The Normal Distribution of $\omega(\varphi(m))$ in Function Fields

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Let $\omega(m)$ be the number of distinct prime factors of m. A celebrated theorem of Erdös-Kac states that the quantity

$$\frac{\omega(m) - \log\log m}{\sqrt{\log\log m}}$$

distributes normally. Let $\varphi(m)$ be Euler's φ-function. Erdös and Pomerance proved that the quantity

$$\frac{\omega(\varphi(m)) - \frac{1}{2}(\log\log m)^2}{\frac{1}{\sqrt{3}}(\log\log m)^{3/2}}$$

also distributes normally. In this thesis, we prove these two results. We also prove a function field analogue of the Erdös-Pomerance Theorem in the setting of the Carlitz module.

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Dedication

This is dedicated to my parents and my husband.

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Chapter 1

Introduction

Let's begin by recalling some definition:

Definition For $m \in \mathbb{N}$, we denote $\omega(m)$ to be the number of distinct prime divisors of m, and $\Omega(m)$ to be the total number of prime divisors of m counting multiplicity.

In 1920, Hardy and Ramanujan proved the following Theorem:

Theorem 1 (Hardy-Ramanujan) *For a given function g_m in m, if $g_m \to \infty$ as $m \to \infty$, we have*

$$\lim_{n\to\infty} \frac{1}{n} \# \left\{ m \leq n : \left| \frac{\omega(m) - \log\log m}{\sqrt{\log\log m}} \right| \leq g_m \right\} = 1$$

This theorem tells us that almost all integers have about $\log\log m$ distinct prime divisors, since we can choose some g_m such that for large m, $g_m\sqrt{\log\log m} = o(\log\log m)$.

In 1934, Turán gave a simplified proof of the Hardy-Ramanujan Theorem by an essentially probabilistic method concerning the frequency, though he didn't really know probability theory at that time. For $n \in \mathbb{N}$, Turán proved that

$$\sum_{m\leq n} (\omega(m) - \log\log n)^2 \ll n \log\log n,$$

from which one can derive Theorem 1. A generalization of this method can be found in [1].

In 1939, Erdös and Kac proved a refinement of the Hardy-Ramanujan theorem:

Theorem 2 (Erdös-Kac)

$$\lim_{n\to\infty} \frac{1}{n}\# \left\{ m \le n : \frac{\omega(m) - \log\log m}{\sqrt{\log\log m}} \le t \right\} = G(t),$$

where

$$G(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\frac{1}{2}u^2} du$$

is the normal distribution.

This showed how the $\omega(m)$ distributed around the central value $\log\log m$. In particular, Erdös and Kac made essential use of the sieve method of Brun and some crude probability theory[1].

We show below the Erdös-Kac Theorem implies the Hardy-Ramanujan theorem. We have that if the Erdös-Kac theorem is true, then

$$\lim_{n\to\infty} \frac{1}{n}\# \left\{ m \le n : \left| \frac{\omega(m) - \log\log m}{g_m\sqrt{\log\log m}} \right| > \varepsilon \right\} = 0,$$

since $\frac{\omega(m)-\log\log m}{g_m\sqrt{\log\log m}}$ has a limiting distribution function $G(t)$, and

$$\lim_{n\to\infty} \frac{1}{n}\# \left\{ m \le n : \left| \frac{1}{g_m} \right| > \varepsilon \right\} = 0,$$

That is,

$$\lim_{n\to\infty} \frac{1}{n}\# \left\{ m \le n : \left| \frac{\omega(m) - \log\log m}{g_m\sqrt{\log\log m}} \right| > \epsilon \right\} = 0$$

for any given ϵ. Let $\epsilon = 1$, we can get

$$\lim_{n\to\infty} \frac{1}{n}\# \left\{ m \le n : \left| \frac{\omega(m) - \log\log m}{\sqrt{\log\log m}} \right| < g_m \right\} = 1,$$

which is the Hardy-Ramanujan theorem.

Definition For a function $f(x)$, we say it is *strongly additive* if for any two numbers a and b, $(a,b) = 1$, $f(ab) = f(a) + f(b)$ and $f(p^\alpha) = f(p)$ for all $\alpha \ge 1$, p a prime number; It is *additive* if for any two numbers a and b, $(a,b) = 1$, $f(ab) = f(a)+f(b)$.

In 1954-1955, Kubilius and Shapiro proved a generalization of the Erdös-Kac Theorem:

Theorem 3 (Kubilius-Shapiro) *Let $f(m)$ be a real valued function and suppose that f is strongly additive. Let*

$$A(n) = \sum_{p \le n} \frac{f(p)}{p}, \qquad B(n) = \left(\sum_{p \le n} \frac{f(p)^2}{p} \right)^{1/2}.$$

Suppose that for any $\varepsilon > 0$, we have

$$\lim_{n \to \infty} \frac{1}{B(n)^2} \cdot \sum_{\substack{p \le n \\ |f(p)| > \varepsilon B(n)}} \frac{f(p)^2}{p} = 0.$$

Then for any real number n,

$$\lim_{n \to \infty} \frac{1}{n} \# \{ m \le n : \ f(m) - A(n) \le t B(n) \} = G(t).$$

That is, the normal value for $m \le n$ of $f(m)$ is $A(n)$ and the standard deviation is $B(n)$.

We can see many applications of Theorem 3 in the book of Elliot [1]. One can also consider to apply this theorem to functions which are not strongly additive. In 1985, by applying Brun's method, Erdös and Pomerance[2] proved a theorem regarding to the distribution of $\omega(\varphi(m))$, where φ is Euler's φ-function.

Theorem 4 (Erdös-Pomerance)

$$\lim_{n \to \infty} \frac{1}{n} \# \left\{ m \le n : \frac{\omega(\varphi(m)) - \frac{1}{2}(\log \log n)^2}{\frac{1}{\sqrt{3}}(\log \log n)^{3/2}} \le t \right\} = G(t).$$

In Chapter 2, we will prove the Erdös-Kac Theorem. Let

$$\delta_p(m) = \begin{cases} 1 & p \mid m \\ 0 & p \nmid m \end{cases}$$

where p is prime. Then

$$\omega(m) = \sum_p \delta_p(m).$$

We define independent random variables $\{X_p, \ p \text{ is prime}\}$ satisfying

$$X_p = \begin{cases} 1 & \text{with probability } \frac{1}{p}; \\ 0 & \text{with probability } 1 - \frac{1}{p}. \end{cases}$$

Then we can see that $\delta_p(m)$ and X_p behave similarly. Thus applying the Central Limit Theorem for $\sum_{p\le m} X_p$, it is possible that $\omega(m)$ is normally distributed.

In Chapter 3, we will give the proof of the Erdös-Pomerance Theorem, with the application of the Bombiere-Vinogradov Theorem. In order to apply this theorem, we will calculate

$$\sum_{p\le x}\Omega(p-1) \text{ and } \sum_{p\le x}\Omega^2(p-1)$$

at first. Then we apply the Kubilius-Shapiro Theorem to show that

$$\frac{\Omega(\varphi(m))-\frac{1}{2}(\log\log n)^2}{\frac{1}{\sqrt{3}}(\log\log n)^{3/2}}$$

distributed normally. Then since

$$\Omega(\varphi(m))-\omega(\varphi(m))$$

is small enough, Theorem 4 follows.

In Chapter 4, we introduce the Carlitz module and Euler's φ-function in the function field:

Definition Let R be a principal ideal domain, M be a finite R-module. Then we can write

$$M=\oplus_{i=1}^k R/c_iR, \text{ where } c_i\in R, c_i|c_{i-1}, i=2,3,\cdots,k.$$

For $a\in M$, We define

$$\varphi(M)=\Pi_{i=1}^k c_i.$$

Let $A=\mathbb{F}_q[T]$ be the polynomial ring over the finite field $\mathbb{F}_q$, where $q=p^m$ for some prime number p and $m\in\mathbb{N}$. To define the φ-function for $n\in A=\mathbb{F}_q[T]$, we need to define a non-trivial A-Module associated to n.

Definition Let $k=\mathbb{F}_q(T)$ be the rational function field over $\mathbb{F}_q$. Let τ be the Frobenius element defined by $\tau(X)=X^q$. We denote $k\{\tau\}$ the *twisted polynomial ring*, whose multiplication is defined by

$$\tau b=b^q\tau, \forall b\in k.$$

The *A-Carlitz module* C is the $\mathbb{F}_q$-algebra homomorphism

$$C: A\longrightarrow k\{\tau\}, f\mapsto C_f,$$

characterized by

$$C_T=T+\tau.$$

Definition Let B be a commutative k-algebra, B_+ the additive group of B. Using this A-Carlitz module, we can define a new multiplication of A on B as follows: for $f \in A$, $u \in B$,

$$f \cdot u := C_f(u),$$

denoted by $C(B)$, which is still an A module.

Given an $n \in A \setminus \{0\}$, the new A-module is $C(A/nA)$. If n is monic and $n = p_1^{r_1} \cdots p_u^{r_u}$, we have

$$C(A/nA) = C(A/p_1^{r_1}A) \times \cdots C(A/p_u^{r_u}A)$$

Then we have following facts: for p prime in A(Here and below, we will say p is prime instead), we have [1]

$$C(A/pA) \cong A/(p-1)A.$$

Also if $q \neq 2$, or $q = 2$ with $p \nmid t(t+1)$, we have

$$C(A/p^rA) \cong A/(p^r - p^{r-1})A;$$

If $q = 2$ with $p \mid t(t+1)$, then

$$C(A/p^rA) \cong \begin{cases} A/(p-1)A & r = 1; \\ A/t(t-1)A & r = 2; \\ A/t(t-1)A \oplus A/p^{r-2}A & r \geq 3. \end{cases}$$

Definition Under A-Carlitz module, in this chapter we still denote the corresponding *Euler's φ-function* by φ. For a prime polynomial $p \in A$ and $r \in \mathbb{N}$, we define

$$\varphi(p^r) := p^r - p^{r-1}.$$

Then for a general $n \in A$, we define

$$\varphi(n) = \prod_{i=1}^{r} (p_i - 1)p_i^{\alpha_i - 1}, \text{when } n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_r^{\alpha_r}.$$

Note that the $\varphi(n)$ is again a polynomial.

Definition For $n \in A$, we define the number of distinct prime factors of n by $\omega(n)$ and the number of prime factors of n counting multiplicity by $\Omega(n)$.

We will prove a function field analogue for the Erdös-Kac Theorem:

Theorem 5 (Prime Analogue of The Erdös-Kac Theorem)

$$\lim_{x\to\infty} \frac{1}{q^x} \# \left\{ p : \deg(p) = x, \frac{\omega(p-1) - \log x}{\sqrt{\log x}} \leq t \right\} = G(t).$$

We also prove a function field analogue for the Erdös-Pomerance Theorem:

Theorem 6 (Normal Distribution of $\omega(\varphi(m))$) *Let m be a monic polynomial in $\mathbb{F}_q[X]$ over the finite field $\mathbb{F}_q$, we have*

$$\lim_{x\to\infty} \frac{1}{q^x} \# \left\{ m \leq n : \deg(m) = x, \frac{\omega(\varphi(m)) - \frac{1}{2}(\log x)^2}{\frac{1}{\sqrt{3}}(\log x)^{\frac{3}{2}}} \leq t \right\} = G(t).$$

We will apply an analogue of the Kubilius-Shapiro theorem in the function field [9] to prove the function field analogue of the Erdös-Pomerance Theorem, following roughly the same procedures in the proof of Erdös-Pomerance Theorem.

Finally we remark that, since the difference between ω and Ω is very small, all theorems still hold if we change ω to Ω.

Chapter 2

The Erdös-Kac Theorem

In the original proof of the Erdös-Kac theorem, they used sieve methods which are difficult. Later, Halberstam proved this theorem using the method of moments [4][5]. His proof was further simplified by Billingsley[8]. Here we will follow the approach of Billingsley to prove the Erdös-Kac theorem:

Theorem 2(Erdös-Kac) We have

$$\lim_{n\to\infty} \frac{1}{n} \# \left\{ m \leq n : \frac{\omega(m) - \log\log m}{\sqrt{\log\log m}} \leq t \right\} = G(t),$$

where

$$G(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\frac{1}{2}u^2} du$$

is the normal distribution function.

In the following, we will give a heuristic explanation for the Erdös-Kac Theorem. Let P_n be the probability measure on the space of positive integers that places mass $1/n$ at each of $1, 2, ..., n$. Then the Erdös-Kac theorem can be represented as

$$\lim_{n\to\infty} P_n \left\{ m \leq n : \frac{\omega(m) - \log\log m}{\sqrt{\log\log m}} \leq t \right\} = G(t).$$

We let

$$\delta_p(m) = \begin{cases} 1 & p \mid m \\ 0 & p \nmid m \end{cases}$$

where p is prime. Then

$$\omega(m) = \sum_p \delta_p(m).$$

From the Central Limit Theorem, if $\delta_p(m)$'s are "independent", then $\omega(m)$ is normally distributed.

To see why the Erdös-Kac theorem is true, we note that

$$P_n\{m:\ a|m\} = \frac{1}{n}\left[\frac{n}{a}\right] \sim \frac{1}{a}, \qquad \text{when } n \text{ is large.}$$

For distinct primes $p_1, \ldots, p_k$,

$$p_i|m, \text{ for } i = 1, \ldots, k. \Longleftrightarrow \prod_i p_i|m.$$

Hence

$$\begin{aligned} P_n\left\{\bigcap_i \{m:\ \delta_{p_i}(m) = 1\}\right\} &= P_n\{m:\ \delta_{\prod_i p_i}(m) = 1\} \\ &= \frac{1}{n}\left[\frac{n}{\prod_i p_i}\right] \\ &\sim \prod_i \frac{1}{n}\left[\frac{n}{p_i}\right] \qquad \text{when } n \text{ is large} \\ &= \prod_i P_n\{m:\ \delta_i(m) = 1\}. \end{aligned}$$

Therefore when m is chosen randomly from $1, \ldots, n$ and n is large, the random variables $\delta_{p_1}(m), \ldots, \delta_{p_k}(m)$ are nearly independent and hence it is possible that they are normally distributed by the Central Limit Theorem.

2.1 Review of Probability Theory

Definition Let X be a random variable with a probability measure P. For $t \in \mathbb{R}$, the function $F(t) = P(X \leq t)$ is the *distribution function* of X, and $E\{X\} = \int_{-\infty}^{\infty} t dF(t)$ is the *expectation* of X.

Definition We say a sequence of random variables $\{D_n\}$ *converges in probability* P_n *to* 0, if for any $\epsilon > 0$, $\lim_{n\to\infty} P_n\{|D_n| > \epsilon\} = 0$, denoted by $D_n \xrightarrow{P_n} 0$.

Definition $\Phi(t)$ is the *limiting distribution function* for a sequence of random variables $\{D_n\}$ with distribution functions $F_n(t)$ respectively, if for any t where $\Phi(t)$ is continuous, we have

$$\lim_{n\to\infty} F_n(t) = \Phi(t).$$

We need to know some probability facts before proving the Erdös-Kac Theorem. We have the following lemmas:

Lemma 7 *Given a sequence of random variables* $\{D_n\}$*, if* $\lim\limits_{n\to\infty} E\{|D_n|\} = 0$*, then* $D_n \xrightarrow{P_n} 0$*.*

Lemma 8 *1)Given two sequences of random variables* $\{D_n\}$ *and* $\{U_n\}$*, if* $\lim_{n\to\infty} E\{|D_n|\} = 0$*, then* $\{U_n\}$ *has a given limiting distribution function* $\Phi(x)$ *if and only if* $U_n + D_n$ *does.*
2)If $D_n \xrightarrow{P_n} 0$*,* U_n *has a distribution function* Φ*, then* $D_nU_n \xrightarrow{P_n} 0$*.*
3)If random variables $A_n \xrightarrow{P_n} 1$*,* $B_n \xrightarrow{P_n} 0$*, then* U_n *has limiting distribution* Φ *if and only if* $A_nU_n + B_n$ *does.*

Lemma 9 $\Phi(t)$ *is determined by its moments* $\mu_r = \int_{-\infty}^{\infty} t^r d\Phi(t), r = 0, 1, 2, ...,$ *i.e, if the distribution function* F_n *satisfy* $\int_{-\infty}^{\infty} t^r dF_n(t) \to \mu_r$ *for* $r = 0, 1, 2, ...,$ *then* $F_n(t) \to \Phi(t)$ *for each* t*.*

Lemma 10 *If* $F_n(t) \to \Phi(t)$ *for each* t*, and if* $\int_{-\infty}^{\infty} t^{r+\epsilon} dF_n(t)$ *is bounded in* n *for some positive* ϵ*, then* $\int_{-\infty}^{\infty} t^r dF_n(t) \to \mu_r$*.*

Lemma 11 *Let* $\{U_n\}$ *be a sequence of independent uniformly bounded variables with mean* 0 *and finite variances* σ_i^2*. If* $\sum\limits_{i=1}^{n} \sigma_i^2$ *diverges to* ∞*, then the distribution of*

$$\sum_{i=1}^{n} \frac{U_i}{\left(\sum\limits_{i=1}^{n} \sigma_i^2\right)^{\frac{1}{2}}}$$

converges to $\Phi(t)$ *which is a special case of the central limit theorem.*

2.2 Outline of The Proof

Let $\{\alpha_n\}$ be a sequence of positive real numbers such that

$$\alpha_n = o(n^\epsilon), \qquad \text{for any } \epsilon > 0$$

and

$$\sum_{\alpha_n < p \leq n} \frac{1}{p} = o\left((\log\log)^{1/2}\right).$$

For example, we can choose α_n to be $n^{1/\log\log n}$. Then using the fact that $\sum_{p\leq x}\frac{1}{p} = \log\log x + O(1)$ we have

$$\sum_{\alpha_n < p \leq n} \frac{1}{p} = \log\log\log n + O(1).$$

Let $\{X_p,\ p \text{ is prime}\}$ be independent random variables satisfying

$$X_p = \begin{cases} 1 & \text{with probability } \frac{1}{p}; \\ 0 & \text{with probability } 1-\frac{1}{p}. \end{cases}$$

Let

$$S_n = \sum_{p\leq\alpha_n} X_p,$$

then

$$c_n = \sum_{p\leq\alpha_n}\frac{1}{p}, \qquad s_n = \sum_{p\leq\alpha_n}\frac{1}{p}\left(1-\frac{1}{p}\right)$$

are the mean and variance of S_n. Here S_n is correspondent to

$$\omega_n(m) = \sum_{p\leq\alpha_n}\delta_p(m)$$

.

To prove the Erdös-Kac Theorem, we will prove first that the following statements are equivalent(Part I of the Proof):
(1) The Erdös-Kac theorem:

$$\lim_{n\to\infty} P_n\left\{m : \frac{\omega(m)-\log\log m}{\sqrt{\log\log m}} \leq t\right\} = G(t).$$

(2) $\lim_{n\to\infty} P_n \left\{ m : \frac{\omega(m) - \log\log n}{\sqrt{\log\log n}} \le t \right\} = G(t).$

(3) $\lim_{n\to\infty} P_n \left\{ m : \frac{\omega_n(m) - \log\log n}{\sqrt{\log\log n}} \le t \right\} = G(t).$

(4) $\lim_{n\to\infty} P_n \left\{ m : \frac{\omega_n(m) - c_n}{s_n} \le t \right\} = G(t).$

Finally(Part II of the Proof) we will show that as $n \to \infty$,

$$E_n \left\{ \frac{(\omega_n - c_n)^r}{s_n^r} \right\} \to \mu_r$$

From the method of moments, the above claim implies that (4) in the Part I follows.

2.3 The Proof

Lemma 12

$$\lim_{n\to\infty} P_n \left\{ m : \frac{\omega(m) - \log\log m}{\sqrt{\log\log m}} \le t \right\} = G(t)$$

if and only if

$$\lim_{n\to\infty} P_n \left\{ m : \frac{\omega(m) - \log\log n}{\sqrt{\log\log n}} \le t \right\} = G(t).$$

Proof: We have

$$\frac{\omega(m) - \log\log n}{\sqrt{\log\log n}} = \frac{\omega(m) - \log\log m}{\sqrt{\log\log m}} \frac{\sqrt{\log\log m}}{\sqrt{\log\log n}} + \frac{\log\log m - \log\log n}{\sqrt{\log\log n}}.$$

From Lemmas 8 and 9, it is sufficient to show that

$$\lim_{n\to\infty} P_n \left\{ m : \left| \frac{\sqrt{\log\log m}}{\sqrt{\log\log n}} - 1 \right| > \epsilon \right\} = 0,$$

and

$$\lim_{n\to\infty} P_n \left\{ m : \left| \frac{\log\log m - \log\log n}{\sqrt{\log\log n}} \right| > \epsilon \right\} = 0,$$

for any $\epsilon > 0$.

If $n^{1/2} < m \le n$, we have

$$\left| \frac{\sqrt{\log\log m}}{\sqrt{\log\log n}} - 1 \right| = \left| \frac{\sqrt{\log\log m} - \sqrt{\log\log n}}{\sqrt{\log\log n}} \right| > \epsilon,$$

which implies that

$$\sqrt{\log\frac{1}{2}+\log\log n} < \sqrt{\log\log m} < \sqrt{\log\log n} - \epsilon\sqrt{\log\log n},$$

which is $\log\log n < \frac{\log\frac{1}{2}}{1-(1-\epsilon)^2} = c_1(\epsilon)$;
$\left|\frac{\log\log m - \log\log n}{\sqrt{\log\log n}}\right| > \epsilon$ implies that

$$\sqrt{\log\log n} < \log\log m < \log\log n - \epsilon\sqrt{\log\log n},$$

that is,

$$\log\log n < \epsilon^{-2}(\log 2)^2 = c_2(\epsilon).$$

So when n is bigger than $e^{e^{max\{c_1(\epsilon),c_2(\epsilon)\}}}$, we have that the above two probabilities are both smaller than $P_n\left\{m : m \le n^{1/2}\right\}$, which tends to 0 as $n \to \infty$. Thus the Erdös-Kac Theorem is equivalent to

$$\lim_{n\to\infty} P_n\left\{m : \frac{\omega(m) - \log\log n}{\sqrt{\log\log n}} \le x\right\} = G.$$

Lemma 13

$$\lim_{n\to\infty} P_n\left\{m : \frac{\omega(m) - \log\log n}{\sqrt{\log\log n}} \le t\right\} = G(t)$$

if and only if

$$\lim_{n\to\infty} P_n\left\{m : \frac{\omega_n(m) - \log\log n}{\sqrt{\log\log n}} \le t\right\} = G(t).$$

Proof: For a function f of positive integers, let

$$E_n\{f\} = \frac{1}{n}\sum_{m=1}^{n} f(m)$$

denote its expected value with respect to P_n. Since

$$E_n\left\{\sum_{\alpha_n<p}\delta_p\right\} = \sum_{\alpha_n<p} P_n\{m : \delta_p(m)=1\} \le \sum_{\alpha_n<p\le n}\frac{1}{p} = o((\log\log n)^{1/2}),$$

Then from Lemma 7,

$$\frac{\sum_{\alpha_n < p} \delta_p}{(\log\log n)^{1/2}} \xrightarrow{P_n} 0.$$

Therefore since

$$\frac{\omega_n(m) - \log\log n}{(\log\log n)^{1/2}} = \frac{\omega(m) - \log\log n}{(\log\log n)^{1/2}} - \frac{\sum_{\alpha_n < p} \delta_p(m)}{(\log\log n)^{1/2}},$$

from Lemma 8, the lemma follows.

Lemma 14

$$\lim_{n\to\infty} P_n \left\{ m : \frac{\omega_n(m) - \log\log n}{\sqrt{\log\log n}} \leq t \right\} = G(t)$$

if and only if

$$\lim_{n\to\infty} P_n \left\{ m : \frac{\omega_n(m) - c_n}{s_n} \leq t \right\} = G(t).$$

Proof: We have that

$$c_n = \log\log n + O(1), \quad s_n^2 = \log\log n + O(1),$$

and

$$\begin{aligned} \frac{\omega_n(m) - \log\log n}{\sqrt{\log\log n}} &= \frac{\omega_n(m) - c_n}{s_n} + \left(\frac{\omega_n(m) - \log\log n}{\sqrt{\log\log n}} - \frac{\omega_n(m) - c_n}{s_n} \right) \\ &= \frac{\omega_n(m) - c_n}{s_n} + \frac{O(\omega_n(m) - \log\log n)}{\sqrt{\log\log n + O(1)}\sqrt{\log\log n}}. \end{aligned}$$

From Lemma 3, we need only to show that

$$\frac{O(\omega_n(m) - \log\log n)}{\sqrt{\log\log n + O(1)}\sqrt{\log\log n}} \xrightarrow{P_n} 0,$$

which is true since

$$\omega_n(m) - \log\log n = o(\log\log n).$$

Lemma 15 *We have*

$$\lim_{n\longrightarrow\infty} E_n \left\{ \frac{(\omega_n - c_n)^r}{s_n^r} \right\} - E_n \left\{ \frac{(S_n - c_n)^r}{s_n^r} \right\} = 0.$$

Proof: By the multinomial theorem and $S_n = \sum_{p\le\alpha_n} X_p$ we get that $E\left\{S_n^r\right\}$ is the sum

$$\sum_{u=1}^{r}\sum{}' \frac{r!}{r_1!\cdots r_u!}\sum{}'' E\left\{X_{p_1}^{r_1}\cdots X_{p_u}^{r_u}\right\},$$

where $\sum'$ denotes summing for all tuples $(r_1,\ldots,r_u)$ with $r_1,\ldots,r_u \ge 0$ and $r_1+\cdots+r_u = r$; $\sum''$ denotes summing for all $(p_1,\ldots,p_u)$ with $0\le p_1\le\cdots\le p_u\le\alpha_n$.

Notice that $X_p = 0$ or 1 and they are independent. Also note that p_i's are distinct. We have

$$E\left\{X_{p_1}^{r_1}\cdots X_{p_u}^{r_u}\right\} = E\left\{X_{p_1}\cdots X_{p_u}\right\} = \frac{1}{p_1\cdots p_u}.$$

Since $\omega_n(m) = \sum_{p\le\alpha_n}\delta_p(m)$, we get

$$E\left\{\omega_n^r(m)\right\} = \sum_{u=1}^{r}\sum{}' \frac{r!}{r_1!\cdots r_u!}\sum{}'' E_n\left\{\delta_{p_1}^{r_1}\cdots\delta_{p_u}^{r_u}\right\},$$

which is just the above one with the summand replaced by $E_n\left\{\delta_{p_1}^{r_1}\cdots\delta_{p_u}^{r_u}\right\}$. And similarly,

$$E_n\left\{\delta_{p_1}\cdots\delta_{p_u}\right\} = \frac{1}{n}\left[\frac{n}{p_1\cdots p_u}\right],$$

but

$$\left|\frac{1}{p_1\cdots p_n} - \frac{1}{n}\left[\frac{n}{p_1\cdots p_n}\right]\right| \le \frac{1}{n}.$$

Hence, for any r we have

$$\begin{aligned}\left|E\left\{S_n^r\right\} - E_n\left\{\omega_n^r\right\}\right| &\le \sum_{u=1}^{r}\sum{}' \frac{r!}{r_1!\cdots r_u!}\sum{}'' \frac{1}{n}\\ &\le \frac{1}{n}\left(\sum_{p\le\alpha_n} 1\right)^r \le \frac{\alpha_n^r}{n}.\end{aligned}$$

Now

$$E\left\{(S_n - c_n)^r\right\} = \sum_{k=0}^{r}\binom{r}{k} E\left\{S_n^k\right\}(-c_n)^{r-k},$$

$$E_n\left\{(\omega_n - c_n)^r\right\} = \sum_{k=0}^{r}\binom{r}{k} E_n\left\{\omega_n^k\right\}(-c_n)^{r-k}.$$

Thus

$$\begin{aligned}
|E\{(S_n - c_n)^r\} - E_n\{(\omega_n - c_n)^r\}| &\leq \sum_{k=0}^{r} \binom{r}{k} (-c_n)^{r-k} \frac{\alpha_n^k}{n} \\
&= \frac{1}{n}(\alpha_n + c_n)^r \longrightarrow 0 \qquad \text{as } n \to \infty,
\end{aligned}$$

since $\alpha_n = o(n^\epsilon)$ for any $\epsilon > 0$ and $c_n \leq \alpha_n$.

Now by Lemma 9, in order to prove that the distribution of $\frac{S_n - c_n}{s_n}$ converges to G, we need only to prove that $E\left\{\frac{(S_n-c_n)^r}{s_n^r}\right\} \to \mu_r$. Again by Lemma 10, it's true if the moment $\int_{-\infty}^{\infty} x^{r+\epsilon} dF_n(x)$,is bounded in n.

Lemma 16

$$\lim_{n\to\infty} E_n \left\{\frac{(S_n - c_n)^r}{s_n^r}\right\} \longrightarrow 0.$$

Proof: Actually we will show that for every r, we have

$$\lim_{n\to\infty} \sup_n \left| E\left\{\frac{(S_n - c_n)^r}{s_n^r}\right\}\right| < \infty.$$

Put $Y_p = X_p - \frac{1}{p}$. We have $E\left\{Y_p^2\right\} = E\left\{X_p^2 + \frac{1}{p^2} - \frac{2X_p}{p}\right\}$. Then $\{Y_p\}$ are independent. Hence

$$E\{(S_n - c_n)^r\} = \sum_{n=1}^{r} \sum{}' \frac{r!}{r_1! \cdots r_n!} \sum{}'' E\left\{Y_{p_1}^{r_1}\right\} \cdots E\left\{Y_{p_n}^{r_n}\right\},$$

where $\sum'$ extends over those u-tuples $(r_1, ..., r_u)$ satisfying $r_1 + ... + r_u = r$ and $\sum''$ extends over those u-tuples $(p_1, ..., p_u)$ of primes satisfying $p_1 < ... < p_u \leq \alpha_n$. Since $E\{Y_p\} = 0$, we can require in $\sum'$ above that $r_1, \ldots, r_n \geq 1$. Since $|Y_p| \leq 1$, $r_i \geq 2 \Rightarrow |E\left\{Y_p^{r_i}\right\}| \leq E\left\{Y_p^2\right\}$, the inner sum has modulus at most

$$\sum{}'' E\left\{Y_{p_1}^2\right\} \cdots E\left\{Y_{p_u}^2\right\} \leq \left(\sum_{p\leq\alpha_n} E\left\{Y_p^2\right\}\right)^n = s_n^{2n}.$$

But if $r_1, \ldots, r_u$ add to r, and each is at least 2, then $2u \leq r$. For n large enough that $s_n \geq 1$ now we have

$$|E\{(S_n - c_n)^r\}| \leq s_n^r \sum_{u=1}^{r} \sum{}' \frac{r!}{r_1! \cdots r_u!}.$$

Then

$$\sum_n \left| E\left\{\frac{(S_n - c_n)^r}{s_n^r}\right\}\right| < \infty.$$

Let

$$F_n(t) = P_n\left\{m : \frac{\omega_n(m) - c_n}{s_n} \le t\right\}, \quad \mu_r = \int_{-\infty}^{\infty} t^r dG(t), r = 0, 1, 2, ...$$

Since

$$\begin{aligned}
&\int_{-\infty}^{\infty} t^r dF_n(t) \\
=\ & \sum_{t=-\infty}^{\infty}\left\{\lim_{u\to\infty}\sum_{i=1}^{u}\left(t+\frac{i}{u}\right)^r\left(F_n\left(t+\frac{i}{u}\right) - F_n\left(t+\frac{i-1}{u}\right)\right)\right\} \\
=\ & \sum_{t=-\infty}^{\infty}\left\{\lim_{u\to\infty}\sum_{i=1}^{u}\left(t+\frac{i}{u}\right)^r P_n\left\{m :\ t+\frac{i-1}{u} < \frac{\omega_n(m)-c_n}{s_n} \le t+\frac{i}{u}\right\}\right\} \\
=\ & \sum_{m=1}^{n}\frac{1}{n}\left(\frac{\omega_n(m)-c_n}{s_n}\right)^r \\
=\ & E_n\left\{\frac{(\omega_n - c_n)^r}{s_n^r}\right\},
\end{aligned}$$

Then the rth moment of $F_n(x)$ is $E_n\left\{\frac{(\omega_n-c_n)^r}{s_n^r}\right\}$. Thus we have that the rth moment of $F_n(x)$ converges to μ_r. Combining Lemmas 12, 13, 14, 15 and 16, the Erdös-Kac theorem follows.

2.4 From $\omega(m)$ to $\Omega(m)$

It can be shown that Erdös-Kac and Hardy-Ramanujan Theorems hold also if each prime divisor is counted according to its multiplicity. That is

Corollary 17

$$\lim_{n\to\infty}\left\{m : \frac{\Omega(m) - \log\log m}{(\log\log m)} \le t\right\} = G(t).$$

Proof: Let $\delta'_p(m)$ be the exponent of p in the prime factorization of m, that is, $m = \prod_p p^{\delta'_p(m)}$. Define $\Omega(m) = \sum_p \delta'_p(m)$. For $k \geq 1$, $\delta'_p(m) - \delta_p(m) \geq k$ if and only if $p^{k+1}|m$, which is an event with P_n measure at most p^{-k-1}. Hence

$$E_n \left\{\delta'_p - \delta_p\right\} = \sum_{k=1}^{\infty} P_n \left\{m : \delta'_p(m) - \delta_p(m) \geq k\right\} \leq \frac{2}{p^2},$$

which implies $E_n \{\Omega - \omega\} = O(1)$.

So from Lemma 8, the Erdös-Kac theorem persists if $\omega(m)$ is replaced by $\Omega(m)$ and similarly we can successively deduce other parts with $\Omega(m)$ in place of $\omega(m)$.

Chapter 3

The Erdös-Pomerence Theorem

In this chapter, we will prove the Erdös-Pomerence Theorem:
Theorem 4(Erdös-Pomerence)

$$\lim_{x\to\infty}\frac{1}{x}\#\left\{n\le x:\frac{\Omega(\varphi(n))-\frac{1}{2}(\log\log x)^2}{\frac{1}{\sqrt{3}}(\log\log x)^{3/2}}\le t\right\}=G(t).$$

3.1 Preliminaries

We recall some results from analytic number theory and probability number theory which we needed for our proof of the normal distribution of $\omega(\varphi(n))$:

Lemma 18 (Partial Summation) *Given a sequence* $\{c_n\}, n=1,2,\ldots,$ *let* $C(x)=\sum_{i\le 1}c_i$. *Let* $f(x)$ *be a differentiable function. Then*

$$\sum_{i=1}^{x}c_if(i)=C(x)f(x)-\int_1^x C(u)df(u).$$

Lemma 19 *[3] If* $2\le k\le x$, *then*

$$\sum_{\substack{p\le x\\ p\equiv 1(\text{mod k})}}\frac{1}{p}=\frac{\log\log x}{\varphi(k)}+O\left(\frac{\log k}{\varphi(k)}\right).$$

Theorem 20 (Brun-Titchmarsh) *[11] Let a and q be relatively prime positive integers and let $\pi(x;q,a)$ denote the number of primes $p < x$ congruent to a mod q. Then we have for some constant $C(\varepsilon)$ only depends on some ε arbitrarily small,*

$$\pi(x,q,a) < \frac{C(\varepsilon)x}{\varphi(q)\log x}$$

as $x \longrightarrow \infty$, uniformly in a and q, subject to

$$q < x^{1-\varepsilon}.$$

Theorem 21 (Bombieri-Vinogradov) *For any $A > 0$, there exists a positive real number B=B(A) such that*

$$\sum_{d \le \frac{x^{1/2}}{(\log x)^B}} \max_{y \le x} \max_{(n,d)=1} \left| \pi(y;d,a) - \frac{\mathrm{li}(y)}{\varphi(d)} \right| \ll \frac{x}{(\log x)^A},$$

where

$$\pi(y;d,a) = \sum_{\substack{p \le y \\ p \equiv a (\mathrm{mod}\ d)}} 1, \mathrm{li}(x) = \int_2^x \frac{1}{\log t} dt.$$

Definition For a function $f(x)$, we say it is *strongly additive* if for any two numbers a and b, $f(a+b) = f(a) + f(b)$; It is *additive* if for any two numbers a and b, $(a,b) = 1$, $f(ab) = f(a) + f(b)$.

Theorem 3(Kubilius-Shapiro) Let $f(n)$ be a real valued function and suppose that f is strongly additive. Let

$$A(x) = \sum_{p \le x} \frac{f(p)}{p}, \qquad B(x) = \left(\sum_{p \le x} \frac{f(p)^2}{p} \right)^{1/2}.$$

Suppose that for any $\varepsilon > 0$, we have

$$\lim_{x \to \infty} \frac{1}{B(x)^2} \cdot \sum_{\substack{p \le x \\ |f(p)| > \varepsilon B(x)}} \frac{f(p)^2}{p} = 0.$$

Then for any real number u,

$$\lim_{x \to \infty} \frac{1}{x} \cdot \# \{ n \le x : \ f(n) - A(x) \le tB(x) \} = G(t),$$

that is, the expected value for $n \le x$ of $f(n)$ is $A(x)$ and the standard deviation is $B(x)$.

Lemma 22 (Turán-Kubilius Inequality) *[12] Let $f(n)$ be a real valued function and suppose that f is strongly additive. Let*

$$E(x) = \sum_{p \le x} \frac{f(p)}{p}, \qquad D(x) = \left(\sum_{p \le x} \frac{f(p)^2}{p} \right)^{1/2}.$$

Let λ_x denote the smallest number for which the inequality

$$\sum_{n=1}^{x} |f(n) - E(x)|^2 \le x \lambda_x D(x)^2$$

is always satisfied, and set

$$\lambda = \limsup_{x \to \infty} \lambda_x.$$

Then we have

$$1.47 \le \lambda \le 2.08.$$

3.2 Lemmas

Let $\Omega_y(n)$ be the total number of prime factors $p \le y$ of n, counting multiplicity. Note that $\Omega_y(n)$ is a completely additive function. Let $P(n)$ denote the largest prime factor of n and let p, q, r always denote primes.

Lemma 23 *We have*

$$\sum_{\substack{q^a \le x \\ a \ge 2 \\ q \le y}} \frac{1}{\varphi(q^a)} = O(1),$$

and

$$\sum_{p < x} \frac{1}{\varphi(p^2)} = C + O\left(\frac{1}{x \log x} \right) \text{ for some constant } C.$$

Proof:

$$\begin{aligned}
\sum_{\substack{q^a \le x \\ a \ge 2 \\ q \le y}} \frac{1}{\varphi(q^a)} &= \sum_{\substack{q^a \le x \\ a \ge 2 \\ q \le y}} \frac{1}{q^{a-1}(q-1)} \\
&\ll \sum_{\substack{q^a \le x \\ a \ge 2 \\ q \le y}} \frac{1}{q^a} \ll \sum_{q \le y} \left(\frac{1}{q^2} + \frac{1}{q^3} + \ldots + \frac{1}{q^{\frac{\log x}{\log 2}}} \right) \\
&\ll \sum_{q \le y} \frac{1}{q^2} = O(1).
\end{aligned}$$

Since from partial summation and $\pi(x) \sim \frac{x}{\log x}$,

$$\begin{aligned}
\sum_{p<x} \frac{1}{\varphi(p^2)} &\ll \sum_{p<x} \frac{1}{p^2} \\
&= \frac{x}{\log x} \frac{1}{x^2} - \int_2^x \frac{x}{\log x} d\frac{1}{x^2} \\
&= C + \frac{1}{x \log x}.
\end{aligned}$$

Lemma 24 *If* $3 \le y \le x$, *then*

$$\sum_{p \le x} \Omega_y(p-1) = \frac{x \log\log y}{\log x} + O\left(\frac{x}{\log x}\right).$$

Proof:

$$\begin{aligned}
\sum_{p \le x} \Omega_y(p-1) &= \sum_{p \le x} \sum_{\substack{q^a | p-1 \\ q \le y}} 1 \\
&= \sum_{\substack{q^a \\ q \le y}} \pi(x; q^a, 1) \\
&= \sum_{q \le y} \pi(x; q, 1) + \sum_{\substack{q^a,\, a \ge 2 \\ q \le y}} \pi(x; q^a, 1) \\
&= S_1 + S_2.
\end{aligned}$$

For S_1, consider two ranges for q: $q \le \min\left\{y, x^{1/3}\right\}$ and $\min\left\{y, x^{1/3}\right\} < q \le y$. We estimate the first range by the Bombieri-Vinogradov Theorem. Thus we have

$$\begin{aligned}\sum_{q\le \min\left\{y,x^{1/3}\right\}} \pi(x;q,1) &= \sum_{q\le \min\left\{y,x^{1/3}\right\}} \frac{\mathrm{li}x}{\varphi(q)} + O\left(\frac{x}{\log 2x}\right) \\ &= \frac{x\log\log y}{\log x} + O\left(\frac{x}{\log x}\right).\end{aligned}$$

For the second range of S_1, we have

$$\begin{aligned}\sum_{\min\left\{y,x^{1/3}\right\}<q\le y} \pi(x;q,1) &\le \sum_{q>x^{1/3}} \pi(x;q,1) \\ &= \sum_{p\le x}\sum_{\substack{q\mid p-1\\ q>x^{1/3}}} 1 \\ &\le 2\pi(x) \qquad \left(\text{since } \sum_{\substack{q\mid p-1\\ q>x^{1/3}}} 1\le 2\right) \\ &= O\left(\frac{x}{\log x}\right).\end{aligned}$$

For S_2 we break it into two parts also: $q^a \le x^{1/3}$ and $x^{1/3} < q^a \le x$.

$$\begin{aligned}\sum_{\substack{q^a\le x^{1/3},a\ge 2\\ q\le y}} \pi(x,q^a,1) &\ll \frac{x}{\log x}\sum_{\substack{q^a\le x^{1/3},a\ge 2\\ q\le y}} \frac{1}{\varphi(q^a)} (\text{from the Brun-Titchmarsh Theorem}) \\ &\ll \frac{x}{\log x} (\text{by Lemma 23}).\end{aligned}$$

Also we have

$$\sum_{\substack{q^a\ge x^{1/3},a\ge 2\\ q\le y}} \pi(x,q^a,1) \le \sum_{\substack{q^a\ge x^{1/3},a\ge 2\\ q\le y}} \frac{x}{q^a} \ll x^{5/6}.$$

Combine above, the lemma is proved.

Lemma 25 *If $3 \le y \le x$, then*

$$\sum_{p\le x} \Omega_y(p-1)^2 = \frac{x(\log\log y)^2}{\log x} + O\left(\frac{x\log\log y}{\log x}\right),$$

where the implied constant is uniform.

Proof: Let u range over the integers with $\omega(u) = 2$ and $P(u) \leq y$. Then

$$\begin{aligned}
\sum_{p\leq x} \Omega_y(p-1)^2 &= \sum_{p\leq x} \sum_{\substack{q^a||p-1 \\ q\leq y}} a^2 + \sum_{\substack{p\leq x \\ l,q\leq y \\ l\neq q}} \sum_{\substack{q^a||p-1 \\ l^b||p-1}} ab \\
&= S_3 + S_4.
\end{aligned}$$

We have

$$\begin{aligned}
S_3 &= \sum_{p\leq x} \Omega_y(p-1) + \sum_{p\leq x} \sum_{\substack{q^a||p-1 \\ q\leq y,\, a\geq 2}} (a^2 - a) \\
&\leq \sum_{p\leq x} \Omega_y(p-1) + \sum_{\substack{q^a\leq x^{1/3} \\ q\leq y,\, a\geq 2}} (a^2-a)\pi(x; q^a, 1) + \sum_{\substack{q^a > x^{1/3} \\ q\leq y,\, a\geq 2}} (a^2-a)\pi(x; q^a, 1) \\
&= O\left(\frac{x \log\log y}{\log x}\right).
\end{aligned}$$

Let $\mu(d)$ be the Möbius function. Then

$$\begin{aligned}
S_4 &= \sum_{\substack{p\leq x \\ l,q\leq y \\ l\neq q}} \sum_{\substack{q^a||p-1 \\ l^b||p-1}} ab = \sum_{\substack{l,q\leq y \\ u=q^a l^b}} \sum_{\substack{p\leq x \\ u|p-1 \\ (u,(p-1)/u)=1}} ab \\
&= \sum_{\substack{l,q\leq y \\ u=q^a l^b}} \sum_{\substack{p\leq x \\ u|p-1}} \left(\sum_{d|(u,(p-1)/u)} \mu(d) \right) ab \\
&= \sum_{\substack{l,q\leq y \\ u=q^a l^b}} \sum_{d|u} \mu(d) \sum_{\substack{p\leq x \\ u|p-1 \\ d|(p-1)/u}} ab = \sum_{\substack{l,q\leq y \\ u=q^a l^b}} \sum_{d|u} \mu(d)\pi(x, du, 1)ab \\
&= \sum_{\substack{u\leq x^{1/6} \\ u=q^a l^b \\ l,q\leq y}} \sum_{d|u} \mu(d)\pi(x, du, 1)ab + \sum_{\substack{x^{1/6}<u \\ u=q^a l^b \\ l,q\leq y}} \sum_{d|u} \mu(d)\pi(x, du, 1)ab \\
&= S_{4.1} + S_{4.2}.
\end{aligned}$$

For $S_{4.1}$, we have

$$\begin{aligned} S_{4.1} &= \sum_{\substack{u\le x^{1/6}\\ u=q^a l^b\\ l,q\le y}} \sum_{d|u} \mu(d)\pi(x,du,1)ab \\ &= \sum_{\substack{u\le x^{1/6}\\ u=ql\\ l,q\le y}} \sum_{d|u} \mu(d)\pi(x,du,1)1 + \sum_{\substack{u\le x^{1/6}\\ u=q^a l^b\\ l,q\le y\\ a\ge 2 \text{ or } b\ge 2}} \sum_{d|u} \mu(d)\pi(x,du,1)ab \\ &= S'_{4.1} + S''_{4.1}. \end{aligned}$$

From the Theorem 21, we have

$$\begin{aligned} S'_{4.1} &= \operatorname{li}(x) \sum_{\substack{u\le x^{1/6}\\ u=ql\\ l,q\le y}} \sum_{d|u} \frac{\mu(d)}{\varphi(du)} + O\left(\frac{x}{\log^2 x}\right) \\ &= \operatorname{li}(x) \sum_{\substack{u\le x^{1/6}\\ u=ql\\ l,q\le y}} \frac{1}{u} + O\left(\frac{x}{\log^2 x}\right) \\ &= \frac{x(\log\log y)^2}{\log x} + O\left(\frac{x\log\log y}{\log x}\right). \end{aligned}$$

Also from the Theorem 21,

$$\begin{aligned} S''_{4.1} &= \sum_{\substack{u\le x^{1/6}\\ u=q^a l^b\\ l,q\le y\\ a\ge 2 \text{ or } b\ge 2}} \sum_{d|u} \mu(d)\pi(x,du,1)ab \\ &\ll \operatorname{li}(x) \sum_{\substack{u\le x^{1/6}\\ u=q^a l^b\\ l,q\le y\\ a\ge 2, b\ge 2}} \frac{ab}{u} + 2\operatorname{li}(x) \sum_{\substack{ql^b\le x^{1/6}\\ q,l\le y\\ b\ge 2}} \frac{b}{ql^b}. \end{aligned}$$

Since

$$\begin{aligned}
\sum_{\substack{l^b\le x^{1/6}\\ l\le y\\ b\ge 2}} \frac{b}{l^b} &\le \sum_{l\le y}\sum_{2<b<\frac{\log x}{6\log l}} \frac{b}{l^b} \\
&\ll \sum_{l\le y} \frac{\left(\frac{\log x}{\log l}\right)^2}{l^{\frac{\log x}{\log l}}} \\
&\le \sum_{l\le y} \frac{2}{l^2} (\text{since } \frac{\log x}{\log l}\ge 2) \\
&= O(1),
\end{aligned}$$

We have

$$\begin{aligned}
S''_{4.1} &\ll \operatorname{li}(x)\sum_{q\le y}\frac{1}{q} \\
&\ll \frac{x}{\log x}.
\end{aligned}$$

For $S_{4.2}$, We can assume $du = q^a r^b$ for some natural numbers q, r, a, b, since $\omega(u) = 2$. Then

$$\begin{aligned}
S_{4.2} &\le 2\sum_{\substack{q^a<r^b\\ q^a r^b > x^{1/6}\\ q,r\le y}} \pi(x; q^a r^b, 1)ab \\
&\le 2\sum_{\substack{q^a\\ q\le y}}\sum_{\substack{r^b>x^{1/6}\\ b\ge 2}} \pi(x; q^a r^b, 1)ab + 2\sum_{\substack{q^a\\ q\le y}}\sum_{r>x^{1/12}} \pi(x; q^a r, 1)a.
\end{aligned}$$

Since we have

$$\begin{aligned}
2\sum_{\substack{q^a\\ q\le y}}\sum_{\substack{r^b>x^{1/6}\\ r\le y\\ b\ge 2}} \pi(x; q^a r^b, 1)ab &\le 2x\sum_{\substack{q^a\\ q\le y}}\sum_{\substack{r^b>x^{q/12}\\ r\le y\\ b\ge 2}} \frac{ab}{q^a r^b} \\
&\ll x^{23/24}\log\log y \\
&= o\left(\frac{x\log\log y}{\log x}\right),
\end{aligned}$$

and

$$\begin{aligned}
2\sum_{\substack{q^a\\ q\le y}}\sum_{r>x^{1/12}}\pi(x;q^ar,1)a &\le 2\sum_{p\le x}\sum_{\substack{q^a|p-1\\ q\le y}}\sum_{\substack{r|p-1\\ r>x^{1/12}}} a \ll \sum_{p\le x}\sum_{\substack{q^a|p-1\\ q\le y}} a \qquad \left(\text{since} \sum_{\substack{r>x^{1/12}\\ r|p-1}} 1\le 12\right)\\
&= \sum_{p\le x}\sum_{\substack{q^a\|p-1\\ q\le y\\ a>1}}(2+\cdots+a) = \sum_{p\le x}\sum_{\substack{q^a\|p-1\\ q\le y\\ a>1}}\frac{(a+2)(a-1)}{2}\\
&\ll \sum_{p\le x}\sum_{\substack{q^a\|p-1\\ q\le y\\ a>1}} a^2 \ll \sum_{\substack{q\le y\\ a>1}}\sum_{p\le x, q^a|p-1} a^2\pi x, q^a, 1\\
&= \sum_{\substack{q\le y\\ a\ge 2}} a^2\frac{x}{\varphi(q^a)\log x} \ll \frac{x}{\log x},
\end{aligned}$$

Then $S_{4.2}=O\left(\frac{x\log\log y}{\log x}\right)$. Combine above we get the lemma.

Lemma 26 *If* $3\le y\le x$, *then*

$$\sum_{p\le x}\frac{\Omega_y(p-1)}{p} = \log\log x\log\log y - \frac{1}{2}(\log\log y)^2 + O(\log\log x),$$

where the implied constant is uniform.

Proof: By the partial summation and Lemma 24, we have

$$\begin{aligned}
&\sum_{p\le x}\frac{\Omega_y(p-1)}{p}\\
=\ &\frac{1}{x}\sum_{p\le x}\Omega_y(p-1) + \int_2^x \frac{1}{t^2}\sum_{p\le t}\Omega_y(p-1)\mathrm{d}t\\
=\ &O\left(\frac{\log\log y}{\log x}\right) + \int_2^y \frac{\log\log t}{t\log t}\mathrm{d}t + \int_y^x\frac{\log\log y}{t\log t}\mathrm{d}t + O\left(\int_2^x\frac{\mathrm{d}t}{t\log t}\right)\\
=\ &O\left(\frac{\log\log y}{\log x}\right) + \frac{1}{2}\left((\log\log y)^2-(\log\log 2)^2\right)\\
+\ &(\log\log y\log\log x - \log\log y\log\log y) + O(\log\log x - \log\log 2)\\
=\ &\log\log x\log\log y - \frac{1}{2}(\log\log y)^2 + O(\log\log x).
\end{aligned}$$

Lemma 27 *If $3 \le y \le x$, then*

$$\sum_{p\le x} \frac{\Omega_y(p-1)^2}{p} = \log\log x(\log\log y)^2 - \frac{2}{3}(\log\log y)^3 + O(\log\log x \log\log y),$$

where the implied constant is uniform.

Proof: By partial summation and Lemma 25, we have

$$\begin{aligned}
& \sum_{p\le x} \frac{\Omega_y^2(p-1)}{p} \\
= \quad & \frac{1}{x}\sum_{p\le x}\Omega_y^2(p-1) + \int_2^x \frac{1}{t^2}\sum_{p\le t}\Omega_y^2(p-1)\mathrm{d}t \\
= \quad & O\left(\frac{(\log\log y)^2}{\log x}\right) + \int_2^y \frac{(\log\log t)^2}{t\log t}\mathrm{d}t + \int_y^x \frac{(\log\log y)^2}{t\log t}\mathrm{d}t + O\left(\int_2^x \frac{\log\log y}{t\log t}\mathrm{d}t\right) \\
= \quad & O\left(\frac{(\log\log y)^2}{\log x}\right) + \frac{1}{3}\left((\log\log y)^3 - (\log\log 2)^3\right) \\
+ \quad & \left((\log\log y)^2\log\log x - (\log\log y)^3\right) + O(\log\log x\log\log y - \log\log 2\log\log y) \\
= \quad & \log\log x(\log\log y)^2 - \frac{2}{3}(\log\log y)^3 + O(\log\log x\log\log y).
\end{aligned}$$

3.3 The Erdös-Pomerance Theorem

In this section we use the Kubilius-Shapiro Theorem to the additive function $f(n) = \sum_{p|n}\Omega(p-1)$ to prove the Erdös-Pomerance Theorem for $\Omega(\varphi(n))$. As for $\omega(\varphi(n))$, we can prove that but for $o(x)$ choices of $n \le x$,

$$\Omega(\varphi(n)) - \omega(\varphi(n)) = O(\log\log x \log\log\log\log x).$$

So $\Omega(\varphi(n))$ and $\omega(\varphi(n))$ differ not much and then we can get the same result for $\omega(\varphi(n))$.

Theorem 4 (Erdös-Pomerance(For $\Omega(\varphi(n))$)) For any real number u, we have

$$\lim_{x\to\infty}\frac{1}{x}\cdot\#\left\{n \le x : \Omega(\varphi(n)) - \frac{1}{2}(\log\log x)^2 \le \frac{t}{\sqrt{3}}(\log\log x)^{3/2}\right\} = G(t).$$

We want to apply the Kubilius-Shapiro Theorem. But notice that $\Omega(\varphi(n))$ is not strongly additive, we can't apply the theorem directly. Let $f(n) = \sum_{p|n}\Omega(p-1)$.

Then $f(n)$ is strongly additive and does not differ very much from $\Omega(\varphi(n))$. To see this, write $n = p_1^{k_1} \cdots p_s^{k_s}$. We have

$$\begin{aligned} \Omega(\varphi(n)) &= (k_1 - 1) + \cdots + (k_s - 1) + \sum_{p|n} \Omega(p-1) \\ &= f(n) + (k_1 + \cdots + k_s) - s \\ &= f(n) + \Omega(n) - \omega(n). \end{aligned}$$

Note that $\Omega(n) - \omega(n)$ is normally $o(\log\log x)$ by the Hardy-Remanujan Theorem. So to prove the Erdös-Pomerance Theorem, we need only to prove

$$\lim_{x\to\infty} \frac{1}{x} \cdot \# \left\{ n \le x : \ f(n) - \frac{1}{2}(\log\log x)^2 \le \frac{t}{\sqrt{3}}(\log\log x)^{3/2} \right\} = G(t).$$

Apply the Kubilius-Shapiro Theorem to $f(n)$, by Lemma 24, we have

$$\begin{aligned} A(x) &= \sum_{p\le x} \frac{f(p)}{p} = \sum_{p\le x} \frac{\Omega(p-1)}{p} \\ &= \frac{1}{2}(\log\log x)^2 + O(\log\log x). \end{aligned}$$

By Lemma 25, we have

$$\begin{aligned} B(x)^2 &= \sum_{p\le x} \frac{f(p)^2}{p} = \sum_{p\le x} \frac{\Omega(p-1)^2}{p} \\ &= \frac{1}{3}(\log\log x)^3 + O((\log\log x)^2). \end{aligned}$$

Now we need only to verify

$$\lim_{x\to\infty} \frac{1}{\frac{1}{3}(\log\log x)^3} \sum_{\substack{p\le x \\ |\Omega(p-1)| > \varepsilon \frac{1}{\sqrt{3}}(\log\log x)^{3/2}}} \frac{\Omega(p-1)^2}{p} = 0.$$

Let $\varepsilon > 0$ be fixed. Let $T = \varepsilon(\log\log x)^{3/2}/\sqrt{3}$. From Erdös and Sarkozy [6], it follows that for any $y \ge 2$

$$\sum_{\substack{n\le y \\ \Omega(n)\ge T}} 1 \ll 2^{-T} T^4 y \log y.$$

Hence by the partial summation,

$$\begin{aligned}
\sum_{\substack{p\le x\\ \Omega(p-1)\ge T}} \frac{\Omega(p-1)^2}{p} &\le \sum_{\substack{n\le x\\ \Omega(n)\ge T}} \frac{\Omega(n)^2}{n} \\
&= x^{-1}\sum_{\substack{n\le x\\ \Omega(n)\ge T}} \Omega(n)^2 + \int_2^x t^{-2} \sum_{\substack{n\le t\\ \Omega(n)\ge T}} \Omega(n)^2 \mathrm{d}t \\
&\ll x^{-1}(\log x)^2 \sum_{\substack{n\le x\\ \Omega(n)\ge T}} 1 + \int_2^x t^{-1}(\log t)^2 \sum_{\substack{n\le t\\ \Omega(n)\ge T}} 1\mathrm{d}t \\
&\ll 2^{-T}T^4(\log x)^3 + 2^{-T}T^4 \int_2^x t^{-1}(\log t)^3 \mathrm{d}t \\
&\ll 2^{-T}T^4(\log x)^4 \\
&= o(1).
\end{aligned}$$

So the condition is satisfied and Erdös-Pomerance Theorem follows.

Let $\omega_y(n)$ denote the number of distinct prime factors of n which do not exceed y. To prove the theorem for $\omega(vi(n))$, we need to prove the two lemmas below. From now on, we always take $y = (\log\log x)^2$.

Lemma 28 *For all but $o(x)$ choices of $n \le x$,*

$$\Omega(\varphi(n)) - \Omega_y(\varphi(n)) = \omega(\varphi(n)) - \omega_y(\varphi(n)).$$

Proof: Write $n = p_1^{k_1}\cdots p_l^{k_l}$. Then we have

$$\varphi(n) = p_1^{k_1-1}\cdots p_l^{k_l-1}(p_1-1)\cdots(p_l-1).$$

If $\Omega(\varphi(n)) - \Omega_y(\varphi(n)) \ne \omega(\varphi(n)) - \omega_y(\varphi(n))$, suppose $p^2|\varphi(n)$ where $p > y$ and $u \le x$. Then p and n satisfy one of the below cases:

1) $p^3|n$;

2)there is some $q|n$ with $q \equiv 1(\mathrm{mod} p^2)$, then $p^2|p_1 - 1$, where $p_i - 1 = q$;

3) there are distinct q_1, q_2 with $q_1q_2|n$ and $q_1 \equiv q_2 \equiv 1(\mathrm{mod} p)$, then $p|p_i-1,\ p|p_j-1$, where $p_i - 1 = q_1,\ p_j - 1 = q_2$.

In the first case, the number of $n \le x$ is at most

$$\sum_{y>p} \frac{x}{p^3} = o\left(\frac{x}{y^2}\right) = o(x).$$

In the second case, by Lemma 19, the number of $n \le x$ is at most

$$\sum_{y<p<x} \sum_{\substack{q\le x\\ q\equiv 1 \pmod{p^2}}} \frac{x}{q} = \sum_{y<p<x} \frac{x\log\log x}{\varphi(p^2)} + O\left(\sum_{y<p<x} \frac{x\log p}{\varphi(p^2)}\right)$$

Thus by Lemma 23,

$$\begin{aligned}\sum_{y<p<x} \sum_{\substack{q\le x\\ q\equiv 1 \pmod{p^2}}} \frac{x}{q} &= O\left(\frac{x\log\log x}{y\log y}\right) + O\left(\frac{x}{y}\right)\\ &= o(x).\end{aligned}$$

In the third case, by partial summation and Lemma 19, the number of $n \le x$ is at most

$$\begin{aligned}\sum_{p>y} \sum_{\substack{q_1<q_2\le x\\ q_1\equiv q_2\equiv 1 \pmod{p}}} \frac{x}{q_1q_2} &\le \frac{1}{2}x\sum_{p>y}\left(\sum_{\substack{q\le x\\ q\equiv 1 \pmod{p}}} \frac{1}{q}\right)^2\\ &= \frac{1}{2}x\sum_{p>y}\left(\frac{\log\log x}{\varphi(p)} + O\left(\frac{\log p}{p}\right)\right)^2\\ &= O\left(\frac{x(\log\log x)^2}{y\log y}\right) + O\left(\frac{x\log\log x}{y}\right) + O\left(\frac{x\log\log y}{y}\right)\\ &= o(x).\end{aligned}$$

Thus combine above we proved this lemma.

Lemma 29 *For all but $o(x)$ choices of $n \le x$,*

$$0 \le \Omega_y(\varphi(n)) - \omega_y(\varphi(n)) \le 2\log\log x \log\log\log\log x.$$

Proof: Here we need to apply Lemma 22 to the additive function $\Omega_y(\varphi(n))$, with

$$E(x) := E_y(x) = \sum_{p^k\le x} \frac{\Omega_y(\varphi(p^k))}{p^k}\left(1-\frac{1}{p}\right), D(x) := D_y(x) = \sum_{p^k\le x} \frac{\Omega_y(\varphi(p^k))^2}{p^k}.$$

Then

$$\begin{aligned}
E_y(x) &= \sum_{p^k\le x} \frac{\Omega_y(\varphi(p^k))}{p^k}\left(1-\frac{1}{p}\right)\\
&= \sum_{p\le x} \frac{\Omega_y(p-1)}{p}\left(1-\frac{1}{p}\right) + \sum_{\substack{p^k\le x\\ k>1}} \frac{\Omega_y(\varphi(p^k))}{p^k}\left(1-\frac{1}{p}\right)\\
&= \log\log x\log\log y - \frac{1}{2}(\log\log y)^2 + O(\log\log x)\\
&\quad -\sum_{p\le x}\frac{\Omega_y(p-1)}{p^2} + \sum_{\substack{p^k\le x\\ k>1}} \frac{\Omega_y((p-1)p^{k-1})}{p^k}\left(1-\frac{1}{p}\right).
\end{aligned}$$

To estimate $\sum_{p\le x}\frac{\Omega_y(p-1)}{p^2}$, we have

$$\begin{aligned}
\sum_{p\le x}\frac{\Omega_y(p-1)}{p^2} &= \frac{1}{x^2}\sum_{p\le x}\Omega_y(p-1) - \int_1^x \sum_{p\le t}\Omega_y(p-1)\mathrm{d}\left(\frac{1}{t^2}\right)\\
&= \frac{\log\log y}{x\log x} + O\left(\frac{1}{x\log x}\right)\\
&\quad +2\int_1^x\left(\frac{t\log\log y}{\log t} + O\left(\frac{t}{\log t}\right)\right)t^{-3}\mathrm{d}t\\
&= O\left(\frac{\log\log y}{x\log x}\right),
\end{aligned}$$

also we have

$$\begin{aligned}
\sum_{\substack{p^k\le x\\ k>1}} \frac{\Omega_y((p-1)p^{k-1})}{p^k} &= O\left(\sum_{\substack{p^k\le x\\ k>1}}\left(\frac{k-1}{p^k} + \frac{\Omega_y(p-1)}{p^k}\right)\right)\\
&\ll O(\log\log x).
\end{aligned}$$

Thus

$$E_y(x) = \log\log x\log\log y_{\frac{1}{2}}(\log\log y)^2 + O(\log\log x).$$

Also

$$\begin{aligned} D_y(x) &= \sum_{p^k \le x} \frac{\Omega_y(\varphi(p^k))^2}{p^k} \\ &= \sum_{p \le x} \frac{\Omega_y(p-1)^2}{p} + O\left(\sum_{\substack{p \le x \\ k>1}} \frac{\Omega(\varphi(p^k))^2}{p^k} \right) \\ &= \log\log x (\log\log y)^2 - \frac{2}{3} (\log\log y)^3 + O(\log\log x \log\log y). \end{aligned}$$

Thus, by the Turán-Kubilius inequality ($\sum_{n \le x} (\Omega_y(\varphi(n)) - E_y(x))^2 \le 32 x D_y(x)^2$), we have

$$\Omega(\varphi(n)) - \omega(\varphi(n)) = \log\log x \log\log\log\log x + O(\log\log x).$$

Now we can prove the Erdös-Pomerance Theorem for $\omega(\varphi(n))$. If we can show that but for $o(x)$ choices of n with $n \le x$, we have $\Omega(\varphi(n)) - \omega(\varphi(n)) = o((\log\log x)^{3/2})$, then this theorem follows from the previous one immediately, and it is true from Lemma 28 and Lemma 29. Thus we finished the proof of the Erdös-Pomerance Theorem.

Remark. Let $\lambda(n)$ be the smallest positive integer such that $a^{\lambda(n)} \equiv 1 \mathrm{mod} n$ for all a with $\gcd(a, n) = 1$. We have

$$\prod_{p|\varphi(n)} p | \lambda(n), \qquad \lambda(n) | \varphi(n).$$

Then $\omega(\varphi(n)) = \omega(\lambda(n)) \le \Omega(\lambda(n)) \le \Omega(\varphi(n))$. Thus we can see that it is still true if we replace $\Omega(\varphi(n))$ in the Erdös-Pomerance theorem with $\omega(\lambda(n))$ or $\Omega(\lambda(n))$.

Chapter 4

A Function Field Analogue of Erdös-Pomerence Theorem

Since the Bombieri-Vinogradov theorem we used in the proof of the Erdös-Pomerence Theoremis an strong unconditional replacement for the GRH bound, and we do not need GRH in the function field to get a similar result, it is natural to ask whether we have an analogue of the Erdös-Pomerance theorem in the function field.

We need to introduce some definitions in the function field at first.

Definition Let R be a principal ideal domain, M be a finite R-module. Then we can write

$$M = \oplus_{i=1}^{k} R/c_i R, \ where \ c_i \in R, c_i | c_{i-1}, i = 2, 3, \cdots, k.$$

For $a \in M$, We define

$$\varphi(M) = \Pi_{i=1}^{k} c_i.$$

Let $A = \mathbb{F}_q[T]$ be the polynomial ring over the finite field $\mathbb{F}_q$, where $q = p^m$ for some prime number p and $m \in \mathbb{N}$. To define the φ-function for $n \in A = \mathbb{F}_q[T]$, we need to define a non-trivial A-Module associated to n.

Definition Let $k = \mathbb{F}_q(T)$ be the rational function field over $\mathbb{F}_q$. Let τ be the Frobenius element defined by $\tau(X) = X^q$. We denote $k(\tau)$ the *twisted polynomial ring*, whose multiplication is defined by

$$\tau b = b^q \tau, \forall b \in k.$$

The A-*Carlitz module* C is the $\mathbb{F}_q$-algebra homomorphism

$$C : A \longrightarrow k\{\tau\}, f \mapsto C_f,$$

characterized by

$$C_T = T + \tau.$$

Definition Let B be the commutative k-algebra, B_+ the additive group of B. Using this A-Carlitz module, we can define a new multiplication of A on B as follows: For $f \in A$, $u \in B$,

$$f \cdot u := C_f(u),$$

denoted by $C(B)$, which is still an A module.

Given an $n \in A \setminus \{0\}$, the new A-module is $C(A/nA)$. If n is monic and $n = p_1^{r_1} \cdots p_u^{r_u}$, we have

$$C(A/nA) = C(A/p_1^{r_1}A) \times \cdots C(A/p_u^{r_u}A)$$

Then we have following facts: for p prime in A, we have [1]

$$C(A/pA) \cong A/(p-1)A.$$

Also if $q \neq 2$, or $q = 2$ with $p \nmid t(t+1)$, we have

$$C(A/p^rA) \cong A/(p^r - p^{r-1})A;$$

If $q = 2$ with $p \mid t(t+1)$, then

$$C(A/p^rA) \cong \begin{cases} A/(p-1)A & r = 1; \\ A/t(t-1)A & r = 2; \\ A/t(t-1)A \oplus A/p^{r-2}A & r \geq 3. \end{cases}$$

Definition Under A-Carlitz module, in this chapter we still denote the corresponding *Euler's φ-function* by φ. Now we can define

$$\varphi(p^r) := p^r - p^{r-1}, \text{for any prime polynomial} p \in A, r \in \mathbb{N}.$$

Then we have

$$\varphi(n) = \Pi_{i=1}^r (p_i - 1)p_i^{\alpha_i - 1}, \forall n \in A = \mathbb{F}_q[T], and\ \ n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_r^{\alpha_r}.$$

We denote by $\omega(n)$ *the number of distinct prime divisors* of n for $n \in A$, and $\Omega(n))$ *the number of distinct prime divisors counting multiplicity* of n for $n \in A$. and

$$\omega(\varphi(p)) = \omega(p-1), \forall p \in A, p \text{ a prime polynomial}.$$

Definition For $x \in \mathbb{N}$, define

$$M(x) = \{m \in M, \deg(m) = x\}.$$

Let

$$P_x\{m : m \text{ satisfies some conditions}\}$$

denote the quantity

$$\frac{1}{|M(x)|}\#\{m \in M(x) : m \text{ satisfies some conditions}\}.$$

Notice that P_x is a probability measure on $M(x)$. Let f be a function from $M(x)$ to $\mathbb{R}$, then the expectation of f with respect to P_x is denoted by

$$E_x\{m : f(m)\} := \frac{1}{|M(x)|}\sum_{m\in M(x)} f(m).$$

In this chapter, we will prove

Theorem 5(Prime Analogue of The Erdös-Kac Theorem)

$$\lim_{x\to\infty} P_x\left\{p : \deg(p) = x, \frac{\omega(p-1) - \log x}{\sqrt{\log x}} \le t\right\} = G(t).$$

Theorem 6(Normal Distribution of $\omega(\varphi(m))$) Let m be a monic polynomial in $\mathbb{F}_q[T]$ over the finite field $\mathbb{F}_q$, we have

$$\lim_{x\to\infty} \frac{1}{q^x}\#\left\{m : \deg(m) = x, \frac{\omega(\varphi(m)) - \frac{1}{2}(\log x)^2}{\frac{1}{\sqrt{3}}(\log x)^{\frac{3}{2}}} \le t\right\} = G(t).$$

In this chapter, we will often use the lemma below :

Lemma 30 *[1] Let $A = \mathbb{F}_q[T]$, and p a prime polynomial in A. Suppose $a, m \in A$ are relatively prime and that m has positive degree. Consider the set of primes*

$$S_x(a, m) = \{p \in A \mid p \equiv a(\ m), \deg(p) = x\},$$

we let $\pi(x, a, m)$ denote the number of such primes. Then we have

$$\pi(x, a, m) = \frac{1}{\Phi(m)}\frac{q^x}{x} + O\left(\frac{q^{\frac{x}{2}}}{x}\right),$$

and

$$\pi(x) = \frac{q^x}{x} + O\left(\frac{q^{\frac{x}{2}}}{x}\right),$$

where $\pi(x)$ is the number of the set of prime polynomials in A of degree x.

In this chapter p will always denote a prime polynomial, m a monic polynomial in A, where $A = \mathbb{F}_q[T]$.

4.1 $\omega(p-1)$ to $\omega_y(p-1)$

To prove the Theorem 5, we consider to transform $\omega_y(p-1)$ of $\omega(p-1)$, where $\omega_y(\varphi(n)) = \sum_{\substack{\deg(p)\le y \\ p|\varphi(n)}} 1.$:

Lemma 31 *If $3 \le y \le x/2$, then*

$$\sum_{\deg(p)=x} \omega_y(p-1) = \frac{q^x}{x}\log y + O\left(\frac{q^x}{x}\right),$$

$$\sum_{\deg(p)=x} \omega_y^2(p-1) = \frac{q^x}{x}(\log y)^2 + O\left(\frac{q^x}{x}\log y\right),$$

and

$$\sum_{\deg(p)=x} (\omega_y(p-1) - \log y)^2 = O\left(\frac{q^x}{x}\log y\right).$$

Proof: Let l denote a prime polynomial too, we have

$$\begin{aligned}
\sum_{\deg(p)=x} \omega_y(p-1) &= \sum_{\deg(p)=x} \sum_{\substack{l|(p-1) \\ \deg(l)\le y}} 1 \\
&= \sum_{\deg(l)\le y} \sum_{\substack{p\equiv 1(l) \\ \deg(p)=x}} 1 \\
&= \sum_{\deg(l)\le y} \left(\frac{q^x}{x}\frac{1}{q^{\deg(l)}-1} + O\left(\frac{q^{x/2}}{x}\right)\right) \\
&= \frac{q^x}{x}\sum_{\deg(l)\le y}\frac{1}{q^{\deg(l)}-1} + \sum_{\deg(l)\le y} O\left(\frac{q^{x/2}}{x}\right).
\end{aligned}$$

Note that

$$\begin{aligned}\sum_{\deg(l)\le y} \frac{1}{q^{\deg(l)}-1} &= \sum_{n\le y} \frac{1}{q^n-1}\left(\frac{q^n}{n}+O\left(\frac{q^{n/2}}{n}\right)\right)\\ &= \sum_{n\le y}\left(\frac{1}{n}+O\left(\frac{1}{nq^{n/2}}\right)\right)\\ &= \log y + O(1).\end{aligned}$$

Also,

$$\begin{aligned}\sum_{\deg(l)\le y} \frac{q^{x/2}}{x} &= \sum_{n\le y} \frac{q^{x/2}}{x}\left(\frac{q^n}{n}\right)\\ &= \left(\frac{q^{x/2}}{x}\right) O\left(q^y\right)\\ &= O\left(\frac{q^x}{x}\right),\end{aligned}$$

Thus, by combining the above estimates, we get

$$\sum_{\deg(p)=x} \omega_y(p-1) = \frac{q^x}{x}\log y + O\left(\frac{q^x}{x}\right).$$

We let u range over the polynomials with $\omega(u)=2$, i.e, $u=l_1l_2$, l_1 and l_2 are prime polynomials, $l_1\neq l_2$. Then

$$\begin{aligned}\sum_{\deg(p)=x} \omega_y^2(p-1) &= \sum_{\deg(p)=x}\left(\sum_{\substack{l\mid p-1\\ \deg(l)\le y}} 1\right)^2\\ &= \sum_{\deg(l)\le y}\sum_{\substack{l\mid p-1\\ \deg(p)=x}} 1 + \sum_{\substack{\deg(l_1)\le y\\ \deg(l_2)\le y}}\sum_{\substack{u\mid p-1\\ u=l_1l_2, l_1\neq l_2\\ \deg(p)=x}} 1.\end{aligned}$$

For the first part, we have

$$\begin{aligned}
\sum_{\deg(l)\le y}\ \sum_{\substack{l|p-1\\ \deg(p)=x}} 1 &= \sum_{\deg(l)\le y} \pi(x,l,1)\\
&= \sum_{n\le y}\left(\left(\frac{q^n}{n}\right)+O\left(\frac{q^{n/2}}{n}\right)\right)\frac{1}{\Phi(l)}\frac{q^x}{x}\\
&= \frac{q^x}{x}\sum_{n\le y}\left(\frac{q^n}{n}+O\left(\frac{q^{n/2}}{n}\right)\right)\frac{1}{q^n-1}\\
&= \frac{q^x}{x}O\left(\sum_{n\le y}\frac{1}{n}\right)=O\left(\frac{q^x}{x}\log y\right);
\end{aligned}$$

For the second part, we have

$$\begin{aligned}
\sum_{\substack{\deg(l_1)\le y\\ \deg(l_2)\le y}}\ \sum_{\substack{u|p-1\\ u=l_1l_2,l_1\ne l_2\\ \deg(p)=x}} 1 &= \sum_{l_1,l_2}\ \sum_{u|p-1,u=l_1l_2} 1-\sum_{l_1,l_2}\ \sum_{\substack{u|p-1\\ u=l_1l_2\\ l_1\ne l_2}} 1\\
&= \sum_{m\le y}\sum_{n\le y}\frac{q^x}{x(q^{m+n}-1)}\left(\frac{q^n}{n}+O\left(\frac{q^{n/2}}{n}\right)\right)\left(\frac{q^m}{m}+O\left(\frac{q^{m/2}}{m}\right)\right)\\
&- \sum_{n\le y}\frac{q^x}{x(q^{2n-1})}\left(\frac{q^n}{n}+O\left(\frac{q^{n/2}}{n}\right)\right)^2\\
&= \frac{q^x}{x}\sum_{m\le y}\left(\frac{q^m}{m}\sum_{n\le y}\frac{q^n}{n(q^{m+n}-1)}\right)\\
&+ O\left(\frac{q^x}{x}\sum_{m\le y}\left(\frac{q^{m/2}}{m}\sum_{n\le y}\frac{q^n}{n(q^{m+n}-1)}\right)\right)\\
&+ O\left(\frac{q^x}{x}\sum_{m\le y}\left(\frac{q^m}{m}\sum_{n\le y}\frac{1}{q^{m+n}-1}\frac{q^{n/2}}{n}\right)\right)\\
&+ O\left(\frac{q^x}{x}\right).
\end{aligned}$$

Here we have

$$\begin{aligned}
\sum_{m\le y}\left(\frac{q^m}{m}\sum_{n\le y}\frac{q^n}{n(q^{m+n}-1)}\right) &= \sum_{m\le y}\left(\frac{q^m}{m}\sum_{n\le y}\frac{1}{q^m n}\right)+O\left(\sum_{m\le y}\frac{q^m}{m}\sum_{n\le y}\frac{1}{nq^{2m+n}}\right)\\
&= \log^2 y+O(\log y).
\end{aligned}$$

Also,

$$\sum_{m\le y}\left(O\left(\frac{q^{m/2}}{m}\right)\sum_{n\le y}\frac{q^n}{n(q^{m+n}-1)}\right)\ll O(\log y),$$

similarly,

$$\sum_{m\le y}\left(\frac{q^m}{m}\sum_{n\le y}\frac{1}{q^{m+n}-1}O\left(\frac{q^{n/2}}{n}\right)\right)\ll O(\log y).$$

So combine above we get

$$\sum_{\deg(p)=x}\omega_y^2(p-1)=\frac{q^x}{x}(\log y)^2+O\left(\frac{q^x}{x}\log y\right).$$

Then from above 2 results, we can get

$$\begin{aligned}\sum_{\deg(p)=x}(\omega_y(p-1)-\log y)^2 &= \sum_{\deg(p)=x}(\omega_y^2(p-1)+\log^2 y-2\omega_y(p-1)\log y)\\ &=\frac{q^x}{x}\log^2 y+O\left(\frac{q^x}{x}\log y\right)\\ &+\sum_{\deg(p)=x}\log^2 y-\left(2\log y\frac{q^x}{x}\log y+O\left(\frac{q^x}{x}\log y\right)\right)\\ &=O\left(\frac{q^x}{x}\log y\right).\end{aligned}$$

From now on, we let $y=\frac{x}{\log x}$. So then we have

$$\sum_{x\ge n\ge y}1/n=o(\sqrt{\log x}).$$

Let

$$\delta_l(p-1)=\begin{cases}1 & \text{if } l\mid p-1,\\ 0 & \text{if } l\nmid p-1,\end{cases}$$

So $\omega_y(p-1)=\sum_{deg(l)\le y}\delta_l(p-1)$.

Lemma 32 *Let p be a prime polynomial in $\mathbb{F}_q[T]$ over finite field $\mathbb{F}_q$, we have*

$$\lim_{x\to\infty} P_x \left\{ p : \deg(p) = x, \frac{\omega(p-1) - \log x}{\sqrt{\log x}} \le t \right\} = G(t)$$

if and only if

$$\lim_{x\to\infty} P_x \left\{ p : \deg(p) = x, \frac{\omega_y(p-1) - \log x}{\sqrt{\log x}} \le t \right\} = G(t).$$

Proof:

Note that

$$\frac{\omega(p-1) - \log x}{\sqrt{\log x}} = \frac{\omega_y(p-1) - \log x}{\sqrt{\log x}} + \frac{\omega(p-1) - \omega_y(p-1)}{\sqrt{\log x}}$$

and

$$\frac{\omega(p-1) - \omega_y(p-1)}{\sqrt{\log x}} = \frac{\sum_{\deg(l)\ge y} \delta_l(p-1)}{\sqrt{\log x}}.$$

Then from Lemma 2 in Chapter 1 we need only to prove that

$$\frac{\sum_{\deg(l)\ge y} \delta_l(p-1)}{\sqrt{\log x}} \xrightarrow{P_x} 0.$$

It's true since

$$\begin{aligned}
\omega(p-1) - \omega_y(p-1) &= \sum_{\substack{l \mid p-1 \\ \deg(l)\ge y}} 1 \\
&= \sum_{\deg(l)\ge y} P_x \left\{ p-1 : \delta_l(p-1) = 1 \right\} \\
&= \sum_{x\ge \deg(l)\ge y} \frac{1}{q^{\deg(l)} - 1} \\
&= \sum_{x\ge n\ge y} \frac{1}{q^n - 1} \left(\frac{q^n}{n} + O\left(\frac{q^{n/2}}{n} \right) \right) \\
&= O\left(\sum_{x\ge n\ge y} \frac{1}{n}. \right)
\end{aligned}$$

Since we have already $\sum_{x \geq n \geq y} 1/n = o(\sqrt{\log x})$ from the choice of y, thus we get the desired result.

4.2 The r-th moment of $\lim_{x\to\infty} P_x \left\{ \omega : \frac{\omega_y(p-1)-c_x}{s_x} \leq t \right\}$

For l a prime polynomial in A, let X_l be random variables which satisfy

$$P(X_l = 1) = \frac{1}{q^{\deg(l)} - 1},$$

$$P(X_l = 0) = 1 - \frac{1}{q^{\deg(l)} - 1}.$$

We let $S_y = \sum_{\deg(l) \leq y} X_l$, and let

$$c_x = E\{S_y\} = \log x + O(1),$$

$$s_x^2 = Var\{S_y\} = \log x + O(1).$$

Lemma 33

$$\lim_{x\to\infty} P_x \left\{ p : \deg(p) = x, \frac{\omega_y(p-1) - \log x}{\sqrt{\log x}} \leq t \right\} = G(t)$$

if and only if

$$\lim_{x\to\infty} P_x \left\{ p : \deg(p) = x, \frac{\omega_y(p-1) - c_x}{s_x} \leq t \right\} = G(t).$$

Proof: Since

$$\begin{aligned}
\frac{\omega_y(p-1) - \log x}{\sqrt{\log x}} &= \frac{\omega_y(p-1) - c_x}{s_x} + \left(\frac{\omega_y(p-1) - \log x}{\sqrt{\log x}} - \frac{\omega_y(p-1) - c_x}{s_x} \right) \\
&= \frac{\omega_y(p-1) - c_x}{s_x} + \frac{O(\omega_y(p-1) - \log x)}{\sqrt{\log x}(\sqrt{\log x + O(1)})} \\
&= \frac{\omega_y(p-1) - c_x}{s_x} + \frac{O(\omega_y(p-1) - \log x)}{\sqrt{\log x}(\sqrt{\log x} + O(1))}, \qquad (4.1)
\end{aligned}$$

then from lemma 2, we need only that

$$\frac{O(\omega_y(p-1))}{\sqrt{\log x}(\sqrt{\log x}+O(1))} \xrightarrow{P_x} 0.$$

This is true since $\omega_y(p-1) = o(y)$, and from the choice of y we get the desired result.

Now it remains to prove the lemma below for theorem 5:

Lemma 34 *For y given as before, we have*

$$\lim_{x\to\infty} P_x \left\{ p : \deg(p) = x, \frac{\omega_y(p-1) - c_x}{s_x} \leq t \right\} = G(t).$$

Proof: We will need to use the method of moments. Let

$$F(t) = \lim_{x\to\infty} P_x \left\{ \omega : \frac{\omega_y(p-1) - c_x}{s_x} \leq t \right\}.$$

The r-th moment of $F(t)$ is

$$\begin{aligned}
\int_{-\infty}^{+\infty} t^r F(t) &= \sum_{t=-\infty}^{\infty} \left\{ \lim_{x\to\infty} \sum_{i=1}^{x} \left(t + \frac{i}{x}\right)^r \left(F_x\left(t+\frac{i}{x}\right) - F_x\left(t + \frac{i-1}{x}\right)\right)\right\} \\
&= \sum_{t=-\infty}^{\infty} \lim_{x\to\infty} \sum_{i=1}^{x} \left(t+\frac{i}{x}\right)^r P_x \left\{ p : \left(t+\frac{i}{x}\right) < \frac{\omega_y(p-1)-c_x}{s_x} \leq t + \frac{i-1}{x}\right\} \\
&= \sum_{m=1}^{x} \frac{1}{x} \left(\frac{\omega_y(p-1)-c_x}{s_x}\right) \\
&= E_x \left\{ \left(\frac{\omega_y(p-1)-c_x}{s_x}\right)^r \right\}.
\end{aligned}$$

We know that

$$E\left\{S_y^r\right\} = \sum_{u=1}^{r} \sum{}' \frac{r!}{r_1!\ldots r_u!} \sum{}'' E\left\{ \left(\sum_{\deg(l)=p_1} X_l \right)^{r_1} \ldots \left(\sum_{\deg(l)=p_u} X_l \right)^{r_u} \right\},$$

where $\sum'$ denotes u-tuples $(r_1,\ldots,r_u)$ with $r_1+\ldots+r_u = r$, $r_i's \in \mathbb{N}\cup 0$, and $\sum''$ denotes u-tuples $(p_1,\ldots,p_u)$ with $p_1 < \ldots < p_u \leq y$.

Notice that

$$
\begin{aligned}
E\left\{\left(\sum_{\deg(l)=k_1} X_l\right)\cdots\left(\sum_{\deg(l)=k_u} X_l\right)\right\} &= \frac{\frac{q^{k_1}}{k_1}+O\left(\frac{q^{k_1/2}}{k_1}\right)}{q^{k_1}-1}\cdots\frac{\frac{q^{k_u}}{k_u}+O\left(\frac{q^{k_u/2}}{k_u}\right)}{q^{k_u}-1} \\
&= \left(\frac{1}{k_1}+O\left(\frac{1}{k_1q^{k_1/2}}\right)\right)\cdots\left(\frac{1}{k_u}+O\left(\frac{1}{k_uq^{k_u/2}}\right)\right),
\end{aligned}
$$

and

$$
\begin{aligned}
E_x\left(\sum_{\deg(l)=k_1}\delta_l\right) &= \sum_{\deg(l)=k_1} P_x\{p : l|p-1\} \\
&= \left(\frac{q^{k_1}}{k_1}+O\left(\frac{q^{k_1/2}}{k_1}\right)\right)\cdot\left(\frac{\frac{q^x}{q^{k_1}-1}+O\left(q^{\frac{x}{2}}\right)}{q^x}\right) \\
&= \frac{1}{k_1}+O\left(\frac{1}{k_1q^{k_1/2}}\right)+\frac{1}{q^{x/2}}\cdot\frac{q^{k_1/2}}{k_1},
\end{aligned}
$$

we have

$$
\begin{aligned}
&E_x\left\{\left(\sum_{\deg(l)=k_1}\delta_l\right)^{r_1}\cdots\left(\sum_{\deg(l)=k_u}\delta_l\right)^{r_u}\right\}-E\left\{\left(\sum_{\deg(l)=k_1}X_l\right)^{r_1}\cdots\left(\sum_{\deg(l)=k_u}X_l\right)^{r_u}\right\} \\
\ll\ & O\left(\frac{1}{q^{x/2}}\frac{q^{r_1k_1/2}}{k_1^{r_1}}\cdot\frac{q^{r_2k_2/2}}{k_2^{r_2}}\cdots\frac{q^{r_uk_u/2}}{k_u^{r_u}}\right) \\
\leq\ & O(q^{ry-x}).
\end{aligned}
$$

So

$$
\begin{aligned}
E(S_y^r)-E_x(\omega_y^r) &\leq \frac{1}{q^{x-ry}}\left(\sum_{n=1}^{y}\frac{q^y}{y}\right)^r \\
&\leq \frac{1}{q^{x-ry}}\left(\frac{q^{y+1}-1}{q-1}\right)^r,
\end{aligned}
$$

and then

$$
\begin{aligned}
|E(S_x-c_x)^r-E(\omega_x-c_x)^r| &\leq \sum_{k=0}^{r}\binom{r}{k}\left(\frac{1}{q^{x-ry}}\frac{q^{y+1}-1}{q-1}\right)^k c_x^{r-k} \\
&= \frac{1}{q^x}\left(\frac{(q^{y+1}-1)q^y}{q-1}+c_x\right)^r \\
&\longrightarrow 0, \text{as } x\longrightarrow\infty, \text{from the choice of } y.
\end{aligned}
$$

From Lemma 3 in Chapter 1, We now only need to prove

$$\sup_{y=y(x)}\left|E\left\{\left(\frac{S_y-E\left\{S_y\right\}}{\sqrt{Var\left\{S_y\right\}}}\right)^r\right\}\right|<\infty.$$

Lemma 35

$$\sup_{y=y(x)}\left|E\left\{\left(\frac{S_y-E\left\{S_y\right\}}{\sqrt{Var\left\{S_y\right\}}}\right)^r\right\}\right|<\infty.$$

Proof: Let $Y_l = X_l - \frac{1}{q^{\deg(l)}-1}$, Then

$$E\left\{(S_y-c_x)^r\right\}=\sum_{u=1}^{r}\sum{}' \frac{r!}{r_1!\ldots r_u!}\sum{}'' E\left\{\left(\sum_{\deg(l)=k_1}Y_l\right)^{r_1}\ldots\left(\sum_{\deg(l)=k_u}Y_l\right)^{r_u}\right\}.$$

Let $Z_{k_i}=\sum_{\deg(l)=k_i}Y_l$. Then

$$\begin{aligned}E(Z_{k_i}) &= E(Y_l)\left(\frac{q^{k_i}}{k_i}+O\left(\frac{q^{k_i/2}}{k_i}\right)\right)\\ &= \frac{q^{k_i}}{k_i}\left(1-\frac{2}{q^{k_i}-1}\right)\\ &= \frac{q^{k_i}}{k_i}-\frac{2}{k_i}.\end{aligned}$$

So from above we have

$$E\left\{(Z_{k_1})^{r_1}\ldots(Z_{k_u})^{r_u}\right\}=\frac{q^{k_1r_1+\ldots+k_ur_u}}{k_1^{r_1}\ldots k_u^{r_u}}$$

Then

$$E\left\{S_y^r\right\}=\left(\sum_{i=1}^{y}\frac{q^i}{i}\right)^r.$$

And

$$\begin{aligned}
s_x^2 &= Var\{S_y\} = E\left(S_y - E(S_y)^2\right) \\
&= E\left\{\left(\sum_{\deg(l)\leq y} Y_l\right)^2\right\} = E\left\{\left(\sum_{i=1}^{y} Z_i\right)^2\right\} \\
&= E\left\{\left(\sum_{1\leq i_1,i_2\leq y} Z_{i_1}Z_{i_2}\right)\right\} \\
&\sim \sum_{1\leq i_1,i_2\leq y} \frac{q^{i_1}q^{i_2}}{i_1 i_2} \\
&= \left(\sum_{i=1}^{y} \frac{q^i}{i}\right)^2.
\end{aligned}$$

Thus $\frac{E\{S_y^r\}}{S_x^r} = 1 + O(1) < \infty$ and the Lemma follows.

Lastly, we can combine above lemmas and then get the analogue for the Erdös-Kac Theorem in function field as we did in Chapter 1, which is Theorem 5:
Theorem 5 (Prime Analogue of The Erdös-Kac Theorem)

$$\lim_{x\to\infty} P_x\left\{p : \deg(p) = x, \frac{\omega(p-1) - \log x}{\sqrt{\log x}} \leq t\right\} = G(t).$$

4.3 The Normal Distribution of $\Omega(\phi(n))$

In this section, we prove the following theorem:
Theorem 6 (Normal Distribution of $\Omega(\varphi(m))$) Let m be a monic polynomial in $\mathbb{F}_q[X]$ over finite field $\mathbb{F}_q$, we have

$$\lim_{x\to\infty} \frac{1}{q^x}\#\left\{m : \deg(m) = x, \frac{\Omega(\varphi(m)) - \frac{1}{2}(\log x)^2}{\frac{1}{\sqrt{3}}(\log x)^{\frac{2}{3}}} \leq t\right\} = G(t).$$

To prove this theorem, we need a theorem which is similar to the Kubilius-Shapiro theorem, but is effective in function field [9]:

Theorem 36 (Zhang) *Let $h(m)$ be a strongly additive function on $\mathbb{F}_q[T]$. For $x \in \mathbb{N}$,*

$$A(x) = \sum_{\deg(p)\le x} \frac{h(p)}{q^{\deg(p)}},\ B(x) = \left(\sum_{\deg(p)\le x} \frac{h(p)^2}{q^{\deg(p)}} \right)^{\frac{1}{2}}.$$

If $\forall \varepsilon > 0$, we have

$$\lim_{x\to\infty} \frac{1}{B^2(x)} \sum_{\substack{\deg(p)\le x,\\ |h(p)|\ge \varepsilon B(x)}} \frac{h^2(p)}{q^{\deg(p)}} = 0,$$

then

$$\lim_{x\to\infty} \frac{1}{q^x} \# \left\{ \deg(m) = x, \frac{h(m) - A(x)}{B(x)} \le t \right\} = G(t).$$

To prove Theorem 6, We want to apply this theorem with

$$h(m) = \Omega(\varphi(m)).$$

However, to apply this theorem, $h(m)$ has to be additive. So instead of $\Omega(\varphi(m))$, we apply the theorem with additive function $f(n) = \sum_{p|n} \Omega(p-1)$, and we will prove that the difference between them is small enough. In order to apply this theorem, we also need $A(x)$ and $B(x)$ to satisfy the requirements:

$$\lim_{x\to\infty} \frac{1}{B(x)^2} \sum_{\substack{q^{\deg(p)} \le x\\ |f(p)| > \varepsilon B(x)}} \frac{f(p)^2}{q^{\deg(p)}} = 0,$$

where

$$A(x) = \sum_{\deg(p)\le x} \frac{f(p)}{q^{\deg(p)}},\ \ B(x) = \left(\sum_{deg(p)\le x} \frac{f(p)^2}{q^{\deg(p)}} \right)^{\frac{1}{2}}.$$

4.4 Lemmas

At first we need to calculate for $\sum_{p\le x} \Omega_y(p-1)$ and $\sum_{p\le x} \Omega_y^2(p-1)$,

Lemma 37 *Let n be a fixed polynomial in $\mathbb{F}_q[T]$, $a \ge 2$ and $\pi(x, n, 1) = \sum_{\substack{\deg(p)=x\\ x\equiv 1(n)}} 1$.*

Then we have

$$\sum_{\deg(n)\le y} \pi(x, n^a, 1) = O(1).$$

Proof: Since we have

$$\pi(x,n,1)=\frac{1}{\Phi(n)}\frac{q^x}{x}+O\left(\frac{q^{x/2}}{x}\right),$$

and

$$\frac{1}{\Phi(n^a)}=\frac{1}{q^{(a-1)\deg(n)}(q^{\deg(n)}-1)},$$

then

$$\begin{aligned}\sum_{\deg(n)\le y}\pi(x,n^a,1) &\ll \sum_{n=1}^{y}\frac{1}{q^{(a-1)n}(q^n-1)}\left(\frac{q^n}{n}\right)^2\\ &\ll \sum_{n=1}^{y}\frac{q^n}{n^2(q^n-1)}\\ &\ll \sum_{n=1}^{y}\frac{1}{n^2}=O(1).\end{aligned}$$

Lemma 38

$$\sum_{\deg(p)\le x}\frac{\omega(p-1)}{q^{\deg(p)}}=\frac{1}{2}(\log n)^2+O(\log x),$$

and

$$\sum_{\deg(p)\le x}\frac{\omega^2(p-1)}{q^{\deg(p)}}=\frac{1}{3}\log^3 x+O(\log^2 x).$$

Proof: By Lemma 31

$$\begin{aligned}\sum_{\deg(p)\le x}\frac{\omega(p-1)}{q^{\deg(p)}} &= \sum_{n=1}^{x}\frac{\sum_{\deg(p)=n}\omega(p-1)}{q^n}\\ &= \sum_{n=1}^{x}\frac{\left(\frac{q^n}{n}\log n+O\left(\frac{q^n}{n}\right)\right)}{q^n}\\ &= \sum_{n=1}^{x}\left(\frac{\log n}{n}+O\left(\frac{1}{n}\right)\right)\\ &= \frac{1}{2}(\log x)^2+O(\log x).\end{aligned}$$

Also, we have

$$\begin{aligned}
\sum_{\deg(p)\le x} \frac{\omega^2(p-1)}{q^{\deg(p)}} &= \sum_{n=1}^{x} \frac{\sum_{\deg(p)=n} \omega^2(p-1)}{q^n} \\
&= \sum_{n=1}^{x} \frac{\frac{q^n}{n}\log^2 n + O\left(\frac{q^n}{n}\log n\right)}{q^n} \\
&= \sum_{n=1}^{x} \left(\frac{\log^2 n}{n} + O\left(\frac{\log n}{n}\right)\right) \\
&= \frac{1}{3}\log^3 x + O(\log^2 x).
\end{aligned}$$

Lemma 39 *For $2 < y \le x$, we have*

$$\sum_{\deg(p)=x} \Omega_y(p-1) = \frac{q^x}{x}\log y + O\left(\frac{q^x}{x}\right),$$

and

$$\sum_{\deg(p)=x} \Omega_y^2(p-1) = \frac{q^x}{x}(\log y)^2 + O\left(\frac{q^x}{x}\log y\right).$$

Proof: We have

$$\begin{aligned}
\sum_{\deg(p)=x} \Omega_y(p-1) &= \sum_{\deg(p)=x} \sum_{\substack{l^a|p-1 \\ \deg(l)\le y}} 1 \\
&= \sum_{\substack{l^a \\ \deg l\le y}} \pi(x, l^a, 1) \\
&= \sum_{\deg(l)\le y} \pi(x, l, 1) + \sum_{\substack{\deg(l)\le y \\ l^a,\, a\ge 2}} \pi(x, l^a, 1) \\
&= S_1 + S_2.
\end{aligned}$$

By Lemma 38,

$$S_1 = \sum_{\deg(p)=x} \omega_y(p-1) = \frac{q^x}{x}\log y + O\left(\frac{q^x}{x}\right).$$

We consider S_2 in two cases, $\deg(l^a) \le x^{1/3}$ and $\deg(l^a) > x^{1/3}$:

$$\begin{aligned} S_2 &= \sum_{\substack{\deg(l)\le y \\ \deg(l^a)\le x^{1/3}, a\ge 2}} \pi(x, l^a, 1) + \sum_{\substack{\deg(l)\le y \\ \deg(l^a)> x^{1/3}, a\ge 2}} \pi(x, l^a, 1) \\ &= S_{2,1} + S_{2,2}. \end{aligned}$$

For $S_{2,1}$, by Lemma 30 and 37, we have

$$S_{2,1} \ll \sum_{\substack{\deg(l^a)\le x^{1/3} \\ a\ge 2}} 1 \ll \frac{q^x}{x} \sum \frac{1}{\varphi(l^a)} \ll \frac{q^x}{x}.$$

For $S_{2,2}$, by Lemma 30, we have

$$\begin{aligned} S_{2,2} &= \sum_{\substack{\deg(l^a)> x^{1/3} \\ \deg l \le y \\ a\ge 2}} \pi(x, l^a, 1) \\ &= \sum_{2y\ge 2n\ge x^{1/3}} \left(\frac{1}{q^{(a-1)n}(q^n-1)} \cdot \frac{q^n}{n} + O\left(\frac{q^{n/2}}{n}\right)\right) \cdot \left(\frac{q^n}{n} + O\left(\frac{q^{n/2}}{n}\right)\right) \\ &\ll \sum_{2y\ge 2n\ge x^{1/3}} \frac{1}{q^{(a-1)n}(q^n-1)} \cdot \frac{q^{2n}}{n^2} \\ &\ll \sum_{2y\ge 2n\ge x^{1/3}} \frac{1}{q^{(a-3)n} n^2 (q^n-1)} \\ &= O(1). \end{aligned}$$

So we have that

$$\sum_{\deg(p)=x} \Omega_y(p-1) = \frac{q^x}{x} \log y + O\left(\frac{q^x}{x}\right).$$

Let u range over the polynomials in $\mathbb{F}_q[T]$ with $\omega(u) = 2$. Then

$$\sum_{\deg(p)=x} \Omega_y^2(p-1) = \sum_{\deg(p)=x} \sum_{\substack{l^a || p-1 \\ \deg(l)\le y}} a^2 + \sum_{\deg(p)=x} \sum_{u|p-1} k_1 k_2 = S_3 + S_4, \tag{4.2}$$

where $u = l_1^{k_1} l_2^{k_2}$ with l_1, l_2 distinct primes and $\deg(l_1)$, $\deg(l_2) \le y$.
Below we calculate S_3 and S_4 separately:

$$S_3 \le \sum_{\deg(p)=x} \Omega_y(p-1) + \sum_{\substack{\deg(l^a)\le x^{1/3} \\ \deg(l)\le y, a\ge 2}} (a^2-a)\pi(x, l^a, 1) + \sum_{\substack{\deg(l^a)> x^{1/3} \\ \deg(l)\le y, a\ge 2}} (a^2-a)\pi(x, l^a, 1). \tag{4.3}$$

For the first part of this sum, we apply Lemma 39; For the second sum, by Lemma 37 we have

$$\begin{aligned} \sum_{\substack{\deg(l^a)\le x^{1/3} \\ \deg(l)\le y, a\ge 2}} (a^2-a)\pi(x, l^a, 1) &\le \sum_{\substack{\deg(l)\le y \\ x^{1/3}\ge a\ge 2}} (a^2-a)\pi(x, l^a, 1) \\ &\le \sum_{x^{1/3}\ge a\ge 2} (a^2-a) = o(x); \end{aligned}$$

For the third part we have

$$\begin{aligned} \sum_{\substack{\deg(l^a)> x^{1/3} \\ \deg(l)\le y, a\ge 2}} (a^2-a)\pi(x, l^a, 1) &\ll \sum_{\substack{\deg(l^a)> x^{1/3} \\ \deg(l)\le y, a\ge 2}} (a^2-a)\frac{q^x}{x(q^{(a-1)\deg(l)})(q^{\deg(l)}-1)} \\ &\le \sum_{a\ge 2}(a^2-a) \sum_{x^{\frac{1}{3}}<n<y} \frac{q^x}{x(q^{(a-1)n})(q^n-1)} \\ &\le \frac{q^x}{x}\sum_{a\ge 2}(a^2-a)\frac{1}{q^{ax^{\frac{1}{3}}}} = O\left(\frac{q^x}{x}\right). \end{aligned}$$

Thus we have

$$S_3 = O\left(\frac{q^x}{x}\log y\right).$$

Also,

$$
\begin{aligned}
S_4 &= \sum_{\deg(p)=x} \sum_{u|p-1} k_1 k_2 \\
&= \sum_{\substack{\deg u \le x^{1/6} \\ \deg(l_1),\deg(l_2)\le y}} \sum_{\substack{u|p-1 \\ \deg(p)=x \\ u=l_1^{k_1} l_2^{k_2}}} k_1 k_2 + \sum_{\deg u \ge x^{1/6}} \sum_{\substack{u|p-1 \\ \deg(p)=x \\ \deg(l_1),\deg(l_2)\le y \\ u=l_1^{k_1} l_2^{k_2}}} k_1 k_2 \\
&\quad - \sum_{\substack{l^k \| p-1 \\ \deg(p)=x}} \sum_{i=1}^{k-1} i(k-i) \\
&= S_{4,1} + S_{4,2} - S_{4,3}.
\end{aligned}
$$

Here $S_{4,1}$ and $S_{4,2}$ are the component in S_4 containing u satisfying above requirements with degree smaller than and bigger than $x^{\frac{1}{6}}$ respectively.
We have

$$
S_{4,3} = \sum_{\substack{l^k \| p-1 \\ \deg(p)=x}} O(k^3),
$$

Since

$$
\begin{aligned}
\sum_{\substack{l^k \| p-1 \\ \deg(p)=x}} k^3 &= \sum_{\deg(p)=x} \Omega_y(p-1) + \sum_{\substack{\deg(l^k)\le x^{1/3} \\ \deg(l)\le y, k\ge 2}} (k^3-k)\pi(x, l^k, 1) \\
&\quad + \sum_{\substack{\deg(l^k)> x^{1/3} \\ \deg(l)\le y, k\ge 2}} (k^3-k)\pi(x, l^k, 1),
\end{aligned}
$$

From similar argument for S_3, we can get that $S_{4,3} = O\left(\frac{q^x}{x}\log y\right)$ too. Also we have

$$\begin{aligned}
S_{4,1} &= \sum_{\substack{\deg(u)\le x^{1/6} \\ \deg(l_1),\deg(l_2)\le y}} \sum_{\substack{d|u \\ u=l_1^{k_1}l_2^{k_2}}} k_1k_2\pi(x,du,1)\mu(d) \\
&= \frac{q^x}{x}\sum_{\substack{deg(u)\le x^{\frac{1}{6}} \\ \deg(l_1),\deg(l_2)\le y}} \sum_{d|u} \left\{k_1k_2\frac{\mu(d)}{\varphi(du)} + O\left(\frac{k_1k_2}{xq^{\frac{x}{2}}}\mu(d)\right)\right\} \\
&= \frac{q^x}{x}\sum_{\substack{\deg(u)\le x^{\frac{1}{6}} \\ \deg(l_1),\deg(l_2)\le y}} \frac{k_1k_2}{q^{k_1\deg(l_1)+k_2\deg(l_2)}} \\
&= \frac{q^x}{x}\sum_{n\le y}\sum_{m\le y}\frac{1}{q^{n+m}}\left\{\frac{q^{n+m}}{n+m} + O\left(\frac{q^{\frac{n+m}{2}}}{n+m}\right)\right\} \\
&= \frac{q^x}{x}(\log y)^2 + O\left(\frac{q^x}{x}\log y\right).
\end{aligned}$$

For $S_{4,2}$, we let $du = l_1^a l_2^b$. Since $\deg(u) > x^{1/6}$, we can assume that $\deg(l_2) > x^{1/12}$. Then

$$S_{4,2} = 2\sum_{\substack{l_1^a\|p-1 \\ l_2^b\|p-1 \\ \deg(l_1)\le y \\ a>1}} \sum_{\substack{y\ge\deg(l_2)>x^{1/12} \\ b\ge 2}} ab + 2\sum_{\deg(p)=x}\sum_{\substack{l_1^a\|p-1 \\ \deg(l_1)\le y \\ a>1}} \sum_{\substack{l_2\|p-1 \\ y\ge\deg(l_2)>x^{1/12}}} a.$$

For the first part we have

$$\begin{aligned}
\sum_{\substack{l_1^a\|p-1 \\ l_2^b\|p-1 \\ \deg(l_1)\le y \\ >xa>1}} \sum_{\substack{y\ge\deg(l_2)>x^{1/12} \\ x>b\ge 2}} ab &\ll \sum_{\substack{\deg(l_1)\le y \\ x>a\ge 2}} a\pi(x,l_1^a,1)\sum_{\substack{\deg(l_2)\le y \\ x>b\ge 2}} b\pi(x,l_2^b,1) \\
&\ll \left(\sum_{\substack{\deg(l_1)\le y \\ x>a\ge 2}} a\pi(x,l_1^a,1)\right)^2 \\
&\ll O\left(\frac{q^x}{x}\right) \text{ by Lemma 37.}
\end{aligned}$$

For the second part we have

$$\begin{aligned}
\sum_{\deg(p)=x} \sum_{\substack{l_1^a|p-1\\ \deg(l_1)\le y\\ a>1}} \sum_{\substack{l_2|p-1\\ \deg(l_2)>x^{1/12}}} a &\ll \sum_{\deg(p)=x} \sum_{\substack{l_1^a|p-1\\ \deg(l_1)\le y\\ a>1}} a \\
&= \sum_{\deg(p)=x} \sum_{\substack{l_1^a\|(p-1)\\ \deg(l_1)\le y\\ a>1}} (2+3+\cdot+a) \\
&\ll \sum_{\deg(p)=x} \sum_{\substack{l_1^a\|(p-1)\\ \deg(l_1)\le y\\ a>1}} a^2 + \sum_{\deg(p)\le x} \Omega_y(p-1).
\end{aligned}$$

Since

$$\sum_{\deg(p)\le x} \Omega_y(p-1) = O\left(\frac{q^x}{x}\log y\right),$$

also

$$\begin{aligned}
\sum_{\deg(p)=x} \sum_{\substack{l_1^a\|(p-1)\\ \deg(l_1)\le y\\ a>1}} a^2 &\ll \sum_{\deg(l_1)\le y} a^2 \sum_{\substack{\deg(p)=x\\ p\equiv 1(l_1^a)}} 1 \\
&= \sum_{\substack{\deg(l_1)\le y\\ 1<a<x}} a^2\pi(x, l_1^a, 1) \\
&\ll xO(1) \\
&= O(x),
\end{aligned}$$

We can have that

$$\sum_{\deg(p)=x} \Omega_y^2(p-1) = \frac{q^x}{x}(\log y)^2 + O\left(\frac{q^x}{x}\log y\right).$$

Lemma 40

$$\sum_{\substack{\deg(p)\le x\\ p\equiv 1(m)}} \frac{1}{q^{\deg(p)}} = \frac{1}{\Phi(m)}\log x + O(1),$$

$$\sum_{\deg(p)\le x} \frac{\Omega(p-1)}{q^{\deg(p)}} = \frac{1}{2}\log^2 x + O(\log x),$$

and

$$\sum_{\deg(p)\le x} \frac{\Omega^2(p-1)}{q^{\deg(p)}} = \frac{1}{3}\log^3 x + O(\log^2 x).$$

Proof:

$$\begin{aligned}
\sum_{\substack{\deg(p)\le x\\ p\equiv 1(m)}} \frac{1}{q^{\deg(p)}} &= \sum_{n=1}^{x} \sum_{\substack{\deg(p)=n\\ \omega\equiv 1(m)}} \frac{1}{q^{\deg(p)}} \\
&= \sum_{n=1}^{x} \left(\frac{1}{q^n} \sum_{\substack{\deg(p)=n\\ \omega\equiv 1(m)}} 1 \right) \\
&= \sum_{n=1}^{x} \left(\frac{1}{q^n} \left(\frac{1}{\Phi(m)} \frac{q^n}{n} + O\left(\frac{q^{n/2}}{n} \right) \right) \right) \\
&= \sum_{n=1}^{x} \left(\frac{1}{n\Phi(m)} + O\left(\frac{1}{nq^{n/2}} \right) \right) \\
&= \frac{1}{\Phi(m)} \log x + O(1).
\end{aligned}$$

Also,

$$\begin{aligned}
\sum_{\deg(p)\le x} \frac{\Omega(p-1)}{q^{\deg(p)}} &= \sum_{n\le x} \sum_{\deg(p)=x} \frac{\Omega(p-1)}{q^n} \\
&= \sum_{n\le x} \frac{\frac{q^n}{n}\log n + O\left(\frac{q^n}{n}\right)}{q^n} \\
&= \sum_{n\le x} \frac{\log n}{n} + O(\log x) \\
&= \frac{1}{2}\log^2 x + O(\log x) \text{ from partial summation.}
\end{aligned}$$

Lastly,

$$\begin{aligned}
\sum_{\deg(p)\le x} \frac{\Omega^2(p-1)}{q^{\deg(p)}} &= \sum_{n=1}^{x} \frac{\frac{q^n}{n}\log^2 n + O\left(\frac{q^n}{n}\log n\right)}{q^n} \\
&= \sum_{n=1}^{x} \frac{1}{n}\log^2 n + O\left(\frac{1}{n}\log x\right) \\
&= \frac{1}{3}\log^3 x + O(\log^2 x) \text{ from partial summation.}
\end{aligned}$$

Lemma 41 *Let* $g(m) = \sum_{p|m} \Omega(p-1)$, *then*

$$\lim_{x\to\infty} \frac{1}{q^x} \# \left\{ m : \deg(m) = x, \frac{\Omega(\varphi(m)) - \frac{1}{2}(\log x)^2}{\frac{1}{\sqrt{3}}(\log x)^{\frac{3}{2}}} \le t \right\} = G(t)$$

if and only if

$$\lim_{x\to\infty} \frac{1}{q^x} \# \left\{ m : \deg(m) = x, \frac{g(m) - \frac{1}{2}(\log x)^2}{\frac{1}{\sqrt{3}}(\log x)^{\frac{3}{2}}} \le t \right\} = G(t).$$

Proof: Note that for $m = p_1^{r_1} \cdots p_u^{r_u}$, we have

$$\varphi(m) = \prod_{i=1}^{u} (p_i - 1) p_i^{r_i - 1}.$$

Thus

$$\Omega(\varphi(m)) = \Omega\left(\prod_{i=1}^{u}(p_i - 1)\right) + \Omega\left(\prod_{i=1}^{u} p_i^{r_i-1}\right),$$

which gives us

$$\sum_{p|m} \Omega(p-1) \le \Omega(\varphi(m)) \le \sum_{p|m} \Omega(p-1) + \Omega(m).$$

Also we have $\Omega(m) \le x$ when $\deg(m) = x$. Then from Lemma 3 in chapter 1, we need that

$$\lim_{x\to\infty} \frac{1}{q^x} \# \left\{ m : \deg(m) = x, \left| \frac{\Omega(m)}{\frac{1}{\sqrt{3}}(\log x)^{\frac{3}{2}}} \right| > \varepsilon \right\} = 0.$$

Since the normal order of $\Omega(m)$ is $\log x$, we have that but for $o(q^x)$ number of monic polynomials m, with $\deg(m) = x$

$$\Omega(m) = (1 + o(1)) \log x = o\left(\frac{1}{\sqrt{3}}(\log x)^{\frac{3}{2}}\right).$$

Then we have

$$\frac{\Omega(x)}{(\log x)^{\frac{3}{2}}} = o\left(\frac{1}{\log^{\frac{1}{2}} x}\right) \longrightarrow 0 \text{ as } x \longrightarrow \infty,$$

and the result follows.

Lemma 42

$$\sum_{\deg(m)=x} \Omega(\varphi(m)) = \frac{1}{2} q^x (\log x)^2 + O(q^x \log x),$$

and

$$\sum_{\deg(m)=x} \Omega^2(\varphi(m)) = \frac{1}{4} q^x (\log x)^4 + O(q^x (\log x)^3).$$

Proof: First let us prove

$$\sum_{\deg(m)=x} \left(\sum_{p|m} \Omega(p-1) \right) = \frac{1}{2} q^x (\log x)^2 + O(q^x \log x),$$

and

$$\sum_{\deg(m)=x} \left(\sum_{p|m} \Omega(p-1) \right)^2 = \frac{1}{4} q^x (\log x)^4 + O(q^x (\log x)^3).$$

Then we can use the inequality

$$\sum_{p|m} \Omega(p-1) \leq \Omega(\varphi(m)) \leq \sum_{p|m} \Omega(p-1) + \Omega(m),$$

to get the result.

For $\sum_{\deg(m)=x}\left(\sum_{p|m}\Omega(p-1)\right)$, we have

$$\begin{aligned}\sum_{\deg(m)=x}\left(\sum_{p|m}\Omega(p-1)\right) &= \sum_{\deg(p)\le x}\sum_{\substack{\deg(m)=x\\ p|m}} 1\\ &= \sum_{\deg(p)\le x}\Omega(p-1)\sum_{\substack{\deg(m)=x\\ p|m}} 1\\ &= q^x\sum_{\deg(p)\le x}\frac{\Omega(p-1)}{q^{\deg(p)}}\\ &= \frac{1}{2}q^x(\log x)^2+O(q^x\log x).\end{aligned}$$

For $\sum_{\deg(m)=x}\left(\sum_{p|m}\Omega(p-1)\right)^2$, we have

$$\sum_{\deg(m)=x}\left(\sum_{p|m}\Omega(p-1)\right)^2 = \sum_{\deg(m)=x}\sum_{p|m}\Omega^2(p-1)+\sum_{\deg(m)\le x}\sum_{p|m}\sum_{\substack{l|m\\ l\neq p}}\Omega(p-1)\Omega(l-1).$$

Since

$$\begin{aligned}\sum_{\deg(m)=x}\sum_{p|m}\Omega^2(p-1) &\le q^x\sum_{\deg(p)\le x}\frac{\Omega^2(p-1)}{q^{\deg(p)}}\\ &= q^x\left(\frac{1}{3}\log^3 x\right)\\ &= O(q^x\log^3 x),\end{aligned}$$

and

$$\begin{aligned}
&\sum_{\deg(m)\le x}\sum_{p|m}\sum_{\substack{l|m\\ l\neq p}}\Omega(p-1)\Omega(l-1)\\
=&\sum_{\deg(l)\le x}\sum_{\substack{\deg(p)\le x\\ p\neq l}}\Omega(p-1)\Omega(l-1)\sum_{\substack{\deg(m)=x\\ pl|m}}1\\
=&\sum_{\deg(l)\le x}\sum_{\substack{\deg(p)\le x\\ p\neq l}}\Omega(p-1)\Omega(l-1)\frac{1}{q^{\deg(p)+\deg(l)}}\\
=&\ q^x\left(\sum_{\deg(p)\le x}\frac{\Omega(p-1)}{q^{\deg(p)}}\right)^2+O\left(\sum_{\deg(p)\le x}(\frac{\Omega(p-1)}{q^{\deg(p)}})^2\right),
\end{aligned}$$

and since

$$\sum_{\deg(p)\le x}\left(\frac{\Omega(p-1)}{q^{\deg(p)}}\right)^2=O\left(\sum_{n\le x}\frac{q^n}{n}\log^2 x\right)=O\left(q^x\log^3 x\right),$$

and

$$q^x\left(\sum_{\deg(p)\le x}\frac{\Omega(p-1)}{q^{\deg(p)}}\right)^2=\frac{1}{4}q^x\log^4 x+O(q^x\log^3 x),$$

then

$$\sum_{\deg(m)=x}\left(\sum_{p|m}\Omega(p-1)\right)^2=\frac{1}{4}q^x(\log x)^4+O(q^x(\log x)^3).$$

Since we have

$$\sum_{\deg(m)=x}\Omega(m)=O(q^x\log x).$$

Then with the inequality

$$\sum_{p|m}\Omega(p-1)\le\Omega(\varphi(m))\le\sum_{p|m}\Omega(p-1)+\Omega(m),$$

we have

$$\sum_{\deg(m)=x}\Omega(\varphi(m))=\frac{1}{2}q^x(\log x)^2+O(q^x\log x).$$

Also, since

$$\begin{aligned}
& \sum_{\deg(m)=x} \left(\sum_{p|m} \Omega(p-1) + \Omega(m) \right)^2 \\
= & \sum_{\deg(m)=x} \left(\sum_{p|m} \Omega(p-1) + O(\log x) \right)^2 \\
= & \sum_{\deg(m)=x} \left(\sum_{p|m} \Omega(p-1) \right)^2 + O\left(\sum_{\deg(m)=x} \sum_{p|m} \Omega(p-1) O(\log x) \right) \\
= & \frac{1}{4} q^x (\log x)^4 + O(q^x (\log x)^3),
\end{aligned}$$

again from the inequality, we have

$$\sum_{\deg(m)=x} \Omega^2(\varphi(m)) = \frac{1}{4} q^x (\log x)^4 + O(q^x (\log x)^3).$$

Similarly, we have the lemma below:

Lemma 43

$$\sum_{\deg(m)=x} \Omega_y(\varphi(m)) = q^x \log x \log y + O(q^x \log y),$$

and

$$\sum_{\deg(m)=x} \Omega_y^2(\varphi(m)) = q^x \log^2 x \log^2 y + O(q^x \log^2 x \log y).$$

Proof: First let us prove

$$\sum_{\deg(m)=x} \left(\sum_{p|m} \Omega_y(p-1) \right) = q^x \log x \log y + O(q^x \log y),$$

and

$$\sum_{\deg(m)=x} \left(\sum_{p|m} \Omega_y(p-1) \right)^2 = q^x \log^2 x \log^2 y + O(q^x \log x \log^2 y).$$

Then we can use the inequality

$$\sum_{p|m} \Omega_y(p-1) \le \Omega_y(\varphi(m)) \le \sum_{p|m} \Omega_y(p-1) + \Omega_y(m),$$

to get the result.

For $\sum_{\deg(m)=x} \left(\sum_{p|m} \Omega_y(p-1) \right)$, by Lemma 39 we have

$$\begin{aligned}
\sum_{\deg(m)=x} \left(\sum_{p|m} \Omega_y(p-1) \right) &= \sum_{\deg(p)\le x} \sum_{\substack{\deg(m)=x \\ p|m}} 1 \\
&= \sum_{\deg(p)\le x} \Omega_y(p-1) \sum_{\substack{\deg(m)=x \\ p|m}} 1 \\
&= q^x \sum_{\deg(p)\le x} \frac{\Omega_y(p-1)}{q^{\deg(p)}} \\
&= q^x \sum_{n=1}^{x} \left(\frac{\frac{q^n}{n} \log y}{q^n} + O\left(\frac{\frac{q^n}{n}}{q^n} \right) \right) \\
&= q^x \log x \log y + O(q^x \log x) \text{ from partial summation.}
\end{aligned}$$

For $\sum_{\deg(m)=x} \left(\sum_{p|m} \Omega_y(p-1) \right)^2$, by Lemma 39, we have

$$\sum_{\deg(m)=x} \left(\sum_{p|m} \Omega_y(p-1) \right)^2 = \sum_{\deg(m)=x} \sum_{p|m} \Omega_y^2(p-1) + \sum_{\deg(m)\le x} \sum_{p|m} \sum_{\substack{l|m \\ l\ne p}} \Omega_y(p-1)\Omega_y(l-1).$$

Since

$$\begin{aligned}
\sum_{\deg(m)=x} \sum_{p|m} \Omega_y^2(p-1) &\le q^x \sum_{\deg(p)\le x} \frac{\Omega_y^2(p-1)}{q^{\deg(p)}} \\
&= q^x \sum_{n=1}^{x} \left(\frac{\frac{q^n}{n} \log^2 y}{q^n} \right) + O\left(\frac{\frac{q^n}{n} \log y}{q^n} \right) \\
&= O(q^x \log x \log^2 y),
\end{aligned}$$

and

$$\sum_{\deg(m)\le x}\sum_{p|m}\sum_{\substack{l|m\\ l\neq p}}\Omega_y(p-1)\Omega_y(l-1)$$

$$= \sum_{\deg(l)\le x}\sum_{\substack{\deg(p)\le x\\ p\neq l}}\Omega_y(p-1)\Omega_y(l-1)\sum_{\substack{\deg(m)=x\\ pl|m}}1$$

$$= \sum_{\deg(l)\le x}\sum_{\substack{\deg(p)\le x\\ p\neq l}}\Omega_y(p-1)\Omega_y(l-1)\frac{1}{q^{\deg(p)+\deg(l)}}$$

$$= q^x\left(\sum_{\deg(p)\le x}\frac{\Omega_y(p-1)}{q^{\deg(p)}}\right)^2+q^xO\left(\sum_{\deg(p)\le x}\left(\frac{\Omega_y(p-1)}{q^{\deg(p)}}\right)^2\right),$$

and since by Lemma 38

$$\sum_{\deg(p)\le x}\left(\frac{\Omega_y(p-1)}{q^{\deg(p)}}\right)^2=O\left(\sum_{n\le x}\frac{q^n}{n}\log^2 y\right)=O\left(\log x\log^2 y\right),$$

and

$$q^x\left(\sum_{\deg(p)\le x}\frac{\Omega_y(p-1)}{q^{\deg(p)}}\right)^2 = q^x\log^2 x\log^2 y+O(q^x\log^2 y\log x),$$

then

$$\sum_{\deg(m)=x}\left(\sum_{p|m}\Omega_y(p-1)\right)^2=q^x\log^2 x\log^2 y+O(q^x\log x\log^2 y).$$

Since we have

$$\sum_{\deg(m)=x}\Omega_y(m)=O(q^x\log y).$$

Then with the inequality

$$\sum_{p|m}\Omega_y(p-1)\le\Omega_y(\varphi(m))\le\sum_{p|m}\Omega_y(p-1)+\Omega_y(m),$$

we have

$$\sum_{\deg(m)=x}\Omega_y(\varphi(m))=q^x\log x\log y+O(q^x\log y).$$

Also, since

$$\begin{aligned}
&\sum_{\deg(m)=x}\left(\sum_{p|m}\Omega_y(p-1)+\Omega_y(m)\right)^2\\
= &\sum_{\deg(m)=x}\left(\sum_{p|m}\Omega(p-1)\right)^2+O\left(\sum_{\deg(m)=x}\sum_{p|m}\Omega(p-1)O(\log x)\right)\\
= &\ q^x\log^2 x\log y+O(q^x\log^2 x\log y),
\end{aligned}$$

again from the inequality, we have

$$\sum_{\deg(m)=x}\Omega^2(\varphi(m))=q^x\log^2 x\log^2 y+O(q^x\log^2 x\log y).$$

From Lemma 42, we have a result below:

Lemma 44

$$\sum_{\deg(m)=x}\left(\Omega(\varphi(m))-\frac{1}{2}\log^2 x\right)^2\ll q^x\log^3 x.$$

Proof:

$$\begin{aligned}
&\sum_{\deg(m)=x}\left(\Omega(\varphi(m))-\frac{1}{2}\log^2 x\right)^2\\
= &\sum_{\deg(m)=x}\left(\Omega^2(\varphi(m))-\Omega(\varphi(m))\log^2 x+\frac{1}{4}\log^4 x\right)\text{ by Lemma 42}\\
= &\ \frac{1}{4}\log^4 x+O(q^x\log^3 x)+\frac{1}{4}\sum_{\deg(m)=x}\log^4 x\\
- &\ \log^2 x\left(\frac{1}{2}q^x\log^2 x+O(q^x\log x)\right)\\
= &\ \frac{1}{2}q^x\log^4 x-\frac{1}{2}q^x\log^4 x+O(q^x\log^3 x)\\
\ll &\ q^x\log^3 x.
\end{aligned}$$

Lemma 45

$$\sum_{\deg(m)=x} (\Omega(\varphi(m)) - \omega(\varphi(m)))^2 = o(q^x \log^{2+\varepsilon} x) \text{ for any } \varepsilon > 0.$$

Proof: Let $\omega_y^+(\varphi(m))$ be the number of distinct prime divisors of $\varphi(m)$ whose degrees are $> y$ and $\Omega_y^+(\varphi(m))$ be defined similarly. Then

$$\begin{aligned}
& \sum_{\deg(m)=x} (\Omega(\varphi(m)) - \omega(\varphi(m)))^2 \\
= & \sum_{\deg(m)=x} \left(\Omega_y(\varphi(m)) + \Omega_y^+(\varphi(m)) - \omega_y(\varphi(m)) - \omega_y^+(\varphi(m))\right)^2 \\
\ll & \sum_{\deg(m)=x} \left(\left(\Omega_y^+(\varphi(m)) - \omega_y^+(\varphi(m))\right)^2 + (\Omega_y(\varphi(m)))^2 + (\omega_y(\varphi(m)))^2\right).
\end{aligned}$$

We then claim that but for $o(q^x)$ choices of $\deg(m) = x$ we have

$$\Omega_y^+(\varphi(m)) - \omega_y^+(\varphi(m)) = 0.$$

To prove this claim, first notice that if there exists some prime p such that $p^2|\varphi(m)$, when $\deg p > y$ and $\deg m = x$, Then p and m satisfy one of three bellow cases:
1) $p^3|m$;
2) There exists some prime polynomial $l|m$ with $l \equiv 1(p^2)$;
3) There exists some prime polynomials l_1, l_2 with $l_1 \neq l_2$, $l_1 l_2|m$, and $l_1 \equiv l_2 \equiv 1(p)$.

In the first case, the number of possible m is

$$\begin{aligned}
& \sum_{n>y} \#\{m : p^3|m, \deg(m) = x, \deg(p) = n\} \\
\ll & \sum_{n>y} \left\{ \left(\frac{q^n}{n} + O\left(\frac{q^{n/2}}{n}\right)\right) \left(q^{x-3n} + O(1)\right) \right\} \\
\leq & \sum_{n>y} \left\{ \left(\frac{q^n}{n} + O\left(\frac{q^{n/2}}{n}\right)\right) q^{x-3n} \right\} \\
\leq & \sum_{n>y} \left\{ \frac{q^{x-2n}}{n} + O\left(\frac{q^{x-\frac{5}{2}n}}{n}\right) \right\} \\
= & \; o(q^x).
\end{aligned}$$

In the second case, by Lemma 30, the number of possible m is

$$
\begin{aligned}
& \sum_{n>y} \sum_{\substack{l\equiv 1(p^2)\\ \deg(l)\le x}} \#\{m : l|m, \deg(m)=x, \deg(p)=n\} \\
\ll & \sum_{n>y} \frac{q^n}{n} \left(\sum_{\substack{l\equiv 1(p^2)\\ \deg(l)\le x}} \frac{q^x}{q^{\deg(l)}} \right) = \sum_{n>y} \frac{q^n}{n} \cdot \frac{q^x \log x}{\varphi(p^2)} \\
= & \sum_{n>y} \frac{q^x \log x}{n q^n} \ll q^x \log x \cdot \frac{1}{y q^y} \\
= & \ o(q^x).
\end{aligned}
$$

For the third case, the number of possible m, is at most

$$
\begin{aligned}
& \sum_{n>y} \frac{q^n}{n} \left(\sum_{\substack{\deg(l_1)\le \deg(l_2)\le x\\ l_1\equiv l_2\equiv 1(p)}} \frac{q^x}{q^{\deg(l_1)} q^{\deg(l_2)}} \right) \\
\ll & \ q^x \sum_{n>y} \frac{q^n}{n} \left(\sum_{\substack{l\equiv 1(p)\\ \deg(l)\le x}} \frac{1}{q^{\deg(l)}} \right)^2 \text{(from Lemma 40)} \\
= & \ q^x \sum_{n>y} \frac{q^n}{n} \cdot \frac{(\log x)^2}{(q^{\deg(p)})^2} \\
= & \ q^x \sum_{n>y} \frac{(\log x)^2}{n} \cdot \frac{1}{q^n} \ \le \ q^x (\log x)^2 \frac{1}{y q^y} \\
= & \ o(q^x).
\end{aligned}
$$

So the claim is proved, and as an instant corollary, we have

$$
\sum_{\deg(m)=x} \left(\Omega_y^+(\varphi(m)) - \omega_y^+(\varphi(m))\right)^2 \ll q^x \log^2 x, \text{ with } \Omega(\varphi(m)) \le \Omega(m) < \log x.
$$

Thus we have

$$
\begin{aligned}
& \sum_{\deg(m)=x} (\Omega(\varphi(m)) - \omega(\varphi(m)))^2 \\
\ll & \sum_{\deg(m)=x} \left(\left(\Omega_y^+(\varphi(m)) - \omega_y^+(\varphi(m))\right)^2 + (\Omega_y(\varphi(m)))^2 + (\omega_y(\varphi(m)))^2 \right) \\
\ll & \; q^x \log^2 x + 2q^x \log^2 x \log^2 y) \\
= & \; o(q^x \log^{2+\varepsilon} x).
\end{aligned}
$$

Lemma 46

$$
\sum_{\deg(m)=x} \left(\omega(\varphi(m)) - \frac{1}{2}\log^2 x \right)^2 \ll q^x \log^3 x
$$

if and only if

$$
\sum_{\deg(m)=x} \left(\Omega(\varphi(m)) - \frac{1}{2}\log^2 x \right)^2 \ll q^x \log^3 x.
$$

Proof: This is true since

$$
\begin{aligned}
& \sum_{\deg(m)=x} \left(\Omega(\varphi(m)) - \frac{1}{2}\log^2 x \right)^2 - \sum_{\deg(m)=x} \left(\omega(\varphi(m)) - \frac{1}{2}\log^2 x \right)^2 \\
= & \sum_{\deg(m)=x} (\Omega(\varphi(m)) - \omega(\varphi(m))) \left(\Omega(\varphi(m)) + \omega(\varphi(m)) - \log^2 x\right) \\
= & \sum_{\deg(m)=x} (\Omega(\varphi(m)) - \omega(\varphi(m)))^2 + 2 \sum_{\deg(m)=x} (\Omega(\varphi(m)) - \omega(\varphi(m))) \left(\omega(\varphi(m)) - \frac{1}{2}\log^2 x \right) \\
\ll & \; q^x \log^3 x + 2 \sqrt{ \sum_{\deg(m)=x} (\Omega(\varphi(m)) - \omega(\varphi(m)))^2 \sum_{\deg(m)=x} \left(\omega(\varphi(m)) - \frac{1}{2}\log^2 x \right)^2 } \\
\ll & \; q^x \log^3 x + 2q^x \log^{5+\varepsilon/2} x \text{ (for any } \varepsilon > 0\text{, from previous Lemma)} \\
\ll & \; q^x \log^3 x.
\end{aligned}
$$

Lemma 47

$$
\lim_{x\to\infty} \frac{1}{q^x} \# \left\{ m : \deg(m) = x, m \text{ satisfies } \frac{\omega(\varphi(m)) - \frac{1}{2}\log^2 x}{\sqrt{\frac{1}{3}\log^3 x}} \le t \right\} = G(t)
$$

if and only if

$$\lim_{x\to\infty}\frac{1}{q^x}\#\left\{m:\deg(m)=x, m\ satisfies\ \frac{\Omega(\varphi(m))-\frac{1}{2}\log^2 x}{\sqrt{\frac{1}{3}\log^3 x}}\le t\right\}=G(t).$$

Proof: Since

$$\frac{\omega(\varphi(m))-\frac{1}{2}\log^2 x}{\sqrt{\frac{1}{3}\log^3 x}}=\frac{\Omega(\varphi(m))-\frac{1}{2}\log^2 x}{\sqrt{\frac{1}{3}\log^3 x}}-\frac{\Omega(\varphi(m))-\omega(\varphi(m))}{\sqrt{\frac{1}{3}\log^3 x}},$$

then by Lemma 3 we need only to prove that for any $q>0$,

$$\lim_{x\to\infty}\frac{1}{q^x}\#\left\{m:\deg(m)=x,\left|\frac{\Omega(\varphi(m))-\omega(\varphi(m))}{\sqrt{\frac{1}{3}\log^3 x}}\right|>\varepsilon\right\}=0.$$

It is obvious since we have

$$\sum_{\deg(m)=x}\big(\Omega(\varphi(m))-\omega(\varphi(m))\big)^2\ll q^x\log^2 x, \text{from Lemma 45.}$$

4.5 Proof

In this section, we will finish the proof of the Theorem 2:

Theorem 2 (Normal Distribution of $\omega(\varphi(m))$) Let m be a polynomial in $\mathbb{F}_q[T]$ over finite field $\mathbb{F}_q$, we have

$$\lim_{x\to\infty}\frac{1}{q^x}\#\left\{m:\deg(m)=x,\frac{\omega(\varphi(m))-\frac{1}{2}(\log x)^2}{\frac{1}{\sqrt{3}}(\log x)^{\frac{2}{3}}}\le t\right\}=G(t).$$

Now we can apply Zhang's theorem to get our goal. Recall that in order to apply this theorem, we need to check for any $\varepsilon>0$,

$$\lim_{x\to\infty}\frac{1}{B(x)^2}\sum_{\substack{q^{\deg(p)}\le x\\ |f(p)|>\varepsilon B(x)}}\frac{f^2(p)}{q^{\deg(p)}}=0.$$

To apply this theorem, we need to let $f(p) = \Omega(\varphi(m))$. But from lemma 41, we know that we can change $\Omega(\varphi(m))$ to $g(m) = \sum_{p|m} \Omega(p-1)$. We already proved that

$$A(x) = \sum_{\deg(p)\le x} \sum_{p|m} \frac{\Omega(p-1)}{q^{\deg(p)}} = \frac{1}{2}\log^2 x + O(\log x),$$

$$B(x) = \left(\sum_{\deg(p)\le x} \sum_{p|m} \frac{\Omega^2(p-1)}{q^{\deg(p)}} \right)^{\frac{1}{2}} = \frac{1}{\sqrt{3}} \log^{\frac{3}{2}} x + O(\log x).$$

So now we let

$$\alpha(p) = \begin{cases} 1 & \text{if } \Omega(p-1) \ge \varepsilon B(x), \\ 0 & \text{otherwise,} \end{cases}$$

and then the requirement becomes

$$\lim_{x\to\infty} \frac{1}{B(x)^2} \sum_{\substack{\deg p \le x \\ |g(p)| \ge \varepsilon B(x)}} \frac{g^2(p)}{q^{\deg(p)}} = \lim_{x\to\infty} \frac{1}{B(x)^2} \sum_{\deg(p)\le x} \alpha(p) \frac{\Omega^2(p-1)}{q^{\deg(p)}} = 0.$$

Since

$$\sum_{\deg(p)\le x} \alpha(p) \frac{\Omega^2(p-1)}{q^{\deg(p)}} \le \left(\sum_{\deg(p)\le x} \frac{\alpha^2(p)}{q^{\deg(p)}} \right)^{\frac{1}{2}} \left(\sum_{\deg(p)\le x} \frac{\Omega^4(p-1)}{q^{\deg(p)}} \right)^{\frac{1}{2}},$$

we can verify as follow:
From Lemma 39, we have

$$\sum_{\deg(p)=x} (\Omega(p-1) - \log x)^2 \ll q^x \log x.$$

Then we can get

$$\sum_{\deg(p)=x} \alpha(p) = \#\{\deg(p) = x : p \text{ satisfies } \Omega(p-1) > \varepsilon B(x)\} \ll \frac{q^x}{\log^2 x}.$$

Since

$$\sum_{\deg(p)=x} \alpha(p) = \sum_{\deg(p)=x} \alpha(p)^2,$$

we have

$$
\begin{aligned}
\sum_{\deg(p)\le x} \frac{\alpha^2(p)}{q^{\deg(p)}} &\ll \left(\sum_{\deg(p)\le x} \alpha^2(p)\right)\frac{1}{q^x} + \int_1^x \left(\sum_{\deg(p)\le t} \alpha^2(p)\right)\frac{\log q}{q^t}dt \\
&\ll \frac{1}{\log^2 x} + \int_1^x \frac{q^t \log t}{q^t t \log^2 t}dt \\
&= O(1).
\end{aligned}
$$

Form previous lemmas, we have

$$
\sum_{\deg(m)=x} \omega(m-1) = \frac{q^x}{x}\log x + O\left(\frac{q^x}{x}\right),
$$

and

$$
\sum_{\deg m=x} \omega^2(m-1) = \frac{q^x}{x}(\log x)^2 + O\left(\frac{q^x \log x}{x}\right).
$$

So

$$
\begin{aligned}
\sum_{\deg(m)=x} \omega^4(m-1) &= \sum_{\deg(m)=x}\left(\sum_{l|m-1} 1\right)^4 = \sum_{\deg(m)=x}\sum_{l_1,l_2,l_3,l_4|m-1} 1 \\
&= \sum_{\substack{l_1,l_2,l_3,l_4 \\ \deg(l_i)\le x, \\ i=1,2,3,4}} \sum_{\substack{m\equiv 1(l_i), \\ i=1,2,3,4 \\ \deg(m)=x}} 1 \\
&\ll \sum_{\deg(l)\le x} \sum_{\substack{m\equiv 1(l) \\ \deg(m)=x}} 1 + \sum_{\substack{l_1,l_2 \\ \deg(l_1)\le x \\ \deg(l_2)\le x}} \sum_{\substack{m\equiv 1(l_1l_2) \\ \deg(m)=x}} 1 + \sum_{\substack{l_1,l_2,l_3 \\ \deg(l_i)\le x, \\ i=1,2,3}} \sum_{\substack{m\equiv 1(l_1l_2l_3) \\ \deg(m)=x}} 1 \\
&\quad + \sum_{\substack{l_1,l_2,l_3,l_4 \\ l_1\ne l_2\ne l_3\ne l_4}} \sum_{\substack{m\equiv 1(l_1l_2l_3l_4) \\ \deg(m)=x}} 1 \\
&\ll \sum_{m\le x}\frac{q^m}{m}\sum_{n\le x}\frac{q^n}{n}\sum_{k\le x}\frac{q^k}{k}\sum_{d\le x}\frac{q^d}{d}\cdot\frac{\frac{q^x}{x}}{q^m q^n q^k q^d} \\
&= \frac{q^x}{x}\sum_{m\le x}\frac{1}{m}\sum_{n\le x}\frac{1}{n}\sum_{k\le x}\frac{1}{k}\sum_{d\le x}\frac{1}{d} \\
&\ll \frac{q^x}{x}(\log x)^4.
\end{aligned}
$$

Then by partial summation, we get

$$\sum_{\deg(p)\le x} \frac{\Omega^4(p-1)}{q^{\deg(p)}} \ll \sum_{n\le x} \frac{1}{q^n}\frac{q^x}{x}\log^4 n \ll \log^5 x.$$

Combine above, we have

$$\sum_{\substack{\deg p\le x\\ |g(p)|\ge \varepsilon B(x)}} \frac{g^2(p)}{q^{\deg(p)}} \ll \log^{5/2} x = o\left(B^2(x)\right).$$

Then the requirement from Zhang's theorem is satisfied, and we have

$$\lim_{x\to\infty} \frac{1}{q^x}\#\left\{m : \deg(m) = x, m \text{ satisfies } \frac{g(m)-\frac{1}{2}\log^2 x}{\sqrt{\frac{1}{3}\log^3 x}} \le t\right\} = G(t),$$

which completes the proof of Theorem 6.

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