The search for an excluded minor characterization of ternary Rayleigh matroids

by

Stephanie Phillips

A thesis
presented to the University of Waterloo
in fulfilment of the
thesis requirement for the degree of
Master of Mathematics
in
Combinatorics and Optimization

Waterloo, Ontario, Canada, 2007

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

ABSTRACT

Rayleigh matroids are a class of matroids with sets of bases that satisfy a strong negative correlation property. Interesting characteristics include the existence of an efficient algorithm for sampling the bases of a Rayleigh matroid [7]. It has been conjectured that the class of Rayleigh matroids satisfies Mason's conjecture [14]. Though many elementary properties of Rayleigh matroids have been established, it is not known if this class has a finite set of minimal excluded minors. At this time, it seems unlikely that this is the case. It has been shown that there is a single minimal excluded minor for the smaller class of binary Rayleigh matroids [5]. The aim of this thesis is to detail our search for the set of minimal excluded minors for ternary Rayleigh matroids. We have found several minimal excluded minors for the above class of matroids. However, our search is incomplete. It is unclear whether the set of excluded minors for this set of matroids is finite or not, and, if finite, what the complete set of minimal excluded minors is. For our method to answer this question definitively will require a new computer program. This program would automate a step in our process that we have done by hand: writing polynomials in at least ten indeterminates as a sum with many terms, squared.

ACKNOWLEDGEMENTS

Contents

Chapter 1

Introduction

Rayleigh matroids are a class of matroids with sets of bases that satisfy a strong weighted negative correlation property. Rayleigh matroids are a subset of the larger class of balanced matroids. It has been shown that if a matroid $\mathcal{M}$ is balanced, then the set of bases of $\mathcal{M}$ can be sampled efficiently [7]. It has been conjectured that the class of Rayleigh matroids satisfies Mason's conjecture [14]. A characterization of binary Rayleigh matroids by means of excluded minors was presented in 1999 [12] and the proof refined and corrected in 2006 [5]. The aim of this thesis is to extend this technique to the class of ternary Rayleigh matroids in an attempt to characterize them in terms of excluded minors. The outline of the thesis is as follows.

Section 2 contains the basic concepts of matroid theory that are needed for the thesis. The source is Oxley [13]. Topics covered are excluded minor characterizations of binary, ternary and quaternary matroids; definitions of the classes of regular, half-plane property and sixth root of unity matroids; connectivity; unique representability of certain classes of matroids; and Seymour's Splitter Theorem.

In Section 3, Rayleigh matroids and the more general class of balanced matroids are defined and their properties discussed. Most material in this section has been published in [5] and [7]. In addition to providing background material, the results in this section are used to pare down the set of ternary matroids we must consider in our search for minimal non-Rayleigh matroids. We review established properties for both classes of matroids: closure under duals, minors and direct sums. Rayleigh matroids (but not balanced ma-

troids) are also closed under 2-sums. These properties are important for our purposes, as they imply a minimal excluded minor for the class of ternary, Rayleigh matroids must be 3-connected. Similarly, the fact that matroids of rank three or less are Rayleigh [19] is used to show that a minimal excluded minor for the class of ternary, Rayleigh matroids must have at least 8 elements. The Rayleigh property is also proven for regular matroids, half-plane property matroids, and sixth-root of unity matroids. Special emphasis is placed on the proof of the excluded minor characterization of binary matroids [5] and the use of Seymour's Splitter Theorem to obtain the result.

Section 4 describes and justifies our search technique. In Section 4.1, the search technique is explained in detail, here we review it briefly. In Section 3, we established a number of properties which a minimal excluded minor for the class of ternary, Rayleigh matroids must possess. In particular, such matroids must have at least 8 elements. Thus our first step is to use Oid (a computer program) to generate our set of 8-element "candidates": all ternary, non-isomorphic matroids on 8 elements that have the required properties. Each matroid of this set is tested for the Rayleigh property using Maple, and the set is divided into two classes: Rayleigh matroids and non-Rayleigh matroids. All non-Rayleigh matroids are added to our set of minimal excluded minors for ternary, Rayleigh matroids. Next, all non-isomorphic, one-element extensions and coextensions of the Rayleigh matroids are generated. Of this new set of matroids, all those having a non-Rayleigh minor are eliminated. The resulting set of matroids are all our 9-element "candidates", which are now tested for the Rayleigh property using Maple. Our method is a reiteration of the above steps for matroids of increasing ground set size. In Section 4.2, we prove the validity of our method, using a corollary of the Splitter Theorem. The last two sections explain algorithms used for checking for isomorphisms between matroids [10], testing a matroid for orbits of its automorphism group acting on 2-element subsets of $E(\mathcal{M})$, and generating extensions and coextensions of matroids [10]. The code for some of these procedures can be found in Appendix A.

In Section 5, we present our "candidates"- sets of ternary, non-isomorphic and potentially non-Rayleigh matroids on 8, 9 and 10 elements. We identify the minimal ternary, non-Rayleigh matroids that our search has yielded thus far. Proof of our claims can be found in Appendix B (8-element matroids) and Appendix C (9-element matroids). Our 10-element candidates have been

tested for balance, but not for the stronger Rayleigh condition. Proof of our claims of balance for 10-element candidates is given in Appendix D.

In Section 6, we discuss some related open problems.

Chapter 2

Matroids

In this section we review key concepts of matroid theory that will be used to demonstrate later results. For the basic concepts of matroid theory we refer the reader to Oxley's book [13]. We discuss the structure of matroids we will be working with, fields where unique representation holds, connectivity, and Seymour's Splitter Theorem. Proofs are omitted- we refer the interested reader to [13].

2.1 Structural results

We review some structural results: excluded minor characterizations of binary, ternary and quaternary matroids. We define three other classes of matroids: regular matroids, half-plane property matroids, and sixth-root of unity matroids.

We now review excluded minor characterizations for three classes of matroids: binary, ternary and quaternary. The set of excluded minors for a class of matroids may be viewed as the "seeds" from which all matroids not of that class can be "grown". Let F be a field and $\mathcal{M}$ a matroid. Then $\mathcal{M}$ is said to be *F-representable* if $\mathcal{M}$ is isomorphic to the vector matroid of a matrix D over F. A matroid is binary if and only if it is $GF(2)$-representable. Similarly, ternary and quaternary matroids are exactly those matroids which are $GF(3)$- and $GF(4)$-representable, respectively. Our first result was established by Tutte in 1958 [18].

Theorem 2.1 A matroid is binary if and only if it has no $U_{2,4}$-minor.

The excluded minor characterization of ternary matroids was announced by Ralph Reid in 1971, but his proof was never published. The first published proofs were given independently by Bixby [1] and Seymour [16] in 1979.

Theorem 2.2 A matroid is ternary if and only if it has no minor isomorphic to any of the matroids $U_{2,5}, U_{3,5}, \mathcal{F}_7$, or $\mathcal{F}_7{}^*$.

The excluded minor characterization of $GF(4)$-representable matroids was published by Geelen, Gerards and Kapoor in 1992 [8].

Theorem 2.3 A matroid is quaternary if and only if it has no minor isomorphic to any of the matroids $U_{2,6}, U_{4,6}, \mathcal{F}_7-, (\mathcal{F}_7-)^*, \mathcal{P}_6$ and $\mathcal{P}_8$.

We now introduce three more classes of matroids: regular, half-plane property (HPP for short), and sixth-root of unity matroids.

A matroid is said to be *regular* if it can be represented over $\mathbb{R}$ by a *totally unimodular matrix*- a matrix over $\mathbb{R}$ for which every square submatrix has a determinant in 1,0,-1. The following theorem offers another characterization of regular matroids.

Theorem 2.4 The following statements are equivalent for a matroid $\mathcal{M}$:

(i) $\mathcal{M}$ is regular.
(ii) $\mathcal{M}$ is representable over every field.
(iii) $\mathcal{M}$ is binary and, for some field F of characteristic other than two, $\mathcal{M}$ is F-representable.

That (i) implies (ii) follows after a few simple steps by considering a totally unimodular matrix representation of $\mathcal{M}$; that (ii) implies (iii) is clear. Brylawski established that (iii) implies (i) in 1975 [2].

By Theorem 2.4 above, it follows that $\mathcal{M}$ is regular if and only if it is both binary and ternary. Thus,a regular matroid must contain none of the excluded minors of $GF(2)$- or $GF(3)$-representable matroids. Observe that both $U_{2,5}$ and $U_{3,5}$ contain $U_{2,4}$ as a minor. From this, Theorem 2.1 and Theorem 2.2, it follows that the complete set of excluded minors for the class of regular matroids is $U_{2,4}, \mathcal{F}_7$, and $\mathcal{F}_7{}^*$.

We now define a term that will be used in our definition of half-plane property matroids: the *basis generating polynomial* of a matroid, denoted $M(y)$. Let $E(\mathcal{M})$ denote the ground set of a matroid $\mathcal{M}$. Fix indeterminates $y = \{y_e | e \in E\}$ indexed by $E(\mathcal{M})$. For a subset X of $E(\mathcal{M})$, let

$$y^X = \prod_{e \in X} y_e,$$

where $y^X = 1$ if X is the empty set. Let $B(\mathcal{M})$ denote the set of bases of $\mathcal{M}$. We define the basis-generating polynomial of $\mathcal{M}$ as follows:

$$M(y) = \sum_{B \in B(\mathcal{M})} y^B.$$

We now define the class of half-plane property matroids. A polynomial $P(y) = \sum_a c_a y^a$ in complex variables $y = \{y_e | e \in E\}$ has the *half-plane property* if when $\operatorname{Re}(y_e) > 0$ for all $e \in E$, then $P(y) \neq 0$. A matroid $\mathcal{M}$ is a *half-plane property matroid* or *HPP matroid* if its basis-generating polynomial $M(y)$ has the half-plane property.

A third class of matroids that is important for our purposes is sixth-root of unity matroids.A matrix A is a *sixth-root of unity* matrix if every nonzero entry of A is a sixth-root of unity. A matroid $\mathcal{M}$ is a *sixth-root of unity matroid* if it can be represented over the complex numbers by a sixth-root of unity matrix. The following characterization of sixth-root of unity matroids is due to Whittle [20].

Theorem 2.5 A matroid is a sixth-root of unity matroid if and only if it is both $GF(3)-$ and $GF(4)-$ representable.

Choe, Oxley, Sokal and Wagner [4] demonstrated that all sixth-root of unity matroids have the HPP.

Corollary 2.6 A matroid is a sixth-root of unity matroid if and only if it has no minor isomorphic to any of the matroids $\{U_{2,5}, U_{3,5}, \mathcal{F}_7, \mathcal{F}_7{}^*, \mathcal{F}_7-, F_7-^*, \mathcal{P}_6, \mathcal{P}_8\}$.

2.2 Connected and 3-connected Matroids

Let $\mathcal{M}$ and $\mathcal{N}$ be matroids with respective ground sets $E(\mathcal{M})$ and $E(\mathcal{N})$, where $E(\mathcal{M}) \cap E(\mathcal{N})$ is empty. Then their direct sum, $\mathcal{M} \oplus \mathcal{N}$, is the matroid with ground set $E(\mathcal{M}) \cup E(\mathcal{N})$ and basis set

$$B(M \oplus N) = \{B_1 \cup B_2 | B_1 \text{ is a basis of } \mathcal{M}, B_2 \text{ is a basis of } \mathcal{N}\}.$$

A matroid $\mathcal{M}$ is *connected* if it cannot be written as the direct sum of two non-empty matroids. A matroid is said to be *1-separable* if there exists a partition $\{X, Y\}$ of $E(\mathcal{M})$ satisfying

$$\min\{|X|, |Y|\} \geq 1$$

and

$$r(X) + r(Y) - r(M) \leq 0.$$

Observe that this implies $\mathcal{M}$ is 1-separable if and only if it is not connected. Since the second inequality above can only hold with equality, we have $r(X) + r(Y) - r(M) = 0$ for some partition $\{X, Y\}$ of $E(\mathcal{M})$, where both X and Y are non-empty. But then $\mathcal{M}$ may be written as $\mathcal{X} \oplus \mathcal{Y}$, where $\mathcal{X}$ is the matroid with ground set X and basis set $B(\mathcal{X}) = \{B \subseteq X | r(B) = r(X)\}$, and $\mathcal{Y}$ is defined similarly.

Now let $\mathcal{M}$ and $\mathcal{N}$ be matroids such that $E(\mathcal{M}) \cap E(\mathcal{N}) = \{p\}$, where p is neither a loop nor a coloop of $\mathcal{M}$ or $\mathcal{N}$. Then the 2-sum of $\mathcal{M}$ and $\mathcal{N}$, $\mathcal{M} \oplus_2 \mathcal{N}$, is the matroid with ground set $E(\mathcal{M}) \cup E(\mathcal{N}) \backslash \{p\}$ and basis set

$$B(\mathcal{M} \oplus_2 \mathcal{N}) = \{D \cup C | D \text{ is a basis of } \mathcal{M} \backslash p, C \text{ is a basis of } \mathcal{N}/p\}$$
$$\cup \{C \cup D | C \text{ is a basis of } \mathcal{M}/p, D \text{ is a basis of } \mathcal{N} \backslash p\}.$$

The above matroid is also called the *2-sum of $\mathcal{M}$ and $\mathcal{N}$ along p*.

A matroid $\mathcal{M}$ is *3-connected* if it is connected and cannot be written as a 2-sum of two non-empty matroids, both of which are proper minors of $\mathcal{M}$. Let $\mathcal{M}$ be a matroid. Then $\mathcal{M}$ is said to be *2-separable* if there exists a partition $\{X, Y\}$ of $E(\mathcal{M})$ satisfying

$$\min\{|X|, |Y|\} \geq 2$$

and

r(X) + r(Y) - r(M) $\leq$ 1.

2.3 Unique representability

As we will see later, some of our search methods are valid only if the class of matroids we are considering is uniquely representable over a given field. Here we review the results we will need to justify our technique.

The next two results follow from a theorem by Brylawski and Lucas, published in 1976 [3].

Theorem 2.7 Binary matroids are uniquely representable over $GF(2)$.

Theorem 2.8 Ternary matroids are uniquely representable over $GF(3)$.

Our final result was proven by Kahn in 1988 [9].

Theorem 2.9 Let $\mathcal{M}$ be a $GF(4)$-representable matroid. Then $\mathcal{M}$ is uniquely $GF(4)$-representable if and only if $\mathcal{M}$ cannot be written as a direct sum or 2-sum of two non-binary matroids. In particular, if $\mathcal{M}$ is 3-connected, then it is uniquely $GF(4)$-represenatble

2.4   Seymour's Splitter Theorem

The following result plays an important role both in the excluded minor characterization of binary Rayleigh matroids and in our search technique.

Let **M** be a class of matroids which is closed under minors and isomorphism. A 3-connected member $\mathcal{N}$ of **M** is called a *splitter* for **M** if, whenever $\mathcal{M}$ is a member of **M** having a proper $\mathcal{N}$-minor, either $\mathcal{M}$ is isomorphic to $\mathcal{N}$, or $\mathcal{M}$ is 1- or 2- separable. Thus $\mathcal{N}$ is a splitter for **M** if and only if **M** has no 3-connected member with a proper $\mathcal{N}$-minor. Intuitively, $\mathcal{N}$ is a splitter for **M** if $\mathcal{N}$ is a maximal 3-connected "building block" from which members of **M** may be constructed.

We now define two classes of matroids that are used in Theorem 2.10: wheels and whirls.

The *r-spoked wheel graph* (for $r \geq 2$), denoted W_r, is the graph with vertex set $\{v_0, v_1, \ldots, v_r\}$ and edge set $\{v_0v_1, v_0v_2, \ldots, v_0v_r\} \cup \{v_1v_2, v_2, v_3, \ldots, v_rv_1\}$. The *r-spoked wheel matroid* (for $r \geq 2$), denoted $\mathcal{M}(W_r)$, is the cycle matroid of the r-spoked wheel graph. We note that while $\mathcal{M}(W_r)$ exists for $r \geq 2$, it is 3-connected if and only if $r \geq 3$ [13].

The *rank r whirl* (for $r \geq 2$), denoted $\mathcal{W}^r$, is the matroid obtained from $\mathcal{M}(W_r)$ by relaxing the unique circuit-hyperplane $\{v_1v_2, v_2v_3, \ldots, v_rv_1\}$. By relaxing, we mean that this circuit in $\mathcal{M}(W_r)$ becomes a basis of $\mathcal{W}^r$. The rank r whirl is 3-connected for all $r \geq 2$ [13].

We now state the version of the Splitter Theorem given in [13]:

Theorem 2.10 Let $\mathcal{N}$ be a connected, simple, cosimple minor of a 3-connected matroid $\mathcal{M}$. Suppose that if $\mathcal{N}$ is a wheel, then $\mathcal{M}$ has no larger wheel as a minor, while if $\mathcal{N}$ is a whirl, then $\mathcal{M}$ has no larger whirl as a minor. Then either $\mathcal{M} = \mathcal{N}$, or $\mathcal{M}$ has a connected, simple, cosimple minor $\mathcal{M}_1$ such that some single-element deletion or some single-element contraction of $\mathcal{M}_1$ is isomorphic to $\mathcal{N}$. Moreover, if $\mathcal{N}$ is 3-connected, so too is $\mathcal{M}_1$.

Chapter 3

Rayleigh Matroids

The notation used throughout this section is taken from Choe and Wagner [5].

In this section we introduce Rayleigh matroids and the more general class of balanced matroids. Proofs of some elementary properties of both classes are given. Most complex results are stated without proof. Our focus is on how these facts may be applied to facilitate our search for minor-minimal ternary, non-Rayleigh matroids. However, the characterization of binary Rayleigh matroids is of particular interest to us. The proof relies on Theorem 2.10; we will adapt the technique to the class of ternary matroids. Thus, we review the main points of this proof in detail.

3.1 Balanced Matroids

The concept of a balanced matroid was first introduced by Feder and Mihail in 1992 [7]. We begin by defining a related notion: negative correlation. Let $\mathcal{M}$ be a matroid with ground set E, and let M denote the number of bases of $\mathcal{M}$. For disjoint subsets $I, J \subset E$, let $\mathcal{M}_I^J$ denote the minor of $\mathcal{M}$ obtained by deleting J and contracting I. $\mathcal{M}$ is said to be *negatively correlated* if the following holds for all pairs of distinct $e, f \in E$

$$M_f^e M_e^f - M^{ef} M_{ef} \geq 0.$$

The number on the left is abbreviated as $\Delta M\{e, f\}$. Observe that M_I^J equals the number of bases of $\mathcal{M}$ that contain I but no element of J. So $M_e = M_e^f + M_{ef}$, $M_f = M_f^e + M_{ef}$, and $M = M_{ef} + M_e^f + M_f^e + M^{ef}$. Using these relations, it can be shown that the above inequality is equivalent to

$$\frac{M_f}{M} \geq \frac{M_{ef}}{M_e}$$

whenever e is not a loop. This equation helps illustrate an intuitive definition of negative correlation: a matroid $\mathcal{M}$ is negatively correlated if the presence of any fixed element in a basis of $\mathcal{M}$ chosen uniformly at random can only make the presence of any other element in that basis less likely. In the above inequality, the left side is the probability that a random basis of $\mathcal{M}$ contains f, and the right side is the probability that a basis contains f given that it contains e. $\mathcal{M}$ is *balanced* if $\mathcal{M}$ and all its minors are negatively correlated. So by definition, the class of balanced matroids is closed under minors.

Feder and Mihail prove two main results about balanced matroids: all regular matroids are balanced, and the basis exchange graph of a balanced matroid has cutset expansion at least one. Proofs will not be provided here. Both results are of interest to us because they motivate some of the work which follows. A stronger version of the first result was proven in 2004 [5]. The second result demonstrates some "practical" applications for balanced matroids. We now explain this finding in more detail.

The *basis-exchange graph* of a matroid $\mathcal{M}$, denoted $G(\mathcal{M})$, was introduced by Edmonds in 1969 [6]. The vertex set of $G(\mathcal{M})$ is the set of bases of $\mathcal{M}$, while the edge set connects all pairs of bases B_1, B_2 such that $B_2 = B_1 \cup \{e\} \backslash \{f\}$ for some $\{e, f\} \subseteq E(\mathcal{M})$. For a balanced matroid $\mathcal{M}$, that $G(\mathcal{M})$ has cutset expansion at least one means that for every bipartition of the vertices of $G(\mathcal{M})$, the number of edges adjacent to both classes is at least the size of the smaller class. One result of this expansion of $G(\mathcal{M})$ is that there is an efficient randomized algorithm which can be used to approximate the number of bases of $\mathcal{M}$. For proof and details of this method, refer to Feder and Mihail [7].

3.2   Rayleigh Matroids

We now introduce the similar concept of a Rayleigh matroid. Let $\mathcal{M}$ be a matroid with ground set $E(\mathcal{M})$ and let $y = \{y_c | c \in E(\mathcal{M})\}$ be a set of indeterminates indexed by the elements of $E(\mathcal{M})$. Recall from Section 2.2 that $M(y)$ represents the basis-generating polynomial of a matroid $\mathcal{M}$. We say that $\mathcal{M}$ is Rayleigh if whenever $y_c > 0$ for all $c \in E$, the following holds for all pairs of distinct $e, f \in E$:

$$\Delta M\{e, f\}(y) = M_f^e(y)M_e^f(y) - M^{ef}(y)M_{ef}(y) \geq 0.$$

The polynomial on the left is called the *Rayleigh difference of $\{e,f\}$ in* $\mathcal{M}$.

This terminology is motivated by the Rayleigh monotonicity property of linear resistive electrical networks. An electrical network can be viewed as a multigraph $G = (V, E)$ with a set of positive weights $y = \{y_e | e \in E\}$ assigned to its edges. Here y_e, the weight of an edge e, corresponds to the electrical conductance of e in the network. Now we choose any two vertices $a, b \in V$ and consider $\mathcal{Y}_{ab}$, the conductance of the network between these two poles. Kirchoff proved that

$$\mathcal{Y}_{ab}(G; y) = \frac{T(G;y)}{T(G/ab;y)}$$

in which $T(G; y) := \sum_T y^T$ over all spanning trees of T. G/ab denotes the graph in which a and b are merged into a single vertex and $T(G/ab; y)$ is defined similarly. The *Rayleigh monotonicity* property is as follows: if $y_c > 0$ for all $c \in E$ and y_e is increased, then the conductance of the network between a and b cannot decrease. Now let H be the multigraph derived from G by adding an edge f joining a and b. Since $\partial \mathcal{Y}_{ab}(G; y)/\partial y_e$ is nonnegative, we can rewrite the above equation in terms of H and f as

$$\frac{\partial}{\partial y_e} \frac{T(H^f)}{T(H_f)} \geq 0.$$

By the quotient rule and some simple expansion and cancellation, we find

$$T_e^f(H;y)T_f^e(H;y) - T^{ef}(H;y)T_{ef}(H;y) \geq 0,$$

where $T(H, y)$ is the sum of y^T over all spanning trees T of H. Replacing $T(H;y)$ with the basis generating polynomial for a matroid $\mathcal{M}$ yields he defining property of Rayleigh matroids.

3.3 Basic Properties of Balanced and Rayleigh Matroids

We now prove some basic facts about both classes of matroids.

Proposition 3.1 If $\mathcal{M}$ is Rayleigh, then $\mathcal{M}$ is negatively correlated.

Proof: Let $\mathcal{M}$ be a Rayleigh matroid, and let $y = \{y_c | y_c = 1 \text{ for all } c \in E\}$. Then

$$\Delta M\{e, f\} = \Delta M\{e, f\}(y) > 0$$

for all $e, f \in E$. Hence $\mathcal{M}$ is negatively correlated. $\square$

Proposition 3.2 If $\mathcal{M}$ is a Rayleigh matroid, then so is $\mathcal{M}^*$.

Proof: Let $\mathcal{M}$ be a Rayleigh matroid. Let e, f be any pair of distinct elements of $E(\mathcal{M}^*)$ and let $y = \{y_c | c \in E(\mathcal{M}^*)\{e, f\}\}$ be a fixed set of positive real numbers. (Since $\Delta M\{e, f\}(y)$ is independent of y_e and y_f, for simplicity we may disregard them). Define a second set of positive reals $1/y = \{1/y_c \text{ for all } c \in E(\mathcal{M})e, f\}\}$. Observe that if B^* is a basis of M^{*e}_f, then $B = E(\mathcal{M}) - (B^* \cup \{e, f\})$ is a basis of M_e^f. Thus $y^{B^*} = y^{E(\mathcal{M})-\{e,f\}}(1/y)^B$, so $M^{*e}_f(y) = y^{E(\mathcal{M})-\{e,f\}}M_e^f(1/y)$. The analogous result holds for M^{*ef} and

M^*_{ef}, etcetera. From this, the result follows:

$$\Delta M^*\{e,f\}(y) = M^{*e}_f(y)M^{*f}_e(y) - M^{*e,f}(y)M^*_{e,f}(y)$$
$$= y^{E(\mathcal{M})-\{e,f\}}y^{E(\mathcal{M})-\{e,f\}}\{M^f_e(1/y)M^e_f(1/y) - M_{e,f}(1/y)M^{e,f}(1/y)\}$$
$$= y^{E(\mathcal{M})-\{e,f\}}y^{E(\mathcal{M})-\{e,f\}}\{\Delta M\{e,f\}(1/y)\}$$
$$\geq 0$$

$\square$

Proposition 3.3 If $\mathcal{M}$ is a Rayleigh matroid and $\mathcal{N}$ is a minor of $\mathcal{M}$, then

$\mathcal{N}$ is Rayleigh.

Proof: First assume that $\mathcal{M}$ is Rayleigh and that $\mathcal{N}$ is derived from $\mathcal{M}$ by the deletion or contraction of a single element g. Take y to be any set of positive reals indexed by the elements of $E(\mathcal{M})$. Let I and J be any pair of disjoint subsets of $E(\mathcal{M})$. Then

$$\mathcal{M}^J_I(y) = y_g M^J_{Ig}(y) + M^{Jg}_I(y).$$

Now, applying the above equality to the expansion of the Rayleigh difference of $\{e,f\}$ in $\mathcal{M}$ yields

$$\Delta M\{e,f\}(y) = M^e_f(y)M^f_e(y) - M^{ef}(y)M_{ef}(y)$$
$$= (y_g M^e_{fg} + M^{eg}_f)(y_g M^f_{eg} + M^{fg}_e) - (y_g M^{ef}_g + M^{efg})(y_g M_{efg} + M^g_{ef})$$
$$= y_g{}^2 \Delta M_g\{e,f\}(y) + y_g \Theta M\{e,f|g\}(y) + \Delta M^g\{e,f\}(y)$$

where

$$\Theta M\{e,f|g\}(y) =$$
$$(M^e_{fg}(y)M^{fg}_e(y) + M^{eg}_f(y)M^f_{eg}(y)) - (M^{ef}_g(y)M^g_{ef}(y) + M^{efg}(y)M_{efg}(y)).$$

We now examine the above result, which will be used again in Theorem 3.6.

$$\Delta M\{e,f\}(y) = y_g{}^2 \Delta M_g\{e,f\}(y) + y_g \Theta M\{e,f|g\}(y) + \Delta M^g\{e,f\}(y) \quad (3.1)$$

Since $\mathcal{M}$ is Rayleigh, the above equation is nonnegative for all $e, f, g \in E(\mathcal{M})$. Taking the limit of $y_g^{-2} \Delta M\{e, f\}(y)$ as $y_g \to \infty$, we see that $\Delta M_g\{e, f\}(y)$ is nonnegative. Since $e, f, g \in E(\mathcal{M})$ and $y = \{y_c | c \in E(M)\}$ were chosen arbitrarily, $\mathcal{M}_g$ is Rayleigh. Similarly, taking $\lim_{y_g \to 0} \Delta M\{e, f\}$ shows that $\mathcal{M}^g$ is Rayleigh. Finally, closure under minors follows from repeated application of the above two steps. $\square$

Proposition 3.4 If $\mathcal{M}$ is Rayleigh, then $\mathcal{M}$ is balanced.

Proof: By Proposition 3.3, all minors of $\mathcal{M}$ are Rayleigh. By Proposition 3.1, all Rayleigh matroids are negatively correlated. Therefore $\mathcal{M}$ is balanced. $\square$

It is worthwhile to note that while all Rayleigh matroids are balanced, the converse does not hold. The following example is taken from Choe and Wagner [5]. Below is a matroid $\mathcal{J}'$ which is balanced but not Rayleigh. $\mathcal{J}'$ is represented over $\mathbb{R}$ by the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 3 \\ 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 2 & 3 \end{pmatrix}$$

In Corollary 3.9 we will show that all matroids with at most seven ground set elements are Rayleigh. For the above example, we will assume this result. Therefore it is enough to show that $\mathcal{J}'$ is negatively correlated but not Rayleigh. Let $E(\mathcal{J}') = \{1, 2, \ldots 8\}$ label the columns of the above matrix. Using Maple, it is easy to show that $J'^{\{e,f\} \geq 0}$ for all distinct pairs $e, f \in E(\mathcal{J}')$. However, if the elements are given weights $y_2 = y_3 = y_4 = t$ and $y_5 = y_6 = y_7 = 1$, we have

$$\Delta \mathcal{J}'\{1, 8\}(y) = (t + 1)^3 (t - 1)(t^2 + t - 1).$$

Thus $\Delta \mathcal{J}'\{1, 8\} < 0$ if $(\sqrt{5} - 1)/2 < t < 1$. Hence, $\mathcal{J}'$ is balanced but not

Rayleigh.

Proposition 3.5 If $\mathcal{N}$ and $\mathcal{P}$ are Rayleigh matroids and $\mathcal{M} = \mathcal{N} \oplus \mathcal{P}$, then $\mathcal{M}$ is Rayleigh.

Proof: Suppose $\mathcal{M} = \mathcal{N} \oplus \mathcal{P}$ and both factors of $\mathcal{M}$ are Rayleigh. Let $y = \{y_c | c \in E(\mathcal{M})\}$ be a fixed set of positive reals. Define y_n to be the subset of y containing all elements indexed by $E(\mathcal{N})$, and define y_p similarly. We have two possibilities for the pair of ground set elements $e, f \in E(\mathcal{M})$. Either e and f belong to the same factor of $\mathcal{M}$, or not. In the first case, say $e, f \in E(\mathcal{N})$, a simple calculation shows $\Delta M\{e, f\}(y) = \Delta N\{e, f\}(y_n)P(y_p) > 0$. On the other hand, if e and f belong to different factors of $\mathcal{M}$ (say $e \in E(\mathcal{N}), f \in E(\mathcal{P})$), then since $M_e^f = N_e P^f$, etc, we have

$$\Delta M\{e, f\}(y) = M_f^e(y)M_e^f(y) - M_{ef}(y)M^{ef}(y)$$
$$= N^e(y_n)P_f(y_p)N_e(y_n)P^f(y_p) - N_e(y_n)P_f(y_p)N^e(y_n)P^f(y_p)$$
$$= 0$$

Therefore $\Delta M\{e, f\}(y) = 0$. $\square$

Proposition 3.6 If $\mathcal{N}$ and $\mathcal{P}$ are Rayleigh and $\mathcal{M} = \mathcal{N} \oplus_2 \mathcal{P}$, then $\mathcal{M}$ is Rayleigh.

Proof: Let $E(\mathcal{N}) \cap E(\mathcal{P}) = g$, and suppose $\mathcal{M}$ is the 2-sum of $\mathcal{N}$ and $\mathcal{P}$ along g. Let $y = \{y_c | c \in E(\mathcal{M})\}$ be any set of positive reals. We have the following relation among the sets of bases of $\mathcal{M}, \mathcal{N}$ and $\mathcal{P}$:

$$B(\mathcal{M}) = B(\mathcal{N}_g) \times B(\mathcal{P}^g) \cup B(\mathcal{N}^g) \times B(\mathcal{P}_g).$$

As in Proposition 3.5, there are two cases to consider. Either $e \in \mathcal{N}$ and $f \in \mathcal{P}$, or $\{e, f\} \subseteq \mathcal{N}$. In the first case, expanding $\Delta M\{e, f\}(y)$ using the above fact yields the equality

$$\Delta M\{e, f\}(y) = \Delta N\{e, g\}(y_n)\Delta P\{f, g\}(y_p).$$

17

The sets y_n and y_p are defined as in the proof of Proposition 3.5. Since both $\mathcal{N}$ and $\mathcal{P}$ are Rayleigh and the sets y_n and y_p consist of positive reals, each term on the right is nonnegative. Hence $\Delta M\{e, f\}(y)$ is nonnegative.

In the second case, we use Equation 3.1 below:

$$\Delta M\{e, f\}(y) = y_g{}^2 \Delta M_g\{e, f\}(y) + y_g \Theta M\{e, f|g\}(y) + \Delta M^g\{e, f\}(y).$$

We define a set of positive reals $w = \{w_c | c \in E(\mathcal{M})\}$ as follows. Let $w_c = y_c$ for all $c \in E(\mathcal{M})$ and let $w_g = \dfrac{P^g(y)}{P_g(y)}$. A short calculation shows that

$$
\begin{aligned}
\Delta M\{e, f\}(y) &= M_f^e(y)M_e^f(y) - M_{ef}(y)M^{ef}(y) \\
&= (N_f^{eg}(w)P_g(w) + N_{fg}^e(w)P^g(w))(N_e^{fg}(w)P_g(w) + N_{eg}^f(w)P^g(w)) \\
&\quad - (N_{efg}(w)P^g(w) + N_{ef}^g(w)P_g(w))(N^{efg}(w)P_g(w) + N_g^{ef}(w)P^g(w)) \\
&= P^g(w)^2 \Delta N_g\{e, f\}(w) + P^g(w)P_g(w)\Theta N\{e, f|g\}(w) + P_g(w)^2 \Delta N^g\{e, f\}(w) \\
&= P_g(w)^2 (w_g^2 \Delta N_g\{e, f\}(w) + w_g \Theta N\{e, f|g\}(w) + \Delta N^g\{e, f\}(w)) \\
&= P_g(w)^2 \Delta N\{e, f\}(w) \\
&\geq 0,
\end{aligned}
$$

since $w > 0$ and $\mathcal{N}$ is Rayleigh. Therefore $\mathcal{M} = \mathcal{N} \oplus_2 \mathcal{P}$ is Rayleigh. $\square$

It is worthwhile to note that balanced matroids, like Rayleigh matroids, are closed under minors, duals, and direct sums. However, closure under 2-sums fails. Closure under minors is immediate from the definition of balance. To see that the the class of balanced matroids is closed under duals, note that $M_f^e = M^{*f}_e$ and $M_{ef} = M^{*ef}$. Hence $\Delta M\{e, f\} = \Delta M^*\{e, f\}$. If $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$, we have two possibilities for our pair e and f. Either they belong to a common factor of $\mathcal{M}$, or not. In the first case, say e and f are in $E(\mathcal{M}_1)$, an easy calculation shows that $\Delta M\{e, f\} = \Delta M_1\{e, f\}M_2$. So both factors must be balanced for their direct sum to be balanced. This condition is both necessary and sufficient for $\mathcal{M}$ to be balanced, as the second case illustrates. If e and f are in different factors of $\mathcal{M}$, then the deletion (or contraction) of e is independent of the deletion (or contraction) of f. In this case $\Delta M\{e, f\} = 0$, as a simple calculation will show. Thus if all factors of

$\mathcal{M}$ are balanced, then so is $\mathcal{M}$.

A counterexample to the closure of balanced matroids under 2-sums is constructed in [5]; we now review the technique. It has been shown that there exist balanced matroids that are not Rayleigh; choose such a matroid $\mathcal{M}$ as a starting point. Next, fix elements e and f of $E(\mathcal{M})$ and a set of positive real numbers $y = \{y_c| \ c \ \in \ E(\mathcal{M})\}$ that satisfy $\Delta M\{e, f\}(y) < 0$. Since the rationals are dense in the reals, there exist positive rationals fulfilling the same inequality, thus we may assume without loss of generality that $\{y_c|c \in E(\mathcal{M})\} \subset \mathbb{Q}$. Let D be the least positive common denominator of $\{y_c|c \in E(\mathcal{M}) - \{e, f\}\}$ and $m = \{m_c|m_c = Dy_c\}$. Now let $\mathcal{M}[m]$ be the matroid obtained from $\mathcal{M}$ by replacing each $c \in E(\mathcal{M}) - \{e, f\}$ with a set of m_c parallel elements. Note that this is equivalent to adjoining $U_{1,1+m_c}$ as a 2-sum along c. Now suppose r is the rank of $\mathcal{M}$ and B is any basis of $\mathcal{M}_f^e$, say $B = \{c_1, c_2, \ldots, c_{r-1}\}$. For each such basis B, consider the following set of bases of $\mathcal{M}[m]_f^e$: $\{\bar{B} = \{\bar{c}_1, \bar{c}_2, \ldots, c_{r-1}^-\}|\bar{c}_i$ is one of the parallel elements replacing $c_i \in \mathcal{M}\}$. From the construction of $\mathcal{M}[m]$ from $\mathcal{M}$, it is clear for each basis B of $\mathcal{M}$, there are exactly $m_{c_1}m_{c_2} \cdots m_{c_{r-1}}$ bases $\bar{B}$ of $\mathcal{M}[m]_f^e$. Using this relation, we have

$$
\begin{aligned}
M_f^e(y) &= \sum_{B \in B(\mathcal{M}_f^e)} y^B \\
&= \sum_{B \in B(\mathcal{M}_f^e)} y_{c_1}y_{c_2} \cdots y_{c_{r-1}} \\
&= \frac{1}{D^{r-1}} \sum_{B \in B(\mathcal{M}_f^e)} m_{c_1}m_{c-2} \cdots m_{c_{r-1}} \\
&= \frac{1}{D^{r-1}} \sum_{\bar{B} \in B(\mathcal{M}[m]_f^e)} 1 \\
&= \frac{1}{D^{r-1}} M[m]_f^e.
\end{aligned}
$$

Multiplying both sides of the equation by D^{r-1}, we have $M[m]_f^e = D^{r-1}M_f^e(y)$.

Similarly, $M[m]^{ef} = D^r M^{ef}(y)$ and $M[m]_{ef} = D^{r-2} M_{ef}(y)$. Hence

$$\begin{aligned}
\Delta M[m]\{e, f\} &= M[m]^e_f M[m]^f_e - M[m]_{ef} M[m]^{ef} \\
&= D^{r-1} M^e_f(y) D^{r-1} M^f_e(y) - D^{r-2} M_{e,f}(y) D^r M^{ef}(y) \\
&= D^{2r-2} \Delta M\{e, f\}(y) \\
&< 0
\end{aligned}$$

establishing the contradiction.

3.4   Consequences of Basic Properties

We now explore how the results of the previous section may be used to simplify our search for minimal ternary, non-Rayleigh matroids. Many of these properties enable us to eliminate groups of matroids from the set of matroids we must consider.

By Proposition 3.4, the class of Rayleigh matroids is contained in the class of balanced matroids. Clearly, it is easier to check if a matroid is balanced than if a matroid is Rayleigh. In the first case we need only check whether $\Delta M\{e, f\}$ is positive for $y = \{y_c | y_c = 1 \text{ for all } c \in E\}$, in the second case we must see if the inequality holds for all sets of positive reals y. So checking if a matroid is not balanced is a quick way to identify a non-Rayleigh matroid.

Since Rayleigh matroids are closed under direct sums (Proposition 3.5) and 2-sums (Proposition 3.6), a matroid which is not 3-connected is Rayleigh if and only if all of its factors are Rayleigh. So we can limit our search for minimal ternary, non-Rayleigh matroids to 3-connected matroids.

An important observation is that in studying the Rayleigh property, we may choose to restrict our attention to the class of simple, cosimple matroids.

Proposition 3.7 A matroid is Rayleigh if and only if its underlying simple matroid is Rayleigh.

Proof: Clearly we may assume that $\mathcal{M}$ is loopless; since no basis of $\mathcal{M}$ contains a loop, their presence has no impact on $M(y)$. We may also assume that $\mathcal{M}$ contains no parallel elements. If $a, a_1, \ldots a_k$ are parallel in $\mathcal{M}$, let

$\mathcal{N}$ be the matroid obtained from $\mathcal{M}$ by deleting $a_1, \ldots a_k$. Since Rayleigh matroids are closed under minors by Theorem 2.3, if $\mathcal{M}$ is Rayleigh then so is $\mathcal{N}$. For the converse, we consider two cases: either both $e, f \in E(\mathcal{M})$ are in $\{a_1, \ldots a_k\}$, or not. In the first case, note that no basis of $\mathcal{M}$ contains two distinct elements of $\{a_1, \ldots, a_k\}$. Thus $\Delta M\{a_i, a_j\} = M_{a_j}^{a_i} M_{a_i}^{a_j} \geq 0$ for all pairs of distinct a_i, a_j. In the second case we fix a set of indeterminates $y = \{y_c | c \in E(\mathcal{M})\}$. From y, construct $w = \{w_c | c \in E(\mathcal{N})\}$ as follows. For $c \in E(\mathcal{N}) - \{a\}$ let $w_c = y_c$, and let $w_a = y_a + y_{a_1} + \ldots + y_{a_k}$. Clearly $M(y) = N(w)$. Hence if $\mathcal{N}$ is Rayleigh so is $\mathcal{M}$. Therefore a matroid is Rayleigh if and only if its underlying simple matroid is Rayleigh. $\square$

By this fact and the closure of the class of Rayleigh matroids under duals, we may limit our attention to the set of simple, cosimple matroids.

3.5 Matroids of rank at most three are Rayleigh

In this section we review key points in the proof that matroids of rank at most three are Rayleigh. The complete proof is not presented. Our focus is on how these findings can be applied in our search for minimal ternary, non-Rayleigh matroids. For details, refer to Wagner [19].

Theorem 3.8 Matroids of rank at most three are Rayleigh.

Proof: By Proposition 3.7, when considering the Rayleigh property we may restrict our attention to the class of simple matroids. It is easy to see that matroids of rank one or two are Rayleigh. A simple, rank one matroid has ground set $E(\mathcal{M}) = \{1, 2, \ldots, m\}$ and basis-generating polynomial $M(y) = y_1 + y_2 + \ldots + y_m$. Since the sum over all bases of $\mathcal{M}_{ef}$ is empty, $M_{ef}(y) = 0$ for all distinct $e, f \in E(\mathcal{M})$. So $\Delta M\{e, f\}(y) = \Delta M_e^f(y) \Delta M_f^e(y) > 0$. If $\mathcal{M}$ is a simple rank two matroid, then $\{e, f\}$ is independent for all distinct pairs of ground set elements e and f. Thus by symmetry, it suffices to prove that $\Delta M\{e, f\}(y) > 0$ holds for a single pair $\{e, f\} \subseteq E(\mathcal{M})$. A short calculation

shows that

$$\Delta M\{1,2\}(y) = M_2^1(y)M_1^2(y) - M_{12}(y)M^{12}(y)$$

$$= (\sum_{i=3}^{m} y_i)(\sum_{i=3}^{m} y_i) - (1)(\sum_{i=3}^{m-1} \sum_{j>i}^{m} y_i y_j)$$

$$= \sum_{3 \le j \le i \le m} y_i y_j$$

$$> 0$$

Therefore $\mathcal{M}$ is Rayleigh.

The case for rank three matroids is more complicated. We will not prove that rank three matroids are Rayleigh, but review the construction used in the proof in [19]. Let $\mathcal{M}$ be a simple matroid of rank three. For a set $X \in E(\mathcal{M})$, let $cl(X)$ denote the closure of X in $E(\mathcal{M})$. For $a \in E(\mathcal{M}) - \{e, f\}$ let $L(a, e) = cl\{a, e\} - \{a, e\}$, $L(a, f) = cl\{a, f\} - \{a, f\}$, and let

$$U(a) = E(\mathcal{M}) - (cl\{a, e\} \cup cl\{a, f\} \cup cl\{e, f\}).$$

Define linear polynomials $B(a) = \sum_{b \in U(a)} y_b$, $C(a) = \sum_{c \in L(a,e)} y_c$, and $D(a) = \sum_{d \in L(a,f)} y_d$, and the quartic polynomials

$$T(\mathcal{M}; e, f, a; y) = (y_a B(a) - C(a)D(a))^2$$

for each $a \in E(\mathcal{M}) - \{e, f\}$ and

$$P(\mathcal{M}; e, f; y) = \frac{1}{4} \sum_{a \in E(\mathcal{M}) - cl\{e,f\}} T(\mathcal{M}; e, f, a; y).$$

Notice that both $T(\mathcal{M}; e, f, a; y)$ and $P(\mathcal{M}; e, f; y)$ are always nonnegative, as the former is the square of a polynomial and the latter is a sum of squares. It is shown that for all distinct pairs $e, f \in E(\mathcal{M})$, every coefficient of $\Delta M\{e, f\}(y) - P(\mathcal{M}; e, f; y)$ is nonnegative. Hence $\Delta M\{e, f\}(y)$ is nonnegative for all $y \in \mathbb{R}^{E - \{e,f\}}$, for all distinct pairs of $e, f \in E(\mathcal{M})$.$\square$

We now make a few observations about the above proof. First, note that $\mathcal{M}$ is shown to be Rayleigh by exhibiting a lower bound for $\Delta M\{e, f\}(y)$ that is a sum of squared polynomials. We will continue to prove that matroids are Rayleigh in this manner (by bounding $\Delta M\{e, f\}(y)$ below by a positive sum of squares) for matroids of rank four and greater. However, since not all matroids of rank at least four are Rayleigh, our calculations will be done on a case by case basis. More details follow in Section 4. Secondly, by Theorem 3.8 and closure of the class of Rayleigh matroids under duals, we have the following corollary:

Corollary 3.9 All matroids $\mathcal{M}$ with $|E(\mathcal{M})| \leq 7$ are Rayleigh.

Thus, we may begin searching for minor-minimal, non-Rayleigh matroids among 8-element, rank 4 matroids.

3.6 Half-Plane property matroids are Rayleigh

The following three results are presented without proof. We explain how they yield the result that half-plane property matroids (HPP matroids for short) are Rayleigh.

1. The class of HPP matroids is closed under duals, minors, direct sums and 2-sums [4].

2. All HPP matroids are Rayleigh [5].

3. All sixth-root of unity matroids have the HPP [4].

The impact of the above results on our search is as follows. Since all GF(4)-representable, ternary matroids are Rayleigh by Theorem 2.5, 2 and 3, we may limit our search for ternary non-Rayleigh matroids to matroids that are not representable over GF(4). Corollary 2.6 is then the strong point of a structural characterization of this class of matroids.

3.7 Excluded Minor Characterzation of Binary Rayleigh Matroids

In this section, we present an excluded minor characterization of binary matroids and review key points of the proof. We are interested in the details of this result because of the crucial role Theorem 2.10 plays in the proof. We will apply the theorem in a similar manner to achieve our result.

The problem of characterizing binary Rayleigh matroids by means of excluded minors was opened by Seymour, Feder and Mihail. The first published proof of our result appeared in Merino's thesis [12], however his argument contained an error. Specifically, it relied on the closure of the set of balanced matroids under 2-sums, which has since been shown to be false. Choe and Wagner [5] refined the proof- we summarize this result below.

Theorem 3.10 A binary matroid is Rayleigh if and only if it does not contain $\mathcal{S}_8$ as a minor.

Proof: The matroid $\mathcal{S}_8$ is represented over $GF(2)$ by the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & b \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}$$

when $b = 0$. When $b = 1$ the above matrix represents $\mathcal{A}_8 = \mathcal{AG}(3,2)$ over $GF(2)$. We now outline the key points in the proof.

1. The matroid $\mathcal{S}_8$ is not negatively correlated. This was observed by Seymour and Welsh and is easily verified. Recall from Section 3.1 that a matroid $\mathcal{M}$ is negatively correlated if and only if it satisfies

$$\frac{M_f}{M} \geq \frac{M_{ef}}{M_e}$$

for all distinct pairs $e, f \in E(\mathcal{M})$ where e is not a loop. Labelling the columns of the above matrix as $\{1, \ldots, 8\}$, we have $(S_8)_1 = 28$, $(S_8)_8 = 20$, $(S_8)_{1,8} = 12$, and $S_8 = 48$. Thus

$$\frac{(S_8)_8}{S_8} = \frac{20}{48} < \frac{12}{28} = \frac{(S_8)_{1,8}}{(S_8)_1}.$$

2. The matroid $\mathcal{A}_8$ is a splitter for the class of binary matroids that do not contain an $\mathcal{S}_8$ minor. This is an unpublished result of Seymour's and is proven by Merino [12].

3. Every binary matroid that does not contain $\mathcal{S}_8$ or $\mathcal{A}_8$ as a minor can be constructed from regular matroids, the Fano matroid $\mathcal{F}_7$ and its dual $\mathcal{F}_7^*$ by taking direct sums and 2-sums. This is due to Seymour [17].

4. The matroids $\mathcal{A}_8, \mathcal{F}_7$ and $\mathcal{F}_7^*$ are Rayleigh. That $\mathcal{A}_8$ is Rayleigh can be verified using Maple. We will explore how this is done in detail in the upcoming chapter. The latter two statements follow from Corollary 3.9.

We now explain briefly how these four points yield the result. Since $\mathcal{S}_8$ is not negatively correlated, it is not Rayleigh. As Rayleigh matroids are closed under minors, no matroid having an $\mathcal{S}_8$ minor is Rayleigh. For the converse, let $\mathcal{M}$ be a matroid with no $\mathcal{S}_8$ minor. Recall from Section 3.2 that Rayleigh matroids are closed under direct sums and 2-sums. So as explained in Section 3.3, we may assume without loss of generality that $\mathcal{M}$ is 3-connected. Now if $\mathcal{M}$ has an $\mathcal{A}_8$ minor, it follows from the definition of a splitter that $\mathcal{M} = \mathcal{A}_8$. It follows that $\mathcal{M}$ is Rayleigh. Now suppose $\mathcal{M}$ has no $\mathcal{A}_8$ minor. Then by our third fact and our assumption that $\mathcal{M}$ is 3-connected, either $\mathcal{M} = \mathcal{F}_7, \mathcal{M} = \mathcal{F}_7^*$, or $\mathcal{M}$ is regular. In the first two cases, the fact that $\mathcal{M}$ is Rayleigh is immediate from our fourth fact. In the latter case, if $\mathcal{M}$ is regular then clearly $\mathcal{M}$ is both $GF(3)-$ and $GF(4)-$ representable. So as stated in the last section, $\mathcal{M}$ is an HPP matroid and is therefore Rayleigh. $\square$

Chapter 4

Searching for Minimal Ternary non-Rayleigh Matroids

As in the excluded minor characterization of binary Rayleigh matroids, Seymour's Splitter Theorem plays a vital role in our search. In the binary case, all matroids with an $\mathcal{S}_8$ minor are non-Rayleigh. The 'splitter' $\mathcal{A}_8$ neatly divides the remaining matroids of GF(2) into two classes. For those matroids with an $\mathcal{A}_8$ minor, the Rayleigh property followed easily. A nice result of Seymour's gives us a way of constructing all remaining binary matroids by taking direct sums and 2-sums of "building blocks" known to be Rayleigh. However, in the case of ternary matroids, our work is not as straightforward. There is no similar choice of splitter which divides the class of ternary matroids as conveniently for our purposes. Nor is there a theorem for constructing a large subset of ternary matroids from a set of 3-connected, Rayleigh building blocks. Thus, we require a different version of Seymour's Splitter Theorem to obtain our result. The structure of this section is as follows. First we describe our search technique. We then explain why our technique is valid. The Splitter Theorem plays an important role in justifying our methods. Finally, we present detailed explanations of the computer programs used in our search and how they work.

4.1   Search Technique

We now describe our method for identifying minor-minimal, non-Rayleigh ternary matroids. As shown previously, we know that the following matroids

are Rayleigh: matroids of rank or corank at most three, matroids with a ground set of at most seven elements, and matroids which are ternary and $GF(4)$-representable. For our purposes we may choose to consider only simple, cosimple and 3-connected matroids. Therefore the first set of "candidate" matroids we examine is precisely the set of 8-element, rank 4, 3-connected, simple and cosimple, ternary but not $GF(4)-$ representable matroids. For the sake of simplicity we refer to matroids having the above properties as "candidates". The list of 8-element candidates was generated using software developed by Sandra Kingan. Details of her program will be discussed in Section 4.3. Next the basis-generating polynomial of each candidate is determined using Maple. For each matroid $\mathcal{N}$ an 8×8 matrix $M(N)$ is defined as follows. The rows and columns of $M(\mathcal{N})$ are indexed by the elements of the ground set $E(\mathcal{N})$ and if $e \neq f$ then

$$ M(\mathcal{N})(e, f) = N_f^e N_e^f - N^{ef} N_{ef}. $$

where $M(\mathcal{N})(e, f)$ is an integer- specifically, the polynomial $\Delta N\{e, f\}(y)$ evaluated for $y_c = 1$ for all $c \in E(\mathcal{N}) - \{e, f\}$. The above computations are done using Maple. If $M(N)(e, f) < 0$ for some $e \neq f \in E(\mathcal{N})$, then $\mathcal{N}$ is not balanced. This matrix helps us quickly identify candidates which are not negatively correlated: any $\mathcal{N}$ such that $M(N)$ has a negative entry. These matroids are therefore non-Rayleigh. The rest of our candidates are negatively correlated, but not necessarily Rayleigh. For each of these matroids, the set of orbits of the automorphism group of $\mathcal{M}$ acting on 2-element subsets of $E(\mathcal{M})$ is determined. Details on the program and how it works follow in Section 4.3. Since the class of Rayleigh matroids is closed under isomorphism, we need only show that $\Delta N\{e, f\}(y) > 0$ (for all $y > 0$) for one pair of elements e, f in each equivalence class to prove that $\mathcal{N}$ is Rayleigh.

Our final step is as follows. For each candidate $\mathcal{N}$, we generate $\Delta N\{e, f\}(y)$ for a general set of indeterminates y. We do this for each non-equivalent pair $e, f \in E(\mathcal{N})$. Now given $\Delta N\{e, f\}(y)$ for a fixed matroid $\mathcal{N}$ and pair $e, f \in E(\mathcal{N})$, we identify all negative terms in the sum. If all terms are positive, clearly $\Delta N\{e, f\}(y) > 0$ for all sets of positive reals $y = \{y_e | e \in E\}$. Now recall Equation 3.1:

$$\Delta M\{e,f\}(y) = y_g{}^2 \Delta M_g\{e,f\}(y) + y_g \Theta M\{e,f|g\}(y) + \Delta M^g\{e,f\}(y).$$

Equation 3.1 gives us a quick way to check for the Rayleigh property. If, in the polynomial $\Delta M\{e,f\}(y)$, there is an indeterminate y_g which never appears to the first power in a term with a negative coefficient, then we have

$$\Delta M\{e,f\}(y) \geq y_g{}^2 \Delta M_g\{e,f\}(y) + \Delta M^g\{e,f\}(y).$$

Since both $\mathcal{M}_g$ and $\mathcal{M}^g$ are Rayleigh, both $\Delta M_g\{e,f\}(y)$ and $\Delta M^g\{e,f\}(y)$ are nonnegative for all sets of reals $y = \{y_e | e \in E(\mathcal{M})\}$. Hence the left side is nonnegative, and $\mathcal{M}$ is Rayleigh.

If the above method fails, we seek to minimize $\Delta M\{e,f\}(y)$ and, if possible, exhibit a set of positive reals $\{y_e | e \in E(\mathcal{M})\}$ satisfying $\Delta M\{e,f\}(y) < 0$ for some pair $\{e,f\} \in E(\mathcal{M})$. Clearly this is sufficient to show that $\mathcal{M}$ is not Rayleigh.

If there are negative terms and the above methods do not work, $\mathcal{M}$ still might be Rayleigh. In order to prove this, we seek to "cover" all negative terms by writing them as a polynomial with many terms squared. For instance, if our polynomial contains the terms $+y_1{}^2 y_2{}^2 y_3{}^2, -y_1 y_2 y_3 y_4 y_5 y_6, +y_4{}^2 y_5{}^2 y_6{}^2$, we may replace these three terms with $0.5(y_1 y_2 y_3 - y_4 y_5 y_6)^2 + 0.5 y_1{}^2 y_2{}^2 y_3{}^2 + 0.5 y_4{}^2 y_5{}^2 y_6{}^2$. Repeating this process to cover all negative terms yields a sum of positive terms and squares. Certainly this equivalent expression is non-negative if $y > 0$. If we complete this process for all non-isomorphic pairs $e, f \in E(\mathcal{N})$, then it follows that $\mathcal{N}$ is Rayleigh. Note that while this method is clearly sufficient to show that a matroid $\mathcal{M}$ is Rayleigh, it is not necessary. However, it has worked thus far for our purposes.

Once we have established which of our 8-element condidates are Rayleigh and which are non-Rayleigh, we use a computer program, Oid, to generate the set of 9-element candidates we must test. Details on Oid follow in Sections 4.3 and 4.4. We now repeat the process used to test our 8-element candidates for negative correlation and the Rayleigh property. Having now found all non-Rayleigh matroids in our 9-element list, we repeat the whole

procedure to generate the set of 10-element candidates and determine which are non-Rayleigh. Our method is a reiteration of the above step for matroids of increasing ground-set size. The hope is that after a certain number of iterations, our new set of candidates will be the empty set. At this point, we will have determined an excluded minor characterization of ternary, non-Rayleigh matroids.

4.2 Seymour's Splitter Theorem and Validity of Search Technique

In this section we present results which will be used to demonstrate that our method is valid. The following corollary of Theorem 2.10 is taken from [13].

Theorem 4.1 Let $\mathcal{M}$ and $\mathcal{N}$ be 3-connected matroids such that $\mathcal{N}$ is a minor of $\mathcal{M}$, $|E(\mathcal{N})| \geq 4$, and if $\mathcal{N}$ is a wheel, then $\mathcal{M}$ has no larger wheel as a minor, while if $\mathcal{N}$ is a whirl, then $\mathcal{M}$ has no larger whirl as a minor. Then there is a sequence $\mathcal{M}_0, \mathcal{M}_1, \ldots, \mathcal{M}_n$ of 3-connected matroids such that $\mathcal{M}_0$ is isomorphic to $\mathcal{N}$, $\mathcal{M}_n = \mathcal{M}$, and, for all i in $\{0, 1, \ldots, n-1\}, \mathcal{M}_i$ is a single-element deletion or a single-element contraction of $\mathcal{M}_{i+1}$.

This theorem shows that there exist a sequence of 3-connected matroids which transform $\mathcal{N}$ to $\mathcal{M}$. In our proof we will choose $\mathcal{M}$ to be a minor-minimal, non-Rayleigh ternary matroid. Next we exhibit a 3-connected minor of $\mathcal{M}$ which appears on our initial list of candidates. We show that the series of matroids which convert $\mathcal{N}$ to $\mathcal{M}$ in the theorem correspond to a series of candidates generated by our method. It was shown earlier that our process correctly identifies both Rayleigh and non-Rayleigh matroids. Note that though our method is sufficient, it is not known if it is necessary. Demonstrating that the method generates all minor-minimal, non-Rayleigh matroids $\mathcal{M}$ for our consideration completes our proof. We now explain why our method is valid in detail.

Let $\mathcal{M}$ be any minor-minimal, ternary, non-Rayleigh matroid. Then $\mathcal{M}$ is simple, cosimple and 3-connected, and all proper minors of $\mathcal{M}$ are Rayleigh. Since $\mathcal{M}$ is ternary but not $GF(4)$-representable, it must contain an excluded minor for $GF(4)$-representability which is also ternary. Thus $\mathcal{M}$ must contain

29

$\mathcal{F}_7^-, \mathcal{F}_7^{-*}$ or $\mathcal{P}_8$ as a minor. All of these matroids are Rayleigh. We now apply the Splitter theorem, taking $\mathcal{N}$ to be one of the above minors of $\mathcal{M}$. Clearly all the necessary conditions for the theorem apply. Since $\mathcal{M}$ is not Rayleigh, it must contain a 3-connected, one-point extension or coextension $\mathcal{M}_1$ of $\mathcal{N}$ by the theorem. All such extensions and coextensions of $\mathcal{F}_7^-$ and $\mathcal{F}_7^{-*}$, together with $\mathcal{P}_8$, form our list of 8-element candidates. Some of these are Rayleigh, some are not. If $\mathcal{M}$ contains a non-Rayleigh candidate $\mathcal{P}$ on the list, then $\mathcal{M}=\mathcal{P}$ and the process terminates. If $\mathcal{M}$ does not contain any of the non-Rayleigh ones, then another application of the Theorem 2.10 shows that $\mathcal{M}$ must contain a one-point extension or coextension of an 8-element candidate. This brings us to our list of 9-element candidates. Again either $\mathcal{M}$ or a proper Rayleigh minor of $\mathcal{M}$ appears on our list. In the first case the process terminates, in the second case we apply Theorem 2.10 again in a similar manner. We continue in this way until $\mathcal{M}$ is eventually generated. As our list of non-Rayleigh matroids grows longer, the hope is that we will reach a stage where all viable candidates of a fixed ground set size have a non-Rayleigh minor.

4.3 Algorithms for identifying redundant Rayleigh differences and isomorphic matroids

Since much of our search depends on software, in the next two sections we review the programs we use and how they work. In this section, we discuss a program for testing a matroid for orbits of the automorphism group acting on 2-element subsets of $E(\mathcal{M})$ and Sandra Kingan's program, Oid, which checks matroids for isomorphisms. The first algorithm enables us to take a matroid $\mathcal{M}$ and identify a minimal set of pairs $\{e, f\} \in E(\mathcal{M})$ that must be tested to verify that $\mathcal{M}$ is Rayleigh. The second algorithm identifies isomorphic matroids, so we need only consider one matroid from each equivalence class when checking for the Rayleigh property. These two algorithms are similar, so we consider both in this section. Oid's isomorphism checker works together with an extension/coextension generator to find all non-isomorphic extensions and coextensions of a matroid. We will discuss the generator in detail in the next section.

First we consider the program for testing a matroid for orbits of its au-

tomorphism group acting on 2-element subsets. Input for this program is a matroid $\mathcal{M}$ as its set of bases. There are four major steps in the algorithm. The program generates the set of circuits of $\mathcal{M}$, $C(\mathcal{M})$; uses $C(\mathcal{M})$ to find the set of all automorphisms on $\mathcal{M}$ which preserve its circuit structure; tests each of these functions to determine the set of all automorphisms of $\mathcal{M}$; and finally outputs a set of representatives of the orbits of $Aut(\mathcal{M})$ acting on the 2-element subsets of $E(\mathcal{M})$. Note that this final output is the set of all pairs we must test in order to determine if $\mathcal{M}$ is Rayleigh. We now review the program in more detail.

The program generates the circuit set of $\mathcal{M}$, $C(\mathcal{M})$, as follows. First, all subsets of $\{1, 2, \ldots, n\}$ are generated, where $n = |E(\mathcal{M})|$ and the columns of the matrix representation of $\mathcal{M}$ over $GF(3)$ are indexed by $\{1, 2, \ldots, n\}$. Next, all circuits of $\mathcal{M}$ are generated in the following manner. Initialize the set of circuits of $\mathcal{M}$ as the empty set. Input consists of the basis-generating polynomial and rank r of $\mathcal{M}$. For s such that $1 \leq s \leq r$, we consider each subset X of $\{1, 2, \ldots, r\}$ of size s. If any circuit on our list thus far is contained in X, then X is dependent, our circuit set remains unchanged, and we move on to the next subset. Otherwise, we check to see if X is contained in any basis of $\mathcal{M}$. If so, then X is independent and we move on leaving our circuit set unchanged. If not, then X is a circuit and we add it to our set. In this way, the set of all circuits of $\mathcal{M}$, $C(\mathcal{M})$, is generated.

Next, the program generates a list, L, which records the point-circuit incidence structure of $\mathcal{M}$. The entries of L are indexed by the elements of $E(\mathcal{M})$. Each entry of L records, for the corresponding $e \in E(\mathcal{M})$, the number of circuits of each size e is in. This information is recorded as a monomial in indeterminates $X_1, X_2, \ldots X_k$. For each entry of L, the lower subscript of each X_j term represents circuits of size j, while the power of X_j records the number of circuits of size j a given element is in. For example, if the third entry of the list is $X_4{}^6 X_3{}^2$, then the third element of $E(\mathcal{M})$ is in six circuits of size four and two circuits of size three. Clearly, any automorphism of $\mathcal{M}$ must preserve this circuit structure. Thus, all automorphisms of $\mathcal{M}$ are contained in a Young subgroup of L: the set of all permutations on L which permute like elements amongst themselves. The program generates the Young subgroup of L and then tests each permutation to see if it is in fact an automorphism of $\mathcal{M}$. (Note that this algorithm is not best possible in terms of efficiency, however it works for our purposes when dealing with

31

small matroids.) Permutations are tested by simply applying a permutation σ to $C(\mathcal{M})$ and checking if $C(\sigma \circ (\mathcal{M})) = C(\mathcal{M})$. Clearly, this is the case if and only if σ is an automorphism of $\mathcal{M}$. In this manner the set of all automorphisms of $\mathcal{M}$, $Aut(\mathcal{M})$, is found. Finally, the program determines all distinct pairs of elements of $E(\mathcal{M})$ which are equivalent under automorphism as follows. A list of all two-element subsets of $E(\mathcal{M})$, TS, is generated. While TS is nonempty, an element $\{i, j\}$ of TS is chosen and its orbit determined by applying each permutation $\sigma \in A(\mathcal{M}$ to $\{i, j\}$. The set of all pairs equivalent to $\{i, j\}$ under automorphism is $[i, j] = \{\{k, l\} | \{k, l\} = \sigma \circ (i, j)\}$ for some $\sigma \in Aut(\mathcal{M})\}$. We add $\{i, j\}$ to the set of pairs we need check to verify the Rayleigh condition, and TS is updated by setting $TS = TS - [i, j]$. The process is repeated until TS is empty. The final output of the program is a set of pairs of elements $\{i, j\} \in E(\mathcal{M})$, one representing each orbit of $Aut(\mathcal{M})$ acting on 2-sets of $E(\mathcal{M})$. We need only check that the Rayleigh condition holds for each of these pairs to verify that $\mathcal{M}$ is Rayleigh.

Next we consider Oid. We begin with a general description of the program, then focus our attention on its isomorphism checker and how it works.

Oid is an interactive program used for matroid generation. Matroids can be input as matrices over a field, sets of bases, or for graphic matroids, the cycle matroid of the corresponding graph. Given a matroid, Oid can determine its circuit set, bases, hyperplanes, independent sets, spanning sets, etc. Furthermore, Oid has a number of different routines for computing the above sets. For a given matroid, Oid selects the most efficient method for determining a desired property, or the user may specify an algorithm from a list. Oid's choice of method may depend on the form in which a matroid is input, or properties of the matroid such as size.

Oid's isomorphism checker works in a manner similar to our first algorithm. In comparing two matroids, the checker computes for each classes of elements based on membership in circuits, independent sets and spanning sets. If the matroids are a match with respect to these classes as well as the numbers of circuits, independent sets and spanning sets of diferent sizes, then there may exist an isomorphism between the two matroids. The checker now generates all bijections from one matroid onto the other which preserve these classes. Each map is checked for isomorhpism by applying it and testing to see if it preserves all bases.

4.4 Algorithms for Generating Extensions and Coextensions of a Matroid

In order to generate all single element extensions of a simple matroid over GF(q), Oid works as follows. A rank r, n-element simple matroid is entered in standard form, as a matrix $[I_r|D]$ with entries from $GF(q)$, where D has $(n-r)$ columns. Note that all simple matroids of this form can be viewed as minors of $PG(r-1, q)$. To see this, observe that $PG(r-1, q)$ may be represented in matrix form by an r by $(q^r - 1)/(q - 1)$ matrix over $GF(q)$, with columns consisting of all nonzero vectors from $V(r, q)$, excepting scalar multiples. To find all single-element extensions of $\mathcal{M}$ over $GF(q)$, simply generate all matrices of the form $[I_r|D|v]$, where v is a column of the matrix representative of $PG(r-1, q)$ such that v (and no scalar multiples of v) is not a column of $[I_r|D]$.

Now we explain how Oid combines single-element extensions and the isomorphism checker to generate all isomorph-free, GF(q)-representable matroids. The input consists of the size of the field and the set of all simple rank r, n-element $GF(q)$-representable matroids in standard form. These matrices are referred to as seed matrices. The set of seed matrices is enumerated as $\{M_1, M_2, \ldots, M_k\}$. For i from 1 to k, the program generates all single-element extensions of M_i over $GF(q)$ as described above. Next, for each single-element extension M of M_i, all single-element deletions of M are computed. For each one-column deletion N of M, the isomorphism checker compares N to the set of previous seed matrices $\{M_1, M_2, \ldots, M_{i-1}\}$. If an isomorphism is found between any one-element deletion of M and a previous seed matrix M_j, then M will have already been generated as an extension of M_j, and is discarded. Otherwise, M is not isomorphic to any extensions found thus far, and is added to the list. This algorithm is exhaustive and irredundant, provided that $GF(q)$ is a field where unique representability holds [10]. In this manner, the program enables us to generate an isomorph-free list of ternary matroids of a given size.

Recall that for our purposes we are interested in matroids which are $GF(3)-$ but not $GF(4)-$ representable. The set of minimal excluded minors for $GF(4)$-representability are $\{U_{2,6}, U_{4,6}, \mathcal{F}_7-, \mathcal{F}_7-^*, \mathcal{P}_6, \mathcal{P}_8\}$. Thus each of our candidate matroids must contain one of the above minors that is also representable over $GF(3)$- namely one of $\{\mathcal{F}_7-, \mathcal{F}_7-^*, \mathcal{P}_8\}$. As all matroids with rank or corank three, or fewer than 7 elements, are Rayleigh, our initial list of candidates for testing consists of $\mathcal{P}_8$ and all rank four, 8-element, single element extensions and coextensions of $\mathcal{F}_7-$ and $\mathcal{F}_7-^*$. The latter matroids are generated by Oid using the technique described above. Once our set of 8-element candidates has been divided into two sets, Rayleigh versus non-Rayleigh matroids, we generate our next set of candidates as follows. Using all the matroids found to be Rayleigh as our set of seed matrices, Oid computes S_1, the set of all non-isomorphic, single-element extensions of the Rayleigh matroids. Similarily, Oid computes S_2 from the set of non-Rayleigh matroids. Now, for each element $\mathcal{M}$ of S_1 , $\mathcal{M}$ is compared to elements of S_2 by Oid's isomorphism checker. If an isomorphism is found between an element s of S_1 and any element of t of S_2, s is eliminated from S_1, and the next element of S_1 is tested. Otherwise, S_1 and S_2 remain unchanged. Once this process is complete, our next set of candidates consists of all matroids remaining in S_1.

Chapter 5

Minimal Ternary non-Rayleigh matroids found thus far

Our search has yielded three minimal excluded minors so far for the class of ternary Rayleigh matroids: $\mathcal{F}_8$, $\mathcal{H}_8$ and $\mathcal{J}_8$. Matrix representations of these three matroids, along with the rest of our 8-element candidates, are found in Section 5.1. All candidates on 9 elements are Rayleigh, see Section 5.2. Our ten element candidates, given in Section 5.3, have all been shown to be balanced. Which are Rayleigh has yet to be determined. Calculations for 8-, 9- and 10-element candidates can be found in Appendices B, C, and D, respectively.

5.1   Candidates with 8 elements

Our set of 8-elements candidate consists of $\mathcal{P}_8$ along with all non-isomorphic extensions and coextensions of $\mathcal{F}_7-$ and $\mathcal{F}_7-^*$. We list these matroids below, represented as matrices over $GF(3)$, and identify each as not balanced, balanced but not Rayleigh, or Rayleigh. Our calculations are found in Appendix B. The matroids below are assigned naes randomly, with the exception of $\mathcal{P}_8$, which is the $\mathcal{P}_8$ in the set of excluded minors of quaternary matroids.

$$D_8 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 2 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 0 & 2 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 & 2 & 0 & 1 \end{pmatrix}$$

$\mathcal{D}_8$ is Rayleigh

$$F_8 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 2 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 0 & 2 & 0 & 2 & 2 \\ 0 & 0 & 0 & 1 & 2 & 2 & 0 & 1 \end{pmatrix}$$

$\mathcal{F}_8$ is balanced, but not Rayleigh.

$$H_8 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 2 & 2 & 1 \\ 0 & 1 & 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 2 & 2 & 0 & 2 \end{pmatrix}$$

$\mathcal{H}_8$ is not balanced.

$$J_8 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 2 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 2 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 2 & 2 & 1 \end{pmatrix}$$

$\mathcal{J}_8$ is not balanced.

$$K_8 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 2 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 2 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 2 & 2 & 2 \end{pmatrix}$$

$\mathcal{K}_8$ is Rayleigh.

36

$$L_8 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 2 & 2 & 1 \\ 0 & 1 & 0 & 0 & 2 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 & 2 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 2 & 2 & 0 \end{pmatrix}$$

$\mathcal{L}_8$ is Rayleigh.

$$M_8 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 2 & 2 & 1 \\ 0 & 1 & 0 & 0 & 2 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 & 2 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 2 & 2 & 2 \end{pmatrix}$$

$\mathcal{M}_8$ is Rayleigh.

$$P_8 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 & 1 & 1 \end{pmatrix}$$

$\mathcal{P}_8$ is Rayleigh.

$$Q_8 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 2 & 2 & 1 \\ 0 & 1 & 0 & 0 & 2 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 & 2 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 2 & 1 & 0 \end{pmatrix}$$

$\mathcal{Q}_8$ is Rayleigh.

5.2 Candidates with 9 elements

All of our 9-element candidates are Rayleigh. As in the previous section, matroids are represented as matrices over $GF(3)$. The names assigned to all 9-element candidates are random. For calculations, refer to Appendix C.

$$A_9 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 2 & 2 & 2 & 2 \end{pmatrix}$$

$$B_9 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 & 2 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 & 2 & 2 & 2 & 2 \end{pmatrix}$$

$$C_9 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 & 2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 2 & 2 & 2 & 2 \end{pmatrix}$$

$$D_9 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 & 2 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 & 2 & 2 & 2 & 1 \end{pmatrix}$$

$$E_9 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 2 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & 2 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 & 2 & 1 & 0 & 1 \end{pmatrix}$$

$$F_9 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 2 & 2 & 1 & 1 \\ 0 & 1 & 0 & 0 & 2 & 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 0 & 2 & 2 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 & 2 & 1 & 0 & 2 \end{pmatrix}$$

$$G_9 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 2 & 0 & 1 & 2 \end{pmatrix}$$

$$H_9 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 2 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 2 & 0 & 1 & 0 \end{pmatrix}$$

$$I_9 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 2 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 2 & 0 & 1 & 2 \end{pmatrix}$$

$$J_9 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 2 & 0 & 1 & 1 \end{pmatrix}$$

39

5.3 Candidates with 10 elements

All the matroids below have been found to be balanced. Refer to Appendix D for calculations. However, it has yet to be determined which are Rayleigh. The 10-element candidates have been named reandomly.

$$A_{10}=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 2 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 2 & 2 & 1 \end{pmatrix}$$

$$C_{10}=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 2 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 2 & 2 & 2 \end{pmatrix}$$

$$E_{10}=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 2 & 2 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 2 & 2 & 0 \end{pmatrix}$$

$$F_{10}=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 2 & 2 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 2 & 1 & 2 \end{pmatrix}$$

$$G_{10}=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 2 & 2 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 2 & 1 & 1 \end{pmatrix}$$

$$I_{10}=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 2 & 2 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 2 & 0 & 2 \end{pmatrix}$$

$$K_{10}=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 2 & 2 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 2 & 0 & 1 \end{pmatrix}$$

$$M_{10}=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 2 & 2 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 2 & 2 & 0 \end{pmatrix}$$

$$N_{10}=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 2 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 2 & 2 & 0 \end{pmatrix}$$

$$P_{10}=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 2 & 2 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 2 & 2 & 1 \end{pmatrix}$$

$$R_{10}= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 2 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 2 & 0 & 1 \end{pmatrix}$$

$$T_{10}= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 2 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 2 & 0 & 2 \end{pmatrix}$$

$$X_{10}= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 2 & 2 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 2 & 0 & 2 & 1 & 2 & 2 \\ 0 & 0 & 1 & 0 & 2 & 2 & 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 & 2 & 1 & 0 & 2 & 1 \end{pmatrix}$$

Chapter 6

Outstanding Questions

In this section we highlight some outstanding questions.

Our first question is whether the set of minimal excluded minors for ternary Rayleigh matroids is finite or not. Informal calculations suggest that there are 2 non-Rayleigh, ternary matroids on ten elements, and none on eleven or twelve. However, proof that the remaining candidates of these sizes are Rayleigh has not been found, so there may be matroids among the remaining 10-, 11- and 12-element candidates that are balanced, but not Rayleigh.

Second, if the set of all excluded minors for this class is finite, what are the matroids?

In this thesis, one of our methods for proving that a balanced matroid is Rayleigh is to write $\Delta M\{e, f\}(y)$ as a positive sum of monomials and squares of polynomials. However, might there exist matroids for which this method fails, but are nonetheless Rayleigh?

Our final question is if a computer program may be written which can automate the process of writing Rayleigh differences as sums with many terms squared. The reformulations of the Rayleigh differences in this thesis were done by hand, which became increasingly difficult as the size of the matroids grew. Clearly, a program which automates this process would greatly reduce the time required to verify if a matroid is Rayleigh. Such a program would be useful not only in the case of ternary matroids, but for verifying if any matroid exhibits the Rayleigh property. There is a program for writ-

ing polynomials in many terms as sums of squares: SOSTOOLS, developed by Parrilo [15]. However, this program has not been sufficient for our purposes. It is able to identify matroids which are not Rayleigh, but is often unable to produce a useful reformulation of $\Delta M\{e, f\}(y)$ for Rayleigh matroids. We note that this program was designed for general polynomials and does not take advantage of the special form of a matroid's basis-generating polynomial. An effective program for our purposes may use the fact that $\Delta M\{e, f\}(y)$ is quadratic in each y_g. Furthermore, a program for identifying minimal non-Rayleigh matroids can capitalise on the following fact. Recall that

$$\Delta M\{e, f\}(y) = y_g{}^2 \Delta M_g\{e, f\}(y) + y_g \Theta M\{e, f|g\}(y) + \Delta M^g\{e, f\}(y).$$

If $\mathcal{M}_g$ and $\mathcal{M}^g$ have been found to be Rayleigh and the algorithm has produced a sum-of-squares reformulation of both $\Delta M_g\{e, f\}(y)$ and $\Delta M^g\{e, f\}(y)$, then perhaps these two polynomials may be used as a starting point from which a cover of $\Delta M\{e, f\}(y)$ can be constructed.

Appendix A

Code

A.1 Calculating the set of orbits of Aut($\mathcal{M}$) acting on 2-element subsets of E($\mathcal{M}$)

In this section we present the code used to generate a complete set of 2-element subsets of E($\mathcal{M}$) which we must test to check if $\mathcal{M}$ is Rayleigh.

```
nextSet:= proc(S,m) local k,T:
k:=nops(S):
if k=0 then T:=[1]: else
if k=m then T:=[]: else
if S[1]+k-1=m then T:=[1..(k+1)]: else
if S[k] < m then T:=[op(1..(k-1),S),S[k]+1]: else
T:=nextSet([op(1..(k-1),S)],m-1):
T:=[op(T),1+T[nops(T)]]:
fi:
fi:
fi:
fi:
RETURN(T): end:


Circuits:=proc(B,m,r) local Cs,S,implied,K,Q;
Cs:=[];
```

```
S:=[1]; while (nops(S) <=r) do;
implied := false; for K in Cs do;
if (nops(op(S) minus K) = nops(S) - nops(K)) then
implied := true; break; fi; od;
if (not implied) then
Q:=map(j → y[j],S);
if (diff(B,op(Q)) = 0) then
Cs := [op(Cs),op(S)]; fi; fi;
S:=nextSet(S,m); od;
RETURN(Cs); end:


aux:=proc(S,i) local A;
A:=1;
if (member(i,S)) then A:=X[nops(S)]: fi:
RETURN(A); end:


# Sigs(A_circ,m) records the circuit structure of A
# as a list of monomials in X[j], j=2,...m
# L[i] corresponds to the entry of M[A] represented by
# the i^th column of A. The power of X[j] for
# each i records the number of circuits of size j
# the i^th element of M[A] is in
Sigs:=proc(Circs,m);
map(v → convert(map(aux,[op(Circs)],v),'*'),[1..m]);
RETURN(%): end:


# Young computes the Young subgroup of M[A]
Young:=proc(L) local m,G,i,j:
m:=nops(L); G:={}:
for i from 1 to m-1 do:
for j from i+1 to m do:
if (L[i]=L[j]) then G:= G union {[i,j]}: fi:
od: od:
```

```
RETURN(permgroup(m,G)): end:


ApplyPerm2Set:=proc(S,P) RETURN(map(v → P[v],S)): end:
ApplyPerm2SetSet:=proc(Cs,P)
RETURN(map(S → ApplyPerm2Set(S,P),Cs)): end:


# Aut computes all automorphisms of E(A)
Aut:=proc(Cs,m) local G,Y,p,q:
G:=[]: Y:=elements(Young(Sigs(Cs,m))):
for p in Y do:
q:=convert(p,permlist,m):
if evalb(Cs = ApplyPerm2SetSet(Cs,q)) then
G:=[op(G),q]: fi: od: RETURN(G): end:


TwoSets:=proc(m) local TS,S:
TS:={}: S:=[1,2]: while nops <3 do:
TS:= TS union op(S):
S:=nextSet(S,m): od:
RETURN(TS): end:


# PairOrbitReps computes a minimal complete set
# of pairs of ground set elements that must be checked
# to prove that M[A] is Rayleigh
PairOrbitReps:=proc(G) local m,TS,Reps,P,Orbit,perm:
m:=nops(G[1]): TS:=TwoSets(m): Reps:={}:
while (nops(TS) >0) do:
P:=op(1,TS): Orbit:=P:
for perm in G do:
Orbit:= Orbit union ApplyPerm2Set(P,perm): od:
Reps:= Reps union P:
TS:= TS minus Orbit: od:
RETURN(Reps): end:
```

A.2   Calculating the basis-generating polynomial of a matroid, testing for balance, verifying the Rayleigh condition

In this section we present the code used to test matroids for balance and verify that a matroid is Rayleigh.

Ymatrix is the diagonal matrix of indeterminates.

```
Ymatrx:=proc(m) local Y, i:
Y:=matrix(m,m,0):
for i from 1 to m do Y[i,i]:=y[i] od:
RETURN(Y): end:
```

```
# Bases produces the bases generating polynomial of the
# GF(3) matroid represented by A.
Bases:=proc(A) local r, m, Y:
r:=nops(A): m:=nops(A[1]):
Y:=Ymatrix(m):
det(evalm(% * Y * transpose(%))) mod 3:
RETURN(%): end:
```

```
# Rayleigh computes all the Rayleigh differences M[e]M[f] - M[ef] M.
Rayleigh:=proc(M) local i,j,Delta,Mi,Mj,Mij:
for i from 1 to m-1 do:
for j from i+1 to m do:
Mi:=coeff(M,y[i]): Mij:=coeff(Mi,y[j]): Mj:=coeff(M,y[j]):
Delta[i,j]:=simplify(Mi*Mj-Mij*M): od: od:
RETURN(Delta): end:
```

```
# AtOne evaluates an expression substituting the value 1 for each
# variable.
AtOne:=proc(P) local Q,L,v:
L:=indets(P): Q:=P:
for v in L do: Q:=subs(v=1,Q): od:
```

RETURN(Q): end:

Balance computes the matrix of Rayleigh differences
all evaluated at y:=1.
Balance:=proc(Delta) local i,j,Del1:
Del1:=matrix(m,m,0):
for i from 1 to m-1 do: for j from i+1 to m do:
Del1[i,j]:=AtOne(Delta[i,j]):
Del1[j,i]:=Del1[i,j]: od: od:
RETURN(op(Del1)): end:

NegTerms lists the monmials with negative coefficients.
NegTerms:=proc(P) local L,N,W,i:
W:=[]: L:=convert(P,list): N:=map(AtOne,L):
for i from 1 to nops(L) do:
if N[i]¡0 then W:=[op(W),L[i]]: fi: od:
RETURN(W): end:

SqTerms lists the monomials that have exactly k variables
occuring to the second power.
sq is an auxiliary subroutine.
sq:=proc(m) local v,b:
B:=0: for v in indets(m) do:
if degree(m,v)=2 then b:=b+1: fi: od:
RETURN(b): end:

SqTerms:=proc(P,k) local L, W, m:
L:=convert(P, list): W:=[]: for m in L do: if sq(m)=k then W:=[op(W),m]:
fi: od: RETURN(W): end:

Minor Check is to see if the negative terms are covered by
deletionof another element.
MinorCheck:=proc(Delta,e,f,m local P,g,N1:

49

```
P:=expand(Delta[e,f]):
for g from 1 to m do:
if ((g¡¿e) and (g¡¿f)) then:
N1:=nops(NegTerms(coeff(P,y[g],1))):
if (N1=0) then print("Negative Terms covered by deletion/contraction
of",g):
fi:
f: od: end:
```

```
# FirstTest does the easy checking– we need only find patches for
# the other "bad" cases.
FirstTest:=proc(Delta, Pairs,r,m) local S,i,j:
for S in Pairs do: i:=min(op(S)): j:=max(op(S)):
print([i,j],nops(NegTerms(Delta[i,j])),
nops(SqTerms(Delta[i,j]r-1)),nops(Delta[i,j])):
MinorCheck(Delta,i,j,m): od: end:
```

```
# SecondTest checks the patches for the "bad" cases.
SecondTest :=proc(Delta,BadPairs) local S,i,j:
for S in BadPairs do: i:=min(op(S)): j:-max(op(S)):
print([i,j], NegTerms(expand(Delta[i,j]-Patch[i,j]))): od: end:
```

```
# SquaresData is a utility to help find patches.
# P is a Rayleigh difference (maybe minus a partial patch)
# look for terms with os coeff and exactly k squared variables
SquaresData:=proc(P,k) local S,T,v:
S:=indets(P):
for v in S do: T:=SqTerms(coeff(P,v^2), k) :
T:=map(w-¿map(op,indets(w)),T):
lprint(WITH,op(v),":",op(T)); od:
end:
```

```
# Check is a utility that helps find patches.
Check :=proc(P,k) local N:
```

```
N:=NegTerms(P);
print(N); print(AtOne(N)); print(nops(N));
N:=SqTerms(P,k);
print(N); print(AtOne(N)); print(nops(N));
SquaresData(P,k-1): end:
```

Appendix B

Calculatons for 8-element candidates

$$
> \mathcal{D}_8 =
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 2 & 2 & 0 \\
0 & 1 & 0 & 0 & 0 & 2 & 1 & 1 \\
0 & 0 & 1 & 0 & 2 & 0 & 2 & 1 \\
0 & 0 & 0 & 1 & 2 & 2 & 0 & 1
\end{bmatrix}
$$

> Delta := Rayleigh(Bases(D8, Y), AllPairs)

$$
> \text{Balance(Delta)} =
\begin{bmatrix}
0 & 106 & 106 & 52 & 48 & 193 & 168 & 52 \\
106 & 0 & 82 & 82 & 138 & 52 & 78 & 190 \\
106 & 82 & 0 & 190 & 138 & 52 & 78 & 82 \\
52 & 82 & 190 & 0 & 138 & 106 & 78 & 82 \\
48 & 138 & 138 & 138 & 0 & 48 & 72 & 138 \\
193 & 52 & 52 & 106 & 48 & 0 & 168 & 106 \\
168 & 78 & 78 & 78 & 72 & 168 & 0 & 78 \\
52 & 190 & 82 & 82 & 138 & 106 & 78 & 0
\end{bmatrix}
$$

> $\mathcal{D}8\text{pairs} := \{\{6,7\}, \{3,7\}, \{3,6\}, \{2,3\}, \{1,6\}, \{1,5\}, \{6,8\}, \{3,5\}, \{3,4\}, \{2,4\}, \{5,7\}\}$

> FirstTest(Delta, D8pairs, 4, 8);

$$\{6,7\},\ 1,\ 0,\ 93$$

"Negative Terms covered by deletion/contraction of", 1
"Negative Terms covered by deletion/contraction of", 5

$$\{3,7\},\ 4,\ 0,\ 57$$
$$\{3,6\},\ 6,\ 0,\ 49$$
$$\{2,3\},\ 2,\ 0,\ 57$$

"Negative Terms covered by deletion/contraction of", 7

$$\{1,6\},\ 0,\ 0,\ 101$$

"Negative Terms covered by deletion/contraction of", 2
"Negative Terms covered by deletion/contraction of", 3
"Negative Terms covered by deletion/contraction of", 4
"Negative Terms covered by deletion/contraction of", 5
"Negative Terms covered by deletion/contraction of", 7
"Negative Terms covered by deletion/contraction of", 8

$$\{1,5\},\ 5,\ 0,\ 43$$
$$\{6,8\},\ 1,\ 0,\ 65$$

"Negative Terms covered by deletion/contraction of", 1
"Negative Terms covered by deletion/contraction of", 3

$$\{3,5\},\ 3,\ 0,\ 75$$

"Negative Terms covered by deletion/contraction of", 4

$$\{3,4\},\ 2,\ 0,\ 94$$

"Negative Terms covered by deletion/contraction of", 5

$$\{2,4\},\ 4,\ 0,\ 56$$
$$\{5,7\},\ 2,\ 0,\ 48$$

"Negative Terms covered by deletion/contraction of", 1
"Negative Terms covered by deletion/contraction of", 6

> D8badpairs := $\{\{3,7\}, \{3,6\}, \{2,4\}\}$;

> Patch[3, 7] := $(y[1] * y[4] * (y[6] + .5 * y[8]) - (y[2] + y[5]) * (y[6] + .5 * y[1]) * y[8])^2$;

53

> Patch[3, 6] := $((y[1] + .5 * y[7]) * (y[2] + y[8]) * y[5] - (y[1] + .5 * y[2]) * y[4] * y[7])^2 + (y[1] * (y[2] + y[4]) * y[8] - .5 * y[2] * y[4] * y[7])^2$;

> Patch[2, 4] := $0.5 * (y[3] * y[6] * y[8] + y[3] * y[7] * y[8] - y[5] * y[6] * y[7])^2 + .5 * (y[1] * y[3] * y[8] + y[3] * y[7] * y[8] - y[1] * y[5] * y[7])^2$;

> SecondTest(Delta, D8badpairs);

$$\{3, 7\}, []$$
$$\{3, 6\}, []$$
$$\{2, 4\}, []$$

THEREFORE, $\mathcal{D}_8$ IS RAYLEIGH

> $\mathcal{F}_8 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 2 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 0 & 2 & 0 & 2 & 2 \\ 0 & 0 & 0 & 1 & 2 & 2 & 0 & 1 \end{bmatrix}$

> Delta := Rayleigh(Bases(F8, Y), AllPairs)

> Balance(Delta); $\begin{bmatrix} 0 & 84 & 82 & 82 & 55 & 198 & 198 & 84 \\ 84 & 0 & 112 & 112 & 84 & 112 & 112 & 168 \\ 82 & 112 & 0 & 172 & 198 & 52 & 52 & 112 \\ 82 & 112 & 172 & 0 & 198 & 52 & 52 & 112 \\ 55 & 84 & 198 & 198 & 0 & 82 & 82 & 84 \\ 198 & 112 & 52 & 52 & 82 & 0 & 172 & 112 \\ 198 & 112 & 52 & 52 & 82 & 172 & 0 & 112 \\ 84 & 168 & 112 & 112 & 84 & 112 & 112 & 0 \end{bmatrix}$

54

However, $\mathcal{F}_8$ is not Rayleigh as there exist positive reals $\{y_1, y_2, ..., y_8\}$ satisfying $\Delta F8\{1, 5\} < 0$.

> expr := subs(y[1] = 1, y[2] = t, y[3] = 1, y[4] = 1, y[5] = 1, y[6] = t*t, y[7] = t*t, y[8] = t, Delta[1, 5]);

$$expr := 4 * t^{10} + 24 * t^8 - 5 * t^6 + 16 * t^7 + 16 * t^9$$

>subs(t = .2, expr);

-0.0000451584

THEREFORE, F_8 IS BALANCED BUT NOT RAYLEIGH

$$> \mathcal{J}_8 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 2 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 2 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 2 & 2 & 1 \end{bmatrix}$$

> Delta := Rayleigh(Bases(J8, Y), AllPairs):

$$> \text{Balance(Delta)}:= \begin{bmatrix} 0 & 37 & 37 & 61 & 61 & 208 & 108 & 88 \\ 37 & 0 & 37 & 61 & 208 & 61 & 108 & 88 \\ 37 & 37 & 0 & 208 & 61 & 61 & 108 & 88 \\ 61 & 61 & 208 & 0 & 37 & 37 & 88 & 108 \\ 61 & 208 & 61 & 37 & 0 & 37 & 88 & 108 \\ 208 & 61 & 61 & 37 & 37 & 0 & 88 & 108 \\ 108 & 108 & 108 & 88 & 88 & 88 & 0 & -8 \\ 88 & 88 & 88 & 108 & 108 & 108 & -8 & 0 \end{bmatrix}$$

Since $DeltaJ\{7,8\} < 0$ for $y = \{1,1,1,1,1,1,1,1\}$, $\mathcal{J}_8$ is not balanced.

THEREFORE $\mathcal{J}_8$ IS NOT RAYLEIGH

$$> \mathcal{H}_8 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 2 & 2 & 1 \\ 0 & 1 & 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 2 & 2 & 0 & 2 \end{bmatrix}$$

> Delta := Rayleigh(Bases(H8, Y), AllPairs);

$$> \text{Balance(Delta)} = \begin{bmatrix} 0 & 100 & 52 & 136 & 52 & 100 & 72 & 160 \\ 100 & 0 & 52 & 72 & 52 & 160 & 136 & 100 \\ 52 & 52 & 0 & 130 & 208 & 52 & 130 & 52 \\ 136 & 72 & 130 & 0 & 130 & 72 & -9 & 136 \\ 52 & 52 & 208 & 130 & 0 & 52 & 130 & 52 \\ 100 & 160 & 52 & 72 & 52 & 0 & 136 & 100 \\ 72 & 136 & 130 & -9 & 130 & 136 & 0 & 72 \\ 160 & 100 & 52 & 136 & 52 & 100 & 72 & 0 \end{bmatrix}$$

Since $\Delta H_8\{4, 7\} < 0$ for $y = \{1, 1, 1, 1, 1, 1, 1, 1\}$, $\mathcal{H}_8$ is not balanced.

THEREFORE $\mathcal{H}_8$ IS NOT RAYLEIGH

$$> \mathcal{L}_8 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 2 & 2 & 1 \\ 0 & 1 & 0 & 0 & 2 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 & 2 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 2 & 2 & 0 \end{bmatrix}$$

> Delta := Rayleigh(Bases(L8, Y), AllPairs);

$$> \text{Balance(Delta)} := \begin{bmatrix} 0 & 133 & 133 & 133 & 162 & 103 & 103 & 103 \\ 133 & 0 & 133 & 133 & 103 & 162 & 103 & 103 \\ 133 & 133 & 0 & 133 & 103 & 103 & 162 & 103 \\ 133 & 133 & 133 & 0 & 103 & 103 & 103 & 162 \\ 162 & 103 & 103 & 103 & 0 & 133 & 133 & 133 \\ 103 & 162 & 103 & 103 & 133 & 0 & 133 & 133 \\ 103 & 103 & 162 & 103 & 133 & 133 & 0 & 133 \\ 103 & 103 & 103 & 162 & 133 & 133 & 133 & 0 \end{bmatrix}$$

> L8pairs := {{5, 8}, {1, 4}, {1, 8}, {2, 6}};

> FirstTest(Delta, L8pairs, 4, 8);

$$\{5, 8\}, 4, 0, 88$$
$$\{1, 4\}, 4, 0, 88$$
$$\{1, 8\}, 8, 0, 79$$
$$\{2, 6\}, 3, 0, 94$$

> L8Badpairs := {{5, 8}, {1, 4}, {1, 8}, {2, 6}};

> Patch[5, 8] := .5*((y[1]*y[3]*y[6]-y[1]*y[2]*y[7])^2 + (y[6] * y[3] * y[4] - y[2] * y[7] * y[4])^2) + (y[1] * y[2] * y[4] - y[2] * y[3] * y[6])^2 + (y[1] * y[3] * y[4] - y[2] * y[7] * y[3])^2

> Patch[1, 4] := .5*((y[2]*y[5]*y[7]-y[5]*y[3]*y[6])^2 + (y[2] * y[8] * y[7] - y[8] * y[3] * y[6])^2) + (y[2] * y[7] * y[3] - y[5] * y[3] * y[8])^2 + (y[2] * y[3] * y[6] - y[2] * y[5] * y[8])^2

> Patch[1, 4] := .5*((y[2]*y[5]*y[7]-y[5]*y[3]*y[6])^2 + (y[2] * y[8] * y[7] - y[8] * y[3] * y[6])^2) + (y[2] * y[7] * y[3] - y[5] * y[3] * y[8])^2 + (y[2] * y[3] * y[6] -

$y[2] * y[5] * y[8])^2$

> Patch[2, 6] := .5*((y[1]*y[8]*y[4]-y[1]*y[3]*y[7])^2 + (y[7] * y[3] * y[4] - y[1] * y[5] * y[4])^2 + (y[1] * y[3] * y[5] - y[8] * y[3] * y[4])^2)$

> SecondTest(Delta, L8Badpairs);

$$\{5, 8\}, [] $$
$$\{1, 4\}, [] $$
$$\{1, 8\}, [] $$
$$\{2, 6\}, [] $$

THEREFORE $\mathcal{L}_8$ IS RAYLEIGH

> $\mathcal{M}_8 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 2 & 2 & 1 \\ 0 & 1 & 0 & 0 & 2 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 & 2 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 2 & 2 & 2 \end{bmatrix}$

> Delta := Rayleigh(Bases(M8, Y), AllPairs);

$$> \text{Balance(Delta);} \quad \begin{bmatrix} 0 & 87 & 87 & 145 & 145 & 145 & 145 & 87 \\ 87 & 0 & 87 & 145 & 145 & 145 & 145 & 87 \\ 87 & 87 & 0 & 145 & 145 & 145 & 145 & 87 \\ 145 & 145 & 145 & 0 & 87 & 87 & 87 & 145 \\ 145 & 145 & 145 & 87 & 0 & 87 & 87 & 145 \\ 145 & 145 & 145 & 87 & 87 & 0 & 87 & 145 \\ 145 & 145 & 145 & 87 & 87 & 87 & 0 & 145 \\ 87 & 87 & 87 & 145 & 145 & 145 & 145 & 0 \end{bmatrix}$$

> M8pairs := {{5, 8}, {1, 3}}

> FirstTest(Delta, M8pairs, 4, 8);

$$\{5, 8\}, \ 6, \ 0, \ 87$$
$$\{1, 3\}, \ 6, \ 0, \ 73$$

> M8badpairs := {{5, 8}, {1, 3}}

> Patch[5, 8] := .5*((y[6]*y[3]*y[4]-y[2]*y[7]*y[4])^2 + (y[2] * y[3] * y[7] − y[1]*y[3]*y[4])^2+(y[1]*y[7]*y[4]−y[7]*y[3]*y[6])^2+(y[2]*y[3]*y[6]−y[1]*y[2]*y[4])^2+(y[1]*y[6]*y[4]−y[2]*y[6]*y[7])^2+(y[1]*y[3]*y[6]−y[1]*y[2]*y[7])^2)

> Patch[1, 3] := (y[8]*y[6]*y[4]-y[5]*y[8]*y[7])^2 + (y[2] * y[5] * y[8] − y[5] * y[6]*y[4])^2+(y[2]*y[6]*y[4]−y[2]*y[5]*y[7])^2+(y[5]*y[6]*y[7]−y[2]*y[6]* y[8])^2+(y[7]*y[5]*y[4]−y[2]*y[8]*y[4])^2+(y[2]*y[8]*y[7]−y[7]*y[6]*y[4])^2

> SecondTest(Delta, M8badpairs);

$$\{5, 8\}, []$$
$$\{1, 3\}, []$$

THEREFORE $\mathcal{M}_8$ IS RAYLEIGH

$$> \mathcal{P}_8 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 & 1 & 1 \end{bmatrix}$$

> Delta := Rayleigh(Bases(P8, Y), AllPairs);

$$> \text{Balance(Delta)} := \begin{bmatrix} 0 & 103 & 103 & 162 & 103 & 133 & 133 & 133 \\ 103 & 0 & 133 & 133 & 133 & 103 & 162 & 103 \\ 103 & 133 & 0 & 133 & 133 & 162 & 103 & 103 \\ 162 & 133 & 133 & 0 & 133 & 103 & 103 & 103 \\ 103 & 133 & 133 & 133 & 0 & 103 & 103 & 162 \\ 133 & 103 & 162 & 103 & 103 & 0 & 133 & 133 \\ 133 & 162 & 103 & 103 & 103 & 133 & 0 & 133 \\ 133 & 103 & 103 & 103 & 162 & 133 & 133 & 0 \end{bmatrix}$$

> P8pairs := {{1, 2}, {5, 8}, {2, 6}}

> FirstTest(Delta, P8pairs, 4, 8);

$$\{1, 2\}, 8, 0, 79$$
$$\{5, 8\}, 3, 0, 94$$
$$\{2, 6\}, 8, 0, 79$$

> P8badpairs := {{1, 2}, {5, 8}, {2, 6}}

> Patch[1, 2] := .5*(y[5]*y[6]*y[4]-y[8]*y[3]*y[4])^2 + .5 * (y[7] * y[3] * y[4] − y[5] * y[3] * y[6])^2 + .5 * (y[7] * y[6] * y[4] − y[5] * y[3] * y[6])^2 + (y[5] * y[3] * y[7] − y[8] * y[6] * y[7])^2 + .5 * (y[5] * y[3] * y[8] − y[7] * y[4] * y[5])^2 + .5 * (y[5] * y[3] * y[8] − y[7] * y[8] * y[4])^2 + .5 * (y[7] * y[3] * y[4] − y[8] * y[3] * y[6])^2 + .5 * (y[8] * y[5] * y[6] − y[7] * y[4] * y[5])^2

> Patch[5, 8] := .5*((y[1]*y[7]*y[4]-y[7]*y[3]*y[6])^2 + (y[1] * y[6] * y[4] − y[2] * y[6] * y[7])^2 + (y[1] * y[3] * y[6] − y[1] * y[2] * y[7])^2)

> Patch[2, 6] := .5*((y[7]*y[3]*y[4]-y[1]*y[8]*y[4])^2 + (y[7] * y[3] * y[4] − y[1] * y[5] * y[4])^2 + (y[1] * y[3] * y[7] − y[1] * y[5] * y[4])^2 + (y[8] * y[3] * y[4] − y[1] * y[3] * y[5])^2 + (y[5] * y[3] * y[7] − y[5] * y[8] * y[4])^2 + (y[5] * y[8] * y[4] − y[7] * y[3] * y[8])^2 + 2 * (y[1] * y[8] * y[7] − y[7] * y[5] * y[4])^2 + (y[5] * y[3] * y[7] − y[1] * y[5] * y[8])^2)

> SecondTest(Delta, P8badpairs);

$$\{1, 2\}, \; []$$
$$\{5, 8\}, \; []$$
$$\{2, 6\}, \; []$$

THEREFORE $\mathcal{P}_8$ IS RAYLEIGH

$$> \mathcal{Q}_8 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 2 & 2 & 1 \\ 0 & 1 & 0 & 0 & 2 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 & 2 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 2 & 1 & 0 \end{bmatrix}$$

> Delta := Rayleigh(Bases(Q8, Y), AllPairs);

$$
> \text{Balance(Delta):=}
\begin{bmatrix}
0 & 133 & 103 & 133 & 103 & 162 & 103 & 133 \\
133 & 0 & 103 & 133 & 162 & 103 & 103 & 133 \\
103 & 103 & 0 & 162 & 133 & 133 & 133 & 103 \\
133 & 133 & 162 & 0 & 103 & 103 & 103 & 133 \\
103 & 162 & 133 & 103 & 0 & 133 & 133 & 103 \\
162 & 103 & 133 & 103 & 133 & 0 & 133 & 103 \\
103 & 103 & 133 & 103 & 133 & 133 & 0 & 162 \\
133 & 133 & 103 & 133 & 103 & 103 & 162 & 0
\end{bmatrix}
$$

> Q8pairs := {{6, 7}, {5, 8}, {1, 4}, {2, 5}}

> FirstTest(Delta, Q8pairs, 4, 8);

$$
\begin{array}{l}
\{5, 8\}, \ 8, \ 0, \ 79 \\
\{1, 4\}, \ 4, \ 0, \ 88 \\
\{2, 5\}, \ 3, \ 0, \ 94 \\
\{6, 7\}, \ 4, \ 0, \ 88
\end{array}
$$

> Q8badpairs := {{6, 7}, {5, 8}, {1, 4}, {2, 5}}

> Patch[5, 8] := $(y[2]*y[3]*y[6]-y[1]*y[2]*y[4])^2 + .5 * ((y[6]*y[3]*y[4] - y[2]*y[7]*y[4])^2 + (y[1]*y[3]*y[7] - y[7]*y[6]*y[4])^2 + (y[2]*y[3]*y[7] - y[1]*y[3]*y[4])^2 + (y[1]*y[3]*y[6] - y[2]*y[6]*y[7])^2 + (y[1]*y[6]*y[4] - y[2]*y[6]*y[7])^2 + (y[1]*y[3]*y[6] - y[1]*y[2]*y[7])^2 + (y[6]*y[3]*y[4] - y[2]*y[3]*y[7])^2)$

> Patch[1, 4] := $.5*((y[8]*y[5]*y[6]-y[2]*y[6]*y[7])^2 + (y[2]*y[3]*y[7] - y[5]*y[3]*y[8])^2) + (y[2]*y[3]*y[6] - y[2]*y[5]*y[8])^2 + (y[2]*y[8]*y[7] - $

63

$y[8] * y[3] * y[6])^2$

> Patch[2, 5] := .5*((y[8]*y[3]*y[4]-y[1]*y[8]*y[6])^2 + (y[1] * y[8] * y[7] - y[1] * y[3] * y[4])^2 + (y[1] * y[6] * y[4] - y[7] * y[8] * y[4])^2)$

> Patch[6, 7] := (y[1]*y[2]*y[8]-y[2]*y[5]*y[4])^2 + (y[1] * y[8] * y[4] - y[2] * y[3] * y[4])^2 + .5 * ((y[5] * y[8] * y[4] - y[2] * y[3] * y[8])^2 + (y[1] * y[2] * y[3] - y[1] * y[5] * y[4])^2)$

> SecondTest(Delta, Q8badpairs);

$$\{5, 8\}, []$$
$$\{1, 4\}, []$$
$$\{2, 5\}, []$$
$$\{6, 7\}, []$$

THEREFORE Q_8 IS RAYLEIGH

Appendix C

Calculations for 9-element candidates

$$
> \mathcal{A}_9 := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 2 & 2 & 2 & 2 \end{bmatrix}
$$

> Delta := Rayleigh(Bases(A9, Y), AllPairs);

$$
> \text{Balance(Delta)} := \begin{bmatrix} 0 & 225 & 225 & 270 & 135 & 495 & 495 & 90 & 90 \\ 225 & 0 & 261 & 414 & 261 & 135 & 225 & 54 & 414 \\ 225 & 261 & 0 & 414 & 261 & 225 & 135 & 414 & 54 \\ 270 & 414 & 414 & 0 & 54 & 90 & 90 & 306 & 306 \\ 135 & 261 & 261 & 54 & 0 & 225 & 225 & 414 & 414 \\ 495 & 135 & 225 & 90 & 225 & 0 & 495 & 270 & 90 \\ 495 & 225 & 135 & 90 & 225 & 495 & 0 & 90 & 270 \\ 90 & 54 & 414 & 306 & 414 & 270 & 90 & 0 & 306 \\ 90 & 414 & 54 & 306 & 414 & 90 & 270 & 306 & 0 \end{bmatrix}
$$

> A9pairs := {{3, 5}, {2, 4}, {5, 6}, {1, 5}, {1, 6}, {7, 8}, {3, 9}, {4,

9}, {1, 4} }

> FirstTest(Delta, A9pairs, 4, 9);

$$\{3,5\},\ 7,\ 0,\ 147$$
$$\{2,4\},\ 1,\ 0,\ 194$$
"Negative Terms covered by deletion/contraction of", 6
"Negative Terms covered by deletion/contraction of", 8
"Negative Terms covered by deletion/contraction of", 9
$$\{5,6\},\ 6,\ 0,\ 132$$
$$\{1,5\},\ 8,\ 0,\ 95$$
$$\{1,6\},\ 5,\ 0,\ 211$$
"Negative Terms covered by deletion/contraction of", 7
$$\{7,8\},\ 14,\ 0,\ 88$$
$$\{3,9\},\ 24,\ 0,\ 90$$
$$\{4,9\},\ 0,\ 0,\ 155$$
"Negative Terms covered by deletion/contraction of", 1
"Negative Terms covered by deletion/contraction of", 2
"Negative Terms covered by deletion/contraction of", 3
"Negative Terms covered by deletion/contraction of", 5
"Negative Terms covered by deletion/contraction of", 6
"Negative Terms covered by deletion/contraction of", 7
"Negative Terms covered by deletion/contraction of", 8
$$\{1,4\},\ 1,\ 0,\ 142$$
"Negative Terms covered by deletion/contraction of", 5
"Negative Terms covered by deletion/contraction of", 8
"Negative Terms covered by deletion/contraction of", 9

> A9badpairs := {{3, 5}, {5, 6}, {1, 5}, {7, 8}, {3, 9}}

> Patch[3, 5] := $y[6]^2 * (y[1] * y[7] + .5 * y[4] * y[7] + .5 * y[1] * y[9] - y[2] * y[8] - .5 * y[2] * y[9] - .5 * y[2] * y[4])^2 + .5 * y[1]^2 * (y[2] * y[8] - y[6] * y[9])^2 + .5 * y[7]^2 * (y[2] * y[8] - y[4] * y[6])^2$

> Patch[5, 6] := $y[3]^2 * (y[1] * y[2] + .5 * y[1] * y[9] - y[7] * y[8] - y[4] * y[7] - .5 * y[7] * y[9])^2 + .5 * y[1]^2 * (y[3] * y[9] - y[7] * y[8])^2$

> Patch[1, 5] := $.5*y[4]^2 * (y[2] * y[6] + y[6] * y[9] - y[7] * y[8] - y[3] * y[7])^2 + y[8]^2 * (y[2] * y[6] - y[3] * y[7] - .5 * y[4] * y[7])^2 + y[9]^2 * (y[3] * y[7] - y[2] * y[6] - .5 * y[4] * y[6])^2$

> Patch[7, 8] := .5*((y[1]*y[2]*y[3]+y[1]*y[3]*y[9]+y[1]*y[6]*y[9]-y[5]*y[3]*y[6]-y[5]*y[6]*y[4]-y[1]*y[5]*y[4]-y[2]*y[6]*y[9]-y[1]*y[2]*y[5])^2 + $(y[1] * y[2] * y[3] + y[1] * y[2] * y[5] - y[5] * y[3] * y[6] - y[5] * y[6] * y[4] - y[2] * y[6] * y[4] - y[9] * y[3] * y[6] - y[9] * y[6] * y[4])^2 + (y[1] * y[5] * y[4] + y[1] * y[5] * y[6] - y[9] * y[3] * y[6] - y[2] * y[6] * y[4] - y[1] * y[3] * y[9] - y[9] * y[6] * y[4])^2$)

> Patch[3, 9] := .5*((y[7]*y[6]*y[4]+y[1]*y[6]*y[7]-y[2]*y[5]*y[6]-y[2]*y[8]*y[7]-y[2]*y[5]*y[7]-y[2]*y[6]*y[8]-y[1]*y[2]*y[8]-y[1]*y[2]*y[6])^2 + $(y[7] * y[6] * y[4] + y[1] * y[5] * y[4] + y[7] * y[5] * y[4] + y[5] * y[6] * y[4] + y[1] * y[5] * y[6] - y[1] * y[8] * y[7] - y[2] * y[8] * y[7] - y[2] * y[6] * y[8] - y[1] * y[2] * y[8])^2 + (y[5] * y[6] * y[4] + y[2] * y[5] * y[7] + y[1] * y[2] * y[5] + y[1] * y[5] * y[4] + y[7] * y[5] * y[4] - y[1] * y[8] * y[7] - y[1] * y[6] * y[7])^2$)

> SecondTest(Delta, A9badpairs);

$\{3, 5\}$, []
$\{5, 6\}$, []
$\{1, 5\}$, []
$\{7, 8\}$, []
$\{3, 9\}$, []

THEREFORE $\mathcal{A}_9$ IS RAYLEIGH

$$> \mathcal{B}_9 := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 & 2 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 & 2 & 2 & 2 & 2 \end{bmatrix}$$

> Delta := Rayleigh(Bases(B9, Y), AllPairs);

$$> \text{Balance(Delta)} := \begin{bmatrix} 0 & 174 & 174 & 174 & 174 & 630 & 486 & 174 & 174 \\ 174 & 0 & 219 & 219 & 498 & 174 & 198 & 126 & 498 \\ 174 & 219 & 0 & 498 & 219 & 174 & 198 & 498 & 126 \\ 174 & 219 & 498 & 0 & 126 & 174 & 198 & 498 & 219 \\ 174 & 498 & 219 & 126 & 0 & 174 & 198 & 219 & 498 \\ 630 & 174 & 174 & 174 & 174 & 0 & 486 & 174 & 174 \\ 486 & 198 & 198 & 198 & 198 & 486 & 0 & 198 & 198 \\ 174 & 126 & 498 & 498 & 219 & 174 & 198 & 0 & 219 \\ 174 & 498 & 126 & 219 & 498 & 174 & 198 & 219 & 0 \end{bmatrix}$$

> B9pairs := {{3, 5}, {5, 6}, {7, 8}, {3, 9}, {6, 7}, {2, 5}, {1, 6}}

> ¿ FirstTest(Delta, B9pairs, 4, 9);

$$\{6, 7\}, 9, 0, 214$$
"Negative Terms covered by deletion/contraction of", 1
$$\{2, 5\}, 2, 0, 237$$
"Negative Terms covered by deletion/contraction of", 7
"Negative Terms covered by deletion/contraction of", 9
$$\{3, 5\}, 7, 0, 142$$
$$\{5, 6\}, 9, 0, 121$$
$$\{1, 6\}, 6, 0, 259$$
"Negative Terms covered by deletion/contraction of", 7

$$\{7,8\},\ 11,\ 0,\ 125$$
$$\{3,9\}]\ 16,\ 0,\ 119$$

> B9badpairs := {{3, 5}, {5, 6 {7, 8}, {3, 9}}

> Patch[3, 5] := .5*((y[1]*y[6]*y[7]+y[7]*y[6]*y[4]-y[2]*y[6]*y[8]-y[9]*y[6]*y[8])^2+ $(y[1] * y[6] * y[7] + y[7] * y[6] * y[9] - y[2] * y[6] * y[8] - y[2] * y[6] * y[4])^2 +$ $(y[2] * y[8] * y[7] - y[7] * y[6] * y[9] - y[7] * y[6] * y[4])^2)$

> Patch[5, 6] := .5*(y[2]*y[7]*y[4]+y[1]*y[2]*y[7]-y[7]*y[3]*y[9])^2 + $(y[1] *$ $y[2] * y[3] + .5 * y[1] * y[9] * y[3] + .5 * y[1] * y[9] * y[4] - .5 * y[7] * y[3] * y[9] -$ $y[7] * y[3] * y[8] - y[7] * y[3] * y[4] - y[7] * y[8] * y[4])^2$

> Patch[7, 8] := .5*((y[2]*y[6]*y[4]+y[5]*y[6]*y[4]+y[5]*y[3]*y[6]-y[1]*y[9]*y[4]- y[1]*y[9]*y[3]-y[1]*y[2]*y[3])^2 + $(y[2] * y[6] * y[4] - y[1] * y[9] * y[3] - y[1] * y[9] *$ $y[4] - y[9] * y[3] * y[6] - y[1] * y[5] * y[4])^2 + (y[1] * y[2] * y[3] - y[5] * y[6] *$ $y[4] - y[5] * y[3] * y[6] - y[9] * y[3] * y[6] - y[1] * y[5] * y[4])^2)$

> Patch[3, 9] := .5*((y[1]*y[6]*y[7]+y[1]*y[7]*y[4]+y[1]*y[2]*y[7]-y[1]*y[5]*y[8]- y[2]*y[6]*y[4]-y[5]*y[6]*y[4]-y[1]*y[5]*y[4])^2 + $(y[1] * y[6] * y[7] + y[7] * y[6] *$ $y[8] + y[5] * y[6] * y[7] - y[1] * y[5] * y[8] - y[2] * y[6] * y[4] - y[2] * y[6] * y[8] -$ $y[1] * y[2] * y[8])^2 + (y[2] * y[7] * y[4] + y[1] * y[7] * y[4] + y[1] * y[2] * y[7] -$ $y[7] * y[6] * y[8] - y[5] * y[6] * y[7] - y[5] * y[8] * y[7])^2)$

> SecondTest(Delta, B9badpairs);

$$\{3,5\},\ []$$
$$\{5,6\},\ []$$
$$\{7,8\},\ []$$
$$\{3,9\},\ []$$

THEREFORE $\mathcal{B}_9$ IS RAYLEIGH

$$> C_9 := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 & 2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 2 & 2 & 2 & 2 \end{bmatrix}$$

> Delta := Rayleigh(Bases(C9, Y), AllPairs);

$$> Balance(Delta) := \begin{bmatrix} 0 & 316 & 186 & 286 & 316 & 516 & 316 & 232 & 316 \\ 316 & 0 & 186 & 286 & 516 & 316 & 316 & 232 & 316 \\ 186 & 186 & 0 & 481 & 286 & 286 & 286 & 522 & 186 \\ 286 & 286 & 481 & 0 & 186 & 186 & 186 & 522 & 286 \\ 316 & 516 & 286 & 186 & 0 & 316 & 316 & 232 & 316 \\ 516 & 316 & 286 & 186 & 316 & 0 & 316 & 232 & 316 \\ 316 & 316 & 286 & 186 & 316 & 316 & 0 & 232 & 516 \\ 232 & 232 & 522 & 522 & 232 & 232 & 232 & 0 & 232 \\ 316 & 316 & 186 & 286 & 316 & 316 & 516 & 232 & 0 \end{bmatrix}$$

> C9pairs := {{3, 5}, {5, 6}, {1, 5}, {7, 8}, {3, 9}, {3, 4}, {1, 6}, {4, 8}}

> FirstTest(Delta, C9pairs, 4, 9);

$$\{3, 4\}, 6, 0, 246$$
"Negative Terms covered by deletion/contraction of", 8
$$\{4, 8\}, 3, 0, 256$$
"Negative Terms covered by deletion/contraction of", 3
$$\{3, 5\}, 11, 0, 177$$
$$\{5, 6\}, 12, 0, 195$$
$$\{1, 5\}, 18, 0, 193$$
$$\{1, 6\}, 10, 0, 248$$

$$\{7, 8\}, 16, 0, 167$$
$$\{3, 9\}, 20, 0, 157$$

> C9badpairs := {{3, 5}, {5, 6}, {1, 5}, {7, 8}, {3, 9}, {1, 6}}

> Patch[3, 5] := (y[7]*y[6]*y[9]+.5*y[7]*y[6]*y[4]-y[2]*y[8]*y[7]-.5*y[2]*y[7]*y[4])^2+
$(y[2]*y[8]*y[6]+.5*y[2]*y[6]*y[4]-y[1]*y[6]*y[7]-.5*y[7]*y[6]*y[4])^2+$
$.5*(y[9]*y[6]*y[8]-y[1]*y[8]*y[7])^2+.5*(y[7]*y[6]*y[4]-y[9]*y[6]*y[4])^2+$
$.5*(y[1]*y[2]*y[7]-y[2]*y[6]*y[9])^2+.5*(y[1]*y[7]*y[9]-y[2]*y[9]*y[4])^2+$
$.5*(y[9]*y[6]*y[4]-y[1]*y[7]*y[4])^2+.5*(y[1]*y[6]*y[9]-y[1]*y[2]*y[4])^2$

> Patch[5, 6] := (y[1]*y[2]*y[9]-y[7]*y[9]*y[4])^2 $+ (y[7]*y[3]*y[9]-y[1]*$
$y[2]*y[7])^2 + (y[1]*y[2]*y[4]-y[9]*y[3]*y[4]-y[9]*y[8]*y[4])^2 + (y[1]*$
$y[2]*y[3]-y[7]*y[3]*y[4]-y[7]*y[3]*y[8])^2 + (y[1]*y[3]*y[9]+.5*y[1]*$
$y[8]*y[9]-y[1]*y[7]*y[4]-.5*y[1]*y[8]*y[7])^2 + (y[2]*y[3]*y[9]+.5*$
$y[2]*y[8]*y[9]-.5*y[2]*y[8]*y[7]-y[2]*y[7]*y[4])^2$

> Patch[1, 5] := .5*((y[2]*y[6]*y[9]-y[7]*y[3]*y[9])^2 $+ (y[2]*y[6]*y[9]-$
$y[7]*y[8]*y[9])^2+(y[2]*y[6]*y[7]-y[7]*y[8]*y[9])^2+(y[2]*y[6]*y[7]-y[7]*y[9]*$
$y[4])^2+(y[7]*y[3]*y[4]-y[2]*y[6]*y[4])^2+(y[2]*y[6]*y[4]-y[7]*y[8]*y[4])^2+$
$(y[2]*y[6]*y[8]-y[7]*y[8]*y[4]-y[7]*y[3]*y[8])^2+(y[2]*y[6]*y[8]-y[9]*y[8]*$
$y[4]-y[9]*y[3]*y[8])^2+(y[2]*y[3]*y[6]-y[9]*y[3]*y[8])^2+(y[9]*y[3]*y[4]-y[2]*$
$y[3]*y[6])^2)+(y[2]*y[9]*y[4]+.5*y[2]*y[3]*y[9]-y[2]*y[8]*y[7]-.5*y[2]*y[3]*$
$y[7])^2+(y[7]*y[3]*y[6]+.5*y[7]*y[6]*y[4]-y[9]*y[6]*y[8]-.5*y[9]*y[6]*y[4])^2$

> Patch[1, 6] := .5*((y[2]*y[3]*y[5]-y[7]*y[3]*y[9])^2 $+ (y[2]*y[5]*y[4]-$
$y[7]*y[9]*y[4])^2 + (y[2]*y[3]*y[4]-y[2]*y[9]*y[7])^2 + (y[7]*y[3]*y[4]-$
$y[2]*y[5]*y[7])^2+(y[2]*y[8]*y[4]-y[2]*y[9]*y[7])^2+(y[5]*y[3]*y[4]-y[7]*$
$y[5]*y[9])^2+(y[9]*y[3]*y[4]-y[2]*y[5]*y[9])^2+(y[5]*y[3]*y[8]-y[7]*y[5]*$
$y[9])^2+(y[9]*y[8]*y[4]-y[2]*y[5]*y[9])^2+(y[2]*y[5]*y[7]-y[7]*y[3]*y[8])^2)$

> Patch[7, 8] := (y[1]*y[2]*y[3]+.5*y[1]*y[2]*y[5]+.5*y[1]*y[2]*y[6]-.5*y[2]*y[9]*y[4]-
.5*y[2]*y[6]*y[4]-y[5]*y[6]*y[4]-y[5]*y[3]*y[6]-.5*y[1]*y[5]*y[4]-.5*y[1]*y[9]*y[4])^2+

$.25 * (y[1] * y[2] * y[5] - y[2] * y[9] * y[4])^2 + .75 * (y[2] * y[6] * y[4] - y[1] * y[5] * y[4])^2 + .5 * (y[1] * y[2] * y[5] - y[2] * y[9] * y[4] - y[2] * y[3] * y[9] - y[5] * y[3] * y[9])^2 + .5 * (y[2] * y[3] * y[9] - y[2] * y[5] * y[6])^2 + .5 * (y[1] * y[5] * y[9] - y[2] * y[6] * y[9])^2 + .5 * (y[1] * y[5] * y[6] - y[1] * y[3] * y[9])^2 + .5 * (y[1] * y[2] * y[6] - y[9] * y[3] * y[6] - y[1] * y[3] * y[9] - y[1] * y[9] * y[4])^2 + .25 * (y[1] * y[2] * y[6] - y[1] * y[9] * y[4])^2$

> Patch[3, 9] := $(y[1]*y[2]*y[7]-y[5]*y[6]*y[7])^2 + .5 * (y[1] * y[5] * y[6] + y[1] * y[2] * y[6] - y[1] * y[7] * y[4] - y[1] * y[8] * y[7] - y[7] * y[6] * y[4] - y[7] * y[6] * y[8])^2 + .5 * (y[7] * y[6] * y[4] + y[1] * y[2] * y[4] - y[2] * y[6] * y[8] - y[1] * y[2] * y[6] - y[5] * y[6] * y[8] - y[5] * y[6] * y[4])^2 + .5 * (y[1] * y[2] * y[4] + y[1] * y[7] * y[4] - y[1] * y[5] * y[8] - y[5] * y[6] * y[8] - y[5] * y[6] * y[4] - y[1] * y[5] * y[6] - y[5] * y[8] * y[7])^2 + .5 * (y[2] * y[7] * y[4] + y[1] * y[5] * y[8] + y[1] * y[2] * y[5] - y[2] * y[5] * y[6] - y[2] * y[6] * y[8] - y[7] * y[5] * y[4])^2 + .5 * (y[2] * y[5] * y[6] + y[1] * y[2] * y[5] - y[7] * y[5] * y[4] - y[5] * y[8] * y[7] - y[2] * y[8] * y[7] - y[2] * y[7] * y[4])^2$

> SecondTest(Delta, C9badpairs);

$$\{3, 5\}, []$$
$$\{5, 6\}, []$$
$$\{1, 5\}, []$$
$$\{1, 6\}, []$$
$$\{7, 8\}, []$$
$$\{3, 9\}, []$$

THEREFORE $\mathcal{C}_9$ IS RAYLEIGH

> $\mathcal{D}_9 :=$ $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 & 2 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 & 2 & 2 & 2 & 1 \end{bmatrix}$

> Delta := Rayleigh(Bases(D9, Y), AllPairs);

$$\text{> Balance(Delta):=} \begin{bmatrix} 0 & 297 & 180 & 258 & 393 & 393 & 258 & 258 & 258 \\ 297 & 0 & 180 & 258 & 393 & 393 & 258 & 258 & 258 \\ 180 & 180 & 0 & 360 & 180 & 180 & 360 & 360 & 360 \\ 258 & 258 & 360 & 0 & 258 & 258 & 132 & 612 & 132 \\ 393 & 393 & 180 & 258 & 0 & 297 & 258 & 258 & 258 \\ 393 & 393 & 180 & 258 & 297 & 0 & 258 & 258 & 258 \\ 258 & 258 & 360 & 132 & 258 & 258 & 0 & 132 & 612 \\ 258 & 258 & 360 & 612 & 258 & 258 & 132 & 0 & 132 \\ 258 & 258 & 360 & 132 & 258 & 258 & 612 & 132 & 0 \end{bmatrix}$$

> D9pairs := {{3, 5}, {2, 4}, {5, 6}, {1, 5}, {7, 8}, {3, 9}, {7, 9}}

> FirstTest(Delta, D9pairs, 4, 9);

$$\{7, 9\}, 0, 0, 268$$
"Negative Terms covered by deletion/contraction of", 1
"Negative Terms covered by deletion/contraction of", 2
"Negative Terms covered by deletion/contraction of", 3
"Negative Terms covered by deletion/contraction of", 4
"Negative Terms covered by deletion/contraction of", 5
"Negative Terms covered by deletion/contraction of", 6
"Negative Terms covered by deletion/contraction of", 8
$$\{3, 5\}, 15, 0, 134$$
$$\{2, 4\}, 11, 0, 160$$
$$\{5, 6\}, 4, 0, 167$$
"Negative Terms covered by deletion/contraction of", 3
$$1, 5\}, 10, 0, 201$$
$$\{7, 8\}, 17, 0, 117$$
$$\{3, 9\}, 2, 0, 183$$
"Negative Terms covered by deletion/contraction of", 4

"Negative Terms covered by deletion/contraction of", 7
"Negative Terms covered by deletion/contraction of", 8

> D9badpairs := {{2, 4}, {1, 5}, {3, 5}, {7, 8}}

> Patch[2, 4] := .5*(y[9]*y[6]*y[8]-y[1]*y[7]*y[9]-y[1]*y[8]*y[7]-y[1]*y[3]*y[9]-y[7]*y[5]*y[9])^2 + .5*(y[1]*y[6]*y[9] + y[9]*y[6]*y[8] - y[5]*y[3]*y[6] - y[5]*y[6]*y[8] - y[5]*y[6]*y[7] - y[5]*y[3]*y[9])^2 + .5*(y[1]*y[6]*y[7] + y[1]*y[6]*y[9] - y[5]*y[3]*y[6] - y[1]*y[5]*y[8])^2

> Patch[1, 5] := .5*((y[2]*y[9]*y[4]-y[2]*y[8]*y[7])^2 + (y[7]*y[6]*y[4] - y[9]*y[6]*y[8])^2 + (y[2]*y[6]*y[4] + y[2]*y[6]*y[8] - y[9]*y[8]*y[4] - y[9]*y[3]*y[8])^2 + (y[2]*y[6]*y[4] - y[7]*y[3]*y[4] - y[9]*y[3]*y[4] - y[7]*y[8]*y[4])^2 + (y[2]*y[6]*y[8] - y[7]*y[3]*y[8] - y[7]*y[8]*y[4])^2)

> Patch[3, 5] := .5*(y[1]*y[6]*y[9]+y[1]*y[6]*y[7]-y[2]*y[6]*y[4]-y[2]*y[6]*y[8])^2+ .5*(y[9]*y[6]*y[8] + y[1]*y[9]*y[4] + y[2]*y[6]*y[8] + y[1]*y[8]*y[9] - y[2]*y[7]*y[4] - y[1]*y[7]*y[4] - y[7]*y[6]*y[4] - y[1]*y[6]*y[7] - y[1]*y[8]*y[7])^2 + .5*(y[9]*y[6]*y[8] + y[1]*y[6]*y[9] + y[2]*y[8]*y[7] + y[2]*y[8]*y[9] - y[7]*y[6]*y[4] - y[2]*y[9]*y[4] - y[2]*y[6]*y[4])^2

> Patch[7, 8] := .5*(y[1]*y[2]*y[3]-y[2]*y[9]*y[4]-y[9]*y[6]*y[4]-y[5]*y[6]*y[4]-y[5]*y[6]*y[9]-y[5]*y[3]*y[6]-y[1]*y[9]*y[4]-y[5]*y[9]*y[4]-y[2]*y[6]*y[4]-y[1]*y[6]*y[9])^2+ .5*(y[1]*y[2]*y[3] + y[1]*y[2]*y[5] + y[1]*y[2]*y[6] - y[2]*y[9]*y[4] - y[9]*y[6]*y[4] - y[5]*y[6]*y[4] - y[5]*y[6]*y[9] - y[5]*y[3]*y[6] - y[1]*y[9]*y[4] - y[5]*y[9]*y[4] - y[2]*y[5]*y[9] - y[1]*y[5]*y[4])^2 + .5*(y[1]*y[6]*y[9] - y[2]*y[5]*y[9])^2 + .5*(y[2]*y[6]*y[4] - y[1]*y[5]*y[4])^2

> SecondTest(Delta, D9badpairs);

$$\{3, 5\}, []$$
$$\{2, 4\}, []$$
$$\{1, 5\}, []$$

74

$$\{7, 8\}, []$$
THEREFORE $\mathcal{D}_9$ IS RAYLEIGH

> $\mathcal{E}_9 :=$
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 2 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & 2 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 & 2 & 1 & 0 & 1 \end{bmatrix}$$

> Delta := Rayleigh(Bases(E9, Y), AllPairs);

> Balance(Delta) :=
$$\begin{bmatrix} 0 & 313 & 174 & 174 & 217 & 601 & 250 & 250 & 300 \\ 313 & 0 & 174 & 174 & 601 & 217 & 250 & 250 & 300 \\ 174 & 174 & 0 & 612 & 174 & 174 & 300 & 300 & 360 \\ 174 & 174 & 612 & 0 & 174 & 174 & 300 & 300 & 360 \\ 217 & 601 & 174 & 174 & 0 & 313 & 250 & 250 & 300 \\ 601 & 217 & 174 & 174 & 313 & 0 & 250 & 250 & 300 \\ 250 & 250 & 300 & 300 & 250 & 250 & 0 & 580 & 120 \\ 250 & 250 & 300 & 300 & 250 & 250 & 580 & 0 & 120 \\ 300 & 300 & 360 & 360 & 300 & 300 & 120 & 120 & 0 \end{bmatrix}$$

> E9pairs := {{3, 5}, {5, 6}, {1, 5}, {1, 6}, {7, 8}, {3, 9}, {6, 7}, {3, 8}, {7, 9}, {3, 4}, {2, 9}}

> FirstTest(Delta, E9pairs, 4, 9);

$$\{3, 4\}, 0, 0, 268$$
"Negative Terms covered by deletion/contraction of", 1
"Negative Terms covered by deletion/contraction of", 2
"Negative Terms covered by deletion/contraction of", 5

"Negative Terms covered by deletion/contraction of", 6
"Negative Terms covered by deletion/contraction of", 7
"Negative Terms covered by deletion/contraction of", 8
"Negative Terms covered by deletion/contraction of", 9
$\{6, 7\}$, 11, 0, 159
$\{7, 9\}$, 14, 0, 112
$\{2, 9\}$, 10, 0, 159
"Negative Terms covered by deletion/contraction of", 5
$\{3, 8\}$, 8, 0, 181
$\{3, 5\}$, 14, 0, 136
$\{5, 6\}$, 2, 0, 169
"Negative Terms covered by deletion/contraction of", 3
"Negative Terms covered by deletion/contraction of", 4 $\{1, 5\}$, 16, 0, 150
$\{1, 6\}$, 6, 0, 258
"Negative Terms covered by deletion/contraction of", 9
$\{7, 8\}$, 0, 0, 268
"Negative Terms covered by deletion/contraction of", 1
"Negative Terms covered by deletion/contraction of", 2
"Negative Terms covered by deletion/contraction of", 3
"Negative Terms covered by deletion/contraction of", 4
"Negative Terms covered by deletion/contraction of", 5
"Negative Terms covered by deletion/contraction of", 6
"Negative Terms covered by deletion/contraction of", 9
$\{3, 9\}$, 2, 0, 179
"Negative Terms covered by deletion/contraction of", 4
"Negative Terms covered by deletion/contraction of", 7
"Negative Terms covered by deletion/contraction of", 8

> E9badpairs := $\{\{3, 5\}, \{1, 5\}, \{6, 7\}, \{3, 8\}, \{7, 9\}\}$

> Patch[3, 5] := .5*(y[1]*y[8]*y[9]+y[9]*y[6]*y[8]-y[2]*y[8]*y[4]-y[2]*y[6]*y[4]-y[2]*y[8]*y[7]-y[1]*y[2]*y[4])^2+.5*$(y[1]*y[8]*y[4]+y[1]*y[7]*y[9]+y[1]*y[8]*y[9]+y[1]*y[6]*y[8]-y[2]*y[8]*y[7]-y[7]*y[6]*y[8]-y[7]*y[6]*y[4]-y[1]*y[2]*y[7]-y[1]*y[2]*y[4])^2$+.5*$(y[1]*y[6]*y[8]-y[2]*y[6]*y[4]-y[1]*y[2]*y[7])^2$

> Patch[1, 5] := (y[7]*y[6]*y[4]-y[8]*y[3]*y[6])^2 + .5 * $(y[2]*y[3]*y[7]-$

76

$y[2] * y[8] * y[4])^2 + .5 * (y[7] * y[9] * y[4] + y[7] * y[3] * y[9] + y[9] * y[8] * y[4] - y[2] * y[6] * y[7] - y[2] * y[6] * y[4] - y[2] * y[6] * y[8])^2 + .5 * (y[2] * y[3] * y[6] - y[9] * y[3] * y[8] - y[7] * y[3] * y[9] - y[9] * y[8] * y[4] - y[7] * y[9] * y[4])^2 + .5 * (y[2] * y[3] * y[6] - y[7] * y[3] * y[4] - y[8] * y[3] * y[4])^2 + .5 * (y[2] * y[6] * y[8] + y[2] * y[6] * y[4] - y[9] * y[3] * y[8] - y[8] * y[3] * y[4] - y[7] * y[3] * y[4])^2$

> Patch[6, 7] := .5*(y[1]*y[5]*y[4]-y[1]*y[2]*y[3]-y[2]*y[5]*y[8])^2 + .5 * $(y[1] * y[5] * y[4] + y[1] * y[3] * y[8] + y[1] * y[8] * y[4] - y[5] * y[8] * y[9] - y[1] * y[2] * y[8] - y[2] * y[9] * y[4] - y[2] * y[3] * y[4])^2 + .5 * (y[1] * y[8] * y[4] - y[5] * y[8] * y[9] - y[5] * y[3] * y[8] - y[5] * y[3] * y[4] - y[5] * y[9] * y[4])^2$

> Patch[3, 8] := .5*(y[2]*y[5]*y[7]-y[1]*y[6]*y[7])$^2 + .5 * (y[1] * y[5] * y[6] - y[7] * y[6] * y[4])^2 + .5 * (y[1] * y[5] * y[6] + y[5] * y[6] * y[9] - y[7] * y[5] * y[4])^2 + .5 * (y[7] * y[6] * y[4] - y[5] * y[6] * y[9] - y[2] * y[5] * y[6])^2 + .5 * (y[7] * y[5] * y[4] - y[2] * y[5] * y[6])^2 + .5 * (y[2] * y[6] * y[4] - y[1] * y[5] * y[4])^2$

> Patch[7, 9] := y[3]*y[4]*(y[2]*y[6]-y[1]*y[5])$^2 + .5 * (y[5] * y[6] * y[8] + y[5] * y[6] * y[4] + y[2] * y[6] * y[4] - y[1] * y[2] * y[8] - y[1] * y[2] * y[3] - y[1] * y[3] * y[5])^2 + .5 * (y[5] * y[6] * y[8] + y[5] * y[6] * y[4] + y[1] * y[5] * y[4] - y[1] * y[2] * y[8] - y[1] * y[2] * y[3] - y[2] * y[3] * y[6])^2 + .5 * (y[1] * y[3] * y[5] - y[2] * y[3] * y[6])^2 + .5 * (y[1] * y[5] * y[4] - y[2] * y[6] * y[4])^2$

> SecondTest(Delta, E9badpairs);

$$\{6, 7\}, []$$
$$\{7, 9\}, []$$
$$\{3, 8\}\} []$$
$$\{3, 5\}, []$$
$$\{1, 5\}, []$$

THEREFORE $\mathcal{E}_9$ IS RAYLEIGH

$$> \mathcal{F}_8 \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 2 & 2 & 1 & 1 \\ 0 & 1 & 0 & 0 & 2 & 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 0 & 2 & 2 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 & 2 & 1 & 0 & 2 \end{bmatrix}$$

> Delta := Rayleigh(Bases(F9, Y), AllPairs);

$$> \text{Balance(Delta)} := \begin{bmatrix} 0 & 360 & 360 & 360 & 360 & 360 & 360 & 360 & 360 \\ 360 & 0 & 360 & 360 & 360 & 360 & 360 & 360 & 360 \\ 360 & 360 & 0 & 360 & 360 & 360 & 360 & 360 & 360 \\ 360 & 360 & 360 & 0 & 360 & 360 & 360 & 360 & 360 \\ 360 & 360 & 360 & 360 & 0 & 360 & 360 & 360 & 360 \\ 360 & 360 & 360 & 360 & 360 & 0 & 360 & 360 & 360 \\ 360 & 360 & 360 & 360 & 360 & 360 & 0 & 360 & 360 \\ 360 & 360 & 360 & 360 & 360 & 360 & 360 & 0 & 360 \\ 360 & 360 & 360 & 360 & 360 & 360 & 360 & 360 & 0 \end{bmatrix}$$

> F9pairs := {{3, 5}}

> FirstTest(Delta, F9pairs, 4, 9);

$$\{3, 5\},\ 18,\ 0,\ 239$$

> Patch[3, 5] := .5*(y[1]*y[6]*y[8]-y[9]*y[6]*y[4]-y[1]*y[2]*y[7])^2 + .5 * $(y[1]*y[6]*y[8] - y[1]*y[2]*y[4] - y[2]*y[6]*y[4])^2 + .5*(y[7]*y[8]*y[9] - y[1]*y[7]*y[4] - y[2]*y[6]*y[9] - y[2]*y[9]*y[4])^2 + .5*(y[7]*y[8]*y[9] + y[1]*y[8]*y[9] + y[1]*y[2]*y[9] - y[2]*y[7]*y[4] - y[2]*y[8]*y[7])^2 + .5*(y[2]*y[8]*y[7] + y[7]*y[6]*y[8] - y[1]*y[8]*y[4] - y[9]*y[8]*y[4])^2 + .5*(y[9]*y[8]*y[4] - y[7]*y[6]*y[4] - y[2]*y[6]*y[8])^2 + .5*(y[1]*y[8]*y[9] + y[1]*y[2]*y[9] - y[2]*y[6]*y[8])^2 + .5*(y[1]*y[8]*y[4] - y[7]*y[6]*y[4])^2$

> SecondTest(Delta, F9pairs);

$$\{3,5\}, []$$
THEREFORE $\mathcal{F}_8$ IS RAYLEIGH

> $\mathcal{G}_9$:=
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 2 & 0 & 1 & 2 \end{bmatrix}$$

> Delta := Rayleigh(Bases(G9, Y), AllPairs);

> Balance(Delta):=
$$\begin{bmatrix} 0 & 238 & 186 & 97 & 94 & 514 & 514 & 238 & 97 \\ 238 & 0 & 154 & 154 & 252 & 238 & 238 & 532 & 154 \\ 186 & 154 & 0 & 424 & 354 & 97 & 97 & 154 & 424 \\ 97 & 154 & 424 & 0 & 354 & 186 & 97 & 154 & 424 \\ 94 & 252 & 354 & 354 & 0 & 94 & 94 & 252 & 354 \\ 514 & 238 & 97 & 186 & 94 & 0 & 514 & 238 & 97 \\ 514 & 238 & 97 & 97 & 94 & 514 & 0 & 238 & 186 \\ 238 & 532 & 154 & 154 & 252 & 238 & 238 & 0 & 154 \\ 97 & 154 & 424 & 424 & 354 & 97 & 186 & 154 & 0 \end{bmatrix}$$

> G9pairs := {{2, 5}, {7, 9}, {3, 5}, {2, 4}, {5, 6}, {1, 6}, {7, 8}, {3, 9}, {1, 4}, {2, 8}}

>¿ FirstTest(Delta, G9pairs, 4, 9);

$$\{2, 5\}, 9, 0, 135$$
"Negative Terms covered by deletion/contraction of", 8
$$\{7, 9\}, 2, 0, 111$$
$$\{3, 5\}, 3, 0, 178$$
"Negative Terms covered by deletion/contraction of", 4
"Negative Terms covered by deletion/contraction of", 9
$$\{2, 4\}, 9, 0, 110$$
$$\{5, 6\}, 11, 0, 82$$
$$\{1, 6\}, 2, 0, 237$$
"Negative Terms covered by deletion/contraction of", 5
"Negative Terms covered by deletion/contraction of", 7
$$\{7, 8\}, 7, 0, 144$$
$$\{3, 9\}, 2, 0, 206$$
"Negative Terms covered by deletion/contraction of", 4
"Negative Terms covered by deletion/contraction of", 5
$$\{1, 4\}, 15, 0, 92$$
$$\{2, 8\}, 6, 0, 227$$
"Negative Terms covered by deletion/contraction of", 5

> G9badpairs := {{2, 4}, {5, 6}, {7, 8}, {1, 4}, {7, 9}}

> Patch[7, 9] := .5*(y[1]*y[3]*y[6]-y[1]*y[2]*y[5])^2 + .5 * (y[1] * y[6] * y[4] - y[5] * y[6] * y[8])^2

> Patch[2, 4] := .5*(y[1]*y[7]*y[5]-y[7]*y[3]*y[8]-y[1]*y[3]*y[8])^2 + .5 * (y[7] * y[3] * y[6] - y[1] * y[7] * y[9])^2 + .5 * (y[7] * y[3] * y[6] + y[5] * y[6] * y[7] - y[9] * y[6] * y[8] - y[7] * y[8] * y[9] - y[8] * y[3] * y[6])^2 + .5 * (y[5] * y[6] * y[7] - y[9] * y[6] * y[8] - y[7] * y[8] * y[9] - y[7] * y[3] * y[8])^2

> Patch[5, 6] := .5*(y[1]*y[8]*y[4]+y[1]*y[8]*y[9]+y[7]*y[8]*y[9]-y[1]*y[2]*y[3]-y[2]*y[3]*y[7]-y[2]*y[7]*y[4])^2 + .5 * (y[1] * y[8] * y[4] - y[7] * y[3] * y[8] - y[1] * y[2] * y[3] - y[2] * y[3] * y[7] - y[1] * y[2] * y[7] - y[1] * y[2] * y[9])^2 + .5 * (y[7] * y[3] * y[8] + y[1] * y[2] * y[9] + y[7] * y[8] * y[9] + y[1] * y[8] * y[9] - y[2] * y[7] * y[4])^2

> Patch[7, 8] := .5*(y[2]*y[3]*y[6]-y[1]*y[3]*y[9]-y[1]*y[2]*y[9])^2 + .5 *
$(y[2]*y[3]*y[6] + y[1]*y[2]*y[6] - y[1]*y[3]*y[4] - y[1]*y[3]*y[5])^2$ + .5 *
$(y[1]*y[2]*y[6] - y[1]*y[3]*y[5] - y[5]*y[3]*y[6] - y[5]*y[6]*y[9])^2$

> Patch[1, 4] := .5*(y[7]*y[3]*y[8]+y[7]*y[3]*y[6]-y[5]*y[6]*y[8]-y[9]*y[6]*y[8]-
y[2]*y[3]*y[6]-y[2]*y[6]*y[8])^2+.5*$(y[7]*y[3]*y[8]+y[5]*y[8]*y[7]+y[2]*y[8]*$
$y[7]-y[9]*y[6]*y[8]-y[2]*y[6]*y[9]-y[2]*y[7]*y[9]-y[6]*y[7]*y[9])^2$+.5*$(y[5]*$
$y[8]*y[7]+y[5]*y[6]*y[8]+y[2]*y[5]*y[7]-y[7]*y[3]*y[6]-y[6]*y[7]*y[9])^2$

> SecondTest(Delta, G9badpairs);

$$\{7, 9\}, []$$
$$\{2, 4\}, []$$
$$\{5, 6\}, []$$
$$\{7, 8\}, []$$
$$\{1, 4\}, []$$
THEREFORE $\mathcal{G}_9$ IS RAYLEIGH

> $\mathcal{H}_9$:= $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 2 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 2 & 0 & 1 & 0 \end{bmatrix}$

> Delta := Rayleigh(Bases(H9, Y), AllPairs);

81

$$> \text{Balance(Delta):=} \begin{bmatrix} 0 & 262 & 262 & 82 & 174 & 496 & 540 & 82 & 126 \\ 262 & 0 & 379 & 109 & 363 & 82 & 135 & 289 & 342 \\ 262 & 379 & 0 & 289 & 363 & 82 & 135 & 109 & 342 \\ 82 & 109 & 289 & 0 & 363 & 262 & 135 & 379 & 342 \\ 174 & 363 & 363 & 363 & 0 & 174 & 135 & 363 & 54 \\ 496 & 82 & 82 & 262 & 174 & 0 & 540 & 262 & 126 \\ 540 & 135 & 135 & 135 & 135 & 540 & 0 & 135 & 270 \\ 82 & 289 & 109 & 379 & 363 & 262 & 135 & 0 & 342 \\ 126 & 342 & 342 & 342 & 54 & 126 & 270 & 342 & 0 \end{bmatrix}$$

> H9pairs := {{3, 4}, {4, 8}, {6, 7}, {7, 9}, {3, 5}, {2, 4}, {5, 6}, {1, 6}, {7, 8}, {3, 9}, {1, 4}, {4, 6}, {1, 9}, {5, 9}, {5, 7}}

> FirstTest(Delta, H9pairs, 4, 9);

$\{3, 4\}$, 7, 0, 149
"Negative Terms covered by deletion/contraction of", 5
$\{4, 8\}$, 2, 0, 182
"Negative Terms covered by deletion/contraction of", 7
"Negative Terms covered by deletion/contraction of", 9
$\{6, 7\}$, 1, 0, 238
"Negative Terms covered by deletion/contraction of", 1
"Negative Terms covered by deletion/contraction of", 5
"Negative Terms covered by deletion/contraction of", 9
$\{7, 9\}$, 0, 0, 147
"Negative Terms covered by deletion/contraction of", 1
"Negative Terms covered by deletion/contraction of", 2
"Negative Terms covered by deletion/contraction of", 3
"Negative Terms covered by deletion/contraction of", 4
"Negative Terms covered by deletion/contraction of", 5
"Negative Terms covered by deletion/contraction of", 6
"Negative Terms covered by deletion/contraction of", 8
$\{3, 5\}$, 6, 0, 170

"Negative Terms covered by deletion/contraction of", 4

$\{2,4\}$, 14, 0, 99

$\{5,6\}$, 7, 0, 113

$\{1,6\}$, 0, 0, 219

"Negative Terms covered by deletion/contraction of", 2

"Negative Terms covered by deletion/contraction of", 3

"Negative Terms covered by deletion/contraction of", 4

"Negative Terms covered by deletion/contraction of", 5

"Negative Terms covered by deletion/contraction of", 7

"Negative Terms covered by deletion/contraction of", 8

"Negative Terms covered by deletion/contraction of", 9

$\{7,8\}$, 10, 0, 107

$\{3,9\}$, 2, 0, 168

"Negative Terms covered by deletion/contraction of", 2

"Negative Terms covered by deletion/contraction of", 7

$\{1,4\}$, 15, 0, 87

$\{4,6\}$, 2, 0, 141

"Negative Terms covered by deletion/contraction of", 1

"Negative Terms covered by deletion/contraction of", 2

$\{1,9\}$, 10, 0, 106

$\{5,9\}$, 22, 0, 88

$\{5,7\}$, 6, 0, 97

"Negative Terms covered by deletion/contraction of", 9

> H9badpairs := $\{\{2, 4\}, \{5, 6\}, \{7, 8\}, \{1, 4\}, \{1, 9\}, \{5, 9\}\}$

> Patch[2, 4] := .5*(y[1]*y[3]*y[8]-y[1]*y[5]*y[9]-y[1]*y[7]*y[5])^2 + .5 * $(y[7]*y[3]*y[8] - y[5]*y[6]*y[7] - y[1]*y[7]*y[5] - y[7]*y[5]*y[9])^2 + .5 * (y[7]*y[5]*y[9] - y[7]*y[3]*y[8] - y[1]*y[8]*y[7] - y[7]*y[3]*y[6])^2 + .5 * (y[1]*y[5]*y[9] - y[1]*y[8]*y[7] - y[1]*y[3]*y[8])^2 + .5 * (y[8]*y[3]*y[6] - y[5]*y[6]*y[7] - y[5]*y[6]*y[9])^2 + .5 * (y[5]*y[6]*y[9] - y[8]*y[3]*y[6] - y[7]*y[3]*y[6])^2 + 2*y[1]*y[7]*(y[5]*y[9] - y[8]*y[3])^2 + 2*y[6]*y[7]*(y[8]*y[3] - y[5]*y[9])^2 + 2*y[1]*y[6]*(y[8]*y[3] - y[5]*y[9])^2$

> Patch[5, 6] := .5*(y[2]*y[7]*y[4]-y[7]*y[3]*y[8])^2 + .5 * $(y[7]*y[3]*y[8] + y[1]*y[2]*y[3] + y[2]*y[3]*y[7] - y[1]*y[8]*y[4] - y[1]*y[8]*y[9])^2 + .5 * (y[1]*$

83

$$y[2]*y[3]+y[2]*y[3]*y[7]+y[2]*y[7]*y[4]-y[1]*y[8]*y[4]-y[1]*y[9]*y[4])^2$$

> Patch[7, 8] := .5*(y[1]*y[2]*y[4]-y[1]*y[5]*y[9])^2 $+.5*(y[1]*y[2]*y[6]-$
$y[1]*y[3]*y[5]-y[1]*y[5]*y[9])^2+.5*(y[1]*y[2]*y[6]+y[9]*y[3]*y[6]+y[1]*$
$y[6]*y[9]+y[2]*y[3]*y[6]-y[1]*y[3]*y[5]-y[5]*y[3]*y[6])^2+.5*(y[1]*y[3]*$
$y[4]-y[9]*y[3]*y[6]-y[2]*y[3]*y[6])^2+.5*(y[1]*y[3]*y[4]-y[1]*y[6]*y[9])^2$

> Patch[1, 4] := y[5]*y[3]*(y[6]*y[9]-y[8]*y[7])^2+$.5*(y[5]*y[8]*y[7]+y[5]*$
$y[6]*y[8]+y[7]*y[5]*y[9]-y[9]*y[3]*y[6]-y[7]*y[3]*y[6]-y[2]*y[6]*y[9]-y[2]*$
$y[6]*y[8])^2+.5*(y[7]*y[3]*y[6]+y[7]*y[3]*y[8]+y[2]*y[6]*y[7]-y[2]*y[5]*y[7]-$
$y[5]*y[6]*y[9]-y[5]*y[6]*y[8]-y[7]*y[5]*y[9])^2+.5*(y[7]*y[3]*y[8]-y[2]*y[3]*$
$y[6]-y[2]*y[6]*y[8]-y[9]*y[3]*y[6])^2+.5*(y[5]*y[8]*y[7]-y[5]*y[6]*y[9])^2$

> Patch[1, 9] := y[6]*y[7]*(y[2]*y[8]-y[3]*y[4])^2 $+ .5 * (y[2] * y[7] * y[4] -$
$y[5] * y[6] * y[4])^2 + .5 * (y[2] * y[7] * y[4] + y[2] * y[6] * y[8] - y[7] * y[3] * y[8] -$
$y[6] * y[3] * y[4])^2 + .5 * (y[6] * y[3] * y[4] + y[7] * y[3] * y[6] - y[2] * y[6] * y[8] -$
$y[2] * y[6] * y[7])^2 + .5 * (y[2] * y[6] * y[7] - y[5] * y[6] * y[4])^2 + .5 * (y[5] * y[6] *$
$y[8] - y[7] * y[3] * y[6] - y[7] * y[3] * y[8])^2$

> Patch[5,9] := 0.5*(((y[2]*y[6]*y[4]+y[2]*y[7]*y[4]+y[1]*y[7]*y[4]+y[1]*y[2]*y[4])-
(y[1]*y[3]*y[8]+y[7]*y[3]*y[8]+ y[8]*y[3]*y[6]+y[7]*y[3]*y[6]+y[1]*y[3]*y[6]+y[1]*y[2]*y[6]))^2+
$((y[1] * y[7] * y[4] + y[1] * y[6] * y[4] + y[1] * y[8] * y[7] + y[1] * y[6] * y[8]) - (y[2] *$
$y[6] * y[7] + y[1] * y[2] * y[6] + y[7] * y[3] * y[6]))^2 + ((y[1] * y[6] * y[4] + y[2] *$
$y[6] * y[4] + y[2] * y[6] * y[7] + y[1] * y[2] * y[4] + y[2] * y[7] * y[4]) - (y[1] * y[3] *$
$y[8] + y[8] * y[3] * y[6] + y[1] * y[8] * y[7] + y[7] * y[3] * y[8] + y[1] * y[3] * y[6]))^2)$

> SecondTest(Delta, H9badpairs);

$$\{2,4\}, []$$
$$\{5,6\}, []$$
$$\{7,8\}, []$$
$$\{1,4\}, []$$

$$\{1, 9\}, []$$
$$\{5, 9\}, []$$

THEREFORE $\mathcal{H}_9$ IS RAYLEIGH

$$> \mathcal{I}_9 := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 2 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 2 & 0 & 1 & 2 \end{bmatrix}$$

> Delta := Rayleigh(Bases(I9, Y), AllPairs);

$$> \text{Balance(Delta)} := \begin{bmatrix} 0 & 238 & 238 & 100 & 232 & 508 & 465 & 100 & 97 \\ 238 & 0 & 354 & 234 & 372 & 78 & 118 & 412 & 154 \\ 238 & 354 & 0 & 412 & 372 & 78 & 118 & 234 & 154 \\ 100 & 234 & 412 & 0 & 264 & 130 & 78 & 390 & 316 \\ 232 & 372 & 372 & 264 & 0 & 88 & 124 & 264 & 192 \\ 508 & 78 & 78 & 130 & 88 & 0 & 649 & 130 & 313 \\ 465 & 118 & 118 & 78 & 124 & 649 & 0 & 78 & 348 \\ 100 & 412 & 234 & 390 & 264 & 130 & 78 & 0 & 316 \\ 97 & 154 & 154 & 316 & 192 & 313 & 348 & 316 & 0 \end{bmatrix}$$

> I9pairs := {{3, 4}, {4, 8}, {6, 7}, {7, 9}, {3, 5}, {2, 4}, {5, 6}, {1, 5}, {1, 6}, {7, 8}, {3, 9}, {4, 9}, {1, 4}, {1, 9}, {5, 9}, {5, 7}, {4, 5}, {2, 6}, {2, 7}, {2, 3}, {6, 9}, {1, 7}, {1, 2}, {4, 6}}

> FirstTest(Delta, I9pairs, 4, 9);

$$\{3, 4\}, 4, 0, 193$$
"Negative Terms covered by deletion/contraction of", 5

$\{4, 8\}$, 2, 0, 182
"Negative Terms covered by deletion/contraction of", 7
"Negative Terms covered by deletion/contraction of", 9
$\{6, 7\}$, 1, 0, 268
"Negative Terms covered by deletion/contraction of", 1
"Negative Terms covered by deletion/contraction of", 5
"Negative Terms covered by deletion/contraction of", 9
$\{7, 9\}$, 0, 0, 162
"Negative Terms covered by deletion/contraction of", 1
"Negative Terms covered by deletion/contraction of", 2
"Negative Terms covered by deletion/contraction of", 3
"Negative Terms covered by deletion/contraction of", 4
"Negative Terms covered by deletion/contraction of", 5
"Negative Terms covered by deletion/contraction of", 6
"Negative Terms covered by deletion/contraction of", 8
$\{3, 5\}$, 7, 0, 178
"Negative Terms covered by deletion/contraction of", 4
$\{2, 4\}$, 10, 0, 142
$\{5, 6\}$, 11, 0, 79
$\{1, 5\}$, 5, 0, 123
"Negative Terms covered by deletion/contraction of", 9
$\{1, 6\}$, 0, 0, 218
"Negative Terms covered by deletion/contraction of", 2
"Negative Terms covered by deletion/contraction of", 3
"Negative Terms covered by deletion/contraction of", 4
"Negative Terms covered by deletion/contraction of", 5
"Negative Terms covered by deletion/contraction of", 7
"Negative Terms covered by deletion/contraction of", 8
"Negative Terms covered by deletion/contraction of", 9
$\{7, 8\}$, 14, 0, 79
$\{3, 9\}$, 9, 0, 109
$\{4, 9\}$, 2, 0, 151
"Negative Terms covered by deletion/contraction of", 7
"Negative Terms covered by deletion/contraction of", 8
$\{1, 4\}$, 10, 0, 87
$\{4, 6\}$, 5, 0, 89
"Negative Terms covered by deletion/contraction of", 2
$\{1, 9\}$, 16, 0, 90

{5,9}, 6, 0, 116
{5,7}, 4, 0, 86
{4,5}, 8, 0, 146
"Negative Terms covered by deletion/contraction of", 3
{2,6}, 13, 0, 78
{2,7}, 7, 0, 88
{2,3}, 2, 0, 182
"Negative Terms covered by deletion/contraction of", 7
"Negative Terms covered by deletion/contraction of", 9
{6,9}, 1, 0, 154
"Negative Terms covered by deletion/contraction of", 1
"Negative Terms covered by deletion/contraction of", 5
"Negative Terms covered by deletion/contraction of", 7
{1,7}, 1, 0, 209
"Negative Terms covered by deletion/contraction of", 5
"Negative Terms covered by deletion/contraction of", 6
"Negative Terms covered by deletion/contraction of", 9
{1,2}, 2, 0, 128
"Negative Terms covered by deletion/contraction of", 4
"Negative Terms covered by deletion/contraction of", 6

> I9badpairs := {{2, 4}, {5, 6}, {7, 8}, {3, 9}, {1, 4}, {1, 9}, {5, 9}, {5, 7}, {2, 6}, {2, 7}}

> Patch[2, 4] := (y[7]*y[3]*y[8]-y[5]*y[6]*y[7]-y[5]*y[7]*y[9]-y[1]*y[5]*y[7]+.5*y[1]*y[3]*y[8]-y[1]*y[5]*y[9]-y[5]*y[6]*y[9]+y[6]*y[3]*y[8])^2

> Patch[5, 6] := (y[1]*y[2]*y[3]+.5*y[7]*y[3]*y[8]+y[2]*y[3]*y[7]+.5*y[2]*y[4]*y[7]-y[1]*y[4]*y[8]-y[1]*y[3]*y[9]-y[1]*y[8]*y[9]-y[1]*y[4]*y[9]-y[1]*y[2]*y[9])^2+.75*$ $(y[2] * y[4] * y[7] - y[7] * y[3] * y[8])^2$

> Patch[7, 8] := .5*((y[1]*y[3]*y[9]+y[1]*y[3]*y[4]+y[1]*y[2]*y[9]-y[5]*y[3]*y[6]-y[1]*y[6]*y[4]-y[2]*y[3]*y[6])$^2 + (y[1] * y[3] * y[9] + y[1] * y[3] * y[5] - y[1] * y[2] *$ $y[6] - y[6] * y[3] * y[4] - y[5] * y[6] * y[4] - y[2] * y[3] * y[6])^2 + (y[5] * y[3] *$ $y[6] + y[1] * y[5] * y[4] + y[1] * y[3] * y[5] - y[1] * y[2] * y[6] - y[1] * y[2] * y[9])^2)$

> Patch[3, 9] := (y[1]*y[6]*y[4]+y[1]*y[2]*y[6]+.5*y[1]*y[4]*y[7]-y[2]*y[5]*y[7]-y[5]*y[7]*y[8]-y[5]*y[6]*y[8]-y[1]*y[5]*y[8]-y[1]*y[2]*y[5]-.5*y[1]*y[7]*y[8])^2

> Patch[1, 4] := (y[2]*y[6]*y[8]+y[5]*y[6]*y[8]+y[2]*y[5]*y[7]+.5*y[5]*y[7]*y[8]-.5*y[3]*y[7]*y[8]-y[3]*y[6]*y[9]-y[3]*y[6]*y[7]-y[3]*y[7]*y[9])^2+.5*(y[2]*y[3]*$ $y[6] - y[3]*y[7]*y[8])^2$

> Patch[1, 9] := y[6]*y[7]*(y[3]*y[4]-y[2]*y[8])$^2 + .5 * ((y[2]*y[3]*y[6] +$ $y[5]*y[3]*y[6]+y[6]*y[3]*y[4]+y[5]*y[6]*y[4]+y[2]*y[5]*y[6]-y[3]*y[7]*y[8]-$ $y[2]*y[7]*y[4]-y[2]*y[8]*y[7]-y[5]*y[3]*y[7])^2+(y[2]*y[3]*y[6]+y[5]*y[6]*$ $y[8]+y[2]*y[6]*y[8]-y[3]*y[7]*y[8]-y[2]*y[7]*y[4]-y[7]*y[3]*y[4]-y[2]*y[5]*$ $y[7])^2+(y[2]*y[6]*y[8]-y[6]*y[3]*y[4])^2+(y[2]*y[8]*y[7]-y[7]*y[3]*y[4])^2)$

> Patch[5, 9] := (y[1]*y[3]*y[8]+y[3]*y[6]*y[8]+.5*y[3]*y[7]*y[8]-y[2]*y[7]*y[4]-y[1]*y[2]*y[4]-.5*y[2]*y[4]*y[6])^2

> Patch[5, 7] := .5*((y[1]*y[2]*y[9]-y[2]*y[3]*y[6])$^2 + (y[1]*y[2]*y[4] -$ $y[1]*y[3]*y[8])^2 + (y[2]*y[4]*y[6] - y[3]*y[6]*y[8])^2 + (y[1]*y[3]*y[9] -$ $y[2]*y[3]*y[6])^2)$

> Patch[2, 6] := .5*((y[1]*y[3]*y[9]+y[1]*y[3]*y[4]+y[1]*y[3]*y[5]-y[1]*y[8]*y[7]-y[7]*y[3]*y[4]-y[7]*y[3]*y[8])$^2 + (y[5]*y[3]*y[7] + y[7]*y[3]*y[8] - y[1]*y[8]*$ $y[9] - y[1]*y[3]*y[9] - y[1]*y[8]*y[4])^2 + (y[1]*y[3]*y[5] + y[5]*y[3]*$ $y[7] + y[7]*y[5]*y[4] - y[1]*y[8]*y[7] - y[1]*y[8]*y[9])^2)$

> Patch[2, 7] := (y[1]*y[6]*y[8]+.5*y[1]*y[8]*y[9]+.5*y[1]*y[8]*y[4]+.5*y[1]*y[4]*y[9]-y[4]*y[5]*y[6]-.5*y[1]*y[4]*y[5])$^2 + .5 * (y[1]*y[8]*y[4] - y[3]*y[4]*y[6])^2 +$ $.5 * (y[1]*y[4]*y[9] - y[3]*y[4]*y[6])^2$

> SecondTest(Delta, I9badpairs);

$$\{2,4\}, [] $$
$$\{5,6\}, [] $$
$$\{7,8\}, [] $$
$$\{3,9\}, [] $$
$$\{1,4\}, [] $$
$$\{1,9\}, [] $$
$$\{5,9\}, [] $$
$$\{5,7\}, [] $$
$$\{2,6\}, [] $$
$$\{2,7\}, [] $$

THEREFORE $\mathcal{J}_9$ IS RAYLEIGH

$$
> \mathcal{J}_9 := \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 2 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 2 & 0 & 2 & 1 & 0 \\
0 & 0 & 0 & 1 & 2 & 2 & 0 & 1 & 1
\end{bmatrix}
$$

> Delta := Rayleigh(Bases(J9, Y), AllPairs);

$$
> \text{Balance(Delta)} := \begin{bmatrix}
0 & 150 & 276 & 429 & 150 & 276 & 342 & 90 & 429 \\
150 & 0 & 150 & 126 & 312 & 429 & 198 & 429 & 312 \\
276 & 150 & 0 & 429 & 429 & 90 & 342 & 276 & 150 \\
429 & 126 & 429 & 0 & 312 & 150 & 198 & 150 & 312 \\
150 & 312 & 429 & 312 & 0 & 150 & 198 & 429 & 126 \\
276 & 429 & 90 & 150 & 150 & 0 & 342 & 276 & 429 \\
342 & 198 & 342 & 198 & 198 & 342 & 0 & 342 & 198 \\
90 & 429 & 276 & 150 & 429 & 276 & 342 & 0 & 150 \\
429 & 312 & 150 & 312 & 126 & 429 & 198 & 150 & 0
\end{bmatrix}
$$

89

> J9pairs := {{3, 5}, {2, 4}, {5, 6}, {1, 6}, {7, 8}, {4, 9}, {7, 9}, {1, 8}}

> FirstTest(Delta, J9pairs, 4, 9);

$$\{7,9\}, 12, 0, 133$$
$$\{3,5\}, 6, 0, 199$$
"Negative Terms covered by deletion/contraction of", 4
$$\{2,4\}, 10, 0, 95$$
$$\{5,6\}, 10, 0, 111$$
$$\{1,6\}, 6, 0, 156$$
$$\{7,8\}, 5, 0, 184$$
$$\{4,9\}, 7, 0, 158$$
"Negative Terms covered by deletion/contraction of", 1
$$\{1,8\}, 20, 0, 103$$

> J9badpairs := {{2, 4}, {5, 6}, {1, 6}, {7, 8}, {7, 9}, {1, 8}}

> Patch[2, 4] := .5*(y[7]*y[8]*y[9]+y[7]*y[3]*y[9]-y[1]*y[6]*y[8]-y[1]*y[6]*y[7])^2+
$.5 * (y[5] * y[6] * y[7] - y[6] * y[3] * y[8] - y[7] * y[3] * y[8] - y[7] * y[8] * y[9])^2 +$
$.5 * (y[7] * y[3] * y[9] - y[1] * y[3] * y[6] - y[1] * y[5] * y[7])^2 + .5 * (y[1] * y[5] *$
$y[7] - y[1] * y[3] * y[8] - y[7] * y[3] * y[8])^2$

> Patch[5, 6] := (y[1]*y[8]*y[4]+y[9]*y[8]*y[4]+y[7]*y[8]*y[9]+y[9]*y[3]*y[8]+y[1]*y[8]*y[9]-
.5*y[2]*y[7]*y[4]-.5*y[7]*y[3]*y[8]-y[1]*y[2]*y[3]-y[2]*y[3]*y[7])^2 $+ .75 * (y[2] *$
$y[7] * y[4] - y[7] * y[3] * y[8])^2$

> Patch[1, 6] := (y[2]*y[3]*y[5]-.5*y[7]*y[3]*y[9])^2 $+ (y[2] * y[3] * y[4] -$
$.5 * y[7] * y[3] * y[9])^2 + .5 * (y[7] * y[3] * y[9] - y[2] * y[7] * y[4])^2 + (y[2] * y[8] *$
$y[4] - .5 * y[7] * y[8] * y[9])^2 + (y[5] * y[8] * y[4] - .5 * y[7] * y[8] * y[9])^2 + .5 *$
$(y[2] * y[7] * y[4] - y[7] * y[8] * y[9])^2$

90

> Patch[7, 8] := .5*(y[1]*y[2]*y[6]-y[1]*y[3]*y[5]-y[5]*y[3]*y[6])^2 + .5 * $(y[1] * y[3] * y[5] - y[1] * y[2] * y[6] - y[2] * y[3] * y[6])^2 + .5 * (y[2] * y[3] * y[6] - y[1] * y[3] * y[4])^2 + .5 * (y[5] * y[3] * y[6] - y[1] * y[6] * y[9])^2$

> Patch[7, 9] := y[4]*y[1]*(y[3]*y[6]-y[2]*y[5])^2 + .5 * $(y[1] * y[3] * y[6] - y[1] * y[2] * y[5] - y[2] * y[3] * y[4] - y[2] * y[3] * y[5])^2 + .5 * (y[1] * y[2] * y[5] + y[1] * y[5] * y[8] + y[1] * y[6] * y[8] + y[1] * y[2] * y[8] - y[5] * y[3] * y[6] - y[1] * y[3] * y[6] - y[5] * y[6] * y[4])^2 + .5 * (y[1] * y[6] * y[8] - y[5] * y[8] * y[4] - y[5] * y[6] * y[4] - y[2] * y[8] * y[4] - y[2] * y[5] * y[4])^2$

>Patch[1, 8] := .5*((y[5]*y[6]*y[4]-y[2]*y[9]*y[7]-y[6]*y[9]*y[7]-y[2]*y[6]*y[7]-y[2]*y[7]*y[4]-y[7]*y[5]*y[9])^2 + $(y[5] * y[6] * y[4] + y[7] * y[3] * y[4] + y[5] * y[3] * y[7] + y[7] * y[4] * y[5] - y[2] * y[9] * y[7] - y[6] * y[9] * y[7] - y[2] * y[6] * y[7] - y[2] * y[3] * y[9])^2 + (y[2] * y[3] * y[9] + y[6] * y[3] * y[4] + y[2] * y[3] * y[6] + y[5] * y[3] * y[6] + y[9] * y[3] * y[6] - y[7] * y[4] * y[5] - y[7] * y[3] * y[4] - y[5] * y[3] * y[7] - y[2] * y[7] * y[4] - y[7] * y[5] * y[9])^2)$

> SecondTest(Delta, J9badpairs);

$$\{7, 9\}, \ []$$
$$\{2, 4\}, \ []$$
$$\{5, 6\}, \ []$$
$$\{1, 6\}, \ []$$
$$\{7, 8\}, \ []$$
$$\{1, 8\}, \ []$$

THEREFORE $\mathcal{J}_9$ IS RAYLEIGH

Appendix D

Calculations for 10-element candidates

```
> Balance(Rayleigh(Bases(A10,Y),AllPairs));
```

$$
\begin{bmatrix}
0 & 345 & 345 & 345 & 144 & 1377 & 1377 & 345 & 345 & 345 \\
345 & 0 & 307 & 307 & 648 & 345 & 345 & 1012 & 307 & 1012 \\
345 & 307 & 0 & 1012 & 648 & 345 & 345 & 307 & 1012 & 307 \\
345 & 307 & 1012 & 0 & 648 & 345 & 345 & 307 & 1012 & 307 \\
144 & 648 & 648 & 648 & 0 & 144 & 144 & 648 & 648 & 648 \\
1377 & 345 & 345 & 345 & 144 & 0 & 1377 & 345 & 345 & 345 \\
1377 & 345 & 345 & 345 & 144 & 1377 & 0 & 345 & 345 & 345 \\
345 & 1012 & 307 & 307 & 648 & 345 & 345 & 0 & 307 & 1012 \\
345 & 307 & 1012 & 1012 & 648 & 345 & 345 & 307 & 0 & 307 \\
345 & 1012 & 307 & 307 & 648 & 345 & 345 & 1012 & 307 & 0
\end{bmatrix}
$$

```
> Balance(Rayleigh(Bases(C10,Y),AllPairs));
```

$$\begin{bmatrix}
0 & 345 & 345 & 345 & 144 & 1377 & 1377 & 345 & 345 & 345 \\
345 & 0 & 307 & 307 & 648 & 345 & 345 & 1012 & 307 & 1012 \\
345 & 307 & 0 & 1012 & 648 & 345 & 345 & 307 & 1012 & 307 \\
345 & 307 & 1012 & 0 & 648 & 345 & 345 & 307 & 1012 & 307 \\
144 & 648 & 648 & 648 & 0 & 144 & 144 & 648 & 648 & 648 \\
1377 & 345 & 345 & 345 & 144 & 0 & 1377 & 345 & 345 & 345 \\
1377 & 345 & 345 & 345 & 144 & 1377 & 0 & 345 & 345 & 345 \\
345 & 1012 & 307 & 307 & 648 & 345 & 345 & 0 & 307 & 1012 \\
345 & 307 & 1012 & 1012 & 648 & 345 & 345 & 307 & 0 & 307 \\
345 & 1012 & 307 & 307 & 648 & 345 & 345 & 1012 & 307 & 0
\end{bmatrix}$$

```
> Balance(Rayleigh(Bases(E10,Y),AllPairs));
```

$$\begin{bmatrix}
0 & 345 & 345 & 67 & 259 & 1594 & 1294 & 67 & 634 & 195 \\
345 & 0 & 655 & 655 & 669 & 67 & 271 & 516 & 553 & 757 \\
345 & 655 & 0 & 516 & 669 & 67 & 271 & 655 & 553 & 757 \\
67 & 655 & 516 & 0 & 669 & 345 & 271 & 655 & 553 & 757 \\
259 & 669 & 669 & 669 & 0 & 259 & 475 & 669 & 561 & 360 \\
1594 & 67 & 67 & 345 & 259 & 0 & 1294 & 345 & 634 & 195 \\
1294 & 271 & 271 & 271 & 475 & 1294 & 0 & 271 & 135 & 546 \\
67 & 516 & 655 & 655 & 669 & 345 & 271 & 0 & 553 & 757 \\
634 & 553 & 553 & 553 & 561 & 634 & 135 & 553 & 0 & 234 \\
195 & 757 & 757 & 757 & 360 & 195 & 546 & 757 & 234 & 0
\end{bmatrix}$$

```
> Balance(Rayleigh(Bases(F10,Y),AllPairs));
```

$$\begin{bmatrix}
0 & 289 & 289 & 103 & 316 & 1468 & 1113 & 103 & 600 & 121 \\
289 & 0 & 655 & 892 & 597 & 73 & 210 & 759 & 514 & 277 \\
289 & 655 & 0 & 759 & 597 & 73 & 210 & 892 & 514 & 277 \\
103 & 892 & 759 & 0 & 453 & 165 & 154 & 739 & 382 & 535 \\
316 & 597 & 597 & 453 & 0 & 124 & 364 & 453 & 516 & 660 \\
1468 & 73 & 73 & 165 & 124 & 0 & 1393 & 165 & 424 & 465 \\
1113 & 210 & 210 & 154 & 364 & 1393 & 0 & 154 & 112 & 700 \\
103 & 759 & 892 & 739 & 453 & 165 & 154 & 0 & 382 & 535 \\
600 & 514 & 514 & 382 & 516 & 424 & 112 & 382 & 0 & 472 \\
121 & 277 & 277 & 535 & 660 & 465 & 700 & 535 & 472 & 0
\end{bmatrix}$$

```
> Balance(Rayleigh(Bases(G10,Y),AllPairs));
```

$$\begin{bmatrix}
0 & 316 & 316 & 316 & 264 & 1476 & 736 & 316 & 856 & 736 \\
316 & 0 & 640 & 1084 & 604 & 316 & 316 & 1084 & 604 & 316 \\
316 & 640 & 0 & 1084 & 604 & 316 & 316 & 1084 & 604 & 316 \\
316 & 1084 & 1084 & 0 & 604 & 316 & 316 & 640 & 604 & 316 \\
264 & 604 & 604 & 604 & 0 & 264 & 856 & 604 & 336 & 856 \\
1476 & 316 & 316 & 316 & 264 & 0 & 736 & 316 & 856 & 736 \\
736 & 316 & 316 & 316 & 856 & 736 & 0 & 316 & 264 & 1476 \\
316 & 1084 & 1084 & 640 & 604 & 316 & 316 & 0 & 604 & 316 \\
856 & 604 & 604 & 604 & 336 & 856 & 264 & 604 & 0 & 264 \\
736 & 316 & 316 & 316 & 856 & 736 & 1476 & 316 & 264 & 0
\end{bmatrix}$$

```
> Balance(Rayleigh(Bases(I10,Y),AllPairs));
```

$$
\begin{bmatrix}
0 & 544 & 544 & 136 & 544 & 1156 & 1224 & 136 & 204 & 136 \\
544 & 0 & 1012 & 280 & 876 & 138 & 220 & 552 & 634 & 172 \\
544 & 1012 & 0 & 552 & 876 & 138 & 220 & 280 & 634 & 172 \\
136 & 280 & 552 & 0 & 552 & 304 & 120 & 808 & 760 & 712 \\
544 & 876 & 876 & 552 & 0 & 274 & 220 & 552 & 90 & 444 \\
1156 & 138 & 138 & 304 & 274 & 0 & 1594 & 304 & 105 & 586 \\
1224 & 220 & 220 & 120 & 220 & 1594 & 0 & 120 & 330 & 540 \\
136 & 552 & 280 & 808 & 552 & 304 & 120 & 0 & 760 & 712 \\
204 & 634 & 634 & 760 & 90 & 105 & 330 & 760 & 0 & 666 \\
136 & 172 & 172 & 712 & 444 & 586 & 540 & 712 & 666 & 0
\end{bmatrix}
$$

```
> Balance(Rayleigh(Bases(K10,Y),AllPairs));
```

$$
\begin{bmatrix}
0 & 493 & 493 & 493 & 357 & 1105 & 918 & 493 & 510 & 918 \\
493 & 0 & 1033 & 268 & 762 & 493 & 351 & 880 & 915 & 351 \\
493 & 1033 & 0 & 880 & 762 & 493 & 351 & 268 & 915 & 351 \\
493 & 268 & 880 & 0 & 762 & 493 & 351 & 1033 & 915 & 351 \\
357 & 762 & 762 & 762 & 0 & 357 & 837 & 762 & 36 & 837 \\
1105 & 493 & 493 & 493 & 357 & 0 & 918 & 493 & 510 & 918 \\
918 & 351 & 351 & 351 & 837 & 918 & 0 & 351 & 378 & 1215 \\
493 & 880 & 268 & 1033 & 762 & 493 & 351 & 0 & 915 & 351 \\
510 & 915 & 915 & 915 & 36 & 510 & 378 & 915 & 0 & 378 \\
918 & 351 & 351 & 351 & 837 & 918 & 1215 & 351 & 378 & 0
\end{bmatrix}
$$

```
> Balance(Rayleigh(Bases(M10,Y),AllPairs));
```

$$
\begin{bmatrix}
0 & 274 & 274 & 138 & 304 & 1249 & 1468 & 138 & 105 & 649 \\
274 & 0 & 1012 & 604 & 552 & 138 & 112 & 876 & 226 & 634 \\
274 & 1012 & 0 & 876 & 552 & 138 & 112 & 604 & 226 & 634 \\
138 & 604 & 876 & 0 & 552 & 274 & 112 & 1012 & 634 & 226 \\
304 & 552 & 552 & 552 & 0 & 304 & 160 & 552 & 624 & 624 \\
1249 & 138 & 138 & 274 & 304 & 0 & 1468 & 274 & 649 & 105 \\
1468 & 112 & 112 & 112 & 160 & 1468 & 0 & 112 & 508 & 508 \\
138 & 876 & 604 & 1012 & 552 & 274 & 112 & 0 & 634 & 226 \\
105 & 226 & 226 & 634 & 624 & 649 & 508 & 634 & 0 & 577 \\
649 & 634 & 634 & 226 & 624 & 105 & 508 & 226 & 577 & 0
\end{bmatrix}
$$

```
> Balance(Rayleigh(Bases(N10,Y),AllPairs));
```

$$
\begin{bmatrix}
0 & 262 & 1012 & 390 & 562 & 946 & 604 & 262 & 262 & 1204 \\
262 & 0 & 514 & 480 & 814 & 262 & 838 & 814 & 514 & 838 \\
1012 & 514 & 0 & 930 & 814 & 262 & 88 & 514 & 364 & 838 \\
390 & 480 & 930 & 0 & 930 & 390 & 210 & 930 & 930 & 210 \\
562 & 814 & 814 & 930 & 0 & 262 & 388 & 814 & 514 & 238 \\
946 & 262 & 262 & 390 & 262 & 0 & 1204 & 562 & 1012 & 604 \\
604 & 838 & 88 & 210 & 388 & 1204 & 0 & 238 & 838 & 1021 \\
262 & 814 & 514 & 930 & 814 & 562 & 238 & 0 & 814 & 388 \\
262 & 514 & 364 & 930 & 514 & 1012 & 838 & 814 & 0 & 88 \\
1204 & 838 & 838 & 210 & 238 & 604 & 1021 & 388 & 88 & 0
\end{bmatrix}
$$

```
>  Balance(Rayleigh(Bases(P10,Y),AllPairs));
```

$$
\begin{bmatrix}
0 & 417 & 417 & 313 & 316 & 1009 & 609 & 313 & 760 & 1201 \\
417 & 0 & 1009 & 609 & 760 & 417 & 313 & 1201 & 316 & 313 \\
417 & 1009 & 0 & 1201 & 760 & 417 & 313 & 609 & 316 & 313 \\
313 & 609 & 1201 & 0 & 552 & 313 & 225 & 1113 & 700 & 225 \\
316 & 760 & 760 & 552 & 0 & 316 & 700 & 552 & 336 & 700 \\
1009 & 417 & 417 & 313 & 316 & 0 & 1201 & 313 & 760 & 609 \\
609 & 313 & 313 & 225 & 700 & 1201 & 0 & 225 & 552 & 1113 \\
313 & 1201 & 609 & 1113 & 552 & 313 & 225 & 0 & 700 & 225 \\
760 & 316 & 316 & 700 & 336 & 760 & 552 & 700 & 0 & 552 \\
1201 & 313 & 313 & 225 & 700 & 609 & 1113 & 225 & 552 & 0
\end{bmatrix}
$$

```
>  Balance(Rayleigh(Bases(R10,Y),AllPairs));
```

$$
\begin{bmatrix}
0 & 564 & 384 & 792 & 384 & 564 & 396 & 852 & 972 & 852 \\
564 & 0 & 316 & 564 & 1096 & 508 & 828 & 1096 & 600 & 316 \\
384 & 316 & 0 & 852 & 913 & 1096 & 621 & 289 & 411 & 913 \\
792 & 564 & 852 & 0 & 852 & 564 & 396 & 384 & 972 & 384 \\
384 & 1096 & 913 & 852 & 0 & 316 & 621 & 913 & 411 & 289 \\
564 & 508 & 1096 & 564 & 316 & 0 & 828 & 316 & 600 & 1096 \\
396 & 828 & 621 & 396 & 621 & 828 & 0 & 621 & 1071 & 621 \\
852 & 1096 & 289 & 384 & 913 & 316 & 621 & 0 & 411 & 913 \\
972 & 600 & 411 & 972 & 411 & 600 & 1071 & 411 & 0 & 411 \\
852 & 316 & 913 & 384 & 289 & 1096 & 621 & 913 & 411 & 0
\end{bmatrix}
$$

```
>  Balance(Rayleigh(Bases(T10,Y),AllPairs));
```

$$\begin{bmatrix} 0 & 355 & 1150 & 355 & 355 & 864 & 355 & 405 & 1150 & 405 \\ 355 & 0 & 199 & 592 & 1039 & 355 & 1039 & 795 & 795 & 199 \\ 1150 & 199 & 0 & 795 & 795 & 405 & 199 & 418 & 865 & 418 \\ 355 & 592 & 795 & 0 & 1039 & 355 & 1039 & 199 & 199 & 795 \\ 355 & 1039 & 795 & 1039 & 0 & 355 & 592 & 795 & 199 & 199 \\ 864 & 355 & 405 & 355 & 355 & 0 & 355 & 1150 & 405 & 1150 \\ 355 & 1039 & 199 & 1039 & 592 & 355 & 0 & 199 & 795 & 795 \\ 405 & 795 & 418 & 199 & 795 & 1150 & 199 & 0 & 418 & 865 \\ 1150 & 795 & 865 & 199 & 199 & 405 & 795 & 418 & 0 & 418 \\ 405 & 199 & 418 & 795 & 199 & 1150 & 795 & 865 & 418 & 0 \end{bmatrix}$$

```
>  Balance(Rayleigh(Bases(X10,Y),AllPairs));
```

$$\begin{bmatrix} 0 & 864 & 864 & 864 & 864 & 864 & 864 & 864 & 864 & 864 \\ 864 & 0 & 864 & 864 & 864 & 864 & 864 & 864 & 864 & 864 \\ 864 & 864 & 0 & 864 & 864 & 864 & 864 & 864 & 864 & 864 \\ 864 & 864 & 864 & 0 & 864 & 864 & 864 & 864 & 864 & 864 \\ 864 & 864 & 864 & 864 & 0 & 864 & 864 & 864 & 864 & 864 \\ 864 & 864 & 864 & 864 & 864 & 0 & 864 & 864 & 864 & 864 \\ 864 & 864 & 864 & 864 & 864 & 864 & 0 & 864 & 864 & 864 \\ 864 & 864 & 864 & 864 & 864 & 864 & 864 & 0 & 864 & 864 \\ 864 & 864 & 864 & 864 & 864 & 864 & 864 & 864 & 0 & 864 \\ 864 & 864 & 864 & 864 & 864 & 864 & 864 & 864 & 864 & 0 \end{bmatrix}$$

Bibliography

[1] R. E. Bixby, *On Reid's characterisation of the Ternary Matroids*, Journal of Combinatorial Theory (B) **26** (1979) 174-203.

[2] T. H. Brylawski, *A note on Tutte's unimodular representation theorem*, Proc. Amer. Math. Soc. **52** (1975), 499-502.

[3] T. H. Brylawski and D. Lucas, *Uniquely representable combinatorial geometries*, Teorie Combinatoire in "Proc. 1973 International Colloquim" Accademia Nazionale dei Lincei, Rome, 1976.

[4] Y.-B. Choe, J. G. Oxley, A. D. Sokal and D. G. Wagner, *Homogeneous polynomials with the half-plane property*, Advanced Applied Math 32 (2004), 88–187.

[5] Y.-B. Choe and D.G. Wagner, *Rayleigh matroids*, Combinatorics, Probability and Computing **15** (2006), 765–781.

[6] J. Edmonds, *Submodular functions, matroids and certain polyhedra*, Combinatorial structures and their applications, Proc. Calgary International Conference, 1969.

[7] T.Feder and M. Mihail, *Balanced matroids*, in "Proceedings of the 24th annual ACM (STOC)", Victoria B.C., ACM Press, New York, 1992.

[8] J. F. Geelen, B. Gerards, and A. Kapoor, *The excluded minors for GF(4)-representable matroids*, Journal of Combinatorial Theory (B) **80** (2000), 247-299.

[9] J. Kahn, *On uniqueness of matroid representations over GF(4)*, Bull. London Math. Soc. **20**, (1988), 5–10.

[10] R.J.Kingan, S.R. Kingan, and Wendy Myrvold, *On matroid generation*, Congressus Numerantium **164** (2003), 95–109.

[11] R.J. Kingan and S.R. Kingan, *A software system for matroids*, Graphs and Discovery, DIMACS Series in Discrete Mathematics and Theoretical Computer Science, (2005), 287–296.

[12] C. Merino, *Matroids, the Tutte polynomial and the chip firing game*, Ph.D. thesis, Somerville College, University of Oxford (1999).

[13] J.G. Oxley, "Matroid Theory," Oxford U.P., New York, 1992.

[14] R. Penmantle, *Towards a theopry of negative dependence*, "Probabilistic techniques in equilibrium and nonequilibrium statistical physics," J. Math. Physics **41** (2000), 1371-1390.

[15] S. Prajna, A. Papachristddoulou, and P. A. Parillo, *Introducing SOSTOOLS: A general purpose sum of squares programming solver*, in Proc. of the 41st IEEE Conference on Decision and Control, Las Vegas, Nevada (2002).

[16] P. D. Seymour, *Matroid representation over GF(3)*, Journal of Combinatorial Theory (B) **26** (1979), 159-173.

[17] P. D. Seymour, *Matroids and multicommodity flows*, European Journal of Combinatorics **2** (1980), 257-290.

[18] W. T. Tutte, *A homnotopy theorem for matroids, I, II*, Trans. Amer. Math. Soc. **88** (1958), 144-174.

[19] D.G. Wagner, *Rank-three matroids are Rayleigh*, The Electronic Journal of Combinatorics **12** (2005), N8.

[20] G. Whittle, *On matroids representable over GF(3) and other fields*, Trans. Amer. Math. Soc. **349** (1997), 579–603.