

Bandlimited functions, curved manifolds, and self-adjoint extensions of symmetric operators

by

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Abstract

Sampling theory is an active field of research that spans a variety of disciplines from communication engineering to pure mathematics. Sampling theory provides the crucial connection between continuous and discrete representations of information that enables one to store continuous signals as discrete, digital data with minimal error. It is this connection that allows communication engineers to realize many of our modern digital technologies including cell phones and compact disc players.

This thesis focuses on certain non-Fourier generalizations of sampling theory and their applications. In particular, non-Fourier analogues of bandlimited functions and extensions of sampling theory to functions on curved manifolds are studied. New results in bandlimited function theory, sampling theory on curved manifolds, and the theory of self-adjoint extensions of symmetric operators are presented. Besides being of mathematical interest in itself, the research contained in this thesis has applications to quantum physics on curved space and could potentially lead to more efficient information storage methods in communication engineering.

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Introduction

Sampling theory can be described as the study of function spaces whose elements have certain special reconstruction and interpolation properties. The classic example of such a function space is the Paley-Wiener space $B(\Omega)$ of Ω -bandlimited functions. The space $B(\Omega)$ is that subspace of $L^2(\mathbb{R})$ which is the image of $L^2[-\Omega, \Omega]$ under the Fourier transform. The finite number $\Omega > 0$ is called the bandlimit. Given any $\phi \in B(\Omega)$, if we identify ϕ with a certain special member of its equivalence class (which happens to be the restriction of an entire function to the real line), then given any equidistantly spaced sequence of points $(x_n)_{n \in \mathbb{Z}}$ with spacing $x_{n+1} - x_n = \frac{\pi}{\Omega}$, the following reconstruction formula holds:

$$(1.0.1) \quad \phi(x) = \sum_{n \in \mathbb{Z}} \phi(x_n) \frac{\sin(\Omega(x - x_n))}{\Omega(x - x_n)}.$$

This shows that ϕ is completely determined and perfectly reconstructible at any $x \in \mathbb{R}$ from the values it takes on certain discrete sets of points which have no finite accumulation point. The values $\{\phi(x_n)\}_{n \in \mathbb{Z}}$ are called the ‘samples’ of ϕ taken on the points x_n , and the function $G(x, y) := \frac{\sin(\Omega(x-y))}{\Omega(x-y)}$ is called the sampling or reconstruction kernel. The above remarkable reconstruction formula (1.0.1) is called the Shannon sampling formula, as it was famously applied by C.E. Shannon in his theory of communication to provide an important link between discrete and continuous representations of information [65].

Sampling theory has found many practical applications in a wide variety of fields including pure mathematics, communication engineering, signal processing, computer graphics, medical imaging, and more recently, mathematical physics. A significant portion of the research contained in this thesis is motivated by two particular applications of sampling theory. First, sampling theory is used extensively throughout communication engineering to provide a method for storing and reconstructing continuous signals (*e.g.* a music recording, or a voice on a cellphone) from discrete values; and secondly, it has been recently observed that sampling theory, suitably generalized to curved manifolds, could provide a description of space-time that better suits the needs of both quantum field theory and general relativity. This motivation will be discussed in greater detail in Section 1.1.

In sampling theory, one usually wishes to consider spaces of functions whose values at points are well-defined. Hence, it is natural that much of sampling theory uses the framework of reproducing kernel Hilbert spaces. Recall that a Hilbert space of functions on a set $X \subset \mathbb{C}$ is called a reproducing kernel Hilbert space (RKHS) if point evaluation at any $x \in X$ is a bounded linear functional.¹ Given such a space \mathcal{H} , the Riesz representation theorem immediately implies that for each $x \in X$ there is a point evaluation vector, δ_x , such that $\langle \phi, \delta_x \rangle = \phi(x)$ for any $\phi \in \mathcal{H}$. The function $K(x, y) := \langle \delta_x, \delta_y \rangle$ is called the reproducing kernel for the RKHS \mathcal{H} . The kernel function will be called positive definite if $K(x, x) > 0$ for all $x \in X$. If a RKHS of functions on X has a positive definite reproducing kernel, it is clear that all the point evaluation vectors δ_x are non-zero for $x \in X$. Here, and throughout this thesis, if \mathcal{H} is a Hilbert space, $\langle \cdot, \cdot \rangle$ will denote the inner product on \mathcal{H} . We will take the convention that the inner product is linear in its first argument and conjugate linear in the second argument.

¹More generally, it could be that X is a Hausdorff topological space.

As we will see, $B(\Omega)$ is a RKHS with point evaluation vectors $\delta_t(x) = \frac{1}{2\Omega} \frac{\sin(\Omega(x-t))}{\Omega(x-t)}$ where $\|\delta_t\|^2 = \frac{1}{2\Omega}$ for each $t \in \mathbb{R}$. It follows that the Shannon sampling formula (1.0.1) can be rewritten:

$$(1.0.2) \quad f(x) = \langle f, \delta_x \rangle = \sum_{n \in \mathbb{Z}} \langle f, \delta_{x_n} \rangle \langle \delta_{x_n}, \delta_x \rangle \frac{1}{\|\delta_{x_n}\|^2}.$$

More generally, if \mathcal{H} is any RKHS on a set $X \subset \mathbb{C}$, and if there exists a countable total orthogonal subset $\{\delta_{x_n}\}_{n \in \mathbb{Z}}$ of point evaluation vectors, then \mathcal{H} will obey a reconstruction formula of this type. Explicitly, in this case, $\{\frac{\delta_{x_n}}{\|\delta_{x_n}\|}\}_{n \in \mathbb{Z}}$ is an orthonormal basis so that $I_{\mathcal{H}} = \sum_{n \in \mathbb{Z}} \langle \cdot, \delta_{x_n} \rangle \langle \delta_{x_n}, \cdot \rangle \frac{1}{\|\delta_{x_n}\|^2}$. Substituting this expression into $f(x) = \langle f, \delta_x \rangle$ yields a sampling formula of the same form as equation (1.0.2). The fact that any RKHS which has a total orthogonal set of point evaluation vectors obeys such a reconstruction formula is called Kramer's abstract sampling theorem, and was first observed in [38]. We will refer to this property as the Kramer sampling property, or to be concise, the **sampling property**. This thesis focuses on the study of reproducing kernel Hilbert spaces with the sampling property.

1.1. Motivation

As mentioned above, much of the work in this thesis is motivated by two particular applications of sampling theory. This section provides further details on the applications I have in mind.

1.1.0.1. *Efficient storage and reconstruction of a continuous signal from discrete information.* A ubiquitous problem in communication engineering is that of storing a continuous signal as discrete, digital information and then reconstructing it with minimal error. An example of this problem is that of storing a continuous music signal as discrete bits, and then reconstructing it from this discrete data. Communication engineers typically take advantage of the fact that most of the continuous signals they wish to discretize, including music signals, are well approximated by a special class of functions which can be reconstructed perfectly from discrete values. Since the sensitivity of the human ear to frequencies greater than 20kHz is very low, any music signal can be filtered to remove frequencies greater in magnitude than this value with negligible loss in sound quality. It follows that the post-filtered signal is Ω -bandlimited where the bandlimit is $\Omega = 20\text{kHz}$. By the Shannon sampling formula, (1.0.1), any Ω -bandlimited signal is completely determined by, and can be reconstructed perfectly from, the values it takes on the set of points in time $\{\frac{n\pi}{\Omega}\}$, where n runs through the integers [65]. It follows that if one measures and records the amplitudes of the music signal once every $\frac{20000}{\pi}$ times a second, then the resulting discrete sequence of numbers can be used to reconstruct the original signal in a stable fashion.

Most of the work that appears in Part III aims to develop non-Fourier generalizations of bandlimited function theory which provide a more efficient way of storing and reconstructing signals obeying what could intuitively be described as a 'time-varying bandlimit'. Consider a music signal which varies or oscillates rapidly for a short time period, and is slowly-varying outside of that time interval. The highest frequency components in the Fourier transform of this signal may be quite high since large frequencies are needed to resolve the sharp features that occur in the time period of high activity. Since a significant amount of the energy, or L^2 norm, of the signal is concentrated at high frequencies, it follows that if one wishes to approximate the signal by a bandlimited function, one will need to choose a large bandlimit to obtain a good approximation. Hence, in spite of the fact that the signal is slowly-varying on average, the bandlimit, and hence the rate at which the signal must be sampled to ensure stable reconstruction, is quite large. Intuitively, this is a very inefficient approach to converting this signal into discrete information. One would expect that such a large number of samples in the time intervals where the signal is slowly varying should not be needed to stably reconstruct it.

Such a music signal could be said to obey a 'time-varying bandlimit' since in certain short time intervals it contains a lot of rapid oscillations or high 'frequencies', while on other time

intervals it is slowly-varying and consists of only low ‘frequencies’. This problem of constructing more efficient methods for the sampling and reconstruction of functions obeying a time-varying bandlimit motivates the search for reproducing kernel Hilbert spaces with the sampling property which are better suited than $B(\Omega)$ for approximating or modelling such functions.

1.1.0.2. *Quantum physics on curved manifolds.* Although much is known about the sampling theory of bandlimited functions on \mathbb{R}^n , the study of the sampling and reconstruction properties of bandlimited functions on manifolds is a very new field [55].

A fully developed sampling theory on curved manifolds would be of particular interest for quantum gravity. This follows from the idea that sampling theory could provide a crucial link between quantum theory and general relativity [35]. Quantum field theory is well-defined on discrete space, and is, in general, ill-defined space if is continuous. Conversely, general relativity requires space-time to be a continuous, smooth manifold, *i.e.*, not discrete. Sampling theory could provide a description of space-time which is effectively both discrete and continuous and hence a framework in which the needs of both theories are satisfied.

The debate as to whether space-time is fundamentally discrete or continuous is still unresolved. On one hand, general relativity is formulated on a smooth differentiable manifold, and if space-time were just a discrete set of points with a finite spacing, this would seem to violate symmetries of the manifold, *e.g.*, Lorentz invariance in flat space-time. For reasons including these, the idea of space-time discreteness is not mathematically or physically attractive.

Quantum field theory, on the other hand, is not really well-defined without an ultra-violet cutoff. Forces between physical charges diverge as the separation between those charges vanishes. In quantum field theory, these divergencies spoil the calculations of many physically relevant quantities. These divergencies are the infamous ‘ultra-violet divergencies’ of quantum field theory, and most of these divergencies disappear if one assumes that space-time is discrete. Furthermore, naïvely combining Heisenberg’s uncertainty relation with general relativity suggests that there should exist a smallest observable length in nature. The argument proceeds as follows. The observation of increasingly smaller volumes or lengths requires one to fire particles with increasingly smaller wavelengths at the region of space one wishes to resolve. Such particles will have increasingly smaller position uncertainties, and hence, by Heisenberg’s uncertainty principle, increasingly larger momentum and energy uncertainties. Einstein’s field equations show that energy curves space and time so that an increasingly large energy uncertainty creates an increasingly large uncertainty in curvature which contributes to the uncertainty in position. This suggests that the attempt to measure increasingly smaller volumes is self-limiting and that there is a smallest volume that one is able to observe. This heuristic argument is known as Heisenberg’s microscope, and has led some physicists to conjecture that space-time is fundamentally discrete.

A possible resolution to this debate that would satisfy both general relativity and quantum field theory is that this expected smallest length in nature could turn out to be a smallest wavelength [35]. That is, it could be that space-time is fundamentally a smooth manifold, and yet at the same time effectively discrete for the physical fields which describe particles in quantum theory, if these physical fields are bandlimited functions. In this case, any physical theory could be described as living on the smooth manifold, or, equivalently, written in terms of the discrete values that the fields take on certain sufficiently dense discrete sets of points. This is an attractive possibility, as this framework could satisfy the needs of both quantum field theory and general relativity. Observe, however, that the physical fields of nature, if they are bandlimited, can not be bandlimited in the usual sense. First, general relativity asserts that space-time is a manifold with curvature. Furthermore, since the mathematical representation of physical laws is necessarily co-ordinate system independent, the physical fields cannot just be band-limited in the usual Fourier sense in a fixed given co-ordinate system for the manifold. Instead, they must be band-limited in a co-ordinate system independent sense. It follows that if one wishes to apply sampling theory to quantum field theory and general relativity, it will be necessary to first develop a co-ordinate system independent notion of a bandlimit and sampling theory for a general manifold.

1.2. Outline

This thesis is split into three main parts. Part I, which consists of Chapters 2-3, deals with the classical sampling theory of bandlimited functions. In Chapter 2, a brief introduction to some of the key aspects of sampling theory for the Paley-Wiener spaces of bandlimited functions is provided. This includes the Beurling-Landau density theorems for sets of sampling and interpolation, as well as the celebrated Paley-Wiener theorem which identifies $B(\Omega)$ with entire functions of exponential type at most Ω which belong to $L^2(\mathbb{R})$. Chapter 3 contains many of the main results of Part I. Namely, in this chapter, I show that any Ω -bandlimited function can be seen as the limit of sequences of Ω -bandlimited trigonometric polynomials in a variety of topologies. In particular, this limit holds in L^2 of any line parallel to \mathbb{R} , in the topology of uniform convergence on compacta, and I further prove convergence of the sample values of the trigonometric polynomials taken on certain discrete sets of points to the sample values of the bandlimited function with respect to a l^2 norm.

In part II, which consists of Chapters 4-8, the recent generalization of $B(\Omega)$ to Riemannian and pseudo-Riemannian manifolds is studied. Chapter 4 introduces the theory of self-adjoint extensions of unbounded symmetric operators, a theory which will be employed extensively throughout Parts II and III. Chapters 5, 6 and 8 generalize the results of Part I to apply to bandlimited functions on manifolds. In particular, in Chapter 5, a much shorter and elegant proof of a result which generalizes one of the main results of Chapter 3 is established using simple operator theoretic techniques. Chapters 6 and 8 study sampling theory on pseudo-Riemannian manifolds (manifolds with an indefinite metric), which is of particular interest to quantum theory on curved space and quantum gravity. It is found that sampling theory on a pseudo-Riemannian manifold has some new features that are not present in sampling theory on Riemannian manifolds. These new features become apparent even in flat Minkowski space-time which has no curvature. To deal with more general space-times, Chapter 7 introduces the theory of symmetric Sturm-Liouville differential operators. Also in this chapter, I prove several simple, but useful results on the spectra of symmetric operators. In particular, I study how the spectra of self-adjoint extensions of a symmetric operator depend on the minimum uncertainty of the symmetric operator (this is joint work with Prof. Achim Kempf), and I show how the essential spectra of symmetric operators behaves under strong graph convergence. Here, strong graph convergence is a generalized notion of strong convergence for closed unbounded operators. Furthermore, I apply these results to the theory of symmetric Sturm-Liouville differential operators of the form $D_n\phi := -(p_n\phi)' + q_n\phi$ in $L^2(a_n, b_n)$, where p_n, q_n are suitable measurable functions, to show how the essential spectrum behaves as $p_n, q_n \rightarrow p, q$ and $a_n, b_n \rightarrow a, b$ in suitable topologies. Chapter 8 then applies the mathematical tools of the preceding chapters to the study of bandlimited functions on de Sitter space-time, a physically relevant space-time which models a universe expanding at an exponential rate.

Part III focuses on the search for concrete realizations of Kramer's abstract sampling theorem, *i.e.*, for reproducing kernel Hilbert spaces with the sampling property. In this part, we discuss the relationship between M.G. Krein's theory of entire operators, de Branges spaces, and Kramer's sampling formula. Chapters 9 -10 discuss the fascinating result that if S is a simple, regular, closed symmetric operator with deficiency indices $(1, 1)$, then there exists a unitary transformation under which S becomes a symmetric multiplication operator acting on a reproducing kernel space of meromorphic functions \mathcal{H} which has the sampling property and is a subspace of $L^2(\mathbb{R}; d\mu)$ for some measure μ .

Chapters 11-12 describe my search for subspaces of $L^2(\mathbb{R}, d\mu)$ which are reproducing kernel Hilbert spaces with the sampling property. Earlier, in Section 4.4, I show that if a reproducing kernel Hilbert space \mathcal{H} of functions on \mathbb{R} with positive definite reproducing kernel is such that the multiplication operator M has a symmetric restriction to a dense domain in \mathcal{H} which is simple, regular and has deficiency indices $(1, 1)$, then \mathcal{H} has the sampling property. With this motivation, Chapter 12 describes a first attempt at determining when a self-adjoint operator A has a densely

defined restriction to a given subspace S . In particular, I prove the sufficient (but restrictive) condition that if P projects onto S , and $U(t) := e^{itA}$, $t \in \mathbb{R}$, is the one-parameter unitary group generated by A , that A has a densely defined symmetric restriction to S provided $V(t) := PU(t)P$ is a semi-group of partial isometries. Chapter 11 seeks to show that the invariant subspaces of certain Sturm-Liouville differential operators are RKHS with the sampling property by proving that they are de Branges spaces. This idea is motivated by the fact that the space $B(\Omega)$ of bandlimited functions is both the invariant subspace of the second derivative operator on $L^2(\mathbb{R})$ and a de Branges space. This approach appears promising, but this is at the edge of my current research, so that my studies here are not complete. Finally, the second last chapter, Chapter 13 studies the compact, convex set of generalized resolvents of a symmetric operator. There is a bijective correspondence between the set of all positive operator valued measures (POVMs) which diagonalize a symmetric operator and this set of generalized resolvents. In this chapter I prove that the set of all POVMs corresponding to a single symmetric operator S is a closed face in the compact, convex set of all contractive POVMs, and that if S has finite deficiency indices, that the set of generalized resolvents is compact with respect to a certain stronger topology than is usually considered.

1.2.0.3. *Remark.* Although it should be pretty clear from the context, I have used a * or a ', respectively, to indicate whether a Theorem, Corollary, etc. is a something new that I have proven or something for which I have provided a new proof. For example, where it is written *Theorem, this indicates that this is a new theorem of my own while 'Proposition would indicate a known proposition for which I have provided a new proof.

Part 1

Bandlimited functions

The classical Paley-Wiener spaces of bandlimited functions

There are two main questions that sampling theory attempts to address. Both are of significant theoretical and practical interest. The first is a question regarding sampling and reconstruction: Given a function space V , and a discrete set of points $\Lambda := \{y_n\}$, what properties must Λ have in order that any $f \in V$ be perfectly reconstructible from the values it takes at the points of Λ ? The second question is about interpolation: Given a fixed function space V , a fixed class of sequences (e.g. $l^2(\mathbb{Z})$), and a discrete set of points Λ , what properties must Λ possess in order that given any sequence of the class there is an $f \in V$ that takes the values of that sequence on the points of Λ ? We will be primarily concerned with the first question in the case where $V = B(\Omega)$.

2.1. Frequency-limited functions and entire functions of exponential type

2.1.1. Entire functions of exponential type. Given an entire function f , *i.e.* a function which is analytic on all of \mathbb{C} , consider its maximum modulus function:

$$(2.1.1) \quad M(r) := \max\{|f(z)| : |z| = r\}$$

Unless f is a constant, the maximum modulus principle implies that M must be a strictly increasing function of r . Furthermore, for any non-constant f , $\lim_{r \rightarrow \infty} M(r) = \infty$ by Liouville's theorem.

2.1.1.1. *Definition.* An entire function f is said to be of exponential type if there exist positive constants, A and B for which

$$|f(z)| \leq Ae^{B|z|}$$

for all z .

The exponential type of an entire function is the infimum of all positive values B for which there is an $A < \infty$ such that $|f(z)| \leq Ae^{B|z|} \forall z \in \mathbb{C}$. If $B < \infty$, then f is said to be an entire function of exponential type B . For example, $\sin(Bz)$ is an entire function of exponential type B .

2.1.2. Frequency-limited functions. Let $S \subset \mathbb{R}^n$ be a compact set. The Hilbert space $B(S)$ which is the image of $L^2(S)$ under the Fourier transform will be called the Hilbert space of functions frequency limited by S . The inner product on $B(S)$ is the usual L^2 inner product. In the case where $S = [-\Omega, \Omega] \subset \mathbb{R}$ is a single interval $B(S) = B(\Omega)$ is the Hilbert space of Ω -bandlimited functions. In general, the volume, or Lebesgue measure $\mu(S)$ of S will be called the bandwidth volume of the space $B(S)$.

In this chapter, the Fourier transform $f \in L^2(\mathbb{R})$ of $F \in L^2(\mathbb{R})$ will be defined as

$$(2.1.2) \quad f(x) = \int_{-\infty}^{\infty} F(w)e^{-iwx} dw \quad x \in \mathbb{R} \quad \text{a.e.}$$

With this asymmetric definition, the Fourier transform is not an isometry. If $f \in L^2(\mathbb{R})$ is the transform of $F \in L^2(\mathbb{R})$, then $\|f\| = \sqrt{2\pi}\|F\|$.

2.1.2.1. *Remark.* Technically, each $f \in B(S)$ is an equivalence class of functions defined up to a set of Lebesgue measure zero. However, if $S \subset \mathbb{R}$ is compact, it is not hard to see from formula (2.1.2) that given $f \in B(S)$, there is a unique function in the equivalence class of f which is the restriction of an entire function to the real line. Indeed, if S is compact and one defines

$$(2.1.3) \quad f(z) = \int_S F(w)e^{-iwz} dw,$$

for all $z \in \mathbb{C}$, where $F \in L^2(S)$, then a simple application of the theorems of Fubini and Morera show that f is an entire function. For the remainder of this thesis, if $f \in B(S)$, and F is its Fourier transform, we will identify f with that member of its equivalence class which is the entire function defined by equation (2.1.3).

The space $B(\Omega)$ is closed under differentiation.

$$(2.1.4) \quad \|f^{(n)}\|^2 = \int_{-\Omega}^{\Omega} w^{2n} |F(w)|^2 dw \leq \frac{\Omega^{2n}}{2\pi} \|f\|^2.$$

This is known as Bernstein's inequality and it shows that $f^{(n)} \in B(\Omega)$ for any $n \in \mathbb{N}$. Thus the derivative operator is bounded on the Hilbert space $B(\Omega)$.

The following bound on the derivative of a function $f \in B(\Omega)$ at a point also holds:

$$(2.1.5) \quad |f^{(n)}(z)|^2 = \left| \int_{-\Omega}^{\Omega} w^n F(w) e^{-iwz} dw \right|^2 \leq \|F\|^2 \int_{-\Omega}^{\Omega} w^{2n} e^{2y|w|} dw \leq \|f\|^2 \frac{\Omega}{\pi} \Omega^{2n} e^{2\Omega|y|},$$

where $y = \text{Im}(z)$. Here, $f^{(n)}$ denotes the n^{th} derivative of f . Thus,

$$(2.1.6) \quad |f^{(n)}(z)| \leq \Omega^n \sqrt{\frac{\Omega}{\pi}} e^{\Omega|y|} \|f\|$$

The case $n = 1$ shows that point evaluation at any $z \in \mathbb{C}$ is a bounded linear functional on $B(\Omega)$, and hence that $B(\Omega)$ is a RKHS. Furthermore, the case $n = 1$, shows that every function bandlimited by Ω is of exponential type at most Ω . In fact, every entire function of exponential type at most Ω that is square integrable on the real axis belongs to $B(\Omega)$.

THEOREM 2.1.1. (Paley-Wiener) *Let f be an entire function such that $\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty$ and $|f(z)| \leq Ae^{\Omega|z|}$, for all $z \in \mathbb{C}$. Then f is the Fourier transform of a function $F \in L^2[-\Omega, \Omega]$.*

The non-trivial half of this theorem can be proven by considering an entire, square integrable function f of exponential type at most Ω , and computing $\int_{-B}^B f(x)e^{ix\Lambda} dx$ using a rectangular contour in \mathbb{C} . If $|\Lambda| > \Omega$, it can be shown that this integral vanishes as $B \rightarrow \infty$.

2.2. Sampling of bandlimited functions

Let $\Lambda := \{y_n\}_{n=-\infty}^{\infty}$ be a discrete set of points. Λ is called a uniformly discrete set if there exists an $\epsilon > 0$ for which $|y_n - y_m| > \epsilon$ for all $n, m \in \mathbb{Z}$. An interesting property of $B(\Omega)$ is that if $\Lambda = (\lambda_n)_{n \in \mathbb{Z}}$ is a uniformly discrete sequence, then for any $f \in B(\Omega)$, the sequence of samples of f on the points of Λ , $(f(\lambda_n))_{n \in \mathbb{Z}}$, is a square summable sequence.

Let $B^p(\Omega)$ be the space of entire functions of exponential type at most Ω whose restrictions to \mathbb{R} belong to $L^p(\mathbb{R})$.

THEOREM 2.2.1. (Plancherel-Pólya) *Let $\Lambda = \{y_n\}_{n=-\infty}^{\infty}$, $|y_n - y_m| \geq \epsilon > 0$ for all $n, m \in \mathbb{Z}$, be a uniformly discrete set of real numbers. There exists a constant $K = K(p, \epsilon, \Omega)$ such that*

$$\sum_n |f(y_n)|^p \leq K \int_{-\infty}^{\infty} |f(x)|^p dx$$

for all $f \in B^p(\Omega)$.

In particular, if $\Lambda = \{y_n\}$ is any uniformly discrete set, then there is an $M > 0$ such that $\sum_n |f(y_n)|^2 \leq M \|f\|^2$ for every $f \in B(\Omega)$. Intuitively this result makes sense. If it were not true, then there would be a sequence of bandlimited functions $\{f_m\}_{m=1}^\infty$ for which the square sum of the samples $\{f_m(y_n)\}$ becomes arbitrarily greater than the norm squared of f_m as $m \rightarrow \infty$. This would mean that the sample values of f_m would have to be arbitrarily large in magnitude in comparison to the average magnitude of $f_m(x)$. This would then imply that the functions f_m become arbitrarily peaked about their sample values as $m \rightarrow \infty$. Clearly the Fourier transforms of the members of such a sequence would contain increasingly high frequencies, contradicting the assumption that every member of the sequence is bandlimited by a fixed, finite value.

2.2.1. Sampling and interpolation of bandlimited functions. As discussed in Chapter 1, bandlimited functions have very special reconstruction properties.

Given $\Omega > 0$, define $e_y \in L^2[-\Omega, \Omega]$ by $e_y(w) := e^{iyw}$ a.e. It is a known fact about Fourier series that the set $\{\frac{1}{\sqrt{2\Omega}} e_{x_n}\}_{n \in \mathbb{Z}}$ is an orthonormal basis for $L^2[-\Omega, \Omega]$ whenever $\Lambda = \{x_n\}_{n \in \mathbb{Z}} \subset \mathbb{R}$ is any equidistantly spaced set of points with spacing $x_n - x_{n-1} = \frac{\pi}{\Omega}$. It is further straightforward to calculate that if the Fourier transform of $F \in L^2(\mathbb{R})$ is defined by equation (2.1.2), that the Fourier transform $\tilde{\delta}_z \in B(\Omega)$ of e_z is given by

$$(2.2.1) \quad \tilde{\delta}_z(w) = 2\Omega \frac{\sin(\Omega(z-w))}{\Omega(z-w)}$$

for all $w, z \in \mathbb{C}$. Further observe that given any $f \in B(\Omega)$ and $z \in \mathbb{C}$ that

$$(2.2.2) \quad \frac{1}{2\pi} \langle f, \tilde{\delta}_z \rangle = \langle F, e_z \rangle = \int_{-\Omega}^{\Omega} F(w) e^{-iwz} dw = f(z).$$

This shows that the vector $\delta_z = \frac{\tilde{\delta}_z}{2\pi}$ is the point evaluation vector of the RKHS $B(\Omega)$ at the point z . Using these facts, it is straightforward to establish Shannon's sampling formula, equation (1.0.1).

CLAIM 2.2.2. *Suppose that $f \in B(\Omega)$ and that $\Lambda = (x_n)_{n \in \mathbb{Z}}$ is any equidistantly spaced set of points with spacing $x_n - x_{n-1} = \frac{\pi}{\Omega}$. Then if*

$$(2.2.3) \quad f_N(z) = \sum_{n=-N}^N f(x_n) \frac{\sin(\Omega(z-x_n))}{\Omega(z-x_n)},$$

then f_N converges to f both in norm, and uniformly on compacta as $N \rightarrow \infty$.

PROOF. Since the vectors $\{e_{x_n}\}_{n \in \mathbb{Z}}$ are a total orthogonal set in $L^2[-\Omega, \Omega]$, it follows that the point evaluation vectors $\{\delta_{x_n}\}_{n \in \mathbb{Z}}$ form a total orthogonal set in $B(\Omega)$. It is straightforward to check that $\|\delta_{x_n}\|^2 = \frac{\Omega}{\pi}$, so that

$$(2.2.4) \quad f = \sum_{n \in \mathbb{Z}} \langle f, \delta_{x_n} \rangle \delta_{x_n} \frac{1}{\|\delta_{x_n}\|^2},$$

while,

$$(2.2.5) \quad f_N = \sum_{n=-N}^N \langle f, \delta_{x_n} \rangle \delta_{x_n} \frac{1}{\|\delta_{x_n}\|^2},$$

where δ_{x_n} is defined by equation (2.2.1). Clearly, $f_N \in B(\Omega)$ for each $n \in \mathbb{N}$, and since $\{\frac{\delta_{x_n}}{\|\delta_{x_n}\|}\}_{n \in \mathbb{Z}}$ is an orthonormal basis of $B(\Omega)$, f_N converges to f in norm.

Furthermore, for any $z \in \mathbb{C}$ and $f \in B(\Omega)$,

$$(2.2.6) \quad \begin{aligned} |f(z) - f_N(z)| &= |\langle f - f_N, \delta_z \rangle| \\ &\leq \|f - f_N\| C e^{\Omega|z|}. \end{aligned}$$

Since $f_N \rightarrow f$ in norm, it follows that the above vanishes uniformly on compacta in the limit as $N \rightarrow \infty$. \square

A natural question that may have already occurred to the reader is the following. Are there other uniformly discrete discrete sets of points $\Lambda := \{\lambda_n\}_{n \in \mathbb{Z}}$, other than the equidistantly spaced ones described above, such that any bandlimited function f can be reconstructed from its samples $\{f(\lambda_n)\}$ taken on the points of Λ ? The answer to this question is yes, and in fact, a sufficient condition for a discrete set of points have this property is that it be ‘dense’ enough using a suitable notion of density.

A uniformly discrete set of points $\Lambda = \{y_n\}_{n \in \mathbb{Z}}$ is called a set of interpolation if, given any square summable sequence $\{a_n\}_{n \in \mathbb{Z}}$, there is a $f \in B(\Omega)$ for which $f(y_n) = a_n$. A uniformly discrete set is called a set of sampling if the norm of every bandlimited function is bounded above by the square sum of its samples taken on the points of Λ . By definition, and by Theorem (2.2.1), $\Lambda = \{y_n\}$ is a set of sampling if and only if the following double inequality holds:

$$(2.2.7) \quad b \sum_n |f(y_n)|^2 \leq \|f\|^2 \leq B \sum_n |f(y_n)|^2.$$

This double inequality implies that the set of square summable sequences that are the sample values of a bandlimited function is a closed subspace of $l^2(\mathbb{Z})$. By the upper inequality, no two distinct bandlimited functions can have the same sample values on the points of Λ . It follows that the linear operator L which maps the sample values of every bandlimited function $\{f(y_n)\}$ onto the bandlimited function f will be a well defined linear operator. Furthermore, the double inequality (2.2.7) shows that L is bounded above and below.

From a practical viewpoint, the fact that L exists and is bounded above means that every bandlimited function can be stably reconstructed from the values it takes on the points of any set of sampling Λ . The reconstruction is stable in the following sense. Suppose that $f \in B(\Omega)$ and that there is an error in the measurement of the sample values of f . Suppose that the actual measured values are $\{g(y_n)\}$ where $g(y_n) = f(y_n) + \epsilon_n$, $\sum_n |\epsilon_n|^2 = \epsilon^2 < \infty$ and the values $\{g(y_n)\}$ are the sample values of a different bandlimited function $g \neq f$. Then, the difference in norm squared between f and g is

$$(2.2.8) \quad \|f - g\|^2 \leq B \sum |f(y_n) - g(y_n)|^2 = B\epsilon^2.$$

This shows that a bounded error of this type in the measurement of the samples of a bandlimited function can yield at most a bounded error in the reconstructed function.

Intuitively, a uniformly discrete set of points Λ will be a set of sampling provided that it is ‘dense enough’ on the real line. That is, if the points of Λ are close enough together, the existence of a sequence of bandlimited functions $\{f_n\}$ for which $\|f_n\|^2 > n \sum_{m=-\infty}^{\infty} |f(y_m)|^2$ would require that the functions f_n become arbitrarily peaked between the sample points y_n , again requiring their Fourier transforms to contain support for increasingly large frequencies, and violating the fact that the f_n are bandlimited. Similar logic suggests that Λ will be a set of interpolation provided its points are ‘sparse enough’ on the real line. Of course these density conditions will depend on the bandlimit Ω .

This intuition is indeed correct. Given a uniformly discrete set of points $\Lambda = \{y_n\}$, let $n_{\pm}(r)$ be the largest and smallest number of points of Λ in any interval of length r . The upper and lower Beurling densities of Λ are then defined as

$$(2.2.9) \quad D_{\pm}(\Lambda) := \lim_{r \rightarrow \infty} \frac{n_{\pm}(r)}{r}$$

Given these definitions and the Hilbert space $B(\Omega)$, the following results hold

THEOREM 2.2.3. *Let Λ be a uniformly discrete set of points:*

(a) *Λ is a set of sampling for $B(\Omega)$ provided that*

$$(2.2.10) \quad D_-(\Lambda) > \frac{\Omega}{\pi}$$

(b) *If Λ is a set of sampling for $B(\Omega)$ then*

$$(2.2.11) \quad D_-(\Lambda) \geq \frac{\Omega}{\pi}$$

(c) *Λ is a set of interpolation for $B(\Omega)$ provided that*

$$(2.2.12) \quad D_+(\Lambda) < \frac{\Omega}{\pi}$$

(d) *If Λ is a set of interpolation for $B(\Omega)$ then*

$$(2.2.13) \quad D_+(\Lambda) \leq \frac{\Omega}{\pi}$$

This theorem is due to A. Beurling [43], although parts (a) and (b) are really just a slight refinement and restatement of results achieved by Duffin and Shaeffer in [18] [19].

Note that if $D_{\pm}(\Lambda) = \frac{\Omega}{\pi}$ then one cannot use Theorem (2.2.3) to conclude that Λ is either a set of sampling or a set of interpolation for $B(\Omega)$. One example of such a set which is in fact both a set of sampling and a set of interpolation is $\Lambda = \{x_n := \frac{n\pi}{\Omega}\}_{n \in \mathbb{Z}}$. That it is a set of sampling is easy to see since $\{e_n\}_{n=-\infty}^{\infty}$ where $e_n(x) := \frac{1}{\sqrt{2\Omega}} e^{iwx_n}$ is an orthonormal basis for $L^2[-\Omega, \Omega]$, and $\sum_n |f(x_n)|^2 = 2\Omega \sum_n |(F, e_n)|^2 = 2\Omega \|F\|^2 = \frac{\Omega}{\pi} \|f\|^2$. It is also straightforward to see that it is a set of interpolation. Given any square summable sequence $\{a_n\}$,

$$(2.2.14) \quad f(x) := \sum_n a_n \frac{\sin(\Omega(x - x_n))}{\Omega(x - x_n)}$$

belongs to $B(\Omega)$ since the vectors $\delta_{x_n}(x) = \frac{\pi \sin(\Omega(x_n - x))}{\Omega(x_n - x)}$ are an orthonormal basis for $B(\Omega)$. Since δ_{x_n} is the point evaluation vector at x_n , $f(x_n) = a_n$. Since f is bandlimited and the sequence $\{a_n\} \in l^2$ was arbitrary, this shows that $\Lambda = \{x_n\}$ is also a set of interpolation.

The following necessary sampling and interpolation conditions have been established for general frequency limited functions [43].

THEOREM 2.2.4. *Let $S \subset \mathbb{R}^n$ be compact.*

(a) *If Λ is a set of sampling for $B(S)$ then*

$$(2.2.15) \quad D^-(\Lambda) \geq \frac{\mu(S)}{(2\pi)^n}$$

(b) *If Λ is a set of interpolation for $B(S)$ then*

$$(2.2.16) \quad D^+(\Lambda) \leq \frac{\mu(S)}{(2\pi)^n}$$

The analogues of the sufficiency conditions of Theorem 2.2.3 do not hold in general. Here μ denotes Lebesgue measure.

For example, even in the simple case where $S := [-b, -a] \cup [a, b]$ is a union of two intervals, one can show that the analogues of the sufficiency conditions of Theorem 2.2.3 do not hold.

'CLAIM 2.2.5. *Let $S := [-b, -a] \cup [a, b]$ where $a > 0$ and $a < b$, and let $m \in \mathbb{N}$ be the largest natural number for which $(m - 1)(b - a) \leq a$. The sequence $\Lambda := (x_n)_{n \in \mathbb{Z}}$, where $x_n := \frac{n\pi}{b-a}$, is set of uniqueness for $B(S)$ if and only if $a = (m - 1)(b - a)$*

Here, a uniformly discrete set of points $\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}}$ is called a set of uniqueness for $B(S)$ if the condition that $f \in B(S)$ and $f(\lambda_n) = 0 \quad \forall n \in \mathbb{Z}$ implies that $f = 0$. It is not hard to see that Λ will be a set of uniqueness if and only if the set of plane waves $\Lambda' := \{e_n\}_{n \in \mathbb{Z}} \subset L^2(S)$, $e_n(w) := e^{ix_n w}$ a.e., is a complete set. It is clear that every set of sampling is a set of uniqueness, while a set of uniqueness is not necessarily a set of sampling.

PROOF. Let $B := b - a$. The set Λ will be a set of uniqueness for $B(S)$, if and only if the set of plane waves $\Lambda' := \{F_n\}_{n \in \mathbb{Z}}$, $F_n(w) := e^{ix_n w}$, is complete in $L^2(S)$. Given $F \in L^2(S)$, consider the following bounded functional:

$$(2.2.17) \quad \Phi[F] := \sum_{n=-\infty}^{\infty} |\langle F_n, F \rangle|^2.$$

The functional Φ is bounded since if $f \in B(S)$ is the Fourier transform of F , it follows from Theorem 2.2.1 that $\Phi[F] = \sum_{n \in \mathbb{Z}} |f(x_n)|^2 \leq C \|f\|^2 = C' \|F\|^2$. To show that Λ is not a set of uniqueness (and hence not a set of sampling), it now suffices to show that there is a non-zero $G \in L^2(S)$ for which $\Phi[G] = 0$.

To find such a G , extremize the functional Φ . First, Φ can be written as:

$$(2.2.18) \quad \Phi[F] = \sum_{n \in \mathbb{Z}} \int_S F(w) \left(\int_S \overline{F}(w') e^{ix_n(w'-w)} dw' \right) dw.$$

Setting the functional derivative of Φ with respect to G to zero yields the Euler-Lagrange equation:

$$(2.2.19) \quad 0 = \int_S G(w) \sum_{n \in \mathbb{Z}} e^{in2\pi(w-w')} 2B.$$

Finally, it is useful to use the identity $\sum_{n \in \mathbb{Z}} e^{\frac{i2\pi nu}{c}} = |c| \sum_{n \in \mathbb{Z}} \delta(u - nc)$ [31], so that the Euler-Lagrange equation becomes

$$(2.2.20) \quad 0 = \int_S G(w) 2B \sum_{n \in \mathbb{Z}} \delta((w-w') - n2B) dw.$$

Let $m \in \mathbb{N}$ be the largest natural number such that $(m-1)B \leq a$. There are four separate cases. If a and $B = b - a$ are such that $(m-1)B < a < (m-1/2)B$ then the Euler-Lagrange equation (2.2.20) becomes

$$(2.2.21) \quad G(w) + G(w + 2m(b-a)) + G(w - 2m(b-a)) = 0.$$

In this case it is easy to show that if one chooses $c := -b + 2mB$, and any $G(w)$ such that $G(w) = G(w + 2m(b-a))$ for $w \in [-b, -c]$, and $G(w) = 0$ for $w \in [-c, -a] \cup [a, c]$, then G satisfies equation (2.2.20) and $\Phi[G] = 0$ so that G is perpendicular to the closed linear span of Λ' . This proves that Λ' is not complete, and in fact that there is an infinite dimensional subspace of $L^2(S)$ which is orthogonal to the closed linear span of Λ' . Similar conclusions can be reached for the cases where $(m - \frac{1}{2})B < a < mB$ and $a = (m - \frac{1}{2})B$. For the case in which $a = (m-1)B$, however, the Euler-Lagrange equation (2.2.20) becomes $G(w) = 0$, so that there is no non-zero solution. This proves that there is no non-zero $F \in L^2(S)$ for which $\Phi[F] = 0$, so that in this particular case Λ' is complete, and Λ is a set of uniqueness. \square

Even more dramatically, using the same techniques as in the proof of the above claim, one can show that for any $\epsilon > 0$, there exists sets $S \subset \mathbb{R}$, such that $\mu(S) = \epsilon$ and uniformly discrete sets Λ for which $D_-(\Lambda) = 1$ and yet are not sets of sampling for $B(S)$.

*CLAIM 2.2.6. *Let $S := [-\pi, -\pi + \epsilon] \cup [\pi, \pi + \epsilon]$ for $\epsilon > 0$. Then the set $\Lambda := \mathbb{Z}$ is not a set of uniqueness for $B(S)$.*

PROOF. \mathbb{Z} is a set of uniqueness if and only if the set $\Lambda' := e^{i\mathbb{Z}w}$ is complete in $L^2(S)$. Using the same techniques as in the previous claim, it is not difficult to arrive at the fact that if $G \in L^2(S)$ is such that $G(w) = -G(w + 2\pi)$ for all $w \in [-\pi, -\pi + \epsilon]$, then G is orthogonal to the closed linear span of Λ' . Indeed, once one has this example, it is easy to verify directly that such a G is orthogonal to each element of Λ' . \square

Although I am not aware of a result that proves it, it seems reasonable that there should always exist sets of interpolation and sets of sampling for $B(S)$ that achieve the lower bounds of Theorem 2.2.4.

For example, suppose that $S_c := [-(b+c), -(a+c)] \cup [(a+c), (b+c)]$ and $c > 0$. Then the uniformly discrete set $\Lambda = \{y_n\}_{n \in \mathbb{Z}}$ where $y_n := \frac{n\pi}{(b-a)}$ if n is even and $y_n = \frac{(n+\alpha)\pi}{(b-a)}$ if n is odd, and $\alpha \in (0, 1)$ is always a set of sampling for $B(S_c)$ [46]. Such a set achieves the minimum possible density for a set of sampling for $B(S_c)$, $D^-(\Lambda) = \frac{(b-a)}{\pi} = \frac{\mu(S_c)}{2\pi}$.

2.2.2. Riesz bases and frames of plane waves. The problem of finding sets of sampling for $B(\Omega)$ is dual to the problem of finding what are called frames of plane waves for $L^2[-\Omega, \Omega]$.

A set of vectors $\{f_n\}$ for a Hilbert space \mathcal{H} is called a Riesz basis if it is the image of an orthonormal basis $\{e_n\}$ for \mathcal{H} under a linear operator T which is both bounded above and bounded below. Every Riesz basis is clearly complete, and the removal of any element of a Riesz basis leaves an incomplete set. It is not difficult to show that if $\{f_n\}$ is a Riesz basis, then given any $f \in \mathcal{H}$, the following double inequality holds

$$(2.2.22) \quad b \sum_n |\langle f, f_n \rangle|^2 \leq \|f\|^2 \leq B \sum_n |\langle f, f_n \rangle|^2.$$

A frame is any set of vectors that obeys the above inequality (2.2.22). Unlike a Riesz basis, a frame can be overcomplete. Removal of a vector from a frame may still leave a frame. If a frame is overcomplete, then not all vectors have a unique representation in terms of the members of the frame [19] [72].

Now consider the set of plane waves $\{F_n\} \subset L^2[-\Omega, \Omega]$ where $F_n(w) := e^{i\alpha_n w}$. If $f \in B(\Omega)$ and F is its Fourier transform, then $\langle F, F_n \rangle = f(\alpha_n)$. It follows from (2.2.22) and (2.2.7) that the set $\{F_n\}$ will be a frame for $L^2[-\Omega, \Omega]$ if and only if $\{\alpha_n\}$ is a set of sampling for $B(\Omega)$.

Here is a classical result which establishes a sufficient condition for a set of plane waves in $L^2[-\pi, \pi]$ to be a Riesz basis.

THEOREM 2.2.7. (Kadec's 1/4 theorem) *If $\{\lambda_n\}_{n=-\infty}^{\infty}$ is a sequence of real numbers for which $|\lambda_n - n| < \frac{1}{4}$, then the set of plane waves $\{e_n\}$ where $e_n(w) = e^{i\lambda_n w}$ is a Riesz basis of $L^2[-\pi, \pi]$*

Note that if Λ is a set of uniqueness, then as before, a linear operator which maps the values of every bandlimited function (taken on the points of Λ) to the bandlimited function can be defined. Although this shows that every bandlimited function can be reconstructed from its values taken on any set of uniqueness, this reconstruction will not be stable unless L is bounded, *i.e.*, unless Λ is a set of sampling. As discussed previously, $\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}}$ is a set of uniqueness for $B(\Omega)$ if and only if $e^{i\Lambda}$ is a complete set in $L^2[-\Omega, \Omega]$.

The following is a classical completeness result about sets of plane waves in $L^2[-\pi, \pi]$ [45].

THEOREM 2.2.8. (Levinson) *If $|\lambda_n| \leq |n| + \frac{m}{2} + \frac{1}{4}$, then the set of plane waves $\{e_n\}_{n \in \mathbb{Z}} \subset L^2[-\pi, \pi]$, where $e_n(w) = e^{i\lambda_n w}$, if not complete, becomes complete upon the addition of at most m new distinct plane waves $\{e^{i\alpha_k w}\}_{k=1}^m$. (In particular, if $m = 0$, then the set of plane waves is complete).*

2.2.3. Distribution of zeroes, rate of growth and completeness. There is a connection between the distribution of zeroes of an entire function and its rate of growth as one travels outward from the origin in the complex plane. For example, a polynomial of degree k has k zeroes, and the higher the degree, the faster its growth, *i.e.* the faster its maximum modulus function $M(r)$ grows as $r \rightarrow \infty$.

The following formula makes this relationship more explicit.

THEOREM 2.2.9. (*Jensen's Formula*) *If f is analytic on $|z| < R$ and $f(0) \neq 0$ then for any $r < R$,*

$$(2.2.23) \quad \frac{1}{2\pi} \int_0^{2\pi} \ln |f(re^{i\theta})| d\theta = \ln |f(0)| + \int_0^r \frac{n(t)}{t} dt,$$

where $n(t)$ is the number of zeroes of f in the region $|z| \leq t$.

For any bandlimited function f , if f has a zero of order k at the origin, then one can always just consider the bandlimited function $g(z) = \frac{1}{z^k} f(z)$ which will have all the zeroes f has except for the ones at the origin. Note that $g \in B(\Omega)$ since it is still square integrable and has the same exponential type as f . Furthermore, by rescaling the function, it can always be assumed that $f(0) = 1$ to simplify Jensen's formula. Since any bandlimited function $f \in B(\Omega)$ is an entire function of exponential type at most Ω it follows from equation (3.4.3) that $|f(re^{i\theta})| \leq Ae^{\Omega r |\sin \theta|}$. Using Jensen's formula, and the fact that $n(r)$ is non-decreasing,

$$(2.2.24) \quad \begin{aligned} n(r) \ln(a) &\leq n(r) \int_r^{ar} \frac{1}{t} dt \leq \int_0^{ar} \frac{n(t)}{t} dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} \ln |f(are^{i\theta})| d\theta \leq \ln A + \frac{2\Omega ar}{\pi}, \end{aligned}$$

for any $a > 1$. It follows that $\limsup_{r \rightarrow \infty} \frac{n(r)}{r} \leq 2\frac{\Omega}{\pi} \frac{a}{\ln a}$, for any $a > 1$. In fact, it has been shown that the limit, $\lim_{r \rightarrow \infty} \frac{n(r)}{r}$, exists for any $f \in B(\Omega)$, and that it is always less than or equal to $\frac{2\Omega}{\pi}$ ([45], pgs. 25-26).

Now suppose, for example, that Λ is any set of real values for which $\frac{n(r)}{r} \geq 2\frac{\Omega}{\pi} + \epsilon$, $\epsilon > 0$ for all $r > 0$ and where $n(r)$ is the number of points of Λ in $[-r, r]$. Then no bandlimited function except for the zero function can vanish on all the points of Λ . Therefore such a set Λ will be a set of uniqueness.

Approximation of bandlimited functions by bandlimited trigonometric polynomials

3.1. Bandlimited trigonometric polynomials

An Ω -bandlimited trigonometric polynomial p of period $2L$ is a linear combination of the plane waves $e^{ik_n x}$, $k_n := \frac{n\pi}{L}$ for which $|k_n| \leq \Omega$. If $|k_n| \leq \Omega$, this implies that $|n| \leq \frac{\Omega L}{\pi}$. Let $N := \lfloor \frac{\Omega L}{\pi} \rfloor$. Any such trigonometric polynomial p can be written as:

$$(3.1.1) \quad p(x) = \sum_{n=-N}^N p_n e^{ik_n x}.$$

Here $\lfloor x \rfloor$ denotes the integer part of x . The set of all Ω -bandlimited trigonometric polynomials of period $2L$ forms a $2\lfloor \frac{\Omega L}{\pi} \rfloor + 1$ dimensional subspace $B_L(\Omega)$ of $L^2[-L, L]$. Any Ω -bandlimited trigonometric polynomial of period $2L$ is clearly holomorphic on the entire complex plane. Since the functions $\{e_n\}_{n \in \mathbb{Z}}$, where $e_n(x) := \frac{1}{\sqrt{2L}} e^{ik_n x}$, are an orthonormal basis for $L^2[-L, L]$, it follows that $p_n = \frac{1}{2L} \int_{-L}^L p(x) e^{-ik_n x} dx$.

Let $B(\Omega)$ denote the Hilbert space of functions bandlimited by Ω . The Fourier transform of any element of this Hilbert space is an element of $L^2[-\Omega, \Omega]$. Given a bandlimited function $f \in B(\Omega)$, consider its Fourier series on an interval $[-L, L]$:

$$(3.1.2) \quad f(x) = \sum_{n \in \mathbb{Z}} f_n e^{ik_n x}.$$

Here $k_n := \frac{n\pi}{L}$ and $f_n := \frac{1}{2L} \int_{-L}^L f(x) e^{-ik_n x} dx$. Now truncate the Fourier series of f on this interval to remove all the plane waves with frequencies $|k_n| > \Omega$. This resulting trigonometric polynomial,

$$(3.1.3) \quad f_N(z) := \sum_{n=-N}^N f_n e^{ik_n z},$$

where $z \in \mathbb{C}$ and $N := \lfloor \frac{\Omega L}{\pi} \rfloor$, will be called the Ω -bandlimited trigonometric polynomial (TP_Ω) version of f on the interval $[-L, L]$. It will be convenient to also consider the functions $\phi_N := f_N \chi_L$ where χ_L is the characteristic function of the vertical strip $|\text{Re}(z)| \leq L$ (*i.e.* χ_L is 1 on this strip and vanishes outside of it). The functions ϕ_N will be called the L -truncated TP_Ω versions of f . These functions are analytic on the vertical strip $|\text{Re}(z)| < L$. These L -truncated TP_Ω versions of a bandlimited function clearly belong to the Hilbert spaces of Ω -bandlimited trigonometric polynomials $B_L(\Omega) \subset L^2[-L, L]$.

Let Δ_L denote the operator on $L^2(\mathbb{R})$ which acts as the self-adjoint Laplacian (*i.e.* minus the second derivative operator) with periodic boundary conditions on $L^2[-L, L]$, and as the zero operator on the orthogonal complement of $L^2[-L, L]$ in $L^2(\mathbb{R})$. Further let χ_L denote the projection of $L^2(\mathbb{R})$ onto $L^2[-L, L]$. Then the L -truncated TP_Ω version of a bandlimited function $f \in B(\Omega)$ is just the image of f under the projection operator

$$(3.1.4) \quad P_{L,\Omega} := \chi_L \chi_{[0,\Omega^2]} (\Delta_L) \chi_L,$$

and $B_L(\Omega) = P_{L,\Omega}L^2(\mathbb{R})$.¹ This is clear since the eigenfunctions of Δ_L which have support only on $[-L, L]$ are just the plane waves $e^{ik_n x}$ truncated to the interval $[-L, L]$ with eigenvalues k_n^2 . Further notice that $B_L(\Omega)$ is a natural generalization of the space of Ω -bandlimited functions to the spatially-limited, finite interval $[-L, L]$ since $B(\Omega) = \chi_{[0,\Omega^2]}(\Delta)L^2(\mathbb{R})$ where Δ is the Laplacian or minus the second derivative operator on $L^2(\mathbb{R})$.

Since a bandlimited function $f \in B(\Omega)$ contains no frequencies larger in magnitude than Ω , one may intuitively expect that the TP_Ω version of f on an interval $[-L, L]$ will become an increasingly better approximation to f as L (or equivalently as $N = \lfloor \frac{\Omega L}{\pi} \rfloor$) approaches infinity. The results of the following sections will justify this intuition.

3.2. Trigonometric polynomial approximation and superoscillations

It is known that the space of Ω -bandlimited functions is dense in L^2 norm on any finite interval $[a, b]$ [67]. In particular, for any $\Omega > 0$, there exist so-called superoscillating Ω -bandlimited functions which can oscillate arbitrarily quickly on any finite interval of arbitrary length.

For example, given any finite interval $[a, b]$, and any positive value $\Omega > 0$, one can construct a sequence of ‘spheroidal prolate wave functions’ which are Ω -bandlimited, form an orthonormal basis for the Hilbert space of Ω -bandlimited functions, $B(\Omega)$, and which are simultaneously a complete orthogonal set in $L^2[a, b]$ [67]. This means that given any finite interval, one can draw a continuous function that oscillates arbitrarily quickly, and then find a sequence of Ω -bandlimited functions that converge to it in L^2 norm on that interval. Furthermore, one can specify any values at any finite (but arbitrarily large) number of points and find a bandlimited function which achieves those values at those points [9] [23].² By choosing alternating positive and negative values on a densely packed finite set of points, one can thus construct a bandlimited function which displays ‘superoscillatory’ behaviour. That is, the resulting bandlimited function undergoes rapid changes on length scales much smaller than the shortest wavelength corresponding to the highest frequency component in its Fourier spectrum.

The phenomenon of superoscillations follows from long-known results of [43] [44] [67]. More recently superoscillations have been rediscovered in the field of mathematical physics [6], and are currently a subject of some interest in both the mathematical physics and sampling theory communities, see, *e.g.*, [2] [9] [23] [34].

The existence of superoscillations shows that in some sense the space of Ω -bandlimited functions has an arbitrarily large number of ‘degrees of freedom’ in any finite interval $[-L, L]$. Conversely, the set of all Ω -bandlimited trigonometric polynomials of period $2L$ with the L^2 inner product on $[-L, L]$ forms a finite $2\lfloor \frac{\Omega L}{\pi} \rfloor + 1$ dimensional Hilbert space, $B_L(\Omega)$, so that any Ω -bandlimited trigonometric polynomial of period $2L$ has only a finite number of ‘degrees of freedom’ in the interval $[-L, L]$. Moreover, since Ω -bandlimited functions are L^2 dense on $[-L, L]$, there exists a sequence $(f_n)_{n=1}^\infty$ of Ω -bandlimited functions which converge in L^2 norm to $e^{ik_l x}$ on $[-L, L]$, $k_l := \frac{l\pi}{L}$ where $|l|$ can be chosen large enough so that $|k_l| > \Omega$. Since all Ω -bandlimited trigonometric polynomials of period $2L$ on $[-L, L]$ are finite linear combinations of the mutually orthogonal plane waves $e^{ik_m x}$ for $|k_m| \leq \Omega$, it follows that the f_n are converging in norm on $L^2[-L, L]$ to a function orthogonal to the subspace $B_L(\Omega)$. That is, for any interval $[-L, L]$ there exist Ω -bandlimited functions whose projections onto $L^2[-L, L]$ are arbitrarily close to being orthogonal to the subspace $B_L(\Omega) \subset L^2[-L, L]$ of Ω -bandlimited trigonometric polynomials of period $2L$. Even more dramatically, for any interval $[-L, L]$ there exist Ω -bandlimited functions which are in fact orthogonal to $B_L(\Omega)$ on $[-L, L]$. This will be demonstrated in Section 3.2.1.

¹Here, if A is a self-adjoint operator and $\chi_{[0,\Omega^2]}(x)$ is the characteristic function of $[0, \Omega^2]$, the spectral projection $\chi_{[0,\Omega^2]}(A)$ is defined by the functional calculus.

²Note that this fact follows immediately from the results of Beurling in Theorem 2.2.3 since any finite set of points is a set of interpolation for Ω -bandlimited functions.

These facts may appear to cast doubt on the idea that any Ω -bandlimited function is, in some sense, the limit of a sequence of Ω -bandlimited trigonometric polynomials whose periods become infinite in length. Nevertheless, for any fixed strictly Ω -bandlimited function f , this chapter shows that the Ω -bandlimited trigonometric polynomials $(f_N)_{N \in \mathbb{N}}$, where f_N is the TP_Ω version of f , converge to the bandlimited function f uniformly on any compact set in \mathbb{C} . It will be further shown that the L -truncated TP_Ω versions of f , $\phi_N := P_{L,\Omega}f$, $N = \lfloor \frac{\Omega L}{\pi} \rfloor$, converge to f in L^2 norm on any line parallel to the real axis in the complex plane \mathbb{C} . For any fixed Ω -bandlimited function, sequences of Ω -bandlimited trigonometric polynomials which converge to it have been constructed in the past [29] [63]. The sequences considered here, however, can be seen as a more natural approximation of the original Ω -bandlimited function, since they are directly the image of the original function under a sequence of spectral projections of self-adjoint Laplacians on spatial intervals of increasing size.

To understand how these results can be consistent with what is known about superoscillations, consider the following. As shown in [23], superoscillatory behaviour comes at a price. Superoscillations are ‘expensive in norm’. Roughly speaking, the amplitude of the superoscillations in a bandlimited signal is very small relative to the amplitude of the signal outside of the superoscillating interval, and an increase in the length of the superoscillating interval or the rapidity of the superoscillations corresponds to a large increase in the norm of the signal. For a large class of superoscillatory Ω -bandlimited signals, if the norm of the bandlimited signal is fixed, the amplitude of superoscillations decreases polynomially with their wavelength, and if the wavelength of the superoscillations is fixed, their amplitude decreases exponentially with the size of the superoscillating interval [23]. This suggests that for a fixed bandlimited function with a fixed, finite norm, there is an upper bound on how much it can ‘superoscillate’.

The results of this chapter support this. Namely, for any fixed Ω -bandlimited function, no matter how wild its local behaviour is in a given finite interval, if one views the function on a sequence of intervals of increasing length L , the functions $f_N := P_{L,\Omega}f$ which belong to the $2\lfloor \frac{\Omega L}{\pi} \rfloor + 1$ dimensional subspaces $B_L(\Omega)$ of $L^2[-L, L]$ become arbitrarily good approximations to the original bandlimited function in the limit as $L \rightarrow \infty$.

Most of the results of this chapter have already been published in [47]. See also [64], which extends the results of [47] to L^p spaces.

3.2.1. Proof that $B_L(\Omega) = P_{L,\Omega}B(\Omega)$. Before proceeding to state the main results of this chapter, it will first be proven that for any $\Lambda > 0$, $B_L(\Omega)$ is the image of $B(\Lambda)$ under the spectral projection operator $P_{L,\Omega}$. This will prove, in particular, that any Ω -bandlimited trigonometric polynomial can be seen as the truncated TP_Ω version of an Ω -bandlimited function. It will further be shown, as was claimed in the previous subsection, that there exist functions $f \in B(\Omega)$ for which $P_{L,\Omega}f \perp \chi_L f$ in $L^2[-L, L]$. Here χ_L denotes the projection operator onto the subspace $L^2[-L, L]$ of $L^2(\mathbb{R})$. The method of proof used here is very similar to that used to construct superoscillating bandlimited functions in [23].

It needs to be shown that given any $2N + 1$ complex values $\{a_n\}_{n=-N}^N$, and any $\Lambda > 0$ there exists an $f \in B(\Lambda)$ such that

$$(3.2.1) \quad \frac{1}{2L} \int_{-L}^L f(x) e^{-ik_n x} dx = a_n$$

for all $|n| \leq N$ where $N := \lfloor \frac{\Omega L}{\pi} \rfloor$ and $k_n := \frac{n\pi}{L}$. This will show $P_{L,\Omega}f(x) = \sum_{n=-N}^N a_n e^{ik_n x}$, proving that $B_L(\Omega) \subset P_{L,\Omega}B(\Lambda)$. That $B_L(\Omega) \supset P_{L,\Omega}B(\Lambda)$ is obvious.

A Λ -bandlimited function of minimum norm satisfying the $2\lfloor \frac{\Omega L}{\pi} \rfloor + 1$ constraints (3.2.1) will now be constructed using variational methods. The Fourier transform of $f \in L^2(\mathbb{R})$ is defined to be

$$(3.2.2) \quad F(w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-ixw} dx.$$

The functional to be extremized can be written

$$\begin{aligned}
\Phi[F] &:= \int_{-\Lambda}^{\Lambda} F(w)\overline{F}(w)dw + \frac{1}{2L} \sum_{n=-N}^N \lambda_n \int_{-L}^L \left(\int_{-\Lambda}^{\Lambda} F(w)e^{iw x} dw \right) e^{-ik_n x} dx \\
(3.2.3) \quad &= \int_{-\Lambda}^{\Lambda} F(w)\overline{F}(w)dw + \sum_{n=-N}^N \lambda_n \int_{-\Lambda}^{\Lambda} F(w) \frac{\sin L(w - k_n)}{L(w - k_n)} dw.
\end{aligned}$$

Setting the functional derivative of Φ to zero yields the Euler-Lagrange equation:

$$(3.2.4) \quad F(w) + \sum_{n=-N}^N \overline{\lambda}_n \frac{\sin L(w - k_n)}{L(w - k_n)} = 0.$$

Using equation (3.2.1), this becomes

$$(3.2.5) \quad a_j = - \sum_{n=-N}^N \overline{\lambda}_n \int_{-\Lambda}^{\Lambda} \frac{\sin L(w - k_j)}{L(w - k_j)} \frac{\sin L(w - k_n)}{L(w - k_n)} dw.$$

Define

$$(3.2.6) \quad S_{jn} := \int_{-\Lambda}^{\Lambda} \frac{\sin L(w - k_j)}{L(w - k_j)} \frac{\sin L(w - k_n)}{L(w - k_n)} dw.$$

This defines a $(2N + 1) \times (2N + 1)$ matrix S .

Given any $p \in B_L(\Omega)$, $p(x) = \sum_{n=-N}^N p_n e^{ik_n x}$ on $[-L, L]$, and vanishes outside of $[-L, L]$. Now consider the norm of $P(w)\chi_{[-\Lambda, \Lambda]}$ where $P(w)$ is the Fourier transform of $p(x)$ and $\chi_{[-\Lambda, \Lambda]}$ is the characteristic function of the interval $[-\Lambda, \Lambda]$. Then,

$$(3.2.7) \quad P(w) = \frac{L}{\pi} \sum_{n=-N}^N p_n \frac{\sin L(w - k_n)}{L(w - k_n)},$$

and,

$$\begin{aligned}
\int_{-\Lambda}^{\Lambda} |P(w)|^2 dw &= \frac{L^2}{\pi^2} \sum_{n=-N}^N \sum_{m=-N}^N p_n \overline{p}_m \int_{-\Lambda}^{\Lambda} \frac{\sin L(w - k_n)}{L(w - k_n)} \frac{\sin L(w - k_m)}{L(w - k_m)} dw \\
(3.2.8) \quad &= \frac{L^2}{\pi^2} \sum_{n,m=-N}^N p_n \overline{p}_m S_{nm}.
\end{aligned}$$

Note that since $p(x)$ vanishes outside of $[-L, L]$, that $P(w)$ is holomorphic on the entire complex plane. Therefore, if $p(x) \neq 0$, $P(w)$ can only be zero on a discrete number of points in $[-\Omega, \Omega]$ which have no limit point so that $\int_{-\Lambda}^{\Lambda} |P(w)|^2 dw$ must be positive. Thus equation (3.2.8) shows that S is a positive definite matrix and hence is invertible.

Using the inverse of S in equation (3.2.5) now yields:

$$(3.2.9) \quad \overline{\lambda}_k = - \sum_{j=-N}^N S_{kj}^{-1} a_j$$

Calculating the Lagrange multipliers λ_k and substituting them into equation (3.2.4) then yields the Fourier transform of the desired Λ -bandlimited function:

$$(3.2.10) \quad F(w) = \sum_{n=-N}^N \sum_{j=-N}^N S_{nj}^{-1} a_j \frac{\sin L(w - k_n)}{L(w - k_n)} \quad w \in [-\Lambda, \Lambda]$$

In summary, given any $2N + 1$ complex values $\{a_n\}_{n=-N}^N$, one can explicitly construct a bandlimited function $f \in B(\Lambda)$ such that its Fourier coefficients $f_n = a_n$ for all $n \in \{-N, \dots, N\}$. That is, given any $p \in B_L(\Omega)$ and $\Lambda > 0$ there is an f in $B(\Lambda)$ for which $P_{[0, \Omega^2]}(\Delta_L)f = p$.

Furthermore, the above result also shows that $P_{L,\Lambda}B_L(\Omega) = B_L(\Lambda)$ for any $\Lambda > \Omega$. Suppose Λ is chosen such that $M := \lfloor \frac{\Lambda L}{\pi} \rfloor > \lfloor \frac{\Omega L}{\pi} \rfloor = N$. It follows that given the sequence $\{a_n\}_{n=-M}^M$, where $a_M = 1$, $a_n = 0$ for $n \neq M$, there exists an $f \in B(\Omega)$ for which its Fourier coefficients on $[-L, L]$ obey $f_n = a_n$ for all $n \in \{-M, \dots, M\}$. Hence, $P_{L,\Omega}f = 0$ so that f is orthogonal to $B_L(\Omega)$. This shows that there exist Ω -bandlimited functions which are orthogonal to all Ω -bandlimited trigonometric polynomials on $[-L, L]$ for any $L > 0$.

3.3. Statement of main results

**PROPOSITION 3.3.1. Any strictly Ω -bandlimited function f is the limit of any sequence of truncated TP_Ω versions of itself, $(\phi_N)_{N=1}^\infty$ where convergence is with respect to the norm of $L^2(\mathbb{R})$.*

A function is strictly bandlimited by Ω if it is bandlimited by Λ where Λ is strictly less than Ω . Note that there is some freedom in the choice of the sequence (ϕ_N) . Since $N = \lfloor \frac{\Omega L}{\pi} \rfloor$, one is free to choose ϕ_N to be the L -truncated TP_Ω version of f on any interval $[-L, L]$ where $L \in [\frac{\pi}{\Omega}N, \frac{\pi}{\Omega}(N+1))$. It is assumed that Ω is fixed so that as $N \rightarrow \infty$, $L \rightarrow \infty$. In terms of the projection operators $P_{L,\Omega}$ and $\chi_{[0,\Omega^2]}(\Delta)$ discussed in the previous sections, *Proposition 3.3.1 can be rephrased in the following way. The operator $P_{L,\Omega}\chi_{[0,\Omega^2]}(\Delta)$ converges strongly to $\chi_{[0,\Lambda^2]}(\Delta)$ in the limit as $L \rightarrow \infty$ for any $\Lambda < \Omega$. Since Δ has purely continuous spectrum, it is not difficult to show that this implies that $(P_{L,\Omega} - 1)\chi_{[0,\Omega^2]}(\Delta)$ converges strongly to zero as $L \rightarrow \infty$.

This proposition will be used to establish the following stronger result.

**PROPOSITION 3.3.2. Given any strictly Ω -bandlimited f , any sequence of its truncated TP_Ω versions $(\phi_N)_{N=1}^\infty$ converge to f in L^2 norm on any line parallel to the real axis in the complex plane. Furthermore, this L^2 convergence is uniform in any horizontal strip in \mathbb{C} .*

By the L^2 convergence being uniform, it is meant that given any horizontal strip $\text{Im}(z) \leq B$, and any $\epsilon > 0$, there is an $N' \in \mathbb{N}$ such that for all $N > N'$, $\|f - \phi_N\|_y < \epsilon$ for all $|y| \leq B$. Here, $\|f\|_y^2 := \int_{-\infty}^{\infty} |f(x + iy)|^2 dx$.

The following corollary is a straightforward application of *Proposition 3.3.2.

**COROLLARY 3.3.3. Given a strictly Ω -bandlimited function f , any sequence $(\phi_N)_{N=1}^\infty$ where $N := \lfloor \frac{\Omega L}{\pi} \rfloor$ and ϕ_N is a truncated TP_Ω version of f on $[-L, L]$ converges uniformly to f on any horizontal strip in the complex plane.*

In particular, this shows that the sequence $(f_N)_{N \in \mathbb{N}}$ converges uniformly to the original strictly bandlimited f on compacta. This next corollary establishes l^2 convergence of the samples taken on uniformly discrete sets.

**COROLLARY 3.3.4. Suppose f is strictly bandlimited by Ω . If $\{\phi_N\}_{N=1}^\infty$ where $N := \lfloor \frac{\Omega L}{\pi} \rfloor$ is any sequence of truncated TP_Ω versions of f and $\Lambda := \{y_n\}_{n \in \mathbb{Z}} \subset \mathbb{R}$ is a uniformly discrete set of points, then the square summable sequence $(\phi_N(y_n))_{n \in \mathbb{Z}}$ converges to the sequence $(f(y_n))_{n \in \mathbb{Z}}$ in $l^2(\mathbb{Z})$.*

3.4. Proof of Results

The following basic facts and inequalities for bandlimited functions will be useful in establishing the results.

For the remainder of this chapter, the Fourier transform $f \in B(\Omega)$ of $F \in L^2[-\Omega, \Omega]$ will be defined as

$$(3.4.1) \quad f(x) = \int_{-\Omega}^{\Omega} F(w)e^{iwx} dw \quad x \in \mathbb{R}.$$

If $f \in B(\Omega)$ is the transform of $F \in L^2(-\Omega, \Omega)$, then $\|f\| = \sqrt{2\pi}\|F\|$.

Recall, as shown in Subsection 2.1.2, that

$$(3.4.2) \quad \|f^{(n)}\| \leq \frac{\Omega^n}{\sqrt{2\pi}} \|f\|,$$

and that if $z = x + iy$ then,

$$(3.4.3) \quad |f^{(n)}(z)| \leq \Omega^n \sqrt{\frac{\Omega}{\pi}} e^{\Omega|y|} \|f\|.$$

Similarly, if f_N is the TP_Ω version of f on $[-L, L]$ then

$$(3.4.4) \quad \begin{aligned} |f_N^{(j)}(z)|^2 &= \left| \sum_{n=-N}^N f_n(ik_n)^j e^{ik_n z} \right|^2 \leq \sum_{n=-N}^N |f_n|^2 \sum_{n=-N}^N |k_n|^{2j} e^{-2k_n y} \\ &\leq (2N+1) \Omega^{2j} e^{2\Omega|y|} \sum_{n=-N}^N |f_n|^2. \end{aligned}$$

Using the fact that $N = \lfloor \frac{\Omega L}{\pi} \rfloor$ it follows that $2N+1 \leq 2\frac{\Omega L}{\pi} + 1$. Therefore for all N ,

$$(3.4.5) \quad |f_N^{(j)}(z)|^2 \leq e^{2\Omega|y|} \Omega^{2j} \left(\frac{\Omega}{\pi} + \frac{1}{2L} \right) 2L \sum_{n=-N}^N |f_n|^2.$$

Using the fact that $2L \sum_{n=-N}^N |f_n|^2 \leq 2L \sum_{n=-\infty}^{\infty} |f_n|^2 = \int_{-L}^L |f(x)|^2 dx \leq \|f\|^2$ for all $N \in \mathbb{N}$, it follows that

$$(3.4.6) \quad |f_N^{(j)}(z)| \leq C \Omega^j e^{\Omega|y|} \|f\|$$

where $C^2 := \left(\frac{\Omega}{\pi} + \frac{1}{2L} \right)$.

3.4.1. Proof of *Proposition 3.3.1. The following basic facts about uniform convergence and interchanging limits will be needed ([62], pgs. 148-149).

THEOREM 3.4.1. (Weierstrass M-test) *Let $\{f_n\}_{n=1}^\infty$ be a sequence of functions defined on a set E . Suppose that $|f_n(x)| \leq M_n$ for all $x \in E$ and all $n \in \mathbb{N}$. Then the sequence of partial sums $\sum_{n=1}^N f_n(x)$ converges uniformly on E if $\sum_{n=1}^\infty M_n$ converges.*

THEOREM 3.4.2. *Suppose that $f_n \rightarrow f$ uniformly on E . If t is a limit point of E and $\lim_{x \rightarrow t} f_n(x) = A_n$ for every $n \in \mathbb{N}$ then $(A_n)_{n=1}^\infty$ converges and $\lim_{x \rightarrow t} f(x) = \lim_{n \rightarrow \infty} A_n$. That is,*

$$\lim_{x \rightarrow t} \lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} \lim_{x \rightarrow t} f_n(x).$$

Using these two results it is straightforward to establish the following fact that will be used in the proof of *Proposition 3.3.1.

THEOREM 3.4.3. *Suppose $(f_{nm})_{n,m=1}^\infty$ is a doubly infinite sequence of functions defined for all $x \in E$ and $|f_{nm}(x)| \leq M_{nm}$ for all $x \in E$ where $\sum_{n,m=1}^\infty M_{nm} < \infty$. Then if t is a limit point of E ,*

$$\lim_{x \rightarrow t} \sum_{n,m=1}^\infty f_{nm}(x) = \lim_{N \rightarrow \infty} \lim_{M \rightarrow \infty} \sum_{n=1}^N \sum_{m=1}^M \lim_{x \rightarrow t} f_{nm}(x).$$

The following lemmas will be helpful in establishing *Proposition 3.3.1.

*LEMMA 3.4.4. Let $N := \lfloor \frac{\Omega L}{\pi} \rfloor$. If $|n| > N$, then the n^{th} Fourier coefficient in the Fourier series of a function bandlimited by Ω on the interval $[-L, L]$ is given by the formula

$$f_n := \frac{(-1)^{(n+1)}}{2L} \sum_{j=1}^{\infty} \frac{f^{(j-1)}(L) - f^{(j-1)}(-L)}{(ik_n)^j}.$$

In other words, the Fourier coefficient of any plane wave $e^{iw x}$ in the Fourier series of $f \in B(\Omega)$ on $[-L, L]$ with frequency $|w| > \Omega$ is given by the above formula.

PROOF.

$$(3.4.7) \quad f_n = \frac{1}{2L} \int_{-L}^L f(x) e^{-ik_n x} dx$$

Integrating by parts yields

$$(3.4.8) \quad \begin{aligned} 2L f_n &= \frac{f(x) e^{-ik_n x}}{-ik_n} \Big|_{-L}^L + \frac{1}{ik_n} \int_{-L}^L f'(x) e^{-ik_n x} dx \\ &= \frac{f(L) e^{-in\pi} - f(-L) e^{in\pi}}{-ik_n} + \frac{1}{ik_n} \int_{-L}^L f'(x) e^{-ik_n x} dx. \end{aligned}$$

Repeatedly integrating by parts gives

$$(3.4.9) \quad \begin{aligned} 2L f_n &= \frac{(-1)^{(n+1)} (f(L) - f(-L))}{ik_n} + \frac{(-1)^{(n+1)} (f'(L) - f'(-L))}{(ik_n)^2} + \dots \\ &\quad + \frac{(-1)^{(n+1)} (f^{(j-1)}(L) - f^{(j-1)}(-L))}{(ik_n)^j} + \frac{1}{(ik_n)^j} \int_{-L}^L f^{(j)}(x) e^{-ik_n x} dx \\ &= (-1)^{n+1} \sum_{m=1}^j \frac{f^{(m-1)}(L) - f^{(m-1)}(-L)}{(ik_n)^m} + \frac{1}{(ik_n)^j} \int_{-L}^L f^{(j)}(x) e^{-ik_n x} dx. \end{aligned}$$

Define $S_j := (-1)^{n+1} \sum_{m=1}^j \frac{f^{(m-1)}(L) - f^{(m-1)}(-L)}{(ik_n)^m}$. Then,

$$(3.4.10) \quad |2L f_n - S_j| = \left| \frac{1}{(ik_n)^j} \int_{-L}^L f^{(j)}(x) e^{-ik_n x} dx \right| \leq \frac{1}{|k_n|^j} \int_{-L}^L |f^{(j)}(x)| dx.$$

Assume without loss of generality that $\|f\| = 1$. Using the bound (3.4.3) on the j^{th} derivative of f at any point on the real line, equation (3.4.10) becomes

$$(3.4.11) \quad |2L f_n - S_j| \leq \sqrt{\frac{\Omega}{\pi}} \left(\frac{\Omega}{|k_n|} \right)^j \int_{-L}^L dx = \sqrt{\frac{\Omega}{\pi}} 2L \left(\frac{\Omega}{|k_n|} \right)^j.$$

Since $|k_n| > \Omega$ for all $|n| > N$, it follows that

$$(3.4.12) \quad \lim_{j \rightarrow \infty} |2L f_n - S_j| \leq 2L \sqrt{\frac{\Omega}{\pi}} \lim_{j \rightarrow \infty} \left(\frac{\Omega}{|k_n|} \right)^j = 0,$$

establishing the claim. \square

LEMMA 3.4.5. The functions $g_n(x) = x^n (-1)^{(n+1)} \psi_n(x)$ are monotonically decreasing functions of x for $x > 0$, for every $n \in \mathbb{N}$. Here $\psi_n(x) = \frac{d^{n+1}}{dx^{n+1}} \ln \Gamma(x)$ is the n^{th} polygamma function

PROOF. For all $x > 0$ the n^{th} polygamma function is given by the formula [1]:

$$(3.4.13) \quad \psi_n(x) = (-1)^{n+1} \int_0^{\infty} \frac{t^n e^{-xt}}{1 - e^{-t}} dt.$$

Hence, the function $g_n(x)$ is given by the formula

$$(3.4.14) \quad \int_0^\infty \frac{(xt)^n e^{-xt}}{1 - e^{-t}} dt = \int_0^\infty \frac{y^n e^{-y}}{x(1 - e^{-y/x})} dy,$$

where the last equality follows by making the change of integration variable $y = tx$ which is valid for all $x > 0$. Now consider the function $h(x) = \frac{1}{x(1 - e^{-a/x})}$ where $a \geq 0$. This function appears in the integrand of (3.4.14), and is monotonically decreasing for $x \in (0, \infty)$. To prove this first note that this is obvious for the trivial case $a = 0$. Now assume $a > 0$. The derivative of h is

$$(3.4.15) \quad h'(x) = \frac{-x(1 - e^{-a/x}) + ae^{-a/x}}{x^3(1 - e^{-a/x})}.$$

The function h is monotonically decreasing for $x > 0$ if and only if h' is negative for all $x > 0$ which happens if and only if the numerator of h' is negative for $x > 0$ since the denominator is always positive for $x, a > 0$. This happens if and only if

$$(3.4.16) \quad x \geq (x + a)e^{-a/x}$$

for all $x > 0$. Dividing through by a and letting $w = x/a$ shows that this inequality is satisfied if and only if $w \geq (w + 1)e^{-1/w}$ or equivalently if and only if $e^{1/w}w \geq (1 + w)$ for all $w > 0$. This inequality is easily established since $we^{1/w} = w(1 + 1/w + \sum_{n=2}^\infty \frac{1}{n!w^n}) \geq w + 1$ for all $w > 0$. Thus, since h is monotonically decreasing for all $x > 0$, it follows that

$$(3.4.17) \quad g_n(x + \epsilon) = \int_0^\infty y^n e^{-y} \frac{1}{(x + \epsilon)(1 - e^{y/(x + \epsilon)})} dy \leq \int_0^\infty y^n e^{-y} \frac{1}{x(1 - e^{-y/x})} dy = g_n(x),$$

which shows that $g_n(x)$ is monotonically decreasing on $(0, \infty)$. \square

*Proposition 3.3.1 will now be proven.

PROOF. (of *Proposition 3.3.1) Suppose f is bandlimited by $\Lambda < \Omega$, and assume without loss of generality that $\|f\| = 1$. Let ϕ_N be the truncated TP_Ω version of f on an interval $[-L, L]$, $N = \lfloor \frac{\Omega L}{\pi} \rfloor$.

Then

$$(3.4.18) \quad \|f - \phi_N\|^2 = \int_{|x| > L} |f(x)|^2 dx + 2L \sum_{|n| > N} |f_n|^2.$$

The coefficient $f_n = \frac{1}{2L} \int_{-L}^L f(x) e^{-ik_n x} dx$ is the n^{th} coefficient in the Fourier series of f on $[-L, L]$. As $N \rightarrow \infty$, $L \rightarrow \infty$, and the integral in (3.4.18) vanishes in this limit as f is square integrable. Thus, to show that ϕ_N converges to f as $N \rightarrow \infty$, one needs to show only that $\lim_{L \rightarrow \infty} 2L \sum_{|n| > N} |f_n|^2 = 0$. It will be proven that $\lim_{L \rightarrow \infty} 2L \sum_{n > N} |f_n|^2 = 0$. Showing that the limit of the other half of the sum vanishes uses similar logic.

Let $S(L) := 2L \sum_{n=N+1}^\infty |f_n|^2$. Using the formula from *Lemma 3.4.4 this becomes

$$(3.4.19) \quad S(L) = \frac{1}{2L} \sum_{n > N} \sum_{j=1}^\infty \sum_{r=1}^\infty \frac{(f^{(j-1)}(L) - f^{(j-1)}(-L)) (f^{*(r-1)}(L) - f^{*(r-1)}(-L))}{ij+r(k_n)^{j+r}}.$$

It will be useful to interchange the orders of summation in S. To show that this is valid, it must be shown that S converges absolutely, *i.e.*, it must be shown that

$$(3.4.20) \quad T := \frac{1}{2L} \sum_{n > N} \sum_{j,r=1}^\infty \frac{|f^{(j-1)}(L) - f^{(j-1)}(-L)| |f^{*(r-1)}(L) - f^{*(r-1)}(-L)|}{(k_n)^{j+r}} < \infty,$$

for any fixed value of $L > 0$. Now,

$$(3.4.21) \quad T \leq \frac{1}{2L} \sum_{n > N} \sum_{j,r=1}^\infty \frac{(|f^{(j-1)}(L)| + |f^{(j-1)}(-L)|) (|f^{(r-1)}(L)| + |f^{(r-1)}(-L)|)}{(k_n)^{j+r}}.$$

Using the fact that $|f^{(j)}(x)| \leq \sqrt{\frac{\Omega}{\pi}} \Omega^j \|f\| = \sqrt{\frac{\Omega}{\pi}} \Omega^j$ as shown in equation (3.4.3), it follows that

$$\begin{aligned} T &\leq \frac{1}{2\pi L} \sum_{n>N} \sum_{j,r=1}^{\infty} \frac{2(\sqrt{\Omega}\Omega^{j-1})2(\sqrt{\Omega}\Omega^{r-1})}{k_n^{r+j}} \\ (3.4.22) \quad &= \frac{2}{\pi L \Omega} \sum_{n>N} \sum_{j,r=1}^{\infty} \left(\frac{\Omega}{k_n}\right)^{r+j}. \end{aligned}$$

Again, since $k_n > \Omega$ for all $|n| > N$, the above sums in j and r are convergent geometric series. These series are easily evaluated, and yield the expression

$$(3.4.23) \quad T \leq \frac{2}{\pi L \Omega} \sum_{n>N} \left(\frac{\Omega}{k_n(1-\Omega/k_n)}\right)^2 = \frac{2\Omega}{\pi L} \sum_{n>N} \frac{1}{(k_n - \Omega)^2}.$$

Now $N := \lfloor \frac{\Omega L}{\pi} \rfloor$ so that $N+1 = \frac{\Omega L}{\pi} + \delta$ where $1 \geq \delta > 0$. Letting $n = s+N+1 = (s+\delta) + \frac{\Omega L}{\pi}$ and using that $k_n = \frac{n\pi}{L}$ the sum in (3.4.23) becomes

$$\begin{aligned} T &\leq \frac{2\Omega}{\pi L} \sum_{s=0}^{\infty} \frac{1}{((s+\delta)\frac{\pi}{L} + \Omega - \Omega)^2} \leq \frac{2\Omega L}{\pi^3} \sum_{s=0}^{\infty} \frac{1}{(s+\delta)^2} \\ (3.4.24) \quad &\leq \frac{2\Omega L}{\pi^3} \left(\frac{1}{\delta^2} + \sum_{s=1}^{\infty} \frac{1}{s^2}\right) = \frac{2\Omega L}{\pi} \left(\frac{1}{\delta^2 \pi^2} + \frac{1}{6}\right) < \infty. \end{aligned}$$

This shows that $S(L)$ converges absolutely for any fixed L so that the orders of summation can be interchanged. Therefore

$$\begin{aligned} S(L) &= \frac{1}{2L} \sum_{j,r=1}^{\infty} \frac{(f^{(j-1)}(L) - f^{(j-1)}(-L)) (f^{*(r-1)}(L) - f^{*(r-1)}(-L))}{i^{j+r}} \sum_{s=0}^{\infty} \frac{1}{k_{s+N+1}^{j+r}} \\ (3.4.25) \quad &:= \sum_{j,r=1}^{\infty} S_{jr}(L). \end{aligned}$$

The sum farthest to the right can be written

$$(3.4.26) \quad \tilde{S}(L) := \sum_{s=0}^{\infty} \frac{1}{k_{s+N+1}^{j+r}} = \sum_{s=0}^{\infty} \frac{1}{((s+\delta)\frac{\pi}{L} + \Omega)^{j+r}} = \left(\frac{L}{\pi}\right)^{j+r} \sum_{s=0}^{\infty} \frac{1}{(s+\delta + \frac{\Omega L}{\pi})^{j+r}}.$$

The n^{th} polygamma function can be expressed as $\psi_n(z) = (-1)^{n+1} n! \sum_{k=0}^{\infty} \frac{1}{(z+k)^{n+1}}$ [1]. Using the polygamma functions, \tilde{S} can be written as

$$\begin{aligned} \tilde{S}(L) &= \left(\frac{L}{\pi}\right)^{j+r} (-1)^{j+r} \frac{1}{(j+r-1)!} \psi_{j+r-1}\left(\frac{\Omega L}{\pi} + \delta\right) \\ (3.4.27) \quad &\leq \left(\frac{L}{\pi}\right)^{j+r} (-1)^{j+r} \frac{1}{(j+r-1)!} \psi_{j+r-1}\left(\frac{\Omega L}{\pi}\right). \end{aligned}$$

Substituting this expression into that for $S_{jr}(L)$ yields

$$\begin{aligned} |S_{jr}(L)| &\leq \left| \frac{(f^{(j-1)}(L) - f^{(j-1)}(-L)) (f^{*(r-1)}(L) - f^{*(r-1)}(-L))}{(\Omega)^{j+r-1} (j+r-1)! (i)^{j+r} 2\pi (-1)^{j+r}} \left(\frac{\Omega L}{\pi}\right)^{j+r-1} \right| \\ (3.4.28) \quad &\times \left| \psi_{j+r-1}\left(\frac{\Omega L}{\pi}\right) \right|. \end{aligned}$$

To show that the limit as $L \rightarrow \infty$ of the double sum $S(L) = \sum_{j,r=1}^{\infty} S_{jr}(L)$ vanishes it will be shown that the conditions of Theorem 3.4.3 are satisfied so that this limit can be interchanged with the summations. That is, it will be shown that $S_{jr}(L) \leq M_{jr}$ for all $L \in [\frac{\pi}{\Omega}, \infty)$ where $\sum_{j,r=1}^{\infty} M_{jr} < \infty$.

By *Lemma 3.4.5, the expression $g_{j+r-1}(\frac{\Omega L}{\pi}) := (\frac{\Omega L}{\pi})^{j+r-1} (-1)^{j+r} \psi_{j+r-1}(\frac{\Omega L}{\pi})$ is a monotonically decreasing function of $\frac{\Omega L}{\pi}$ for all $\frac{\Omega L}{\pi} > 0$. Hence, for fixed Ω and all $L > 0$, this expression is a monotonically decreasing function of L . In particular $g_{j+r-1}(\frac{\Omega L}{\pi}) \leq g_{j+r-1}(1) = (-1)^{j+r} \psi_{j+r-1}(1)$ for all L in the interval $[\frac{\pi}{\Omega}, \infty)$. Using this fact and that $|f^{(j)}(x)| \leq \sqrt{\frac{\Lambda}{\pi}} \Lambda^j$, since f is bandlimited by $\Lambda < \Omega$, the summand $S_{jr}(L)$ is bounded by

$$(3.4.29) \quad \begin{aligned} S_{jr}(L) &\leq \frac{1}{2\pi} 2 \left(\sqrt{\frac{\Lambda}{\pi}} \Lambda^{j-1} \right) 2 \left(\sqrt{\frac{\Lambda}{\pi}} \Lambda^{r-1} \right) \frac{(-1)^{j+r} \psi_{j+r-1}(1)}{\Omega^{j+r-1} (j+r-1)!} \\ &= \frac{2}{\pi^2} \left(\frac{\Lambda}{\Omega} \right)^{j+r-1} \frac{(-1)^{j+r} \psi_{j+r-1}(1)}{(j+r-1)!} \end{aligned}$$

for all $L \in [\frac{\pi}{\Omega}, \infty)$. Finally, using the identity $\psi_n(1) = (-1)^{n+1} n! \zeta(n+1)$ [1] for all $n \in \mathbb{N}$ and the fact that the Riemann zeta function $\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n}$ is clearly monotonically decreasing for all $n \in \mathbb{N}$, we have that $\psi_n(1) \leq (-1)^{n+1} n! \zeta(2) = (-1)^{n+1} n! \frac{\pi^2}{6}$ for all $n \geq 1$. Therefore,

$$(3.4.30) \quad \frac{(-1)^{j+r} \psi_{j+r-1}(1)}{(j+r-1)!} \leq \frac{\pi^2}{6},$$

so that,

$$(3.4.31) \quad S_{jr}(L) \leq \frac{1}{3} \left(\frac{\Lambda}{\Omega} \right)^{j+r-1} := M_{jr},$$

for all $L \in [\frac{\pi}{\Omega}, \infty)$, and

$$(3.4.32) \quad \sum_{j,r=1}^{\infty} M_{jr} = \frac{1}{3} \sum_{j,r=1}^{\infty} \left(\frac{\Lambda}{\Omega} \right)^{j+r-1} = \frac{\Lambda}{3\Omega} \left(\frac{1}{1 - \frac{\Lambda}{\Omega}} \right)^2 < \infty.$$

This shows that the conditions of Theorem 3.4.3 are satisfied so that the limit as $L \rightarrow \infty$ can be interchanged with the double sum:

$$(3.4.33) \quad \begin{aligned} \lim_{L \rightarrow \infty} S(L) &= \lim_{L \rightarrow \infty} \sum_{j,r=1}^{\infty} S_{jr}(L) = \lim_{J \rightarrow \infty} \lim_{R \rightarrow \infty} \sum_{j=1}^J \sum_{r=1}^R \lim_{L \rightarrow \infty} S_{jr}(L) \\ &\leq \lim_{J,R \rightarrow \infty} \sum_{j,r=1}^{J,R} \lim_{L \rightarrow \infty} \frac{\left(|f^{(j-1)}(L)| + |f^{(j-1)}(-L)| \right) \left(|f^{(r-1)}(L)| + |f^{(r-1)}(-L)| \right) g_{j+r-1}(\frac{\Omega L}{\pi})}{2\pi \Omega^{j+r-1} (j+r-1)!} \\ &\leq \lim_{J,R \rightarrow \infty} \sum_{j,r=1}^{J,R} \frac{\pi}{12 \Omega^{j+r-1}} \lim_{L \rightarrow \infty} \left(|f^{(j-1)}(L)| + |f^{(j-1)}(-L)| \right) \left(|f^{(r-1)}(L)| + |f^{(r-1)}(-L)| \right). \end{aligned}$$

This limit is zero since each $f^{(j)}$ is bandlimited so that $\lim_{L \rightarrow \infty} |f^{(j)}(\pm L)| = 0$. Since $S \geq 0$ this shows that $\lim_{L \rightarrow \infty} S = 0$. \square

The above proposition can be applied to any bandlimited function by noting that if f is bandlimited by Ω it is strictly bandlimited by any $\Gamma = \Omega + \epsilon$ where $\epsilon > 0$ is arbitrary.

3.4.2. Proof of Proposition 3.3.2. The following theorems will be needed in the proof of this proposition ([72], pgs. 83, 93-94).

THEOREM 3.4.6. *If f is an entire function of exponential type and if $f(x) \rightarrow 0$ as $|x| \rightarrow \infty$ then $f(x + iy) \rightarrow 0$ as $|x| \rightarrow \infty$ uniformly in every horizontal strip.*

THEOREM 3.4.7. *(Plancherel - Pólya) If g is an entire function of exponential type Ω and if for some $p > 0$,*

$$\int_{-\infty}^{\infty} |g(x)|^p dx < \infty$$

then

$$\int_{-\infty}^{\infty} |g(x + iy)|^p dx \leq e^{p\Omega|y|} \int_{-\infty}^{\infty} |g(x)|^p dx.$$

Given a strictly Ω -bandlimited function f , and $b \in \mathbb{R}$, define $g_b(z) := f(z + ib)$. Then g_b is an entire function since f is. Let $g_{b,N}(z)$ be the Ω bandlimited TP version of g_b on $[-L, L]$, i.e.

$$(3.4.34) \quad g_{b,N}(z) := \sum_{n=-N}^N g_{b,n} e^{ik_n z}$$

where $N := \lfloor \frac{\Omega L}{\pi} \rfloor$ and $g_{b,n} := \frac{1}{2L} \int_{-L}^L g_b(x) e^{-ik_n x} dx$. Finally let $\gamma_{b,n}(z) := \chi_L(z) g_{b,N}(z)$.

*LEMMA 3.4.8. *Given g_b as described above, any sequence of the truncated TP $_{\Omega}$ functions $\{\gamma_{b,N}\}_{N=1}^{\infty}$ converges to g_b with respect to the L^2 norm on \mathbb{R} . That is,*

$$(3.4.35) \quad \lim_{N \rightarrow \infty} \int_{-\infty}^{\infty} |g_b(x) - \gamma_{b,N}(x)|^2 dx = 0.$$

Furthermore, this L^2 convergence is uniform on any horizontal strip i.e. $\forall |b| \leq B$ where $B > 0$ is fixed, in the same sense as described following the statement of *Proposition 3.3.2.

Since g_b is just a vertical translation of the strictly bandlimited function f , an equality similar to (3.4.3) holds for g_b :

$$(3.4.36) \quad |g_b^{(j)}(z)| = |f^{(j)}(z + iB)|^2 \leq \|f\| e^{\Omega|B+y|} \sqrt{\frac{\Omega}{\pi}} \Omega^j.$$

Since the proof of *Lemma 3.4.8 is very similar to that of *Proposition 3.3.1, its proof will be merely sketched here.

3.4.2.1. *Sketch of proof of *Lemma 3.4.8.* Using the bound (3.4.36), it is straightforward to verify that the following formula directly analogous to the one proven in *Lemma 3.4.4 holds for all $|n| > \lfloor \frac{\Omega L}{\pi} \rfloor$:

$$(3.4.37) \quad g_{b,n} = \frac{(-1)^{n+1}}{2L} \sum_{j=1}^{\infty} \frac{1}{(ik_n)^j} \left(f^{(j-1)}(L + ib) - f^{(j-1)}(-L + ib) \right).$$

Now the difference in norm between g_b and $\gamma_{b,N}$ is

$$(3.4.38) \quad \|g_b - \gamma_{b,N}\|^2 = \int_{|x|>L} |g_b(x)|^2 dx + 2L \sum_{|n|>N} |g_{b,n}|^2.$$

By Theorem 3.4.7, on any fixed horizontal strip $|\text{Im}(z)| \leq B$ in the complex plane, the integrals $\int_{|x|>L} |g_b(x)|^2 dx$ converge to zero uniformly as $L \rightarrow \infty$. Repeating the same steps as in *Proposition 3.3.1 then leads to the following equation, similar to equation (3.4.34):

$$(3.4.39) \quad \begin{aligned} & \lim_{L \rightarrow \infty} 2L \sum_{n>N} |g_{b,n}|^2 \\ & \leq \lim_{J,R \rightarrow \infty} \sum_{j,r=1}^{J,R} \frac{\pi}{12\Omega^{j+r-1}} \lim_{L \rightarrow \infty} \left(|f^{(j-1)}(L + ib)| + |f^{(j-1)}(-L + ib)| \right) \\ & \times \left(|f^{(r-1)}(L + ib)| + |f^{(r-1)}(-L + ib)| \right). \end{aligned}$$

By Theorem 3.4.6 it follows that this quantity also converges uniformly to zero for all $|b| \leq B$ for any fixed B .

*Proposition 3.3.2 will now be proven. For convenience, let $\|f\|_y^2 := \int_{-\infty}^{\infty} |f(x + iy)|^2 dx$.

PROOF. (of *Proposition 3.3.2) Let $B > 0$ be arbitrary. To show that the sequence of truncated TP_Ω functions $\{\phi_N = \chi_L f_N\}$ converge in norm to f on any line $x + ib$ parallel to the real axis, uniformly for all $|b| \leq B$, it suffices to show that $\int_{-\infty}^{\infty} |\phi_N(x + ib) - \gamma_{b,N}(x)|^2 dx \rightarrow 0$ as $N \rightarrow \infty$ uniformly for $|b| \leq B$ where $\gamma_{b,N}$ is the truncated Ω -bandlimited TP version of $g_b(z) = f(z + ib)$. This follows since by the previous lemma,

$$(3.4.40) \quad \int_{-\infty}^{\infty} |f(x + ib) - \gamma_{b,N}(x)|^2 dx \rightarrow 0$$

uniformly for $|b| \leq B$, as $N \rightarrow \infty$. Now,

$$(3.4.41) \quad f_N(x + ib) = \sum_{n=-N}^N f_n e^{ik_n(x+ib)} = \sum_{n=-N}^N f_n e^{-k_n b} e^{ik_n x}$$

where $f_n = \frac{1}{2L} \int_{-L}^L f(x) e^{-ik_n x} dx$.

Therefore,

$$(3.4.42) \quad \int_{-\infty}^{\infty} |\phi_N(x + ib) - \gamma_{b,N}(x)|^2 dx = 2L \sum_{n=-N}^N |f_n e^{-k_n b} - g_{b,n}|^2.$$

Recall that $g_{b,n} = \frac{1}{2L} \int_{-L}^L f(x + ib) e^{-ik_n x} dx$. Thus,

$$(3.4.43) \quad \begin{aligned} |f_n e^{-k_n b} - g_{b,n}| &= \left| \frac{e^{-k_n b}}{2L} \int_{-L}^L f(x) e^{-ik_n x} dx - \frac{1}{2L} \int_{-L}^L f(x + ib) e^{-ik_n x} dx \right| \\ &= \frac{1}{2L} \left| e^{-k_n b} \int_{-L}^L f(x) e^{-ik_n x} dx - \int_{-L}^L f(x + ib) e^{-ik_n(x+ib)} dx \right| \\ &\leq \frac{e^{\Omega B}}{2L} \left| \int_{-L}^L f(x) e^{-ik_n x} dx - \int_{-L}^L f(x + ib) e^{-ik_n(x+ib)} dx \right|. \end{aligned}$$

Now consider a counterclockwise oriented rectangular contour S in the complex plane with vertices $(-L, 0)$, $(L, 0)$, (L, b) and $(-L, b)$. The function $h(z) = f(z) e^{-ik_n z}$ is entire, and so by Cauchy's theorem, $\int_S h(z) dz = \int_S f(z) e^{-ik_n z} dz = 0$. Therefore,

$$(3.4.44) \quad \begin{aligned} &\int_{-L}^L f(x) e^{-ik_n x} dx - \int_{-L}^L f(x + ib) e^{-ik_n(x+ib)} dx \\ &= \int_0^b f(-L + iy) e^{-ik_n(-L+iy)} dy - \int_0^b f(L + iy) e^{-ik_n(L+iy)} dy. \end{aligned}$$

Substituting this into equation (3.4.43) yields

$$(3.4.45) \quad \begin{aligned} &|f_n e^{-k_n b} - g_n| \\ &\leq \frac{e^{\Omega B}}{2L} \left| \int_0^b f(-L + iy) e^{-ik_n(-L+iy)} dy - \int_0^b f(L + iy) e^{-ik_n(L+iy)} dy \right| \\ &\leq \frac{B e^{2\Omega B}}{2L} \left(\max_{b \in [-B, B]} |f(-L + ib)| + \max_{b \in [-B, B]} |f(L + ib)| \right). \end{aligned}$$

Substituting this inequality (3.4.45) into equation (3.4.42) yields

$$\begin{aligned}
& \int_{-\infty}^{\infty} |\phi_N(x+ib) - \gamma_{b,N}(x)|^2 dx = 2L \sum_{n=-N}^N |f_n e^{-k_n b} - g_{b,n}|^2 \\
& \leq 2L \sum_{-N}^N \frac{B^2 e^{4\Omega B}}{(2L)^2} \left(\max_{b \in [-B, B]} |f(-L+ib)| + \max_{b \in [-B, B]} |f(L+ib)| \right)^2 \\
& = \frac{2N+1}{2L} B^2 e^{4\Omega B} \left(\max_{b \in [-B, B]} |f(-L+ib)| + \max_{b \in [-B, B]} |f(L+ib)| \right)^2 \\
(3.4.46) \quad & < \left(\frac{\Omega}{\pi} + \frac{3}{2L} \right) B^2 e^{4\Omega B} \left(\max_{b \in [-B, B]} |f(-L+ib)| + \max_{b \in [-B, B]} |f(L+ib)| \right)^2.
\end{aligned}$$

This same upper bound holds for all $|b| \leq B$. Since f is bandlimited, Theorem 3.4.6 implies that this goes to zero uniformly for all $|b| \leq B$ in the limit as L (or equivalently as N) goes to infinity. □

Using this result, it is straightforward to establish uniform convergence on any horizontal strip in the complex plane.

3.4.3. Proof of corollaries.

PROOF. (of *Corollary 3.3.3) Assume the contrary. That is, assume that there is a horizontal strip $S := \{z \in \mathbb{C} \mid |\operatorname{Im}(z)| \leq B\}$ on which the sequence $\{\phi_N\}$ of L -truncated TP_Ω versions of a strictly bandlimited $f \in B(\Omega)$ do not converge uniformly to f . Then there exists a number $\epsilon > 0$ such that for every $N \in \mathbb{N}$ there is an $N' > N$ and a point $z_{N'} = x_{N'} + iy_{N'} \in S$ for which $|f(z_{N'}) - \phi_{N'}(z_{N'})| > 2\epsilon$. Using the bounds (3.4.6) and (3.4.3) it follows that

$$(3.4.47) \quad |f'(z) - \phi'_{N'}(z)| \leq |f'(z)| + |\phi'_{N'}(z)| \leq K\|f\| := M$$

for all $z \in S$, where $M < \infty$. Now choose $\tilde{N} := \lfloor \frac{\Omega \tilde{L}}{\pi} \rfloor$ so large that for $|x| > \tilde{L}$, $|f(x+iy)| < \epsilon$ for all $|y| \leq B$. This can be done by Theorem 3.4.6. This shows that for all $N > \tilde{N}$, if $|x| > L_N$ and $x+iy \in S$ then $|f(x+iy) - \phi_N(x+iy)| = |f(x+iy)| < \epsilon$. Therefore the points $z_{N'}$ lie in the rectangles $|x| \leq L_{N'}$, $|y| \leq B$ for all $N' > \tilde{N}$.

Now the difference function $g_{N'} = f - \phi_{N'}$ is analytic for all $z = x+iy$ such that $|x| < L'$ and continuous for $|x| \leq L'$. Consider a point $w = x+iy_{N'} \in S$ for which $|w - z_{N'}| < \frac{\epsilon}{M}$.

Now consider

$$(3.4.48) \quad |g_{N'}(w) - g_{N'}(z_{N'})| = \left| \int_x^{x_{N'}} g'_{N'}(t+iy_{N'}) dt \right| \leq |w - z_{N'}| M < \epsilon,$$

where the bound (3.4.47) was used.

Therefore,

$$(3.4.49) \quad \epsilon > |g_{N'}(w) - g_{N'}(z_{N'})| \geq |g_{N'}(z_{N'})| - |g_{N'}(w)| > 2\epsilon - |g_{N'}(w)|,$$

so that $|g_{N'}(w)| > \epsilon$ for all $w = x+iy_{N'}$ such that $|x - x_{N'}| < \frac{\epsilon}{M}$. It then follows that

$$\begin{aligned}
(3.4.50) \quad & \int_{-\infty}^{\infty} |f(x+iy_{N'}) - \phi_{N'}(x+iy_{N'})|^2 dx \\
& \geq \int_{x_{N'}}^{x_{N'} + \frac{\epsilon}{2M}} |g_{N'}(x+iy_{N'})|^2 dx \geq \frac{\epsilon^2}{2M} > 0.
\end{aligned}$$

In conclusion, for every $N \in \mathbb{N}$ there is an $N' > N$ for which $\|f - \phi_{N'}\|_{y_{N'}} \geq \frac{\epsilon^2}{2M} > 0$ where $|y_{N'}| \leq B$ for all N' . This is a contradiction since $\|f - \phi_N\|_y$ goes to zero uniformly for all $y \leq B$ in the limit as $N \rightarrow \infty$ by *Proposition 3.3.2. \square

*Proposition 3.3.2 can again be applied to prove *Corollary 3.3.4. The proof of this corollary is very similar to that of Theorem 17 in [72].

PROOF. (of *Corollary 3.3.4) Let f be strictly bandlimited by Ω and let $\{f_N\}_{N=1}^\infty$ be a sequence of TP_Ω versions of f . Let $\Lambda := \{y_n\}_{n \in \mathbb{Z}}$ be a uniformly discrete set of real points. Then Λ has no limit points, and there exists an $\epsilon > 0$ such that $|y_n - y_m| \geq \epsilon$ for all $n, m \in \mathbb{Z}$.

Now

$$(3.4.51) \quad \sum_{n \in \mathbb{Z}} |f(y_n) - \phi_N(y_n)|^p = \sum_{y_n \notin [-L, L]} |f(y_n)|^2 + \sum_{y_n \in [-L, L]} |f(y_n) - \phi_N(y_n)|^p.$$

Recall that $\phi_N = \chi_L f_N$. Since the sequence $(f(y_n))_{n \in \mathbb{Z}}$ is square summable, the first sum in equation (3.4.51) will vanish in the limit as N (or equivalently L) approaches infinity. Therefore, to prove the corollary, it needs to be shown that the second sum also vanishes in this limit.

Let $g_N := f_N - f$. Since $f_N - f$ is entire, $|g_N|^p$ is subharmonic for any $p \in \mathbb{N}$. Given any z_0 in \mathbb{C} it follows that

$$(3.4.52) \quad |g_N(z_0)|^p \leq \frac{1}{2\pi} \int_0^{2\pi} |g_N(z_0 + re^{i\theta})|^p d\theta.$$

For a description of subharmonic functions and their properties, see for example [60]. Multiplying both sides by r , integrating from 0 to δ and then switching from polar to cartesian co-ordinates gives

$$(3.4.53) \quad |g_N(z_0)|^p \leq \frac{1}{\pi\delta^2} \int \int_{|z-z_0| \leq \delta} |g_N(z)|^p dx dy.$$

Here $x = \text{Re}(z)$ and $y = \text{Im}(z)$.

Therefore,

$$(3.4.54) \quad \begin{aligned} \sum_{y_n \in [-L, L]} |g_N(y_n)|^2 &\leq \frac{1}{\pi\delta^2} \sum_{y_n \in [-L, L]} \int \int_{|z| \leq \delta} |g_N(y_n + z)|^2 dx dy \\ &\leq \frac{1}{\pi\delta^2} \sum_{y_n \in [-L, L]} \int_{-\delta}^{\delta} \int_{y_n - \delta}^{y_n + \delta} |g_N(x + iy)|^2 dx dy. \end{aligned}$$

Letting $\delta = \frac{\epsilon}{2}$, the integrals in the above sum (3.4.54) become pairwise disjoint so that

$$(3.4.55) \quad \begin{aligned} \sum_{y_n \in [-L, L]} |g_N(y_n)|^2 &\leq \frac{1}{\pi\delta^2} \int_{-\delta}^{\delta} \int_{-(L+\delta)}^{L+\delta} |g_N(x + iy)|^2 dx dy \\ &= \frac{1}{\pi\delta^2} \int_{-\delta}^{\delta} \int_{-L}^L |f_N(x + iy) - f(x + iy)|^2 dx dy \\ &\quad + \frac{1}{\pi\delta^2} \int_{-\delta}^{\delta} \int_{L+\delta \geq |x| \geq L} |f_N(x + iy) - f(x + iy)|^2 dx dy. \end{aligned}$$

By *Proposition (3.3.2) the first double integral in equation (3.4.55) converges to 0 in the limit as N approaches infinity. The second double integral can be bounded in the following way.

First observe that the triangle inequality implies that

$$\begin{aligned}
(3.4.56) \quad & \left(\int_{L+\delta \geq |x| \geq L} |f_N(x+iy) - f(x+iy)|^2 dx \right)^{\frac{1}{2}} \\
& \leq \left(\int_{L+\delta \geq |x| \geq L} |f_N(x+iy)|^2 dx \right)^{\frac{1}{2}} + \left(\int_{L+\delta \geq |x| \geq L} |f(x+iy)|^2 dx \right)^{\frac{1}{2}} \\
& = \left(\int_{L \geq |x| \geq L-\delta} |\phi_N(x+iy)|^2 dx \right)^{\frac{1}{2}} + \left(\int_{L+\delta \geq |x| \geq L} |f(x+iy)|^2 dx \right)^{\frac{1}{2}}.
\end{aligned}$$

In the last line the fact that f_N is the $2L$ periodic version of ϕ_N was used. Using Theorem 3.4.6 and *Proposition 3.3.2, it is not difficult to see that this vanishes in the limit as N (and L) approach infinity, uniformly for $|y| \leq \delta$. It follows that the second double integral in the last line of equation (3.4.55) vanishes in this same limit, proving the claim. \square

*Corollary 3.3.4 and *Proposition 3.3.1 immediately imply the following sufficiency and necessity conditions for a uniformly discrete set of real points Λ to be a set of sampling for $B(\Omega)$.

*COROLLARY 3.4.9. *Let f , Λ and the sequence (ϕ_N) be as in the previous corollary. Suppose that there is an $N' \in \mathbb{N}$ such that for all $N > N'$,*

$$(3.4.57) \quad B \sum_{n=-\infty}^{\infty} |\phi_N(y_n)|^2 \geq \int_{-L}^L |\phi_N(x)|^2 dx \geq b \sum_{n=-\infty}^{\infty} |\phi_N(y_n)|^2$$

where $0 < b \leq B < \infty$ are independent of the choice of strictly bandlimited $f \in B(\Omega)$. Then $\Lambda = \{y_n\}_{n \in \mathbb{Z}}$ is a set of sampling for $B(\Omega)$.

Conversely suppose there exists a non-zero strictly bandlimited $f \in B(\Omega)$ such that for any $C > 0$ there is an N' such that if $N > N'$, then

$$(3.4.58) \quad \int_{-L}^L |\phi_N(x)|^2 dx > C \sum_{n=-\infty}^{\infty} |\phi_N(y_n)|^2.$$

Then Λ is not a set of sampling for $B(\Omega)$.

The sum in equation (3.4.57) contains only a finite number of non-zero terms since for each fixed $N = \lfloor \frac{\Omega L}{\pi} \rfloor$ the functions ϕ_N vanish outside of $[-L, L]$ and there is a smallest non-zero distance between any two points of Λ .

PROOF. By *Proposition 3.3.1, the middle term of equation (3.4.57) converges to the norm squared of f as $N \rightarrow \infty$, while the left and right hand sides converge to the square sum of the samples $\{f(\lambda_n)\}$ times B and b respectively in the same limit by *Corollary 3.3.4. This proves that

$$(3.4.59) \quad B \sum_{n \in \mathbb{Z}} |f(y_n)|^2 \geq \|f\|^2 \geq b \sum_{n \in \mathbb{Z}} |f(y_n)|^2$$

for all $f \in B(\Omega)$ that are strictly bandlimited. Since strictly Ω -bandlimited functions are dense in $B(\Omega)$ it is not difficult to show that the inequality (3.4.59) holds for all $f \in B(\Omega)$. The converse statement is similarly straightforward to establish. \square

3.5. Discussion

Corollary (3.4.9) of the previous section shows that uniformly discrete sets of points which are sets of sampling for the subspaces $B_L(\Omega)$ of Ω -bandlimited trigonometric polynomials for all L will also be sets of sampling for $B(\Omega)$ provided certain conditions are satisfied. This relationship between the reconstruction and interpolation properties of Ω -bandlimited trigonometric polynomials and those of Ω -bandlimited functions will be discussed in more detail here.

3.5.1. Reconstruction and interpolation of bandlimited trigonometric polynomials. The study of the reconstruction and interpolation properties of Ω -bandlimited functions has been an active area of research for many years. Determining what properties a discrete set of real or complex values $\Lambda := \{y_n\}_{n \in \mathbb{Z}}$ must possess in order to be a set of sampling, uniqueness, or interpolation is in general very difficult (see, *e.g.*, [43] [19]). On the other hand, finding sets of sampling and interpolation for Ω -bandlimited trigonometric polynomials is easy. Interpolation and reconstruction of Ω -bandlimited trigonometric polynomials is straightforward and intuitive since they form a finite dimensional Hilbert space. Any element p of $B_L(\Omega)$ can be written as $p(z) = \sum_{n=-N}^N p_n e^{ik_n z}$ where $k_n := \frac{n\pi}{L}$ and the sequence of complex numbers $\{p_n\}_{n=-N}^N$ is arbitrary. Since each element of $B_L(\Omega)$ has $2N+1$ ‘degrees of freedom’, the $2N+1$ Fourier coefficients $\{p_n\}_{n=-N}^N$, one may intuitively expect that any $p \in B_L(\Omega)$ should be perfectly reconstructible from its values taken on any $2N+1$ values in the interval $[-L, L]$. Indeed, if $p \in B_L(\Omega)$ then letting $w := e^{ik_n z}$, it follows that $w^N p_n(w) = \sum_{n=0}^{2N} p_n w^n$. Since this is a polynomial of degree $2N$, it follows that it has at most $2N$ zeroes, and hence that the function $p_n(w)$ has at most $2N$ zeroes, so that $p_n(z)$ has at most $2N$ zeroes in the interval $[-L, L]$. It follows that the linear operator which maps the $2N+1$ Fourier coefficients $\{p_n\}_{n=-N}^N$ of $p \in B_L(\Omega)$ onto the $2N+1$ sample values $\{p(y_n)\}_{n=-N}^N$, where $\{y_n\}_{n=-N}^N$ are any $2N+1$ points in $[-L, L]$, is invertible. Therefore, given any $2N+1$ points $\{y_n\}$ in $[-L, L]$, any element of $B_L(\Omega)$ is perfectly reconstructible from the values it takes on those points, and given any $2N+1$ complex values $\{a_n\}$, there is an element of $B_L(\Omega)$ that takes those values on the points y_n , $p(y_n) = a_n$. In summary, any $2N+1$ points in the interval $[-L, L]$ is both a set of sampling and a set of interpolation for $B_L(\Omega)$. Note that since $B_L(\Omega)$ is finite dimensional, any set of uniqueness is automatically a set of sampling.

3.5.2. Condition number of the reconstruction matrix. *Corollary 3.4.9 provides some information on the relationship between sets of sampling for $B(\Omega)$ and sets of sampling for $B_L(\Omega)$. For example, let $\Lambda := \{y_n\}$ be a uniformly discrete set of real values such that for all $N > N' \in \mathbb{N}$ there are at least $2N+1$ points of Λ in the interval $[-L_N, L_N]$ where $\frac{\pi N}{\Omega} \leq L_N < \frac{\pi}{\Omega}(N+1)$. Then for each $N > N'$ the set of points $\Lambda \cap [-L_N, L_N]$ is a set of sampling for $B_{L_N}(\Omega)$. Now consider the matrix M which maps the $2N+1$ sample values of $\phi \in B_{L_N}(\Omega)$ onto the $2N+1$ Fourier coefficients, $\{\phi_n\}_{n=-N}^N$ of ϕ . Here

$$(3.5.1) \quad \phi(x) := \sum_{-N}^N \phi_n \frac{e^{ik_n x}}{\sqrt{2L}}$$

and

$$(3.5.2) \quad \phi_n := \int_{-L}^L \phi(x) \frac{e^{-ik_n x}}{\sqrt{2L}} dx.$$

It follows that $M_{jl}^{-1} := \frac{e^{ik_j y_l}}{\sqrt{2L}}$. Note that the Fourier coefficients are scaled differently here than in previous sections. Now suppose that Λ satisfies the assumptions of the first part of *Corollary 3.4.9 so that

$$(3.5.3) \quad \frac{1}{b} \sum_{-N}^N |\phi_n|^2 \geq \sum_{-N}^N |\phi(y_n)|^2 \geq \frac{1}{B} \sum_{-N}^N |\phi_n|^2 \quad \forall \phi \in B_{L_N}(\Omega)$$

where b, B are independent of $N > N'$. This shows that the condition numbers (the ratio of the largest to the smallest eigenvalue) of the matrices M which map the sample values $\{\phi(y_n)\}_{n=-N}^N$ onto the Fourier coefficients $\{\phi_n\}_{n=-N}^N$ where $\phi \in B_{L_N}(\Omega)$, are bounded above by $B/b < \infty$. Thus the first part of *Corollary 3.4.9 can be restated in the following way. If Λ is a uniformly discrete set of real points, and there exists $N' \in \mathbb{N}$ such that for all $N > N'$, the condition numbers of the matrices which map the sample values of $\phi \in B_{L_N}(\Omega)$ onto the Fourier coefficients of ϕ (as defined in equation (3.5.2)) are bounded above by some $C < \infty$, then Λ is a set of sampling for $B(\Omega)$. In other words, if the condition numbers of these finite dimensional matrices which reconstruct elements of $B_{L_N}(\Omega)$ from their samples taken on points of Λ do not diverge in the limit as $N \rightarrow \infty$, then Λ is a set of sampling for $B(\Omega)$. This is exactly what one would expect.

3.5.3. Outlook. My original motivation for approximating bandlimited functions with trigonometric polynomials was to see whether I could prove necessary and sufficient conditions for a uniformly discrete sequence of points $\Lambda := (\lambda_n)_{n \in \mathbb{Z}} \subset \mathbb{R}$ to be a set of sampling for $B(\Omega)$. Although the result of Beurling, Theorem 2.2.3 characterizes sets of sampling for $B(\Omega)$ almost completely in terms of a suitable notion of density, $D_-(\Lambda)$, this theorem makes no conclusion about whether or not Λ is a set of sampling if Λ has the critical density $D_-(\Lambda) = \frac{\Omega}{\pi}$. Since, as discussed in Subsection 3.5.1, sampling theory for the finite dimensional space $B_L(\Omega)$ is particularly simple, my original aim was to see whether facts about sampling theory and sets of sampling for $B(\Omega)$ could be derived from those of the simpler $B_L(\Omega)$ in the limit as $L \rightarrow \infty$. Note that my goal here was not just to rederive and extend known results about sets of sampling for $B(\Omega)$, but also to try to develop a method for proving such results that would readily generalize to prove analogous results for bandlimited functions on manifolds.

The proof of *Proposition 3.3.1 showed that the operators $(P_L(\Omega) - 1)\chi_{[0, \Omega^2]}(\Delta)$ converge strongly to zero as $L \rightarrow \infty$. Here Δ_L was the direct sum of the Laplacian with periodic boundary conditions on $[-L, L]$ and the zero operator outside of this interval, $P_L(\Omega)$ was the projector $\chi_L \chi_{[0, \Omega^2]}(\Delta_L) \chi_L$, χ_L was the projection of $L^2(\mathbb{R})$ onto $L^2[-L, L]$ and Δ was the Laplacian (minus the second derivative operator) on $L^2(\mathbb{R})$. Later, in Chapter 5, we will actually prove that $\chi_L \chi_{[0, \Omega^2]}(\Delta_L) \chi_L$ converges strongly to $\chi_{[0, \Omega^2]}(\Delta)$ in the limit as $L \rightarrow \infty$. One would expect that the choice of boundary conditions on $[-L, L]$ should not matter provided the resulting Laplacian is self-adjoint. Furthermore, one may expect that for any $k \in \mathbb{N}$ and $f \in B(\Omega)$, the k^{th} derivative of the TP_Ω approximation f_N of f on $[-L, L]$ should converge in norm to $f^{(k)}$ as $N \rightarrow \infty$. Both of these conjectures are true, and although they can be proven using the elementary techniques of this chapter, we will delay their proof until Chapter 5 where they will be immediate consequences of a more general result.

Since any $2N + 1$ points in $[-L, L]$, where $N := \lfloor \frac{\Omega L}{\pi} \rfloor$, is a set of sampling for $B_L(\Omega) := P_{L, \Omega} L^2(\mathbb{R})$, and (as will be proven in Chapter 5), $P_{L, \Omega} \xrightarrow{s} P_\Omega$, this suggests that any uniformly discrete set of points Λ which has at least $2N + 1$ points in any interval $[-L, L]$ should be a set of uniqueness for $B(\Omega)$ and that if Λ is such that it always has fewer than $2N + 1$ points in any subinterval of length $2L$, that it is not a set of sampling for $B(\Omega)$. While this may seem intuitive, and is in fact true by Theorem 2.2.3, the strong convergence of the $P_{L, \Omega}$ to P_Ω does not appear to be enough to establish this.

For example, suppose that Λ is such a set of points as described above which has fewer than $2N + 1$ points in any subinterval of length $2L$. Then $\Lambda \cap [-L, L]$ is not a set of sampling for $B_L(\Omega)$ for any $L > 0$. This means that for any $L > 0$ there is a function $0 \neq p_L \in B_L(\Omega)$ such that $p_L(\lambda_n) = 0 \quad \forall \lambda_n \in \Lambda \cap [-L, L]$. Without loss of generality, it can be assumed that for each such L , $\|p_L\| = 1$. To prove that Λ is not a set of uniqueness for $B(\Omega)$, one would need to show that there exists a $0 \neq f \in B(\Omega)$ such that $f(\lambda_n) = 0 \quad \forall \lambda_n \in \Lambda$. If it were true that $P_{L, \Omega}$ converged to P_Ω in norm, this would imply that given any $\epsilon > 0$ and $f \in B(\Omega)$ there is a $L' > 0$ such that $L > L'$ would imply that $\|f - P_{L, \Omega} f\| < \epsilon$ so that $\|f\| > 1 - \epsilon$. In this case, choose $L > L'$ and $f \in B(\Omega)$ such that $P_{L, \Omega} f = p_L$. From equation (3.4.55) in the proof of *Corollary 3.3.4,

it would then follow that $\sum_{n \in \mathbb{Z}} |f(\lambda_n)|^2 = \sum_{n \in \mathbb{Z}} |p_L(\lambda_n) - f(\lambda_n)|^2 < \epsilon^2$. This would then prove that Λ is not a set of sampling, as given any $B > 0$, one could choose $\epsilon > 0$ small enough so that $\|f\|^2 > (1 - \epsilon)^2 > B\epsilon^2 > B \sum_{n \in \mathbb{Z}} |f(\lambda_n)|^2$.

However, I can only show that $P_{L,\Omega} \xrightarrow{s} P_\Omega$ in the strong operator topology, and it is not difficult to see that $P_{L,\Omega}$ cannot converge in norm to P_Ω . To see this, recall that given any $g \in L^2[a, b]$ and $\epsilon > 0$, one can find an $f \in B(\Omega)$ such that $\|(f - g)\chi_{[a,b]}\| < \epsilon$. Given any $L > 0$ and $\epsilon > 0$ choose $g \in L^2[-2L, 2L]$ such that $\|g\| = 1$ and such that $\|g\chi_{[-L,L]}\| < \epsilon$. Then one can find an $f \in B(\Omega)$ such that $\|(f - g)\chi_{[-2L,2L]}\| < \epsilon$. It follows that $\|f\| > 1 - 2\epsilon$ and that $\|P_\Omega - P_{L,\Omega}\| \geq \|(P_\Omega - P_{L,\Omega})f\| \geq \|f\chi_{(-\infty, L) \cup (L, \infty)}\| \geq \|f\| - 2\epsilon \geq 1 - 4\epsilon$. Since $\epsilon > 0$, it follows that norm convergence is not possible.

In summary, although it may seem intuitively clear that the results of this section should be useful for proving facts about necessary and sufficient density for a uniformly discrete set of points to be a set of sampling for $B(\Omega)$, actually proving such results using the techniques of this chapter has turned out to be more subtle and difficult than I had originally anticipated.

For a possible approach to strengthening the results of this chapter to prove results on necessary density of sets of sampling for $B(\Omega)$ and for bandlimited functions on manifolds, see Appendix A. It may be useful to first read Chapter 4 and Chapter 5 before doing this.

Part 2

Sampling theory on curved manifolds

A co-ordinate system independent bandlimit

In the Introduction, Chapter 1, physical motivation was provided for a co-ordinate system independent generalization of bandlimited functions and sampling theory to curved manifolds.

Such a generalization is easily accomplished if one first makes the following observation. The space $B(\Omega)$ of Ω -bandlimited functions is clearly an invariant subspace of any power of the self-adjoint derivative operator $D := i\frac{d}{dx}$ in $L^2(\mathbb{R})$. In particular, if $\Delta := -\frac{d^2}{dx^2}$, it is straightforward to see that $B(\Omega)$ is the range of the spectral projection $\chi_{[0, \Omega^2]}(\Delta)$. Roughly speaking, this projection projects onto the subspace spanned by the non-normalizable ‘eigenvectors’ to Δ whose ‘eigenvalues’ lie in the interval $[0, \Omega^2]$. Indeed, the non-normalizable eigenvectors of Δ to the eigenvalues $|w| > 0$ are the plane waves $e^{\pm iw x}$, and if $f \in B(\Omega)$, then f can be seen as an uncountable ‘linear combination’ of these eigenvectors:

$$(3.5.4) \quad f(x) = \int_{-\Omega}^{\Omega} F(w)e^{iw x} dw.$$

Since the operator $\Delta = -\frac{d^2}{dx^2}$ is the Laplacian of the real line, it is clear how the notion of a bandlimit can be naturally generalized to an arbitrary manifold in a co-ordinate system free manner. Given an arbitrary C^∞ Riemannian or pseudo-Riemannian manifold M , simply define $B(M, \Omega)$ to be the image of $L^2(M)$ under the spectral projection $\chi_{[-\Omega^2, \Omega^2]}(\Delta)$ of the Laplacian or d’Alembertian Δ of the manifold M . The subspaces $B(M, \Omega)$, where M is a Riemannian manifold with bounded curvature were first studied in the context of sampling theory by Pesenson [55]. Pesenson has proved that elements of the spaces $B(M, \Omega)$ obey special reconstruction formulas. Namely, he has shown that there exist uniformly discrete sets of points Λ on the manifold M which have a finite proper density, and such that any $f \in B(M, \Omega)$ is perfectly reconstructible from the values it takes on the points of Λ .

The Laplacian or D’Alembertian of a manifold M is always an unbounded symmetric operator. Furthermore, as we will see, if M is a compact C^∞ Riemannian manifold with boundary, or if M is de Sitter space-time, then there is no unique choice of self-adjoint Laplacian on the manifold. Instead, one can define a symmetric, non self-adjoint Laplacian on the domain $C_0^\infty(M) \subset L^2(M)$ of infinitely differentiable functions with compact support in M , and then construct different self-adjoint Laplacians which can be seen as extensions of this original symmetric Laplacian.

Before studying $B(M, \Omega)$, it will therefore be convenient to first introduce unbounded linear operators, closed operators, and the theory of self-adjoint extensions of symmetric operators. This will be the content of the next chapter.

Closed operators and self-adjoint extensions of symmetric operators

Let \mathcal{H} be a separable Hilbert space, and let $B(\mathcal{H})$ denote the Banach space of bounded linear operators on \mathcal{H} . Recall that the norm for $B \in B(\mathcal{H})$ is $\|B\| := \sup_{\phi \in \mathcal{H} \text{ } \|\phi\|=1} \|B\phi\|$. If $B \in B(\mathcal{H})$, then it is continuous at any point $x \in \mathcal{H}$ since if $x_n \rightarrow x$ in norm then $\|Bx_n - Bx\| \leq \|B\| \|x_n - x\| \rightarrow 0$. It is an elementary exercise to show that the boundedness of a linear operator is equivalent to continuity ([57], pg.9):

THEOREM 4.0.1. *Let B be a bounded linear map between two normed linear spaces. The following are equivalent:*

- (a) B is continuous at one point.
- (b) B is continuous at all points.
- (c) B is bounded.

A linear operator T defined on a dense linear manifold $\mathfrak{D}(T)$ in \mathcal{H} is said to be unbounded if it is not bounded. Here, the term linear manifold denotes a subspace of \mathcal{H} which is not necessarily closed. If T is unbounded, this means that $\sup_{\phi \in \mathfrak{D}(T) \text{ } \|\phi\|=1} \|T\phi\| = \infty$, and, in particular, that one can find a sequence of vectors $\phi_n \in \mathfrak{D}(T)$ such that $\|T\phi_n\| > n$. Typically, the linear manifold $\mathfrak{D}(T)$ is defined as the set of all $\phi \in \mathcal{H}$ such that $T\phi$ belongs to the Hilbert space, *i.e.*, such that $\|T\phi\| < \infty$. The linear manifold $\mathfrak{D}(T)$ on which T is defined is called the domain of T . It follows immediately from the above theorem, Theorem 5.2.2, that if T is unbounded, it is discontinuous at every point $x \in \mathcal{H}$.

Unbounded operators are generally more difficult to deal with than bounded ones. For example, since unbounded operators are generally not defined on the whole Hilbert space, but only a dense subspace, composition of unbounded operators is in general not well-defined. That is if S, T are unbounded operators with domains $\mathfrak{D}(S)$ and $\mathfrak{D}(T)$ respectively, then in general there can be elements $S\phi$ in the range of S that are not in the domain of T so that $TS\phi$ does not make sense. Despite the fact that they are more difficult to deal with, unbounded operators occur frequently throughout physics and applied mathematics, and so it is important to study them. For example, differential operators, which are used ubiquitously in applied mathematics for mathematical modelling, fluid dynamics, etc. are generally unbounded operators acting on a normed linear space or an inner product space. The Laplacian of a Riemannian manifold or the D'Alembertian of a pseudo-Riemannian manifold are always unbounded linear operators, and the study of these operators yields important information about the geometry and topology of the manifold. Furthermore, many operators in physics, in particular quantum mechanics are unbounded. In quantum mechanics, one seeks to represent position and momenta as linear operators \mathbf{x} and \mathbf{p} on a Hilbert space \mathcal{H} , which obey the canonical commutation relations:

$$(4.0.5) \quad [\mathbf{x}, \mathbf{p}] := \mathbf{x}\mathbf{p} - \mathbf{p}\mathbf{x} = iI.$$

It can be concluded immediately that not both \mathbf{x} and \mathbf{p} are bounded. If both \mathbf{x} and \mathbf{p} were both bounded, then iterating equation (4.0.5) yields $[\mathbf{x}, \mathbf{p}^n] = in\mathbf{p}^{n-1}$. This would imply

$$(4.0.6) \quad n\|\mathbf{p}\|^{n-1} = n\|\mathbf{p}^{n-1}\| \leq 2\|\mathbf{x}\|\|\mathbf{p}\|^n$$

so that for every $n \in \mathbb{N}$, $\|\mathbf{x}\|\|\mathbf{p}\| \geq n$, a contradiction.

Unbounded operators are used extensively throughout this thesis. The purpose of this chapter is to collect some basic facts about unbounded operators that we will use repeatedly.

4.1. Closed operators

When dealing with unbounded operators, the concept of a closed, or closable operator is often useful. A linear operator T with domain $\mathfrak{D}(T) \subset \mathcal{H}$ is called closed if its graph,

$$(4.1.1) \quad \Gamma(T) := \{(\phi, T\phi) \in \mathcal{H} \oplus \mathcal{H} \mid \phi \in \mathfrak{D}(T)\},$$

is a closed subspace of $\mathcal{H} \oplus \mathcal{H}$ ([3], section 46). In other words, T is closed if and only if the condition that $(\phi_n)_{n \in \mathbb{N}} \subset \mathfrak{D}(T)$, $\phi_n \rightarrow \phi$ and $T\phi_n \rightarrow \psi$ implies that $\phi \in \mathfrak{D}(T)$ and $T\phi = \psi$. An operator S is called an extension of T if $\mathfrak{D}(S) \supset \mathfrak{D}(T)$ and $S|_{\mathfrak{D}(T)} = T$. In this case one writes $T \subset S$ to denote that S is an extension of T . An operator T is called closable if it has a closed extension. If T is closable, then one can show that it has a smallest closed extension \overline{T} which is called the closure of T ([3], section 38). If T is closable then one obtains its closure \overline{T} by taking the closure of $\Gamma(T)$ in $\mathcal{H} \oplus \mathcal{H}$. In this case, $\Gamma(\overline{T}) = \overline{\Gamma(T)}$.

If a linear operator T is densely defined, *i.e.*, if its domain $\mathfrak{D}(T)$ is dense in \mathcal{H} , then one can uniquely define the adjoint operator T^* of T . Namely, one first defines

$$(4.1.2) \quad \mathfrak{D}(T^*) := \{\psi \in \mathcal{H} \mid \exists \psi^* \in \mathcal{H} \text{ s.t. } \langle T\phi, \psi \rangle = \langle \phi, \psi^* \rangle \forall \phi \in \mathfrak{D}(T)\},$$

and then one defines T^* by $T^*\psi = \psi^*$ for all $\psi \in \mathfrak{D}(T^*)$. Of course, in order for T^* to be a well-defined linear operator, one needs that the vectors ψ^* such that $\langle T\phi, \psi \rangle = \langle \phi, \psi^* \rangle$ for all $\phi \in \mathfrak{D}(T)$ be unique. It is elementary to verify that this will be the case if and only if $\mathfrak{D}(T)$ is dense in \mathcal{H} .

The following basic facts about adjoints and closed operators are easily established, see *e.g.* ([3], Sections 38 and 39).

PROPOSITION 4.1.1. *Let T and S be densely defined linear operators in a separable Hilbert space \mathcal{H} .*

- (a) *If T is closed so is $T - \lambda$ (for any $\lambda \in \mathbb{C}$) and so is T^{-1} (if it exists).*
- (b) *The adjoint T^* is closed whether or not T is.*
- (c) *If T is closable, then $\overline{T}^* = T^*$.*
- (d) *If $T^{**} \exists$, then $T \subset T^{**}$.*
- (e) *If T is bounded, it is closed if and only if $\mathfrak{D}(T)$ is closed.*
- (f) *If $T \subset S$, then $S^* \subset T^*$.*

4.1.0.1. *Remark.* Here, the bi-adjoint, T^{**} of T is defined as $T^{**} := (T^*)^*$. Of course, this operator exists if and only if $\mathfrak{D}(T^*)$ is dense in \mathcal{H} . Note that if T is closed and $T - \lambda$ is bounded below, then parts (a) and (e) of the above proposition imply that the range of $T - \lambda$, $\mathfrak{R}(T - \lambda)$ is a closed subspace of \mathcal{H} . Here, and throughout this thesis, if T is a linear operator defined on $\mathfrak{D}(T)$, $\mathfrak{R}(T) := T\mathfrak{D}(T)$ denotes the range of T .

Using the concept of the graph of a linear operator T , the following theorem which characterizes the closure of a closable operator can be established ([3], Section 46).

THEOREM 4.1.2. *If T is a closable linear operator such that $\overline{\mathfrak{D}(T)} = \mathcal{H}$, then T^{**} exists and $T^{**} = \overline{T}$.*

This theorem implies that if T is densely defined and closable, then the domain of its adjoint T^* is dense in \mathcal{H} .

4.1.0.2. *Definition.* If T is densely defined and closed, and $\mathcal{D} \subset \mathfrak{D}(T)$ is a dense set such that $\overline{T|_{\mathcal{D}}} = T$, then \mathcal{D} is called a **core** for the operator T .

4.1.1. The spectrum of a closed operator. Let A be a densely defined closed operator in \mathcal{H} . As usual, the spectrum, $\sigma(A)$ of A is defined to be the set of all $\lambda \in \mathbb{C}$ for which $(A - \lambda)$ does not have a bounded inverse defined on all of \mathcal{H} .

We will let $\sigma(A)$, $\sigma_p(A)$, $\sigma_c(A)$, $\sigma_r(A)$, and $\sigma_e(A)$ denote the spectrum, and the point, continuous, residual and essential spectrum of A respectively. The point spectrum $\sigma_p(A)$ is defined as the set of all eigenvalues, $\sigma_c(A)$ is here defined as the set of all λ such that $\mathfrak{R}(A - \lambda)$ is not closed, $\sigma_r(A)$ is defined as the set of all λ such that $\lambda \notin \sigma_p(A)$ and $\mathfrak{R}(A - \lambda)$ is not dense, and $\sigma_e(A)$ is the set of all λ such that $A - \lambda$ is not Fredholm. Recall that a closed, densely defined operator T is called Fredholm if $\mathfrak{R}(T)$ is closed and if the dimension of $\mathfrak{Ker}(T)$ and the co-dimension of $\mathfrak{R}(T)$ are both finite. Here, $\mathfrak{Ker}(T)$ denotes the kernel, or nullspace of T , $\mathfrak{Ker}(T) := \{\phi \in \mathfrak{D}(T) \mid T\phi = 0\}$. Since T is closed, $\mathfrak{Ker}(T)$ is always closed. If T is unbounded, we include the point at infinity as part of the essential spectrum. Clearly all the above sets are subsets of $\sigma(A)$, and $\sigma(A) = \sigma_p(A) \cup \sigma_c(A) \cup \sigma_r(A)$.

Note that if S is closed, the closed graph theorem can be applied to show that our definition of $\sigma_c(A)$ is equivalent to the set of all $\lambda \in \mathbb{C}$ for which $A - \lambda$ is not bounded below on $\mathfrak{Ker}(A - \lambda)^\perp$. One often defines the absolutely continuous spectrum $\sigma_{ac}(A)$ as the set of all $\lambda \in \mathbb{C}$ for which $A - \lambda$ is not bounded below. Here, we have decomposed the absolutely continuous spectrum into $\sigma_p(A)$ and $\sigma_c(A)$ (these sets are not disjoint in general), so that $\sigma_{ac}(A) = \sigma_p(A) \cup \sigma_c(A)$. The resolvent set $\rho(T)$ of T is defined as $\mathbb{C} \setminus \sigma(T)$. Hence, $\rho(T)$ is the set of all $z \in \mathbb{C}$ for which $(T - z)^{-1}$ is a bounded linear operator defined on all of \mathcal{H} .

4.2. Symmetric vs. self-adjoint

For unbounded operators there is a distinction between the concepts of a symmetric and a self-adjoint operator. This distinction is extremely important for our purposes and will be used repeatedly in this thesis. Later, in Section 4.4, it will be shown how this distinction is particularly relevant for sampling theory. A densely defined linear operator S is called symmetric if

$$(4.2.1) \quad \langle S\phi, \psi \rangle = \langle \phi, S\psi \rangle \quad \forall \phi, \psi \in \mathfrak{D}(S).$$

From the general definition of the domain of the adjoint of S , Equation (4.1.2), it is clear that $S \subset S^*$, and by Proposition 4.1.1, S^* is closed so that S^* is a closed extension of S , and hence S is closable, $\overline{S} = S^{**}$. It is straightforward to verify that S^{**} is also symmetric.

If S is bounded, then the concepts of symmetric and self-adjoint are equivalent. Furthermore the following theorem shows that if S is symmetric and $\mathfrak{D}(S) = \mathcal{H}$, then S must be bounded.

THEOREM 4.2.1. (Hellinger-Toeplitz) *Let S and T be linear operators on a separable Hilbert space. If $\mathfrak{D}(S) = \mathcal{H} = \mathfrak{D}(T)$ and if $\langle S\phi, \psi \rangle = \langle \phi, T\psi \rangle$ for all $\phi, \psi \in \mathcal{H}$, then S is bounded.*

The proof is provided for the reader's convenience.

PROOF. This is a straightforward application of the uniform boundedness principle.

Suppose S is not bounded. Then one can find a sequence $(\phi_n)_{n \in \mathbb{N}} \subset \mathcal{H}$ such that $\|\phi_n\| = 1$ for all $n \in \mathbb{N}$ and $\|S\phi_n\| > n$. Define linear functionals Φ_n by $\Phi_n(\varphi) := \langle \varphi, S\phi_n \rangle$. Then,

$$(4.2.2) \quad |\Phi_n(\varphi)| = |\langle S\phi_n, \varphi \rangle| = |\langle \phi_n, T\varphi \rangle| \leq \|\phi_n\| \|T\varphi\| = \|T\varphi\| < \infty,$$

for any $\varphi \in \mathcal{H}$. This shows that the numerical sequences $(|\Phi_n(\varphi)|)_{n=1}^\infty$ are bounded for each $\varphi \in \mathcal{H}$. The uniform boundedness principle then implies that there is a $B < \infty$ such that $\|\Phi_n\| \leq B$ for all $n \in \mathbb{N}$. This in turn implies that $\|S\phi_n\|^2 = \langle S\phi_n, S\phi_n \rangle = |\Phi_n(S\phi_n)| \leq B\|S\phi_n\|$ so that $\|S\phi_n\| \leq B$. This contradicts the assumption that $\|S\phi_n\| > n$ for all $n \in \mathbb{N}$. \square

The above theorem shows, in particular, that if S is an unbounded symmetric operator, its domain cannot be all of \mathcal{H} .

4.2.0.1. *Definition.* If S is a densely defined unbounded symmetric operator, it can be that $\mathfrak{D}(S) \subsetneq \mathfrak{D}(S^*)$ so that $S \neq S^*$. If $S = S^*$, then S is called self-adjoint. If S is not closed, but $\overline{S} = S^*$, then S is said to be **essentially self-adjoint** since the fact that $\overline{S}^* = S^*$ (see Proposition 4.1.1) implies that the closure of S , $\overline{S} = S^{**}$, is self-adjoint.

4.2.1. An example of a symmetric operator which is not self-adjoint. For an example of a closed operator which is symmetric but not self-adjoint, consider the following derivative operator D on a dense domain in $L^2[a, b]$. This example is taken from ([3], Section 49). Define

$$(4.2.3) \quad \mathfrak{D}(D) := \{f \in L^2[a, b] \mid f \in AC[a, b]; f' \in L^2[a, b]; f(a) = 0 = f(b)\},$$

and then let $Df = if'$ for all $f \in \mathfrak{D}(D)$ where $f'(x) := \frac{d}{dx}f(x)$ a.e. Here, $AC[a, b]$ denotes the set of all absolutely continuous functions on $[a, b]$. Recall that $f \in AC[a, b]$ if and only if there exists a $g \in L^1[a, b]$ such that

$$(4.2.4) \quad f(x) = f(a) + \int_a^x g(t)dt$$

([7], pg. 339). For the purposes of this section, this will be taken as the definition of an absolutely continuous function. For such an f , the derivative of f exists and is equal to g almost everywhere ([7], pg. 341).

Clearly, D is an unbounded operator on $L^2[a, b]$. That the operator D is symmetric is also elementary to verify using integration by parts: given $f, g \in \mathfrak{D}(D)$,

$$(4.2.5) \quad \begin{aligned} \langle Df, g \rangle &= \int_a^b if'(x)\overline{g(x)}dx = \left[f(x)\overline{g(x)} \right]_a^b - \int_a^b if(x)\overline{g'(x)}dx \\ &= 0 + \int_a^b f(x)\overline{ig'(x)}dx = \langle f, Dg \rangle. \end{aligned}$$

The boundary term above vanishes since all elements in $\mathfrak{D}(D)$ vanish at the endpoints a and b . This proves that D is a symmetric operator. Although it is symmetric, it will now be shown that D is not self adjoint unless $(a, b) = (-\infty, \infty)$.

First consider the case where both a and b are finite. In this case it is straightforward to check that any $f \in AC[a, b]$ such that $f' \in L^2[a, b]$ is in the domain of D^* . If f is such a function, then for all $g \in \mathfrak{D}(D)$, it is easy to check that $\langle Dg, f \rangle = \langle g, f^* \rangle$, where $f^* = if'$. The calculation is exactly the same as in (4.2.5), and the boundary term again vanishes since any g in the domain of D vanishes at the endpoints. This proves that in this case, D is not self-adjoint since, for example, the plane waves $f_t(x) = e^{itx}$ a.e. $x \in [a, b]$, where $t \in \mathbb{R}$, are absolutely continuous, with square integrable derivatives on $[a, b]$, but do not vanish at the endpoints. Hence, $f_t \in \mathfrak{D}(D^*) \notin \mathfrak{D}(D)$.

In fact, it can be shown that $\mathfrak{D}(D^*)$ is precisely those elements of $L^2[a, b]$ which are absolutely continuous on $[a, b]$, and whose image under D is in $L^2[a, b]$:

$$(4.2.6) \quad \mathfrak{D}(D^*) = \{f \in L^2[a, b] \mid f \in AC[a, b]; f' \in L^2[a, b]\}.$$

Suppose $g \in \mathfrak{D}(D^*)$, $D^*g = g^*$. Given any $f \in \mathfrak{D}(D)$

$$(4.2.7) \quad \begin{aligned} \langle Df, g \rangle &= \langle f, g^* \rangle = \int_a^b f(x)\overline{g^*(x)}dx \\ &= \int_a^b f(x)\overline{\frac{d}{dx} \left(\int_a^x g^*(t)dt + c \right)}dx \\ &= \left[f(x)\overline{\frac{d}{dx} \left(\int_a^x g^*(t)dt + c \right)} \right]_a^b - \int_a^b f'(x)\overline{\int_a^x g^*(t)dt + c}dx. \end{aligned}$$

The boundary term vanishes in the last line above since $f \in \mathfrak{D}(D)$ is zero at a, b . Thus,

$$\begin{aligned}
0 &= \langle Df, g \rangle - \langle f, g^* \rangle \\
&= \int_a^b i \frac{d}{dx} f(x) \overline{g(x)} dx - \int_a^b i \frac{d}{dx} f(x) \overline{\left(-i \int_a^x g^*(t) dt + c \right)} dx \\
(4.2.8) \quad &= i \int_a^b f'(x) \overline{\left(g(x) + i \int_a^x g^*(t) dt - c \right)} dx.
\end{aligned}$$

The integration constant c can be chosen such that

$$(4.2.9) \quad \int_a^b \left(g(x) + \int_a^x i g^*(t) dt - c \right) dx = 0.$$

With this choice of c , define

$$(4.2.10) \quad h(x) = \int_a^x \left(g(t) + \int_a^t i g^*(s) ds - c \right) dt.$$

The function h is absolutely continuous on $[a, b]$, and by the above choice of c , $h(a) = h(b) = 0$. Therefore, $h \in \mathfrak{D}(D)$. Substituting h for f in (4.2.8) shows that

$$(4.2.11) \quad \int_a^b \left| g(x) + \int_a^x i g^*(t) dt - c \right|^2 dx = 0.$$

Thus,

$$(4.2.12) \quad g(x) = - \int_a^x i g^*(t) dt + c.$$

This shows that g is absolutely continuous on $[a, b]$, since $g^* \in L^2[a, b] \subset L^1[a, b]$. Furthermore, this shows that $ig'(x) = Dg(x) = g^*(x) = D^*g(x)$ a.e. Since $g \in \mathfrak{D}(D^*)$ was arbitrary, this proves that $\mathfrak{D}(D^*) = \{f \in L^2[a, b] \mid f \in AC[a, b]; f' \in L^2[a, b]\}$.

One can further show that D is a closed operator by showing that $D = D^{**}$. That $D^{**} \supset D$ always holds, so it remains to verify that $D \supset D^{**}$. Since $D \subset D^{**}$ (by part (d) of the Proposition 4.1.1), D^{**} is densely defined so that $\overline{D} = D^{**} \subset D^{***} = \overline{D^*} = D^*$. Here, we have applied Theorem 4.1.2. Thus, every element $f \in \mathfrak{D}(D^{**}) \subset \mathfrak{D}(D^*)$ is absolutely continuous on $[a, b]$, $f' \in L^2[a, b]$ and $D^{**}f = if'$

Hence, for every $g \in \mathfrak{D}(D^*)$,

$$\begin{aligned}
\int_a^b g(x) \overline{if'(x)} dx &= \langle g, D^{**}f \rangle = \langle D^*g, f \rangle \\
(4.2.13) \quad &= \int_a^b ig'(x) \overline{f(x)} dx = i \left[g(x) \overline{f(x)} \right]_a^b + \int_a^b g(x) \overline{if'(x)} dx.
\end{aligned}$$

It follows that $g(b) \overline{f(b)} - g(a) \overline{f(a)} = 0$. Since $g \in \mathfrak{D}(D^*)$ is arbitrary, and the values it takes at the endpoints a, b are arbitrary, this equation can only hold if $f(a) = f(b) = 0$. Thus f vanishes at a, b and hence $f \in \mathfrak{D}(D)$. This proves that $D^{**} \subset D$ so that $D^{**} = D$. Although it was assumed in this example that a, b were finite, the above arguments are easily modified to the cases where either $a = -\infty$, $b = \infty$ or both $a = -\infty$ and $b = \infty$. In particular, the same arguments for the case where $(a, b) = (-\infty, \infty)$ show that D on $L^2(\mathbb{R})$ is an unbounded, closed, self-adjoint operator, since elements of $L^2(\mathbb{R})$ automatically vanish at the end-points $a = -\infty$ and $b = \infty$.

4.3. Self-adjoint extensions

Given a symmetric operator S with dense domain $\mathfrak{D}(S)$, a symmetric operator S' such that $S \subset S'$ is said to be a symmetric extension of S . If such an S' is self-adjoint, it is called a self-adjoint extension of S . If $S \subset T$, then by Proposition 4.1.1, $T^* \subset S^*$. For a symmetric operator it is always true that $S \subset S^*$. This suggests that by suitably enlarging the domain

of a symmetric operator S , and hence shrinking the domain of its adjoint, one may be able to construct a symmetric operator whose domain is equal to the domain of its adjoint, *i.e.*, a self-adjoint extension of S .

Not all symmetric operators densely defined in \mathcal{H} have extensions to self-adjoint operators densely defined in \mathcal{H} . The example of the derivative operator D on an interval can provide a good illustration of when a symmetric operator has self-adjoint extensions.

If $[a, b]$ is a finite interval, then one can define

$$(4.3.1) \quad \mathfrak{D}(D_\beta) := \{\phi \in \mathfrak{D}(D^*) \mid \phi(a) = e^{i2\pi\beta}\phi(b)\}$$

for each $\beta \in [0, 1)$. Using integration by parts, it is again easy to check that D_β is a symmetric operator, and by definition $D \subset D_\beta$. For each $\beta \in [0, 1)$, D_β is in fact self-adjoint:

Given $g \in \mathfrak{D}(D_\beta^*)$ and any $f \in \mathfrak{D}(D_\beta)$

$$(4.3.2) \quad \begin{aligned} \langle D_\beta f, g \rangle &= \int_a^b i f'(x) \overline{g(x)} dx \\ &= \left[f(x) \overline{g(x)} \right]_a^b + \int_a^b f(x) i g'(x) dx \\ &= \left[f(x) \overline{g(x)} \right]_a^b + \langle f, D_\beta^* g \rangle \end{aligned}$$

Since $\langle D_\beta f, g \rangle = \langle f, D_\beta^* g \rangle$ for all $f \in \mathfrak{D}(D_\beta)$, and $g \in \mathfrak{D}(D_\beta^*)$, it follows that

$$(4.3.3) \quad \left[f(x) \overline{g(x)} \right]_a^b = f(b)g(b) - f(a)g(a) = 0 \quad \forall f \in \mathfrak{D}(D_\beta).$$

Using the fact that $f(a) = e^{i2\pi\beta}f(b)$ for all $f \in \mathfrak{D}(D_\beta)$, it follows that

$$(4.3.4) \quad 0 = f(b) \left(\overline{g(b)} - e^{i2\pi\beta} \overline{g(a)} \right).$$

Hence, $g(a) = e^{i2\pi\beta}g(b)$ so that $g \in \mathfrak{D}(D_\beta)$. Since $g \in \mathfrak{D}(D_\beta^*)$ was arbitrary, this shows $\mathfrak{D}(D_\beta^*) \subset \mathfrak{D}(D_\beta)$ and that D_β is indeed self adjoint for any $\beta \in [0, 1)$.

A more systematic method for obtaining all self-adjoint extensions of a symmetric operator can be developed using the so-called Cayley transform of a symmetric operator.

4.3.1. Deficiency indices and the Cayley transform. The following result provides a criterion for determining whether or not a symmetric operator S is self-adjoint. See for example, [57], pg. 256.

CLAIM 4.3.1. *The following are equivalent:*

- (a) S is self-adjoint
- (b) $\mathfrak{R}(S \pm i) = \mathcal{H}$
- (c) S is closed and $\mathfrak{R}(S^* \mp i) = \{0\}$

PROOF. Clearly (a) implies (b) and (c). To see that (b) and (c) are equivalent observe that $\mathfrak{R}(S^* \mp i) = \mathfrak{R}(S \pm i)^\perp$. This shows that (b) implies (c), and that $\mathfrak{R}(S^* \pm i) = \{0\}$ implies that $\mathfrak{R}(S \mp i)$ is dense. If it is further assumed that S is closed, then, by Remark 4.1.0.1, the range of $S \pm i$ is closed so that $\mathfrak{R}(S \pm i) = \mathcal{H}$ and (c) implies (b). The proof will be complete if it can be shown that (b) implies (a). Choose any $\psi \in \mathfrak{D}(S^*)$. Since $\mathfrak{R}(S + i) = \mathcal{H}$ there is a $\phi \in \mathfrak{D}(S)$ such that $(S + i)\phi = (S^* + i)\psi$. Since $S \subset S^*$ this shows that $(S^* + i)(\psi - \phi) = 0$. Hence, $\psi - \phi \in \mathfrak{R}(S^* + i) = \mathfrak{R}(S - i)^\perp = \{0\}$. This shows that $\psi = \phi$ so that $\mathfrak{D}(S^*) \subset \mathfrak{D}(S)$ and S is self-adjoint. \square

The numbers $n_\pm := \dim(\mathfrak{R}(S \pm i))^\perp$ are called the deficiency indices of the symmetric operator S . Here, $\dim V$ denotes the dimension of a subspace V .

4.3.1.1. *Remark.* The above Claim 4.3.1, further implies that S is essentially self-adjoint if $\Re(S \pm i)$ is dense, and that S is essentially self-adjoint if $\Re(S^* \pm i) = \{0\}$.

4.3.1.2. *Remark.* For a closed operator T , a point $z \in \mathbb{C}$ is said to be a point of regularity for the operator if $T - z$ is bounded below. This does not mean that the point z belongs to the resolvent set of T , as $\Re(T - z)$ may not be dense in \mathcal{H} . The set of all such points is called the field of regularity of T . For a symmetric operator S it is clear that any $z \in \mathbb{C} \setminus \mathbb{R}$ belongs to the field of regularity of S since $\|(S - z)\phi\|^2 \geq (\text{Im}(z))^2 \|\phi\|^2$ for any $\phi \in \mathfrak{D}(S)$. Note that it is not difficult to use this inequality to show that $\Re(S - z)$ is closed whenever S is closed for any $z \in \mathbb{C} \setminus \mathbb{R}$.

The following theorem shows that the definition of the deficiency indices of a symmetric operator does not depend on the choice of point z in the open upper half plane ([3], section 78, pg. 92).

THEOREM 4.3.2. *If Γ is a connected subset of the field of regularity of a closed operator T , then $\dim(\Re(T - z)^\perp)$ is constant for all $\lambda \in \Gamma$.*

Given any $\lambda \in \mathbb{C} \setminus \mathbb{R}$, consider the Möbius transformation

$$(4.3.5) \quad \mu_\lambda(z) = \frac{z - \lambda}{z - \bar{\lambda}}.$$

It has an inverse transformation,

$$(4.3.6) \quad \mu_\lambda^{-1}(z) = \frac{\bar{\lambda}z - \lambda}{z - 1}.$$

The transformation μ_λ is a bijection of \mathbb{R} onto $\mathbb{T} \setminus 1$, where \mathbb{T} denotes the unit circle. If S is a self-adjoint operator, then the functional calculus immediately implies that $U_\lambda = \mu_\lambda(S)$ is a unitary operator, and it can be shown that $\mu_\lambda^{-1}(U_\lambda) = S$.

If $z = i$, then this Möbius transform $\mu(S) := \mu_i(S)$, is called the Cayley transform of S and $S = \mu^{-1}(U)$, where $U = \mu(S)$, is called the Cayley transform of U . The Cayley transform can be defined for an arbitrary symmetric, closed, densely defined S whether it is self-adjoint or not. It turns out that the Cayley transform of a closed symmetric operator S is a partially defined transformation which is an isometry from its domain onto its range, both of which, by Remark 4.3.1.2, are closed subspaces in \mathcal{H} . The Cayley transform is a bijection between the set of all self-adjoint operators on a Hilbert space and the set of all unitary operators U such that $U - 1$ has dense range. More generally, it is a bijection between the set of all densely defined, closed, symmetric operators on a Hilbert space \mathcal{H} and the set of all partially defined isometries V in \mathcal{H} which are such that $\Re(V - 1) = (V - 1)\mathfrak{D}(V)$ is dense in \mathcal{H} .

This correspondence is very useful, as many questions about unbounded self-adjoint or symmetric operators are more tractable upon translation into questions involving bounded partially defined isometries.

THEOREM 4.3.3. *(Cayley transform)*

(a) *Suppose S is a densely defined, closed, symmetric linear operator on a separable Hilbert space \mathcal{H} . Then, given any $\lambda \in \mathbb{C} \setminus \mathbb{R}$, the linear transformation $V := \mu_\lambda(S) = (S - \lambda)(S - \bar{\lambda})^{-1}$ is an isometry from the closed subspace $\Re(S - \bar{\lambda})$ onto the closed subspace $\Re(S - \lambda)$ and $S = (\bar{\lambda}V - \lambda)(V - I)^{-1} = \mu_\lambda^{-1}(V)$.*

(b) *If V is an isometry from $\mathfrak{D}(V)$ onto $\Re(V)$, where $\mathfrak{D}(V)$ and $\Re(V)$ are closed subspaces of \mathcal{H} , and $\Re(V - 1)$ is dense in \mathcal{H} , then $S := \mu_\lambda^{-1}(V) = (\bar{\lambda}V - \lambda)(V - 1)^{-1}$ is a closed, densely defined symmetric operator in \mathcal{H} , and $V = \mu_\lambda(S)$.*

In this section, the linear transformation V is called a partially defined isometry instead of a partial isometry, as it is actually not defined on the orthogonal complement of its domain.

PROOF. (a) By Remark 4.3.1.2, $\mathfrak{R}(S - \lambda)$ and $\mathfrak{R}(S - \bar{\lambda})$ are closed. Since λ is not real, $(S - \bar{\lambda})^{-1}$ exists on $\mathfrak{R}(S - \bar{\lambda})$ and is bounded. It is straightforward to verify that V is an isometry: Given $f_1, f_2 \in \mathfrak{D}(V) = \mathfrak{R}(S - \bar{\lambda})$, $f_1 = (S - \bar{\lambda})h_1$, $f_2 = (S - \bar{\lambda})h_2$, where $h_1, h_2 \in \mathfrak{D}(S)$, and

$$\begin{aligned}
\langle Vf_1, Vf_2 \rangle &= \langle (S - \lambda)h_1, (S - \lambda)h_2 \rangle = \langle Sh_1, Sh_2 \rangle - \lambda \langle h_1, Sh_2 \rangle - \bar{\lambda} \langle Sh_1, h_2 \rangle + |\lambda|^2 \langle h_1, h_2 \rangle \\
&= \langle Sh_1, Sh_2 \rangle - \lambda \langle Sh_1, h_2 \rangle - \bar{\lambda} \langle h_1, Sh_2 \rangle + |\lambda|^2 \langle h_1, h_2 \rangle \\
(4.3.7) \quad &= \langle (S - \bar{\lambda})h_1, (S - \bar{\lambda})h_2 \rangle = \langle f_1, f_2 \rangle
\end{aligned}$$

This proves that V is an isometry from its domain onto its range.

Now given any $f \in \mathfrak{D}(V) = \mathfrak{R}(S - \bar{\lambda})$,

$$(4.3.8) \quad f = (S - \bar{\lambda})h$$

and

$$(4.3.9) \quad Vf = (S - \lambda)h$$

for some $h \in \mathfrak{D}(S)$. It is a matter of simple algebra to verify that these two equations imply that

$$(4.3.10) \quad h = \frac{(V - 1)f}{\bar{\lambda} - \lambda},$$

and,

$$(4.3.11) \quad Sh = \frac{(\bar{\lambda}V - \lambda)f}{\bar{\lambda} - \lambda}.$$

Together, these two equations show that $S = (\bar{\lambda}V - \lambda)(V - 1)^{-1}$.

Since equation (4.3.10) holds for all $h \in \mathfrak{D}(S)$, which is dense, the range of $V - I$ is dense. This in turn implies that the inverse operator $(V - I)^{-1}$ exists (although it may be unbounded). If it did not, then 1 would be an eigenvalue of V , $Vg = g$ for some $g \in \mathfrak{D}(V)$. In this case, given any $f \in \mathfrak{D}(V)$,

$$(4.3.12) \quad \langle Vf - f, g \rangle = \langle Vf, g \rangle - \langle f, g \rangle = \langle Vf, Vg \rangle - \langle f, g \rangle = 0.$$

This implies that $g \perp \overline{\mathfrak{R}(V - I)}$, contradicting the density of the range of $V - I$.

(b) The proof of part (b) is merely sketched here: Since $(V - 1)$ has dense range, the inverse operator $(V - 1)^{-1}$ exists (but is not necessarily bounded). Since $(V - I)^{-1}$ exists, so does the operator $S = (\bar{\lambda}V - \lambda)(V - 1)^{-1}$. Furthermore, the domain of S is the range of $V - I$, which is dense.

It is now a matter of simple algebra to verify that S is symmetric. It is also easy to show that S is closed, and proving that $V = (S - \lambda)(S - \bar{\lambda})^{-1}$ is very similar to the proof of part (a) above. \square

4.3.1.3. *Remark.* It is also clear from the above theorem that if S and S' are symmetric operators with Cayley transforms V and V' respectively, then $S \subset S'$ if and only if $V \subset V'$. This shows, in particular, that a closed self-adjoint operator has no non-trivial self-adjoint extensions. Further observe that if S' is a symmetric extension of a symmetric operator S , then $S \subset S' \subset S^*$. This is clear, as if $\varphi \in \mathfrak{D}(S')$, then for any $\phi \in \mathfrak{D}(S)$, $\langle S\phi, \varphi \rangle = \langle \phi, S'\varphi \rangle$. By definition of the adjoint this shows that $\varphi \in \mathfrak{D}(S^*)$ and $S^*\varphi = S'\varphi$.

4.3.2. The Cayley transform and self-adjoint extensions. The Cayley transform provides an efficient method for constructing all self-adjoint extensions of a symmetric operator. If S is a symmetric operator and the partially defined isometry V is its Cayley transform, $V := \mu(S) = \mu_i(S)$, then by the definition of the deficiency indices of S , $n_{\pm} = \mathfrak{R}(S \pm i)$ so that $n_+ = \dim(\mathfrak{D}(V)^{\perp})$ and $n_- = \dim(\mathfrak{R}(V)^{\perp})$. The numbers n_{\pm} are also called the deficiency indices of V . The closed subspaces $\mathfrak{D}_+ := \mathfrak{R}(S + i)^{\perp} = \mathfrak{D}(V)^{\perp}$ and $\mathfrak{D}_- := \mathfrak{R}(S - i)^{\perp} = \mathfrak{R}(V)^{\perp}$ are called the deficiency subspaces of S or V .

With this viewpoint, it is clear when a symmetric operator S has self-adjoint extensions. If S' is a self-adjoint extension of S , then $V \subset U$ where V is the Cayley transform of S and U , a unitary operator, is the Cayley transform of S' . This shows that if S has such a self-adjoint extension, its Cayley transform must be extendable to a unitary operator acting on all of \mathcal{H} . If the deficiency indices of S are equal, then one can extend its Cayley transform V to a unitary operator U by adding to V an arbitrary isometry W from \mathfrak{D}_+ to \mathfrak{D}_- . Then, $U_W = V \oplus W$ on $\mathcal{H} = \mathfrak{D}(V) \oplus \mathfrak{D}_+$ is unitary. Since $\mathfrak{R}(V - 1)$ is dense, so is $\mathfrak{R}(U_W - 1)$, so that the inverse Cayley transform S_W of U_W is well defined, and is a self adjoint extension of S . Conversely, if the deficiency indices of S are not equal, then there is no way to extend its Cayley transform to a unitary operator acting on all of \mathcal{H} , which means that S has no self-adjoint extensions acting on a dense domain in \mathcal{H} . It is, however, always possible to construct a self-adjoint extension \tilde{S} of any symmetric operator S which is densely defined on a larger Hilbert space $\tilde{\mathcal{H}} \supset \mathcal{H}$ ([3], pg. 127).

If S has equal and finite deficiency indices (n, n) , fix an orthonormal basis $\{\phi_i\}_{i=1}^n$ of \mathfrak{D}_+ and an orthonormal basis $\{\psi_i\}_{i=1}^n$ of \mathfrak{D}_- . Then all unitary extensions of the Cayley transform V of S are in 1 – 1 correspondence with $U(n)$, the set set of all $n \times n$ unitary matrices. Namely, given any $n \times n$ unitary matrix W' , define W by $W \sum_{i=1}^n c_i \phi_i = \sum_{i=1}^n (W'c)_i \psi_i$. Then the set of all $U_W := V \oplus W$ on $\mathfrak{D}(V) \oplus \mathfrak{D}(V)^{\perp} = \mathcal{H}$ is the set of all unitary extensions of V . Here, the n component vector $W'c$ is the image of the n component vector c whose components in the basis $\{\phi_i\}$ are $(c_i)_{i=1}^n$ under W' . The inverse Cayley transform of this set of all the U_W is the set of all self-adjoint extensions S_W of S . In the particular case where the deficiency indices of S are $(1, 1)$, $\mathfrak{D}_+ = \mathbb{C}\{\phi_+\}$ and $\mathfrak{D}_- = \mathbb{C}\{\phi_-\}$ where $\|\phi_+\| = \|\phi_-\| \neq 0$, and the family of all unitary extensions of V can be labelled by a single real parameter $\alpha \in [0, 1)$. Namely, the family of all unitary extensions of V are given by $U(\alpha) = V \oplus e^{i2\pi\alpha} \phi_- \otimes \phi_+^* = V \oplus e^{i2\pi\alpha} \langle \cdot, \phi_+ \rangle \phi_-$ as α ranges in the interval $[0, 1)$. The self-adjoint extension of S corresponding to the value $\alpha \in [0, 1)$ will be labelled as $S(\alpha)$.

As a final remark, if S has deficiency indices (n_+, n_-) where $n_+ \neq n_-$, then although it does not have self-adjoint extensions, it can always be extended to a symmetric operator with deficiency indices $(n_+ - n_-, 0)$ if $n_+ > n_-$ or $(0, n_- - n_+)$ if $n_- > n_+$.

Recall that in Section 4.3, a one-parameter family of self-adjoint extensions D_{β} of a symmetric derivative operator D in $L^2[a, b]$ were constructed by enlarging its domain to include vectors ϕ which obey the boundary conditions $\phi(b) = e^{i2\pi\beta} \phi(a)$ for $\beta \in [0, 1)$. In the paragraphs above, we described a different method for constructing all self-adjoint extensions $D(\alpha)$ of D , by extending $\mu(D)$ to the unitary operator $U(\alpha) := \mu(D) \oplus e^{i2\pi\alpha} \phi_- \langle \cdot, \phi_+ \rangle$ on $\mathcal{H} = \mathfrak{D}(\mu(D)) \oplus \mathfrak{D}_+$, where ϕ_{\pm} were fixed unit norm vectors in \mathfrak{D}_{\pm} , and then taking the inverse Cayley transform of $U(\alpha)$. In general, if D' is a self-adjoint extension of D , then $D' = D_{\beta} = D(\alpha)$ where $\alpha \neq \beta$. However, there is a bijection between α and β that is easily calculated as follows.

Eigenvectors of D^* to eigenvalues $\pm i$ are $\varphi_{\pm}(x) = e^{\pm x}$ a.e. $[a, b]$. Let $\phi_+(x) = e^x$, then as one can check, the vector $\phi_-(x) = e^{(a+b)-x} \in \mathfrak{D}_-$ has the same norm as ϕ_+ . Given $\mathfrak{D}(D(\alpha)) := \mathfrak{D}(D) \dot{+} \mathbb{C}(e^{i2\pi\alpha} \phi_- - \phi_+)$, the goal is to find an $\beta \in [0, 1)$ such that $\mathfrak{D}(D(\alpha)) = \mathfrak{D}(D_{\beta})$. Here, $\dot{+}$ denotes the direct sum of two linearly independent linear manifolds, see the next subsection for a more detailed explanation.

Let $\Theta = e^{i2\pi\alpha}$ and $\Gamma = e^{i2\pi\beta}$. Then for any $\phi \in \mathfrak{D}(D(\alpha))$,

$$(4.3.13) \quad \phi(a) = c(\Theta e^b - e^a) \text{ and } \phi(b) = c(\Theta e^a - e^b).$$

It follows that if one chooses

$$(4.3.14) \quad \Gamma := \frac{\Theta e^a - e^b}{\Theta e^b - e^a}$$

then $\mathfrak{D}(D(\alpha)) \subset \mathfrak{D}(D_\beta)$. This relationship is invertible,

$$(4.3.15) \quad \Theta = \frac{\Gamma e^a - e^b}{\Gamma e^b - e^a}.$$

Since the set of all $D(\alpha)$ for $\alpha \in [0, 1)$ is all of the self-adjoint extensions of D , and $D(\alpha) \subset D_\beta$ if and only if β and α are related by the above equations, it follows by Remark 4.3.1.3 that $D(\alpha) = D_\beta$ if β and α are related by equations (4.3.14) and (4.3.15), where $\Theta = e^{i2\pi\alpha}$ and $\Gamma = e^{i2\pi\beta}$.

4.3.3. The Neumann formulas. If M_i , $1 \leq i \leq n$ are linear manifolds of \mathcal{H} , *i.e.* subspaces which are not necessarily closed, one says that they are linearly independent if the condition that $\sum_{i=1}^n f_i = 0$, where $f_i \in M_i$, implies that $f_i = 0 \ \forall 1 \leq i \leq n$. If the M_i are linearly independent, then the notation $M_1 \dot{+} M_2 \dots \dot{+} M_n$ will be used to denote their direct sum. If $M := \dot{+}_{i=1}^n M_i$ and $f = \dot{+}_{i=1}^n f_i \in M$, then this representation of f is unique. For, if it is also true that $f = \dot{+}_{i=1}^n g_i$, then $\dot{+}_{i=1}^n (f_i - g_i) = 0$ so that $f_i - g_i = 0$ for all $1 \leq i \leq n$ by the linear independence of the M_i . Also, given two linearly independent linear manifolds M_1 and M_2 , define the dimension of M_1 modulo M_2 to be the maximum number of vectors in M_1 which are linearly independent modulo M_2 . Using this notion of a direct sum of linearly independent manifolds, the following useful theorem holds ([3], pg. 98).

THEOREM 4.3.4. (von Neumann) *If S is a densely defined symmetric operator then the linear manifolds $\mathfrak{D}(S)$, \mathfrak{D}_+ and \mathfrak{D}_- are linearly independent and $\mathfrak{D}(S^*) = \mathfrak{D}(S) \dot{+} \mathfrak{D}_+ \dot{+} \mathfrak{D}_-$.*

Although these subspaces are not orthogonal and not all closed with respect to the inner product and norm of \mathcal{H} , they are closed and orthogonal with respect to a different inner product. For the closed symmetric operator S , consider the graph inner product $\langle \cdot, \cdot \rangle_S := \langle \cdot, \cdot \rangle + \langle S^* \cdot, S^* \cdot \rangle$ defined on the linear manifold $\mathfrak{D}(S^*)$. Since S^* is closed, the linear manifold $\mathfrak{D}(S^*)$ with this inner product is a Hilbert space. Furthermore, it is easy to check that with respect to this inner product and the norm it generates, $\mathfrak{D}(S)$ and \mathfrak{D}_\pm are closed, mutually orthogonal subspaces of $\mathfrak{D}(S^*)$. For this reason, it will be written that $\mathfrak{D}(S^*) = \mathfrak{D}(S) \oplus_S \mathfrak{D}_+ \oplus_S \mathfrak{D}_-$ where \oplus_S denotes the orthogonal direct sum of these closed subspaces of the Hilbert space $(\mathfrak{D}(S^*), \langle \cdot, \cdot \rangle_S)$. Subspaces of $\mathfrak{D}(S^*)$ closed and orthogonal with respect to this inner product will be called S -closed and S -orthogonal.

If S has equal deficiency indices $n_\pm = n$, and S' is a self-adjoint extension of S , then the Cayley transform U of S' is given by $U = V \oplus W$ on $\mathcal{H} = \mathfrak{D}(V) \oplus \mathfrak{D}_+$, where V is the Cayley transform of S and W is an isometry from \mathfrak{D}_+ onto \mathfrak{D}_- that uniquely defines the extension S' . From the formula for the inverse Cayley transform it is clear that

$$(4.3.16) \quad \begin{aligned} \mathfrak{D}(S') &= \Re(U - 1) = (V - 1)\mathfrak{D}(V) \dot{+} (W - 1)\mathfrak{D}_+ \\ &= \mathfrak{D}(S) \oplus_S (W - 1)\mathfrak{D}_+. \end{aligned}$$

For any $\phi = \varphi \dot{+} (W - 1)\phi_+ \in \mathfrak{D}(S')$ it follows that

$$(4.3.17) \quad S'\phi = S\phi - i\phi_+ - iW\phi_+,$$

since $\phi_+ \in \Re(S^* - i)$ and $W\phi_+ \in \Re(S^* + i)$. These formulas for the domain of the self-adjoint extension S' and its action on such elements are called the Neumann formulas.

4.4. Self-adjoint extensions and sampling theory

The fact that elements of $B(\Omega)$ obey the Shannon sampling formula is a consequence of the fact that the multiplication operator in $B(\Omega)$ is a closed, densely defined, simple symmetric and regular operator with deficiency indices $(1, 1)$. Before establishing this, it will be necessary to first discuss some of the basic properties of the spectra of symmetric operators, and to define the concepts of simple and regular.

4.4.1. Basic spectral properties of symmetric operators and their self-adjoint extensions.

4.4.1.1. *Remark.* If S is symmetric, then by Remark 4.3.1.2, $S - z$ is bounded below by $\text{Im}(z)^{-1}$. This shows that any non-real $z \in \sigma(S)$ must belong to the residual spectrum $\sigma_r(S)$ of S . If S has finite deficiency indices, then the orthogonal complement of $\mathfrak{R}(S - z)$ is finite dimensional for any $z \in \mathbb{C} \setminus \mathbb{R}$, which shows that $\sigma_e(S) \subset \mathbb{R}$.

4.4.1.2. *Remark.* Let T be a closed operator. Recall that if $\lambda \in \sigma_r(T)$, then $\bar{\lambda} \in \sigma_p(T^*)$ since if $\lambda \in \sigma_r(T)$ there exists a $\phi \in \mathfrak{R}(T - \lambda)^\perp$ so that $0 = \langle (T - \lambda)\psi, \phi \rangle$ and hence $\langle \psi, \bar{\lambda}\phi \rangle = \langle T\psi, \phi \rangle$ for all $\psi \in \mathfrak{D}(T)$. By the definition of the adjoint, $\phi \in \mathfrak{D}(T^*)$ and $T^*\phi = \bar{\lambda}\phi$ so that $\bar{\lambda} \in \sigma_p(T^*)$. In particular, this means that if A is a closed self-adjoint operator, then $\sigma_r(A) = \emptyset$ is empty, since if there is a $0 \neq \psi \in \mathfrak{R}(A - \lambda)^\perp$ then $\bar{\lambda} \in \sigma_p(A)$, and hence $\bar{\lambda} = \lambda \in \mathbb{R}$. Recall that $\sigma_r(A)$ is defined to not include any eigenvalues of A (see Subsection 4.1.1). It follows that $\sigma_r(A) = \emptyset$ and that $\sigma(A) = \sigma_p(A) \cup \sigma_c(A)$.

4.4.1.3. *Remark.* By Theorem 4.3.4, the domain of the adjoint S^* of S can be decomposed as

$$(4.4.1) \quad \mathfrak{D}(S^*) = \mathfrak{D}(S) \dot{+} \mathcal{D}_+ \dot{+} \mathcal{D}_-.$$

Here, the linear manifolds $\mathfrak{D}(S)$, \mathcal{D}_+ and \mathcal{D}_- are non-orthogonal, linearly independent, non-closed subspaces of \mathcal{H} . If S has finite deficiency indices, and if the co-dimension of $\mathfrak{R}(S - \lambda)$ is infinite, then $\lambda \in \mathbb{R}$. Furthermore, if $\lambda \in \mathfrak{R}(S - \lambda)^\perp$, then $\bar{\lambda}$ is an eigenvalue to S^* . This and the fact that the dimension of $\mathfrak{D}(S^*)$ modulo $\mathfrak{D}(S)$ is finite (by the above equation (4.4.1)) allows us to conclude that λ must be an eigenvalue of infinite multiplicity to S . Hence, if $\lambda \in \sigma_e(S)$ then $\lambda \in \mathbb{R}$, and it is either an eigenvalue of infinite multiplicity or it belongs to the continuous spectrum of S .

CLAIM 4.4.1. *If S is a symmetric operator with finite and equal deficiency indices, then $\sigma_e(S) = \sigma_e(S')$ for any self-adjoint extension S' of S within \mathcal{H} .*

PROOF. (sketch) Using the Neumann formula (4.3.16), the domain of any self-adjoint extension S' of S can be written as

$$(4.4.2) \quad \mathfrak{D}(S') = \mathfrak{D}(S) \dot{+} (U - 1)\mathcal{D}_+.$$

Here, U is the isometry from \mathcal{D}_+ onto \mathcal{D}_- that defines the self-adjoint extension S' . Since the domain of S and S' differ by a finite dimensional subspace, so do the range of S' and S . Using these facts it is straightforward to establish the claim. \square

A symmetric operator S with dense domain $\mathfrak{D}(S) \subset \mathcal{H}$ is called **simple** if there is no subspace of \mathcal{H} such that the restriction of S to this subspace is self-adjoint. A symmetric operator is called **regular** if its field of regularity is all of \mathbb{C} . Recall here that the field of regularity of a closed operator T is the set of all $z \in \mathbb{C}$ for which $T - z$ is bounded below. Note that if there is any $\lambda \in \mathbb{R}$ that is regular for a symmetric operator S , then by Theorem 4.3.2, S has equal deficiency indices. It also follows from the same theorem that if S is regular, then every $\lambda \in \mathbb{C}$ belongs to the residual spectrum of S .

The following elementary facts about the spectra of symmetric operators and their self-adjoint extensions are proven in [3], Section 83.

THEOREM 4.4.2. *If λ is a real point of regular type of a symmetric operator S with finite deficiency indices (n, n) , then there exists a self-adjoint extension S' of S for which λ is an eigenvalue of multiplicity n .*

THEOREM 4.4.3. *If S is a symmetric operator with finite deficiency indices (n, n) , $\lambda \in \mathbb{R}$, $\lambda \notin \sigma_p(S)$, then the dimension of $\ker(S^* - \lambda)$ does not exceed n .*

THEOREM 4.4.4. *If S has finite and equal deficiency indices, the continuous spectrum $\sigma_c(S')$ of any self-adjoint extension S' of S is equal to the continuous spectrum of S .*

With these results, it can be shown that if S is a simple symmetric, regular operator with deficiency indices $(1, 1)$, then the spectra of all its self-adjoint extensions cover the real line exactly once [36]:

THEOREM 4.4.5. *Let S be a closed symmetric operator densely defined in \mathcal{H} . If S is simple, regular and has deficiency indices $(1, 1)$, then the spectra of any one of its self-adjoint extensions consists of eigenvalues of multiplicity one with no finite accumulation point. Furthermore, the spectra of all of its self-adjoint extensions covers \mathbb{R} exactly once.*

Recall, as discussed in Subsection 4.1.1, that $\sigma(A) = \sigma_p(A) \cup \sigma_c(A)$, for any self-adjoint operator A . Further recall that all self-adjoint extensions of a densely defined symmetric operator with deficiency indices $(1, 1)$ can be labelled by $\alpha \in [0, 1)$. That is, S' is a self-adjoint extension of S if and only if $S' = S(\alpha)$ for some $\alpha \in [0, 1)$, where $S(\alpha)$ is defined as the inverse Cayley transform of $U(\alpha) := V \oplus e^{i2\pi\alpha} \phi_- \langle \cdot, \phi_+ \rangle$ on $\mathcal{H} = \mathfrak{D}(V) \oplus \mathfrak{D}_+$. Here ϕ_{\pm} are fixed unit norm vectors in \mathfrak{D}_{\pm} and V is the Cayley transform of S .

PROOF. (of Theorem 4.4.5) First, by Theorem 4.4.2, since S is regular, any self-adjoint extension $S(\alpha)$ of S has no continuous spectrum, and given any point $\lambda \in \mathbb{R}$, there is an extension $S(\alpha)$ of S for which λ is an eigenvalue of multiplicity one. Secondly, by Theorem 4.4.3, any $\lambda \in \mathbb{R}$ is not an eigenvalue of multiplicity greater than one for any fixed self-adjoint extension of S . Finally, if $\lambda \in \mathbb{R}$ was an eigenvalue to two different self-adjoint extensions $S(\alpha)$ and $S(\beta)$ of S , Theorem 4.4.3 implies that any eigenvector of $S(\alpha)$ to λ must also be an eigenvector of $S(\beta)$ to λ . The Neumann formula (4.3.16) would then imply that

$$(4.4.3) \quad \phi_{\lambda} = \phi_S + c_1(e^{i2\pi\alpha} \phi_- - \phi_+) = \varphi_S + c_2(e^{i2\pi\beta} \phi_- - \phi_+)$$

for some non-zero $c_1, c_2 \in \mathbb{C}$ and $\phi_S, \varphi_S \in \mathfrak{D}(S)$ so that,

$$(4.4.4) \quad 0 = (\phi_S - \varphi_S) + (c_1 e^{i2\pi\alpha} - c_2 e^{i2\pi\beta}) \phi_- + (c_2 - c_1) \phi_+$$

in $\mathfrak{D}(S^*) = \mathfrak{D}(S) \dot{+} \mathfrak{D}_- \dot{+} \mathfrak{D}_+$. Since these three linear manifolds are linearly independent it follows that $\phi_S = \varphi_S$, $c_1 e^{i2\pi\alpha} = c_2 e^{i2\pi\beta}$ and that $c_1 = c_2$. This shows, in particular, that $e^{i2\pi\alpha} = e^{i2\pi\beta}$. Since $\alpha, \beta \in [0, 1)$ this proves that $\alpha = \beta$ so that $S(\alpha) = S(\beta)$, contradicting the assumption that these are two different self-adjoint extensions of S .

The fact that the eigenvalues of any $S(\alpha)$ cannot have a finite accumulation point follows from the assumption that S is regular. If λ was an accumulation point of $\sigma(S(\alpha))$, then $\lambda \in \sigma_e(S(\alpha)) = \sigma_e(S)$. By remark 4.4.1.3, such a λ would belong to either $\sigma_p(S)$ or $\sigma_c(S)$, contradicting the regularity of S . \square

4.4.1.4. *Remark.* If B is a symmetric operator satisfying the assumptions of the above theorem, then it must be unbounded. Theorem 4.2.1 implies that any bounded symmetric operator is self-adjoint and hence cannot have non-zero deficiency indices. Together with the above theorem, this implies that the spectrum of each $B(\alpha)$ must be an infinite sequence of eigenvalues of multiplicity one with no finite accumulation point. It follows that the spectrum of $B(\alpha)$ can be arranged as a non-decreasing sequence $(\lambda_n(\alpha))_{n \in \mathbb{M}}$ where here \mathbb{M} is either \mathbb{N} , $-\mathbb{N}$, or \mathbb{Z} depending on whether the spectrum of $B(\alpha)$ is bounded below, above, or neither bounded above nor below.

4.4.1.5. *Remark.* Note that the same argument as in the proof above shows that the intersection of the domains of any two different self-adjoint extensions of S is equal to the domain of S .

4.4.2. Symmetric multiplication operators and sampling theory.

4.4.2.1. *The example of $B(\Omega)$.* Consider the space $B(\Omega)$ of Ω -bandlimited functions. The image of $B(\Omega)$ under the unitary Fourier transform \mathcal{F} is $L^2[-\Omega, \Omega]$. As was shown earlier in Subsection 4.2.1, one can define a closed symmetric derivative operator $D = i \frac{d}{dx}$ on the following dense domain $\mathfrak{D}(D) \subset L^2[-\Omega, \Omega]$:

$$(4.4.5) \quad \mathfrak{D}(D) := \{f \in L^2[-\Omega, \Omega] \mid f \in AC[-\Omega, \Omega]; f' \in L^2[-\Omega, \Omega]; f(\Omega) = 0 = f(-\Omega)\}.$$

The image of D under the inverse Fourier transform is a symmetric operator which acts as multiplication by the independent variable on a dense domain in $B(\Omega)$.

The symmetric operator D has deficiency indices $(1, 1)$ and is simple. To see this note that the deficiency indices (n_+, n_-) of D are equal to the number of solutions ϕ to the equation $D^*\phi = \pm i\phi$, i.e., to $i\phi' = \pm i\phi$ or $\phi' = \pm \phi$. The solutions to these equations are $\phi_{\pm}(x) = e^{\pm x}$ a.e. $[-\Omega, \Omega]$. This shows that both i and $-i$ are eigenvalues of multiplicity one to D^* so that $n_+ = 1 = n_-$.

Furthermore, the operator D is both simple and regular. This will be proven by showing that the minimum uncertainty of D is bounded below:

DEFINITION 4.4.6. *The **uncertainty** of a symmetric operator S with respect to a unit-length vector $\phi \in \mathfrak{D}(S)$ is denoted by $\Delta S[\phi] := \sqrt{\langle S\phi, S\phi \rangle - \langle S\phi, \phi \rangle^2}$. The overall lower bound on the uncertainty of S will be denoted by $\Delta S := \inf_{\phi \in \mathfrak{D}(S), \|\phi\|=1} \Delta S[\phi]$.*

Consider the multiplication operator \widetilde{M} on $L^2[-\Omega, \Omega]$. This is a bounded, self-adjoint operator defined on the whole space. It is a simple algebraic exercise to prove the following lower bound on the product of the uncertainties for two symmetric operators S and T for unit norm vectors $\phi \in \mathfrak{D}(T) \cap \mathfrak{D}(S)$:

$$(4.4.6) \quad \Delta S[\phi] \Delta T[\phi] \geq \frac{1}{2} |\langle S\phi, T\phi \rangle - \langle T\phi, S\phi \rangle|.$$

This above inequality is often referred to as the Heisenberg uncertainty relation. Observe that \widetilde{M} maps $\mathfrak{D}(D)$ into itself since it preserves the boundary conditions, the function $f(x) = x$ a.e. is absolutely continuous, and the product of any two absolutely continuous functions is itself absolutely continuous ([7], pg. 337). It is clear that for all unit length vectors ϕ , $\Delta \widetilde{M}[\phi] \leq \|\widetilde{M}\| = \Omega$ so for all unit length $\phi \in \mathfrak{D}(D)$, it follows that

$$(4.4.7) \quad \Delta D[\phi] \geq \frac{1}{2\Omega} \left| \langle \phi, (D\widetilde{M} - \widetilde{M}D)\phi \rangle \right| = \frac{1}{2\Omega} > 0.$$

This shows that $\Delta D \geq \frac{1}{2\Omega} > 0$. It follows that the symmetric operator D can have no eigenvalues and no continuous spectrum on the real line as otherwise there would be unit length vectors ϕ for which $\Delta D[\phi]$ is either 0 or arbitrarily small. This shows that $D - \lambda$ is bounded below for any $\lambda \in \mathbb{R}$ so that D is regular. Furthermore, D must also be simple. Otherwise, if there were a subspace S of $L^2[-\Omega, \Omega]$ such that the restriction of D to S was self-adjoint, then D would have eigenvalues or continuous spectra. The relationship between the minimum uncertainty of a symmetric operator and its spectrum will be analyzed in greater detail in Section 7.2.1. In conclusion, the closed, symmetric operator D is simple, regular and has deficiency indices $(1, 1)$. An application of Theorem 4.4.5 now shows that the spectra of all self-adjoint extensions of D cover the real line exactly once.

Consider the family of self-adjoint extensions $D(\alpha)$ of D , defined in Section 4.3. This family covers all self-adjoint extensions of D as discussed in Subsection 4.3.2. It follows that as α ranges in the interval $[0, 1)$, the spectra of the $D(\alpha)$ cover \mathbb{R} exactly once.

Now consider the space $B(\Omega)$ of bandlimited functions, which is the image of $L^2[-\Omega, \Omega]$ under the Fourier transform. It follows that the Fourier transform of D , M , is a symmetric multiplication operator on a dense domain in $B(\Omega)$, and that it has a one parameter family of self-adjoint extensions $M(\alpha)$. As was shown in Chapter 2, $B(\Omega)$ is a reproducing kernel Hilbert space. Let δ_x denote the point evaluation vector at the point $x \in \mathbb{R}$. Since there is no $x \in \mathbb{R}$ at which every bandlimited function vanishes, it follows that $\delta_x \neq 0$ for any $x \in \mathbb{R}$. The vector δ_x is an eigenvector of M^* to eigenvalue x since for any $\phi \in \mathfrak{D}(M)$, $\langle M\phi, \delta_x \rangle = x\phi(x) = \langle \phi, x\delta_x \rangle$. Since this is true for all $\phi \in \mathfrak{D}(M)$, it follows from the definition of the adjoint that $M^*\delta_x = x\delta_x$. If $M(\alpha)$ is a self-adjoint extension of M , it has an orthogonal eigenbasis of vectors $(\tilde{\delta}_{\lambda_n(\alpha)})_{n \in \mathbb{M}}$ where here they are arranged in order of increasing $\lambda_n(\alpha)$ ¹. Since any eigenvector to $M(\alpha)$ is also an eigenvector to M^* , and since the multiplicity of any eigenvalue $x \in \mathbb{R}$ to M^* is exactly 1, it follows that $\tilde{\delta}_{\lambda_n(\alpha)} = c_n(\alpha)\delta_{\lambda_n(\alpha)}$ for all $n \in \mathbb{M}$ and $\alpha \in [0, 1)$. This immediately proves that $B(\Omega)$ obeys a sampling formula! If $\phi \in B(\Omega)$ then letting $\delta_{\lambda_n} := \delta_{\lambda_n(\alpha)}$ for some fixed $\alpha \in [0, 1)$,

$$\begin{aligned}
\phi(x) &= \langle \phi, \delta_x \rangle = \sum_{n \in \mathbb{M}} \langle \phi, \frac{\delta_{\lambda_n}}{\|\delta_{\lambda_n}\|} \rangle \langle \frac{\delta_{\lambda_n}}{\|\delta_{\lambda_n}\|}, \delta_x \rangle \\
&= \sum_{n \in \mathbb{M}} \phi(\lambda_n) \langle \delta_{\lambda_n}, \delta_x \rangle \frac{1}{\|\delta_{\lambda_n}\|^2} \\
(4.4.8) \quad &= \sum_{n \in \mathbb{M}} \phi(\lambda_n) \frac{\delta_{\lambda_n}(x)}{\delta_{\lambda_n}(\lambda_n)}.
\end{aligned}$$

In this particular case, the eigenvalues of $D(\alpha)$ are $\lambda_n(\alpha) = \frac{(n+\alpha)\pi}{\Omega}$, $\mathbb{M} = \mathbb{Z}$, and the vectors $\delta'_x(x) = \frac{\Omega}{\pi} \frac{\sin \Omega(x-x')}{\Omega(x-x')}$, have norm squared $\|\delta_x\|^2 = \frac{\Omega}{\pi}$ so that the above reconstruction formula, equation (4.4.8), becomes the usual Shannon sampling formula:

$$(4.4.9) \quad \phi(x) = \sum_{n \in \mathbb{Z}} \phi(\lambda_n(\alpha)) \frac{\sin \Omega(x - \lambda_n(\alpha))}{\Omega(x - \lambda_n(\alpha))} \quad \lambda_n(\alpha) = \frac{(n + \alpha)\pi}{\Omega}.$$

4.4.2.2. *A sufficient condition for a RKHS to have the sampling property.* In fact, we have proven something more general. We have proven the following:

*THEOREM 4.4.7. *Let \mathcal{H} be a reproducing kernel Hilbert space of functions on \mathbb{R} with positive definite kernel function (i.e., $K(x, x) = \|\delta_x\|^2 > 0$ for all $x \in \mathbb{R}$). Suppose that the operator of multiplication by the independent variable, M , is densely defined in \mathcal{H} , and is symmetric, regular and simple with deficiency indices $(1, 1)$. Then \mathcal{H} has the sampling property.*

In particular, if $\sigma(M(\alpha)) = (\lambda_n(\alpha))_{n \in \mathbb{M}}$ and $\delta_{\lambda_n(\alpha)}$ is the point evaluation vector at the point $\lambda_n(\alpha) \in \mathbb{R}$, then for any $\phi \in \mathcal{H}$, the vectors $\phi_N := \sum_{|n| \leq N; n \in \mathbb{M}} \langle \phi, \delta_{\lambda_n(\alpha)} \rangle \delta_{\lambda_n(\alpha)} \frac{1}{\|\delta_{\lambda_n(\alpha)}\|^2}$ converge to ϕ as $N \rightarrow \infty$ both pointwise and in norm. If the map $x \mapsto \delta_x$ is continuous, this pointwise convergence is uniform on compact subsets of \mathbb{R} .

PROOF. The proof is the same as above. The assumptions on M imply that the spectra of all self-adjoint extensions of M cover \mathbb{R} exactly once, and consist of eigenvalues of multiplicity one. Since \mathcal{H} is a reproducing kernel Hilbert space, let δ_x denote the point evaluation vector at $x \in \mathbb{R}$. Since $\delta_x \neq 0$ for any $x \in \mathbb{R}$, we conclude, as above, that δ_x is an eigenvector of M^* to eigenvalue x . Namely, $\langle M\phi, \delta_x \rangle = x\phi(x) = x\langle \phi, \delta_x \rangle = \langle \phi, x\delta_x \rangle$ which implies that $M^*\delta_x = x\delta_x$, by the definition of the adjoint. It follows that if $(\lambda_n(\alpha))_{n \in \mathbb{M}}$ are the sequences of eigenvalues of the self-adjoint extensions $M(\alpha)$ of M , that $\{\delta_{\lambda_n(\alpha)}\}_{n \in \mathbb{M}}$ is a total orthogonal set of eigenvectors to $M(\alpha)$ for each $\alpha \in [0, 1)$. This proves that \mathcal{H} has the Kramer sampling property.

The rest of the theorem is similarly straightforward to establish, and is similar to the proof of Claim 2.2.2. \square

¹Here $\mathbb{M} = -\mathbb{N}, \mathbb{N}$ or \mathbb{Z} , see Remark 4.4.1.4

In particular, if $\mathcal{H} := L^2(\mathbb{R}, d\mu)$, where μ is a regular, countably additive Borel measure, then the multiplication operator M is a densely defined self-adjoint operator in this Hilbert space. To see this, define M' to be the multiplication operator on the set $\mathfrak{D}(M')$ of all $\phi \in L^2(\mathbb{R}, d\mu)$ which have compact support. I claim that this operator is essentially self-adjoint so that its closure is a densely defined self-adjoint multiplication operator, M . First, it is clear that M' is densely defined, and that for any $\phi, \psi \in \mathfrak{D}(M')$ that

$$(4.4.10) \quad \langle M\phi, \psi \rangle = \int_{-\infty}^{\infty} x\phi(x)\overline{\psi(x)}d\mu(x) = \int_{-\infty}^{\infty} \phi(x)\overline{x\psi(x)}d\mu(x) = \langle \phi, M'\psi \rangle,$$

so that M' is symmetric. To prove that M' is essentially self-adjoint, it suffices to show, by Remark 4.3.1.1, that $\mathfrak{R}(M' \pm i)$ is dense in \mathcal{H} . Given $[-L, L] \subset \mathbb{R}$, we can view $\mathcal{H}_L := L^2([-L, L]; \mu)$ as a subspace of \mathcal{H} . It is clear that $M'|_{\mathcal{H}_L}$ is both symmetric and bounded, and hence is self-adjoint. By Claim 4.3.1, it follows that $(M' \pm i)\mathcal{H}_L = \mathcal{H}_L$ for any $L > 0$. This shows in turn that $(M' \pm i)\mathfrak{D}(M') \supset \mathcal{H}_L$ for any $L > 0$. In conclusion, $\mathfrak{R}(M' \pm i)$ is dense in \mathcal{H} so that M' is essentially self-adjoint.

4.4.2.3. *Remark.* The discussion of the above paragraph shows that *Theorem 4.4.7 can be applied to reproducing kernel subspaces S of $\mathcal{H} := L^2(\mathbb{R}, d\mu)$ which have the property that there is a dense linear subspace $\mathfrak{D}(\mathcal{M}) \subset S$, such that $\mathfrak{D}(\mathcal{M}) \subset \mathfrak{D}(M)$ and such that $\mathcal{M} := M|_{\mathfrak{D}(\mathcal{M})}$ is a symmetric operator satisfying the conditions the theorem.

I expect that it should be possible to further refine *Theorem 4.4.7. Namely, I expect that under certain suitable additional assumptions on the reproducing kernel Hilbert space \mathcal{H} , the assumption that the multiplication operator M is densely defined and symmetric on \mathcal{H} should automatically imply that it is simple, regular and has deficiency indices $(1, 1)$. I further expect that if the multiplication operator in a reproducing kernel Hilbert space has the properties assumed in *Theorem 4.4.7, that the point evaluation vectors δ_x of the RKHS must all be non-zero. These refinements are something that I am currently working on.

Later, in the final part, Part III, of this thesis, we will return to this idea that if the multiplication operator in a reproducing kernel Hilbert space satisfies the conditions of *Theorem 4.4.7, then the space has the sampling property.

Bandlimited functions on Riemannian manifolds

5.1. Bandlimited functions on compact manifolds

As discussed previously, a natural generalization of the space of bandlimited functions to a manifold M is $B(M, \Omega) := \chi_{[0, \Omega^2]}(\Delta)L^2(M)$ where Δ is the Laplacian on M [55]. If, however, M is a manifold that has compact closure and a boundary, then there is no unique choice of a self-adjoint Laplacian on M . For example, by choosing different boundary conditions on the boundary of M , *e.g.* Dirichlet or Neumann boundary conditions, one can define different self-adjoint Laplacians on M . This means that one can define $B(M, \Omega) := \chi_{[0, \Omega^2]}(\Delta')L^2(M)$ where Δ' is any choice of a self-adjoint Laplacian on M . In other words, if M has compact closure, there is no unique natural choice for the subspace of Ω -bandlimited functions. This will be discussed in detail in Subsection 5.1.1. Sampling theory on Riemannian manifolds with compact closure is particularly simple. Indeed, if M is such a manifold, an appropriate choice of self-adjoint Laplacian on M yields a finite dimensional subspace $B(M, \Omega)$. This will be discussed in detail in Section 5.3.

Now suppose that M is an oriented C^∞ complete Riemannian manifold. Let K_n be a sequence of nested, connected submanifolds of M , $K_n \subset K_{n+1}$, with compact closures and smooth, C^∞ boundaries, and whose union, $\cup_{n=1}^\infty K_n$ is all of M . A natural question that will be addressed in this chapter is whether the projectors $P_{n, \Omega}$ onto the subspaces $B(K_n, \Omega)$ converge in a suitable sense to the projector P_Ω onto $B(M, \Omega)$ in the limit as $n \rightarrow \infty$ if these subspaces are viewed as subspaces of $L^2(M)$. Here, $P_{n, \Omega} := \chi_n \chi_{[0, \Omega^2]}(\Delta_n) \chi_n$, $P_\Omega := \chi_{[0, \Omega^2]}(\Delta)$, χ_n denotes the projector of $L^2(M)$ onto $L^2(K_n)$, and Δ_n denotes an arbitrary choice of self-adjoint Laplacian on K_n . The affirmative answer to this question will be given by the proof of the following proposition.

*PROPOSITION 5.1.1. *If $a, b \in \mathbb{R}$, $a < b$; $a, b \notin \sigma_p(\Delta)$, then $\chi_n \chi_{(a, b)}(\Delta_n) \chi_n$ converges strongly to $\chi_{[a, b]}(\Delta)$.*

This proposition appears in [48] which has recently been accepted for publication.

Here, Δ_n is an arbitrary choice of a self-adjoint Laplacian on K_n and $\sigma_p(\Delta)$ denotes the set of all eigenvalues of Δ . In particular, this proposition claims that P_n converges strongly to P_Ω provided that 0 and Ω^2 are not eigenvalues of Δ . The proof of the above proposition is given in Section 5.2 and is one of the main results of this chapter.

For example, if $M = \mathbb{R}$, then $B(\mathbb{R}, \Omega) = B(\Omega)$ is the regular space of Ω -bandlimited functions. In this case, one can choose a sequence of nested intervals (connected submanifolds of \mathbb{R} with compact closures) $K_n := (-L_n, L_n)$, where $\lim_{n \rightarrow \infty} L_n = \infty$. One can further choose the particular self-adjoint Laplacian Δ_n whose domain consists of functions which obey periodic boundary conditions. Then, given any $f \in L^2(\mathbb{R})$ it is not hard to see that

$$(5.1.1) \quad P_{n, \Omega} f(x) = \chi_{[-L_n, L_n]}(x) \sum_{j=-N_n}^{N_n} f_j e^{ik_j x}.$$

Here $k_j := \frac{j\pi}{L_n}$, $f_j := \frac{1}{2L_n} \int_{-L_n}^{L_n} f(x) e^{-ik_j x} dx$, $N_n := \lfloor \frac{\Omega L_n}{\pi} \rfloor$, and $\chi_I(x)$ denotes the characteristic function of the interval I . This image (5.1.1) of f under the projector $P_{n, \Omega}$ is just that Ω -bandlimited trigonometric polynomial which we called the L -truncated TP_Ω version

of $f \in B(\Omega)$ in Chapter 3. *Proposition 5.1.1 then implies that $P_{n,\Omega}f$ converges to $P_\Omega f$ as $n \rightarrow \infty$. In particular, if $f = P_\Omega f$ is already bandlimited, then $f_n \rightarrow f$. Hence *Proposition 5.1.1 implies *Proposition 3.3.1, and is a generalization of *Proposition 3.3.1 to an orientable, C^∞ , complete Riemannian manifold.

5.1.1. A Reminder of some basic facts about Riemannian manifolds. Let M be an n -dimensional C^∞ manifold. This means that there is an atlas $\{(U_\alpha, x_\alpha)\}_{\alpha \in I}$ where $\{U_\alpha\}_{\alpha \in I}$ is an open cover of M , and each x_α is a homeomorphism of U_α onto an open subset of \mathbb{R}^n such that whenever $U_\alpha \cap U_\beta \neq \emptyset$, the map $x_\alpha \circ x_\beta^{-1} : x_\beta(U_\alpha \cap U_\beta) \rightarrow x_\alpha(U_\alpha \cap U_\beta)$ is a C^∞ diffeomorphism. Each pair (U_α, x_α) is called a chart. Suppose that (U, x) is a chart and $p \in U$ so that $x(p) = (x^1(p), x^2(p), \dots, x^n(p))$. The functions $x^i(p)$ are called co-ordinate functions, and given a point $p \in M$, the points $x^i(p)$ are called its co-ordinates.

Recall that a function $f : M \rightarrow \mathbb{R}$ is said to be C^k if given an atlas (U_α, x_α) of M , each of the functions $f \circ x_\alpha^{-1} : x_\alpha(U_\alpha) \rightarrow \mathbb{R}$ is a C^k function of $x_\alpha(U_\alpha) \subset \mathbb{R}^n$. If $f : M \rightarrow \mathbb{C}$, f is said to be C^∞ if both its real and imaginary parts are C^∞ .

Given any point $p \in M$, let $T_p(M)$ denote the tangent space at the point p . Recall that $T_p(M)$ is a vector space consisting of all linear maps $\xi : C^\infty(M) \rightarrow \mathbb{R}$ which obey the Leibniz rule:

$$(5.1.2) \quad \xi(fg) = f(p)\xi(g) + \xi(f)g(p),$$

for all $f, g \in C^\infty(M)$. Let $TM := \bigsqcup_{p \in M} T_p(M)$ denote the disjoint union of the tangent spaces and $\Gamma(TM)$ denote the vector space of all vector fields on M . Recall that a vector field X on M is a linear map $X : C^\infty(M) \rightarrow C^\infty(M)$ which obeys the Leibniz rule $X(fg) = X(f)g + fX(g)$ for all $f, g \in C^\infty(M)$. In particular for each $p \in M$, $X|_p \in T_p(M)$ where $X|_p f := (Xf)(p)$.

A Riemannian metric g on M is an inner product g_p on $T_p(M)$ for each $p \in M$ such that for any two vector fields X, Y the function f defined by $f(p) := g_p(X|_p, Y|_p)$ is C^∞ . Given any two vector fields X, Y , let $\langle X, Y \rangle$ denote the C^∞ function defined by $\langle X, Y \rangle(p) = g_p(X, Y)$.

An n -dimensional C^∞ Riemannian manifold, is a pair (M, g) where M is a C^∞ manifold, and g is a Riemannian metric on M .

For $f \in C^1(M)$, the gradient of f , $\text{grad}(f)$ is the vector field which satisfies $\langle \text{grad}(f), Y \rangle = Yf$ for each $Y \in \Gamma(TM)$. Let (U, x) be a chart with co-ordinate functions x^i , $1 \leq i \leq n$. Define $\partial_i|_p \in T_p(M)$, where $p \in U$ by

$$(5.1.3) \quad \partial_i|_p f = \frac{\partial(f \circ x^{-1})}{\partial x^i} \Big|_{x(p)}.$$

The set of all $\partial_i|_p$ form a basis for the vector space $T_p(M)$.

The Laplacian Δ of a Riemannian manifold can be defined as an operator $\Delta : C^\infty(M) \rightarrow C^\infty(M)$ uniquely determined by (and which in turn uniquely determines) the metric g of M . Consider a chart (U, x) . If x^i are the co-ordinate functions and ∂_i are the corresponding co-ordinate vector fields, then if $g_{ij} := g(\partial_i, \partial_j)$ and $g := \det(g_{ij})$, then

$$(5.1.4) \quad \Delta f = \frac{1}{\sqrt{g}} \sum_{i,j=1}^n \partial_i(\sqrt{g} g^{ij} \partial_j f).$$

Clearly one can actually define Δ on all $f \in C^2(M)$. One can take the above formula as the definition of the Laplacian, and then show that this definition is independent of the choice of co-ordinates.

An oriented Riemannian manifold is a Riemannian manifold, together with an atlas (x_α, U_α) such that whenever $U_\alpha \cap U_\beta \neq \emptyset$, the diffeomorphism $x_\alpha \circ x_\beta^{-1} : x_\beta(U_\alpha \cap U_\beta) \rightarrow x_\alpha(U_\alpha \cap U_\beta)$ is orientation preserving, *i.e.* if the determinant of the Jacobian associated with this co-ordinate transformation in \mathbb{R}^n is positive.

Recall that given a Riemannian manifold, there is a canonical volume measure dV on its Borel sets which is defined using the metric. Let (U, x) be a chart, and $f : U \rightarrow \mathbb{R}$ be a measurable function. Then, in this chart,

$$(5.1.5) \quad \int_U f dV = \int_{x(U)} (f \circ x^{-1}) \sqrt{g \circ x^{-1}} dx^1 \dots dx^n.$$

For an oriented manifold, M , one can extend this notion of integration to the entire manifold M ([59], pgs. 14-15). If f is complex valued, we can clearly define the above integral by splitting it into its real and imaginary parts. Using this notion of integration, one can define the Banach spaces $L^p(M)$ with the norms

$$(5.1.6) \quad \|f\| := \int_M \|f\|^p dV.$$

In particular $L^2(M)$ is a Hilbert space with the inner product $\langle f, g \rangle := \int_M f \bar{g} dV$. As usual, $L^2(M)$ can be seen as the completion of $C^\infty(M)$ with respect to the norm generated by this inner product.

For the remainder of this chapter, we will assume that (M, g) is an oriented, C^∞ Riemannian manifold without boundary. Choose $f, g \in C^\infty(M)$ such that at least one of f, g is compactly supported in M . Here, let $C_0^\infty(M)$ denote the set of all smooth functions with support contained in a compact set which lies inside M . Then, Green's formula, ([17], pg. 12), holds:

$$(5.1.7) \quad \int_M f \Delta^k g dV = \int_M g \Delta^k f,$$

for any $k \in \mathbb{N}$. This shows that the Laplacian, defined on $C_0^\infty(M)$ is a densely defined, symmetric operator in $L^2(M)$.

Now let $K \subset M$ be an open, connected set with a smooth boundary ∂K . Then ∂K and K are C^∞ oriented Riemannian manifolds with metric given by the restriction of the metric from M . In this case one can show that there is a unique outward-pointing normal unit vector field ν on ∂K . That is, for every $p \in \partial K$, if $n = \dim(M)$, then the dimension of $T_p(\partial K)$ is $n - 1$, and there is a vector $\nu_p \in T_p(M)$ which is perpendicular to all the vectors in $T_p(\partial K) \subset T_p(M)$. In a chart, one can choose this vector so that it points outwards from the compact set K , and such that $\langle \nu_p, \nu_p \rangle = 1$. This uniquely defines the vector ν_p at p , and the vector field ν such that for $p \in \partial K$, $\nu|_p = \nu_p$. Let dA denote the Riemannian measure of ∂K . In this case, the following version of Green's formula holds

$$(5.1.8) \quad \int_K (f \Delta g - g \Delta f) dV = \int_{\partial K} (f \langle \nu, \text{grad}(g) \rangle - g \langle \nu, \text{grad}(f) \rangle) dA.$$

It follows that if $f \in C^\infty(M)$, and $g \in C_0^\infty(K)$, that

$$(5.1.9) \quad \int_K f \Delta^k g dV = \int_K g \Delta^k f.$$

Let Δ_K denote the Laplacian of the manifold K . By the above formula, Δ'_K , the restriction of Δ_K to the domain $C_0^\infty(K)$ is a densely defined symmetric operator in $L^2(K)$. Also, since $K \subset M$ is a Riemannian submanifold of M , it follows that $\Delta'_K f = \Delta' f$ for all $f \in C_0^\infty(K)$. This symmetric Laplacian Δ'_K is not self-adjoint, since equation (5.1.9) shows that the restriction of any $f \in C^\infty(M)$ to \bar{K} belongs to the domain of $(\Delta'_K)^*$, but not to the domain of K .

One can, however, construct self-adjoint extensions of Δ'_K by extending its domain to include all those $f \in C^\infty(\bar{K})$ which obey certain boundary conditions on ∂K . For example, let

$$(5.1.10) \quad \mathfrak{D}(\Delta_N) := \{\phi \in C^2(K) \cap C^1(\bar{K}) \mid \langle \nu, \text{grad} \phi \rangle = 0\}$$

and let

$$(5.1.11) \quad \mathfrak{D}(\Delta_D) := \{\phi \in C^2(K) \cap C^0(\bar{K}) \mid \phi|_{\partial K} = 0\}.$$

The Laplacian Δ_K is defined on both $\mathfrak{D}(\Delta_D)$ and $\mathfrak{D}(\Delta_N)$ of elements obeying Dirichlet and Neumann boundary conditions respectively. Let $\Delta_D := \Delta_K|_{\mathfrak{D}(\Delta_D)}$ and let $\Delta_N := \Delta_K|_{\mathfrak{D}(\Delta_N)}$ be the Dirichlet and Neumann Laplacians on K respectively. Equation (5.1.8) clearly shows that both Δ_N and Δ_K are symmetric extensions of Δ'_K . In fact, their closures in $L^2(M)$ are both self-adjoint extensions ([11], pg. 8).

5.1.2. Complete Riemannian manifolds and compact submanifolds. Let M be a complete C^∞ Riemannian manifold. Completeness of a Riemannian manifold is characterized by the following theorem ([17], pg. 18).

THEOREM 5.1.2. (*Hopf-Rinow*) *Let M be a Riemannian manifold. The following are equivalent.*

- (a) *M is complete as a metric space.*
- (b) *Closed and bounded subsets of M are compact.*
- (c) *M is geodesically complete.*

Given any point $p \in M$, and any tangent vector ξ at that point, there is maximal open interval $I_\xi \subset \mathbb{R}$ and a unique geodesic $\gamma_\xi(t)$, $t \in I_\xi$, which passes through p , $\gamma_\xi(0) = p$, whose tangent vector at p is ξ . M is said to be geodesically complete if $I_\xi = \mathbb{R}$ for any tangent vector ξ ([17], pg. 18).

The following theorem shows that the assumption of completeness ensures the essential self-adjointness of the Laplacian on the domain of infinitely differentiable functions with compact support [12] [16].

THEOREM 5.1.3. *Every power of the Laplacian $-\Delta$ of a complete C^∞ Riemannian manifold is essentially self-adjoint on the dense domain $C_0^\infty(M) \subset L^2(M)$.*

Since we are dealing with a complete C^∞ Riemannian manifold, we will let Δ' denote the essentially self-adjoint Laplacian whose domain is $C_0^\infty(M)$ and $\Delta := \overline{\Delta'}$ denote the unique self-adjoint Laplacian which is the closure of Δ' .

Now let $\{K_n\}_{n \in \mathbb{N}}$ be a sequence of open, connected submanifolds of M with compact closures and smooth, C^∞ boundaries. That is, $\overline{K_n}$ is compact for each $n \in \mathbb{N}$. Further assume that the K_n are nested and form an open cover of M : $K_n \subset K_{n+1}$ and $\cup_n K_n = M$.

Observe that $L^2(\overline{K_n})$ can be viewed as a subspace of $L^2(M)$ in the same way that $L^2[a, b]$ can be viewed as a subspace of $L^2(\mathbb{R})$. Namely, we identify $L^2(\overline{K_n})$ with that subspace of square integrable functions on M which have support contained in $\overline{K_n}$. This identification makes sense as the measures of $L^2(\overline{K_n})$ and $L^2(M)$ are defined by the metrics on $\overline{K_n}$ and M respectively, and the metric on $\overline{K_n}$ is just the restriction of the metric on M to $\overline{K_n}$. Hence, the measure of $L^2(\overline{K_n})$ is just the restriction of the measure on $L^2(M)$ to Borel subsets of $\overline{K_n}$. On each submanifold K_n , let $\tilde{\Delta}_n$ be an arbitrary self-adjoint extension of the symmetric Laplacian Δ'_n which is defined on the dense domain $C_0^\infty(K_n) \subset L^2(\overline{K_n})$. Then $\tilde{\Delta}_n$ is a densely defined self-adjoint operator in $L^2(\overline{K_n})$. We view the operators $\tilde{\Delta}_n$ as operators acting on a dense domain in the larger Hilbert space $L^2(M)$ in the following natural way. Given $\mathfrak{D}(\Delta_n) := \mathfrak{D}(\tilde{\Delta}_n) \oplus L^2(\overline{K_n})^\perp$ in $L^2(M) = L^2(\overline{K_n}) \oplus L^2(\overline{K_n})^\perp$, where $L^2(\overline{K_n})^\perp$ denotes the orthogonal complement of $L^2(\overline{K_n})$ in $L^2(M)$, define $\Delta_n := \tilde{\Delta}_n \oplus 0$ on $\mathfrak{D}(\Delta_n)$. Then Δ_n is a natural extension of $\tilde{\Delta}_n$ to a dense domain in $L^2(M)$, and as a subspace of $L^2(M)$, define $B(K_n, \Omega) := \chi_n \chi_{[0, \Omega^2]}(\Delta_n) \chi_n L^2(M)$ where χ_n is the self-adjoint projection of $L^2(M)$ onto $L^2(\overline{K_n})$. The operator $P_{n, \Omega} := \chi_n \chi_{[0, \Omega^2]}(\Delta_n) \chi_n = \chi_n \chi_{[0, \Omega^2]}(\tilde{\Delta}_n) \chi_n$ is the self-adjoint projector of $L^2(M)$ onto $B(K_n, \Omega)$. Note that this definition of bandlimited functions on K_n depends on the choice of boundary conditions (*i.e.* on the choice of self-adjoint extension $\tilde{\Delta}_n$ of the symmetric Laplacian Δ'_n) on K_n . Finally, observe that if $\phi \in C_0^\infty(K_n)$ then $\Delta_n \phi = \Delta \phi$, so that $\Delta'_n \subset \Delta'$.

5.2. Proof of strong graph convergence of the Laplacians

*Proposition 5.1.1 will now be established using the assumptions of the previous section. This proposition is restated below for convenience.

*PROPOSITION 5.2.1. *If $a, b \in \mathbb{R}$, $a < b$; $a, b \notin \sigma_p(\Delta)$ then $\chi_n \chi_{(a,b)}(\Delta_n) \chi_n$ converges strongly to $\chi_{[a,b]}(\Delta)$.*

Recall that $\sigma_p(A)$ denotes the set of eigenvalues of A . In particular, if $0, \Omega^2$ are not eigenvalues of Δ , then $P_{n,\Omega} \xrightarrow{s} P_\Omega$. Here, \xrightarrow{s} denotes convergence in the strong operator topology.

To prove *Proposition 5.1.1, it will be shown that the Laplacians Δ_n on K_n converge to the Laplacian Δ on the full manifold using a suitable notion of convergence for unbounded self-adjoint operators. One says that a sequence of self-adjoint operators A_n converges to a self-adjoint operator A in the strong resolvent sense, $A_n \xrightarrow{sr} A$, if there is a $z \in \mathbb{C} \setminus \mathbb{R}$ for which $(A_n - z)^{-1}$ converges strongly to $(A - z)^{-1}$ [57]. It is known that if f is any bounded continuous function on \mathbb{R} and $A_n \xrightarrow{sr} A$, then $f(A_n) \xrightarrow{s} f(A)$. In particular, if $A_n \xrightarrow{sr} A$, then $(A_n - z)^{-1} \xrightarrow{s} (A - z)^{-1}$ for every $z \in \mathbb{C} \setminus \mathbb{R}$ [57]. Let $\text{BC}(\mathbb{R})$ denote the set of bounded continuous functions on \mathbb{R} .

THEOREM 5.2.2. *If A_n, A are self adjoint operators, $A_n \xrightarrow{sr} A$, and $f \in \text{BC}(\mathbb{R})$, then $f(A_n) \xrightarrow{s} f(A)$.*

Even more useful for our purposes, the following theorem [57] shows that if $A_n \xrightarrow{sr} A$ then certain spectral properties of A are related to those of the A_n .

THEOREM 5.2.3. *Suppose $A_n \rightarrow A$ in the strong resolvent sense then*
 (a) *Of $a, b \in \mathbb{R}$, $a < b$ and $(a, b) \cap \sigma(A_n) = \emptyset$ for all n , then $(a, b) \cap \sigma(A) = \emptyset$. That is, if $\lambda \in \sigma(A)$, then there are $\lambda_n \in \sigma(A_n)$ such that $\lambda_n \rightarrow \lambda$.*
 (b) *If $a, b \in \mathbb{R}$, $a < b$ and $a, b \notin \sigma_p(A)$, then $\chi_{(a,b)}(A_n)$ converges strongly to $\chi_{[a,b]}(A)$.*

The second part of the above theorem will imply *Proposition 5.1.1 provided that it can be shown that $\Delta_n \xrightarrow{sr} \Delta$. If this can be shown, it will follow that $\chi_{[a,b]}(\Delta_n) \xrightarrow{s} \chi_{[a,b]}(\Delta)$. Using this and the fact that $\chi_n \xrightarrow{s} I$, it is then elementary to prove *Proposition 5.1.1.

5.2.0.1. *Remark.* Combining Theorems 5.2.2 and 5.2.3, it is easy to see that if $A_n \xrightarrow{sr} A$, f is continuous on $[a, b]$, the support of f is contained in $[a, b]$, and $a, b \notin \sigma_p(A)$, then $f(A_n) \xrightarrow{s} f(A)$.

To establish that $\Delta_n \xrightarrow{sr} \Delta$, it will be easier to first show that the Δ_n converge to Δ in another sense which is in fact equivalent to strong resolvent convergence for self-adjoint operators ([57], pg. 293).

5.2.0.2. *Definition.* Let S_n be a sequence of operators defined in a Hilbert space \mathcal{H} . A pair of elements $(\phi, \psi) \in \mathcal{H} \times \mathcal{H}$ is said to belong to the **strong graph limit**, $\Gamma^\infty(S_n)$, of this sequence if one can find pairs $(\phi_n, S_n \phi_n)$, where $\phi_n \in \mathfrak{D}(S_n)$, such that $(\phi_n, S_n \phi_n) \rightarrow (\phi, \psi)$ in $\mathcal{H} \oplus \mathcal{H}$.

Such a sequence of operators S_n is said to converge to an operator S in the strong graph sense if $\Gamma^\infty(S_n) = \Gamma(S)$ where $\Gamma(S)$ denotes the graph of S . The notation $S_n \xrightarrow{sg} S$ will be used to denote the **strong graph convergence** of the S_n to S . If A_n, A are self-adjoint operators, then $A_n \xrightarrow{sg} A$ if and only if $A_n \xrightarrow{sr} A$ ([57], pg. 293).

The following fact about the strong graph limits of symmetric operators will be used in the proof of *Proposition 5.1.1.

THEOREM 5.2.4. *Let $(A_n)_{n \in \mathbb{N}}$ be a sequence of symmetric operators. Let $\mathfrak{D}_\infty^s := \{\phi \mid \exists \psi \in \mathcal{H} \text{ s.t. } (\phi, \psi) \in \Gamma^\infty(A_n)\}$. If \mathfrak{D}_∞^s is dense, then $\Gamma^\infty(A_n) =: \Gamma(A)$ is the graph of a closed symmetric operator A . In particular, this means that $A_n \xrightarrow{sg} A$ and that $\Gamma(A) = \Gamma^\infty(A_n)$ is closed in $\mathcal{H} \oplus \mathcal{H}$.*

It is a fact that for self adjoint operators A_n and A , strong graph convergence is equivalent to strong resolvent convergence [57]. It follows that *Proposition 5.1.1 will be proven if we can show that $\Delta_n \xrightarrow{sg} \Delta$. In fact, we will prove the following more general result.

Let M be a complete, oriented, connected C^∞ Riemannian manifold. Let $K_n \subset M$ be a sequence of nested, open submanifolds of M with smooth boundary as described previously.

***PROPOSITION 5.2.5.** *Let M and K_n be as above. Let D be a closed, self-adjoint linear operator defined on a dense domain $\mathfrak{D}(D) \subset L^2(M)$. Suppose that $C_0^\infty(M) \subset \mathfrak{D}(D)$, that $D' := D|_{C_0^\infty(M)}$ is essentially self-adjoint, and that $D'_n := D|_{C_0^\infty(K_n)}$ is a symmetric operator in $L^2(\overline{K_n})$ for every $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, let $D_n := \tilde{D}_n \oplus 0$ in $L^2(M) = L^2(\overline{K_n}) \oplus L^2(\overline{K_n})^\perp$ where \tilde{D}_n denotes an arbitrary symmetric extension of D'_n in $L^2(\overline{K_n})$ (in particular, it could be that $\tilde{D}_n = D'_n$ is the trivial extension). Then $D_n \xrightarrow{sg} D$.*

Recall that if \mathfrak{D} is a core for a self-adjoint operator A , then $\overline{A|_{\mathfrak{D}}} = A$. In the assumptions of the above proposition, $C_0^\infty(M)$ is a core for the self-adjoint operator D .

In the literature, the above and related results are known for the special case where D is the Laplacian, and $M = \mathbb{R}^d$ [68] [71].

The proof of the following lemma is a simple application of the definition of a compact set, and is omitted.

LEMMA 5.2.6. *If B is a compact subset of M , then there exists $N \in \mathbb{N}$ such that $B \subset K_n$ for all $n \geq N$.*

PROOF. (of *Proposition 5.2.5)

To prove the proposition, it suffices to prove that $\Gamma(D') \subset \Gamma^\infty(D_n)$. To see this, observe that Theorem 5.2.4 states that if the set \mathfrak{D}_∞ of all ϕ for which there is a $\psi \in L^2(M)$ such that $(\phi, \psi) \in \Gamma^\infty(D_n)$ is dense in $L^2(M)$, then $\Gamma^\infty(D_n)$ is closed, and is the graph of a closed, densely defined symmetric operator. Hence, if it can be shown that $\Gamma(D') \subset \Gamma^\infty(D_n)$, this will imply that $\mathfrak{D}_\infty \supset \mathfrak{D}(D') = C_0^\infty(M)$ is dense, so that $\Gamma^\infty(D_n)$ will be the closed graph of a closed symmetric operator. Since $\overline{\Gamma(D')} = \Gamma(D)$, it will then follow that $\Gamma^\infty(D_n) \supset \Gamma(D)$, so that $\Gamma^\infty(D_n)$ is the graph of a closed symmetric extension \tilde{D} of the symmetric operator D . However, since D is actually self-adjoint, it has no non-trivial symmetric extensions, which will imply that $\tilde{D} = D$ and that $\Gamma^\infty(D_n) = \Gamma(D)$.

It remains to prove that $\Gamma(D') \subset \Gamma^\infty(D_n)$. This is easily accomplished. If $(\phi, D'\phi) \in \Gamma(D')$, then $\phi \in C_0^\infty(M)$ so that the support of ϕ is contained in some compact subset $B \subset M$. By Lemma 5.2.6, there is an $N \in \mathbb{N}$ such that $n > N$ implies that $B \subset K_n$ for all $n > N$. For $n > N$, $\phi \in C_0^\infty(K_n)$ so that $(\phi, D_n\phi) = (\phi, D'_n\phi) = (\phi, D'\phi)$ for all such n . Let $\phi_n = 0$ for $n \leq N$ and $\phi_n = \phi$ for $n > N$. Then $(\phi_n, D_n\phi_n) \rightarrow (\phi, D'\phi)$ in $L^2(M) \oplus L^2(M)$. Hence, $(\phi, D'\phi) \in \Gamma^\infty(D_n)$ and the proof is complete. □

Observe that the K_n do not need to have compact closures in order for the above proof to work. However, for our purposes, it is convenient to choose the K_n so that their closures are compact, since in this case the subspaces $B(K_n, \Omega)$ can be chosen to be finite dimensional.

*Proposition 5.1.1 is now an immediate corollary of the above result if we take $D = \Delta$. In fact, in *Proposition 5.2.5, one can take $D = \Delta^k$ for any $k \in \mathbb{N}$. Even more can be said. Let $\Delta'_n := \Delta|_{C_0^\infty(K_n)}$. *Proposition 5.1.1 implies that if Δ_n is an arbitrary choice of self-adjoint extension of Δ'_n then $\Delta_n \xrightarrow{sg} \Delta$. By Theorem 5.2.2, and Remark 5.2.0.1 it then follows that if $f \in BC(\mathbb{R})$ or if $a, b \notin \sigma_p(\Delta)$ and g is continuous on $[a, b]$, that $f(\Delta_n) \xrightarrow{s} f(\Delta)$ and $g(\Delta_n) \xrightarrow{s} g(\Delta)$.

For example, consider the special case $M := \mathbb{R}$. Let $D := i\frac{d}{dx}$ be the closure of the essentially self-adjoint derivative operator on $C_0^\infty(\mathbb{R})$, and choose D'_L to be the symmetric derivative operator

with domain $C_0^\infty(-L, L)$ in $L^2[-L, L]$. Choose D_L to be the self-adjoint extension of D'_L which obeys periodic boundary conditions. In this case, as in the introductory section, Section 5.1, it is not hard to see that given any $f \in L^2(\mathbb{R})$, that $\chi_{[-\Omega, \Omega]}(D_L)f$ is that Ω -bandlimited trigonometric polynomial on $[-L, L]$ which is the largest partial sum of the Fourier series of f on $[-L, L]$ containing no terms $e^{ik_L x}$ with frequencies $|k_L| > \Omega$:

$$(5.2.1) \quad \phi_N(x) = \chi_{[-L, L]}(x) \sum_{n=-N}^N f_n e^{ik_n x} \quad N := \lfloor \frac{\Omega L}{\pi} \rfloor \quad k_n := \frac{n\pi}{L},$$

$$(5.2.2) \quad f_n := \frac{1}{2L} \int_{-L}^L f(x) e^{-ik_n x} dx.$$

If $f \in B(\Omega)$, the functions $\phi_N := \chi_{[-\Omega, \Omega]}(D_L)f$ are the L -truncated TP_Ω versions of f first considered in Chapter 3. The above results applied to this situation, show that $D_L^k \chi_{[-\Omega, \Omega]}(D_L) \xrightarrow{s} D^k \chi_{[-\Omega, \Omega]}(D)$. This proves that if $f \in B(\Omega)$, then the k^{th} derivative of ϕ_N converges in L^2 norm to the k^{th} derivative of f , for any $k \in \mathbb{N}$. This will now be stated as a corollary.

**COROLLARY 5.2.7. If $f \in B(\Omega)$ and ϕ_N is the L -truncated, TP_Ω version of f , where $N = \lfloor \frac{\Omega L}{\pi} \rfloor$, then $\phi_N^{(k)} \rightarrow f^{(k)}$ in norm as $N \rightarrow \infty$.*

*Proposition 5.2.5 can in fact be generalized even further to prove the following:

**PROPOSITION 5.2.8. Let M and K_n be as in *Proposition 5.2.5. Let D be a closed self-adjoint linear operator defined on a dense domain $\mathfrak{D}(D) \subset L^2(M)$. For each $n \in \mathbb{N}$, let $\mathcal{D}_n \subset L^2(\overline{K_n})$ be a dense linear manifold in $L^2(\overline{K_n})$. Suppose that $\mathcal{D}_n \subset \mathfrak{D}(D)$, that $\mathcal{D}_n \subset \mathcal{D}_{n+1}$, that $D'_n := D|_{\mathcal{D}_n}$ is a symmetric operator acting in $L^2(\overline{K_n})$, and that $D' := D|_{\mathfrak{D}}$, where $\mathfrak{D} := \cup_{n=1}^\infty \mathcal{D}_n$, is essentially self-adjoint and densely defined in $L^2(M)$.*

For each $n \in \mathbb{N}$, let $D_n := \tilde{D}_n \oplus 0$ in $L^2(M) = L^2(\overline{K_n}) \oplus L^2(\overline{K_n})^\perp$, where \tilde{D}_n denotes an arbitrary symmetric extension of D'_n in $L^2(\overline{K_n})$. In particular, it could be that $\tilde{D}_n = D'_n$ is the trivial extension. Then $D_n \xrightarrow{sg} D$.

The proof of this proposition is very similar to that of *Proposition 5.2.5, and is omitted.

5.3. Outlook

Choose Δ_n to be that Laplacian in $L^2(\overline{K_n})$ whose domain consists of functions obeying either Neumann or Dirichlet boundary conditions on K_n . In this case, a convenient property of the subspaces $B(K_n, \Omega)$ is that they are finite dimensional. This follows from the known fact that the self-adjoint Laplacian on a compact manifold K obeying Dirichlet or Neumann boundary conditions has a purely discrete spectrum consisting of eigenvalues of finite multiplicity with no finite accumulation point ([11], pg.8). Hence $P_{n, \Omega} = \chi_n \chi_{[0, \Omega^2]}(\Delta) \chi_n$ will project onto a finite dimensional subspace spanned by eigenfunctions to Δ_n , and the dimension of this subspace $B(K_n, \Omega)$ is equal to $N(K_n, \Omega)$, where $N(K_n, \Omega)$ is the number of eigenvalues to Δ_n which lie in the interval $[0, \Omega^2]$.

5.3.1. Sampling in a finite dimensional function space is trivial. Such a finite dimensional function space always trivially obeys a sampling theorem. For example, consider an N -dimensional function space, \mathcal{F} , spanned by some generic basis functions $\{b_i(x)\}_{i=1 \dots N}$, *i.e.*, all $f \in \mathcal{F}$ can be written $f(x) = \sum_{i=1}^N \lambda_i b_i(x)$ for some $\{\lambda_i\}_{i=1}^N \subset \mathbb{C}$. Assume we know of a function $f \in \mathcal{F}$ only its amplitudes $a_n = f(x_n)$, for $n = 1 \dots N$, at some N generically chosen points x_n , *i.e.*,

$$(5.3.1) \quad f(x_n) = a_n = \sum_{i=1}^N \lambda_i b_i(x_n).$$

Then, equation (5.3.1) generally allows us to determine the coefficients λ_i and therefore $f(x)$ for all x . This is because for generic basis functions b_i , if the sample points x_n are chosen such that the $N \times N$ matrix $B = (b_i(x_n))_{i,n=1\dots N}$ has a non-vanishing determinant, then: $\lambda_i = \sum_{j=1}^N B_{ij}^{-1} a_j$, and therefore

$$f(x) = \sum_{n=1}^N f(x_n)G(x_n, x) \text{ for all } x,$$

where the reconstruction kernel G reads: $G(x_n, x) = \sum_{i=1}^N B_{ni}^{-1} b_i(x)$. Since the b_i are linearly independent functions, such a set of N points x_n must exist. It follows that the minimum number of points needed for a set of sampling is N , the dimension of \mathcal{F} .

5.3.2. Estimating the dimension of $B(K_n, \Omega)$ and generalizing Landau's theorem.

There are various results from the field of spectral geometry that can be used to estimate $N(K_n, \Omega)$, the number of points needed for a set of sampling for $B(K_n, \Omega)$. For example Weyl's asymptotic formula [17] states that

$$(5.3.2) \quad N(K_n, \Omega) \sim \frac{\Omega^d V(K) V(B_d)}{(2\pi)^d}$$

for large Ω , where $V(K)$ is the volume of K and $V(B_d)$ is the volume of the d -dimensional unit ball in \mathbb{R}^d .

An interesting question to ask is whether the density $\frac{N(K_n, \Omega)}{V(K_n)}$ has a finite limit ρ as $n \rightarrow \infty$. Since $N(K_n, \Omega)$ is the number of points needed for a set of sampling for $B(K_n, \Omega)$, and since $B(K_n, \Omega)$ converges strongly to $B(M, \Omega)$, this would then seem to suggest that the necessary density a discrete set of points $\Lambda := \{\lambda_n\} \subset M$ must have in order to be a set of sampling for $B(M, \Omega)$ is ρ . Here, the density of the countable set of points $\Lambda \subset M$ may be defined analogously to Beurling-Landau density of a discrete set of points in \mathbb{R}^n . Namely, one could define $n_-(r)$ to be the minimum number of the points of Λ in any $\dim(M)$ dimensional ball of proper radius r in M , and then define the lower Beurling-Landau density of Λ to be $D_-(\Lambda) := \lim_{r \rightarrow \infty} \frac{n_-(r)}{r}$.

If these ideas could be made rigorous, this could then provide an approach for generalizing H.J. Landau's theorem, Theorem 2.2.4, on necessary density for sets of sampling for bandlimited functions to manifolds. Such a generalized result would be of great interest, in particular, in mathematical physics [35].

Pesenson has already shown that functions in $B(M, \Omega)$ are stably reconstructible from their values taken on certain sets of points Λ , which have a finite proper 'density', in the case where M has bounded geometry [55]. It therefore seems reasonable that the numbers $\frac{N(K_n, \Omega)}{V(K_n)}$ should have a finite limit, or at least be bounded above for such manifolds. Methods from the field of spectral geometry, including those used to calculate Weyl's asymptotic formula should be useful for investigating these ideas. The fact that correction terms to Weyl's asymptotic formula can be calculated in terms of integrals of scalars formed from the curvature tensor and its covariant derivatives, which is also used in non-commutative geometry, should be useful here [10].

5.4. Applications to quantum theory on curved space

The material contained in this section has been recently published in [37].

Let M be complete C^∞ Riemannian manifold and K_n be a sequence of nested submanifolds with compact closures, as in the previous sections. In the previous section, it was observed that if one defines the subspaces $B(K_n, \Omega)$ using the Dirichlet or Neumann Laplacian, then one obtains a finite dimensional function space.

This is extremely useful and convenient for quantum field theory. Restricting the space of physical fields on the manifold M to $L^2(\overline{K}_n)$ is a form of infrared cutoff, and is a common tool

employed by physicists to make quantum field theoretical calculations more tractable. Restricting the space of physical fields to $B(K_n, \Omega)$ corresponds to imposing both an infrared and an ultraviolet cutoff. Such a cutoff is very convenient. Since $B(K_n, \Omega)$ is finite dimensional, it has a finite number of spatial degrees of freedom, so that if one restricts the set of physical fields to this space, the path integral will reduce to a well-defined finite number of ordinary spatial integrations. Whether or not the number of spatial degrees of freedom will approach a finite density as the infrared cutoff is removed for a given manifold is the question discussed in the previous section.

An ultraviolet (UV) cutoff is expected to exist in nature, while an infrared (IR) cutoff is merely a calculational tool that is to be temporarily imposed to make calculations simpler, and then be removed subsequently. At this point one needs to consider that, in the presence of a UV cutoff, it is in fact non-trivial to keep the imposition and subsequent removal of an IR cutoff under control. Removing the IR cutoff corresponds to considering the nested sequence of ever larger submanifolds K_i (whose closures are compact), and whose union is all of M . One possibility would be to impose the UV cutoff first, *i.e.*, to restrict the space of fields that is being integrated over in the path integral to $B(M, \Omega)$, then to impose an IR cutoff, perform calculations, and finally remove the IR cutoff. Technically, one would work with the image of $L^2(M)$ under the operators $\chi_n P_\Omega$ and then take the limit as $n \rightarrow \infty$, where P_Ω projects onto $B(M, \Omega)$ and χ_n projects onto $L^2(\overline{K_n})$. This procedure, however, is not practical.

First notice that the operator $\chi_n P_\Omega$ is not a projector because χ_n and P_Ω do not commute. In fact, the range of $\chi_n P_\Omega$ is not closed and is therefore not the image of $L^2(M)$ under any projector. In the path integral it would not be straightforward to restrict the fields to this subspace so that the UV cutoff on the full manifold is regained as the IR cutoff is removed. In fact, the subspace resulting from imposing first the UV and then the IR cutoff does not obey an UV cutoff on K_n , *i.e.*, performing the path integral will be no simpler than with no UV cutoff on the full manifold. The reason can be traced to the existence of superoscillations [67] [9] [23] [34]: even for the simple case where M is the real line, it is known that the space of Ω -bandlimited functions contains functions that oscillate arbitrarily fast on any given finite interval. This means that the projection of $B(\Omega) = B(\mathbb{R}, \Omega)$ onto the space of functions with support on a finite interval I does not yield a space $B(I, \Omega)$ of bandlimited functions on that finite interval. Instead, it yields a linear subspace which is dense in $L^2(I)$ [67], suggesting that imposing first an UV and then an IR cutoff will yield an infinite dimensional subspace of functions even on the compact submanifolds K_n .

Instead, it is more practical to first restrict the fields to $L^2(K_n)$ and then to cut off the spectrum of the Laplacian on K_n , namely, to project $L^2(M)$ with the projector $P_{n,\Omega} := \chi_n P_{n,\Omega} \chi_n$ onto $B(K_n, \Omega)$. This is what we did in the previous section and we know that the resulting space of fields, $B(K_n, \Omega)$, is a closed, finite-dimensional subspace, so that the path integral in the presence of both the UV and IR cutoffs is then simple and well defined. It is necessary to show, however, that the removal of the IR cutoff is under control, *i.e.*, that one recovers the full theory with just the ultraviolet cutoff as $n \rightarrow \infty$.

To this end, consider functionals on fields ϕ , such as the action functional $S[\phi]$ in the path integral formulation or such as the state functionals $\Psi[\phi]$ in the Schrödinger formulation of quantum field theory. It needs to be shown that the evaluation of such functionals in the full (*i.e.* only UV cutoff) theory agrees with the limit of evaluating these functionals on successively larger submanifolds. Concretely, if Ψ is such a functional then in order that the removal of the IR cutoff be safe it is necessary that

$$(5.4.1) \quad \Psi[P_{n,\Omega}\phi] \rightarrow \Psi[P_\Omega\phi]$$

as $n \rightarrow \infty$ for any $\phi \in L^2(M)$. For any continuous Ψ , equation (5.4.1) will hold provided that $P_{n,\Omega}\phi \rightarrow P_\Omega\phi$ for all $\phi \in L^2(M)$, where $P_\Omega := \chi_{[0,\Omega^2]}(\Delta)$, *i.e.*, provided that $P_{n,\Omega}$ converges strongly to P_Ω . *Proposition 5.1.1 has already established this fact, in spite of the above-discussed

superoscillations and in spite of the non-uniqueness of the boundary conditions (as well as self-adjoint extensions, eigenvectors and spectra) of the IR-cutoff Laplacians. Hence, we have proven that imposing and removing an infrared cutoff in the presence of such an ultraviolet cutoff is valid.

Bandlimited functions on flat space-time

In the case where M is a pseudo-Riemannian manifold, the sampling and reconstruction properties of elements of $B(M, \Omega) := \chi_{[-\Omega^2, \Omega^2]}(\square)L^2(M)$ where \square denotes the D'Alembertian of M , are fundamentally different from the case where M is Riemannian. Contrary to the case of a Riemannian manifold, there is in general no discrete set of points of finite density on a pseudo-Riemannian manifold with the property that any bandlimited function can be reconstructed perfectly from the values it takes on this set. The basic reason for this difference, as will be made apparent in this chapter for the case where M is flat space-time, is that the D'Alembertian, unlike the Laplacian of a Riemannian manifold, is not elliptic.

Despite the fact that there is no overall, global reconstruction formula for elements of $B(M, \Omega)$ if M is a space-time, elements of this subspace still have special sampling and reconstruction properties. This will be demonstrated in this chapter for the case of flat space-time and will be demonstrated later in Chapter 8 for the case of de Sitter space-time with a finite end-time.

6.1. Sampling on Minkowski space-time

The D'Alembertian for Minkowski space-time can be represented as

$$(6.1.1) \quad \square := -\frac{\partial^2}{\partial t^2} - \Delta,$$

acting on fields $\phi(x_0, \mathbf{x})$ in $L^2(\mathbb{R}^{1+k})$, where $\Delta := -\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ is the spatial Laplacian for \mathbb{R}^k , $1 \leq k \leq 3$.

As discussed in Chapter 2, any $f \in B(\mathbb{R}^k, \Omega)$ is perfectly reconstructible from the values it takes on certain sufficiently dense discrete sets of points in \mathbb{R}^k . A natural question to ask is whether elements of $B(M, \Omega)$ where M is $1+k$ dimensional Minkowski spacetime, $1 \leq k \leq 3$, have similar properties, *i.e.*, whether they are perfectly reconstructible from their values taken on certain sufficiently dense discrete sets of points in this space-time. The answer is no, at least not if the density is defined as in Theorem 2.2.3 using the Euclidean metric, identifying points on the $1+k$ dimensional spacetime with points in \mathbb{R}^{1+k} .

To see this, observe that the D'Alembertian on this space-time is unitarily equivalent under Fourier transform to the multiplication operator $p_0^2 - \mathbf{p}^2$ acting on fields $\phi(p_0, \mathbf{p})$ in \mathbb{R}^{1+k} . In this representation, the projector $\chi_{[-\Omega^2, \Omega^2]}(\square)$ onto the space of bandlimited functions is the projector of $L^2(\mathbb{R}^{1+k})$ onto $L^2(S)$ where $S \subset \mathbb{R}^{1+k}$ is the set of all $(p_0, \mathbf{p}) \in \mathbb{R}^{1+k}$ obeying

$$(6.1.2) \quad |p_0^2 - \mathbf{p}^2| \leq \Omega^2.$$

The equations $p_0^2 - \mathbf{p}^2 = \pm\Omega^2$ describe hyperboloids in \mathbb{R}^{1+k} . The set S is the interior region bounded by these hyperboloids.

Hence, $\phi \in B(M, \Omega)$ if and only if its Fourier transform has support contained in S . Now Landau's density theorem, Theorem 2.2.4, states that if $\Lambda := \{\lambda_n\}_{n \in \mathbb{Z}}$ is a set of sampling for the frequency limited functions $\phi(x_0, \mathbf{x}) \in B(M, \Omega) = \chi_{[-\Omega^2, \Omega^2]}(\square)L^2(\mathbb{R}^{k+1})$, then the Beurling-Landau density of the set of points Λ must be proportional to the volume of the set S on which the frequencies of these functions have support. However, as is easily checked, the volume, or Lebesgue measure of the set S is infinite in \mathbb{R}^{1+k} for $1 \leq k \leq 3$. For example, in the case where

$k = 1$, it is straightforward to show that the volume of the set S is proportional to the area under the curve $f(x) = \frac{1}{x}$ for $x \in [\Omega, \infty)$. This proves that there are no sets of sampling for $B(M, \Omega)$ which have finite Beurling-Landau density.

Even though there is no overall reconstruction formulas of the type described for bandlimited functions in Chapter 2, it is not difficult to show that fixed modes of fields in $B(M, \Omega)$ will obey special reconstruction formulas. For example, given a field $\phi \in B(M, \Omega)$, if one performs a Fourier transform with respect to the time variable t , the result is a function $\varphi(p_0, \mathbf{x})$. For a fixed temporal frequency $p_0 \in \mathbb{R}$, the function $\varphi_{p_0}(\mathbf{x}) := \varphi(p_0, \mathbf{x})$ of \mathbf{x} will be called the p_0 temporal mode of ϕ . Since the global Fourier transform of $\phi \in B(M, \Omega)$ has support contained in the set described by the inequality $|p_0^2 - \mathbf{p}^2| \leq \Omega^2$, it is clear that for a fixed temporal mode p_0 , the spatial Fourier transform of φ_{p_0} has support contained in a compact set S_{p_0} of finite volume described by the inequality

$$(6.1.3) \quad R(p_0) := \sqrt{p_0^2 + \Omega^2} \geq |\mathbf{p}| \geq \sqrt{p_0^2 - \Omega^2} := r(p_0).$$

For the case of 1 + 1 dimensional flat space-time, as is illustrated in Figure 6.1 below, S_{p_0} is an interval for $p_0^2 \leq \Omega^2$ and a union of two intervals if $p_0^2 > \Omega^2$.

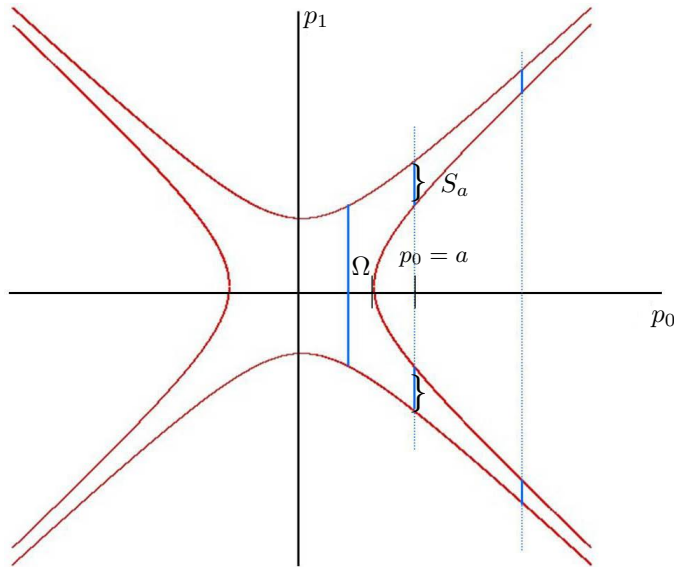


FIGURE 1. The set S_{p_0} in 1 + 1 dimensions

For 1 + 2 dimensional flat space-time this region is the area between two circles of radius $R(p_0)$ and $r(p_0)$ (see Figure 2), and in 1 + 3 dimensions this region is that contained between two spheres of radius $R(p_0)$ and $r(p_0)$.

It follows that each temporal mode of any $\phi \in B(M, \Omega)$ is spatially bandlimited, and hence will obey a spatial reconstruction formula.

Recall that the bandwidth volume of the subspaces $B(S)$ of functions whose Fourier transforms have support contained in a fixed compact set $S \subset \mathbb{R}^n$ is defined as the volume $\mu(S)$ of S , where μ denotes Lebesgue measure. Further recall that Landau's density theorem, Theorem 2.2.4, states that the minimum density a discrete set of points Λ must have to be a set of sampling for a set of frequency limited functions $B(S)$ is proportional to the volume of S . For this reason, it

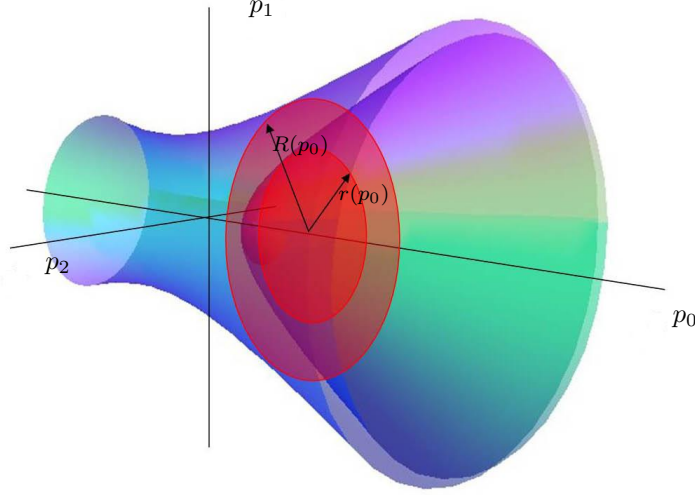


FIGURE 2. The set S_{p_0} in $1 + 2$ dimensions

is interesting to investigate how the spatial bandwidth volume of the fixed temporal mode p_0 behaves as $p_0 \rightarrow \infty$. Computing the spatial bandwidth volume of the space of fixed temporal modes φ_{p_0} , *i.e.*, the volume of the region S_{p_0} described by the inequality (6.1.3), will determine how the necessary density a set of spatial points $\Lambda = \{\mathbf{x}_n\}$ must have to be a set of sampling for the fixed temporal mode p_0 behaves as $p_0 \rightarrow \infty$.

The volume of the set S_{p_0} described by the inequality (6.1.3) depends on the number of spatial dimensions k of M :

$$\lim_{p_0 \rightarrow \infty} V(S_{p_0}) = \begin{cases} \lim_{p_0 \rightarrow \infty} (\sqrt{p_0^2 + \Omega^2} - \sqrt{p_0^2 - \Omega^2}) = 0 & \text{if } k = 1 \\ \pi \lim_{p_0 \rightarrow \infty} (p_0^2 + \Omega^2 - (p_0^2 - \Omega^2)) = 2\pi\Omega^2 & \text{if } k = 2 \\ \frac{4\pi}{3} \lim_{p_0 \rightarrow \infty} ((p_0^2 + \Omega^2)^{\frac{3}{2}} - (p_0^2 - \Omega^2)^{\frac{3}{2}}) = \infty & \text{if } k = 3 \end{cases} .$$

For the cases $k = 1$ and $k = 2$, there is an upper bound on the spatial bandwidth volume for any fixed temporal mode p_0 . Hence, it is consistent with Landau's theorem that there could exist sets of spatial points $\Lambda = \{\mathbf{x}_n\}_{n \in \mathbb{Z}}$, $\Lambda \subset \mathbb{R}^k$, $k = 1, 2$ which are sets of sampling for every fixed temporal mode $p_0 \in \mathbb{R}$. Indeed, such sets of points do exist. For example, if $k = 1$, then the space of all p_0 temporal modes is $B(S_{p_0})$ where $S(p_0) = [-R(p_0), r(p_0)] \cup [r(p_0), R(p_0)]$. The volume of this set is $V(S_{p_0}) = 2(R(p_0) - r(p_0))$. It is known, as was discussed in Subsection 2.2.1, that any set of spatial points $\Lambda := \{\lambda_n\}_{n \in \mathbb{Z}} = \{x_n\}_{n \in \mathbb{Z}} \cup \{y_n\}_{n \in \mathbb{Z}}$ where $x_{n+1} - x_n = \frac{4\pi}{V(p_0)} = y_{n+1} - y_n$ for all $n \in \mathbb{Z}$ is a set of sampling for $B(S(p_0))$ provided that $|x_n - y_n| \neq \frac{2\pi}{V(p_0)}$ [46]. This depends only on the volume $V(S_{p_0})$ of S_{p_0} . It follows that if B is the upper bound on $V(S_{p_0})$ for the case $k = 1$, then any set of points $\Lambda := \{x_n\}_{n \in \mathbb{Z}} \cup \{y_n\}_{n \in \mathbb{Z}}$ where $x_{n+1} - x_n = \frac{2\pi}{B} = y_{n+1} - y_n$ and $|x_n - y_n| \neq \frac{\pi}{B}$ is a set of sampling for $B([-B - r(p_0), -r(p_0)] \cup [r(p_0), r(p_0) + B]) \supset B(S(p_0))$ for any $p_0 \in \mathbb{R}$. This means that any temporal mode ϕ_{p_0} of a bandlimited field $\phi \in B(M, \Omega)$ where $M = 1 + 1$ dimensional flat space-time can be stably reconstructed at any spatial point on the manifold from the knowledge of the values $\{\varphi_{p_0}(\lambda_n)\}_{n \in \mathbb{Z}}$ it takes on any one of these discrete sets of points Λ . For the precise reconstruction formula, see ([46], pg. 1221). It is simple to calculate that the Beurling-Landau density of any one of these sets Λ is $\frac{B}{\pi}$. In conclusion, if $\phi \in B(M, \Omega)$ is any bandlimited field on this space-time, then the knowledge of the values $\{\phi(t, \lambda_n)\}_{t \in \mathbb{R}; n \in \mathbb{Z}}$ is sufficient to reconstruct it everywhere on the manifold. This follows since

the values $\{\varphi_{p_0}(\lambda_n)\}_{p_0 \in \mathbb{R}; n \in \mathbb{Z}}$ can be computed from the values $\{\phi(t, \lambda_n)\}_{t \in \mathbb{R}; n \in \mathbb{Z}}$, so that one can then compute $\varphi_{p_0}(x)$ for all $p_0, x \in \mathbb{R}$ and then compute $\phi(t, x)$ by inverse Fourier transform.

Since there is an upper bound on the spatial bandwidth volume of any fixed temporal mode when the number of spatial dimensions is $k = 2$, a similar reconstruction formula should hold for this case as well. That is, there should be sets of spatial points $\Lambda := \{\lambda_n\}_{n \in \mathbb{Z}} \subset \mathbb{R}^2$ such that any $\phi \in B(M, \Omega)$ can be stably reconstructed from the values $\{\phi(t, \lambda_n)\}_{t \in \mathbb{R}; n \in \mathbb{Z}}$. In this case the space of temporal modes φ_{p_0} is the space $B(S_{p_0})$, where S_{p_0} is the region contained between the circles of radius $R(p_0)$ and $r(p_0)$. One might imagine that this problem is reducible to the case of one spatial dimension by enclosing the region S_{p_0} by a region $\tilde{S}(p_0)$ between two squares, and then considering each co-ordinate separately. This, however, does not work, as to do this one would need to choose the side length of the inner square less than or equal to $\frac{r(p_0)}{\sqrt{2}}$ and the side length of the outer square to be greater or equal to $R(p_0)$. As is easy to check, there is no upper bound on the volume of this region $\tilde{S}(p_0)$ as $p_0 \rightarrow \infty$. Although I am not currently aware of an example, I still expect that sets of spatial points Λ which are sets of sampling for every fixed temporal mode of a $\phi \in B(M, \Omega)$, where $M = 1 + 2$ dimensional flat spacetime, should exist.

A similar reconstruction formula cannot hold for $1 + 3$ dimensional Minkowski space-time since for this space-time there is no upper bound on the spatial bandwidth volume for a fixed temporal mode. In this case, given $\phi \in B(M, \Omega)$ one can consider the function $\tilde{\phi}_{p_0, p_1}(x_2, x_3) := \tilde{\phi}(p_0, p_1, x_2, x_3)$ where $\tilde{\phi}$ is that function obtained by performing Fourier transform with respect to t and a given spatial co-ordinate x_1 . In this case, one can again show that the spatial bandwidth volume of $\tilde{\phi}_{p_0, p_1}$ is bounded above for all $p_0, p_1 \in \mathbb{R}$ so that there should exist discrete sets of points of finite density $\Lambda := \{\mathbf{x}_n\}_{n \in \mathbb{Z}} \subset \mathbb{R}^2$ such that $\phi \in B(M, \Omega)$ should be perfectly reconstructible from the information $\{\phi(t, x_1, \mathbf{x}_n)\}_{t, x_1 \in \mathbb{R}; n \in \mathbb{Z}}$. This shows that any bandlimited function in this $1 + 3$ dimensional space-time can be reconstructed perfectly from the values it takes on certain discrete sets of two-dimensional hypersurfaces of the space-time.

All of the analysis so far has involved fixing temporal modes of bandlimited fields $\phi \in B(M, \Omega)$. Conversely, one can define $\Phi_{\mathbf{p}}(t) := \Phi(t, \mathbf{p})$ to be the function of time defined by taking the spatial Fourier transform of ϕ and fixing a spatial frequency \mathbf{p} . This function of time $\Phi_{\mathbf{p}}$ will be called the \mathbf{p} spatial mode of the bandlimited field ϕ . In symmetry with the previous results, one can show that any fixed spatial mode is temporally bandlimited, *i.e.* the temporal Fourier transform of any fixed spatial mode \mathbf{p} is confined to a compact set, so that every fixed spatial mode obeys a temporal reconstruction formula. That is, there exist discrete sets of points in time, $\Lambda := \{\lambda_n\} \subset \mathbb{R}$ such that any spatial mode $\Phi_{\mathbf{p}}$ is stably reconstructible from the values $\{\Phi_{\mathbf{p}}(\lambda_n)\}_{n \in \mathbb{Z}}$. Furthermore, since there is only one time dimension, and the temporal bandwidth volume of any fixed spatial mode \mathbf{p} vanishes in the limit as $\|\mathbf{p}\| \rightarrow \infty$, it follows, as before, that any $\phi \in B(M, \Omega)$ is stably reconstructible from the values $\{\phi(\lambda_n, \mathbf{x})\}_{n \in \mathbb{Z}; \mathbf{x} \in \mathbb{R}^k}$ where M is $1 + k$ dimensional flat space-time and $k \in \mathbb{N}$.

6.2. Physical interpretation

Consider the case of $M = 1 + 1$ dimensional flat space-time. Any fixed temporal mode φ_{p_0} of a $\phi \in B(M, \Omega)$ can be reconstructed perfectly from the values $\{\varphi_{p_0}(\lambda_n)\}_{t \in \mathbb{R}; n \in \mathbb{Z}}$ where $\Lambda := \{\lambda_n\}_{n \in \mathbb{Z}}$ is a certain discrete set of spatial points with Beurling-Landau density $\frac{V(S_{p_0})}{\pi}$. Now given that this density is defined with respect to a fixed co-ordinate system, one may wonder how this is consistent with Lorentz invariance. Intuitively, in the co-ordinate system of another inertial observer, due to time dilation, the wavelength of the temporal mode p_0 will appear longer, so that to the new observer this temporal mode will appear to be composed of lower temporal frequencies than p_0 . Since lower temporal frequencies correspond to larger spatial bandwidth volume, it seems that a discrete set of points with larger density than $\frac{V(S_{p_0})}{\pi}$ is needed to sample and reconstruct φ_{p_0} in this new co-ordinate system. However, due to length contraction, the set of points Λ will appear to be denser to the new inertial observer. Hence, it must be that this denser set of points

which is the image of Λ in the new co-ordinate system is dense enough to reconstruct the function φ_{p_0} in this new co-ordinate system.

Our goal now is to study bandlimited functions on more general space-times. Before doing this, it will be useful to first introduce the theory of symmetric Sturm-Liouville differential operators, and to prove some elementary facts about the spectra of symmetric operators. This will be done in the next chapter.

Uncertainty, strong convergence, and the spectra of symmetric operators

7.1. Second order symmetric differential operators

Consider the Sturm-Liouville differential equation

$$(7.1.1) \quad -(p(x)\phi'(x))' + q(x)\phi(x) = \lambda w(x)\phi(x),$$

on the interval (a, b) . The values (a, b) are allowed to be $-\infty$ or $+\infty$ respectively. We will assume that the functions p, q, w are real valued functions such that $1/p, q, w$ are Lebesgue measurable on (a, b) , that they belong to L^1 of any compact subinterval of (a, b) , and that $w > 0$ almost everywhere with respect to Lebesgue measure on (a, b) .

Given a second order Sturm-Liouville differential equation with these properties, define the differential expression $L[\cdot]$ by

$$(7.1.2) \quad L[\phi] := \frac{1}{w(t)} \left(-\frac{d}{dt} \left(p(t) \frac{d}{dt} \phi \right) + q(t)\phi \right),$$

on the interval $(a, b) \subset \mathbb{R}$.

Following ([52], pg. 60), see also ([4], pages 137-171, and Sections 1-4), one can use the differential expression (7.2.14) to define a symmetric operator on a dense domain in $L^2(a, b; w)$. Here $L^2(a, b; w)$ denotes the Hilbert space of functions on (a, b) which are square integrable with respect to the measure $w(x)dx$. First let

$$(7.1.3) \quad \mathfrak{D}(D^*) := \{ \phi \in L^2(a, b; w) \mid \phi, p(t)\phi' \in AC_{loc}(a, b); L[\phi] \in L^2(a, b; w) \},$$

where $AC_{loc}(a, b)$ denotes the set of all functions which are absolutely continuous on any compact subinterval of (a, b) . The set $\mathfrak{D}(D^*)$ is the largest domain in $L^2(a, b; w)$ on which this differential expression, L can be defined. Define the operator D^* by $D^*\phi = L[\phi]$ for all $\phi \in \mathfrak{D}(D^*)$. Next, define $\mathfrak{D}(D')$ to be the set of all $\phi \in \mathfrak{D}(D^*)$ which have compact support in the open interval (a, b) . Using integration by parts, it is easy to see that if $D' := D^*|_{\mathfrak{D}(D')}$, then D' is symmetric, and it can be shown that D^* is the adjoint of D' ([52], Section 17). Let D denote the closure of D' .

The symmetric differential operator D is said to be generated by the differential expression L , (7.2.14), and always has equal deficiency indices (n, n) where $n \leq 2$. The fact that D must have equal deficiency indices is a consequence of the following theorem of von Neumann ([58], pgs. 143-144):

THEOREM 7.1.1. *Let S be a symmetric operator and $C : \mathfrak{D}(S) \rightarrow \mathfrak{D}(S)$ be a conjugation map such that $SC = CS$. Then S has equal deficiency indices.*

Here a conjugation map C is an idempotent, norm-preserving, anti-linear map. For example, if $\phi, \psi \in L^2(\mathbb{R})$, the complex conjugation map $C, C(\alpha\phi + \beta\psi) = \bar{\alpha}C\phi + \bar{\beta}C\psi$ where $C\phi(x) = \overline{\phi(x)}$ a.e. is a conjugation map.

PROOF. Since C is idempotent, $C^2 = I$, and $C\mathfrak{D}(S) \subset \mathfrak{D}(S)$, it follows that $\mathfrak{D}(S) = C^2\mathfrak{D}(S) \subset C\mathfrak{D}(S)$ so that $C\mathfrak{D}(S) = \mathfrak{D}(S)$.

Choose any $\phi_+ \in \mathfrak{D}_+$ and $\phi \in \mathfrak{D}(S)$. Since C takes $\mathfrak{D}(S)$ onto $\mathfrak{D}(S)$, it follows that

$$(7.1.4) \quad \begin{aligned} 0 &= \overline{\langle (S+i)\phi, \phi_+ \rangle} \\ &= \langle C(S+i)\phi, C\phi_+ \rangle = \langle (S-i)C\phi, C\phi_+ \rangle. \end{aligned}$$

This shows that $C\phi_+ \in \mathfrak{D}_-$, and that C maps \mathfrak{D}_+ into \mathfrak{D}_- . An analogous argument shows that C maps \mathfrak{D}_- into \mathfrak{D}_+ . Since C is norm preserving it then follows that $n_+ = \dim(\mathfrak{D}_+) = \dim(\mathfrak{D}_-) = n_-$ \square

The following existence-uniqueness theorem holds for differential equations of the form (7.1.1), ([52], pg. 54).

THEOREM 7.1.2. *For any interior point $x_0 \in (a, b)$ and arbitrary constants c_1 and c_2 , the differential equation (7.1.1) on (a, b) has one and only one solution $\phi(x)$ which satisfies the initial conditions $\phi(x_0) = c_1$ and $p(x_0)\phi'(x_0) = c_2$.*

Here, a function ϕ is said to be a solution to (7.1.1) if it belongs to $AC_{\text{loc}}(a, b)$ and satisfies (7.1.1) almost everywhere. The above theorem can be proven using the method of Picard iterates.

The fact that the deficiency index $n = n_+ = n_-$ of a second order symmetric differential operator D is less than or equal to 2 follows from the above existence-uniqueness theorem for solutions to the differential equation $L\phi = \lambda\phi$, $\lambda \in \mathbb{C}$, and the form of the adjoint operator D^* . Here, L is the differential expression which generates D .

The deficiency index n is, by definition, equal to the number of linearly independent solutions $\phi \in L^2(a, b; w)$ to the equation $D^*\phi = i\phi$. That is, n is equal to the number of linearly independent solutions to the differential equation $L\phi = i\phi$ which belong to $L^2(a, b; w)$. By the existence-uniqueness theorem, there are exactly two linearly independent solutions to this equation, so that there are at most two such solutions which belong to $L^2(a, b; w)$. In conclusion, the deficiency indices of any second order symmetric differential operator D are (n, n) where $n \leq 2$.

7.1.1. End-points. Let D be a second-order symmetric differential operator in $L^2(a, b; w)$ generated by the differential expression L with coefficient functions p, q . The end-point a is called regular if $a > -\infty$ and if $1/p$ and q belong to $L^1[a, c]$ for any $c \in (a, b)$. Similarly, b is called regular if $b < \infty$ and $1/p, q$ are in $L^1[c, b]$ for any $c \in (a, b)$. If an end point is not regular, it is called singular.

A function $\phi : (a, b) \rightarrow \mathbb{C}$ is said to lie left (*resp.* right) in $L^2(a, b; w)$ if the restriction of ϕ to $(a, c]$ (*resp.* $[c, b)$) belongs to $L^2(a, c; w)$ (*resp.* $L^2(c, b; w)$) for any $c \in (a, b)$.

The differential expression L is said to be of the limit circle case at a (*resp.* b) if all solutions to $L[\phi] = z\phi$, $z \in \mathbb{C}$, lie left (*resp.* right) in $L^2(a, b; w)$.

7.1.2. A spectral theorem for symmetric second-order differential operators. For a symmetric differential operator D , one can explicitly construct a unitary operator which is an integral operator whose integral kernel is expressed in terms of solutions to the differential equations $L\phi = \lambda\phi$ for $\lambda \in \mathbb{R}$, and which maps the differential operator D onto a multiplication operator on a space of vector-valued functions which are square integrable with respect to a certain matrix-valued measure.

Let $\mathcal{B}(\mathbb{R})$ denote the Borel σ -algebra of \mathbb{R} , and $B(\mathcal{H})^+$ denote the cone of bounded positive operators on \mathcal{H} .

DEFINITION 7.1.3. *A positive operator valued measure (POVM) on \mathbb{R} is a map $Q : \mathcal{B}(\mathbb{R}) \rightarrow B(\mathcal{H})^+$ from the σ -algebra $\mathcal{B}(\mathbb{R})$ of Borel subsets of \mathbb{R} into the set of positive operators in $B(\mathcal{H})$ which has the following properties:*

- (1) $Q(\emptyset) = 0$, $Q(\mathbb{R}) = I$
- (2) if $\Omega = \cup_{n=1}^{\infty} \Omega_n$ with $\Omega_n \cap \Omega_m = \emptyset$ for $n \neq m$ then $\sum_{i=1}^N Q(\Omega_i) \xrightarrow{s} Q(\Omega)$

Given a self-adjoint operator A on a separable Hilbert space \mathcal{H} , the spectrum of A is said to be simple if there is a vector $x \in \mathcal{H}$ such that the linear span of all the vectors $\chi_\Omega(A)x$ where Ω runs through $\mathcal{B}(\mathbb{R})$, the Borel subsets of \mathbb{R} , is dense in \mathcal{H} . Such a vector x is called a generating vector for A . For example, the multiplication operator M in $L^2(\mathbb{R})$ is simple, and any $\phi \in L^2(\mathbb{R})$ such that $\phi(x) \neq 0$ a.e. is a generating vector for M . If A is not simple but there is a set of vectors $\{x_i\}_{i=1}^n$ with the property that the linear span of the $\chi_\Omega(A)x_i$ for $1 \leq i \leq n$ is dense in \mathcal{H} , then the set $\{x_i\}$ is called a generating basis for A . If n is the minimum number of vectors in any generating basis for A , then the spectrum of A is said to have multiplicity n , or to be n -fold degenerate. One form of the spectral theorem for self-adjoint operators states that if A is a self-adjoint operator of multiplicity n , then it is unitarily equivalent to multiplication by the independent variable on a space $L^2(\mathbb{R}^n, d\sigma)$ of n -component vector valued functions $\phi = (\phi_1, \dots, \phi_n)$ on \mathbb{R} which are square integrable with respect to a positive $n \times n$ matrix valued measure $\sigma : \mathcal{B}(\mathbb{R}) \rightarrow \mathbb{M}_n(\mathbb{C})^+$ ([52], Sections 20.4-20.6).

For a symmetric second order differential operator D , the following theorem is a generalization of the Fourier transform, and gives a concrete realization of the spectral theorem described in the above paragraph for the case where the deficiency indices of D are $(0, 0)$, ([52], pg. 111). For the statement of the theorem below, assume that $w \equiv 1$.

THEOREM 7.1.4. *Let D be a closed, second-order, symmetric differential operator generated by the differential expression L on the interval $(a, b) \subset \mathbb{R}$, and let D' be any self-adjoint extension of D . Let $\phi_i(x; \lambda)$ be solutions of the differential equation $L\phi = \lambda\phi$ which satisfy the boundary conditions*

$$(7.1.5) \quad \begin{pmatrix} \phi_1(x_0; \lambda) & p(x_0)\phi_1'(x_0; \lambda) \\ \phi_2(x_0; \lambda) & p(x_0)\phi_2'(x_0; \lambda) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then, there is a positive 2×2 matrix-valued measure $\sigma = (\sigma_{j,k})$ on \mathbb{R} such that the formulas

$$(7.1.6) \quad \varphi_j(\lambda) := \int_a^b f(x)\phi_j(x; \lambda)dx$$

and

$$(7.1.7) \quad f(x) = \sum_{i,j=1}^2 \int_{-\infty}^{\infty} \varphi_i(\lambda)\phi_j(x; \lambda)d\sigma_{i,j}(\lambda)$$

define a unitary transformation U from $L^2(a, b)$ onto $L^2(\mathbb{R}^2, \sigma)$ such that $UD'U^{-1}$ acts as multiplication by λ in $L^2(\mathbb{R}^2, \sigma)$.

Actually, in [52], a more general version of this theorem which applies to symmetric differential operators of any even order is established.

7.2. Uncertainty, strong convergence, and the spectrum of symmetric operators

Before returning to the problem of studying bandlimited functions on expanding FRW space-time, it will be convenient to prove some simple and general results about the spectra of symmetric operators. The relationship between the spectrum of a symmetric operator and its minimum uncertainty, and the behaviour of the essential spectrum of a symmetric operator under strong convergence will be studied here.

7.2.1. Minimum uncertainty and spectra of self-adjoint extensions. As was first pointed out in [36], there exists a close relationship between the finite lower bound ΔS_t on the uncertainty of a symmetric operator and the spectra of its self-adjoint extensions. Our aim now is to refine those results, including new results, in particular, on the dependence of the density of eigenvalues on the operator's deficiency indices. This subsection is joint work with Prof. Kempf. Several of the results in this subsection are part of a manuscript that has been recently submitted for publication [49].

7.2.1.1. *Remark and Definition.* Recall the definition of the uncertainty of a symmetric operator S . The expectation value and the uncertainty of a symmetric operator S with respect to a unit-length vector $\phi \in \mathfrak{D}(S)$ is denoted by $\bar{S}_\phi := \langle \phi, S\phi \rangle$ and by $\Delta S[\phi] := \sqrt{\langle S\phi, S\phi \rangle - (\langle S\phi, \phi \rangle)^2}$ respectively. For a fixed expectation value $t \in \mathbb{R}$, the quantity $\Delta S_t := \inf_{\phi \in \mathfrak{D}(S), \langle S\phi, \phi \rangle = t, \|\phi\|=1} \Delta S[\phi]$ will be called the minimum uncertainty of S at t . Further recall that the overall lower bound on the uncertainty of S is denoted by $\Delta S := \inf_{t \in \mathbb{R}} \Delta S_t$.

'THEOREM 7.2.1. *If S is a symmetric operator with unequal deficiency indices, then $\Delta S = 0$.*

PROOF. Suppose S has deficiency indices (m, n) , $n \neq m$. Then it has a symmetric extension S' with deficiency indices either $(|m - n|, 0)$ or $(0, |m - n|)$. Accordingly, the Cayley transform U of S' is either an isometry or the adjoint of an isometry. By the Wold decomposition theorem, U is then isometrically isomorphic to k copies of the left shift operator on $\oplus_{i=1}^k l^2(\mathbb{N})$ or j copies of the right shift operator on $\oplus_{i=1}^j l^2(\mathbb{N})$. It follows that $\sigma(U) = \sigma(R)$ or $= \sigma(L)$ respectively, where R and L are the right and left shift operators on $l^2(\mathbb{N})$. It is known that the unit circle lies in the continuous spectrum of both the right and left shift operators. It follows that the continuous spectrum of S (which is a subset of \mathbb{R}) is non-empty and hence there exist $\phi \in \mathfrak{D}(S)$ for which $\Delta S[\phi]$ is arbitrarily small. \square

*THEOREM 7.2.2. *Let S be a densely defined, closed symmetric operator with finite and equal deficiency indices (n, n) . If $\Delta S > 0$, then any self-adjoint extension S' of S has a purely discrete spectrum, $\sigma(S') = \sigma_p(S')$. In particular, if $\Delta S_t > \epsilon > 0$ for all $t \in I \subset \mathbb{R}$, then S' can have no more than n eigenvalues (including multiplicities) in any interval $J \subset I$ of length less than or equal to ϵ*

This theorem shows, in particular, that if $\Delta S > \epsilon$, then any self-adjoint extension of S has no more than n eigenvalues in any interval of length ϵ .

PROOF. If $\Delta S > 0$, then by 'Theorem 7.2.1 and Remark 4.3.1.2, we conclude that every $z \in \mathbb{C}$ is a point of regular type for S . Since S has finite and equal deficiency indices, if S' is any self-adjoint extension of S , it follows that $\sigma_e(S') = \sigma_e(S)$ consists only of the point at infinity. This implies that the spectrum of S' consists solely of eigenvalues of finite multiplicity with no finite accumulation point.

Suppose that there is a self-adjoint extension S' of S which has $n + 1$ eigenvectors ϕ_i to eigenvalues λ_i where $\lambda_i \in J \subset I$, and the length of J is less than or equal to ϵ . Then, since the dimension of $\mathfrak{D}(S')$ modulo $\mathfrak{D}(S)$ is n , there is a non-zero linear combination of these orthogonal eigenvectors, $\psi = \sum_{i=1}^{n+1} c_i \phi_i$ which has unit norm and which belongs to $\mathfrak{D}(S)$. The expectation value of the symmetric operator S in the state ψ lies in J , $t := \bar{S}_\psi \in J$, since ψ is a linear combination of orthogonal eigenvectors to S' whose eigenvalues all lie in J . Now it is straightforward to verify that since $|\lambda_i| < |t| + \epsilon$ for all $1 \leq i \leq n + 1$, that

$$(7.2.1) \quad (\Delta S[\psi])^2 = \sum_{i=1}^{n+1} \lambda_i^2 |c_i|^2 - t^2 \leq \sum_{i=1}^{n+1} (|t| + \epsilon)^2 |c_i|^2 - t^2 = 2|t|\epsilon + \epsilon^2.$$

First suppose that $0 \in J$ and that $t := \bar{S}_\psi = 0$. Then in this case, equation (7.2.1) contradicts the fact that $\Delta S_0 > \epsilon$, proving the claim for this case.

If $t \neq 0$, then consider the symmetric operator $S(t) := S - t$ on $\mathfrak{D}(S)$. Given any $\phi \in \mathfrak{D}(S)$ which has unit norm and expectation value $\bar{S}_\phi = \langle S\phi, \phi \rangle = t$, it is not hard to see that $\overline{S(t)}_\phi = \langle S(t)\phi, \phi \rangle = 0$ and that

$$(7.2.2) \quad \begin{aligned} \Delta S(t)[\phi]^2 &= \langle S(t)\phi, S(t)\phi \rangle = \langle S\phi, S\phi \rangle - 2t\langle S\phi, \phi \rangle + t^2 \\ &= \langle S\phi, S\phi \rangle - t^2 = \Delta S[\phi]. \end{aligned}$$

This shows that $\Delta S(t)_0 = \Delta S_t > \epsilon$. Now let S' be any self-adjoint extension of S . Applying the above argument for the expectation value 0 to the symmetric operator $S(t)$, we conclude that the self-adjoint extension $S'(t) := S' - t$ of $S(t)$ can have no more than n eigenvalues in the interval $J - t$. This in turn implies that S' can have no more than n eigenvalues in the interval J . \square

**COROLLARY 7.2.3. If S is a symmetric operator with finite deficiency indices such that $\Delta S = \epsilon > 0$, then S is simple and regular, the deficiency indices (n, n) of S are equal, and the spectrum of any self-adjoint extension of S is purely discrete and consists of eigenvalues of finite multiplicity at most n with no finite accumulation point.*

If S is a closed, densely defined simple symmetric operator with equal and finite deficiency indices (n, n) , then, by Theorem 4.4.3, the multiplicity of any eigenvalue of any self-adjoint extension S' of S does not exceed n . *Corollary 7.2.3 is an immediate consequence of this fact, and *Theorems 7.2.1 and 7.2.2.

7.2.1.2. *Example.* Consider the symmetric differential operator $S' := -\frac{d}{dx} \left(x \frac{d}{dx} \cdot \right) + x$ defined on the dense domain $C_0^\infty(0, \infty) \subset L^2[0, \infty)$ of infinitely differentiable functions with compact support in $(0, \infty)$. Let S be the closure of S' . Let D be the closed symmetric derivative operator on $L^2[0, \infty)$ which is the closure of the symmetric derivative operator $D' := i \frac{d}{dx}$ on the domain $\mathfrak{D} := C_0^\infty(0, \infty)$. Recall that if T is a closed operator, then $\mathcal{D} \subset \mathfrak{D}(T)$ is called a core for T if $\overline{T|_{\mathcal{D}}} = T$. It follows that \mathfrak{D} is a core for both D and for S . For all $\phi \in \mathfrak{D}$, it is easy to verify that $[D, S]\phi := (DS - SD)\phi = i(D^2 + 1)\phi$. By the uncertainty principle, it follows that for any $\phi \in \mathfrak{D}$,

$$(7.2.3) \quad \Delta S[\phi] \Delta D[\phi] \geq \frac{1}{2} |\langle \phi, [D, S]\phi \rangle| = \frac{1}{2} \langle \phi, (D^2 + 1)\phi \rangle = \frac{1}{2} \left[(\Delta D[\phi])^2 + \langle \phi, D\phi \rangle + \langle \phi, \phi \rangle \right].$$

Using the fact that the function $f(x) = \frac{x^2+1}{2x}$ is concave up for all $x \in (0, \infty)$ and has a global minimum $f(1) = 1$, we conclude that $\Delta S[\phi] \geq 1$ for any unit norm $\phi \in \mathfrak{D}$. Since \mathfrak{D} is a core for S , given any unit norm $\psi \in \mathfrak{D}(S)$, one can find a sequence $\psi_n \in \mathfrak{D}$ such that $\|\psi_n\| = 1$, $\psi_n \rightarrow \psi$ and $S\psi_n \rightarrow S\psi$. It follows that $\Delta S[\psi] = \lim_{n \rightarrow \infty} \Delta S[\psi_n] \geq 1$. Now S is a second order symmetric differential operator of the type discussed in Section 2.1. As discussed in that subsection, the deficiency indices of such an operator are equal and do not exceed $(2, 2)$ ([52], Section 17). Since $\Delta D \geq 1$, *Corollary 7.2.3 also implies that the deficiency indices of D must be equal and non-zero. Hence, D has deficiency indices $(1, 1)$ or $(2, 2)$. Applying *Theorem 7.2.2 one can now conclude that D can have at most two eigenvalues in any interval of unit length.

Conversely, as the next theorem shows, if S has finite deficiency indices and is simple and regular, then $\Delta S > 0$.

**THEOREM 7.2.4. Suppose that S is a regular, simple symmetric operator with finite and equal deficiency indices. Let \mathfrak{S} denote the set of all self-adjoint extensions of S within \mathcal{H} . Then*

$$(7.2.4) \quad \Delta S_t \geq \max_{S' \in \mathfrak{S}} \Delta S'_t = \max_{S' \in \mathfrak{S}} \left(\min_{\lambda, \mu \in \sigma(S'); \lambda \neq \mu} \sqrt{|\lambda - t| |\mu - t|} \right).$$

PROOF. Note that if S is simple and regular with finite deficiency indices (m, n) , then these indices must be equal, otherwise, by Theorem 7.2.1, S would have continuous spectra and would not be regular.

Since $\mathfrak{D}(S) \subset \mathfrak{D}(S')$ and $S'|_{\mathfrak{D}(S)} = S$ for any $S' \in \mathfrak{S}$, it is clear that $\Delta S_t \geq \max_{S' \in \mathfrak{S}} \Delta S'_t$. It remains to prove that $\Delta S'_t = \min_{\lambda, \mu \in \sigma(S')} \sqrt{|\lambda - t| |\mu - t|}$ for any $S' \in \mathfrak{S}$. Since we assume S is regular, simple, and has finite deficiency indices, the essential spectrum of S is empty. Hence, by Claim 4.4.1, $\sigma_e(S')$ is empty for any $S' \in \mathfrak{S}$. This shows that the spectrum of any S' consists solely of eigenvalues of finite multiplicity with no finite accumulation point. Order the eigenvalues as a non-decreasing sequence $(t_n)_{n \in \mathbb{M}}$. Here, as in Section 4.4, $\mathbb{M} = \pm\mathbb{N}$, or \mathbb{Z} . For convenience,

assume $\mathbb{M} = \mathbb{Z}$, and let $\{b_n\}_{n \in \mathbb{Z}}$ be the orthonormal eigenbasis such that $S'b_n = t_n b_n$. To calculate $\Delta S'_t$, we need to minimize the functional

$$(7.2.5) \quad \Phi'[\phi] := \langle S'\phi, S'\phi \rangle - t^2$$

over the set of all normalized $\phi \in \mathfrak{D}(S')$ which satisfy $\langle S'\phi, \phi \rangle = t$. Let us assume that t is not an eigenvalue of S' as in this case $\Delta S'_t = 0$ and (7.2.4) holds trivially. Expanding ϕ in the basis b_n , $\phi = \sum_{n \in \mathbb{Z}} \phi_n b_n$, we see that to find the extreme points of Φ' subject to these constraints we need to set the functional derivative of

$$(7.2.6) \quad \Phi[\phi] := \sum_{n \in \mathbb{Z}} \phi_n \bar{\phi}_n ((t_n^2 - t^2) - \alpha_1 t_n - \alpha_2)$$

to zero. Here, α_1 and α_2 are Lagrange multipliers. Setting this functional derivative Φ with respect to $\bar{\phi}$ to 0 yields:

$$(7.2.7) \quad 0 = \phi_n ((t_n^2 - t^2) - \alpha_1 t_n - \alpha_2).$$

Formula (7.2.7) leads to the conclusion that if ϕ is an extreme point, it must be a linear combination of two eigenvectors to S' corresponding to two distinct eigenvalues. To see this note that if the decomposition of ϕ in the eigenbasis $\{b_n\}$ had three non-zero coefficients, say ϕ_{j_i} , $i = 1, 2, 3$, all of which correspond to eigenvectors b_{j_i} with distinct eigenvalues, $t_i \neq t_j$, $1 \leq i, j \leq 3$, then Equation (7.2.7) leads to the conclusion that $\alpha_1 = t_{j_1} + t_{j_2} = t_{j_2} + t_{j_3}$. This would imply that $t_{j_1} = t_{j_3}$, a contradiction. Furthermore, ϕ cannot just be a linear combination of eigenvectors b_j to one eigenvalue, as such a linear combination cannot satisfy the constraint $\langle S\phi, \phi \rangle = t$. Let $\lambda := t_i$ and $\mu := t_j$ for any $j, i \in \mathbb{Z}$ for which $t_i \neq t_j$. Choose $\varphi \in \mathfrak{Rer}(S^* - \lambda)$ and $\psi \in \mathfrak{Rer}(S^* - \mu)$. We have shown that any ϕ which extremizes Φ has the form $\phi = c_1 \varphi + c_2 \psi$. Using the constraints that $\langle \phi, \phi \rangle = 1$ and $\langle S\phi, \phi \rangle = t$ uniquely determines c_1 and c_2 up to complex numbers of modulus one:

$$(7.2.8) \quad |c_1| = \sqrt{\frac{|\lambda - t|}{|\lambda - \mu|}} \quad \text{and} \quad |c_2| = \sqrt{\frac{|\mu - t|}{|\lambda - \mu|}}.$$

The phases of c_1 and c_2 do not affect the value of $\Phi[\phi]$. It follows that if ϕ extremizes Φ , then $\Delta S'[\phi] = \sqrt{|\mu - t||\lambda - t|}$ so that $\Delta S'_t = \min_{\mu, \lambda \in \sigma(S'); \lambda \neq \mu} \sqrt{|\mu - t||\lambda - t|}$. \square

7.2.1.3. Remark. Observe that the curve $f(t) = \sqrt{|\mu - t||\lambda - t|}$ describes the upper half of a circle of radius $\frac{|\lambda - \mu|}{2}$ centred at the point $\frac{\lambda + \mu}{2}$.

7.2.2. Strong convergence and the essential spectrum of symmetric operators. In this subsection, a result on the behaviour of the essential spectrum of a sequence of symmetric operators $(S_n)_{n \in \mathbb{N}}$ which have finite deficiency indices (n, n) and which converge in a certain strong sense to a symmetric operator S with finite and equal deficiency indices will be established. This result will be applied later to the study of bandlimited functions on expanding de Sitter space-time.

Recall that the set of all compact operators $K(\mathcal{H})$ is a two-sided norm-closed ideal in $B(\mathcal{H})$, and that the Calkin algebra $\frac{B(\mathcal{H})}{K(\mathcal{H})}$ with the norm $\|\pi(B)\| := \inf_{K \in K(\mathcal{H})} \|B + K\|$ is a C^* -algebra. Here, $\pi : B(\mathcal{H}) \rightarrow \frac{B(\mathcal{H})}{K(\mathcal{H})}$ denotes the canonical $*$ -homomorphism of $B(\mathcal{H})$ onto the Calkin algebra. It is a well known fact that the essential spectrum $\sigma_e(L)$ of any $L \in B(\mathcal{H})$ is equal to the spectrum of the image, $\pi(L)$, of L in the Calkin algebra ([15], pgs. 358-362). If S has equal deficiency indices and S' is a self-adjoint extension within \mathcal{H} of S , then the essential spectrum of S' is equal to $f(\sigma_e(R_\lambda(S')))$ where $f(z) := 1/z + \lambda$. This follows from the spectral mapping theorem for closed self-adjoint operators, see, for example, Theorems 5.12.1 and 5.12.2 of [21].

7.2.3. Strong graph convergence. Here we make a simple generalization of results which describe the relationship between spectrum and strong graph convergence of self-adjoint operators as described in ([57], section VIII.7).

Let $(S_n)_{n \in \mathbb{N}}$ be a sequence of densely defined, closed, symmetric operators on \mathcal{H} . Let S be a fixed, closed, densely defined symmetric operator on \mathcal{H} . Given $\lambda \in \mathbb{C} \setminus \mathbb{R}$, let $P_n(\lambda)$ denote the projector onto $\mathfrak{R}(S_n - \lambda)$, and $P(\lambda)$ denote the projector onto $\mathfrak{R}(S - \lambda)$. Given any closed, densely defined symmetric operator A and $z \in \mathbb{C} \setminus \mathbb{R}$, $A - z$ has a bounded inverse defined on $\mathfrak{R}(A - z)$. Hence, the operator $(A - z)^{-1}P(z)$, where $P(z)$ projects onto the closed subspace $\mathfrak{R}(A - z)$, is a well-defined bounded linear operator on \mathcal{H} for any such A .

The main result of this subsection is the following theorem:

**THEOREM 7.2.5. Suppose that S_n, S are closed, densely defined symmetric operators. For any fixed n , assume that S_n has finite and equal deficiency indices, and that the deficiency indices of S are equal. Suppose that $(S_n - z)^{-1}P_n(z)P(z) \xrightarrow{s} (S - z)^{-1}P(z)$ for any fixed $z \in \mathbb{C} \setminus \mathbb{R}$.*

If $a < b$, $a, b \in \mathbb{R}$ and $\sigma_e(S_n) \cap [a, b] = \emptyset$ then either $\sigma_e(S) \cap (a, b) = \emptyset$, or, for any sequence $(S'_n)_{n=1}^\infty$ of self-adjoint extensions of the S_n , the number of eigenvalues of S'_n in the interval (a, b) diverges as $n \rightarrow \infty$.

Before proving *Theorem 7.2.5, it will be useful to first establish a few basic facts. Recall that a dense set $\mathcal{D} \subset \mathcal{H}$ is called a core for a symmetric operator S if $\overline{S|_{\mathcal{D}}} = S$.

**LEMMA 7.2.6. Let S_n and S be closed, symmetric operators which are densely defined in \mathcal{H} . If either*

(a) \mathcal{D} is a core for S , each S_n is defined on \mathcal{D} , and $S_n\phi \rightarrow S\phi$ for every $\phi \in \mathcal{D}$,

or,

(b) $\Gamma(S) \subset \Gamma^\infty(S_n)$,

then for any $\lambda \in \mathbb{C} \setminus \mathbb{R}$ the operators $(S_n - \lambda)^{-1}P_n(\lambda)P(\lambda)$ converge strongly to $(S - \lambda)^{-1}P(\lambda)$.

PROOF. Fix $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

(a) For any $\phi \in \mathcal{D}$ let $\psi = (S - \lambda)\phi$. We have

$$\begin{aligned} ((S_n - \lambda)^{-1}P_n(\lambda) - (S - \lambda)^{-1})\psi &= (S_n - \lambda)^{-1}P_n(\lambda)(S - S_n + S_n - \lambda)\phi - \phi \\ (7.2.9) \qquad \qquad \qquad &= (S_n - \lambda)^{-1}(S - S_n)\phi. \end{aligned}$$

This vanishes in the limit as $n \rightarrow \infty$ since $(S_n - \lambda)^{-1}$ is uniformly bounded by $\frac{1}{\text{Im}(\lambda)}$. It follows that $((S_n - \lambda)^{-1}P_n(\lambda) - (S - \lambda)^{-1}P(\lambda))\psi \rightarrow 0$ for all $\psi \in (S - \lambda)\mathcal{D}$. Since $S := \overline{S|_{\mathcal{D}}}$, one can conclude that this also vanishes for all $\psi \in \mathfrak{R}(S - \lambda) = P(\lambda)\mathcal{H}$. This proves (a).

(b) If $\Gamma(S) \subset \Gamma^\infty(S_n)$, then for any $\phi \in \mathfrak{D}(S)$ one can find $\phi_n \in \mathfrak{D}(S_n)$ such that $\phi_n \rightarrow \phi$ and $S_n\phi_n \rightarrow S\phi$. Let $P_n := P_n(\lambda)$ and $P := P(\lambda)$. Then,

$$\begin{aligned} ((S_n - \lambda)^{-1}P_nP - (S - \lambda)^{-1}P)(S - \lambda)\phi &= ((S_n - \lambda)^{-1}P_n - (S - \lambda)^{-1})(S - \lambda)\phi \\ (7.2.10) \qquad \qquad \qquad &= (S_n - \lambda)^{-1}P_n((S - \lambda)\phi - (S_n - \lambda)\phi_n) - (\phi - \phi_n). \end{aligned}$$

Since $\phi_n \rightarrow \phi$, and $(S_n - \lambda)\phi_n \rightarrow (S - \lambda)\phi$, and $(S_n - \lambda)^{-1}$ is uniformly bounded by $\frac{1}{\text{Im}(\lambda)}$, the last line above vanishes in the limit. Since the set of all $(S - \lambda)\phi$ for $\phi \in \mathfrak{D}(S)$ is all of $P\mathcal{H}$, we conclude that $(S_n - \lambda)^{-1}P_nP \xrightarrow{s} (S - \lambda)^{-1}P$. \square

The following simple fact will be used in the proof of *Theorem 7.2.5. Its proof is elementary and is omitted.

LEMMA 7.2.7. *Let $(K_n)_{n=1}^\infty$ be a uniformly bounded sequence of finite rank operators whose rank is uniformly bounded by n . This sequence contains a weakly convergent subsequence which converges to a bounded operator K of finite rank at most n .*

Let $(S_n)_{n \in \mathbb{N}}$ and S be symmetric operators that satisfy at least one of the sets of assumptions (a) or (b) of *Lemma 7.2.6. Further suppose that the deficiency indices of the S_n are finite and uniformly bounded for all $n \in \mathbb{N}$, and that the deficiency indices of S are also finite. Also assume that for each S_n , the deficiency indices $n_+(S_n) = n_-(S_n)$ are equal and that the deficiency indices of S are equal. In this case, the projection operators $P_n(z)$ and $P(z)$ which project onto $\mathfrak{R}(S_n - z)$ and $\mathfrak{R}(S - z)$ respectively for $z \in \mathbb{C} \setminus \mathbb{R}$ are such that $I - P_n(z)$ and $I - P(z)$ have finite rank.

For each n , let S'_n denote an arbitrary self-adjoint extension of S_n within \mathcal{H} and let S' similarly denote such an extension of S . It follows that,

$$\begin{aligned}
(S'_n - z)^{-1} &= (S'_n - z)^{-1}((I - P_n(z)) + P_n(z))((I - P(z)) + P(z)) \\
&= (S'_n - z)^{-1}(P_n(z)P(z) + P_n(z)(I - P(z)) \\
&\quad + (I - P_n(z))P(z) + (I - P_n(z))(I - P(z))) \\
(7.2.11) \quad &= (S_n - z)^{-1}P_n(z)P(z) + F_n,
\end{aligned}$$

where F_n is a finite rank operator. Also observe that if $z \in \mathbb{C} \setminus \mathbb{R}$, then the norms of the F_n are uniformly bounded since $\|(A - z)^{-1}\| \leq \frac{1}{|\operatorname{Im}(z)|}$ for any symmetric operator A .

Now all of the facts needed for the proof of *Theorem 7.2.5 have been gathered.

PROOF. (of *Theorem 7.2.5) Let $\lambda_0 := \frac{a+b}{2} + i\frac{b-a}{2}$ and let $P_n := P_n(\lambda_0)$ and $P := P(\lambda_0)$.

Consider $\pi[R_{\lambda_0}(S'_n)]$, where S'_n is any choice of self-adjoint extension of the S_n , and π is the canonical *-homomorphism of $B(\mathcal{H})$ onto the Calkin algebra. By the spectral mapping theorem,

$$(7.2.12) \quad \sigma(\pi[R_{\lambda_0}(S'_n)]) = \sigma_e(R_{\lambda_0}(S'_n)) = \frac{1}{(\sigma_e(S'_n)) - \lambda_0}.$$

Since $\pi[R_{\lambda_0}(S'_n)]$ is normal in the Calkin algebra, and the essential spectrum of an operator is equal to the spectrum of its image in the Calkin algebra, it follows that

$$(7.2.13) \quad \|\pi[R_{\lambda_0}(S'_n)]\| = \operatorname{spr}(\pi[R_{\lambda_0}(S'_n)]) = \sup_{\lambda \in \sigma_e(S'_n)} \frac{1}{\lambda - \lambda_0} \leq \frac{\sqrt{2}}{b-a},$$

where $\operatorname{spr}(\cdot)$ denotes the spectral radius.

It follows that for any $\epsilon > 0$, and each n , one can find compact operators K_n with the property that $\|R_{\lambda_0}(S'_n) + K_n\| \leq \frac{\sqrt{2}}{b-a} + \epsilon$. Furthermore, the K_n can be chosen so that they are all of finite rank and such that their norms are uniformly bounded. To see this note that if $\lambda \in [a, b] \cap \sigma(S'_n)$, then λ must be an eigenvalue of finite multiplicity since we assume that $[a, b] \cap \sigma_e(S'_n) = \emptyset$ and S'_n is self-adjoint. (Recall that a self-adjoint operator has no residual spectrum.) Furthermore, the set of all such eigenvalues $\lambda \in [a, b]$ can have no limit point in the interval, so let Q_n be the finite rank projector whose range is the direct sum of all the eigenspaces corresponding to the eigenvalues of S'_n in $[a, b]$. It follows that if we choose $K_n := -R_{\lambda_0}(S'_n)Q_n$, then $\|R_{\lambda_0}(S'_n) + K_n\| \leq \frac{\sqrt{2}}{b-a} + \epsilon$, that the norms of the K_n are uniformly bounded by $\frac{1}{\operatorname{Im}(\lambda_0)}$, and each K_n has finite rank.

Assume that the rank of the K_n is uniformly bounded. By equation (7.2.11), $R_{\lambda_0}(S'_n) = R_{\lambda_0}(S_n)P_nP + F_n$, where the F_n are uniformly bounded finite rank operators whose rank is uniformly bounded. Therefore, $\|R_{\lambda_0}(S'_n) + K_n\| = \|R_{\lambda_0}(S_n)P_nP + K'_n\| \leq \frac{\sqrt{2}}{b-a} + \epsilon$ where $K'_n = K_n + F_n$. Since both K_n and F_n are uniformly bounded in norm and rank, so is K'_n . By Lemma 7.2.7, there is a weakly convergent subsequence, K'_{n_j} that converges to some finite rank operator $K \in B(\mathcal{H})$. It follows that $R_{\lambda_0}(S_{n_j})P_{n_j}P + K'_{n_j}$ converges weakly to $R_{\lambda_0}(S)P + K$. Since $\|R_{\lambda_0}(S_{n_j})P_{n_j}P + K'_{n_j}\| \leq \frac{\sqrt{2}}{b-a} + \epsilon$ it is easy to see that $\|R_{\lambda_0}(S)P + K\| \leq \frac{\sqrt{2}}{b-a} + \epsilon$. The difference

between $R_{\lambda_0}(S)P + K$ and $R_{\lambda_0}(S')$ is finite rank which implies that there exists a finite rank operator K' such that $\|R_{\lambda_0}(S') + K'\| \leq \frac{\sqrt{2}}{b-a} + \epsilon$. Since this is true for any $\epsilon > 0$, one can now conclude that $\|\pi[R_{\lambda_0}(S')]\| \leq \frac{\sqrt{2}}{b-a}$. Applying the spectral mapping theorem again shows that $(a, b) \notin \sigma_\epsilon(S)$. Observe that the same argument as above can be used to show that if the rank of any subsequence of the K_n is uniformly bounded, then $(a, b) \notin \sigma_\epsilon(S)$.

Alternatively, if the rank of the $K_n := -R_{\lambda_0}(S'_n)Q_n$ is not uniformly bounded then the rank of the Q_n diverges as $n \rightarrow \infty$. This just means that the number of eigenvalues that each S'_n has in the interval $[a, b]$ diverges as $n \rightarrow \infty$. \square

7.2.4. Application to differential operators.

Consider the differential expression

$$(7.2.14) \quad L[y] := -\frac{d}{dt} \left(p(t) \frac{d}{dt} y \right) + q(t)y$$

on the interval $(a, b) \subset \mathbb{R}$. The values (a, b) are allowed to be $-\infty$ or $+\infty$ respectively. In addition to the usual assumptions on p and q described in Section 7.1, assume that the functions p, q belong to L^2 of any compact subinterval of (a, b) .

Now let $L_n[y]$ be a second order differential expression defined by the functions p_n and q_n and $L[y]$ an expression defined by p, q . Let D_n, D be the corresponding symmetric second order differential operators generated by L_n and L , respectively.

The following theorems allow one to apply the results of the previous subsection to second order differential operators of this type.

**THEOREM 7.2.8. Let D_n and D be second order symmetric differential operators in $L^2(a, b)$ defined by the coefficient functions p_n, q_n and p, q respectively. Assume that $1/p_n, 1/p$ and q_n, q belong to L^2 of any compact subinterval of (a, b) . If $1/p_n \rightarrow 1/p$ and $q_n \rightarrow q$ in L^2 of any compact subinterval of the open interval (a, b) , then $\Gamma(D) \subset \Gamma^\infty(D_n) \subset \Gamma(D^*)$, and $\Gamma^\infty(D_n)$ is the graph of a closed symmetric extension of D .*

The proof of this theorem will make use of Theorem 5.2.4.

PROOF. (of *Theorem 7.2.8)

As in Section 7.1, let D' denote the non-closed symmetric operator defined as the restriction of D^* to the set of all elements in its domain which are compactly supported in the open interval (a, b) .

By Theorem 5.2.4, it follows that if we can show that $\Gamma(D') \subset \Gamma^\infty(D_n)$, then $\Gamma^\infty(D_n)$ will be the graph of a closed symmetric extension of D' . In particular, since $D = \overline{D'}$ this will show that $\Gamma(D) \subset \Gamma^\infty(D_n) \subset \Gamma(D^*)$. That $\Gamma^\infty(D_n) \subset \Gamma(D^*)$ follows from the fact that $\Gamma^\infty(D_n)$ is the graph of a closed symmetric extension of D , and Remark 4.3.1.3.

Suppose $(\phi, D'\phi) \in \Gamma(D')$. Then, by the definition of the domain of D' , ϕ has support contained in some subinterval (a', b') of (a, b) , where $[a', b']$ is compact. Now pick c' such that $b' < c' < \infty$ and choose $c \in (b', c')$, and $\epsilon > 0$ so that $b' < c - \epsilon < c + \epsilon < c'$.

Choose $\psi \in C_0^2(a', c')$ such that $\psi(x) = 0$ for all $x \in (a, b') \cup [c', b)$, such that $\psi(x) = 1$ for all $x \in [c - \epsilon, c + \epsilon]$, and such that $\psi'(x) \leq 0$ for all $x \in [c, c']$. Then define

$$(7.2.15) \quad \phi'_n(x) := \frac{p(x)}{p_n(x)} \phi'(x) + C_n \frac{\psi'(x)}{p_n(x)} \chi_{[c, c']}(x) \quad x \in (a, b),$$

where

$$(7.2.16) \quad C_n := -\frac{\int_{a'}^{b'} \frac{p(x)}{p_n(x)} \phi'(x) dx}{\int_c^{c'} \frac{1}{p_n(x)} \psi'(x) dx}.$$

Since $1/p_n \in L^2[a', c']$, and $p\phi', \psi'$ are absolutely continuous on $[a', c']$, it follows that C_n is finite for each $n \in \mathbb{N}$. Furthermore, since ψ' is differentiable and $p\phi' \in AC[a', c']$ it follows that $\psi, p\phi' \in L^\infty[a', c']$. Recall here that the domain $\mathfrak{D}(D)$ of a differential operator D defined with coefficient functions p, q is defined such that if $\phi \in \mathfrak{D}(D)$, then $p\phi'$ belongs to $AC_{\text{loc}}(a, b)$ (see Section 7.1). Since $1/p, 1/p_n$ belong to $L^2[a', c'] \subset L^1[a', c']$, this shows that $\phi'_n \in L^1[a, b]$. Note also that by the definition of ϕ and ψ that ϕ'_n vanishes almost everywhere for $x \in (a, a') \cup [b', c + \epsilon] \cup [c', b)$.

In particular, if we define $\phi_n(x) := \int_{a'}^x \phi'_n(t)dt$, then $\phi_n \in AC_{\text{loc}}(a, b)$. Clearly $\phi_n(x) = 0$ for $x \leq a'$. Also, for $x \geq c'$, it follows that $\phi_n(x) = \int_{a'}^{b'} \frac{p(t)}{p_n(t)} \phi'(t)dt + C_n \int_c^{c'} \frac{1}{p_n(t)} \psi'(t)dt = 0$ by the definition of C_n . Furthermore, $p_n(x)\phi'_n(x) = p(x)\phi'(x) + C_n\psi'(x)\chi_{[c, c']}(x)$ a.e. Since $\psi'(x) = 0$ for all $x \in [c - \epsilon, c]$, it follows that $\psi'\chi_{[c, c']}$ is differentiable. Also, $p\phi' \in AC_{\text{loc}}(a, b)$ since $\phi \in \mathfrak{D}(D^*)$. This proves that $p_n\phi'_n \in AC_{\text{loc}}(a, b)$. To show that $\phi_n \in \mathfrak{D}(D_n)$ it remains to verify that $\|D_n\phi_n\| < \infty$. This is easily accomplished since

$$\begin{aligned}
\|D_n\phi_n\| &= \|(p\phi')' + C_n\psi''\chi_{[c, c']} + q_n\phi_n\| \\
&\leq \|D\phi\| + \|q\phi\| + \|C_n\chi_{[c', c']}\psi''\| + \|q_n\phi_n\| \\
(7.2.17) \quad &\leq \|D\phi\| + \|q\|_{L^2[a', b']}\|\phi\|_\infty + |C_n|(c' - c)\|\psi''\|_\infty + \|q_n\|_{L^2[a', c']}\|\phi_n\|_\infty
\end{aligned}$$

which is finite for each $n \in \mathbb{N}$. Hence $\phi_n \in \mathfrak{D}(D_n)$. In the above, note that for $\phi \in \mathfrak{D}(D')$, $\|\phi\|_\infty < \infty$ by definition.

Now it will be shown that $\phi_n \rightarrow \phi$. Since ϕ_n and ϕ are continuous for any $x \in [a', b']$, it follows that for any fixed $x \in [a', b']$,

$$\begin{aligned}
|\phi_n(x) - \phi(x)| &= \left| \int_{a'}^x \frac{p(t)}{p_n(t)} \phi'(t)dt - \int_{a'}^x \phi'(t)dt \right| \\
&= \left| \int_{a'}^x \left(\frac{p(t)}{p_n(t)} - 1 \right) \phi'(t)dt \right| \\
&= \left| \int_{a'}^x \frac{p(t) - p_n(t)}{p(t)p_n(t)} p(t)\phi'(t)dt \right| \\
&\leq \int_{a'}^{b'} \left| \frac{1}{p_n(t)} - \frac{1}{p(t)} \right| |p(t)\phi'(t)|dt \\
&\leq \int_{a'}^{b'} \left| \frac{1}{p_n(t)} - \frac{1}{p(t)} \right|^2 dt \int_{a'}^{b'} |p(t)\phi'(t)|^2 dt \\
(7.2.18) \quad &\leq (b' - a')\|p\phi'\|_\infty \|1/p_n - 1/p\|_{L^2[a', b']} \xrightarrow{n \rightarrow \infty} 0.
\end{aligned}$$

This proves that $\|(\phi_n - \phi)\chi_{[a', b']}\|_\infty \xrightarrow{n \rightarrow \infty} 0$. The above further implies that for $x = b'$, $\int_{a'}^{b'} \phi'_n(t)dt \rightarrow \int_{a'}^{b'} \phi'(t)dt = \phi(b') - \phi(a') = 0$ as $n \rightarrow \infty$. Furthermore, for $x \in [c, c']$,

$$(7.2.19) \quad \left| \int_c^x \left(\frac{1}{p_n(t)} - \frac{1}{p(t)} \right) \psi'(t)dt \right| \leq \|\psi'\|_\infty (c' - c) \|1/p_n - 1/p\|_{L^2[c, c']} \rightarrow 0.$$

In particular since we assume $\psi'(x) \leq 0$ for all $x \in [c, c']$, this means that $\int_c^{c'} \frac{1}{p(t)} \psi'(t)dt < 0$, and,

$$(7.2.20) \quad \lim_{n \rightarrow \infty} C_n = - \frac{\lim_{n \rightarrow \infty} \int_{a'}^{b'} \frac{p(t)}{p_n(t)} \phi'(t)dt}{\lim_{n \rightarrow \infty} \int_c^{c'} \frac{1}{p_n(t)} \psi'(t)dt} = 0.$$

It follows that

$$(7.2.21) \quad \|\phi_n - \phi\| \leq (b' - a')\|(\phi_n - \phi)\chi_{[a', b']}\|_\infty + |C_n|(c' - c)\|\psi'\|_\infty \|1/p_n\|_{L^2[c, c']}.$$

Since $1/p_n \rightarrow 1/p$ in $L^2[c, c']$ by assumption, it follows that there is a $B > 0$ such that $\|1/p_n\|_{L^2[c, c']} < B$ for all $n \in \mathbb{N}$. Hence $\|\phi_n - \phi\| \rightarrow 0$ as $n \rightarrow \infty$. Furthermore, the above arguments actually prove that $\|\phi_n - \phi\|_\infty \xrightarrow{n \rightarrow \infty} 0$.

Finally, it will be shown that $D_n\phi_n \rightarrow D\phi$. Now,

$$(7.2.22) \quad \|D\phi_n - D\phi\| = \|q_n\phi_n - q\phi + C_n\psi''\| \leq \|(q_n - q)\phi_n\| + \|q(\phi_n - \phi)\| + |C_n|\|\psi''\|,$$

and since ϕ_n, ϕ have support contained in $[a', c']$, it follows that

$$(7.2.23) \quad \|D\phi_n - D\phi\| \leq \|q_n - q\|_{L^2[a', c']} \|\phi_n\|_\infty + \|q\|_{L^2[a', c']} \|\phi_n - \phi\|_\infty + (c' - c)|C_n|\|\psi''\|_\infty.$$

Since $\phi_n \rightarrow \phi$ uniformly on $[a', c']$, it follows that there exists a $B < \infty$ such that $\|\phi_n\|_\infty \leq B$ for all $n \in \mathbb{N}$. Hence, $D_n\phi_n \rightarrow D\phi$. This proves that $\Gamma(D') \subset \Gamma^\infty(D_n)$. As discussed at the beginning of the proof, it follows that $\Gamma(D) \subset \Gamma^\infty(D_n) \subset \Gamma(D^*)$ and that $\Gamma^\infty(D_n)$ is the graph of a closed symmetric extension of D .

□

**THEOREM 7.2.9. Let D be a second order symmetric differential operator defined by the coefficient functions p and q on (a, b) . For each $n \in \mathbb{N}$, let D_n be the second order symmetric differential operator defined by p and q on the subintervals (a_n, b_n) , where $a_n < b_n$, $a_n \rightarrow a$, and $b_n \rightarrow b$ as $n \rightarrow \infty$. Then $\Gamma(D) \subset \Gamma^\infty(D_n) \subset \Gamma(D^*)$.*

The proof of this theorem is very similar to that of *Proposition 5.2.5 which was applied to prove the strong graph convergence of the Laplacian on submanifolds of a complete Riemannian manifold to the Laplacian on the full manifold.

PROOF. Recall that D is the closure of the operator D' whose domain consists of all those $\phi \in \mathfrak{D}(D^*)$ which have compact support in the open interval (a, b) (see Section 7.1). Now it is quite easy to see that $\Gamma(D') \subset \Gamma^\infty(D_n)$. Given any $\phi \in \mathfrak{D}(D')$, the support of ϕ is contained in some compact $[a', b'] \subset (a, b)$ so that there exists $N \in \mathbb{N}$ such that $n > N$ implies that $\phi \in \mathfrak{D}(D_n)$ and that $D_n\phi = D'\phi$. Choosing $\phi_n = 0$ for $n \leq N$ and $\phi_n = \phi$ for $n > N$ yields a sequence of elements $\phi_n \in \mathfrak{D}(D_n)$ such that $(\phi_n, D_n\phi_n) \rightarrow (\phi, D'\phi)$. This proves that $\Gamma(D') \subset \Gamma^\infty(D_n)$.

Since $D = \overline{D'}$, Theorem 5.2.4 then implies that $\Gamma(D) \subset \Gamma^\infty(D_n) \subset \Gamma(D^*)$, and that $\Gamma^\infty(D_n)$ is the graph of a closed symmetric extension of D .

□

Combining *Theorems 7.2.5, 7.2.8 and 7.2.9 yields the following corollary:

**COROLLARY 7.2.10. Let D_n, D be second order symmetric differential operators defined by the coefficient functions p_n, q_n and p, q on the intervals (a_n, b_n) and (a, b) respectively. Assume that $1/p_n, 1/p$ and q_n, q belong to L^2 of any compact subinterval of (a, b) . Suppose that $a_n \rightarrow a$, $b_n \rightarrow b$ and that both $1/p_n \rightarrow 1/p$ and $q_n \rightarrow q$ in L^2 of any compact subinterval of (a, b) .*

If $\lambda < \mu$, $\lambda, \mu \in \mathbb{R}$ and $\sigma_e(D_n) \cap [\lambda, \mu] = \emptyset$ then either $\sigma_e(D) \cap (\lambda, \mu) = \emptyset$ or for any self-adjoint extension D'_n of D_n , the number of eigenvalues of D'_n in the interval (λ, μ) diverges as $n \rightarrow \infty$.

This shows, in particular, that for any $\lambda \in \sigma_e(D)$, one can find $\lambda_n \in \sigma(D_n)$ such that $\lambda_n \rightarrow \lambda$.

PROOF. Choose any $\phi \in \Gamma(D')$. Then the support of ϕ is contained in some compact $[a', b'] \subset (a, b)$. Choose $N \in \mathbb{N}$ such that for $n > N$, $[a', c'] \subset (a'_n, b'_n)$ where $c' > b'$ is fixed. As in the proof of *Theorem 7.2.8, for $n > N$, one can construct a sequence of elements $\phi_n \in \mathfrak{D}(D_n)$ such that $(\phi_n, D_n\phi_n) \rightarrow (\phi, D\phi)$ in $\mathcal{H} \oplus \mathcal{H}$. This proves that $\Gamma(D') \subset \Gamma^\infty(D_n)$, and hence by Theorem 5.2.4 that $\Gamma(D) \subset \Gamma^\infty(D_n) \subset \Gamma(D^*)$, and that $\Gamma^\infty(D_n)$ is the graph of a closed symmetric extension of D . Applying *Lemma 7.2.6 and *Theorem 7.2.5 now yields the claim. □

7.2.4.1. *Example.* Now consider a specific example in which differential expressions $L_k[\cdot]$ and symmetric differential operators D_k are defined by $p(t) = t^2$, $q_k(t) = \frac{-k^2}{t^2}$ and $(a, b) = (-1, 0)$ for $k \in [0, \infty)$. For $k > 0$ the deficiency indices of D_k are $(2, 2)$. Indeed, one can verify that if $\lambda \in \mathbb{C}$, then two linearly independent solutions to $L_k[\phi] = \lambda[\phi]$ are $f_{\pm}(\lambda; t) = (\eta(t))^{\frac{1}{2}} J_{\pm\sqrt{\frac{1}{4}-\lambda}}(k\eta(t))$ where $\eta(t) := \frac{-1}{t}$. Choosing $\lambda \in \mathbb{C} \setminus \mathbb{R}$ it is easy to show that both solutions $f_{\pm}(\lambda; \cdot)$ belong to $L^2(-1, 0)$. Hence, the deficiency indices are $(2, 2)$ (see Section 7.1). Furthermore, it is known that if D is a second order symmetric differential operator with deficiency indices $(2, 2)$, then given any self-adjoint extension \tilde{D} of D , the operators $(\tilde{D} - z)^{-1}$ are Hilbert-Schmidt operators for any $z \in \mathbb{C} \setminus \mathbb{R}$ [52]. This implies that the spectrum of any self-adjoint extension of the operator D_k , where $k > 0$ is fixed, is a sequence of eigenvalues of finite multiplicity with no finite accumulation point.

If $k = 0$, the situation is different. One can again verify that two linearly independent solutions to the differential equation $L_0[\phi] = \lambda[\phi]$ where $\lambda \in \mathbb{C}$ are $f_{\pm}(\lambda; t) = (\eta(t))^{\frac{1}{2} \pm \sqrt{\frac{1}{4}-\lambda}}$ where $\eta(t) = \frac{-1}{t}$. Recall here that L_0 is the differential expression that generates D_0 (see Section 7.1). In this case, however, if $\lambda \in \mathbb{C} \setminus \mathbb{R}$ it is not hard to show that $f_+(\lambda; \cdot)$ is not square integrable while $f_-(\lambda; \cdot)$ is. This proves that D_0 has deficiency indices $(1, 1)$. Furthermore, if $\lambda \in (\frac{1}{4}, \infty)$, it is easy to verify that both solutions $f_{\pm}(\lambda; \cdot)$ are non-normalizable. Applying Theorem 3 of ([52], pg. 93), we conclude that $[\frac{1}{4}, \infty) \subset \sigma_c(D_0)$, and hence that $[\frac{1}{4}, \infty)$ belongs to the continuous spectrum of every self-adjoint extension of D_0 .

Now one would expect that the spectrum of D_0 should depend in some way on the spectrum of the D_k in the limit as $k \rightarrow 0$. By applying *Corollary 7.2.10 it will be shown that this is indeed the case.

Observe that $p_k = p$, that $q_k(t) = -\frac{k^2}{t^2}$, and that as $k \rightarrow 0$, $q_k \rightarrow q = 0$ in L^2 of any compact subinterval of $(-1, 0)$. Thus, the conditions of *Theorem 7.2.8 and *Corollary 7.2.10 are satisfied so that these results can be applied here. Namely, since the interval $[\frac{1}{4}, \infty)$ belongs to the essential spectrum of D_0 and each D_k has no essential spectrum for $k > 0$, if D'_k is any fixed self-adjoint extension of D_k for each $k > 0$, *Corollary 7.2.10 implies that the number of eigenvalues each D_k has in any subinterval of $[\frac{1}{4}, \infty)$ diverges as $k \rightarrow 0$. As will be shown later, in Chapter 8, this example appears in a certain physical context when studying bandlimited functions on de Sitter space-time.

I expect that the results of this section, *Theorems 7.2.8, 7.2.9 and *Corollary 7.2.10 will generalize to the case of symmetric differential operators of arbitrary finite even order (see [52], Section 17, for a formal description of such operators). There are several results already in the literature which describe how the spectra of self-adjoint differential operators D_n behave as their coefficient functions p_n , q_n converge to p, q in some suitable topology ([4], pgs. 75-98; [5]; [73] Chapter 2, Section 5 and Chapter 10, Section 9). The results established here in this section are, in my opinion, of particular interest in the case where the deficiency indices of the D_n and of the limit operator D are different (as in the example above), so that it is not clear which, if any, of the self-adjoint extensions of the symmetric operators D_n converge to self-adjoint extensions of D in the strong graph sense. It will be interesting to compare these results with those that have already appeared in the literature.

7.2.5. Essential norm resolvent convergence. Suppose S_n and S are closed, densely defined symmetric operators with equal deficiency indices. If there is a sequence of compact operators $(K_n)_{n \in \mathbb{N}} \in K(\mathcal{H})$ and a $z \in \mathbb{C} \setminus \mathbb{R}$ such that $R_z(S'_n) + K_n$ converges in operator norm to $R_z(S')$, then by definition of the norm in the Calkin algebra, $\pi(R_z(S'_n))$ converges to $\pi(R_z(S'))$ in the Calkin algebra. In this case, we will say that S_n converges to S in the essential norm resolvent sense.

S_n converges to S in this sense if, for example, the deficiency indices of the S_n are finite and uniformly bounded, the deficiency indices of S are finite and $R_z(S_n)P_n(z)P(z)$ converges in

operator norm to $R_z(S)P(z)$ as $n \rightarrow \infty$. Here $P(z)$ denotes the projection onto $\mathfrak{R}(S - z)$ and $P_n(z)$ denotes the projection onto $\mathfrak{R}(S_n - z)$.

Using the fact that the Calkin algebra is a C^* -algebra, many of the results in Section VIII.7 of [57] on norm resolvent convergence for self-adjoint operators generalize directly with minimal modification to the case of essential norm resolvent convergence.

For example, the following theorem lists just a few of the results that can be obtained through simple modifications of results on norm resolvent convergence in [57].

**THEOREM 7.2.11. Let $(S_n)_{n \in \mathbb{N}}$ be a sequence of closed symmetric operators with equal deficiency indices, densely defined in a separable Hilbert space \mathcal{H} . Let S be a closed symmetric operator with equal deficiency indices. If $S_n \rightarrow S$ in the essential norm resolvent sense, then for arbitrary self-adjoint extensions S'_n and S' of S_n and S within \mathcal{H} , the following statements hold:*

- (a) $\pi(R_z(S'_n)) \rightarrow \pi(R_z(S'))$ for all $z \in \mathbb{C} \setminus \mathbb{R}$.
- (b) $\pi(f(S'_n)) \rightarrow \pi(f(S'))$ for any $f \in C_\infty(\mathbb{R})$, the continuous functions vanishing at infinity.
- (c) If $\lambda \notin \sigma_e(S)$, then $\lambda \notin \sigma_e(S_n)$ for n sufficiently large and $\pi(R_\lambda(S'_n)) \rightarrow \pi(R_\lambda(S'))$.
- (d) If $a, b \in \mathbb{R}$, $a < b$ and $a, b \notin \sigma_e(S)$, then $\pi(\chi_{(a,b)}(S'_n)) \rightarrow \pi(\chi_{(a,b)}(S'))$.

Bandlimited functions on de Sitter space-time

Let M be an expanding FRW space-time. In this case the line element for the space-time is given by

$$(8.0.24) \quad ds^2 = -dt^2 + a^2(t)dx^2.$$

The function $a(t) > 0$ is called the scale factor, and describes the expansion of the universe as time increases. The goal of this chapter is to study the subspaces of bandlimited functions $B(M, \Omega)$ on these more general and physically interesting FRW space-times M , and to see whether similar results to those of Chapter 6 still hold.

For simplicity, much of this chapter will assume that the manifold is 1 + 1 dimensional. Defining the conformal time co-ordinate η by $\frac{d\eta}{dt} = \frac{1}{a(t)}$ yields the new line element,

$$(8.0.25) \quad ds^2 = a^2(\eta) (-d\eta^2 + dx^2),$$

in the co-ordinates (η, x) . In this co-ordinate system, for the 1+1 dimensional case, the d'Alembertian is simply:

$$(8.0.26) \quad \square = \frac{1}{a^2(\eta)} \left(-\frac{\partial^2}{\partial \eta^2} + \frac{\partial^2}{\partial x^2} \right).$$

8.1. Sampling theory on expanding FRW space-time: Reducing the problem

In this section, the strategy we will pursue for studying $B(M, \Omega)$ will be the following. In the η, x co-ordinates, $\sqrt{|g(x, \eta)|} = a^{-2}(\eta)$. Here $g(x, \eta)$ is the determinant of the metric at (x, η) . Hence, the D'Alembertian for this space-time can be represented as:

$$(8.1.1) \quad \square = a^{-2}(\eta) \left(-\frac{\partial^2}{\partial \eta^2} + \frac{\partial^2}{\partial x^2} \right),$$

acting on a dense domain in the Hilbert space $L^2((\eta_i, \eta_f) \times \mathbb{R}; a^{-2}(\eta)d\eta dx)$, where $\eta_i = \eta(t)|_{t=-\infty}$ and $\eta_f = \eta(t)|_{t=+\infty}$. Under Fourier transform with respect to the spatial variable x , \square becomes

$$(8.1.2) \quad \square = -a^{-2}(\eta)(\partial_\eta^2 + k^2)$$

acting on a dense domain in $L^2((\eta_i, \eta_f) \times \mathbb{R}; a^{-2}(\eta)d\eta dk)$. For each fixed $k \in \mathbb{R}$, consider the symmetric differential operator

$$(8.1.3) \quad \square_k := -a^{-2}(\eta)(\partial_\eta^2 + k^2),$$

acting on an appropriate dense domain in the Hilbert space $L^2((\eta_i, \eta_f); a^{-2}(\eta)d\eta)$.

The basic strategy for studying $B(M, \Omega)$ will be to study the subspaces $B_k(\Omega) := \mathfrak{R} \left(\chi_{[-\Omega^2, \Omega^2]}(\square_k) \right) \subset L^2((\eta_i, \eta_f); a^{-2}(\eta)d\eta)$ for each fixed value of k . Intuitively, given any ϕ in $B(M, \Omega)$, the functions $\varphi_k := \varphi(\cdot, k)$ a.e. for a fixed $k \in \mathbb{R}$, where φ is the spatial Fourier transform of ϕ , should belong to $B_k(\Omega)$. The goal of the next few paragraphs is to motivate this idea. The function φ_k will be called the k^{th} spatial mode of the bandlimited field $\phi \in B(M, \Omega)$. This reduces the problem of studying the invariant subspace $B(M, \Omega)$ of \square to that of studying invariant subspaces of the symmetric second-order differential operators \square_k for each $k \in \mathbb{R}$. Symmetric differential operators of even order have been well studied in the literature, see *e.g.*

[52], so we will be able to employ the extensive tools and theory already developed for such operators to study the invariant subspaces $B_k(\Omega)$ of the operators \square_k .

Assume for now that each \square_k is self-adjoint. Then, intuitively, the subspaces $B_k(\Omega) := \chi_{[-\Omega^2, \Omega^2]}(\square_k)$ are spanned by the (in general non-normalizable) eigenvectors to \square_k whose eigenvalues lie in the interval $[-\Omega^2, \Omega^2]$. To be more precise, these invariant subspaces of the \square_k can be seen as uncountable linear combinations, *i.e.* integrals, of the formal solutions to the differential expression,

$$(8.1.4) \quad \square_k \phi = \lambda \phi,$$

multiplied with coefficient functions of λ , with respect to some measure. This is the content of the explicit form of the spectral theorem for second-order symmetric differential operators, Theorem 7.1.4, stated in the previous chapter. Again, intuitively, one would expect that it should be possible to express the full space $B(M, \Omega)$ as the linear span of the formal eigenvectors, or solutions to the family of differential equations

$$(8.1.5) \quad \square \phi(\eta, k) = \lambda \phi(\eta, k)$$

for $\lambda \in \mathbb{R}$. If $\phi_\lambda(\eta, k)$ is a solution to this equation, then the function in η , $\phi_{\lambda, k}(\eta) := \phi_\lambda(\eta, k)$ for each $\lambda \in \mathbb{R}$ and a fixed value of k , is a formal solution to the differential equation $\square_k \phi(\eta) = \lambda \phi(\eta)$. One would therefore expect that if $\phi \in B(M, \Omega)$, then the function $\phi_k := \phi(\cdot, k)$ a.e. for a fixed value of k should belong to $B_k(\Omega)$.

Conversely, suppose that for each $k \in \mathbb{R}$, $\varphi_k(\eta) \in B_k(\Omega)$. Then consider the function $\Phi(\eta, k) := f(k)\varphi_k(\eta)$ where $f(k)$ is chosen to decay fast enough so that $g(k) := |f(k)|\|\varphi_k\|$ is square integrable with respect to k for each fixed η . Then,

$$(8.1.6) \quad \|\Phi\|^2 = \int_{-\infty}^{\infty} \int_{\eta_i}^{\eta_f} |f(k)|^2 |\varphi_k(\eta)|^2 a^{-2}(\eta) d\eta dk = \int_{-\infty}^{\infty} |f(k)|^2 \|\varphi_k\|^2 dk = \|g\|^2 < \infty.$$

It follows that for any $j \in \mathbb{N}$,

$$(8.1.7) \quad \begin{aligned} \|\square^j \Phi\|^2 &= \int_{-\infty}^{\infty} \int_{\eta_i}^{\eta_f} |\square^j f(k) \varphi_k(\eta) a^{-2} \eta|^2 dx \\ &= \int_{-\infty}^{\infty} \int_{\eta_i}^{\eta_f} |f(k)|^2 |\square_k^j \varphi_k(\eta)|^2 a^{-2}(\eta) d\eta dk \\ &\leq \Omega^{4j} \int_{-\infty}^{\infty} |g(k)|^2 dk = \Omega^{4j} \|\Phi\|^2. \end{aligned}$$

Since this is true for any $j \in \mathbb{N}$, one would expect that this should imply that $\Phi \in B(M, \Omega)$.

Decomposing the subspace $B(M, \Omega)$ into the subspaces $B_k(\Omega)$ will be the strategy pursued in the following sections. This strategy still needs further justification, the arguments above need to be made more precise. This is work in progress. Despite this, it is intuitively clear that one can obtain information about $B(M, \Omega)$ by studying the simpler subspaces $B_k(\Omega)$. This is what we will do for the remainder of this chapter.

8.2. de Sitter space-time

A particularly simple, and yet physically interesting expanding FRW space-time that will be studied here in detail is de Sitter space-time. This is the space-time with scale factor $a(t) = e^{Ht}$ where H is a constant called the Hubble constant. Here, for convenience, it will be assumed that $H = 1$. To render the calculations more tractable, many of the calculations in this section will assume that the time co-ordinate ends at the finite value $t = 0$ so that $t \in (-\infty, 0]$ and $x \in \mathbb{R}$. To reduce the number of minus signs, the conformal time co-ordinate η will be defined by $\eta'(t) = -a^{-1}(t) = -e^{-t}$ so that $\eta(t)$ can be taken to be $\eta(t) = e^{-t}$, and $a(\eta) = \frac{1}{\eta}$. Since $t \in (-\infty, 0]$, $\eta \in [1, \infty)$. Here the point $\eta = \infty$ corresponds to the infinite past $t = -\infty$ while the point $\eta = 1$ corresponds to $t = 0$ which could, for example, represent the present day. In

terms of this conformal time co-ordinate, time runs from right to left. If one considers the full space-time, $t \in \mathbb{R}$, then $\eta \in (0, \infty)$ and 0 corresponds to the infinite future. The line element for this space-time is

$$(8.2.1) \quad ds^2 = \eta^2(-d\eta^2 + dx^2)$$

so that the covariant volume measure for this space-time manifold is given by $dV = \sqrt{|g(\eta, x)|}d\eta dx = \eta^{-2}d\eta dx$.

In summary, the D'Alembertian \square for this space-time will be represented as a symmetric second order differential operator which is generated by the differential expression

$$(8.2.2) \quad L_{\square}[\phi] := -\eta^2(\partial_{\eta}^2 - \partial_x^2)$$

and which acts on a suitable dense domain in the Hilbert space $L^2([1, \infty) \times (-\infty, \infty); \eta^{-2}d\eta dx)$. For example, a symmetric D'Alembertian \square can be defined as the closure of the operator $\tilde{\square}$ which is defined by $\tilde{\square}\phi = L_{\square}[\phi]$ for all $\phi \in \mathcal{D}(\tilde{\square}) := \{\phi \in \mathcal{H} \mid \phi \in C_0^{\infty}((1, \infty) \times (-\infty, \infty))\}$. The space of Ω -bandlimited functions on this space-time is then defined by $B(M, \Omega) := \chi_{[-\Omega^2, \Omega^2]}(\square')$ where \square' is a fixed self-adjoint extension of the symmetric operator \square . If $\tilde{\square}$ is essentially self-adjoint on its domain $\mathcal{D}(\tilde{\square})$, then \square would be its unique self-adjoint extension. As we will see however, the operator \square is not self-adjoint, and has infinite deficiency indices so that there is no unique choice of a self-adjoint D'Alembertian for this space-time.

8.3. Deficiency indices of the operators $-\square_k$

Assume that $t \in (t_i, t_f)$, where $-\infty \leq t_i < t_f \leq \infty$. In order to completely define the operators \square_k one needs to first specify their domains in $\mathcal{H} := L^2((a, b); \eta^{-2}d\eta)$, where $0 \leq a = \eta(t_f) < b = \eta(t_i) < \infty$.

To be precise, as in Chapter 7, Section 7.1, let

$$(8.3.1) \quad L_k[\phi] := -\eta^2(\phi'' + k^2\phi)$$

be a differential expression on (η_i, η_f) , and then define

$$(8.3.2) \quad \mathcal{D}(\square_k^*) := \{\phi \in \mathcal{H} \mid \phi' \in AC_{loc}[a, b]; L_k[\phi] \in \mathcal{H}\}.$$

Then define $\square_k^*\phi = L_k[\phi]$ for all $\phi \in \mathcal{D}(\square_k^*)$.

A symmetric operator \square_k' can then be defined as the restriction of \square_k^* to the set of all $\phi \in \mathcal{D}(\square_k^*)$ which have compact support in the conformal time interval (a, b) . Let \square_k denote the symmetric closure of \square_k' .

Let us now determine the deficiency indices of the symmetric operators \square_k in three different cases:

- (α) $a = 1$ and $b = \infty$
- (β) $a = 0$ and $b = 1$
- (γ) $a = 0$ and $b = \infty$

As discussed in Section 7.1, in each case, \square_k has equal deficiency indices (n, n) where $n \leq 2$, and therefore has self-adjoint extensions in each case.

To determine the deficiency indices, one needs to compute the solutions to the differential equation $L_k[\phi] = \lambda\phi$ for $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

For now assume that $k \neq 0$. The second order ordinary differential equation $L_k[\phi] = \lambda\phi$ is:

$$(8.3.3) \quad \eta^2\phi''(\eta) + (k^2\eta^2 + \lambda)\phi(\eta) = 0.$$

The solutions to this equation have the form $\eta^{1/2}w_{p(\lambda)}(|k|\eta)$, where w is a Bessel function of order $p(\lambda) := \sqrt{1/4 - \lambda}$. For $p(\lambda) \notin \mathbb{Z}$ two linearly independent solutions can be chosen to be

$$(8.3.4) \quad f_\lambda(\eta) := \eta^{1/2}J_{\sqrt{1/4-\lambda}}(|k|\eta)$$

and

$$(8.3.5) \quad g_\lambda(\eta) := \eta^{1/2}J_{-\sqrt{1/4-\lambda}}(|k|\eta).$$

Here $J_p(x)$ is the Bessel-J function of order p . Let $\lambda = 1/4 - i$ so that $p(\lambda) = e^{i\pi/4} = \frac{1+i}{\sqrt{2}}$

The case $k = 0$ must be considered separately. The eigenfunctions of \square_0^* are solutions to the ordinary differential equation:

$$(8.3.6) \quad \eta^2\phi''(\eta) + \lambda\phi(\eta) = 0.$$

Since we want to consider solutions to complex λ , it can be assumed that $\lambda \neq \frac{1}{4}$ so that a linearly independent set of solutions is formed by $f_\lambda(\eta) := \eta^{1/2+p(\lambda)}$ and $g_\lambda(\eta) := \eta^{1/2-p(\lambda)}$. Again, let $\lambda = 1/4 - i$ so that $p(\lambda) = \frac{1+i}{\sqrt{2}}$. It follows that $|f_\lambda(\eta)| = \eta^{1/2+1/\sqrt{2}}$ while $|g_\lambda(\eta)| = \eta^{1/2-1/\sqrt{2}}$.

8.3.0.1. *Case (α).* The Bessel function $J_p(x)$ behaves asymptotically like $J_p(x) \sim \sqrt{\frac{2}{\pi x}} \cos(x - p\pi/2 - \pi/4)$ for large $|x|$. Hence, for large enough $\eta > 0$,

$$(8.3.7) \quad |f_\lambda(\eta)| \sim \left| \eta^{1/2} \sqrt{\frac{2}{\pi|k|\eta}} \cos(|k|\eta - p(\lambda)\pi/2 - \pi/4) \right|.$$

This shows that,

$$(8.3.8) \quad \int_B^\infty |f_\lambda(\eta)|^2 \eta^{-2} d\eta \sim C \int_B^\infty \cos^2(|k|\eta + z) \eta^{-2} d\eta < \infty.$$

Furthermore, for fixed $p \in \mathbb{C}$, the function $J_p(z)$ is analytic on any region not containing $z = 0$, and is hence bounded in the interval $[1, B]$ for any fixed $B > 0$. We conclude that $f_\lambda(\eta)$ is normalizable. Similarly one can conclude that $g_\lambda(\eta)$ is also normalizable. Since both solutions to $\square_k^*\phi = \lambda\phi$ for $\lambda = 1/4 - i$ are normalizable this means that \square_k has deficiency indices $(2, 2)$ for $k \neq 0$.

For the zero mode, $k = 0$,

$$(8.3.9) \quad \|f_\lambda\|^2 = \int_1^\infty \eta^{1+\sqrt{2}-2} d\eta = \infty$$

while $\|g_\lambda\|^2 = \int_1^\infty \eta^{1-\sqrt{2}-2} d\eta < \infty$. Hence, the deficiency indices of \square_0 are $(1, 1)$.

8.3.0.2. *Case (β).* Again, for $k \neq 0$, the following asymptotic formula holds for $J_p(x)$:

$$(8.3.10) \quad J_p(x) \sim \frac{1}{\Gamma(p+1)} \left(\frac{x}{2}\right)^p \quad x \rightarrow 0^+,$$

where Γ is the Euler Gamma function.

It follows that $|f_\lambda(\eta)| \sim \eta^{1/2} \left| e^{\frac{1+i}{\sqrt{2}} \ln \frac{\eta}{2}} \right| \sim \eta^{\frac{1}{2} + \frac{1}{\sqrt{2}}}$ as $\eta \rightarrow 0$. Therefore,

$$(8.3.11) \quad \int_0^\epsilon \frac{|f_\lambda(\eta)|^2}{\eta^2} d\eta \sim \int_0^\epsilon \eta^{\sqrt{2}-1} d\eta < \infty,$$

showing that f_λ is normalizable. On the other hand, $g_\lambda(\eta) \sim \eta^{\frac{1}{2} - \frac{1}{\sqrt{2}}}$ as $\eta \rightarrow 0$,

$$(8.3.12) \quad \int_0^\epsilon |g_\lambda(\eta)|^2 \eta^{-2} d\eta \sim \int_0^\epsilon \eta^{-1-\sqrt{2}} d\eta = \infty,$$

so that g_λ is not normalizable. In conclusion, the deficiency indices for \square_k in this case are $(1, 1)$.

For the zero mode, \square_0 , $\eta^{-2}|f_\lambda(\eta)|^2 = \eta^{\sqrt{2}-1}$ so that f_λ is normalizable, and $\eta^{-2}|g_\lambda(\eta)|^2 = \eta^{-1-\sqrt{2}}$ so that g_λ is not normalizable. So in this case the deficiency indices for \square_0 are also $(1, 1)$.

8.3.0.3. *Case (γ)*. In this final case of the full de Sitter spacetime, it is again not difficult to use the asymptotic formulas for the Bessel functions to determine that the deficiency indices for $k \neq 0$ are $(1, 1)$ and for $k = 0$ are $(0, 0)$. This conclusion is also implied by the following theorem ([52], Section 17.5).

THEOREM 8.3.1. *Let L be a symmetric differential expression of order $2n$ on $L^2(a, b)$. Let D_+ and D_- be the symmetric differential operators generated by L on $L^2(a, c]$ and $L^2[c, b)$. Then the deficiency indices of the differential operator D generated by L in $L^2(a, b)$ are (m, m) where $m = m_- + m_+ - 2n$ and m_{\pm} are the deficiency indices of D_{\pm} .*

By this theorem, the deficiency indices for the \square_k in the case (γ) are determined by the other two cases. Namely, for $k \neq 0$, $m = 2 + 1 - 2 = 1$ and for $k = 0$, $m = 1 + 1 - 2 = 0$.

8.3.0.4. *Deficiency indices of the full d'Alembertian*. The fact that the deficiency indices of \square_k are non-zero means that the full operator \square is not self-adjoint. Indeed, let $g_{\lambda}(\eta)$ be the normalizable solution to $\square_k \phi = \lambda g_{\lambda} \phi$ for some fixed $\lambda \in \mathbb{C} \setminus \mathbb{R}$ which obeys a fixed set of initial conditions at some point η_0 and every $k \in \mathbb{R}$. Then choose any $f(k)$ such that $f \in C_0^{\infty}(\mathbb{R})$. It follows that $f(k)g_{\lambda}(\eta; k)$ will be a normalizable solution to $\square^* \phi = \lambda \phi$. From this it is clear that the deficiency indices of the full symmetric operator \square are infinite.

This is different from flat Minkowski spacetime. In the case of flat space-time, the d'Alembertian is unitarily equivalent to $\square = p_0^2 - \mathbf{p}^2$ in $L^2(\mathbb{R}^4)$. It is not difficult to check that this operator is essentially self-adjoint on $C_0^{\infty}(\mathbb{R}^4)$, and hence has a unique self-adjoint extension.

8.4. The case (α), de Sitter with finite end-time

In this section, the case where $\eta \in [1, \infty)$ will be studied in detail.

Recall that if $k \neq 0$, one can immediately make some conclusions about the spectra of the \square_k and their self-adjoint extensions. For this case, the deficiency indices of the \square_k are $(2, 2)$. It is known that the spectrum of any self-adjoint extension of a second order symmetric differential operator with deficiency indices $(2, 2)$ is purely discrete with no finite accumulation points ([52], pg. 90). This implies that the spectrum of any self-adjoint extension of the \square_k consists of eigenvalues which have no finite accumulation point, and that each \square_k has no continuous spectrum. Furthermore, since the point $\eta = 1$ is a regular end-point of the differential expression L_k that generates \square_k (see Section 7.1 for the definition of a regular end-point), it follows ([52], pg. 93) that none of the \square_k have any eigenvalues. This allows us to conclude that the symmetric operators \square_k are both simple and regular, so that by Theorem 4.4.3, the spectra of any self-adjoint extension of these operators is purely discrete, and consists of eigenvalues of multiplicity at most 2 with no finite accumulation point.

By the conclusions of the previous paragraph, if one puts a cutoff on the spectrum of any self-adjoint extension, \square'_k , of \square_k , one obtains a finite dimensional subspace. That is, the subspaces $B_k(\Omega) := \mathfrak{R} \left(\chi_{[-\Omega^2, \Omega^2]}(\square'_k) \right)$ have finite dimension N_k for any $k > 0$. Note also that the projections onto the subspaces $B_k(\Omega) = \mathfrak{R} \left(\chi_{[-\Omega^2, \Omega^2]}(\square'_k) \right)$ are strongly continuous functions of the 2×2 unitary matrix that indexes the choice of self-adjoint extension.

8.4.0.5. *Remark*. The above implies that any fixed non-zero spatial mode $\phi_k(\eta)$ of a bandlimited function ϕ in this space-time has a finite number of degrees of freedom in time, *i.e.*, $\phi_k(\eta)$ belongs to a finite N_k -dimensional subspace of $\mathcal{H} = L^2([1, \infty); \eta^{-2} d\eta)$, and obeys a finite reconstruction formula. See Subsection 5.3.1 for a description of how any finite dimensional function space trivially obeys a finite sampling and reconstruction formula. That is, given any $k \neq 0$ the fixed spatial mode ϕ_k can be reconstructed everywhere, for all conformal time, simply from the knowledge of the values it takes on certain finite sets of N_k points in conformal time η .

8.4.0.6. *Remark.* Although this may seem very dramatic and remarkable, it is supported by physical intuition. We are considering de Sitter space-time with finite end-time, a space-time which expands exponentially until time ends at $t = 0$. This means that a fixed co-moving spatial mode ϕ_k , for most values of $t \in (-\infty, 0]$ corresponds to a proper wavelength that is vanishingly small, or equivalently to extremely high frequencies, *i.e.*, to large values of $|k|$. In our studies of flat space-time, we observed that large spatial modes $|k|$ have a vanishingly small temporal bandwidth in $1 + 1$ dimensions. Therefore, it is conceivable, that since for most $t \in (-\infty, 0]$ a co-moving spatial mode ϕ_k in $1 + 1$ dimensional de Sitter space-time corresponds to exponentially large proper spatial frequencies which have a vanishingly small temporal bandwidth, that these co-moving spatial modes can have merely a finite number of degrees of freedom in time.

The main goal of the following sections is to determine how the number, N_k , of sample points in conformal time needed to reconstruct the k^{th} spatial mode $\phi_k \in B_k(\Omega)$ of a bandlimited function $\varphi \in B(M, \Omega)$ behaves as a function of k . It will be shown that $N_k \rightarrow \infty$ as $k \rightarrow 0$, and that as $|k| \rightarrow \infty$, N_k approaches a number less than or equal to 2, depending on the choice of self-adjoint extension \square'_k used to define the subspaces $B_k(\Omega)$. Again, this is in agreement with the analysis of fixed spatial modes of bandlimited functions in flat space-time where we saw that larger spatial modes have a smaller density of degrees of freedom in time.

8.4.1. The behaviour of N_k as $|k| \rightarrow 0$. Recall here that $N_k := \dim(B_k(\Omega))$ where $\Omega > 0$ is the bandlimit. Define the variable s by $s = \eta^{-1}$. In terms of this co-ordinate, the operator \square_k becomes a symmetric operator D_k , given by

$$(8.4.1) \quad D_k = -\frac{d}{ds} \left(s^2 \frac{d}{ds} \cdot \right) - \frac{k^2}{s^2}.$$

This operator D_k acts on a dense domain in $L^2[-1, 0]$. This D_k is the same symmetric differential operator considered in Example 7.2.4.1. As was shown in Example 7.2.4.1, the number of eigenvalues of any self-adjoint extension of D_k in any finite subinterval of $[-\frac{1}{4}, \infty)$ diverges as $k \rightarrow 0$. This shows, in particular, that $N_k \rightarrow \infty$ as $|k| \rightarrow 0$ for any fixed bandlimit Ω .

8.4.2. The behaviour of N_k as $|k| \rightarrow \infty$. For this subsection let $\Omega > 0$ be a fixed bandlimit. The number $N_k = \dim B_k(\Omega)$, where $B_k(\Omega) := \chi_{[-\Omega^2, \Omega^2]}(\square'_k)\mathcal{H}$, depends on the choice \square'_k of self-adjoint extension of \square_k . The goal of this subsection is to prove the following proposition:

*PROPOSITION 8.4.1. *For any fixed $\Omega > 0$, there is a $K > 0$ such that $|k| > K$ implies that $N_k = N_k(\square'_k) \leq 2$ for any self-adjoint extension \square'_k of \square_k .*

Choose a $\lambda' \in \mathbb{R}$ such that $|\lambda'| \leq \Omega^2$. By Theorem 4.4.2, given any $k \neq 0$, there is a self-adjoint extension \square'_k of \square_k for which λ' is an eigenvalue of multiplicity 2. To avoid writing $|k|$, we will assume for the remainder of this section, without loss of generality, that $k > 0$ so that $|k| = k$.

Let $f_\lambda(\eta) := \eta^{1/2} J_{\sqrt{1/4-\lambda}}(k\eta)$ and $g_\lambda(\eta) = \eta^{1/2} Y_{\sqrt{1/4-\lambda}}(k\eta)$. These functions form a basis for the eigenspace of \square_k^* to eigenvalue λ . Here $J_p(x)$ and $Y_p(x)$ are the Bessel-J and Bessel-Y functions of order p . Also recall that if $f, g \in \mathcal{H} = L^2([1, \infty); \eta^{-2}d\eta)$, the inner product for this Hilbert space is $\langle f, g \rangle := \int_1^\infty f(\eta)\overline{g(\eta)}\eta^{-2}d\eta$.

*CLAIM 8.4.2. *If $\lambda \neq \lambda'$ is an eigenvalue of \square'_k , then it is a zero of the following function.*

$$(8.4.2) \quad \Lambda(\lambda) := \langle g_\lambda, f_{\lambda'} \rangle \langle f_\lambda, g_{\lambda'} \rangle - \langle f_\lambda, f_{\lambda'} \rangle \langle g_\lambda, g_{\lambda'} \rangle$$

Notice that the function $\Lambda(\lambda) = \Lambda(\lambda, \lambda', k)$ also depends on k and λ' . To simplify the notation, this dependence will be suppressed, and we will simply write $\Lambda(\lambda)$.

PROOF. To see this note that if λ is another eigenvalue of \square'_k then there is a linear combination $c_1 f_\lambda - c_2 g_\lambda$ which is an eigenvector to \square'_k , and hence must be perpendicular to both $f_{\lambda'}$ and $g_{\lambda'}$, since both are eigenvectors of \square'_k . That is, both $\langle c_1 f_\lambda - c_2 g_\lambda, f_{\lambda'} \rangle = 0$ and $\langle c_1 f_\lambda - c_2 g_\lambda, g_{\lambda'} \rangle = 0$ so that

$$(8.4.3) \quad c_1 \langle f_\lambda, f_{\lambda'} \rangle = c_2 \langle g_\lambda, f_{\lambda'} \rangle,$$

and,

$$(8.4.4) \quad c_1 \langle f_\lambda, g_{\lambda'} \rangle = c_2 \langle g_\lambda, g_{\lambda'} \rangle.$$

It can be assumed that either $c_1 \neq 0$ or $c_2 \neq 0$. Suppose $c_2 \neq 0$. It follows that

$$(8.4.5) \quad \begin{aligned} c_2 \Lambda(\lambda) &= c_2 \langle g_\lambda, f_{\lambda'} \rangle \langle f_\lambda, g_{\lambda'} \rangle - c_2 \langle f_\lambda, f_{\lambda'} \rangle \langle g_\lambda, g_{\lambda'} \rangle \\ &= c_1 \langle f_\lambda, f_{\lambda'} \rangle \langle f_\lambda, g_{\lambda'} \rangle - c_1 \langle f_\lambda, f_{\lambda'} \rangle \langle f_\lambda, g_{\lambda'} \rangle = 0. \end{aligned}$$

This shows that if λ is another eigenvalue of \square'_k , then it is a zero of $\Lambda(\lambda)$. \square

To simplify notation, let $u := \sqrt{\lambda - 1/4}$ and $u' := \sqrt{\lambda' - 1/4}$. Here, for $z = |z|e^{i\arg(z)} \in \mathbb{C}$, define $\sqrt{z} := +|z|^{1/2}e^{i\frac{\arg(z)}{2}}$ to be the positive square root. As a function of u , $\Lambda(u) = \langle g_u, f_{u'} \rangle \langle f_u, g_{u'} \rangle - \langle f_u, f_{u'} \rangle \langle g_u, g_{u'} \rangle$, where $f_u(\eta) = \eta^{1/2}J_u(k\eta)$ and $g_u(\eta) = \eta^{1/2}Y_u(k\eta)$.

*CLAIM 8.4.3. *The function $\Lambda(u)$ is entire.*

This proof is a simple application of Fubini's theorem and Morera's theorem. Recall that Fubini's theorem states when it is valid to interchange orders of integration. Further recall that Morera's theorem is a converse statement to Cauchy's theorem. Namely, Morera's theorem states that if a function is continuous in a region of the complex plane, and if the integral of that function over any closed triangular contour in the region vanishes, then the function is analytic in that region ([14], pg. 88).

In the proof below, we will use the fact that the Bessel function $J_p(z)$ is an entire function of p for fixed $z \neq 0$, and that for fixed $p \in \mathbb{C}$, it is analytic in any region not containing $z = 0$ ([70], pg. 44).

PROOF. It will be shown that $g(u) := \langle f_u, f_{u'} \rangle$ is entire. Showing that the full function $\Lambda(u)$ is entire uses similar logic. Recall that,

$$(8.4.6) \quad \langle f_u, f_{u'} \rangle = \int_1^\infty J_u(\eta)J_{u'}(\eta) \frac{1}{\eta} d\eta.$$

Let Γ be any finite length straight line contour in \mathbb{C} , so that it can be represented as $\Gamma(t) = a + te^{i\theta}$ where $t \in [t_1, t_2]$. Then,

$$(8.4.7) \quad \int_\Gamma g(u)du = \int_{t_1}^{t_2} g(\Gamma(t))\Gamma'(t)dt = \int_{t_1}^{t_2} g(\Gamma(t))e^{i\theta}dt.$$

Now,

$$(8.4.8) \quad \begin{aligned} \left| \int_\Gamma g(u)du \right| &\leq \int_{t_1}^{t_2} \int_1^\infty |J_{\Gamma(t)}(\eta)| |J_{u'}(\eta)| \frac{1}{\eta} d\eta dt \\ &= \int_{t_1}^{t_2} \langle |f_{\Gamma(t)}|, |f_{u'}| \rangle dt \\ &\leq \int_{t_1}^{t_2} \|f_{\Gamma(t)}\| \|f_{u'}\| dt. \end{aligned}$$

It is not difficult to show that $f_{\Gamma(t)}$ is a continuous Hilbert space valued function of t , so that $\|f_{\Gamma(t)}\|$ is a continuous function of t , and is hence bounded above for $t \in [t_1, t_2]$. Therefore,

$$(8.4.9) \quad \int_{t_1}^{t_2} \int_1^\infty |J_{\Gamma(t)}(\eta)| |J_{u'}(\eta)| \frac{d\eta}{\eta} dt < \infty.$$

By Fubini's theorem, the orders of integration can be interchanged. It follows that for any closed triangular contour Γ in \mathbb{C} ,

$$(8.4.10) \quad \int_{\Gamma} \int_1^{\infty} J_u(\eta) J_{u'}(\eta) \frac{1}{\eta} d\eta du = \int_1^{\infty} \left(\int_{\Gamma} J_u(\eta) du \right) J_{u'}(\eta) \frac{1}{\eta} d\eta = 0,$$

as Γ is a closed contour and $J_u(\eta)$ is an entire function of u for any fixed $\eta \in [1, \infty)$. By Morera's theorem, we conclude that $g(u)$ is also an entire function of u . \square

The following asymptotic formulas for the Bessel functions hold for large $|x|$ ([70], pg. 199):

$$(8.4.11) \quad J_p(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{2}p - \frac{\pi}{4}\right)$$

and

$$(8.4.12) \quad Y_p(x) \sim \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\pi}{2}p - \frac{\pi}{4}\right)$$

as $|x| \rightarrow \infty$. To further simplify the notation, let $a(\eta; u) := k\eta - \frac{u\pi}{2} - \frac{\pi}{4}$, let $C_u(\eta) = \cos(a(\eta; u))$, and let $S_u(\eta) := \sin(a(\eta; u))$. It follows that $f_u(\eta) \sim \sqrt{\frac{2}{\pi k}} C_u(\eta)$ and $g_u(\eta) \sim \sqrt{\frac{2}{\pi k}} S_u(\eta)$ as $|k\eta| \rightarrow \infty$. Since $\eta \in [1, \infty)$, these asymptotic formulas become increasingly accurate for large $|k|$. These formulas also follow from Lemma 8.4.8 which will be proven in Subsection 8.4.3.

Let $\Phi(u) := \langle S_u, C_{u'} \rangle \langle C_u, S_{u'} \rangle - \langle C_u, C_{u'} \rangle \langle S_u, S_{u'} \rangle$. Notice that the function $\Phi = \Phi(u, u', k)$ also depends on u' and k . This dependence is suppressed to simplify the notation. The asymptotic formulas (8.4.11) and (8.4.12) suggest that the function $\Phi(u)$ should be a good approximation to $\frac{k^2 \pi^2}{4} \Lambda(u)$ for large $|k|$. The following lemma asserts that this is indeed the case.

**LEMMA 8.4.4. Given any compact set $K \subset \mathbb{C}$ there is a $B > 0$ such that $|k| > B$ implies that $\left| \Phi(u) - \frac{k^2 \pi^2}{4} \Lambda(u) \right| \leq \frac{B}{k}$ for all $u \in K$*

To streamline the proof of *Proposition 8.4.1, the proof of this lemma will also be delayed until Subsection 8.4.3.

In fact, the function $\Phi(u)$ is constant with respect to u as the next claim will show.

**CLAIM 8.4.5. $\Phi(u, u', k) = \Phi(u', k)$, is constant with respect to u and depends only on u' and k .*

By the definition of $\Phi(u)$, it is clear that $\Phi(u', k) = \Phi(u'+4, k)$ for all $u' \in \mathbb{C}$ and $k \in (0, \infty)$. The next claim describes the behaviour of $\Phi(0, k) = \Phi(4n, k)$, $n \in \mathbb{N}$ in the limit as $k \rightarrow \infty$.

**CLAIM 8.4.6. $\Phi(0, k) \rightarrow \frac{1}{4}$ as $k \rightarrow \infty$.*

The proofs of both of the above claims will be provided in upcoming subsections. The strategy for proving *Proposition 8.4.1 is now clear. Any eigenvalue $\lambda \neq \lambda'$ of \square'_k is a zero of $\Lambda(\lambda)$ so that $u := +\sqrt{\lambda - \frac{1}{4}}$ is a zero of the entire function $\Lambda(u)$. By *Lemma 8.4.4, the function $\Phi(u) = \Phi(u, u', k)$, which is also entire in u for fixed u' and k , becomes a good approximation to $\Lambda(u) = \Lambda(u, u', k)$ in the limit as $k \rightarrow \infty$. Hence, it is certainly plausible that for these fixed values of u' , $\Lambda(u, u', k)$ and $\Phi(u, u', k)$ will have the same number of zeroes as functions of u in a given compact set K in the limit as $k \rightarrow \infty$. Furthermore for $u' = 4n$, $n \in \mathbb{N}$, *Claims 8.4.5 and 8.4.6 show that $\Phi(u, u', k)$ approaches a non-zero constant independent of u in this limit. This suggests that $\Lambda(u, u', k)$, for $u' = 4n$, $n \in \mathbb{N}$, will have no zeroes as a function of u in K , if k is sufficiently large. Using this it will be shown that the eigenvalues of the self-adjoint extensions \square'_k of \square_k become increasingly further apart as $k \rightarrow \infty$.

The final ingredient needed for the proof of *Proposition 8.4.1 is Rouché's theorem ([14], pg. 121):

THEOREM 8.4.7. (*Rouché's theorem*) Suppose f and g are analytic in a region Ω and $K \subset \Omega$ is compact. If f and g have no zeroes on the boundary of K and $|f(z) - g(z)| < |g(z)|$ on the boundary of K , then f and g have the same number of zeroes in K

We now proceed with the proof of *Proposition 8.4.1.

PROOF. (*Proposition 8.4.1) For any fixed $n \in \mathbb{N}$, choose a compact subset $K \subset \mathbb{C}$ such that $u(\lambda) \subset K$ for all $|\lambda - \lambda'| \leq 5n$ where $\lambda' = \frac{1}{4}$ so that $u' = u(\lambda') = 0$. By *Claim 8.4.6 and *Lemma 8.4.4 there is a $B > 0$ such that if $k > B$ then both $|\Phi(0, k) - \frac{1}{4}| < \frac{1}{8}$ and $\left| \Phi(0, k) - \frac{k^2 \pi^2}{4} \Lambda(u) \right| < \frac{1}{8}$. It follows that

$$(8.4.13) \quad \left| \Phi(0, k) - \frac{k^2 \pi^2}{4} \Lambda(u) \right| < |\Phi(0, k)|$$

for all $u \in K$ so that Rouché's theorem implies that $\Phi(0, k)$ and $\Lambda(u)$ have the same number of zeroes as functions of u in K for $k > B$. Since Φ is a non-zero constant, we can conclude that $\Lambda(u)$ has no zeros in K , and hence that $\Lambda(\lambda) = \Lambda(u(\lambda))$ has no zeroes in the interval $(-4n, 4n)$. This shows that the self-adjoint extension \square'_k which has $\lambda' = \frac{1}{4}$ as an eigenvalue of multiplicity 2 has no zeroes in the open interval $(-4n, 4n)$. It follows from the formula of *Theorem 7.2.4 that $\Delta(\square_k)_t \geq \Delta(\square'_k)_t \geq 2\sqrt{3}n$ for all $t \in (-2n, 2n)$. In particular $\Delta(\square_k)_t \geq 2n$ for all $t \in (-2n, 2n)$. This fact and *Theorem 7.2.2 then imply that any self-adjoint extension $\tilde{\square}_k$ of \square_k can have at most 2 eigenvalues in the interval $[-n, n]$. Since the fixed value of $n \in \mathbb{N}$ was arbitrary, we conclude that for any $\Omega > 0$, there is a $B > 0$ such that $k > B$ implies that any self-adjoint extension of \square_k has at most two eigenvalues in the interval $[-\Omega^2, \Omega^2]$. This proves the proposition. \square

8.4.3. Proof of claims and lemmas.

8.4.3.1. *Proof of *Lemma 8.4.4.* Before proving this claim, it will be convenient to first prove the following auxiliary lemma.

LEMMA 8.4.8. Let $H_v^{(1)}(z)$ be the Hankel function of the first kind of order $v \in \mathbb{C}$. Let $K \subset \mathbb{C}$ be compact. Then there is an $M < \infty$ such that

$$(8.4.14) \quad \left| \sqrt{\frac{\pi z}{2}} H_v^{(1)}(z) - e^{i(z - \frac{\pi v}{2} - \frac{\pi}{4})} \right| \leq \frac{M}{|z|}$$

for all $z \in [1, \infty)$ and $v \in K$.

Although the assertion of the above lemma is not explicitly stated in [70], its proof is achieved by examining the details of Section 7.2 of this reference.

PROOF. From [70], pg. 197, for any $p \in \mathbb{N}$,

$$(8.4.15) \quad H_v^{(1)}(z) = \sqrt{\frac{2}{\pi z}} e^{i(z - v \frac{\pi}{2} - \frac{\pi}{4})} \left[\sum_{m=0}^{p-1} \frac{(1/2 - v)_m \Gamma(v + m + 1/2)}{m! \Gamma(v + 1/2) (2iz)^m} + Q_p^{(1)}(v, z) \right].$$

Here $(1/2 - v)_m = (1/2 - v)(3/2 - v) \dots (\frac{2m-1}{2} - v)$ is a polynomial of degree m in v and $\Gamma(z)$ is the gamma function. The remainder term is given by

$$(8.4.16) \quad Q_p^{(1)}(z, v) = \frac{(1/2 - v)_p}{\Gamma(v + 1/2) (2iz)^p (p-1)!} \int_0^\infty \exp(i\beta t) e^{-u} u^{v-1/2+p} \int_0^1 (1-t)^{p-1} \left(1 - \frac{ut}{2iz}\right)^{v-p-1/2} dt du.$$

Here, $|\beta| < \pi/2$, $\pi/2 + \beta < \arg z < \frac{3\pi}{2} + \beta$, and $\operatorname{Re}(v + 1/2) > 0$.

It follows that,

$$(8.4.17) \quad \left| \sqrt{\frac{\pi z}{2}} H_v^{(1)}(z) - e^{i(z - \frac{\pi v}{2} - \frac{\pi}{4})} \right| = \left| \frac{e^{i(z - \frac{\pi v}{2} - \frac{\pi}{4})}}{2iz} \right| \left| \sum_{m=1}^{p-1} \frac{(1/2 - v)_m \Gamma(v + m + 1/2)}{\Gamma(v + 1/2) (2iz)^{m-1}} + R_p^{(1)}(z, v) \right|,$$

where $R_p^{(1)}(z, v) = (2iz) Q_p^{(1)}(z, v)$.

Let $L := \max_{v \in K} |\operatorname{Re}(v)|$ and $W := \max_{v \in K} |\operatorname{Im}(v)|$, and Let S be the rectangle of length $2L$ and width $2W$ centred at the origin in the complex plane. Then $S \supset K$ is compact. Clearly, there is a constant C such that

$$(8.4.18) \quad \left| \frac{e^{i(z - \frac{\pi v}{2} - \frac{\pi}{4})}}{(2iz)} \right| \leq \frac{c_1}{|z|} e^{\frac{\pi}{2}|v|} \leq \frac{C}{|z|},$$

for all $v \in S$.

The lemma will be established if it can be shown that the quantity

$$(8.4.19) \quad \left| \sum_{m=1}^{p-1} \frac{(1/2 - v)_m \Gamma(v + m + 1/2)}{\Gamma(v + 1/2)(2iz)^{m-1}} + R_p^{(1)}(z, v) \right|$$

is bounded for all $v \in K$ and $z \in [1, \infty)$. Using the identity $\Gamma(z + 1) = z\Gamma(z)$ obeyed by the gamma function, it is not hard to see that the term

$$(8.4.20) \quad B(v, z) := \sum_{m=1}^{p-1} \frac{(1/2 - v)_m \Gamma(v + m + 1/2)}{\Gamma(v + 1/2)(2iz)^{(m-1)}},$$

is analytic in v for any fixed $z \in [1, \infty)$, so that there is a constant B_1 such that $|B(v, z)| \leq B_1$ for all $v \in K$ and $z \in [1, \infty)$.

It remains to show that the remainder term,

$$(8.4.21) \quad R_p^{(1)}(z, v) = \frac{(1/2 - v)_p}{\Gamma(v + 1/2)(2iz)^{p-1}(p-1)!} \int_0^\infty \exp(i\beta) e^{-u} u^{v-1/2+p} \int_0^1 (1-t)^{p-1} \left(1 - \frac{ut}{2iz}\right)^{v-p-1/2} dt du,$$

is bounded for $v \in K$ and $z \in [1, \infty)$.

Choose $p \in \mathbb{N}$ such that for all $v \in S$, $-p - 1/2 \leq \operatorname{Re}(v) \leq p + 1/2$. Recall that the formula we are using here for $H_v^{(1)}(z)$ is valid for $|\beta| < \frac{\pi}{2}$ and $-\pi/2 + \beta < \arg(z) < 3\pi/2 + \beta$. Choose $\delta > 0$ such that $|\beta| \leq \frac{\pi}{2} - \delta$ so that $-\pi + \delta \leq \arg z - \pi/2 - \beta \leq \pi - \delta$. Let $w := 1 + \frac{iut}{2z}$. Then it is a straightforward calculation to verify that

$$(8.4.22) \quad \begin{aligned} |w|^2 &= 1 + \frac{|u|t}{|z|} \cos(\beta + \pi/2 - \arg(z)) + \frac{|u|^2 t^2}{4|z|^2} \\ &\geq 1 + \frac{|u|t}{|z|} \cos(\pi - \delta) + \frac{|u|^2 t^2}{4|z|^2} \\ &= 1 - \frac{|u|t}{|z|} \cos \delta + \frac{|u|^2 t^2}{4|z|^2} \\ &= \left(1 - \frac{|u|t}{2|z|} \cos \delta\right)^2 + \frac{|u|^2 t^2}{4|z|^2} \sin^2 \delta. \end{aligned}$$

It follows that if $\frac{|u|^2 t^2}{4|z|^2} \geq 1$ then $|w|^2 \geq \sin^2 \delta > 0$ while if $\frac{|u|^2 t^2}{4|z|^2} \leq 1$ then $|w|^2 \geq (1 - \frac{|u|t}{2|z|} \cos \delta)^2 \geq (1 - \cos \delta)^2 > 0$. Either way, for all such β and z , there is an $\epsilon > 0$ such that $|w| = \left|1 + \frac{iut}{2z}\right| \geq \epsilon$. We conclude that there is a constant C_2 such that

$$(8.4.23) \quad \left| \left(1 - \frac{ut}{2iz}\right)^{v-p-1/2} \right| \leq e^{|\arg(w)||\operatorname{Im}(v)|} \epsilon^{\operatorname{Re}(v-p-1/2)} =: B_p(v) \leq C_2$$

for all $z \in [1, \infty)$ and $v \in S$ such that $\operatorname{Re}(v + 1/2) > 0$.

Hence,

$$(8.4.24) \quad |R_p^{(1)}(z, v)| \leq \frac{C_2}{(p-1)! |2z|^{p-1}} \left| \frac{(1/2 - v)_p}{\Gamma(v + 1/2)} \right| \left| \int_0^1 (1-t)^{p-1} dt \right| \left| \int_0^\infty \exp(i\beta) e^{-u} u^{v+p-1/2} du \right|.$$

Finally, observe that

$$(8.4.25) \quad \left| \int_0^\infty \exp(i\beta) e^{-u} u^{v-p-1/2} du \right| \leq \int_0^\infty \exp(i\beta) e^{-\operatorname{Re}(u)} |u|^{\operatorname{Re}(v+p-1/2)} e^{\frac{\pi}{2} \operatorname{Im}(v)} |du|.$$

Notice that $\operatorname{Re}(u) > 0$ since $u = |u|e^{i\beta}$, and $|\beta| < \pi/2$. Also, $\operatorname{Re}(v+p-1/2) > 0$ since $\operatorname{Re}(v+p-1/2) > \operatorname{Re}(v+1/2) > 0$. It follows that the above is a continuous function of v for $|\operatorname{Re}(v)| \leq L$, *i.e.*, for all $v \in S$, so that the above is bounded by some constant C_3 for $v \in S$ such that $\operatorname{Re}(v+1/2) \geq 0$.

In conclusion, there is a finite constant B such that $|R_p^{(1)}(z, v)| \leq B$ for all $v \in S$ such that $\operatorname{Re}(v+1/2) > 0$ and all $z \in [1, \infty)$ so that there is a constant $B_1 < \infty$ such that,

$$(8.4.26) \quad \left| \sqrt{\frac{\pi z}{2}} H_v^{(1)}(z) - e^{i(z-\pi v/2-\pi/4)} \right| \leq \frac{B_1}{|z|},$$

for all v in this range and $z \in [1, \infty)$. Using the identity $H_{-v}^{(1)}(z) = e^{i\pi v} H_v^{(1)}(z)$, we conclude that

$$(8.4.27) \quad \left| \sqrt{\frac{\pi z}{2}} H_{-v}^{(1)}(z) - e^{i(z+\pi v/2-\pi/4)} \right| = |e^{-iv\pi}| \left| \sqrt{\frac{\pi z}{2}} H_v^{(1)}(z) - e^{i(z-\pi v/2-\pi/4)} \right|.$$

With the aid of this identity, it follows that for all $v \in S \supset K$, and $z \in [1, \infty)$, there is a constant M such that

$$(8.4.28) \quad \left| \sqrt{\frac{\pi z}{2}} H_v^{(1)}(z) - e^{i(z-\pi v/2-\pi/4)} \right| \leq \frac{M}{|z|}.$$

□

A similar inequality holds for $H_v^{(2)}(z)$. Using the identities $J_v(z) = \frac{H_v^{(1)}(z) + H_v^{(2)}(z)}{2}$ and $Y_v(z) = \frac{H_v^{(1)}(z) - H_v^{(2)}(z)}{2i}$, we conclude that given any compact set $K \subset \mathbb{C}$ there exist constants $M_1, M_2 < \infty$ such that

$$(8.4.29) \quad \left| \sqrt{\frac{\pi z}{2}} J_v(z) - \cos(z - v\pi/2 - \pi/4) \right| \leq \frac{M_1}{|z|},$$

and,

$$(8.4.30) \quad \left| \sqrt{\frac{\pi z}{2}} Y_v(z) - \sin(z - v\pi/2 - \pi/4) \right| \leq \frac{M_2}{|z|},$$

for all $z \in [1, \infty)$ and $v \in K$.

Lemma 8.4.8 can also be applied to prove *Lemma 8.4.4. To shorten the presentation, we will only prove the following half of the lemma:

*LEMMA 8.4.9. *Given any compact set $K \subset \mathbb{C}$ there is a $B > 0$ such that $|k| > B$ implies that $|\frac{k\pi}{2} \langle f_u, f_{u'} \rangle - \langle C_u, C_{u'} \rangle| < \frac{N}{k}$ for all $u \in K$*

PROOF. (of *Lemma 8.4.9)

Consider the following:

$$(8.4.31) \quad \left| \frac{k\pi}{2} \langle f_u, f_{u'} \rangle - \langle C_u, C_{u'} \rangle \right| \leq \left| \frac{k\pi}{2} \langle f_u, f_{u'} \rangle - \sqrt{\frac{k\pi}{2}} \langle C_u, f_{u'} \rangle \right| + \left| \sqrt{\frac{k\pi}{2}} \langle C_u, f_{u'} \rangle - \langle C_u, C_{u'} \rangle \right|.$$

Now,

$$(8.4.32) \quad \begin{aligned} \left| \frac{k\pi}{2} \langle f_u, f_{u'} \rangle - \sqrt{\frac{k\pi}{2}} \langle C_u, f_{u'} \rangle \right| &= \left| \sqrt{\frac{k\pi}{2}} \int_1^\infty \left(\sqrt{\frac{k\pi}{2}} \eta^{1/2} J_u(k\eta) - \cos(k\eta - u\pi/2 - \pi/4) \right) \eta^{1/2} J_{u'}(k\eta) \eta^{-2} d\eta \right| \\ &\leq \sqrt{\frac{k\pi}{2}} \int_1^\infty \left| \sqrt{\frac{k\pi}{2}} \eta^{1/2} J_u(k\eta) - \cos(k\eta - u\pi/2 - \pi/4) \right| \left| \eta^{1/2} J_{u'}(k\eta) \right| \eta^{-2} d\eta. \end{aligned}$$

Let $y = k\eta$. Applying equation (8.4.29), the above becomes

$$(8.4.33) \quad \sqrt{\frac{\pi}{2}}k \int_k^\infty \left| \sqrt{\frac{y\pi}{2}} J_u(y) - \cos(y - u\pi/2 - \pi/4) \right| \left| y^{1/2} J_{u'}(y) \right| y^{-2} dy \leq Ck \int_k^\infty \frac{M_1}{y^3} dy.$$

This integral is equal to $\frac{M_1 C}{k}$ which is arbitrarily small as k gets large. The proof that $\left| \sqrt{\frac{k\pi}{2}} \langle C_u, f_{u'} - C_{u'} \rangle \right|$ also vanishes in this limit uses a similar argument. \square

Proving the full Lemma 8.4.4 uses similar logic.

8.4.3.2. Proof of Claim 8.4.5.

PROOF. Using the simple trigonometric identities $\sin(a + \pi/2) = \cos(a)$ and $\cos(a + \pi/2) = -\sin(a)$, it can be concluded that $C_u(\eta) = S_{u-1}(\eta)$ and $S_u(\eta) = -C_{u-1}(\eta)$.

Therefore,

$$(8.4.34) \quad \Phi(u) = \langle C_u, S_{u'} \rangle \langle S_u, C_{u'} \rangle + \langle S_{u-1}, C_{u'} \rangle \langle C_{u-1}, S_{u'} \rangle =: \phi(u) + \phi(u-1).$$

Let $a(\eta) := k\eta - \pi/4 = a(\eta; u) + u\pi/2$. Standard trigonometric identities show that

$$(8.4.35) \quad C_u(\eta) = \cos(a - u\pi/2) = \cos(a) \cos(u\pi/2) + \sin(a) \sin(u\pi/2),$$

while

$$(8.4.36) \quad S_u(\eta) = \sin(a) \cos(u\pi/2) - \cos(a) \sin(u\pi/2).$$

Letting $c(\eta) := \cos(a(\eta))$ and $s(\eta) := \sin(a(\eta))$,

$$(8.4.37) \quad \langle C_u, S_{u'} \rangle = \cos(u\pi/2) \langle c, S_{u'} \rangle + \sin(u\pi/2) \langle s, S_{u'} \rangle,$$

while

$$(8.4.38) \quad \langle S_u, C_{u'} \rangle = \cos(u\pi/2) \langle s, C_{u'} \rangle - \sin(u\pi/2) \langle c, C_{u'} \rangle.$$

This yields

$$(8.4.39) \quad \begin{aligned} \phi(u) &= \cos^2(u\pi/2) \langle c, S_{u'} \rangle \langle s, C_{u'} \rangle - \cos(u\pi/2) \sin(u\pi/2) \langle c, S_{u'} \rangle \langle c, C_{u'} \rangle \\ &\quad + \sin(u\pi/2) \cos(u\pi/2) \langle s, S_{u'} \rangle \langle s, C_{u'} \rangle - \sin^2(u\pi/2) \langle s, S_{u'} \rangle \langle c, C_{u'} \rangle. \end{aligned}$$

Finally,

$$(8.4.40) \quad \begin{aligned} \Phi(u) &= \phi(u) + \phi(u-1) = (\cos^2(u\pi/2) + \cos^2((u-1)\pi/2)) \langle c, S_{u'} \rangle \langle s, C_{u'} \rangle \\ &\quad + (\sin(u\pi/2) \cos(u\pi/2) + \sin((u-1)\pi/2) \cos((u-1)\pi/2)) (\langle s, S_{u'} \rangle \langle s, C_{u'} \rangle - \langle c, S_{u'} \rangle \langle c, C_{u'} \rangle) \\ &\quad - (\sin^2(u\pi/2) + \sin^2((u-1)\pi/2)) \langle s, S_{u'} \rangle \langle c, C_{u'} \rangle. \end{aligned}$$

This can be simplified further by noting that $\cos((u-1)\pi/2) = \sin(u\pi/2)$ and $\sin((u-1)\pi/2) = -\cos(u\pi/2)$

It follows that

$$(8.4.41) \quad \Phi(u) = \langle c, S_{u'} \rangle \langle s, C_{u'} \rangle - \langle s, S_{u'} \rangle \langle c, C_{u'} \rangle =: \Phi,$$

which is independent of u . \square

8.4.3.3. *Proof of *Claim 8.4.6.* The proof of *Claim 8.4.6, follows from the proof of:

*CLAIM 8.4.10. *If $0 < \eta_i < \eta_f \leq \infty$ then*

$$\lim_{k \rightarrow \infty} \int_{\eta_i}^{\eta_f} \sin^2(k\eta - \pi/4) \eta^{-2} d\eta = \frac{1}{2} \left(\frac{1}{\eta_i} - \frac{1}{\eta_f} \right)$$

PROOF. Let $\epsilon > 0$ be arbitrary. We will prove the claim for the case where $\eta_i = 1$ and $\eta_f = \infty$, the more general claim can be proven using similar logic. Choose B so large that

$$(8.4.42) \quad \int_B^\infty \eta^{-2} < \epsilon.$$

Consider the following:

$$(8.4.43) \quad \begin{aligned} I &= \int_1^B \sin^2(k\eta - \pi/4) \eta^{-2} d\eta \\ &= \sum_{n=0}^{N_k-1} \int_{1+2\pi n/k}^{1+2\pi(n+1)/k} \sin^2(k\eta - \pi/4) \eta^{-2} d\eta + \int_{1+2\pi N_k/k}^B \sin^2(k\eta - \pi/4) \eta^{-2} d\eta. \end{aligned}$$

Here, $N_k \in \mathbb{N}$ is the largest natural number such that $2\pi N_k/k + 1 \leq B$.

Now, η^{-2} is uniformly continuous on $[1, B]$. Choose $\delta > 0$ such that $|\eta - \eta'| < \delta$ implies that $|\eta^{-2} - (\eta')^{-2}| < \epsilon/B$, for $\eta, \eta' \in [1, B]$. Choose k large enough so that both $\frac{2\pi}{k} < \delta$ and $\frac{2}{1+2\pi N_k} < \epsilon$. Then,

$$(8.4.44) \quad \begin{aligned} I &= \int_{1+2\pi N_k/k}^B \sin^2(k\eta + \pi/4) \eta^{-2} d\eta + \sum_{n=0}^{N_k-1} (1 + 2\pi n/k)^{-2} \int_{1+2\pi n/k}^{1+2\pi(n+1)/k} \sin^2(k\eta - \pi/4) d\eta \\ &+ \sum_{n=0}^{N_k-1} \int_{1+2\pi n/k}^{1+2\pi(n+1)/k} \sin^2(k\eta - \pi/4) (\eta^{-2} - (2\pi n/k + 1)^{-2}) d\eta. \end{aligned}$$

First observe that

$$(8.4.45) \quad \left| \int_{1+2\pi N_k/k}^B \sin^2(k\eta + \pi/4) \eta^{-2} d\eta \right| \leq \left| -\eta^{-1} \Big|_{1+2\pi N_k/k}^B \right| = \left| \frac{1}{B} - \frac{1}{1 + 2\pi N_k/k} \right| \leq \frac{2}{1 + 2\pi N_k} < \epsilon.$$

The last term in equation (8.4.44) is bounded by

$$(8.4.46) \quad \begin{aligned} \left| \sum_{n=0}^{N_k} \int_{1+\frac{2\pi n}{k}}^{1+\frac{2\pi(n+1)}{k}} \sin^2(k\eta - \frac{\pi}{4}) \left(\eta^{-2} - \left(\frac{2\pi n}{k} + 1 \right)^{-2} \right) d\eta \right| &\leq \frac{\epsilon}{B} \sum_{n=0}^{N_k} \int_{1+\frac{2\pi n}{k}}^{1+\frac{2\pi(n+1)}{k}} \sin^2(k\eta - \frac{\pi}{4}) d\eta \\ &= \sum_{n=0}^{N_k} \frac{\epsilon}{2B} \frac{2\pi}{k} = (N_k + 1) \frac{\epsilon\pi}{Bk} \\ &< \left(\frac{2\pi N_k}{k} + 1 \right) \frac{\epsilon\pi}{B} \leq \pi\epsilon. \end{aligned}$$

The middle term of equation (8.4.44) can be expressed as

$$(8.4.47) \quad \begin{aligned} \sum_{n=0}^{N_k} (1 + 2\pi n/k)^{-2} \int_{1+2\pi n/k}^{1+2\pi(n+1)/k} \sin^2(k\eta - \pi/4) d\eta &= \frac{1}{2} \left(\frac{k}{2\pi} \sum_{n=0}^{N_k} \frac{1}{(k/2\pi + n)^2} \right) \\ &= \frac{1}{2} \frac{k}{2\pi} \Psi^{(1)}\left(\frac{k}{2\pi}\right) - \frac{1}{2} \frac{k}{2\pi} \sum_{n=N_k}^{\infty} \frac{1}{(k/2\pi + n)^2}, \end{aligned}$$

where $\Psi^{(1)}$ is the first polygamma function.

It is a known fact that

$$(8.4.48) \quad \lim_{x \rightarrow \infty} x \Psi^{(1)}(x) = 1,$$

so to establish the claim for $\eta_i = 1$ and $\eta_f = \infty$, it remains to verify that

$$(8.4.49) \quad \lim_{k \rightarrow \infty} \frac{k}{2\pi} \sum_{n=N_k}^{\infty} \frac{1}{(k/2\pi + n)^2} = \frac{1}{B}.$$

Letting $m = n - N_k$ this remainder term is

$$(8.4.50) \quad \frac{k}{2\pi} \sum_{m=0}^{\infty} \frac{1}{(k/2\pi + N_k + m)^2} = \frac{k}{2\pi} \Psi^{(1)}(k/2\pi + N_k) = \frac{1}{B} \frac{kB}{2\pi} \Psi^{(1)}\left(\frac{k}{2\pi} \left(1 + \frac{2\pi N_k}{k}\right)\right)$$

Equation (8.4.48), and the fact that $\lim_{k \rightarrow \infty} (1 + \frac{2\pi N_k}{k}) = B$ then show that

$$(8.4.51) \quad \frac{1}{B} \lim_{k \rightarrow \infty} \frac{kB}{2\pi} \Psi^{(1)}\left(\frac{k}{2\pi} \left(1 + \frac{2\pi N_k}{k}\right)\right) = \frac{1}{B} < \epsilon.$$

We can now conclude that $\lim_{k \rightarrow \infty} \int_1^{\infty} \sin^2(k\eta - \pi/4)\eta^{-2}d\eta = \frac{1}{2}$. This proves the claim for the case $\eta_i = 1$ and $\eta_f = \infty$. □

Similarly, one can prove that $\langle c_k, c_k \rangle_{\eta_i, \eta_k}$ converges to $\frac{1}{2} \left(\frac{1}{\eta_i} - \frac{1}{\eta_k}\right)$ and $\langle s_k, c_k \rangle_{\eta_i, \eta_k}$ converges to 0 in the limit as $k \rightarrow \infty$. We conclude that $|\Phi_{\eta_i, \eta_f}(0, k)| \rightarrow \frac{1}{4}$ as $k \rightarrow \infty$, independently of η_i and η_f . This establishes *Claim 8.4.6.

8.4.4. The zero spatial mode. The spectral analysis of the symmetric operator \square_k in the previous subsections assumed that $k \neq 0$. In this subsection the case when $k = 0$ is studied.

As was shown in Section 8.3, the deficiency indices of \square_0 are $(1, 1)$. Also, as discussed in Subsection 7.2.4, and Subsection 8.4.1, the interval $(\frac{1}{4}, \infty) \subset \sigma_c(\square_0)$. From the definition of regular and singular end-points of a symmetric differential operator (see Section 7.1), it follows that for \square_0 , the point $a = 1$ is a regular end-point while the point $b = \infty$ is irregular.

In this section it will be shown that that the infinite dimensional subspace $B_0(\Omega) = \mathfrak{R}\left(\chi_{[-\Omega^2, \Omega^2]}(\square_0)\right)$, also obeys a sampling or reconstruction formula. This will be accomplished with the aid of a particular form of the spectral theorem for second order symmetric differential operators whose deficiency indices are $(1, 1)$, and which are defined in $L^2(a, b; w)$, where a is a regular end-point and b is a singular end-point. This spectral theorem will allow us to explicitly write down a unitary transformation that transforms \square_0 into a multiplication operator acting on a dense domain in $L^2([\frac{1}{4}, \infty); \sigma)$. This unitary transformation will be an integral operator whose kernel is expressed in terms of the solutions to the differential equations $L_0[\phi] = \lambda\phi$ for $\lambda \in \mathbb{R}$. (Recall that L_0 denotes the differential expression that generates \square_0 .) Using this explicit expression, one can also write down a concrete expression for the projector onto the subspace $B_0(\Omega)$. The particular form of this projector will allow us to derive a sampling formula for $B_0(\Omega)$.

8.4.4.1. *A particular spectral theorem.* Let D be a symmetric, second-order differential operator of the type described in Section 7.1, which has deficiency indices $(1, 1)$, and is defined on $L^2(a, b)$ where a is a regular end point and b is singular.

One can obtain self-adjoint extensions $D(\theta)$, $\theta \in \mathbb{R}$, of D as follows. Define

$$(8.4.52) \quad \mathfrak{D}(D(\theta)) := \{\phi \in D^* \mid p(1)\phi'(1) = \theta\phi(1)\}.$$

For each $\theta \in \mathbb{R}$, $D(\theta) := D^*|_{\mathfrak{D}(D(\theta))}$ is a different self-adjoint extension of D . By convention, let $\mathfrak{D}(D(\infty)) := \{\phi \in \mathfrak{D}(D^*) \mid \phi(1) = 0\}$. The operator $D(\infty) := D^*|_{\mathfrak{D}(D(\infty))}$ is also a self-adjoint extension of D . It is straightforward to verify that the operators $D(\theta)$, $\theta \in \mathbb{R} \cup \{\infty\}$ are indeed self-adjoint extensions of D using integration by parts.

Let $L := L_D$ denote the differential expression associated with D (see Section 7.1). Let $u_1(\eta; \lambda)$ be that solution to the equation $L[\phi] = \lambda\phi$ that satisfies the initial conditions $u_1(a; \lambda) = 1$

and $p(a)u'_1(a; \lambda) = 0$, and $u_2(\eta; \lambda)$ the solution to the same equation satisfying $u_2(a; \lambda) = 0$ and $p(a)u'_2(a; \lambda) = 1$.

For each $\theta \in \mathbb{R}$, let $u_\theta(\eta; \lambda) := u_1(\eta; \lambda) + \theta u_2(\eta; \lambda)$ For $\theta = \infty$, define $u_\infty(\lambda) = u_2(\eta; \lambda)$.

With the above definitions, the following version of the spectral theorem for second order differential operators, Theorem 7.1.4, holds ([3], pgs. 192-197).

THEOREM 8.4.11. (*Krein*) *For each $\theta \in \mathbb{R} \cup \{\infty\}$, there is a regular Borel measure σ_θ such that*

$$(8.4.53) \quad \Phi_\theta(\lambda) = (U_\theta \phi)(\lambda) := \int_a^b f(t) u_\theta(t; \lambda) dt,$$

and

$$(8.4.54) \quad \phi(t) = (U_\theta^{-1} \Phi_\theta)(t) := \int_{-\infty}^{\infty} \Phi(\lambda) u_\theta(t; \lambda) d\sigma_\theta(\lambda)$$

define a unitary transformation U from $L^2[a, b]$ onto $L^2((-\infty, \infty); d\sigma_\theta)$. The image of D_θ under this transformation acts as multiplication by λ .

In the above, $\sigma_\theta(\lambda)$ is a non-decreasing function of λ which is continuous from the left and which obeys $\sigma(-\infty) = 0$. This function can be determined uniquely as follows. For $\text{Im}(z) > 0$, define the function $m_\theta(z)$ by the requirement that

$$(8.4.55) \quad m_\theta(z) u_\theta(t; z) - u_2(t; z) \in L^2[a, b].$$

Since D has deficiency indices $(1, 1)$, this uniquely determines $m_\theta(z)$. The resolvent operators $(D_\theta - z)^{-1}$ for $\text{Im}(z) > 0$ can be represented as integral operators ([3], Appendix II), ([52], Section 19). By studying the form of the integral kernel of these integral operators, and using the Stieltjes inversion formula, one can show that for each $\lambda \in \mathbb{R}$,

$$(8.4.56) \quad \frac{\sigma_\theta(\lambda^-) + \sigma_\theta(\lambda^+)}{2} = C + \lim_{y \rightarrow 0} \frac{1}{\pi} \int_0^\lambda \text{Im}(m_\theta(x + iy)) dx.$$

where $\sigma_\theta(\lambda^\pm)$ denotes the limits of $\sigma_\theta(t)$ as $t \rightarrow \lambda$ from the right and left respectively.

8.4.4.2. *A sampling formula for $B_0(\Omega)$.* Applying the tools and theorems described in the previous subsection, one can write down a concrete expression for the projector of $L^2([1, \infty); \eta^{-2} d\eta)$ onto $B_0(\Omega)$.

Consider the differential equation defined by $L_0[\phi] = \lambda\phi$,

$$(8.4.57) \quad -\eta^2 \phi''(\eta) = \lambda\phi(\eta),$$

generated by the differential expression L_0 associated with the symmetric differential operator \square_0 on $L^2([1, \infty); \eta^{-2} d\eta)$. For any $\lambda \neq \frac{1}{4}$, two linearly independent solutions to the differential equation (8.4.57) are $f_\lambda(\eta) := \eta^{\frac{1}{2} + \sqrt{\frac{1}{4} - \lambda}}$ and $g_\lambda(\eta) := \eta^{\frac{1}{2} - \sqrt{\frac{1}{4} - \lambda}}$. If $\lambda = \frac{1}{4}$ then $f_{\frac{1}{4}}(\eta) = \eta^{\frac{1}{2}}$ and another linearly independent solution is $g_{\frac{1}{4}} := \eta^{\frac{1}{2}} \ln(\eta)$.

Under the change of variables $y = -\frac{1}{\eta}$, the differential operator \square_0 becomes the differential operator D , defined by

$$(8.4.58) \quad D\phi = -(y^2 \phi)'$$

for all ϕ in a dense domain in $L^2[-1, 0]$. The end point $y(1) = -1$ is a regular end-point and the point 0 is singular. This operator D is of the type described in the subsection above, so that the tools and theorems described there can be applied to it.

In particular, using the notation of the previous subsection, Subsection 8.4.4.1, one can calculate that

$$(8.4.59) \quad u_1(\eta; \lambda) = \begin{cases} \frac{1}{2}(1 - c(\lambda))\eta^{\frac{1}{2} + \sqrt{\frac{1}{4} - \lambda}} + \frac{1}{2}(1 + c(\lambda))(\lambda)\eta^{\frac{1}{2} - \sqrt{\frac{1}{4} - \lambda}} & \lambda \neq \frac{1}{4} \\ \eta^{\frac{1}{2}}(1 - \frac{1}{2} \ln \eta) & \lambda = \frac{1}{4} \end{cases},$$

and

$$(8.4.60) \quad u_2(\eta; \lambda) = \begin{cases} c(\lambda)\eta^{\frac{1}{2}+\sqrt{\frac{1}{4}-\lambda}} - c(\lambda)(\lambda)\eta^{\frac{1}{2}-\sqrt{\frac{1}{4}-\lambda}} & \lambda \neq \frac{1}{4} \\ \eta^{\frac{1}{2}} \ln \eta & \lambda = \frac{1}{4} \end{cases},$$

where $c(\lambda) := (1 - 4\lambda)^{-\frac{1}{2}}$. Furthermore, as is straightforward to calculate, for $\theta \in \mathbb{R}$, the function $m_\theta(z)$ is given by the formula

$$(8.4.61) \quad m_\theta(z) := \frac{2}{2\theta - 1 - \sqrt{1 - 4z}}.$$

It follows that,

$$(8.4.62) \quad \frac{\sigma_\theta(\lambda - 0) + \sigma_\theta(\lambda + 0)}{2} = C_\theta + \frac{2}{\pi} \lim_{y \rightarrow 0^+} \int_0^\lambda \operatorname{Im} \left(\frac{1}{2\theta - 1 - \sqrt{1 - 4(x + iy)}} \right) dx.$$

Observe that the function $m(z)$ is differentiable for all $z \neq \frac{1}{4}$.

It is not difficult to see that $\tilde{\sigma}_y(\lambda) := \frac{1}{\pi} \int_0^\lambda \operatorname{Im} (m(x + iy)) dx$ is a differentiable function of λ for each $y > 0$ and that $\lim_{y \rightarrow 0^+} \tilde{\sigma}'_y(\lambda) = \frac{d}{d\lambda} \lim_{y \rightarrow 0^+} \tilde{\sigma}_y(\lambda)$. It follows that $\sigma(\lambda)$ is differentiable for $\lambda \neq \frac{1}{4}$ and that

$$(8.4.63) \quad \sigma'(\lambda) = \frac{2}{\pi} \lim_{y \rightarrow 0} \operatorname{Im} \left(\frac{1}{2\theta - 1 - \sqrt{1 - 4(\lambda + iy)}} \right) = \operatorname{Im} \left(\frac{1}{2\theta - 1 - \sqrt{1 - 4(\lambda)}} \right).$$

The formula (8.4.63) shows that $\sigma'(\lambda) = 0$ for $\lambda < \frac{1}{4}$, and that

$$(8.4.64) \quad \sigma'(\lambda) = \frac{2}{\pi} \frac{\sqrt{4\lambda - 1}}{(2\theta - 1)^2 + (4\lambda - 1)^2}$$

for $\lambda > \frac{1}{4}$.

For example, choose $\theta = \frac{1}{2}$. In this case, one can verify that

$$(8.4.65) \quad \sigma'(\lambda) = \frac{2}{\pi} \frac{1}{\sqrt{4\lambda - 1}},$$

and that,

$$(8.4.66) \quad u_\theta(\eta, \lambda) = \frac{1}{2} \eta^{\frac{1}{2}(1+i\sqrt{4\lambda-1})} + \frac{1}{2} \eta^{\frac{1}{2}(1-i\sqrt{4\lambda-1})}.$$

In this case, Theorem 8.4.11 implies that the formulas

$$(8.4.67) \quad \Phi(\lambda) = (U\phi)(\lambda) := \int_1^\infty \phi(\eta) \frac{\eta^{\frac{1}{2}(1+i\sqrt{4\lambda-1})} + \eta^{\frac{1}{2}(1-i\sqrt{4\lambda-1})}}{2\eta^2} d\eta,$$

and,

$$(8.4.68) \quad \phi(\eta) = \int_{\frac{1}{4}}^\infty \Phi(\lambda) \frac{\eta^{\frac{1}{2}(1+i\sqrt{4\lambda-1})} + \eta^{\frac{1}{2}(1-i\sqrt{4\lambda-1})}}{\pi\sqrt{4\lambda-1}} d\lambda,$$

define a unitary transformation from $L^2([1, \infty); \eta^{-2} d\eta)$ onto $L^2([\frac{1}{4}, \infty); d\sigma(\lambda))$.

For convenience let $u := \frac{2}{\pi\sqrt{4\lambda-1}}$. In terms of the variable u , the above formulas become

$$(8.4.69) \quad \Phi(u) := \int_1^\infty \phi(\eta) \eta^{\frac{1}{2}} \frac{\eta^{i\pi u} + \eta^{-i\pi u}}{2} \eta^{-2} d\eta,$$

and,

$$(8.4.70) \quad \phi(\eta) = \int_0^\infty \Phi(u) \eta^{\frac{1}{2}} \frac{\eta^{i\pi u} + \eta^{-i\pi u}}{2} du.$$

These formulas define a unitary transformation U from $L^2([1, \infty); \eta^{-2} d\eta)$ onto $L^2[0, \infty)$ given by $(U\phi) := \Phi \in L^2[0, \infty)$ and $\phi = U^{-1}\Phi$ for any $\phi \in L^2([1, \infty); \eta^{-2} d\eta)$. Furthermore, the operator $UD_{\frac{1}{2}}U^{-1}$ is the self-adjoint multiplication operator by the variable u on $L^2[0, \infty)$.

Fix the self-adjoint extension $D_{\frac{1}{2}}$ of D . Let $\square_0(\frac{1}{2})$ be the self-adjoint extension of \square_0 which is unitarily equivalent to $D_{\frac{1}{2}}$ under the change of variables $y = -\frac{1}{\eta}$. Let $B_0(\Omega) := \chi_{[-\Omega^2, \Omega^2]}(-\square_0(\frac{1}{2}))L^2([1, \infty); \eta^{-2} d\eta)$. Let $P_\Omega := \chi_{[\frac{1}{4}, \Omega^2]}(\square_0(\frac{1}{2}))$ denote the projector onto $B_0(\Omega)$. Choose $\Omega^2 = \frac{1}{4} + B^2$. Then it follows that $\phi \in B_0(\Omega)$ if and only if $\Phi(u) \in L^2[0, B]$. Using the formulas (8.4.69) and (8.4.70), it is not difficult to calculate that for $\phi \in L^2([1, \infty); \eta^{-2} d\eta)$,

$$(8.4.71) \quad P_\Omega \phi(\eta) := \int_1^\infty \phi(\nu) K(\nu, \eta) \nu^{-2} d\nu,$$

where the integral kernel $K(\nu, \eta)$ is given by the formula

$$(8.4.72) \quad K(\nu, \eta) := \frac{\eta^{\frac{1}{2}} \nu^{\frac{1}{2}}}{2} \left[\frac{\sin(B\pi(\ln(\nu\eta)))}{\pi \ln(\nu\eta)} + \frac{\sin\left(B\pi(\ln(\frac{\nu}{\eta}))\right)}{\pi \ln(\frac{\nu}{\eta})} \right].$$

A reconstruction formula can now be obtained using an argument which is similar to that provided in the proof of the Shannon sampling formula, Claim 2.2.2. If $\phi \in B_k(\Omega)$ where $\Omega^2 = \frac{1}{4} + B^2$, then $\Phi \in L^2[0, B]$. It follows that

$$(8.4.73) \quad \Phi(u) = \sum_{n=0}^\infty \Phi_n \cos\left(\frac{n\pi u}{B}\right),$$

where

$$(8.4.74) \quad \Phi_n := \frac{2}{B} \int_0^B \Phi(u) \cos\left(\frac{n\pi u}{B}\right) du.$$

Let $\eta_n := e^{\frac{n}{B}}$ for $n \in \mathbb{N} \cup \{0\}$. Then, from the formula (8.4.70), it follows that $\Phi_n = \frac{2\phi(\eta_n)}{\eta_n^{1/2} B}$. Hence, for any $\phi \in B_0(\Omega)$,

$$(8.4.75) \quad \begin{aligned} \phi(\eta) &= \int_0^\infty \eta^{\frac{1}{2}} \Phi(u) \cos(\pi u \ln \eta) du \\ &= \sum_{n=0}^\infty \frac{2\phi(\eta_n)}{B\eta_n} K(\eta_n, \eta), \end{aligned}$$

where $K(\eta_n, \eta)$ is given by the formula (8.4.72). In conclusion, any $\phi \in B_0(\Omega)$, $\Omega > \frac{1}{2}$, is reconstructible from the values it takes on the set of points $\{e^{\sqrt{\Omega^2 - \frac{1}{4}} n}\}_{n \in \mathbb{N} \cup \{0\}}$.

8.4.5. Higher spatial dimensions. Similar results to those of the previous subsections also hold for 1 + 3 dimensional de Sitter spacetime.

The D'Alembertian for 1 + 3 dimensional de Sitter spacetime with finite end-time can be expressed as the differential operator

$$(8.4.76) \quad -\square = -\eta^{-2} \left(\partial_\eta^2 - \frac{2}{\eta} d\eta - \Delta \right),$$

acting on a dense domain in $L^2([1, \infty) \times \mathbb{R}^3; \eta^{-4} d\eta dx_1 dx_2 dx_3)$, where $\Delta := \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$ is the spatial Laplacian.

Let $k^2 := \sqrt{k_1^2 + k_2^2 + k_3^2}$ denote the magnitude of the spatial frequency vector $\mathbf{k} = (k_1, k_2, k_3)$. Following the same procedure as for the case of 1 + 1 dimensional de Sitter space-time, define a

symmetric operator \square_k acting on a suitable dense domain in $\mathcal{H} := L^2([1, \infty); \eta^{-4} d\eta)$ with the aid of the formal differential expression

$$(8.4.77) \quad L_k[\phi] := -\eta^{-2} \left(\partial_\eta^2 - \frac{2}{\eta} d\eta + k^2 \right) \phi.$$

8.4.5.1. *Deficiency indices of the \square_k .* The formal solutions of the differential equation $L_k[\phi] = \lambda\phi$ are again Bessel functions for $k \neq 0$. For $k \neq 0$, a linearly independent set of solutions is given by $f_\lambda(\eta) := \eta^{\frac{3}{2}} J_{\sqrt{\frac{9}{4}-\lambda}}(k\eta)$ and $g_\lambda(\eta) := \eta^{\frac{3}{2}} Y_{\sqrt{\frac{9}{4}-\lambda}}(k\eta)$.

Using the asymptotic formula, $J_p(x) \sim \sqrt{\frac{2}{\pi x}} \cos(x - p\pi/2 - \pi/4)$ as $|x| \rightarrow \infty$, as in Section 8.3, it is straightforward to verify that the deficiency indices of \square_k are $(2, 2)$ for $k > 0$. For the case $k = 0$, two linearly independent solutions (provided $\lambda \neq \frac{9}{4}$) are given by $f_\lambda(\eta) := \eta^{\frac{3}{2} + \sqrt{9/4 - \lambda}}$ and $g_\lambda(\eta) := \eta^{\frac{3}{2} - \sqrt{9/4 - \lambda}}$. In both cases $k = 0$ and $k \neq 0$, these solutions are qualitatively similar to those for the case of 1 + 1 dimensional de Sitter space-time. As in Section 8.3, one can check that if $\lambda = \frac{9}{4} + i$, then g_λ is normalizable but f_λ is not so that \square_0 has deficiency indices $(1, 1)$.

Using Theorem 3 on page 97 of [52], it again follows that $[\frac{9}{4}, \infty) \subset \sigma_c(\square_0)$. Similar results to those derived in Subsections 8.4.1 and 8.4.2 also hold in this case. Namely, using similar methods to those of before, it can be shown that if N_k is the finite dimension of the subspace $B_k(\Omega) := \chi_{[-\Omega^2, \Omega^2]}(\square'_k)\mathcal{H}$ where $k > 0$, and \square'_k is a fixed choice of self-adjoint extension of \square_k , then $N_k \rightarrow \infty$ as $k \rightarrow 0$ and for any fixed Ω there is $B > 0$ such that $k > B$ implies that $N_k \leq 2$. Furthermore, I expect that it should be possible to calculate a sampling formula for the case $k = 0$, similar to the one found in the previous subsection for the case of 1 + 1 dimensions.

8.5. More general FRW space-times

A natural and physically important question to ask is whether the qualitative features of the results of the previous section hold for more general FRW space-times.

Consider a general 1 + 3 dimensional FRW space-time with line element

$$(8.5.1) \quad ds^2 = -dt^2 + a^2(t)(dx_1^2 + dx_2^2 + dx_3^2),$$

and scale factor $a(t)$. In this case, as one can calculate, the D'Alembertian $\square(a)$ is a differential operator, which depends on the function a and which acts as

$$(8.5.2) \quad -\square(a)\phi = -(\partial_t^2 + 3\frac{a'}{a}\partial_t + \Delta)\phi$$

on a suitable domain. Here, Δ is the spatial Laplacian. In this case the operators $\square_k(a)$ can be represented as symmetric operators which act as

$$(8.5.3) \quad -\square_k(a)\phi = -a^{-3}(t) ((a^3\phi')' + ak^2\phi)$$

for fixed k on a dense domain in $L^2((t_1, t_2), a^3(t)dt)$.

Let D_n and D be second order symmetric differential operators defined by the coefficient functions p_n, q_n and p, q on the intervals (a_n, b_n) and (a, b) . There are many results, (including my results of Subsection 7.2.4) which describe how the spectra, and spectral projections of the self-adjoint extensions of the D_n converge to those of self-adjoint extensions of D if $a_n \rightarrow a, b_n \rightarrow b$ and $1/p_n \rightarrow 1/p, q_n \rightarrow q$ in a suitable topology, e.g in L^1 or L^2 of any compact sub-interval of (a, b) . It should be possible to apply results of this kind to show how the spectra and spectral projections of the $\square_k(a_n)$ converge to those of $\square_k(a)$ if the scale factors $a_n(t)$ converge to $a(t)$ in a suitable sense.

For example, suppose that D is a second order symmetric differential operator in $L^2(a, b; w)$ generated by a differential expression L which is of the limit circle case at both end points (see Section 7.1 for the definition of limit circle case). Then it is known that if one chooses D'_n to be a suitable self-adjoint extension of the symmetric operator D_n defined by L on (a_n, b_n) where

$a_n \rightarrow a$ and $b_n \rightarrow b$, then for any $z \in \mathbb{C} \setminus \mathbb{R}$, $(D_n - z)^{-1} \chi_{[a_n, b_n]} \rightarrow (D - z)^{-1}$ in Hilbert-Schmidt norm. This implies, in particular, that the spectral projections of the D_n converge in norm to the spectral projections of D ([4], pg. 86).

Suppose that D'_n are self-adjoint extensions of a symmetric second order differential operator D_n , defined by the coefficient functions p_n, q_n which is defined in $L^2(a, b; w)$ and which is of the limit circle case at both end points. If $1/p_n \rightarrow 1/p$ and $q_n \rightarrow q$ in L^2 of any compact sub-interval of (a, b) , I expect that if the self-adjoint extensions D'_n are chosen suitably, that the spectral projections of D'_n will converge in norm to those of the D_n . I expect that such a result may already exist in the literature, although I have not yet found it. Related results of this kind can be found in ([73], Chapter 3, Section 5, and Chapter 10, Section 9).

Let $a(t) = e^t$ be the scale factor of de Sitter space-time. If results of the kind discussed above can be found or established, then one should be able to conclude that the spectral projections $P_\Omega(n) := \chi_{[-\Omega^2, \Omega^2]}(\square_k(a_n))$ converge in the norm or strong sense to the projector $P_\Omega := \chi_{[\Omega^2, \Omega^2]}(\square_k(a))$ if $a_n \rightarrow a$ in a suitable sense. This could be used to show that if a_n is ‘sufficiently close to a ’, then the subspaces projected onto by the $P_\Omega(n)$ and P_Ω will have similar properties, including the same dimension. This would be useful for showing that the results of the previous subsections are robust, and are not unique to de Sitter space-time.

8.5.1. The Behaviour of $N_k(\Omega)$ for large $|\Omega|$. Let L be a second order symmetric differential expression with coefficient functions p and q , and weight function w which defines a differential operator D in $L^2(a, b; w)$ where $-\infty < a < b < \infty$ are both regular end-points. Let

$$(8.5.4) \quad B := \int_a^b \sqrt{\frac{w(t)}{p(t)}} dt.$$

It is known that if both a and b are both regular, then D has deficiency indices $(2, 2)$, ([52], pg 66). As discussed previously, the spectrum of any self-adjoint extension of such an operator is purely discrete and consists of eigenvalues of multiplicity at most 2 with no finite limit point. It can further be shown that the operator D has a smallest eigenvalue λ_0 , and that for sufficiently large n , the n^{th} eigenvalue of D lies between $\frac{n - \frac{1}{2}\pi}{B}$ and $\frac{n + \frac{3}{2}\pi}{B}$ ([25], pg. 319, pg. 303).

If one considers the operators \square_k for $t \in [t_i, t_f]$ in a $1 + 3$ dimensional FRW space-time, then $p(t) = a^3(t)$ and $w(t) = a^3(t)$ so that $B = t_f - t_i$ is independent of the choice of scale factor. Hence for any fixed k , the dimension N_k of the subspace spanned by the k^{th} spatial modes of Ω -bandlimited functions on these space-times, *i.e.*, the subspace of $L^2([t_i, t_f]; a^3(t)dt)$ projected onto by $\chi_{[-\Omega^2, \Omega^2]}(-\square_k(a))$, will behave asymptotically like $\frac{t_f - t_i}{\pi} \Omega^2$, for large values of Ω .

Part 3

Self-adjoint extensions of symmetric operators and sampling theory

Introduction: Symmetric operators, de Branges spaces and sampling theory

This part of the thesis deals with my most recent and current research. Consequently, much of the work presented in this part has yet to be fully developed. I welcome any advice the reader may have on how this should be done.

In the Introduction, Chapter 1, we motivated the idea that non-Fourier generalizations of bandlimited function theory could improve the efficiency of information storage in communication engineering. For example, modelling a music signal as an element of a more general reproducing kernel Hilbert space with the Kramer sampling property could lead to more efficient sampling and reconstruction of the music signal.

With this motivation in mind, this part of the thesis deals with the study of reproducing kernel Hilbert spaces with the sampling property.

9.0.1.1. *Definition.* As we proved in Chapter 4, if \mathcal{H} is a reproducing kernel Hilbert space of functions on \mathbb{R} for which every point evaluation vector δ_x is non-zero, and if the multiplication operator in \mathcal{H} is densely defined, symmetric, regular, and simple with deficiency indices $(1, 1)$, then \mathcal{H} has the Kramer sampling property. For reasons that should already be apparent, we will say that a symmetric operator defined on a domain in a separable Hilbert space \mathcal{H} has the **sampling property** if it is densely defined, closed, regular, simple, and has deficiency indices $(1, 1)$.

Let \mathcal{H} be any separable Hilbert space, and let A be any densely defined symmetric operator on \mathcal{H} which is regular and simple with deficiency indices $(1, 1)$. Then it can be shown that there is a unitary transformation which maps \mathcal{H} onto a reproducing kernel Hilbert space \mathcal{H}' of meromorphic functions which are square integrable with respect to a certain Borel measure on \mathbb{R} , and such that $M := UAU^{-1}$ is a symmetric operator which acts as multiplication by the independent variable in \mathcal{H}' [36] [66] [26] [28]. This statement is, in a sense, a generalized spectral theorem that applies to this particular class of symmetric operators.

The fact that such a unitary U exists which maps \mathcal{H} onto a RKHS \mathcal{H}' with the sampling property was first proven by Kempf in [36]. What is remarkable is that this same class of operators, namely regular symmetric operators with finite deficiency indices, were already studied in detail over 60 years ago by the famous Russian mathematician M.G. Krein [39] [40] [41] [42]. However, these results were not widely known, and the full theory developed by Krein on these operators was not published in english until very recently in the book [28] in 1997. In [28], Krein shows, in particular, that given any densely defined regular, simple, symmetric operator B with deficiency indices $(1, 1)$ in a Hilbert space \mathcal{H} , *i.e.*, a symmetric operator B with the sampling property, that one can define a unitary transformation which takes \mathcal{H} onto a reproducing kernel Hilbert space of meromorphic functions which is a subspace $L^2(\mathbb{R}; d\mu)$ for some measure μ , such that the image of A under this transformation is the operator of multiplication by the independent variable. The authors of [66], independently of Kempf, exploit the theory developed by Krein to show that the resulting reproducing kernel Hilbert space actually has the sampling property. A related, but less general result was also proved earlier in [26]. We will later observe that the result of [66] can also be attained from Krein's theory as a simple consequence of *Theorem 4.4.7. Namely, if the symmetric operator of multiplication by the independent variable in a reproducing kernel Hilbert

space \mathcal{H} of functions on \mathbb{R} has the sampling property, and if every point evaluation vector $\delta_x \in \mathcal{H}$ is non-zero, then \mathcal{H} has the sampling property.

The following section provides a brief description of Krein's representation theory for simple symmetric operators with deficiency indices $(1, 1)$.

9.1. Krein's theory of symmetric operators

Let \mathcal{H} be a separable Hilbert space and let B be a simple symmetric operator defined on a dense domain $\mathfrak{D}(B) \subset \mathcal{H}$ with deficiency indices $(1, 1)$. Let B' denote an arbitrary self-adjoint extension of B , and for $z, z' \in \mathbb{C} \setminus \mathbb{R}$ let $\mu_{zz'}(w) := \frac{w-z}{w-z'}$. Define

$$(9.1.1) \quad U_{zz'} := \mu_{zz'}(B') = (B' - z)(B' - z')^{-1} = I + (z - z')(B' - z)^{-1}.$$

Observe that $U_{\bar{z}z}(B') = \mu_z(B')$ is a Cayley transform of B' . For $z \in \mathbb{C} \setminus \mathbb{R}$, let $\mathfrak{D}_z := \mathfrak{ker}(B^* - z)$.

LEMMA 9.1.1. $\mu_{wz}(B')$ maps \mathfrak{D}_w into \mathfrak{D}_z

This easily provable fact can be found in [28] or [3].

PROOF. Given $\phi \in \mathfrak{D}_w$, and any $\varphi \in \mathfrak{D}(B)$,

$$(9.1.2) \quad 0 = \langle (B - \bar{w})\varphi, \phi \rangle = \langle \mu_{\bar{w}z}(B' - \bar{z})\varphi, \phi \rangle = \langle (B - \bar{z})\varphi, \mu_{wz}(B')\phi \rangle$$

This proves the lemma. □

Fix $w \in \mathbb{C} \setminus \sigma(B')$, and a non-zero $\psi_w \in \mathfrak{D}_w$. It follows from the above lemma that

$$(9.1.3) \quad \psi(z) := U_{wz}\psi_w = \psi_w + (z - w)(B' - z)^{-1}\psi_w,$$

is a \mathcal{H} -valued function for $z \in \mathbb{C} \setminus \mathbb{R}$ such that $\psi(z) \in \mathfrak{D}_z$. Furthermore, it is clear from the definition of $\psi(z)$ that it is analytic on $\mathbb{C} \setminus \sigma(B')$, that $\psi(w) = \psi_w$, and that $\psi(z) \neq 0$ for any $z \in \mathbb{C} \setminus \sigma(B')$. Using the first resolvent formula, it is further not difficult to see that for any $w' \in \mathbb{C} \setminus \sigma(B')$ that $\psi(z) = U_{w'z}\psi(w') = \psi(w') + (z - w')(B' - z)^{-1}\psi(w')$.

Now given any $z_1 \notin \sigma(B')$, choose $u \in \mathcal{H}$ so that $\langle \psi(z_1), u \rangle \neq 0$. Since $\psi(z)$ is analytic on $\mathbb{C} \setminus \sigma(B')$ it follows that $f(z) = \langle \psi(z), u \rangle$ is an analytic function in $\mathbb{C} \setminus \mathbb{R}$, and hence has at most a countable number of zeroes in that region with no finite accumulation point. Let S_u denote the zeroes of this function. It follows from the fact that $\psi(z) \in \mathfrak{D}_z$ that S_u is that subset of \mathbb{C} for which the linear span of $\{\mu\} \cup \mathfrak{R}(B - z)$ is not dense in \mathcal{H} . Krein calls the vector $u \in \mathcal{H}$ a gauge for the symmetric operator B .

Borrowing the notation from [66], let

$$(9.1.4) \quad \xi(z) := \frac{\psi(\bar{z})}{\langle \psi(\bar{z}), u \rangle}.$$

Then $\bar{\xi}$ is clearly a meromorphic function of z with poles at the points of S_u . Also observe that $\xi(z) \neq 0$ for any $z \in \mathbb{C}$. Furthermore, it is not difficult to prove [66]:

LEMMA 9.1.2. *The \mathcal{H} valued function $\xi(z)$ does not depend on the choice of self-adjoint extension B' used to define $\psi(z)$.*

Now for each $\phi \in \mathcal{H}$, define $\phi_u(z)$ by

$$(9.1.5) \quad \phi_u(z) := \langle \phi, \xi(z) \rangle.$$

It follows that each ϕ_u is a meromorphic function in $\mathbb{C} \setminus \mathbb{R}$ with poles at the points of S_u that lie in this region.

Krein calls a point $z \in \mathbb{C}$ u -regular for the operator B , if there exists a ball of non-zero radius about z such that the functions ϕ_u are analytic in this ball, for all $\phi \in \mathcal{H}$. Krein has proven ([28], pg. 56) that a point $z \in \mathbb{C}$ is u -regular if and only if $z \notin S_u$ and z is a regular point of B .

9.1.0.2. *Remark.* It follows that if B is regular, then every point in \mathbb{C} is regular for B so that every point in $\mathbb{C} \setminus S_u$ is u -regular for B . Krein asserts that for a regular operator B , one can always choose u so that $S_u \cap \mathbb{R} = \emptyset$ so that the functions ϕ_u for $\phi \in \mathcal{H}$ are meromorphic functions on \mathbb{C} with poles that lie off of the real axis [41] [66].

9.1.0.3. *POVMs.* Recall the definition of a positive operator-valued measure given in Subsection 7.1.2. A contractive positive operator valued measure (POVM) on \mathbb{R} is a map $Q : \mathcal{B}(\mathbb{R}) \rightarrow B(\mathcal{H})$ with the properties:

- (1) $Q(\Omega) \geq 0$
- (2) If $\Omega = \cup_{n=1}^{\infty} \Omega_n$ with $\Omega_n \cap \Omega_m = \emptyset \quad \forall n \neq m$, then $\sum_{n=1}^N Q(\Omega_n) \xrightarrow{s} Q(\Omega)$.
- (3) $Q(\emptyset) = 0$ and $Q(\mathbb{R}) = I$.

It is easy to see, ([57], pgs. 234-235), that if Q is a POVM, then $\mu_\phi(\Omega) := \langle Q(\Omega)\phi, \phi \rangle$, for $\Omega \in \mathcal{B}(\mathbb{R})$, defines a regular, countably additive Borel measure for any $\phi \in \mathcal{H}$. Furthermore, using a POVM Q , and the measures μ_ϕ , one can define the integral

$$(9.1.6) \quad \Phi[f] := \int_{-\infty}^{\infty} f(\lambda)Q(d\lambda),$$

for any $f \in L^\infty(\mathbb{R})$ which has compact support by

$$(9.1.7) \quad \langle \Phi[f]\phi, \phi \rangle = \int_{-\infty}^{\infty} f(\lambda)d\mu_\phi(\lambda) =: \int_{-\infty}^{\infty} f(\lambda)\langle Q(d\lambda)\phi, \phi \rangle.$$

Using the polarization identity and the Riesz representation theorem, it is straightforward to see that this does in fact define a unique bounded linear operator. In fact, even if f is just a measurable function which is not bounded, *e.g.* $f(\lambda) = \lambda$, then $\Phi[f]$ often still defines an unbounded linear operator on a dense domain in \mathcal{H} . In the case where $Q(\Omega)$ is a projection for each $\Omega \in \mathcal{B}(\mathbb{R})$, Q is called a projection valued measure. Recall that one form of the spectral theorem asserts that there is a 1 – 1 correspondence between projection valued measures on $\mathcal{B}(\mathbb{R})$ and self-adjoint operators. This follows immediately from the L^∞ functional calculus for self-adjoint operators.

Returning to the discussion regarding the symmetric operator B , let $\mathcal{P}(B)$ denote the set of all unital POVMs such that

$$(9.1.8) \quad \|B\phi\|^2 = \int_{-\infty}^{\infty} \lambda^2 d\mu_\phi(\lambda) =: \int_{-\infty}^{\infty} \lambda^2 \langle Q(d\lambda)\phi, \phi \rangle \quad \text{and} \quad B\phi = \int_{-\infty}^{\infty} \lambda Q(d\lambda)\phi$$

for all $\phi \in \mathfrak{D}(B)$. In particular, the projection-valued measures (PVMs) $Q(\Omega) := \chi_\Omega(B')$ defined using any self-adjoint extension B' of B belong to $\mathcal{P}(B)$. The set $\mathcal{P}(B)$ can be thought of as the set of all POVMs that ‘diagonalize’ the symmetric operator B .

Now suppose that B is regular. Then, as mentioned above, the gauge u can be chosen so that the functions $\phi_u(z)$ associated with each $\phi \in \mathcal{H}$ are meromorphic on \mathbb{C} with poles that lie off of the real axis. In this case it can be shown that if $Q \in \mathcal{P}(B)$ then ([28], pg. 49),

$$(9.1.9) \quad \langle \phi, \psi \rangle = \int_{-\infty}^{\infty} \phi_u(\lambda) \overline{\psi_u(\lambda)} \langle Q(d\lambda)u, u \rangle.$$

Let σ_Q denote the measure $\langle Q(\cdot)u, u \rangle$. It follows that the set of all ϕ_u , $\phi \in \mathcal{H}$ belongs to $L^2(\mathbb{R}; d\sigma_Q)$, and one can prove ([28], pg. 51 and pg. 12):

THEOREM 9.1.3. *Let B be a regular symmetric operator with deficiency indices $(1, 1)$. Let u be a gauge for B such that $S_u \cap \mathbb{R} = \emptyset$. Then the map Φ from \mathcal{H} into $L^2(\mathbb{R}, d\sigma_Q)$ defined by $\Phi[\phi] = \phi_u$ is an isometry. The map Φ is onto if and only if the POVM Q used to define σ_Q is a PVM. Furthermore, $\Phi B \Phi^{-1}$ acts as multiplication by the independent variable on its dense domain in $\Phi\mathcal{H} \subset L^2(\mathbb{R}, d\sigma_Q)$.*

Given a choice of $Q \in \mathcal{P}(B)$, let $\mathcal{H}' := \Phi\mathcal{H}$ denote the range of Φ which is some subspace of $L^2(\mathbb{R}, d\sigma_Q)$.

Finally, this next theorem of ([28], pg. 55) leads to the conclusion that \mathcal{H}' is a reproducing kernel Hilbert space. Let $B_r(z)$ denote the ball of radius $r > 0$ about $z \in \mathbb{C}$.

THEOREM 9.1.4. *Fix $z \in \mathbb{C}$, and $r > 0$. Then there exists a constant $c(z, r)$ such that $|\phi_u(z)| \leq c(z, r)\|\phi\|$ for all $z \in B_r(z)$, and for all ϕ for which $\phi_u(z)$ is analytic in $B_r(z)$.*

Since we assume B is regular, this means, by Remark 9.1.0.2, that the functions $\phi_u(z)$ are all meromorphic in \mathbb{C} , and that their poles all belong to a fixed set S_u where $S_u \cap \mathbb{R} = \emptyset$. Hence, given any $x \in \mathbb{R}$, there exists an $\epsilon > 0$ such that $B_\epsilon(x) \cap S_u = \emptyset$ so that for any $\phi \in \mathcal{H}$, ϕ_u is analytic in $B_\epsilon(x)$. Applying the above theorem, there is a $c = c(x, \epsilon) > 0$ such that $|\phi_u(z)| \leq c\|\phi\| = c\|\phi_u\|_{\mathcal{H}'}$ for all $\phi \in \mathcal{H}$. This proves that \mathcal{H}' is a reproducing kernel Hilbert space. Furthermore, we observe that for any $\phi \in \mathcal{H}$, $\phi_u(x) := \langle \phi, \xi(x) \rangle$. Since $\xi(x) \in \mathcal{H}$ is a non-zero vector for each $x \in \mathbb{R}$, it follows that there is no point $x \in \mathbb{R}$ such that $\phi_u(x) = 0$ for all $\phi \in \mathcal{H}$. Hence, the point evaluation vectors δ_x , $x \in \mathbb{R}$ for this RKHS are all non-zero and the reproducing kernel for \mathcal{H}' , $K(x, y) = \langle \delta_x, \delta_y \rangle$, is positive definite, $K(x, x) = \|\delta_x\|^2 > 0$.

Since by Theorem 9.1.3, the operator $\Phi B \Phi^{-1}$ is a densely defined, closed, simple, regular, symmetric linear multiplication operator with deficiency indices $(1, 1)$, and there is no point $x \in \mathbb{R}$ at which all functions $\phi_u \in \mathcal{H}'$ vanish, *Theorem 4.4.7 now implies that \mathcal{H}' has the sampling property:

COROLLARY 9.1.5. *Under the same assumptions as in Theorem 9.1.3, $\mathcal{H}' = \Phi \mathcal{H}$ is a reproducing kernel Hilbert space with the sampling property.*

9.2. de Branges spaces

There is a connection between Krein's theory of regular symmetric operators and a theory of special reproducing kernel Hilbert spaces of entire functions called de Branges spaces. De Branges spaces were first introduced by Louis de Branges, [8], and were used most notably in his famous proof of the Bieberbach conjecture. The Bieberbach conjecture concerns a necessary condition for a holomorphic function to map the open unit disc injectively into the complex plane. More recently, over the past decade or so, de Branges has been using his theory of Hilbert spaces of entire functions in an attempt to prove the Riemann hypothesis. In fact, he claims to have a proof of the Riemann hypothesis. This unverified proof has been posted on his website for the past few years.

The Paley-Wiener space of Ω -bandlimited functions is an example of a de Branges space. In fact, it appears that $B(\Omega)$ is the canonical example that de Branges generalized to arrive at his theory of Hilbert spaces of entire functions ([8], pg. 50). Our interest in de Branges spaces lies in the fact that most de Branges spaces have the sampling property.

It will be necessary to introduce a few concepts from complex function theory. Given a region $\Omega \subset \mathbb{C}$, let $\text{Hol}(\Omega)$ denote the set of functions which are holomorphic in Ω .

9.2.0.4. Definition. A function f , holomorphic in a region Ω is said to be of bounded type in that region if there exist functions $p, q \in \text{Hol}(\Omega)$ such that $f(z) = \frac{p(z)}{q(z)}$, $q \neq 0$, and q, p are bounded in Ω .

If f is analytic in the upper half plane (UHP), then the mean type $h[f]$ of f can be defined by

$$(9.2.1) \quad h[f] := \limsup_{y \rightarrow \infty} \frac{1}{y} \ln |f(iy)|.$$

Mean type for functions analytic in the lower half plane is defined analogously. The notion of mean type is a measure of growth in the upper half plane, and is clearly a generalization of the notion of exponential type (see Chapter 2) to functions analytic in the upper half plane. One can show that if f is of bounded type in the upper half plane, then $h[f] < \infty$ ([8], Chapter 9). In fact, if f is an entire function, that is of bounded type in both UHP and LHP, then it is of exponential

type. In this case, the exponential type of the function is equal to the maximum of its mean types in the LHP and the UHP. This fact is a theorem due to Krein ([8], pg. 26, pg. 38).

Given an entire function f , let f^* denote the entire function defined by $f^*(z) := \overline{f(\bar{z})}$. An entire function E is called a de Branges function if it obeys $|E(x - iy)| < |E(x + iy)|$ for all $y > 0$. This inequality implies, in particular, that E has no zeroes in the upper half plane. Given such a function E , the de Branges space $\mathcal{H}(E)$ is defined as the set of all entire functions F such that F/E and F^*/E are of bounded type and non-positive mean type in the upper half plane, and which are square integrable with respect to the norm generated by the inner product:

$$(9.2.2) \quad \langle F, G \rangle := \int_{-\infty}^{\infty} F(t) \overline{G(t)} \frac{1}{|E(t)|^2} dt.$$

The space $\mathcal{H}(E)$ is complete with respect to this inner product ([8], pg. 53).

Let $A := \frac{1}{2}(E + E^*)$ and $B := \frac{i}{2}(E - E^*)$. Then the following theorem shows that $\mathcal{H}(E)$ is a reproducing kernel Hilbert space whose reproducing kernel can be expressed in terms of E and E^* ([8], pg. 50).

THEOREM 9.2.1. (*de Branges*) *Given any entire function E such that $|E(x - iy)| < |E(x + iy)|$ for $y > 0$, let $K(w, z) := \frac{B(z)A(w) - A(z)B(w)}{\pi(z - \bar{w})}$. Then δ_w , where $\delta_w(z) := K(w, z)$, belongs to $\mathcal{H}(E)$ for every $w \in \mathbb{C}$ and $F(w) = \langle F, \delta_w \rangle$ for any $F \in \mathcal{H}(E)$.*

Note that $B(\Omega)$ is the de Branges space defined by the function $E(z) := e^{-i\Omega z}$. It is a fact that many de Branges spaces have the sampling property. The following equivalent axiomatic definition of de Branges spaces makes this fact more apparent ([8], pgs. 56-57).

THEOREM 9.2.2. *A Hilbert space of entire functions \mathcal{H} is isometrically equivalent to a de Branges space $\mathcal{H}(E)$ if and only if the following three axioms are satisfied:*

- (A1) *Point evaluation at every $z \in \mathbb{C} \setminus \mathbb{R}$ is a bounded linear functional.*
- (A2) *If $F \in \mathcal{H}$, then $F^* \in \mathcal{H}$, and $\|F\| = \|F^*\|$.*
- (A3) *If $F \in \mathcal{H}$ and $F(w) = 0$ for some $w \in \mathbb{C} \setminus \mathbb{R}$, then $G(z) := F(z) \frac{z - \bar{w}}{z - w} \in \mathcal{H}$, and $\|G\| = \|F\|$.*

Notice that axiom (A3) immediately implies that we can define multiplication by the function $\mu_w(z) := \frac{z - \bar{w}}{z - w}$ on a certain subspace of a de Branges space for any $w \in \mathbb{C} \setminus \mathbb{R}$, and that this resulting multiplication operator V_w is an isometry from its domain to its range. It is not difficult to further prove the following

THEOREM 9.2.3. *Let \mathcal{H} be any Hilbert space of entire functions satisfying the axioms (A1), (A2), and (A3) of Theorem 9.2.2. Then multiplication by z is a closed, symmetric operator in \mathcal{H} with deficiency indices $(1, 1)$.*

PROOF. Let V_w denote the operator of multiplication by the function $\mu_w(z) := \frac{z - \bar{w}}{z - w}$. Then, by assumption, V_w is defined on the subspace $\mathfrak{D}(V_w)$ of all $F \in \mathcal{H}$ for which $F(w) = 0$. Property (A1) implies that $\mathfrak{D}(V_w)$ is closed, and (A3) implies that V_w is an isometry from its domain onto its range, $\mathfrak{R}(V_w)$. Now I claim that for any $w \in \mathbb{C} \setminus \mathbb{R}$ that $n := \dim(\mathfrak{D}(V_w)^\perp) = 1$.

If $n > 1$, then there exist 2 linearly independent functions F_1 and F_2 which are orthogonal to $\mathfrak{D}(V_w)$, and hence do not vanish at w . But then $F := F_1 - \frac{F_1(w)}{F_2(w)} F_2$ must belong to $\mathfrak{D}(V_w)^\perp$ since it is a subspace, and yet $F(w) = 0$ which means $F \in \mathfrak{D}(V_w)$. Hence $F = 0$ so that F_1 and F_2 are linearly dependent. This proves that $n \leq 1$.

Now $n \neq 0$ since if $G \in \mathfrak{D}(V_w)$ then G has a zero of finite order k at w . By property (A3), $V_w^k G \in \mathcal{H}(E)$ is non-zero, and has no zero at w . Hence, $V_w^k G \notin \mathfrak{D}(V_w)$. Since $\mathfrak{D}(V_w)$ is closed this means that $\mathfrak{D}(V_w)^\perp$ is non-empty so that $n > 0$. We conclude that $n = 1$.

Let M denote the operator which acts as multiplication by z . Then for any $w \in \mathbb{C} \setminus \mathbb{R}$ we have that $M = (\bar{w}V_w - w)(V_w - 1)^{-1}$ where $\mathfrak{D}(M) := \mathfrak{R}(V_w - 1)$. As observed previously, (A1)

implies that $\mathfrak{D}(V_w)$ is closed, so that V_w is a closed linear transformation. As in the proof of Theorem 4.3.3 (b), it is not difficult to show that this implies that M is closed.

Now observe that the range of V_w is equal to the domain of $V_{\bar{w}}$. To see this note that if F is in the range of V_w then $F(\bar{w}) = 0$ so that $F \in \mathfrak{D}(V_{\bar{w}})$ and that $V_{\bar{w}}$ is just the inverse of V_w . Furthermore, it is elementary to check that $\Re(M - w) = \Re(V_w) = \mathfrak{D}(V_{\bar{w}})$ and that $\Re(M - \bar{w}) = \mathfrak{D}(V_w)$. By the previous arguments, $\dim(\Re(M - w)^\perp) = \dim(\Re(M - \bar{w})^\perp) = 1$ so that M has deficiency indices $(1, 1)$. \square

Let $\mathcal{H}(E)$ be a de Branges space, and let M denote the multiplication operator by the independent variable, defined in the proof of the previous theorem.

THEOREM 9.2.4. The multiplication operator M on $\mathcal{H}(E)$ has no eigenvalues. Furthermore, if $E(\lambda) \neq 0$, where $\lambda \in \mathbb{R}$, then λ does not belong to the continuous spectrum of M . In particular, if E has no real zeroes, then M is both simple and regular.

Although this proof is my own, this theorem follows immediately from more powerful results of [8].

PROOF. If λ is an eigenvalue of M , it must be a finite real value. If λ is such an eigenvalue then $\mu = \frac{\lambda - w}{\lambda - \bar{w}}$ is an eigenvalue of V_w which lies on the unit circle. Let $F \neq 0$ be the corresponding eigenfunction. Then $V_w^k F = \lambda^k F$, so that $0 \neq V_w^k F \in \mathfrak{D}(V_w)$ for every $k \in \mathbb{N}$. This implies that F has a zero of infinite order at w , which is impossible as $F \neq 0$ is entire.

Suppose that $\lambda \in \sigma_c(M)$. Since M is symmetric it follows that $\lambda \in \mathbb{R}$. Assume that $E(\lambda) \neq 0$. Then since $\lambda \in \sigma_c(M)$, there exists a sequence $(f_n)_{n \in \mathbb{N}} \subset \mathfrak{D}(M)$ such that $\|f_n\| = 1$ and $(M - \lambda)f_n \rightarrow 0$. Now f_n is a bounded sequence, and so it has a weakly convergent subsequence $(g_k = f_{n_k})_{k=1}^\infty$, $g_k \xrightarrow{w} g$.

I claim that $\|g\| = 1$. To see this, first choose $B > 0$ arbitrary. For any $\epsilon > 0$, there is an $n \in \mathbb{N}$ such that $n > N$ implies that $\|g_n\|_{\mathbb{R} \setminus [\lambda - B, \lambda + B]}^2 := \|g_n\|^2 - \int_{[\lambda - B, \lambda + B]} \left| \frac{g_n(x)}{E(x)} \right|^2 dx < \epsilon$. If this were not true, then there would be an $\epsilon > 0$ such that for any $N \in \mathbb{N}$ there is an $n > N$ for which $\|g_n\|_{\mathbb{R} \setminus [\lambda - B, \lambda + B]}^2 > \epsilon$. For such an n ,

$$(9.2.3) \quad \|(M - \lambda)g_n\|^2 \geq \int_{-\infty}^{\lambda - B} (x - \lambda)^2 \left| \frac{g_n(x)}{E(x)} \right|^2 dx + \int_{\lambda + B}^{\infty} (x - \lambda)^2 \left| \frac{g_n(x)}{E(x)} \right|^2 dx \geq B^2 \epsilon.$$

This would contradict the fact that $(M - \lambda)g_n \rightarrow 0$.

Since $g_n \xrightarrow{w} g$, and $|F(z)|^2 \leq \|F\|^2 K(z, z)$ for all $F \in \mathcal{H}(E)$, where $K(z, w)$ is the reproducing kernel of $\mathcal{H}(E)$, and $K(z, z)$ is continuous for all $z \in \mathbb{C}$ ([8], pg. 50), it follows that $g_n \rightarrow g$ uniformly on compact subsets of \mathbb{C} . Furthermore, by the proof of ([8], Theorem 19), given any $F \in \mathcal{H}(E)$, F/E is continuous on \mathbb{R} .

Since $E(\lambda) \neq 0$ and E is entire, choose δ small enough so that $E \neq 0$ on $[\lambda - \delta, \lambda + \delta]$. Then $1/E$ is continuous on $[\lambda - \delta, \lambda + \delta]$, and it follows that g_n/E converges to g/E uniformly on this interval.

Given any $\epsilon > 0$ choose $N \in \mathbb{N}$ such that $n > N$ implies that $\|g_n\|_{x \notin [\lambda - \delta, \lambda + \delta]} < \frac{\epsilon}{2}$. Now choose $N' \in \mathbb{N}$ such that $n > N'$ implies that $|(g_n(x) - g(x))/E(x)| \leq \frac{\epsilon}{4\delta}$ for all $x \in [\lambda - \delta, \lambda + \delta]$. It follows that for any $n > M := \max\{N, N'\}$ we have that

$$(9.2.4) \quad \begin{aligned} \|g\| &\geq \|g\|_{[\lambda - \delta, \lambda + \delta]} \geq \|g_n + (g - g_n)\|_{[\lambda - \delta, \lambda + \delta]} \geq \|g_n\|_{[\lambda - \delta, \lambda + \delta]} - \|g - g_n\|_{[\lambda - \delta, \lambda + \delta]} \\ &> \left(1 - \frac{\epsilon}{2}\right) - \frac{\epsilon}{2} = 1 - \epsilon \end{aligned}$$

Since $\epsilon > 0$ was arbitrary, it follows that $\|g\| \geq 1$. Conversely, $|\langle g_n, g \rangle| \leq \|g_n\| \|g\| = \|g\|$ and $|\langle g_n, g \rangle| \rightarrow \|g\|^2$, so that $\|g\|^2 \leq \|g\|$. This implies that $\|g\| \leq 1$ and hence that $\|g\| = 1$.

The fact that $\|g\| = 1$ means that g_n actually converges strongly to g . This follows because

$$(9.2.5) \quad \|g_n - g\|^2 = \langle g_n - g, g_n - g \rangle = \|g_n\|^2 + \|g\|^2 - 2\operatorname{Re}(\langle g_n, g \rangle) = 2(1 - \operatorname{Re}(\langle g_n, g \rangle)) \rightarrow 0$$

Since $g_n \rightarrow g \neq 0$, $(M - \lambda)g_n \rightarrow 0$, and M is a closed operator, it follows that $g \in \mathfrak{D}(M)$ and $(M - \lambda)g = 0$. In other words, g is actually an eigenvector of M to eigenvalue λ . We have already proven that this is not possible. We conclude that $\lambda \notin \sigma_c(M)$. \square

In conclusion, if E is a de Branges function which has no real zeroes, then the multiplication operator M in $\mathcal{H}(E)$ is simple, regular, symmetric, and closed with deficiency indices $(1, 1)$. That is, M has the sampling property. Since $\mathcal{H}(E)$ is a RKHS, in order to prove that it has the sampling property, it remains to show that M is densely defined. De Branges has characterized exactly when this happens ([8], pg. 84):

THEOREM 9.2.5. *A necessary and sufficient condition for a function $S \in \mathcal{H}(E)$ to be orthogonal to the domain $\mathfrak{D}(M)$ of the multiplication operator in $\mathcal{H}(E)$ is that $S = aE + bE^*$ for some $a, b \in \mathbb{C}$. In particular, if no such function belongs to $\mathcal{H}(E)$, then M is densely defined.*

As the next theorem shows, de Branges is well aware of the fact that most $\mathcal{H}(E)$ have the sampling property, although his proof does not use the theory of self-adjoint extensions.

First, it can be shown that given any de Branges function E , that there is a continuous phase function φ such that $E(x)e^{i\varphi(x)}$ is real for all $x \in \mathbb{R}$, and that $\varphi'(x) = \pi \frac{K(x,x)}{|E(x)|^2} > 0$.

THEOREM 9.2.6. *(de Branges) Consider $\mathcal{H}(E)$ and let φ be a phase function for E . For each $\alpha \in \mathbb{R}$ define the set $\{t_n\}_{n \in \mathbb{Z}}$ by $\varphi(t_n) = \alpha \bmod \pi$. Then the set $\{\frac{K(t_n, z)}{E(t_n)}\}_{n \in \mathbb{Z}}$ is orthogonal. If $F \in \mathcal{H}(E)$ is orthogonal to this set then $F \in \mathbb{C}\{e^{i\alpha}E - e^{-\alpha}E^*\}$.*

Notice that if M is densely defined, then by Theorem 9.2.5, the sets in the above theorem are total for each $\alpha \in \mathbb{R}$. In this case $\mathcal{H}(E)$ will have the sampling property.

9.2.0.5. Remark. Let B be a densely defined simple, regular symmetric operator with deficiency indices $(1, 1)$ in a Hilbert space \mathcal{H} . As discussed in the previous section, there exists unitary transformations U which ‘diagonalize’ B in the sense that they map B onto a multiplication operator. Furthermore, as was discussed, these unitary transformations can be chosen so that the image of B is a multiplication operator on a reproducing kernel space of meromorphic functions with the sampling property. As a final remark, it is not difficult to show that if this RKHS of meromorphic functions actually consists of only entire functions, then this space is a de Branges space [66]. Regular, simple, symmetric, closed operators with deficiency indices $(1, 1)$ which have such a representation as a multiplication operator on a reproducing kernel Hilbert space of entire functions, are called entire operators, and Krein has developed a whole theory of such operators, including generalizations to arbitrary deficiency indices [28].

Later, in Chapter 11 we will discuss the strategy of proving that the invariant subspaces of certain differential operators are reproducing kernel Hilbert spaces with the sampling property by proving that they are de Branges spaces. The axiomatic characterization of de Branges spaces given in Theorem 9.2.2 will be particularly useful for this endeavour.

Differentiability of the spectra of extensions of a regular, simple, symmetric operator with deficiency indices $(1, 1)$

For this chapter, let \mathcal{H} denote a separable Hilbert space and B denote a closed, symmetric, simple and regular operator with deficiency indices $(1, 1)$ defined on a dense domain $\mathfrak{D}(B) \subset \mathcal{H}$. That is, let B be a symmetric operator with the sampling property.

Recall that in *Theorem 4.4.7, we proved that if $\mathcal{M} := B$ is the operator of multiplication by the independent variable in a reproducing kernel Hilbert space \mathcal{H} of functions on \mathbb{R} with positive definite reproducing kernel, then this RKHS has the sampling property.

Now suppose that \mathcal{H} is a subspace of $\tilde{\mathcal{H}} := L^2(\mathbb{R}, d\mu)$ where μ is absolutely continuous with respect to Lebesgue measure. In this case \mathcal{M} can be seen as a symmetric restriction of the self-adjoint multiplication operator M in $\tilde{\mathcal{H}}$ to a dense domain in \mathcal{H} . From the previous chapter, we know that there is an isometry U that maps \mathcal{M} onto a multiplication operator in a reproducing kernel Hilbert space of meromorphic functions. Since \mathcal{M} is already a multiplication operator on functions, one might expect that in many circumstances, there is such an isometry which, in fact, simply acts as multiplication by a measurable function. That is, that there is an isometry which acts as multiplication by a measurable function and which maps \mathcal{H} onto a reproducing kernel Hilbert space. If this measurable function is non-zero almost everywhere, this would imply that the original Hilbert space, \mathcal{H} , can itself be seen as a RKHS with the sampling property.

In this chapter, we explore one approach to establishing this conjecture. Although this approach has yet to provide a complete proof, it is of some interest in itself as it establishes new results about the spectra of the self-adjoint extensions of an arbitrary symmetric operator B with the sampling property. In particular, it will be shown that the eigenvalues $(\lambda_n(\alpha))_{n \in \mathbb{M}}$ are smooth, differentiable functions of the parameter $\alpha \in [0, 1)$ that indexes the self-adjoint extensions $B(\alpha)$ of B . Here, $\mathbb{M} = \mathbb{Z}$, or $\pm\mathbb{N}$ (see Remark 4.4.1.4).

This fact is of particular interest in Kempf's approach [36] to sampling theory using self-adjoint extensions of symmetric operators. In this approach, Kempf proves that if $(\lambda_n(\alpha))_{n \in \mathbb{M}}$ are the spectra of all self-adjoint extensions of a simple, regular symmetric operator, and if the λ_n are differentiable with respect to α , then the knowledge of the values $\lambda_n(\beta)$ and $\lambda'_n(\beta)$ for any fixed $\beta \in [0, 1)$ and all $n \in \mathbb{M}$ completely determines the $\lambda_n(\alpha)$ for all $\alpha \in [0, 1)$ and $n \in \mathbb{M}$. The results of this chapter will prove that the $\lambda_n(\alpha)$ are, in fact, always infinitely differentiable.

10.1. Properties of the spectra of the $B(\alpha)$

Recall that if B is a symmetric, regular, simple, closed operator densely defined in \mathcal{H} with deficiency indices $(1, 1)$, then the family of all self-adjoint extensions of B can be labelled by a single real parameter $\alpha \in [0, 1)$ (see Section 4.3.2). Namely, we fix unit norm vectors $\phi_{\pm} \in \mathfrak{D}_{\pm}$ and then define

$$(10.1.1) \quad U(\alpha) := V \oplus e^{i2\pi\alpha} \phi_- \langle \cdot, \phi_+ \rangle$$

on $\mathcal{H} := \mathfrak{D}(V) \oplus \mathfrak{D}_+$ where V is the Cayley transform of B . The self-adjoint extensions $B(\alpha)$ of B are defined as the inverse Cayley transforms of the $U(\alpha)$. All self-adjoint extensions of B are obtained in this manner.

Recall that the spectrum of each self-adjoint extension $B(\alpha)$ of B can be arranged as a non-decreasing sequence of eigenvalues $(\lambda_n(\alpha))_{n \in \mathbb{M}}$ where $M = -\mathbb{N}, \mathbb{N}$ or \mathbb{Z} , and that the spectra $\sigma(B_\alpha)$ of the self-adjoint extensions do not intersect and cover \mathbb{R} exactly once (see Remark 4.4.1.4 and Theorem 4.4.5). In fact, even more can be said ([28], pg. 19):

THEOREM 10.1.1. *Let B be a closed simple symmetric operator in \mathcal{H} with deficiency indices $(1, 1)$. Suppose that the interval $I \subset \mathbb{R}$ consists of regular points of B . Then, the eigenvalues of any two self-adjoint extensions B' and B'' of B in I alternate.*

In our case, we assume B is regular so that every point in \mathbb{R} is regular for B . It follows that the eigenvalues of any two self-adjoint extensions $B(\alpha)$ and $B(\beta)$ of B alternate. That is, given any two consecutive eigenvalues, $\lambda_n(\alpha)$ and $\lambda_{n+1}(\alpha)$ of $B(\alpha)$, every other self-adjoint extension $B(\beta)$ of B , $\beta \neq \alpha$ has exactly one eigenvalue in the interval $(\lambda_n(\alpha), \lambda_{n+1}(\alpha))$. In particular, this means that if $\sigma(B(\alpha))$ is bounded above or below, or is not bounded above or below, then the same is true of the spectrum of every other self-adjoint extension of B . This means that $\sigma(B(\alpha)) = (\lambda_n(\alpha))_{n \in \mathbb{M}}$ where \mathbb{M} is equal to $\pm\mathbb{N}$ or \mathbb{Z} , and is the same for every $\alpha \in [0, 1)$. For convenience, we will assume from now on that $\mathbb{M} = \mathbb{Z}$.

Consider $\{\lambda_n(0)\}_{n \in \mathbb{Z}}$. Given any $\alpha \in (0, 1)$, define $\lambda_n(\alpha)$ to be that unique eigenvalue of $B(\alpha)$ in the interval $(\lambda_n(0), \lambda_{n+1}(0))$. Then for each $n \in \mathbb{Z}$ the map $\lambda_n(\alpha)$ is a bijection from $[0, 1]$ onto $[\lambda_n(0), \lambda_{n+1}(0)]$, and $\sigma(B(\alpha)) = (\lambda_n(\alpha))_{n \in \mathbb{Z}}$. Using the functions λ_n , we can define $\hat{\lambda}, \tilde{\lambda} : \mathbb{R} \rightarrow \mathbb{R}$ by $\hat{\lambda}(x) := \lambda_{\lfloor x \rfloor}(x - \lfloor x \rfloor)$ and $\tilde{\lambda}(x) := \lambda_{\lfloor x \rfloor}(1 - (x - \lfloor x \rfloor))$. It follows that both $\hat{\lambda}$ and $\tilde{\lambda}$ are bijections, and that $\hat{\lambda}(x) \in \sigma(B(x))$ while $\tilde{\lambda}(x) \in \sigma(B(-x))$ for each $x \in \mathbb{R}$.

Our goal is to prove that each $\lambda_n(\alpha)$ is an infinitely differentiable function of α . Using this fact we will show that either $\tilde{\lambda}$ or $\hat{\lambda}$ is an infinitely differentiable, bijective homeomorphism of \mathbb{R} onto \mathbb{R} :

***PROPOSITION 10.1.2.** *Either $\hat{\lambda}$ or $\tilde{\lambda}$ is an infinitely differentiable homeomorphism of \mathbb{R} onto \mathbb{R} .*

10.2. The spectral function of a symmetric operator with the sampling property

The proof of *Proposition 10.1.2 will be broken into several smaller claims.

Consider the Möbius transform $\mu(z) := \frac{z-i}{z+i}$ and its inverse $\mu^{-1}(z) := i\frac{1+z}{1-z}$. Let

$$(10.2.1) \quad U(z) := \mu(B(z)) = \mu(B) \oplus e^{i2\pi z} \phi_- \langle \cdot, \phi_+ \rangle$$

for any $z \in \mathbb{C}$. For $x \in \mathbb{R}$, $U(x) = U(\lfloor x \rfloor) = U(x + k)$ for any $k \in \mathbb{Z}$ is the Cayley transform of $B(x)$. The spectral mapping theorem implies that the spectrum of $U(\alpha)$ is $(\kappa_n(\alpha))_{n \in \mathbb{Z}}$ where $\kappa_n(\alpha) := \mu(\lambda_n(\alpha))$ so that $\hat{\kappa}(x) := \mu(\hat{\lambda}(x)) = \mu(\lambda_{\lfloor x \rfloor}(x - \lfloor x \rfloor)) = \kappa_{\lfloor x \rfloor}(x - \lfloor x \rfloor)$. Now since μ^{-1} is a meromorphic function with a simple pole at $z = 1$, and $\hat{\lambda}(x) = \mu^{-1}(\hat{\kappa}(x))$ it follows that $\hat{\lambda}$ will be infinitely differentiable for $x \in \mathbb{R}$ if $\hat{\kappa}$ is. Similarly, we define $\tilde{\kappa}(x) = \mu(\tilde{\lambda}(x)) = \kappa_{\lfloor x \rfloor}(1 - (x - \lfloor x \rfloor))$. Again, observe that $\hat{\kappa}(x) \in \sigma(U(x))$ and $\tilde{\kappa}(x) \in \sigma(U(-x))$ for each $x \in \mathbb{R}$. Further note that for $n \in \mathbb{Z}$, $\hat{\kappa}(n) = \kappa_n(0)$ while $\tilde{\kappa}(n) = \kappa_n(1) = \kappa_n(0)$ since $U(0) = U(1)$.

The fact that $\hat{\kappa}, \tilde{\kappa}$ are continuous functions of x follows from the discreteness of the spectra of each $U(x)$, the continuity of the operator valued function $U(x)$, and Newburgh's theorem [53]:

THEOREM 10.2.1. *Let \mathbb{A} be a unital Banach algebra and let $a \in \mathbb{A}$. Suppose that $\sigma(a) \subset U \cup V$ where U, V are open and disjoint, $U \cap V = \emptyset$ and $U \cap \sigma(a) \neq \emptyset$. Then, there is an $\epsilon > 0$ such that $\|x - a\| < \epsilon$ implies that $\sigma(x) \cap U \neq \emptyset$.*

10.2.0.6. *Notation.* Given $z, w \in \mathbb{T}$, the unit circle in the complex plane, we will write (z, w) to denote the arc of the circle \mathbb{T} which lies between z and w , and doesn't include the point 1. That is (z, w) is the image of the open interval $(\mu^{-1}(z), \mu^{-1}(w)) \subset \mathbb{R}$ under the Möbius transformation μ . Similarly, if $z, w \in \mathbb{T}$, we will say that $z \leq w$ if $\mu^{-1}(z) \leq \mu^{-1}(w)$. Furthermore, we will say that $\hat{\kappa}$ or $\tilde{\kappa}$ is monotonically increasing on $(z, w) \subset \mathbb{T}$ if $\hat{\lambda}$ or $\tilde{\lambda}$ is monotonically increasing on $(\mu^{-1}(z), \mu^{-1}(w))$.

First consider the functions $\kappa_n(\alpha)$ for $\alpha \in (0, 1)$ and $n \in \mathbb{Z}$. Recall that $\kappa_n(\alpha)$ is the unique eigenvalue to $U(\alpha)$ in the open arc $(\kappa_n(0), \kappa_{n+1}(0))$.

*CLAIM 10.2.2. *For each $n \in \mathbb{Z}$, $\kappa_n(\alpha)$ is a continuous map from $(0, 1)$ onto $(\kappa_n(0), \kappa_{n+1}(0))$.*

PROOF. We already know that for each $n \in \mathbb{Z}$, $\kappa_n(\alpha)$ is a bijection from $(0, 1)$ onto $(\kappa_n(0), \kappa_{n+1}(0))$. It remains to establish continuity. Choose $\alpha' \in (0, 1)$ and $n \in \mathbb{Z}$. Let $\epsilon > 0$ be arbitrary and consider $S := B_\epsilon(\kappa_n(\alpha')) \cap (\kappa_n(0), \kappa_{n+1}(0))$. Since $U(\alpha)$ is a continuous operator-valued function of $\alpha \in (0, 1)$, it follows from Newburgh's theorem, Theorem 10.2.1 that there is a $\delta > 0$ such that if $\alpha \in (0, 1)$ satisfies $|\alpha - \alpha'| < \delta$ then $\sigma(U(\alpha)) \cap S \neq \emptyset$. For such an α , $\sigma(U(\alpha)) \cap B_\epsilon(\kappa_n(\alpha')) = \kappa_n(\alpha)$ so that $|\kappa_n(\alpha) - \kappa_n(\alpha')| < \epsilon$. Since $\epsilon > 0$ was arbitrary, this proves the claim. \square

*CLAIM 10.2.3. *For each $n \in \mathbb{Z}$, $\kappa_n(\alpha)$ is a monotonic strictly increasing or monotonic strictly decreasing function of $\alpha \in (0, 1)$. If $\kappa_n(\alpha)$ is increasing then $\lim_{\alpha \rightarrow 0^+} \kappa_n(\alpha) = \kappa_n(0)$ and $\lim_{\alpha \rightarrow 1^-} \kappa_n(\alpha) = \kappa_{n+1}(0)$. Conversely, if κ_n is decreasing then $\lim_{\alpha \rightarrow 0^+} \kappa_n(\alpha) = \kappa_{n+1}(0)$ and $\lim_{\alpha \rightarrow 1^-} \kappa_n(\alpha) = \kappa_n(0)$.*

PROOF. Suppose that for some $n \in \mathbb{Z}$ that $\kappa_n(\alpha)$ was not monotonic. Then there would exist $\alpha_i \in (0, 1)$, $1 \leq i \leq 3$ such that $\kappa_n(\alpha_1) < \kappa_n(\alpha_2)$ and $\kappa_n(\alpha_3) < \kappa_n(\alpha_2)$. Let $M := \max\{\kappa_n(\alpha_1), \kappa_n(\alpha_3)\}$. Since $\kappa_n(\alpha)$ is continuous on $[\alpha_1, \alpha_3]$ by *Claim 10.2.2, the intermediate value theorem then implies that there exists a $c_1 \in [\alpha_1, \alpha_2)$ and a $c_2 \in (\alpha_2, \alpha_3]$ such that $\kappa(c_1) = M = \kappa(c_2)$. This contradicts the fact that $\kappa_n(\alpha)$ is injective. This proves that each $\kappa_n(\alpha)$ is either monotonically strictly increasing or decreasing.

Now suppose that $\kappa_n(\alpha)$ is monotonically decreasing on $(0, 1)$. Suppose, contrary to the claim, that

$$(10.2.2) \quad \lim_{\alpha \rightarrow 0^+} \kappa_n(\alpha) \neq \kappa_{n+1}(0).$$

It follows that there is an $\epsilon > 0$ such that for each $k \in \mathbb{N}$, one can find $\alpha_k \in (0, 1)$ so that $\alpha_k \rightarrow 0$, and $\kappa_{n+1}(0) - \kappa_n(\alpha_k) > \epsilon$. Since $\kappa_n(\alpha)$ is a bijection of $(0, 1)$ onto $(\kappa_n(0), \kappa_{n+1}(0))$, it follows that there is an $\alpha' \in (0, 1)$ such that $\kappa_{n+1}(0) - \kappa_n(\alpha') < \frac{\epsilon}{2}$. It follows that $\kappa_n(\alpha') > \kappa_n(\alpha_k)$ for all $k \in \mathbb{N}$. Choosing k large enough so that $\alpha_k < \alpha'$ contradicts our assumption that κ_n is monotonically decreasing. The rest of the claim is proved in a similar fashion. \square

*CLAIM 10.2.4. *The functions $\kappa_n(\alpha)$, $n \in \mathbb{Z}$, are either all monotonically increasing, or all monotonically decreasing.*

PROOF. Suppose that for some fixed $m \in \mathbb{Z}$, that $\kappa_m(\alpha)$ is monotonically decreasing. Then, by *Claim 10.2.3, it follows that $\lim_{\alpha \rightarrow 0^+} \kappa_m(\alpha) = \kappa_{m+1}(0)$, and $\lim_{\alpha \rightarrow 1^-} \kappa_m(\alpha) = \kappa_m(0)$. Let $\epsilon := \min\{\frac{\kappa_{m+1}(0) - \kappa_m(0)}{2}, \frac{\kappa_m(0) - \kappa_{m-1}(0)}{2}\}$. Choose $\delta > 0$ such that $\alpha \in (0, 1)$ and $\alpha < \delta$ implies that $\kappa_{m+1}(0) - \kappa_m(\alpha) < \epsilon$. By Newburgh's theorem, Theorem 10.2.1, for any sufficiently large $k \in \mathbb{N}$, there is a $\delta_k > 0$ so that $|\alpha| < \delta_k$ implies that $\sigma(U(\alpha)) \cap B_{1/k}(\kappa_m(0)) \neq \emptyset$. Choose $\delta'_k := \min\{\delta, \delta_k\}$, and $K \in \mathbb{N}$ so that $\frac{1}{K} < \epsilon$. It follows that when $k > K$, if $\alpha \in (0, 1)$ and $\alpha < \delta'_k$ then $\kappa_m(\alpha) \notin B_{1/k}(\kappa_m(0))$. Since $\sigma(U(\alpha)) \cap B_{1/k}(\kappa_m(0)) \neq \emptyset$ it follows that for each such α , $\kappa_{m-1}(\alpha) \in B_{1/k}(\kappa_m(0))$. It follows that $\lim_{\alpha \rightarrow 0^+} \kappa_{m-1}(\alpha) \neq \kappa_{m-1}(0)$. By *Claim 10.2.3, it follows that $\lim_{\alpha \rightarrow 0^+} \kappa_{m-1}(\alpha) = \kappa_m(0)$, and that $\kappa_{m-1}(\alpha)$ is monotonically decreasing on $(0, 1)$.

Proceeding in a similar fashion, it is not difficult to show that for every $n \in \mathbb{Z}$, $\kappa_n(z)$ is monotonically decreasing. Proving the other half of the claim is directly analogous. \square

Recall the functions $\hat{\kappa}, \tilde{\kappa} : \mathbb{R} \rightarrow \mathbb{R}$ defined by $\hat{\kappa}(x) := \kappa_{\lfloor x \rfloor}(x - \lfloor x \rfloor)$ and $\tilde{\kappa}(x) := \kappa_{\lfloor x \rfloor}(1 - (x - \lfloor x \rfloor))$. Recall that $\hat{\kappa}(x)$ is the unique eigenvalue to the operator $U(x)$ in the arc $[\kappa_{\lfloor x \rfloor}(0), \kappa_{\lfloor x \rfloor + 1}(0)] \subset \mathbb{T}$ while $\tilde{\kappa}(x)$ is the unique eigenvalue to the operator $\tilde{U}(x) := U(-x)$ in the arc $[\kappa_{\lfloor x \rfloor}(0), \kappa_{\lfloor x \rfloor + 1}(0)]$. Furthermore, recall that $\sigma(U(0)) = (\kappa_n(0))_{n \in \mathbb{Z}}$, and that $\kappa_n(1) := \kappa_n(0)$, so that $\tilde{\kappa}(n) = \kappa_n(1) = \kappa_n(0) = \hat{\kappa}(n)$, for any $n \in \mathbb{Z}$. Also, remember that $\kappa_n(0) < \kappa_{n+1}(0)$ for all $n \in \mathbb{Z}$.

*CLAIM 10.2.5. *Either $\hat{\kappa}$ or $\tilde{\kappa}$ is a homeomorphism of \mathbb{R} onto \mathbb{R} which is strictly monotonically increasing.*

PROOF. By *Claim 10.2.4, either

$$(10.2.3) \quad \lim_{\alpha \rightarrow 0^+} \kappa_n(\alpha) = \kappa_n(0) \text{ and } \lim_{\alpha \rightarrow 1^-} \kappa_n(\alpha) = \kappa_{n+1}(0)$$

for all $n \in \mathbb{Z}$, or

$$(10.2.4) \quad \lim_{\alpha \rightarrow 0^+} \kappa_n(\alpha) = \kappa_{n+1}(0) \text{ and } \lim_{\alpha \rightarrow 1^-} \kappa_n(\alpha) = \kappa_n(0)$$

for all $n \in \mathbb{Z}$.

In the first case where equation (10.2.3) holds, it is clear that $\hat{\kappa}$ satisfies the requirements of the claim. In the second case of equation (10.2.4), it is not difficult to verify that $\tilde{\kappa}$ satisfies the requirements of the claim since, for example,

$$(10.2.5) \quad \begin{aligned} \lim_{x \rightarrow n^+} \tilde{\kappa}(x) &= \lim_{x \rightarrow 0^+} \kappa_n(1 - (x - \lfloor x \rfloor)) \\ &= \lim_{\alpha \rightarrow 1^-} \kappa_n(\alpha) = \kappa_n(0) = \tilde{\kappa}(n). \end{aligned}$$

\square

In lieu of the above result,

10.2.0.7. *Definition.* For a symmetric operator B with the sampling property, define κ to be the choice of the two functions $\hat{\kappa}, \tilde{\kappa}$ in the above claim which is continuous on all of \mathbb{R} . If $\kappa = \tilde{\kappa}$, redefine $U(x) := V \oplus e^{-i2\pi x} \langle \cdot, \phi_+ \rangle \phi_-$ so that for each $x \in \mathbb{R}$, $\kappa(x)$ is the unique eigenvalue to $U(x)$ in the arc $[\kappa_{\lfloor x \rfloor}(0), \kappa_{\lfloor x \rfloor + 1}(0)]$ of the unit circle, \mathbb{T} .

The function $\kappa(x)$ will be called the **spectral function** of the isometric operator $V = \kappa(B)$, and $\lambda := \mu^{-1}(\kappa)$ will be called the spectral function of the symmetric operator B .

10.2.0.8. *Remark.* The definition of $U(x)$, and hence of $\kappa(x)$, depends on the arbitrary choice of unit norm $\phi_{\pm} \in \mathfrak{D}_{\pm}$. If $\varphi_{\pm} \in \mathfrak{D}_{\pm}$ are a different choice of deficiency vectors then, since the deficiency subspaces \mathfrak{D}_{\pm} are one dimensional, $\varphi_{\pm} = e^{i2\pi\theta_{\pm}} \phi_{\pm}$, for some $\theta_{\pm} \in [0, 1)$. If one then defines $\tilde{U}(x) := V \oplus e^{i2\pi x} \varphi_- \langle \cdot, \varphi_- \rangle$, it follows that $\tilde{U}(x) = U(x - \theta_+ + \theta_-)$. If one uses \tilde{U} to define a spectral function $\tilde{\kappa}$, then $\tilde{\kappa}(x) = \kappa(x - \theta_+ + \theta_-)$, for all $x \in \mathbb{R}$. For this reason, we will say that two spectral functions κ_1 and κ_2 are equivalent if there is a $c \in \mathbb{R}$ such that $\kappa_1(x) = \kappa_2(x + c)$ for all $x \in \mathbb{R}$.

10.3. Infinite differentiability and analyticity of the spectral function

Using standard functional calculus techniques, this section will show that the functions $\kappa(x)$ and $\lambda(x)$ are infinitely differentiable.

*CLAIM 10.3.1. *Let λ be an eigenvalue of a self-adjoint extension B' of the symmetric operator B . Then there is an $\epsilon > 0$ such that $B_{\epsilon}(\lambda) \cap \sigma(B(\alpha))$ contains at most one point for each $\alpha \in [0, 1)$.*

PROOF. Choose $\epsilon > 0$ so that $B_\epsilon(\lambda) \cap \sigma(B') = \{\lambda\}$. Suppose that the claim does not hold. Then there would be a sequence of values $\alpha_k \in (0, 1)$ such that for each $k \in \mathbb{N}$, $B(\alpha_k)$ has at least two eigenvalues in $B_{1/k}(\lambda)$. Choose $K \in \mathbb{N}$ so that $\frac{1}{K} < \epsilon$. For $k > K$, the alternating eigenvalue theorem, Theorem 10.1.1, implies that each such $B(\alpha_k)$ can have at most two eigenvalues λ_k, μ_k in $B_\epsilon(\lambda)$ where $\lambda_k < \lambda < \mu_k$. Otherwise B' would have more than one eigenvalue in $B_\epsilon(\lambda)$, which is a contradiction. It follows that $\lambda_k \rightarrow \lambda$ and that $\mu_k \rightarrow \lambda$. Fix a self-adjoint extension $\tilde{B} \neq B'$ of B . By the alternating eigenvalue theorem, it follows that \tilde{B} has a sequence of eigenvalues α_k such that $\lambda_k < \alpha_k < \mu_k$ for each $k > K$. Since $\sigma(\tilde{B})$ is closed, it follows that $\lambda \in \sigma(\tilde{B})$ which, by Theorem 4.4.5, is a contradiction. \square

The goal now is to show that κ , and hence λ , is infinitely differentiable. Fix $y \in \mathbb{R}$. We will show that $\kappa^{(k)}(y) = \frac{d^k}{dx^k} \kappa(x)|_{x=y}$ exists for any $k \in \mathbb{Z}$. Since y is arbitrary, this will establish *Proposition 10.1.2.

By *Claim 10.3.1, there is an $\epsilon > 0$ so that $\overline{B_{2\epsilon}(\kappa(y))} \cap \sigma(U(x))$ contains at most one point for each $x \in \mathbb{R}$. Since κ is continuous, choose $\delta' > 0$ so that $|x - y| < \delta'$ implies that $\kappa(x) \in B_\epsilon(\kappa(y))$. It follows that for all $|x - y| < \delta'$ that

$$(10.3.1) \quad \sigma(U(x)) \cap \left(\overline{B_{2\epsilon}(\kappa(y))} \setminus B_\epsilon(\kappa(y)) \right) = \emptyset$$

and that

$$(10.3.2) \quad \sigma(U(x)) \cap B_\epsilon(\kappa(y)) = \kappa(x).$$

For each x such that $|x - y| < \delta'$, let $P(x)$ denote the projection onto the eigenspace of $U(x)$ to eigenvalue $\kappa(x)$. This is a one-dimensional subspace, spanned by some normalized eigenvector which we denote $\phi_{\kappa(x)}$. For each such x , the spectrum of $U(x)$ is purely discrete and is contained in the union of open sets $V' := \mathbb{C} \setminus \overline{B_{2\epsilon}(\kappa(y))}$ and $B_\epsilon(\kappa(y))$. Let $S := V' \cup B_\epsilon(\kappa(y))$. Then S is an open set containing the spectrum of $U(x)$ for all $|x - y| < \delta'$.

Recall that the spectrum of a bounded operator is upper semi-continuous [51]:

THEOREM 10.3.2. (*upper semi-continuity of the spectrum*) *Let \mathbb{A} be a Banach algebra. Then if $a \in \mathbb{A}$, and U is an open set such that $\sigma(a) \subset U$, then there exists a $\delta > 0$ such that $\|b - a\| < \delta$ implies that $\sigma(b) \subset U$.*

Since $U(w)$ is an entire operator-valued function for $w \in \mathbb{C}$, it follows that there is a $\delta_1 > 0$ such that $|w - y| < \delta_1$ implies that $\sigma(U(w)) \subset S$. Choose a simple, smooth, counterclockwise contour Γ that lies in the interior of $\mathbb{C} \setminus S$, i.e., so that Γ lies between the balls of radius ϵ and 2ϵ about $\kappa(y)$. Let $\delta_2 := \min\{\delta_1, \delta'\}$. For $|w - y| < \delta_2$, the Riesz holomorphic functional calculus can be used to define the following operators $P(w)$ and $U(w)P(w)$:

$$(10.3.3) \quad P(w) = \frac{1}{2\pi i} \int_{\Gamma} (z - U(w))^{-1} dz,$$

and,

$$(10.3.4) \quad U(w)P(w) = \frac{1}{2\pi i} \int_{\Gamma} z(z - U(w))^{-1} dz.$$

It follows from the Riesz decomposition theorem that for each w such that $|w - y| < \delta_2$, the operators $P(w)$ are idempotents such that $\sigma(U(w)|_{P(w)\mathcal{H}}) = \sigma(U(w)) \cap B_\epsilon(\kappa(y))$. In particular, when $w = x \in \mathbb{R}$ so that $U(x)$ is a unitary operator, $P(w) = P(x)$ is the self-adjoint projection onto the eigenspace of $U(x)$ to eigenvalue $\kappa(x)$.

Now since $U(w) \rightarrow U(y)$ in operator norm as $w \rightarrow y$, and since the spectrum of $\sigma(U(w)) \subset S$ for all $|w - y| < \delta_2$, standard functional calculus techniques show that $P(w) \rightarrow P(y)$ in operator norm as $x \rightarrow y$:

PROPOSITION 10.3.3. *Let $a \in \mathbb{A}$, a unital Banach algebra, and let $\{a_n\}_{n \in \mathbb{Z}} \subset \mathbb{A}$ be a sequence such that $a_n \rightarrow a$. Let $U \supset \sigma(a)$, suppose that $\sigma(a_n) \subset U$ for all $n \in \mathbb{Z}$, and that f is analytic on U . Then $f(a_n) \rightarrow f(a)$.*

It follows from the above proposition that $P(w) \rightarrow P(y)$ as $w \rightarrow y$. Hence, there is a $\tilde{\delta} > 0$ so that $|w - y| < \tilde{\delta}$ implies that $|\langle P(w)\phi_{\kappa(y)}, \phi_{\kappa(y)} \rangle > 0|$. Choose $\delta := \min\{\tilde{\delta}, \delta_2\}$.

For all $w \in \mathbb{C}$ such that $|w - y| < \delta$, define

$$(10.3.5) \quad \kappa(w) := \frac{\langle U(w)P(w)\phi_{\kappa(y)}, \phi_{\kappa(y)} \rangle}{\langle P(x)\phi_{\kappa(y)}, \phi_{\kappa(y)} \rangle}.$$

If $w = x \in \mathbb{R}$, then $U(x)P(x) = \kappa(x)P(x)$, and the above agrees with our original definition of $\kappa(x)$. Hence, this definition of $\kappa(w)$ is an extension of $\kappa(x)$ to a neighbourhood of y in the complex plane.

Using this representation of $\kappa(w)$, equations (10.3.3) and (10.3.4) can now be applied to show that $\kappa(w)$ is analytic in $B_\delta(y)$, and hence is infinitely differentiable at y .

Let

$$(10.3.6) \quad f(w) := \langle P(w)\phi_{\kappa(y)}, \phi_{\kappa(y)} \rangle,$$

and,

$$(10.3.7) \quad g(w) := \langle U(w)P(w)\phi_{\kappa(y)}, \phi_{\kappa(y)} \rangle.$$

The fact that f and g are analytic functions of w for $z \in \mathbb{C} \setminus S$ will follow from the fact that $U(w)$ is an entire $B(\mathcal{H})$ -valued function.

Now $U(w)$ is clearly an entire operator-valued function of $w \in \mathbb{C}$. Indeed, if $\mathcal{U}(w) := i2\pi e^{i2\pi w} \langle \cdot, \phi_+ \rangle \phi_-$, it is easy to check that

$$(10.3.8) \quad \lim_{z \rightarrow 0} \left\| \frac{1}{z} (U(w+z) - U(w)) - \mathcal{U}(w) \right\| = 0,$$

so that $\mathcal{U}(w) = U'(w)$ for all $w \in \mathbb{C}$.

*CLAIM 10.3.4. *The operators $P(w)$ and $U(w)P(w)$ are analytic for $w \in B_\delta(y)$.*

The following lemma will be used in the proof of *Claim (10.3.4).

LEMMA 10.3.5. *Let $A(z)$ be an operator-valued function that is differentiable at w . Suppose that for all z in some neighbourhood U_w of w , each $A(z)$ has a bounded inverse $A(z)^{-1}$, and that $\|A(z)^{-1}\|$ is uniformly bounded for z in this neighbourhood. Then $A(z)^{-1}$ is differentiable at w , and $\frac{d}{dz} A(w)^{-1} = A(w)^{-1} A'(w) A(w)^{-1}$.*

PROOF. First it will be shown that $A(z)^{-1}$ is continuous on U_w . To see this, consider a fixed point $w' \in U_w$, and consider a compact neighbourhood $U' \subset U_w$ of w' . Since $A(z)$ is continuous on U' , it follows that $\inf_{\|\phi\|=1} \|A(z)\phi\|$ is continuous on the compact set U' , and hence takes its minimum value at some point $z' \in U'$. By the assumption that $A(z)^{-1}$ is a bounded operator for all $z \in U_w \supset U'$, it follows that there exists a $\theta > 0$ such that $\inf_{\|\phi\|=1} \|A(z)\phi\| \geq \theta > 0$ for all $z \in U'$. Hence, $A(z)^{-1}$ is bounded above uniformly by θ^{-1} for all $z \in U'$. Now it is matter of simple algebra to show that

$$(10.3.9) \quad A(z)^{-1} - A(w')^{-1} = A(w')^{-1} (A(w') - A(z)) A(z)^{-1}.$$

Using this identity, and the fact that $A(z)^{-1}$ is uniformly bounded above for $z \in U'$, it is straightforward to show that $\|A(z)^{-1} - A(w')^{-1}\|$ vanishes as $z \rightarrow w'$. Since $w' \in U_w$ was arbitrary, we conclude that $A(z)^{-1}$ is continuous in U_w .

Applying the identity (10.3.9), it follows by the continuity of $A(z)^{-1}$ at w and the differentiability of $A(z)$ that

$$(10.3.10) \quad \begin{aligned} \lim_{z \rightarrow 0} \frac{A(w+z)^{-1} - A(w)^{-1}}{z} &= \lim_{z \rightarrow 0} \frac{A(w)^{-1}(A(w) - A(w+z))A(w+z)^{-1}}{z} \\ &= -A(w)^{-1}A'(w)A(w)^{-1}. \end{aligned}$$

□

PROOF. (of *Claim 10.3.4) By the previous lemma, and the fact that for each $z \in \mathbb{C} \setminus S$, $(z - U(w))$ is an analytic function of w for $w \in B_\delta(y)$, it follows that for each such z , $(z - U(w))^{-1}$ and $U(w)(z - U(w))^{-1}$ are also analytic as functions of $w \in B_\delta(y)$. To show that $P(w)$ and $U(w)P(w)$ are analytic, we will use Morera's theorem.

Let Γ_1 be a closed, finite, straight line contour in $\mathbb{C} \setminus S$ and Γ_2 be a closed, finite straight line contour in $B_\delta(y)$. That is, the curve Γ_1 is described by $\Gamma_1(r) = re^{i\alpha} + a$ for $r \in [r_1, r_2]$ while $\Gamma_2(s) = se^{i\beta} + c$ for $s \in [s_1, s_2]$. Given any $\phi, \psi \in \mathcal{H}$, consider the integral

$$(10.3.11) \quad \int_{\Gamma_2} \int_{\Gamma_1} \langle (z - U(w))^{-1} \phi, \psi \rangle dz dw = \int_{s_1}^{s_2} \int_{r_1}^{r_2} \langle (re^{i\alpha} + a) - U(se^{i\beta} + c)^{-1} \phi, \psi \rangle dz dw.$$

Since $\|(z - U(w))^{-1}\|$ is a continuous function of z and w , for $(z, w) \in \mathbb{C} \setminus S \times B_\delta(y)$, it follows that $\|(z - U(w))^{-1}\| \leq M$ for all (z, w) in the compact set $\Gamma_1 \times \Gamma_2$. Hence, it follows that

$$(10.3.12) \quad \int_{\Gamma_2} \int_{\Gamma_1} |\langle (z - U(w))^{-1} \phi, \psi \rangle| dz dw \leq (r_2 - r_1)(s_2 - s_1)M \|\phi\| \|\psi\| < \infty.$$

By Fubini's theorem ([57], pg. 25), it follows that we can interchange the order of integration so that

$$(10.3.13) \quad \int_{\Gamma_2} \int_{\Gamma_1} \langle (z - U(w))^{-1} \phi, \psi \rangle dz dw = \int_{\Gamma_1} \int_{\Gamma_2} \langle (z - U(w))^{-1} \phi, \psi \rangle dw dz.$$

Since any finite length contour can be approximated arbitrarily well by a finite number of straight line contours, it follows that equation (10.3.13) holds for all finite, closed, smooth contours $\Gamma_1 \in \mathbb{C} \setminus V$ and $\Gamma_2 \in B_\delta(y)$. In particular, it follows that for any such a contour Γ_2 ,

$$(10.3.14) \quad \begin{aligned} \int_{\Gamma_2} P(w) dw &= \frac{1}{2\pi i} \int_{\Gamma_2} \int_{\Gamma} (z - U(w))^{-1} dz dw \\ &= \frac{1}{2\pi i} \int_{\Gamma} \int_{\Gamma_2} (z - U(w))^{-1} dw dz = 0, \end{aligned}$$

since for each $z \in \Gamma$, $(z - U(w))^{-1}$ is analytic in w . It then follows from Morera's theorem ([14], pg. 88), that $\langle P(w)\phi, \psi \rangle$ is an analytic function of w for any $\phi, \psi \in \mathcal{H}$. This proves that $P(w)$ is an analytic operator-valued function of w . Similar arguments show that $U(w)P(w)$ is analytic.

□

In summary, it can be concluded that both the functions $f(w) := \langle P(w)\phi_{\kappa(y)}, \phi_{\kappa(y)} \rangle$ and $g(w) := \langle U(w)P(w)\phi_{\kappa(y)}, \phi_{\kappa(y)} \rangle$ are analytic in $B_\delta(y)$, and that f does not vanish on $B_\delta(y)$. Since $y \in \mathbb{R}$ was arbitrary, we can immediately conclude:

*THEOREM 10.3.6. *The spectral function $\lambda = \mu^{-1}(\kappa)$ of a symmetric operator B with the sampling property is a monotonically strictly increasing homeomorphism, and is infinitely differentiable at any point $x \in \mathbb{R}$. Furthermore, at any point $x \in \mathbb{R}$, it has an analytic extension to a neighbourhood of x .*

10.3.0.9. *Remark.* The above theorem implies, in particular, that $\lambda'(x)$ can only vanish on a countable set of points with no finite accumulation point. The results of [36] further imply that $\lambda'(x)$ cannot vanish at any point at $x \in \mathbb{R}$. Otherwise, this would imply $\lambda(x) \in \sigma(B)$, contradicting the simplicity of B . Hence, $\lambda'(x) > 0$ for all $x \in \mathbb{R}$, and λ is a diffeomorphism.

10.3.0.10. *Remark.* The fact that any spectral function λ of a symmetric operator B with the sampling property is locally extendible to an analytic function in some neighbourhood of any real point, raises the question as to whether it can be extended to an analytic function on a much larger region, such as a half-plane or the whole complex plane.

Here is one way that one might be able to accomplish this. The operators

$$(10.3.15) \quad U(z) := V \oplus e^{i2\pi z} \langle \cdot, \phi_+ \rangle \phi_-$$

are extensions of the partially defined isometry $V = \mu(B)$ for any $z \in \mathbb{C}$. Let $B(z)$ be that extension of B which is the inverse Cayley transform of $U(z)$. The following theorem about the spectra of such arbitrary extensions of B is proven in ([3], pg. 149).

THEOREM 10.3.7. (*Lifschitz*) *The spectrum of any extension B' of a simple symmetric operator B with deficiency indices $(1, 1)$ consists of $\sigma_c(B)$ and eigenvalues of finite multiplicity. The set of eigenvalues lies entirely in either the upper half plane or lower half plane. Omitting the special case where the whole half plane belongs to $\sigma_p(B')$, the only possible limit points of $\sigma_p(B')$ belong to $\sigma_c(B)$.*

Ignoring the special case referred to in the above theorem, it follows that if B is regular and simple, with deficiency indices $(1, 1)$, then for any $z \in \mathbb{C}$, the spectrum of $B(z)$ is purely discrete with no finite limit point. Hence, if $x \in \mathbb{R}$, I think it could follow from standard holomorphic functional calculus techniques like those used in this chapter that there is a $\delta > 0$ such that $|z - x| < \delta$ implies that our definition of $\kappa(z)$ is actually the unique eigenvalue of $U(z)$ in the region $B_\delta(x)$. Again, using similar techniques to those used in this chapter, it may then be possible to extend $\kappa(z)$ to a function which is analytic in a neighbourhood $B_{\delta'}(z)$ of z . Iterating this procedure could yield a function analytic on the whole complex plane.

***THEOREM 10.3.8.** *Let B, \hat{B} be two symmetric operators with the sampling property on \mathcal{H} and $\hat{\mathcal{H}}$ respectively. Let $\lambda, \hat{\lambda}$ be their spectral functions. The operators B and \hat{B} are unitarily equivalent if and only if λ and $\hat{\lambda}$ are equivalent, i.e., if and only if there is a $c \in \mathbb{R}$ such that $\lambda(x) = \hat{\lambda}(x + c)$ for all $x \in \mathbb{R}$.*

PROOF. Sufficiency is easy. If $\lambda(x) = \hat{\lambda}(x + c)$ for all $x \in \mathbb{R}$ then it follows that $\sigma(\hat{B}(x + c)) = \sigma(B(x)) = \{\lambda(x + n)\}_{n \in \mathbb{Z}}$. In this case, for any fixed x one can define a unitary transformation that maps an eigenbasis of $B(x)$ onto an eigenbasis for $\hat{B}(x + c)$. Hence, $B(x)$ and $\hat{B}(x + c)$ are unitarily equivalent, so that B and \hat{B} are too.

Conversely, if \hat{B} is unitarily equivalent to B , let W be the unitary transformation such that $\hat{B} = WBW^{-1}$. Let $U(x) = V \oplus e^{\pm i2\pi x} \phi_- \langle \cdot, \phi_+ \rangle$ be the Cayley transform of $B(x)$ and $\hat{U}(x) = \hat{V} \oplus e^{\pm i2\pi x} \hat{\phi}_- \langle \cdot, \hat{\phi}_+ \rangle$ be the Cayley transform of $\hat{B}(x)$. The fact that W is unitary implies that $W\phi_\pm = e^{i2\pi\theta_\pm} \hat{\phi}_\pm$ for some $\theta_-, \theta_+ \in [0, 1)$, so that

$$(10.3.16) \quad WU(x)W^{-1} = \hat{V} \oplus e^{i2\pi(\pm x + \theta_- - \theta_+)} \hat{\phi}_- \langle \cdot, \hat{\phi}_+ \rangle = \hat{U}(\pm x + \theta_- - \theta_+).$$

The above implies that $\hat{\lambda}(x) = \lambda(\pm x + c)$ where $c = \theta_- - \theta_+$. However, since both λ and $\hat{\lambda}$ are strictly monotonically increasing, it must be that $\hat{\lambda}(x) = \lambda(x + c)$ for all $x \in \mathbb{R}$. \square

10.4. Symmetric restrictions of the multiplication operator and sampling theorems

Let τ denote the monotonically increasing function which is the inverse of the infinitely differentiable monotonically increasing diffeomorphism λ . Since, by Remark 10.3.0.9, $\lambda'(x) > 0$ for all $x \in \mathbb{R}$, it follows that $\tau'(\lambda(x)) = \frac{1}{\lambda'(x)} > 0$ for all $x \in \mathbb{R}$. Let $\delta_n(\alpha)$ denote a fixed, non-zero eigenvector of $B(\alpha)$ to eigenvalue $\lambda_n(\alpha)$ for $\alpha \in [0, 1)$. Recall that $\sigma(B(\alpha)) = \{\lambda(\alpha + n)\}_{n \in \mathbb{Z}}$, $\alpha \in [0, 1)$. For every $x \in \mathbb{R}$, let δ_x denote a fixed unit norm vector of B^* to eigenvalue x .

It follows that given any $\phi \in \mathcal{H}$,

$$\begin{aligned}
\langle \phi, \phi \rangle &= \sum_{n \in \mathbb{Z}} \langle \phi, \delta_{\lambda_n(\alpha)} \rangle \langle \delta_{\lambda_n(\alpha)}, \phi \rangle \\
&= \sum_{n \in \mathbb{Z}} \int_0^1 \langle \phi, \delta_{\lambda_n(\alpha)} \rangle \langle \delta_{\lambda_n(\alpha)}, \phi \rangle d\alpha \\
&= \int_{-\infty}^{\infty} \langle \phi, \delta_{\lambda(x)} \rangle \langle \delta_{\lambda(x)}, \phi \rangle dx \\
&= \int_{-\infty}^{\infty} \langle \phi, \delta_y \rangle \langle \delta_y, \phi \rangle \frac{1}{\lambda'(\tau(y))} dy \\
(10.4.1) \quad &= \int_{-\infty}^{\infty} \langle \phi, \delta_y \rangle \langle \delta_y, \phi \rangle \tau'(y) dy.
\end{aligned}$$

It follows from equation (10.4.1), that the map $U : \mathcal{H} \rightarrow L^2(\mathbb{R}, d\tau)$ defined by

$$(10.4.2) \quad U\phi(y) := \langle \phi, \delta_y \rangle \quad \text{a.e. } \tau,$$

is an isometric transformation that takes \mathcal{H} onto a subspace \mathcal{H}' of $L^2(\mathbb{R}; d\tau)$, and that the measure defined by τ is absolutely continuous with respect to Lebesgue measure.

CLAIM 10.4.1. *The image $\mathcal{H}' \subset L^2(\mathbb{R}, d\tau)$ of \mathcal{H} under U is a reproducing kernel Hilbert space, and UBU^{-1} acts as multiplication by the independent variable.*

Observe that since $y(x) = \lambda(x)$, δ_y is an eigenvector of B^* to eigenvalue $\lambda(x) = y$.

PROOF. To see that \mathcal{H}' is a reproducing kernel Hilbert space, let $K_x \in \mathcal{H}'$ be defined by $K_x = U\delta_x$. Then, for any $\psi = U\phi \in \mathcal{H}'$,

$$\begin{aligned}
\langle \psi, K_x \rangle_{\mathcal{H}'} &= \int_{-\infty}^{\infty} \psi(y) \overline{K_x(y)} \tau'(y) dy \\
&= \int_{-\infty}^{\infty} \langle \phi, \delta_y \rangle \langle \delta_y, \delta_x \rangle \tau'(y) dy \\
(10.4.3) \quad &= \langle \phi, \delta_x \rangle = \psi(x).
\end{aligned}$$

This shows that point evaluation at any $x \in \mathbb{R}$ is a bounded linear functional so that \mathcal{H}' is a reproducing kernel Hilbert subspace of $L^2(\mathbb{R}; d\tau)$.

Now consider the symmetric operator $M := UBU^{-1}$ with domain $\mathfrak{D}(M) = U\mathfrak{D}(B)$. It follows that given any $\psi = U\phi \in \mathfrak{D}(M) \subset \mathcal{H}'$ that

$$\begin{aligned}
M\psi(y) &= (UB\phi)(y) = \langle B\phi, \delta_y \rangle \\
(10.4.4) \quad &= \langle \phi, B^*\delta_y \rangle = y\langle \phi, \delta_y \rangle = y(U\phi)(y) = y\psi(y).
\end{aligned}$$

This proves the claim. □

Since each of the point evaluation vectors $K_x = U\delta_x \in \mathcal{H}'$ is non-zero, *Theorem 4.4.7 now shows that the following is a simple consequence of the above claim.

THEOREM 10.4.2. *If B is a regular, simple symmetric densely defined operator on \mathcal{H} with deficiency indices $(1, 1)$, then there is an isometry U from \mathcal{H} onto a reproducing kernel subspace \mathcal{H}' of $L^2(\mathbb{R}, d\tau)$. Here τ is a Lebesgue-Stieltjes measure which is absolutely continuous with respect to Lebesgue measure, and is determined uniquely by B . Furthermore, UBU^{-1} acts as multiplication by the independent variable in \mathcal{H}' , and \mathcal{H}' has the sampling property.*

This theorem has already been established by slightly different methods in [36] and [66]. Note, however, that the assertion that τ is absolutely continuous with respect to Lebesgue measure is new, and follows from my results of the previous section and results of [36] (see Remark 10.3.0.9).

PROOF. By Claim 10.4.1, it remains to show that \mathcal{H}' has the sampling property. First note that for any $x \in \mathbb{R}$ the point evaluation vector $K_x := U\delta_x \neq 0$ since δ_x was taken to be a non-zero eigenvector of B^* to eigenvalue x . Since UBU^{-1} is a symmetric multiplication operator with the sampling property in the reproducing kernel Hilbert space \mathcal{H}' , the statement of the theorem follows immediately from *Theorem 4.4.7. \square

10.4.1. The case where $\mathcal{H} \subset L^2(\mathbb{R}, d\sigma)$. Let us now return to the question posed at the beginning of this chapter. Namely, if \mathcal{M} is a symmetric multiplication operator with the sampling property in $\mathcal{H} \subset L^2(\mathbb{R}, d\sigma)$, is there an isometry from \mathcal{H} onto a RKHS $\mathcal{H}' \subset L^2(\mathbb{R}, d\sigma)$ such that U takes \mathcal{M} onto a symmetric multiplication operator \mathcal{M}' , and U acts as multiplication by a locally measurable function? In other words, can we multiply every element of \mathcal{H} by a suitable measurable function to obtain a reproducing kernel Hilbert space?

Assume that the measure σ is absolutely continuous with respect to Lebesgue measure so that $d\sigma(x) = \sigma'(x)dx$ for some locally measurable function $\sigma'(x)$. In this case, instead of the isometry of Claim 10.4.1, define the map U from $\mathcal{H} \subset L^2(\mathbb{R}, d\sigma)$ into another subspace $\mathcal{H}' \subset L^2(\mathbb{R}, d\sigma)$ by

$$(10.4.5) \quad (U\phi)(y) = \langle \phi, \delta_y \rangle \sqrt{\frac{\tau'(y)}{\sigma'(y)}}.$$

Clearly $\|U\phi\|_{L^2(\mathbb{R}, d\sigma)} = \|\phi\|_{L^2(\mathbb{R}, d\sigma)}$ since

$$(10.4.6) \quad \|U\phi\|^2 = \int_{-\infty}^{\infty} \langle \phi, \delta_y \rangle \langle \delta_y, \phi \rangle \frac{\tau'(y)}{\sigma'(y)} \sigma'(y) dy = \langle \phi, \phi \rangle = \|\phi\|^2.$$

As before, U maps \mathcal{M} onto a multiplication operator $\mathcal{M}' := U\mathcal{M}U^{-1}$, and \mathcal{H}' is a reproducing kernel Hilbert space with the sampling property. However, in this case, \mathcal{H} and \mathcal{H}' are subspaces of the same Hilbert space $\tilde{\mathcal{H}} := L^2(\mathbb{R}, d\sigma)$. It follows that if M is the self-adjoint multiplication operator on $L^2(\mathbb{R}, d\sigma)$ that $M|_{\mathfrak{D}(\mathcal{M})} = \mathcal{M}$ and $M|_{\mathfrak{D}(\mathcal{M}')} = \mathcal{M}'$. This means that $MU\phi = UM\phi$ for all $\phi \in \mathfrak{D}(\mathcal{M})$.

This raises the question as to whether the isometry U is extendible to a unitary \tilde{U} defined on all of $\tilde{\mathcal{H}} = L^2(\mathbb{R}, d\sigma)$, such that \tilde{U} commutes with M on the whole Hilbert space \tilde{H} . Since the multiplication operator M is multiplicity free, it would follow that $\tilde{U} = f(M)$ for some $f \in L^\infty(\mathbb{R})$ so that for any $\phi \in \tilde{\mathcal{H}}$, $\tilde{U}\phi(x) = f(x)\phi(x)$ a.e. This would show that the question raised at the beginning of this section has an affirmative answer.

I have not yet determined under what conditions, if any, such an extension \tilde{U} of U exists. Here is what I can say so far.

*CLAIM 10.4.3. *If the isometric linear map $U : \mathcal{H} \rightarrow \mathcal{H}'$ is extendible to a unitary operator \tilde{U} that commutes with $\mu(M)$ on $\tilde{\mathcal{H}}$, then $UP_{\mathcal{H}}\mu(M)\phi_+ = P_{\mathcal{H}'}\mu(M)U\phi_+$. Here, ϕ_+ is any fixed unit vector spanning the deficiency subspace $\mathfrak{D}_+ := (M + i)\mathfrak{D}(\mathcal{M})$.*

PROOF. The fact that $UM\phi = MU\phi$ for all ϕ in $\mathfrak{D}(\mathcal{M})$ implies that $\mu(M)U\phi = U\mu(M)\phi$ for all $\phi \in (M + i)\mathfrak{D}(\mathcal{M}) = \mathfrak{R}(\mathcal{M} + i)$. Now $\mathcal{H} := \mathfrak{R}(\mathcal{M} + i) \oplus \mathfrak{D}_+ = \mathfrak{R}(\mathcal{M} - i) \oplus \mathfrak{D}_-$ where $\mathfrak{D}_\pm = \mathbb{C}\{\phi_\pm\}$ and ϕ_\pm are fixed unit norm eigenvectors of \mathcal{M}^* to eigenvalues $\pm i$. Since $\mu(M)$ is unitary and U is an isometry from \mathcal{H} onto \mathcal{H}' , it follows that

$$(10.4.7) \quad \mu(M)\phi_+ = c_1\phi_- + c_2\psi$$

where $\psi \in \tilde{\mathcal{H}} \ominus \mathcal{H}$. Similarly, it follows that

$$(10.4.8) \quad \mu(M)U\phi_+ = a_1U\phi_- + a_2\varphi$$

where $\varphi \in \tilde{\mathcal{H}} \ominus \mathcal{H}'$.

If \tilde{U} is a unitary extension of U that commutes with $\mu(M)$, then

$$(10.4.9) \quad (\tilde{U}\mu(M) - \mu(M)\tilde{U})\phi_+ = (c_1 - a_1)U\phi_- + (c_2\tilde{U}\psi + a_2\varphi).$$

The first term in the above expression belongs to \mathcal{H}' while the second belongs to $\tilde{\mathcal{H}} \ominus \mathcal{H}'$. Since $\mu(M)$ and \tilde{U} are assumed to commute, it follows that $c_1 = a_1$, and hence that $UP_{\mathcal{H}}\mu(M)\phi_+ = P_{\mathcal{H}'}\mu(M)U\phi_+$.

□

I am currently trying to determine when the isometry U is extendible to a bounded operator that commutes with $\mu(M)$ on all of $\tilde{\mathcal{H}}$.

Invariant subspaces of Sturm-Liouville differential operators and de Branges spaces

11.1. Time-varying bandlimits, Sturm-Liouville operators and de Branges spaces

Recall that the Paley-Wiener space of Ω -bandlimited functions, $B(\Omega)$, is a de Branges space (see Chapter 9), and is also the invariant subspace of a differential operator. Namely, $B(\Omega)$ is the range of the projection $\chi_{[0, \Omega^2]}(D)$ acting on $L^2(\mathbb{R})$, where $D := \frac{d^2}{dt^2}$. This raises the following natural question: Given a more general self-adjoint Sturm-Liouville differential operator of the form

$$(11.1.1) \quad D_{pq} := -\frac{d}{dt} \left(p(t) \frac{d}{dt} \cdot \right) + q(t),$$

defined on some suitable dense domain in $L^2(\mathbb{R})$, are the invariant subspaces $B(D_{pq}, a) := \chi_{[0, a]}(D_{pq})$ also de Branges spaces?

This question, which is of significant mathematical interest in itself, may also be of interest for practical reasons. In the Introduction chapter, Chapter 1, we motivated the idea that it would be useful to find reproducing kernel Hilbert spaces with the sampling property, which are better suited to the reconstruction of ‘time-varying bandlimited’ functions. Consider a differential operator D_{pq} of the form (11.1.1). If $q = 0$ and $p = 1$, then $D_{10} = D$, and $B(D_{10}, \Omega) = B(\Omega^{1/2})$, is a usual space of bandlimited functions. Now suppose that $q \equiv 0$ and that $p(t)$ is slowly varying on \mathbb{R} , so that it is essentially constant on large intervals. In particular, suppose that $p(t) \simeq M > 0$ is essentially constant on a given interval I . Then, on this interval I , D_{pq} acts approximately as $-M \frac{d^2}{dt^2} = -MD$. Intuitively, one would expect that if M is large relative to the values of p in $\mathbb{R} \setminus I$, then if $f \in B(D_{pq}, \Omega)$, the high ‘frequencies’ of f on I will be suppressed. That is, one would expect that if $f \in B(D_{pq}, \Omega)$, then it must be relatively slowly varying on I , relative to other places on the real axis. With this intuitive argument, it seems reasonable that the space $B(D_{pq}, \Omega)$ could be a good formalization of the notion of a space of functions obeying a ‘time-varying bandlimit’. The function $p(t)$ would seem to describe how the ‘bandlimit’ changes with respect to t , since where $p(t)$ is large, elements of $B(D_{pq}, \Omega)$ should be slowly-varying, while where $p(t)$ is small, elements of $B(D_{pq}, \Omega)$ could contain relatively rapid oscillations.

If one allows $q(t)$ to be non-zero, then this could provide further control over both the relative amplitudes that elements of $B(D_{pq}, \Omega)$ have on different subintervals of \mathbb{R} , as well as more precise control over the time-varying frequency content of $B(D_{pq}, \Omega)$. Namely, suppose that $q(t) \geq 0$ is large on an interval $I \subset \mathbb{R}$, and let $\|\cdot\|_I := \|\chi_I \cdot\|$, $\langle \cdot, \cdot \rangle_I := \langle \chi_I \cdot, \chi_I \cdot \rangle$ where χ_I denotes the projector of $L^2(\mathbb{R})$ onto $L^2(I)$. If $\phi \in B(D_{pq}, \Omega)$, then,

$$(11.1.2) \quad \min_{t \in I} q(t) \|\phi\|_I^2 \leq \langle q\phi, \phi \rangle_I \leq \langle D_{pq}\phi, \phi \rangle_I \leq \|D_{pq}\phi\|_I \|\phi\|_I.$$

It follows that,

$$(11.1.3) \quad \|D_{pq}\phi\|_I \geq \|D_{pq}\phi\|_I \geq \min_{t \in I} q(t) \|\phi\|_I.$$

If $\min_{t \in I} q(t) = \Omega c$, where $c > 1$, then it must be that $\|\phi\|_I < \frac{\|\phi\|}{c}$. Otherwise, $\|D_{pq}\phi\| > (\Omega c) \frac{\|\phi\|}{c} = \Omega \|\phi\|$, which would contradict the fact that $\phi \in B(D_{pq}, \Omega)$. In conclusion, if q is large

everywhere on some subinterval I , then the amplitudes of any $\phi \in B(D_{pq}, \Omega)$ will be suppressed on I .

To see how q can provide further control over the time-varying ‘frequency’ content, observe that if both $p(t)$, and $q(t)$ are essentially constant on an interval $I \subset \mathbb{R}$, $p(t) \approx p_0$ and $q(t) \approx q_0$ on I , then $D_{pq} \approx p_0 D + q_0$ on I . The constant p_0 magnifies frequencies, while q_0 generates a frequency shift. If $e_w(x) := e^{iwx}$ is a plane wave on I , then, formally, $De_w = w^2 e_w$ and $D_{pq} e_w = (p_0 w^2 + q_0) e_w$. In this sense, e_w is approximately an eigenvector of D_{pq} on I . Intuitively, if $\phi \in B(D_{pq}, \Omega)$, one would expect that ϕ can be well-approximated on I by a linear combination of the plane waves e_w , whose ‘frequencies’ w are such that $p_0 w^2 + q_0 \in [0, \Omega]$, *i.e.*,

$$(11.1.4) \quad -\frac{q_0}{p_0} \leq w^2 \leq \frac{\Omega - q_0}{p_0}.$$

The above inequality suggests that choosing $q_0 = 0$ and $p_0 \gg 0$ large will suppress the high frequency content of $B(D_{pq}, \Omega)$ on I , while choosing $p_0 \geq 0$, and $q_0 < 0$ will restrict the ‘frequencies’ w to subintervals of length $\frac{\Omega}{p_0}$ centred at the frequencies $\pm \frac{\Omega - 2q_0}{p_0}$.

In summary, it appears that by suitably choosing the coefficient functions p and q , one can control the rate of variation, and the amplitudes of elements of the function space $B(D_{pq}, \Omega)$ in different subsets of the real line. Hence, by suitably choosing p and q , it seems reasonable that one could tailor the space $B(D_{pq}, \Omega)$ to be a better model than $B(\Omega)$ for certain ensembles of time-varying bandlimited functions like music signals.

In order that the spaces $B(D_{pq}, \Omega)$ be useful for sampling and reconstruction, however, it is necessary that they be, like the usual spaces $B(D, \Omega)$ of bandlimited functions, reproducing kernel Hilbert spaces with the sampling property. If $B(D_{pq}, \Omega)$ has these properties, one would expect that in order to reconstruct an element of $B(D_{pq}, \Omega)$, one would need to know its values on fewer sample points in regions where elements of $B(D_{pq}, \Omega)$ are relatively slowly-varying (where $p(t)$ is large), and on more sample points in regions where elements of $B(D_{pq}, \Omega)$ are relatively quickly varying (where $p(t)$ is relatively small). If, for example, $f(t)$ is a given music signal, which contains rapid oscillations or high frequencies in certain time intervals, and relatively slow oscillations or low frequencies in other time intervals, by modelling f as an element of a subspace $B(D_{pq}, \Omega)$ for a suitably chosen p, q , this could provide a more efficient method for sampling and reconstructing ensembles of such music signals.

Namely, as discussed in Chapter 1, suppose that f is rapidly oscillating on some time intervals and relatively slowly varying on others. If one wishes to model f as a bandlimited function, then in general, one needs to choose the bandlimit to be relatively large, since large frequencies are needed to resolve the sharp features and rapid oscillations that occur in the time periods of high activity. Since the minimum rate at which one needs to sample a bandlimited function in order to stably reconstruct it is proportional to the bandlimit, it follows that one needs a high density of sample points to reconstruct f if they wish to model f as a bandlimited function. The idea here, is that by modelling the music signal f as an element of a suitable $B(D_{pq}, \Omega)$, one may be able to achieve a more efficient method of sampling and reconstructing the music signal. That is, if one models f as an element of a suitable $B(D_{pq}, \Omega)$, one may need fewer sample points in order to reconstruct f to the same level of accuracy.

The above intuitive arguments provide motivation for attempting to show that the subspaces $B(D_{pq}, \Omega)$ are reproducing kernel Hilbert spaces with the sampling property. The strategy presented here for accomplishing this is to prove that, under suitable assumptions on p and q , the space $B(D_{pq}, \Omega)$ is a de Branges space. Recall that de Branges spaces are always reproducing kernel Hilbert spaces, and that de Branges spaces often have the sampling property.

11.1.0.1. *Relation to time-varying bandlimits as described in [36].* The formalization of the notion of a time varying bandlimit first appeared in [36]. In this paper, as discussed in the previous chapter, Prof. Kempf showed that if B is a symmetric operator with the sampling property, then there is an isometry U which takes the Hilbert space \mathcal{H} onto a reproducing kernel Hilbert space

\mathcal{H}' of functions on \mathbb{R} with the sampling property. Let $B(\alpha)$ denote the self-adjoint extension of B which is the inverse Cayley transform of $U(\alpha) := V \oplus e^{i2\pi\alpha}\phi_{-}\langle \cdot, \phi_{+} \rangle$ for fixed unit vectors $\phi_{\pm} \in \mathfrak{D}_{\pm}$. As discussed in the previous chapter, the eigenvalues of $B(\alpha)$ can be labelled as $(\lambda_n(\alpha))_{n \in \mathbb{M}}$ where $\lambda_n(\alpha) < \lambda_{n+1}(\alpha)$. Assume here that $\mathbb{M} = \mathbb{Z}$. Further recall that $\lambda_n(\alpha)$ is a differentiable function of α for $\alpha \in [0, 1)$, and that the eigenvalues $\lambda_n(\alpha)$ of all the self-adjoint extensions cover the real line exactly once. Kempf observed that for each $\phi \in \mathcal{H}$, if one defines the functions $\tilde{\phi}(t) := \langle \phi, \delta_t \rangle$ where δ_t is a normalized eigenvector to B^* to eigenvalue $t \in \mathbb{R}$, then one immediately obtains a sampling theorem for such functions. To see this, note that each set of vectors $\{\delta_{\lambda_n(\alpha)}\}_{n \in \mathbb{Z}}$ is a total orthonormal set for $\alpha \in [0, 1)$ so that

$$(11.1.5) \quad \tilde{\phi}(t) = \langle \phi, \delta_t \rangle = \sum_{n \in \mathbb{Z}} \langle \phi, \delta_{\lambda_n(\alpha)} \rangle \langle \delta_{\lambda_n(\alpha)}, \delta_t \rangle \frac{1}{\|\delta_{\lambda_n(\alpha)}\|^2} = \sum_{n \in \mathbb{Z}} \tilde{\phi}(\lambda_n(\alpha)) K(\lambda_n(\alpha), t),$$

where for each $\alpha \in [0, 1)$, the sampling kernel $K(y, t)$ is given by,

$$(11.1.6) \quad K(y, t) = \frac{\tilde{\delta}_y(t)}{\tilde{\delta}_y(y)}.$$

It was further shown that the values $(\lambda(\alpha + n))_{n \in \mathbb{Z}}$ and $(\lambda'(\alpha + n))_{n \in \mathbb{Z}}$ completely determine the function $\lambda(t)$ for all $t \in \mathbb{R}$.

Kempf used the eigenvalues $\lambda_n(\alpha)$ to define a time-varying bandlimit. Consider the canonical example where B is the symmetric multiplication operator in $B(\Omega)$. In this case, as was verified in Chapter 4, this operator has the sampling property and the eigenvalues of its self-adjoint extensions are the equidistantly spaced sets of points $\{t_n\}$ where $t_{n+1} - t_n = \frac{\pi}{\Omega}$. This shows that the bandlimit Ω is inversely proportional to the spacing of the eigenvalues of the $B(\alpha)$, $\Omega = \frac{t_{n+1} - t_n}{\pi}$. Using this fact, in [36], the time varying-bandwidth as a function of $t \in \mathbb{R}$ is defined by $w(t) = \frac{1}{\lambda(t+1) - \lambda(t)}$, where λ is the spectral function of the operator B . The idea is that, given a music signal $f(t)$, by choosing an appropriate spectral function $\lambda(t)$, and hence an appropriate time-varying bandlimit $w(t)$, one may be able to achieve a more efficient method of sampling and reconstructing ensembles \mathcal{F} of such music signals $f(t)$. Intuitively, one should choose the time-varying bandlimit to be large in regions where elements of \mathcal{F} are rapidly varying, and to be small in regions where elements of \mathcal{F} are slowly varying. Note that by the definition of the time varying bandlimit, the magnitude of $w(t)$ in a region is proportional to the density of sample points in that region.

The only problem with this approach is that there is no known efficient algorithm for determining what an appropriate choice of $w(t)$ is for a given music signal f . One wishes to choose $w(t)$ so that f is well-approximated by a time-varying bandlimited function with time-varying bandlimit $w(t)$. While it is clear that $w(t)$ should be relatively large in regions where f is rapidly varying, it is not clear what the optimal $w(t)$ should be.

Further note that there is a slight tradeoff here. Suppose that one wishes to sample and reconstruct a given music signal $f(t)$ by modelling it as a time-varying bandlimited function. If one wishes to do this, one needs to know the time-varying bandlimit, $w(t)$, as well as the values of f on the sample points which are essentially determined by $w(t)$. It follows that one needs to record the values of $w(t) = \frac{1}{\lambda(t+1) - \lambda(t)}$ and its derivative on the discrete set of points $\lambda(\alpha' + n)$, $\alpha' \in [0, 1)$ fixed, as well as the values of f on the discrete set of points t_n in order to reconstruct it. Note that if one knows $w(t_n)$ and $w'(t_n)$, then one essentially knows $\lambda(t_n)$ and $\lambda'(t_n)$ which determines $\lambda(t)$ (see [36]).

Now suppose that it can be shown that under suitable conditions on the coefficient functions p, q , the spaces $B(D_{pq}, \Omega)$ are in fact de Branges spaces in which the multiplication operator is densely defined. Then this would provide a concrete representation of the spaces of time-varying bandlimited functions defined in [36]. This concrete representation could provide insight into the problem of determining the appropriate time-varying bandlimit to efficiently sample and reconstruct a given music signal.

11.1.0.2. *Remark.* Suppose that, for appropriate p, q , the spaces $B(D_{pq}, \Omega)$ can be shown to be de Branges spaces with the sampling property. A further interesting question to ask would be whether any symmetric operator B with the sampling property, can always be seen as multiplication by the independent variable in a subspace $B(D_{pq}, \Omega)$ for appropriate p and q .

11.2. Showing $B(D_{pq}, \Omega)$ is a de Branges space

Let D_{pq} be a Sturm-Liouville differential operator,

$$(11.2.1) \quad D_{pq} := -\frac{d}{dx} \left(p(x) \frac{d}{dx} \right) + q(x),$$

defined on a dense domain $\mathfrak{D}(D_{pq}) \subset L^2(\mathbb{R})$. To ensure that D_{pq} is symmetric, choose p and q to be real-valued on \mathbb{R} . Choose $p(x) > 0$, assume that p, q are the restriction of entire functions to the real axis, and further assume that p, q are suitably chosen so that D_{pq} is self-adjoint on its domain.

The goal is to show that $B(D_{pq}, \Omega) := \chi_{[0, \Omega]}(D_{pq})L^2(\mathbb{R})$ is a de Branges space. To accomplish this we will try to show that $B(D_{pq}, \Omega)$ satisfies the axiomatic definition of a de Branges space given by Theorem 9.2.2. Recall that this theorem states that a Hilbert space of entire functions is isometrically equivalent to a de Branges space $\mathcal{H}(E)$ if and only if the following three axioms are satisfied:

- (A1) Point evaluation at every $z \in \mathbb{C} \setminus \mathbb{R}$ is a bounded linear functional.
- (A2) If $F \in \mathcal{H}$ then $F^* \in \mathcal{H}$ and $\|F\| = \|F^*\|$.
- (A3) If $F \in \mathcal{H}$ and $F(w) = 0$ for some $w \in \mathbb{C} \setminus \mathbb{R}$ then $G(z) := F(z) \frac{z-\bar{w}}{z-w} \in \mathcal{H}$ and $\|G\| = \|F\|$.

Since p and q are chosen to be entire functions, consider the differential equation $L[\phi] = \lambda\phi$ where

$$(11.2.2) \quad L[\phi] := -(p\phi')' + q\phi,$$

and $\phi'(z) := \frac{d}{dz}\phi(z)$, acting on functions $\phi(z)$ in the complex plane. The existence-uniqueness theorem for ordinary differential equations asserts that given any $z_0 \in \mathbb{C}$, there is a unique solution $u(z; \lambda)$ to this equation obeying the initial conditions $u(z; \lambda) = a$ and $p(z)u'(z; \lambda) = b$, for $a, b \in \mathbb{C}$. Furthermore, using the method of Picard iterates that is often employed to prove the existence-uniqueness theorem, it is not difficult to show that if the initial conditions are held fixed, this solution is an entire function in λ for fixed z , and for fixed λ is analytic as a function of z everywhere that both p and q are analytic. See for example ([30], Section 2.3), and ([52], pgs. 51-56).

Further recall that, as discussed in Subsection 7.1.2, one can define a generalized Fourier transform for the differential operator D_{pq} . Explicitly, Theorem 7.1.4, of that subsection shows the following. Choose any point $x_0 \in \mathbb{R}$, and let $\phi_i(x; \lambda)$, $1 \leq i \leq 2$ be the two solutions to the differential equation (11.2.2) which obey the initial conditions

$$(11.2.3) \quad \begin{pmatrix} \phi_1(x_0; \lambda) & p(x_0)\phi_1'(x_0; \lambda) \\ \phi_2(x_0; \lambda) & p(x_0)\phi_2'(x_0; \lambda) \end{pmatrix} = I.$$

Then there is a positive 2×2 matrix-valued measure, $\sigma = (\sigma_{j,k})$ on \mathbb{R} , such that the formulas,

$$(11.2.4) \quad \varphi_j(\lambda) := \int_{-\infty}^{\infty} f(x)\phi_j(x; \lambda)dx$$

and,

$$(11.2.5) \quad f(x) = \sum_{i,j=1}^2 \int_{-\infty}^{\infty} \varphi_i(\lambda)\phi_j(x; \lambda)d\sigma_{i,j}(\lambda),$$

define a unitary transformation U from $L^2(\mathbb{R})$ onto $L^2(\mathbb{R}^2, \sigma)$ such that $UD'U^{-1}$ acts as multiplication by λ in $L^2(\mathbb{R}^2, \sigma)$.

This generalized Fourier transform maps D_{pq} onto a multiplication operator on the space $L^2(\mathbb{R}^2, \sigma)$ of vector valued functions $\varphi(\lambda) = (\varphi_1(\lambda), \varphi_2(\lambda))$. It follows that $f \in B(D_{pq}, \Omega)$ if and only if its generalized Fourier transform $Uf = \varphi = (\varphi_1, \varphi_2)$ is such that the supports of the component functions φ_i are contained in $[0, \Omega]$, so that $\varphi \in L^2([0, \Omega]^2; \sigma)$. Hence, any f in $B(D_{pq}, \Omega)$ can be written as

$$(11.2.6) \quad f(x) = \sum_{i,j=1}^2 \int_0^\Omega \varphi_i(\lambda) \phi_j(x; \lambda) d\sigma_{i,j}(\lambda).$$

Since each $\phi_j(z; \lambda)$ is an entire function of $z \in \mathbb{C}$, a simple application of the theorems of Morera and Fubini proves that each $f \in B(D_{pq}, \Omega)$ is in fact an entire function. Furthermore, it follows from equation (11.2.6) that for any $f \in B(D_{pq}, \Omega)$ and $z \in \mathbb{C}$,

$$(11.2.7) \quad \begin{aligned} |f(z)| &= \left| \sum_{i,j=1}^2 \int_0^\Omega \varphi_i(\lambda) \phi_j(z; \lambda) d\sigma_{i,j}(\lambda) \right| \\ &\leq \|\varphi\| \|(\phi_1(\cdot, \lambda)\chi_{[0,\Omega]}, \phi_2(\cdot, \lambda)\chi_{[0,\Omega]})\| \\ &\leq \|f\| \max_{i=1,2; \lambda \in [0,\Omega]} |\phi_i(z, \lambda)| \sigma([0, \Omega] \times [0, \Omega]) = C\|f\|. \end{aligned}$$

The above was achieved by a straightforward application of the Cauchy-Schwarz inequality in $L^2(\mathbb{R}^2, \sigma)$ and the fact that U is an isometry so that $\|\varphi\| = \|f\|$. Equation (11.2.7) shows explicitly that point evaluation at any point $z \in \mathbb{C}$ is a bounded linear functional on $B(D_{pq}, \Omega)$. It follows that $B(D_{pq}, \Omega)$ is a Hilbert space of entire functions, and that axiom (A1) of Theorem 9.2.2 is satisfied. To show that $B(D_{pq}, \Omega)$ is a de Branges space, it remains to verify the other two axioms.

The second axiom, (A2) is even easier to establish. Recall that given an entire function $f(z)$, one defines $f^*(z) = \overline{f(\bar{z})}$. Note that if f is real valued on the real line, then $f(x) = f^*(x)$ for all $x \in \mathbb{R}$ so that $f \equiv f^*$. It is clear that the map C defined on $B(D_{pq}, \Omega)$ by $Cf = f^*$ is an anti-linear, idempotent, isometric linear map. Furthermore, the coefficient functions p, q which define D_{pq} are such that $p = p^*$ and $q = q^*$ and so it is clear that C commutes with D_{pq} . It is not difficult to see that this means that C commutes with any L^∞ function of D_{pq} so that in particular, $\chi_{[0,\Omega]}(D_{pq})C = C\chi_{[0,\Omega]}(D_{pq})$. Hence, if $f \in B(D_{pq}, \Omega)$, $Cf = C\chi_{[0,\Omega]}(D_{pq})f = \chi_{[0,\Omega]}(D_{pq})Cf$ so that $Cf = f^* \in \chi_{[0,\Omega]}(D_{pq})L^2(\mathbb{R}) = B(D_{pq}, \Omega)$ for any $f \in B(D_{pq}, \Omega)$. Since C is isometric, the axiom (A2) holds for $B(D_{pq}, \Omega)$.

Thus, to prove that $B(D_{pq}, \Omega)$ is a de Branges space, it remains to verify the third and final axiom (A3). This is more difficult and is a current research project of mine.

The strategy I have been pursuing is the following. To prove that $B(D_{pq}, \Omega)$ satisfies axiom (A3), it needs to be shown that if $f \in B(D_{pq}, \Omega)$ and $f(w) = 0$ where $w \in \mathbb{C} \setminus \mathbb{R}$, then $g(z) := \frac{z-\bar{w}}{z-w}f(z) \in B(D_{pq}, \Omega)$ and has the same norm as f . Since the norm on $B(D_{pq}, \Omega)$ is just the usual L^2 norm and $\left| \frac{z-\bar{w}}{z-w} \right| = 1$, the fact that $\|g\| = \|f\|$ is immediate. It remains to verify that $g \in B(D_{pq}, \Omega)$. To show this, it is sufficient to show that the generalized Fourier transform Ug of g , defined by equation (11.2.4) is such that its component functions $(Ug)_i$, $1 \leq i \leq 2$ vanish outside of the interval $[0, \Omega]$.

By definition,

$$(11.2.8) \quad \frac{1}{2\pi i} (Ug)_j(\lambda) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{(z-\bar{w})f(x)\phi_j(x; \lambda)}{z-w} dz =: \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\Phi(z)}{z-w} dz.$$

This looks like Cauchy's formula! Intuitively, if it can be shown that the assumptions that $f \in B(D, \Omega)$ and $|\lambda| > \Omega$ imply that the entire function $\Phi(z) := (z-\bar{w})f(z)\phi_j(z; \lambda)$ decays

sufficiently quickly as one moves away from the real axis in either the upper half plane or the lower half plane, one would expect (11.2.8) to satisfy:

$$(11.2.9) \quad \frac{1}{2\pi i}(Ug)_j(\lambda) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\Phi(z)}{z-w} dz = \Phi(w) = (w-\bar{w})f(w)\phi_j(x; \lambda) = 0,$$

since we assume $f(w) = 0$. This would imply that $g(z) := \frac{z-\bar{w}}{z-w}f(z) \in B(D_{pq}, \Omega)$, verifying the third and final axiom (A3).

Substituting the expression (11.2.5) for f in terms of its generalized Fourier transform φ into equation (11.2.8) shows that

$$(11.2.10) \quad (Ug)_j(\lambda) = \sum_{i,k=1}^2 \int_{-\infty}^{\infty} \varphi_i(\lambda') \left(\int_{-\infty}^{\infty} \frac{(x-\bar{w})\phi_k(x; \lambda')\phi_j(x; \lambda)}{x-w} dx \right) \sigma_{i,j}(\lambda).$$

It follows that in order to verify axiom (A3), it should be sufficient to show that the assumptions $|\lambda'| \leq \Omega$ and $|\lambda| > \Omega$ imply that the product $\Phi(z) = \phi_k(z; \lambda')\phi_j(z; \lambda)$ of the solutions to the differential equation (11.2.2) decay sufficiently quickly as one moves away from the real axis in either the upper half plane or the lower half plane.

More precisely, recall the definitions of bounded type, and mean type for functions analytic in the upper half plane or lower half plane. These definitions were introduced in Section 9.2. The following theorem shows when Cauchy's formula for the upper half plane or lower half plane is valid ([8], pg. 32):

THEOREM 11.2.1. *Let $f(z)$ be a function which is analytic, of bounded type, and non-positive mean type in the upper half plane, and which has a continuous extension to the real axis. If $f \in L^2(\mathbb{R})$ and $z = x + iy$ then,*

$$(11.2.11) \quad f(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t)}{t-z} dt,$$

for $y > 0$ and,

$$(11.2.12) \quad 0 = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t)}{t-z} dt,$$

for $y < 0$.

It follows that in order to verify axiom (A3), it is sufficient to show that the assumptions $|\lambda'| \leq \Omega$ and $|\lambda| > \Omega$ imply that the product $\Phi(z) = \phi_k(z; \lambda')\phi_j(z; \lambda)$ of the solutions $\phi_k(z; \lambda')$, and $\phi_j(z; \lambda)$ to the differential equation (11.2.2) are of bounded type and non-positive mean type in either the upper half plane or the lower half plane.

11.2.1. Example of $B(\Omega)$. For example, consider the case where $p = 1$ and $q = 0$, so that $B(D_{pq}, \Omega^2) = B(\Omega)$ is the usual space of bandlimited functions.

In this case the approach of the previous subsection can be applied to prove that $B(\Omega)$ is a de Branges space.

If $f \in B(\Omega)$ and $v \in \mathbb{C} \setminus \mathbb{R}$ is such that $f(v) = 0$, let $g(z) := \frac{z-\bar{v}}{z-v}f(z)$. To show that $g \in B(\Omega)$, consider its Fourier transform $G(w)$,

$$(11.2.13) \quad \begin{aligned} G(w) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iw x} g(x) dx \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iw x} \frac{x-\bar{v}}{x-v} \int_{-\Omega}^{\Omega} F(w') e^{iw' x} dw' dx \\ &= \int_{-\Omega}^{\Omega} F(w') \left(\int_{-\infty}^{\infty} \frac{(x-\bar{v})e^{ix(w'-w)}}{x-v} dx \right) dw'. \end{aligned}$$

Suppose that $|w| > \Omega$. Then if $w > \Omega$, it is easy to verify that $(z - w)e^{i(w' - w)z}$ is of bounded type and non-positive mean type in the upper half plane for all $|w'| < \Omega$ while if $w < \Omega$ then $(z - w)e^{i(w' - w)z}$ is of bounded type and non-positive mean type in the lower half plane. In either case, Cauchy's formula for the half plane shows that equation (11.2.13) vanishes whenever $|w| > \Omega$. This proves that $G \in L^2[-\Omega, \Omega]$, and hence that $g \in B(\Omega)$. The other axioms, as before, are straightforward to verify. It follows that $B(\Omega)$ is a de Branges space.

11.3. Outlook

I am currently considering Sturm-Liouville operators D_{pq} with $p = 1$, and $q \geq 0$ chosen such that q is meromorphic on \mathbb{C} , and such that $q(z) \rightarrow 0$ as $|z| \rightarrow \infty$ at least as fast as $\frac{1}{|z|^2}$. For example $q(z) = \frac{1}{z^2 + 1}$ is such a function. Under these assumptions D_{pq} is a positive operator. The reason for choosing q to approach 0 for large $|z|$, is that in this case the solutions to the complex ordinary differential equation,

$$(11.3.1) \quad -(\phi')' + q\phi = \lambda\phi,$$

behave asymptotically, for large $|z|$, like the solutions to the usual sine differential equation $-(\phi')' = \lambda\phi$ associated with the usual differential operator $D = -\frac{d^2}{dx^2}$. In this case, it may be possible to exploit the fact that there are solutions to equation (11.3.1) which behave asymptotically like $e^{\pm i\sqrt{\lambda}z}$ to show that $B(D_{pq}, \Omega)$ satisfies the third of the axioms (A1)-(A3) which characterize de Branges spaces.

Note that since q is not entire, the solutions to the equation (11.3.1), and hence elements of $B(D, \Omega)$ are not necessarily entire, and can have singularities at any point where q has a pole.

Despite this fact, it is not hard to see that if the axioms (A1)-(A3) of Theorem 9.2.2 are satisfied, then the multiplication operator in $B(D_{pq}, \Omega)$ is a simple symmetric regular operator with deficiency indices $(1, 1)$. This can be shown using the same arguments as in Section 9.2. Thus, to prove that $B(D, \Omega)$ has the sampling property, it would remain to show that the operator of multiplication by the independent variable, M , is densely defined. Since the elements of $B(D_{pq}, \Omega)$ in this case still have nice analytic properties, it seems possible that the methods and results of de Branges may be generalizable to this case of spaces of meromorphic functions satisfying (A1)-(A3).

In this subsection, since we choose D_{pq} to be positive, its generalized Fourier transform will be such that the matrix-valued measure σ vanishes on subsets of the negative real axis, and hence we need only consider solutions to the differential equation (11.3.1) to positive eigenvalues $\lambda^2 \geq 0$:

$$(11.3.2) \quad -(\phi')' + q\phi = \lambda^2\phi.$$

Define a new variable $w = \lambda z$. Then if $\Phi(w) := \phi(z(w)) = \phi(\frac{w}{\lambda})$, equation (11.3.2) becomes

$$(11.3.3) \quad \Phi''(w) + \left(1 - \frac{q(\frac{w}{\lambda})}{\lambda^2}\right)\Phi(w) = 0.$$

Since $q(w)$ vanishes as $|w| \rightarrow \infty$ like $|w|^{-2}$, the following theorem applies ([30], pg. 181).

THEOREM 11.3.1. *Let $\phi(z)$ be a non-zero solution of the differential equation:*

$$(11.3.4) \quad \phi''(z) + (1 - Q(z))\phi(z) = 0.$$

Suppose that $Q(z)$ is a meromorphic function of z whose poles lie in a horizontal strip $S := \{z = z + iy \mid |y| \leq B\}$, and which has a zero at infinity of at least the second order. Then there is a solution $\phi_0(z)$ of the sine equation,

$$(11.3.5) \quad \phi''(z) + \phi(z) = 0.$$

such that for all $z \in \mathbb{C} \setminus S$,

$$(11.3.6) \quad |\phi(x + iy) - \phi_0(x + iy)| \leq M(y) \left(\exp \left(\int_x^\infty |Q(t + iy)| dt \right) - 1 \right).$$

This above theorem, and another of ([30], pg. 184), can be applied to show that solutions of (11.3.2) behave like $e^{i\lambda z}$ for large $|z|$. There is yet another obstacle, however, to proving axiom (A3) for the case under consideration in this subsection. Namely, even though it appears that the solutions to (11.3.2) have the required decay properties to apply Cauchy's formula in the upper half plane or lower half plane, these solutions are not necessarily analytic in these regions, and could have singularities at the poles of $q(z)$. Hence, even if Cauchy's formula does hold, it could pick up residues. I am currently trying to determine whether the third axiom (A3) can hold in these circumstances.

Semigroups of contractions and symmetric operators

12.1. A question about invariant linear manifolds for self-adjoint operators

The previous chapter dealt with trying to prove that certain subspaces of $L^2(\mathbb{R})$ are reproducing kernel Hilbert spaces with the sampling property. Recall that *Theorem 4.4.7 showed that a sufficient condition for a reproducing kernel Hilbert space \mathcal{H} of functions on \mathbb{R} with positive definite reproducing kernel to have the sampling property is that the operator M of multiplication by the independent variable be a densely defined symmetric, regular, simple operator with deficiency indices $(1, 1)$ in \mathcal{H} . This raises a more general operator theoretic question:

Let \mathcal{H} be a separable Hilbert space and A be a closed self-adjoint operator defined on a dense domain $\mathfrak{D}(A) \subset \mathcal{H}$. Let $S \subset \mathcal{H}$ be a subspace which is the range of a projection P . If A is fixed, what conditions does P need to satisfy in order that there be a dense domain $\mathfrak{D}(B) \subset S$ so that $\mathfrak{D}(B) \subset \mathfrak{D}(A)$, and $B := A|_{\mathfrak{D}(B)}$ is a densely defined symmetric operator in S ? A more restrictive, but useful condition to require is that $A\mathfrak{D}(B) \subset \mathfrak{D}(B)$, in which case one could call $\mathfrak{D}(B)$ an invariant linear manifold, or a non-closed invariant subspace for the operator A .

The answer to this question for the case where M is the operator of multiplication by the independent variable in $L^2(\mathbb{R}, d\mu)$ would help determine when a given subspace is a reproducing kernel Hilbert space with the sampling property.

This chapter contains my first attempt at answering this question. The canonical example of a self-adjoint operator which has a symmetric restriction to a dense domain in a proper subspace is the example of the derivative operator $D := i \frac{d}{dx}$ in $L^2(\mathbb{R})$ which has a densely defined symmetric restriction to $L^2[a, b]$ where $[a, b]$ is any subinterval of \mathbb{R} . Initially, I observed that the compression of the strongly continuous one parameter unitary group $U(t) = e^{itD}$ to $L^2[a, b]$ is a semigroup of partial isometries. With this example as motivation, I proved that whenever the unitary group generated by a self-adjoint operator is such that its compression to a given subspace is a semigroup of partial isometries, then the self-adjoint operator has a symmetric restriction to a dense domain in that subspace. The proof of this result is provided in this chapter. However, as we will see, the above condition for a self-adjoint operator to have a symmetric restriction, while sufficient, is not necessary, and is in fact very restrictive.

12.2. Introduction

In this section we investigate the relationship between semi-groups of contractions and symmetric operators.

Given a separable Hilbert space \mathcal{H} , let A be a closed, self-adjoint operator defined on a dense domain $\mathfrak{D}(A) \subset \mathcal{H}$. Consider the operator valued function $U(t) := e^{itA}$ for $t \in \mathbb{R}$. This function is an example of a strongly continuous one-parameter unitary group ([57], pg. 265):

DEFINITION 12.2.1. *A strongly continuous one-parameter unitary group is a strongly continuous map $U : \mathbb{R} \rightarrow B(\mathcal{H})$ such that $U(0) = I$ and $U(t)U(s) = U(t + s)$ for all $s, t \in \mathbb{R}$.*

Using the functional calculus, it is straightforward to verify that $U(t) = e^{itA}$ satisfies the above definition. The fact that every strongly continuous one-parameter unitary group arises in this way is the content of Stone's theorem ([57], pg. 268):

THEOREM 12.2.2. *(Stone) Let $U(t)$ be a strongly continuous one-parameter unitary group on a Hilbert space \mathcal{H} . There is a closed self-adjoint operator A , densely defined in \mathcal{H} , such that $U(t) = e^{itA}$.*

In the proof of Stone's theorem, the self-adjoint operator A which satisfies the statement of the theorem is constructed by taking the strong derivative of $U(t)$ at 0 on a suitable dense domain of vectors $\mathfrak{D}(A)$. Namely, it is shown that there is a dense linear manifold of vectors $\mathfrak{D}(A)$ such that for $\phi \in \mathfrak{D}(A)$, $\lim_{t \rightarrow 0} \frac{U(t) - I}{t} \phi$ exists in \mathcal{H} . The self adjoint operator A defined by $A\phi = -i \lim_{t \rightarrow 0} \frac{U(t) - I}{t} \phi$ for all $\phi \in \mathfrak{D}(A)$ then satisfies the statement of Stone's theorem.

The definition of a strongly continuous one-parameter semi-group of operators is an immediate generalization of Definition 12.2.1

DEFINITION 12.2.3. *A strongly continuous one-parameter operator semigroup is a strongly continuous map $V : [0, \infty) \rightarrow B(\mathcal{H})$ which obeys $V(0) = I$ and which has the semi-group property: $V(t)V(s) = V(t+s)$ for all $t, s \geq 0$.*

Recall that a bounded operator $V \in B(\mathcal{H})$ is called a partial isometry if it is an isometry from the orthogonal complement of its kernel onto its range. It is not hard to verify that if V is a partial isometry, then V^*V projects onto the orthogonal complement of $\mathfrak{ker}(V)$ while VV^* projections onto the range, $\mathfrak{R}(V)$. It is further not difficult to prove the following: (see, for example [32], pg. 400)

PROPOSITION 12.2.4. *The following are equivalent:*

- (a) V is a partial isometry
- (b) V^*V is a projection
- (c) VV^* is a projection

In this chapter, a result which generalizes Stone's theorem will be proven. Namely, it will be shown that if $V : \mathbb{R} \rightarrow B(\mathcal{H})$ is a strongly continuous map such that $V(-t) = V^*(t)$ for all $t \geq 0$ and such that $V(t)$ is a strongly continuous semi-group of partial isometries for $t \geq 0$ (and hence also for $t \leq 0$), then one can take the strong derivative of V at 0 on a suitable dense domain of vectors $\mathfrak{D}(B)$ to obtain a closed symmetric operator B . This operator B will be self-adjoint if and only if $V(t)$ is in fact a unitary group.

This result is motivated by the following example:

12.3. A motivating example

Let $\mathcal{H} := L^2(\mathbb{R})$ and let D' and M' be the essentially self-adjoint derivative and multiplication operators on the domain $C_0^\infty(\mathbb{R})$ defined by $D'\phi = i\phi'$ and $M'\phi(x) = x\phi(x)$ a.e for all $\phi \in C_0^\infty(\mathbb{R})$. Let D and M be the self-adjoint closures of D' and M' respectively. It is not hard to check that the unitary group $U(t) := e^{itD}$ generates right translations. That is, if $f \in C_0^\infty(\mathbb{R}) \subset L^2(\mathbb{R})$, using the fact that f has a Taylor series with a non-zero radius of convergence about any point $x \in \mathbb{R}$, it is not hard to verify that $U(t)f = g$ where $g(x) = f(x-t)$. That is, g is the translation of f to the right by t . From this fact it follows that for any $f \in L^2(\mathbb{R})$, $U(t)f(x) = f(x-t)$ a.e.

Now let $P := \chi_{[a,b]}(M)$ be the projector onto $L^2[a,b]$. Let $V : \mathbb{R} \rightarrow B(\mathcal{H})$ be the strongly continuous operator-valued map defined by $V(t) := PU(t)P$. Then, for any $f \in L^2(\mathbb{R})$, and any $s, t \geq 0$ it follows that

$$(12.3.1) \quad V(s)V(t)f(x) = \chi_{[a,b]}(x)\chi_{[a,b]}(x-s)f(x-s-t)\chi_{[a,b]}(x-s-t) \quad \text{a.e,}$$

while,

$$(12.3.2) \quad V(s+t)f(x) = \chi_{[a,b]}(x)f(x-t-s)\chi_{[a,b]}(x-s-t).$$

Using that $\chi_{[a,b]}(x-s) = \chi_{[a+s,b+s]}(x)$ and the fact that $s, t \geq 0$, leads directly to the conclusion that $V(s)V(t) = V(s+t)$ so that V has the semi-group property for $s, t \geq 0$. Since $V(-t) = PU(-t)P = PU^*(t)P = V^*(t)$, it follows that V has the semi-group property for $t \leq 0$ as well. It is straightforward to visualize the action of $V(t)$ for $t \geq 0$ as translation to the right truncated to the interval $[a, b]$. It is further easy to verify that $V(t)V^*(t) = \chi_{[a,b-t]}(M)$ and that $V^*(t)V(t) = \chi_{[a+t,b]}(M)$. Hence, for each $t \in \mathbb{R}$, $V(t)$ is in fact a partial isometry and $V(t)$ is a strongly continuous semigroup of partial isometries. This particular semigroup $V(t)$ for $t \geq 0$ is called a truncated shift for obvious reasons.

Further observe that if f is an arbitrary element of $C_0^\infty(a, b)$, then

$$(12.3.3) \quad \frac{1}{i} \lim_{t \rightarrow 0^+} \frac{V(t) - I}{t} f = \frac{1}{i} \lim_{t \rightarrow 0^-} \frac{V(t) - I}{t} f = if' = Df.$$

This shows that the strong, two-sided derivative of $V(t)$ exists for any $f \in C_0^\infty(a, b)$ which is a dense linear manifold in $L^2[a, b]$. This shows that the self-adjoint derivative operator D on $L^2(\mathbb{R})$ has a restriction to a dense domain in $L^2[a, b]$ for any interval $[a, b]$. It is clear that this restriction is a symmetric operator.

Let A and $U(t) = e^{itA}$ denote a closed self-adjoint operator and its unitary group. The above example raises the following natural question: If $S \subset \mathcal{H}$ is a closed subspace with self-adjoint projector P , and $V(t) := PU(t)P$ is a semi-group of partial isometries, does this imply that A has a symmetric restriction to a dense domain in S ? The results of this chapter will show that the answer is yes.

12.4. Semi-groups of contractions

Before proceeding further, it will be convenient to first give a brief review of some basic results on operator semi-groups. For brevity, a one-parameter strongly continuous semigroup of operators will just be called an operator semigroup.

Given an operator semi-group $V(t)$, $t \geq 0$, in analogy with Stone's theorem, one defines the **generator** T of the semigroup by taking the strong one-sided derivative of $V(t)$ at zero. Namely, one defines $\mathfrak{D}(T) := \{\phi \in \mathcal{H} \mid \lim_{t \rightarrow 0^+} \frac{V(t) - I}{t} \phi \exists\}$, and then T is defined by $T\phi = \lim_{t \rightarrow 0^+} \frac{V(t) - I}{t} \phi$ for all $\phi \in \mathfrak{D}(T)$.

PROPOSITION 12.4.1. *Let $V(t)$ be an operator semigroup with generator T on $\mathfrak{D}(T)$ where T and $\mathfrak{D}(T)$ are defined as above. Then, $\mathfrak{D}(T)$ is dense in \mathcal{H} , and T is closed.*

For a proof of this proposition, see for example ([58], pgs. 236-237). Given the generator T of the semigroup $V(t)$, one says that T generates $V(t)$ and that by definition, $e^{tT} := V(t)$.

Recall that an operator W is called a contraction if $\|W\| \leq 1$. An operator semigroup $V(t)$ will be called a contraction semigroup if $V(t)$ is a contraction for all $t \geq 0$. Consider the formal Laplace transform equation,

$$(12.4.1) \quad \frac{1}{\lambda - T} = - \int_0^\infty e^{-\lambda s} e^{sT} ds,$$

where $e^{sT} := V(s)$. Since $\|V(t)\| \leq 1$; it follows that for all $t \geq 0$, and for any λ such that $\text{Re}(\lambda) > 0$, the integral on the right converges to a bounded operator R_λ of norm at most $\frac{1}{\text{Re}(\lambda)}$. One can in fact show that R_λ is a left and right inverse for $\lambda - T$ so that $R_\lambda = (\lambda - T)^{-1}$. This proves that the spectrum of T is contained in the closed left half plane.

Furthermore, if a closed operator T satisfies the above conditions, then the following theorem of Hille and Yosida shows that T is the generator of a contraction semigroup.

THEOREM 12.4.2. (Hille-Yosida) *A closed, densely defined linear operator T on a Banach space X generates a contraction semi-group if and only if: (a) $(0, \infty) \subset \rho(T)$ and (b) $\|(T - \lambda)^{-1}\| \leq \frac{1}{\lambda}$ for all $\lambda \in (0, \infty)$.*

This theorem is directly analogous to Stone's theorem, Theorem 12.2.2, for unitary groups.

12.4.0.1. *Remark.* Observe that if $V(t)$ is a contraction semi-group, so is the adjoint semigroup $V^*(t)$. Furthermore, if T is the generator of $V(t)$ and T' is the generator of $V^*(t)$ then $T^* = T'$. To see this, observe that if $\psi \in \mathfrak{D}(T')$, then given any $\phi \in \mathfrak{D}(T)$,

$$(12.4.2) \quad \langle T\phi, \psi \rangle = \lim_{s \rightarrow 0^+} \left\langle \frac{V(s) - I}{s} \phi, \psi \right\rangle = \lim_{s \rightarrow 0^+} \left\langle \phi, \frac{V^*(s) - I}{s} \psi \right\rangle = \langle \phi, T'\psi \rangle.$$

By definition of the adjoint of T , this shows that $\psi \in \mathfrak{D}(T^*)$ and that $T^*\psi = T'\psi$. Hence, $T' \subset T^*$. To show that $T' = T^*$ it remains to show that $\mathfrak{D}(T') = \mathfrak{D}(T^*)$. Since both T and T' generate contraction semi-groups, their spectra are both contained in the closed left half plane. Choose, for example, $\lambda = 1$ in the right half plane. Since $\mathfrak{D}(T') \subset \mathfrak{D}(T^*)$, it follows that $\Re(T' - 1) \subset \Re(T^* - 1)$. But $\Re(T' - 1) = \mathcal{H}$ so that $\Re(T' - 1) = \Re(T^* - 1) = \mathcal{H}$. Suppose that there is a $\phi \in \mathfrak{D}(T^*)$ such that $\phi \notin \mathfrak{D}(T')$. If $\psi = (T^* - 1)\phi$, then there is a $\varphi \in \mathfrak{D}(T')$ such that $\psi = (T' - 1)\varphi = (T^* - 1)\varphi$. This shows that $(T^* - 1)(\phi - \varphi) = 0$, i.e., $\phi - \varphi$ is an eigenvector to T^* with eigenvalue 1. This implies that $\phi - \varphi$ is perpendicular to the range of $T - 1$ which is all of \mathcal{H} since the right half plane does not belong to the spectrum of T . This contradiction proves that $\mathfrak{D}(T^*) = \mathfrak{D}(T')$ so that $T' = T^*$.

12.4.0.2. *The co-generator of a semigroup.* Given the generator T of a semi-group $V(t)$, it will be convenient to define the operator $W := (T + 1)(T - 1)^{-1}$. Since $1 \notin \sigma(T)$, this operator is defined on all of \mathcal{H} . Following [24], this operator will be called the co-generator of the semigroup $V(t)$ for reasons that will soon be apparent. Note that if $V(t) = U(t) = e^{itA}$ is the restriction of a one-parameter unitary group to $t \geq 0$, where A is a self-adjoint operator, then iA is the generator of this semi-group, and the co-generator $W = (iA + 1)(iA - 1)^{-1} = (A - i)(A + i)^{-1} = \mu(A)$ is the Cayley transform of A .

More generally, if T generates a contraction semigroup, the operator W is itself a contraction. To see this, first note that if T generates a contraction semi-group, then $\operatorname{Re}(\langle T\phi, \phi \rangle) \leq 0$ for all $\phi \in \mathfrak{D}(T)$. This follows from the definition of T since

$$(12.4.3) \quad \operatorname{Re}(\langle (V(s) - I)\phi, \phi \rangle) = \operatorname{Re}(\langle V(s)\phi, \phi \rangle) - \|\phi\|^2 \leq \|V(s)\|\|\phi\|^2 - \|\phi\|^2 \leq 0.$$

This in turn implies that for any $\phi \in \mathcal{H}$, $\|(T + I)\phi\|^2 - \|(T - I)\phi\|^2 = 4\operatorname{Re}(\langle T\phi, \phi \rangle) \leq 0$. For any $\phi \in \mathcal{H}$, if $\psi := (T - I)^{-1}\phi$, it follows that

$$(12.4.4) \quad \|W\phi\| = \|(T + I)(T - I)^{-1}\phi\| = \|(T + I)\psi\| \leq \|(T - I)\psi\| = \|\phi\|.$$

This shows that W is a contraction. Since $(W - I) = (T + I)(T - I)^{-1} - (T - I)(T - I)^{-1} = 2(T - I)^{-1}$, and $W + I = 2T(T - I)^{-1}$, it also follows that $(W + I)(W - I)^{-1} = T$. This shows that T and hence $V(s)$ are uniquely determined by the co-generator W . Since W uniquely determines $V(s)$, one writes $V(s) := e^{sT} = e^{s\frac{W+I}{W-I}} = e_s(W)$ where $e_s(z) := e^{s\frac{z+1}{z-1}}$. In fact, since $(W - I)^{-1} = \frac{1}{2}(T - I)$, it follows that $1 \notin \sigma_p(W)$. For such a contraction one can define $e_s(W)$ using the Hardy functional calculus ([24], pgs. 117-118, 141) for contractions. The Hardy functional calculus for a contraction V is an algebra homomorphism from a certain V -dependent subalgebra of $H^\infty(\mathbb{D})$ into $B(\mathcal{H})$, with several nice properties. Using this functional calculus one can show that if a contraction W does not have 1 as an eigenvalue, then $V(s) := e_s(W)$ is in fact a contraction semigroup, and if W is the co-generator of a contraction semigroup $\tilde{V}(s)$ then $\tilde{V}(s) = V(s)$. In particular, one can reformulate the statement of the Hille-Yosida theorem, Theorem 12.4.2, in terms of co-generators ([24], pg. 142):

THEOREM 12.4.3. *Given a contraction V on \mathcal{H} , V is the co-generator of a contraction semi-group $V(s)$ if and only if 1 is not an eigenvalue of V . In this case $V(s)$ and V determine each other by the formulas $V(s) = e_s(V)$ and $V = \lim_{s \rightarrow 0^+} \varphi_s(V(s))$, where $\varphi_s(z) := \frac{z-1+s}{z-1-s}$.*

12.4.0.3. *Unitary dilations of contractions.* Let A be a closed self-adjoint operator and $U(t) = e^{itA}$ be its unitary group. The question raised at the end of the previous subsection asked whether the condition that $V(s) = PU(s)P$ defines a semi-group of partial isometries implies that A has a symmetric restriction B to a dense domain in $S := P\mathcal{H}$.

It is worth remarking here that if $U = (A - i)(A + i)^{-1} = \mu(A)$ is the Cayley transform of A , and if $W = (T + 1)(T - 1)^{-1} = (B' - i)(B' + i)^{-1} = \mu(B')$ is the Cayley transform of the generator T of $V(t)$, where $B' := -iT$, then the statement that $PU(t)P = V(t)$ is a semigroup for $t \geq 0$ is, in fact, equivalent to the statement that $PU^kP = W^k$ for all $k \in \mathbb{N}$. Taking adjoints, this also means that $\mu_{-i}(A) = \mu(A)^* = U^* = (A + i)(A - i)^{-1}$ is such that $P(U^*)^kP = PU^{-k}P = (W^*)^k$ for all $k \in \mathbb{N}$ where $W^* = ((B')^* + i)((B')^* - i)^{-1}$ and $(B')^* = (-iT)^* = iT^*$. These facts are immediate consequences of the dilation theory and the Hardy functional calculus developed for contractions in [24].

In general, if C is a bounded operator on \mathcal{H} and D is a bounded operator on a larger Hilbert space $\mathcal{K} \supset \mathcal{H}$, then D is called a **dilation** of C if $P_{\mathcal{H}}D^k|_{\mathcal{H}} = C^k$ for all $k \in \mathbb{N}$. Here, $P_{\mathcal{H}}$ denotes the projection of \mathcal{K} onto \mathcal{H} . A celebrated theorem of [24], states that any contraction V on a Hilbert space \mathcal{H} has a unitary dilation:

THEOREM 12.4.4. (Nagy-Foias) *For every contraction V on a Hilbert space \mathcal{H} , there exists a unitary dilation U on a space $\mathcal{K} \supset \mathcal{H}$ which is minimal in the sense that $\mathcal{K} = \bigvee_{n \in \mathbb{Z}} U^n \mathcal{H}$. This minimal unitary dilation is determined up to isomorphism, and hence can be called the minimal unitary dilation of V .*

In the language of [24], if $U(t) = e^{itA}$ is a strongly continuous one-parameter unitary group and $PU(t)P = V(t)$, where P projects onto a subspace $S \subset \mathcal{H}$, defines a semi-group for $t \geq 0$, then $U(t)$ is called a unitary dilation of the semi-group $V(t)$. Using Theorem 12.4.3, it is not hard to show that if $U(t) = e^{itA}$ is a unitary dilation of a contraction semi-group $V(t)$, then $\mu(A)$ is a unitary dilation of the co-generator W of $V(t)$. Conversely, the Hardy functional calculus states that if W is a contraction and U is its minimal unitary dilation, then for all f in a certain W -dependent subalgebra of $H^\infty(\mathbb{D})$, $f(W) = Pf(U)P$. Using this Hardy functional calculus one can show that if U is a unitary dilation of W where W is the co-generator of a contraction semi-group, then the unitary group $U(s) = e_s(U)$ is a unitary dilation of the contraction semi-group $V(s) = e_s(W)$ ([24], pg. 146).

12.5. A symmetric restriction

Suppose $U(t) = e^{itA}$ is the unitary group of a self-adjoint operator A , that P is a projection, $P\mathcal{H} = S$, and that $V(t) := PU(t)P$ is a contraction semi-group for $t \geq 0$. Recall that the generator T of $V(t)$ is defined by $T\phi = \lim_{s \rightarrow 0^+} \frac{V(s) - I}{s} \phi$ on the domain of all ϕ for which this limit exists.

In this situation, $V(t)$ is also defined for $t \leq 0$ and $V(-t) = V^*(t)$, so that $V(-t)$ is also a contraction semigroup for $t \geq 0$. The generator of this semigroup is T^* and is defined by $T^*\phi = \lim_{s \rightarrow 0^+} \frac{V(-s) - I}{s} \phi = \lim_{s \rightarrow 0^-} \frac{I - V(s)}{s} \phi$ on the set of all ϕ for which this limit exists.

One can use $V(t)$ to define a symmetric operator as follows. Let $B' := -iT$ and let

$$(12.5.1) \quad \mathfrak{D}(B) := \left\{ \phi \in \mathcal{H} \mid \lim_{s \rightarrow 0} \frac{V(s) - I}{s} \phi \exists \right\} \subset \mathfrak{D}(B') \cap \mathfrak{D}((B')^*),$$

and define $B\phi := \frac{1}{i} \lim_{s \rightarrow 0} \frac{V(s) - I}{s} \phi$ for all $\phi \in \mathfrak{D}(B)$. That is, B is defined as $-i$ times the strong two-sided derivative of $V(t)$ at 0. Then B is symmetric since $\mathfrak{D}(B)$ is, by definition, the set of all

$\phi \in \mathfrak{D}(B') \cap \mathfrak{D}((B')^*)$ such that $B'\phi = (B')^*\phi$. Explicitly, if $\phi \in \mathfrak{D}(B)$ this means that

$$\begin{aligned}
B'\phi &= -i \lim_{s \rightarrow 0^+} \frac{V(s) - I}{s} \phi = B\phi \\
&= -i \lim_{s \rightarrow 0^-} \frac{V(s) - I}{s} \phi = -i \lim_{t \rightarrow 0^+} \frac{V(-t) - I}{-t} \phi \\
(12.5.2) \quad &= i \lim_{t \rightarrow 0^+} \frac{V^*(t) - I}{t} \phi = iT^*\phi = (B')^*\phi.
\end{aligned}$$

Hence for all $\phi, \psi \in \mathfrak{D}(B)$,

$$(12.5.3) \quad \langle B\phi, \psi \rangle = \langle B'\phi, \psi \rangle = \langle \phi, (B')^*\psi \rangle = \langle \phi, B\psi \rangle.$$

In general, $\mathfrak{D}(B)$ will not be dense and may even be empty. The major result of this chapter will be the proof of the fact that if $V(t)$ is a semi-group of partial isometries, then $\mathfrak{D}(B)$ is dense so that B is a densely defined symmetric operator.

*THEOREM 12.5.1. *The operator B is closed and is the restriction of the self-adjoint operator A to $\mathfrak{D}(B) \subset S$, $B = A|_{\mathfrak{D}(B)}$.*

PROOF. The fact that B is closed follows easily from the fact that B' and $(B')^*$ are closed. If $(\phi_n)_{n \in \mathbb{N}} \subset \mathfrak{D}(B)$ is such that $\phi_n \rightarrow \phi$ and $B\phi_n \rightarrow \psi$. Then since $(\phi_n)_{n \in \mathbb{N}} \subset \mathfrak{D}(B') \cap \mathfrak{D}((B')^*)$ and $B\phi_n = B'\phi_n = (B')^*\phi_n$ for all $n \in \mathbb{N}$, the closedness of B' and $(B')^*$ imply that $\phi \in \mathfrak{D}(B') \cap \mathfrak{D}((B')^*)$ and that $B'\phi = \psi = (B')^*\phi$. By definition of B and $\mathfrak{D}(B)$ it follows that $\phi \in \mathfrak{D}(B)$ and that $B\phi = \psi$ so that B is indeed a closed operator.

Let W denote the co-generator of the semi-group $V(t) = Pe^{itA}P$ for $t \geq 0$, and recall that $B' := -iT$, where T is the generator of $V(t)$. Then, as observed previously, $B \subset B'$ and $W = (T+1)(T-1)^{-1} = (iB'+1)(iB'-1)^{-1} = (B'-i)(B'+i)^{-1} = \mu(B')$, where $\mu(z) = \frac{z-i}{z+i}$. Since $B \subset B'$ is symmetric, this means that the partially defined transformation $\mu(B)$, the Cayley transform of B , is an isometry from $\mathfrak{R}(B+i)$ to $\mathfrak{R}(B-i)$. Since $U(t)$ is a dilation of $V(t)$ for $t \geq 0$, it further follows that $\mu(A)$ is a unitary dilation of the contraction $\mu(B')$. Hence, $P_S(A-i)(A+i)^{-1}|_S = (B'-i)(B'+i)^{-1}|_S$ and, in particular, $P_S(A-i)(A+i)^{-1}|_{\mathfrak{R}(B+i)} = (B-i)(B+i)^{-1}|_{\mathfrak{R}(B+i)}$. Here, P_S is the projector onto $S \subset \mathcal{H}$. Since $\mu(A) = (A-i)(A+i)^{-1}$ is unitary and $\mu(B)$ is an isometry from $\mathfrak{R}(B+i)$ to $\mathfrak{R}(B-i)$, it follows that for any $\psi \in \mathfrak{R}(B+i)$ of unit norm, that $1 = \|\mu(B)\psi\|^2 = \|P\mu(A)\psi\|^2$, while $1 = \|\mu(A)\psi\|^2 = \|P\mu(A)\psi\|^2 + \|(I-P)\mu(A)\psi\|^2$. This proves that $P\mu(A)|_{\mathfrak{R}(B+i)} = \mu(A)|_{\mathfrak{R}(B+i)} = \mu(B)$.

Hence, $\mathfrak{D}(B) = (I - \mu(B))\mathfrak{R}(B+i) = (I - \mu(A))\mathfrak{R}(B+i) = 2i(A+i)^{-1}\mathfrak{R}(B+i) \subset \mathfrak{D}(A)$. Furthermore, since $A = i(I + \mu(A))(I - \mu(A))^{-1}$, and $B = i(I + \mu(B))(I - \mu(B))^{-1}$, it follows that given any $\phi \in \mathfrak{D}(B)$, that $\phi = (I - \mu(B))\psi = (I - \mu(A))\psi$ where $\psi \in \mathfrak{R}(B+i)$ and,

$$\begin{aligned}
A\phi &= i(I + \mu(A))\psi = i(I + \mu(B))\psi \\
(12.5.4) \quad &= i(I + \mu(B))(I - \mu(B))^{-1}(I - \mu(B))\psi = B\phi.
\end{aligned}$$

□

Recall that the deficiency indices of a symmetric operator B are defined as $n_{\pm} := \dim(\mathfrak{R}(B \pm i)^{\perp})$.

*THEOREM 12.5.2. *The deficiency indices n_{\pm} of B are equal to the number of vectors in $\mathfrak{D}(B')$ and $\mathfrak{D}((B')^*)$ respectively which are linearly independent modulo $\mathfrak{D}(B)$.*

Observe that the above theorem says that n_+ is equal to the number of vectors in \mathcal{H} for which the positive one-sided strong derivative of $V(t)$ at 0 exists but for which the negative one-sided derivative does not exist, or exists but is not equal to the positive one-sided derivative.

PROOF. Since $B' = -iT$ and $\sigma(T)$ is contained in the closed left half plane, it follows that $\sigma(B') \subset \overline{UHP}$ and that $\sigma((B')^*) \subset \overline{LHP}$. In particular, this shows that $\mathfrak{R}(B' + i) = \mathcal{H}$. By definition $n_+ = \dim(\mathfrak{R}(B + i)^\perp)$. Since B is closed, $\mathfrak{R}(B + i)$ is a closed subspace. Since $(B' + i)^{-1}$ is a bijection from \mathcal{H} onto $\mathfrak{D}(B')$ and $B \subset B'$ it follows that $(B' + i)^{-1}\mathfrak{R}(B + i)^\perp$ is the linear manifold of vectors in $\mathfrak{D}(B')$ which are linearly independent modulo $\mathfrak{D}(B)$. This proves the claim for n_+ . The same logic using $(B')^* \supset B$ proves the analogous statement about n_- . \square

The symmetric operator B defined in this subsection will be called the symmetric operator associated with the contraction semigroup $V(t)$.

12.6. Semigroups of partial isometries

In this subsection, let $V(t)$ denote a strongly continuous one-parameter semigroup of partial isometries. Recall that a partial isometry is a contraction which is an isometry from the orthogonal complement of its kernel onto its range. See Proposition 12.2.4 for a characterization of partial isometries.

As in the previous section, let $V(-t) := V^*(t)$, and define $\mathfrak{D}(B)$ as the set of all vectors ϕ for which $\lim_{s \rightarrow 0} \frac{V(s) - I}{s} \phi$ exists.

The main result of this section is the following:

*PROPOSITION 12.6.1. *If $V : \mathbb{R} \rightarrow B(\mathcal{H})$ is a strongly continuous, one-parameter semigroup of partial isometries, then $\mathfrak{D}(B)$ is dense in \mathcal{H} .*

In particular, combined with *Theorem 12.5.1 of the previous subsection, this will imply:

*THEOREM 12.6.2. *Let $S \subset \mathcal{H}$ be a closed subspace with projector P . Let A be the closed self-adjoint generator of a one-parameter, strongly continuous unitary group $U(t)$ on \mathcal{H} . If $V(t) = PU(t)P$ is a strongly continuous one-parameter semigroup of partial isometries, then A has a symmetric restriction B to a dense domain $\mathfrak{D}(B) \subset S$.*

12.6.0.4. *Basic properties of semigroups of partial isometries.* Let $P(t) := V(t)V^*(t)$ and $Q(t) = V^*(t)V(t)$ for all $t \geq 0$ so that $P(t)$ is the projection onto $\mathfrak{R}(V(t)) = \mathfrak{R}\ker(V^*(t))^\perp$ and $Q(t)$ is the projection onto $\mathfrak{R}\ker(V(t))^\perp = \mathfrak{R}(V^*(t))$.

LEMMA 12.6.3. *If $0 \leq s \leq t$ then $Q(s) \geq Q(t)$ and $P(s) \geq P(t)$*

PROOF. If $t = s$ then the claim holds trivially. if $s < t$ then

$$(12.6.1) \quad Q(s)Q(t) = Q(s)V^*(t)V(t) = Q(s)V^*(s)V^*(t-s)V(t).$$

Since $Q(s)$ is the projector onto $\mathfrak{R}(V^*(s))$, $Q(s)V^*(s) = V^*(s)$. It follows that

$$(12.6.2) \quad Q(s)Q(t) = V^*(s)V^*(t-s)V(t) = Q(t).$$

Furthermore, since $Q(s)$ projects onto $\mathfrak{R}(V^*(s)) = \mathfrak{R}\ker(V(s))^\perp$, $V(s)Q(s) = V(s)$, and,

$$(12.6.3) \quad Q(t)Q(s) = V^*(t)V(t-s)V(s)Q(s) = V^*(t)V(t-s)V(s) = V^*(t)V(t) = Q(t).$$

This proves that $Q(t) \leq Q(s)$.

Similarly,

$$(12.6.4) \quad P(s)P(t) = P(s)V(s)V(t-s)V^*(t) = V(s)V(t-s)V^*(t) = P(t),$$

since $P(s)$ projects onto $\mathfrak{R}(V(s))$ so that $P(s)V(s) = V(s)$. Finally,

$$(12.6.5) \quad P(t)P(s) = V(t)V^*(t-s)V^*(s)P(s) = P(t),$$

since $V^*(s)P(s) = V^*(s)$. The relations $P(s)P(t) = P(t)P(s) = P(t)$ then imply that $P(t) \leq P(s)$. \square

LEMMA 12.6.4. For any $0 \leq s \leq t$, $Q(t-s)V(s)Q(t) = V(s)Q(t)$ and $P(t-s)V^*(s)P(t) = V^*(s)P(t)$

In other words, this claim says that if $\phi \in \mathfrak{Rer}(V(t))^\perp$ then $V(s)\phi \in \mathfrak{Rer}(V(t-s))^\perp$ and if $\phi \in \mathfrak{Rer}(V^*(t))^\perp$ then $V^*(s)\phi \in \mathfrak{Rer}(V^*(t-s))^\perp$, whenever $0 \leq s \leq t$.

PROOF. $V(t)$ is an isometry from $\mathfrak{Rer}(V(t))^\perp$ onto $\mathfrak{R}(V(t))$. Choose any $\phi \in \mathfrak{Rer}(V(t))^\perp$, then $\|\phi\| = \|V(t)\phi\|$. Now if $V(s)\phi \notin \mathfrak{Rer}(V(t-s))^\perp$ then

$$(12.6.6) \quad \|\phi\| > \|V(t-s)(V(s)\phi)\| = \|V(t)\phi\|,$$

which is a contradiction.

Similarly if $\phi \in \mathfrak{Rer}(V^*(t))^\perp$ then $\|\phi\| = \|V^*(t)\phi\|$. If $V^*(s)\phi \notin \mathfrak{Rer}(V^*(t-s))^\perp$ then

$$(12.6.7) \quad \|\phi\| > \|V^*(t-s)V^*(s)\phi\| = \|V^*(t)\phi\|.$$

Again, this is a contradiction. □

12.6.0.5. *Remark.* By Lemma 12.6.4 and by Lemma 12.6.3, it follows that for any $0 \leq s \leq t$, if $Q(t)\phi = \phi$ then $Q(y)V(s)\phi = V(s)\phi$ for any $0 \leq y \leq t-s$, and similarly that if $P(t)\phi = \phi$ then $P(y)V^*(s)\phi = V^*(s)\phi$ for any $0 \leq y \leq t-s$.

LEMMA 12.6.5. $P(t)$ and $Q(t)$ are strongly continuous for $t \geq 0$

PROOF. Given any $t \geq 0$ we need to show that $Q(t)$ is strongly continuous at t . Choose an arbitrary $\phi \in \mathcal{H}$. Then,

$$(12.6.8) \quad \begin{aligned} \|(Q(s) - Q(t))\phi\| &= \|(V^*(s)V(s) - V^*(t)V(t))\phi\| \\ &\leq \|(V^*(s)V(s) - V^*(s)V(t))\phi\| + \|(V^*(s)V(t) - V^*(t)V(t))\phi\| \\ &\leq \|(V(s) - V(t))\phi\| + \|(V^*(s) - V^*(t))\psi\| \end{aligned}$$

where in the last line above $\psi := V(t)\phi$ is fixed since t is fixed and we used that $\|V^*(s)\| \leq 1$ for all $s \geq 0$. The last line above vanishes as $s \rightarrow t$ by the strong continuity of $V(t)$ and $V^*(t)$ for $t \geq 0$.

The proof that $P(t)$ is strongly continuous is directly analogous. It is provided here for completeness. Given $\phi \in \mathcal{H}$ and $t \geq 0$ fixed, consider

$$(12.6.9) \quad \begin{aligned} \|(P(s) - P(t))\phi\| &= \|(V(s)V^*(s) - V(t)V^*(t))\phi\| \\ &= \|(V(s)V^*(s) - V(s)V^*(t) + V(s)V^*(t) - V(t)V^*(t))\phi\| \\ &\leq \|(V^*(s) - V^*(t))\phi\| + \|(V(s) - V(t))\psi\| \end{aligned}$$

where here $\psi := V^*(t)\phi$ is fixed. This last line vanishes in the limit as $s \rightarrow t$ by the strong continuity of $V(t)$ and $V^*(t)$, establishing the strong continuity of $P(t)$. □

The following result of von Neumann can be found, for example, in ([57], pg. 219).

THEOREM 12.6.6. (*von Neumann's alternating projection theorem*) Let P and Q be projection operators. Then the projection onto $\mathfrak{R}(P) \cap \mathfrak{R}(Q)$ is given by

$$s - \lim_{n \rightarrow \infty} (PQ)^n$$

Here $s - \lim$ denotes the limit in the strong operator topology.

*LEMMA 12.6.7. The projector $R(t)$ onto $\mathfrak{R}(Q(t)) \cap \mathfrak{R}(P(t))$ is strongly continuous for $t \geq 0$ and $R(0) = I$.

PROOF. Since $V(0) = V^*(0) = I$, it follows that $P(0) = I = Q(0)$ and that $R(0) = I$. Now by von Neumann's alternating projection theorem

$$(12.6.10) \quad R(t) = s - \lim_{n \rightarrow \infty} (Q(t)P(t))^n$$

If $n = 2$ then

$$(12.6.11) \quad \begin{aligned} (Q(t)P(t))^2 &= Q(t)P(t)Q(t)P(t) \\ &= V^*(t)V(t)V(t)V^*(t)V^*(t)V(t)V(t)V^*(t) \\ &= V^*(t)V(2t)V^*(2t)V(t)P(t) = V^*(t)P(2t)V(t)P(t) \end{aligned}$$

Similarly if $n = 3$ then

$$(12.6.12) \quad \begin{aligned} (Q(t)P(t))^3 &= V^*(t)V(t)V(t)V^*(t)V^*(t)V(t)V(t)V^*(t)V^*(t)V(t)V(t)V^*(t) \\ &= V^*(t)P(2t)P(2t)V(t)P(t) = V^*(t)P(2t)V(t)P(t) = (Q(t)P(t))^2. \end{aligned}$$

By induction, it follows that $(Q(t)P(t))^n = V^*(t)P(2t)V(t)P(t)$ for all $n \in \mathbb{N}$. The alternating projection theorem then implies that $R(t) = V^*(t)P(2t)V(t)P(t)$. Since $R(t)$ is a projection it must be self adjoint so that $R(t) = P(t)V^*(t)P(2t)V(t)$. Since $R(t)^2 = R(t)$ we get that

$$(12.6.13) \quad \begin{aligned} R(t) = R(t)^2 &= P(t)V^*(t)P(2t)V(t)V^*(t)P(2t)V(t)P(t) \\ &= P(t)V^*(t)P(2t)P(t)P(2t)V(t)P(t) \\ &= P(t)V^*(t)P(2t)V(t)P(t). \end{aligned}$$

The last line above used the fact that, by Lemma 12.6.3, $P(t)P(2t) = P(t)$. This form of $R(t)$ is more obviously self-adjoint.

Verifying that $R(t) = V^*(t)P(2t)V(t)P(t)$ is strongly continuous for $t \geq 0$ is straightforward, just as in the proof of Lemma 12.6.5. Given any $\phi \in \mathcal{H}$ and a fixed $t \geq 0$,

$$(12.6.14) \quad \begin{aligned} \|(R(s) - R(t))\phi\| &= \|(V^*(s)P(2s)V(s)P(s) - V^*(t)P(2t)V(t)P(t))\phi\| \\ &\leq \|V^*(s)P(2s)V(s)(P(s) - P(t))\phi\| + \|V^*(s)P(2s)(V(s) - V(t))P(t)\phi\| \\ &\quad + \|V^*(s)(P(2s) - P(2t))V(t)P(t)\phi\| + \|(V^*(s) - V^*(t))P(2t)V(t)P(t)\phi\| \\ &\leq \|(P(s) - P(t))\phi\| + \|(V(s) - V(t))P(t)\phi\| \\ &\quad + \|(P(2s) - P(2t))V(t)P(t)\phi\| + \|(V^*(s) - V^*(t))P(2t)V(t)P(t)\phi\|. \end{aligned}$$

By the strong continuity of $V(t)$, $V^*(t)$ and $P(t)$ this vanishes in the limit as $s \rightarrow t$, establishing the strong continuity of $R(t)$. □

12.6.0.6. *Remark.* By Lemma 12.6.3, $P(t) \leq P(s)$ and $Q(t) \leq Q(s)$ for $0 \leq s \leq t$. Hence, if $\phi = R(t)\phi$ then $\phi = P(t)\phi = Q(t)\phi$ and Lemma 12.6.3 implies that $\phi = P(s)\phi = Q(s)\phi$ for any $0 \leq s \leq t$. This proves that if $\phi = R(t)\phi$ then, $\phi = R(s)\phi$ for all $0 \leq s \leq t$. That is, $R(t) \leq R(s)$ for $0 \leq s \leq t$.

12.6.0.7. *The main proposition, *Proposition 12.6.1.* Consider the following set in \mathcal{H} :

$$(12.6.15) \quad \mathcal{D}' := \{\phi \in \mathcal{H} \mid \exists \epsilon_\phi > 0 \text{ s.t. } R(s)\phi = \phi \ \forall 0 \leq s \leq \epsilon_\phi\}.$$

Given any $\phi \in \mathcal{H}$ and $\epsilon > 0$, since $R(t)$ is strongly continuous and $R(0) = I$, there is a $\delta > 0$ such that $0 \leq s \leq \delta$ implies that $\|R(s)\phi - \phi\| < \epsilon$. Let $\psi = R(\delta)\phi$ so that, by Remark 12.6.0.6, $R(y)\psi = \psi$ for all $0 \leq y \leq \delta$. This shows that $\psi \in \mathcal{D}'$ and that \mathcal{D}' is dense in \mathcal{H} .

LEMMA 12.6.8. *If $\phi \in \mathcal{D}'$, $0 \leq t \leq \epsilon_\phi/2$, and $0 \leq s \leq \epsilon_\phi/2$, then $Q(s)V(t)\phi = V(t)\phi$ and $P(s)V^*(t)\phi = V^*(t)\phi$.*

PROOF. This follows from earlier results. First, by definition, $\phi = R(s)\phi$ for all $0 \leq s \leq \epsilon_\phi$. By Lemma 12.6.4, it follows that $Q(\epsilon_\phi - t)V(t)\phi = V(t)\phi$ for all $t \in [0, \epsilon_\phi/2]$. By Lemma 12.6.3 since $\epsilon_\phi - t \geq \epsilon_\phi/2$ for all $t \in [0, \epsilon_\phi/2]$, and $V(t)\phi \in \mathfrak{R}(Q(\epsilon_\phi - t))$ for all $t \in [0, \epsilon_\phi/2]$, it follows that $V(t)\phi \in \mathfrak{R}(Q(\epsilon_\phi/2))$ for all t in this range. Again, by Lemma 12.6.3, since $Q(\epsilon_\phi/2) \leq Q(s)$ for any $s \in [0, \epsilon_\phi/2]$, one can conclude that if $0 \leq s \leq \epsilon_\phi/2$, then $V(t)\phi \in \mathfrak{R}(Q(s))$, so that $Q(s)V(t)\phi = V(t)\phi$.

Similarly, since $\phi = P(\epsilon_\phi)\phi$, Lemma 12.6.4 implies that $V^*(t)\phi = P(\epsilon_\phi - t)V^*(t)\phi$ for all $t \in [0, \epsilon_\phi/2]$. Since $\epsilon_\phi - t \geq \epsilon_\phi/2$ for all $t \in [0, \epsilon_\phi/2]$, Lemma 12.6.3 then implies that $P(s)V^*(t)\phi = V^*(t)\phi$ for all $s \in [0, \epsilon_\phi/2]$. Finally, again by Lemma 12.6.3 $P(\epsilon_\phi/2) \leq P(s)$ for all s in this range, so that $P(s)V^*(t)\phi = V^*(t)\phi$ for all s in this range. This allows one to conclude that $P(s)V^*(t)\phi = V^*(t)\phi$ for all $s, t \in [0, \epsilon_\phi/2]$. This completes the proof. \square

Now given any $\phi \in \mathcal{D}'$, consider the element

$$(12.6.16) \quad \phi_f := \int_{-\infty}^{\infty} f(t)V(t)\phi dt = \int_{-\frac{\epsilon_\phi}{2}}^{\frac{\epsilon_\phi}{2}} f(t)V(t)\phi dt,$$

where f is any function in $C_0^\infty(-\frac{\epsilon_\phi}{2}, \frac{\epsilon_\phi}{2})$. Let \mathcal{D} denote the linear manifold of all finite linear combinations of such elements.

LEMMA 12.6.9. \mathcal{D} is dense in \mathcal{H} .

This proof just uses a simple resolution of the identity, and is very similar to a lemma used in the proof of Stone's theorem ([57], pg. 266).

PROOF. Let $j(x)$ be any element of $C_0^\infty(-1, 1)$ such that $j(x) \geq 0 \forall x \in [-1, 1]$ and $\int_{-\infty}^{\infty} j(x)dx = \int_{-1}^1 j(x)dx = 1$. Let $j_\epsilon(x) := \epsilon^{-1}j(\epsilon^{-1}x)$. Then $j_\epsilon(x)$ has support only on $[-\epsilon, \epsilon]$. Given any $\phi \in \mathcal{D}'$, the vector

$$(12.6.17) \quad \phi_{j_\epsilon} = \int_{-\infty}^{\infty} j_\epsilon(t)V(t)\phi dt,$$

belongs to \mathcal{D} for any $0 < \epsilon \leq \epsilon_\phi$. Now,

$$(12.6.18) \quad \begin{aligned} \|\phi_{j_\epsilon} - \phi\| &= \left\| \int_{-\infty}^{\infty} j_\epsilon(t) (V(t)\phi - \phi) dt \right\| \\ &\leq \int_{-\infty}^{\infty} |j_\epsilon(t)| \| (V(t)\phi - \phi) \| dt \\ &\leq \sup_{t \in [-\epsilon, \epsilon]} \|V(t)\phi - \phi\| \int_{-\infty}^{\infty} j_\epsilon(t) dt = \sup_{t \in [-\epsilon, \epsilon]} \|V(t)\phi - \phi\|. \end{aligned}$$

This vanishes in the limit as $\epsilon \rightarrow 0$ by the strong continuity of $V(t)$, and the fact that $V(0) = I$. This proves that \mathcal{D} is dense in \mathcal{D}' , and therefore, is dense in \mathcal{H} . \square

All of the tools needed for the proof of *Proposition 12.6.1 have finally been assembled. In the following proof of this proposition it will be shown that the set $\mathcal{D} \subset \mathfrak{D}(B)$ so that $\mathfrak{D}(B)$ is indeed dense.

PROOF. (of Proposition 12.6.1)

Let \mathcal{D} and \mathcal{D}' be the sets of vectors constructed above. Given any $\phi_f \in \mathcal{D}$, assume $s > 0$, and consider

$$(12.6.19) \quad \begin{aligned} \frac{V(s) - I}{s} \phi_f &= \int_0^\infty f(t) \left(\frac{V(s+t) - V(t)}{s} \right) \phi dt + \frac{V(s) - I}{s} \int_0^\infty f(-t) V^*(t) \phi dt \\ &= \int_{-\infty}^\infty f(t) \left(\frac{V(s+t) - V(t)}{s} \right) \phi dt \end{aligned}$$

$$(12.6.20) \quad + \int_0^s f(-t) \left(\frac{V(s-t)P(t) - V^*(t)}{s} \right) \phi dt - \int_{-s}^0 f(t) \frac{V(s+t) - V(t)}{s} \phi dt$$

$$(12.6.21) \quad + \int_s^\infty f(-t) \frac{P(s)V^*(t-s) - V^*(t)}{s} \phi dt - \int_{-\infty}^{-s} f(t) \frac{V(s+t) - V(t)}{s} \phi dt.$$

Consider the lines $A = (12.6.19)$, $B = (12.6.20)$, and $C = (12.6.21)$ separately. First

$$(12.6.22) \quad \begin{aligned} A &= \int_{-\infty}^\infty f(t) \frac{V(s+t) - V(t)}{s} \phi dt \\ &= \int_{-\infty}^\infty \frac{f(t-s) - f(t)}{s} V(t) \phi dt \\ &\xrightarrow{s \rightarrow 0} \phi_{-f'}. \end{aligned}$$

Notice that since f has support only on $[-\epsilon_\phi/2, \epsilon_\phi/2]$, so does $-f'$, so that $\phi_{-f'} \in \mathcal{D}$.

Next, consider

$$(12.6.23) \quad \begin{aligned} B &= \int_0^s f(-t) \frac{V(s-t)P(t) - V^*(t)}{s} \phi dt - \int_0^s f(-t) \frac{V(s-t) - V^*(t)}{s} \phi dt \\ &= \frac{1}{s} \int_0^s f(-t) V(s-t) (P(t) - I) \phi dt. \end{aligned}$$

Since $\phi \in \mathcal{D}'$, as soon as $s \leq \epsilon_\phi$, $P(t)\phi = \phi$ for all $t \in [0, s]$ so that B vanishes in the limit as $s \rightarrow 0$.

Finally,

$$(12.6.24) \quad \begin{aligned} C &= \int_s^\infty f(-t) \frac{P(s)V^*(t-s) - V^*(t)}{s} \phi dt - \int_s^\infty f(-t) \frac{V^*(t-s) - V^*(t)}{s} \phi dt \\ &= \frac{P(s) - I}{s} \int_s^\infty f(-t) V^*(t-s) \phi dt \\ &= \frac{P(s) - I}{s} \int_0^\infty f(-y-s) V^*(y) \phi dy \\ &= \frac{P(s) - I}{s} \int_0^{\epsilon_\phi/2} f(-y-s) V^*(y) dy. \end{aligned}$$

The last line above follows from the fact that $f(t)$ has support only on $[-\epsilon_\phi/2, \epsilon_\phi/2]$ so that $f(-y-s)$ is non-zero only if $-\epsilon_\phi/2 - s \leq y \leq \epsilon_\phi/2 - s \leq \epsilon_\phi/2$. By *Lemma 12.6.8, it follows that as soon as $s \leq \epsilon_\phi/2$, $P(s)V^*(t)\phi = V^*(t)\phi$ for all $t \in [0, \epsilon_\phi/2]$ so that $C = 0$ as soon as $s \leq \epsilon_\phi/2$.

This allows one to conclude that

$$(12.6.25) \quad \lim_{s \rightarrow 0^+} \frac{(V(s) - I)}{s} \phi_f = \phi_{-f'}$$

for all $\phi_f \in \mathcal{D}$.

To show that the strong two-sided limit of $\frac{(V(s) - I)}{s}$ as $s \rightarrow 0$ exists on \mathcal{D} , it remains to show that

$$(12.6.26) \quad \lim_{s \rightarrow 0^-} \frac{V(s) - I}{s} \phi_f = \phi_{-f'}.$$

Let $y > 0$ and $s = -y$. Equation (12.6.26) becomes

$$(12.6.27) \quad \lim_{y \rightarrow 0^+} \frac{V(-y) - I}{-y} \phi_f = \lim_{y \rightarrow 0^+} \frac{I - V^*(y)}{y} \phi_f.$$

For $s > 0$, consider

$$(12.6.28) \quad \frac{I - V^*(s)}{s} \phi_f = \int_{-\infty}^{\infty} f(-t) \frac{V^*(t) - V^*(s+t)}{s} \phi dt$$

$$(12.6.29) \quad + \int_0^s f(t) \frac{V(t) - V^*(s)V(t)}{s} \phi dt - \int_{-s}^0 f(-t) \frac{V^*(t) - V^*(s+t)}{s} \phi dt$$

$$(12.6.30) \quad + \int_s^{\infty} f(t) \frac{V(t) - V^*(s)V(t)}{s} \phi dt - \int_{-\infty}^{-s} f(-t) \frac{V^*(t) - V^*(s+t)}{s} \phi dt$$

As before, the right hand side of (12.6.28) becomes

$$(12.6.31) \quad \int_{-\infty}^{\infty} \frac{f(-t) - f(-t+s)}{s} V^*(t) \phi dt \xrightarrow{s \rightarrow 0} - \int_{-\infty}^{\infty} f'(-t) V^*(t) \phi dt \\ = - \int_{-\infty}^{\infty} f'(t) V(t) \phi dt = \phi_{-f'},$$

and the second line, (12.6.29) becomes

$$(12.6.32) \quad \int_0^s f(t) \frac{V(t) - V^*(s)V(t)}{s} \phi dt - \int_0^s f(t) \frac{V(t) - V^*(s-t)}{s} \phi dt \\ = \int_0^s f(t) \frac{V^*(s-t) - V^*(s-t)V^*(t)V(t)}{s} \phi dt \\ = \frac{1}{s} \int_0^s f(t) V^*(s-t) (I - Q(t)) \phi dt.$$

Since $\phi \in \mathcal{D}'$, this vanishes as soon as s , and hence t , is less than ϵ_ϕ . Similarly, the last line, equation (12.6.30), becomes

$$(12.6.33) \quad \int_s^{\infty} f(t) \frac{V(t) - V^*(s)V(t)}{s} \phi dt - \int_s^{\infty} f(t) \frac{V(t) - V^*(s-t)}{s} \phi dt \\ = \int_s^{\infty} \frac{V(t-s) - V^*(s)V(s)V(t-s)}{s} \phi dt \\ = \frac{I - Q(s)}{s} \int_s^{\infty} f(t) V(t-s) \phi dt \\ = \frac{I - Q(s)}{s} \int_0^{\infty} f(t+s) V(t) \phi dt.$$

Since $f(t)$ has support only on $[-\epsilon_\phi/2, \epsilon_\phi/2]$ it follows that $f(t+s)$ is non-zero only when $-\epsilon_\phi/2 - s \leq t \leq \epsilon_\phi/2 - s \leq \epsilon_\phi/2$. Hence, (12.6.33) is equal to

$$(12.6.34) \quad \frac{1 - Q(s)}{s} \int_0^{\epsilon_\phi/2} f(t+s) V(t) \phi dt.$$

As soon as $s \leq \epsilon_\phi/2$, *Lemma 12.6.8 implies that $Q(s)V(t)\phi = V(t)\phi$. This proves that (12.6.33) vanishes as $s \rightarrow 0$. In conclusion,

$$(12.6.35) \quad \lim_{s \rightarrow 0^+} \frac{I - V^*(s)}{s} \phi_f = \phi_{-f'}.$$

Using that $V(-s) = V^*(s)$ and $V^*(-s) = V(s)$ for all $s \geq 0$, (12.6.25) and (12.6.35) imply that

$$(12.6.36) \quad \lim_{s \rightarrow 0} \frac{I - V^*(s)}{s} \phi_f = \phi_{-f'} = \lim_{s \rightarrow 0} \frac{V(s) - I}{s} \phi_f,$$

for all $\phi_f \in \mathcal{D}$.

This proves that $\mathcal{D} \subset \mathfrak{D}(B)$. □

Recall that the symmetric operator B on $\mathfrak{D}(B)$ is defined by $B\phi := -i \lim_{s \rightarrow 0} \frac{V(s) - I}{s} \phi$ for $\phi \in \mathfrak{D}(B)$. The proof of Proposition 12.6.1 above actually shows that $B\mathcal{D} \subset \mathcal{D}$. This shows that \mathcal{D} is what is called an **analytic domain** for the operator B , since any power of B is defined on \mathcal{D} . Combining this fact with *Theorem 12.5.1 yields the following corollary

*COROLLARY 12.6.10. B^k is defined on the dense domain $\mathcal{D} \subset \mathfrak{D}(B)$ for all $k \in \mathbb{N}$ and $A^k|_{\mathcal{D}} = B^k$.

Combining *Proposition 12.6.1 with *Theorem 12.5.1 now yields *Theorem 12.6.2.

In the particular case where the semigroup $V(t)$ is purely isometric or co-isometric, the following stronger statement can be made.

*COROLLARY 12.6.11. If $V(t)$ is a purely isometric (co-isometric) then the symmetric operator B of the previous theorem has deficiency indices $(0, m)$ (resp. $(n, 0)$), where $m \in \mathbb{N} \cup \{\infty\}$. In this case B is unitarily equivalent to m copies of $i \frac{d}{dx}$ on $L^2[0, \infty)$ (resp. n copies of $i \frac{d}{dx}$ on $L^2(-\infty, 0]$).

Here, by purely isometric, it is meant that $V(t)$ is an isometry for each $t \geq 0$, and that for each $t \geq 0$ there is no subspace S of \mathcal{H} such that the restriction of $V(t)$ to this subspace is unitary.

12.6.0.8. *Remark.* To prove this corollary it is sufficient to show that the assumption that $V(t)$ is purely isometric or co-isometric implies that B has deficiency indices $(0, m)$ or $(n, 0)$ respectively. To see this, note that the symmetric operator B associated with $V(t)$ must be simple, i.e., there cannot be any subspace such that the restriction of B to that subspace is self-adjoint, as this would contradict the fact that $V(t)$ is purely isometric or purely co-isometric. There is a theorem due to von Neumann which shows that any closed, simple symmetric operator with deficiency indices $(0, n)$ or $(m, 0)$ is isomorphic to n copies of $i \frac{d}{dx}$ on $L^2(0, \infty)$ or m copies of $i \frac{d}{dx}$ on $L^2(-\infty, 0)$, respectively ([3], pgs. 104-107).

We will only prove the case where $V(t)$ is purely isometric. The proof of the other case is directly analogous. If $V(t)$ is purely isometric, then $Q(t) = V^*(t)V(t) = I$ for all $t \geq 0$. In this case one can replace the definition (12.6.15) by

$$(12.6.37) \quad \mathcal{D}' := \{\phi \in \mathcal{H} \mid \exists \epsilon_\phi > 0 \text{ s.t. } P(s)\phi = \phi \quad \forall 0 \leq s \leq \epsilon_\phi\},$$

and *Lemma (12.6.8) becomes:

LEMMA 12.6.12. If $\phi \in \mathcal{D}'$, and $0 \leq t, s \leq \frac{\epsilon_\phi}{2}$, then $P(s)V^(t)\phi = V^*(t)\phi$.

Re-define \mathcal{D} to be the linear manifold of all finite linear combination of vectors of the form

$$(12.6.38) \quad \phi_f := \int_{-\infty}^{\infty} f(t)V(t)\phi dt = \int_{-\epsilon_\phi/2}^{\infty} f(t)V(t)\phi dt,$$

where $\phi \in \mathcal{D}'$ and $f(t) \in C_0^\infty[-\epsilon_\phi/2, \infty)$.

*LEMMA 12.6.13. The linear manifold \mathcal{D} is invariant under the semigroup $V(t)$, i.e., $V(t) : \mathcal{D} \rightarrow \mathcal{D}$. Furthermore, \mathcal{D} is an analytic domain for B , and $V(t)B\phi = BV(t)\phi$ for all $\phi \in \mathcal{D}$.

PROOF. Using similar arguments to those in the proof of *Proposition 12.6.1, it is not difficult to show that $\mathcal{D} \subset \mathfrak{D}(B)$, and that $B\mathcal{D} \subset \mathcal{D}$.

To see that \mathcal{D} is also invariant under $V(t)$ for any $t \geq 0$, observe that

$$(12.6.39) \quad V(s)\phi_f = \int_0^\infty f(t)V(s+t)\phi dt + V(s) \int_{-\epsilon_\phi/2}^0 f(t)V(t)\phi dt,$$

and,

$$(12.6.40) \quad \begin{aligned} V(s) \int_{-\epsilon_\phi/2}^0 f(t)V(t)\phi dt &= V(s) \int_0^{\epsilon_\phi/2} f(-t)V^*(t)\phi dt \\ &= V(s) \int_0^s f(-t)V^*(t)\phi dt + V(s) \int_s^{\epsilon_\phi/2} f(-t)V^*(t)\phi dt \\ &= \int_0^s f(-t)V(s-t)P(t)\phi dt + \int_s^{\epsilon_\phi/2} f(-t)P(s)V^*(t-s)\phi dt. \end{aligned}$$

By *Lemma 12.6.12, $P(s)V^*(t-s)\phi = V^*(t-s)\phi$ in the second integral, and $P(t)\phi = \phi$ in the first integral, so that (12.6.40) becomes

$$(12.6.41) \quad \int_0^{\epsilon_\phi/2} f(-t)V(s-t)\phi dt = \int_{-\epsilon_\phi/2}^0 f(t)V(s+t)\phi dt.$$

In the above, the fact that $V^*(t-s) = V(s-t)$ was also used. It follows that,

$$(12.6.42) \quad V(s)\phi_f = \int_{-\epsilon_\phi/2}^\infty f(t)V(s+t)\phi dt = \int_{-\epsilon_\phi/2+s}^\infty f(y-s)V(y)\phi dy = \phi_g,$$

where $g(y) = f(y-s)\chi_{[s-\epsilon_\phi/2, \infty)} \in C_0^\infty[-\epsilon_\phi/2, \infty)$. Hence $V(s)\phi_f = \phi_g \in \mathcal{D}$. This shows \mathcal{D} is invariant for $V(t)$, for any $t \geq 0$.

Since $V(s)\mathcal{D} \subset \mathcal{D}$ for $s \geq 0$, if $\phi \in \mathcal{D}$ and $t \geq 0$,

$$(12.6.43) \quad \begin{aligned} V(t)B\phi &= V(t) \lim_{\epsilon \rightarrow 0} \frac{V(\epsilon)\phi - \phi}{i\epsilon} \\ &= \lim_{\epsilon \rightarrow 0} \frac{V(t+\epsilon)\phi - V(t)\phi}{i\epsilon} = -i \frac{d}{dt} V(t)\phi \\ &= \lim_{\epsilon \rightarrow 0} \frac{V(\epsilon)\psi - \psi}{i\epsilon} = B\psi = BV(t)\phi, \end{aligned}$$

where $\psi = V(t)\phi \in \mathcal{D}$. This shows that $V(t)$ commutes with B on \mathcal{D} , completing the proof. \square

PROOF. (of *Corollary 12.6.11)

Now suppose that B^* had an eigenvector ψ to eigenvalue $+i$. This would imply that for any $t \geq 0$, and any $\phi \in \mathcal{D} \subset \mathfrak{D}(B)$,

$$(12.6.44) \quad \frac{d}{dt} \langle V(t)\phi, \psi \rangle = \langle iBV(t), \psi \rangle = i \langle V(t)\phi, B^*\psi \rangle = \langle V(t)\phi, \psi \rangle$$

In other words, the complex-valued function $f(t) = \langle V(t)\phi, \psi \rangle$ satisfies $f'(t) = f(t)$ for all $t \geq 0$. It follows that $f(t) = f(0)e^t$ for all $t \geq 0$. Since $|f(t)| = |\langle V(t)\phi, \psi \rangle| \leq \|V(t)\phi\| \|\psi\| \leq \|\phi\| \|\psi\| < \infty$, it must be that $0 = f(0) = \langle \phi, \psi \rangle$. Since $\phi \in \mathcal{D}$ was arbitrary and \mathcal{D} is dense, one can conclude that $\psi = 0$ so that B^* has no eigenvalues in the upper half plane. It follows that the deficiency indices of B are $(0, n)$, where $n \in \mathbb{N} \cup \{\infty\}$. The proof now follows from Remark 12.6.0.8. \square

Furthermore, the following result, which has also been proven by different methods in ([24], pg. 151), is a simple consequence of the results of this subsection.

'COROLLARY 12.6.14. If $V(t)$ is purely isometric, then $V(t)$ is unitarily equivalent to n copies of the semi-group of right translations on n copies of $L^2[0, \infty)$. If it is purely co-isometric, it is unitarily equivalent to m copies of right translations on $L^2(-\infty, 0]$

PROOF. Assume that $V(t)$ is purely isometric. By *Corollary 12.6.11, it can be differentiated on a dense domain $\mathfrak{D}(B)$ to yield a closed symmetric operator B with deficiency indices $(0, n)$. It follows that the spectrum of B is contained in the upper half plane. From the Hille-Yosida theorem, it follows that iB generates a contraction semi-group $W(t) = \exp(itB)$ for $t \geq 0$. It follows from the definition of the domain of the generator of the semigroup $W(t)$ that $\mathfrak{D}(B) = \{\phi \in \mathcal{H} \mid \lim_{t \rightarrow 0^+} \frac{I - W(t)}{t} \phi = -W'(t)\phi \exists\}$, and that $W(t) : \mathfrak{D}(B) \rightarrow \mathfrak{D}(B)$ (see Section 12.4). Thus, the strong derivative of $W(t)$ at 0 is $+iB$ on the dense domain $\mathfrak{D}(B)$. We also know that $V(t) : \mathfrak{D}(B) \rightarrow \mathfrak{D}(B)$. Given $\phi \in \mathfrak{D}(B)$, let $w(t) = V(t)\phi - W(t)\phi$. Then,

$$(12.6.45) \quad w'(t) = iBV(t)\phi - iBW(t)\phi = iBw(t),$$

so that,

$$(12.6.46) \quad \frac{d}{dt} \|w(t)\|^2 = \langle iBw(t), w(t) \rangle + \langle w(t), iBw(t) \rangle = 0,$$

since B is symmetric. But $w(0) = 0$ since $V(0) = I = W(0)$. This shows that $w(t) = 0$ for all $t \geq 0$. Since $\mathfrak{D}(B)$ is dense in \mathcal{H} , we conclude that $V(t) = W(t)$ for all $t \geq 0$.

Finally, since B is unitarily equivalent to n copies of $D = i \frac{d}{dt}$ on $L^2[0, \infty)$, it follows that $W(t) = V(t)$ is equal to n copies of e^{itD} on n copies of $L^2[0, \infty)$. Since $V(t) = e^{itD}$ for $t \geq 0$ is the semigroup of right translations on $L^2[0, \infty)$, this proves the first half of the corollary. Proof of the second half is similar, and is omitted. \square

**COROLLARY 12.6.15. Suppose that $V(t)$ is a semi-group of partial isometries which is nilpotent, i.e., $\exists t_0 > 0$ such that $V(t) = 0$ for all $t \geq t_0$. Let t_0 be the smallest number such that this is true. Then the symmetric operator B obtained from V is simple and has equal deficiency indices. If it has deficiency indices $(1, 1)$, then $V(t)$ is unitarily equivalent to the semigroup of truncated shifts on $L^2[0, t_0]$.*

The proof of this corollary relies on the following theorem of Lifschitz [3]:

THEOREM 12.6.16. *Any simple symmetric operator with deficiency indices $(1, 1)$ which has an extension without spectrum is isomorphic to the symmetric derivative operator $D = i \frac{d}{dx}$ on a finite interval.*

In this theorem, we also use the following form of the spectral mapping theorem that holds for closed operators whose spectrum is contained in an open proper subset Δ of the extended complex plane ([21], pg. 199).

THEOREM 12.6.17. *Let Δ be an open proper subset of the extended complex plane. Suppose that f is analytic in Δ and that T is a closed operator such that $\sigma(T) \subset \Delta$. Then $\sigma(f(T)) = f(\sigma(T))$.*

PROOF. (of *Corollary 12.6.15) Since $V(t) = 0$ for all $t \geq t_0$, $V(t)$ is nilpotent for any $t > 0$, so that $\sigma(V(t)) = \{0\}$ for any $t > 0$. This is clear, since if $t > 0$, then there is an $n \in \mathbb{N}$ such that $nt > t_0$, and hence $V(t)^n = V(nt) = 0$. Define the infinitesimal generator A of this semigroup to be i times the strong one-sided derivative of $V(t)$ at 0, and $V(t) = e^{itA}$. The domain of A is defined to be the set of all $\phi \in \mathcal{H}$ for which the limit $\lim_{\epsilon \rightarrow 0^+} \frac{V(\epsilon) - I}{\epsilon} \phi$ exists. By *Proposition 12.6.1, the strong derivative of $V(t)$ on a dense domain $\mathfrak{D}(B)$ yields a closed symmetric operator B . Clearly, $\mathfrak{D}(B) \subset \mathfrak{D}(A)$ and A is an extension of B . The spectrum of A must be empty, since by the spectral mapping theorem, if $\lambda \in \mathbb{C}$ and $\lambda \in \sigma(A)$, then $e^{it\lambda} \in \sigma(V(t))$. This would contradict the fact that $\sigma(V(t)) = \{0\}$. This also implies that B must have equal deficiency indices, for if it did not, then it would have a symmetric extension with deficiency indices $(0, n)$ or $(m, 0)$.

Such symmetric operators have continuous spectra, and hence so would the symmetric operator B . This would in turn imply that the extension A of B would have continuous spectra, which is not possible. We conclude that B must have equal deficiency indices. The same argument also shows that B must be simple, otherwise it would have eigenvalues or continuous spectra which would imply that A has eigenvalues or continuous spectra, again a contradiction.

If B has deficiency indices $(1, 1)$, then by Theorem 12.6.16, B is unitarily equivalent to the symmetric derivative operator D on a finite interval. It is not difficult to show, see Example 12.7.0.10 to come, that the only extension of D that generates a semigroup of partial isometries is the one that generates the semigroup of truncated shifts. Hence, A must be isomorphic to this extension, and $V(t)$ is isomorphic to the semigroup of truncated shifts on a finite interval. Since t_0 is the smallest number such that $V(t_0) = 0$, it follows that $V(t)$ is isomorphic to the semigroup of truncated shifts on $L^2[0, t_0]$. \square

12.7. Observations and Conclusions

It has been proven that a sufficient condition for a self-adjoint operator A in \mathcal{H} to have a symmetric restriction to a dense domain $\mathfrak{D}(B) \subset S$ of a subspace $S \subset \mathcal{H}$ is that $V(t) := Pe^{itA}P$ be a strongly continuous one-parameter semi-group of partial isometries on S where P is the projector onto S . This is the content of *Theorem 12.6.2.

The condition that $V(t)$ consists of partial isometries turns out to be very restrictive, and does not appear to be necessary. To see this, first consider the following.

12.7.0.9. *Definition.* Let A be an arbitrary linear operator defined in a separable Hilbert space \mathcal{H} . The **numerical range** $\mathfrak{N}\text{um}(A)$ of A is defined as the set $\mathfrak{N}\text{um}(A) := \{\langle A\phi, \phi \rangle \mid \|\phi\| = 1\}$. A is called **dissipative** if $\text{Re}(\mathfrak{N}\text{um}(A)) \leq 0$.

The following variation of Theorem 12.4.2 characterizes generators of semi-groups of contractions ([33], pg. 23).

THEOREM 12.7.1. *A generates a strongly continuous one-parameter semigroup of contractions if and only if it is closed, densely defined, dissipative, and $\lambda I - A$ is surjective for all $\lambda > 0$.*

Consider a symmetric operator B which is closed, densely defined on $\mathfrak{D}(B) \subset \mathcal{H}$, and which has equal but arbitrary deficiency indices (n, n) . Let $V = \mu_i(B)$ be the partially defined isometry that is the Cayley transform of B . Then given any bounded linear map W from $\mathfrak{D}(V)^\perp$ to $\mathfrak{R}(V)^\perp$ define a maximal extension V_W of V by

$$(12.7.1) \quad V_W := V \oplus W$$

on $\mathcal{H} = \mathfrak{D}(V) \oplus \mathfrak{D}_+$. Then B_W , defined as the inverse Cayley transform of V_W is an extension of the symmetric operator B . In the special case where W is chosen to be a surjective isometry, then V_W will be unitary and B_W is a self-adjoint extension of B . Since $B_W = i(1 + V_W)(1 - V_W)^{-1}$, it follows that

$$(12.7.2) \quad \mathfrak{D}(B_W) = \mathfrak{D}(B) + (W - 1)\mathfrak{D}(V)^\perp.$$

Note that since $\mathfrak{R}(V - 1)$ is dense, so is $\mathfrak{R}(V_W - 1)$ so that $(V_W - 1)^{-1}$ is well defined. Further observe that since $U_W = (B_W - i)(B_W + i)^{-1}$, that $\mathfrak{R}(B_W + i) = \mathfrak{D}(U_W) = \mathcal{H}$.

I claim that if $\|W\| \leq 1$, then iB_W is the generator of a contraction semigroup. To prove this, the conditions of Theorem 12.7.1 will be verified. First consider the numerical range of B_W . If $\phi \in \mathfrak{D}(B) \subset \mathfrak{D}(B_W)$ then

$$(12.7.3) \quad \text{Im}(\langle B_W\phi, \phi \rangle) = \text{Im}(\langle B\phi, \phi \rangle) = 0.$$

If $\phi \in (W-1)\mathfrak{D}(V)^\perp = (W-1)\mathfrak{Rer}(B^* - i)$ then $\phi = (W-1)\phi_i$ for some $\phi_i \in \mathfrak{Rer}(B^* - i)$, and $W\phi_i \in \mathfrak{Rer}(B^* + i)$. It follows that

$$\begin{aligned}
\langle B_W\phi, \phi \rangle &= \langle B_W(W-1)\phi_i, (W-1)\phi_i \rangle \\
&= -i\langle (W+1)\phi_i, (W-1)\phi_i \rangle = -i\langle (W^* - 1)(W+1)\phi_i, \phi_i \rangle \\
(12.7.4) \quad &= -i[\langle (W^*W - 1)\phi_i, \phi_i \rangle - 2i\text{Im}(\langle W\phi_i, \phi_i \rangle)].
\end{aligned}$$

Hence $\text{Im}(\langle B_W\phi, \phi \rangle) = \langle (1 - W^*W)\phi_i, \phi_i \rangle$.

Now if $\phi \in \mathfrak{D}(B_W)$ is arbitrary then $\phi = \varphi + \psi$ for some $\varphi \in \mathfrak{D}(B)$ and $\psi \in \mathfrak{Rer}(B^* - i)$. This shows that

$$\begin{aligned}
\text{Im}(\langle B_W\phi, \phi \rangle) &= \text{Im}(\langle B\varphi + B_W\psi, \varphi + \psi \rangle) \\
&= \text{Im}(\langle B\phi, \phi \rangle + 2\text{Re}(ipB\varphi\psi) + \langle B_W\psi, \psi \rangle) \\
(12.7.5) \quad &= \text{Im}(\langle B_W\psi, \psi \rangle).
\end{aligned}$$

Since $\psi = (W-1)\phi_i$ for some $\phi_i \in \mathfrak{Rer}(B^* - i)$, it follows, as before from equation (12.7.4) that $\text{Im}(\langle B_W\phi, \phi \rangle) = \langle (1 - W^*W)\phi_i, \phi_i \rangle$. It follows that if $\|W\| \leq 1$ then $\text{Im}(\langle B_W\phi, \phi \rangle) \geq 0$ for all $\phi \in \mathfrak{D}(B_W)$ which means that $\text{Re}(\langle iB_W\phi, \phi \rangle) \leq 0$ for all $\phi \in \mathfrak{D}(B_W)$. This shows that iB_W is dissipative.

Since iB_W is dissipative, it is not hard to see that every z in the lower half plane (LHP) is a regular point for B_W . Otherwise, if $z \in LHP$ and $B_W - z$ was not bounded below, this would mean that there is a sequence $\phi_n \in \mathfrak{D}(B_W)$ such that $\|\phi_n\| = 1$ and $\|(B_W - z)\phi_n\| \rightarrow 0$. This would imply that $\langle B_W\phi_n, \phi_n \rangle - z \rightarrow 0$ which would contradict the fact that $\text{Im}(\mathfrak{Im}(B_W)) \geq 0$. It follows that every $z \in LHP$ belongs to the field of regularity for B_W . Furthermore, as has already been shown, $(B_W - (-i))$ is onto so that $\dim(\mathfrak{R}(B_W + i)^\perp) = 0$. It follows from Theorem 4.3.2, that since every $z \in LHP$ belongs to the field of regularity of B_W and since $\dim \mathfrak{R}(B_W + i)^\perp = 0$, $\mathfrak{R}(B_W - z) = \mathcal{H}$ for all $z \in LHP$. This establishes that iB_W is dissipative and that $\mathfrak{R}(iB_W - z)$ is onto for every z in the open right half plane. Applying Theorem 12.7.1 now proves the following.

PROPOSITION 12.7.2. *Let B be a symmetric operator with equal deficiency indices and $V = (B - i)(B + i)^{-1}$ be its Cayley transform. Also let $V' := (B + i)(B - i)^{-1}$. Suppose that $W : \mathfrak{D}(V)^\perp \rightarrow \mathfrak{R}(V)^\perp$ and $W' : \mathfrak{R}(V)^\perp \rightarrow \mathfrak{D}(V)^\perp$ are contractions. Then iB_W and $-iB'_W$ generate contraction semigroups where B_W is the inverse Cayley transform of $V \oplus W$ and B'_W is the inverse Cayley transform of $V' \oplus W'$.*

In particular, note that if B has deficiency indices $(1, 1)$, then all extensions B_z of B are obtained as the inverse Cayley transform of $U_z := V \oplus z\langle \cdot, \phi_+ \rangle \phi_-$ where $z \in \mathbb{C}$ where ϕ_\pm are fixed unit norm vectors in $\mathfrak{D}(V)^\perp$ and $\mathfrak{R}(V)^\perp$ respectively. In this case, it is not hard to see that if $|z| \leq 1$, then iB_z generates a contraction semigroup while if $|z| \geq 1$ then $-iB_z$ generates a contraction semigroup.

12.7.0.10. *Example.* For example, suppose $V(t)$ is a truncated shift, a semigroup of partial isometries obtained by compressing the unitary group of right translations $U(t)$ on $L^2(\mathbb{R})$ to L^2 of some interval $[a, b]$. This was our motivating example for this chapter, and was discussed in detail in Section 12.3. Recall that the self-adjoint generator of $U(t)$ is the self-adjoint derivative operator \tilde{D} which is the closure of $i\frac{d}{dx}$ defined on $C_0^\infty(\mathbb{R})$. Further recall that this operator \tilde{D} has a densely defined closed symmetric restriction D in $L^2[a, b]$ which is defined as the closure of $\tilde{D}|_{C_0^\infty(a, b)}$. Now recall, from Section 4.3 of Chapter 4, that one can construct the self-adjoint extensions of D by extending the domain of D to include functions which obey certain boundary conditions on $[a, b]$. In particular, recall that

$$(12.7.6) \quad \mathfrak{D}(D^*) = \{f \in L^2[a, b] \mid f \in AC[a, b]; f' \in L^2[a, b]\},$$

that

$$(12.7.7) \quad \mathfrak{D}(D) = \{f \in L^2[a, b] \mid f \in AC[a, b]; f' \in L^2[a, b] \text{ and } f(a) = 0 = f(b)\},$$

and that if

$$(12.7.8) \quad \mathfrak{D}(D_\theta) = \{f \in L^2[a, b] \mid f \in AC[a, b]; f' \in L^2[a, b] \text{ and } f(a) = \theta f(b)\},$$

then $D_\theta := D^*|\mathfrak{D}(D_\theta)$ defines a self-adjoint extension of D if $\theta \in \mathbb{T}$. More generally, choosing $\theta \in \mathbb{C} \setminus \mathbb{T}$ yields non self-adjoint extensions D_θ of D . In this case it is straightforward to see that if one chooses $\theta \in \mathbb{D}$, then the semigroup $V_\theta(t) = e^{itD_\theta}$ generated by D_θ will translate functions to the right so that as the exit through b they reappear at the point a multiplied by the constant θ . In particular if one chooses $\theta = 0$, then it is clear that $V_0(t) = PU(t)P$ is the semi-group of truncated shifts if P is the projector onto $L^2[a, b]$.

Recall that the dilation theory of Nagy and Foias shows that any one-parameter strongly continuous contraction semigroup can be dilated to a unitary group on a larger Hilbert space (see Remark 12.4.0.3). By this fact and Proposition 12.7.2, it follows that if B_W is any extension of the symmetric operator B where $\|W\| \leq 1$, then the semi-group it generates on \mathcal{H} can be seen as the compression of a unitary group on a larger Hilbert space $\tilde{\mathcal{H}}$ to \mathcal{H} . Given such a B_W , let $V_W(t)$ be the semi-group that it generates, and let $U_W(t)$ be the minimal unitary dilation of $V_W(t)$. The generator of $U_W(t)$ is, by Stone's theorem, a self-adjoint operator in $\tilde{\mathcal{H}}$. It is not difficult to see that if A_W is the generator of $U_W(t)$, then $B = A_W|_{\mathfrak{D}(B)}$.

Since it is very unlikely that the semi-group generated by every such extension of B_W is always a semi-group of partial isometries, this indicates that there are many examples of self-adjoint operators on a Hilbert space $\tilde{\mathcal{H}}$ which have symmetric restrictions to dense domains in a subspace \mathcal{H} of $\tilde{\mathcal{H}}$, and such that the compression of their unitary groups to the subspace are not semi-groups of partial isometries.

Furthermore, there are self-adjoint operators which have densely defined symmetric restrictions to a given subspace, and for which the compression of the unitary group they generate to that subspace is not even a semigroup. To see this, consider the example of a Schrodinger operator of the form $D := -\frac{d^2}{dx^2} + V(x)$ which is self-adjoint on a dense domain in $L^2(\mathbb{R})$, and consider the subspace $S = L^2[a, b]$ of $L^2(\mathbb{R})$. Suppose that the potential, $V(x) \geq 0$, is chosen to be 0 on $[a, b]$, and to be increasing on an interval $[c, d]$ where $c > b$. It is easy to see that the closure of D restricted to $C_0^\infty(a, b)$ is a densely defined symmetric operator in $L^2[a, b]$. Imagine $\phi(t) = e^{itD}\phi$ to be the 'time evolution' of an initial function $\phi \in L^2[a, b]$ which is zero outside of $[a, b]$. Further imagine ϕ to be a suspended column of water above $[a, b]$ at the instant in time $t = 0$. As time increases, the water will 'flow' out of the interval $[a, b]$, and some of the water will reflect off the potential V at $[c, d]$ and flow back into the interval $[a, b]$ at some later time. It follows that there exist $t > 0$ so that if P projects onto $L^2[a, b]$ that $PU(t)P\phi = PU(t)\phi \neq PU(t_1)PU(t_2)\phi$ for all $t_1 + t_2 = t$. For example, suppose that $t_1 = t_2 = \frac{t}{2}$ is the instant of time at which the water is about to flow back through the interval $[a, b]$ after reflecting off of the potential barrier at $[c, d]$. Then the action of $PU(\frac{t}{2})PU(\frac{t}{2})P$ on ϕ is as if at the instant of time $\frac{t}{2}$, all of the water outside of the interval $[a, b]$ is removed, so that none of the reflected water flows back through $[a, b]$. This is different from the action of $PU(t)P$ on ϕ , as in this case, some of the water is reflected off of the potential barrier and flows back through $[a, b]$. In particular, this shows that $PU(t)P$ is not a semigroup. This intuitive argument makes it clear that there are self-adjoint operators which have symmetric restrictions to a given subspace, and yet which are such that the compression of the unitary group they generate to that subspace is not even a semigroup.¹ It could still be true that a necessary condition for a self-adjoint operator to have a symmetric restriction with deficiency indices $(1, 1)$ to a subspace is that the compression of its unitary group is a semi-group.

Finally, it should be pointed out that the condition that a semigroup consist of partial isometries is very restrictive. Recall the motivating example of this chapter of the compression of the unitary group of translations on $L^2(\mathbb{R})$ to $L^2[a, b]$ (see Section 12.3). Recall that the semigroup given by this compression for any finite interval $[a, b]$ is called a truncated shift.

¹I'd like to thank William Donnelly for pointing out this example to me.

The following theorem gives a canonical decomposition of a strongly continuous one-parameter semigroups of partial isometries, and shows that, in a sense, the truncated shifts are the only semigroups of partial isometries [22]:

THEOREM 12.7.3. *Let $V(t)$ be a semigroup of partial isometries on \mathcal{H} . Then \mathcal{H} can be decomposed as*

$$(12.7.9) \quad \mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$$

where each \mathcal{H}_i reduces $V(t)$ and $V(t)|_{\mathcal{H}_i}$ is invertible, $V(t)|_{\mathcal{H}_\infty}$ is purely isometric, $V(t)|_{\mathcal{H}_2}$ is purely co-isometric and \mathcal{H}_3 has a direct integral decomposition relative to which $V(t)$ decomposes into truncated shifts.

In particular, as has been shown in this chapter, $V(t)$ restricted to \mathcal{H}_1 is isomorphic to the semigroup generated by the derivative operator $D = i\frac{d}{dx}$ on n copies of $L^2[0, \infty)$, $V(t)|_{\mathcal{H}_2}$ is isomorphic to a semigroup generated by m copies of the derivative operator on $L^2(-\infty, 0]$. Furthermore, since $V_0(t) := V(t)|_{\mathcal{H}_0}$ is invertible, it follows that for each $t \geq 0$, $V_0(t)$ is unitary, and that if one defines $V_0(-t) := V_0^*(t)$, then $V_0(t)$ is a unitary group for $t \in \mathbb{R}$. From these observations it is clear that many of the results of Section 12.6, are an easy consequence of the example of the compression of the unitary group of translations on $L^2(\mathbb{R})$ to subintervals, Stone's theorem, and the above Theorem 12.7.3.

The compact, convex set of generalized resolvents of a symmetric operator

In this chapter, it is shown that the convex set \mathcal{P} of all positive operator-valued measures $Q(\cdot)$ on the Borel subsets of \mathbb{R} which are contractive, *i.e.* $Q(\mathbb{R}) \leq I$, is compact with respect to a certain natural topology. This is accomplished by showing that the set of all generalized resolvents \mathcal{R} which are defined by integrating elements of \mathcal{P} with respect to the functions $f_z(x) = (z - x)^{-1}$ is compact with respect to a certain topology, and then using the fact that there is a bijective correspondence between \mathcal{P} and \mathcal{R} . It is further shown that the set of all generalized resolvents of a single symmetric operator is a closed face of \mathcal{R} . This generalizes the fact that any unital PVM is an extreme point in the convex, compact set of all contractive POVMs. In the case where the symmetric operator has finite deficiency indices, I also prove that set of generalized resolvents is compact with respect to a stronger metric topology.

13.1. Introduction

Let \mathcal{H} be a separable Hilbert space. Let $\mathcal{B}(\mathbb{R})$ denote the set of all Borel subsets of \mathbb{R} . Recall that a contractive positive operator valued measure (POVM) on \mathbb{R} as a map $Q : \mathcal{B}(\mathbb{R}) \rightarrow B(\mathcal{H})^+$ with the properties:

- (1) If $\Omega = \cup_{n=1}^{\infty} \Omega_n$ with $\Omega_n \cap \Omega_m = \emptyset \quad \forall n \neq m$ then $\sum_{n=1}^N Q(\Omega_n) \xrightarrow{s} Q(\Omega)$.
- (2) $Q(\emptyset) = 0$ and $Q(\mathbb{R}) \leq I$.

13.1.0.11. *Remark.* This is the same as our previous definition, except that its no longer required that $Q(\mathbb{R}) = I$. Let \leq denote the usual partial order on positive operators defined by $Q_1 \leq Q_2 \Leftrightarrow \langle (Q_2 - Q_1)\phi, \phi \rangle \geq 0$ for all $\phi \in \mathcal{H}$. It is not difficult to show that if $\langle Q\phi, \phi \rangle = \langle \phi, \phi \rangle$ for all $\phi \in \mathcal{H}$, then $Q = I$ both respect to this partial order, and as elements of $B(\mathcal{H})$. If $Q(\mathbb{R}) < I$, we will call Q strictly contractive, and if $Q(\mathbb{R}) = I$ we will call Q unital. We will let \mathcal{P} denote the set of all contractive POVMs on $\mathcal{B}(\mathbb{R})$ which take their values in $B(\mathcal{H})$.

The following dilation theorem of Naimark gives a concrete representation for any contractive POVM [54][3].

THEOREM 13.1.1. *(Naimark's dilation theorem)*

If $Q : \mathcal{B}(\mathbb{R}) \rightarrow B(\mathcal{H})^+$ is a contractive POVM, where \mathcal{H} is a Hilbert space, then there exists a Hilbert space \mathcal{K} and a PVM, P , on \mathcal{K} such that $Q(\cdot) = VP(\cdot)V^$, where $V : \mathcal{K} \rightarrow \mathcal{H}$ is a contractive linear map.*

If Q is unital, then V, \mathcal{K} can be chosen such that $\mathcal{H} \subset \mathcal{K}$, and $V = P_{\mathcal{H}}$, the self-adjoint projection from \mathcal{K} onto \mathcal{H} .

For any such a POVM Q , one obtains a natural linear map Φ_Q from $L^\infty(\mathbb{R})$ to $B(\mathcal{H})$ via

$$(13.1.1) \quad \langle \Phi_Q[f]\phi, \psi \rangle := \int_{-\infty}^{\infty} f(\lambda) \langle Q(d\lambda)\phi, \psi \rangle.$$

An operator valued function ρ_z on $\mathbb{U} := \mathbb{C} \setminus \mathbb{R}$ will be called a generalized resolvent if $\rho_z = \Phi_Q[(z-x)^{-1}]$ where $Q(\cdot)$ is a contractive POVM. Let \mathcal{R} denote the set of all generalized resolvents.

By Naimark's theorem, it is clear that if $\rho \in \mathcal{R}$, then $\rho = V(z - A)^{-1}V^*$, where A is the self-adjoint operator in \mathcal{K} defined by $A\phi := \lim_{B \rightarrow \infty} \int_{-B}^B \lambda dP_\lambda \phi$ on the dense domain of all vectors ϕ for which this limit exists.

For a densely defined symmetric operator B in \mathcal{H} we define $\mathcal{R}(B)$ to be all the set of generalized resolvents $\rho(z)$ such that $\rho(z) = \rho(A)(z) = P_{\mathcal{H}}(z - A)^{-1}|_{\mathcal{H}}$ where A is a self-adjoint extension of B , in general to a dense domain in a larger space $\tilde{\mathcal{H}} \supset \mathcal{H}$. Naturally, $\mathcal{P}(B)$ will denote the set of all POVMs Q such that $Q(\Omega) = P_{\mathcal{H}}\chi_\Omega(A)|_{\mathcal{H}}$ where A is any self-adjoint extension of B , in general to a larger space $\tilde{\mathcal{H}} \supset \mathcal{H}$. This agrees with the earlier definition of $\mathcal{P}(B)$ given in Chapter 9.

It is not hard to show that the sets \mathcal{P} , $\mathcal{P}(B)$ and \mathcal{R} , $\mathcal{R}(B)$ are convex. In what follows it will be shown that \mathcal{R} is a compact, set with respect to a certain natural topology. The definition of elements of \mathcal{R} defines a natural map Γ between \mathcal{P} and \mathcal{R} , $\Gamma(Q) = \rho$, $\rho(z) = \int_{-\infty}^{\infty} (z - x)^{-1} Q(dx)$ for any $z \in \mathbb{U}$. It will be proven that Γ is a bijection. Using Γ , one can use the topology on \mathcal{R} to induce a topology on \mathcal{P} , so that Γ becomes a homeomorphism between these two topological spaces. This will show that \mathcal{P} is a compact, convex set with respect to this induced topology. That \mathcal{P} is compact with respect to this induced topology is not a new result, but the method of proof contains new features.

13.2. The bijective correspondence between POVMs and generalized resolvents

THEOREM 13.2.1. *The map $\Gamma : \mathcal{P} \rightarrow \mathcal{R}$ is a bijection.*

In the proof of this theorem, we will use the following version of the Riesz-Markov theorem (see, for example, [15], pg. 75):

THEOREM 13.2.2. *Let X be a locally compact Hausdorff space. For any continuous linear functional l on $C_\infty(X)$, the continuous functions vanishing at infinity, there is a unique, regular, countably additive, complex Borel measure μ on X such that*

$$l[f] = \int_X f(x) d\mu(x)$$

for all $f \in C_\infty(X)$. Furthermore, l is positive if and only if μ is non-negative and $\|l\| = |\mu|(X)$, the total variation of μ .

PROOF. (of Theorem 13.2.1) Γ is onto by definition. Given $\rho \in \mathcal{R}$, one can construct a $Q \in \mathcal{P}$ as follows. By Naimark's theorem $\rho(z) = V(z - A)^{-1}V^*$ where A is a densely defined self-adjoint operator in some Hilbert space \mathcal{K} and $V : \mathcal{K} \rightarrow \mathcal{H}$ is contractive linear map. Choose $z \in \mathbb{U}$ arbitrary. Using the holomorphic functional calculus for closed operators [21], we can use ρ to compute $Vp(A)V^*$ where p is any polynomial in $(z - x)^{-1}$ and $(\bar{z} - x)^{-1}$. By the Stone-Weierstrass theorem, such polynomials are dense in $C_\infty(\mathbb{R})$, the continuous functions vanishing at ∞ . This shows that for any $f \in C_\infty(\mathbb{R})$ we can actually compute $\Phi[f] := Vf(A)V^*$ from the knowledge of $\rho(z)$. This defines a map from $C_\infty(\mathbb{R})$ into $B(\mathcal{H})$. It is clear that this map is linear and self adjoint (i.e. $\Phi[f]^* = \Phi[\bar{f}]$). This map is also contractive since $\|\Phi[f]\| = \sup_{\|\phi\|, \|\psi\| \leq 1} |\langle \Phi[f]\phi, \psi \rangle| \leq \|f(A)V^*\phi\| \|V^*\psi\| \leq \|f\|_\infty \|\phi\| \|\psi\|$. This shows that $\|\Phi[f]\| \leq \|f\|_\infty$ so that $\|\Phi\| \leq 1$. It follows that for any fixed $\phi, \psi \in \mathcal{H}$ the map $l : C_\infty(\mathbb{R}) \rightarrow \mathbb{C}$ defined by $l[f] := \langle \Phi[f]\phi, \psi \rangle$ is a bounded linear functional on $C_\infty(\mathbb{R})$. By the Riesz-Markov representation theorem, Theorem 13.2.2, there exists a unique countably additive complex Borel measure $\mu(\phi, \psi; \cdot)$ on \mathbb{R} such that

$$(13.2.1) \quad l(f) = \int_{-\infty}^{\infty} f(x) d\mu(\phi, \psi; x),$$

$\|l\| = |\mu|(\phi, \psi; \mathbb{R})$, and l is positive if and only if μ is non-negative. Note that l is positive if we choose $\phi = \psi$, and that $\|l\| \leq \|\phi\| \|\psi\|$.

To obtain a POVM, Q , we will extend the definition of Φ to all of $L^\infty(\mathbb{R})$, and then define $Q(\Omega)$ by $\Phi[\chi_\Omega]$, where χ_Ω is the characteristic function of the Borel set Ω . This can be done using a standard technique that is often employed to obtain the L^∞ functional calculus for self-adjoint operators, once one has the continuous functional calculus [57].

Using the fact that $[\cdot, \cdot]_f := \langle \Phi[f]\cdot, \cdot \rangle$ defines a bounded, sesquilinear form on $\mathcal{H} \oplus \mathcal{H}$ for any $f \in C_\infty(\mathbb{R})$, and the uniqueness of the spectral measures $\mu(\phi, \psi; \cdot)$ defined by the Riesz-Markov theorem, it is not difficult to verify the identities: $\mu(\alpha\phi_1 + \phi_2, \psi; \Omega) = \alpha\mu(\phi_1, \psi; \Omega) + \mu(\phi_2, \psi; \Omega)$ and $\bar{\alpha}\mu(\phi, \psi_1; \Omega) + \mu(\phi, \psi_2; \Omega) = \mu(\phi, \alpha\psi_1 + \psi_2; \Omega)$ for all $\Omega \in \mathcal{B}(\mathbb{R})$. These relations hold for all $\alpha \in \mathbb{C}$ and $\phi_1, \phi_2, \psi_1, \psi_2 \in \mathcal{H}$.

Using these relations, and given any $f \in L^\infty(\mathbb{R})$, define the following bounded sesquilinear functional:

$$(13.2.2) \quad [\phi, \psi]_f := \int_{-\infty}^{\infty} f(x) d\mu(\phi, \psi; x).$$

Boundedness follows from the fact that $|\langle \phi, \psi \rangle_f| \leq \|f\|_\infty |\mu|(\phi, \psi; \mathbb{R}) = \|f\|_\infty \|\mu\| \leq \|f\|_\infty \|\phi\| \|\psi\|$. Here, $|\mu|(\phi, \psi; \mathbb{R})$ denotes the total variation of the measure $\mu(\phi, \psi; \cdot)$. By the Riesz representation theorem, there is a unique bounded linear operator, which we denote by $\Phi[f]$, such that $[\phi, \psi]_f = \langle \Phi[f]\phi, \psi \rangle$. In this way, the domain of definition of the map Φ is extended to all of $L^\infty(\mathbb{R})$.

Now define $Q(\Omega) := \Phi[\chi_\Omega]$. This can be seen to define a contractive POVM. Since $\|\chi_\Omega\|_\infty = 1$ for any Borel set Ω and $\|\Phi\| \leq 1$, $\|Q(\Omega)\| \leq 1$ for any Borel subset Ω of \mathbb{R} , and hence Q is contractive. If \emptyset denotes the empty set, then clearly $Q(\emptyset) = 0$. Also it is clear that each $Q(\Omega)$ is positive, since given any $\psi \in \mathcal{H}$, $\langle Q(\Omega)\psi, \psi \rangle = \int_\Omega d\mu(\psi, \psi; x)$, and $\mu(\psi, \psi; \cdot)$ is a positive measure by the Riesz-Markov theorem. It remains to check that if $\Omega := \cup_{n=1}^\infty \Omega_n$ with $\Omega_n \cap \Omega_m = \emptyset$ for $n \neq m$, then $\sum_{n=1}^N Q(\Omega_n) \xrightarrow{s} Q(\Omega)$. First note that since the operators $Q_N := \sum_{n=1}^N Q(\Omega_n)$ are positive, non-decreasing and bounded by 1, the strong limit of the Q_N exists, and is a positive operator bounded by 1. To prove that the strong limit is equal to $Q(\Omega)$, it suffices to show that $Q_N \xrightarrow{w} Q(\Omega)$. This is easily verified since,

$$(13.2.3) \quad \begin{aligned} \lim_{N \rightarrow \infty} \langle Q_N \phi, \psi \rangle &= \lim_{N \rightarrow \infty} \sum_{n=1}^N \int_{\Omega_n} d\mu(\phi, \psi; x) \\ &= \lim_{N \rightarrow \infty} \sum_{n=1}^N \mu(\phi, \psi; \Omega_n) = \mu(\phi, \psi; \Omega) = \int_\Omega d\mu(\phi, \psi; x) = \langle Q(\Omega)\phi, \psi \rangle. \end{aligned}$$

In the above, the countable additivity of the measure $\mu(\phi, \psi; \cdot)$ was used. We conclude that Q_Ω is a contractive POVM. By definition, for any $\phi, \psi \in \mathcal{H}$ and $f \in C_\infty(\mathbb{R})$,

$$(13.2.4) \quad \left\langle \int_{-\infty}^{\infty} f(x) Q(dx) \phi, \psi \right\rangle = \int_{-\infty}^{\infty} f(x) \langle Q(dx) \phi, \psi \rangle = \int_{-\infty}^{\infty} f(x) d\mu(\phi, \psi; x) = \langle \Phi[f]\phi, \psi \rangle.$$

This shows that, in particular, $\rho(z) = \int_{-\infty}^{\infty} (z-x)^{-1} Q(dx)$.

Now suppose ρ was defined by $\rho(z) = \int_{-\infty}^{\infty} (z-x)^{-1} dQ'_x$. Then the POVM Q constructed from ρ in the manner described above is equal to Q' . To see this note that given any polynomial p in $(z-x)^{-1}$ and $(\bar{z}-x)^{-1}$ for some fixed $z \in \mathbb{U}$, one can use the holomorphic functional calculus to show that

$$(13.2.5) \quad \begin{aligned} Vp(A)V^* &= \frac{1}{2\pi i} \int_\Gamma p(\lambda) V(\lambda - A)^{-1} V^* d\lambda = \frac{1}{2\pi i} \int_\Gamma p(\lambda) \rho(\lambda) d\lambda \\ &= \frac{1}{2\pi i} \int_\Gamma p(\lambda) V'(\lambda - A')^{-1} (V')^* d\lambda = V'p(A')(V')^*. \end{aligned}$$

Here A and A' are the self-adjoint operators and V, V' are the contractive linear maps which are determined by dilations of the POVMs Q and Q' to projection valued measures on different Hilbert spaces as described in Theorem 13.1.1. Using that such polynomials are dense in $C_\infty(\mathbb{R})$ it follows that for any $f \in C_\infty(\mathbb{R})$, $Vf(A)V^* = \int_{-\infty}^{\infty} f(x) dQ_x = \int_{-\infty}^{\infty} f(x) dQ'_x = V'f(A')(V')^*$.

Given any Borel set Ω , χ_Ω can be seen as the pointwise limit of such functions f . Using this fact it is not difficult to show that $Q(\Omega) = Q'(\Omega)$ for any Borel set Ω , so that $Q = Q'$.

In particular, if $\tilde{Q}, Q' \in \mathcal{P}$ and if $\tilde{\rho} = \rho'$ where $\rho'(z) := \int_{-\infty}^{\infty} (z-x)^{-1} Q'(dx)$, then, by the preceding arguments, $Q' = Q = \tilde{Q}$, where Q is the POVM constructed from $\rho' = \tilde{\rho}$. This shows that the map Γ is 1-1. Since it is surjective by definition, we conclude that Γ is a bijection. \square

13.2.0.12. *Remark.* Observe that Γ and Γ^{-1} respect convex combinations. That is $\Gamma(tQ_1 + (1-t)Q_2) = t\Gamma(Q_1) + (1-t)\Gamma(Q_2)$ for all $Q_1, Q_2 \in \mathcal{P}$, and $t \in [0, 1]$, and Γ^{-1} has the same properties.

13.3. Pre-compactness of the set of generalized resolvents

Consider the set \mathcal{A} of all analytic operator-valued functions on $\mathbb{U} := \mathbb{C} \setminus \mathbb{R}$ with the topology of uniform weak operator convergence on compacta. The set \mathcal{A} with this topology is a locally convex linear Hausdorff space. Given any fixed compact set $K \subset \mathbb{U}$, with non-empty interior, let \mathcal{A}_K denote the set of all operator-valued functions on K which are analytic in the interior of K , with the topology of uniform weak-operator convergence on K . As we will see, these two topologies are equivalent on the set \mathcal{R}

Recall that for a densely defined symmetric operator B in \mathcal{H} $\mathcal{R}(B)$, is defined to be all the set of generalized resolvents $\rho(z)$ such that $\rho(z) = P_{\mathcal{H}}(z-A)^{-1}|_{\mathcal{H}}$ where A is a self-adjoint extension of B , in general to a larger space $\tilde{\mathcal{H}} \supset \mathcal{H}$.

In [13] it is established that the set $\mathcal{R}(B) \subset \mathcal{A}$ of all generalized resolvents of a fixed densely defined symmetric operator B is pre-compact in the topology of \mathcal{A} . Here a set is called pre-compact if it has compact closure. Their argument extends without modification to show that all of \mathcal{R} is pre-compact in this topology. For the convenience of the reader, their proof is reproduced here.

THEOREM 13.3.1. *$\mathcal{R} \subset \mathcal{A}$ is pre-compact*

LEMMA 13.3.2. *If X is a reflexive Banach space, then the unit ball in $B(X)$ is compact in the weak operator topology.*

PROOF. This appears as a problem in ([21], pg. 53). Its proof is a straightforward exercise using the Banach-Alaoglu theorem, and Tychonoff's theorem. \square

LEMMA 13.3.3. *The set \mathcal{R} , viewed as a subset of the continuous functions from any closed half plane $V_\delta := \overline{UHP} + i\delta$, $\delta > 0$, into $B(\mathcal{H})$ (with the supremum norm), is uniformly equicontinuous and uniformly bounded. In particular, \mathcal{R} is uniformly equicontinuous in $C(K, B(\mathcal{H}))$ where K is any fixed compact subset of \mathbb{U} .*

PROOF. It is easy to see that $\|\rho(z)\| \leq \frac{1}{\text{Im}(z)}$. This implies uniform boundedness. Since the functions $\rho(z)$ are analytic, the fact that they are uniformly bounded on compacta automatically implies that they are uniformly equicontinuous on any compact subset $\mathcal{K} \subset \mathbb{U}$. In this special case, however, we can also use the resolvent formula to establish equicontinuity.

Given $\rho(A) \in \mathcal{R}$ and $z, w \in V_\delta$,

$$\begin{aligned} \|\rho(A)(z) - \rho(A)(w)\| &= \|V(R_z(A) - R_w(A))V^*\| \leq \|R_z(A) - R_w(A)\| \\ &= |z-w| \|R_z(A)R_w(A)\| \leq |z-w| \|R_z(A)\| \|R_w(A)\| \\ (13.3.1) \quad &\leq |z-w| \frac{1}{\text{Im}(z)\text{Im}(w)} \leq |z-w| \frac{1}{\delta^2}. \end{aligned}$$

It follows that given any $\epsilon > 0$, if $|z-z'| < \epsilon\delta^2$, then $\|\rho(A)(z) - \rho(A)(z')\| < \epsilon$ for any $\rho(A) \in \mathcal{R}$. This shows that \mathcal{R} is uniformly equicontinuous on V_δ . \square

13.3.0.13. *Remark.* Since \mathcal{R} is uniformly equicontinuous on any compact $K \subset \mathbb{U}$, if any net $(\rho_\alpha)_{\alpha \in \Lambda} \subset \mathcal{R}$ converges with respect to the topology of pointwise weak-operator convergence on \mathbb{U} , it automatically converges with respect to the topology of \mathcal{A} . That is, for \mathcal{R} , the topology of pointwise weak operator convergence on \mathbb{U} is equivalent to the topology of uniform weak operator convergence on compact subsets of \mathbb{U} .

PROOF. (of Theorem 13.3.1) By Lemma 13.3.2, for each $z \in \mathbb{U}$, the ball $B(z)$ in $B(\mathcal{H})$ of radius $\frac{1}{\text{Im}(z)}$ is compact in the weak-operator topology. By Tychonoff's theorem, the Cartesian product $\pi := \prod_{z \in \mathbb{U}} B(z)$ is compact in the product topology. This product is the set of all functions f from \mathbb{U} to $B(\mathcal{H})$ with the property that $\|f(z)\| \leq \frac{1}{\text{Im}(z)}$, and the topology is just that of pointwise weak-operator convergence.

Now let $(\rho_\alpha)_{\alpha \in \Lambda} \subset \mathcal{R}$ be an arbitrary net. Since this net belongs to π , and π is compact, there is a Cauchy subnet $(\rho_\beta)_{\beta \in \Omega}$ which converges to an element of π . By Remark 13.3.0.13, we conclude that this subnet is Cauchy with respect to the topology of \mathcal{A} . This proves that \mathcal{R} is pre-compact. \square

13.4. Closedness of the set of generalized resolvents

To conclude that \mathcal{R} is compact with respect to the topology of \mathcal{A} , it remains to show that it is closed. Let $K \subset \mathbb{U}$ be any compact subset with non-empty interior, and let \mathcal{A}_K denote the set of operator-valued functions on K which are analytic on the interior of K , endowed with the topology of uniform weak-operator convergence on K . It will be shown that the set \mathcal{R} , viewed as a subset of \mathcal{A}_K , or as a subset of \mathcal{A} , is compact. Pre-compactness of \mathcal{R} in \mathcal{A}_K follows from the pre-compactness of \mathcal{R} in \mathcal{A} .

'THEOREM 13.4.1. \mathcal{R} is a closed subset of \mathcal{A}_K .

LEMMA 13.4.2. Let A be a densely defined self-adjoint operator on a Hilbert space \mathcal{H} . Fix $z' \in \mathbb{U} = \mathbb{C} \setminus \mathbb{R}$. Let Γ be a closed contour in \mathbb{U} such that $\text{Ind}_\Gamma(z') = 1$. Then

$$(z' - A)^{-k} = \frac{1}{2\pi i} \int_\Gamma (z' - z)^{-k} (z - A)^{-1} dz.$$

PROOF. This follows immediately from the holomorphic functional calculus for closed operators whose spectrum is confined to a sector of the complex plane [21]. It can also be proven with a straightforward application of the residue theorem. \square

LEMMA 13.4.3. Given a self-adjoint operator A , the following formula holds for all $z \neq z' \in \mathbb{U}$.

$$\begin{aligned} (z - A)^{-k} (z' - A)^{-j} &= (z - z')^{-k+1} \sum_{i=0}^{j-1} (-1)^i (z - z')^{-i-1} (z' - A)^{j-i} (1 + i(k-1)) \\ &\quad - (z - z')^{-j+1} \sum_{i=0}^{k-1} (z - z')^{-i-1} (z - A)^{k-i} (1 + i(j-1)). \end{aligned}$$

PROOF. This is easily established with the first resolvent formula:

$$(13.4.1) \quad (z - A)^{-1} (z' - A)^{-1} = \frac{1}{z - z'} ((z' - A)^{-1} - (z - A)^{-1})$$

for $z \neq z'$, and induction. \square

PROOF. (of 'Theorem 13.4.1)

Let $(\rho_\alpha)_{\alpha \in \Lambda}$ be a Cauchy net in $\mathcal{R} \subset \mathcal{A}$. Since each of the nets of operators $\rho_\alpha(z)$, for $z \in K$, is weakly convergent and uniformly bounded, ρ_α converges pointwise in the weak operator topology to a bounded operator valued function ρ . Since $(\rho_\alpha)_{\alpha \in \Lambda} \subset \mathcal{R}$, this convergence is equivalent to uniform convergence on compacta (see Remark 13.3.0.13), so that $\rho \in \mathcal{A}$.

Let K be a fixed compact set in \mathbb{U} . Let λ be an interior point of K and let Γ be a closed contour inside K satisfying $\text{Ind}_\Gamma(\lambda) = 1$. By Naimark's theorem, each ρ_α can be expressed as $\rho_\alpha(z) = V_\alpha(z - A_\alpha)^{-1}V_\alpha^*$ for some self-adjoint operator A_α on a Hilbert space \mathcal{K}_α and where V_α is a contractive linear map from \mathcal{K}_α to \mathcal{H} . Given any $f \in L^\infty(\mathbb{R})$, let $\Phi_\alpha[f] := V_\alpha f(A_\alpha)V_\alpha^*$. It follows that for any $\phi, \psi \in \mathcal{H}$, and $k \in \mathbb{N}$ we have

$$\begin{aligned}
|\langle (\Phi_\alpha[(\lambda - x)^{-k}] - \Phi_\beta[(\lambda - x)^{-k}])\phi, \psi \rangle| &= \left| \frac{1}{2\pi i} \int_\Gamma (\lambda - z)^{-k} \langle (V_\alpha(z - A_\alpha)^{-1}V_\alpha^* \phi - V_\beta(z - A_\beta)^{-1}V_\beta^* \phi) \langle \psi \rangle dz \right| \\
&\leq \frac{1}{2\pi} \int_\Gamma |\lambda - z|^{-k} |\langle (\rho_\alpha(z) - \rho_\beta(z))\phi, \psi \rangle| |dz| \\
(13.4.2) \quad &\leq \frac{1}{2\pi} C \int_\Gamma |\langle (\rho_\alpha(z) - \rho_\beta(z))\phi, \psi \rangle| |dz|,
\end{aligned}$$

where $C := \max_{z \in \Gamma} |z - \lambda|^{-k} < \infty$. Since $(\rho_\alpha)_{\alpha \in \Lambda}$ is Cauchy in the topology of uniform weak operator convergence on K , and $\|\Gamma\| := \int_\Gamma |dz| < \infty$, we conclude that for each $k \in \mathbb{N}$, the net of operators $(\Phi_\alpha[(\lambda - x)^{-k}])_{\alpha \in \Lambda}$ is Cauchy in the weak operator topology. Since for any $\phi, \psi \in \mathcal{H}$,

$$(13.4.3) \quad \langle V_\alpha R_\lambda(A_\alpha)^k V_\alpha^* \phi, \psi \rangle = \langle \phi, V_\alpha R_{\bar{\lambda}}(A_\alpha)^k V_\alpha^* \psi \rangle,$$

it follows that the nets $(\Phi_\alpha[(\bar{\lambda} - x)^{-1}])_{\alpha \in \Lambda}$ are also Cauchy in the weak operator topology. Finally, by applying Lemma 13.4.3 we conclude that given any polynomial p in $(\lambda - x)^{-1}$ and $(\bar{\lambda} - x)^{-1}$, that $(\Phi_\alpha[p])_{\alpha \in \Lambda}$ is Cauchy in the weak operator topology.

The Stone-Weierstrass theorem implies that the polynomials in $(\lambda - x)^{-1}$ and $(\bar{\lambda} - x)^{-1}$ are dense in $C_\infty(\mathbb{R})$, the continuous functions vanishing at ∞ . Using this fact it is easy to show that for any $f \in C_\infty(\mathbb{R})$, the nets $(\Phi_\alpha[f])_{\alpha \in \Lambda}$ are Cauchy in the weak operator topology, and hence converge to bounded operators which we will denote $\Phi[f]$.

As in the proof of Theorem 13.2.1, it is straightforward to show that the map $\Phi : C_\infty(\mathbb{R}) \rightarrow B(\mathcal{H})$ is linear, contractive, and self-adjoint. Using the exact same technique as in the proof of Theorem 13.2.1, one can use the spectral measures defined by the Riesz-Markov theorem to construct a contractive POVM $Q(\cdot)$ such that for any $f \in C_\infty(\mathbb{R})$, $\Phi[f] = \int_{-\infty}^{\infty} f(\lambda)Q(d\lambda)$. In particular, we conclude that $\rho(w) = \Phi[(w - x)^{-1}] = \int_{-\infty}^{\infty} (w - \lambda)^{-1}Q(d\lambda)$. Using this formula we can extend $\rho(z)$ to a unique analytic function on \mathbb{U} . This shows that $\rho \in \mathcal{R}$. We conclude that \mathcal{R} is closed in this topology \square

'COROLLARY 13.4.4. \mathcal{R} is a closed subset of \mathcal{A}

PROOF. This is a straightforward consequence of the previous result, 'Theorem 13.4.1. Indeed, if $(\rho_\alpha)_{\alpha \in \Lambda} \subset \mathcal{R}$ is Cauchy in the topology of \mathcal{A} , it converges to an element $\rho \in \mathcal{A}$ since \mathcal{A} is closed with respect to this topology. This net will also be Cauchy in the topology of \mathcal{A}_K . Since \mathcal{R} is a closed subset of \mathcal{A}_K , this net converges to an element $\rho' \in \mathcal{R} \subset \mathcal{A}_K$. Since $\rho(z) = \rho'(z)$ for all $z \in K$, we have that $\rho(z) = \rho'(z)$ for all $z \in \mathbb{U}$ since they are analytic on \mathbb{U} . Since, as in the proof of 'Theorem 13.4.1 we can define $\rho'(z) = \int_{-\infty}^{\infty} (z - \lambda)^{-1}Q'(d\lambda)$ for all $z \in \mathbb{U}$, it follows that $\rho = \rho' \in \mathcal{R}$ and that \mathcal{R} is closed in \mathcal{A} . \square

We can now conclude that the set of generalized resolvents \mathcal{R} is compact in the topology of uniform weak operator convergence on compacta of \mathbb{U} , or equivalently with respect to pointwise weak operator convergence on some fixed compact $K \subset \mathbb{U}$ with non-empty interior.

13.5. The topology on \mathcal{R} and the induced topology on \mathcal{P}

The previous results show that a net $(\rho_\alpha)_{\alpha \in \Lambda} \subset \mathcal{R}$ converges to an element $\rho \in \mathcal{R}$ with respect to the topology of \mathcal{A}_K if and only if $\Phi_\alpha[f] \xrightarrow{w} \Phi[f]$ for every $f \in C_\infty(\mathbb{R})$. Here \xrightarrow{w} denotes convergence in the weak operator topology. In particular, if $f(x) = (z - x)^{-1}$ for any $z \in \mathbb{U}$ then $\rho_\alpha(z) = \Phi_\alpha[f] \xrightarrow{w} \Phi[f] = \rho(z)$. This shows that for the set \mathcal{R} , the topology of \mathcal{A}_K is equivalent to that of pointwise convergence on $\mathbb{U} = \mathbb{C} \setminus \mathbb{R}$. By Remark 13.3.0.13, this topology is equivalent to the topology of \mathcal{A} on \mathcal{R} . That is, ρ_α converges with respect to the topology of pointwise weak-operator convergence on a fixed $K \subset \mathbb{U}$ with non-empty interior, if and only if it converges uniformly (in the weak-operator topology) on all compact subsets of \mathbb{U} .

Observe that the set \mathcal{R}' of all resolvents such that $\Gamma^{-1}(\rho)$ is unital is not closed in this topology. Indeed, if I is the identity operator on \mathcal{H} , consider the sequence $(\rho_n)_{n=1}^\infty \subset \mathcal{R}'$ defined by $\rho_n(z) := (z - nI)^{-1}$. It is not hard to see that $\rho_n(z) \xrightarrow{s} \rho(z) = 0$ for each $z \in \mathbb{U}$. Since Γ^{-1} is injective we conclude that $\Gamma^{-1}(\rho) = \Gamma^{-1}(0) = 0$. Thus the limit of $\Gamma^{-1}(\rho_n)$ is not a unital POVM, even though each $\Gamma^{-1}(\rho_n)$ is unital.

The topology on \mathcal{R} induces a natural topology on \mathcal{P} . The proof of Theorem 13.4.1 has shown that $Q_\alpha \rightarrow Q \in \mathcal{P}$ in this topology if and only if $\Phi_\alpha[f] \xrightarrow{w} \Phi[f]$ for any $f \in C_\infty(\mathbb{R})$, where $\Phi_\alpha[f] := \int_{-\infty}^\infty f(x)Q_\alpha(dx)$ and $\Phi[f] := \int_{-\infty}^\infty f(x)Q(dx)$. Furthermore, one can even say certain things about the convergence of the $\Phi_\alpha[f]$ for more general $f \in L^\infty(\mathbb{R})$, as shown below.

THEOREM 13.5.1. *Suppose that $(\rho_\alpha)_{\alpha \in \Lambda} \subset \mathcal{R}$ is Cauchy and $\rho_\alpha \rightarrow \rho$. Let $Q_\alpha := \Gamma^{-1}(\rho_\alpha)$ and $\Phi_\alpha[f] := \int_{-\infty}^\infty f(\lambda)Q_\alpha(d\lambda)$, as before. Let A be one of the self-adjoint operators such that $\rho(z) = V(z - A)^{-1}V^*$. If a, b are not eigenvalues of A , then $\Phi_\alpha[\chi_{(a,b)}] \xrightarrow{w} \Phi[\chi_{(a,b)}]$.*

This theorem is a simple modification of the results of ([57], pgs. 290-291) and is omitted.

As a final remark, note that the set \mathcal{R} and hence \mathcal{P} with these topologies is first countable ([13], pg. 221).

13.6. Closed convex subsets and faces of the set of generalized resolvents

Consider the subset \mathcal{R}_a of all unital POVMs Q such that $Q([-a, a]) = I$ for some fixed $a > 0$. A subset S of a convex set C is called a **face** if given $s = tc_1 + (1 - t)c_2$, $t \in [0, 1]$ where $s \in S$, $c_1, c_2 \in C$ implies that $c_1, c_2 \in S$. A point $c \in C$ is called an extreme point if $\{c\}$ is a face of C .

CLAIM 13.6.1. \mathcal{R}_a is a face of \mathcal{R} .

PROOF. Suppose that $Q \in \mathcal{R}_a$ and that $Q = tQ_1 + (1 - t)Q_2$ for some $t \in (0, 1)$. Since each Q_i is contractive, it is clear that for any $\phi \in \mathcal{H}$, $\langle Q_i([-a, a])\phi, \phi \rangle \leq \langle \phi, \phi \rangle$. I claim that equality holds for every $\phi \in \mathcal{H}$. Otherwise, if there exists $\phi \in \mathcal{H}$ such that $\langle Q_1([-a, a])\phi, \phi \rangle < \|\phi\|^2$ then,

$$\begin{aligned} \langle \phi, \phi \rangle &= \langle Q([-a, a])\phi, \phi \rangle \\ &= t\langle Q_1([-a, a])\phi, \phi \rangle + (1 - t)\langle Q_2([-a, a])\phi, \phi \rangle \\ (13.6.1) \quad &< t\langle \phi, \phi \rangle + (1 - t)\langle \phi, \phi \rangle = \langle \phi, \phi \rangle, \end{aligned}$$

which is a contradiction. The argument in Remark 13.1.0.11 allows one to conclude that $Q_i([-a, a]) = I$ for each $i = 1, 2$. Hence $Q_1, Q_2 \in \mathcal{R}_a$, and \mathcal{R}_a is a face. \square

COROLLARY 13.6.2. \mathcal{R}_a is a closed subset of \mathcal{R}

PROOF. Consider a Cauchy net $\rho_\alpha \subset \mathcal{R}_a$. By Theorem (13.4.1), $\rho_\alpha \rightarrow \rho \in \mathcal{R}$. It remains to verify that $Q := \Gamma^{-1}(\rho)$ obeys $Q([-a, a]) = I$. Let $Q_\alpha := \Gamma^{-1}(\rho_\alpha)$. Choose a sequence of positive functions $(f_n)_{n \in \mathbb{N}} \subset C_\infty(\mathbb{R})$ such that $f_n(x) = 1$ for all $x \in [-a, a]$, $f_n(x) \leq 1$ for all $x \in \mathbb{R}$, each

f_n has support contained in $[-a - 1/n, a + 1/n]$, $f_n \geq f_{n+1}$ and $f_n \rightarrow \chi_{[-a,a]}$ pointwise. As in the proof of Theorem 13.2.1, let $\Phi_\alpha[f] := \int_{-\infty}^{\infty} f(\lambda)Q_\alpha(d\lambda)$ for any $f \in L^\infty(\mathbb{R})$. Then,

$$(13.6.2) \quad \Phi_\alpha[f_n] = \int_{-\infty}^{\infty} f_n(x)Q_\alpha(dx) = \int_{-a}^a Q_\alpha(dx) = \chi_{[-a,a]}(A_\alpha) = Q_\alpha([-a, a]) = I$$

for every n . Also, $\Phi_\alpha[f_n] = I$ converges weakly to $\Phi[f_n] = \int_{-\infty}^{\infty} f(x)dQ_x$ for each $n \in \mathbb{N}$. It follows that $\Phi[f_n] = I$ for every $n \in \mathbb{N}$.

Given any $\phi \in \mathcal{H}$,

$$(13.6.3) \quad \begin{aligned} \left| \langle (\Phi[f_n] - \Phi[\chi_{[-a,a]}])\phi, \phi \rangle \right| &= \int_{-\infty}^{\infty} (f_n(x) - \chi_{[-a,a]}(x))d\mu(\phi, \phi; x) \\ &\leq \int_{-a-1/n}^{-a-0} + \int_{a+0}^{a+1/n} d\mu(\phi, \phi; x) \rightarrow 0 \end{aligned}$$

It can be concluded that $\langle (I - \Phi(\chi_{[-a,a]}))\phi, \phi \rangle = 0$ for all $\phi \in \mathcal{H}$, and hence by Remark 13.1.0.11, that $Q([-a, a]) = \Phi(\chi_{[-a,a]}) = I$. This shows that $\rho \in \mathcal{R}_a$, establishing the claim. \square

13.6.1. The convex subset of generalized resolvents of a single symmetric operator. Recall that for a densely defined symmetric operator B in \mathcal{H} , the set of generalized resolvents $\mathcal{R}(B)$ is defined to be the set of all ρ such that $\rho(z) = P_{\mathcal{H}}(A - z)^{-1}|_{\mathcal{H}}$, where A is a self-adjoint extension of B , in general to a larger space $\tilde{\mathcal{H}} \supset \mathcal{H}$. Let \mathbb{A} denote the set of all such self-adjoint extensions of B . A generalized resolvent $\rho \in \mathcal{R}(B)$ will be called canonical if it corresponds to a self-adjoint extension of B within \mathcal{H} , i.e., if $\rho(z) = (z - A)^{-1}$, where $A \supset B$ is self-adjoint. The proof of the fact that $\mathcal{R}(B)$ is convex is elementary, and is omitted.

*THEOREM 13.6.3. $\mathcal{R}(B)$ is a face in \mathcal{R} .

PROOF. Suppose $\rho \in \mathcal{R}(B)$ and $\rho = t\rho_1 + (1-t)\rho_2$, where $\rho_1, \rho_2 \in \mathcal{R}$. Since $\Gamma^{-1}(\rho)$ is a unital POVM, the same argument as in Claim 13.6.1 can be used to conclude that the POVMs $\Gamma^{-1}(\rho_1)$ and $\Gamma^{-1}(\rho_2)$ are unital, so that by Naimark's theorem 13.1.1, $\rho_i := P_i R_z(A_i)|_{\mathcal{H}} = P_i(A_i - z)^{-1}|_{\mathcal{H}}$, $i = 1, 2$. Here, A_i are densely defined self-adjoint operators on $\tilde{\mathcal{H}}_i$ and P_i are projectors from $\tilde{\mathcal{H}}_i \supset \mathcal{H}$ onto \mathcal{H} . Similarly, $\rho = P R_z(A)|_{\mathcal{H}}$ for some self-adjoint A which is densely defined in a Hilbert space $\tilde{\mathcal{H}} \supset \mathcal{H}$ and P is the projector from $\tilde{\mathcal{H}}$ onto \mathcal{H} . Note that $\rho = t\rho_1 + (1-t)\rho_2$ implies that $P R_z(A)|_{\mathcal{H}} = tP_1 R_z(A_1)|_{\mathcal{H}} + (1-t)P_2 R_z(A_2)|_{\mathcal{H}}$.

Fix $z \in \mathbb{U}$. We will first show that $P_1 R_z(A_1)\phi = P_2 R_z(A_2)\phi$ for all $\phi \in \mathfrak{R}(B - z)$. Since $A \in \mathbb{A}$, it follows that $P R_z(A)\mathfrak{R}(B - z) = R_z(A)\mathfrak{R}(B - z) = R_z(B)\mathfrak{R}(B - z)$. Hence, for this z , and $\phi \in \mathfrak{R}(B - z)$,

$$(13.6.4) \quad (\bar{z} - z)\langle \rho(z)\phi, \rho(z)\phi \rangle = (\bar{z} - z)\langle R_{\bar{z}}(A)R_z(A)\phi, \phi \rangle$$

$$(13.6.5) \quad = \langle (R_{\bar{z}}(A) - R_z(A))\phi, \phi \rangle.$$

Equation (13.6.5) can be written as:

$$(13.6.6) \quad \begin{aligned} &\langle (t(R_{\bar{z}}(A_1) - R_z(A_1)) + (1-t)(R_{\bar{z}}(A_2) - R_z(A_2)))\phi, \phi \rangle \\ &= (\bar{z} - z)\langle (tR_{\bar{z}}(A_1)R_z(A_1) + (1-t)R_{\bar{z}}(A_2)R_z(A_2))\phi, \phi \rangle \\ &= (\bar{z} - z)\langle (tR_{\bar{z}}(A_1)P_1R_z(A_1) + (1-t)R_{\bar{z}}(A_2)P_2R_z(A_2))\phi, \phi \rangle \end{aligned}$$

$$(13.6.7) \quad + (\bar{z} - z)\langle (t(I_1 - P_1)R_z(A_1)\phi, (I_1 - P_1)R_z(A_1)\phi) + (1-t)\langle (I_2 - P_2)R_z(A_2)\phi, (I_2 - P_2)R_z(A_2)\phi) \rangle.$$

Here, I_i are the identity operators on $\tilde{\mathcal{H}}_i$. Since $t \in (0, 1)$, observe that line (13.6.7) divided by $\bar{z} - z$ is positive.

Equation (13.6.4) is also equal to:

$$(13.6.8) \quad \begin{aligned} &(\bar{z} - z)\langle (tR_{\bar{z}}(A_1) + (1-t)R_{\bar{z}}(A_2))P(tR_z(A_1) + (1-t)R_z(A_2))\phi, \phi \rangle \\ &= (\bar{z} - z)\langle (t^2R_{\bar{z}}(A_1)P_1R_z(A_1) + t(1-t)R_{\bar{z}}(A_1)P_2R_z(A_2) \\ &\quad + t(1-t)R_{\bar{z}}(A_2)P_1R_z(A_1) + (1-t)^2R_{\bar{z}}(A_2)P_2R_z(A_2))\phi, \phi \rangle. \end{aligned}$$

Since Equation (13.6.4) is equal to Equation (13.6.5), it follows that subtracting line (13.6.6) from (13.6.4) and dividing by $\bar{z} - z$ yields (13.6.7) which is positive. However, by equation (5) ((13.6.4) - (13.6.6)) divided by $(\bar{z} - z)$ is equal to

$$\begin{aligned}
& ([t(t-1)R_{\bar{z}}(A_1)P_1R_z(A_1) + t(1-t)(R_{\bar{z}}(A_1)P_2R_z(A_2) \\
& \quad + R_{\bar{z}}(A_2)P_1R_z(A_1)) + t(t-1)R_{\bar{z}}(A_2)P_2R_z(A_2)]\phi)\langle\phi\rangle \\
& = t(t-1)([R_{\bar{z}}(A_1)(P_1R_z(A_1) - P_2R_z(A_2)) - R_{\bar{z}}(A_2)(P_1R_z(A_1) - P_2R_z(A_2))]\phi, \phi) \\
& = t(t-1)\langle(R_{\bar{z}}(A_1) - R_{\bar{z}}(A_2))(P_1R_z(A_1) - P_2R_z(A_2))\phi, \phi\rangle \\
(13.6.9) \quad & = -t(1-t)\langle(P_1R_z(A_1) - P_2R_z(A_2))\phi, (P_1R_z(A_1) - P_2R_z(A_2))\phi\rangle \leq 0.
\end{aligned}$$

Since (13.6.9) must be ≥ 0 and $t(1-t) \neq 0$ for any $t \in (0, 1)$, we conclude that $P_1R_z(A_1)\phi = P_2R_z(A_2)\phi$ for any $\phi \in \mathfrak{R}(B-z)$.

To show that $\rho_1, \rho_2 \in \mathcal{R}(B)$, it is sufficient to show that $\rho_i(z)\phi = R_z(A)\phi$ for all $\phi \in \mathfrak{R}(B-z)$. So far it has been shown that $R_z(A)\phi = PR_z(A)\phi = P(tR_z(A_1) + (1-t)R_z(A_2))\phi = P_1R_z(A_1)\phi = P_2R_z(A_2)\phi$ for all such ϕ . It follows that $R_z(A_i)\phi = R_z(A)\phi + h_i$ where $h_i \in \widetilde{\mathcal{H}}_i \ominus \mathcal{H}$ for $i = 1, 2$. It remains to prove that $h_i = 0$. This is easily accomplished using the following simple argument. First,

$$(13.6.10) \quad A_i(R_z(A)\phi + h_i) = A_iR_z(A_i)\phi = \phi + zR_z(A_i)\phi = \phi + zR_z(A)\phi + zh_i = AR_z(A)\phi + zh_i$$

for any $\phi \in \mathfrak{R}(B-z)$. It then follows that

$$\begin{aligned}
(13.6.11) \quad \mathbb{R} \ni \langle A_iR_z(A_i)\phi, R_z(A_i)\phi \rangle & = \langle AR_z(A)\phi + zh_i, R_z(A)\phi + h_i \rangle \\
& = \langle AR_z(A)\phi, R_z(A)\phi \rangle + z\langle h_i, h_i \rangle.
\end{aligned}$$

Since $z \notin \mathbb{R}$, it must be that each $h_i = 0$. We conclude that $A_1\phi = A_2\phi = A\phi = B\phi$ for all $\phi \in \mathfrak{D}(B)$, so that $\rho_1, \rho_2 \in \mathcal{R}(B)$. □

By Remark 13.2.0.12, we see that $\mathcal{P}(B)$ is a face in \mathcal{P} . In particular,

'COROLLARY 13.6.4. *Any unital PVM is an extreme point of \mathcal{P} .*

PROOF. Given any unital PVM, $Q(\cdot)$, let A be the self-adjoint operator defined by $A\phi = \int_{-\infty}^{\infty} \lambda Q(d\lambda)\phi$ on the dense domain of all $\phi \in \mathcal{H}$ for which this integral exists. Since A is self-adjoint, the face $\mathcal{R}(A)$ reduces to the point $\rho(z) = (z-A)^{-1}$ and so this point is an extreme point of \mathcal{R} . By Remark 13.2.0.12, $\Gamma^{-1}(\rho) = Q$ is an extreme point of \mathcal{P} . □

It has already been shown that $\mathcal{R}(B)$ is compact in the topology of \mathcal{A} [13]. For completeness, alternative proof of this result which shows how it can be obtained as a corollary of 'Theorem 13.4.1 is provided below.

'PROPOSITION 13.6.5. *$\mathcal{R}(B)$ is a closed subset of \mathcal{R} .*

Since \mathcal{R} is compact, this will prove that $\mathcal{R}(B)$ is a convex, compact subset of \mathcal{R} with the topology of pointwise weak operator convergence on compact subsets of \mathbb{U} .

PROOF. Given a Cauchy net $(\rho_\alpha)_{\alpha \in \Lambda} \subset \mathcal{R}(B)$, $\rho_\alpha \rightarrow \rho \in \mathcal{R}$ by Theorem 13.4.1. To conclude that $\rho \in \mathcal{R}(B)$ it remains to verify that $Q = \Gamma^{-1}(\rho)$ is unital and that $\int_{-\infty}^{\infty} \lambda dQ_\lambda\phi = B\phi$ for all $\phi \in \mathfrak{D}(B)$.

To show that $Q(\mathbb{R}) = I$, it will be shown that $Q_\lambda := Q(-\infty, \lambda] \xrightarrow{s} I$. Since this limit clearly exists, and is a positive operator of norm less than or equal to 1, it is sufficient to show that

$$(13.6.12) \quad \langle Q_\lambda\phi, \phi \rangle \rightarrow \langle \phi, \phi \rangle$$

for all vectors ϕ in a dense set. For this purpose, consider $\phi \in \mathfrak{D}(B)$. Since $\langle \rho(z)\phi, \phi \rangle = \int_{-\infty}^{\infty} \frac{1}{z-\lambda} dQ_x$, equation (13.6.12) will hold provided that $\lim_{y \rightarrow \infty} iy \langle \rho(iy)\phi, \phi \rangle = \langle \phi, \phi \rangle$. Let $Q_\alpha = \Gamma^{-1}(\rho_\alpha)$. Then,

$$\begin{aligned}
|iy \langle \rho_\alpha(iy)\phi, \phi \rangle - \langle \phi, \phi \rangle| &= \left| \int_{-\infty}^{\infty} \frac{\lambda}{\lambda - iy} \langle Q_\alpha(d\lambda)\phi, \phi \rangle \right| \\
&\leq \left(\int_{-\infty}^{\infty} \lambda^2 \langle Q_\alpha(d\lambda)\phi, \phi \rangle \right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} \frac{1}{|\lambda - iy|^2} \langle Q_\alpha(d\lambda)\phi, \phi \rangle \right)^{\frac{1}{2}} \\
(13.6.13) \quad &\leq \frac{1}{y} \|B\phi\| \|\phi\|.
\end{aligned}$$

This shows that $iy \langle \rho_\alpha(iy)\phi, \phi \rangle \rightarrow \langle \phi, \phi \rangle$ uniformly for $\alpha \in \Lambda$. Therefore, given any $\epsilon > 0$, choose $B > 0$ such that $y > B$ implies that $|iy \langle \rho_\alpha(iy)\phi, \phi \rangle - \langle \phi, \phi \rangle| < \frac{\epsilon}{2}$ for all $\alpha \in \Lambda$. Then, for any fixed $y > B$, choose $\alpha_y \in \Lambda$ so that $\alpha > \alpha_y$ implies that $|\langle (\rho(iy) - \rho_\alpha(iy))\phi, \phi \rangle| < \frac{\epsilon}{2}$. It follows that for this y ,

$$(13.6.14) \quad |iy \langle \rho(iy)\phi, \phi \rangle - \langle \phi, \phi \rangle| < \epsilon.$$

Since $y > B$ was arbitrary, equation (13.6.14) holds for all $y > B$. That is, given any $\epsilon > 0$ there is a $B > 0$ such that $y > B$ implies that equation (13.6.14) holds. We conclude that $iy \langle \rho(iy)\phi, \phi \rangle \rightarrow \langle \phi, \phi \rangle$. Since $\phi \in \mathfrak{D}(B)$ was arbitrary, $Q(\mathbb{R}) = I$. To complete the proof, it needs to be shown that if A is the self-adjoint operator such that $\rho(z) = P_{\mathcal{H}}(A - z)^{-1}|_{\mathcal{H}}$, then A is an extension of B . This is equivalent to showing that $R_z(A)\phi = R_z(B')\phi$, where B' is an arbitrary canonical extension of B and $\phi \in \mathfrak{R}(z - B)$. First of all, by assumption, we have that for any such ϕ , and any $\psi \in \mathcal{H}$,

$$(13.6.15) \quad \langle \rho(z)\phi, \psi \rangle = \lim_{\alpha} \langle \rho_\alpha(z)\phi, \psi \rangle = \langle R_z(B')\phi, \psi \rangle$$

from which it can be concluded that given any such ϕ , $R_z(A)\phi = R_z(B')\phi + h$ where $h \in \tilde{\mathcal{H}} \ominus \mathcal{H}$, and $\tilde{\mathcal{H}}$ is the Hilbert space on which A is densely defined. It remains to prove that $h = 0$. This can be accomplished using the exact same argument as in the end of *Theorem 13.6.3. Applying this argument proves the proposition. \square

13.6.1.1. *The case of finite deficiency indices.* In the case where B is a symmetric operator with finite deficiency indices, $\mathcal{R}(B)$ is compact with respect to a much stronger topology. From now on, B will denote a closed, densely defined symmetric operator with finite deficiency indices (m, n) .

Consider the set \mathcal{A} of analytic operator-valued functions on \mathbb{U} , this time endowed with the stronger topology of uniform operator-norm convergence on compacta. This is a metrizable topology.

Let $G \subset \mathbb{C}$ be an open set, and let (X, d) be a complete metric space. Define the following metric on $C(G, X)$. First let $(K_n)_{n=1}^{\infty}$ be a sequence of compact, nested subsets of G whose union is all of G . That is $K_n \subset K_{n+1}$ and $\cup K_n = G$. On each K_n , consider the metric generated by the supremum norm on continuous functions from K_n into X , i.e., $d_n(f, g) := \|(f - g)\chi_n\|_{\infty}$, where χ_n is the characteristic function of K_n . The following formula defines a metric d on $C(G, X)$ [14]:

$$(13.6.16) \quad d(f, g) := \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{d_n(f, g)}{1 + d_n(f, g)}.$$

The following facts are proven in [14], pgs. 138 – 143.

- PROPOSITION 13.6.6. (a) $(C(G, X), d)$ is a complete metric space.
(b) A sequence (f_n) in $(C(G, X), d)$ converges to f if and only if it converges to f uniformly on all compact subsets of G .

Define $G := \mathbb{U}$, and $X := B(\mathcal{H})$ with the metric given by the operator norm. Then, $C(\mathbb{U}, B(\mathcal{H}))$ will be a complete metric space with the metric d , and (\mathcal{A}, d) will be a closed subspace of this metric space. Also, given any compact subset $K \subset \mathbb{U}$ with non-empty interior, \mathcal{A}_K , the set of all operator-valued functions on K which are analytic on the interior of K , is a closed subspace of the Banach space $C(K, B(\mathcal{H}))$ with the supremum norm on K .

By 'Theorem 13.4.1 and 'Corollary 13.4.4, the set of operator-valued functions $\mathcal{R}(B)$ will be a closed subset of the metric space \mathcal{A} with the metric d , and of the Banach space \mathcal{A}_K with the supremum norm. To show that $\mathcal{R}(B)$ is in fact a compact subset of these spaces, it now suffices to show that it has compact closure.

These results will be achieved by a straightforward application of the following version of half of the Arzela-Ascoli theorem.

THEOREM 13.6.7. (Arzela-Ascoli) *Let (X, χ) be a compact metric space and (Y, ρ) a metric space. A subset \mathcal{F} of $C(X, Y)$ has compact closure with respect to the supremum norm topology if it is equicontinuous and pointwise pre-compact.*

The above can, for example, be proven with a slight modification of the arguments in ([61] pgs. 369-370).

CLAIM 13.6.8. *The set $\mathcal{R} \subset \mathcal{A}_K$ is pointwise precompact.*

Before proceeding with the proof, observe the following. Let A be an arbitrary self-adjoint extension of B . Consider the generalized Cayley transform $\mu_{wz}(A) := P_{\mathcal{H}}U_{wz}(A)|_{\mathcal{H}}$ where $U_{wz}(A) := (w-A)(z-A)^{-1} = I + (w-z)R_z(A)$. For $z \in \mathbb{C}$, let $\mathfrak{D}_z := \mathfrak{R}(\bar{z} - B)^{\perp} = \mathfrak{ker}(z - B^*)$. Here, we assume that $z \in \mathbb{U}$.

As in Lemma 9.1.1, it is straightforward to verify that $\mu_{wz}(A)$ maps \mathfrak{D}_w into \mathfrak{D}_z .

Suppose that A' is an arbitrary fixed self-adjoint extension of B and $A \in \mathbb{A}$ is arbitrary. Recall that $\mathcal{H} = \mathfrak{R}(B - z) \oplus \mathfrak{D}_{\bar{z}}$. It is clear that if $\phi \in \mathfrak{R}(B - z)$ then $(\rho_z(A') - \rho_z(A))\phi = 0$. If $\phi \in \mathfrak{D}_{\bar{z}}$ then

$$\begin{aligned} (\bar{z} - z)(\rho_z(A') - \rho_z(A))\phi &= (\bar{z} - z)(PR_z(A') - PR_z(A))\phi \\ &= P(I + (\bar{z} - z)R_z(A'))\phi - P(I + (\bar{z} - z)R_z(A))\phi \\ (13.6.17) \qquad \qquad \qquad &= \mu_{\bar{z}z}(A')\phi - \mu_{\bar{z}z}(A)\phi \end{aligned}$$

Since both $\mu_{\bar{z}z}(A')$ and $\mu_{\bar{z}z}(A)$ map $\mathfrak{D}_{\bar{z}}$ into \mathfrak{D}_z it follows that the difference $\rho_z(A') - \rho_z(A)$ lies in \mathfrak{D}_z . Thus for any $A \in \mathbb{A}$ and $z \in \mathbb{U}$ there is a linear map $K(A, z) : \mathfrak{D}_{\bar{z}} \rightarrow \mathfrak{D}_z$ such that $\rho_z(A) = \rho_z(A') + K(A, z)$. Since it is assumed that B has finite deficiency indices (m, n) , it follows that $K(A, z)$ is a finite rank operator of rank at most $k := \max\{m, n\}$.

PROOF. (of Claim 13.6.8) Given any $z \in \mathbb{U}$, consider the set $S_z := \{\rho(z) : \rho \in \mathcal{R}(B)\}$. It needs to be shown that S_z is precompact in $B(\mathcal{H})$ for any $z \in \mathbb{U}$. Note that we need only show that every sequence has a convergent subsequence as we work in a metric space setting.

Let A' be an arbitrary fixed self-adjoint extension of B . Then given any $A \in \mathcal{A}$, and a fixed non-real z ,

$$(13.6.18) \qquad \qquad \qquad \rho(A)(z) = \rho(A')(z) + K(A, z),$$

where $K(A, z)$ is a finite rank operator from $\mathfrak{D}_+ := \mathfrak{D}_{\bar{z}}$ to $\mathfrak{D}_- = \mathfrak{D}_z$. For fixed z , $K(A, z) = K(A)$ depends only on $A \in \mathcal{A}$. Furthermore it is easy to see that $\|K(A, z)\| = \|\rho_z(A) - \rho_z(A')\| \leq \|\rho_z(A)\| + \|\rho_z(A')\| \leq 2C$ where $C := (\min_{z \in K} \text{Im}(z))^{-1}$ so that the $K(A, z)$ are uniformly bounded in norm.

Notice that for this fixed z , each $K(A) = K(A, z)$ belongs to the Banach space of linear operators $B(\mathfrak{D}_+, \mathfrak{D}_-)$ from \mathfrak{D}_+ into \mathfrak{D}_- . Since this is a finite dimensional Banach space, the ball of radius $2C$ in this space is compact. It follows that given any sequence $\rho(A_n)(z) = \rho_z(A') + K(A_n)$,

there is a subsequence $K_k := K(A_{n_k})$ such that $K_k \rightarrow F \in B(\mathfrak{D}_+, \mathfrak{D}_-)$, and hence $K_k \rightarrow F$ with respect to the norm of $B(\mathcal{H})$ as well. It follows that the subsequence $\rho(A_{n_k})(z) = R_z(B') + K_k$ is Cauchy. This proves that $\mathcal{R}(B)$ is pointwise precompact. \square

COROLLARY 13.6.9. $\mathcal{R}(B)$ is a precompact subset of (\mathcal{A}, ρ) .

PROOF. Since (\mathcal{A}, ρ) is a metric space it suffices to show that every sequence of elements in $\mathcal{R}(B)$ has a Cauchy subsequence. This is easily accomplished by taking a countable sequence of compact subsets $K_n \subset \mathbb{U}$ such that $K_n \subset K_{n+1}$ and $\cup_n K_n = \mathbb{U}$, applying the previous result, Claim 13.6.8, and then using a diagonal argument. \square

The following result is an immediate consequence of Lemma 13.3.3, Claim 13.6.8, Theorem 13.6.7 and Corollary 13.6.9.

COROLLARY 13.6.10. $\mathcal{R}(B)$ is a convex, compact subset of \mathcal{A}_K with the topology of uniform operator-norm convergence, and of \mathcal{A} with the topology of uniform operator-norm convergence on compacta.

Since \mathcal{A}_K and \mathcal{A} are metric spaces with these topologies, $\mathcal{R}(B)$ is a separable set in these spaces. In fact, it is not hard to see that the topologies induced on \mathcal{R} by \mathcal{A} and \mathcal{A}_K are equivalent as before. If $\mathcal{P}(B)$ denotes the image of $\mathcal{R}(B)$ under Γ^{-1} , it follows that $Q_\alpha \rightarrow Q$ in the induced topology if and only if $\Phi_\alpha[f] \rightarrow \Phi[f]$ in the norm topology for any $f \in C_\infty(\mathbb{R})$.

13.7. Discussion and Outlook

My original motivation for studying the convex set $\mathcal{R}(B)$ of generalized resolvents of a symmetric operator was again generated by the example of the symmetric multiplication operator M in $\mathcal{H} := B(\Omega)$.

The self-adjoint multiplication operator \widetilde{M} in $\widetilde{\mathcal{H}} = L^2(\mathbb{R})$ is a self-adjoint extension of M . Consider the POVM Q obtained by the compression of the PVM of \widetilde{M} to \mathcal{H} , $Q(\Lambda) := P_{\mathcal{H}}\chi_\Lambda(\widetilde{M})|_{\mathcal{H}}$ for any $\Lambda \in \mathcal{B}(\mathbb{R})$. It is not difficult to verify that $Q(\Lambda)$ can be represented as

$$(13.7.1) \quad Q(\Lambda) = \int_{\Lambda} \delta_x \langle \cdot, \delta_x \rangle dx,$$

where $\delta_y(x) = \frac{\sin \Omega(x-y)}{\Omega(x-y)}$. Now consider the projection valued measures P_α , $\alpha \in [0, 1)$, where $P_\alpha(\Lambda) := \chi_\Lambda(M(\alpha))$. Here, $M(\alpha)$ is that self-adjoint extension of M which is the Fourier transform of the self-adjoint extension of the symmetric derivative operator D on $L^2[-\Omega, \Omega]$ obtained by extending the domain of D to include functions f which obey the boundary conditions $f(-\Omega) = e^{i2\pi\alpha} f(\Omega)$. It is again not hard to check that

$$(13.7.2) \quad P_\alpha(\Lambda) = \frac{\pi}{\Omega} \sum_{x_n(\alpha) \in \Lambda} \langle \cdot, \delta_{x_n(\alpha)} \rangle \delta_{x_n(\alpha)},$$

where, as before, $x_n(\alpha) = \frac{(n+\alpha)\pi}{\Omega}$. It follows that for any $\Lambda \in \mathcal{B}(\mathbb{R})$,

$$(13.7.3) \quad Q(\Lambda) = \int_0^1 P_\alpha(\Lambda) d\alpha.$$

This shows that the POVM $Q \in \mathcal{P}(B)$ is a convex combination of the PVM's P_α of the canonical self-adjoint extensions M_α of M . Recall here, that a self-adjoint extension of a symmetric operator B in \mathcal{H} is called canonical if it is defined on a dense domain of the same Hilbert space \mathcal{H} .

This observation led me to investigate whether it is true that the projection valued measures of the canonical self-adjoint extensions of a symmetric operator B with equal deficiency indices are all of the extreme points of $\mathcal{P}(B)$. It turns out that this is not the case. Naimark has proven that if A is any self-adjoint extension of B to a larger space $\widetilde{\mathcal{H}} \supset \mathcal{H}$ such that $\widetilde{\mathcal{H}} \ominus \mathcal{H}$ is finite dimensional, then the POVM obtained from the compression of the PVM of A to \mathcal{H} is an extreme

point of $\mathcal{P}(B)$ [27]. Furthermore, Gilbert has proven that if S has finite and equal deficiency indices, then given any $\rho \in \mathcal{R}(B)$, one can find a sequence ρ_n corresponding to extensions of B to spaces $\mathcal{H}_n \supset \mathcal{H}$ where each $\mathcal{H}_n \ominus \mathcal{H}$ is finite dimensional, such that $\rho_n \rightarrow \rho$ uniformly in norm on compacta of \mathbb{U} [13]. Combining these two facts, it is not difficult to see in the above example of $B(\Omega)$ that $\mathcal{R}(B)$ and hence $\mathcal{P}(B)$ contain extreme points which are generalized resolvents or POVMs obtained from non-canonical self-adjoint extensions of B .

Nevertheless, what is interesting about the above example of $B(\Omega)$ is that, not only is Q in the convex hull of the PVMs corresponding to the canonical extensions of M , Q is also the POVM associated to a self-adjoint extension \widetilde{M} of M which has the special property that its Cayley transform $\mu(\widetilde{M})$ is a unitary dilation of its compression to \mathcal{H} . That is, as shown in Chapter 12, $\mu(\widetilde{M})$ is a unitary dilation of $P_{\mathcal{H}}\mu(\widetilde{M})|_{\mathcal{H}}$. This means, in particular that if $\tilde{\rho}(z) = P_{\mathcal{H}}R_z(\widetilde{M})|_{\mathcal{H}}$ is the generalized resolvent of M obtained from \widetilde{M} , then $\tilde{\rho}$ obeys the first resolvent formula, $(z-w)\tilde{\rho}(z)\tilde{\rho}(w) = \tilde{\rho}(w) - \tilde{\rho}(z)$. Clearly, this is a special property that most generalized resolvents do not have. This raises the question, when and how does the subset of generalized resolvents of a symmetric operator B that has this property intersect with the convex hull of the canonical resolvents of B ? In the case of the multiplication operator M on $B(\Omega)$ we see that this set is non-empty. In the time since completing the first draft of this thesis, I have made some progress on this question, but as my results are not yet complete, I will save them for a future paper.

Krein has established a formula that establishes a bijective correspondence between the set of all generalized resolvents of a symmetric operator with deficiency indices $(1, 1)$ and a certain convex subset of functions which are analytic in the upper half plane [28]. I look forward to applying this formula to the study of the questions raised above.

Conclusions and Outlook

I would like to answer the question regarding invariant linear manifolds of self-adjoint operators raised in Section 12.1. Recall that this question was the following. Let \mathcal{H} be a separable Hilbert space and A be a closed self-adjoint operator defined on a dense domain $\mathfrak{D}(A) \subset \mathcal{H}$. Let $S \subset \mathcal{H}$ be a subspace which is the range of the projection P . If A is fixed, what conditions does P need to satisfy in order that there be a dense domain $\mathfrak{D}(B) \subset S$ so that $\mathfrak{D}(B) \subset \mathfrak{D}(A)$, and $B := A|_{\mathfrak{D}(B)}$ is a densely defined symmetric operator in S ?

In particular it would be great to have a sufficiently simple condition that would characterize when A has such a restriction, and which would be useful for determining when the multiplication operator M on $L^2(\mathbb{R}; d\mu)$ has such restrictions.

Here are a few observations concerning this question. Given the self-adjoint operator A , let $U := (A + i)(A - i)^{-1}$ be its unitary Cayley transform. Then $A = i(1 + U)(1 - U)^{-1}$. Now if S is such that $(U - 1)S \subset S$ is dense in S and $(U + 1)S \subset S$, then clearly, in this case, A has a densely defined restriction B to $\mathfrak{D}(B) := (U - 1)S \subset S$. However, this is not terribly illuminating.

Another observation that is more interesting is the following. Suppose that one is interested to know, as is the case for the multiplication operator $A = M$, when A has a symmetric restriction with finite deficiency indices to a dense domain in a subspace S projected onto by a projector P . If A has such a restriction, it follows that for any $\lambda \in \mathbb{C} \setminus \mathbb{R}$, that $P(A - \lambda)^{-1}P - (A - \lambda)^{-1}P$ is of finite rank. Since this is true for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$, taking adjoints shows that $P(A - \lambda)^{-1}P - P(A - \lambda)^{-1}$ is of finite rank for each $\lambda \in \mathbb{C} \setminus \mathbb{R}$. In particular, $[P, (A - \lambda)^{-1}] = P(A - \lambda)^{-1} - (A - \lambda)^{-1}P$ is of finite rank for each $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Since, for any fixed $\lambda \in \mathbb{C} \setminus \mathbb{R}$, the polynomials in $\frac{1}{x - \lambda}$ and $\frac{1}{x - \bar{\lambda}}$ are dense in the continuous functions vanishing at infinity, it follows that $[P, f(A)]$ is compact for any continuous function f that vanishes at infinity. This shows that if \mathbb{A} is the C^* -algebra of continuous functions of A vanishing at ∞ , then the subspace S is ‘essentially invariant’ for \mathbb{A} , and that the compression of any $f(A) \in \mathbb{A}$ to S is ‘essentially normal’. If $A = M$ is the multiplication operator on $L^2(\mathbb{R}, d\mu)$, it will be interesting to see what additional conditions, if any, are needed to conclude that M has a symmetric restriction to S , or that S has the sampling property.

Now consider the case where M is the multiplication operator in $L^2(\mathbb{R})$, and $D := i \frac{d}{dx}$ is the self-adjoint derivative operator in $L^2(\mathbb{R})$. Then, in this case, it is straightforward to show that if I_1 and I_2 are finite subintervals of \mathbb{R} , then $Q(I_1, I_2) := \chi_{I_1}(D)\chi_{I_2}(M)\chi_{I_1}(D)$ is an integral operator with a square integrable kernel, so that it is Hilbert-Schmidt. Furthermore, it is not difficult to show that $\text{Tr}(Q(I_1, I_2)) = \frac{\mu(I_1)\mu(I_2)}{2\pi}$, where μ denotes Lebesgue measure. It follows that

$$(14.0.4) \quad \lim_{B \rightarrow \infty} \frac{\text{Tr}(Q([- \Omega, \Omega], [-B, B]))}{\mu([-B, B])2\pi} = \lim_{B \rightarrow \infty} \frac{\text{Tr}(\chi_{[- \Omega, \Omega]}(D)\chi_{[-B, B]}(M)\chi_{[- \Omega, \Omega]}(D))}{\mu([-B, B])2\pi} = \frac{\Omega}{\pi}.$$

Intuitively, this is to be expected. If a projector P projects onto an n -dimensional subspace, then $\text{Tr}(P) = n$. Further recall that the dimension of a finite dimensional function space is equal to the minimum number of sample points needed for stable reconstruction of any element of the space. In equation (14.0.4), since $\chi_{[-B, B]}(M)$ converges strongly to the identity as $B \rightarrow \infty$, it follows that $Q([- \Omega, \Omega], [-B, B])$ converges strongly to $\chi_{[- \Omega, \Omega]}(D)$, the projector onto $B(\Omega)$. Hence, it seems intuitively reasonable that the quantity $\frac{\text{Tr}(Q([- \Omega, \Omega], [-B, B]))}{\mu([-B, B])2\pi}$ is a measure of the spatial density of degrees of freedom, or the density of points needed for a set of sampling of

$B(\Omega)$ as B gets large. This raises the following question. Let M be a Riemannian manifold, and consider a nested sequence of compact submanifolds $K_n \subset M$, which have compact closures, and form an open cover of M , as in Chapter 5. Let χ_n denote the projector of $L^2(M)$ onto $L^2(\overline{K_n})$. If S is a subspace of $L^2(M)$ and P projects onto S , suppose that $Q_n := P\chi_n P$ is a Hilbert-Schmidt operator for each $n \in \mathbb{N}$, and that $\lim_{n \rightarrow \infty} \frac{\text{Tr}(Q_n)}{V(\overline{K_n})} < \infty$, where $V(\overline{K_n})$ denotes the proper volume of $\overline{K_n}$. If this is the case, this would seem to indicate, as before, that the subspace S has a finite proper spatial density of degrees of freedom. Does this imply that the subspace S has the sampling property? In particular, note that if $P = \chi_{[-\Omega, \Omega]}(D_{pq})$ where D_{pq} is a Sturm-Liouville differential operator acting on a dense domain in $L^2(\mathbb{R})$, then for most choices of D_{pq} , Q_n will in fact be a Hilbert-Schmidt operator.

The above questions, as well as the questions raised in Chapters 10, 11 and 13 are sure to provide me with challenging problems for some time to come. I look forward to solving these problems.

APPENDIX A

An approach to density theorems for sampling of $B(M, \Omega)$

A.1. Introduction and Motivation

Consider the real line. We have seen that if one considers any sequence of nested compact intervals $I_n \subset I_{n+1}$ such that $\cup_n I_n = \mathbb{R}$, and any sequence of self-adjoint derivative operators D_n on I_n , that this sequence of operators converges in the strong resolvent sense to $D := i \frac{d}{dx}$, the self-adjoint derivative operator in $\mathcal{H} := L^2(\mathbb{R})$. From this fact, known results show that the projection operators $P_n := \chi_{[-\Omega, \Omega]}(D_n)$ converge strongly to $P := \chi_{[-\Omega, \Omega]}(D)$. Now one might hope that sampling properties of $P\mathcal{H}$ would follow from those of the $P_n\mathcal{H}$ and the strong convergence of these projectors. However, as discussed in the final section of Chapter 3, such results appear to be more difficult to establish than one might expect.

In an attempt to remedy this problem, in this chapter it will be shown that given any $\epsilon > 0$, if one chooses a suitable smooth function f such that $[-\Omega, \Omega] \subset \text{supp}(f) \subset [-\epsilon - \Omega, \Omega + \epsilon]$ and such that $f(x)\chi_{[-\Omega, \Omega]} = \chi_{[-\Omega, \Omega]}$, that one can construct a sequence of bounded positive operators $\Phi_n[f]$ such that $\Phi_n[f] \rightarrow P$ in norm and such that the $\Phi_n[f]$ are a suitable average of f evaluated on restrictions of D to suitable tilings of the real line. This will be explained in full detail in the upcoming section.

In Chapter 5, we saw that an appropriate notion of strong convergence for unbounded self-adjoint operators is strong resolvent convergence. One can also define norm resolvent convergence for unbounded self-adjoint operators ([57], pg. 284).

DEFINITION A.1.1. *A sequence A_n of self-adjoint operators is said to converge to a self-adjoint operator A in the norm resolvent sense if there is a $\lambda \in \mathbb{C} \setminus \mathbb{R}$ such that $(A_n - \lambda)^{-1} \rightarrow (A - \lambda)^{-1}$ in norm.*

Of particular interest is the following result

THEOREM A.1.2. *If $A_n \rightarrow A$ in the norm resolvent sense, and if f is a continuous function on \mathbb{R} that vanishes at ∞ , then $\|f(A_n) - f(A)\| \rightarrow 0$.*

A.2. A tiling of \mathbb{R}

Let $D := i \frac{d}{dx}$ be the self-adjoint derivative operator in $L^2(\mathbb{R})$. For each $n \in \mathbb{N}$, $\mathbb{R} = \cup_{m \in \mathbb{Z}} I_{nm}$ where $I_{nm} := [nm, n(m+1)]$. Let D_{nm} be the self-adjoint derivative operator $i \frac{d}{dx}$ on $L^2(I_{nm})$ with periodic boundary conditions. Then define $D_n := \bigoplus_{m \in \mathbb{Z}} D_{nm}$ on $\bigoplus_{m \in \mathbb{Z}} L^2(I_{nm}) = L^2(\mathbb{R})$. D_n is clearly self-adjoint. In words, we are breaking the real line into equal sized subintervals of length n , and then considering a particular self-adjoint extension of the symmetric operator which is obtained as the direct sum of the symmetric restrictions of D to dense domains in L^2 of each of these subintervals.

Let $U_n(0; t) := e^{itD_n} = \bigoplus_{m \in \mathbb{Z}} e^{itD_{nm}}$. The idea behind considering the operator D_n is the following. Consider the projection operators $\chi_{[-\Omega, \Omega]}(D_n) = \bigoplus_{m \in \mathbb{Z}} \chi_{[-\Omega, \Omega]}(D_{nm})$. These projection operators project onto the direct sum of spaces of Ω -bandlimited trigonometric polynomials on intervals of length n . It follows that it is still easy to determine sets of sampling for $\chi_{[-\Omega, \Omega]}(D_n)\mathcal{H}$.

Namely, Λ is a set of sampling for this subspace if and only if each Λ has at least $2\lfloor \frac{\Omega n}{2\pi} \rfloor + 1$ members in each subinterval I_{nm} . Thus, if it could be proven that the projectors $\chi_{[-\Omega, \Omega]}(D_n)$ converge to $\chi_{[-\Omega, \Omega]}(D)$ in norm, then, as discussed at the end of Chapter 3, one could use this fact to prove results on necessary density for sets of sampling for $B(\Omega)$. Actually using suitable approximation arguments, and using Theorem A.1.2, it would in fact be sufficient to show that D_n converges to D in the norm resolvent sense.

It turns out that the sequence of operators D_n do not converge in the norm resolvent sense to D , but the proof of this fact will provide us with an idea of how to construct a better way of approximating $B(\Omega)$.

Recall that if D_α is the self-adjoint derivative operator $i\frac{d}{dx}$ in $L^2[a, b]$ with domain $\mathfrak{D}(D_\alpha) := \{f \in L^2[a, b] \mid f \in AC[a, b], f(a) = e^{i\alpha} f(b)\}$ then e^{itD_α} acts as translation to the right by t , where anything translated past b appears at a , multiplied with the phase $e^{i\alpha}$. Observe that if the function f has support only on $[a, b - t']$ then $e^{itD_\alpha} f = e^{itD} f$ for all $t \leq t'$. It is straightforward to visualize the action of the one parameter, strongly continuous unitary group e^{itD_n} .

*CLAIM A.2.1. *The operators D_n do not converge to D in the norm resolvent sense.*

For an operator A , and $z \notin \sigma(A)$, let $R_z(A)$ denote the resolvent operator $(A - z)^{-1}$.

PROOF. Given any $\phi \in \mathcal{H}$ and $\mu \in LHP$, the following Laplace formula holds for the resolvent $R_z(A)$ of A , ([58], pg. 237):

$$(A.2.1) \quad \|(R_\mu(D_n) - R_\mu(D))\phi\| = \left\| i \int_0^\infty e^{-it\mu} (e^{itD_n} - e^{itD}) \phi dt \right\|.$$

Simply choose, for example $\phi_\delta \in \mathcal{H}$ such that $\phi_\delta(t) = \frac{1}{\sqrt{\delta}} \chi_{[n-\delta, n]}$. These are all norm 1 vectors. Now given any $\epsilon > 0$, choose $\delta > 0$ so that both $\|R_\mu(D_n)\phi - i \int_\delta^\infty e^{-it\mu} e^{itD_n} \phi dt\| < \epsilon$ and $\|R_\mu(D)\phi - i \int_\delta^\infty e^{-it\mu} e^{itD} \phi dt\| < \epsilon$. Let $R_\mu(D_n; \delta)\phi := \int_\delta^\infty e^{-it\mu} e^{itD_n} \phi dt$ and $R_\mu(D; \delta)\phi := \int_\delta^\infty e^{-it\mu} e^{itD} \phi dt$. Then it follows that,

$$(A.2.2) \quad \begin{aligned} \|R_\mu(D_n)\phi_\delta - R_\mu(D)\phi_\delta\| &= \|R_\mu(D_n)\phi_\delta - R_\mu(D_n; \delta)\phi_\delta \\ &\quad + R_\mu(D; \delta)\phi_\delta - R_\mu(D)\phi_\delta + R_\mu(D_n; \delta)\phi_\delta - R_\mu(D; \delta)\phi_\delta\| \\ &\geq \|R_\mu(D_n; \delta)\phi_\delta - R_\mu(D; \delta)\phi_\delta\| - 2\epsilon. \end{aligned}$$

For all $t \in [\delta, \infty)$, $e^{itD_n}\phi_\delta$ has support in $[0, n]$, while $e^{itD}\phi$ has support in $[n, \infty)$. It follows that $\|R_\mu(D_n; \delta)\phi_\delta - R_\mu(D; \delta)\phi_\delta\| = \|R_\mu(D_n; \delta)\phi_\delta\| + \|R_\mu(D; \delta)\phi_\delta\|$ since these vectors are orthogonal. Hence,

$$(A.2.3) \quad \begin{aligned} \|R_\mu(D_n; \delta)\phi_\delta - R_\mu(D; \delta)\phi_\delta\| &= \|R_\mu(D_n; \delta)\phi_\delta\| + \|R_\mu(D; \delta)\phi_\delta\| \\ &\geq \frac{2}{\text{Im}(\mu)} \|\phi_\delta\| = \frac{2}{\text{Im}(\mu)}. \end{aligned}$$

In conclusion, $\|R_\mu(D_n) - R_\mu(D)\| \geq \frac{2}{\text{Im}(\mu)} - 2\epsilon$ for any $\epsilon > 0$, and the claim is proven. \square

A.3. An average over all equidistant tilings

For each $n \in \mathbb{N}$, define $I_{nm}(\alpha) := [nm + \alpha, n(m+1)\alpha]$ where $\alpha \in [0, n)$. $D_n(\alpha)$ is defined by $I_{nm}(\alpha)$ in the same way that $D_n := D_n(0)$ is defined using $I_{nm}(0) := I_{nm}$. That is, each $D_n(\alpha)$ is constructed by breaking the real line into subintervals of length n , and then considering the direct sum of the self-adjoint extensions of the symmetric derivative operator on each of these subintervals which obey periodic boundary conditions. As α runs through the interval $[0, n)$, $I_{nm}(\alpha)$ runs through all such possible tilings of the the real line.

The idea in this section is the following. Although bounded continuous functions f of the $D_n(\alpha)$ converge to $f(D)$ only in the strong operator topology, one would expect that if the support of $\phi \in L^2(\mathbb{R})$ is contained in an interval which is small enough, and far enough away from any of the end points of the subintervals in the tiling $I_{nm}(\alpha)$, that $\|(f(D_n(\alpha)) - f(D(\alpha))\phi)\|$ should be small, independently of the choice of ϕ with these properties. In other words, the reason that the $f(D_n(\alpha))$ do not converge in norm may be due to boundary effects that occur near the end-points of the subintervals of the tiling $I_{nm}(\alpha)$. In this section we will show that by taking an average with respect to α of the $f(D_n(\alpha))$ we can remove these ‘boundary effects’ and construct a bounded positive operator, $\Phi_n[f] := \frac{1}{n} \int_0^n f(D_n(\alpha))d\alpha$ which converges to $f(D)$ in norm.

Given any bounded continuous function, $f \in BC(\mathbb{R})$, define the operator

$$(A.3.1) \quad \Phi_n[f]\phi := \frac{1}{n} \int_0^n f(D_n(\alpha))\phi d\alpha.$$

In particular, if $e_t(x) := e^{itx}$, then

$$(A.3.2) \quad \Phi_n[e_t]\phi = \frac{1}{n} \int_0^n e^{itD_n(\alpha)}\phi d\alpha =: \frac{1}{n} \int_0^n U_n(t; \alpha)\phi d\alpha,$$

is the ‘average’ of the unitary groups, $U_n(t; \alpha)$, of the operators $D_n(\alpha)$, $\alpha \in [0, n]$. It is easy to check that the map $\alpha \mapsto D_n(\alpha)$ is continuous in the strong resolvent sense. It follows that for any $f \in BC(\mathbb{R})$, the map $\alpha \mapsto f(D_n(\alpha))$ is strongly continuous. Hence, $\alpha \mapsto f(D_n(\alpha))\phi$ is a continuous Hilbert space valued function for each $\phi \in \mathcal{H}$ and the integral in equation (A.3.1) is well-defined, and defines a bounded linear operator for each $f \in BC(\mathbb{R})$.

Consider the elements $g_\alpha := U_n(t; \alpha)\phi \in L^2(\mathbb{R})$ where t, n are fixed. For each α , g_α is a square integrable function. Furthermore, if we choose $\phi \in BC(\mathbb{R}) \cap L^2(\mathbb{R})$, it follows from the action of $U_n(t, \alpha)$ on ϕ that g_α is piecewise continuous, and that there is a member of its L^2 equivalence class which is continuous from the left. Let f_α denote this member.

*CLAIM A.3.1. $f(\alpha) := f_\alpha(x)$ where $x \in \mathbb{R}$ is fixed, is a bounded, measurable function of α for $\alpha \in [0, n]$, for any $\phi \in BC(\mathbb{R}) \cap L^2(\mathbb{R})$.

PROOF. $f(\alpha)$ is measurable if and only if $|f(\alpha)|$ is. It is sufficient to show that the set $S := \{\alpha \in [0, n] \mid |f(\alpha)| > c\}$ is Borel for each $c > 0$. Choose $\alpha \in S$. Then $|f(\alpha)| > c + 3\epsilon$ for some $\epsilon > 0$. Now choose $\delta > 0$ so that $|t - y| < \delta \Rightarrow |\phi(t) - \phi(y)| < \epsilon$ for all $t, y \in [x - n, x + n]$. It follows from the left continuity of the f_α that if $y < x$ and $x - y < \delta$ we have that $|f_\beta(x) - f_\beta(y)| < \epsilon$ for all $\beta \in [0, n]$.

Using this fact,

$$(A.3.3) \quad \frac{1}{\delta} \int_{x-\delta}^x |f_\alpha(y)|^2 dy > (c + 2\epsilon)^2$$

so that $\|U_n(t, \alpha)\phi\chi_{[x-\delta, x]}\| > (c + 2\epsilon)\sqrt{\delta}$.

By the strong continuity of $U_n(t, \alpha)$ for fixed n and t , there is a neighbourhood V_α of α such that $\beta \in V_\alpha$ implies that

$$(A.3.4) \quad \epsilon\sqrt{\delta} > \|(U_n(t, \alpha) - U_n(t, \beta))\chi_{[x-\delta, x]}\phi\| > \left| \|U_n(t, \alpha)\chi_{[x-\delta, x]}\phi\| - \|U_n(t, \beta)\chi_{[x-\delta, x]}\phi\| \right|.$$

Hence, for $\beta \in V_\alpha$,

$$(A.3.5) \quad \|U_n(t, \beta)\chi_{[x-\delta, x]}\phi\| > (c + \epsilon)\sqrt{\delta}$$

which is equivalent to

$$(A.3.6) \quad \int_{x-\delta}^x |U_n(t, \beta)\phi(y)|^2 dy > \delta(c + \epsilon)^2.$$

But $|U_n(t, \beta)\phi(y)| < |U_n(t, \beta)\phi(x)| + \epsilon$. This implies that

$$(A.3.7) \quad \delta(|U_n(t, \beta)\phi(x)| + \epsilon)^2 > \delta(c + \epsilon)^2,$$

which is the same as $|f(\beta)| > c$. It follows that $V_\alpha \subset S$. Since α was arbitrary this shows that S is open, so that $f(\alpha)$ is measurable. That it is bounded is obvious, since, by assumption, ϕ is bounded. \square

Let $U(t) := e^{itD}$.

*CLAIM A.3.2. For any compact interval $[a, b]$, $\|\Phi_n[e_t] - U(t)\| \rightarrow 0$ uniformly for $t \in [a, b]$ as $n \rightarrow \infty$

PROOF. Choose $n > |b|$. If $\phi \in \mathcal{H} = L^2(\mathbb{R})$ is a C^∞ function of unit norm then $U_n(t; \alpha)\phi = \phi(x-t)$ a.e. for $x \in \cup_{m \in \mathbb{Z}} [nm + \alpha + t, n(m+1) + \alpha]$. For any $x \in \mathbb{R}$, define $\beta \in [0, n]$ by $x = nj + \beta$. It follows that $U_n(t; \alpha)\phi(x) = \phi(x-t)$ for all $\alpha \in [0, n] \cap ([\beta, n-t+\beta] \cup [-n+\beta, \beta-t])$. For any x this is a compact set K_x of length $n-t$. It follows from the definition of $\Phi_n[e_t]$ that

$$\begin{aligned} \psi(x) &= \Phi_n[e_t]\phi(x) - U(t)\phi(x) = \frac{1}{n} \int_{\alpha \notin K_x} U_n(t; \alpha)\phi(x) d\alpha + \left(\frac{n-t}{n} - 1\right) \phi(x-t) \\ (A.3.8) \quad &=: \psi_1(x) + \psi_2(x). \end{aligned}$$

Let $I_x = [0, n] \setminus K_x$, it follows that

$$(A.3.9) \quad \|\psi_1(x)\|^2 = \frac{1}{n^2} \int_{-\infty}^{\infty} \left| \int_{I_x} U_n(t; \alpha)\phi(x) d\alpha \right|^2 dx.$$

Note that

$$\begin{aligned} |\psi_1(x)|^2 &= \left| \int_0^n \chi_{I_x}(\alpha) U_n(t; \alpha)\phi(x) d\alpha \right|^2 \\ &\leq \left(\int_{I_x} 1^2 d\alpha \right) \left(\int_0^n |U_n(t; \alpha)\phi(x)|^2 d\alpha \right) \\ (A.3.10) \quad &= t \int_0^n |U_n(t; \alpha)\phi(x)|^2 d\alpha \end{aligned}$$

by the Cauchy-Schwarz inequality for elements in $L^2[0, n]$. It follows that

$$\begin{aligned} \|\psi_1(x)\|^2 &\leq \frac{t}{n^2} \int_0^n \int_{-\infty}^{\infty} |U_n(t; \alpha)\phi(x)|^2 dx d\alpha \\ &= \frac{t}{n^2} \int_0^n \|U_n(t; \alpha)\phi\|^2 d\alpha \\ (A.3.11) \quad &= \frac{t}{n} \|\phi\|^2. \end{aligned}$$

This shows that $\|\psi_1\| \leq \sqrt{\frac{t}{n}} \|\phi\|$. It is also clear that $\|\psi_2\| = \frac{t}{n} \|\phi\|$. Therefore, $\|\psi\| \leq \|\psi_1\| + \|\psi_2\| \leq \left(\frac{t}{n} + \sqrt{\frac{t}{n}}\right) \|\phi\|$. This is true for arbitrary $\phi \in C^\infty(\mathbb{R})$ which is a dense linear subspace of \mathcal{H} . Hence, $\|\Phi_n[e_t] - U(t)\| \leq \left(\frac{t}{n} + \sqrt{\frac{t}{n}}\right)$ which vanishes as $n \rightarrow \infty$. \square

We can now use this result to show that the averages $\Phi_n[f]$ converge in norm to $f(D)$ for certain sufficiently well-behaved functions f . It is known that the Fourier transform \mathcal{F} maps $L^1(\mathbb{R})$ into $C_\infty(\mathbb{R})$, the continuous functions vanishing at ∞ [58].

*PROPOSITION A.3.3. If $f \in C_\infty(\mathbb{R})$ is the Fourier transform of an element in $L^1(\mathbb{R})$, then $\Phi_n[f] \rightarrow f(D)$ in operator norm as $n \rightarrow \infty$.

PROOF. For such an f , $f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(w)e^{iw x} dw$ so that $f(D_n(\alpha)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(w)e^{iw D_n(\alpha)} dw$. With the aid of Fubini's theorem,

$$(A.3.12) \quad (\Phi_n[f] - f(D))\phi = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(w) (\Phi_n[e_w] - e_w(D))\phi dw,$$

so that

$$(A.3.13) \quad \begin{aligned} \|\Phi_n[f] - f(D)\| &\leq \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(w)| \|\Phi_n[e_w] - e_w(D)\| dw \\ &\leq \frac{1}{2\pi} \int_{-B}^B |F(w)| \|\Phi_n[e_w] - e_w(D)\| dw + \int_{|w|>B} |F(w)| \|\Phi_n[e_w] - e_w(D)\| dw \\ &\leq \frac{B}{n} \|F\|_1 + 2 \int_{|w|>B} |F(w)| dw. \end{aligned}$$

In the above, recall that $e_w(D) := e^{iwD}$. Given any $\epsilon > 0$, choose B large enough so that the second term is less than $\epsilon/2$. Then n can be chosen large enough so that the first term is less than $\epsilon/2$, proving the claim. \square

A.4. Outlook

The Fourier transform of any characteristic function does not belong to $L^1(\mathbb{R})$, so that *Proposition A.3.3 cannot be applied to show that $\Phi_n[\chi_{[-\Omega, \Omega]}]$ converges in norm to $P = \chi_{[-\Omega, \Omega]}(D)$. However, as discussed in Section A.1, for any $\epsilon > 0$ one can always choose a function f so that its Fourier transform $F \in L^1(\mathbb{R})$ and so that $[-\Omega, \Omega] \subset \text{supp}(f) \subset [-\Omega - \epsilon, \Omega + \epsilon]$. For such an f it follows that $\mathfrak{R}(\chi_{[-\Omega, \Omega]}(D)) \subset \mathfrak{R}(f(D)) \subset \mathfrak{R}(\chi_{[-\Omega - \epsilon, \Omega + \epsilon]}(D))$. This shows that one can approximate the range $B(\Omega)$ of P as well as one likes by the range of the $\Phi_n[f]$ as $n \rightarrow \infty$.

Now for such a function f , the range of the positive operators $f(D_n(\alpha))$ is contained in the range of $P_{[-\Omega - \epsilon, \Omega + \epsilon]}(D_n(\alpha))$, and as discussed in Section A.1, it is easy to determine what properties a discrete set of points Λ needs to have in order to be a set of sampling for this subspace. Namely it must contain at least $2 \lfloor \frac{(\Omega + \epsilon)n}{2\pi} \rfloor + 1$ points in each subinterval of the tiling of \mathbb{R} given by the subintervals $I_{nm}(\alpha)$. Recall that $\Phi_n[f]$ is the average of the $f(D_n(\alpha))$. Now consider the range of $\Phi_n[f]$. If it could also be shown that Λ is a set of sampling for elements of this linear manifold only if it has at least $2 \lfloor \frac{(\Omega + \epsilon)n}{2\pi} \rfloor + 1$ points in any subinterval of length n , then using a similar method to the one outlined in the end of Chapter 3, one could combine this fact with the norm convergence of the $\Phi_n[f]$ to $f(D)$ to provide a new proof of necessity part of Beurling's theorem, Theorem 2.2.3.

Moreover, what is of greater interest, is that if this approach could be made viable, then by suitably tiling a given manifold M , it may be possible to generalize this approach to prove results about sets of sampling for $B(M, \Omega)$.

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