}|}{|O^1_{A^3}|}\cdot\frac{|\overline{B^2}|}{|O^2_B|} > 3.95 \tag{B.64}$$

by (B.55) and (B.56),

$$\frac{|\overline{A^2}|}{|\overline{O^2_A}\urcorner|}\cdot\frac{|\overline{B^2}\urcorner|}{|O^2_B|} > 1.02 \tag{B.65}$$

by (B.57) and (B.58), and

$$\frac{|\overline{A^2A^3}\urcorner|}{|\overline{A^1}|}\cdot\frac{|O^2_B|}{|\overline{O^2_B}\urcorner|} > 1.02 \tag{B.66}$$

by (B.59) and (B.60). Also

$$\frac{|A^{23}|}{|\overline{\ulcorner A^2}|} \geq \mathrm{fr}([2,3,\overline{3,1}],[2,\overline{3,1}],[2,\overline{1,2}],[2,\overline{3,1}],4/5) = 0.1035\ldots$$

and

$$\frac{|O^2_B|+|B^{213L}|}{|\overline{O^2_{B^{21}}}|} \geq \mathrm{fr}([3,\overline{3,1}],[2,1,3,\overline{1,2}],[2,1,3,\overline{2,1}],[2,\overline{1,2}],1/4) = 10.30\ldots$$

therefore

$$\frac{|A^{23}|}{|\overline{\ulcorner A^2}|}\cdot\frac{|O^2_B|+|B^{213L}|}{|\overline{O^2_{B^{21}}}|} > 1.06\ . \tag{B.67}$$

Furthermore

$$\frac{|\overline{\ulcorner A^2}|}{|\overline{O^2_{A^{21}}}|} \geq \mathrm{fr}([2,\overline{1,2}],[2,\overline{3,1}],[2,1,3,\overline{2,1}],[2,\overline{1,2}],4/5) = 14.02\ldots$$

and

$$\frac{|B^{33}|}{|O_B^2|+|B^{213L}|} \geq \mathrm{fr}([3,3,\overline{3,1}],[3,\overline{3,1}],[3,\overline{3,1}],[2,1,3,\overline{1,2}],1/4) = 0.07667\ldots \tag{B.68}$$

thus

$$\frac{|\ulcorner\overline{A^2}|}{|\overline{O^2_{A^{21}}}|} \cdot \frac{|B^{33}|}{|O_B^2|+|B^{213L}|} > 1.07\,. \tag{B.69}$$

Now,

$$\frac{|A^2|-|A^{213L}|}{|O_A^2|+|A^{213L}|} \geq \mathrm{fr}([2,1,3,\overline{1,2}],[2,\overline{3,1}],[3,\overline{3,1}],[2,1,3,\overline{1,2}],4/5) = 1.465\ldots\,,$$

$$\frac{|\overline{B^3}\urcorner|}{|O_B^2|+|B^{213L}|} \geq \mathrm{fr}([3,\overline{1,2}],[3,\overline{3,1}],[3,\overline{3,1}],[2,1,3,\overline{1,2}],1/4) = 0.7521\ldots\,,$$

$$\frac{|A^{23}|}{|O_A^2|+|A^{213L}|} \geq \mathrm{fr}([2,3,\overline{3,1}],[2,\overline{3,1}],[3,\overline{3,1}],[2,1,3,\overline{1,2}],4/5) = 0.1375\ldots$$

and

$$\frac{|\ulcorner\overline{B^3}|}{|\overline{O^2_{B^{21}}}|} \geq \mathrm{fr}([3,\overline{1,2}],[3,\overline{3,1}],[2,1,3,\overline{2,1}],[2,\overline{1,2}],1/4) = 7.749\ldots$$

hence

$$\frac{|A^2|-|A^{213L}|}{|O_A^2|+|A^{213L}|} \cdot \frac{|\overline{B^3}\urcorner|}{|O_B^2|+|B^{213L}|} > 1.10 \tag{B.70}$$

and

$$\frac{|A^{23}|}{|O_A^2|+|A^{213L}|} \cdot \frac{|\ulcorner\overline{B^3}|}{|\overline{O^2_{B^{21}}}|} > 1.06\,. \tag{B.71}$$

Also

$$\frac{|O^2_B|}{|O^1_{B^3}|} \geq \mathrm{fr}([3,\overline{3,1}],[2,\overline{1,3}],[3,1,\overline{1,3}],[3,2,\overline{3,1}],4/5) = 5.136\ldots, \tag{B.72}$$

$$\frac{|A^2|}{|\overline{A^1}|} \geq \mathrm{fr}([2,\overline{1,3}],[2,\overline{3,1}],[1,\overline{1,2}],[1,\overline{2,1}],1/4) = 0.6038\ldots, \tag{B.73}$$

$$\frac{|\overline{A^2}|}{|O^1_{A^2}|} \geq \mathrm{fr}([2,\overline{1,2}],[1,\overline{2,1}],[2,\overline{3,1}],[1,\overline{1,3}],4/5) = 3.000\ldots, \tag{B.74}$$

$$\frac{|\overline{A^2}|}{|O^1_{A^{23}}|} \geq \mathrm{fr}([2,\overline{1,2}],[1,\overline{2,1}],[3,1,\overline{1,3}],[3,2,\overline{3,1}],4/5) = 28.32\ldots \tag{B.75}$$

and

$$\frac{|\overline{B^{213}}|}{|O^2_B|} \geq \mathrm{fr}([2,1,3,\overline{1,2}],[2,1,3,\overline{2,1}],[3,\overline{3,1}],[2,\overline{1,3}],4/5) = 0.04270\ldots \tag{B.76}$$

so

$$\frac{|\overline{A^2}|}{|\overline{A^1}|}\cdot\frac{|O^2_B|}{|O^1_{B^3}|} > 2.10 \quad \text{and} \quad \frac{|\overline{A^2}|}{|\overline{O^1_{A^2}}|}\cdot\frac{|B^3|}{|O^2_B|} > 1.89 \tag{B.77}$$

by (B.49), (B.72), (B.51) and (B.54),

$$\frac{|A^2|}{|\overline{A^1}|}\cdot\frac{|O^2_B|}{|\overline{O^1_{B^2}}|} > 1.30 \quad \text{and} \quad \frac{|\overline{A^2}|}{|O^1_{A^2}|}\cdot\frac{|\overline{B^2}|}{|O^2_B|} > 3.03 \tag{B.78}$$

by (B.73), (B.61), (B.74) and (B.56), and

$$\frac{|\overline{A^2}|}{|O^1_{A^{23}}|}\cdot\frac{|\overline{B^{213}}|}{|O^2_B|} > 1.20 \tag{B.79}$$

by (B.75) and (B.76). Further, we have

$$\frac{|B^2B^3|}{|O^1_{B^1}|} \geq \mathrm{fr}([3,\overline{1,3}],[2,\overline{3,1}],[1,1,\overline{1,3}],[1,2,\overline{3,1}],1/4) = 3.947\ldots \tag{B.80}$$

so

$$\frac{|\overline{A^2}|}{|\overline{O^1_A}|}\cdot\frac{\min\{|B^1|,|B^2B^3|\}}{|O^1_{B^1}|} > 1.51 \tag{B.81}$$

by (B.6), (B.28) and (B.80). Now

$$\frac{|B^2|}{|\overline{O^1_{B^3}}|} \geq \mathrm{fr}([2,\overline{1,3}],[2,\overline{3,1}],[3,1,\overline{1,2}],[3,2,\overline{2,1}],4/5) = 5.728\ldots, \tag{B.82}$$

$$\frac{|\overline{A^2}|}{|O^1_{A^3}|} \geq \mathrm{fr}([2,\overline{1,2}],[2,\overline{2,1}],[3,1,\overline{1,3}],[3,2,\overline{3,1}],4/5) = 5.193\ldots, \tag{B.83}$$

$$\frac{|\overline{B^3}|}{|O^2_B|} \geq \mathrm{fr}([3,\overline{1,2}],[3,\overline{2,1}],[3,\overline{3,1}],[2,\overline{1,3}],1/4) = 0.5749\ldots, \tag{B.84}$$

$$\frac{|\overline{A^{213}}|}{|\overline{O^1_A}|} \geq \mathrm{fr}([2,1,3,\overline{1,2}],[2,1,3,\overline{2,1}],[2,\overline{2,1}],[1,\overline{1,2}],1/4) = 0.01578\ldots, \tag{B.85}$$

$$\frac{|B^2|}{|O^1_{B^{33}}|} \geq \mathrm{fr}([2,\overline{1,3}],[2,\overline{3,1}],[3,3,2,\overline{3,1}],[3,3,1,\overline{1,3}],4/5) = 84.12\ldots, \tag{B.86}$$

$$\frac{|\overline{A^2A^3}^{\urcorner}|}{|\overline{O^1_A}|} \geq \mathrm{fr}([3,\overline{1,3}],[2,\overline{2,1}],[2,\overline{2,1}],[1,\overline{1,2}],1/4) = 1.102\ldots \tag{B.87}$$

and

$$\frac{|B^2|}{|\overline{{}^{\ulcorner}O^2_B}|} \geq \mathrm{fr}([2,\overline{1,3}],[2,\overline{3,1}],[3,\overline{2,1}],[2,\overline{1,3}],4/5) = 1.248\ldots \tag{B.88}$$

thus

$$\frac{|A^3|}{|\overline{O^1_A}|} \cdot \frac{|B^2|}{|\overline{O^1_{B^3}}|} > 1.74 \quad \text{and} \quad \frac{|\overline{A^2}|}{|O^1_{A^3}|} \cdot \frac{|\overline{B^3}|}{|O^2_B|} > 2.98 \tag{B.89}$$

by (B.46), (B.82), (B.83) and (B.84),

$$\frac{|\overline{A^{213}}|}{|\overline{O^1_A}|} \cdot \frac{|B^2|}{|O^1_{B^{33}}|} > 1.32 \tag{B.90}$$

by (B.85) and (B.86) and

$$\frac{|\overline{A^2A^3}^{\urcorner}|}{|\overline{O^1_A}|} \cdot \frac{|B^2|}{|\overline{{}^{\ulcorner}O^2_B}|} > 1.38 \tag{B.91}$$

by (B.87) and (B.88). Further,

$$\frac{|\overline{{}^{\ulcorner}A^1}|}{|\overline{O^2_{A^{11}}}|} \geq \mathrm{fr}([3,\overline{1,3}],[1,\overline{2,1}],[1,\overline{2,1}],[1,3,\overline{2,1}],1/4) = 13.46\ldots, \tag{B.92}$$

$$\frac{|{}^{\ulcorner}\overline{A^1}|}{|O^1_A| + |A^{113L}|} \geq \mathrm{fr}([1,\overline{1,2}],[1,\overline{3,1}],[2,\overline{3,1}],[1,1,3,\overline{1,2}],4/5) = 1.496\ldots \tag{B.93}$$

and

$$\frac{|\ulcorner\overline{B^3}|}{|O_B^2| + |B^{213L}|} \geq \mathrm{fr}([3, \overline{1,2}], [3, \overline{3,1}], [3, \overline{3,1}], [2, 1, 3, \overline{1,2}], 1/4) = 0.7521\ldots \quad \text{(B.94)}$$

whence

$$\frac{|\ulcorner\overline{A^1}|}{|\overline{O_{A^{11}}^2}|} \cdot \frac{|B^{33}|}{|O_B^2| + |B^{213L}|} > 1.03 \quad \text{(B.95)}$$

by (B.92) and (B.68), and

$$\frac{|\ulcorner\overline{A^1}|}{|O_A^1| + |A^{113L}|} \cdot \frac{|\ulcorner\overline{B^3}|}{|O_B^2| + |B^{213L}|} > 1.12 \quad \text{(B.96)}$$

by (B.93) and (B.94). Also we have

$$\frac{|\overline{A^1}|}{|O_{A^{13}}^1|} \geq \mathrm{fr}([2, \overline{1,2}], [1, \overline{2,1}], [3, 1, \overline{1,3}], [3, 2, \overline{3,1}], 4/5) = 28.32\ldots, \quad \text{(B.97)}$$

$$\frac{|\overline{B^{213}}|}{|O_B^2|} \geq \mathrm{fr}([2, 1, 3, \overline{1,2}], [2, 1, 3, \overline{2,1}], [3, \overline{3,1}], [2, \overline{1,3}], 4/5) = 0.04270\ldots, \quad \text{(B.98)}$$

$$\frac{|A^{13}|}{|\overline{O_A^1}|} \geq \mathrm{fr}([1, 3, \overline{3,1}], [1, \overline{3,1}], [2, \overline{2,1}], [1, \overline{1,2}], 4/5) = 0.1240\ldots, \quad \text{(B.99)}$$

$$\frac{|B^3|}{|\overline{O_{B^{21}}^2}|} \geq \mathrm{fr}([3, \overline{1,3}], [3, \overline{3,1}], [2, 1, 3, \overline{2,1}], [2, \overline{1,2}], 1/4) = 8.601\ldots, \quad \text{(B.100)}$$

$$\frac{|A|}{|\overline{O_A^1}|} \geq \mathrm{fr}([3, \overline{1,3}], [1, \overline{3,1}], [2, \overline{2,1}], [1, \overline{1,2}], 0.5617\ldots) = 3.369\ldots, \quad \text{(B.101)}$$

$$\frac{|B^3|}{|\overline{O_{B^2}^1}|} \geq \mathrm{fr}([3, \overline{1,3}], [3, \overline{3,1}], [2, 1, \overline{1,2}], [2, 2, \overline{2,1}], 1/4) = 1.814\ldots \quad \text{(B.102)}$$

and

$$\frac{|\overline{B^2}|}{|O_B^2|} \geq \mathrm{fr}([2, \overline{1,2}], [2, \overline{2,1}], [3, \overline{3,1}], [2, \overline{1,3}], 4/5) = 1.011\ldots \quad \text{(B.103)}$$

therefore

$$\frac{|\overline{A^1}|}{|\overline{O_A^1}|} \cdot \frac{|B^3|}{|O_{B^3}^1|} > 3.77 \quad \text{and} \quad \frac{|B^3|}{|O_B^2|} \cdot \frac{|\overline{A}|}{|\overline{O_A^1}|} > 1.89 \quad \text{(B.104)}$$

by (B.1), (B.28), (B.54) and (B.51),

$$\frac{|\overline{A^1}|}{|O^1_{A^{13}}|}\cdot\frac{|\overline{B^{213}}|}{|O^2_B|} > 1.20 \tag{B.105}$$

by (B.97) and (B.98),

$$\frac{|A^{13}|}{|\overline{O^1_A}|}\cdot\frac{|B^3|}{|\overline{O^2_{B^{21}}}|} > 1.06 \tag{B.106}$$

by (B.99) and (B.100), and

$$\frac{|A|}{|\overline{O^1_A}|}\cdot\frac{|B^3|}{|\overline{O^1_{B^2}}|} > 6.11 \quad\text{and}\quad \frac{|\overline{A^1}|}{|O^1_A|}\cdot\frac{|B^2|}{|O^2_B|} > 1.13 \tag{B.107}$$

by (B.101), (B.102), (B.11) and (B.103). Finally,

$$\frac{|\overline{A^{113}}|}{|\overline{O^1_A}|} \geq \mathrm{fr}([1,1,3,\overline{1,2}],[1,1,3,\overline{2,1}],[2,\overline{2,1}],[1,\overline{1,2}],4/5) = 0.03366\dots\,, \tag{B.108}$$

$$\frac{|B^3|}{|\overline{O^1_{B^{33}}}|} \geq \mathrm{fr}([3,\overline{1,3}],[3,\overline{3,1}],[3,3,2,\overline{3,1}],[3,3,1,\overline{1,3}],1/4) = 48.81\dots\,, \tag{B.109}$$

$$\frac{|A^2|}{|\overline{O^1_A}|} \geq \mathrm{fr}([2,\overline{1,3}],[2,\overline{3,1}],[2,\overline{2,1}],[1,\overline{1,2}],1/4) = 0.5643\dots\,, \tag{B.110}$$

$$\frac{|B^3|}{|\overline{O^1_{B^3}}|} \geq \mathrm{fr}([3,\overline{1,3}],[1,\overline{3,1}],[2,\overline{2,1}],[1,\overline{1,2}],0.5617\dots) = 3.369\dots \tag{B.111}$$

and

$$\frac{|\overline{A^1}|}{|O^1_{A^2}|} \geq \mathrm{fr}([1,\overline{1,2}],[1,\overline{2,1}],[2,1,\overline{1,3}],[2,2,\overline{3,1}],4/5) = 6.235\dots \tag{B.112}$$

hence

$$\frac{|\overline{A^{113}}|}{|\overline{O^1_A}|}\cdot\frac{|B^3|}{|O^1_{B^{33}}|} > 1.64 \tag{B.113}$$

by (B.108), (B.109), and

$$\frac{|A^2|}{|\overline{O^1_A}|}\cdot\frac{|B^3|}{|O^1_{B^3}|} > 1.90 \quad\text{and}\quad \frac{|\overline{A^1}|}{|O^1_{A^2}|}\cdot\frac{|\overline{B^3}|}{|O^2_B|} > 3.58 \tag{B.114}$$

by (B.110), (B.111), (B.112) and (B.84).

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