On the classification of the R-separable webs for the Laplace equation in $\mathbf{E}^3$

by

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A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Doctor of Philosophy
in
Physics

Waterloo, Ontario, Canada, 2008

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

In the first two Chapters I outline the theory and background of separation of variables as an ansatz for solving fundamental partial differential equations (pdes) in Mathematical Physics. Two fundamental approaches will be highlighted, and more modern approaches discussed. In Chapter 3 I calculate the general trace-free conformal Killing tensor defined in Euclidean space - from the sum of symmetric tensor products of conformal Killing vectors. In Chapter 4 I determine the subcases with rotational symmetry and recover known examples pertaining to classical rotational coordinates. In Chapter 5 I obtain the induced action of the conformal group on the space of trace-free conformal Killing tensors. In Chapter 6 I use the invariants of trace-free conformal Killing tensors under the action of the conformal group to characterize, up to equivalence, the symmetric R-separable webs in $\mathbf{E}^3$ that permit conformal separation of variables of the fundamental pdes in Mathematical Physics. In Chapter 7 the asymmetric R-separable metrics are obtained via a study of the separability conditions for the conformally invariant Laplace equation.

Acknowledgments

I thank my supervisor Ray McLenaghan for Ph.D funding, guidance and ideas in the two main research projects that comprise this thesis, and also for patient help during the thesis write-up prior to official submission. While on my first exchange period abroad in Turin, Italy during the summer of 2005, in a high intensity period when I had in a hurry to learn more background formalism and current tools in the research on separation of variables theory - in parallel with adjusting to a new place, new culture and not to mention learning a new language, our collaborator Claudia Chanu greatly assisted me by hours of patient guidance, tutoring and inspiration which in a few month's time put me on track. She is an excellent teacher and one to discuss ideas and problems in research with. Claudia was a magnificent help to me during high stress upon seeing the wealth of formalism I had to warm up to in so short a time.

I wish to thank Luca Degiovanni and Giovanni Rastelli for helpful discussions on the background theory, as well as Luca's patient explanation of first principles theory I still needed to learn. I appreciate the hospitality of the University of Turin during my three exchange periods (not to mention the privilege of subsidized mensa that was extended to me, and accommodation in friendly student residences (especially Residenza Cavour) where I was exposed to the charm and openness of Italian culture through the many friends I met there which certainly boosted my energy of research during my stays), and the help of the exchange co-ordinate Lorenzo Fatibene and Enrico Bibbona in logistics during the course of my research. I am also grateful to Stepf Czapor for providing me and my collaborators with useful theory of polynomial resultants that helped solidify our main conjecture of the research I collaborated on. I am grateful for the support my parents - Peter and Heidi - gave me throughout my Ph.D studies.

Recently I owe my strong energy level to German Club and International Pub Night events at the University of Waterloo - the people and their ambiance certainly helped during the months I spent writing up this manuscript.

I am grateful for additional funding which assisted me, in the University of Waterloo and in Universita di Torino, provided by numerous bursaries from the Graduate Student Office in UW. I am also indebted to C.M Lerici foundation in Stockholm, Sweden which provided me with a scholarship to assist me during my 3rd exchange period in Turin.

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NOTATIONS AND CONVENTIONS

R : Set of real numbers

M : Riemannian manifold

[,] : Lie-Schouten bracket

$_{(\, , \,)}$: Symmetrization of the indices

$_{[\, , \,]}$: Anti-symmetrization of the indices

∂_i : Partial derivative w.r.t x^i (also denoted by $_{,i}$)

∇_i : Covariant differentiation operator (also denoted by $_{;i}$)

$\odot$: Symmetric tensor product

g : contravariant metric tensor

g : determinant of the metric tensor

Γ^i_{jk} : Christoffel symbol

Γ_i : Contracted Christoffel symbol

R_{ijkl} : Riemann curvature tensor

$X_{i;jk} - X_{i;kj} = X_l R^l_{ijk}$: Ricci identity

$R_{ij} = g^{kl} R_{kijl}$: Ricci tensor

$R_s = g^{ij} R_{ij}$: Curvature scalar, usually in the literature denoted as R

R_{ijk} : Cotton tensor

δ_{ij} : Kroenicker delta

K_{ij} : Valence two Killing tensor

ρ : Eigenvalue of a Killing tensor

$h^i{}_a$: i'th component of the eigenvector corresponding to the eigenvalue ρ_a

p : Valence of Killing tensor

$C\hat{K}^p(M)$: Linear space of valence p trace-free conformal Killing tensors

$\mathbf{k}$: Some symmetric tensor of type $(p-1,\ 0)$ or Killing vector, depending on context.

S : Determinant of a Stäckel matrix

Q : Conformal factor in Stäckel matrix

R : Modulation factor of R-separability

log : Natural logarithm function, base e

S_{ij} : Stäckel operator

H : Hamiltonian of Classical System

Δ : Laplace-Beltrami operator

W : Solution of the Hamilton-Jacobi equation

E : Energy of a Classical System

V : Potential of a Classical System

E^3 : Euclidean space

x^i, u^i, q^i : i'th coordinate, depending on context

p_i : i'th component of the generalized momenta

I : Invariant function

$C(M)$: Conformal Group

ϕ_t : One parameter group of transformations

f_* : The push forward map of the diffeomorphism $f : M \to N$

$\mathcal{L}$: Lie derivative operator

G : Connected Lie group of transformations

$\mathbf{v}$: Infinitesimal generator of Lie group action

k : Jacobi Elliptic parameter

$\square$: End of Proof symbol

Chapter 1

Introduction

The method of separation of variables is a classical tool to split a partial differential equation into systems of ordinary differential equations, or decouple systems of partial differential equations (pdes) into classes of ordinary differential equations each depending on one variable only. This is a standard approach to solve important boundary value problems in mathematical physics. There are many known examples of coordinate systems admitting separation of variables for the Laplace equation. These include Cartesian, polar, spherical and cylindrical coordinates the choice of which depends on the type of initial conditions specified. Less known examples (applied especially in electromagnetic theory) include elliptic-hyperbolic, ellipsoidal, paraboloidal and conical coordinates to tackle boundary problems that might otherwise be solved only numerically by those unaware of the separation of variables properties such coordinates admit.

Clearly an exhaustive search for other coordinate systems possessing this property would facilitate calculations in many fields where analytic solutions of partial differential equations are desired over numerical ones, despite the very complicated boundary value conditions that may arise. The standard theory of separation of variables has been broadened to include a weaker R-separation of variables method where the product solution ansatz contains a non-constant factor (denoted by R, also called the modulation factor) depending on the coordinates. This relaxation of strict separability permits a broader class of coordinate systems where equations can be separated in this way.

The most familiar coordinate system admitting this property are toroidal coordinates. Less familiar are the Jacobi-elliptic coordinates that arise from the solution of confocal quartics (defined in [39]) and contain degree four surfaces. Other examples with rotational symmetry are bi-cyclide coordinates, flat-ring cyclide coordinates and disk-cyclide coordinates. Two examples are known to exist with no coordinate symmetries whatsoever - but have yet to be named.

This thesis focuses on R-separation of variables of the Laplace equation in Euclidean space. The seventeen separable and R-separable coordinates in Euclidean space were first determined by Bôcher, Eisenhart, Weinacht and Blaschke ([5], [20],

[49], [4]). Much later they were classified using group theoretic methods by Boyer, Kalnins and Miller [6]. We provide an exhaustive classification of the extra set of R-separable coordinates based on differential invariants of valence-two conformal Killing tensors under the action of the conformal group, which is an extension of the ordinary isometry group. The approach is an extension of that employed by Horwood, McLenaghan and Smirnov [25] who give an invariant classification of the eleven simply separable coordinate systems (also known as coordinate webs) for the Hamilton Jacobi and Helmholtz equation in E^3 in terms of the invariants and reduced invariants (with respect to the isometry group) of valence-two Killing tensors.

Associated to R-separable coordinates are certain valence-two conformal Killing tensors. Associated to each conformal Killing tensor is a trace-free representation, this subset has finite dimension and is the geometric object characterizing R-separation of variables. In this thesis these will be calculated for all known rotational R-separable coordinates and given in standard form (expressed in Cartesian coordinates) which will facilitate future researchers who wish to study boundary value problems with a potential.

In the study of boundary value problems for Schrödinger's equation in $\mathbf{E}^3$, one would like to know whether the problem may be solved by separation of variables. The potential function is usually given in terms of Cartesian coordinates. The existence of a potential restricts the number of possible coordinate systems with respect to which the equation separates or R-separates. This determination may be made in terms of the general valence-two Killing or conformal Killing tensors admitted by $\mathbf{E}^3$. In this thesis we shall restrict ourselves to analysis of rotationally symmetric coordinate systems.

1.1 Definitions of Separation of Variables

The linear partial differential equations (pdes) considered in this thesis for *separation of variables theory* are the time-independent Schrödinger (S) equation and the Helmholtz/Laplace (H/L) equation, defined on an n dimensional Riemannian manifold (M, g). These are often considered for separability in a class of coordinate systems to facilitate the solution of various boundary value problems.

All of the above pdes are special cases of the pde

$$\Delta\varphi + C\varphi = 0, \tag{1.1.1}$$

where Δ is the Laplace-Beltrami operator defined by

$$\Delta\varphi = g^{ij}\nabla_i\nabla_j\varphi = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\frac{\sqrt{g}g^{ij}\partial\varphi}{\partial x^j}\right), \tag{1.1.2}$$

where ∇_i is the covariant derivative with respect to the Levi-Civita connection of the metric tensor $\mathbf{g}$, g is the determinant of the metric tensor and C is an infinitely differentiable function defined on M. For $C \neq const$, Eq. (1.1.1) yields the time-independent Schrödinger equation. For $C = const \neq 0$, Eq. (1.1.1) reduces to the Helmholtz equation. For $C = 0$, Eq. (1.1.1) reduces to the Laplace equation. Note that for the time-independent Schrödinger equation, it is customary to write $C = V - E$, where V denotes the potential and E the energy of the system.

Separation of variables theory for Eq. (1.1.1) is closely related to that of the Hamilton-Jacobi (HJ) equation for a natural Hamiltonian defined on (M, g). Such an equation may be written as

$$g^{ij}\partial_i W \partial_j W + V = E, \tag{1.1.3}$$

where ∂_i denotes $\frac{\partial}{\partial x^i}$ and V denotes the potential and E the energy of the system. In this thesis u^i or q^i or x^i will be understood to represent the i^{th} coordinate, this is because certain symbols for the coordinate are more prevalent in certain proofs and definitions in the literature than others.

The fundamental definition of R-separation of variables for the equation (1.1.1), given by Moon & Spencer [38] and Morse & Feshbach [39], is as follows:

Definition 1.1.1 *If the ansatz for trial solutions*

$$\varphi = \frac{U^1(u^1, \mathbf{a}) \cdot U^2(u^2, \mathbf{a}) \cdot \quad \cdots \quad \cdot U^n(u^n, \mathbf{a})}{R(u^1, \cdots, u^n)} \qquad \mathbf{a} \in \mathbf{R}^{2n-1}$$

for some analytic $R \neq const$ permits the separation of

$$\Delta\varphi + C\varphi = 0$$

*into n ordinary differential equations, then the equation is said to be conformally or R-separable. The function R is called the **modulation factor**.*

Remark 1.1.2 *A completeness condition on the parameters $\mathbf{a}$ exists to ensure non-degeneracy of the solutions and is given in Chapter 2. If in the above definition $R = const$, then the equation is said to be simply separable. One can describe R-separability as a relaxation of the ad-hoc simple separability to permit solutions of partial differential equations in a broader class of coordinate systems - thus motivating the research into classifying which coordinates admit this property. Furthermore if R is a product of functions of a single variable - that is $\partial_{ij}log(R) = 0$ for $i \neq j$, we have the case of **trivial** R**-separation** [28]. Since coordinates are trivially R-separable iff they are separable, we regard trivial R-separation as equivalent to ordinary separation [31].*

The fundamental definition of separation of variables for the equation (1.1.3) for *fixed* values of the energy is as follows:

Definition 1.1.3 *If the ansatz for trial solutions*

$$W = U^1(u^1, \mathbf{a}) + U^2(u^2, \mathbf{a}) + \cdots + U^n(u^n, \mathbf{a}) \qquad \mathbf{a} \in \mathbf{R}^n$$

permits the separation of

$$g^{ij}\partial_i W \partial_j W + V = E$$

for fixed values of the energy E into n ordinary differential equations, then the equation is said to be conformally separable.

Note that the HJ equation for free ranges of the energy admits simple (sum) separation of variables, where the parameter $\mathbf{a}$ depends on one more arbitrary constant - that is $\mathbf{a} \in \mathbf{R}^{n+1}$. The completeness condition on $\mathbf{a}$, to ensure non-degeneracy of the solutions, is well known [23].

There is indeed a strong connection between the conditions of separability for the equations (1.1.1) and (1.1.3). Necessary conditions for separation of the HJ equation are identical with those required for separation of the H/L/S equation, despite the fact that the HJ equation admits *sum separability* whereas the H/L/S equation admits *product separability* of the variables.

The theory of separation of variables goes back well over 200 years. In 1905 a significant theorem by Levi-Civita was formulated [26]:

Theorem 1.1.4 *The HJ equation equation, which can be re-written as*

$$H(q^1, \ldots, q^n, \frac{\partial W}{\partial q^1}, \ldots, \frac{\partial W}{\partial q^n}) = E,$$

where H is the Hamiltonian operator and $W(q^1, \ldots, q^n)$ is the solution of the equation for coordinates q^i, admits (simple) sum separation in a coordinate system if and only if the following equation holds true:

$$\begin{aligned} \frac{\partial H}{\partial p_j}\frac{\partial H}{\partial p_i}\frac{\partial^2 H}{\partial q^i \partial q^j} &- \frac{\partial H}{\partial p_j}\frac{\partial H}{\partial q^i}\frac{\partial^2 H}{\partial q^j \partial p_i} - \frac{\partial H}{\partial q^j}\frac{\partial H}{\partial p_i}\frac{\partial^2 H}{\partial p_j \partial q^i} \\ &+ \frac{\partial H}{\partial q^j}\frac{\partial H}{\partial q^i}\frac{\partial^2 H}{\partial p_j \partial p_i} = 0, \quad i \neq j \end{aligned} \tag{1.1.4}$$

where $p_i \equiv \frac{\partial W}{\partial q^i}$ represents the generalized momentum.

Since the classical Levi-Civita criterion is almost always stated without a clear proof in the literature, an outline of a proof is provided in Appendix A. The Levi-Civita criterion can be generalized to encompass the case of fixed energy, as derived in [3]. It can also be formulated to encompass R-separation of the L and S equation, as in [29]. However direct reference to it can be circumvented by considering conformal transformations to metrics admitting simple sum separability of the HJ equation; this result will be described in Section 2.1.

In this thesis we restrict ourselves to orthogonal separability. In fact in spaces of constant curvature separability of the HJ equation is necessarily orthogonal, a proof in the literature is given in [26]. Orthogonal separability was once assumed to be a strict condition for separation of the Helmholtz equation in any space, until Kalnins and Miller proved this to be false by a counter-example, as well as B. Carter studying the HJ and S equation on the (non-orthogonal) Kerr metric [10], [9]. In this thesis only orthogonal separability will be considered. It should be noted that the general Laplace equation reduces for flat space (and Cartesian coordinates) to the well-known *classical Laplace* equation:

$$\Delta\varphi = \sum_{i=1}^{n} \frac{\partial^2 \varphi}{\partial (u^i)^2} = 0 \tag{1.1.5}$$

Clearly different orthogonal metrics determine the different forms that $\Delta\varphi$ takes. R-separation is a weaker condition but the advantage is clearly that a larger class of coordinate systems admit this property. Research is being undertaken to *characterize* these extra coordinate systems based on a classification scheme - this thesis comprises such research for $\mathbf{E}^3$. In Euclidean space there are in fact eleven inequivalent orthogonal coordinate systems affording simple separability [25], and an additional six admitting R-separability which will be addressed in the coming chapters. Future research is required for higher dimensional flat spaces pertinent to complicated boundary value problems in mechanics and electromagnetism.

1.2 The theory of Stäckel matrices

Powerful tools exist to determine whether or not a coordinate system admits simple separation of variables. One method was formulated by P. Stäckel in 1896 [45]:

Definition 1.2.1 *Associated with each separable metric is a non-singular* ***Stäckel matrix****, which is an $n \times n$ array where the i^{th} row is a function of the coordinate q^i only and the first row of its inverse yields the diagonal components of the contravariant metric tensor.*

Proving the existence of a Stäckel matrix is non-algorithmical. However this matrix determines *everything we need to know* about separated equations [37]. Explicitly it is, for the general case of n-dimensions:

$$[S] = \begin{bmatrix} \phi_{11}(q^1) & \phi_{12}(q^1) & \dots & \phi_{1n}(q^1) \\ \phi_{21}(q^2) & \phi_{22}(q^2) & \dots & \phi_{2n}(q^2) \\ \vdots & \vdots & \vdots & \vdots \\ \phi_{n1}(q^n) & \phi_{n2}(q^n) & \dots & \phi_{nn}(q^n) \end{bmatrix} \tag{1.2.1}$$

Moon and Spencer, in their papers [36], [37], formulated necessary and sufficient conditions - connecting the metric tensor expressed in the separable coordinates with the associated Stäckel matrix - for separation of the Helmholtz equation:

$$g_{ii} \;=\; \frac{S}{M_{i1}} \tag{1.2.2}$$

$$\frac{g^{\frac{1}{2}}}{S} \;=\; f_1(q^1)\cdot f_2(q^2)\cdot\ldots\cdot f_n(q^n), \tag{1.2.3}$$

where M_{i1} is the associated co-factor of the Stäckel matrix, and S is its determinant. The first of the above conditions on the metric tensor is necessary and sufficient for (sum) separation of the HJ equation in Classical Mechanics, whereas both conditions are required for product separation of the Helmholtz as well as the Schrödinger equation. A metric satisfying the first condition is also denoted to be in *Stäckel form*[31]. The second condition is also known as the *Robertson condition* - named after the mathematician who discussed it in a 1927 paper [44]. Remarkably there is a geometric relation with the associated Ricci tensor of the metric (in separable coordinates) for the Robertson condition: $R_{ij} = 0, \;\; i \neq j$ [20], which was discovered by Eisenhart in 1934.

For the case of the Helmholtz equation reducing to the (special case) Laplace equation, the following conditions are special cases of the previous:

$$\frac{g_{ii}}{g_{jj}} \;=\; \frac{M_{j1}}{M_{i1}} \tag{1.2.4}$$

$$\frac{g^{\frac{1}{2}}}{g_{ii}} \;=\; f_1(q^1)\cdot f_2(q^2)\cdot\ldots\cdot f_n(q^n)\cdot M_{i1} \tag{1.2.5}$$

The above equations become more complicated for the case of R-separation of variables, where for $R \neq const$ simple separation of variables is no longer possible. An additional non-constant function $Q(q^1, q^2, \ldots, q^n)$ must be introduced such that, for R-separation of both the Helmholtz and Laplace equation,

$$g_{ii} \;=\; \frac{SQ}{M_{i1}} \tag{1.2.6}$$

$$\frac{g^{\frac{1}{2}}}{S} \;=\; f_1(q^1)\cdot f_2(q^2)\cdot\ldots\cdot f_n(q^n)\cdot R^2 Q \tag{1.2.7}$$

$$\alpha_1 \;\equiv\; \frac{-Q}{R}\sum_{i=1}^{3}\frac{1}{f_i(q^i)g_{ii}}\frac{\partial}{\partial q^i}\left(f_i(q^i)\frac{\partial R}{\partial q^i}\right) = const \tag{1.2.8}$$

A useful formula resulting from the above relates the determinant of the Stäckel matrix S to the function Q [37]:

$$S = \frac{g_{ii} M_{i1}}{Q} \tag{1.2.9}$$

This may be seen as a definition for Q. Clearly the search for the 'right' function $Q(q^1, q^2, \ldots, q^n)$ that satisfies all of the above conditions is also a non-algorithmical endeavor. The function Q is sometimes referred to as the *conformal factor*. An orthogonal metric (with associated Riemann tensor not neccesarily vanishing) admitting separation of variables for the HJ equation with fixed energy is always conformal to an orthogonal metric admitting simple separability of the HJ equation (with free ranges of the energy). This is not necessarily the case for the H/L/S equation. In exceptional circumstances when this is true, a simple relationship exists between R and Q - a connection that becomes apparent when later the conformally invariant Laplace equation is considered. We digress here to prove this relationship, which is a relationship between metrics admitting R-separation of variables for the H/L/S equation, and those satisfying $R_{ij} = 0, \quad i \neq j$.

Theorem 1.2.2 *In orthogonal metrics admitting R-separation of variables for the H/L/S equation that are conformal to orthogonal metrics admitting simple separability of the H/L/S equation, the modulation factor R satisfies*

$$R = Q^{\frac{n-2}{4}} \tag{1.2.10}$$

Proof: Let $g^{'}_{ij}$ be the metric satisfying simple separability, and g_{ij} be the metric admitting R-separability. Thus we have from this starting assumption:

$$\frac{\sqrt{g^{'}}}{S} = f_1(q^1) \cdot f_2(q^2) \cdot \ldots \cdot f_n(q^n) \tag{1.2.11}$$

$$\frac{\sqrt{g}}{S} = f_1(q^1) \cdot f_2(q^2) \cdot \ldots \cdot f_n(q^n) \cdot R^2 Q \tag{1.2.12}$$

Now by definition $g_{ii} = g^{'}_{ii} Q$ which implies, in general for n-dimensions, $g = g^{'} Q^n$ or $g^{'} = g/Q^n$. Therefore if $\frac{\sqrt{g^{'}}}{S} = f_1(q^1) \cdot f_2(q^2) \cdot \ldots \cdot f_n(q^n)$ this implies $\frac{\sqrt{g}}{Q^{n/2} S} = f_1(q^1) \cdot f_2(q^2) \cdot \ldots \cdot f_n(q^n)$. So $\frac{\sqrt{g}}{S} = f_1(q^1) \cdot f_2(q^2) \cdot \ldots \cdot f_n(q^n) \cdot Q^{n/2}$, which implies $Q^{n/2} = R^2 Q$. Hence $Q^{\frac{n-2}{2}} = R^2$ and after taking square roots we obtain the required formula. □

In the special case of 3-dimensions, this reduces to $R = Q^{1/4}$. Examples are numerous in [38]. With 6-sphere coordinates $R = (u^2 + v^2 + w^2)^{-1/2}$ whereas $Q = (u^2 + v^2 + w^2)^{-2}$. The same pattern holds true for inverse oblate coordinates, inverse prolate coordinates and tangent sphere coordinates. Toroidal and bispherical coordinates satisfy this relationship too however they are unique in that they are conformal to non-flat (albeit Ricci diagonal) metrics. The above relation does not hold for the cyclide coordinates, which are conformal to coordinates that do not admit simple separability of the H/L/S equation - indeed these coordinates do not admit an orthogonal Ricci tensor.

Non-trivial R-separation for the Helmholtz equation does not occur in flat spaces or spaces of constant curvature - only the Laplace equation R-separates. Also, any R-separable solution of the Laplace equation on a conformally flat space corresponds to a regular separable solution of the Helmholtz equation on a space of constant curvature [43]. An example of a metric admitting R-separation of the Helmholtz equation on a conformally flat space is given by [31]:

$$\begin{aligned} ds^2 &= (x+y+z)[(x-y)(x-z)dx^2 + (y-z)(y-x)dy^2 \\ &+ (z-x)(z-y)dz^2] \\ R &= (x+y+z)^{-\frac{1}{4}} \end{aligned} \tag{1.2.13}$$

Another example is

$$\begin{aligned} ds^2 &= -dx^2 + dy^2 + (y-x)^{-1}dz^2 \\ R &= (x-y)^{\frac{1}{4}} \end{aligned} \tag{1.2.14}$$

Having introduced Stäckel matrices it must be pointed out that each coordinate system admitting separation of variables does not admit a unique Stäckel matrix: they are not in 1:1 correspondence. Two such matrices are understood to be *equivalent* if their ratios $\frac{S}{M_{i1}}$ are the same - indeed if they generate the same metric tensor. This allows for a flexibility of operations on the columns except for one (usually the first or the last in the literature). Operations on the rows are generally not permitted. The allowed actions on the columns are [37]

1. Interchanging of the i^{th} and the j^{th} column, where we understand $i,\ j = 2, 3, \cdots, n$.

2. Multiplication of the j^{th} column by a non-zero constant $c \in \mathbf{R}$.

3. Addition of each element in the i^{th} column by the corresponding element in the j^{th} column multiplied by a common non-zero factor $c \in \mathbf{R}$.

We conclude this section with a brief discussion of the equivalence between separability of the HJ equation and the existence of a Stäckel matrix, which was achieved by Paul Stäckel in 1893. We start by constructing this non-singular $n \times n$ matrix which we denote as φ_i^k, the index k labeling the column and the index i labeling the row. We define each row of the Stäckel matrix, of course, to only depend on the corresponding coordinate q^i, thus $\frac{\partial}{\partial q^j}\varphi_i^k = 0$ if $i \neq j$. This means then $\varphi_i^k = \varphi_i^k(q^i)$. For notational simplicity we denote $S^{-1} = \varphi_k^i$, taking care not to confuse this with the transpose operation.

Theorem 1.2.3 *The HJ equation of a conservative Hamiltonian (no time-dependence) is separable in the q^i iff:*

1) $(g^{11}, g^{22}, \ldots, g^{nn})$ is a row of the inverse Stäckel matrix.

2) The potential $V = g^{11}U_1(q^1) + g^{22}U_2(q^2) + \ldots + g^{nn}U_n(q^n)$ for some arbitrary set of functions $U_i(q^i)$.

Proof: A known result is that separability of the Hamiltonian implies separability of $\frac{1}{2}g^{ii}p_i^2$, the so called geodesic part. This means that since $p_i = \frac{\partial W}{\partial q^i} = \tilde{\phi}_i(q^i, \alpha_k)$ that $(p^i)^2 = \phi_i(q^i, \alpha_k)$, which is a function of q^i only. The squared variable is still only dependent on q^i. So now we use the fact that, without loss of generality, we may set the total energy E of the Hamiltonian to equal α_n, one of the arbitrary constants [23]. Thus we have $\frac{1}{2}g^{ii}\phi_i(q^i, \alpha_k) = E = \alpha_n$.

How the components of the Stäckel matrix are constructed, based on the above, is given in Appendix C.

1.3 Tensorial formulation of separation of variables theory

A first principles approach to separation and conformal separation of variables for the HJ, H and L equation in terms of the separable coordinates was described in Section 1.2. In this section we shall discuss a coordinate invariant approach based on the theory of valence-two symmetric Killing tensors and valence-two symmetric conformal Killing tensors as discovered by Eisenhart [20], [21] during research in the 1930's. Following [20] we make the definition,

Definition 1.3.1 *A valence-two symmetric tensor K_{ij} is a **Killing tensor** if it satisfies:*

$$K_{ij;l} + K_{jl;i} + K_{li;j} = 0, \tag{1.3.1}$$

where ; denotes the covariant derivative.

Note that in Eisenhart's 1934 paper K_{ij} is denoted as a_{ij}. From the definition one can see that Killing tensors are unique up to a scalar c times the metric tensor. In the literature Killing tensors are often denoted as *simple* Killing tensors to distinguish them clearly from conformal Killing tensors. If the Killing tensor has real pointwise simple eigenvalues and normal eigenvectors, then there exists a system of orthogonal coordinates such that [20]:

$$K_{ij} = \rho_i g_{ij} \text{ (no sum)}, \tag{1.3.2}$$

where the eigenvalues ρ_i satisfy the following system of equations (also known as the *Eisenhart equations*):

$$\frac{\partial \rho_i}{\partial x^i} = 0 \tag{1.3.3}$$

$$\frac{\partial \rho_i}{\partial x^j} = (\rho_i - \rho_j)\frac{\partial \log(g_{ii})}{\partial x^j}, \quad i \neq j \tag{1.3.4}$$

The integrability conditions for the above equations (also called the *Eisenhart integrability conditions*) are

$$\frac{\partial^2 \log(g_{ii})}{\partial x^i \partial x^j} + \frac{\partial \log(g_{ii})}{\partial x^j}\frac{\partial \log(g_{jj})}{\partial x^i} = 0, \quad i \neq j \tag{1.3.5}$$

and

$$\begin{aligned}\frac{\partial^2 \log(g_{ii})}{\partial x^j \partial x^k} &- \frac{\partial \log(g_{ii})}{\partial x^j}\frac{\partial \log(g_{ii})}{\partial x^k} + \frac{\partial \log(g_{ii})}{\partial x^j}\frac{\partial \log(g_{jj})}{\partial x^k} \\ &+ \frac{\partial \log(g_{ii})}{\partial x^k}\frac{\partial \log(g_{kk})}{\partial x^j} = 0, \quad \mathrm{i, j, k\ all\ distinct}\end{aligned} \tag{1.3.6}$$

Whether or not a particular metric admits separation of variables depends on whether or not the above two conditions are satisfied. Indeed it is proven in [2] and [3] that the HJ equation for null geodesics is separable in orthogonal coordinates on an n-dimensional Riemannian space if and only if there exist n Killing tensors $(K_i) = (K_1, K_2, \ldots, K_n)$ such that they are pointwise linearly independent and with common eigenvectors. As already mentioned the metrics admitting separability of the HJ equation for fixed values of the energy are conformally related to metrics allowing separability of the HJ equation for free ranges of the energy via the conformal transformation equation

$$\tilde{g_{ii}} = e^{2\sigma} g_{ii} \tag{1.3.7}$$

for a well behaved function of the coordinates σ. Indeed σ is related to the aforementioned Q factor in determining a Stäckel matrix for proving R-separability of the Laplace equation.

A coordinate invariant characterization of R-separability of the H/L/S equation requires the following definition:

Definition 1.3.2 *A valence-two symmetric tensor K_{ij} is a **conformal Killing tensor** if it satisfies*

$$K_{ij;l} + K_{jl;i} + K_{li;j} = k_i g_{jl} + k_j g_{li} + k_l g_{ij}, \tag{1.3.8}$$

where k_i is some vector field.

The property of being a conformal Killing tensor is preserved under addition of a smooth function f times the metric, a fact that will come into play in the sequel. Indeed, two conformal Killing tensors K and L are said to be *equivalent*, or in the same *equivalence class*, if $K_{ij} = L_{ij} + f g_{ij}$, for some scalar function f. Not all conformal Killing tensors, with pointwise real and distinct eigenvalues and normal eigenvectors, in an equivalence class are simple Killing tensors with respect to some conformally related metric. However, there exists at least one such *representative element* in each equivalence class [3]. This will be proved in the next section. With respect to orthogonal coordinates, the diagonal components of the conformal Killing tensor obey the following relations:

$$K_{ij} = \rho_i g_{ij}, \quad \frac{\partial \rho_i}{\partial x^i} = k_i \quad \text{(no sum)}, \tag{1.3.9}$$

$$\frac{\partial \rho_i}{\partial x^j} = (\rho_i - \rho_j)\frac{\partial \log(g_{ii})}{\partial x^j} + \frac{\partial \rho_j}{\partial x^j} \tag{1.3.10}$$

Eisenhart proved a fundamental theorem linking Killing tensors with Stäckel matrices [20]:

Theorem 1.3.3 *A necessary and sufficient condition for the existence of a Stäckel matrix in some coordinate system - making it a separable coordinate system [30], is that there exists a valence-two Killing tensor with real distinct eigenvalues and normal eigenvectors.*

Remark 1.3.4 *The existence of such a Killing tensor implies that there exists a system of coordinates such that the contravariant components of the diagonalized Killing tensor are elements in a row of the inverse Stäckel matrix, defined for the same coordinates. These coordinates for which this property holds are often called* ***separable coordinates****. This is equivalent to the existence of n Killing tensors, in involution and pairwise commutation, associated with each separable metric - one member always being the metric tensor itself [3].*

The above property is fundamental to this thesis, since we used the connection between Stäckel matrices and Killing tensors to compute Killing tensors for known coordinate systems admitting separation of variables for the Laplace equation [27], [30]. In Appendix B is our own extension of the proof in Eisenhart's 1934 paper, noting that many steps were skipped or left to the reader to verify.

1.4 Example of R-separation of the Laplace equation for toroidal coordinates

A common boundary value problem in electromagnetic theory requiring R-separation of variables are toroidal coordinates in E^3, especially for problems with toroidal boundary conditions such as the 3-Torus. In these coordinates, where $x_1 = \eta$, $x_2 = \theta$ and $x_3 = \psi$, the metric tensor components are

$$\begin{aligned} g_{11} &= g_{22} = \frac{a^2}{(\cosh(\eta) - \cos(\theta))^2} \\ g_{33} &= \frac{a^2 \sinh(\eta)^2}{(\cosh(\eta) - \cos(\theta))^2} \end{aligned} \tag{1.4.1}$$

The denominator in the metric proves an impossible difficulty in the search for a Stäckel matrix associated with simple separation for toroidal coordinates. It can be shown by first principles methods that simple separation of toroidal coordinates is impossible - hence by extension no Stäckel matrix exists. However, if we propose R-separation, and let

$$Q = \frac{a^2}{(\cosh(\eta) - \cos(\theta))^2}$$

then we obtain

$$\frac{M_{11}}{S} = \frac{M_{21}}{S} = 1$$

and

$$\frac{M_{31}}{S} = \frac{1}{\sinh(\eta)^2}$$

The determinant S of the Stäckel matrix is then simply unity and the matrix itself takes on the simple form:

$$\begin{bmatrix} 1 & -1 & -1/\sinh(\eta)^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.4.2}$$

The condition

$$\alpha_1 \equiv \frac{-Q}{R} \sum_{i=1}^{3} \frac{1}{f_i(q^i) g_{ii}} \frac{\partial}{\partial q^i} \left(f_i(q^i) \frac{\partial R}{\partial q^i} \right) = const \tag{1.4.3}$$

is satisfied since

$$R^2 = (\cosh(\eta) - \cos(\theta))^{-1}$$

(recall the relationship between R and Q in the sections prior). For the solutions (f_1, f_2, f_3) in the condition

$$\frac{g^{\frac{1}{2}}}{S} = f_1(q^1) \cdot f_2(q^2) \cdot \ldots \cdot f_n(q^n) \cdot R^2 Q$$

we finally obtain $\alpha_1 = 1/4$. Therefore requirements for R-separation of the Laplace equation are satisfied.

1.5 Symmetry operator approach to separation of variables theory

In the classic papers by Boyer, Kalnins and Miller [6], [7], a connection with the existence of separable coordinates of the H/L equation and commuting pairs of second order *symmetry operators* is elucidated. The separated solutions for orthogonal coordinate systems are characterized as common eigenfunctions of pairs of commuting symmetry operators. These operators are linear and differential. A first order symmetry operator is of the form

$$L = \sum a_j(x)\partial_j + b(x), \tag{1.5.1}$$

where a_j and b are analytic functions of the coordinates in some domain D on the manifold such that $L\psi$ is a solution of the Helmholtz equation in D for any analytic solution ψ of the Helmholtz equation in D. The set of all such symmetry operators forms a Lie algebra under the operations of scalar multiplication and commutator bracket $[L_1, L_2] = L_1L_2 - L_2L_1$. Second order symmetry operators are constructed from products of first order symmetry operators. This is also known as an *enveloping* algebra of the space. Each separable system is associated with a two-dimensional subspace of commuting operators with S_1, S_2 a non-unique basis for the subspace. The Euclidean group of isometries acts on the set of all two-dimensional subspaces of commuting operators and decomposes this set into orbits of equivalent subspaces. Separable coordinates associated with equivalent subspaces are regarded as equivalent, as one can obtain any such system from any other by a Euclidean isometric transformation.

To study R-separable coordinates, the Euclidean group of isometries is extended to encompass the full *conformal group* of $\mathbf{E}^3$, as well as the discrete inversion and space reflection. R-separable coordinates are regarded as equivalent to separable coordinates if one can obtain the other and vice versa by a group transformation belonging to the above set. An R-separable coordinate is deemed 'additional' if this cannot be done. By this measure there are six additional R-separable coordinates found by [7] along with the eleven simply separable ones that E^3 admits with respect to the full conformal group as well as the discrete inversion and reflection. It should be noted that the extended group of transformations does not make equivalent any of the eleven simply separable coordinates classified with respect to the Euclidean isometry group.

In their analysis Boyer, Kalnins and Miller used the isomorphism between the conformal group and the isometry group defined on Minkowski space of dimension $(n + 2)$ where the group action becomes linear [6]. This approach was anticipated by Bôcher who constructed R-separable coordinates for the n-dimensional Laplace equation by a method in going up two dimensions and considering functions on the $(n + 2)$ cone. Modern conformal geometers have recently denoted this as the *ambient space*. This is beyond the scope of this thesis however, as we restrict ourselves directly to Euclidean space.

In the paper of [7], the above formalism is applied to the Helmholtz equation defined on *complex Riemannian* manifolds. A discussion of this formalism is beyond the scope of this thesis. Results from this article will however be correlated with the additional R-separable coordinates found in the course of my research by means of the invariance of conformal Killing tensors. Indeed the coefficients A^{ij} of the second order part of the symmetry operators S characterizing each type of R-separable rotationally symmetric coordinates, with respect to Cartesian coordinates, listed in Table 2 of [6] when written as

$$S = \partial_i A^{ij} \partial_j$$

correspond to the components of conformal Killing tensors equivalent to those that will be calculated in this thesis for bi-cyclide, flat-ring cyclide, disk cyclide and toroidal coordinates. It should be added that the simple Killing tensors found for the eleven simply separable orthogonal coordinates in Euclidean space [25] correspond to the coefficients A^{ij} of the second order part of the symmetry operators S listed in Table 1 of [6].

Referring to the previous example for toroidal coordinates, the two second order symmetry operators associated with that coordinate system are:

$$\begin{aligned} S_1 &= (x^2\partial_1 - x^1\partial_2)^2 \\ S_2 &= \frac{1}{4}(\partial_3 + x^3 + ((x^3)^2 - (x^1)^2 - (x^2)^2)\partial_3 \\ &+ 2x^3x^1\partial_1 + 2x^3x^2\partial_2)^2 \end{aligned} \tag{1.5.2}$$

1.6 Outline of the thesis

The remainder of the thesis will be organized as follows. In Chapter 2 is an illustration of the modern tools used in the proof of the connection between Stäckel formalism and conformal Killing tensors, as well as the very important link between conditions for sum separation of variables of the HJ equation and product separation of variables of the H/L and Schrödinger equations. Killing vectors and conformal Killing vectors are also introduced, as well as the definition of isometries and group transformations. In Chapter 3, we confirm using symmetric products of conformal Killing vectors that the number of arbitrary constants for the most general trace-free conformal Killing tensor defined in $\mathbf{E}^3$ is thirty five, as is stated in the literature [18, 46, 47]. All independent relations are listed. Initially there are twenty constants too many; this requires one to impose fourteen conditions that result from the trace-free assumption. There are an additional six conditions that arise from relationships among the basis elements themselves. In Chapter 4, the rotationally invariant subset of all conformal Killing tensors is given, using the known coordinate systems in [38] admitting conformal separation of variables. The general rotationally symmetric Killing tensor is deduced from the most general Killing tensor by two equivalent means. These two approaches in consideration of the normality (integrability) of the eigenvectors (the TSN conditions) of the coordinate surfaces [48] are presented and make the study of the rotational webs considerably simpler. After this the characteristic Killing tensors of all the known R-separable coordinates are given. Their representations, in terms of symmetric tensor products of conformal Killing vectors, are also discussed and in difficult cases derived first (especially for fourth degree surfaces defined in terms of Jacobi elliptic functions).

In Chapter 5 the group transformations preserving the rotationally invariant subset of all conformal Killing tensors are given. The conformal group, as well as discrete operations not continuously connected with the identity, are discussed along with their effect on the transformed conformal Killing tensor components. In Chapter 6, we present a proof based on the set of defined group transformations that the R-separable webs known thus far are either related to simple separable webs or are otherwise inequivalent. The classical theory of invariants [34] is a useful tool in this question and applied to characterize all the rotationally symmetric coordinates tabulated in [38]. Although the main contents in Chapter 6 have been published in the Journal of Mathematical Physics [13], the formalism presented here differs from the formalism of the paper in that we gave an alternate characterization conceived in 2006. For historical reasons, we chose to present this 'first principles' approach which is admittedly not as compact as the formalism of invariants and covariants of bi-quartic polynomials which is given in the paper. A simple proof will be shown that the only additional R-separable coordinates admitting symmetries are the rotational ones, leaving the asymmetrical cases to be considered next. In Chapter 7 the general Laplace equation is modified to include the property of invariance of solutions under conformal transformations. This is also called the *conformally invariant Laplace equation* and an ansatz for a Stäckel matrix associated with it is used to derive metrics of asymmetric coordinates expressed in canonical Cartesian coordinates. These coordinates are discussed in light of the results of [5] and [7]. Finally, we draw conclusions in Chapter 8 and discuss directions for future research. Some classical proofs not easily found in the literature are given in the Appendices for the interested reader.

The reader will no doubt realize that the classification of the coordinate webs and the algorithm for determining characteristic conformal Killing tensors for all coordinates considered is highly computational. Nevertheless, all computations are *purely algebraic* in nature, and this allowed all tasks to be performed in Maple 9.

Chapter 2

Theory of separation of variables

2.1 Link between Stäckel formalism and conformal Killing tensors

A beautiful geometric result is that all separable webs are defined by valence-two symmetric Killing tensors, with pointwise simple eigenvalues and normal eigenvectors. Such Killing tensors are said to be characteristic. The separable webs are the families of $(n-1)$-dimensional hypersurfaces orthogonal to each eigenvector field of the Killing tensor. This geometrical property is why characterizing all Killing tensors in a certain dimension is fundamental to this research. The goal is to express the Killing and conformal Killing tensors in canonical Cartesian coordinates, not in terms of canonical separable coordinates. This is because in physical problems involving potentials, where the method of separation of variables is used, the potential is usually expressed in Cartesian coordinates. To this end the Jacobian of the tensor transformation law must be calculated for every coordinate system studied, and with the known (contravariant) Killing tensor diagonalized in the separable coordinates, the following equation applied:

$$\mathbf{K} = \mathbf{J}^T\mathbf{D}\mathbf{J}, \tag{2.1.1}$$

where $\mathbf{J}$ is the Jacobian calculated from the coordinate transformation from Cartesian to separable coordinates and $\mathbf{D}$ is the diagonalized Killing tensor. The tensorial expression can be written as

$$K^{ij} = \frac{\partial x^i}{\partial u^k}\frac{\partial x^j}{\partial u^l}D^{kl}, \tag{2.1.2}$$

where x^i are the canonical Cartesian coordinates and u^i are the canonical separable coordinates. In component form the Killing tensor $\mathbf{K}$ is:

$$\begin{bmatrix} \frac{\partial x^1}{\partial u^1} & \frac{\partial x^1}{\partial u^2} & \cdots & \frac{\partial x^1}{\partial u^n} \\ \frac{\partial x^2}{\partial u^1} & \frac{\partial x^2}{\partial u^2} & \cdots & \frac{\partial x^2}{\partial u^n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial x^n}{\partial u^1} & \frac{\partial x^n}{\partial u^2} & \cdots & \frac{\partial x^n}{\partial u^n} \end{bmatrix} \begin{bmatrix} \rho^1 g^{11} & 0 & \cdots & 0 \\ 0 & \rho^2 g^{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \rho^n g^{nn} \end{bmatrix} \begin{bmatrix} \frac{\partial x^1}{\partial u^1} & \frac{\partial x^1}{\partial u^2} & \cdots & \frac{\partial x^1}{\partial u^n} \\ \frac{\partial x^2}{\partial u^1} & \frac{\partial x^2}{\partial u^2} & \cdots & \frac{\partial x^2}{\partial u^n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial x^n}{\partial u^1} & \frac{\partial x^n}{\partial u^2} & \cdots & \frac{\partial x^n}{\partial u^n} \end{bmatrix}^T \tag{2.1.3}$$

where x^i are Cartesian coordinates in terms of the separable coordinates u^i. Clearly to calculate the Jacobian explicitly the coordinate transformation law must be known. After the above equation is applied, the result is initially expressed in separable coordinates (albeit with the matrix form's basis being the standard canonical basis) - the last step is to calculate or guess the result in Cartesian coordinates. To assist the reader in parallel calculations, for every separable coordinate system studied in this thesis, the coordinate transformation law and the associated Stäckel matrix will be provided. Although the Killing tensors can be computed by solving the Eisenhart equations, we chose the route of first calculating the diagonalized Killing tensors through Stäckel theory and then using the (proven) link given by Eisenhart. The recipe for finding conformal Killing tensors is precisely the same save for the different Jacobian arising from the conformal transformation law associated with simple separability - recall they share the same Stäckel matrix modulo the functions R and Q.

Recent research by S. Benenti, C. Chanu and G. Rastelli has yielded additional interpretations of conditions for metrics to admit separation of variables of the H/L as well as the HJ equation. In [15] they showed that R-separation of the H/L equation is equivalent to additive R-separation of the HJ equation for fixed value of the energy (instead of a free range parameter of the energy). Furthermore they show that R-separation of variables corresponds to separability of the HJ equation for fixed values of the energy, whereas simply separable coordinates correspond to (simple) separability of the HJ equation for free ranges of the energy. This gives a physical insight into what happens when simple separability is relaxed. A fundamental concept they introduced to prove the above results is the so called *Stäckel operator*:

Definition 2.1.1 *A Stäckel operator is a linear second order differential operator defined on any real function* $f(Q) \to \mathbf{R}$ *such that:*

$$S_{ij}(f) = \partial^2_{ij} f - \partial_j \ln(g^{ii})\partial_i f - \partial_i \ln(g^{jj})\partial_j f \tag{2.1.4}$$

Stäckel operators satisfy the following properties [2], [3]:

$$S_{ij}(c) \;=\; 0$$

$$
\begin{aligned}
S_{ij}(A+B) &= S_{ij}(A)+S_{ij}(B) \\
S_{ij}(cA) &= cS_{ij}(A) \\
S_{ij}(AB) &= AS_{ij}(B)+BS_{ij}(A)+\partial_i A\partial_j B+\partial_j A\partial_i B \\
S_{ij}(A^{-1}) &= 2A^{-3}\partial_i A\partial_j A - A^{-2}S_{ij}(A) \qquad (2.1.5)
\end{aligned}
$$

Stäckel operators $\tilde{S}_{ij}$ corresponding to a conformal orthogonal metric $\tilde{g}^{ii} = e^{-2\sigma}g^{ii}$ satisfy the above as well as:

$$
\begin{aligned}
\tilde{S}_{ij}(A) &= S_{ij}(A)+e^{-2\sigma}\cdot(\partial_i e^{2\sigma}\partial_j A+\partial_i A\partial_j e^{2\sigma}) \\
\tilde{S}_{ij}(\tilde{g}^{kk}) &= e^{-2\sigma}S_{ij}(g^{kk})-g^{kk}e^{-4\sigma}S_{ij}(e^{2\sigma}) \\
&= e^{-2\sigma}g^{kk}\left(\frac{1}{g^{kk}}S_{ij}(g^{kk})-e^{-2\sigma}S_{ij}(e^{2\sigma})\right) \qquad (2.1.6)
\end{aligned}
$$

Proposition 2.1.2 *An orthogonal coordinate system admitting simple separability of the HJ equation satisfies*

$$
S_{ij}(g^{hh}) = 0 \qquad (2.1.7)
$$

Proof: Consider a natural Hamiltonian in orthogonal coordinates of the form:

$$
H(q,p) = \frac{1}{2}g^{ii}p_i^2 + V(q) \equiv G+V \qquad (2.1.8)
$$

It can be shown, by some algebra, that the Levi-Civita separability criterion on H is equivalent to the equation:

$$
L_{ij}(H) = g^{ii}g^{jj}p_i p_j\left(\frac{1}{2}S_{ij}(g^{kk})p_k^2+S_{ij}(V)\right) = 0 \quad \text{(n.s)} \qquad (2.1.9)
$$

and this is satisfied if and only if $\frac{1}{2}S_{ij}(g^{kk})p_k^2+S_{ij}(V)=0$. Indeed g^{kk} is denoted as a *Stäckel metric* iff $S_{ij}(g^{kk})=0$, and a potential is simply separable in these coordinates iff $S_{ij}(V)=0$. This completes the proof [3] that an orthogonal coordinate system admitting simple separability of the HJ equation satisfies Eq. (2.1.7). □

In the formalism of [28] and [33], T is a Stäckel multiplier if $S_{ij}(T) = 0$. Given a metric ds^2 in Stäckel form, the function $T(x,y,z)$ is a Stäckel multiplier if $d\hat{s}^2 = Tds^2$ is also in Stäckel form. A Stäckel transform is one that is a conformal transformation preserving the Stäckel form of the separable system.

Remark 2.1.3 *In the previous formula about $\tilde{S}_{ij}$, if we in particular choose $e^{2\sigma}$ to be any one of $(g^{11},\ldots,g^{nn})$, then one recovers a theorem of [7]: if g^{ii} is a Stäckel metric, then all of $\left(\frac{g^{ii}}{g^{11}},\ldots,\frac{g^{ii}}{g^{nn}}\right)$ are Stäckel metrics. Remarkably the equation $S_{ij}(g^{kk})=0$ is also equivalent to equations:*

$$
\begin{aligned}
\partial^2_{ij}|g_{kk}| &- \partial_i\ln|g_{kk}|\partial_j\ln|g_{kk}|+\partial_i\ln|g_{kk}|\partial_j\ln|g_{ii}| \\
&+ \partial_j\ln|g_{kk}|\partial_i\ln|g_{jj}| = 0 \qquad (2.1.10)
\end{aligned}
$$

If in the above one makes the substitution $g_{ii} = e_iH_i^2, e_i = \pm 1$, one then recovers the famous Eisenhart's equations. Thus we see the power of the Stäckel operators.

Theorem 2.1.4 *The HJ equation for fixed value of the energy, namely* $\frac{1}{2}g^{ii}p_i^2 + (V - E) = 0$ *where* $p_i \equiv \partial_i W$ *is separable in orthogonal coordinates, for* $E \in R$*, if and only if*

$$\frac{S_{ij}(g^{hh})}{g^{hh}} - \frac{S_{ij}(g^{kk})}{g^{kk}} = 0 \tag{2.1.11}$$

and

$$S_{ij}(V) = \frac{(V - E)}{g^{hh}} S_{ij}(g^{hh}) \tag{2.1.12}$$

for all indices $h,\ k$ *and* $i \neq j$ *[3].*

Remark 2.1.5 *This condition is conformally invariant, and will be exploited fully in Chapter 7. This is shown in [3] to be equivalent to the existence of a function* $e^{2\sigma}$ *such that the conformal metric* $\tilde{g}^{ii} = e^{-2\sigma}g^{ii}$ *is a Stäckel metric, that is* $\tilde{S}_{ij}(\tilde{g}^{kk}) = 0$*.* ***Conformally separable coordinates*** *are orthogonal coordinates* $q = q^i$ *for which Eq. (2.1.11) or Eq. (2.1.12) holds. Indeed the conformally separable coordinates are useful because they are the only ones in which a natural Hamiltonian with fixed value of the energy can be solved by additive separation of variables.*

An important property is that coordinates q^i are conformally separable if and only if there exists a Stäckel matrix, with elements of the inverse denoted by $\varphi^i_{(n)}$, such that

$$\exists\ e^{2\sigma} \ \mid\ e^{-2\sigma}g^{ii} = \varphi^i_{(n)} \Leftrightarrow \frac{g^{ii}}{\varphi^i_{(n)}} = \frac{g^{jj}}{\varphi^j_{(n)}} \tag{2.1.13}$$

for all indices i and j.

The Eq. (2.1.11) and Eq. (2.1.12) of Theorem 2.1.4 for fixed value of the energy are useful in the proofs of the following two theorems:

Theorem 2.1.6 *The HJ equation*

$$\frac{1}{2}g^{ii}p_i^2 = E, \tag{2.1.14}$$

with $E \neq 0$ *fixed, is separable in orthogonal coordinates* q^i *iff* g^{ii} *is a Stäckel metric, that is iff it is separable in the ordinary sense for all values of* E*.*

Proof: Since $V = 0$, Eq. (2.1.12) yields $S_{ij}(g^{kk}) = 0$. For the other direction, if the equation is separable in the ordinary sense then $S_{ij}(g^{kk}) = 0$ and thus both Eq. (2.1.11) and Eq. (2.1.12) are trivially satisfied. □

Theorem 2.1.7 *The HJ equation of the null geodesics*

$$g^{ii}p_i^2 = 0 \tag{2.1.15}$$

is separable in the orthogonal coordinates q^i iff these coordinates are conformally separable.

Proof: For $V = E = 0$, the Eq. (2.1.12) is trivially satisfied, hence only Eq. (2.1.11) characterizes the equation. □

Theorem 2.1.8 *The HJ equation*

$$\frac{1}{2}g^{ii}p_i^2 + V - E = 0 \quad (V - E) \neq 0, \tag{2.1.16}$$

is separable if and only if the conformal metric

$$\tilde{g}^{ii} = \frac{1}{E - V}g^{ii} \tag{2.1.17}$$

is a Stäckel metric, or equivalently, if and only if for all indices h, k and $i \neq j$,

$$\frac{1}{g^{kk}}S_{ij}(g^{kk}) = \frac{1}{V - E}S_{ij}(V). \tag{2.1.18}$$

Thus the coordinates are conformally separable, but the conformal factor $e^{2\sigma}$ must be equal to the function $V - E$.

Proof: Eq. (2.1.12) is equivalent to $\frac{1}{g^{kk}}S_{ij}(g^{kk}) = \frac{1}{V-E}S_{ij}(V)$. Note this is just an instance of the Stäckel transform [8]. □

Proposition 2.1.9 *The HJ equation is separable for two distinct values of the energy E if and only if it is separable in the ordinary sense. Alternatively - if a natural Hamiltonian $H = G + V$ is not simply separable, then there exists at most one value of the energy E such that $H = E$ is separable.*

Consider one very important case of the HJ equation with fixed value of the energy,

$$\frac{1}{2}g^{ii}p_i^2 + (V - E) = 0, \qquad (V - E) \neq 0. \tag{2.1.19}$$

This is separable if and only if $\tilde{g}^{ii} = \frac{1}{(E-V)}g^{ii}$ is a Stäckel metric, or equivalently $\frac{S_{ij}(g^{kk})}{g^{kk}} = \frac{1}{(E-V)}S_{ij}(V)$. Coordinates are conformally separable, with conformal factor $e^{2\sigma}$ equal to $(E - V)$ since with $e^{2\sigma} = (E - V)$,

$$\tilde{S}_{ij}(\tilde{g}^{kk}) = \frac{1}{(E - V)}S_{ij}(g^{kk}) - \frac{g^{kk}}{(E - V)^2}S_{ij}(E - V) \tag{2.1.20}$$

hence $\tilde{S}_{ij}(\tilde{g}^{kk}) = 0$. The metric $\tilde{g}^{ii} = (E - V)^{-1}g^{ii}$ is called the *Jacobi metric* of the Hamiltonian $H = G + V$ with fixed value of the energy E.

Proposition 2.1.10 *If a conformal Jacobi metric is a Stäckel metric for two distinct values $E_1 \neq E_2$ of the energy, then it is a Stäckel metric for all energy E.*

Proof: Indeed

$$\frac{1}{(V-E_1)} S_{ij}(V) = \frac{1}{(V-E_2)} S_{ij}(V) \Rightarrow S_{ij}(V) = 0 \Rightarrow S_{ij}(g^{kk}) = 0. \quad \square \quad (2.1.21)$$

Conditions for separability of the HJ equation for fixed energy correspond to those for separation of the Schrödinger equation for fixed value of the energy. If one wishes to impose conditions on an arbitrary metric tensor, such that R-separation of the H/L equation is satisfied, one needs to consider the final 'compatibility' condition [14]:

$$\begin{aligned} S_{ij}(\chi) g^{11} &= S_{ij}(g^{11}) \chi = 0 \\ S_{ij}(\chi) g^{22} &= S_{ij}(g^{22}) \chi = 0 \\ &\vdots \\ S_{ij}(\chi) g^{nn} &= S_{ij}(g^{nn}) \chi = 0 \\ i &\neq j \end{aligned} \quad (2.1.22)$$

where

$$\chi \equiv \frac{g^{hh}}{4} \left(2\partial_h \Gamma_h - \Gamma_h^2 + \frac{1}{2} R_{hh} \right) \quad (2.1.23)$$

R_{hh} are the diagonal components of the Ricci tensor associated with the orthogonal metric g^{ij}. Furthermore,

$$\Gamma_h \equiv g_{ih} \Gamma^i, \Gamma^i \equiv g^{hh} \Gamma^i_{hh}, \quad (2.1.24)$$

where Γ^i_{hh} is the standard Christoffel symbol of the metric.

Remark 2.1.11 *For the special case of three dimensions this yields nine pdes that the metric coefficients must satisfy - this is considered and with these tools the general metric of a totally asymmetric coordinate web is integrated in the last chapter. The Γ_h symbols are useful in other respects as it can be shown that $R_{ij} = \frac{3}{2}\partial_j \Gamma_i$. Hence the Robertson condition, namely $R_{ij} = 0$ for $i \neq j$, is equivalent to $\partial_i \Gamma_j = 0$. The Γ_h symbols also share an explicit relationship with the metric tensor that conformally separable coordinates satisfy:*

$$\partial_i \Gamma_j = \partial_j \Gamma_i \Leftrightarrow \frac{S_{ij}(g^{jj})}{g^{jj}} = \frac{S_{ij}(g^{ii})}{g^{ii}} \quad (2.1.25)$$

The material introduced in this section thus far is sufficient to prove two propositions, one for the modulation factor R and the other for the conformal factor Q:

Proposition 2.1.12 *The modulation factor R satisfies the relation*

$$\partial_i \log(R) = \frac{1}{2} \cdot \partial_i \left(\log \left(\frac{g_{ii}}{\sqrt{g}} \right) \right) + q^i(x^i), \qquad (2.1.26)$$

where $q^i(x^i)$ is some arbitrary function of the i^{th} coordinate only, denoted here as x^i.

Proof: From Stäckel theory:

$$\frac{\sqrt{g}}{\varphi} = R^2 Q \prod_{i=1}^{n} f_i(x^i),$$

where f_i are functions of the coordinate x^i only and R, Q are in general functions of all variables. It is sometimes customary to denote by φ the determinant of the non-singular Stäckel matrix, and g is the determinant of the non-singular metric tensor. The above expression can be inverted to give an equivalent expression:

$$\frac{\varphi}{\sqrt{g}} = \frac{\prod_{i=1}^{n} \Psi_i(x^i)}{R^2 Q}$$

Here trivially $\Psi_i(x^i) \equiv f_i(x^i)^{-1}$. Substituting the known expression for the determinant φ, we arrive at:

$$\frac{g_{ii} M_{i1}}{Q\sqrt{g}} = \frac{\prod_{i=1}^{n} \Psi_i(x^i)}{R^2 Q}$$

Canceling out the common factor Q and taking the logarithm of both sides yields:

$$\log \left(\frac{g_{ii}}{\sqrt{g}} \right) + \log(M_{i1}) = \sum_{i=1}^{n} \log(\Psi_i(x^i)) - 2\log(R)$$

Performing the derivative of both sides with respect to x^i and using the fact that the cofactor M_{i1} is independent of x^i we arrive at:

$$\partial_i \log \left(\frac{g_{ii}}{\sqrt{g}} \right) = -2\partial_i \log(R) + \partial_i \log(f_i^{-1}(x^i))$$

Re-arranging and division by two yields:

$$\partial_i \log(R) = -\frac{1}{2}\partial_i \log(f_i(x^i)) - \frac{1}{2}\partial_i \log \left(\frac{g_{ii}}{\sqrt{g}} \right)$$

A simple exercise with Christoffel symbols, assuming orthogonal metrics, will yield $\Gamma_i = \partial_i \left(\log \left(\frac{g_{ii}}{\sqrt{g}} \right) \right)$. We arrive then at the desired formula of separability conditions pertaining to an R-separability test:

$$\partial_i \log(R) = \frac{1}{2}\Gamma_i + q^i(x^i) \qquad \square$$

The $q^i(x^i)$ are arbitrary functions to be determined. At first glance this appears contrary to the above formalism by a factor of minus unity, however in the literature formulae for R are sometimes the reciprocal of the modulation factor defined in [38]. Thus the equation is satisfied from the above derivation and furthermore the form of the arbitrary function $q^i(x^i)$ is uniquely determined, up to the factor Q used in the Stäckel matrix definition, by the existence of the separable functions $f_i(x^i)$. Specifically the equation amounts to:

$$q^i(x^i) = -\frac{1}{2}\partial_i \log(f_i(x^i))$$

Proposition 2.1.13 *The reciprocal of the conformal Q factor in Stäckel theory can be expanded in the following way:*

$$\frac{1}{Q} = g^{11}f_1(u^1) + g^{22}f_2(u^2) + g^{33}f_3(u^3) + \cdots + g^{nn}f_n(u^n), \tag{2.1.27}$$

where each f_i is a function of the i^{th} coordinate only, denoted here as u^i.

Proof: The conformal Q function is defined in [38] to satisfy

$$g_{ii} = \frac{SQ}{M_{i1}} \Rightarrow g^{ii} = \frac{M_{i1}}{SQ} \Rightarrow \frac{1}{Q} = g^{ii}\frac{S}{M_{i1}}, \tag{2.1.28}$$

where M_{i1} denotes the determinant of the matrix co-factor. The determinant S appearing in Eq.(2.1.28) can be expanded in terms of the elements of the first column of the Stäckel matrix, which we know from Stäckel theory to be functions of the corresponding i^{th} variable only. Explicitly:

$$g^{ii}\frac{S}{M_{i1}} = g^{ii}\frac{f_1(u^1)M_{11} + f_2(u^2)M_{21} + f_3(u^3)M_{31} + \cdots + f_n(u^n)M_{n1}}{M_{i1}} \tag{2.1.29}$$

As we know from [38], the ratio of minors yields the inverse ratio of the corresponding covariant metric terms. This was used for studying separability of the Laplace equation; now we use the fact that the ratio of the minors yields the ratio of the corresponding contravariant metric terms. Explicitly:

$$\begin{aligned} g^{ii} &= \frac{M_{i1}}{SQ} \Rightarrow \frac{g^{ii}}{g^{jj}} = \frac{M_{i1}}{M_{j1}} \\ \Rightarrow \frac{1}{Q} &= g^{ii}\frac{S}{M_{i1}} \\ &= g^{11}f_1(u^1) + g^{22}f_2(u^2) + g^{33}f_3(u^3) + \cdots + g^{nn}f_n(u^n) \end{aligned} \tag{2.1.30}$$

as is required to show. □

The Schrödinger equation can also be handled by the formalism introduced in this section. By [14], there is a one to one correspondence between the solutions of

$$-\frac{\hbar^2}{2}\Delta\psi + (V - E)\psi = 0, \tag{2.1.31}$$

of the form $\psi = R\prod_i \phi_i(q^i)$ and the additively separated solutions $u = \ln\phi$ of

$$g^{ij}u_iu_j + g^{ii}u_{ii} - \hat{\Gamma}^i u_i + \frac{2}{\hbar^2}E - U = 0, \tag{2.1.32}$$

where $u_i = \partial_i u, u_{ii} = \partial_i^2 u$ and U is the modified potential

$$U = -\left(\frac{\Delta R}{R} - \frac{2}{\hbar^2}V\right), \tag{2.1.33}$$

and

$$\hat{\Gamma}^i = g^{ij}(\Gamma_j - 2\partial_j \ln R). \tag{2.1.34}$$

The following proposition is proven in [14]:

Theorem 2.1.14 *Equation (2.1.32) is separable in orthogonal coordinates q^i if and only if for all $i \neq j$*

$$\begin{aligned} \partial_j\hat{\Gamma}^i - \hat{\Gamma}^i\partial_j \ln(g^{ii}) &= 0, \\ \frac{S_{ij}(g^{hh})}{g^{hh}} - \frac{S_{ij}(g^{kk})}{g^{kk}} &= 0, \quad \forall h, k, \\ S_{ij}(U)g^{hh} - S_{ij}(g^{hh})\left(U - \frac{2}{\hbar^2}E\right) &= 0, \quad \forall h. \end{aligned} \tag{2.1.35}$$

Note that the potential V is arbitrary: in Chapter 7 a very specific choice for V is made to ensure conformal invariance, however the above conditions will still hold.

2.2 Invariant theory of conformal Killing tensors

In this section the theory of conformal Killing tensors defined on a Riemannian manifold $(M, \mathbf{g})$ is described. We begin this section with a definition of the *conformal group* acting on this space. This group of transformations and the corresponding Lie algebra of infinitesimal transformations are fundamental for the classification scheme that will be constructed.

Definition 2.2.1 *A diffeomorphism $\phi : M \to M$ with the property that $\phi_*\mathbf{g} = f\mathbf{g}$, where ϕ_* is the push forward of ϕ and f some positive function, is said to be conformal.*

The set of all such transformations forms a Lie group of maximal dimension $\frac{1}{2}(n+1)(n+2)$, provided $n \geq 3$, called the **conformal group** of transformations of $(M, \mathbf{g})$ which we'll denote by $C(M)$. If ϕ is a homothetic transformation, then f is a positive number not equal to unity. If ϕ is an isometry, then $f = 1$.

Proposition 2.2.2 *Let V be an infinitesimal generator of the one-parameter group of conformal transformations ϕ_t. Then*

$$\mathcal{L}_V \mathbf{g} = h\mathbf{g}, \tag{2.2.1}$$

where h is some function.

If ϕ_t denotes a one-parameter group of homothetic transformations then the function h is a non-zero constant. If ϕ_t denotes a one-parameter group of isometries then the function h is zero.

We now proceed to give the general definition of a conformal Killing tensor on $(M, \mathbf{g})$.

Definition 2.2.3 *A conformal Killing tensor of valence p defined on $(M, \mathbf{g})$ is a symmetric $(p, 0)$ tensor $\mathbf{K}$ which satisfies the conformal Killing tensor equation*

$$[\mathbf{g}, \mathbf{K}] = 2\mathbf{k} \odot \mathbf{g}, \tag{2.2.2}$$

where $[\ ,\ ]$ denotes the Schouten bracket, $\mathbf{k}$ is some symmetric tensor of type $(p-1, 0)$ and $\odot$ denotes the symmetric tensor product.

The tensor $\mathbf{k}$ can be determined by contracting Eq.(2.2.2) with the covariant metric. Let K and L be symmetric tensors of types $(p, 0)$ and $(q, 0)$ respectively. The Schouten bracket of K and L denoted by $[K, L]$ is a tensor of type $(p + q - 1, 0)$ and is defined in terms of local coordinates x^i, $i = 1, \ldots, n$, by

$$\begin{aligned} [K, L]^{i_1 \ldots i_{p+q-1}} &= -qK^{(i_1 \ldots i_p}{}_{,\,k} L^{i_{p+1} \ldots i_{p+q-1})k} \\ &+ pK^{k(i_1 \ldots i_{p-1}} L^{i_p \ldots i_{p+q-1})}{}_{,\,k} \end{aligned} \tag{2.2.3}$$

It can be shown that $[K, L]$ has the following properties:

$$[K, L] = -[L, K]$$

$$[K, L + M] = [K, L] + [K, M]$$
$$[K, L \odot M] = [K, L] \odot M + L \odot [K, M]$$
$$[K, [L, M]] + [M, [K, L]] + [L, [M, K]] = 0$$

Special cases:
If, in the definition of the Schouten bracket $p = q = 1$, then $[K, L]$ is the standard Lie bracket of the vector fields K and L. For the case $p = 1$, q arbitrary:

$$[K, L]^{i_1 \dots i_q} = (\mathcal{L}_K L)^{i_1 \dots i_q}$$

which is the Lie derivative of L with respect to K. When $p = 1$, $\mathbf{K}$ is said to be a **conformal Killing vector** (CKV) and Eq. (2.2.2) reads

$$\mathcal{L}_{\mathbf{K}} \mathbf{g} = f \mathbf{g}, \tag{2.2.4}$$

where $\mathcal{L}$ denotes the Lie derivative operator. With respect to a local system of coordinates x^i Eq. (2.2.2) may be written as

$$\nabla_{(i_1} K_{i_2 \dots i_{p+1})} = k_{(i_1 \dots i_{p-1}} g_{i_p i_{p+1})}, \tag{2.2.5}$$

where ∇ denotes the covariant derivative with respect to the Levi-Civita connection of $\mathbf{g}$. If $\mathbf{k} = 0$ in Eq.(2.2.2), then $\mathbf{K}$ is said to be a **Killing tensor**.

It follows from the properties of the Schouten bracket that the set $CK^p(M)$ of all conformal Killing tensors of type $(p, 0)$ forms a generally infinite dimensional vector space. However, it's important to note that

$$\mathbf{K}' = \mathbf{K} + \mathbf{l} \odot \mathbf{g}, \tag{2.2.6}$$

where $\mathbf{l}$ is any symmetric tensor of type $(p-2, 0)$, also defines a CKT. This property may be used to define the following equivalence relation on $CK^p(M)$:

$$\mathbf{K}' \sim (K) \Leftrightarrow \mathbf{K}' = \mathbf{K} + \mathbf{l} \odot \mathbf{g}, \tag{2.2.7}$$

Let $C\hat{K}^p(M)$ denote the set of equivalence classes of $CK^p(M)$. One may equip $C\hat{K}^p(M)$ with the structure of a vector space over the reals. Let $\hat{\mathbf{K}}_1$ and $\hat{\mathbf{K}}_2 \in C\hat{K}^p(M)$. Let $\mathbf{K}_1$ and $\mathbf{K}_2$ be representative elements of $\hat{\mathbf{K}}_1$ and $\hat{\mathbf{K}}_2$ respectively. Then $\hat{\mathbf{K}}_1 + \hat{\mathbf{K}}_2$ is defined to be the equivalence class represented by $\mathbf{K}_1 + \mathbf{K}_2$. Let $\mathbf{K}$ be representative of $\hat{\mathbf{K}}$ and $a \in \mathbf{R}$. Then $a\hat{\mathbf{K}}$ is defined to be the equivalence class represented by $a\mathbf{K}$. It is easy to check that these operations are well defined. Let $TCK^p(M)$ denote the vector space of trace-free conformal Killing tensors of type $(p, 0)$. It is easily verified that $TCK^p(M)$ is canonically isomorphic to $C\hat{K}^p(M)$. A necessary and sufficient condition for an element of $C\hat{K}^p(M)$ to be represented by a Killing tensor is that there exists a type $(p-2, 0)$ tensor $\mathbf{l}$ such that

$$[\mathbf{l}, \mathbf{g}] = 2\mathbf{k}. \tag{2.2.8}$$

For $p = 2$ the above equation may be written as

$$dl = -\mathbf{k}. \tag{2.2.9}$$

The integrability condition for this equation is

$$d\mathbf{k} = 0. \tag{2.2.10}$$

By solving Eq.(2.2.2) for $\mathbf{k}$ one may write the integrability condition in component form as

$$K_{k[i;}{}^{k}{}_{j]} = 0. \tag{2.2.11}$$

This is a necessary and sufficient condition for $\hat{\mathbf{K}}$ to be represented by a Killing tensor.

We now study the behavior of the conformal Killing tensor $\mathbf{K}$ under a conformal transformation, which is again

$$\tilde{\mathbf{g}} = e^{-2\sigma}\mathbf{g}. \tag{2.2.12}$$

By an easy calculation we find that

$$[\tilde{\mathbf{g}}, \mathbf{K}] = 2\tilde{\mathbf{k}} \odot \tilde{\mathbf{g}}, \tag{2.2.13}$$

where

$$\tilde{\mathbf{k}} = (\mathbf{k} - [\sigma, \mathbf{K}]). \tag{2.2.14}$$

This result shows that $\mathbf{K}$ is also a conformal Killing tensor for the conformally related metric $\tilde{\mathbf{g}}$. Therefore we can prove the following:

Proposition 2.2.4 *A necessary and sufficient condition that* $\mathbf{K}$ *is a Killing tensor with respect to the conformal metric, that is*

$$[\tilde{\mathbf{g}}, \mathbf{K}] = 0 \tag{2.2.15}$$

is that there exists a function σ *such that*

$$\mathbf{k} = [\sigma, \mathbf{K}]. \tag{2.2.16}$$

Note this proposition holds true for Killing tensors of any valence p. For the special case $p = 2$ we prove immediately Proposition 7.1 in [3]. For the remainder of this section and chapter we assume $p = 2$.

Definition 2.2.5 *A conformal Killing tensor* $\mathbf{K}$ *is of* ***self-gradient type*** *if there exists a continuous function* U *such that in the definition* $[\mathbf{K}, \mathbf{g}] = 2\mathbf{k} \odot \mathbf{g}$*,* $\mathbf{k} = [\mathbf{K}, U]$*.*

Indeed by Prop. (2.2.4) self-gradient conformal Killing tensors are simple Killing tensors with respect to the conformally related metric $\tilde{\mathbf{g}} = e^{-U}\mathbf{g}$.

To determine the transformation equations of the eigenvectors and eigenvalues of the Killing tensor under a conformal transformation (2.2.12), we present some general results that hold true for symmetric tensors and then successively add mathematical conditions to those corresponding to the conformal Killing tensor equation.

Proposition 2.2.6 *A symmetric tensor satisfying $\tilde{K}^{ij} = K^{ij}$ under a conformal transformation has the same eigenvectors with respect to the conformally related metric and the corresponding eigenvalues satisfy $\tilde{\rho}_i = e^{2\sigma}\rho_i$.*

Proof: Note that implicit in Eq. (2.2.13) is that

$$\tilde{K}^{ij} = K^{ij},$$

however this identity could well be satisfied by other symmetric tensors of valence-two, or type $(2,0)$, under a conformal transformation. Lowering an index to generate a type $(1,1)$ tensor we arrive at

$$\tilde{K}^i{}_j = \tilde{K}^{ik}\tilde{g}_{kj} = e^{2\sigma}K^{ik}g_{kj}.$$

Therefore

$$\tilde{K}^i{}_j = e^{2\sigma}K^i{}_j.$$

Now we consider the eigenvalue problem for $K^i{}_j$:

$$\begin{aligned} K^i{}_jX^j &= \rho X^i \\ (e^{2\sigma}K^i{}_j)X^j &= e^{2\sigma}\rho X^i \\ \tilde{K}^i{}_jX^j &= \tilde{\rho}X^i \end{aligned} \qquad (2.2.17)$$

The conclusion is that X^j is an eigenvector of $K^i{}_j$ corresponding to the eigenvalue ρ if and only if X^j is an eigenvector of $\tilde{K}^i{}_j$ corresponding to the eigenvalue $\tilde{\rho} = e^{2\sigma}\rho$. □

This property was arrived at by Eisenhart but with more mathematical assumptions. A proof outlining his method but with more steps is given in Appendix E. Now we assume that the symmetric tensor not only satisfies $\tilde{K}^{ij} = K^{ij}$, but also has pointwise real and distinct eigenvalues.

Proposition 2.2.7 *A symmetric tensor K^{ij}, with pointwise real and distinct eigenvalues satisfies*

$$K_{ab} = \rho_a g_{ab},$$

where the components are with respect to a basis of normalized eigenvectors.

Proof: Since the eigenvalues of K_{ab} are real and distinct (point-wise K_{ab} can be described then as Hermitian), it admits n orthogonal eigenvectors $h^i{}_a$, where i are the component indices and a is the label for the eigenvector. Thus

$$K_{ij}h^j{}_a = \rho_a g_{ij}h^j{}_a, \qquad (2.2.18)$$

where $h^j{}_a$ is the eigenvector corresponding to the eigenvalue ρ_a. Since the eigenvalues are real and distinct, the eigenvectors are orthogonal with respect to the metric g_{ij}. Namely

$$g_{ij}h^i{}_ah^j{}_b = 0, \quad a \neq b \qquad (2.2.19)$$

The eigenvectors can be normalized, such that

$$g_{ij}{h^i}_a{h^j}_a = e_a, \tag{2.2.20}$$

where $e_a^2 = 1$. For Riemannian geometry the e's are always plus unity. Thus we can write

$$g_{ab} = g_{ij}{h^i}_a{h^j}_b = e_a\delta_{ab} \tag{2.2.21}$$

where there is no sum on the a. Note that the term ${h^i}_a$ may be interpreted as a change of basis transformation from the natural basis $\frac{\partial}{\partial x^i}$ to the basis of eigenvectors $E_a = {h^i}_a\frac{\partial}{\partial x^i}$. Now contract (2.2.18) with ${h^i}_b$ to obtain

$$\begin{aligned} K_{ij}{h^i}_b{h^j}_a &= \rho_a g_{ij}{h^i}_b{h^j}_a \\ K_{ba} &= \rho_a g_{ba} \\ \Rightarrow K_{ab} &= \rho_a g_{ab} \quad \square \end{aligned} \tag{2.2.22}$$

Now we extend to the case of symmetric tensor *fields* with *normal* eigenvectors:

Proposition 2.2.8 *Let K^{ij} be a symmetric tensor field with pointwise real and distinct eigenvalues and normal eigenvectors. Then there exists a coordinate system u^i such that*

$$\begin{aligned} g_{ij} &= 0, \quad i \neq j \\ K_{ij} &= \rho_i g_{ij} \end{aligned} \tag{2.2.23}$$

Proof: We assume that K_{ij} is a symmetric tensor field with pointwise real distinct eigenvalues and normal (integrable) eigenvectors. Then K_{ij} defines n mutually orthogonal eigenvector fields ${h^i}_a$. These can be written as

$$E_a = {h^i}_a\frac{\partial}{\partial x^i} \tag{2.2.24}$$

with respect to a general coordinate system on M. The E_a define a basis of the tangent space of M at each point. Let E^a denote the dual basis of 1-forms. We can write

$$E^a = {h^a}_i dx^i, \tag{2.2.25}$$

where ${h^a}_i$ is the inverse of ${h^i}_a$. Since each eigenvector field is assumed normal (integrable), there exist functions f_a and u^a such that

$$E^a = f_a du^a, \tag{2.2.26}$$

where there is no sum assumed on the a, which ranges from 1 to n. The u^a define a coordinate system on M. Write (2.2.26) as

$$E^a = f_a\delta^a_i du^i, \tag{2.2.27}$$

where again no sum on the a is assumed. Comparison with (2.2.25) yields

$$h^a{}_i = f_a \delta^a_i \tag{2.2.28}$$

Now we compute the inverse of $h^a{}_i$:

$$h^i{}_a = f_a^{-1} \delta^i_a \tag{2.2.29}$$

This implies that

$$\begin{aligned} E_a &= h^i{}_a \frac{\partial}{\partial u^i} \\ &= f_a^{-1} \frac{\partial}{\partial u^a} \end{aligned} \tag{2.2.30}$$

We now need to prove the identity $e_i f_i^2 \delta_{ij} = g_{ij}$. To do so we write the metric in terms of the coordinates u^i. Starting from (2.2.21), which is again

$$g_{ab} = e_a \delta_{ab} \tag{2.2.31}$$

contract this with $h^a{}_i h^b{}_j$ to obtain

$$\begin{aligned} g_{ab} h^a{}_i h^b{}_j &= e_a \delta_{ab} f_a \delta^a_i f_b \delta^b_j \\ \Rightarrow g_{ij} &= e_i \delta_{ij} f_i f_j \\ g_{ij} &= e_i f_i^2 \delta_{ij} \end{aligned} \tag{2.2.32}$$

Thus the metric has the form

$$\begin{aligned} ds^2 &= g_{ij} du^i du^j \\ &= e_i f_i^2 \delta_{ij} du^i du^j \\ &= e_i f_i^2 (du^i)^2 \end{aligned} \tag{2.2.33}$$

We then write (2.2.22) in terms of the coordinate u^i by contracting (2.2.22) with $h^a{}_i h^b{}_j$ using (2.2.28)

$$\begin{aligned} K_{ab} h^a{}_i h^b{}_j &= \rho_a g_{ab} h^a{}_i h^b{}_j \\ K_{ij} &= \rho_a g_{ab} f_a \delta^a_i f_b \delta^b_j \\ K_{ij} &= \rho_i e_i f_i^2 \delta_{ij} \\ \Rightarrow K_{ij} &= \rho_i g_{ij} \end{aligned} \tag{2.2.34}$$

Thus we have shown that for symmetric tensors with pointwise real and distinct eigenvalues, and normal (integrable) eigenvectors, there exists a coordinate system such that simultaneously $K_{ij} = 0$ and $g_{ij} = 0$ for $i \neq j$. □

Definition 2.2.9 *A conformal Killing tensor with pointwise real and distinct eigenvalues and normal eigenvector fields is called a* ***characteristic*** *conformal Killing tensor.*

We now impose the conformal Killing tensor equation (2.2.5) for $p = 2$, which reads:

$$K_{ij;l} + K_{jl;i} + K_{li;j} = k_i g_{jl} + k_j g_{li} + k_l g_{ij} \tag{2.2.35}$$

The derivations outlined are therefore not valid for non characteristic Killing tensors. However, such tensors are not useful in the characterization of separable coordinates as will be explained later. In the remainder of this thesis non-characteristic Killing tensors will not be considered.

Proposition 2.2.10 *The eigenvalues ρ_i of a characteristic conformal Killing tensor K_{ij} when expressed in terms of coordinates for which the conditions of Prop. 2.2.8 hold, satisfy the differential equations:*

$$\begin{aligned} \frac{\partial \rho_i}{\partial x^i} &= k_i \\ \frac{\partial \rho_i}{\partial x^j} &= (\rho_i - \rho_j)\frac{\partial \log(g_{ii})}{\partial x^j} + \frac{\partial \rho_j}{\partial x^j} \end{aligned} \tag{2.2.36}$$

Proof: The first equation follows from setting $i = j = l$ in the definition of the conformal Killing tensor equation which yields

$$\frac{\partial K_{ii}}{\partial x^i} - \frac{\partial \log(g_{ii})}{\partial x^i} K_{ii} = k_i g_{ii} \tag{2.2.37}$$

Then substitute $K_{ii} = \rho_i g_{ii}$ to get the required result. Note that in the simple Killing tensor case, where $k_i = 0$, we obtain the Eisenhart result that the i^{th} eigenvalue is independent of the i^{th} coordinate. An alternative proof of this is given in Appendix E for the interested reader. The second equation follows from setting $j \neq i$, $l = j$ in the definition and arriving at

$$\frac{\partial K_{jj}}{\partial x^i} - 2\frac{\partial \log(g_{jj})}{\partial x^i} K_{jj} + \frac{1}{g_{ii}}\frac{\partial g_{jj}}{\partial x^i} K_{ii} = k_i g_{jj} \tag{2.2.38}$$

Substituting $K_{ii} = \rho_i g_{ii}$ and $K_{jj} = \rho_j g_{jj}$ in the above yields the second formula. □

Proposition 2.2.11 *(i) A CKT* $\mathbf{K}$ *which is diagonalized in orthogonal coordinates is equivalent to a CKT* $\mathbf{K}'$ *of self-gradient type. (ii) For any given orthogonal coordinate system there exists a function U such that any CKT* $\mathbf{K}$ *which is diagonalized in these coordinates is equivalent to a CKT* $\mathbf{K}'$ *of self-gradient type such that* $[\mathbf{g}, \mathbf{K}'] = 2[\mathbf{K}', U] \odot \mathbf{g}$, *that is to a simple Killing tensor of the conformal metric* $\tilde{\mathbf{g}} = e^{-U}\mathbf{g}$. *(iii) The n functions $U_k = \log(g_{kk})$ satisfy (ii).*

Proof: If $g_{ij} = 0$ and $K_{ij} = 0$ for $i \neq j$, then $K_{ii} = \rho_i g_{ii}$. Furthermore, by the proof of Prop. 2.2.10 the CKT equation $[\mathbf{g}, \mathbf{K}] = 2\mathbf{k} \odot \mathbf{g}$ is equivalent to $k_j = \frac{\partial}{\partial x^j}\rho_j$ and the formula

$$\frac{\partial \rho_i}{\partial x^j} = (\rho_i - \rho_j)\frac{\partial \log(g_{ii})}{\partial x^j} + \frac{\partial \rho_j}{\partial x^j}.$$

Let us consider the equivalent tensor $\mathbf{K}' = \mathbf{K} - \rho_n \mathbf{g}$ that has eigenvalues $\tilde{\rho_i} = \rho_i - \rho_n$. By using the above, one can easily show that

$$\frac{\partial \tilde{\rho_i}}{\partial x^j} = (\tilde{\rho_i} - \tilde{\rho_j}) \frac{\partial \log(g_{ii})}{\partial x^j} + \tilde{\rho_j} \frac{\partial \log(g_{nn})}{\partial x^j}.$$

This shows that $\mathbf{K}'$ is a CKT with $\tilde{k}_j = \tilde{\rho_j} \frac{\partial \log(g_{nn})}{\partial x^j}$, thus of self-gradient type with $U = \log(g_{nn})$ and a simple Killing tensor for the conformal metric $e^{-U}\mathbf{g}$. □

The connection with conformal Killing tensors and the existence of R-separation of variables will now be described.

As discussed before it is well known that Killing tensors are deeply related with additive separation of variables for the HJ equation for the geodesics or a natural Hamiltonian in orthogonal coordinates ([7], [1])

$$H = \frac{1}{2} g^{ii} p_i p_i + V = E, \qquad E \in \mathbf{R},$$

which reads

$$\frac{1}{2} g^{ii} \left(\frac{\partial W}{\partial x^i} \right)^2 + V = E.$$

They are also connected to multiplicative separation of the Schrödinger equation [7], [2]

$$\Delta \psi + (E - V)\psi = 0, \qquad E \in \mathbf{R},$$

where Δ is the Laplace-Beltrami operator. We have [25]

Theorem 2.2.12 *The Hamiltonian $H = (\frac{1}{2} g^{ii} p_i p_i + V)$ is orthogonally separable if and only if there exists a valence-two characteristic Killing tensor $\mathbf{K}$ (the properties of which have been elucidated earlier) such that*

$$d(\mathbf{K}dV) = 0.$$

Note that $d(\mathbf{K}dV) = 0$ is equivalent to the formula $S_{ij}(V) = 0$ in Prop. 2.1.7 and the metric components g^{hh} in the above theorem must satisfy $S_{ij}(g^{hh}) = 0$.

Finally, for the multiplicative separation of the Schrödinger equation the so-called Robertson condition must also hold: the Ricci tensor is diagonalized in the separable coordinates ([20]) (geometrically, this means that $\mathbf{K}$ and the Ricci tensor share the same eigenvectors [2]). The condition that the eigenvalues are real is automatically satisfied for positive definite metrics; recently, KTs with complex conjugate eigenvalues have also been used to separate variables for a natural HJ equation [17].

Similar results also hold for conformal Killing tensors.

Remark 2.2.13 Any CKT equivalent to a characteristic one is characteristic, which is a consequence of the eigenvectors remaining invariant within an equivalence class. Hence, it is always possible to choose a representative characteristic CKT which is trace-free. Furthermore any CKT $\mathbf{K}$ which is characteristic with respect to the metric $\mathbf{g}$ is also characteristic with respect to any conformally related metric $\tilde{\mathbf{g}} = e^{-2\sigma}\mathbf{g}$. This is a consequence of Prop. 2.2.6: clearly real and pointwise distinct eigenvalues remain real and pointwise distinct after any conformal transformation. The invariance of the eigenvectors themselves guarantees invariance of their normality.

The following important result holds:

Theorem 2.2.14 *There exists an orthogonal coordinate system in which additive separation for the null geodesic HJ equation,*

$$g^{ii}(\partial_i W)^2 = 0$$

occurs, if and only if there exists a characteristic CKT $\mathbf{K}$ *on* M*. By construction the coordinate hypersurfaces will be orthogonal to the eigenvectors of* $\mathbf{K}$*.*

Proof: According to the intrinsic characterization of the orthogonal separation of a geodesic Hamiltonian [25], a metric $\tilde{\mathbf{g}}$ defined on M is orthogonally separable if and only if it admits a simple characteristic Killing tensor, note this is a special case of Thm. 2.2.12. This simple Killing tensor is a conformal Killing tensor with respect to any conformally related metric $\mathbf{g}$ also defined on M, by Eq. (2.2.13). That this conformal Killing tensor is also characteristic has been explained in Remark 2.2.13, which summarizes Prop. 2.2.7 to Prop. 2.2.11, therefore the theorem is proved. □

Definition 2.2.15 *We call a* ***conformally separable web*** *the set of hypersurfaces orthogonal to the eigenvectors of a characteristic CKT. Any coordinates associated with a conformally separable web are called* ***conformally separable coordinates****.*

Remark 2.2.16 *Note the connection here with conformally separable coordinates defined in the previous section if Eq. (2.1.11) or Eq. (2.1.12) holds.*

Theorem 2.2.17 *There exists an orthogonal coordinate system in which additive separation for the HJ equation with fixed value of the energy* E*,*

$$g^{ii}(\partial_i W)^2 + V - E = 0,$$

occurs, if and only if there exists a characteristic CKT $\mathbf{K}$ *on* M *satisfying the compatibility condition*

$$[\mathbf{g}, \mathbf{K}] = \frac{1}{E - V}[\mathbf{K}, V] \odot \mathbf{g}. \tag{2.2.39}$$

Note this is equivalent to the formula $\frac{1}{g^{kk}} S_{ij}(g^{kk}) = \frac{1}{V-E} S_{ij}(V)$ in Thm. 2.1.8.

Remark 2.2.18 *In the compatibility condition (2.2.39) for the potential V, the characteristic CKT $\mathbf{K}$ is in general not trace-free and the formula does not hold for all the CKT equivalent to $\mathbf{K}$. Indeed, if we consider the equivalent characteristic CKT $\hat{\mathbf{K}} = \mathbf{K} + f\mathbf{g}$ the compatibility condition becomes*

$$[\mathbf{g}, \hat{\mathbf{K}}] = \frac{1}{E-V}\left([\mathbf{K}, V] + [f, \mathbf{g}]\right) \odot \mathbf{g}. \tag{2.2.40}$$

In spite of the fact that the null geodesic equation is trivial for a positive definite metric, the conformally separable coordinates are useful because they are the only ones in which a natural Hamiltonian with fixed value of the energy can be solved by additive separation of variables. Moreover, they are the only ones in which R-separation of the Laplace equation can occur. This is a subject for the next section. Having discussed the uses of a single CKT, an important characterization is associated with n CKTs, which is Theorem 7.2 in [3]:

Theorem 2.2.19 *The n characteristic conformal Killing tensors $(K_i) = (K_1, K_2, \ldots, K_n)$ associated with an orthogonal metric g_{ij} are (i) point-wise linearly independent, (ii) with common eigenvectors, (iii) mutually commutative and (iv) in involution.*

Proof: Since the rows of the inverse Stäckel matrix are linearly independent (this follows trivially from the definition that the determinant is non-zero), by construction the n Killing tensors produced (one of them being the metric tensor itself) are point-wise linearly independent, being in the same (normal) eigenbasis of the separable coordinates by Eisenhart theory. They are all then simultaneously diagonalized. So are the n conformal Killing tensors conformally related to them, as well as the n equivalent conformal Killing tensors. Then by definition these conformal Killing tensors in any orthogonal coordinate system share common eigenvectors and by sharing eigenvectors they commute. That they are in involution is proven in [3]. □

2.3 Relation of CKT's to existence of R-separable webs

Recall Definition 1.1.1 [14]:

Definition 2.3.1 *We say that multiplicative R-separation of the Laplace equation $\Delta\psi = 0$ or Schrödinger equation $-\frac{\hbar^2}{2}\Delta\psi + (V-E)\psi = 0$ occurs in a coordinate* *system* (q^i) *if there exists a solution* ψ *of the form*

$$\psi = R(q^1, \ldots, q^n) \prod_i \phi_i(q^i, c_a) \qquad (c_a) \in \mathbf{R}^{2n-1}; \tag{2.3.1}$$

satisfying the ***completeness condition***

$$\text{rank}\left[\frac{\partial}{\partial c_a}\left(\frac{\phi_i'}{\phi}\right) \; \frac{\partial}{\partial c_a}\left(\frac{\phi_i''}{\phi}\right)\right] = 2n-1, \qquad a = 1, \ldots, 2n-1, \quad i = 1, \ldots, n.$$

From ([14]) we have

Theorem 2.3.2 *Necessary and sufficient conditions for R-separation of Schrödinger's equation*

$$-\frac{\hbar^2}{2}\Delta\psi + (V-E)\psi = 0 \tag{2.3.2}$$

in a given coordinate system q^i *are:*

i: the coordinates are orthogonal;
ii: the coordinates are conformally separable;
iii: the function

$$\frac{2}{\hbar^2}(E-V) + \frac{g^{ii}}{4}\left(2\partial_i\Gamma_i - \Gamma_i^2\right) \tag{2.3.3}$$

is a pseudo-Stäckel factor, in that it can be written in the form $f = g^{ii}\phi_i(q^i)$ *where* g^{ii} *is a conformal Stäckel metric. Furthermore in this case the modulation factor* R *is any solution of*

$$2\partial_i \ln R = \Gamma_i - \xi_i(q^i) \quad (i = 1, \ldots, n), \tag{2.3.4}$$

where $\xi_i(q^i)$ *is a function of one variable.*

For the proof, see ([14]). Furthermore we also have ([7], [14])

Theorem 2.3.3 *On a flat manifold, R-separation of the Laplace equation occurs in a coordinate system* (q^i) *if and only if the coordinates* (q^i) *are orthogonal conformally separable coordinates. The function* R *is (up to separated factors) a solution of the first order system*

$$\partial_i \ln R = \frac{1}{2}\Gamma_i,$$

Remark 2.3.4 If the manifold is not flat the conformal separability is a necessary (but no longer sufficient) condition: to guarantee R-separation we also need that the function $\frac{\Delta R}{R}$ be of the form $g^{ii}f_i(q^i)$ for suitable functions of a single variable f_i.

Definition 2.3.5 *We call an R-separable web a conformally separable web if R-separation for the Laplace equation occurs in any associated coordinate system.*

Remark 2.3.6 *In* $\mathbf{E}^3$*, every conformally separable web is an R-separable web for the Laplace equation. This means that R-separable webs are defined by any characteristic CKT. R-separable coordinates of* $\mathbf{E}^3$ *have been extensively studied by many authors (see Bôcher[5], Moon and Spencer [38], Boyer et al.[6]). The webs consist of families of confocal cyclides.*

In later chapters of this thesis we restrict ourselves to the webs and associated characteristic CKTs admitting a rotational symmetry. To make the notion of web-symmetry precise, we start with the definition of invariance of conformal Killing tensors under one parameter groups of conformal transformations [13].

Definition 2.3.7 *Let* $\mathbf{K}$ *denote a characteristic conformal Killing tensor on* $(\mathbf{M}, \mathbf{g})$*. Let* ϕ_t *denote a one parameter group of conformal transformations. The R-separable webs defined by* $\mathbf{K}$ *are said to be* ϕ_t*-symmetric iff*

$$\phi_{t*}\mathbf{K} = f\mathbf{K}, \tag{2.3.5}$$

where f *is some function.*

The infinitesimal version of the above definition is given by the following proposition [13]:

Proposition 2.3.8 *Let* $\mathbf{V}$ *be an infinitesimal generator of the one parameter group of conformal transformations* ϕ_t*. Then* ϕ_t *is a web-symmetry of the R-separable web defined by a conformal Killing tensor* $\mathbf{K}$ *if and only if*

$$\mathcal{L}_{\mathbf{V}}\mathbf{K} = h\mathbf{K}, \tag{2.3.6}$$

where h *is some function.*

If ϕ_t denotes a one-parameter group of homothetic transformations then the functions f and h are non-zero constants. If ϕ_t denotes a one-parameter group of isometries then the functions f and h are zero.

2.4 Conformal Killing tensors in spaces of zero curvature

We now assume that the Riemann curvature tensor R_{ijkl} of $\mathbf{g}$ vanishes. In this case it has been shown by Eastwood [19] that $C\hat{K}^p(M)$ is finite dimensional and that its dimension d is given by

$$d = \frac{(n+p-3)!(n+p-2)!(n+2p-2)(n+2p-1)(n+2p)}{p!(p+1)!(n-2)!n!} \tag{2.4.1}$$

for $n \geq 3, p \geq 1$. Thus the general element of $C\hat{K}^p(M)$ is represented by d arbitrary parameters $a^1, \ldots, a^d$, with respect to an appropriate basis.

Each element h of the conformal group $C(M)$ induces, by a push forward map, a non-singular linear transformation $\zeta(h)$ of $C\hat{K}^p(M)$. It is implicit in the work of [19] that the map

$$\zeta : C(M) \rightarrow GL(C\hat{K}^p(M)) \tag{2.4.2}$$

defines a representation of $C(M)$. Once the form of the general element $\hat{\mathbf{K}}$ of $C\hat{K}^p(M)$ is available with respect to some convenient coordinate system on M, the explicit form of the transformation $\zeta(h)\hat{\mathbf{K}}$ (written more succinctly as $h \cdot \hat{\mathbf{K}}$) may be written in terms of the parameters $a^1, \ldots, a^d$. We shall be particularly concerned with the smooth real-valued functions on $C\hat{K}^p(M)$ that are invariant under the group $C(M)$. The precise definition of such $C(M)$-invariant functions of $C\hat{K}^p(M)$ is as follows.

Definition 2.4.1 *Let $(\mathbf{M}, \mathbf{g})$ be a Riemannian manifold with zero curvature. Let $p \geq 1$ be fixed. A smooth function $F : C\hat{K}^p(M) \rightarrow \mathbf{R}$ is said to be an $C(M)$-invariant of $C\hat{K}^p(M)$ iff it satisfies the condition*

$$F(h \cdot \hat{\mathbf{K}}) = F(\hat{\mathbf{K}}), \tag{2.4.3}$$

for all $\hat{\mathbf{K}} \in C\hat{K}^p(M)$ and for all $h \in C(M)$.

The above can also be formulated for pseudo-Riemannian manifolds with zero curvature, however they are beyond the scope of this thesis. The main problem of invariant theory is to describe the whole space of invariants of a vector space under the action of the group. To achieve this one has to determine the set of *fundamental invariants* with the property that any other invariant is an analytic function of the fundamental invariants (see [41]). The fundamental theorem of invariants for a regular Lie group action [41] determines the number of fundamental invariants needed to define the whole of the space of $C(M)$-invariants.

Theorem 2.4.2 *Let G be a Lie group acting regularly on an n-dimensional manifold M with s-dimensional orbits. Then, in a neighborhood N of each point $x \in M$, there exist $(n - s)$ functionally independent G-invariants $\Delta_1, \ldots, \Delta_{n-s}$. Any other G-invariant I defined near x can be locally uniquely expressed as an analytic function of the fundamental invariants namely $I = F(\Delta_1, \ldots, \Delta_{n-s})$.*

One of the standard methods for determining the invariants of $C\hat{K}^p(M)$ is to use the fact that the invariants of a function under an entire Lie group is equivalent to the invariants of the function under the infinitesimal transformation of the group given by the corresponding Lie algebra. The precise result is as follows [40]:

Proposition 2.4.3 *Let G be a connected Lie group of transformations acting regularly on a manifold M. A smooth real valued function $F : M \to \mathbf{R}$ is G-invariant iff*

$$\boldsymbol{v}(F) = 0, \tag{2.4.4}$$

for all $x \in M$ and for every infinitesimal generator $\boldsymbol{v}$ of G.

In our application G is the representation ζ defined by Eq.(2.4.2) where the condition (2.4.4) reads

$$\mathbf{U}_i(F) = 0, \quad i = 1, \ldots, r, \tag{2.4.5}$$

where the U_i are vector fields which form a basis of the Lie algebra of the representation and $r = dim\ C(M) = \frac{1}{2}(n+1)(n+2)$. This Lie algebra is isomorphic to the Lie algebra of $C(M)$. Such a basis may be computed directly as the basis of the tangent space to $\zeta(C(M)$ at the identity if an explicit form of the representation is available. According to Theorem 2.4.2 the general solution of the system of first-order pdes (2.4.5) is an analytic function F of a set of fundamental $C(M)$-invariants. The number of fundamental invariants is $d - s$, where d is given by Eq. (2.4.1) and s is the dimension of the orbits of $\zeta(C(M))$ acting regularly on the space $C\hat{K}^p(M)$.

Chapter 3

Construction of the general CKT in $\mathbf{E}^3$

3.1 Killing vector formalism

Now we wish to calculate the general thirty-five dimensional trace-free conformal Killing tensor Euclidean space admits, and express it in terms of Cartesian coordinates. To this end we now specialize the general theory of the previous chapter to the vector space $C\hat{K}^2(M)$ of conformal Killing tensors of type $(2,0)$ defined in Euclidean space $\mathbf{E}^3$.

It is well known [42] that in $\mathbf{E}^3$, any conformal Killing tensor is expressible modulo a multiple of the metric as a sum of symmetrized products of conformal Killing vectors. A canonical basis of the Lie algebra of conformal Killing vectors in $\mathbf{E}^3$ with respect to a system of Cartesian coordinates x^i may be written as

$$\begin{aligned} \mathbf{X}_i &= \frac{\partial}{\partial x^i} \\ \mathbf{R}_i &= \epsilon_{ijk} x^j \mathbf{X}_k \\ \mathbf{D} &= x^i \mathbf{X}_i \\ \mathbf{I}_i &= (2x^i x^k - \delta_{ik} x^j x^j)\mathbf{X}_k \end{aligned} \tag{3.1.1}$$

for $i = 1, 2, 3$, and where ϵ_{ijk} is the Levi-Civita tensor. We also note the commutation relations

$$\begin{aligned} [\mathbf{X}_i, \mathbf{X}_j] &= 0 \\ [\mathbf{X}_i, \mathbf{R}_j] &= -\epsilon_{ijk}\mathbf{X}_k \\ [\mathbf{R}_i, \mathbf{R}_j] &= -\epsilon_{ijk}\mathbf{R}_k \\ [\mathbf{X}_i, \mathbf{D}] &= \mathbf{X}_i \\ [\mathbf{R}_i, \mathbf{D}] &= 0 \\ [\mathbf{I}_i, \mathbf{I}_j] &= 0 \end{aligned}$$

$$\begin{aligned}
[\mathbf{X}_i, \mathbf{I}_j] &= 2(\delta_{ij}\mathbf{D} - \epsilon_{ijk}\mathbf{R}_k) \\
[\mathbf{R}_i, \mathbf{I}_j] &= -\epsilon_{ijk}\mathbf{I}_k \\
[\mathbf{D}, \mathbf{I}_i] &= \mathbf{I}_i
\end{aligned} \tag{3.1.2}$$

We now determine the form of the general element of $TCK^2(M)$. By Eq.(2.4.1) $d = 35$. It is clear that a sum of symmetrized products of conformal Killing vectors is a conformal Killing tensor. It will be shown that all trace-free conformal Killing tensors may be obtained in this way. One begins by writing

$$\begin{aligned}
\mathbf{K} \;=\; & A_{ij}\mathbf{X}_i \odot \mathbf{X}_j + B_{ij}\mathbf{X}_i \odot \mathbf{R}_j + C_{ij}\mathbf{R}_i \odot \mathbf{R}_j + D_i\mathbf{X}_i \odot \mathbf{D} + E_{ij}\mathbf{X}_i \odot \mathbf{I}_j \\
+\; & F_i\mathbf{R}_i \odot \mathbf{D} + G_{ij}\mathbf{R}_i \odot \mathbf{I}_j + H\mathbf{D} \odot \mathbf{D} + L_i\mathbf{D} \odot \mathbf{I}_i + M_{ij}\mathbf{I}_i \odot \mathbf{I}_j
\end{aligned} \tag{3.1.3}$$

The coefficients in Eq.(3.1.3) obey the following symmetry relations

$$A_{ij} = A_{ji}, \quad C_{ij} = C_{ji}, \quad M_{ij} = M_{ji} \tag{3.1.4}$$

Thus the apparent dimension of $TCK^2(M)$ is fifty five, which exceeds the required dimension by twenty. Indeed there exist the following six relations among the basis set of symmetric tensor products of Killing vectors:

$$\begin{aligned}
\mathbf{X}_i \odot \mathbf{R}_i &= 0 \\
\mathbf{R}_i \odot \mathbf{I}_i &= 0 \\
\mathbf{D} \odot \mathbf{D} &= \mathbf{X}_i \odot \mathbf{I}_i + \mathbf{R}_i \odot \mathbf{R}_i \\
2\mathbf{R}_i \odot \mathbf{D} + \epsilon_{ikl}\mathbf{X}_k \odot \mathbf{I}_l &= 0
\end{aligned} \tag{3.1.5}$$

Consequently, the general element of $TCK^2(M)$ may be written as

$$\begin{aligned}
\mathbf{K} \;=\; & A_{ij}\mathbf{X}_i \odot \mathbf{X}_j + B_{ij}\mathbf{X}_i \odot \mathbf{R}_j + C_{ij}\mathbf{R}_i \odot \mathbf{R}_j + D_i\mathbf{X}_i \odot \mathbf{D} \\
+\; & E_{ij}\mathbf{X}_i \odot \mathbf{I}_j + G_{ij}\mathbf{R}_i \odot \mathbf{I}_j + L_i\mathbf{D} \odot \mathbf{I}_i + M_{ij}\mathbf{I}_i \odot \mathbf{I}_j
\end{aligned} \tag{3.1.6}$$

where the coefficients B_{ij} and G_{ij} may be chosen to satisfy

$$\begin{aligned}
B_{ii} &= 0 \\
G_{ii} &= 0
\end{aligned} \tag{3.1.7}$$

This follows from the fact that in the expression for the Killing tensor $\mathbf{K}$, one can add the terms $k_a\mathbf{X}_i \odot \mathbf{R}_i$ and $k_b\mathbf{R}_i \odot \mathbf{I}_i$ since we know these are essentially zero for any arbitrary $k_a, k_b \in \mathbf{R}$. Since by definition the Kronecker delta δ_{ij} vanishes for $i \neq j$, the above expressions can be modified to $k_a\delta_{ij}\mathbf{X}_i \odot \mathbf{R}_j$ and $k_b\delta_{ij}\mathbf{R}_i \odot \mathbf{I}_j$. These will still vanish and hence can be added to the expression for the Killing tensor $\mathbf{K}$ in terms of the basis of symmetric tensor products of Killing vectors. Collecting coefficients in front of $\mathbf{X}_i \odot \mathbf{R}_j$ and $\mathbf{R}_i \odot \mathbf{I}_j$, we define $\widetilde{B}_{ij} = B_{ij} + k_a\delta_{ij}$ and $\widetilde{G}_{ij} = G_{ij} + k_b\delta_{ij}$. Setting $i = j$ and enacting a summation, we arrive at $\widetilde{B}_{ii} = B_{ii} + 3k_a$ and $\widetilde{G}_{ii} = G_{ii} + 3k_b$. As k_a and k_b are arbitrary, we have the freedom to set them such that $\widetilde{B}_{ii} = 0$ and $\widetilde{G}_{ii} = 0$, hence $k_a = -\frac{1}{3}B_{ii}$ and $k_b = -\frac{1}{3}G_{ii}$.

Finally, the tilde sign is dropped and the trace-free result is proven for the B_{ij} and G_{ij} coefficients.

In terms of the natural basis, $X_i \odot X_j$, the components of $\mathbf{K}$ are given by

$$\begin{aligned} K_{ij} &= A_{ij} + (B_{(i|k}\epsilon_{kl|j)} + D_{(i}\delta_{j)l})x^l \\ &+ (C_{mn}\epsilon_{mk(i}\epsilon_{|nl|j)} + 2E_{(i|k|}\delta_{j)l} - E_{(ij)}\delta_{lk})x^l x^k \\ &+ (2G_{mn}\epsilon_{mk(i}\delta_{j)l} - G_{m(i}\epsilon_{j)mk}\delta_{ln} + 2L_k\delta_{in}\delta_{jl} - L_{(i}\delta_{j)n}\delta_{lk})x^l x^k x^n \\ &+ (4M_{kl}\delta_{jn} - 4M_{k(i}\delta_{j)n}\delta_{li} + M_{ij}\delta_{kn}\delta_{li})x^l x^k x^n x^i \end{aligned} \tag{3.1.8}$$

Next we impose the trace-free condition namely

$$K_{ii} = 0. \tag{3.1.9}$$

This procedure yields the following additional fourteen relations among the coefficients of $\mathbf{K}$:

$$\begin{aligned} M_{11} &= -M_{22} - M_{33} \\ A_{11} &= -A_{22} - A_{33} \\ E_{12} &= C_{12} - E_{21} \\ E_{13} &= C_{13} - E_{31} \\ E_{23} &= C_{23} - E_{32} \\ E_{11} &= C_{11} + 1/2(C_{22} + C_{33}) \\ E_{22} &= C_{22} + 1/2(C_{11} + C_{33}) \\ E_{33} &= C_{33} + 1/2(C_{11} + C_{22}) \\ L_1 &= G_{32} - G_{23} \\ L_2 &= G_{13} - G_{31} \\ L_3 &= G_{21} - G_{12} \\ D_1 &= B_{32} - B_{23} \\ D_2 &= B_{13} - B_{31} \\ D_3 &= B_{21} - B_{12} \end{aligned} \tag{3.1.10}$$

The above formulae may be written compactly as follows:

$$\begin{aligned} A_{ii} &= 0, \quad D_i = B_{jk}\epsilon_{kji} \\ E_{(ij)} - \frac{1}{3}E_{kk}\delta_{ij} &= \frac{1}{2}(C_{ij} - \frac{1}{3}C_{kk}\delta_{ij}) \\ L_i &= G_{lm}\epsilon_{mli}, \quad M_{ii} = 0 \\ E_{kk} &= 2C_{kk} \end{aligned} \tag{3.1.11}$$

We chose, among the A_{ii}, B_{ii}, G_{ii} and M_{ii}, the coefficients with index (1,1) to be written in terms of the other two, for example: $A_{11} = -A_{22} - A_{33}$. There are now twenty required relations among the coefficients. Implementing them, one obtains the conformal Killing tensor in $\mathbf{E}^3$ which we present first in compact form, and then in fully expanded form in components. We use the above conditions to remove the D_i, L_i, and (temporarily) C_{ij}. Note that the matrix coefficients A_{ij}, M_{ij}, B_{ij} and G_{ij} must be trace-free.

3.2 Compact and expanded form of the general CKT

In terms of the natural basis the components of $\mathbf{K}$ are given by:

$$\begin{aligned} K_{ij} &= A_{ij} + (B_{(i|k}\epsilon_{kl|j)} + B_{ab}\epsilon_{ba(i}\delta_{j)l})x^l \\ &+ ((2E_{(mn)} - 1/2E_{aa}\delta_{mn})\epsilon_{mk(i}\epsilon_{|nl|j)} + 2E_{(i|k|}\delta_{j)l} - E_{(ij)}\delta_{lk})x^l x^k \\ &+ (2G_{mn}\epsilon_{mk(i}\delta_{j)l} - G_{m(i}\epsilon_{j)mk}\delta_{ln} + 2G_{ab}\epsilon_{bak}\delta_{in}\delta_{jl} - G_{ab}\epsilon_{ba(i}\delta_{j)n}\delta_{lk})x^l x^k x^n \\ &+ (4M_{kl}\delta_{jn} - 4M_{k(i}\delta_{j)n}\delta_{li} + M_{ij}\delta_{kn}\delta_{li})x^l x^k x^n x^i \end{aligned} \quad (3.2.1)$$

Moreover, any CKT of $\mathbf{E}^3$ is equivalent to

$$\mathbf{K} \sim A_{ij}\mathbf{X}_i \odot \mathbf{X}_j + B_{ij}\mathbf{X}_i \odot \mathbf{R}_j + E_{ij}\mathbf{X}_i \odot \mathbf{I}_j + G_{ij}\mathbf{R}_i \odot \mathbf{I}_j + M_{ij}\mathbf{I}_i \odot \mathbf{I}_j,$$

where A_{ij}, M_{ij}, B_{ij} and G_{ij} must be trace-free matrices. The CKT coefficients K_{ij}, found by collecting all polynomials with common factor $X_i \odot X_j$, may be written as follows:

$X_1 \odot X_1$:

$K_{11} = -A_{22} - A_{33} - 4M_{12}xy^3 + (-G_{32} + 3G_{23})z^2x + (-2M_{22} - 2M_{33})z^2y^2 + (2C_{13} - 2E_{31})zx - 2C_{23}yz + 8M_{23}zyx^2 + (6M_{22} + 2M_{33})x^2y^2 + B_{12}z - B_{13}y + (B_{32} - B_{23})x + (-M_{22} - M_{33})x^4 + (G_{32} - G_{23})x^3 + 4M_{13}zx^3 + 4M_{12}yx^3 + (C_{11} + 1/2C_{22} + 1/2C_{33})x^2 + (3G_{21} - 2G_{12})zx^2 + (6M_{33} + 2M_{22})z^2x^2 + (-3G_{31} + 2G_{13})x^2y + (-3G_{32} + G_{23})xy^2 - 4M_{13}zxy^2 + (2C_{12} - 2E_{21})xy + (2G_{22} - 2G_{33})zxy - 4M_{12}z^2xy - 4M_{13}z^3x - G_{21}z^3 + (-M_{22} - M_{33})z^4 + G_{31}y^3 + (-M_{22} - M_{33})y^4 + (-C_{11} - 1/2C_{22} + 1/2C_{33})y^2 - G_{21}zy^2 + G_{31}z^2y + (-C_{11} + 1/2C_{22} - 1/2C_{33})z^2$

$X_2 \odot X_2$:

$K_{22} = A_{22} + 4M_{12}xy^3 + (2C_{23} - 2E_{32})zy + 2E_{21}xy + (C_{22} + 1/2C_{11} + 1/2C_{33})y^2 + 4M_{23}zy^3 + (G_{13} - G_{31})y^3 + (-C_{22} + 1/2C_{33} - 1/2C_{11})x^2 - 4M_{23}zyx^2 + (4M_{33} - 2M_{22})z^2y^2 + (-3G_{12} + 2G_{21})zy^2 - 4M_{12}yx^3 + 2M_{22}z^2x^2 + 8M_{13}zxy^2 - 4M_{12}z^2xy + G_{12}zx^2 - B_{21}z + B_{23}x - G_{32}z^2x + (B_{13} - B_{31})y - 2C_{13}xz + (-6M_{22} - 4M_{33})x^2y^2 + (-G_{13} + 3G_{31})x^2y + (3G_{32} - 2G_{23})xy^2 + (4G_{33} + 2G_{22})zxy + M_{22}x^4 - G_{32}x^3 + M_{22}z^4 + G_{12}z^3 + M_{22}y^4 + (-3G_{13} + G_{31})z^2y - 4M_{23}z^3y + (-C_{22} + 1/2C_{11} - 1/2C_{33})z^2$

$X_3 \odot X_3$:

$K_{33} = 2E_{32}zy + A_{33} + (1/2C_{22} - C_{33} - 1/2C_{11})x^2 - 4M_{23}zy^3 - 4M_{23}zyx^2 + G_{23}x^3 + 2E_{31}xz + (-3G_{21} + G_{12})zx^2 + (-G_{21} + 3G_{12})zy^2 - 4M_{13}zx^3 - 4M_{13}zxy^2 + 8M_{12}z^2xy + (1/2C_{11} - C_{33} - 1/2C_{22})y^2 + 4M_{13}z^3x + (-6M_{33} - 4M_{22})z^2x^2 - 2C_{12}xy + 4M_{23}z^3y - B_{32}x + B_{31}y + (B_{21} - B_{12})z + (-4G_{22} - 2G_{33})zxy + 2M_{33}x^2y^2 - G_{13}x^2y + M_{33}x^4 + (-3G_{23} + 2G_{32})z^2x + (4M_{22} - 2M_{33})z^2y^2 + (3G_{13} - 2G_{31})z^2y + G_{23}xy^2 + M_{33}z^4 + (G_{21} - G_{12})z^3 + (C_{33} + 1/2C_{11} + 1/2C_{22})z^2 + M_{33}y^4 - G_{13}y^3$

$X_1 \odot X_2$:

$K_{12} = (-G_{32} + 1/2G_{23})y^3 - 2M_{23}z^3x + (-1/2G_{13} + G_{31})x^3 + (2C_{13} - E_{31})zy + 3/2G_{23}yz^2 + (B_{22} + 1/2B_{33})z + (-E_{21} + 1/2C_{12})y^2 + (-4M_{22} - 2M_{33})yx^3 - 2M_{13}z^3y + (E_{21} - 1/2C_{12})x^2 - 2M_{13}zy^3 + 6M_{12}y^2x^2 - 3/2G_{13}xz^2 + (3G_{32} - 3/2G_{23})yx^2 - 2M_{23}zx^3 + (1/2B_{32} - B_{23})y - 3/2G_{33}zy^2 + 3/2G_{33}zx^2 + (2C_{23} - E_{32})zx + (-G_{22} - 1/2G_{33})z^3 + A_{12} - 3/2C_{12}z^2 + 6M_{33}xyz^2 + 6M_{13}zyx^2 + (4M_{22} + 2M_{33})xy^3 + (3/2G_{13} - 3G_{31})xy^2 + 6M_{23}zxy^2 + (-3G_{12} + 3G_{21})zxy - M_{12}x^4 + (3/2C_{11} + 3/2C_{22})xy + (B_{13} - 1/2B_{31})x - M_{12}y^4 + M_{12}z^4$

$X_1 \odot X_3$:

$K_{13} = (-1/2C_{13} + E_{31})x^2 + (-2M_{22} - 4M_{33})zx^3 + (-G_{21} + 1/2G_{12})x^3 + (3G_{21} - 3/2G_{12})z^2x + (2C_{12} - E_{21})zy + 3/2G_{12}xy^2 + (1/2G_{22} + G_{33})y^3 + (3/2C_{11} + 3/2C_{33})zx - 2M_{23}xy^3 - 3/2G_{32}zy^2 + (1/2B_{21} - B_{12})x + 6M_{22}xzy^2 - 3/2C_{13}y^2 + A_{13} + 3/2G_{22}yz^2 - 2M_{23}yx^3 + (3/2G_{32} - 3G_{23})zx^2 - 3/2G_{22}yx^2 + 6M_{13}z^2x^2 - M_{13}x^4 + 6M_{12}zyx^2 + (E_{32} + C_{23})xy + (3G_{13} - 3G_{31})zxy + 6M_{23}z^2xy + (2M_{22} + 4M_{33})z^3x - M_{13}z^4 + M_{13}y^4 - 2M_{12}zy^3 - 2M_{12}z^3y + (B_{32} - 1/2B_{23})z + (1/2C_{13} - E_{31})z^2 + (G_{23} - 1/2G_{32})z^3 + (-1/2B_{22} - B_{33})y$

$X_2 \odot X_3$:

$K_{23} = -3/2C_{23}x^2 + (3/2G_{22} + 3/2G_{33})z^2x + 3/2G_{31}zx^2 - 3/2G_{21}yx^2 + (1/2B_{33} - 1/2B_{22})x - 2M_{13}yx^3 - 2M_{12}zx^3 - 2M_{13}xy^3 + (1/2G_{22} - 1/2G_{33})x^3 + (G_{12} - 1/2G_{21})y^3 - 2M_{12}z^3x + 6M_{23}z^2y^2 + (3G_{13} - 3/2G_{31})zy^2 + (-1/2C_{23} + E_{32})y^2 + (-2M_{33} + 2M_{22})zy^3 + (1/2B_{13} - B_{31})z + (-2M_{22} + 2M_{33})z^3y + (3/2G_{21} - 3G_{12})z^2y + (3/2C_{22} + 3/2C_{33})zy + (B_{21} - 1/2B_{12})y + (-G_{13} + 1/2G_{31})z^3 + (-E_{32} + 1/2C_{23})z^2 + (-6M_{33} - 6M_{22})zyx^2 + (-3/2G_{22} - 3/2G_{33})xy^2 + 6M_{12}zxy^2 + (E_{31} + C_{13})xy + M_{23}x^4 + (E_{21} + C_{12})zx + (3G_{32} - 3G_{23})zxy + 6M_{13}z^2xy - M_{23}z^4 - M_{23}y^4 + A_{23}$

Knowing the form of the general conformal Killing tensor allows one to consider lower dimensional sub-sets. These are often representative of symmetries of coordinate webs which will be studied in the next chapters.

Chapter 4

The set of rotationally symmetric characteristic CKTs in $\mathbf{E}^3$

4.1 Definitions and constructions of rotationally symmetric webs

Now we begin the task of finding characteristic conformal Killing tensors corresponding to each of the three dimensional known rotational R-separable webs given in [38]. Later we address the question as to whether they describe inequivalent coordinate webs or not. Because we restrict ourselves to the R-separable webs admitting a rotational symmetry, to describe them we must find the most general rotational conformal Killing tensor sub-set of the conformal Killing tensor calculated in the previous chapter. Rotational coordinate webs means that one foliation of the web consists of half planes with common intersection forming the z-axis. This we label the rotational axis. Without loss of generality we can restrict ourselves to the webs having the z-axis as their rotational axis. Up to an isometry, a characteristic conformal Killing tensor of such a web admits the Killing vector $\mathbf{R}_3$ as an eigenvector.

The **continuous** operation to characterize a conformal tensor $\mathbf{T}$ representing a *symmetric web* is given by the solutions of

$$\mathcal{L}_{\mathbf{k}}\mathbf{T} = h\mathbf{T}, \tag{4.1.1}$$

where h is any real scalar and $\mathcal{L}$ is the Lie derivative operator with respect to the conformal Killing vector $\mathbf{k}$ which generates a group action under which the web is invariant. Note the above is Prop. 2.3.8, where $\mathbf{k}$ is the infinitesimal generator of the one parameter group action. This is a property of all conformal Killing vectors. We use from now on the Lie derivative formula for contravariant rank two tensors $\mathbf{T}$ which is:

$$(\mathcal{L}_{\mathbf{k}}T)^{ij} = k^l\partial_l T^{ij} - T^{lj}\partial_l k^i - T^{il}\partial_l k^j \tag{4.1.2}$$

The condition (4.1.1) is not sufficient on its own because it does not imply normality of the eigenvectors of T^{ij}. Hence the solution set is not the set of rotationally symmetric characteristic conformal Killing tensors. Thus not only must one set the rotational Lie derivative of the general conformal Killing tensor to zero, namely Eq. (4.1.1) for Killing vector $\mathbf{k} = \mathbf{R}_3$, but in addition impose the three Tonolo-Schouten-Nijenhuis (TSN) conditions which are both necessary and sufficient for a given symmetric (Killing) tensor field to have integrable eigenvectors. These conditions read

$$
\begin{aligned}
N^l{}_{[jk}g_{i]l} &= 0 \\
N^l{}_{[jk}K_{i]l} &= 0 \\
N^l{}_{[jk}K_{i]m}K^m{}_l &= 0,
\end{aligned}
\tag{4.1.3}
$$

where $N^i{}_{jk}$ are the components of the Nijenhuis tensor of K^{ij} given by

$$
N^i{}_{jk} = K^i{}_l K^l{}_{[j,k]} + K^l{}_{[j}K^i{}_{k],l}. \tag{4.1.4}
$$

Lie differentiation leaves nine independent coefficients of the conformal Killing tensor solution and the tensor (not characteristic yet) is:

$$
\begin{aligned}
K_{11} &= -\frac{1}{2}A_{33} + 6G_{22}xyz - M_{33}x^2y^2 + G_{12}z^3 - B_{21}z + G_{12}zy^2 - 2E_{21}xy \\
&- \frac{1}{2}M_{33}x^4 + 5M_{33}z^2x^2 - \frac{1}{2}M_{33}z^4 - \frac{1}{2}M_{33}y^4 - 5G_{12}zx^2 - M_{33}z^2y^2 \\
&+ (-1/2C_{22} - 1/2C_{33})z^2 + (3/2C_{22} + 1/2C_{33})x^2 \\
&+ (-3/2C_{22} + 1/2C_{33})y^2 \\
K_{22} &= -\frac{1}{2}A_{33} - 6G_{22}xyz - M_{33}x^2y^2 + G_{12}z^3 - B_{21}z - 5G_{12}zy^2 \\
&+ 2E_{21}xy - \frac{1}{2}M_{33}x^4 - M_{33}z^2x^2 - \frac{1}{2}M_{33}z^4 - \frac{1}{2}M_{33}y^4 \\
&+ G_{12}zx^2 + 5M_{22}z^2y^2 + (-3/2C_{22} + 1/2C_{33})x^2 + (3/2C_{22} + 1/2C_{33})y^2 \\
&+ (-1/2C_{22} - 1/2C_{33})z^2 \\
K_{33} &= -C_{33}y^2 + A_{33} + 2M_{33}x^2y^2 - 2G_{12}z^3 + 2B_{21}z - C_{33}x^2 + 4G_{12}zy^2 \\
&+ M_{33}x^4 - 4M_{33}z^2x^2 + M_{33}z^4 + M_{33}y^4 + 4G_{12}zx^2 - 4M_{33}z^2y^2 \\
&+ (C_{22} + C_{33})z^2 \\
K_{12} &= 3C_{22}xy - 3G_{22}zx^2 - 6G_{12}xyz + 3G_{22}y^2z + 6M_{33}xyz^2 \\
&+ E_{21}x^2 - E_{21}y^2 \\
K_{13} &= (3/2C_{33} + 3/2C_{22})zx - 9/2G_{12}z^2x + 3M_{33}z^3x - E_{21}zy \\
&+ 3/2G_{12}xy^2 - 3M_{33}xzy^2 + 3/2G_{22}yz^2 - 3/2G_{22}yx^2 - 3M_{33}zx^3 \\
&+ 3/2B_{21}x + 3/2G_{12}x^3 - 3/2G_{22}y^3 + 3/2B_{22}y \\
K_{23} &= 3M_{33}z^3y - 9/2G_{12}z^2y - 3/2G_{22}z^2x - 3M_{33}zy^3 + 3/2G_{22}xy^2 \\
&+ 3/2B_{21}y - 3/2B_{22}x + E_{21}xz + 3/2G_{12}yx^2 + (3/2C_{33} + 3/2C_{22})zy \\
&- 3M_{33}zyx^2 + 3/2G_{22}x^3 + 3/2G_{12}y^3
\end{aligned}
\tag{4.1.5}
$$

The imposition of the TSN conditions implies that the coefficients E_{21}, B_{22} and G_{22} must vanish. The resulting six-dimensional rotational characteristic Killing tensor thus takes the form:

$$
\begin{aligned}
K_{11} &= -1/2M_{33}x^4 - M_{33}x^2y^2 + (3/2C_{22} + 1/2C_{33})x^2 - 5G_{12}zx^2 + 5M_{33}x^2z^2 \\
&- 1/2M_{33}y^4 + (1/2C_{33} - 3/2C_{22})y^2 + G_{12}zy^2 - M_{33}y^2z^2 - 1/2A_{33} - B_{21}z \\
&+ (-1/2C_{22} - 1/2C_{33})z^2 - 1/2M_{33}z^4 + G_{12}z^3 \\
K_{22} &= -1/2M_{33}x^4 - M_{33}x^2y^2 + (1/2C_{33} - 3/2C_{22})x^2 + G_{12}zx^2 - M_{33}x^2z^2 \\
&- 1/2M_{33}y^4 + (3/2C_{22} + 1/2C_{33})y^2 - 5G_{12}zy^2 + 5M_{33}y^2z^2 - 1/2A_{33} - B_{21}z \\
&+ (-1/2C_{22} - 1/2C_{33})z^2 - 1/2M_{33}z^4 + G_{12}z^3 \\
K_{33} &= M_{33}x^4 + 2M_{33}x^2y^2 - C_{33}x^2 + 4G_{12}zx^2 - 4M_{33}x^2z^2 + M_{33}y^4 - C_{33}y^2 + 4G_{12}zy^2 \\
&- 4M_{33}y^2z^2 + 2B_{21}z + (C_{33} + C_{22})z^2 - 2G_{12}z^3 + M_{33}z^4 + A_{33} \\
K_{12} &= 3C_{22}yx - 6G_{12}xyz + 6M_{33}xyz^2 \\
K_{13} &= 3/2G_{12}x^3 - 3M_{33}zx^3 + 3/2G_{12}xy^2 - 3M_{33}xzy^2 + 3/2B_{21}x + 3M_{33}z^3x \\
&+ (3/2C_{22} + 3/2C_{33})zx - 9/2G_{12}xz^2 \\
K_{23} &= 3/2G_{12}yx^2 - 3M_{33}zyx^2 + 3/2G_{12}y^3 - 3M_{33}zy^3 + 3/2B_{21}y + 3M_{33}z^3y \\
&+ (3/2C_{22} + 3/2C_{33})zy - 9/2G_{12}yz^2
\end{aligned}
\tag{4.1.6}
$$

A more compact way of expressing the above, in terms of symmetric tensor products of CKVs expressed as linear combinations of the chosen six parameters, is:

$$
\begin{aligned}
\mathbf{K} &= -\frac{A_{33}}{2}\mathbf{X}_1 \odot \mathbf{X}_1 - \frac{A_{33}}{2}\mathbf{X}_2 \odot \mathbf{X}_2 + A_{33}\mathbf{X}_3 \odot \mathbf{X}_3 \\
&- B_{21}\mathbf{X}_1 \odot \mathbf{R}_2 + B_{21}\mathbf{X}_2 \odot \mathbf{R}_1 + C_{22}\mathbf{R}_1 \odot \mathbf{R}_1 \\
&+ C_{22}\mathbf{R}_2 \odot \mathbf{R}_2 + C_{33}\mathbf{R}_3 \odot \mathbf{R}_3 + 2B_{21}\mathbf{X}_3 \odot \mathbf{D} \\
&+ (3/2C_{22} + C_{33}/2)\mathbf{X}_1 \odot \mathbf{I}_1 + (3/2C_{22} + C_{33}/2)\mathbf{X}_2 \odot \mathbf{I}_2 \\
&+ (C_{33} + C_{22})\mathbf{X}_3 \odot \mathbf{I}_3 + G_{12}\mathbf{R}_1 \odot \mathbf{I}_2 - G_{12}\mathbf{R}_2 \odot \mathbf{I}_1 \\
&- 2G_{12}\mathbf{D} \odot \mathbf{I}_3 - \frac{M_{33}}{2}\mathbf{I}_1 \odot \mathbf{I}_1 - \frac{M_{33}}{2}\mathbf{I}_2 \odot \mathbf{I}_2 + M_{33}\mathbf{I}_3 \odot \mathbf{I}_3
\end{aligned}
\tag{4.1.7}
$$

There is an elegant alternate approach which is computationally easier than the method outlined above.

Remark 4.1.1 *The linear space of all possible CKTs which are characteristic CKTs of rotational webs in Euclidean space is the subspace of the general thirty five parameter CKT defined by the* ***discrete*** *linear operation:*

$$(\mathbf{K} \cdot \boldsymbol{R}_3) \times \boldsymbol{R}_3 = 0 \tag{4.1.8}$$

The result by definition forces the third rotational Killing vector to be an eigenvector of the modified Killing tensor. Indeed the normality of the eigenvectors is ensured by the fact that $\mathbf{R}_3$ is normal and that the second linearly independent eigenvector is tangent to the half-planes and can be considered planar. Clearly the

pair are surface forming and hence normal. In any case we checked that the three conditions making up the Tonolo-Schouten-Nijenhuis test for integrability of the eigenvectors are satisfied. Note that Eq. (4.1.8) works only for 3-dimensional Euclidean space and not on higher dimensional manifolds. Thus the discrete method cannot be taken as a universal approach to find subsets of conformal Killing tensors indicative of symmetries of the corresponding coordinate webs.
It was shown that each discrete operation analogous to Eq.(4.1.8), but along all *canonical* conformal Killing vectors, results in Killing tensor subspaces of dimension six instead of thirty five that describes the most general CKT. Application of the condition Eq.(4.1.8) confirms that rotational webs are six dimensional webs characterized by the following conditions on the Killing tensor coefficients:

$$\begin{aligned}
A_{22} &= -\frac{A_{33}}{2} \\
B_{12} &= -B_{21} \\
C_{11} &= C_{22} \\
G_{12} &= -G_{21} \\
M_{22} &= -\frac{M_{33}}{2} \\
A_{12} &= A_{13} = A_{23} = 0 \\
B_{13} &= B_{31} = B_{23} = B_{32} = 0 \\
B_{22} &= B_{33} = 0 \\
C_{12} &= C_{13} = C_{23} = 0 \\
E_{12} &= E_{21} = E_{13} = E_{31} = E_{23} = E_{32} = 0 \\
G_{13} &= G_{31} = G_{23} = G_{32} = 0 \\
G_{22} &= G_{33} = 0
\end{aligned} \tag{4.1.9}$$

All other parameters vanish except for those that are linear combinations of the six free independent parameters ($A_{33}, B_{21}, C_{22}, C_{33}, G_{12}$ and M_{33}) as required by the trace-free condition explained in the previous section. The same general rotational conformal Killing tensor then results after applying the above criterion. This proves that the Lie derivative and discrete method of finding rotationally symmetric webs are equivalent for Euclidean space. We have verified that (4.1.8) and (4.1.1) with TSN conditions are equivalent for all canonical conformal Killing vectors modulo cases of Killing tensors with constant components.

4.2 Characteristic Killing tensors for rotational R-separable coordinates

In this subsection we discuss the derivation of characteristic Killing tensors for R-separable webs. As explained in the Introduction this method relies on the observation of Eisenhart that the associated Stäckel matrix contains in its inverse information about the characteristic Killing tensors unique to the coordinate system [20]. Namely, the three rows of the inverse Stäckel matrix are the contravariant components of the linearly independent Killing tensors expressed in the eigenbasis that generates the coordinate web. One row is the usual contravariant metric tensor. This is a fundamental property of all Stäckel matrices and in [38] the first row of the inverse is defined to represent the diagonalized contravariant metric tensor. The second and third rows are the diagonalized Killing tensors; one will be common to all rotational systems but the other unique only to the coordinate web.

One is interested in Killing tensors expressed in Cartesian coordinates, thus the coordinate transformation law is needed to calculate the Jacobian matrix. Recall from Chapter 2 that when the Jacobian is left-multiplied by the diagonalized Killing tensor and the transpose of the Jacobian, that this yields the tensor expressed in Cartesian coordinates albeit with variables belonging to the original R-separable coordinate definition. So far the technique is algorithmic especially when the associated Stäckel matrix and coordinate transformation are already known (in this thesis we provide them for each coordinate case). The difficult step is guessing the Killing tensor in canonical Cartesian variables. Although an algorithm is outlined in [25], it becomes very unwieldy for the cyclidic coordinates where solving for one separable coordinate in terms of the Cartesian coordinates involves solving quartic equations. This, as well as writing the tensor as a symmetrized product of Conformal Killing vectors (CKVs), will be discussed in each case. Of particular difficulty were the Jacobi-elliptic coordinate systems that comprised the last four coordinate systems in Ch.4 of [38].

For *6-sphere* and *tangent sphere coordinates* the form of the characteristic tensor can be determined by inspection. 6-sphere coordinates are the only example in Ch.4 of [38] that are not rotational so we briefly digress from the main theme of this chapter to discuss them. The coordinate transformation law from Cartesian coordinates to canonical R-separable coordinates is given by

$$\begin{aligned} x &= \frac{u}{u^2+v^2+w^2} \\ y &= \frac{v}{u^2+v^2+w^2} \\ z &= \frac{w}{u^2+v^2+w^2} \end{aligned} \tag{4.2.1}$$

The resulting covariant metric coefficients in the separable coordinates are

$$g_{11} = g_{22} = g_{33} = \frac{1}{(u^2+v^2+w^2)^2} \tag{4.2.2}$$

The associated Stäckel matrix is:

$$\begin{bmatrix} 0 & -1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \tag{4.2.3}$$

with conformal Q-factor $(u^2+v^2+w^2)^{-2}$ and modulation R-factor the fourth root of Q. Using this information and the Eisenhart theory the corresponding conformal Killing tensors are:

$$\begin{bmatrix} 4x^2z^2 & 4xyz^2 & 2xz(-x^2-y^2+z^2) \\ 4xyz^2 & 4y^2z^2 & 2yz(-x^2-y^2+z^2) \\ 2xz(-x^2-y^2+z^2) & 2yz(-x^2-y^2+z^2) & (-x^2-y^2+z^2)^2 \end{bmatrix} \quad (4.2.4)$$

$$\begin{bmatrix} 4x^2y^2 & -2(x^2-y^2+z^2)xy & 4xy^2z \\ -2(x^2-y^2+z^2)xy & (x^2-y^2+z^2)^2 & -2(x^2-y^2+z^2)yz \\ 4xy^2z & -2(x^2-y^2+z^2)yz & 4z^2y^2 \end{bmatrix} \quad (4.2.5)$$

They are not however trace-free. In order to determine the coefficients used in symmetric tensor products of CKVs by comparing the above fourth degree expressions with the general formula for the thirty five parameter conformal Killing tensor - the trace must be removed. This is accomplished by addition of the identity matrix multiplied by one third of the negative of the trace. For the above two tensors the coefficients of symmetric tensor products of CKVs are: $M_{22} = -\frac{1}{3}, M_{33} = \frac{2}{3}$ and $M_{22} = \frac{2}{3}, M_{33} = -\frac{1}{3}$, respectively. All other coefficients are zero. With *tangent sphere coordinates* the coordinate transformation law from Cartesian coordinates to canonical R-separable coordinates is given by

$$\begin{aligned} x &= \frac{\mu \cos\psi}{\mu^2+\nu^2} \\ y &= \frac{\mu \sin\psi}{\mu^2+\nu^2} \\ z &= \frac{\nu}{\mu^2+\nu^2} \end{aligned} \quad (4.2.6)$$

The resulting covariant metric coefficients in the separable coordinates are

$$\begin{aligned} g_{11} &= g_{22} = \frac{1}{(\mu^2+\nu^2)^2} \\ g_{33} &= \frac{\mu^2}{(\mu^2+\nu^2)^2} \end{aligned} \quad (4.2.7)$$

The associated Stäckel matrix is:

$$\begin{bmatrix} 1 & -1 & -1/\mu^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (4.2.8)$$

with conformal Q-factor $(\mu^2+\nu^2)^{-2}$ and modulation R-factor the fourth root of Q. From this the two conformal Killing tensors obtained are:

$$\begin{bmatrix} 4x^2z^2 & 4xyz^2 & 2xz(-x^2-y^2+z^2) \\ 4xyz^2 & 4y^2z^2 & 2yz(-x^2-y^2+z^2) \\ 2xz(-x^2-y^2+z^2) & 2yz(-x^2-y^2+z^2) & (-x^2-y^2+z^2)^2 \end{bmatrix} \quad (4.2.9)$$

$$\begin{bmatrix} y^2 & -xy & 0 \\ -xy & x^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (4.2.10)$$

The second tensor is common to all coordinate webs invariant under rotations. Its trace-free representation is

$$\begin{bmatrix} \frac{2y^2}{3}-\frac{x^2}{3} & -xy & 0 \\ -xy & \frac{2x^2}{3}-\frac{y^2}{3} & 0 \\ 0 & 0 & -\frac{y^2}{3}-\frac{x^2}{3} \end{bmatrix} \quad (4.2.11)$$

The basis in terms of symmetrized products of CKVs is easy to find from the general trace-free rotational CKT expressed in terms of six arbitrary constants, since this tensor is of second degree:

$$C_{22} = -\frac{1}{3}, \quad C_{33} = \frac{1}{3}, \quad A_{33} = B_{21} = G_{12} = M_{33} = 0. \quad (4.2.12)$$

Although largely neglected in this chapter, this tensor will be fundamental in discussions of group operations leaving rotational webs and algebraic quantities invariant. It is required for classifying inequivalent coordinates in the next section. The first characteristic tensor's trace-free representation is purely degree four with basis:

$$M_{33} = \frac{2}{3a^2}, \quad A_{33} = B_{21} = C_{22} = C_{33} = G_{12} = 0. \quad (4.2.13)$$

For *cardioid coordinates* the coordinate transformation law is:

$$\begin{aligned} x &= \frac{\mu\nu\cos\psi}{(\mu^2+\nu^2)^2} \\ y &= \frac{\mu\nu\sin\psi}{(\mu^2+\nu^2)^2} \\ z &= \frac{\mu^2-\nu^2}{2(\mu^2+\nu^2)^2} \end{aligned} \quad (4.2.14)$$

The resulting covariant metric coefficients in the separable coordinates are

$$\begin{aligned} g_{11} &= g_{22} = \frac{1}{(\mu^2+\nu^2)^3} \\ g_{33} &= \frac{\mu^2\nu^2}{(\mu^2+\nu^2)^4} \end{aligned} \quad (4.2.15)$$

The associated Stäckel matrix is:

$$\begin{bmatrix} \mu^2 & -1 & -1/\mu^2 \\ \nu^2 & 1 & -1/\nu^2 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.2.16}$$

with conformal Q-factor $(\mu^2 + \nu^2)^{-4}$, and modulation R factor the fourth root of Q. The characteristic conformal Killing tensor, in canonical Cartesian coordinates, could not be guessed but a formula relating the variables μ, ν and ψ to Cartesian variables x, y and z is given in [38] and can be inserted into the expression. After rearrangement one obtains the conformal Killing tensor written in components:

$$\begin{bmatrix} -8z(x^2 - y^2 - z^2) & -16xyz & 4x(x^2 + y^2 - 3z^2) \\ -16xyz & 8z(x^2 - y^2 + z^2) & 4y(x^2 + y^2 - 3z^2) \\ 4x(x^2 + y^2 - 3z^2) & 4y(x^2 + y^2 - 3z^2) & 16z(x^2 + y^2) \end{bmatrix} \tag{4.2.17}$$

This is the only coordinate system yielding a purely degree three characteristic tensor. The basis of symmetrized tensor products of CKVs is $G_{12} = \frac{8}{3}$. A word of caution is required here. Although there is only one independent basis, this does not mean that G_{12} is the only coefficient involved in the formula for symmetric tensor products of CKVs. Recall the trace-free condition on the CKT also requires that $L_i = G_{lm}\alpha_{mli}$ where L_i was defined as the tensor coefficient of the dilatation vector multiplied with the i^{th} inversion vector. In the case of cardioid coordinates $L_3 = -\frac{16}{3}$, $L_1 = L_2 = 0$.

The remaining rotational webs have the additional feature of a parameter 'a' which appears in the definition of the coordinates. This will be present in the Jacobian and in the final characteristic conformal Killing tensor. An interesting fact is that A_{33} will always have 'units' a^2 and M_{33} units $\frac{1}{a^2}$. It will be clarified later that this parameter naturally arises from the dilatation member of the conformal group acting on the coordinate web. C_{22} and C_{33}, the second degree terms, never depend on this parameter. The algebra of the characteristic tensors (representing coordinates at least in canonical centered form) will show that for all non-cardioid coordinates only four independent coefficients come into play and these are A_{33}, C_{22}, C_{33} and M_{33} with $B_{21} = G_{12} = 0$. This will be expanded on in Chapter 5 and 6.

For *toroidal coordinates* the coordinate transformation law is:

$$\begin{aligned} x &= \frac{a\sinh(\eta)\cos\psi}{\cosh(\eta) - \cos\theta} \\ y &= \frac{a\sinh(\eta)\sin\psi}{\cosh(\eta) - \cos\theta} \\ z &= \frac{a\sin\theta}{\cosh(\eta) - \cos\theta} \end{aligned} \tag{4.2.18}$$

The resulting covariant metric coefficients in the separable coordinates are

$$
\begin{aligned}
g_{11} &= g_{22} = \frac{a^2}{(\cosh(\eta) - \cos\theta)^2} \\
g_{33} &= \frac{a^2 \sinh^2(\eta)}{(\cosh(\eta) - \cos\theta)^2}
\end{aligned} \tag{4.2.19}
$$

The associated Stäckel matrix is:

$$
\begin{bmatrix} 1 & -1 & -1/\sinh(\eta)^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.2.20}
$$

with conformal Q factor $a^2(\cosh(\eta) - \cos\theta)^{-2}$ and modulation R-factor the fourth root of Q. The characteristic conformal Killing tensor in canonical Cartesian coordinates is found by solving quadratic equations relating (η, θ, ψ), given in [38], with the Cartesian variables (x, y, z). One obtains

$$
\begin{bmatrix} \frac{z^2x^2}{a^2} & \frac{xyz^2}{a^2} & -\frac{(x^2+y^2-z^2-a^2)zx}{2a^2} \\ \frac{xyz^2}{a^2} & \frac{z^2y^2}{a^2} & -\frac{(x^2+y^2-z^2-a^2)zy}{2a^2} \\ -\frac{(x^2+y^2-z^2-a^2)zx}{2a^2} & -\frac{(x^2+y^2-z^2-a^2)zy}{2a^2} & \frac{(x^2+y^2-z^2-a^2)^2}{4a^2} \end{bmatrix} \tag{4.2.21}
$$

The independent coefficients of symmetric tensor products of CKVs are

$$
A_{33} = \frac{a^2}{6}, \quad C_{22} = 0, \quad C_{33} = \frac{1}{3}, \quad M_{33} = \frac{1}{6a^2} \tag{4.2.22}
$$

Writing the complete expression in terms of symmetrized tensor products of CKVs again requires the algebra derived in the previous section based on the conditions that resulted from the trace-free assumption.

Bispherical coordinates are handled in a similar way to toroidal coordinates. Their coordinate transformation to Cartesian coordinates are:

$$
\begin{aligned}
x &= \frac{a \sin\theta \cos\psi}{\cosh(\eta) - \cos\theta} \\
y &= \frac{a \sin\theta \sin\psi}{\cosh(\eta) - \cos\theta} \\
z &= \frac{a \sinh(\eta)}{\cosh(\eta) - \cos\theta}
\end{aligned} \tag{4.2.23}
$$

The resulting covariant metric coefficients in the separable coordinates are

$$
\begin{aligned}
g_{11} &= g_{22} = \frac{a^2}{(\cosh(\eta) - \cos\theta)^2} \\
g_{33} &= \frac{a^2 \sin^2\theta}{(\cosh(\eta) - \cos\theta)^2}
\end{aligned} \tag{4.2.24}
$$

The associated Stäckel matrix is:

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1/\sin\theta^2 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.2.25}$$

with conformal Q factor $a^2(\cosh(\eta) - \cos\theta)^{-2}$ and modulation R-factor the fourth root of Q. The characteristic conformal Killing tensor in Cartesian coordinates is expressed in components and is given by:

$$\begin{aligned}
K_{11} &= \frac{1}{4a^2}(x^4 - 2z^2x^2 + 2x^2a^2 + 2x^2y^2 + z^4 - 2z^2a^2 + y^4 + 2y^2a^2 + a^4 + 2z^2y^2) \\
K_{22} &= \frac{1}{4a^2}(x^4 + 2x^2y^2 + 2x^2a^2 + 2z^2x^2 + z^4 - 2z^2a^2 + y^4 + 2y^2a^2 + a^4 - 2z^2y^2) \\
K_{33} &= \frac{(x^2+y^2)z^2}{a^2} \\
K_{12} &= -\frac{xyz^2}{a^2} \\
K_{13} &= \frac{(x^2+y^2-z^2+a^2)zx}{2a^2} \\
K_{23} &= \frac{(x^2+y^2-z^2+a^2)zy}{2a^2}
\end{aligned} \tag{4.2.26}$$

The independent coefficients of symmetric tensor products of CKVs are

$$A_{33} = -\frac{a^2}{6}, \quad C_{22} = 0, \quad C_{33} = \frac{1}{3}, \quad M_{33} = -\frac{1}{6a^2} \tag{4.2.27}$$

These conditions are the same as those for toroidal coordinates except for a sign change in A_{33} and M_{33}; this subtlety will be revisited in the next section when considering inequivalence of coordinates.

Inverse oblate spheroidal coordinates and *inverse prolate spheroidal coordinates* are handled in a similar manner: coordinate relations in [38] can be solved in terms of Cartesian variables by use of the quadratic formula. For *inverse oblate coordinates* the coordinate transformation law is:

$$\begin{aligned}
x &= \frac{a\cosh(\eta)\sin\theta\cos\psi}{\cosh^2(\eta) - \cos^2\theta} \\
y &= \frac{a\cosh(\eta)\sin\theta\sin\psi}{\cosh^2(\eta) - \cos^2\theta} \\
z &= \frac{a\sinh(\eta)\cos\theta}{\cosh^2(\eta) - \cos^2\theta}
\end{aligned} \tag{4.2.28}$$

The resulting covariant metric coefficients in the separable coordinates are

$$\begin{aligned}
g_{11} &= g_{22} = \frac{a^2(\cosh^2(\eta) - \sin^2\theta)}{(\cosh^2(\eta) - \cos^2\theta)^2} \\
g_{33} &= \frac{a^2\cosh^2(\eta)\sin^2\theta}{(\cosh^2(\eta) - \cos^2\theta)^2}
\end{aligned} \tag{4.2.29}$$

The associated Stäckel matrix is:

$$\begin{bmatrix} a^2\cosh^2(\eta) & -1 & 1/\cosh^2(\eta) \\ -a^2\sin^2\theta & 1 & -1/\sin^2\theta \\ 0 & 0 & 1 \end{bmatrix} \tag{4.2.30}$$

with conformal Q factor $(\cosh^2(\eta) - \cos^2\theta)^{-2}$ and modulation R-factor the fourth root of Q. The characteristic conformal Killing tensor for *inverse oblate coordinates* is given in components, written in canonical Cartesian coordinates, by:

$$\begin{aligned}
K_{11} &= \frac{(x^4 - 2z^2x^2 + 2x^2y^2 + y^4 + z^4 + z^2a^2 + y^2a^2 + 2z^2y^2)}{a^2} \\
K_{22} &= \frac{(x^4 + 2z^2x^2 + x^2a^2 + 2x^2y^2 + z^2a^2 + y^4 + z^4 - 2z^2y^2)}{a^2} \\
K_{33} &= \frac{(x^2 + y^2)(4z^2 + a^2)}{a^2} \\
K_{12} &= -\frac{(4z^2 + a^2)xy}{a^2} \\
K_{13} &= -\frac{zx(a^2 - 2x^2 - 2y^2 + 2z^2)}{a^2} \\
K_{23} &= -\frac{zy(a^2 - 2x^2 - 2y^2 + 2z^2)}{a^2}
\end{aligned} \tag{4.2.31}$$

The independent coefficients of symmetric tensor products of CKVs are

$$A_{33} = 0, \qquad C_{22} = -\frac{1}{3}, \qquad C_{33} = -\frac{1}{3}, \qquad M_{33} = -\frac{2}{3a^2} \tag{4.2.32}$$

For *inverse prolate coordinates* the coordinate transformation law is:

$$\begin{aligned}
x &= \frac{a\sinh(\eta)\sin\theta\cos\psi}{\cosh^2(\eta) - \sin^2\theta} \\
y &= \frac{a\sinh(\eta)\sin\theta\sin\psi}{\cosh^2(\eta) - \sin^2\theta} \\
z &= \frac{a\cosh(\eta)\cos\theta}{\cosh^2(\eta) - \sin^2\theta}
\end{aligned} \tag{4.2.33}$$

The resulting covariant metric coefficients in the separable coordinates are

$$\begin{aligned}
g_{11} &= g_{22} = \frac{a^2(\sinh^2(\eta) + \sin^2\theta)}{(\cosh^2(\eta) - \sin^2\theta)^2} \\
g_{33} &= \frac{a^2\sinh^2(\eta)\sin^2\theta}{(\cosh^2(\eta) - \sin^2\theta)^2}
\end{aligned} \tag{4.2.34}$$

The associated Stäckel matrix is:

$$\begin{bmatrix} a^2\sinh^2\eta & -1 & -1/\sinh^2\eta \\ a^2\sin^2\theta & 1 & -1/\sin^2\theta \\ 0 & 0 & 1 \end{bmatrix} \tag{4.2.35}$$

with conformal Q factor $(\cosh^2(\eta) - \sin^2\theta)^{-2}$ and modulation R-factor the fourth root of Q. The characteristic conformal Killing tensor for *inverse prolate coordinates* is given in components by:

$$\begin{aligned} K_{11} &= \frac{(-x^4 + 2z^2x^2 - 2x^2y^2 + z^2a^2 + y^2a^2 - z^4 - y^4 - 2z^2y^2)}{a^2} \\ K_{22} &= \frac{(-x^4 + x^2a^2 - 2x^2y^2 - 2z^2x^2 + z^2a^2 - y^4 - z^4 + 2z^2y^2)}{a^2} \\ K_{33} &= \frac{(-4z^2 + a^2)(x^2 + y^2)}{a^2} \\ K_{12} &= -\frac{xy(-4z^2 + a^2)}{a^2} \\ K_{13} &= -\frac{zx(a^2 + 2x^2 + 2y^2 - 2z^2)}{a^2} \\ K_{23} &= -\frac{zy(a^2 + 2x^2 + 2y^2 - 2z^2)}{a^2} \end{aligned} \tag{4.2.36}$$

The independent coefficients of symmetric tensor products of CKVs are

$$A_{33} = 0, \quad C_{22} = -\frac{1}{3}, \quad C_{33} = -\frac{1}{3}, \quad M_{33} = \frac{2}{3a^2} \tag{4.2.37}$$

4.3 Jacobi elliptic coordinates

The case of the Jacobi elliptic functions involved in the last four rotational coordinate systems given in [38] are much more difficult to decipher their associated CKTs. Solving for μ, ν and ψ directly in terms of x, y and z in [38] using information in [38] implies solving quartic equations. Closed form solutions may exist in theory but are unwieldy. We had the idea of using numerical methods to 'measure' what the unknowns $(A_{33}, B_{21}, C_{22}, C_{33}, G_{12}, M_{33})$ must be by numerically evaluating the components of the characteristic Killing tensor and inverting the 'coefficient' matrix defined in terms of the six unknown parameters.

If the six parameter set were factored out into a column vector, the resulting six by six 'coefficient' matrix inverted and multiplied with the column vector (numerically estimated) of Killing tensor components
$(A_{33}, B_{21}, C_{22}, C_{33}, G_{12}, M_{33})$ would be approximated. The hope was that the numerical output would be very close to repeating decimal expansions hinting at

simple fractions. The fractions chosen would comprise an intelligent guess for the unknowns and their use in the general formula for symmetric tensor products of CKVs would yield the Killing tensor automatically in Cartesian coordinates. The idea is sound except that the determinant of the coefficient matrix is 0! In fact, the rank is only three. Even the guess $B_{21} = G_{12} = 0$ (which turned out to be correct) would leave us with one unknown too many. Moreover, it can be proved that the singular condition implies any formula of symmetric tensor products of CKVs is unique up to a very general 3-parameter non-constant algebraic expression.

The only way around this difficulty was to proceed numerically: either to guess the CKT coefficients (some are now functions of the Jacobi-elliptic parameter k) and judge by numerical estimates if the hypothesis was reasonable or restrict the domain to extreme values such as $\mu = \nu = \psi = 0$. The coefficients are constant independently of where in the manifold the numerical evaluations are done. However, extreme cases vastly simplify the six by six matrix and step by step allowed one to numerically gauge certain unknowns one at a time. For *bi-cyclide*, *flat ring cyclide* and *disk cyclide coordinates* the above procedure was painstakingly applied to the limiting cases $k = 1$ and $k = 0$. For such limiting values the coordinates take on a simpler form, albeit the actual coordinate system in question is only defined for k belonging to the open set $(0, 1)$. The coordinate parameter k' also belongs to the open set $(0, 1)$ with the relationship $k'^2 = 1 - k^2$. For interior values of k, we successfully conjectured - and then tested by the least squares method - that the coefficients involved in the symmetric tensor products of CKVs behave quadratically in k with the prior calculated 'endpoints'. So in these last four coordinate systems calculating the characteristic Killing tensor did not precede calculating its representation in terms of symmetrized tensor products of CKVs! A result was that the characteristic Killing tensor derived is automatically trace-free.

Bi-cyclide coordinates have the coordinate transformation law:

$$\begin{aligned} x &= \frac{a}{\Lambda}\mathrm{cn}(\mu)\mathrm{dn}(\mu)\mathrm{sn}(\nu)\mathrm{cn}(\nu)\cos\psi \\ y &= \frac{a}{\Lambda}\mathrm{cn}(\mu)\mathrm{dn}(\mu)\mathrm{sn}(\nu)\mathrm{cn}(\nu)\sin\psi \\ z &= \frac{a}{\Lambda}\mathrm{sn}(\mu)\mathrm{dn}(\nu) \\ \Lambda &\equiv 1 - \mathrm{dn}^2(\mu)\mathrm{sn}^2(\nu) \end{aligned} \tag{4.3.1}$$

The resulting covariant metric coefficients in the separable coordinates are

$$\begin{aligned} g_{11} &= g_{22} = \frac{a^2(1 - \mathrm{sn}^2(\mu)\mathrm{dn}^2(\nu))(\mathrm{dn}^2(\nu) - k^2\mathrm{sn}^2(\mu))}{\Lambda^2} \\ g_{33} &= \frac{a^2\mathrm{cn}^2(\mu)\mathrm{dn}^2(\mu)\mathrm{sn}^2(\nu)\mathrm{cn}^2(\nu)}{\Lambda^2} \end{aligned} \tag{4.3.2}$$

The associated Stäckel matrix is:

$$\begin{bmatrix} -k^2\mathrm{sn}^2(\mu) & -1 & -\frac{k'^4\mathrm{sn}^2(\mu)}{\mathrm{cn}^2(\mu)\mathrm{dn}^2(\mu)} \\ \\ \mathrm{dn}^2(\nu) & 1 & -\frac{\mathrm{dn}^2(\nu)}{\mathrm{sn}^2(\nu)\mathrm{cn}^2(\nu)} \\ \\ 0 & 0 & 1 \end{bmatrix} \tag{4.3.3}$$

with conformal Q factor $\frac{a^2}{\Lambda^2}(1-\mathrm{sn}^2(\mu)\mathrm{dn}^2(\nu))$, and modulation R-factor equal to $\Lambda^{-1/2}$. For the characteristic conformal Killing tensor of *bi-cyclide coordinates* the $k=0$ limit has the following parameters: $A_{33}=-\frac{2a^2}{3}, C_{22}=0, C_{33}=\frac{2}{3}, M_{33}=0$. The $k=1$ limit has the parameters: $A_{33}=-\frac{2a^2}{3}, C_{22}=0, C_{33}=\frac{4}{3}, M_{33}=-\frac{2}{3a^2}$. Only the A_{33} term is constant; conjecturing a quadratic dependence on the others to fit the 'endpoints', the coefficients of the Killing tensor for bi-cyclide coordinates are

$$A_{33}=-\frac{2a^2}{3}, \quad C_{22}=0, \quad C_{33}=\frac{2}{3}(1+k^2), \quad M_{33}=-\frac{2k^2}{3a^2} \tag{4.3.4}$$

The conformal Killing tensor in components that results is:

$$\begin{aligned}
K_{11} &= \frac{1}{3a^2}\cdot(a^2x^2+a^2x^2k^2+a^2y^2+a^2y^2k^2-a^2z^2-a^2z^2k^2+a^4 \\
&+ k^2x^4+2k^2x^2y^2-10k^2x^2z^2+k^2y^4+2k^2y^2z^2+k^2z^4) \\
K_{22} &= \frac{1}{3a^2}\cdot(a^2x^2+a^2x^2k^2+a^2y^2+a^2y^2k^2-a^2z^2-a^2z^2k^2+a^4 \\
&+ 2k^2x^2y^2+k^2x^4+2k^2x^2z^2+k^2y^4-10k^2y^2z^2+k^2z^4) \\
K_{33} &= -\frac{2}{3a^2}\cdot(a^2x^2+a^2x^2k^2+a^2y^2+a^2y^2k^2-a^2z^2-a^2z^2k^2+a^4 \\
&- 4k^2x^2z^2-4k^2y^2z^2+k^2x^4+2k^2x^2y^2+k^2y^4+k^2z^4) \\
K_{12} &= -\frac{4xyk^2z^2}{a^2} \\
K_{13} &= -\frac{xz(-a^2-a^2k^2-2k^2x^2-2k^2y^2+2k^2z^2)}{a^2} \\
K_{23} &= -\frac{yz(-a^2-a^2k^2-2k^2x^2-2k^2y^2+2k^2z^2)}{a^2}
\end{aligned} \tag{4.3.5}$$

Using Maple one verifies that this is indeed the trace-free characteristic Killing tensor for bi-cyclide coordinates.

Flat-ring cyclide coordinates have the coordinate transformation law:

$$\begin{aligned}
x &= \frac{a}{\Lambda}\mathrm{sn}(\mu)\mathrm{dn}(\nu)\cos\psi \\
y &= \frac{a}{\Lambda}\mathrm{sn}(\mu)\mathrm{dn}(\nu)\sin\psi \\
z &= \frac{a}{\Lambda}\mathrm{cn}(\mu)\mathrm{dn}(\mu)\mathrm{sn}(\nu)\mathrm{cn}(\nu) \\
\Lambda &\equiv 1-\mathrm{dn}^2(\mu)\mathrm{sn}^2(\nu)
\end{aligned} \tag{4.3.6}$$

The resulting covariant metric coefficients in the separable coordinates are

$$\begin{aligned} g_{11} &= g_{22} = \frac{a^2(1-\mathrm{sn}^2(\mu)\mathrm{dn}^2(\nu))(\mathrm{dn}^2(\nu)-k^2\mathrm{sn}^2(\mu))}{\Lambda^2} \\ g_{33} &= \frac{a^2\mathrm{sn}^2(\mu)\mathrm{dn}^2(\nu)}{\Lambda^2} \end{aligned} \tag{4.3.7}$$

The associated Stäckel matrix is:

$$\begin{bmatrix} -k^2\mathrm{sn}^2(\mu) & -1 & -(k^2\mathrm{sn}^2(\mu)+1/\mathrm{sn}^2(\mu)) \\ \mathrm{dn}^2(\nu) & 1 & (\mathrm{dn}^2(\nu)+k^2/\mathrm{dn}^2(\nu)) \\ 0 & 0 & 1 \end{bmatrix} \tag{4.3.8}$$

with conformal Q-factor $\frac{a^2}{\Lambda^2}(1-\mathrm{sn}^2(\mu)\mathrm{dn}^2(\nu))$ and modulation R-factor equal to $\Lambda^{-1/2}$.

For the characteristic conformal Killing tensor of *flat-ring cyclide coordinates* the $k=0$ limit has the parameters: $A_{33}=\frac{2a^2}{3}, C_{22}=\frac{1}{3}, C_{33}=\frac{1}{3}, M_{33}=0$. The $k=1$ limit has the parameters: $A_{33}=\frac{2a^2}{3}, C_{22}=\frac{2}{3}, C_{33}=\frac{2}{3}, M_{33}=\frac{2}{3a^2}$. Only the A_{33} term is constant, again conjecturing a quadratic dependence on the others to fit the 'endpoints', the coefficients of the Killing tensor for flat-ring cyclide coordinates are

$$A_{33}=\frac{2a^2}{3}, \quad C_{22}=\frac{1}{3}(1+k^2), \quad C_{33}=\frac{1}{3}(1+k^2), \quad M_{33}=\frac{2k^2}{3a^2} \tag{4.3.9}$$

This also turned out to be correct. The conformal Killing tensor in components that results is:

$$\begin{aligned} K_{11} &= -\frac{1}{3a^2}\cdot(a^2y^2+a^2z^2-2a^2x^2+a^2k^2y^2+a^2k^2z^2-2a^2k^2x^2+a^4 \\ &+ k^2x^4+2k^2x^2y^2-10k^2x^2z^2+k^2y^4+2k^2y^2z^2+k^2z^4) \\ K_{22} &= -\frac{1}{3a^2}\cdot(a^2x^2+a^2z^2-2a^2y^2+a^2k^2x^2+a^2k^2z^2-2a^2k^2y^2+a^4 \\ &+ 2k^2x^2y^2+k^2x^4+2k^2x^2z^2+k^2y^4-10k^2y^2z^2+k^2z^4) \\ K_{33} &= \frac{1}{3a^2}\cdot(-a^2x^2-a^2y^2+2a^2z^2-a^2k^2x^2-a^2k^2y^2+2a^2k^2z^2+2a^4 \\ &- 8k^2x^2z^2-8k^2y^2z^2+2k^2x^4+4k^2x^2y^2+2k^2y^4+2k^2z^4) \\ K_{12} &= \frac{xy(a^2+a^2k^2+4k^2z^2)}{a^2} \\ K_{13} &= \frac{xz(a^2+a^2k^2-2k^2x^2-2k^2y^2+2k^2z^2)}{a^2} \\ K_{23} &= \frac{yz(a^2+a^2k^2-2k^2x^2-2k^2y^2+2k^2z^2)}{a^2} \end{aligned} \tag{4.3.10}$$

Disk cyclide coordinates have the following coordinate transformation law:

$$\begin{aligned} x &= \frac{a}{\Lambda}\mathrm{cn}(\mu)\mathrm{cn}(\nu)\cos\psi \\ y &= \frac{a}{\Lambda}\mathrm{cn}(\mu)\mathrm{cn}(\nu)\sin\psi \\ z &= \frac{a}{\Lambda}\mathrm{sn}(\mu)\mathrm{dn}(\mu)\mathrm{sn}(\nu)\mathrm{dn}(\nu) \\ \Lambda &\equiv 1-\mathrm{dn}^2(\mu)\mathrm{sn}^2(\nu) \end{aligned} \tag{4.3.11}$$

The resulting covariant metric coefficients in the separable coordinates are

$$\begin{aligned} g_{11} &= g_{22} = \frac{a^2(\mathrm{sn}^2(\nu)+\mathrm{sn}^2(\mu)\mathrm{cn}^2(\nu))(\mathrm{dn}^2(\nu)-k^2\mathrm{sn}^2(\mu))}{\Lambda^2} \\ g_{33} &= \frac{a^2\mathrm{cn}^2(\mu)\mathrm{cn}^2(\nu)}{\Lambda^2} \end{aligned} \tag{4.3.12}$$

The associated Stäckel matrix is:

$$\begin{bmatrix} -k^2\mathrm{sn}^2(\mu) & -1 & (k^2\mathrm{cn}^2(\mu)-k'^2/\mathrm{cn}^2(\mu)) \\ \mathrm{dn}^2(\nu) & 1 & (k'^2\mathrm{cn}^2(\nu)-k^2/\mathrm{cn}^2(\nu)) \\ 0 & 0 & 1 \end{bmatrix} \tag{4.3.13}$$

with conformal Q-factor $\frac{a^2}{\Lambda^2}(\mathrm{sn}^2(\nu)+\mathrm{sn}^2(\mu)\mathrm{cn}^2(\nu))$ and modulation R-factor equal to $\Lambda^{-1/2}$.

For the characteristic conformal Killing tensor of *disk cyclide coordinates* the $k=0$ limit has the parameters: $A_{33}=\frac{2a^2}{3}, C_{22}=\frac{1}{3}, C_{33}=\frac{1}{3}, M_{33}=0$. The $k=1$ limit has the parameters: $A_{33}=0, C_{22}=-\frac{1}{3}, C_{33}=-\frac{1}{3}, M_{33}=-\frac{2}{3a^2}$. This time no coefficient is constant with respect to the k coordinate parameter but the quadratic dependence still holds; the coefficients of the Killing tensor for *disk cyclide coordinates* are

$$\begin{aligned} A_{33} &= -\frac{a^2}{6}(-4+4k^2), \quad C_{22}=\frac{1}{3}(1-2k^2), \quad C_{33}=\frac{1}{3}(1-2k^2) \\ M_{33} &= -\frac{2k^2}{3a^2} \end{aligned} \tag{4.3.14}$$

The conformal Killing tensor in components is:

$$\begin{aligned} K_{11} &= \frac{1}{3a^2}\cdot(2x^2a^2-4x^2a^2k^2-y^2a^2+2y^2a^2k^2-z^2a^2+2z^2a^2k^2-a^4 \\ &+ a^4k^2+k^2x^4+2k^2x^2y^2-10k^2x^2z^2+k^2y^4+2k^2y^2z^2+k^2z^4) \\ K_{22} &= \frac{1}{3a^2}\cdot(-x^2a^2+2x^2a^2k^2+2y^2a^2-4y^2a^2k^2-z^2a^2+2z^2a^2k^2 \\ &- a^4+a^4k^2+2k^2x^2y^2+k^2x^4+2k^2x^2z^2+k^2y^4-10k^2y^2z^2+k^2z^4) \\ K_{33} &= -\frac{1}{3a^2}\cdot(x^2a^2-2x^2a^2k^2+y^2a^2-2y^2a^2k^2-2z^2a^2+4z^2a^2k^2-2a^4 \end{aligned}$$

$$
\begin{aligned}
&+ \quad 2a^4k^2 - 8k^2x^2z^2 - 8k^2y^2z^2 + 2k^2x^4 + 4k^2x^2y^2 + 2k^2y^4 + 2k^2z^4) \\
K_{12} &= -\frac{xy(-a^2 + 2a^2k^2 + 4k^2z^2)}{a^2} \\
K_{13} &= -\frac{xz(-a^2 + 2a^2k^2 - 2k^2x^2 - 2k^2y^2 + 2k^2z^2)}{a^2} \\
K_{23} &= -\frac{yz(-a^2 + 2a^2k^2 - 2k^2x^2 - 2k^2y^2 + 2k^2z^2)}{a^2}
\end{aligned} \tag{4.3.15}
$$

Cap-cyclide coordinates have the following coordinate transformation law:

$$
\begin{aligned}
x &= \frac{\Lambda}{a\Upsilon}\mathrm{sn}(\mu)\mathrm{dn}(\nu)\cos\psi \\
y &= \frac{\Lambda}{a\Upsilon}\mathrm{sn}(\mu)\mathrm{dn}(\nu)\sin\psi \\
z &= \frac{k^{1/2}\Pi}{2a\Upsilon} \\
\Lambda &\equiv 1 - \mathrm{dn}^2(\mu)\mathrm{sn}^2(\nu) \\
\Upsilon &\equiv \mathrm{sn}^2(\mu)\mathrm{dn}^2(\nu) + [\frac{\Lambda}{\sqrt{k}} + \mathrm{cn}(\mu)\mathrm{dn}(\mu)\mathrm{sn}(\nu)\mathrm{cn}(\nu)]^2 \\
\Pi &\equiv \frac{\Lambda^2}{k} - (\mathrm{sn}^2(\mu)\mathrm{dn}^2(\nu) + \mathrm{cn}^2(\mu)\mathrm{dn}^2(\mu)\mathrm{sn}^2(\nu)\mathrm{cn}^2(\nu))
\end{aligned} \tag{4.3.16}
$$

The resulting covariant metric coefficients in the separable coordinates are

$$
\begin{aligned}
g_{11} &= g_{22} = \frac{\Lambda^2(1 - \mathrm{sn}^2(\mu)\mathrm{dn}^2(\nu))(\mathrm{dn}^2(\nu) - k^2\mathrm{sn}^2(\mu))}{a^2\Upsilon^2} \\
g_{33} &= \frac{\Lambda^2\mathrm{sn}^2(\mu)\mathrm{dn}^2(\nu)}{a^2\Upsilon^2}
\end{aligned} \tag{4.3.17}
$$

The associated Stäckel matrix is:

$$
\begin{bmatrix}
-k^2\mathrm{sn}^2(\mu) & -1 & -(k^2\mathrm{sn}^2(\mu) + 1/\mathrm{sn}^2(\mu)) \\
\mathrm{dn}^2(\nu) & 1 & (\mathrm{dn}^2(\nu) + k^2/\mathrm{dn}^2(\nu)) \\
0 & 0 & 1
\end{bmatrix} \tag{4.3.18}
$$

with conformal Q-factor $\frac{\Lambda^2}{a^2\Upsilon^2}(1-\mathrm{sn}^2(\mu)\mathrm{dn}^2(\nu))$ and modulation R-factor $\Lambda^{1/2}\Upsilon^{-1/2}$.

The case of the characteristic conformal Killing tensor of *cap-cyclide coordinates* presented the biggest challenge to find the CKT coefficients. For one, the $k = 0$ limit was impossible to deal with numerically as the coordinates suffer a singularity at that limiting case and consequently analyzing the six by six matrix of coefficients numerically meant dealing with ill conditioned systems. After many failed attempts, we realized the only way to proceed was to make numerical estimations for several values of k close to the known $k = 1$ limit. After seeing that the best quadratic fit through the data points failed to be an accurate estimate for other measured points, the *best cubic fit* was conjectured and that turned out to give the answer. Furthermore some coefficients were defined in terms of others and the overall dependence fails to be a polynomial function in k. One coefficient even undergoes a singularity for the $k = 0$ limit! The coefficients are:

$$\begin{aligned} A_{33} &= \frac{a^2(k + 2k^2 + k^3)}{24}, \quad C_{22} = \frac{2k}{3}, \quad C_{33} = -\frac{1}{3}\cdot(1 - 4k + k^2) \\ M_{33} &= 2\frac{(1+k)^2}{3a^2k} \end{aligned} \tag{4.3.19}$$

The corresponding characteristic conformal Killing tensor in components is:

$$\begin{aligned} K_{11} &= -\frac{1}{48ka^2}\cdot(k^2 + 8x^2a^2k - 8z^2a^2k + 8x^2a^2k^3 + 48z^2a^2k^2 - 8z^2a^2k^3 \\ &+ 16a^4x^4 + 16a^4x^4k^2 + 32a^4x^4k + 16a^4z^4 + k^4 + 2k^3 - 160a^4x^2z^2 \\ &- 160a^4x^2z^2k^2 - 320a^4x^2z^2k + 16a^4z^4k^2 + 32a^4z^4k + 16y^2a^2k^2 + 8y^2a^2k^3 \\ &+ 16a^4y^4 + 16a^4y^4k^2 + 32a^4y^4k - 80x^2a^2k^2 + 8y^2a^2k + 32a^4x^2y^2 + 32a^4x^2y^2k^2 \\ &+ 64a^4x^2y^2k + 32a^4y^2z^2 + 32a^4y^2z^2k^2 + 64a^4y^2z^2k) \\ K_{22} &= -\frac{1}{48ka^2}\cdot(k^2 + 8x^2a^2k - 8z^2a^2k + 8x^2a^2k^3 + 48z^2a^2k^2 - 8z^2a^2k^3 + 16a^4x^4 \\ &+ 16a^4x^4k^2 + 32a^4x^4k + 16a^4z^4 + k^4 + 2k^3 + 32a^4x^2z^2 \\ &+ 32a^4x^2z^2k^2 + 64a^4x^2z^2k + 16a^4z^4k^2 + 32a^4z^4k - 80y^2a^2k^2 + 8y^2a^2k^3 \\ &+ 16a^4y^4 + 16a^4y^4k^2 + 32a^4y^4k + 16x^2a^2k^2 + 8y^2a^2k + 32a^4x^2y^2 + 32a^4x^2y^2k^2 \\ &+ 64a^4x^2y^2k - 160a^4y^2z^2 - 160a^4y^2z^2k^2 - 320a^4y^2z^2k) \\ K_{33} &= \frac{1}{24ka^2}\cdot(k^2 + 8x^2a^2k - 8z^2a^2k + 8x^2a^2k^3 + 48z^2a^2k^2 - 8z^2a^2k^3 \\ &+ 16a^4x^4 + 16a^4x^4k^2 + 32a^4x^4k + 16a^4z^4 + k^4 + 2k^3 - 64a^4x^2z^2 \\ &- 64a^4x^2z^2k^2 - 128a^4x^2z^2k + 16a^4z^4k^2 + 32a^4z^4k - 32y^2a^2k^2 + 8y^2a^2k^3 \\ &+ 16a^4y^4 + 16a^4y^4k^2 + 32a^4y^4k - 32x^2a^2k^2 + 8y^2a^2k + 32a^4x^2y^2 + 32a^4x^2y^2k^2 \\ &+ 64a^4x^2y^2k - 64a^4y^2z^2 - 64a^4y^2z^2k^2 - 128a^4y^2z^2k) \\ K_{12} &= \frac{2xy(k^2 + 4z^2a^2k + 2a^2z^2 + 2z^2a^2k^2)}{k} \\ K_{13} &= -\frac{1}{2k}\cdot(xz(-6k^2 + k + k^3 + 8x^2a^2k + 4x^2a^2 + 4x^2a^2k^2 + 8y^2a^2k \\ &+ 4y^2a^2 + 4y^2a^2k^2 - 8z^2a^2k - 4a^2z^2 - 4z^2a^2k^2)) \\ K_{23} &= -\frac{1}{2k}\cdot(yz(-6k^2 + k + k^3 + 8x^2a^2k + 4x^2a^2 + 4x^2a^2k^2 + 8y^2a^2k \\ &+ 4y^2a^2 + 4y^2a^2k^2 - 8z^2a^2k - 4a^2z^2 - 4z^2a^2k^2)) \end{aligned} \tag{4.3.20}$$

The algebra was too formidable to verify this for the general case except for the off-diagonal components. We verified the cases $\psi = 0$ and $\psi = \frac{\pi}{2}$ knowing that any angle ψ can be redefined by an appropriate rotation in the $x - y$ plane to yield the simpler values tested above. This concludes the calculation of the characteristic conformal Killing tensors associated with the R-separable coordinates in Ch. 4 of [38].

Chapter 5

Group actions preserving rotationally symmetric canonical CKTs

In this chapter we give explicitly the conformal transformations of Euclidean space that leave invariant the linear space of trace-free rotationally symmetric conformal Killing tensors. We also give additional transformations defined directly on this space that are needed in the classification scheme. Invariants of these transformations will be used to classify the rotationally symmetric R-separable coordinate webs of the Laplace equation in Chapter 6.

5.1 Continuous group actions

The conformal transformations on $\mathbf{E}^3$ induce linear transformations on the space of trace-free conformal Killing tensors. We now give conformal transformations that leave invariant the six dimensional space of trace-free conformal Killing tensors that are invariant under rotations about the z-axis. We also give two additional transformations defined directly on this space that also leave it invariant.

We first consider the translations. Translations in the xy plane are ruled out since they will break the rotational symmetry about the z-axis. So are rotations about the y and x axis or any combination thereof. However, a translation in the z-direction, of the form:

$$\begin{aligned} \tilde{x} &= x \\ \tilde{y} &= y \\ \tilde{z} &= z + \alpha, \quad \alpha \in \mathbf{R} \end{aligned} \tag{5.1.1}$$

is an allowed group action leaving the symmetry of the web invariant. The other is the rotation of the coordinates in the xy plane; however, as the web itself is rotationally symmetric this action leaves invariant all the coefficients of the tensor and is thus trivial. Clearly rotations in the xz and yz planes are forbidden. A dilatation of the space described by the equations:

$$\begin{aligned} \tilde{x} &= e^{\alpha} x \\ \tilde{y} &= e^{\alpha} y \qquad \alpha > 0 \\ \tilde{z} &= e^{\alpha} z \end{aligned} \tag{5.1.2}$$

also does not change the geometry of the coordinate surfaces. Here we draw attention to two group transformations on the space of Killing tensors which do not arise from the action of C(M). The first one is the dilatation of the Killing tensor itself:

$$\tilde{\mathbf{K}} = \alpha \mathbf{K}, \quad \alpha > 0 \tag{5.1.3}$$

The second arises by the addition of a scalar multiple of $\mathbf{R}_3 \odot \mathbf{R}_3$ to the Killing tensor. Thus

$$\tilde{\mathbf{K}} = \mathbf{K} + \alpha \mathbf{R}_3 \odot \mathbf{R}_3, \quad \alpha \in \mathbf{R} \tag{5.1.4}$$

is another allowed functionally independent transformation from the first set. The last two transformations are not motivated by geometric considerations, but rather by the allowed operations on Stäckel matrices which were used to find the characteristic Killing tensor studied. Indeed, the last two group actions result directly from the allowed steps 2) and 3) on Stäckel matrices as explained on p. 593 of [36]. That Stäckel matrices are unique up to some defined operations translates to a characteristic Killing tensor being unique up to a multiple of itself and the addition of a scalar multiple of $\mathbf{R}_3 \odot \mathbf{R}_3$. Note that $\mathbf{R}_3 \odot \mathbf{R}_3$ is not a characteristic Killing tensor since two eigenvalues are zero and hence not all distinct.

5.2 Discrete group transformations

The last transformation belonging to the conformal group to be considered is inversion about the unit sphere. It is a discrete transformation defined by:

$$\begin{aligned} x \to \tilde{x} &= \frac{x}{x^2 + y^2 + z^2} \\ y \to \tilde{y} &= \frac{y}{x^2 + y^2 + z^2} \\ z \to \tilde{z} &= \frac{z}{x^2 + y^2 + z^2} \end{aligned} \tag{5.2.1}$$

This transformation preserves the rotational web, inducing the following simultaneous interchanges on the CKT coefficients:

$$\tilde{M}_{33} = A_{33}, \quad \tilde{A}_{33} = M_{33}, \quad \tilde{B}_{21} = -G_{12}, \quad \tilde{G}_{12} = -B_{21} \tag{5.2.2}$$

The second order terms C_{22} and C_{33} are left unchanged. Studying this effect of the discrete inversion on the 6-tuple of rotational CKT coefficients reveals that some tuples (and hence CKTs) are mapped into one another - proving that their associated coordinate webs are equivalent under that operation. In particular this proves that 6-sphere coordinates result from the inverse of Cartesian coordinates, tangent sphere coordinates are the inverse of circular cylindrical coordinates, inverse prolate spheroidal coordinates are the inverse of prolate spheroidal coordinates and inverse oblate spheroidal coordinates are the inverse of oblate spheroidal coordinates. The discrete inversion also puts cardioid coordinates in the same coordinate class as paraboloidal coordinates since the coefficients after inversion differ only by a factor of 4. As we have just discussed Killing tensors differing by a common non-zero factor are deemed equivalent as they admit the same eigenvectors. Therefore from the standpoint of group operations one might say that 6-sphere, tangent sphere, inverse oblate spheroidal, inverse prolate spheroidal and cardioid coordinates are not inequivalent from the set of simple separable metrics. However, these R-separable coordinates are distinguished from the rest in that their A_{33} coefficient is 0.

For $A_{33} \neq 0$, a discrete inversion cannot map an R-separable web into a simply separable web, since after the interchange $M_{33} \neq 0$ and this fourth order term is non-zero only for conformally separable metrics! Furthermore, the claim in [38] that cap-cyclide coordinates are the (discrete) inverse of bi-cyclide coordinates is refuted by noting that interchanging A_{33} and M_{33} does not map the one set of coefficients into the other. Neither does inversion in the unit sphere map the one set of coordinate definitions to the other, as may be verified by Maple. Thus we had to study the five independent group actions permitted by the geometry of the rotational webs. They can be used in various combination to classify the R-separable rotational webs, as well as the simple separable rotational cases already categorized in [25].

The restriction on the group parameter α for the dilatation cases can be lifted if one allow the discrete '-1 switch' of the Killing tensor and of the space, in the form:

$$\tilde{\mathbf{K}} = -\mathbf{K}, \quad (\tilde{x}, \tilde{y}, \tilde{z}) = (-x, -y, -z) \tag{5.2.3}$$

Although these discrete operations preserve the rotational web, the entire group orbits will not all be continuously connected with the identity, since the singular case $\alpha = 0$ must always be avoided. The negative dilatation of the space has the following effect on the Killing tensor coefficients:

$$\begin{aligned} \tilde{A}_{33} &= A_{33}, \quad \tilde{B}_{21} = -B_{21}, \quad \tilde{C}_{22} = C_{22}, \quad \tilde{C}_{33} = C_{33} \\ \tilde{G}_{12} &= -G_{12}, \quad \tilde{M}_{33} = M_{33} \end{aligned} \tag{5.2.4}$$

namely the first and third order terms are mapped to their negative inverses while the zeroth, second and fourth order terms are left unchanged. For the cases satisfying $(B_{21}, G_{12}) = 0$, the negative dilatation of the space plays no further role. Note that the group operation

$$\tilde{\mathbf{K}} = \mathbf{K} + \alpha\mathbf{g}, \quad \alpha \in \mathbf{R} \tag{5.2.5}$$

is not allowed, where $\mathbf{g}$ is the contravariant metric tensor. Conformal Killing tensors are of course only unique up to a functional multiple of the metric tensor. However by setting the trace to vanish we have already fixed this free parameter. The above group operation that was used in simple separable cases to distinguish between inequivalent coordinates [25] is now not available.

Having defined the discrete inversion we next look at a continuous group action that involves it. The definition is given on p.128 of [40] which defines the group as a composition of three maps:

Discrete inversion, an infinitesimal translation along a preferred direction followed by the discrete inversion.

Remark 5.2.1 *Clearly this group action is connected with the identity. One should point out that this infinitesimal group transformation is not independent of the preceding ones as it is constructed by an explicit composition of a discrete inversion and an infinitesimal translation. Unlike the discrete inversion however this group action is continuous and described by a parameter making it more convenient in some cases of study than the discrete inversion.*

After some algebra one can show that the group action on the coordinates, along z, amounts to:

$$\begin{aligned} x \to \tilde{x} &= \frac{x}{1 + 2z\alpha + \alpha^2(x^2 + y^2 + z^2)} \\ y \to \tilde{y} &= \frac{y}{1 + 2z\alpha + \alpha^2(x^2 + y^2 + z^2)} \\ z \to \tilde{z} &= \frac{z + \alpha(x^2 + y^2 + z^2)}{1 + 2z\alpha + \alpha^2(x^2 + y^2 + z^2)} \end{aligned} \tag{5.2.6}$$

Formulae for group inversion along the directions x and y are similar.

These group actions neatly coincide with the inversional conformal Killing vectors: indeed differentiations of the transformed variables at $\alpha = 0$ yield the infinitesimal inversions given in (3.1.1). As predicted these transformations along x and y do not preserve the rotational web which is defined about the z axis. However, the group inversion along z preserves this rotational symmetry. This is a quasi extension of the existing four continuous group operations that leave the rotational web considered invariant.

5.3 Effect of continuous group actions on the Killing tensor coefficients and calculation of invariants

We begin this sub-section by giving a definition for an invariant of a CKT under a continuous group action.

Definition 5.3.1 *An invariant of the rotationally symmetric CKT* $\mathbf{K}$ *is an analytic function* F *of the parameters defining* $\mathbf{K}$ *which satisfies*

$$F(\tilde{A}_{33}, \tilde{B}_{21}, \tilde{C}_{22}, \tilde{C}_{33}, \tilde{G}_{12}, \tilde{M}_{33}) = F(A_{33}, B_{21}, C_{22}, C_{33}, G_{12}, M_{33}) \tag{5.3.1}$$

for all values of the parameters, where the tilded parameters are related to the untilded parameters by the transformation laws induced by an arbitrary group action.

Note that the above is a very special case of (2.4.3). We first consider an infinitesimal translation in the z direction, which induces the following transformations:

$$\begin{aligned}
\tilde{A}_{33} &= A_{33} - 2B_{21}\alpha + (C_{22} + C_{33})\alpha^2 + 2G_{12}\alpha^3 + M_{33}\alpha^4 \\
\tilde{B}_{21} &= B_{21} - (C_{22} + C_{33})\alpha - 3G_{12}\alpha^2 - 2M_{33}\alpha^3 \\
\tilde{C}_{22} &= C_{22} + 2G_{12}\alpha + 2M_{33}\alpha^2 \\
\tilde{C}_{33} &= C_{33} + 4G_{12}\alpha + 4M_{33}\alpha^2 \\
\tilde{G}_{12} &= G_{12} + 2M_{33}\alpha \\
\tilde{M}_{33} &= M_{33}
\end{aligned} \tag{5.3.2}$$

The derivatives of the new coefficients with respect to the transformation parameter α, evaluated at $\alpha = 0$, are given by

$$\begin{aligned}
\frac{\partial \tilde{A}_{33}}{\partial \alpha} &= -2B_{21} \\
\frac{\partial \tilde{B}_{21}}{\partial \alpha} &= -(C_{22} + C_{33}) \\
\frac{\partial \tilde{C}_{22}}{\partial \alpha} &= 2G_{12} \\
\frac{\partial \tilde{C}_{33}}{\partial \alpha} &= 4G_{12} \\
\frac{\partial \tilde{G}_{12}}{\partial \alpha} &= 2M_{33} \\
\frac{\partial \tilde{M}_{33}}{\partial \alpha} &= 0
\end{aligned} \tag{5.3.3}$$

The derivative of Eq. (5.3.1) with respect to α, with the use of the chain rule yields

$$\begin{aligned}\frac{\partial F}{\partial \tilde{A}_{33}}\frac{\partial \tilde{A}_{33}}{\partial \alpha}+\frac{\partial F}{\partial \tilde{B}_{21}}\frac{\partial \tilde{B}_{21}}{\partial \alpha} &+ \frac{\partial F}{\partial \tilde{C}_{22}}\frac{\partial \tilde{C}_{22}}{\partial \alpha}+\frac{\partial F}{\partial \tilde{C}_{33}}\frac{\partial \tilde{C}_{33}}{\partial \alpha}+\frac{\partial F}{\partial \tilde{G}_{12}}\frac{\partial \tilde{G}_{12}}{\partial \alpha}\\ &+ \frac{\partial F}{\partial \tilde{M}_{33}}\frac{\partial \tilde{M}_{33}}{\partial \alpha}=0 \end{aligned} \tag{5.3.4}$$

Evaluation at the identity ($\alpha = 0$) with the use of Eq. (5.3.3) gives, after dropping the tildes, our first determining pde namely

$$\begin{aligned} 0 &= -2\frac{\partial F}{\partial A_{33}}B_{21}-\frac{\partial F}{\partial B_{21}}(C_{22}+C_{33})+2\frac{\partial F}{\partial C_{22}}G_{12}\\ &+ 4\frac{\partial F}{\partial C_{33}}G_{12}+2\frac{\partial F}{\partial G_{12}}M_{33} \end{aligned} \tag{5.3.5}$$

Remark 5.3.2 *We note here that for all R-separable coordinates in [38],* $M_{33} = 0$ *and* $G_{12} \neq 0$ *solely for* ***cardioid*** *coordinates. All other rotational coordinates satisfy* $M_{33} \neq 0$, $G_{12} = 0$ *and* $B_{21} = 0$. *This observation segregates cardioid coordinates from all the others.*

Using the tensor transformation law, it can be shown that transformed coefficients resulting from a dilatation satisfy:

$$\begin{aligned} \tilde{A}_{33} &= e^{2\alpha}A_{33}\\ \tilde{B}_{21} &= e^{\alpha}B_{21}\\ \tilde{C}_{22} &= C_{22}\\ \tilde{C}_{33} &= C_{33}\\ \tilde{G}_{12} &= e^{-\alpha}G_{12}\\ \tilde{M}_{33} &= e^{-2\alpha}M_{33} \end{aligned} \tag{5.3.6}$$

The derivatives of the tilded parameters at $\alpha = 0$ are given by

$$\begin{aligned} \frac{\partial \tilde{A}_{33}}{\partial \alpha} &= 2A_{33}\\ \frac{\partial \tilde{B}_{21}}{\partial \alpha} &= B_{21}\\ \frac{\partial \tilde{C}_{22}}{\partial \alpha} &= 0\\ \frac{\partial \tilde{C}_{33}}{\partial \alpha} &= 0\\ \frac{\partial \tilde{G}_{12}}{\partial \alpha} &= -G_{12}\\ \frac{\partial \tilde{M}_{33}}{\partial \alpha} &= -2M_{33} \end{aligned} \tag{5.3.7}$$

The resulting determining pde associated to the dilatation is found to be

$$2\frac{\partial F}{\partial A_{33}}A_{33} + \frac{\partial F}{\partial B_{21}}B_{21} - \frac{\partial F}{\partial G_{12}}G_{12} - 2\frac{\partial F}{\partial M_{33}}M_{33} = 0 \tag{5.3.8}$$

The third group transformation to be considered is the scalar multiple of the Killing tensor itself, namely $\tilde{\mathbf{K}} = \alpha\mathbf{K}$ for $\alpha \neq 0$. The transformation of the coefficients and the resulting derivatives with respect to the parameter are too trivial to tabulate. A subtle point is that evaluation at the identity element of the group transformation amounts to evaluation at $\alpha = 1$, not $\alpha = 0$ as in all the other examples. This subtlety is easily overlooked since the derivative of the coefficients are the unprimed coefficients themselves and have no dependence on α. The resulting determining pde follows easily:

$$\begin{aligned} 0 &= \frac{\partial F}{\partial A_{33}}A_{33} + \frac{\partial F}{\partial B_{21}}B_{21} + \frac{\partial F}{\partial C_{22}}C_{22} + \frac{\partial F}{\partial C_{33}}C_{33} \\ &+ \frac{\partial F}{\partial G_{12}}G_{12} + \frac{\partial F}{\partial M_{33}}M_{33} \end{aligned} \tag{5.3.9}$$

The fourth group transformation considered is the addition to the Killing tensor of a scalar multiple of the second rotational Killing tensor common to all rotational coordinates. Recall that this transformation has the form $\tilde{\mathbf{K}} = \mathbf{K} + \alpha\mathbf{R}_3 \odot \mathbf{R}_3$. However, it should be pointed out that this is an abuse of notation, since the symmetric tensor product indicated is only true for the non trace-free representation. Explicitly the tensor has two non-zero parameters with respect to our basis set C_{22} and C_{33}: $C_{22} = -\frac{1}{3}$ and $C_{33} = \frac{1}{3}$. The resulting transformation of the coefficients is as follows:

$$\begin{aligned} \tilde{A}_{33} &= A_{33} \\ \tilde{B}_{21} &= B_{21} \\ \tilde{C}_{22} &= C_{22} - \frac{1}{3}\alpha \\ \tilde{C}_{33} &= C_{33} + \frac{1}{3}\alpha \\ \tilde{G}_{12} &= G_{12} \\ \tilde{M}_{33} &= M_{33} \end{aligned} \tag{5.3.10}$$

All derivatives of the coefficients with respect to the transformation parameter α are zero except for $\frac{\partial C_{22}}{\partial \alpha} = -\frac{1}{3}$ and $\frac{\partial C_{33}}{\partial \alpha} = \frac{1}{3}$. Again the last result is from evaluation at the identity element of the group transformation. We thus arrive at the fourth determining pde which is:

$$\frac{\partial F}{\partial C_{22}} - \frac{\partial F}{\partial C_{33}} = 0 \tag{5.3.11}$$

We next look at the consequences of the continuous inversion. One can show the following transformation of the Killing tensor coefficients:

$$\begin{array}{rcl} \tilde{A}_{33} & \to & A_{33} \\ \tilde{B}_{21} & \to & B_{21} - 2\alpha A_{33} \\ \tilde{C}_{22} & \to & C_{22} - 2\alpha B_{21} + 2\alpha^2 A_{33} \\ \tilde{C}_{33} & \to & C_{33} - 4\alpha B_{21} + 4\alpha^2 A_{33} \\ \tilde{G}_{12} & \to & G_{12} + (C_{22} + C_{33})\alpha - 3B_{21}\alpha^2 + 2A_{33}\alpha^3 \\ \tilde{M}_{33} & \to & M_{33} + 2G_{12}\alpha + (C_{22} + C_{33})\alpha^2 - 2B_{21}\alpha^3 + A_{33}\alpha^4 \end{array} \tag{5.3.12}$$

One sees immediately that the term A_{33} is invariant; clearly the discrete inversion cannot be 'reached' by the continuous group inversion. This is perhaps analogous to the fact that dilatation of the Killing tensor itself cannot attain multiplication of the Killing tensor by -1.

The continuous inversion along z admits a well defined pde which is:

$$\begin{array}{rcl} 0 & = & -2\frac{\partial F}{\partial B_{21}} A_{33} - 2\frac{\partial F}{\partial C_{22}} B_{21} - 4\frac{\partial F}{\partial C_{33}} B_{21} \\ & + & \frac{\partial F}{\partial G_{12}}(C_{22} + C_{33}) + 2\frac{\partial F}{\partial M_{33}} G_{12} \end{array} \tag{5.3.13}$$

Now that the pdes resulting from all the infinitesimal group actions considered have been written down, it is instructive to tabulate the vectors of the generators of the solution to verify that they are indeed linearly independent. In terms of the basis

$$\left(\frac{\partial}{\partial A_{33}}, \quad \frac{\partial}{\partial B_{21}}, \quad \frac{\partial}{\partial C_{22}}, \quad \frac{\partial}{\partial C_{33}}, \quad \frac{\partial}{\partial G_{12}}, \quad \frac{\partial}{\partial M_{33}}\right), \tag{5.3.14}$$

the vector generators of the five determining pdes are:

$$\begin{array}{rcl} V_1 & = & [-2B_{21}, -(C_{22} + C_{33}), 2G_{12}, 4G_{12}, 2M_{33}, 0] \\ V_2 & = & [2A_{33}, B_{21}, 0, 0, -G_{12}, -2M_{33}] \\ V_3 & = & [A_{33}, B_{21}, C_{22}, C_{33}, G_{12}, M_{33}] \\ V_4 & = & [0, 0, 1, -1, 0, 0] \\ V_5 & = & [0, 2A_{33}, 2B_{21}, 4B_{21}, -(C_{22} + C_{33}), -2G_{12}] \end{array} \tag{5.3.15}$$

The corresponding Lie algebra is:

[,]	V_1	V_2	V_3	V_4	V_5
V_1	0	V_1	0	0	$2V_2$
V_2	$-V_1$	0	0	0	V_5
V_3	0	0	0	$-V_4$	0
V_4	0	0	V_4	0	0
V_5	$-2V_2$	$-V_5$	0	0	0

Table 5.1: Lie Commutator Table

The set of vectors $(V_1, \ldots, V_5)$ is generically linearly independent. However, there are special cases when it is not. In particular there are coordinates such that the associated dimension of the space spanned by the V_is reduces to two. This closed commutator table proves that the system of pdes is completely integrable. Actual integration of the above system to find group invariants is exceedingly difficult, especially after also imposing the discrete inversion not admitting a pde. It is best to first start with canonical examples of characteristic Killing tensors which satisfy $B_{21} = G_{12} = 0$ for non-cardioid cases in [38]. The study of canonically centered coordinates admitting a restricted set of group transformations will be the topic of the next chapter.

The system of pdes was eventually integrated first by finding two functionally independent invariants with respect to the entire group modulo the dilatation of the conformal Killing tensor. The method of undetermined polynomial coefficients, utilized especially in [25], was used since Maple 9 was unable to perform the integration of Eq. (5.3.15). Thus the derived I_1 and I_2 are polynomial functions of the coefficients:

$$\begin{aligned} I_1 &= -72M_{33}A_{33}(C_{22}+C_{33}) + 108M_{33}B_{21}^2 + 2(C_{22}+C_{33})^3 + 108A_{33}G_{12}^2 \\ &+ 36B_{21}G_{12}(C_{22}+C_{33}) \\ I_2 &= (C_{22}+C_{33})^2 + 12A_{33}M_{33} + 12B_{21}G_{12} \end{aligned} \tag{5.3.16}$$

To construct the single functionally independent invariant with respect to the entire group action is now trivial. The square of I_1 is divided by the cube of I_2 to get the polynomial degrees (of the coefficients) to match (at 6). The resulting quantity is then invariant under dilatation of the conformal Killing tensor itself. This explicit form of the invariant is

$$\begin{aligned} I &= (-72M_{33}A_{33}(C_{22}+C_{33}) + 108M_{33}B_{21}^2 + 2(C_{22}+C_{33})^3 + 108A_{33}G_{12}^2 \\ &+ 36B_{21}G_{12}(C_{22}+C_{33}))^2/((C_{22}+C_{33})^2 + 12A_{33}M_{33} + 12B_{21}G_{12})^3 \end{aligned} \tag{5.3.17}$$

It is easy to see that the above is also invariant with respect to discrete inversion given the invariance of every term in the sum with respect to that discrete action. This solution will be discussed in the next chapter after first considering the *reduced invariant* on canonically centered coordinate webs which avoids the continuous translation and inversion group action. Unfortunately, it seems that I is a poor choice to discriminate between coordinate webs as it lumps together simple and conformally separable cases whereas the reduced invariant was defined only for 'purely' conformal coordinate webs. This will be elucidated in the next chapter.

Chapter 6

Classification of the symmetric R-separable webs

6.1 Classification of rotationally symmetric R-separable coordinates

With the results and methods presented in the previous chapters - we now proceed with the main goal of this thesis which is to *classify* the rotationally symmetric webs and partition them into equivalence classes under the Lie group of conformal transformations derived in the previous chapter. The question of the exhaustiveness of rotationally symmetric R-separable coordinates will finally be answered. The lack of other types of symmetric coordinates admitting R-separation of the Laplace equation will be discussed at the end of this chapter.

We begin this chapter by giving a definition of equivalence of characteristic CKTs under the action of the conformal group.

Definition 6.1.1 *Two characteristic CKTs are said to be* ***equivalent*** *if and only if there exists an element of the transformation group on the space of such tensors which maps one of the tensors into the other. The R-separable webs defined by equivalent tensors are said to be equivalent.*

For the classification of the rotationally symmetric R-separable webs the following transformations need to be considered:

Translation along the z-axis
Continuous inversion along the z-axis
Dilatation of the space
Discrete inversion
Dilatation of the Killing tensor

Addition to the Killing tensor of a scalar multiple of the rotational Killing tensor $\mathbf{R}_3 \odot \mathbf{R}_3$

The effect of these operations on the coefficients of the reduced Killing tensor has already been described in the previous section. Thus we have available degrees of freedom for the change of coefficients, as well as an adjustment of the parameter k that appears in the definition of the four Jacobi-elliptic coordinate systems. It is these last four coordinates in Ch. 4 of [38] that contain this arbitrary parameter which is defined on the open set $(0, 1)$. Indeed k appears in the coefficients for symmetric tensor products of CKVs and its variation within the allowed range must be taken into account. Recall that for elliptic-hyperbolic coordinates in $\mathbf{E}^3$, the parameter used to describe the coordinate surface was defined to be the inter focal distance [25] whose range gives a related family of coordinates. Quantities invariant under group transformations leaving the web unchanged are called 'invariants'. However, usually the invariants are themselves functions of the parameters appearing in the coordinate definitions. In some of the cases presented in this section the invariants will differ for different values of k. If adjusting k in addition to the degrees of freedom afforded by all possible group transformations does not yield an equality of two sets of invariants, then it is reasonable to conclude that the two coordinate systems considered are inequivalent. Note we specifically omit the other coordinate parameter 'a' appearing in most definitions in [38] because this is none other than the dilatation of the space which has already been considered.

The coordinates 6-Sphere, tangent sphere and cardioid coordinate systems are defined without any such parameters. All algebraic invariants will be constant but as mentioned before the discrete inversion puts these in the same equivalence class as simple separable coordinates in $\mathbf{E}^3$.

6.2 Classification of canonically centered rotationally symmetric webs

We call attention to a curious paradox. Consider for the subset of canonically centered webs satisfying $B_{21} = 0$ and $G_{12} = 0$, the three allowed functionally independent group actions acting on the coefficients. These are the dilatation of the Killing tensor, dilatation of the space and addition to the Killing tensor of a scalar multiple of the rotational Killing tensor $\mathbf{R}_3 \odot \mathbf{R}_3$. Recall that the translation and continuous inversion along the z-axis changes the values of B_{21} and G_{12} which is unacceptable since both are zero and must remain so for canonically centered webs. Since there are only four independent unknowns in the coefficients A_{33}, C_{22}, C_{33} and M_{33} and three determining pdes, we obtain one functionally independent (*reduced*) invariant:

$$\left(\frac{C_{22} + C_{33}}{\sqrt{A_{33}M_{33}}}\right) \tag{6.2.1}$$

Note that the product $A_{33}M_{33}$ negates dependence on the coordinate parameter a: the invariant is thus constant for entire families of coordinates. It is easy to see that the invariant is constant with respect to dilatation of the space and dilatation of the Killing tensor itself. Furthermore the product $A_{33}M_{33}$ is invariant under the discrete inversion since A_{33} and M_{33} are simply interchanged by this transformation. We now use this single invariant to partition the known canonical R-separable webs into disjoint equivalence classes.

Proposition 6.2.1 *Inequality of invariants is a sufficient but not a necessary condition for inequivalence of any two R-separable coordinate systems. Contrapositively, equality of invariants to prove equivalence of webs is simply a necessary but not sufficient condition.*

Proof: This follows by analyzing two systems of coordinates. Recall that the list of CKT coefficients identifying *toroidal coordinates* is:

$$A_{33} = \frac{a^2}{6}, \quad C_{22} = 0, \quad C_{33} = \frac{1}{3}, \quad M_{33} = \frac{1}{6a^2} \tag{6.2.2}$$

The list of CKT coefficients identifying *bispherical coordinates* is:

$$A_{33} = \frac{-a^2}{6}, \quad C_{22} = 0, \quad C_{33} = \frac{1}{3}, \quad M_{33} = \frac{-1}{6a^2} \tag{6.2.3}$$

Substituting these values into Eq.(6.2.1) gives the surprising result of $+2$ for both toroidal and bispherical coordinates. Is this proof that, with respect to the three group operations, toroidal and bispherical coordinates are equivalent? A simple argument shows that they cannot be. Recall from the previous section that dilatation of the space amounts to multiplying M_{33} by a positive quantity reciprocal to that which multiplies A_{33} with the variables C_{22} and C_{33} being left unchanged. Dilatation of the Killing tensor multiplies all coefficients, including M_{33}, equally by a greater-than-zero scalar. Addition of a scalar of the second rotational Killing tensor leaves both M_{33} and A_{33} alone. Finally, it is noted that M_{33} depends on the inverse square of the coordinate parameter a. Thus varying a amounts to multiplication of M_{33} by a positive scalar. It follows that no group transformation or coordinate adjustment can change the sign of M_{33} and A_{33} with respect to the other C_{ii} coefficients. Yet to transform the characteristic tensor for toroidal coordinates into the one for bispherical coordinates requires precisely this forbidden operation! This reasoning is the ultimate 'acid test' for determining whether toroidal coordinates are indeed inequivalent to bispherical coordinates. Nonetheless their invariants are precisely the same. What resolves this paradox? Note the factor $(A_{33}M_{33})$ in the denominator completely eliminates information as to whether both A_{33} and M_{33} were positive or negative. It is easy to see knowing the transformation of the coefficients that the invariant listed is indeed constant under the three group transformations that generated it. This completes the proof by example for the pair toroidal and bispherical coordinates. □

The same proof could also be obtained by analyzing bi-cyclide and flat ring cyclide coordinates. The CKT coefficients for *bi-cyclide coordinates* are given by:

$$A_{33} = -\frac{2a^2}{3}, \quad C_{22} = 0, \quad C_{33} = \frac{2}{3}(1+k^2), \quad M_{33} = -\frac{2k^2}{3a^2} \tag{6.2.4}$$

The corresponding coefficients for *flat-ring cyclide coordinates* are:

$$A_{33} = \frac{2a^2}{3}, \quad C_{22} = \frac{1}{3}(1+k^2), \quad C_{33} = \frac{1}{3}(1+k^2), \quad M_{33} = \frac{2k^2}{3a^2} \tag{6.2.5}$$

Substitution of these coefficients into the single invariant yields $\frac{(1+k^2)}{k} \in (+2, \infty)$ for both bi-cyclide and flat ring cyclide coordinates. Note again how their values for M_{33} differ in sign. This criterion proves their inequivalence despite equality of their single invariant. On a reassuring note, for k defined on the open set $(0,1)$ the invariant $\frac{(1+k^2)}{k}$ always differs from $+2$ which is the invariant identifying toroidal and bispherical coordinates, proving positively that these four are an inequivalent set. Of course Eq.(6.2.1) fails to discriminate between webs satisfying $A_{33} = 0$, namely tangent sphere, inverse oblate spheroidal and inverse prolate spheroidal coordinates. Inspection of the coefficients for the above (listed in the previous section) yields $M_{33} = -\frac{2}{3a^2}$ for inverse oblate spheroidal coordinates and $M_{33} = \frac{2}{3a^2}$ for inverse prolate spheroidal coordinates, while their C_{ii} variables are the same - thus setting them apart immediately. Recall from the last chapter that it is not necessary to consider the $A_{33} = 0$ cases alongside the canonical conditions $B_{21} = G_{12} = 0$ as the discrete inversion transformation places tangent sphere, inverse oblate spheroidal and inverse prolate spheroidal coordinates into the same equivalence class as simple separable webs already classified in [25]. Furthermore any canonical Killing tensor identified by $M_{33} = 0$ will admit a characteristic Killing tensor of degree two which is already a subset of the simple separable cases. Cardioid coordinates, with only $G_{12} \neq 0$, is placed in the class of parabolic coordinates by the discrete inversion, that is the degree three tensor is mapped to a degree one tensor and so is not an additional R-separable coordinate system.

Remark 6.2.2 *The proof of Prop 6.2.1 relied on continuous group operations only. It is true that no continuous group operation so far considered, connected with the identity, can map a Killing tensor to its additive inverse. However, if the three group operations can change the tensor coefficients to bring them to the exact negative of another characteristic Killing tensor, then the Killing tensor pairs are equivalent. The previous arguments must be modified somewhat to take this feature, the result of the inclusion of a discrete -1 switch, into account.*

For the coordinate pairs toroidal, bispherical, bi-cyclide and flat ring cyclide, A_{33}, M_{33} differ only by sign. Dilatation of the space, dilatation of the Killing tensor itself and addition of a scalar multiple of the second rotational tensor $\mathbf{R}_3 \odot \mathbf{R}_3$ as well as variation of the parameters a and k amount only to multiplication of A_{33}

and M_{33} by positive non-zero scalars. If the tensors are to be made equivalent, then A_{33} and M_{33} must be left alone. Once that is known, C_{22} and C_{33} must be simultaneously brought to their additive inverses. This is easily seen to be impossible, as one increases at the negative rate of the other under the above group action.

With toroidal and bispherical coordinates, $C_{22} = 0, C_{33} \neq 0$. This relation in size is 'out of phase' with the transformation of the variables under addition of a scalar multiple of $\mathbf{R}_3 \odot \mathbf{R}_3$ which implies $\tilde{C_{22}} = C_{22} + \frac{\alpha}{3}$ and $\tilde{C_{33}} = C_{33} - \frac{\alpha}{3}$. It is impossible to simultaneously bring both variables to their additive inverses - thus proving that bispherical and toroidal coordinates are inequivalent even with the discrete -1 switch added to the list of coefficient transformations.

With bi-cyclide and flat ring cyclide coordinates the A_{33} and M_{33} must be similarly left alone, but the C_{22} coefficient is 0 for bi-cyclide coordinates and non-zero for flat ring cyclides. Adjusting C_{22} accordingly will not make the C_{33} pair additive inverses. This reasoning completes the discussion on possible equivalence between bi-cyclide and flat-ring cyclide coordinates even with the -1 switch degree of freedom on the coefficients.

Disk cyclide coordinates are described by the CKT coefficients:

$$\begin{aligned} A_{33} &= -\frac{a^2}{6}(-4+4k^2), \quad C_{22} = \frac{1}{3}(1-2k^2), \quad C_{33} = \frac{1}{3}(1-2k^2) \\ M_{33} &= -\frac{2k^2}{3a^2} \end{aligned} \tag{6.2.6}$$

These are the only ones with the property $A_{33}M_{33} < 0$. This makes Eq.(6.2.1) complex however its negative square value partitions it from the square value of other invariants discussed thus far. Another argument for their inequivalence to all other coordinate systems is this: A_{33} and M_{33} will always differ in sign as group transformations and adjustment of parameters amount to multiplications by non-zero constants. No operation exists to make their signs equal or their values vanish - so setting them apart from all coordinates considered. It should be noted that this reasoning trivially explains why cardioid coordinates are distinguished from the rest. It is the only coordinate system admitting $M_{33} = 0$ while every other member has $M_{33} \neq 0$.

For *cap-cyclide coordinates* the invariant $\left(\frac{C_{22}+C_{33}}{\sqrt{A_{33}M_{33}}}\right)$ is equal to $\frac{(-2+12k-2k^2)}{(1+k)^2}$. With $k \in (0,1)$ the range of the invariant is the open interval (-2,2). This distinguishes cap-cyclide coordinates from toroidal and bispherical which is fixed at +2, and from bi-cyclide and flat ring cyclide coordinates whose range is $(+2, \infty)$.

Studying the square of Eq.(6.2.1) now becomes useful. Recall that any continuous function of an invariant is an invariant. Firstly the square is invariant to the discrete 'minus 1 switch' that complements the dilatation of the Killing tensor itself. Secondly it is then possible to consider disk cyclide coordinates where $A_{33}M_{33} < 0$.

Disk cyclide coordinates will, from variation of k, admit an invariant in the range $(-\infty, 0]$. Cap-cyclide will have its invariant in the finite interval $[0, 4)$. Toroidal and bispherical have their invariants fixed at $+4$, and finally bi-cyclide and flat-ring cyclides have their invariants in the infinite interval $(4, +\infty)$.

Remark 6.2.3 *For invariants with no intersection in their ranges, even after varying a and k, it is sufficient to prove that the corresponding webs are inequivalent under dilations, discrete inversion and addition of $\boldsymbol{R}_3 \odot \boldsymbol{R}_3$, since we have considered simultaneously the independent group actions on the subset of pertinent coefficients. Note that the singular infinities are excluded as they correspond either to $k = 0$ or $k = 1$ which are not defined for the Jacobi-elliptic coordinates.*

Remark 6.2.4 *The reader might well ask whether disk cyclide coordinates are only one of a pair of inequivalent coordinate systems since the product $A_{33}M_{33} < 0$ destroys information as to which coefficient is less than zero and which is greater than zero. Suppose one had $M_{33} < 0$ and $A_{33} > 0$ instead?*

Recall that dilatation of the space or variation of the coordinate parameter a multiplies M_{33} with the reciprocal of the positive quantity which multiplies A_{33}. Thus the set of coefficients can be continuously mapped to the case: $A_{33} < 0$ and $M_{33} > 0$ for dilatation parameter $a^2 = \frac{A_{33}}{M_{33}}$. In fact since $B_{21} = G_{12} = 0$ the discrete inversion can also be used to perform the same interchange and thus the question of which coefficient is less than or greater than zero is meaningless. Recall once again the discrete inversion and dilatation of the space does not affect the degree two C_{ii} parameters. Note that the above argument does not lump together the cases where both A_{33} and M_{33} are either greater than or less than zero for fixed values of C_{22} and C_{33}. Note further that the seemingly distinct cases $M_{33} < 0$, $A_{33} > 0$, $(C_{22} + C_{33}) > 0$ and $M_{33} > 0$, $A_{33} < 0$, $(C_{22} + C_{33}) < 0$ are related by the -1 switch and hence not representative of inequivalent webs. Thus the range of the invariant $(-\infty, 0)$ is indicative of only one equivalence class of coordinates.

A similar argument (so far!) proves that cap-cyclide coordinates does not appear to have a 'twin' inequivalent coordinate system either, as the reader could again point out that the case $A_{33} > 0$, $M_{33} > 0$ represents another inequivalent case as per the situation for the coordinate pairs toroidal, bispherical and bi-cyclide, flat ring cyclide coordinates. However the variation of the parameter k for cap-cyclide coordinates can force $(C_{22} + C_{33})$ to vanish and at that point the -1 switch can be applied to reverse the signs of A_{33} and M_{33} simultaneously without affecting the numerator of Eq.(6.2.1) since it is 0. Note this subtlety could not hold for invariants not spanning the null element as the -1 switch would reverse all the four signs of the CKT coefficients simultaneously and not two in isolation. The invariant ranges from cap-cyclide and disk-cyclide coordinates seem to intersect at the null element. However, as mentioned before no group action can make A_{33} and M_{33} switch from being equal in sign to opposite. This is yet another example of where equality of invariants is only a necessary condition for equivalence of coordinates.

6.3 The question of non-canonically centered rotationally symmetric R-separable coordinates

We have, with respect to the invariant in Eq.(6.2.1), a total of six additional inequivalent R-separable webs to the known set of eleven simple separable webs in $\mathbf{E}^3$. The entire real line of the invariant in Eq.(6.2.1) is exhausted as well as ambiguities that resulted from the product factor. We must now address the question as to whether this represents the totality of distinct R-separable webs especially since we have restricted ourselves to the canonically centered cases $B_{21} = G_{12} = 0$. What if one was given a 6-tuple of random numbers representing the values for the six CKT coefficients where the previous condition no longer holds? This spells trouble since an addition of two coefficients even with the inclusion of the translation group operation implies that an additional invariant has to be considered. This immediately means there is 'more room' for inequivalent coordinates which will be hard to classify not knowing a priori (i.e from solving the Eisenhart equations) similar formulae for the coordinate parameter k.

An alternative is to consider the translation and *continuous inversion* group actions as a set in its own right used simply to bring the two coefficients B_{21} and G_{12}, one or both assumed non-zero, to vanish irrespective of the effect on the other terms.

Definition 6.3.1 *Any 6-tuple of numbers representing the CKT coefficients are equivalent to some 4-tuple of numbers representing coordinates in canonical centered form if any pair (B_{21}, G_{12}) belonging to the original 6-tuple can be mapped to (0,0) by a composition of continuous inversion and translation group actions.*

If the above holds *uniquely* then the previous section provides proof that the coordinates in [38] represent an *exhaustive* list of all rotationally symmetric R-separable coordinates in $\mathbf{E}^3$, up to equivalence. The issue of uniqueness is a subtle one and the consequence for non-uniqueness will be addressed later on.

We have proceeded with this calculation and deduced that for the above to hold, the following pair of equations resulting from the composition of an inversion with a translation must have a real solution for the unknown transformation parameters (a, b) respectively with given arbitrary B_{21} and G_{12}:

$$\begin{aligned}
0 &= B_{21} - 2aA_{33} - 3(C_{22} + C_{33})ab^2 - (C_{22} + C_{33})b - 2(C_{22} + C_{33})b^3a^2 \\
&+ 4B_{21}a^3b^3 - 6A_{33}ba^2 - 2M_{33}b^3 - 2A_{33}a^4b^3 + 9B_{21}a^2b^2 + 6B_{21}ba \\
&- 4G_{12}ab^3 - 3G_{12}b^2 - 6A_{33}a^3b^2 \\
0 &= G_{12} + 2(C_{22} + C_{33})a^2b + 2A_{33}a^3 + 2A_{33}a^4b + 2M_{33}b - 4B_{21}a^3b \\
&- 3B_{21}a^2 + (C_{22} + C_{33})a + 4G_{12}ab
\end{aligned} \tag{6.3.1}$$

Solving for the group parameter variables (a, b) results however in solutions given in terms of roots of a degree six polynomial expression divided by a quartic polynomial.

By classical algebra solutions are guaranteed but over the complex field Z. We performed stochastic numerical tests to state with a degree of certainty that the roots of the sextic polynomial always contain at least one real pair. The presence of the quartic polynomial in the denominator poses a potential problem, since one must take into account the possibility of a pathological case where the numerator and the denominator share real roots leaving only complex roots for the remainder. Even one such case would signify a new R-separable web. However, its Lebesgue measure is zero in the abstract space of the coefficients, and beyond the reach of random numerical tests. The special case where only the pair B_{21}, G_{12} is not trivial was verified to yield real solutions for $(a,\, b)$ provided the product $G_{12}B_{21}$ is non-zero. Recall that this subcase either represents a simply separable web or can be mapped to one by a discrete inversion, as is the case for cardioid coordinates. Due to this special case yielding a real solution and the numerous numerical tests performed to verify the existence of real solutions to the degree six polynomial, we present

Proposition 6.3.2 *Eq.(6.3.1) contains at least one pair of real roots.*

Proof: Using *resultant theory* of polynomials one can prove the proposition for the special case of $C_{22}, C_{33}, B_{21}, G_{12}$ arbitrary and $A_{33} = M_{33} = 0$. This is made possible since Maple 9 and 10 both give two classes of solutions to $(a,\, b)$ in solving for B_{21} and G_{12} = (0,0). Where one class yields only complex or undefined solutions, the other does. This may be proved with the help of resultant theory of polynomials. By extension one can prove the conjecture for $A_{33} = 0$ or $M_{33} = 0$, since infinitesimal inversion or translation can bring the other term to zero since the equation involved is of degree three in the parameter guaranteeing a real solution. For A_{33}, M_{33} both non-zero however a pure inversion or translation cannot guarantee solving for either of them (the equations involved are of degree four in the parameters). However, a composition of inversion and translation yields the transformation equation for A_{33} being degree four but bivariate:

$$\begin{aligned}\tilde{A}_{33} &= A_{33} + (-2B_{21} + 4A_{33}a)b + (C_{22} + C_{33} - 6B_{21}a + 6A_{33}a^2)b^2 \\ &+ (2C_{33}a + 2C_{22}a - 6B_{21}a^2 + 4A_{33}a^3 + 2G_{12})b^3 \\ &+ (2G_{12}a - 2B_{21}a^3 + A_{33}a^4 + C_{33}a^2 + M_{33} + C_{22}a^2)b^4 \qquad (6.3.2)\end{aligned}$$

Stochastic tests using fifty million random cases have verified that for any 6-tuple of coefficients it is very likely that $\tilde{A}_{33}$ can be made zero in the general case. No pathological case of Lebesgue measure zero exists and so Prop. 6.3.2 appears true for arbitrary values of $(A_{33}, B_{21}, C_{22}, C_{33}, G_{12}, M_{33})$. □

The transformation equations for the other rotational CKT coefficients under composition of inversion and translation parameters (a, b) respectively are given here for completeness, where for compactness $C \equiv C_{22} + C_{33}$:

$$\begin{aligned}\tilde{C} &= C + 6A_{33}a^2 - 6B_{21}a \\ &+ (-18B_{21}a^2 + 12A_{33}a^3 + 6Ca + 6G_{12})b \\ &+ (6Ca^2 + 6M_{33} + 6A_{33}a^4 - 12B_{21}a^3 + 12G_{12}a)b^2 \qquad (6.3.3)\end{aligned}$$

and lastly,

$$\tilde{M}_{33} = M_{33} + 2G_{12}a + Ca^2 - 2B_{21}a^3 + A_{33}a^4 \tag{6.3.4}$$

Should any reader wish to convert a random 6-tuple of rotational CKT coefficients to canonically centered form, namely $(B_{21}, G_{12}) = (0, 0)$ by inversion parameter a and translation parameter b, the other coefficients can then be calculated using the information presented above.

6.4 Canonically centered rotational coordinates related by balanced combination of inversion and translation

With the previously discussed 'projection' from 6-tuple to 4-tuple space of Killing tensor coefficients, one must be aware of the non-uniqueness of real roots to Eq.(6.3.1). This suggests the possibility of applying the continuous inversion and translation group action on a priori canonically centered webs to yield other canonically centered webs. Namely it is a question of the existence of real solutions to Eq.(6.3.1) when B_{21} and G_{12} are zero a priori while the other four coefficients are arbitrary. Of course the trivial solution $(a,b) = (0,0)$ representing the identity transformation is always present but ignored. Solving for Eq.(6.3.1) yields for inversion parameter a:

$$a = \sqrt[4]{\frac{M_{33}}{A_{33}}} \tag{6.4.1}$$

The translation parameter b in terms of the inversion parameter a is given by:

$$b = \frac{-a}{2}\frac{((C_{22} + C_{33}) + 2A_{33}a^2)}{(2M_{33} + (C_{22} + C_{33})a^2)}. \tag{6.4.2}$$

One sees that all coordinates could be affected by the above transformations save for disk-cyclide which is the only one where the product $A_{33}M_{33} < 0$ and so only the trivial case $a = b = 0$ is possible which amounts to the identity transformation. For all other canonically centered rotationally symmetric coordinates one must verify whether the transformed values for A_{33}, C_{22}, C_{33} and M_{33} correspond to the same equivalence class or not by studying the square of the invariant given by Eq.(6.2.1). In so doing we take into account all the members of the conformal group used to classify coordinates. The transformation for A_{33} from canonical form to canonical form is given in Eq.(6.3.2) by setting both B_{21} and G_{12} to zero. We list here the transformation of the other coefficients as a result of the above process, in terms of the solved values for inversion parameter a and translation parameter b:

$$\tilde{M}_{33} = M_{33} + (C_{22} + C_{33})a^2 + A_{33}a^4$$

$$
\begin{aligned}
\tilde{C}_{22} &= C_{22} + 2A_{33}a^2 + (4A_{33}a^3 + 2C_{33}a + 2C_{22}a)b \\
&+ (2C_{33}a^2 + 2C_{22}a^2 + 2A_{33}a^4 + 2M_{33})b^2 \\
\tilde{C}_{33} &= C_{33} + 4A_{33}a^2 + (4C_{22}a + 4C_{33}a + 8A_{33}a^3)b \\
&+ (4M_{33} + 4A_{33}a^4 + 4C_{33}a^2 + 4C_{22}a^2)b^2 \qquad (6.4.3)
\end{aligned}
$$

Applying this to toroidal coordinates one finds that M_{33} is multiplied by 4 while A_{33} is divided by 4, which is merely the effect of a dilation of the space! Toroidal coordinates are therefore not affected by this procedure. Bispherical coordinates are another matter: the inversion parameter b in Eq.(6.4.2) has an indeterminate form $\frac{0}{0}$; if one labels this indeterminate fraction as c one obtains - after applying canonical form to canonical form mapping:

$$
\tilde{A}_{33} = \frac{-k^2}{6} + \frac{k^2 c}{3} - \frac{k^2 c^2}{6} \qquad \tilde{C}_{22} = -\frac{1}{3} \qquad \tilde{C}_{33} = -\frac{1}{3} \qquad \tilde{M}_{33} = 0 \qquad (6.4.4)
$$

Note that all cases with respect to the indeterminate constant c are simple separable cases, but if $M_{33} = 0$ the only way to invert the procedure back to bispherical coordinates is to choose the constant c such that $A_{33} = 0$. If $A_{33} \neq 0$ and $M_{33} = 0$ the canonical form to canonical form mapping will never map that simple separable coordinate web to an R-separable coordinate web. Therefore $c \equiv 1$ and we prove that bispherical coordinates are indeed conformally related to spheroidal coordinates, as is claimed in the literature, such as [6]. Recall that spheroidal coordinate webs are invariant to the discrete inversion - this is the case when $A_{33} = M_{33} = B_{21} = G_{12} = 0$.

Bi-cyclide coordinates are mapped to bi-cyclide coordinates given that the transformed coefficients satisfy the same invariant (with respect to the three other group transformations) as the original ones do as a function of k. However, one can show that flat ring cyclide coordinates and cap-cyclide coordinates are interchanged (by studying the range of the invariants after transformation) and vice-versa as a result of the above process. Hence one loses either flat-ring cyclide or cap-cyclide coordinates as an additional R-separable coordinate system. We prove finally that cap-cyclides are related by inversion to other cyclides although not in the manner implied by [38].

The number of additional rotationally symmetric R-separable coordinates seems to agree, after consideration of canonical form to canonical form mapping, with the results of Miller et al. In p. 70 and 71 of [6] toroidal and three additional cyclide coordinates are listed as exhausting the possibilities for rotationally symmetric R-separable webs.

One should add that canonical form to canonical form mapping does not reduce the set of inequivalent simply separable rotational coordinates classified previously [25], despite that additional transformation group actions could very well do this. Indeed, the property of simple separable webs, namely $M_{33} = 0$, implies that the inversion parameter a vanishes and b is either 0 or undefined - meaning the coefficients are left invariant. The case of paraboloidal coordinates, with $A_{33} = M_{33} = 0$

but $B_{21} \neq 0$, cannot be mapped to any other simple separable rotational coordinate system using the algebraic knowledge that no inversion and/or translation can map B_{21} and G_{12} simultaneously to zero, as mentioned previously.

6.5 Classification scheme of non-canonically centered R-separable coordinates in $\mathbf{E}^3$

Now that we have provided an existential proof of the maximal number of inequivalent rotational coordinates, we discuss here a classification of arbitrary 6-tuples of conformal Killing tensor coefficients *without* a priori mapping to canonical centered form which has been dealt with in the previous sections.

Given a 6-tuple of CKT coefficients, the first step in the classification scheme is to compute the full invariant $I = \frac{I_1^2}{I_2^3}$. A simple calculation reveals that $I \in (-\infty, 0]$ and $I \in (4, +\infty)$ for disk cyclide coordinates. $I \in [0, 4)$ for the inequivalent twin bi-cyclide and flat-ring cyclide coordinates, and finally $I = +4$ for the remaining rotationally symmetric coordinates except for the pairs cardioid-paraboloidal and tangent sphere-circular cylindrical coordinates where its value is indeterminate. As it stands, the full group invariant lumps together toroidal, bispherical-spheroidal, prolate-inverse prolate spheroidal and oblate-inverse oblate spheroidal coordinates when its value is computed to be +4.

For the case $I_1 = 0$, it is possible to distinguish between disk-cyclide and the pairs bi-cyclide/flat-ring cyclide coordinates by the observation that in the case for disk-cyclide coordinates $I_2 < 0$ and in the other case $I_2 > 0$. Note that although on its own I_2 is not invariant to dilation of the Killing tensor, it is by virtue of being a quadratic invariant to the discrete -1 switch and so the sign cannot be altered by positive dilation of the Killing tensor alone! Thus one is able to discriminate disk-cyclide coordinates from all the rest. Another classification scheme is needed to discriminate between the other case which share the same value for the full group invariant.

A first idea was to use the assumption of undetermined polynomial coefficients to a fixed degree in the CKT coefficients, as in [25], to compute the invariants to continuous inversion, translation and addition of $R_3 \odot R_3$. This was motivated by studying how such invariants behave under dilation of the space and thereby see if different cases could be discriminated. The result is the vanishing of the purely linear and purely quartic ansatz, and the quadratic and cubic ansatz being a linear combination of the already found I_2 and I_1 respectively! Under the polynomial ansatz, one arrives at the interesting (but not useful) fact that an invariant to inversion and translation is simultaneously an invariant with respect to dilation of the space.

This approach reveals a deeper pattern. Computing also by undetermined polynomial coefficients the invariant with respect to infinitesimal translation and dilation of the space yields an invariant to infinitesimal inversion. Similarly an invariant with respect to infinitesimal inversion and dilation of the space is invariant with respect to infinitesimal translation. Thus the hope of ignoring one group action when finding invariants, and using that group action to split degenerate pairs of cases fails when the answer is again the total invariant.

Thus one is left with having to consider ignoring two group actions and finding invariants with respect to the remainder. The logical choice was to ignore translation and infinitesimal inversion but consider ratios such as $\frac{B_{21}G_{12}}{A_{33}M_{33}}$, $\frac{B_{21}G_{12}}{(C_{22}+C_{33})^2}$ and $\frac{A_{33}M_{33}}{(C_{22}+C_{33})^2}$ and see how the transformed coefficients affect the values of these ratios when arbitrarily applying inversion and translation to known canonically centered cases. The first such ratio, denoted by I_3, seemed to bear fruit in discriminating some cases.

A simple calculation reveals that for paraboloidal/cardioid coordinates I_3 is indeterminate and for tangent sphere/circular cylindrical coordinates $I_3 = -4$ (the ratio is constant and defined for non-zero inversion/translation; thus it has a limiting value for the null group actions). If I_3 lies in the span of

$$\frac{-4(ba^2 + a + b)^2}{(a^2 + 1)(1 + 2ab + b^2a^2 + b^2)} \tag{6.5.1}$$

for non-zero inversion parameter a and/or translation parameter b (where the span explicitly lies in the open interval $(-4, 0)$), the coordinates are either toroidal, oblate/inverse oblate spheroidal or prolate/inverse prolate spheroidal. If I_3 lies in the (disjoint from all previous) span of

$$\frac{-4(ba^2 + a - b)^2}{(a^2 - 1)(1 + 2ab + b^2a^2 - b^2)} \tag{6.5.2}$$

for non-zero inversion parameter a and/or translation parameter b (where the span explicitly lies in the open intervals $(-\infty, -4)$ and $(0, \infty)$), the coordinates are either bispherical/spheroidal, oblate/inverse oblate spheroidal or prolate/inverse prolate spheroidal. In the case of bispherical and toroidal coordinates, $I_3 = 0$ corresponds uniquely to the case $a = b = 0$.

The invariance of the *rank* of the group action can also be utilized to discriminate between entangled coordinate pairs. A closer inspection of the transformation equations of the coefficients of the CKT reveals that C_{22}, C_{33} always appear in the sum $(C_{22} + C_{33})$ which is invariant to the group action of addition of $R_3 \odot R_3$. This motivates one to consider ignoring that group action on a reduced number of coefficients, namely defining $C \equiv (C_{22} + C_{33})$ and studying only four group actions on the reduced set $(A_{33}, B_{21}, C, G_{12}, M_{33})$. A simple calculation reveals that the modified pde array of this reduced group action set becomes:

$$\begin{aligned} V_1 &= [-2B_{21}, -C, 6G_{12}, 2M_{33}, 0] \\ V_2 &= [0, 2A_{33}, 6B_{21}, -C, -2G_{12}] \end{aligned}$$

$$
\begin{aligned}
V_3 &= [A_{33}, B_{21}, C, G_{12}, M_{33}] \\
V_4 &= [2A_{33}, B_{21}, 0, -G_{12}, -2M_{33}]
\end{aligned}
\tag{6.5.3}
$$

This is advantageous because now not all canonically centered coordinates will obey the maximal rank condition (as they did with the full group action on six coefficients). In particular, the rank of the pde set for the oblate/inverse oblate spheroidal, prolate/inverse prolate spheroidal coordinates is maximal at +4. However the rank for toroidal and bispherical/spheroidal coordinates is not maximal – at +3. This value is also shared for paraboloidal/cardioid coordinates. Finally for tangent sphere/circular cylindrical coordinates the rank is even further reduced at +2. Thus one is able with the invariance of rank under the group action, which was explicitly checked in our case, to discriminate between the pairs inverse oblate spheroidal, inverse prolate spheroidal and toroidal, bispherical coordinates.

Unfortunately for the pairs oblate/prolate spheroidal and bi-cyclide/flat-ring cyclide coordinates, one must use the mapping to canonical centered form, and then use the reduced invariant to classify the transformed coefficients $(\tilde{A}_{33}, \tilde{C}_{22}, \tilde{C}_{33}, \tilde{M}_{33})$ - it will be an either/or scenario. We have tried plotting quantities such as $\tilde{B}_{21}\tilde{G}_{12}$ and $\tilde{A}_{33}\tilde{M}_{33}$ to distinguish a pair of coordinates by the *sign* of the graph which is invariant. Unfortunately one member of each pair does not admit a graph of the above, as a function of inversion parameter a and translation parameter b, with a unique sign. The other member however admits a negative graph for all a, b. Hence for some 6-tuples, the above plots might ascertain which member of the coordinate pair the tuple is associated with by evaluation of the listed products and observation that that product is positive. When this criterion fails, one must then map to canonically centered form to complete the discrimination.

For a complete classification *without* transformation to canonically centered form, recent research was completed using the invariance of the roots of binary quartics and the equivalence of the discussed group actions on the coefficients with the general linear transformation on binary quartics studied extensively in the literature, for example in [41]. For details, see the material placed in Appendix F which is based on [13].

The results of the classification given in this chapter of the rotationally symmetric webs listed by Moon and Spencer are summarized in the following table:

Coordinate web	Equivalent to	Transformation
Cap cyclide	Flat-ring cyclide	cont. inversion + trans.
Inverse Prolate Spheroidal	Prolate Spheroidal	discrete inversion
Inverse Oblate Spheroidal	Oblate Spheroidal	discrete inversion
Bispherical	Spheroidal	cont. inversion + trans.
Cardiod	Paraboloidal	discrete inversion
Tangent sphere	Circular cylindrical	discrete inversion
Toroidal	-	-
Bi-cyclide	-	-
Flat-ring cyclide	-	-

Table 6.1: Equivalence classes of R-separable webs

6.6 Classifying the remaining symmetric R-separable coordinates in $\mathbf{E}^3$

Proposition 6.6.1 *Aside from rotational cases just discussed, there are no additional symmetric R-separable coordinates in $\mathbf{E}^3$.*

Proof: We have applied Eq.(4.1.8) on the general conformal Killing tensor using, instead of the rotational Killing vector R_3, the translational, dilatation and inversion Killing vector. The result after some algebra analogous to Eq.(4.1.9) is a degree two, six coefficient subset of the general Killing tensor for webs admitting a translational symmetry along x. This result holds whether the discrete or continuous method is carried out to characterize translational symmetry. In components the resulting tensor is:

$$\begin{aligned}
K_{11} &= -A_{22} - A_{33} + 1/2B_{21}z - 1/2B_{31}y + C_{33}(y^2 + z^2) \\
K_{22} &= A_{22} - B_{21}z - 1/2B_{31}y + C_{33}(y^2 - 2z^2) \\
K_{33} &= A_{33} + 1/2B_{21}z + B_{31}y + C_{33}(z^2 - 2y^2) \\
K_{12} &= 0
\end{aligned}$$

$$\begin{aligned} K_{13} &= 0 \\ K_{23} &= A_{23} - 3/4B_{31}z + 3/4B_{21}y + 3C_{33}zy \end{aligned} \tag{6.6.1}$$

For dilatational webs, where the Lie derivative of the general conformal Killing tensor is set equal to a real scalar of the Killing tensor before imposing the TSN integrability conditions, we found five integer cases where the result is non-trivial. Each case corresponds to a single degree expression for the Killing tensor subset. The degree two case, after imposing the TSN criterion – is in exact agreement with the discrete operation, Eq.(4.1.8), using the dilational Killing vector. The other cases however, either correspond directly to simple separable webs or can be mapped into one by the discrete inversion, namely all degree four terms to degree zero and all degree three terms to degree one. Immediately one can conclude there are no additional conformal coordinates admitting a translational or dilational symmetry, irrespective of whether one characterizes the symmetries using the standard Lie derivative method or the discrete formula! Note this is in agreement with [6] where the only non-rotationally symmetric R-separable coordinates are asymmetric cases not studied in this chapter. Recall that on p. 234 and 235 of [37] a first principles proof was given that R-separability of the Helmholtz and Laplace equations is never possible for a cylindrical coordinate system. Thus we are in agreement with known results in the literature for null cases of translational conformal coordinates.

Repeating the above procedure using the vector generator of the continuous inversion, for both the discrete and Lie derivative method (set equal to zero for all components) coupled with the TSN impositions, yields a degree four, six coefficient subset of the general Killing tensor with no presence of zero or first order terms. It was checked that the discrete inversion maps this web to that generated by Eq.(4.1.8) using the translational Killing vector, and vice versa. Setting the Lie derivative, with respect to the inversional conformal Killing vector, of the general conformal Killing tensor equal to a non-zero scalar of the Killing tensor yields the trivial result even before applying the TSN conditions. This is analogous to the dilational Lie derivative set equal to an arbitrary multiple of the conformal Killing tensor for values not equal the five integer cases found. Our conclusion is that the only additional conformal coordinates are either rotationally symmetric or asymmetric admitting no symmetries. □

The difficult task of characterizing asymmetric webs remains to be studied. This adds more emphasis on the need to generate the class of R-separable webs by first principles from the method of Eisenhart.

Chapter 7

Asymmetric R-separable webs in $\mathbf{E}^3$

In this chapter we study the remaining case of R-separable webs in $\mathbf{E}^3$ that admit no symmetry. This case is not considered in [38]. However, it has been studied in [6] and [7]. We adopt a different starting point than these papers by studying the *conformally invariant* (CI) Laplace equation rather that the ordinary Laplace equation which is not conformally invariant. It will be seen that there is a close relation of the conformal invariance property of the equation and R-separability which is also a conformally invariant property. Our approach which has been described in [11], [12] is based on the theory of R-separability explained in Chapter 2.

7.1 The conformally invariant Laplace equation

The subject of this chapter is the study of R-separation of variables for the *conformally invariant* (CI) Laplace equation on an n-dimensional Riemannian manifold $(M, \mathbf{g})$, which is:

$$\mathbf{H}\varphi := \Delta\varphi + C\varphi = 0, \tag{7.1.1}$$

where we make a very specific choice of the constant C:

$$C = \frac{n-2}{4(n-1)} R_s \tag{7.1.2}$$

where R_s is the Ricci scalar and n is the dimension of the space. As mentioned before, the closely related problem is additive separation of variables for the HJ equation with null geodesics, which like the classical Laplace equation can also be extended to the pseudo-Riemannian case namely

$$g^{ij}\partial_i W \partial_j W = 0. \tag{7.1.3}$$

The crucial property of both (7.1.1) and (7.1.3) is invariance under conformal transformation of the metric. From this it follows that if φ is any solution of $\mathbf{H}\varphi = 0$, then $\tilde{\varphi} = e^{\frac{2-n}{2}}\varphi$ is a solution of $\tilde{\mathbf{H}}\tilde{\varphi} = 0$ on any conformally related manifold. Consequently, R-separability of the CI-Laplace equation is a conformally invariant property, which is not shared by the classical Laplace equation introduced at the start of this thesis:

$$\Delta\varphi = 0, \tag{7.1.4}$$

which is the equation most often studied in this regard [5, 7, 15].

We digress here to give a proof of the conformal invariance property of the Laplace-Beltrami operator.

7.2 Proof of conformal invariance

Consider a pseudo-Riemannian manifold $(\mathbf{M}, \mathbf{g})$ with corresponding Levi-Civita connection Γ. The covariant derivative ∇ may be written in local coordinates $\{x^i\}$ as follows:

$$\nabla_i A^j = \partial_i A^j + \Gamma^j_{ik} A^k \tag{7.2.1}$$

for a contravariant vector field A^j, where Γ^j_{ik} denotes the Christoffel symbols of the second kind. Contracting Eq. (7.2.1) over i and j we obtain the divergence of A^j namely:

$$\nabla_i A^i = \partial_i A^i + \Gamma^i_{ik} A^k \tag{7.2.2}$$

By a standard result, [22] we have equivalently:

$$\Gamma^i_{ik} = 2\partial_k log\sqrt{g} \tag{7.2.3}$$

Thus (7.2.2) takes the form

$$\begin{aligned}
\nabla_i A^i &= \partial_i A^i + A^i \partial_i log\sqrt{g} \\
&= \partial_i A^i + \frac{1}{\sqrt{g}} A^i \partial_i \sqrt{g} \\
\nabla_i A^i &= \frac{1}{\sqrt{g}}(\sqrt{g}\partial_i A^i + A^i \partial_i \sqrt{g}) \\
\nabla_i A^i &= \frac{1}{\sqrt{g}}\partial_i(\sqrt{g} A^i) \qquad (7.2.4)
\end{aligned}$$

Let ϕ be a function defined on $\mathbf{M}$. The Laplace-Beltrami operator on $\mathbf{M}$ was defined as the divergence of the vector field $g^{ij}\partial_j\phi$. In local coordinates we thus have:

$$\Delta\phi = \frac{1}{\sqrt{g}}\partial_i(\sqrt{g} g^{ij}\partial_j\phi) \tag{7.2.5}$$

Given conformal transformations of the pseudo-Riemannian metric it follows that the determinant of g_{ij} transforms as:

$$\tilde{g} = e^{2n\sigma}g, \tag{7.2.6}$$

where $n = \dim(M)$. Thus

$$\sqrt{\tilde{g}} = e^{n\sigma}\sqrt{g} \tag{7.2.7}$$

Suppose that ϕ transforms as

$$\tilde{\phi} = e^{m\sigma}\phi, \tag{7.2.8}$$

where $m \in R$. We are now in a position to compute the transformation law for the Laplace-Beltrami operator Δ acting on ϕ.

$$\begin{aligned}
\tilde{\Delta}\tilde{\phi} &= \frac{1}{\sqrt{g}}\partial_i(\sqrt{\tilde{g}}\tilde{g}^{ij}\partial_j\tilde{\phi}) \\
&= \frac{e^{-n\sigma}}{\sqrt{g}}\partial_i(e^{(n-2)\sigma}\sqrt{g}g^{ij}\partial_j(e^{m\sigma}\phi)) \\
&= \frac{e^{-n\sigma}}{\sqrt{g}}\partial_i(e^{(n-2)\sigma}\sqrt{g}g^{ij}(e^{m\sigma}(\partial_j\phi + m\phi\partial_j\sigma))) \\
&= \frac{e^{-n\sigma}}{\sqrt{g}}\partial_i(e^{(m+n-2)\sigma}\sqrt{g}(g^{ij}\partial_j\phi + m\phi g^{ij}\partial_j\sigma))) \\
&= \frac{e^{-n\sigma}}{\sqrt{g}}[(m+n-2)e^{(m+n-2)\sigma}\partial_i\sigma\sqrt{g}(g^{ij}\partial_j\phi + m\phi g^{ij}\partial_j\sigma) + e^{(m+n-2)\sigma}\partial_i(\sqrt{g}g^{ij}\partial_j\phi \\
&+ m\sqrt{g}\phi g^{ij}\partial_j\sigma)] \\
\tilde{\Delta}\tilde{\phi} &= \frac{e^{(m-2)\sigma}}{\sqrt{g}}[(m+n-2)\sqrt{g}g^{ij}\partial_i\sigma\partial_j\phi + m(m+n-2)\sqrt{g}\phi g^{ij}\partial_i\sigma\partial_j\sigma + \partial_i(\sqrt{g}g^{ij}\partial_j\phi) \\
&+ m\sqrt{g}g^{ij}\partial_i\phi\partial_j\sigma + m\phi\partial_i(\sqrt{g}g^{ij}\partial_j\sigma)] \\
\tilde{\Delta}\tilde{\phi} &= e^{(m-2)\sigma}[(m+n-2)g^{ij}\partial_i\sigma\partial_j\phi + m(m+n-2)\phi g^{ij}\partial_i\sigma\partial_j\sigma + \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}g^{ij}\partial_j\phi) \\
&+ mg^{ij}\partial_i\phi\partial_j\sigma + m\phi\frac{1}{\sqrt{g}}\partial_i(\sqrt{g}g^{ij}\partial_j\sigma)] \\
\tilde{\Delta}\tilde{\phi} &= e^{(m-2)\sigma}[\Delta\phi + m\phi(\Delta\sigma + (m+n-2)g^{ij}\partial_i\sigma\partial_j\sigma) \\
&+ (m+n-2)g^{ij}\partial_i\sigma\partial_j\phi + mg^{ij}\partial_i\phi\partial_j\sigma] \qquad (7.2.9)
\end{aligned}$$

The next step results from the fact that:

$$g^{ij}\partial_i\phi\partial_j\sigma = g^{ij}\partial_i\sigma\partial_j\phi,$$

which follows from the symmetry of g^{ij} in the indices i and j. The final formula is:

$$\tilde{\Delta}\tilde{\phi} = e^{(m-2)\sigma}[\Delta\phi + (2m+n-2)g^{ij}\partial_i\sigma\partial_j\phi + m\phi(\Delta\sigma \ + (m+n-2)g^{ij}\partial_i\sigma\partial_j\sigma)]$$

The second last term on the RHS may be removed by choosing

$$m = \frac{1}{2}(2-n) \tag{7.2.10}$$

With this choice Eq. (7.2.9) reduces to

$$\tilde{\Delta}\tilde{\phi} = e^{-\frac{1}{2}(2+n)\sigma}[\Delta\phi + \frac{1}{2}(2-n)\phi(\Delta\sigma + \frac{1}{2}(n-2)g^{ij}\partial_i\sigma\partial_j\sigma)] \tag{7.2.11}$$

Now we consider the quantity $\tilde{R}\tilde{\phi}$, recalling that the Ricci scalar R transforms under conformal transformation as [22]:

$$\tilde{R} = e^{-2\sigma}[R + 2(n-1)\Delta\sigma + (n-1)(n-2)g^{ij}\sigma_i\sigma_j]$$

and remembering that ϕ was defined to transform as: $\tilde{\phi} = e^{m\sigma}\phi$. This yields:

$$\tilde{R}\tilde{\phi} = e^{(m-2)\sigma}\phi[R + 2(n-1)\Delta\sigma + (n-1)(n-2)g^{ij}\sigma_i\sigma_j] \tag{7.2.12}$$

Adding (7.2.11) and k times (7.2.12) we obtain the new operator

$$\begin{aligned} \tilde{\Delta}\tilde{\phi} + k\tilde{R}\tilde{\phi} &= e^{(m-2)\sigma}[\Delta\phi + \frac{1}{2}(2-n)\phi(\Delta\sigma + \frac{1}{2}(n-2)g^{ij}\sigma_i\sigma_j) \\ &+ k\phi(R + 2(n-1)\Delta\sigma + (n-1)(n-2)g^{ij}\sigma_i\sigma_j)] \\ &= e^{(m-2)\sigma}[\Delta\phi + kR\phi + \phi((1-\frac{n}{2} + 2k(n-1))\Delta\sigma \\ &+ (-\frac{1}{4}(n-2)^2 + k(n-1)(n-2))g^{ij}\sigma_i\sigma_j)] \end{aligned} \tag{7.2.13}$$

To remove the term containing $\Delta\sigma$ we must choose

$$k = \frac{1}{4}\frac{(n-2)}{(n-1)} \tag{7.2.14}$$

Fortuitously this choice of k also removes the term containing $g^{ij}\sigma_i\sigma_j$!

We conclude that the operator (7.2.13) with k given by Eq. (7.2.14) has the transformation law:

$$\tilde{\Delta}\tilde{\phi} + \frac{1}{4}\frac{(n-2)}{(n-1)}\tilde{\phi}\tilde{R} = e^{-\frac{1}{2}(n+2)\sigma}(\Delta\phi + \frac{1}{4}\frac{(n-2)}{(n-1)}R\phi) \tag{7.2.15}$$

where ϕ and $\tilde{\phi}$ are related by:

$$\tilde{\phi} = e^{m\sigma}\phi = e^{\frac{1}{2}(2-n)\sigma}\phi \tag{7.2.16}$$

The operator is thus invariant under a conformal transformation, as desired. We now state some special cases for lower dimensions:

$$n=3 \qquad \Delta\phi + \frac{1}{8}R\phi \quad , \ \tilde{\phi} = e^{-\frac{\sigma}{2}}\phi \tag{7.2.17}$$

$$n=4 \qquad \Delta\phi + \frac{1}{6}R\phi \quad , \ \tilde{\phi} = e^{-\sigma}\phi \tag{7.2.18}$$

$$n=5 \qquad \Delta\phi + \frac{3}{16}R\phi \quad , \ \tilde{\phi} = e^{-\frac{3}{2}\sigma}\phi \tag{7.2.19}$$

This concludes the proof of conformal invariance of the operator (7.1.1) with C given by (7.1.2). For the rest of this chapter the advantages of studying R-separability for the CI-Laplace equation continuing the work begun in [32] are illustrated. For the flat case (in which the CI-Laplacian reduces to the classical one) we recover the results given by Bôcher [5] and Boyer et al. [7]. Furthermore, these results are applied to provide CI-Laplace R-separable coordinates on other conformally flat manifolds.

7.3 The CI-Laplace equation and R-separation

The study of R-separation of the CI-Laplace equation, instead of the classical equation, is more general [32]. Indeed, the existence of a complete R-separated solution of the CI-Laplace equation is a conformally invariant property that holds on the whole class of conformally related metrics. This follows directly from the form of the solution ansatz and the conformal invariance of Eq. (7.1.1).

The techniques giving differential conditions for the R-separation of a single pde [15] were outlined in Chapter 2 and are repeated here:

Theorem 7.3.1 *Equation* (7.1.1) *admits R-separation in the coordinates* (q^i) *if and only if*

1. *the coordinates are orthogonal:* $g_{ij} = 0, i \neq j$*;*
2. *the coordinates are conformally separable;*
3. *the contravariant components* (g^{ii}) *satisfy the differential condition*

$$\frac{S_{ij}(g^{hh})}{g^{hh}} = \frac{S_{ij}(g^{kk})}{g^{kk}}, \qquad (\forall\ h,k,\ \forall\ i \neq j,\ i,j \text{ n.s.}) \tag{7.3.1}$$

where S_{ij} *are the second order Stäckel operators,* *4. the function R is (up to separated factors) a solution of*

$$\partial_i \ln R = \frac{1}{2}\Gamma_i, \tag{7.3.2}$$

where $\Gamma_i = g^{hk}\Gamma_{hki}$;

Remark 7.3.2 Recall from Chapter 2 that orthogonal coordinates satisfying condition (7.3.1) are called *conformally separable* (see [3]), while orthogonal coordinates satisfying $S_{ij}(g^{hh}) = 0$ are said to be *simply separable*. The additive separation of variables for the null geodesic HJ equation in orthogonal coordinates,

$$g^{ii}(\partial_i W)^2 = 0, \tag{7.3.3}$$

and for the geodesic HJ equation

$$\frac{1}{2}g^{ii}(\partial_i W)^2 = E, \quad (E \in \mathbf{R}), \tag{7.3.4}$$

occurs if and only if the coordinates are conformally separable and simply separable, respectively. This fact shows an important link between Eq. (7.1.1) and Eq. (7.1.3).

Also in the Riemannian case, even if the null geodesics are trivial, the study of conformal separation can be applied effectively to the CI-Laplace equation. Indeed, as for the Laplace equation $\Delta\psi = 0$, we have that

Corollary 7.3.3 *A necessary condition for R-separation of the CI-Laplace equation* (7.1.1) *in a given coordinate system is that the null geodesic equation* (7.1.3) *is additively separable in the same coordinates.*

Remark 7.3.4 Two conditions equivalent to (7.3.1) are

- $\mathbf{g}$ is conformal to a metric which is separable for the geodesic HJ equation (7.3.4) in the same coordinates;
- there exists a Stäckel matrix S such that [35]

$$\frac{g^{ii}}{g^{jj}} = \frac{M_{in}}{M_{jn}}, \tag{7.3.5}$$

where M_{in} is the minor of S obtained by eliminating the i-th row and the n-th column. We remark that the elements of the last column of the Stäckel matrix are not involved in (7.3.5).

7.4 The three-dimensional case

Definition 7.4.1 *A coordinate q^i is said to be* ***conformally ignorable*** *if it appears in the conformal factor of the metric only, that is if*

$$\partial_i(g^{hh}/g^{kk}) = 0,$$

for all h, k. We call a coordinate system ***general*** *if it does not contain any conformally ignorable coordinates.*

Up to a coordinate transformation of the form $\tilde{q}^i(q^i)$, a coordinate q^i is conformally ignorable if and only if ∂_i is a conformal Killing vector, that is an infinitesimal conformal symmetry. In this chapter only general coordinate systems are considered, leaving as a further research project the analysis of the cases involving conformal symmetries.

The form of the general conformally separable coordinates in a three dimensional manifold is given in the following proposition (see [7])

Proposition 7.4.2 *In general conformally separable coordinates (q^i), the form of the (contravariant) metric on a three dimensional manifold is given by*

$$g^{ii} = Qh_i(q^i)(q^{i+2} - q^{i+1}), \qquad i = 1, \ldots, 3 \ (mod\ 3), \tag{7.4.1}$$

where Q is the conformal factor, and h_i three arbitrary functions of a single variable.

Proof: Following [7], without loss of generality, we may choose S to be a 3×3 Stäckel matrix with third column set equal to unity.

$$S = \begin{bmatrix} \phi_1 & \psi_1 & 1 \\ \phi_2 & \psi_2 & 1 \\ \phi_3 & \psi_3 & 1 \end{bmatrix} \tag{7.4.2}$$

Then, we have

$$g^{11} = Q(\psi_3\phi_2 - \psi_2\phi_3), \quad g^{22} = Q(\psi_1\phi_3 - \psi_3\phi_1), \quad g^{33} = Q(\psi_2\phi_1 - \psi_1\phi_2). \tag{7.4.3}$$

In the general case, we can assume that none of the ψ_i and ϕ_i is identically null. Thus

$$g^{11} = Q\phi_2\phi_3\left(\frac{\psi_3}{\phi_3} - \frac{\psi_2}{\phi_2}\right), \ g^{22} = Q\phi_1\phi_3\left(\frac{\psi_1}{\phi_1} - \frac{\psi_3}{\phi_3}\right), \ g^{33} = Q\phi_1\phi_2\left(\frac{\psi_2}{\phi_2} - \frac{\psi_1}{\phi_1}\right). \tag{7.4.4}$$

By transforming each coordinate $\tilde{q}^i = \tilde{q}^i(q^i)$ and the conformal factor such that

$$\tilde{g}^{ii} \to \phi_i g^{ii}, \qquad \tilde{Q} \to Q\phi_1\phi_2\phi_3,$$

we obtain $\tilde{g}^{ii} = \tilde{Q}(F_{i+2} - F_{i+1})$ with $F_i(q^i) = \frac{\psi_i}{\phi_i}$. If none of the F_i is a constant, then we can use them as coordinates; thus we obtain

$$g^{11} = \tilde{Q}h_1(q^1)(q^3 - q^2), \quad g^{22} = \tilde{Q}h_2(q^2)(q^1 - q^3), \quad g^{33} = \tilde{Q}h_3(q^3)(q^2 - q^1),$$

where the h_i are the reciprocal of ϕ_i, and the tilde symbol can be dropped for the conformal factor. □

Remark 7.4.3 If one of the elements of the Stäckel matrix is zero or one of the functions F_i is a constant then, up to a coordinate transformation $\tilde{q}^i(q^i)$, one of the coordinates is conformally ignorable.

By Proposition 7.4.2 and Theorem 7.3.1 we obtain

Theorem 7.4.4 *The form of the metric in general R-separable coordinates for the CI-Laplace equation is*

$$g^{ii} = QP(q^i) \cdot (q^{i+2} - q^{i+1}), \qquad i = 1, \ldots, 3 \ (mod\ 3), \tag{7.4.5}$$

where P is an arbitrary fifth-degree polynomial.

Proof: Computing the modified potential χ for the general conformal separable metric (7.4.1) and imposing the compatibility condition

$$\begin{aligned} S_{ij}(\chi)g^{11} &= S_{ij}(g^{11})\chi = 0 \\ S_{ij}(\chi)g^{22} &= S_{ij}(g^{22})\chi = 0 \\ S_{ij}(\chi)g^{33} &= S_{ij}(g^{33})\chi = 0 \\ i &\neq j, \end{aligned} \tag{7.4.6}$$

where

$$\chi \equiv g^{hh}(2\partial_h\Gamma_h - \Gamma_h^2 + \frac{1}{2}R_{hh}) \tag{7.4.7}$$

we obtain three additional independent differential conditions (out of the nine equations) on the functions h_i that form a linear second order ODE system in three unknowns. Performing the calculation in Maple, where for convenience we denote $q^1 = u,\ q^2 = v,\ q^3 = w$ and $h_1 = U(u),\ h_2 = V(v),\ h_3 = W(w)$ - we write the covariant metric components as the Tensor Package requires the metric tensor to be written with both indices down. The conformal freedom in the metric ansatz allows us now to drop the conformal factor Q and write:

$$g_{11} = \frac{1}{U(u)(v-w)}, \ g_{22} = \frac{1}{V(v)(w-u)}, \ g_{33} = \frac{1}{W(w)(u-v)} \tag{7.4.8}$$

The reduced Christoffel symbols that follow are,
where we write $(U,\ V,\ W)$ instead of $(U(u),\ V(v),\ W(w))$:

$$\begin{aligned}
\Gamma_1 &= \frac{-U_u}{2U} + \frac{(v-2u+w)}{2(v-u)(u-w)} \\
\Gamma_2 &= \frac{-V_v}{2V} + \frac{(2v-u-w)}{2(v-u)(v-w)} \\
\Gamma_3 &= \frac{-W_w}{2W} + \frac{(2w-u-v)}{2(u-w)(v-w)}
\end{aligned} \tag{7.4.9}$$

The diagonal Ricci tensor components R_{hh} and Ricci scalar R needed for a future calculation are easily attained using the Tensor Package and are omitted here. Finally the differential equations that follow are

$$\begin{aligned}
eq_1 &= -\frac{(v-w)U_{u,u}}{4(v-u)} + \frac{(u-w)V_{v,v}}{4(v-u)} + \frac{(v-w)(v-4u+3w)U_u}{2(v-u)^2(u-w)} \\
&- \frac{(u-w)(4v-3w-u)V_v}{2(v-u)^2(v-w)} + \frac{(v-u)^2W}{2(u-w)^2(v-w)^2} \\
&- \frac{(v-w)(v^2-5uv+3vw+10u^2-15wu+6w^2)U}{2(v-u)^3(u-w)^2} \\
&+ \frac{(u-w)(u^2-5uv+3wu+10v^2-15vw+6w^2)V}{2(v-u)^3(v-w)^2}
\end{aligned} \tag{7.4.10}$$

$$\begin{aligned}
eq_2 &= \frac{(v-w)U_{u,u}}{4(u-w)} - \frac{(v-u)W_{w,w}}{4(u-w)} - \frac{(v-w)(3v+w-4u)U_u}{2(v-u)(u-w)^2} \\
&- \frac{(v-u)(3v-4w+u)W_w}{2(u-w)^2(v-w)} - \frac{(u-w)^2V}{2(v-u)^2(v-w)^2} \\
&+ \frac{(v-w)(6v^2+3vw-15uv+w^2-5wu+10u^2)U}{2(v-u)^2(u-w)^3} \\
&- \frac{(v-u)(6v^2-15vw+3uv+10w^2-5wu+u^2)W}{2(u-w)^3(v-w)^2}
\end{aligned} \tag{7.4.11}$$

$$\begin{aligned}
eq_3 &= -\frac{(u-w)V_{v,v}}{4(v-w)} - \frac{(v-u)W_{w,w}}{4(v-w)} + \frac{(u-w)(4v-3u-w)V_v}{2(v-u)(v-w)^2} \\
&- \frac{(v-u)(v-4w+3u)W_w}{2(v-w)^2(u-w)} + \frac{(v-w)^2U}{2(v-u)^2(u-w)^2} \\
&- \frac{(u-w)(6u^2+3uw-15uv+w^2-5vw+10v^2)V}{2(v-u)^2(v-w)^3} \\
&- \frac{(v-u)(6u^2-15uw+3uv+10w^2-5vw+v^2)W}{2(v-w)^3(u-w)^2}
\end{aligned} \tag{7.4.12}$$

Maple successfully integrated the above yielding

$$U(u) = c_1 + c_2u + c_3u^2 + c_4u^3 + c_5u^4 + c_6u^5$$

$$
\begin{aligned}
V(v) &= c_1 + c_2 v + c_3 v^2 + c_4 v^3 + c_5 v^4 + c_6 v^5 \\
W(w) &= c_1 + c_2 w + c_3 w^2 + c_4 w^3 + c_5 w^4 + c_6 w^5 \qquad (7.4.13)
\end{aligned}
$$

Going back to the older notation, we see that the $h_i = P(q^i)$, where P is an arbitrary fifth-degree polynomial. □

Note that the compatibility condition has an intriguing geometrical interpretation.

Theorem 7.4.5 *On a three dimensional manifold, R-separation of the CI-Laplace equation occurs in general conformal separable coordinates if and only if the metric is conformally flat.*

Proof: The conformal flatness conditions for a 3-dimensional Riemannian manifold are (see for example [22])

$$
R_{ijk} = R_{ij;k} - R_{ik;j} + \frac{1}{4}(g_{ik}R_{s;j} - g_{ij}R_{s;k}) = 0, \qquad (7.4.14)
$$

where ; denotes the covariant derivative and R_{ij} the covariant Ricci tensor. By imposing these conditions on the general conformally separable metric (7.4.1), we arrive at nine linear second order ODEs in the h_i. Three of them are trivially zero, while another three are equivalent to the remaining three which are equal to those allowing R-separation. Thus, the conformal flatness condition is equivalent to the compatibility condition for R-separation. □

Remark 7.4.6 If one or more conformally ignorable coordinates appears, then being conformally flat is a sufficient but no longer a necessary condition for R-separation. Hence, in particular, equations (7.1.1) and (7.1.3) separate in the same orthogonal coordinates for all conformally flat 3-manifolds.

We can apply these results to the study of R-separation for the classical Laplace equation. Indeed if a three dimensional manifold satisfies $R_s = 0$, then R-separation of the Laplace equation $\Delta\psi = 0$ occurs in general coordinates if and only if the manifold is conformally flat. This is because since $R_s = 0$, the CI-Laplace equation and Laplace equation coincide. Furthermore on a conformally flat three dimensional manifold, R-separation of the Laplace equation $\Delta\psi = 0$ occurs in general conformally separable coordinates if and only if the Ricci scalar R_s satisfies the compatibility condition

$$
S_{ij}(g^{hh})R_s = S_{ij}(R_s)g^{hh}.
$$

This follows since the manifold is conformally flat, the CI-Laplace equation admits R-separation of variables in general conformally separable coordinates.

7.5 Applications and examples

A fundamental example is the flat case, where the Laplace equation and the CI-Laplace equation become the same. This has been studied by several authors (see [5, 39, 35, 7]).

Example 7.5.1 In order to determine the expression of general R-separable coordinates on $\mathbf{E}^3$ we need to compute the conformal factor Q such that the metric (7.4.5) is flat and the coordinate transformations from a Cartesian coordinate system. Let us denote the R-separable coordinates (q^1, q^2, q^3) by (u, v, w) and by $e_1 < e_2 < e_3 < e_4 < e_5$ the five zeros of the polynomial P (we restrict ourselves to the special case where all zeros e_i are real and distinct, as we use tools from classical Riemannian geometry and not those pertaining to complex manifolds).

We now discuss the special case of

$$g^{ii} = QP(q^i) \cdot (q^{i+2} - q^{i+1}), \qquad i = 1, \ldots, 3 \ (mod\ 3)$$

when $P(q^i)$ can be factored into five real and distinct factors which we denote as in the literature by e_i. An explicit formula for the (covariant) conformal factor $\tilde{Q}$ is given on Eq.(4.31) of [7], as well as coordinate transformations from Cartesian to R-separable ones. The conformal factor $\tilde{Q}$ is written $1/\lambda^2$ where:

$$\begin{aligned}\lambda &= \sqrt{\frac{(q^1 - e_1) \cdot (q^2 - e_1) \cdot (q^3 - e_1)}{(e_1 - e_2) \cdot (e_1 - e_3) \cdot (e_1 - e_4) \cdot (e_1 - e_5)}} \\ &+ \sqrt{\frac{-(q^1 - e_5) \cdot (q^2 - e_5) \cdot (q^3 - e_5)}{(e_5 - e_1) \cdot (e_5 - e_2) \cdot (e_5 - e_3) \cdot (e_5 - e_4)}} \qquad (7.5.1)\end{aligned}$$

This is an adaptation from the pentaspherical coordinate representation given by the formula for ds^2 in Euclidean space on p. 89 of [5], where the (covariant) metric coefficients $g_{ii} = \tilde{Q} \cdot P^{-1}(q^i) \cdot (q^{i+1} - q^i) \cdot (q^{i+2} - q^{i+1})$ appear explicitly - with polynomial $P(q^i)$ described on p. 87. To reconcile with our contravariant formalism for Q, we note the relation $Q = \tilde{Q}^{-1}/(q^{i+1} - q^i)$.

By using pentaspherical coordinates we can derive the following relations linking Cartesian coordinates to the R-separable ones given in [7]:

$$\begin{aligned}\lambda \cdot x &= \sqrt{\frac{(q^1 - e_2) \cdot (q^2 - e_2) \cdot (q^3 - e_2)}{(e_2 - e_1) \cdot (e_2 - e_3) \cdot (e_2 - e_4) \cdot (e_2 - e_5)}} \\ \lambda \cdot y &= \sqrt{\frac{(q^1 - e_3) \cdot (q^2 - e_3) \cdot (q^3 - e_3)}{(e_3 - e_1) \cdot (e_3 - e_2) \cdot (e_3 - e_4) \cdot (e_3 - e_5)}} \\ \lambda \cdot z &= \sqrt{\frac{(q^1 - e_4) \cdot (q^2 - e_4) \cdot (q^3 - e_4)}{(e_4 - e_1) \cdot (e_4 - e_2) \cdot (e_4 - e_3) \cdot (e_4 - e_5)}}, \qquad (7.5.2)\end{aligned}$$

where the following (not unique) relations on the (assumed real and distinct) set of roots is assumed:

$$\begin{aligned} e_1 < e_2, \quad e_2 < e_3, \quad e_3 < e_4, \quad e_4 < e_5 \\ e_1 < u < e_2, \quad e_2 < v < e_3, \quad e_3 < w < e_4 \end{aligned} \tag{7.5.3}$$

As expected the metric g^{ii} resulting from the coordinate transforms of Eq.(7.5.2) is orthogonal and flat, and conformal to the metric $\tilde{g}^{ii} = P(q^i) \cdot (q^{i+2} - q^{i+1})$ but with conformal factor $\tilde{Q} = 1/(4\lambda^2)$ instead of the given $1/(\lambda^2)$ in [5] on p. 89. This is clearly a trivial error as a spatial dilatation of two units would ensure consistency of the formulae and of course not affect the flatness condition.

Therefore there is agreement, assuming the polynomial $P(q^i)$ has been factored into five real and distinct factors, between this derived form of the metric and results given in the literature.

The proof of the above metric coefficients resulting from the coordinate transformation equations given by Kalnins and Miller could not be verified by 'brute force' in Maple 9. This is due to the irrational factors appearing in the algebra which Maple handles poorly. We digress here to provide a simplification of the algebra involved, should the reader wish to reproduce the above results. Let us denote by $(q^i) = (u, v, w)$, $(i = 1, \ldots, 3)$ and $(e_h) = (e_1, ...e_5)$, $(h = 1, \ldots, 5)$. We start from the assumption $e_1 < u < e_2 < v < e_3 < w < e_4 < e_5$ and

$$L_h = \frac{(u - e_h)(v - e_h)(w - e_h)}{\prod_{k \neq h}(e_h - e_k)} = \frac{\prod_{i=1}^{3}(q^i - e_h)}{\prod_{k \neq h}(e_h - e_k)}$$

We have $L_1 > 0$, $L_2 > 0$, $L_3 > 0$, $L_4 > 0$ but $L_5 < 0$. Then we let $(x^i) = (x, y, z)$ be Cartesian coordinates, so that the transformation rules by Kalnins and Miller are in compact form:

$$x^i = \frac{\sqrt{L_{i+1}}}{\sqrt{L_1} + \sqrt{-L_5}},$$

where the λ factor is $\lambda = \sqrt{L_1} + \sqrt{-L_5}$. Let us compute the derivatives of L_h and $\sqrt{L_h}$:

$$\frac{\partial}{\partial q^i} L_h = \frac{\prod_{j \neq i}(q_j - e_h)}{\prod_{k \neq h}(e_h - e_k)} = \frac{L_h}{(q^i - e_h)}$$

$$\frac{\partial}{\partial q^i}\sqrt{L_h} = \frac{1}{2\sqrt{L_h}}\frac{\partial}{\partial q^i}L_h = \frac{1}{2\sqrt{L_h}}\frac{L_h}{(q^i - e_h)} = \frac{\sqrt{L_h}}{2(q^i - e_h)}, \qquad h = 1, \ldots, 4$$

and

$$\frac{\partial}{\partial q^i}\sqrt{-L_5} = -\frac{1}{2\sqrt{-L_5}}\frac{\partial}{\partial q^i}L_5 = -\frac{1}{2\sqrt{-L_5}}\frac{L_5}{(q^i - e_5)} = \frac{\sqrt{-L_5}}{2(q^i - e_5)}$$

Moreover

$$\frac{\partial}{\partial q^i}x^j = \frac{\partial_i\sqrt{L_{j+1}}(\sqrt{L_1} + \sqrt{-L_5}) - \sqrt{L_{j+1}}(\partial_i\sqrt{L_1} + \partial_i\sqrt{-L_5})}{(\sqrt{L_1} + \sqrt{-L_5})^2} =$$

$$= \frac{\frac{\sqrt{L_{j+1}}}{(q^i-e_{j+1})}(\sqrt{L_1}+\sqrt{-L_5}) - \sqrt{L_{j+1}}(\frac{\sqrt{L_1}}{(q^i-e_1)}+\frac{\sqrt{-L_5}}{(q^i-e_5)})}{2(\sqrt{L_1}+\sqrt{-L_5})^2}$$

$$= \frac{\sqrt{L_{j+1}}}{2(\sqrt{L_1}+\sqrt{-L_5})^2}\left(\sqrt{L_1}\frac{(e_{j+1}-e_1)}{(q^i-e_{j+1})(q^i-e_1)} + \sqrt{-L_5}\frac{(e_{j+1}-e_5)}{(q^i-e_{j+1})(q^i-e_5)}\right)$$

Hence

$$\left(\frac{\partial}{\partial q^i}x^j\right)^2 = \frac{L_{j+1}(q^i-e_{j+1})^{-2}}{4(\sqrt{L_1}+\sqrt{-L_5})^4}\cdot$$

$$\left(L_1\frac{(e_{j+1}-e_1)^2}{(q^i-e_1)^2} - L_5\frac{(e_{j+1}-e_5)^2}{(q^i-e_5)^2} + 2\sqrt{-L_1L_5}\frac{(e_{j+1}-e_1)(e_{j+1}-e_5)}{(q^i-e_1)(q^i-e_5)}\right) \quad (7.5.4)$$

Note that $g_{ii}(\sqrt{L_1}+\sqrt{-L_5})^4$ contains only the irrational term $\sqrt{-L_1L_5}$ and it is a first order polynomial in it (with coefficients that are rational functions of q^i and e_h). Thus Maple is able to simplify it (always as a first order polynomial in $\sqrt{-L_1L_5}$). Finally by comparing the expansion of

$$(\sqrt{L_1}+\sqrt{-L_5})^2 = L_1 - L_5 + 2\sqrt{-L_1L_5}$$

as a polynomial in $\sqrt{-L_1L_5}$, we recover the answer.

Remark 7.5.2 The conformal metric

$$\tilde{g}_{ii} = \frac{(q^i-q^{i+1})(q^i-q^{i+2})}{P(q^i)}$$

is the general three-dimensional conformally flat metric allowing multiplicative separation of the Helmholtz equation computed by Eisenhart [21].

The formulae for the flat case can be adapted to a general conformally flat manifold $(M, \mathbf{g}_M)$. Since M is conformally flat, there exists a coordinate system (X^i) such that

$$\mathbf{g}_M = Q_{ME}^{-1}\sum_i dX^i \otimes dX^i,$$

where Q_{ME} is the conformal factor transforming $\mathbf{g}_M$ into the flat Euclidean metric $\mathbf{g}_E$. Then, if we formally replace (x^1, x^2, x^3) by (X^1, X^2, X^3) in the transformations (7.5.2) we obtain the coordinate transformations from (X^i) to the R-separable coordinates (q^i). Indeed, by inserting these relations in the metric $\mathbf{g}_M$, we have

$$\mathbf{g}_M = Q_{ME}^{-1}\sum_i dX^i \odot dX^i = Q_{ME}^{-1}Q_E^{-1}\sum_i [P(q^i)\cdot(q^{i+2}-q^{i+1})dq^i \odot dq^i].$$

Hence, $Q_M = Q_{ME}Q_E$ is the conformal factor that transforms the general conformally flat metric (7.4.5) into a metric on the specific conformally flat manifold M. Then, in order to compute the conformal factor and the coordinate transformation, we only need to know the coordinates X^i on M corresponding to the Cartesian coordinates on $\mathbf{E}_3$.

In the following example we develop explicitly the case of $\mathbf{S}_3$

Example 7.5.3 Let (X^1, X^2, X^3) be stereographic coordinates on $\mathbf{S}_3$, considered as a sub manifold of $\mathbf{E}^4$. They are related to the Cartesian coordinates $(x^1, \ldots, x^4)$ of $\mathbf{E}_4$ by the following equations

$$\begin{aligned} x^a &= \frac{2r^2 X^a}{r^2+\sum_{i=1}^3 (X^i)^2}, \qquad a = 1, \ldots, 3 \\ x^4 &= r - \frac{2r^3}{r^2+\sum_{i=1}^3 (X^i)^2}, \end{aligned}$$

where r is the radius of the sphere. The components of the metric of $\mathbf{S}_3$ in the coordinates (X^i) are (see also [22])

$$g_{ii} = \frac{4r^4}{(r^2 + \sum_{i=1}^3 (X^i)^2)^2}.$$

Hence, the function $Q_{SE} = (r^2 + \sum_{i=1}^3 (X^i)^2)^2/4r^4$ is the conformal factor relating $\mathbf{S}_3$ to $\mathbf{E}_3$. Then,

$$Q_S = \frac{\lambda^2[(r^2 + \sum_{i=1}^3 (X^i(q^1, q^2, q^3))^2)^2]}{r^4 \prod_h (q^h - q^{h+1})}$$

is the conformal factor which makes (7.4.5) the metric of $\mathbf{S}_3$. The coordinates (q^1, q^2, q^3), related to the stereographic coordinates (X^1, X^2, X^3) by

$$\begin{aligned} \lambda \cdot X^1 &= \sqrt{\frac{(q^1-e_2)\cdot(q^2-e_2)\cdot(q^3-e_2)}{(e_2-e_1)\cdot(e_2-e_3)\cdot(e_2-e_4)\cdot(e_2-e_5)}}, \\ \lambda \cdot X^2 &= \sqrt{\frac{(q^1-e_3)\cdot(q^2-e_3)\cdot(q^3-e_3)}{(e_3-e_1)\cdot(e_3-e_2)\cdot(e_3-e_4)\cdot(e_3-e_5)}}, \\ \lambda \cdot X^3 &= \sqrt{\frac{(q^1-e_4)\cdot(q^2-e_4)\cdot(q^3-e_4)}{(e_4-e_1)\cdot(e_4-e_2)\cdot(e_4-e_3)\cdot(e_4-e_5)}}, \end{aligned}$$

with λ given by Eq. (7.5.1), are coordinates on $\mathbf{S}_3$.

Chapter 8

Conclusion

We have shown in this thesis that the known R-separable coordinate webs with symmetry form an exhaustive set of additional coordinates admitting R-separation of variables for the Laplace equation in Euclidean space. The conformal Killing tensors derived for each case expressed in canonical Cartesian coordinates can now be used by future researchers who wish to include a potential in boundary value problems that involve the use of Killing tensors written in canonical Cartesian coordinates. Using geometrical methods very different from the literature we have also independently derived the form of the asymmetric metric tensor for R-separable coordinates of the conformally invariant Laplace equation in Euclidean space admitting no symmetry. The associated Killing tensors for them have yet to be computed.

The canonical rotational R-separable webs known thus far form an exhaustive self contained set based on the study of the square of Eq.(6.2.1). The seemingly infinite room for more inequivalent coordinates was 'filled up' by the infinite ranges of the invariants resulting from variation of the coordinate parameter k appearing in the definitions for Jacobi-elliptic coordinate systems. Without a priori knowledge of these specific examples it would be hard to partition the invariant defined over $\mathbf{R}$ into a finite number of partitions of $\mathbf{R}$ representing disjoint equivalence classes.

This provides a clear motive to solve for the R-separable coordinates in $\mathbf{E}^3$ directly using the method of Eisenhart [20] along the lines of his classic 1935 paper on conformal separation in Euclidean space. Clearly more work is needed in this area. We have been able to discriminate between coordinates known a priori, as well as put others into same equivalence classes or with simple separable cases previously studied in [25]. All this was achieved without systematically solving for all cases of the Eisenhart conditions in conformal Euclidean space which could comprise a future research project of searching for the remaining two asymmetric R-separable webs in Euclidean space.

The last chapter of this thesis gives a derivation of the general asymmetric metrics in $\mathbf{E}^3$, and associated coordinate transforms to Cartesian coordinates. However, they have not been classified under the full conformal group like the known symmetric and conformally symmetric webs have been. Such an invariant classification using the approach employed in Chapter 6 would be a difficult task; all the invariants of the general characteristic conformal Killing tensor components under the entire conformal group would have to be computed. The TSN conditions would have to be solved on the entire thirty five parameter CKT, not just on nine-dimensional subsets that followed Lie differentiations. A direction for further research lies in the cases of metrics with one or more conformal symmetries; these can be found in principle by using the techniques discussed prior. The main obstacle at present in this calculation are the coupled non-linear pdes resulting from solving for the conformal factor representing transformation to flat space. Work has been done in this direction in [7] but using the formalism of complex Riemannian space which is beyond the scope of this thesis.

Appendix A

Proof of Levi-Civita's criterion for separability

For the proof in one direction, we assume sum separability of the solution W to the HJ equation. Assume a conservative system, namely that the Hamiltonian is time independent and equal to a constant which is the system's total energy. Equivalently we assume the Hamiltonian does not explicitly depend on time: $\mathbf{H} = H(q^i, p^i)$. Hence

$$\frac{dE}{dq^i} = \frac{dH}{dq^i} = 0 \Rightarrow \frac{\partial H}{\partial q^j}\frac{\partial q^j}{\partial q^i} + \frac{\partial H}{\partial p^j}\frac{\partial p^j}{\partial q^i} + \frac{\partial H}{\partial t}\frac{\partial t}{\partial q^i} = 0 \tag{A.0.1}$$

As the coordinate system is linearly independent, it is clear that $\frac{\partial q^j}{\partial q^i} = \delta_{j,i}$ and the sum $\frac{\partial H}{\partial q^j}\frac{\partial q^j}{\partial q^i}$ reduces to one term only: $\frac{\partial H}{\partial q^i}$. In general, $\frac{\partial t}{\partial q^i} \neq 0$ however we have assumed $\frac{\partial H}{\partial t} = 0$. Finally $\frac{\partial p^j}{\partial q^i} = 0$ for $i \neq j$ by our assumption of separability. This is so because from HJ theory $p^j = \frac{\partial W}{\partial q^j}$ and since $\frac{\partial W}{\partial q^j}$ is a function of q^j only, due to our starting assumption that $W \equiv \sum_{i=1}^n W^i(q^i, c_1, c_2, \ldots, c_n)$, $\frac{\partial^2 W}{\partial q^i \partial q^j} = 0, i \neq j$.

Thus we obtain

$$\begin{aligned} \frac{\partial H}{\partial q^i} &+ \frac{\partial H}{\partial p^i}\frac{\partial p^i}{\partial q^i} = 0 \\ \Rightarrow &\frac{\partial p^i}{\partial q^i} = -\frac{\frac{\partial H}{\partial q^i}}{\frac{\partial H}{\partial p^i}} = \frac{\partial^2 W}{\partial (q^i)^2}, \frac{\partial}{\partial q^j}(\frac{\partial p^i}{\partial q^i}) = 0. \end{aligned} \tag{A.0.2}$$

It is clear that if p^i is a function of q^i only, then $\frac{\partial p^i}{\partial q^i}$ is a function of q^i only, hence the second statement in the above formula for $i \neq j$. Let us denote $S \equiv \frac{\partial p^i}{\partial q^i}$. We have then

$$\begin{aligned} \frac{dS}{dq^j} &= \frac{\partial S}{\partial q^l}\frac{\partial q^l}{\partial q^j} + \frac{\partial S}{\partial p^l}\frac{\partial p^l}{\partial q^j} + \frac{\partial S}{\partial t}\frac{\partial t}{\partial q^j} \\ \Rightarrow \frac{dS}{dq^j} &= \frac{\partial S}{\partial q^j} + \frac{\partial S}{\partial p^j}\frac{\partial p^j}{\partial q^j} = 0, \end{aligned} \tag{A.0.3}$$

where we have $\frac{\partial S}{\partial t} = 0$ since W has no explicit dependence on time t. Performing the calculations we obtain:

$$\frac{\partial S}{\partial q^j} = \frac{-\frac{\partial H}{\partial p^i}\frac{\partial^2 H}{\partial q^j \partial q^i} + \frac{\partial H}{\partial q^i}\frac{\partial^2 H}{\partial q^j \partial p^i}}{(\frac{\partial H}{\partial p^i})^2}$$

$$\frac{\partial S}{\partial p^j} = \frac{-\frac{\partial H}{\partial p^i}\frac{\partial^2 H}{\partial p^j \partial q^i} + \frac{\partial H}{\partial q^i}\frac{\partial^2 H}{\partial p^j \partial p^i}}{(\frac{\partial H}{\partial p^i})^2}$$

$$\frac{\partial p^j}{\partial q^j} = -\frac{\frac{\partial H}{\partial q^j}}{\frac{\partial H}{\partial p^j}} \tag{A.0.4}$$

The last line follows from Hamilton's canonical equations. Clearly in the identity $\frac{\partial S}{\partial q^j} = 0$ one can multiply away the $(\frac{\partial H}{\partial p^i})^2$ term, and furthermore multiply both sides of the equation by $-\frac{\partial H}{\partial p^j}$ obtaining the classical Levi-Civita criterion, as required to show. □

Next we must show the sufficiency of the Levi-Civita conditions. Let us define $R_i = -\frac{\frac{\partial H}{\partial q^i}}{\frac{\partial H}{\partial p^i}}$, or in more common notation: $R_i = -\frac{\partial_i H}{\partial p^i H}$ as $H = H(p^h, q^h)$. We recognize R_i as a general function of p^h and q^h. If we assume the Levi Civita conditions hold true, we obtain $\partial_j R_i + R_j \frac{\partial}{\partial p^j} R_i = 0, i \neq j$. We seek a possible existence of a continuous, well defined solution to the following ansatz of a first order system: $\partial_i P_h = \delta_{i,h} R_h \Rightarrow \partial_i P_h = 0, i \neq j$ and $\partial_i P_h = R_h, i = h$.

Consider the more generic first order system of the form

$$\frac{\partial}{\partial x^i} y^h(x^1, \ldots, x^n) = F_i^h(x^1, \ldots, x^n, y^1, \ldots, y^n) \tag{A.0.5}$$

A continuous solution y^h must satisfy

$$\frac{d}{dx^i}(\frac{\partial y^h}{\partial x^j}) = \frac{d}{dx^j}(\frac{\partial y^h}{\partial x^i}) \tag{A.0.6}$$

note that total derivatives are needed because F_i^h is defined to depend also on $y^1, \ldots, y^n$ - this will be useful since R_i actually depends on both sets q^h and p^h, not on one set alone! But since complete integrability conditions refers to the demand that $\frac{\partial^2 y^h}{\partial x^i \partial x^j} = \frac{\partial^2 y^h}{\partial x^j \partial x^i}$, we require that

$$\frac{d}{dx^i}(\frac{\partial y^h}{\partial x^j}) = \frac{\partial}{\partial x^i}(\frac{\partial y^h}{\partial x^j}) \tag{A.0.7}$$

namely that $x^1, \ldots, x^n$ form an independent set. We thus assume that no x^i can be functionally dependent on $x^j, j \neq i$. Proceeding from Eq. (A.0.6) we have

$$\frac{d}{dx^i}(F_j^h(x^1, \ldots, x^n, y^1, \ldots, y^n)) = \frac{d}{dx^j}(F_i^h(x^1, \ldots, x^n, y^1, \ldots, y^n)) \tag{A.0.8}$$

Expanding the total derivatives, we obtain

$$\frac{\partial}{\partial x^i}F_j^h + \sum_a \frac{\partial F_j^h}{\partial y^a}\frac{\partial y^a}{\partial x^i} = \frac{\partial}{\partial x^j}F_i^h + \sum_a \frac{\partial F_i^h}{\partial y^a}\frac{\partial y^a}{\partial x^j} \tag{A.0.9}$$

or in more simple Einstein summation notation (where a repeated index above and below implies summation unless stated otherwise) we derive, as a test for complete integrability:

$$\frac{\partial}{\partial x^i}F_j^h + \frac{\partial F_j^h}{\partial y^a}F_i^a = \frac{\partial F_i^h}{\partial x^j} + \frac{\partial F_i^h}{\partial y^a}F_j^a \tag{A.0.10}$$

We want the L.H.S = R.H.S for $F_j^h = \delta_{j,h}R_h$ and $F_i^h = \delta_{i,h}R_h$, and $y^a = p^a$ on account that we seek a solution to $\partial_i P_h = \delta_{i,h}R_h$. Also $F_i^a = \delta_{i,a}R_a$ and $F_j^a = \delta_{j,a}R_a$.

Expanding, we obtain:

$$\partial_i\delta_{j,h}R_h + \frac{\partial}{\partial p^a}(\delta_{j,h}R_h)\delta_{i,a}R_a = \partial_j\delta_{i,h}R_h + \frac{\partial}{\partial p^a}(\delta_{i,h}R_h)\delta_{j,a}R_a \tag{A.0.11}$$

The sums collapse down to one term each:

$$\partial_i\delta_{j,h}R_h + R_i\frac{\partial}{\partial p^i}(\delta_{j,h}R_h) = \partial_j\delta_{i,h}R_h + R_j\frac{\partial}{\partial p^j}(\delta_{i,h}R_h) \tag{A.0.12}$$

There are distinct cases of indices to consider on both sides of the equation. For the case $i \neq h$, clearly the $\delta_{i,h}$ term in the R.H.S are 0, hence $\partial_j(0) + R_j\frac{\partial}{\partial p^j}(0) = 0$. For the case $i = h, j \neq i$ we have, on the R.H.S, $\partial_j R_i + R_j\frac{\partial}{\partial p^j}R_i = 0$ by the Levi-Civita assumption. For the case $i = h = j$ this is the only chance of R.H.S not vanishing; it takes on the form $\partial_i R_i + R_i\frac{\partial}{\partial p^i}R_i$ which is not necessarily zero.

For the L.H.S, case $j \neq h$ yields 0 as clearly the $\delta_{j,h}$ terms all vanish. Hence $\partial_i(0)+R_i\frac{\partial}{\partial p^i}(0) = 0$. For case $j = h, i \neq j$ we have, on the L.H.S, $\partial_i R_j+R_i\frac{\partial}{\partial p^i}R_j = 0$ also by the Levi-Civita assumption. Case $j = h, i = j$ is the only chance for the L.H.S not to vanish; it takes on the form $\partial_i R_i + R_i\frac{\partial}{\partial p^i}R_i$ which is not necessarily zero.

Thus for all possible n functions $y^h, h = (1, 2, \ldots, n)$ and all possible partial derivatives denoted by $\frac{\partial}{\partial x^i}$ and $\frac{\partial}{\partial x^j}$, the complete integrability conditions are satisfied.

Hence $\exists$ n functions $Q_i(q^i)$ such that $P_i = Q_i(q^i)$ are solutions of $\partial_i P_h = \delta_{i,h}R_h$. As $\partial_j P_i = 0, j \neq i$, it is clear that each P_i is a function of q^i alone, hence we write $P_i = Q_i(q^i)$.

The next step is to verify that $H(q^1, \ldots, q^n, Q_1(q^1), \ldots, Q_n(q^n))$ is equal to a constant, namely the total energy, which must be a constant for a conservative system. Then we need $\frac{dH}{dq^j} = 0$ for every $j \in (1, 2, \ldots, n)$.

$$\begin{aligned} \frac{dH}{dq^j} &= \frac{\partial H}{\partial q^j} + \frac{\partial H}{\partial P_l}\frac{\partial Q_l(q^l)}{\partial q^j} = \frac{\partial H}{\partial q^j} + \frac{\partial H}{\partial P_j}\frac{\partial Q_j(q^j)}{\partial q^j} \\ &= \frac{\partial H}{\partial q^j} + \frac{\partial H}{\partial P_j}R_j = \frac{\partial H}{\partial q^j} + \frac{\partial H}{\partial P_j}\left(-\frac{\frac{\partial H}{\partial q^j}}{\frac{\partial H}{\partial p^j}}\right) = 0. \end{aligned} \tag{A.0.13}$$

As the only independent variables of H involve $(q^1, q^2, \ldots, q^n)$, and since we prove that $\frac{dH}{dq^j} = 0 \;\; \forall \;\; j$, we can clearly write

$$H(q^1, \ldots, q^n, Q_1(q^1), \ldots, Q_n(q^n)) = E \tag{A.0.14}$$

But we know that

$$H(q^1, \ldots, q^n, \frac{\partial W}{\partial q^1}, \ldots, \frac{\partial W}{\partial q^n}) = E \tag{A.0.15}$$

Taking

$$\begin{aligned}
\frac{\partial W}{\partial q^1} &= Q_1(q^1) = P_1 \\
\frac{\partial W}{\partial q^2} &= Q_2(q^2) = P_2 \\
&\vdots \\
\frac{\partial W}{\partial q^n} &= Q_n(q^n) = P_n
\end{aligned} \tag{A.0.16}$$

we have constructed a separated solution to the HJ equation of the form:

$$W = \int Q_1(q^1)dq^1 + \int Q_2(q^2)dq^2 + \ldots + \int Q_n(q^n)dq^n - Et \tag{A.0.17}$$

satisfying $\frac{\partial W}{\partial q^i} = P_i$, where P_i is a function of q^i only.

The solution W is unique up to an arbitrary constant, which is fixed by the system's total energy. Thus, specifying E, we found a *unique* solution to the HJ equation that is separable, assuming the L-C criterion holds as well as conservation of energy and independence of the variables $(q^1, q^2, \ldots, q^n)$. This completes the proof of sufficiency of the Levi-Civita conditions. □

Appendix B

Proof of the connection between Stäckel matrices and Killing tensors in Eisenhart's formalism

The proof I present here starts with the fact, proven in Eisenhart's 1934 paper, that the matrix elements φ_{i1} defined in Eq. (2.1) are known to be functions of x^i at most. By definition in [20], $\rho_i^\alpha \equiv \frac{\psi^{i\alpha}}{\psi^{i1}}$ is independent of x^i. The quantity denoted by $\varphi^{i\alpha}$ is the co-factor (determinant of the minor) of φ_{ij}. For two dimensions let the arbitrary matrix φ_{ij} be denoted by:

$$\begin{bmatrix} f_1(x^1) & a \\ f_2(x^2) & b \end{bmatrix} \tag{B.0.1}$$

At this stage a and b are any arbitrary functions over all variables x^1 and x^2. However, from Eisenhart's definitions, ρ_1^2 is independent of x^1; thus in two dimensions ρ_1^2 is a function of x^2 at most. Now $\rho_1^2 = \frac{\varphi^{12}}{\varphi^{11}} = \frac{-f_2(x^2)}{b} \Rightarrow b = \frac{-f_2(x^2)}{\rho_1^2}$; the last equality being a function of x^2 only.

Similarly $\rho_2^2 = \frac{\varphi^{22}}{\varphi^{21}} = \frac{f_1(x^1)}{-a} \Rightarrow a = \frac{-f_1(x^1)}{\rho_2^2}$; the last equality being a function of x^i only. This easy analysis completes the proof for two dimensions that the n^2 φ_{ij} functions defined by $(H_i)^2 = \frac{\varphi}{\varphi^{i1}}$, $\rho_i^\alpha = \frac{\varphi^{i\alpha}}{\varphi^{i1}}$ form a Stäckel matrix whenever the total determinant $\varphi \neq 0$. Recall that in a Stäckel matrix the functions inside the i^{th} row must be functions of x^i only.

In three dimensions let the general 'Stäckel' matrix be denoted by:

$$\begin{bmatrix} f_1(x^1) & a & b \\ f_2(x^2) & c & d \\ f_3(x^3) & e & f \end{bmatrix} \tag{B.0.2}$$

Knowing that the expressions ρ_1^2 and ρ_1^3 are independent of x^1, we arrive in particular at $\frac{d \cdot f_3(x^3) - f_2(x^2) \cdot f}{c \cdot f - d \cdot e}$ not depending on x^1. Hence:

$$
\begin{aligned}
(c \cdot f - d \cdot e)[f_3(x^3)\frac{\partial d}{\partial x^1} - f_2(x^2) \cdot \frac{\partial f}{\partial x^1}] \\
-[d \cdot f_3(x^3) - f \cdot f_2(x^2)][\frac{\partial c}{\partial x^1} \cdot f + c \cdot \frac{\partial f}{\partial x^1} - \frac{\partial d}{\partial x^1} \cdot e - d \cdot \frac{\partial e}{\partial x^1}] = 0
\end{aligned}
\qquad \text{(B.0.3)}
$$

Since g_{11} is finite and non-zero, clearly $(c \cdot f - d \cdot e) \neq 0$ and one can divide by the quantity, obtaining:

$$
f_3(x^3)\frac{\partial d}{\partial x^1} - f_2(x^2)\frac{\partial f}{\partial x^1} - f_3(x^3)d(\frac{(cf - de)'}{(cf - de)}) + ff_2(x^2)(\frac{(cf - de)'}{(cf - de)}) = 0 \qquad \text{(B.0.4)}
$$

This can be re-arranged to yield the more familiar looking form:

$$
f_3(x^3)[\frac{\partial d}{\partial x^1} - d(\frac{(cf - de)'}{(cf - de)})] - f_2(x^2)[\frac{\partial f}{\partial x^1} - f(\frac{(cf - de)'}{(cf - de)})] = 0 \qquad \text{(B.0.5)}
$$

This expression must be satisfied for any arbitrary functional form of $f_3(x^3)$ and $f_2(x^2)$, so it is clear that the expressions in parenthesis must identically vanish. Say the second one is evaluated for some fixed function f and a fixed value for x^1, denoted for now by x_0:

$$
[\frac{\partial f}{\partial x^1}|_{x_0} - f|_{x_0}\frac{\partial}{\partial x^1}\log(cf - de)|_{x_0}] = 0 \qquad \text{(B.0.6)}
$$

This must hold true no matter what functional form c, d and e may take. In particular hold fixed any two of the (c, d, e) triplet and vary the third. For conceptual simplicity, denote $\log(cf - de)$ instead by $z(c, d, e, f)$, where z is a continuous function of the arbitrary functions c, d, e and f. f is already fixed, now fix say the functions d and e.

Thus $\frac{\partial}{\partial x^1}z(c(x^1, x^2, x^3), d_0, e_0, f_0) = 0$ implies the possible variations in the arbitrary functions c cannot involve the independent variable x^1, for then each differentiation with respect to x^1, evaluated at the particular value x_0, would yield a different numerical answer which we hold as impossible. The variation is thus limited to x^2 and x^3, hence $c = c(x^2, x^3)$ alone and the argument is unchanged for e and d; simply fix the other two and perform the above steps. Note all of the above steps necessitated the arbitrary function f being fixed; otherwise the reader may well ask if the possibility exists for the two separate terms in parenthesis to negate each other - but that is overrided by the fixing of f at one particular function it can take.

To prove $f = f(x^2, x^3)$ we now only consider the previous argument applied to the first expression in parenthesis in equation (B.0.5):

$$[\frac{\partial d}{\partial x^1}|_{x_0} - d|_{x_0}\frac{\partial}{\partial x^1}\log(cf - de)|_{x_0}] = 0 \tag{B.0.7}$$

Fix d, c and e to ascertain that variations in f cannot involve the variable x^1. Note there is no need to consider the quantity ρ_1^3 for the same conclusions will follow identically.

Now that the general method has been presented, the reader may well inquire about special cases where the method may break down. For instance, what if the factor $(cf - de) = 0$ leading to $\log(0)$? This case is impossible since Eisenhart already built into the matrices the definition $g_{ii} = \frac{\varphi^{i1}}{\varphi}$ and we know that the metric components can neither vanish nor be infinite, thus $(cf - de) \neq 0$.

What if $(cf - de) < 0$? Then the minus sign can be absorbed, to yield say $[\frac{\partial f}{\partial x^1} + f[\frac{-(cf-de)'}{(cf-de)}]$ but the final argument remains unchanged. Now consider the chain rule application. $\frac{\partial}{\partial x^1}z(c(x^1, x^2, x^3), d_0, e_0, f_0) = z'(c(x^1, x^2, x^3), d_0, e_0, f_0) \cdot \frac{\partial c(x^1,x^2,x^3)}{\partial x^1}$ by the usual chain rule of the composition of two continuous functions, of which $\log(z)$ is one of them. The first derivative in the above product cannot be zero as that would imply $(cf - de) = \infty$, leading to a contradiction considering the non-singular element of the metric component g_{11}.

Thus we finally prove c, d, e and f - all belonging to the second and third row - cannot be functions of x^1. By manipulating the second row, we get by similar argumentation (i.e $(af - be) \neq 0$) that a, b, e and f cannot be functions of x^2. This proves immediately that e, f are functions of x^3 at most; similar steps prove the analogous statements for (a, b) and (c, d). Thus each row of the matrix defined by Eisenhart's paper is a function of one variable only, as required to show.

Note also that by fixing functions, one must choose a form for them and values of x_0 such that the determinant φ of the matrix defined never vanishes. This technically limits the independence of the n^2 arbitrary functions somewhat, but innumerable functions exist within this constraint for which all the above facts hold, and must hold for the 'special cases' considered, hence validating the conclusion. For example, set $f_3(x^3) = x^3$ and $f_2(x^2) = x^2$, and functions to be fixed like e^{x^1}, $e^{(x^2)^2}$ and $e^{(x^3)^2}$ ensuring that the determinant can never vanish for non-trivial values of the independent variables.

If all of the c, d, e and f are independent of x^1, then clearly $\frac{\partial}{\partial x^1}(cf - de) = 0$ and individually $\frac{\partial}{\partial x^1}c$, $\frac{\partial}{\partial x^1}d$, $\frac{\partial}{\partial x^1}e$ and $\frac{\partial}{\partial x^1}f = 0$ thus the coefficients of $f_3(x^3)$ and $f_2(x^2)$ reduce to zero, as needed.

Note that metrics and Stäckel matrices very well exist where one of $f_1(x^1)$, $f_2(x^2)$ and $f_3(x^3)$ vanish, hence negating the necessity of all the coefficients in the above parenthesis vanishing. These are special cases only however; the formulae must work for all of the $f_1(x^1)$, $f_2(x^2)$, $f_3(x^3)$ and all possible metrics (of which the specific components g_{11}, g_{22} and g_{33} can be functions of all the coordinates) thus to encompass all cases satisfied simultaneously the coefficient functions must all vanish. This completes the proof, in two and three dimensions, of the Stäckel form of the matrices defined by Eisenhart – namely that the first row of the inverse form the components of the contravariant metric, and the remaining rows form the diagonal components of the remaining contravariant Killing tensors characteristic to the metric. □

Appendix C

Construction of the Stäckel matrix associated with coordinates separating the HJ equation

We start with the formalism of the components of the Stäckel matrix introduced in Chapter 1, whereby $\varphi_i^k = \frac{1}{2}\frac{\partial \phi_i}{\partial \alpha_k}$. This satisfies $\frac{\partial}{\partial q^j}\varphi_i^k = 0$ if $i \neq j$, since ϕ_i only depends on q^i.

Note that since $\phi_i = (p_i)^2$, we obtain $\frac{\partial \phi_i}{\partial \alpha_k} = 2p_i\frac{\partial p_i}{\partial \alpha_k}$. We know that all the $p_i \neq 0$ since if one were it could not have been a canonical variable [23]. Critically $det(\frac{\partial p_i}{\partial \alpha_k}) \neq 0$, since we have assumed a complete solution of the HJ equation necessitating that $det(\frac{\partial^2 W}{\partial q^i \partial \alpha^k}) \neq 0 \Rightarrow det(\frac{\partial p_i}{\partial \alpha_k}) \neq 0$.

Specifically $det(\frac{\partial \phi_i}{\partial \alpha_k}) = k det(\frac{\partial p_i}{\partial \alpha_k})\prod_i^n p_i \neq 0$ as $\prod_i^n p_i \neq 0$. Each p_i acts on the i^{th} row of the non-singular matrix $\frac{\partial p_i}{\partial \alpha_k}$ hence the presence of the $\prod_i^n p_i$ formula.

We will construct S^{-1} by defining, without loss of generality, $(g^{11}, g^{22}, \ldots, g^{nn})$ to be the last row. Hence $\frac{\partial}{\partial \alpha_k}(\frac{1}{2}g^{ii}\phi_i(q^i, \alpha_k)) = \frac{\partial \alpha_n}{\partial \alpha_k} = \delta_k^n$, which equals $\frac{1}{2}g^{ii}\frac{\partial \phi_i}{\partial \alpha_k} = \sum_i g^{ii}\varphi_i^k = \delta_k^n$. This proves that $(g^{11}, g^{22}, \ldots, g^{nn})$ was the last row of the inverse Stäckel matrix.

Now, if the potential V is separable we want to prove condition 2). Note this condition follows trivially if $V = 0$; simply declare $U_1(q^1) = U_2(q^2) = \ldots = U_n(q^n) = 0$. Now that $V = -\frac{1}{2}g^{ii}p_i^2 + E$, construct $U_i = -\frac{1}{2}\phi_i + E\varphi_i^n$. φ_i^n is simply a column, n, of the Stäckel matrix S.

We have $\sum g^{ii}U_i = -\frac{1}{2}\sum g^{ii}\phi_i + \sum g^{ii}E\varphi_i^n = -\frac{1}{2}g^{ii}p_i^2 + E$, as g^{ii} is the n^{th} row of S^{-1}, clearly $E\sum g^{ii}\varphi_i^n = E$. Thus since $V = -\frac{1}{2}g^{ii}p_i^2 + E$, it follows that $V = g^{11}U_1(q^1) + g^{22}U_2(q^2) + \ldots + g^{nn}U_n(q^n)$, as required to show. □

Now for the other direction, assume we are given a Stäckel matrix such that $(g^{11}, g^{22}, \ldots, g^{nn})$ is the n^{th} row of its inverse. Let $\varphi_i^k(q^i)$ be defined, with $g^{ii} = \varphi_n^i$. We then express $V = \sum g^{ii} U_i(q^i)$. Thus we write (where the Einstein summation rule is assumed to apply over i):

$$\begin{aligned} \frac{1}{2} g^{ii} p_i^2 + g^{ii} U_i(q^i) &= E \\ \frac{1}{2} g^{ii} [p_i^2 + 2U_i(q^i)] &= E \\ \frac{1}{2} \varphi_n^i [p_i^2 + 2U_i(q^i)] &= E \end{aligned} \tag{C.0.1}$$

Next let $\varphi_k^i[p_i^2 + 2U_i(q^i)] = 2\alpha_k$, $k = 1, \ldots, n$ hence we have $\alpha_n = E$, and each $\alpha_k \in \mathbf{R}$. Now

$$\begin{aligned} \varphi_j^k \varphi_k^i (p_i^2 + 2U_i(q^i)) &= 2\varphi_j^k \alpha_k \\ \delta_j^i (p_i^2 + 2U_i(q^i)) &= 2\alpha_k \varphi_j^k (q^j) \\ \Rightarrow p_j^2 + 2U_j(q^j) &= 2\alpha_k \varphi_j^k (q^j) \\ p_j^2 = 2\alpha_k \varphi_j^k(q^j) &- 2U_j(q^j) \end{aligned} \tag{C.0.2}$$

The last line clearly shows that p_j is a function of q^j only, showing that $W(q^i, \alpha_k)$ is indeed sum separable as required to show. □

Appendix D

An equivalent property of the Schouten bracket

Let K and L be symmetric tensors of types $(p, 0)$ and $(q, 0)$ respectively. The Schouten bracket of K and L denoted by $[K, L]$ is a tensor of type $(p + q - 1, 0)$. An equivalent property of $[K, L] = 0$, also known as an *involution* of K and L, is that their contraction into quadratic polynomials in the momenta commute in the standard classical Poisson bracket:

$$\{K, L\} = \sum_{i=1}^{n} \left(\frac{\partial K}{\partial p_i}\frac{\partial L}{\partial q^i} - \frac{\partial K}{\partial q^i}\frac{\partial L}{\partial p_i}\right) \tag{D.0.1}$$

whereby $P_{K_h} = K_h^{ij} p_i p_j$ for $h \neq n$ and $P_{K_n} = g^{ij} p_i p_j$ satisfies, for $(h, j) = 1, \ldots, n$:

$$\left\{P_{K_h}, P_{K_j}\right\} = 0 \tag{D.0.2}$$

The above is the equivalent formulation of the standard Lie-Schouten bracket.

Appendix E

Proof of the eigenvalue equations for characteristic conformal Killing tensors by construction from simple Killing tensors

Consider again the basic eigenvalue equation from Eisenhart theory:

$$K_{ii} = g_{ii}\rho_i, \tag{E.0.1}$$

where K_{ii} is understood to be the diagonal components of the Killing tensor already diagonalized in the normal eigenbasis of its eigenvectors. The proof of this is instructive, however it relies on more mathematical assumptions than in the text and is hence presented here instead. Ordinarily the eigenvector fields of tensors are not *normal* - namely that these eigenvector fields admit a family of hypersurfaces orthogonal to them. In such cases the vector fields are deemed non-integrable or non-surface-forming. Eisenhart assumed that the Killing tensor eigenvectors are normal and hence the hypersurfaces can be taken as parametric. Furthermore he assumed the symmetric Killing tensor to have real *simple eigenvalues*, namely n distinct eigenvalues admitting n orthogonal eigenvectors x^i that are surface-forming:

$$ds^2 = e_1 g_{11}(dx^1)^2 + e_2 g_{22}(dx^2)^2 + \ldots + e_n g_{nn}(dx^n)^2 \tag{E.0.2}$$

where for Riemannian manifolds all the e's are unity, and for non-Riemannian manifolds the e's may take on plus/minus unity. That the fundamental form above can be expressed in terms of the eigenvectors x^i depends crucially on the normality assumption as well as the fact that there are n orthogonal eigenvectors the Killing tensor admits. We perform a coordinate transformation to the coordinates defined by the normal eigenvectors such that $K_{ij} = 0, i \neq j$ and $\tilde{g}_{ij} = 0, i \neq j$. The tilde can be dropped and the Killing tensor equation for $i = j = l$ reduces to

$$\frac{\partial \log(\sqrt{K_{ii}})}{\partial x^i} = \frac{\partial \log(\sqrt{g_{ii}})}{\partial x^i} \tag{E.0.3}$$

This can be easily integrated to

$$\log(\sqrt{K_{ii}}) = \log(\sqrt{g_{ii}}) + a(x^1, x^2, \ldots, x^{i-1}, x^{i+1}, \ldots, x^n) \tag{E.0.4}$$

After performing exponentials the form $K_{ii} = g_{ii}\rho_i$ is arrived at, where

$$\frac{\partial \rho_i}{\partial x^i} = 0. \tag{E.0.5}$$

This eigenvalue property of the diagonal components of the Killing tensor is actually true in general for tensors, without the assumption of the specific Killing tensor equation.

Knowing that $g_{ii} = \tilde{g}_{ii}e^{-2\sigma}$, we can rewrite the previous equation as:

$$\begin{aligned} K_{ii} &= \tilde{g}_{ii}e^{2\sigma}\rho_i \\ K_{ii} &= \tilde{g}_{ii}\tilde{\rho}_i, \end{aligned} \tag{E.0.6}$$

where $\tilde{\rho}_i \equiv e^{2\sigma}\rho_i$. Thus we arrive at two fundamental properties [3] of conformal Killing tensors:

Proposition E.0.4 *The eigenvectors of* $\mathbf{K}$ *are the same with respect to both metrics* $\tilde{\mathbf{g}}$ *and* $\mathbf{g}$. *Furthermore if* $\tilde{\rho}^i$ *are the eigenvalues with respect to (contravariant)* $\tilde{\mathbf{g}}$, *then the eigenvalues with respect to* $\mathbf{g}$ *are* $\rho = e^{2\sigma}\tilde{\rho}^i$.

Note that eigenvalues of two equivalent conformal Killing tensors differ only by the scalar function f; going from one equivalent tensor to another means simply adding or subtracting f from all its eigenvalues, namely:

$$\tilde{\rho}_i = \rho_i \pm f \tag{E.0.7}$$

These transformations still leave invariant the form $K_{ii} = \tilde{g}_{ii}\tilde{\rho}_i$, as well as its eigenvectors, for *all* conformal Killing tensors in an equivalence class.

Remark E.0.5 *In words a special conformal Killing tensor, within its equivalence class, is in terms of some specially conformally related metric a simple Killing tensor, and to calculate the eigenvalues of this conformal Killing tensor, the tools for simple Killing tensors apply provided the conformal factor is a priori known. This was especially important for this thesis, since the connection between Stäckel matrices and the eigenvalues of the ordinary Killing tensors in terms of the separable coordinates are employed at length.*

Knowing how the eigenvalues transform allows us to deduce the differential relationship they satisfy for conformal Killing tensors. The second identity for eigenvalues from [20] is

$$\frac{\partial}{\partial x^j}\log\left(\frac{(\rho_i - \rho_j)}{g_{ii}}\right) = 0, \quad i \neq j, \tag{E.0.8}$$

which is a result for simple Killing tensors. The proof comes from the Killing tensor equation (on the diagonalized Killing tensor in terms of the normal eigenbasis) for the case $j \neq i, l = j$:

$$\frac{\partial K_{ii}}{\partial x^j} - 2K_{ii}\frac{\partial \log(g_{ii})}{\partial x^j} + K_{jj}\frac{1}{g_{jj}}\frac{\partial g_{ii}}{\partial x^j} = 0. \tag{E.0.9}$$

Considering both transformation rules on the eigenvalues, we see immediately that the same form in Eq. (E.0.8) is preserved for transformed metric coefficients and eigenvalues with the only change being that ρ_i can depend on x^i due to the general form of the conformal factor $e^{2\sigma}$. Note trivially that the equivalence function f cancels in the numerator of Eq. (E.0.8). Thus we are able to prove an important proposition in [3]:

Proposition E.0.6 *The eigenvalues ρ_i of a conformal Killing tensor satisfy the set of coupled linear partial differential equations:*

$$\frac{\partial \rho_i}{\partial x^j} = (\rho_i - \rho_j)\frac{\partial \log(g_{ii})}{\partial x^j} + \frac{\partial \rho_j}{\partial x^j} \tag{E.0.10}$$

Proof:

$$\begin{aligned} 0 &= \frac{\partial}{\partial x^j}\log(\frac{(\rho_i - \rho_j)}{g_{ii}}) \\ &= \frac{g_{ii}}{(\rho_i - \rho_j)}((\frac{\partial \rho_i}{\partial x^j} - \frac{\partial \rho_j}{\partial x^j})\frac{1}{g_{ii}} - (\frac{(\rho_i - \rho_j)}{g_{ii}^2}\frac{\partial g_{ii}}{\partial x^j})) \end{aligned} \tag{E.0.11}$$

multiplying through by g_{ii} and discarding the $(\rho_i - \rho_j)^{-1}$ common factor yields the required result. □

A crucial subtlety in the above reasoning is that our proof is restricted to conformal Killing tensors with normal eigenvectors and real simple eigenvalues, allowing them to be diagonalized in orthogonal coordinates. Indeed this is the hypothesis of Proposition 7.2 in [3], which among other statements reads:

Proposition E.0.7 *A CKT $\mathbf{K}$ which is diagonalized in orthogonal coordinates (that is $g_{ij} = 0$ and $K_{ij} = 0$ for $i \neq j$) is equivalent to a CKT $\mathbf{K}'$ that is a simple Killing tensor with respect to a conformally related metric.*

Our proof of the properties of the eigenvalues of a conformal Killing tensor is essentially the reverse direction of Proposition 7.2 in [3], as we first started from a simple diagonalized Killing tensor and then enacted a conformal transformation bringing us to a representative of an equivalence class of characteristic conformal Killing tensors. The proof is therefore not valid for non characteristic Killing tensors, however they are not useful for characterizing separable coordinates anyhow.

Appendix F

Alternate classification scheme using the invariants and covariants of biquartic polynomials

Here is material from [13], firstly please note the change in dictionary of the coefficients of the rotational Killing tensor which is outlined below. This rotational characteristic conformal Killing tensor is equivalent to

$$M_{33}\mathbf{I}_3 \odot \mathbf{I}_3 + L_3\mathbf{D} \odot \mathbf{I}_3 + H\mathbf{D} \odot \mathbf{D} + C_{33}\mathbf{R}_3 \odot \mathbf{R}_3 + D_3\mathbf{D} \odot \mathbf{X}_3 + A_{33}\mathbf{X}_3 \odot \mathbf{X}_3. \quad \text{(F.0.1)}$$

Let $RCK^2(\mathbf{E}^3)$ be the subspace $CK^2(\mathbf{E}^3)$ of CKTs of the form (F.0.1). The free parameters describing a general element $\mathbf{K} \in RCK^2(\mathbf{E}^3)$ are

$$(M_{33}, L_3, H, C_{33}, D_3, A_{33}) \quad \text{(F.0.2)}$$

and all the other forty nine coefficients of the general linear combination of symmetric products of CKVs (3.1.3) are null. Given any CKT in Cartesian coordinates satisfying $\mathcal{L}_{\mathbf{R}_3}\mathbf{K} = 0$, and the TSN-conditions, the value of the parameters (F.0.2) are determined as follows:

- M_{33} is 1/4 of the coefficient of xyz^2 in K_{12};
- L_3 is 1/2 of the coefficient of xyz in K_{12};
- H is the coefficient of xz in K_{13};
- $H - C_{33}$ is the coefficient of xy in K_{12};
- D_3 is twice the coefficient of x in K_{13};
- A_{33} is the constant term of $K_{33} - K_{22}$.

Since we are considering components (or functions of the components) which are not affected by the addition of a multiple of the metric $f\mathbf{g}$, the six parameters are well defined, irrespective of whether one starts from a CKT in $TCK(\mathbf{E}^3)$ or not.

Remark F.0.8 Since $\mathbf{E}^3$ has dimension three, there is an equivalent way to characterize rotational R-separable webs. Any rotational web contains a family of hypersurfaces made of half-planes issued from the rotation axis (the z-axis in our case). These planes are orthogonal to the Killing vector $\mathbf{R}_3$. Hence $\mathbf{R}_3$ must be an eigenvector of the CKT defining the web. Moreover, this condition is also sufficient to ensure that the eigenvectors of $\mathbf{K}$ are normal. Indeed, one of them is the normal vector $\mathbf{R}_3$ and the other two are contained in the two-dimensional planes orthogonal to $\mathbf{R}_3$ and hence they are normal. By imposing the condition

$$(\mathbf{K}\cdot\mathbf{R}_3)\times\mathbf{R}_3 = 0,$$

we find again the six dimensional linear subspace described by (F.0.1).

Finally, in order to prove that the general rotational CKT (F.0.1) is characteristic, we check that the eigenvalues are simple almost everywhere. Since $\mathbf{R}_3$ is orthogonal to $\mathbf{I}_3$, $\mathbf{D}$, $\mathbf{X}_3$, we have

$$\mathbf{K}\cdot\mathbf{R}_3 = C_{33}(x^2+y^2)\mathbf{R}_3.$$

Hence, $\mathbf{R}_3 = \mathbf{E}_1$ is an eigenvector corresponding to the eigenvalue $\lambda_1 = C_{33}(x^2+y^2)$. The other two eigenvectors $\mathbf{E}_2$ and $\mathbf{E}_3$ are orthogonal to $\mathbf{E}_1$; they and their corresponding eigenvalues do not depend on C_{33}. Moreover, the associated eigenvalues are of the form

$$\lambda_{2,3} = \frac{A\pm\sqrt{B}}{2},$$

where

$$A = r^4M_{33} + zr^2L_3 + r^2H + zD_3 + A_{33}, \qquad (r^2 = x^2+y^2+z^2) \tag{F.0.3}$$

$$B = (x^2+y^2)\left[r^2L_3 + 2zH + \frac{4z^2-r^2}{r^2}D_3 + \frac{4z(2z^2-r^2)}{r^4}A_{33}\right]^2 + \tag{F.0.4}$$

$$\left[r^4M_{33}+zr^2L_3 + (2z^2-r^2)H+\frac{z(4z^2-3r^2)}{r^2}D_3+\frac{r^4-8z^2(r^2-z^2)}{r^4}A_{33}\right]^2.$$

Any change of the parameter C_{33} does not affect the web; indeed, $\mathbf{E}_2$ and $\mathbf{E}_3$ do not involve C_{33} (see also Sect. F.2). Thus, it is always possible to choose C_{33} such that λ_1 is different from λ_2 and λ_3 at any point outside of the z-axis. On the contrary for $x = y = 0$ we have

$$\lambda_1 = 0, \qquad \lambda_2 = \frac{1}{2}(q(z)+|q(z)|), \qquad \lambda_3 = \frac{1}{2}(q(z)-|q(z)|),$$

with

$$q(z) = M_{33}z^4 + L_3z^3 + Hz^2 + D_3z + A_{33}. \tag{F.0.5}$$

Thus, (at least) one of λ_2, λ_3 identically vanishes and all points of the z-axis are singular points of all rotational webs. The singular points that are not on the rotation axis are those satisfying $\lambda_2 = \lambda_3$, that is where $B = 0$.

Remark F.0.9 The roots of (F.0.5) are points on the z-axis where the three eigenvalues coincide and $\mathbf{K}$ is proportional to the metric tensor. The number of the roots z_0 of q in $\mathbf{PR}^1$ (so that the point at infinity is also considered) and their multiplicity characterize the web from a geometric point of view.

Remark F.0.10 The knowledge of the eigenvalues of the characteristic tensor in a rotational web allows one to write the equations of the (not planar) hypersurfaces (see [16]) The hypersurfaces S_2 orthogonal to $\mathbf{E}_2$ satisfy the equation

$$\frac{\lambda_1 - \lambda_3}{x^2 + y^2} = h, \qquad h \in \mathbf{R},$$

while the hypersurfaces S_3 orthogonal to $\mathbf{E}_3$ satisfy the equation

$$\frac{\lambda_1 - \lambda_2}{x^2 + y^2} = h, \qquad h \in \mathbf{R}.$$

It follows that the hypersurfaces have the form

$$2(h - C_{33})(x^2 + y^2) + A = \pm\sqrt{B},$$

that is they are both described by the equation

$$[2(h - C_{33})(x^2 + y^2) + A]^2 - B = 0, \tag{F.0.6}$$

but for different ranges of the value of h: we have surfaces of S_2 for $h < h_0$ and surfaces of S_3 for $h > h_0$, respectively, where

$$h_0 = C_{33} - \frac{A}{2(x^2 + y^2)} = C_{33} - \frac{r^4 M_{33} + zr^2 L_3 + r^2 H + zD_3 + A_{33}}{x^2 + y^2}.$$

For $h = h_0$ we do not obtain a surface of the web because this value of the parameter h would imply $B = 0$, that is $\lambda_2 = \lambda_3$. Expanding the equations (F.0.6) we arrive at

$$\begin{gathered}[4(H - C_{33} + h)M_{33} - L_3^2]r^4 + [8M_{33}D_3 - 4(C_{33} - h)L_3]r^2 z + \\ [2L_3 D_3 - 4(C_{33} - h)H]r^2 + 16M_{33}A_{33}z^2 + 4(C_{33} - h)^2(x^2 + y^2) + \\ [8L_3 A_{33} - 4(C_{33} - h)D_3]z - D_3^2 + 4(H - C_{33} + h)A_{33} = 0,\end{gathered} \tag{F.0.7}$$

which represents two families of confocal cyclides, one for $h > h_0$ and one for $h < h_0$.

F.1 Characteristic CKTs of the known R-separable rotational coordinate systems

Table 1 contains the parameters of a characteristic CKT corresponding to each of the rotational R-separable coordinates listed in Moon and Spencer's book [38].

Coordinates	M_{33}	L_3	H	C_{33}	D_3	A_{33}
Bi-cyclide	$-\frac{k^2}{a^2}$	0	$1+k^2$	$1+k^2$	0	$-a^2$
Flat-ring cyclide	$\frac{k^2}{a^2}$	0	$1+k^2$	0	0	a^2
Disk cyclide	$-\frac{k^2}{a^2}$	0	$1-2k^2$	0	0	$a^2(1-k^2)$
Cap cyclide	$\frac{a^2(1+k)^2}{k}$	0	$\frac{4k-(k-1)^2}{2}$	$\frac{-(k-1)^2}{2}$	0	$\frac{k(k+1)^2}{16a^2}$
Toroidal	$\frac{1}{4a^2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{a^2}{4}$
Bispherical	$-\frac{1}{4a^2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{a^2}{4}$
Inverse prolate spheroidal	$\frac{1}{a^2}$	0	-1	0	0	0
Inverse oblate spheroidal	$-\frac{1}{a^2}$	0	-1	0	0	0
Tangent spheres	1	0	0	0	0	0
Cardioid	0	1	0	0	0	0
Prolate spheroidal	0	0	-1	0	0	a^2
Oblate spheroidal	0	0	1	0	0	a^2
Spherical	0	0	1	-1	0	0
Parabolical	0	0	0	0	1	0
Cylindrical	0	0	0	-1	0	1

Table F.1: Characteristic CKT of rotationally symmetric R-separable webs

We briefly describe how they are determined (for further details, such as plots, transformation laws to Cartesian coordinates, components of the metric tensor in these coordinates, separated equations etc., see [38] or [5]). The CKTs are constructed from the Stäckel matrices that are associated with each system of coordinates in [38].

Recall that a *Stäckel matrix* is a regular matrix of functions S_{ij} depending on the single variable q^i corresponding to the row index i of the element. One row (the first in the examples in [38]) of the inverse of the Stäckel matrix contains the components of the contravariant metric tensor in the R-separable coordinates, while the other two rows are made of the components of two CKTs with common eigenvectors orthogonal to the web hypersurfaces. Moreover, there is always a real linear combination of these two tensors which provides a characteristic tensor of the web (see [3]).

For each row of the inverse of the Stäckel matrix we construct the conformal Killing tensors in the R-separable coordinates, then the parameters (F.0.2) are determined by transforming the tensor to Cartesian coordinates and comparing with the Cartesian components of the general rotationally symmetric CKT (F.0.1). For all the coordinate systems considered in [38] the tensor corresponding to the third row of the inverse Stäckel matrix is $\mathbf{R}_3 \odot \mathbf{R}_3$. In most of the examples, the other tensor is a characteristic tensor of the web so its parameters appear unchanged in the Table 1. On the contrary, the tensors arising from the Stäckel matrices given in [38] for Spherical, Tangent spheres and Cylindrical coordinates have $C_{33} = 0$, so they are not characteristic CKTs. In order to get a characteristic CKT associated with these webs we add a suitable multiple of the tensor $\mathbf{R}_3 \odot \mathbf{R}_3$: that is, we change the value of C_{33} in Table 1.

The first four coordinate systems have transformation laws to Cartesian coordinates involving Jacobi elliptic functions. The parameter a is a scaling parameter, while the parameter $k \in (0, 1)$ is the parameter of the Jacobi elliptic functions.

F.2 Group action preserving rotationally symmetric CKTs

F.2.1 The group and its one-parameter subgroups

In order to classify the different types of R-separable webs admitting a rotational symmetry, we consider transformations acting on $CK^2(\mathbf{E}^3)$ which preserve the space $RCK^2(\mathbf{E}^3)$ of the rotationally symmetric CKTs previously discussed in chapter five. For this purpose, we use a group G that is generated by five one-parameter transformations and a discrete transformation. Three of the one-parameter transformations are induced on $RCK^2(\mathbf{E}^3)$ by conformal transformations of $\mathbf{E}^3$ mapping the z-axis into itself. The other two are transformations of the CKT that do not change the corresponding web.

The five continuous transformations to be taken into account are

1. The change of the tensor under a continuous inversion along the z-axis parameterized by a_0:
$$\phi_0 : (x, y, z) \to \left(\frac{x}{1 + 2a_0 z + a_0^2 r^2}, \frac{y}{1 + 2a_0 z + a_0^2 r^2}, \frac{z + a_0 r^2}{1 + 2a_0 z + a_0^2 r^2} \right),$$
where $r^2 = x^2 + y^2 + z^2$.

2. The change of the tensor under a translation along the z-axis parameterized by a_1:
$$\phi_1 : (x, y, z) \to (x, y, z + a_1).$$

3. The change of the tensor under a dilation of the space with singular point at the origin parameterized by a_2:
$$\phi_2 : (x, y, z) \to (a_2 x, a_2 y, a_2 z), \qquad (a_2 \neq 0).$$

4. The multiplication of the tensor by a non-zero scalar a_3:

$$\mathbf{K} \to a_3\mathbf{K}, \qquad (a_3 \neq 0).$$

5. The addition to the tensor of a multiple of $\mathbf{R}_3 \odot \mathbf{R}_3$:

$$\mathbf{K} \to \mathbf{K} + a_4\mathbf{R}_3 \odot \mathbf{R}_3.$$

Moreover, the discrete transformation considered is the one induced by the inversion I with respect to the unit sphere with center at the origin

$$I : (x, y, z) \to \left(\frac{x}{x^2+y^2+z^2}, \frac{y}{x^2+y^2+z^2}, \frac{z}{x^2+y^2+z^2}\right). \tag{F.2.8}$$

Note that $I^{-1} = I$ and that for the continuous inversion ϕ_0 we have $\phi_0 = I^{-1} \circ \phi_1 \circ I$, where ϕ_1 is the translation along the z-axis.

Remark F.2.1 The addition of the metric $\mathbf{g}$ and the transformation induced by the rotation around the z-axis are not relevant, since they do not modify the parameters (F.0.2) defining the tensor.

F.2.2 Group action, invariants and canonical forms

Let G be the group generated by the above described transformations. Since the discrete inversion is included, G is not connected. Moreover, two of the continuous one-parameter transformations are defined only for values of the parameter in $\mathbf{R} - \{0\}$, so that the connected component of G containing the identity is characterized by $a_2 > 0$ and $a_3 > 0$. Two other discrete transformations are implicitly included in G: the change of sign of the tensor (for $a_3 = -1$) and the transformation induced by the symmetry around the origin in $\mathbf{E}^3$ (for $a_2 = -1$).

The effect of the inversion around the unit sphere on the coefficients (F.0.2) of $\mathbf{K} \in RCK^2(\mathbf{E}^3)$ is given by

$$\begin{array}{l} \tilde{M}_{33} = A_{33}, \\ \tilde{L}_3 = D_3, \\ \tilde{H} = H, \\ \tilde{C}_{33} = C_{33}, \\ \tilde{D}_3 = L_3, \\ \tilde{A}_{33} = M_{33}. \end{array} \tag{F.2.9}$$

The equations of the action generated by the five continuous transformations acting on (F.0.2) are

$$\begin{array}{r} \tilde{M}_{33} = a_3\frac{P(a_0)}{a_2^2}, \\ \tilde{L}_3 = a_3\frac{-4a_1P(a_0)-a_2P^{(1)}(a_0)}{a_2^2}, \end{array}$$

$$\tilde{H} = a_3 \tfrac{6a_1^2 P(a_0)+3a_1a_2P^{(1)}(a_0)+a_2^2P^{(2)}(a_0)}{a_2^2},$$
$$\tilde{C}_{33} = a_4 + a_3 C_{33} + a_3 \tfrac{6a_1^2P(a_0)+3a_1a_2P^{(1)}(a_0)+a_2^2(P^{(2)}(a_0)-H)}{3a_2^2},$$
$$\tilde{D}_3 = a_3 \tfrac{-4a_1^3P(a_0)-3a_1^2a_2P^{(1)}(a_0)-2a_1a_2^2P^{(2)}(a_0)-a_2^3P^{(3)}(a_0)}{a_2^2},$$
$$\tilde{A}_{33} = a_3 \tfrac{a_1^4P(a_0)+a_1^3a_2P^{(1)}(a_0)+\ldots+a_1a_2^3P^{(3)}(a_0)+a_2^4P^{(4)}(a_0)}{a_2^2},$$

where

$$P(a_0) = A_{33}a_0^4 - D_3a_0^3 + Ha_0^2 - L_3a_0 + M_{33}, \qquad \text{(F.2.10)}$$

and

$$P^{(n)} = \frac{1}{n!}\frac{d^nP}{(da_0)^n}.$$

Since C_{33} and a_4 are involved only with $\tilde{C}_{33}$, and C_{33} is unchanged by the discrete inversion (F.2.9), we can disregard C_{33} (which can be made equal to any fixed constant by choosing a particular value for a_4). Then we consider the reduced action on the vector subspace of $RCK^2(\mathbf{E}^3)$ defined by the five parameters

$$(M_{33}, L_3, H, D_3, A_{33}) \qquad \text{(F.2.11)}$$

of the subgroup G' of G defined by $a_4 = 0$:

$$\tilde{M}_{33} = a_3 \tfrac{P(a_0)}{a_2^2}, \qquad \text{(F.2.12)}$$
$$\tilde{L}_3 = a_3 \tfrac{-4a_1P(a_0)-a_2P^{(1)}(a_0)}{a_2^2}, \qquad \text{(F.2.13)}$$
$$\tilde{H} = a_3 \tfrac{6a_1^2P(a_0)+3a_1a_2P^{(1)}(a_0)+a_2^2P^{(2)}(a_0)}{a_2^2}, \qquad \text{(F.2.14)}$$
$$\tilde{D}_3 = a_3 \tfrac{-4a_1^3P(a_0)-3a_1^2a_2P^{(1)}(a_0)-2a_1a_2^2P^{(2)}(a_0)-a_2^3P^{(3)}(a_0)}{a_2^2}, \qquad \text{(F.2.15)}$$
$$\tilde{A}_{33} = a_3 \tfrac{a_1^4P(a_0)+a_1^3a_2P^{(1)}(a_0)+\ldots+a_1a_2^3P^{(3)}(a_0)+a_2^4P^{(4)}(a_0)}{a_2^2}. \qquad \text{(F.2.16)}$$

It appears that the building blocks of the action equation is the polynomial (F.2.10) and its derivatives.

Remark F.2.2 If we denote the parameters (F.2.11) by α^i ($i = 0, \ldots, 4$), setting $\alpha^4 = M_{33}$, $\alpha^3 = L_3$, $\alpha^2 = H$, $\alpha^1 = D_3$, $\alpha^0 = A_{33}$), then their transformation laws under the action can be written in a compact formal way as

$$\tilde{\alpha}^{4-i} = \frac{a_3}{a_2^2}\sum_{h=0}^{i}(-1)^i \binom{4-h}{i} P^{(h)}(a_0)a_1^{i-h}a_2^h, \qquad i = 0, \ldots, 4.$$

Theorem F.2.3 *Let G_1 be the subgroup of G' defined by $a_3 > 0$. Then, the action of G_1 on (F.2.11) given by (F.2.12 − −F.2.16) and (F.2.9) is equivalent to the classical action of $GL(2, \mathbf{R})$ on real binary quartics.*

Proof: Consider the following binary quartic constructed from the five coefficients (F.2.11) of the CKT:

$$Q(X,Y) = M_{33}X^4 + L_3X^3Y + HX^2Y^2 + D_3XY^3 + A_{33}Y^4. \qquad \text{(F.2.17)}$$

By inserting the linear transformation of the variables (X, Y)

$$X = \alpha\bar{X} + \beta\bar{Y}, \qquad Y = \gamma\bar{X} + \delta\bar{Y},$$

with $(\alpha\delta - \beta\gamma) \neq 0$, in (F.2.17), we obtain a new quartic $\bar{Q}(\bar{X}, \bar{Y})$ whose coefficients $\bar{M}_{33}, \ldots, \bar{A}_{33}$ depend on the $GL(2, \mathbf{R})$ matrix

$$M = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

and on the coefficients of Q $(M_{33}, \ldots, A_{33})$. Since we assume $a_3 > 0$, by setting

$$\alpha = \sqrt[4]{a_3a_2^2}, \quad \beta = -a_1\sqrt[4]{a_3a_2^2}, \quad \gamma = -a_0\sqrt[4]{a_3a_2^2}, \quad \delta = (a_1a_0 + a_2)\sqrt[4]{a_3a_2^2},$$

we obtain equations $(F.2.12 - -F.2.16)$. The regularity of M follows from $(\alpha\delta - \beta\gamma) = \sqrt{a_3}a_2|a_2| \neq 0$, since $a_2a_3 \neq 0$. Furthermore, setting $\alpha = \gamma = 0$, and $\beta = \delta = 1$, we recover (F.2.9). Conversely, we prove that for any transformation of the quartic we can associate a transformation of G_1. We distinguish two cases: for $\alpha \neq 0$, by setting

$$a_0 = -\gamma\alpha^{-1}, \quad a_1 = -\beta\alpha^{-1}, \quad a_2 = (\alpha\delta - \beta\gamma)\alpha^{-1}, \quad a_3 = (\alpha\delta - \beta\gamma)^2,$$

into $(F.2.12 - -F.2.16)$ we obtain the action of M on the quartic form. The fact that $a_2a_3 \neq 0$ follows from the regularity of the matrix M. If $\alpha = 0$, we apply first the discrete inversion $(F.2.9)$ on the parameters of the CKT, that is we multiply M by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ on the left. In this way we obtain a new matrix M_1 with $\alpha_1 = \gamma \neq 0$ since M is regular, and thus revert to the previous case. □

As an immediate consequence of the theorem we are able to determine the invariant of the action and the list of canonical forms which are given in the following propositions.

Proposition F.2.4 *The only independent differential invariant of the action of G on $RCK^2(\mathbf{E}^3)$ is*

$$F = \frac{I^3}{J^2},$$

where the functions

$$I = 12A_{33}M_{33} - 3L_3D_3 + H^2,$$
$$J = 72A_{33}M_{33}H - 27A_{33}L_3^2 - 27D_3^2M_{33} + 9D_3L_3H - 2H^3,$$

are relative invariants of the action of G and independent differential invariants for the action of the subgroup of G defined by $a_3 = 1$.

Proof: The functions I and J are the fundamental invariants (of weight 4 and 6 respectively) of the binary quartic form (F.2.17) [24],[41]. $\square$

Proposition F.2.5 *Each CKT of* $RCK^2(\mathbf{E}^3)$ *is equivalent under the action of* G *to one of the following representatives:*

$$
\begin{array}{rcll}
I. & \mathbf{I}_3 \odot \mathbf{I}_3 + \mu \mathbf{D} \odot \mathbf{D} + \mathbf{X}_3 \odot \mathbf{X}_3, & \mu \in \mathbf{R}, & \text{(F.2.18)} \\
II. & \mathbf{I}_3 \odot \mathbf{I}_3 + \mu \mathbf{D} \odot \mathbf{D} - \mathbf{X}_3 \odot \mathbf{X}_3, & \mu \in \mathbf{R}, & \text{(F.2.19)} \\
III. & \mathbf{I}_3 \odot \mathbf{I}_3 + \nu \mathbf{D} \odot \mathbf{D}, & \nu = \pm 1, & \text{(F.2.20)} \\
IV. & \mathbf{D} \odot \mathbf{I}_3, & & \text{(F.2.21)} \\
V. & \mathbf{I}_3 \odot \mathbf{I}_3. & & \text{(F.2.22)}
\end{array}
$$

Proof: Starting from the list of canonical forms of real binary quartics (given for instance in [24]), we combine those differing only by sign. We remark that for $\mu = 2$ the canonical form $I.$ is equivalent to $\mathbf{D} \odot \mathbf{D}$. $\square$

Remark F.2.6 The action of G over $RCK^2(\mathbf{E}^3)$ has infinitely many orbits. However, the tensors in (F.2.18) and (F.2.19) are not pairwise inequivalent for all values of μ: for $\mu \neq \pm 2$ there exists a finite number of μ' such that the corresponding tensors are pairwise equivalent (see [24]).

F.3 Invariant classification of the R-separable rotationally symmetric webs

The polynomial P defined in (F.2.10) as the building block of the action equations (F.2.12–F.2.16) is deeply related to the polynomial q (F.0.5). Indeed, we have $P(X) = X^4 q(-1/X)$. Moreover q is the inhomogeneous polynomial corresponding to the quartic binary form Q (F.2.17).

The roots of q are the points on the z-axis where all the eigenvalues of $\mathbf{K}$ coincide (see Remark F.0.9). The conformal transformations ϕ_0, ϕ_1, ϕ_2 and I described in Sect. F.2.1 map the z-axis to itself with a one to one correspondence (if we include also the point at infinity). Thus two distinct points cannot be made coincident or removed. This provides the geometric interpretation of the fact that the invariants of Q are invariants of the CKT defining the web. The meaning of q in terms of invariant theory is made more precise in the following proposition.

Proposition F.3.1 *The polynomial* $q(z) = M_{33}z^4 + L_3 z^3 + Hz^2 + D_3 z + A_{33}$ *is a relative covariant of the induced extended action on* $C\hat{K}^2(\mathbf{E}^3) \times \mathbf{E}^3$ *restricted on the invariant subset* $S_0 = \{x = y = 0\}$.

Proof: The equations of the extended action are (F.2.12–F.2.16) together with

$$\begin{aligned}\tilde{x} &= \frac{a_2 x}{(a_0 z+1)^2 + a_0^2(x^2+y^2)},\\ \tilde{y} &= \frac{a_2 y}{(a_0 z+1)^2 + a_0^2(x^2+y^2)},\\ \tilde{z} &= a_2 \frac{z + a_0^2(x^2+y^2+z^2)}{(a_0 z+1)^2 + a_0^2(x^2+y^2)} + a_1.\end{aligned}$$

The subset $S_0 = \{x = y = 0\}$ is an invariant subset of the extended action. Moreover, on S_0 the transformation law for z reduces to the linear fractional transformation

$$\tilde{z} = \frac{(a_2 + a_1 a_0)z + a_1}{a_0 z + 1} \qquad (a_2 \neq 0), \tag{F.3.23}$$

which is the general linear transformation on $\mathbf{RP}^1$ (see [41]). Let $\tilde{q}(\tilde{z})$ be the polynomial we obtain by inserting (F.2.12–F.2.16) and (F.3.23) in (F.0.5). We obtain

$$(a_0 z + 1)^4 \tilde{q}(\tilde{z}) = a_3 a_2^2 q(z), \tag{F.3.24}$$

that is (up to a_3) a covariant of weight two of the action. For the discrete inversion (mapping z into $\tilde{z} = 1/z$), we immediately see that it maps $q(z)$ to $\tilde{q}(\tilde{z}) = \frac{q(z)}{z^4}$ □

Equation (F.3.24) shows that the number and multiplicity of the real roots of $q(z)$ (that is the number and multiplicity of the real linear factors of Q) are invariant with respect to the group action. Hence they can be used to define and classify the different types of webs.

Definition F.3.2 *We say that two rotationally symmetric R-separable webs are of the same type if the polynomials associated with the corresponding characteristic CKT have the same number and multiplicity of real roots.*

Thus we have reduced the classification of rotational R-separable webs to the classical classification of real binary quartics (see [41], [24]).

We have nine types of webs, listed in Table 2.

The remaining coordinates systems of Table 1 are equivalent to one of the coordinates listed above (correcting a typographical error in [38], where Cap cyclide coordinates are said to be equivalent to Bi-cyclide coordinates), as it is described in Table 3.

Remark F.3.3 The number of the types of rotationally R-separable coordinate systems agree with the results of [6], where the subject is examined from the point of view of symmetry operators. The coefficients A^{ij} of the second order part of the symmetry operators S characterizing each type of R-separable rotationally symmetric coordinates, with respect to Cartesian coordinates, listed in Table 2. of Boyer, et al. [6] when written as

$$S = A^{ij}\partial_i\partial_j + B^i\partial_i$$

correspond to the components of CKTs equivalent to those listed in Table 1 for Bi-cyclide, Flat-ring cyclide, Disk cyclide and Toroidal coordinates, respectively.

Associated web	roots of q	canonical form of $\mathbf{K}$
Bi-cyclide	4 distinct real roots	$I.$ for $\mu < -2$
Flat-ring cyclide	4 distinct complex conjugate roots	$I.$ for $\mu > -2$, $\mu \neq 2$
Disk cyclide	4 distinct roots, 2 real, 2 complex conjugate	$II.$
Inverse prolate spheroidal	1 double real root, 2 distinct real roots	$III.$ for $\nu = -1$
Inverse oblate spheroidal	1 double real root, 2 distinct complex conjugate roots	$III.$ for $\nu = 1$
Toroidal	2 double complex conjugate roots	$I.$ for $\mu = 2$
Bispherical	2 double real roots	$I.$ for $\mu = -2$
Cardioid	1 triple (real) root 1 simple real root	$IV.$
Tangent sphere	1 quartuple (real) root	$V.$

Table F.2: the nine types of inequivalent rotational R-separable webs

Finally, we provide algebraic conditions on the parameters (F.2.11) in order to determine the type of the corresponding web. In order to obtain these conditions, we solve the equivalent problem of determining the number and multiplicity of the linear factors of the corresponding binary quartic form Q which can be done by applying the classical algorithm (see for example [24]) based on the sign and vanishing of relative invariants and covariants of Q.

Together with I and J, the following invariant and covariants are used in the classification scheme: the discriminant of the form (a relative invariant which vanishes if and only if the quartic has a multiple root)

$$\Delta = I^3 - 27J^2,$$

the Hessian of the form (a covariant which vanishes if and only if the quartic has a quadruple root)

$$H(X,Y) = (\partial^2_{XX}Q) \cdot (\partial^2_{YY}Q) - (\partial^2_{XY}Q)^2;$$

the covariants

$$L(X,Y) = IH(X,Y) - 6JQ(X,Y),$$

and

$$M(X,Y) = 12H^2(X,Y) - IQ^2(X,Y).$$

We summarize the classification in Table 4.

Web	equivalent to	transformation
Cap cyclide	Flat-ring cyclide	cont. inversion + trans.
Prolate Spheroidal	Inverse Prolate Spheroidal	discrete inversion
Oblate Spheroidal	Inverse Oblate Spheroidal	discrete inversion
Spherical	Bispherical	cont. inversion + trans.
Parabolical	Cardioid	discrete inversion
Circular Cylindrical	Tangent sphere	discrete inversion

Table F.3: Pairwise conformally equivalent webs

Web	Algebraic condition
Disk cyclide	$\Delta < 0$
Bi-cyclide	$\Delta > 0$ and $H(X,Y) < 0$, and $M(X,Y) > 0$
Flat-ring cyclide	$\Delta > 0$ and $(H(X,Y) > 0$ or $M(X,Y) > 0)$
Inverse prolate spheroidal	$\Delta = 0$ and $L(X,Y) < 0$
Inverse oblate spheroidal	$\Delta = 0$ and $L(X,Y) > 0$
Toroidal	$L(X,Y) = 0$ and $H(X,Y) > 0$
Bispherical	$L(X,Y) = 0$ and $H(X,Y) < 0$
Cardioid	$I = J = 0$ and $H(X,Y) \neq 0$
Tangent sphere	$H(X,Y) = 0$

Table F.4: Invariant classification of the webs

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