# Multivariate Time Series Analysis of the Investment Guarantee in Canadian Segregated Fund Products

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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**Abstract**

In the context of the guarantee liability valuation, the sophisticated fund-of-funds structure, of some Canadian segregated fund products, often requires us to model multiple market indices simultaneously in order to benchmark the return of the underlying fund. In this thesis, we apply multivariate GARCH models with Gaussian and non-Gaussian noise to project the future investment scenarios of the fund. We further conduct a simulation study to investigate the difference, among the proposed multivariate models, in the valuation of the Guaranteed Minimum Maturity Benefit (GMMB) option.

Based on the pre-data analysis, the proposed multivariate GARCH models are data driven. The goodness-of-fit for the models is evaluated through formal statistical tests from univariate and multivariate perspectives. The estimation and associated practical issues are discussed in details. The impact from the innovation distributions is addressed. More importantly, we demonstrate an actuarial approach to manage the guarantee liability for complex segregated fund products.

**Acknowledgements**

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## Dedication

*For dear mom, Jianmin Shu. Your love and care is always the origin of what I have accomplished.*

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# Chapter 1

# Introduction

Segregated fund products, offered by insurance companies, are very popular in Canada. These equity-linked insurance products are similar to mutual funds, but they offer additional protective features for investors. The main distinct advantages include:

- down-side protection on principal,
- right to lock the current investment gain,
- and death benefit with guaranteed amount.

Under a typical segregated fund contract, the premium from an investor is used to purchase mutual funds based on the investor's risk preference, and the accumulated value at maturity, normally in 10 years, is guaranteed to be at least the initial principal. When the market value of the investment exceeds the principal, the investor is provided the choice to put the capital gain into the principal and reset the contract. In addition, as an insurance contract, a guaranteed benefit amount will be paid in the event of the investor's death.

Segregated fund products are attractive to Canadians who wish to enjoy the unlimited up-side equity growth opportunity with a pre-fixed maximum loss. For

Canadian insurers, managing the liabilities of these embedded insurance options becomes a fairly challenging task, especially for the liability of the investment guarantee. Two quantitative approaches are widely used in practice to deal with the guarantee liability. One is the hedging approach, which relies on creating a replicating portfolio that will meet the liability at maturity. This approach from the financial engineering has become increasingly popular in the industry. Alternatively, there is an actuarial approach for coping with the same problem. This approach tries to understand the underlying probability distribution of the guarantee liability through simulation. A "sufficient" amount of assets are reserved according to the simulation results, and put aside in fixed income securities to accumulate. At maturity, the accumulated amount is expected to maintain insurer's solvency on the guarantee liability with a high probability, say 97.5%. In this thesis, we focus on the traditional actuarial approach, which sets aside a fraction of incoming premium to build up a reserve that will be used to meet the liability in the event the accumulated value of the investment is below the guarantee level at maturity. We only consider the liability associated with the guaranteed minimum maturity benefit (GMMB).

As mutual fund products continue to develop over the years, Canadian insurance companies offer increasingly more innovative segregated fund products. For instance, not all segregated funds are managed to trace the performance of a particular single index. To gain more diversification, funds under one contract may consist of multiple sub-funds, such as Canadian equity fund, Canadian fixed income fund, US equity fund, or international equity fund. These sub-funds are themselves also segregated fund available to investors.

Offered by Manulife Financial, the MLI Fidelity True North GIF is a typical single-indexed segregated fund. The underlying index, whose performance is being traced, is S&P/TSX composite index. The single-indexed fund represents a class of the segregated fund. The management expense is relatively low for these funds;

however, the long-term performance is satisfactory[1], as found in many empirical studies.

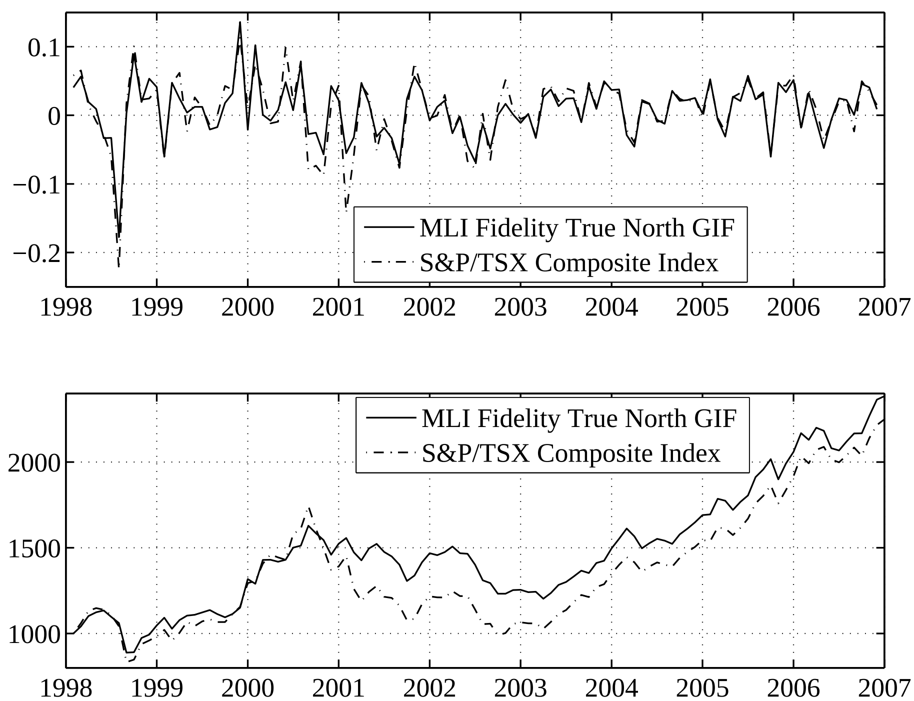


Figure 1.1: Historical performance of Manulife Fidelity True North guaranteed investment fund. The top plot shows the movement of the historical return, and the bottom one gives the history of the market price for an initial investment of one thousand. Overall, the segregated fund closely traces the performance of the S&P/TSX composite index.

Simulating the future guarantee liability distribution requires modeling the return of the fund in the long term. For this single-indexed fund, it requires the calibration of an appropriate univariate model to S&P/TSX composite index, and then simulation based on the estimated model. In general, the variety of portfo-

[1]Compared with the actively managed fund, the indexed fund usually has lower costs and turnovers, which result a smaller expense. On the other hand, empirical studies on persistence in mutual fund performance have shown that the actively managed fund under-perform the market (index). Hence, the indexed fund is likely to generate more return than the actively managed fund in the long term.

lio structures of segregated funds greatly increases the difficulty in modeling the long-term return and further complicates the guarantee liability valuation. As an illustration, we present another segregated fund also from Manulife Financial. The MLI Fidelity Canadian Asset Allocation GIF is a fund whose asset is invested in Canadian equity market, Canadian fixed income market, US equity market, international equity market, and money market. The immediate issue in modeling the return is that the available data of the fund is limited. One way to solve the problem is to create a benchmark portfolio to proxy the performance of the fund, and this may be in fact the only feasible solution for this complicated fund.

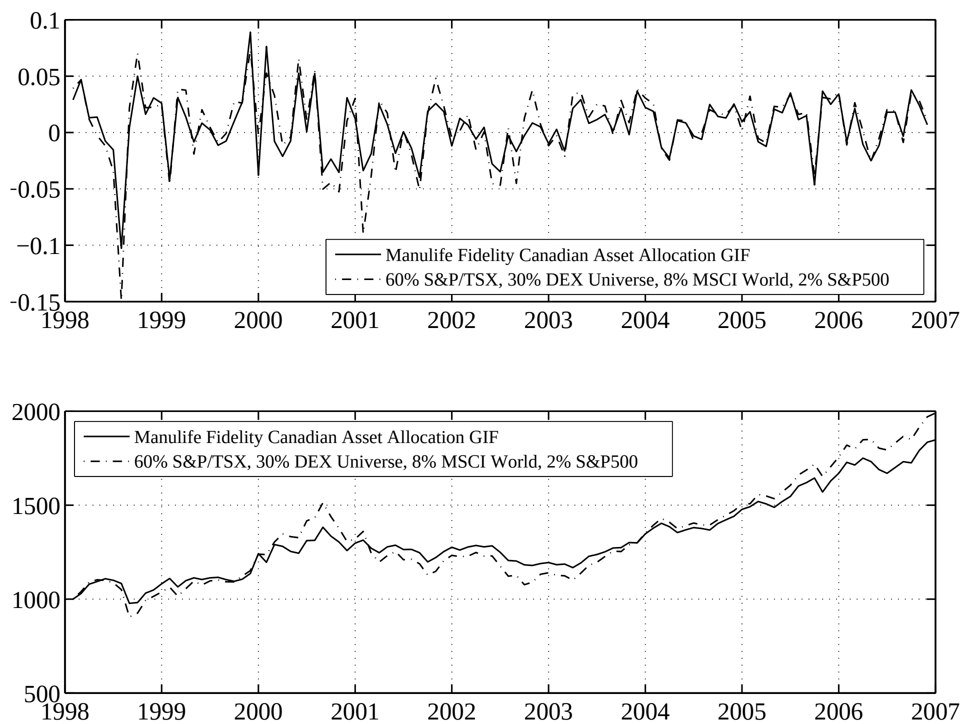


Figure 1.2: Historical performance of Manulife Fidelity Canadian Asset Allocation guaranteed investment fund. The benchmark index is created from three equity indices and one fixed income index based on the current fund structure. As shown in the figure, the benchmark return is highly correlated with the fund return. This further indicates the average change in the asset structure is relatively small during the sample period, and the major event like the “Technology Bubble” has less impact on the asset structure of the fund.

In Figure 1.2, we notice that overall the created benchmark portfolio closely trace the fund performance. In practice, such a sophisticated fund requires quantitative analysts; not only being able to model multiple currency-adjusted market indices simultaneously but also creating an appropriate mapping between actual fund return and index returns.

Empirical studies indicate the index return shows stylized facts, for instance, the volatility of the return is clustering and the return distribution is fat-tailed. Furthermore, some indices move together across markets. To account for all these empirical features, we propose to use a multivariate GARCH model in the context of the investment guarantee valuation.

In this thesis, our main goal is to demonstrate a practical actuarial approach, which is based on the multivariate GARCH model, for managing the financial risk associate with the GMMB option. In Chapter 2, we provide some useful market indices, and perform a relevant statistical analysis of the data. In Chapter 3, the multivariate GARCH models are presented, and their estimation methods are discussed. In Chapter 4, the estimated dynamic of the models are further evaluated through the fitted parameters. The goodness-of-fit is assessed for the estimated models, and the model selection is briefly discussed. In Chapter 5, we present our results in the guarantee liability valuation for the GMMB option based on the simulation analysis.

# Chapter 2

# Data

## 2.1 Introduction

Before we introduce multivariate GARCH models, it is worth spending time for a pre-analysis of the data. Such work can help verify the model assumptions and identify the models that are truly data driven. This chapter is structured as follows. In Section 2.2, we first answer the question why we need to use index data to create benchmark portfolios for modeling and describe in detail the indices used throughout this thesis. Section 2.3 is devoted to basic statistical analysis of historical index returns. The widely used normality assumption for returns is examined both in univariate and multivariate setting in Section 2.4. The autocorrelation of the volatilities in the univariate setting and the autocorrelation of the conditional covariance matrices in the multivariate setting are evaluated in Section 2.5.

## 2.2 Index Data

The available history of a segregated fund return is often too short to be applicable for modeling purpose. Even if we have enough return data, which are calculated from the past market price, using it to fit a model is not appropriate. The main reason for this is that return data of an underlying fund are fairly biased to the investment objects established by fund managers, and these investment objects will change in the future in order to adapt new market conditions. In the CIA's report (see Canadian Institute of Actuaries (CIA) 2001), the recommendation for choosing modeling data is creating an appropriate benchmark portfolio to trace the performance of the underlying fund. One type of possible benchmarks, mentioned in the report, is a linear combination of recognized market indices, which should be rich enough for modeling and inherit a sufficient amount of past market information. In this thesis, we take our benchmark to be a linear combination of the following market indices.

- **S&P/TSX** composite index measures the performance of Canadian equity market. The index is a list of the largest companies traded on the Toronto Stock Exchange.
- **S&P500** includes the current 500 large-cap US corporations traded on New York Stock Exchange and Nasdaq. The index is widely used as an indicator of US equity market performance.
- **MSCI World** is an index for measuring the performance of the international equity market. The index includes securities from 23 countries and maintained by Morgan Stanley Capital International.
- **DEX Universe** is a bond index tracking the performance of Canadian fixed income market.

Taken from Bloomberg, all indices are Canadian-dollar denominated and range from 1/31/1980 to 12/31/2006. In addition, these are recorded on total return level instead of price level; therefore, dividends should assume to be carried over and reinvested when calculating the monthly returns from these indices.

In the following figure, the total return index, $s_t$, is a market-closing quote from the last day of each month, and its value has been scaled by one thousand. In total, we have 324 historical observations for each index series, which implies there will be 323 monthly returns available for modeling.

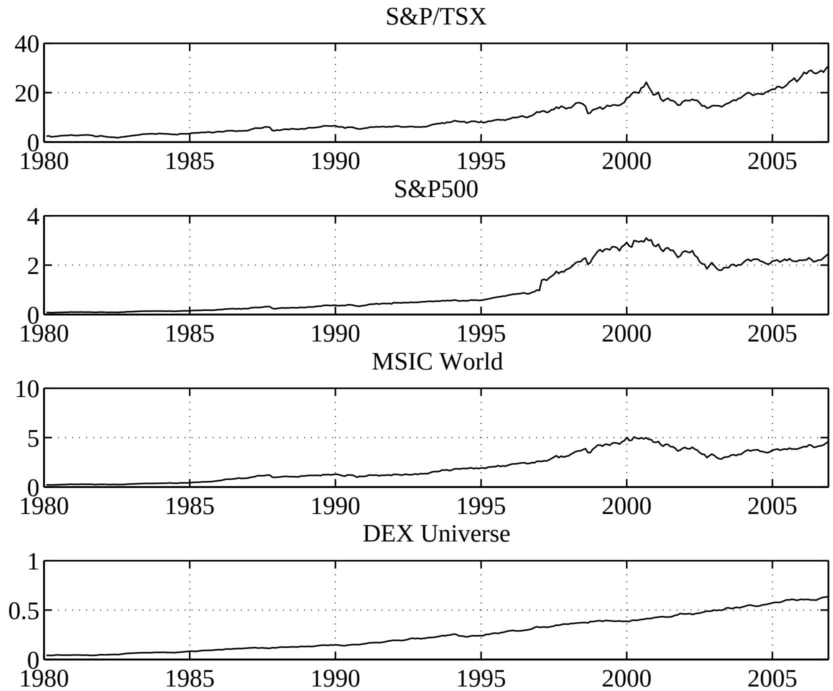


Figure 2.1: Month-end total return indices (in $10^3$) from Jan. 1980 to Dec. 2006

From the figure above, we observe equity indices exhibit strong co-movement with each other over time. US equity market is more likely to offer the largest capital return up to 2001, before the Technology Bubble induces all equity markets to start falling. On the other hand, Canadian fixed income market is progressively growing when the equity markets experience changes; in other words, a negative correlation between equity market and fixed income market appeared during the period between 2001 and 2003. This observation implies the historical correlation between markets is not constant. A data driven multivariate model should have a dynamic interpretation for the market correlation.

## 2.3 Empirical Return

In this section, we will analyze the historical index returns, on which our modeling is based. First, we define the one-period single return from $t-1$ to $t$ as follows:

$$r_t = \frac{s_t}{s_{t-1}} - 1 \in [-1, +\infty). \tag{2.1}$$

The maximum an investor can loss is everything invested at the beginning and obtain a $-100\%$ return; theoretically any security available at the market can be traded at any agreed price so there is no upper bound for $r_t$. For modeling, this is not a desirable property as many probability distributions have an unbounded domain. One way to solve the problem is to make a logarithmic transformation. Applying log function on $1 + r_t$ gives a new measure $x_t$ that can take any value from the real line.

$$x_t = \log(1 + r_t) = \log\left(\frac{s_t}{s_{t-1}}\right) \in \mathbb{R} \tag{2.2}$$

$x_t$ is commonly known as the log-return, and many interesting features of the log-return series at daily and weekly frequency have been uncovered. The following statement is a univariate version of the stylized facts. Many of these futures still remain valid for monthly log-returns.

- Return series are not iid although they show little serial correlation.
- Series of squared returns show profound serial correlation.
- Conditional expected returns are close to zero.
- Volatility appears to vary over time.
- Return series are leptokurtic or heavy-tailed.
- Extreme returns appear in clusters.

In Figure 2.2, we do not observe strong evidence in flavor of the serial correlation in returns. Most of time, an up-movement is likely followed by a down-movement

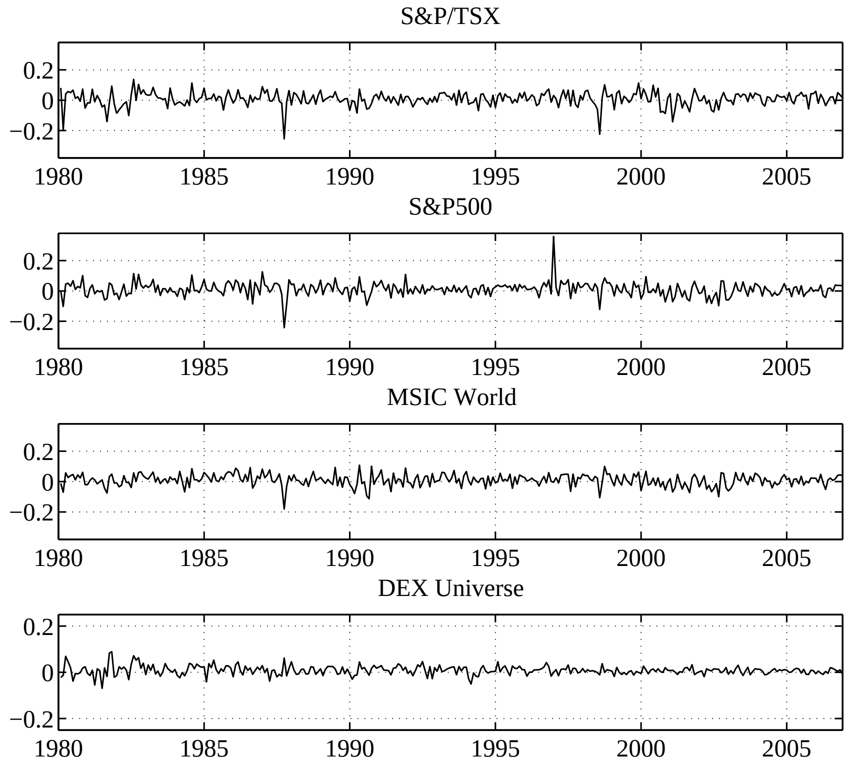


Figure 2.2: Monthly log-returns from Jan. 1980 to Dec. 2006

in each series. Furthermore, the volatility of the returns seems to vary over time, and the bond index exhibits stronger volatility-clustering than any equity index. Besides these observation, we also find that the fluctuations of equity indices are quite similar with each other, and extreme returns often appear at the same time. Based on the historical movement, we believe the returns of equity funds[1]are more positively correlated across markets for the same period.

The next figure shows the accumulation process for a thousand Canadian dollars, invested at February 1, 1980. As expected, US equity fund greatly outperforms its competitors. The accumulation value of the index fund, which traces the performance of S&P500, reached $38,000 approximately in 2001. This is about 1.5 times

[1]Throughout this thesis, a fund is referred to either an indexed fund or a balanced fund, wherever is appropriate.

the value of international equity fund, and about 3.8 times the value of Canadian equity fund or Canadian fixed income fund.

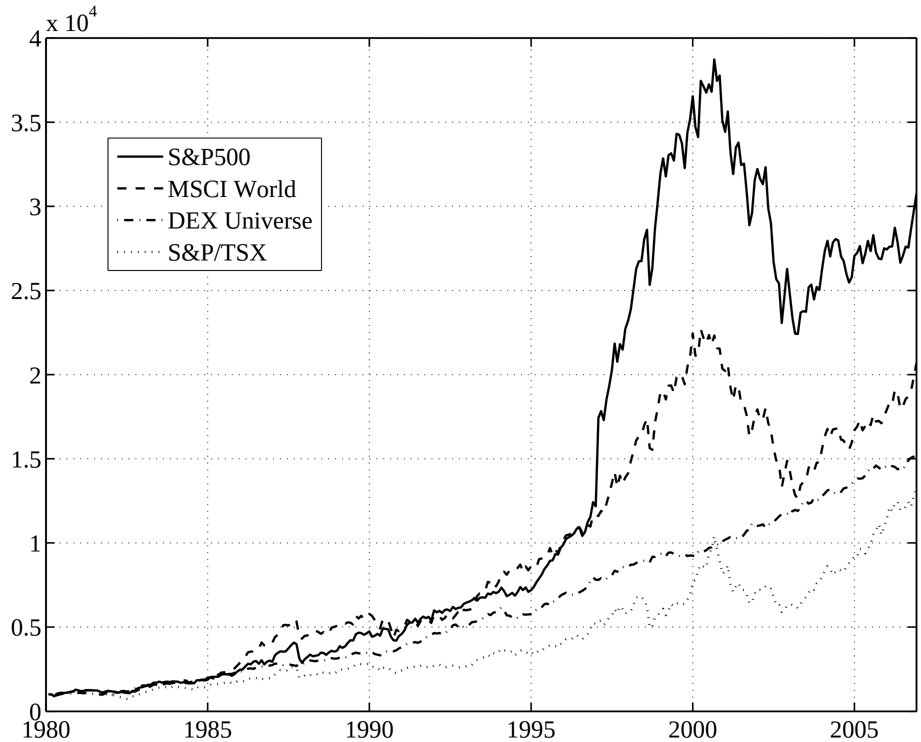


Figure 2.3: Accumulation process of one thousand investment from Feb. 1980 to Dec. 2006

Another observation from the figure is Canadian fixed income fund offers better return than Canadian equity fund, which seems counterintuitive. One possible explanation is that risk-free rates were quite high during the 80's, as a result, the return from fixed income market was pushed up by those high base rates.

We now turn our attention to the empirical distributions of the index returns. In addition to sample mean and standard deviation, sample skewness and kurtosis are statistics used to describe the empirical distributions. The skewness measures symmetry of a distribution, whereas the kurtosis makes a comparison of the thickness of the tail, between an empirical distribution and the normal distribution. Any relatively large nonzero skewness reports evidence of asymmetry, and any kurtosis larger than 3 implies that the empirical distribution has fatter tails than the normal. Statistical definitions of these estimators are given next.

Let $\{x_1, x_2, ..., x_T\}$ be the observed log-returns for each index series over T periods. The sample mean is

$$\hat{\mu}_x = \frac{1}{T}\sum_{t=1}^{T} x_t, \tag{2.3}$$

the sample standard deviation[2] is

$$\hat{\sigma}_x = \Big(\frac{1}{T-1}\sum_{t=1}^{T}(x_t - \hat{\mu}_x)^2\Big)^{1/2}, \tag{2.4}$$

the sample skewness is

$$\hat{S}(x) = \frac{1}{(T-1)\hat{\sigma}_x^3}\sum_{t=1}^{T}(x_t - \hat{\mu}_x)^3, \tag{2.5}$$

and the sample kurtosis[3] is

$$\hat{K}(x) = \frac{1}{(T-1)\hat{\sigma}_x^4}\sum_{t=1}^{T}(x_t - \hat{\mu}_x)^4. \tag{2.6}$$

From Table 2.1, we find the international index fund offers the best diversification among equity funds since it has the lowest sample volatility. The sample percentiles indicate that the returns of US equity fund are more spread out compared to the rest of funds. In addition, US equity fund also has the largest sample mean. On the other hand, the underlying return distribution of Canadian equity fund is slightly skewed to the left as indicated by the sample skewness. In contrast, the rest of the three funds tend to have a symmetric return distribution. The tail of each sample distribution is fatter than the tail of the normal. The US equity fund has the thickest tail.

[2] In Finance, the term volatility refers to the standard deviation of the underlying random variable, such as the asset return.

[3] The calculated kurtosis in Table 2.1 is actually the excess kurtosis.

| Sample Statistics | | | | | |
|---|---|---|---|---|---|
| Index | Min | 1 Q | Median | 3 Q | Max |
| S&P/TSX | -25.52% | -1.48% | 1.17% | 3.70% | 13.71% |
| S&P500 | -24.26% | -1.39% | 1.29% | 3.82% | 35.87% |
| MSIC World | -18.02% | -1.31% | 1.10% | 3.69% | 10.72% |
| DEX Universe | -6.89% | -0.36% | 0.86% | 1.89% | 8.91% |
| Sample Moments | | | | | |
| Index | Average | Volatility | Skewness | Kurtosis | |
| S&P/TSX | 0.80% | 4.70% | -1.27 | 5.20 | |
| S&P500 | 1.07% | 4.62% | 0.67 | 11.64 | |
| MSIC World | 0.95% | 3.92% | -0.57 | 1.52 | |
| DEX Universe | 0.84% | 1.97% | 0.24 | 2.30 | |

Table 2.1: Summary of the log-returns from Feb. 1980 to Dec. 2006

To get an idea of the annual performance of each index fund, we apply a simple scaling rule on monthly returns to get annual return estimates. Table 2.2 is obtained based on the following procedure: the sample average of monthly returns is multiplied by 12, and the sample volatility is scaled up by the square root of 12. According to the table, all markets approximately offer a 10% return on annual basis, which is quite optimistic according to recent market performance.

| Annualized Performance | | | |
|---|---|---|---|
| Index | Average | Volatility | Sharp Ratio |
| S&P/TSX | 9.64% | 16.29% | 0.35 |
| S&P500 | 12.86% | 16.02% | 0.55 |
| MSIC World | 11.40% | 13.57% | 0.55 |
| DEX Universe | 10.10% | 6.84% | 0.89 |

Table 2.2: Summary of the annualized log-returns from Feb. 1980 to Dec. 2006

The Sharp ratio measures an investment's excess return over a risk-free rate for taking an extra unit of risk[4], and it is defined as

$$\text{Sharp Ratio} = \frac{\hat{\mu}_x - r_f}{\hat{\sigma}_x}. \tag{2.7}$$

The sharp ratio based on 4% annual risk-free rate indicates Canadian fixed income fund has the highest risk premium.

To analyze the empirical correlation across markets, we need to compute the sample correlation matrix $\widehat{\Gamma}$, which can be obtained through the sample covariance matrix $\widehat{\Sigma}$. The column vector $\mathbf{x}_t$ is formed by the log-returns of all indices observed in period t, and $\widehat{\mu}$ is the sample average of $\mathbf{x}_t$.

$$\widehat{\Gamma} = \widehat{\Sigma}^{-1} \cdot \widehat{\Sigma} \cdot \widehat{\Sigma}^{-1} \tag{2.8}$$

$$\widehat{\Sigma} = \frac{1}{T-1} \sum_{t=1}^{T} (\mathbf{x}_t - \widehat{\mu})(\mathbf{x}_t - \widehat{\mu})' \tag{2.9}$$

| | S&P/TSX | S&P500 | MSIC World | DEX Universe |
|---|---|---|---|---|
| S&P/TSX | 1.00 | - | - | - |
| S&P500 | 0.67 | 1.00 | - | - |
| MSIC World | 0.65 | 0.73 | 1.00 | - |
| DEX Universe | 0.26 | 0.22 | 0.16 | 1.00 |

Table 2.3: Empirical correlation of the log-returns from Feb. 1980 to Dec. 2006

In Table 2.3, all index funds were positively correlated throughout the period, and the correlation is low between equity fund and fixed income fund.

For simplicity, we assume the expected monthly return is constant in long run; however, we will not use the sample mean as our long run mean because it is not conservative enough for guarantee liability valuation. Instead, we propose a set

[4]The risk is typically measured by the sample volatility.

of lower return rates in Table 2.4, and our multivariate models will explore the conditional dynamics of the excess return $\mathbf{Y}_t|\mathcal{F}_{t-1}$, where

$$\mathbf{Y}_t|\mathcal{F}_{t-1} = \mathbf{X}_t|\mathcal{F}_{t-1} - \mathbb{E}(\mathbf{X}_t|\mathcal{F}_{t-1}) = \mathbf{X}_t|\mathcal{F}_{t-1} - \mathbf{C}. \tag{2.10}$$

The vector $\mathbf{C}$ represents our view on the monthly return, which is based on the market information available at the beginning of the month, and this adjustment is consistent with the third stylized fact that conditional expected returns are close to zero.

| Cash | S&P/TSX | S&P500 | MSIC World | DEX Universe |
|---|---|---|---|---|
| 4% | 6.5% | 7% | 7% | 5% |

Table 2.4: Long-term view of the log-returns. The constant view is on the monthly return, and it has been annualized by multiplying 12.

## 2.4 Normality Assessment

The classical multivariate GARCH models assume the excess return $\mathbf{Y}_t$ follows a multivariate Gaussian process. This assumption is often criticized by the empirical evidences found in daily or weekly returns. In this section, we verify the normality assumption for monthly returns from the perspective of univariate and multivariate distributions. In univariate analysis, the diagnostic tools include QQ-plot, histogram[5], Jarque-Bera test, and Shapiro-Wilks test; whereas for multivariate analysis, a chi-square QQ-plot is used to draw conclusion.

From the histogram, we observe the normal approximation of the true distribution of $Y_{t,i}$[6] is overall reasonable, although there are two issues around this assumption. The primary issue is that it does not adequately provide enough thickness

[5] Each overlayed bell curve in the histogram is the probability density function estimated from the sample under the normality assumption.

[6] The $Y_{t,i}$ denotes the excess log-return of the $i^{th}$ component series at time t.

for the tail, to accommodate the extreme historical observations. In Figure 2.4, all equity markets have extreme values in the left tail. The second issue is a minor amount of skewness. The histograms are slightly skewed to the left to cover the observed low returns.

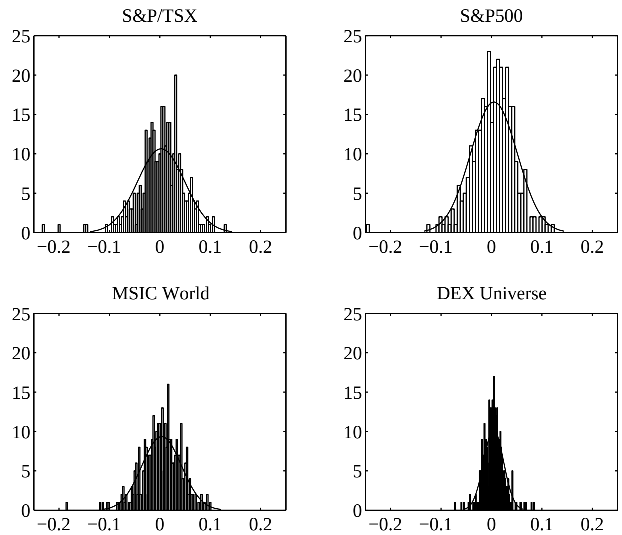


Figure 2.4: Histograms of the excess log-returns

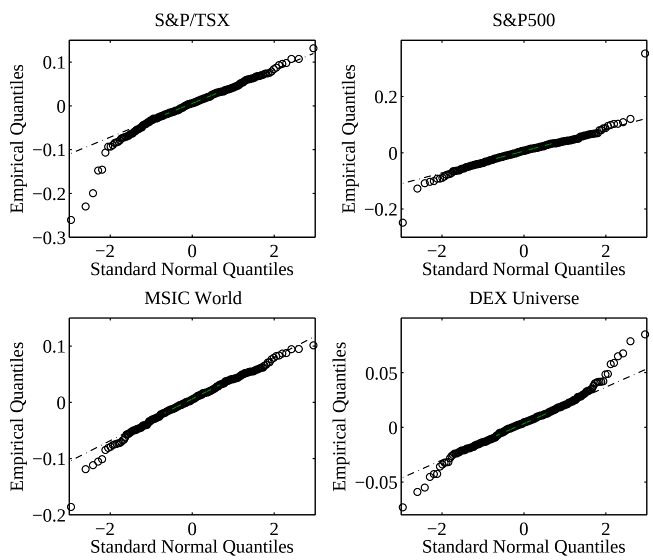


Figure 2.5: QQ-plots of the excess log-returns

The quantile-to-quantile plots suggest the empirical distribution of the excess log-returns corresponds to a normal distribution most of the time except in the tail area, which is consistent with the observation based on the histograms. Therefore, rejecting the normality assumption in the univariate setting is expected in the following statistical tests.

The Jarque-Bera statistic (see Jarque and Bera 1980, 1987), in equation 2.11, is defined in terms of the sample skewness $\hat{S}$ and the sample kurtosis $\hat{K}$. Under the null hypothesis, the statistic is asymptotically chi-square with 2 degrees of freedom; thus, a large value of the test statistic leads to reject normality assumption.

$$JB(y) = \frac{n}{6}\Big(\hat{S}^2(y) + \frac{(\hat{K}(y)-3)^2}{4}\Big) \tag{2.11}$$

The Shapiro-Wilks test (see Shapiro and Wilks 1965) is more involved in terms of computing the statistics. In equation 2.12, $y_{(i)}$ is the $i^{th}$ ordered observation from a sample of size $n$. $M$ is the expected value of the order statistics from the standard normal distribution, and $V$ is the covariance matrix of these order statistics. The Shapiro-Wilks test statistic is normally distributed, and a small value indicates departures from normality.

$$SW(y) = \frac{\sum_{i=1}^{n} a_i y_{(i)}}{\sum_{i=1}^{n}(y_i - \bar{y})^2}\ , \qquad (a_1, ..., a_n)' = \frac{MV^{-1}}{(M'V^{-1}V^{-1}M)^{1/2}} \tag{2.12}$$

| Jarque-Bera Test | | | | |
|---|---|---|---|---|
| | S&P/TSX | S&P500 | MSIC World | DEX Universe |
| Test Statistics | 456.81 | 1818.07 | 48.88 | 72.43 |
| P_Value | 0.00 | 0.00 | 0.00 | 0.00 |
| **Shapiro-Wilks Test** | | | | |
| | S&P/TSX | S&P500 | MSIC World | DEX Universe |
| Test Statistics | 0.929 | 0.904 | 0.981 | 0.971 |
| P_Value | 0.00 | 0.00 | 0.00 | 0.00 |

Table 2.5: Testing results for the normality of the excess log-returns

Both tests reject null hypothesis that the true underlying distribution of the excess log-return is normal; however, the normal approximation provides a overall reasonable fit based on the previously visual analysis. Other distributions, such as $t$ or Hansen's skewed $t$, may lead to a tail improvement.

We turn to our main interest, that is, to test whether or not the random vector $\mathbf{Y}_t$ follows Gaussian process. This is equivalent to asking if $\mathbf{Y}_t$ is independently and identically distributed as a $d$-dimensional multivariate normal, which can be verified by a chi-square test. Given the null hypothesis, the test statistic $\tilde{\mathbf{y}}_t' \cdot \tilde{\mathbf{y}}_t$ should have a chi-square distribution with $d$ degrees of freedom. In the chi-square

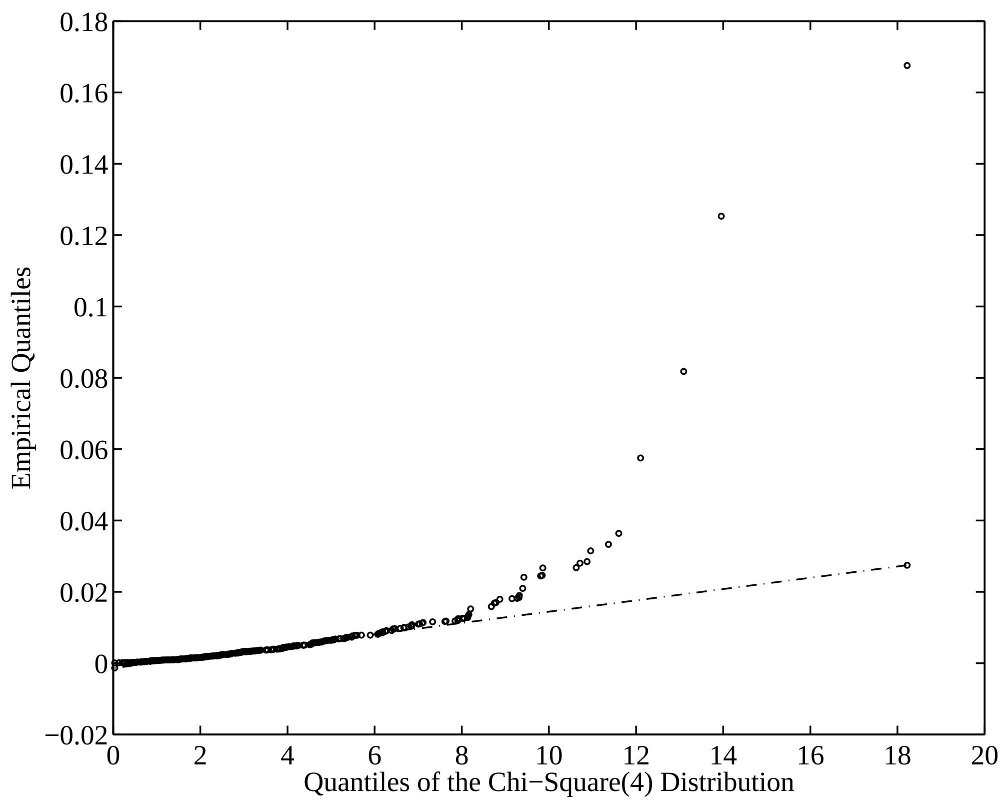


Figure 2.6: Chi-square QQ-plot for multivariate excess log-returns

QQ-plot, we find the same issue as for the univariate series. The extreme $\mathbf{Y}_t$ in our history do not look like they are from a multivariate normal distribution. More probably, it is from a multivariate distribution, which has a similar shape as the multivariate normal but with fatter tails. A good candidate may be a multivariate $t$ distribution.

## 2.5 Autocorrelation Analysis

We check if the return data exhibit heteroscedasticity in this section. The plot of autocorrelation function (ACF) is given for analyzing the serial correlation in the volatility[7] of the component return series. We then provide the results of the univariate portmanteau tests. The serial correlation between covariance matrices[8] is also tested in a multivariate setting.

The *Lag-$l$* autocorrelation $\rho_l$ is used to measure the strength of the underlying correlation of a univariate return series. The sample estimator is in the following form, where $\bar{y}$ is the sample mean.

$$\widehat{\rho}_l = \frac{\sum_{t=l+1}^{T}(y_t - \bar{y})(y_{t-l} - \bar{y})}{\sum_{t=1}^{T}(y_t - \bar{y})^2} \ , \quad 0 \leq l < T - 1 \tag{2.13}$$

Asymptotically $\widehat{\rho}_l$ has a normal distribution with mean zero and variance 1/T for any positive integer $l$, when $(Y_t)_{t\in\mathbb{Z}}$ is truly an iid random sequence satisfying $\mathbb{E}(Y_t^2) < \infty$. This result is often used to test $H_o$: $\rho_l$=0 versus $H_a$: $\rho \neq 0$ with next statistic.

$$\text{t-ratio} = \frac{\widehat{\rho}_l}{\sqrt{(1 + 2\sum_{i=1}^{l-1}\widehat{\rho}_i^2)/T}} \tag{2.14}$$

If $Y_t$ is a covariance stationary[9]univariate Gaussian time series satisfying $\rho_j$=0 for $j > l$, then asymptotically the t-ratio has a standard normal distribution. Therefore, if $\widehat{\rho}_l$ exceeds the upper or lower confidence band in an ACF plot, we find a statistical evidence to reject $H_o$ at a certain confidence level. In practice, this graphical

[7]Strictly speaking, we want to analyze the serial correlation in the conditional variance.

[8]The covariance matrix refers to $\mathbb{COV}(\mathbf{Y}_t|\mathcal{F}_{t-1})$ in Section 3.2.

[9]The covariance stationary means that a time series has constant mean and variance.

method based on the ACF plot is routinely applied to the autocorrelation in return squares, which are sample observations for the underlying conditional variance of the excess log-return; however, we should not entirely rely on this method since the underlying series is often not a Gaussian series, on which the test is assumed.

Verifying the autocorrelation in volatility is critically important as the GARCH models applied in later analysis directly specify a regression-type structure on volatility, which assumes the volatility in next period is a function of the volatility from previous period. In the ACF plots, we do not observe significant deviation from either of the confidence bands, except for the returns from DEX Universe index. This result was anticipated at the discussion of the empirical return.

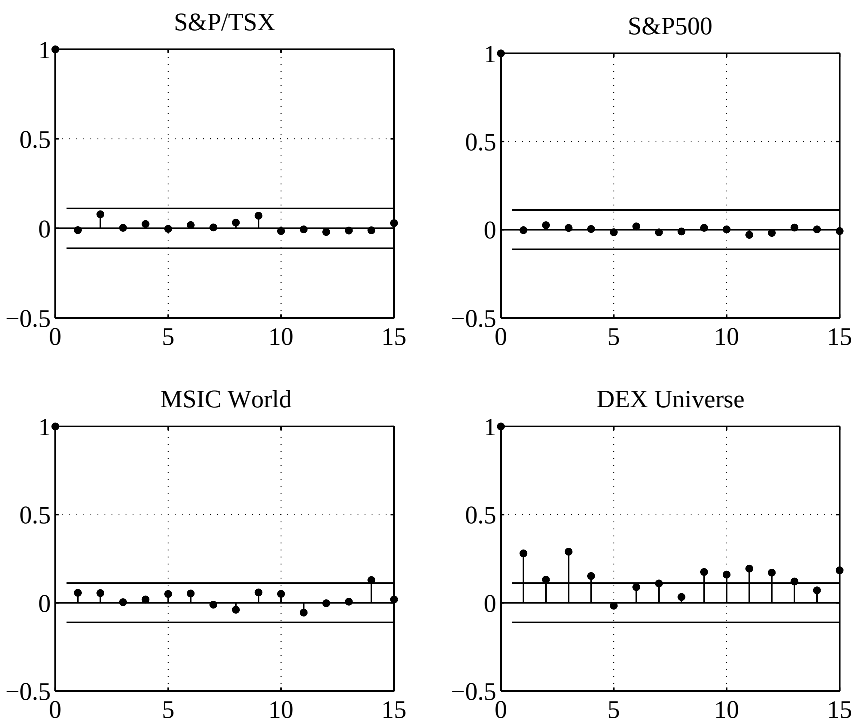


Figure 2.7: Autocorrelation plots of the excess log-return squares

We introduce Box-Pierce test and Ljung-Box test to formally verify the autocorrelation in volatility. Box and Pierce (1970) propose a portmanteau statistic $Q_{BP}(m)$ to test the null hypothesis $H_o$: $\rho_1 = ... = \rho_m = 0$ against the alternative hypothesis $H_a$: not all $\rho_i$s are zero, where $\rho_i$ is the autocorrelation of the volatility for a temporal separation $i$. Under the assumption that $(Y_t)_{t\in\mathbb{Z}}$ is an iid sequence obeying some regularity conditions on moments, $Q_{BP}(m)$ asymptotically has chi-square distribution with $m$ degrees of freedom. Later, Ljung and Box (1978) modify the $Q_{LB}(m)$ statistic to increase the power in small samples. The modified test statistics $Q_{LB}(m)$ follows the same asymptotic distribution.

$$Q_{BP}(m) = T\sum_{l=1}^{m}\hat{\rho}_l^2 \tag{2.15}$$

$$Q_{LB}(m) = T(T+2)\sum_{l=1}^{m}\frac{\hat{\rho}^2 l}{T-l} \tag{2.16}$$

The univariate portmanteau test is further generalized into multivariate setting by Hosking (1980). For a multivariate series, the null hypothesis changes to $H_o$: $\mathbf{P}(1) = ... = \mathbf{P}(m) = \mathbf{0}$, and the alternative hypothesis becomes $H_a$: not all $\mathbf{P}(i)$ are zero matrices. The definition of $\mathbf{P}(i)$ is given in Section 3.2, and it is the correlation matrix for $\mathbf{Y}_t$ and $\mathbf{Y}_{t-i}$. The statistic is now applied to test the hypothesis that there is no autocorrelation in the vector series $(\mathbf{Y}_t^*)_{t\in\mathbb{Z}}$, where

$$\mathbf{Y}_t^* = \text{vech}(\mathbf{Y}_t \cdot \mathbf{Y}_t'). \tag{2.17}$$

The "vech" stands for the vector half operator, which stacks the columns of the lower triangle of a symmetric matrix in a single column vector. Here, $\mathbf{Y}_t^*$ is a vector containing $Y_{t,i} \cdot Y_{t,j}$ where $i \in \{1, 2, ...d\}$ and $i \leq j \leq d$. The realizations of these products are sample observations for the variances and the covariances in the conditional covariance matrix at time t. Applying Hosking's multivariate portmanteau test on $(\mathbf{Y}_t^*)_{t\in\mathbb{Z}}$ will help test whether or not there is any correlation between the conditional covariance matrix of $\mathbf{Y}_t$ and the conditional covariance matrix of $\mathbf{Y}_s$.

Given the null hypothesis is true and some regularity conditions are satisfied,

$Q_H(m)$ is asymptotically a chi-square random variable with $k^2m$ degrees of freedom.

$$Q_H(m) = T^2 \sum_{l=1}^{m} \frac{1}{T-l} \text{trace}\Big(\widehat{\Gamma}_l' \, \widehat{\Gamma}_0^{-1} \, \widehat{\Gamma}_l \, \widehat{\Gamma}_0^{-1}\Big) \tag{2.18}$$

In Table 2.6, the univariate tests show no evidence in flavor of autocorrelations in volatility for equity index returns, whereas there is a strong evidence found in bond index returns. Moreover, the multivariate test indicates there are autocorrelations in the conditional covariance matrices at 5% significance level.

| Ljung-Box Univariate Test | | | | |
|---|---|---|---|---|
| | S&P/TSX | S&P500 | MSIC World | DEX Universe |
| Test Statistics | 4.56 | 0.97 | 7.32 | 114.44 |
| P_Value | 0.97 | 1.00 | 0.84 | 0.00 |
| Box-Pierce Univariate Test | | | | |
| | S&P/TSX | S&P500 | MSIC World | DEX Universe |
| Test Statistics | 4.45 | 0.94 | 7.13 | 111.83 |
| P_Value | 0.97 | 1.00 | 0.85 | 0.00 |
| Hosking Multivariate Test | | | | |
| | $\mathbf{Y}_t^*$ | | | |
| Test Statistics | 230.49 | | | |
| P_Value | 0.03 | | | |

Table 2.6: Testing results for the autocorrelations of the excess log-return squares for 12 lags

## 2.6 Summary

Before proceeding to next chapter, we make a summary of important observations in the following list.

- All index return series have an empirical distribution whose tail is fatter than normal, though the degree of thickness in tail is different.
- For individual return series, the autocorrelation in volatility is not statistically significant, except for the DEX Universe index.
- In multivariate setting, the conditional covariances are serially correlated, indicated by Hosking's multivariate portmanteau test.

# Chapter 3

# Multivariate Models

## 3.1 Introduction

We begin this chapter by a statement taken in a recent survey of multivariate GARCH models by Bauwens *et al.* (2006).

> Understanding and predicting the temporal dependence in the second-order moments of asset returns is important for many issues in financial econometrics. It is now widely accepted that financial volatilities move together over time across assets and markets. Recognizing this feature through a multivariate modeling framework leads to more relevant empirical models than working with separate univariate models. From a financial point of view, it opens the door to better decision tools in various areas, such as asset pricing, portfolio selection, option pricing, hedging and risk management.

Applying multivariate GARCH models in the context of the investment guarantee valuation provides a dynamic interpretation for the complicated movement of volatilities and correlations of asset returns. To better understand these multivariate models, we first introduce the relevant fundamentals from multivariate time series analysis in Section 3.2. We then present some of the most important multivariate GARCH models in Section 3.3. The emphasize of the presentation is on the dynamic structure of $\Sigma_t$, known as the conditional covariance matrix at time $t$. In Section 3.4, the innovation distributions, which may be potentially coupled with the models, are explored with a focus on the tail. In Section 3.5, we introduce the method of maximum likelihood estimation for fitting the multivariate GARCH models, and briefly discuss the existing numerical issues.

## 3.2 Basic Definition

A multivariate time series is a stochastic vector process $(\mathbf{Y}_t)_{t\in\mathbb{Z}}$, indexed by the integers. This process is well defined on a filtered probability space $(\mathit{\Omega}, \mathcal{F}, P, \underline{\mathcal{F}})$. Typically, $\mathit{\Omega}$ is known as the sample space, $\mathcal{F}$ is a sigma algebra of the sample space, $P$ is a probability measure function, and $\underline{\mathcal{F}}$ is the natural filtration $(\mathcal{F}_t)_{t\in\mathbb{Z}}$ generated by the process, which represents the market information available at time $t$.

The mean function $\boldsymbol{\mu}(t)$ and the covariance matrix function $\mathbf{\Gamma}(t,s)$ of the process are defined in terms of the mathematical expectation.

$$\boldsymbol{\mu}(t) = \mathbb{E}(\mathbf{Y}_t) \qquad t \in \mathbb{Z} \tag{3.1}$$

$$\mathbf{\Gamma}(t,s) = \mathbb{E}\Big(\mathbf{Y}_t - \boldsymbol{\mu}(t)\Big)\Big(\mathbf{Y}_s - \boldsymbol{\mu}(s)\Big)' \qquad t,s \in \mathbb{Z} \tag{3.2}$$

Many multivariate time series models are stationary because the stationarity assumption is needed in the mathematics of the models, and in most of cases the assumption is valid from the empirical perspective. The multivariate GARCH models applied in this thesis are stationary in one or both of the following senses: a multivariate time series $(\mathbf{Y}_t)_{t\in\mathbb{Z}}$ is strictly stationary if the next statement holds for all $t_1$,...,$t_n$,$k \in \mathbb{Z}$ and for all $n \in \mathbb{N}$

$$(\mathbf{Y}_{t_1}', ..., \mathbf{Y}_{t_n}') \stackrel{d}{=} (\mathbf{Y}_{t_1+k}', ..., \mathbf{Y}_{t_n+k}'), \tag{3.3}$$

and a multivariate time series $(\mathbf{Y}_t)_{t\in\mathbb{Z}}$ is covariance stationary if the first two moments exist and satisfy

$$\boldsymbol{\mu}(t) = \boldsymbol{\mu} \qquad t \in \mathbb{Z}, \tag{3.4}$$

$$\boldsymbol{\Gamma}(t,s) = \boldsymbol{\Gamma}(t+k, s+k) \qquad t, s, k \in \mathbb{Z}. \tag{3.5}$$

Strictly stationary implies covariance stationary if $\boldsymbol{\Gamma}(t,s)$ is finite for all t and s; however, certain multivariate GARCH models, which have infinite-variance process, are strictly stationary but not covariance stationary.

The covariance stationary assumption implies $\Gamma(t,s) = \Gamma(t-s,0)$ for all $t$ and $s$. In other words, the covariance matrix for $\mathbf{Y}_t$ and $\mathbf{Y}_s$ only depends on their temporal separation $t-s$, known as lag. For a covariance-stationary process we rewrite the covariance matrix function in terms of the temporal separation time $h$: $\Gamma(h) = \Gamma(h,0), \forall h \in \mathbb{Z}$. One important observation is that $\Gamma(0) = \mathbb{COV}(\mathbf{Y}_t)$ for all $t$; in other words, the unconditional covariance matrix of $\mathbf{Y}_t$ remains unchanged throughout the entire stochastic process.

In some cases, the correlation function between $\mathbf{Y}_t$ and $\mathbf{Y}_s$ is of more interest than the covariance function. The following two operators are often applied on a covariance matrix $\Sigma$ to obtain the corresponding correlation matrix.

$$\triangle(\Sigma) = \mathrm{diag}(\sqrt{\sigma_{11}}, ..., \sqrt{\sigma_{dd}}) \tag{3.6}$$

$$\wp(\Sigma) = \Big(\triangle(\Sigma)\Big)^{-1} \cdot \Sigma \cdot \Big(\triangle(\Sigma)\Big)^{-1} \tag{3.7}$$

The $\triangle$ operator extracts a diagonal volatility matrix from $\Sigma$, and the $\wp$ operator extracts the required correlation matrix.

The correlation matrix function $\mathbf{P}(h)$ of a covariance-stationary multivariate time series $(\mathbf{Y}_t)_{t\in\mathbb{Z}}$ is defined by

$$\mathbf{P}(h) = \triangle^{-1}\, \boldsymbol{\Gamma}(h)\, \triangle^{-1} = \wp\Big(\boldsymbol{\Gamma}(h)\Big) \quad \forall h \in \mathbb{Z}. \tag{3.8}$$

The diagonal entries of $\mathbf{P}(h)$ give autocorrelation functions of component series, and the off-diagonal entries give cross-correlation between different component series at different times.

A class of important multivariate time series, known as the multivariate white noise, is defined in term of the correlation matrix function. They often serve as the building blocks for the multivariate GARCH models. A multivariate time series $(\mathbf{Y}_t)_{t\in\mathbb{Z}}$ is a multivariate white noise process if the process is covariance stationary and the correlation matrix function has the following structure

$$\mathbf{P}(h) = \begin{cases} \mathbf{P}, & h = 0 \\ 0, & h \neq 0 \end{cases} \tag{3.9}$$

for some positive-definite correlation matrix $\mathbf{P}$. Except at lag zero, such a process has no cross-correlation between component series. A simple sub-class is known as multivariate strict white noise, which is a multivariate white noise series formed by independent identically distributed random vectors. However, the independence is not a requirement for the multivariate white noise in general.

Later the multivariate white noise with mean zero and covariance matrix $\Sigma = \mathbb{COV}(\mathbf{Y}_t)$ will be denoted as $\mathbf{WN}(\mathbf{0}, \Sigma)$. Correspondingly, the multivariate strict white noise will be denoted as $\mathbf{SWN}(\mathbf{0}, \Sigma)$.

We complete this section by providing multivariate martingale difference property and its relation with multivariate white noise process. A multivariate time series $(\mathbf{Y}_t)_{t\in\mathbb{Z}}$ is said to have the multivariate martingale-difference[1] property with respect to the filtration $(\mathcal{F}_t)_{t\in\mathbb{Z}}$ if $\mathbb{E}|\mathbf{Y}_t| < \infty$ and

$$\mathbb{E}(\mathbf{Y}_t|\mathcal{F}_{t-1}) = 0, \qquad \forall t \in \mathbb{Z}. \tag{3.10}$$

The unconditional expectation of the process $(\mathbf{Y}_t)_{t\in\mathbb{Z}}$ is clearly zero. This further

[1]This definition is equivalent to the usual definition found in many books on advanced stochastic process provided that the process $(\mathbf{X}_t)_{t\in\mathbb{Z}}$ is a martingale, which requires $(\mathbf{X}_t)_{t\in\mathbb{Z}}$ to be adapted to a filtration $(\mathcal{F}_t)_{t\in\mathbb{Z}}$, $\mathbb{E}(|\mathbf{X}_t|) < \infty$ for all t, and $\mathbb{E}(\mathbf{X}_t|\mathcal{F}_s) = \mathbf{X}_s$ for all $t > s$. In the thesis, we treat the process $(\mathbf{X}_t)_{t\in\mathbb{Z}}$ as a martingale.

implies $\mathbf{\Gamma}(t,s) = 0$ for $t > s$ since from the definition we have

$$\begin{aligned}\mathbf{\Gamma}(t,s) &= \mathbb{E}\Big(\mathbf{Y}_t - \boldsymbol{\mu}(t)\Big)\Big(\mathbf{Y}_s - \boldsymbol{\mu}(s)\Big)' = \mathbb{E}\Big(\mathbf{Y}_t - 0\Big)\Big(\mathbf{Y}_s - 0\Big)' \\ &= \mathbb{E}\Big(\mathbb{E}(\mathbf{Y}_t\mathbf{Y}_s'|\mathcal{F}_s)\Big) = \mathbb{E}\Big(\mathbb{E}(\mathbf{Y}_t|\mathcal{F}_s)\mathbf{Y}_s'\Big) \\ &= \mathbb{E}\Big(\mathbf{0}\cdot\mathbf{Y}_s'\Big) = \mathbf{0}. \end{aligned} \tag{3.11}$$

Therefore, if the process has a constant $\mathbb{COV}(\mathbf{Y}_t)$ for all $t$, then it is a multivariate white noise process. To sum up the relation, a multivariate process with martingale difference property and constant $\mathbb{COV}(\mathbf{Y}_t)$ is automatically a multivariate white noise process.

## 3.3 Multivariate GARCH Models

We start this section by defining the **general structure** of the multivariate GARCH model at first. Let $(\mathbf{Z}_t)_{t\in\mathbb{Z}}$ be $\mathbf{SWN}(\mathbf{0}, I_d)$, where $I_d$ is a $d$-dimensional identity matrix. The noise series $(\mathbf{Y}_t)_{t\in\mathbb{Z}}$ follows a multivariate GARCH process if it is strictly stationary and satisfies the equation in the following form:

$$\mathbf{Y}_t = \Sigma_t^{1/2}\,\mathbf{Z}_t, \qquad t\in\mathbb{Z}, \tag{3.12}$$

where $\Sigma_t^{1/2}$ is the Cholesky factor[2] of a positive definite matrix $\Sigma_t$ measurable with respect to $\mathcal{F}_{t-1} = \sigma(\mathbf{Y}_i : i \le t-1)$.

By using the relation between martingale difference property and multivariate white noise, we can verify that any covariance stationary multivariate process satisfying the general structure is a multivariate while noise. First, it is easy to see a covariance stationary process of this type has the multivariate martingale-difference property:

$$\mathbb{E}(\mathbf{Y}_t|\mathcal{F}_{t-1}) = \mathbb{E}(\Sigma_t^{1/2}\,\mathbf{Z}_t|\mathcal{F}_{t-1}) = \Sigma_t^{1/2}\,\mathbb{E}(\mathbf{Z}_t) = \mathbf{0}. \tag{3.13}$$

[2]The Cholesky factor $\Sigma_t^{1/2}$ is the lower triangle matrix with positive diagonal element and satisfies the equation $\Sigma_t = (\Sigma_t^{1/2})\cdot(\Sigma_t^{1/2})'$.

Any covariance stationary process also has constant $\mathbb{COV}(\mathbf{Y}_t)$ as shown in the previous section. Hence, a covariance stationary process following the general specification is a multivariate white noise process with mean zero and covariance matrix $\Sigma$ where

$$\begin{aligned}
\Sigma = \mathbb{COV}(\mathbf{Y}_t) &= \mathbb{E}\Big(\mathbb{COV}(\mathbf{Y}_t|\mathcal{F}_{t-1})\Big) + \mathbb{COV}\Big(\mathbb{E}(\mathbf{Y}_t|\mathcal{F}_{t-1})\Big) \\
&= \mathbb{E}\Big(\mathbb{COV}(\mathbf{Y}_t|\mathcal{F}_{t-1})\Big) + \mathbf{0} = \mathbb{E}\Big(\mathbb{E}(\mathbf{Y}_t\mathbf{Y}_t'|\mathcal{F}_{t-1})\Big) \\
&= \mathbb{E}\Big(\Sigma_t^{1/2}\mathbb{E}(\mathbf{Z}_t\mathbf{Z}_t')(\Sigma_t^{1/2})'\Big) = \mathbb{E}\Big(\Sigma_t^{1/2}(\Sigma_t^{1/2})'\Big) = \mathbb{E}(\Sigma_t).
\end{aligned} \tag{3.14}$$

Furthermore, each multivariate GARCH model, in essence, gives a deterministic specification for $\mathbb{COV}(\mathbf{X}_t|\mathcal{F}_{t-1}) = \Sigma_t$. There are two directions to give such a time-varying structure. The clear direction is to specify the dynamics of $\Sigma_t$, and the multivariate GARCH models developed at the early stage of the literature belong to this family, such as DVEC and BEKK models introduced shortly. The alternative is to specify the dynamics of $\mathbf{P}_t = \wp(\Sigma_t)$ with the component series following the univariate GARCH. The CCC-GARCH and DCC-GARCH models defined later on belong to this category.

The diagonal vector GARCH model, known as DVEC, should be considered when the dimensionality of the modeling is low. It is a simplified version of the most general vector GARCH model (see Bollerslev *et al.* 1988), which involves too many parameters to be practically useful. The simplification is to restrict the parameter matrices to be in a diagonal form.

The process $(\mathbf{Y}_t)_{t\in\mathbb{Z}}$ is a DVEC process if it is a process with general structure, and the dynamics of $\Sigma_t$ is completely governed by the equation

$$\Sigma_t = \mathbf{A}_0 + \sum_{i=1}^{p} \mathbf{A}_i \odot (\mathbf{Y}_{t-i}\mathbf{Y}_{t-i}') + \sum_{j=1}^{q} \mathbf{B}_j \odot \Sigma_{t-j} \qquad t \in \mathbb{Z}, \tag{3.15}$$

where $\odot$ is the element-by-element Hadamard product, and parameter matrices $\mathbf{A}_0, \mathbf{A}_i, \mathbf{B}_j$ are all symmetric and lie in $\mathbb{R}^{d\times d}$. In addition, $\mathbf{A}_0$ must have positive diagonal elements and $\mathbf{A}_i, \mathbf{B}_j$ must all have non-negative diagonal elements.

The specification on $\Sigma_t$ does not guarantee that $\Sigma_t$ will be positive definite for all t. We need to further put restrictions on $\mathbf{A}_i$ and $\mathbf{B}_j$ to ensure the conditional covariance matrices are well defined. A sufficient condition is each parameter matrix having a Cholesky decomposition, which means we can re-parametrize the model in terms of lower-triangular Cholesky factor matrices $\mathbf{A}_0^{1/2}$, $\mathbf{A}_i^{1/2}$ and $\mathbf{B}_j^{1/2}$ satisfying

$$\mathbf{A}_0 = \mathbf{A}_0^{1/2}(\mathbf{A}_0^{1/2})' \quad \mathbf{A}_i = \mathbf{A}_i^{1/2}(\mathbf{A}_i^{1/2})' \quad \mathbf{B}_j = \mathbf{B}_j^{1/2}(\mathbf{B}_j^{1/2})'. \tag{3.16}$$

It is also valid to consider two simpler parameterizations to preserve the positive definite $\Sigma_t$. One is of the following form

$$\mathbf{A}_0 = \mathbf{A}_0^{1/2}(\mathbf{A}_0^{1/2})' \quad \mathbf{A}_i = \mathbf{a}_i \cdot \mathbf{a}_i' \quad \mathbf{B}_j = \mathbf{b}_j \cdot \mathbf{b}_j', \tag{3.17}$$

where $\mathbf{a}_i$ and $\mathbf{b}_j$ are both real-valued $d$-dimensional vectors. A much cruder parametrization would be

$$\mathbf{A}_0 = \mathbf{A}_0^{1/2}(\mathbf{A}_0^{1/2})' \quad \mathbf{A}_i = a_i\mathbf{I}_d \quad \mathbf{B}_j = b_j\mathbf{I}_d, \tag{3.18}$$

where $a_i$ and $b_j$ are simply positive constants.

We select the DVEC(1,1) with the vector re-parametrization as one of the candidate models for our later analysis. To better understand the dynamic implication of the DVEC(1,1), we rewrite such a bivariate model into a system of equations, which governs the structure of time-varying volatilities and covariances.

$$\begin{cases} \sigma_{t,1}^2 = a_{0,11} + a_{1,11}Y_{t-1,1}^2 + b_{11}\sigma_{t-1,1}^2 \\ \sigma_{t,12} = a_{0,12} + a_{1,12}Y_{t-1,1}Y_{t-1,2} + b_{12}\sigma_{t-1,12} \\ \sigma_{t,2}^2 = a_{0,22} + a_{1,22}Y_{t-1,2}^2 + b_{22}\sigma_{t-1,2}^2 \end{cases} \tag{3.19}$$

In equation 3.19, the volatilities of component series follow univariate GARCH(1,1), whereas the evolution of the covariance is driven by lagged values $Y_{t-1,1}$, $Y_{t-1,2}$ and previous covariance $\sigma_{t-1,12}$. This time-varying covariance structure is similar in natural to the GARCH(1,1) specification for volatility.

Unlike the diagonal vector GARCH, the BEKK model (see Baba *et al.* 1989), developed by Baba, Engle, Kroner and Kraft, has an almost surely positive definite conditional covariance matrix $\Sigma_t$.

The process $(\mathbf{Y}_t)_{t\in\mathbb{Z}}$ is a BEKK process if it is a process with general structure, and the dynamics of $\Sigma_t$ is completely governed by

$$\Sigma_t = \mathbf{A}_0 + \sum_{i=1}^{p} \mathbf{A}_i' \mathbf{Y}_{t-i} \mathbf{Y}_{t-i}' \mathbf{A}_i + \sum_{j=1}^{q} \mathbf{B}_j' \Sigma_{t-j} \mathbf{B}_j \qquad t \in \mathbb{Z}, \tag{3.20}$$

where all parameter matrices $\mathbf{A}_i, \mathbf{B}_j \in \mathbb{R}^{d\times d}$, and $\mathbf{A}_0$ is symmetric and positive definite.

The construction of the BEKK model ensures the positive definiteness since for any nonzero $d$-dimensional vector $\mathbf{v}$ the following strict equality always holds due to the strictly positive first term and two non-negative summation terms.

$$\mathbf{v}'\Sigma_t\mathbf{v} = \mathbf{v}'A_0\mathbf{v} + \sum_{i=1}^{p} (\mathbf{v}'\mathbf{A}_i'\mathbf{Y}_{t-i})^2 + \sum_{j=1}^{q} (\mathbf{v}\mathbf{B}_j)'\Sigma_{t-j}(\mathbf{v}\mathbf{B}_j) > \mathbf{0} \tag{3.21}$$

We again investigate the dynamic structures of individual volatility and covariance for a simple bivariate BEKK(1,1). A bivariate BEKK(1,1) can be rewritten as follows:

$$\begin{cases} \sigma_{t,1}^2 = & a_{0,11} + a_{1,11}^2 Y_{t-1,1}^2 + 2a_{1,11}a_{1,12}Y_{t-1,1}Y_{t-1,2} + a_{1,12}^2 Y_{t-1,2}^2 \\ & + b_{11}^2\sigma_{t-1,1}^2 + 2b_{11}b_{12}\sigma_{t-1,12} + b_{12}^2\sigma_{t-1,2}^2 \\ \sigma_{t,12} = & a_{0,12} + (a_{1,11}a_{1,22} + a_{1,12}a_{1,21})Y_{t-1,1}Y_{t-1,2} \\ & + a_{1,11}a_{1,21}Y_{t-1,1}^2 + a_{1,22}a_{1,12}Y_{t-1,2}^2 \\ & + (b_{11}b_{22} + b_{12}b_{21})\sigma_{t-1,12} + b_{11}b_{21}\sigma_{t-1,1}^2 + b_{22}b_{12}\sigma_{t-1,2}^2 \\ \sigma_{t,2}^2 = & a_{0,22} + a_{1,22}^2 Y_{t-1,2}^2 + 2a_{1,22}a_{1,21}Y_{t-1,1}Y_{t-1,2} + a_{1,21}^2 Y_{t-1,1}^2 \\ & + b_{22}^2\sigma_{t-1,2}^2 + 2b_{22}b_{21}\sigma_{t-1,21} + b_{21}^2\sigma_{t-1,1}^2. \end{cases} \tag{3.22}$$

From the most simple BEKK model, we discover a property of "volatility-covariance emission". The volatility of a single component and the covariance between two different components are now affected by the lagged values from all components. Compared with the DVEC model, the dynamics in the BEKK model is richer, and the interaction across markets is more intense.

However, we believe the interaction in the real world is only at moderate level for monthly data, as a result, all off-diagonal entries of the BEKK(1,1) model are set to zero to eliminate any crossover effect in $\Sigma_t$. This modified BEKK(1,1) model serves as the second candidate model.

For some segregated funds, the number of indices, which are required to be modeled simultaneously, is fairly high. The multivariate models from DVEC or BEKK family become impractical since the parameters need to be estimated grows dramatically. Modeling conditional correlation with the component series following univariate GARCH models becomes a popular recipe in high-dimensional situation. In rest of this section, we introduce the constant conditional correlation (CCC) model proposed by Bollerslev (1990) and the dynamic conditional correlation (DCC) model developed by Engle (2002).

The process $(\mathbf{Y}_t)_{t\in\mathbb{Z}}$ is a CCC-GARCH process if it is a process with the general structure, and its conditional covariance matrix function is of the form $\Sigma_t = \triangle_t \mathbf{P}_c \triangle_t$, where

(i) $\mathbf{P}_c$ is a constant and positive-definite correlation matrix,

(ii) $\triangle_t$ is a diagonal volatility matrix with each element $\sigma_{t,k}$ following a univariate GARCH specification

$$\sigma_{t,k}^2 = a_{k0} + \sum_{i=1}^{p_k} a_{ki} y_{t-i,k}^2 + \sum_{j=1}^{q_k} b_{kj} \sigma_{t-j,k}^2 \qquad t \in \mathbb{Z},\ k = 1, ..., d \tag{3.23}$$

where $a_{k0} > 0, a_{ki} \geq 0, i = 1, ..., p_k, b_{kj} \geq 0, j = 1, ..., q_k$.

The CCC-GARCH model is well defined in the sense that $\Sigma_t$ is almost surely positive definite for all t. In addition, it is covariance stationary if and only if $\sum_{i=1}^{p_k} a_{ki} + \sum_{j=1}^{q_k} b_{kj} < 1$ for $k = 1, ..., d$.

One criticism of the CCC-GARCH model is its assumption of a constant conditional correlation, which is unrealistic in some settings since financial markets are constantly driven by news and investor's expectation. A more appropriate improvement is to give a parsimonious time-vary structure on $\mathbf{P}_t$, and this step is made in DCC-GARCH model.

The process $(\mathbf{Y}_t)_{t\in\mathbb{Z}}$ is a DCC-GARCH process if it is a process with the general structure, and its conditional covariance matrix is of the form $\Sigma_t = \triangle_t \mathbf{P}_t \triangle_t$, where

(i) the conditional correlation matrices $\mathbf{P}_t$ satisfies the equation

$$\mathbf{P}_t = \wp\left(\Big(1-\sum_{i=1}^{p}\alpha_i-\sum_{j=1}^{q}\beta_j\Big)\mathbf{P}_c+\sum_{i=1}^{p}\alpha_i(\triangle_t^{-1}\mathbf{y}_t)(\triangle_t^{-1}\mathbf{y}_t)'+\sum_{j=1}^{q}\beta_j\mathbf{P}_{t-j}\right) \quad t\in\mathbb{Z} \tag{3.24}$$

where $\mathbf{P}_c$ is a constant positive-definite correlation matrix, and $\alpha_i \geq 0, \beta_j \geq 0, \sum_{i=1}^{p}\alpha_i - \sum_{j=1}^{q}\beta_j < 1$,

(ii) $\triangle_t$ is a diagonal volatility matrix with each element $\sigma_{t,k}$ following univariate GARCH specification

$$\sigma_{t,k}^2 = a_{k0} + \sum_{i=1}^{p_k} a_{ki} y_{t-i,k}^2 + \sum_{j=1}^{q_k} b_{kj}\sigma_{t-j,k}^2 \qquad t\in\mathbb{Z},\ k=1,...,d$$

where $a_{k0} > 0, a_{ki} \geq 0, i = 1, ..., p_k, b_{kj} \geq 0, j = 1, ..., q_k$.

The $(\triangle_t^{-1}\mathbf{Y}_t)_{t\in\mathbb{Z}}$ is sometimes referred to as a devolatized process. It acts like a "noise" term for time-varying $\mathbf{P}_t$. The specification on conditional correlation is GARCH-type, and $\mathbf{P}_c$ can be viewed as the long-term base correlation. In addition, CCC-GARCH model is a special case of DCC-GARCH model.

The major benefit associated with both of these correlation models is that the estimation can be done in an efficient manner even when the number of parameters is high. There is a two-stage fitting method specially designed for these correlation models. We will leave the model estimation for Section 3.5.

We select CCC-GARCH(1,1) and DCC(1,1)-GARCH(1,1) as our candidate models for the investment guarantee valuation.

## 3.4 Innovation Distributions

In econometric literature, $\mathbf{Z}_t$ often denotes the innovation. Comparing the shape of possible innovation distributions is important since different innovations have different degrees of thickness in the tail.

We first introduce the Hansen's skewed student $t$ and make a comparison with standard normal and student $t$. These three univariate distributions are coupled with the CCC-GARCH(1,1) and DCC(1,1)-GARCH(1,1) models in later analysis. For the DVEC and BEKK models, we propose to use multivariate standard normal and multivariate student $t$ distributions as the underlying innovation.

The skewed student $t$ distribution was introduced by Hansen (1994). It has the following probability density function (pdf)

$$f(z_t|\nu,\lambda) = \begin{cases} bc\Big(1+\frac{1}{\nu-2}\Big(\frac{bz_t+a}{1-\lambda}\Big)^2\Big)^{-(\nu+1)/2} & z_t < -a/b \\ bc\Big(1+\frac{1}{\nu-2}\Big(\frac{bz_t+a}{1+\lambda}\Big)^2\Big)^{-(\nu+1)/2} & z_t \geq -a/b \end{cases} \qquad 2<\nu<\infty, -1<\lambda<1 \tag{3.25}$$

$$a = 4\lambda c\Big(\frac{\nu-2}{\nu-1}\Big) \qquad b^2 = 1+3\lambda^2-a^2 \qquad c = \frac{\Gamma\Big(\frac{\nu+1}{2}\Big)}{\sqrt{(\nu-2)\pi}\Gamma\Big(\frac{\nu}{2}\Big)},$$

where $\Gamma(\cdot)$ is the gamma function, $\nu$ is the degree of freedom controlling the thickness of the tail, and $\lambda$ is a parameter specifying the direction as well as the degree of the skewness. If $\lambda > 0$, the distribution is skewed to the right, and vice-versa when $\lambda < 0$.

The pdf of the standard normal and student $t$ are respectively

$$f(z_t) = \frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{z_t^2}{2}\Big) \qquad z_t \in \mathbb{R}, \tag{3.26}$$

and

$$f(z_t|\nu) = \frac{\Gamma\Big(\frac{\nu+1}{2}\Big)}{\sqrt{(\nu-2)\pi}\Gamma\Big(\frac{\nu}{2}\Big)}\Big(1+\frac{z_t^2}{\nu-2}\Big)^{-(\nu+1)/2} \qquad z_t \in \mathbb{R},\ 2<\nu<\infty. \tag{3.27}$$

The normal distribution is the limiting distribution of the student $t$ when $\nu$ goes to infinity, and the tail of a student $t$ distribution is thicker as $\nu$ becomes smaller.

In order to visualize the small difference in the tail, we plot the pdfs of these distributions in the following figures. Zooming into the left tail in figure 3.2, we discover the student $t$ or the Hansen's skewed[3] student $t$ distribution is better at providing extremely low returns compared to the normal distribution. We also observe the Hansen's skewed student $t$ has more flexibility in modeling, and the student $t$ may be a good alternative innovation if the observed return is fat-tailed and overall symmetric.

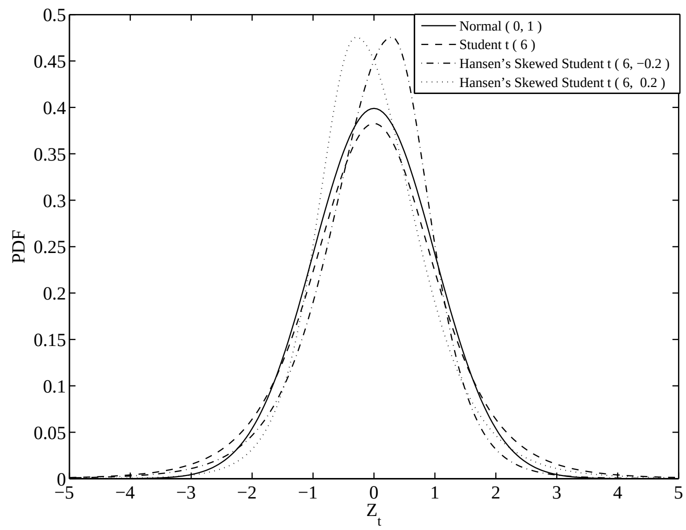


Figure 3.1: Probability density functions of the univariate innovations

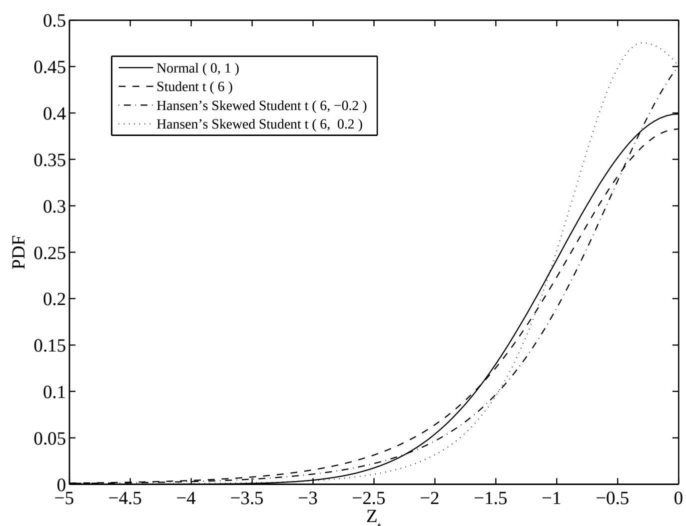


Figure 3.2: Left tail of the probability density functions of the univariate innovations

[3]We only focus on the one that is skewed to the left.

On the other hand, DVEC and BEKK models require a multivariate innovation distribution. In practice, the choice is either multivariate normal or multivariate student $t$. The densities are as follows:

$$f(\mathbf{z}_t) = \frac{1}{(2\pi)^{d/2}} \exp\Big(-\frac{1}{2}\mathbf{z}_t'\mathbf{z}_t\Big) \qquad \mathbf{z}_t \in \mathbb{R}^d, \tag{3.28}$$

$$f(\mathbf{z}_t|\nu) = \frac{\Gamma\Big(\frac{\nu+d}{2}\Big)}{\Big((\nu-2)\pi\Big)^{d/2}\Gamma\Big(\frac{\nu}{2}\Big)}\Big(1+\frac{1}{\nu-2}\mathbf{z}_t'\mathbf{z}_t\Big)^{-(\nu+d)/2} \qquad \mathbf{z}_t \in \mathbb{R}^d,\ 2<\nu<\infty. \tag{3.29}$$

Two comparable bivariate densities are provided in the figure below. The tail surface of the bivariate student $t$ is clearly above the tail surface of the bivariate normal.

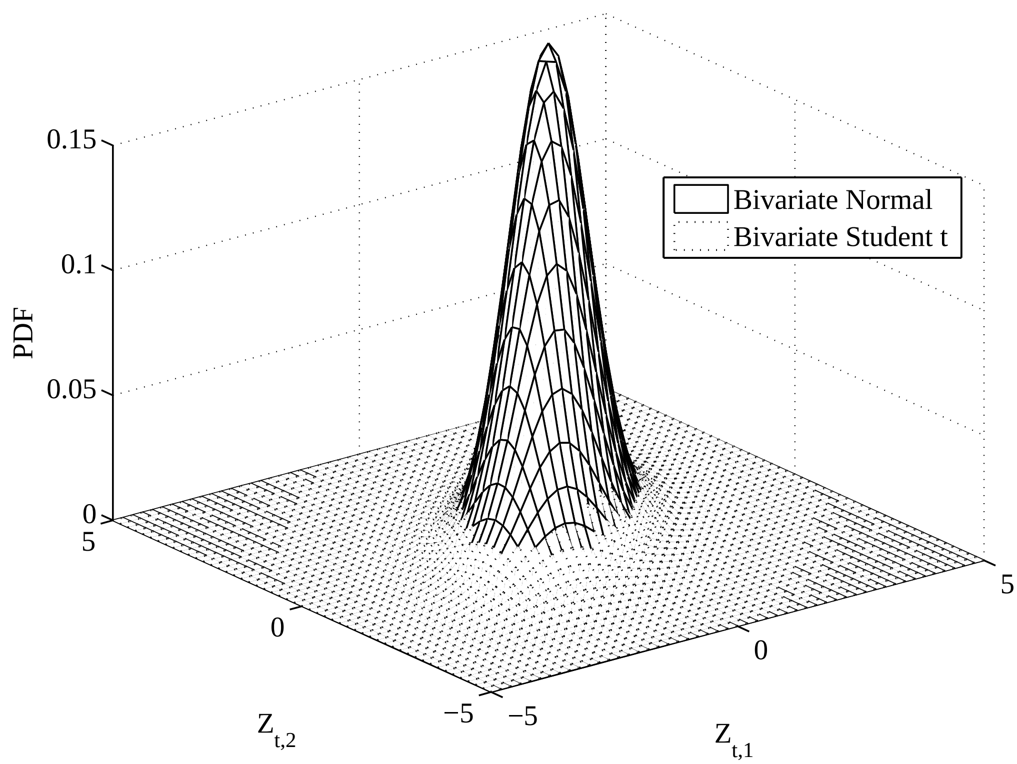


Figure 3.3: Probability density functions of the bivariate innovations

Although the multivariate student $t$ has fatter tail, the degree of thickness in the tail, which is controlled by the single parameter $\nu$, is same for each component series. This mathematical assumption is limiting from the empirical perspective (see Bauwens and Laurent 2005). For instance, it is not clear why the return distribution of a fixed income fund should have the same degree of thickness in the tail as the return distribution of an equity fund. In contrast, the conditional correlation models have the flexibility of incorporating student $t$ innovation with different degrees of freedom in each component.

## 3.5 Model Estimation

All the selected multivariate GARCH models are estimated via the method of maximum likelihood estimation (MLE). The likelihood function for a generic multivariate GARCH model is the conditional joint density function valued at the observed data $\mathbf{y}_0$, $\mathbf{y}_1$, ..., $\mathbf{y}_T$:

$$f_{\mathbf{Y}_1\ ,...,\ \mathbf{Y}_n|\mathbf{Y}_0,\Sigma_0}(\mathbf{y}_1\ ,...,\ \mathbf{y}_n|\mathbf{y}_0,\Sigma_0) = \prod_{t=1}^{T} f_{\mathbf{Y}_t|\mathbf{Y}_{t-1}\ ,...,\ \mathbf{Y}_0,\Sigma_0}(\mathbf{y}_t|\mathbf{y}_{t-1}\ ,...,\ \mathbf{y}_0,\Sigma_0) \quad (3.30)$$

with

$$\prod_{t=1}^{T} f_{\mathbf{Y}_t|\mathbf{Y}_{t-1}\ ,...,\ \mathbf{Y}_0,\Sigma_0}(\mathbf{y}_t|\mathbf{y}_{t-1}\ ,...,\ \mathbf{y}_0,\Sigma_0) = \prod_{t=1}^{T} |\Sigma_t|^{-1/2} g(|\Sigma_t|^{-1/2}\mathbf{y}_t), \quad (3.31)$$

where $|\Sigma_t|^{-1/2}$ is the jacobian from transforming the density, and $g$ is the multivariate innovation density.

Hence, the log-likelihood is of the following form

$$l(\mathbf{\Theta}; \mathbf{y}_1, ..., \mathbf{y}_T) = \sum_{t=1}^{T} \log\left(|\Sigma_t|^{-1/2} g(|\Sigma_t|^{-1/2}\mathbf{y}_t)\right). \quad (3.32)$$

Maximum likelihood estimation involves finding the parameter values $\mathbf{\Theta}$ which maximize the log-likelihood function. However, the log-likelihood function for some multivariate GARCH model is either difficult or time-consuming to maximize. For instance, the log-likelihood of a DVEC(1,1) model with multivariate student $t$ innovation is difficult to maximize. Bollerslev and Wooldridge (1992) propose another method, called quasi-maximum likelihood (QML) estimation, for estimating the parameters of a complex log-likelihood function. They have shown that the QML estimator is a consistent estimator provided the conditional mean and the conditional covariance are correctly specified. Simply speaking, the QML estimation is equivalent to replacing the multivariate innovation density $g$ with the multivariate standard normal density. The hope is that with a large sample the loss in efficiency will not matter.

This is equivalent to maximizing the following simpler function

$$l(\mathbf{\Theta}; \mathbf{y}_1, ..., \mathbf{y}_T) = -\frac{1}{2}\sum_{t=1}^{T} \log|\Sigma_t| - \frac{1}{2}\sum_{t=1}^{T} \mathbf{y}_t' \Sigma_t^{-1} \mathbf{y}_t, \tag{3.33}$$

which can be efficiently optimized if the number of the parameters is modest.

If the number of the parameters required to be estimated is high, this single stage estimation is not practical. For our correlation models we need to maximize the log-likelihood in two steps (see Engle and Sheppard 2001):

- First, estimate the volatility matrix $\triangle_t$ by fitting univariate GARCH models to the component series. Then, obtain the realization of the devolatized process by taking $\widehat{\mathbf{z}}_t = \widehat{\triangle}_t^{-1}\mathbf{y}_t$. Estimate $\mathbf{P}_c$ by applying the sample correlation function[4] to the devolatized data.

- If the model is a DCC model, estimate the remaining parameters, which specify the dynamic of the conditional correlation, by applying either MLE or QMLE method to the devolatized data.

For the proposed multivariate GARCH models in this chapter, the models with non-normal innovations are estimated via the QMLE method, and the rest of the

[4]Other statistical estimators can also be used to estimate the long-term correlation matrix $\mathbf{P}_c$.

models are fitted by the MLE method. Furthermore, the correlation models are estimated in two steps given previously.

During the estimation, we observe the time required for fitting the correlation models is much shorter than for the covariance models. Moreover, for the covariance models, we need to specify an effective error tolerance level and smart initial values in order to "successfully" complete the estimation. Further this indicates that the log-likelihood surface associated with the covariance models are quite flat.

# Chapter 4

# Model Evaluation

## 4.1 Introduction

After fitting any stochastic model, it is crucial to evaluate each fitted model thoroughly. The most important task is to analyze the fitted residuals. A well fitted model should have residuals that look like they are drawn from a normal distribution. Additionally, they should not have strong correlation since the multivariate GARCH models are expected to explain the heteroscedasticity in the return series.

This chapter is structured as follows. In Section 4.2, all fitted parameters are shown and analyzed for better understanding of the model dynamics. In Section 4.3, the standardized residuals are diagnosed for the goodness of fit. Model selection information, which is based on likelihood, are provided in Section 4.4 to help identify the superior models.

## 4.2 Fitted Parameters

The parameters, which are fitted through the method of Maximum Likelihood, need to be carefully checked because the log-likelihood surface[1] of a multivariate GARCH model is sometimes fairly flat, as a result, obtaining misleading parameters becomes easier. Some fitted parameters may give a dynamics that is far from being a good approximation of the true dynamics. In this section, we analyze the fitted parameters for each proposed model. Our focuses are as follows:

- study the heteroscedasticity structure of the estimated models
- identify the potentially "problematic" models

The $d$-dimensional DVEC(1,1) model can be compactly written in the following equation. It should be understood that $\sigma_{t,ij}$ is the covariance between the $i^{th}$ and the $j^{th}$ component of the series at time t:

$$\sigma_{t,ij} = a_{0,ij} + a_{1,ij} y_{t-1,i} y_{t-1,j} + b_{1,ij} \sigma_{t-1,ij} \quad \forall i, j \in \{1, 2, ...d\}. \tag{4.1}$$

Under the DVEC(1,1) model, the volatilities and covariances in $\Sigma_t$ will vary based on the GARCH(1,1) structure. Each $\sigma_{t,ij}$ is only affected by its own lagged value and lagged noise from the previous period.

From the Table 4.1, we can observe that both fitted DVEC(1,1) models regress $\Sigma_t$ more toward to its lagged value than the lagged noise. Such a structure indicates the noise effect[2] is less apparent for the monthly log-returns in the long term. In addition, the fat-tail improvement from the multivariate student $t$ will be small, since overall the noise effect in DVEC(1,1)-MVT model is smaller than in DVEC(1,1)-MVN model.

[1]The log-likelihood surface is sometimes quite flat when it is valued at the observed excess log-returns $\mathbf{y}_0$, $\mathbf{y}_1$, ..., $\mathbf{y}_T$.

[2]In the fitted DVEC(1,1) models, the average news impact from the last month is small from the long-term perspective.

The BEKK(1,1) model can also be characterized by a single indexed equation, which gives a slightly different GARCH(1,1) structure for the volatilities and covariances:

$$\sigma_{t,ij} = a_{0,ij} + a_{1,ii}a_{1,jj}y_{t-1,i}y_{t-1,j} + b_{1,ii}b_{1,jj}\sigma_{t-1,ij} \quad \forall i,j \in \{1,2,...d\}. \tag{4.2}$$

| | DVEC(1,1)-MVN | | | | DVEC(1,1)-MVT | | | |
|---|---|---|---|---|---|---|---|---|
| $\widehat{\mathbf{A}}_0$ | 0.00016 | - | - | - | 0.00008 | - | - | - |
| | 0.00017 | 0.00029 | - | - | 0.00004 | 0.00006 | - | - |
| | 0.00019 | 0.00024 | 0.00032 | - | 0.00005 | 0.00006 | 0.00006 | - |
| | 0.00001 | 0.00000 | 0.00001 | 0.00000 | 0.00001 | 0.00000 | 0.00000 | 0.00000 |
| $\widehat{\mathbf{A}}_1$ | 0.04571 | - | - | - | 0.05371 | - | - | - |
| | 0.09955 | 0.21681 | - | - | 0.07089 | 0.09357 | - | - |
| | 0.08217 | 0.17896 | 0.14772 | - | 0.06883 | 0.09085 | 0.08820 | - |
| | 0.03407 | 0.07420 | 0.06125 | 0.02539 | 0.04546 | 0.06001 | 0.05826 | 0.03848 |
| $\widehat{\mathbf{B}}_1$ | 0.87519 | - | - | - | 0.90588 | - | - | - |
| | 0.78675 | 0.70724 | - | - | 0.89595 | 0.88613 | - | - |
| | 0.77375 | 0.69556 | 0.68407 | - | 0.89903 | 0.88918 | 0.89224 | - |
| | 0.92142 | 0.82830 | 0.81462 | 0.97009 | 0.92628 | 0.91612 | 0.91927 | 0.94713 |
| $\widehat{\nu}$ | | | | | 6.50 | | | |

Table 4.1: Fitted parameters of the DVEC(1,1) models

| | BEKK(1,1)-MVN | | | | BEKK(1,1)-MVT | | | |
|---|---|---|---|---|---|---|---|---|
| $\widehat{\mathbf{A}}_0$ | 0.00138 | - | - | - | 0.00008 | - | - | - |
| | 0.00137 | 0.00169 | - | - | 0.00004 | 0.00006 | - | - |
| | 0.00085 | 0.00127 | 0.00108 | - | 0.00005 | 0.00005 | 0.00006 | - |
| | 0.00004 | 0.00003 | 0.00002 | 0.00000 | 0.00001 | 0.00000 | 0.00000 | 0.00000 |
| $(\widehat{a}_{1,ii} \cdot \widehat{a}_{1,jj})$ | 0.12986 | - | - | - | 0.05352 | - | - | - |
| | 0.18894 | 0.27490 | - | - | 0.06951 | 0.09028 | - | - |
| | 0.19741 | 0.28722 | 0.30010 | - | 0.06735 | 0.08747 | 0.08475 | - |
| | 0.07503 | 0.10916 | 0.11405 | 0.04335 | 0.04277 | 0.05554 | 0.05382 | 0.03417 |
| $(\widehat{b}_{1,ii} \cdot \widehat{b}_{1,jj})$ | 0.24772 | - | - | - | 0.90628 | - | - | - |
| | -0.11200 | 0.05064 | - | - | 0.89879 | 0.89136 | - | - |
| | 0.16504 | -0.07462 | 0.10996 | - | 0.90157 | 0.89412 | 0.89688 | - |
| | 0.48429 | -0.21896 | 0.32266 | 0.94681 | 0.93017 | 0.92248 | 0.92533 | 0.95469 |
| $\widehat{\nu}$ | | | | | 6.68 | | | |

Table 4.2: Fitted parameters of the BEKK(1,1) models

In Table 4.2, we discover the estimated BEKK(1,1) model with multivariate Gaussian noise tend to provide a quite different dynamic for $\Sigma_t$, in comparison with DVEC(1,1) models and BEKK(1,1) model with multivariate student $t$ noise. There is a negative regression effect on covariances between returns on S&P500 and returns on other index, which is difficult to justify; however the estimated BEKK(1,1) model with multivariate student $t$ noise has a dynamic similar to the fitted DVEC(1,1) models.

For the constant conditional correlation models, the fitted long-term correlation matrix is close to the sample correlation matrix, and more of the heteroscedasticity is explained by lagged volatility than by lagged noise.

| CCC-GARCH(1,1)-N | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $\widehat{a}_0$ | $\widehat{a}_1$ | $\widehat{b}_1$ | $\widehat{\nu}$ | $\widehat{\lambda}$ | $\widehat{\mathbf{P}}_c$ | | | |
| 0.00120 | 0.18750 | 0.29350 | | | 1.00 | - | - | - |
| 0.00190 | 0.00050 | 0.13090 | | | 0.66 | 1.00 | - | - |
| 0.00080 | 0.07650 | 0.42460 | | | 0.66 | 0.74 | 1.00 | - |
| 0.00000 | 0.24090 | 0.75100 | | | 0.22 | 0.19 | 0.14 | 1.00 |
| CCC-GARCH(1,1)-T | | | | | | | | |
| $\widehat{a}_0$ | $\widehat{a}_1$ | $\widehat{b}_1$ | $\widehat{\nu}$ | $\widehat{\lambda}$ | $\widehat{\mathbf{P}}_c$ | | | |
| 0.00070 | 0.11150 | 0.56070 | 4.77 | | 1.00 | - | - | - |
| 0.00060 | 0.07040 | 0.64730 | 4.83 | | 0.67 | 1.00 | - | - |
| 0.00080 | 0.07800 | 0.42530 | 11.19 | | 0.65 | 0.74 | 1.00 | - |
| 0.00000 | 0.19590 | 0.79120 | 9.05 | | 0.22 | 0.19 | 0.14 | 1.00 |
| CCC-GARCH(1,1)-SKT | | | | | | | | |
| $\widehat{a}_0$ | $\widehat{a}_1$ | $\widehat{b}_1$ | $\widehat{\nu}$ | $\widehat{\lambda}$ | $\widehat{\mathbf{P}}_c$ | | | |
| 0.00060 | 0.08210 | 0.62570 | 5.17 | -0.214 | 1.00 | - | - | - |
| 0.00070 | 0.09210 | 0.59840 | 4.10 | -0.205 | 0.66 | 1.00 | - | - |
| 0.00100 | 0.08710 | 0.31510 | 11.66 | -0.204 | 0.66 | 0.73 | 1.00 | - |
| 0.00000 | 0.14790 | 0.82100 | 11.39 | -0.118 | 0.24 | 0.18 | 0.12 | 1.00 |

Table 4.3: Fitted parameters of the CCC-GARCH(1,1) models

From the CCC-GARCH(1,1)-N model and CCC-GARCH(1,1)-T model, we should anticipate a fatter tail for the simulated distribution of the Canadian equity return, due to the enhanced noise effect observed from the fitted parameters.

Moreover, the estimated skewness is to the left for each component series under CCC-GARCH(1,1)-SKT model.

| DCC(1,1)-GARCH(1,1)-N | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\widehat{a}_0$ | $\widehat{a}_1$ | $\widehat{b}_1$ | $\widehat{\nu}$ | $\widehat{\lambda}$ | $\widehat{\mathbf{P}}_c$ | | | | $\widehat{\alpha}_1$ | $\widehat{\beta}_1$ |
| 0.00120 | 0.18750 | 0.29350 | | | 1.00 | - | - | - | 0.01800 | 0.95220 |
| 0.00190 | 0.00050 | 0.13090 | | | 0.66 | 1.00 | - | - | | |
| 0.00080 | 0.07650 | 0.42460 | | | 0.66 | 0.74 | 1.00 | - | | |
| 0.00000 | 0.24090 | 0.75100 | | | 0.22 | 0.19 | 0.14 | 1.00 | | |
| DCC(1,1)-GARCH(1,1)-T | | | | | | | | | | |
| $\widehat{a}_0$ | $\widehat{a}_1$ | $\widehat{b}_1$ | $\widehat{\nu}$ | $\widehat{\lambda}$ | $\widehat{\mathbf{P}}_c$ | | | | $\widehat{\alpha}_1$ | $\widehat{\beta}_1$ |
| 0.00070 | 0.11150 | 0.56070 | 4.77 | | 1.00 | - | - | - | 0.01810 | 0.95150 |
| 0.00060 | 0.07040 | 0.64730 | 4.83 | | 0.67 | 1.00 | - | - | | |
| 0.00080 | 0.07800 | 0.42530 | 11.19 | | 0.65 | 0.74 | 1.00 | - | | |
| 0.00000 | 0.19590 | 0.79120 | 9.05 | | 0.22 | 0.19 | 0.14 | 1.00 | | |
| DCC(1,1)-GARCH(1,1)-SKT | | | | | | | | | | |
| $\widehat{a}_0$ | $\widehat{a}_1$ | $\widehat{b}_1$ | $\widehat{\nu}$ | $\widehat{\lambda}$ | $\widehat{\mathbf{P}}_c$ | | | | $\widehat{\alpha}_1$ | $\widehat{\beta}_1$ |
| 0.00060 | 0.08210 | 0.62570 | 5.17 | -0.214 | 1.00 | - | - | - | 0.02150 | 0.94650 |
| 0.00070 | 0.09210 | 0.59840 | 4.10 | -0.205 | 0.66 | 1.00 | - | - | | |
| 0.00100 | 0.08710 | 0.31510 | 11.66 | -0.204 | 0.66 | 0.73 | 1.00 | - | | |
| 0.00000 | 0.14790 | 0.82100 | 11.39 | -0.118 | 0.24 | 0.18 | 0.12 | 1.00 | | |

Table 4.4: Fitted parameters of the DCC(1,1)-GARCH(1,1) models

The dynamic conditional correlation models give the same fitted parameters for the univariate GARCH(1,1) process of the individual component, as well as the base correlation matrix $\mathbf{P}_c$. This is an expected result since both CCC and DCC models have the identical first stage estimation. Furthermore, more heteroscedasticity in correlation is explained by the previous correlation instead of the noise.

## 4.3 Residual Diagnostics

Two criteria are frequently used to rate the quality of fit for multivariate GARCH models. Both of them are based on the standardized residual vector defined as follows:

$$\tilde{\mathbf{z}}_t = \widehat{\triangle}_t^{-1} \cdot \mathbf{y}_t = (z_1, z_2, ..., z_d)', \tag{4.3}$$

where $\widehat{\triangle}_t^{-1}$ is the inverse of the estimated diagonal volatility matrix, and $\mathbf{y}_t$ is the excess log-return vector.

The first criterion asserts that the standardized residual vector $\tilde{\mathbf{z}}_t$ should mimic a $\mathbf{SWN}(\mathbf{0},\mathbf{I}_d)$ process when the model has a good fit. One way to understand this is: multivariate GARCH models are expected to provide enough thickness in the tail to accommodate the extreme market returns, hence in a well fitted model the standardized residuals should have a relatively "normal" tail. A chi-square test, shown in Section 2.4, can be used to verify whether or not the standardized residual vector are iid multivariate standard normal. Other statistical tests in the literature include the Cox-Small test and Smith and Jain's adaptation of the Friedman-Rafsky test. If $\mathbf{Z}_t$ indeed follows $\mathbf{SWN}(\mathbf{0},\mathbf{I}_d)$, each component of $\mathbf{Z}_t$ should be a iid normal random variable, so as a first attempt, we use QQ-plot to diagnose the normality in the standardized residuals for each component series. According to the QQ-plots in appendix, we observe for each proposed model the ordered statistics of the standardized residuals tend to form a distribution that is similar to normal in mid-range but are much fatter in both tails. This observation indicates there is a margin for improvement with respect to the fitting in the tail.

The second criterion assesses the ability of explaining the heteroscedasticity in the return vector process. The underlying principle is that the vector of the standardized residual squares should not be serially correlated since the well fitted model has explained most of the heteroscedasticity. We apply the univariate test on $z_{t,i}^2$ and the multivariate test on $\mathbf{z}_t^*$ to validate the second criterion.

$$\mathbf{z}_t^* = (z_{t,1}^2, z_{t,2}^2, ..., z_{t,d}^2)' \tag{4.4}$$

Ljung-Box Univariate Test

| Models | S&P/TSX | S&P500 | MSIC World | DEX Universe |
|---|---|---|---|---|
| DVEC-MVN | 1.00 | 1.00 | 1.00 | 0.00 |
| DVEC-MVT | 1.00 | 1.00 | 1.00 | 1.00 |
| BEKK-MVN | 0.38 | 1.00 | 0.90 | 0.06 |
| BEKK-MVT | 0.94 | 1.00 | 1.00 | 0.39 |
| CCC-GARCH-N | 0.87 | 1.00 | 0.39 | 0.09 |
| CCC-GARCH-T | 0.75 | 1.00 | 0.46 | 0.11 |
| CCC-GARCH-SKT | 0.72 | 1.00 | 0.30 | 0.29 |
| DCC-GARCH-N | 0.86 | 1.00 | 0.62 | 0.08 |
| DCC-GARCH-T | 0.75 | 1.00 | 0.69 | 0.11 |
| DCC-GARCH-SKT | 0.63 | 1.00 | 0.70 | 0.14 |

Box-Pierce Univariate Test

| Models | S&P/TSX | S&P500 | MSIC World | DEX Universe |
|---|---|---|---|---|
| DVEC-MVN | 1.00 | 1.00 | 1.00 | 0.00 |
| DVEC-MVT | 1.00 | 1.00 | 1.00 | 1.00 |
| BEKK-MVN | 0.41 | 1.00 | 0.91 | 0.07 |
| BEKK-MVT | 0.95 | 1.00 | 1.00 | 0.41 |
| CCC-GARCH-N | 0.89 | 1.00 | 0.41 | 0.10 |
| CCC-GARCH-T | 0.77 | 1.00 | 0.49 | 0.13 |
| CCC-GARCH-SKT | 0.75 | 1.00 | 0.32 | 0.31 |
| DCC-GARCH-N | 0.88 | 1.00 | 0.65 | 0.09 |
| DCC-GARCH-T | 0.77 | 1.00 | 0.71 | 0.12 |
| DCC-GARCH-SKT | 0.66 | 1.00 | 0.72 | 0.15 |

Hosking Multivariate Test

| Models | $\mathbf{z}_t^*$ |
|---|---|
| DVEC-MVN | 0.91 |
| DVEC-MVT | 1.00 |
| BEKK-MVN | 0.97 |
| BEKK-MVT | 1.00 |
| CCC-GARCH-N | 0.87 |
| CCC-GARCH-T | 0.69 |
| CCC-GARCH-SKT | 0.90 |
| DCC-GARCH-N | 0.95 |
| DCC-GARCH-T | 0.87 |
| DCC-GARCH-SKT | 0.96 |

Table 4.5: Testing results for the autocorrelations of the fitted residual squares for 12 lags

At 5% confidence level, all univariate tests lead to not reject null hypothesis $H_o$: $\rho_1 = ... = \rho_{12} = 0$, except for the case with bond index return under the DVEC(1,1)-MVN model. Surprisingly, the null hypothesis, $H_o$: $\mathbf{P}(1) = ... = \mathbf{P}(12) = \mathbf{0}$, is passed by the Hosking test under the same confidence level, which means all proposed multivariate GARCH models have done a good job in explaining the heteroscedasticity in the return vector series.

## 4.4 Likelihood-Based Model Selection

The Akaike Information Criterion (AIC) and the Schwartz Bayes Criterion (SBC) are presented to provide relevant model selection information. The main idea behind two criterions is to penalize the log-likelihood for increasing the number of parameters in a model since extra parameters should only be worth adding if there will be a large improvement in the log-likelihood.

The Akaike Information Criterion (see Akaike 1974) is defined below. The log-likelihood is only improved where adding an extra parameter results at least an unit increase in the log-likelihood.

$$\text{AIC} = \text{Log-Likelihood} - \text{Num of Parameters} \tag{4.5}$$

The AIC penalizes the log-likelihood when increasing the number of the parameters, and the penalty is equal regardless of the sample size; however, adding an extra parameter tends to improve the log-likelihood more for large samples. The Schwartz Bayes Criterion (see Schwartz 1978) defined below takes account of this issue. In SBC, extra parameters are penalized more heavily for large samples.

$$\text{SBC} = \text{Log-Likelihood} - \frac{1}{2} \cdot \text{Num of Parameters} \cdot \ln(\text{Sample Size}) \tag{4.6}$$

Although the correlation model has more parameters than the covariance model, the log-likelihood has a reverse order. As a result, the values of AIC and SBC for covariance models are larger than the values for correlation models.

| Multivariate Covariance Models | | | | |
|---|---|---|---|---|
| Models | Parameters | Log-Likelihood | AIC | SBC |
| DVEC-MVN | 18 | 2754.53 | 2736.53 | 2702.53 |
| DVEC-MVT | 19 | 2890.42 | 2871.42 | 2835.53 |
| BEKK-MVN | 18 | 2757.84 | 2739.84 | 2705.84 |
| BEKK-MVT | 19 | 2890.99 | 2871.99 | 2836.10 |
| Multivariate Correlation Models | | | | |
| Models | Parameters | Log-Likelihood | AIC | SBC |
| CCC-GARCH-N | 18 | 2739.02 | 2721.02 | 2687.02 |
| CCC-GARCH-T | 22 | 2740.85 | 2718.85 | 2677.30 |
| CCC-GARCH-SKT | 26 | 2743.37 | 2717.37 | 2668.26 |
| DCC-GARCH-N | 20 | 2742.15 | 2722.15 | 2684.37 |
| DCC-GARCH-T | 24 | 2744.85 | 2720.85 | 2675.51 |
| DCC-GARCH-SKT | 28 | 2748.68 | 2720.68 | 2667.80 |

Table 4.6: Summary of the model selection information

Among all proposed models, the BEKK(1,1) model with multivariate student $t$ innovation has the largest log-likelihood, AIC, and SBC values. The DVEC model with multivariate student $t$ innovation is nearly indifferent with it in terms of the model selection information. For multivariate correlation models, the discrepancy under each ranking index is small between models. In other words, we do not see any estimated correlation model that absolutely dominates the rest of the correlation models. Therefore, the simplest CCC-GARCH-N or DCC-GARCH-N is the best model in the family of the correlation models, due to fewer parameters.

In a high dimensional setting, we should not rely on the selection information based on the likelihood because estimating a covariance model is not quite executable in terms of the optimization, and also historically the component series are more likely to differ with each other. In that case, the CCC-GARCH and DCC-GARCH models are more flexible in choosing univariate volatility models for component series, and the two-stage estimation is more efficient.

# Chapter 5

# Guarantee Liability Valuation

## 5.1 Introduction

Our primary purpose of creating multivariate GARCH models is using them to find the answers of the following questions.

- What is the probability the segregated fund value will be below the guarantee level at maturity?
- How much on average will the difference be if the fund value is below the guarantee at maturity?

One way to provide answers to these important questions is through a simulation, which is the subject of this chapter.

In Section 5.2, we create hypothetical benchmark portfolios served as our proxies in tracing the performance of the actual mutual funds. In Section 5.3, we then calculate the n-year accumulation factor $S(n)$ of these benchmark portfolios based on the simulated future investment scenarios, and compared them with the CIA's calibration table. In Section 5.4, the frequency and severity of the guarantee liability are further quantified based on the simulation.

## 5.2 Benchmark Portfolios

It is a difficult task to create a benchmark portfolio whose return is highly correlated with the actual fund performance, since the asset mix of a fund will keep changing according to the market condition. Nevertheless, in this thesis we assume the asset mix will change, but on average the magnitude of such a change is small over the period of 10 years. We further assume the fund is re-balanced at monthly frequency to maintain a high degree of mean-variance efficiency for the portfolio. Under the assumptions, we hope assigning constant weights to create benchmark portfolios is appropriate to proxy the performance of the actual fund.

The monthly log-return for a benchmark portfolio is defined in the following equation.

$$F_t = \omega_{t,1} \cdot x_{\text{risk-free}} + \sum_{i=2}^{n} \omega_{t,i} \cdot X_{t,i} \qquad \sum_{i=1}^{n} \omega_{t,i} = 1 \tag{5.1}$$

The weight $\omega_{t,1}$ is 5% for risk-free asset, which has an annual rate of 4% compounded continuously throughout the period of 10 years. The weights of other type of assets in each benchmark portfolio are summarized in the table below.

| Benchmark Portfolio of | S&P/TSX | S&P500 | MSIC World | DEX Universe |
|---|---|---|---|---|
| Canadian Equity Balanced | 65% | 12.5% | 12.5% | 5% |
| Canadian Fixed Income Balanced | 12.5% | 12.5% | 12.5% | 65% |
| Canadian Neutral Balanced | 40% | 12.5% | 12.5% | 30% |
| Global Equity Balanced | 12.5% | 12.5% | 65% | 5% |
| US Equity Balanced | 12.5% | 65% | 12.5% | 5% |

Table 5.1: Composition of the benchmark portfolios

The listed five balanced mutual funds form the main types of the segregated funds available at Manulife Financial. We will valuate the guarantee liability associated with these segregated funds via their benchmark portfolios. The return of each benchmark portfolio will be calculated based on the simulated index return, which is from the estimated multivariate GARCH model. The number of the simulation used for the guarantee liability valuation is 10,000.

## 5.3 Accumulation Factor $S(n)$

The $n$-year accumulation factor measures the growing power for a dollar invested. It is widely used by actuaries to value the actuarial liability in future under a deterministic assumption on return. In valuating the guarantee liability, this accumulation factor $S(n)$ becomes a random variable since the underlying return is technically treated as a stochastic process. We define the $S(n)$ as follows, where $F_t$ is the log-return of a benchmark portfolio.

$$S(n) = \exp\Big\{ \sum_{t=1}^{12 \cdot n} F_t \Big\} \tag{5.2}$$

To avoid an overly optimistic valuation of the guarantee liability, the Segregated Funds Task Force (SFTF) at CIA recommends a calibration table[1] to assess the adequacy of the fit in the tail of historical returns. The focus of this table is placed on the degree of the thickness in the left tail, since low returns have crucial impact on the accumulation factors as opposed to the high historical observations.

In addition to the quantile criterion listed in the table, SFTF also suggests that the mean of 1-year accumulation factor should lie in the range of 1.10 to 1.12, and its volatility should be at least 0.175. Among these recommended values, the most important ones are the quantiles for 10-years accumulation factors, which are directly used to compute the values of a benchmark portfolio at maturity.

[1]The calibration table is recommended by the SFTF only for the equity fund, such as a simple single-indexed segregated fund; however, many complex segregated funds are neither pure equity fund nor pure fixed income fund, and for the conservative reason any stochastic model applied to project the future fund return is forced to be calibrated to Canadian equity index on which the calibration table is based. In the later analysis, we find this recommendation is not appropriate for the fixed income balanced fund because the conservative level defined by the history of the Canadian equity index is really not suitable for the less risky fund.

| Accumulation Period | 2.5th Percentile | 5th Percentile | 10th Percentile |
|---|---|---|---|
| 1-Year | 0.76 | 0.82 | 0.90 |
| 5-Year | 0.75 | 0.85 | 1.05 |
| 10-Year | 0.85 | 1.05 | 1.35 |

Table 5.2: Calibration table from the CIA's report

The corresponding quantiles, means and volatilities are calculated for the five benchmark portfolios in the following tables. For the benchmark portfolio of Canadian equity balanced fund, the covariance models in general have slightly higher quantiles for the 1-year accumulation factor, and the means are not inside but close to the required range whereas the volatilities are lower than expected. Therefore, the simulated 1-year accumulation factors based on the covariance models are less conservative than the calibration requirement. In contrast, the correlation models all satisfy calibration requirements for the 1-year accumulation factor. For 5-year and 10-year accumulation factors, all our models reach the conservative level as suggested by the calibration table. The correlation models are even more conservative than the covariance models. We refer to the appendix for the entire simulated distributions of the accumulation factors.

The left-tail quantiles associated with the other four benchmark portfolios are generally higher than the calibration criterion. In most of the cases, we observe that the correlation models provide more conservative accumulation factors than the covariance models, which can also be seen from the distribution plots in the appendix. The worst violation appears in the benchmark portfolio of Canadian fixed income balanced fund. This is an expected result since the SFTF's criteria is recommended for pure equity fund, and our benchmark has 65% in (median yield and less volatile) fixed income securities, which historically have smaller extreme log-returns in the left tail. Based on our study, we believe the conservative level set for different type of fund should differ, especially for fixed income balanced fund.

| One-Year Accumulation Factor | | | | | |
|---|---|---|---|---|---|
| Models | 2.5%-ile | 5%-ile | 10%-ile | Mean | Volatility |
| DVEC-MVN | 0.80 | 0.84 | 0.89 | 1.08 | 0.159 |
| DVEC-MVT | 0.83 | 0.87 | 0.92 | 1.08 | 0.135 |
| BEKK-MVN | 0.79 | 0.83 | 0.88 | 1.08 | 0.171 |
| BEKK-MVT | 0.84 | 0.87 | 0.92 | 1.08 | 0.137 |
| CCC-GARCH-N | 0.74 | 0.79 | 0.85 | 1.10 | 0.221 |
| CCC-GARCH-T | 0.76 | 0.81 | 0.87 | 1.10 | 0.214 |
| CCC-GARCH-SKT | 0.76 | 0.82 | 0.88 | 1.10 | 0.194 |
| DCC-GARCH-N | 0.75 | 0.80 | 0.86 | 1.09 | 0.198 |
| DCC-GARCH-T | 0.77 | 0.82 | 0.87 | 1.09 | 0.190 |
| DCC-GARCH-SKT | 0.77 | 0.82 | 0.88 | 1.10 | 0.187 |
| Five-Year Accumulation Factor | | | | | |
| Models | 2.5%-ile | 5%-ile | 10%-ile | | |
| DVEC-MVN | 0.72 | 0.80 | 0.92 | | |
| DVEC-MVT | 0.77 | 0.86 | 0.96 | | |
| BEKK-MVN | 0.72 | 0.80 | 0.91 | | |
| BEKK-MVT | 0.78 | 0.87 | 0.97 | | |
| CCC-GARCH-N | 0.60 | 0.70 | 0.82 | | |
| CCC-GARCH-T | 0.63 | 0.73 | 0.85 | | |
| CCC-GARCH-SKT | 0.60 | 0.68 | 0.81 | | |
| DCC-GARCH-N | 0.63 | 0.73 | 0.85 | | |
| DCC-GARCH-T | 0.67 | 0.76 | 0.87 | | |
| DCC-GARCH-SKT | 0.65 | 0.75 | 0.87 | | |
| Ten-Year Accumulation Factor | | | | | |
| Models | 2.5%-ile | 5%-ile | 10%-ile | | |
| DVEC-MVN | 0.76 | 0.90 | 1.07 | | |
| DVEC-MVT | 0.84 | 0.97 | 1.13 | | |
| BEKK-MVN | 0.76 | 0.89 | 1.07 | | |
| BEKK-MVT | 0.84 | 0.98 | 1.17 | | |
| CCC-GARCH-N | 0.59 | 0.73 | 0.94 | | |
| CCC-GARCH-T | 0.64 | 0.79 | 0.98 | | |
| CCC-GARCH-SKT | 0.60 | 0.72 | 0.91 | | |
| DCC-GARCH-N | 0.65 | 0.78 | 0.98 | | |
| DCC-GARCH-T | 0.68 | 0.82 | 1.01 | | |
| DCC-GARCH-SKT | 0.70 | 0.83 | 1.02 | | |

Table 5.3: Quantiles of the accumulation factor of Canadian equity balanced fund

| One-Year Accumulation Factor | | | | | |
|---|---|---|---|---|---|
| Models | 2.5%-ile | 5%-ile | 10%-ile | Mean | Volatility |
| DVEC-MVN | 0.99 | 1.00 | 1.01 | 1.06 | 0.038 |
| DVEC-MVT | 0.98 | 1.00 | 1.01 | 1.06 | 0.040 |
| BEKK-MVN | 0.98 | 0.99 | 1.01 | 1.06 | 0.041 |
| BEKK-MVT | 0.98 | 1.00 | 1.01 | 1.06 | 0.039 |
| CCC-GARCH-N | 0.96 | 0.98 | 1.00 | 1.06 | 0.051 |
| CCC-GARCH-T | 0.97 | 0.99 | 1.00 | 1.06 | 0.048 |
| CCC-GARCH-SKT | 0.97 | 0.99 | 1.01 | 1.06 | 0.047 |
| DCC-GARCH-N | 0.97 | 0.98 | 1.00 | 1.06 | 0.049 |
| DCC-GARCH-T | 0.97 | 0.99 | 1.00 | 1.06 | 0.048 |
| DCC-GARCH-SKT | 0.97 | 0.99 | 1.00 | 1.06 | 0.048 |

| Five-Year Accumulation Factor | | | |
|---|---|---|---|
| Models | 2.5%-ile | 5%-ile | 10%-ile |
| DVEC-MVN | 1.14 | 1.17 | 1.20 |
| DVEC-MVT | 1.12 | 1.15 | 1.19 |
| BEKK-MVN | 1.13 | 1.16 | 1.20 |
| BEKK-MVT | 1.13 | 1.16 | 1.19 |
| CCC-GARCH-N | 1.06 | 1.10 | 1.15 |
| CCC-GARCH-T | 1.07 | 1.11 | 1.16 |
| CCC-GARCH-SKT | 1.05 | 1.10 | 1.15 |
| DCC-GARCH-N | 1.05 | 1.10 | 1.15 |
| DCC-GARCH-T | 1.07 | 1.11 | 1.16 |
| DCC-GARCH-SKT | 1.07 | 1.12 | 1.16 |

| Ten-Year Accumulation Factor | | | |
|---|---|---|---|
| Models | 2.5%-ile | 5%-ile | 10%-ile |
| DVEC-MVN | 1.42 | 1.48 | 1.54 |
| DVEC-MVT | 1.38 | 1.44 | 1.51 |
| BEKK-MVN | 1.40 | 1.46 | 1.53 |
| BEKK-MVT | 1.38 | 1.44 | 1.52 |
| CCC-GARCH-N | 1.26 | 1.34 | 1.43 |
| CCC-GARCH-T | 1.27 | 1.36 | 1.46 |
| CCC-GARCH-SKT | 1.28 | 1.36 | 1.44 |
| DCC-GARCH-N | 1.24 | 1.33 | 1.42 |
| DCC-GARCH-T | 1.28 | 1.36 | 1.46 |
| DCC-GARCH-SKT | 1.30 | 1.37 | 1.46 |

Table 5.4: Quantiles of the accumulation factor of Canadian fixed income balanced fund

| One-Year Accumulation Factor | | | | | |
|---|---|---|---|---|---|
| Models | 2.5%-ile | 5%-ile | 10%-ile | Mean | Volatility |
| DVEC-MVN | 0.88 | 0.91 | 0.95 | 1.07 | 0.103 |
| DVEC-MVT | 0.90 | 0.93 | 0.96 | 1.07 | 0.090 |
| BEKK-MVN | 0.87 | 0.91 | 0.94 | 1.07 | 0.112 |
| BEKK-MVT | 0.91 | 0.93 | 0.96 | 1.07 | 0.090 |
| CCC-GARCH-N | 0.84 | 0.88 | 0.92 | 1.08 | 0.143 |
| CCC-GARCH-T | 0.85 | 0.89 | 0.93 | 1.08 | 0.137 |
| CCC-GARCH-SKT | 0.85 | 0.89 | 0.94 | 1.09 | 0.127 |
| DCC-GARCH-N | 0.85 | 0.88 | 0.92 | 1.08 | 0.130 |
| DCC-GARCH-T | 0.86 | 0.89 | 0.93 | 1.08 | 0.125 |
| DCC-GARCH-SKT | 0.86 | 0.90 | 0.94 | 1.08 | 0.123 |
| Five-Year Accumulation Factor | | | | | |
| Models | 2.5%-ile | 5%-ile | 10%-ile | | |
| DVEC-MVN | 0.90 | 0.97 | 1.05 | | |
| DVEC-MVT | 0.94 | 1.00 | 1.07 | | |
| BEKK-MVN | 0.89 | 0.96 | 1.04 | | |
| BEKK-MVT | 0.94 | 1.01 | 1.08 | | |
| CCC-GARCH-N | 0.79 | 0.88 | 0.98 | | |
| CCC-GARCH-T | 0.82 | 0.91 | 1.00 | | |
| CCC-GARCH-SKT | 0.80 | 0.86 | 0.96 | | |
| DCC-GARCH-N | 0.82 | 0.90 | 0.99 | | |
| DCC-GARCH-T | 0.85 | 0.92 | 1.01 | | |
| DCC-GARCH-SKT | 0.83 | 0.91 | 1.01 | | |
| Ten-Year Accumulation Factor | | | | | |
| Models | 2.5%-ile | 5%-ile | 10%-ile | | |
| DVEC-MVN | 1.03 | 1.15 | 1.29 | | |
| DVEC-MVT | 1.09 | 1.21 | 1.34 | | |
| BEKK-MVN | 1.03 | 1.14 | 1.29 | | |
| BEKK-MVT | 1.10 | 1.21 | 1.36 | | |
| CCC-GARCH-N | 0.88 | 1.01 | 1.19 | | |
| CCC-GARCH-T | 0.92 | 1.06 | 1.22 | | |
| CCC-GARCH-SKT | 0.88 | 0.99 | 1.15 | | |
| DCC-GARCH-N | 0.92 | 1.04 | 1.21 | | |
| DCC-GARCH-T | 0.96 | 1.08 | 1.24 | | |
| DCC-GARCH-SKT | 0.96 | 1.08 | 1.23 | | |

Table 5.5: Quantiles of the accumulation factor of Canadian neutral balanced fund

| One-Year Accumulation Factor | | | | | |
|---|---|---|---|---|---|
| Models | 2.5%-ile | 5%-ile | 10%-ile | Mean | Volatility |
| DVEC-MVN | 0.93 | 0.95 | 0.97 | 1.08 | 0.082 |
| DVEC-MVT | 0.95 | 0.97 | 0.99 | 1.07 | 0.066 |
| BEKK-MVN | 0.92 | 0.94 | 0.97 | 1.08 | 0.084 |
| BEKK-MVT | 0.95 | 0.97 | 0.99 | 1.07 | 0.066 |
| CCC-GARCH-N | 0.90 | 0.93 | 0.96 | 1.08 | 0.098 |
| CCC-GARCH-T | 0.92 | 0.94 | 0.97 | 1.08 | 0.092 |
| CCC-GARCH-SKT | 0.91 | 0.94 | 0.96 | 1.08 | 0.090 |
| DCC-GARCH-N | 0.91 | 0.94 | 0.97 | 1.08 | 0.091 |
| DCC-GARCH-T | 0.91 | 0.94 | 0.97 | 1.08 | 0.089 |
| DCC-GARCH-SKT | 0.91 | 0.94 | 0.97 | 1.08 | 0.092 |
| **Five-Year Accumulation Factor** | | | | | |
| Models | 2.5%-ile | 5%-ile | 10%-ile | | |
| DVEC-MVN | 1.00 | 1.05 | 1.13 | | |
| DVEC-MVT | 1.05 | 1.10 | 1.16 | | |
| BEKK-MVN | 1.01 | 1.06 | 1.14 | | |
| BEKK-MVT | 1.05 | 1.11 | 1.17 | | |
| CCC-GARCH-N | 0.97 | 1.03 | 1.11 | | |
| CCC-GARCH-T | 0.98 | 1.05 | 1.13 | | |
| CCC-GARCH-SKT | 0.95 | 1.01 | 1.09 | | |
| DCC-GARCH-N | 0.96 | 1.02 | 1.10 | | |
| DCC-GARCH-T | 0.98 | 1.04 | 1.12 | | |
| DCC-GARCH-SKT | 0.94 | 1.02 | 1.10 | | |
| **Ten-Year Accumulation Factor** | | | | | |
| Models | 2.5%-ile | 5%-ile | 10%-ile | | |
| DVEC-MVN | 1.22 | 1.31 | 1.44 | | |
| DVEC-MVT | 1.29 | 1.38 | 1.50 | | |
| BEKK-MVN | 1.25 | 1.35 | 1.47 | | |
| BEKK-MVT | 1.29 | 1.40 | 1.52 | | |
| CCC-GARCH-N | 1.20 | 1.30 | 1.44 | | |
| CCC-GARCH-T | 1.23 | 1.33 | 1.46 | | |
| CCC-GARCH-SKT | 1.17 | 1.28 | 1.41 | | |
| DCC-GARCH-N | 1.15 | 1.27 | 1.41 | | |
| DCC-GARCH-T | 1.19 | 1.29 | 1.43 | | |
| DCC-GARCH-SKT | 1.15 | 1.26 | 1.41 | | |

Table 5.6: Quantiles of the accumulation factor of global equity balanced fund

| One-Year Accumulation Factor | | | | | |
|---|---|---|---|---|---|
| Models | 2.5%-ile | 5%-ile | 10%-ile | Mean | Volatility |
| DVEC-MVN | 0.88 | 0.91 | 0.94 | 1.08 | 0.110 |
| DVEC-MVT | 0.88 | 0.91 | 0.95 | 1.08 | 0.108 |
| BEKK-MVN | 0.86 | 0.89 | 0.93 | 1.08 | 0.123 |
| BEKK-MVT | 0.88 | 0.91 | 0.95 | 1.08 | 0.107 |
| CCC-GARCH-N | 0.85 | 0.88 | 0.92 | 1.08 | 0.133 |
| CCC-GARCH-T | 0.87 | 0.90 | 0.94 | 1.08 | 0.121 |
| CCC-GARCH-SKT | 0.87 | 0.90 | 0.94 | 1.09 | 0.120 |
| DCC-GARCH-N | 0.85 | 0.88 | 0.92 | 1.08 | 0.134 |
| DCC-GARCH-T | 0.85 | 0.89 | 0.93 | 1.08 | 0.126 |
| DCC-GARCH-SKT | 0.84 | 0.88 | 0.92 | 1.09 | 0.134 |
| Five-Year Accumulation Factor | | | | | |
| Models | 2.5%-ile | 5%-ile | 10%-ile | | |
| DVEC-MVN | 0.87 | 0.95 | 1.04 | | |
| DVEC-MVT | 0.90 | 0.97 | 1.06 | | |
| BEKK-MVN | 0.86 | 0.93 | 1.02 | | |
| BEKK-MVT | 0.90 | 0.98 | 1.06 | | |
| CCC-GARCH-N | 0.84 | 0.91 | 1.01 | | |
| CCC-GARCH-T | 0.87 | 0.95 | 1.04 | | |
| CCC-GARCH-SKT | 0.84 | 0.91 | 1.01 | | |
| DCC-GARCH-N | 0.83 | 0.91 | 1.00 | | |
| DCC-GARCH-T | 0.86 | 0.94 | 1.03 | | |
| DCC-GARCH-SKT | 0.82 | 0.91 | 1.00 | | |
| Ten-Year Accumulation Factor | | | | | |
| Models | 2.5%-ile | 5%-ile | 10%-ile | | |
| DVEC-MVN | 0.99 | 1.12 | 1.28 | | |
| DVEC-MVT | 1.05 | 1.17 | 1.32 | | |
| BEKK-MVN | 1.00 | 1.13 | 1.29 | | |
| BEKK-MVT | 1.05 | 1.18 | 1.33 | | |
| CCC-GARCH-N | 0.97 | 1.08 | 1.25 | | |
| CCC-GARCH-T | 1.02 | 1.14 | 1.30 | | |
| CCC-GARCH-SKT | 0.94 | 1.06 | 1.25 | | |
| DCC-GARCH-N | 0.96 | 1.09 | 1.24 | | |
| DCC-GARCH-T | 1.00 | 1.11 | 1.28 | | |
| DCC-GARCH-SKT | 0.95 | 1.08 | 1.24 | | |

Table 5.7: Quantiles of the accumulation factor of US equity balanced fund

## 5.4 VaR and CTE

In this section, we use value-at-risk (VaR) and conditional tail expectation (CTE) to measure the investment risk associated with the single premium segregated fund contract. For simplicity, we ignore mortality and lapsation, and we further assume the policyholder will not reset and switch the underlying funds.

The liability for the maturity guarantee can be defined similarly to a European style put option

$$L = Max(G - S(n)e^{-nm}, 0), \tag{5.3}$$

where $G$ is the fixed guarantee level at maturity and $S(n)e^{-nm}$ is the accumulated fund value net the annual management fee. The $m$ is the continuously compounded annual management expense ratio (MER), which is used to deduct the management expense from the fund. The maturity guarantee in the contract is set to be the 100% of the initial premium, which is simply 1$ for our analysis, and the annual management expense ratio (MER) is 3% compounded continuously.

The probability of having a non-zero guarantee liability is denoted by $\xi$. It can be easily computed in the simulation since we only need to count the number of simulated liabilities that are below the guarantee level at maturity date.

$$\xi = P(L > 0) = P(G > S(n)e^{-nm}) \tag{5.4}$$

The VaR is the most widely used risk measure among financial institutions. It is defined as follows:

$$\text{VaR}(\alpha) = inf\{l \in \mathbb{R} : P(L > l) \leqslant 1 - \alpha\} = inf\{l \in \mathbb{R} : F_L(l) \geqslant \alpha\}. \tag{5.5}$$

From the definition, the VaR($\alpha$) is the smallest number $l$ such that the probability that the random guarantee liability $L$ exceeds $l$ is no larger than (1-$\alpha$).

VaR is essentially a quantile of the guarantee liability distribution. It has many issues from the principle of coherent risk measure. The major criticism is VaR violates the subadditivity property required by this principle. For example, for two risks $X$ and $Y$, it is possible under the VaR measure that $\mathrm{VaR}_X(\alpha) + \mathrm{VaR}_Y(\alpha) < \mathrm{VaR}_{X+Y}(\alpha)$. In other words, VaR does not take account of the diversification.

CTE is a better risk measure under such principle. It is defined as the expected value of the guarantee liability given that it falls into the upper (1-$\alpha$) range of the distribution.

$$\mathrm{CTE}(\alpha) = \mathbb{E}(L|L > \mathrm{VaR}(\alpha)) \tag{5.6}$$

Compared with $\mathrm{VaR}(\alpha)$, $\mathrm{CTE}(\alpha)$ is an improved measure since it takes account of the entire tail that is beyond the $\alpha$ quantile. CTE with a conservative $\alpha$, say 97.5%, is used to set up the actuarial reserve for the guarantee liability.

For the contract linked with the Canadian equity balanced fund, we firstly observe that $\xi$, VaR and CTE are lower under the covariance models than those under the correlation models. Within the covariance models, the type of the innovation distribution has impact on the liability distribution and its upper tail. The models with multivariate Gaussian innovation generates more conservative risk measures, compared with the models with multivariate $t$ innovation. In the family of the correlation models, the innovation distribution, however, does not make a big difference. The CCC-GARCH(1,1) gives slightly higher required values. For the rest of benchmark portfolios, we have the similar observations.

| Models | $\xi$ | $\mathrm{VaR}^{90\%}$ | $\mathrm{CTE}^{90\%}$ | $\mathrm{VaR}^{95\%}$ | $\mathrm{CTE}^{95\%}$ | $\mathrm{VaR}^{97.5\%}$ | $\mathrm{CTE}^{97.5\%}$ |
|---|---|---|---|---|---|---|---|
| DVEC-MVN | 0.21 | 0.20 | 0.36 | 0.34 | 0.46 | 0.44 | 0.53 |
| DVEC-MVT | 0.19 | 0.16 | 0.31 | 0.28 | 0.41 | 0.38 | 0.50 |
| BEKK-MVN | 0.22 | 0.21 | 0.36 | 0.34 | 0.46 | 0.43 | 0.53 |
| BEKK-MVT | 0.18 | 0.14 | 0.30 | 0.27 | 0.40 | 0.37 | 0.48 |
| CCC-GARCH-N | 0.25 | 0.30 | 0.48 | 0.46 | 0.59 | 0.56 | 0.66 |
| CCC-GARCH-T | 0.23 | 0.27 | 0.44 | 0.41 | 0.55 | 0.53 | 0.63 |
| CCC-GARCH-SKT | 0.26 | 0.33 | 0.49 | 0.47 | 0.58 | 0.56 | 0.66 |
| DCC-GARCH-N | 0.24 | 0.28 | 0.44 | 0.42 | 0.54 | 0.52 | 0.61 |
| DCC-GARCH-T | 0.23 | 0.25 | 0.42 | 0.39 | 0.52 | 0.49 | 0.59 |
| DCC-GARCH-SKT | 0.22 | 0.25 | 0.41 | 0.39 | 0.50 | 0.48 | 0.58 |

Table 5.8: Risk measures of the investment guarantee in Canadian equity balanced fund

| Models | $\xi$ | $\text{VaR}^{90\%}$ | $\text{CTE}^{90\%}$ | $\text{VaR}^{95\%}$ | $\text{CTE}^{95\%}$ | $\text{VaR}^{97.5\%}$ | $\text{CTE}^{97.5\%}$ |
|---|---|---|---|---|---|---|---|
| DVEC-MVN | 0.01 | 0.00 | 0.04 | 0.00 | 0.04 | 0.00 | 0.04 |
| DVEC-MVT | 0.02 | 0.00 | 0.05 | 0.00 | 0.05 | 0.00 | 0.05 |
| BEKK-MVN | 0.01 | 0.00 | 0.04 | 0.00 | 0.04 | 0.00 | 0.04 |
| BEKK-MVT | 0.02 | 0.00 | 0.05 | 0.00 | 0.05 | 0.00 | 0.05 |
| CCC-GARCH-N | 0.05 | 0.00 | 0.09 | 0.01 | 0.10 | 0.07 | 0.17 |
| CCC-GARCH-T | 0.04 | 0.00 | 0.09 | 0.00 | 0.09 | 0.06 | 0.15 |
| CCC-GARCH-SKT | 0.05 | 0.00 | 0.08 | 0.00 | 0.08 | 0.05 | 0.13 |
| DCC-GARCH-N | 0.06 | 0.00 | 0.10 | 0.02 | 0.12 | 0.08 | 0.19 |
| DCC-GARCH-T | 0.04 | 0.00 | 0.09 | 0.00 | 0.09 | 0.05 | 0.14 |
| DCC-GARCH-SKT | 0.04 | 0.00 | 0.07 | 0.00 | 0.07 | 0.04 | 0.11 |

Table 5.9: Risk measures of the investment guarantee in Canadian fixed income balanced fund

| Models | $\xi$ | $\text{VaR}^{90\%}$ | $\text{CTE}^{90\%}$ | $\text{VaR}^{95\%}$ | $\text{CTE}^{95\%}$ | $\text{VaR}^{97.5\%}$ | $\text{CTE}^{97.5\%}$ |
|---|---|---|---|---|---|---|---|
| DVEC-MVN | 0.13 | 0.04 | 0.17 | 0.15 | 0.26 | 0.24 | 0.33 |
| DVEC-MVT | 0.11 | 0.01 | 0.14 | 0.11 | 0.22 | 0.19 | 0.29 |
| BEKK-MVN | 0.13 | 0.05 | 0.18 | 0.15 | 0.26 | 0.24 | 0.33 |
| BEKK-MVT | 0.10 | 0.00 | 0.13 | 0.10 | 0.21 | 0.18 | 0.28 |
| CCC-GARCH-N | 0.17 | 0.12 | 0.28 | 0.25 | 0.37 | 0.35 | 0.45 |
| CCC-GARCH-T | 0.15 | 0.09 | 0.25 | 0.22 | 0.34 | 0.32 | 0.42 |
| CCC-GARCH-SKT | 0.18 | 0.15 | 0.29 | 0.27 | 0.37 | 0.35 | 0.45 |
| DCC-GARCH-N | 0.16 | 0.10 | 0.25 | 0.23 | 0.34 | 0.32 | 0.42 |
| DCC-GARCH-T | 0.15 | 0.08 | 0.23 | 0.20 | 0.32 | 0.29 | 0.39 |
| DCC-GARCH-SKT | 0.15 | 0.09 | 0.22 | 0.20 | 0.31 | 0.29 | 0.38 |

Table 5.10: Risk measures of the investment guarantee in Canadian neutral balanced fund

| Models | $\xi$ | $\mathrm{VaR}^{90\%}$ | $\mathrm{CTE}^{90\%}$ | $\mathrm{VaR}^{95\%}$ | $\mathrm{CTE}^{95\%}$ | $\mathrm{VaR}^{97.5\%}$ | $\mathrm{CTE}^{97.5\%}$ |
|---|---|---|---|---|---|---|---|
| DVEC-MVN | 0.06 | 0.00 | 0.10 | 0.03 | 0.12 | 0.10 | 0.18 |
| DVEC-MVT | 0.04 | 0.00 | 0.09 | 0.00 | 0.09 | 0.05 | 0.13 |
| BEKK-MVN | 0.05 | 0.00 | 0.10 | 0.00 | 0.10 | 0.07 | 0.16 |
| BEKK-MVT | 0.04 | 0.00 | 0.10 | 0.00 | 0.10 | 0.04 | 0.13 |
| CCC-GARCH-N | 0.07 | 0.00 | 0.11 | 0.04 | 0.15 | 0.11 | 0.22 |
| CCC-GARCH-T | 0.06 | 0.00 | 0.11 | 0.01 | 0.12 | 0.09 | 0.19 |
| CCC-GARCH-SKT | 0.08 | 0.00 | 0.11 | 0.05 | 0.16 | 0.13 | 0.24 |
| DCC-GARCH-N | 0.08 | 0.00 | 0.12 | 0.06 | 0.17 | 0.15 | 0.23 |
| DCC-GARCH-T | 0.07 | 0.00 | 0.11 | 0.04 | 0.14 | 0.12 | 0.21 |
| DCC-GARCH-SKT | 0.07 | 0.00 | 0.13 | 0.06 | 0.17 | 0.15 | 0.24 |

Table 5.11: Risk measures of the investment guarantee in global equity balanced fund

| Models | $\xi$ | $\mathrm{VaR}^{90\%}$ | $\mathrm{CTE}^{90\%}$ | $\mathrm{VaR}^{95\%}$ | $\mathrm{CTE}^{95\%}$ | $\mathrm{VaR}^{97.5\%}$ | $\mathrm{CTE}^{97.5\%}$ |
|---|---|---|---|---|---|---|---|
| DVEC-MVN | 0.13 | 0.05 | 0.20 | 0.17 | 0.29 | 0.26 | 0.36 |
| DVEC-MVT | 0.11 | 0.02 | 0.16 | 0.13 | 0.25 | 0.22 | 0.33 |
| BEKK-MVN | 0.13 | 0.04 | 0.19 | 0.16 | 0.28 | 0.26 | 0.35 |
| BEKK-MVT | 0.11 | 0.01 | 0.16 | 0.13 | 0.24 | 0.22 | 0.32 |
| CCC-GARCH-N | 0.14 | 0.07 | 0.22 | 0.20 | 0.31 | 0.28 | 0.38 |
| CCC-GARCH-T | 0.12 | 0.04 | 0.18 | 0.15 | 0.26 | 0.24 | 0.34 |
| CCC-GARCH-SKT | 0.15 | 0.08 | 0.23 | 0.21 | 0.33 | 0.31 | 0.40 |
| DCC-GARCH-N | 0.14 | 0.08 | 0.23 | 0.20 | 0.32 | 0.29 | 0.39 |
| DCC-GARCH-T | 0.13 | 0.06 | 0.19 | 0.17 | 0.28 | 0.26 | 0.34 |
| DCC-GARCH-SKT | 0.15 | 0.08 | 0.23 | 0.20 | 0.32 | 0.30 | 0.39 |

Table 5.12: Risk measures of the investment guarantee in US equity balanced fund

# Chapter 6

# Conclusion

In this thesis, we apply multivariate GARCH models to analyze the guarantee liability associated with the GMMA option. We first conduct a statistical analysis for the historical index returns. We then fit the proposed multivariate models by the method of maximum or quasi-maximum likelihood estimation. Via simulation, the accumulation factor are computed. The values are further compared with the CIA's calibration table to analyze the conservative level provided from each multivariate GARCH model. The actuarial reserve for the guarantee liability, linked with a single premium segregated contract, is calculated by the conditional tail expectation.

During the analysis of the return series, we find the historical monthly returns are non-normal, either from univariate or multivariate perspective, due to the extreme observations from the left tail; however, using the normal distribution to approximate the return distribution is generally reasonable based on the QQ-plots. In the univariate setting, the autocorrelation of the volatilities in the return series is insignificant for equity indices whereas there is a statistical evidence found in the fixed income index. In contrast, the conditional covariance matrices of the return vectors are serially correlated based on the Hosking's multivariate test. Therefore, a multivariate test should be applied for the vector return data before applying any multivariate GARCH model.

In studying the models in Chapter 3, we discover the dynamics of multivariate GARCH models is fairly rich. For low-dimensional problems, the DVEC models have GARCH specification in the volatility and the covariance. The BEKK models offers the flexible volatility-covariance emission structure. For high-dimensional problems, the CCC-GARCH and DCC-GARCH models are efficient in the estimation and provide more choices in modeling the returns of the component series. Comparing with the normal and student $t$, the Hansen's skewed student $t$ is capable of capturing the skewness and thickness of the empirical return distribution.

In analyzing the standardized residuals, we find multivariate GARCH models have done a good job in explaining the heteroscedasticity in the return vector series, although the fit in the tail still has some margin for improvement. In addition, the widely used model selection criteria, AIC and SBC, are in flavor of the covariance models, but we experienced that estimating covariance models correctly is not a easy task since the log-likelihood surface sometimes is quite fat.

We also find that for Canadian equity balanced fund the CCC-GARCH and DCC-GARCH models can provide accumulation factors comparable with the CIA's calibration table, which indicates that the conservative level set by the correlation models may be sufficient for the purpose of the guarantee liability valuation. As a result, the VaR and CTE calculated based on the correlation models are fairly high for the benchmark portfolio of Canadian equity balanced fund. On the other hand, the improvement of applying the student $t$, Hansen's skewed student $t$, and multivariate student $t$ is small in terms of the VaR and CTE values.

Based on our research, we recommend the flexible CCC-GARCH and DCC-GARCH models with the normal innovation for the guarantee liability valuation of the complex segregated fund products. The direction of future research may be extending the simulation setting to include the mortality and the lapsation, which are important extra risk factors for the practical actuarial problem.

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# Appendices

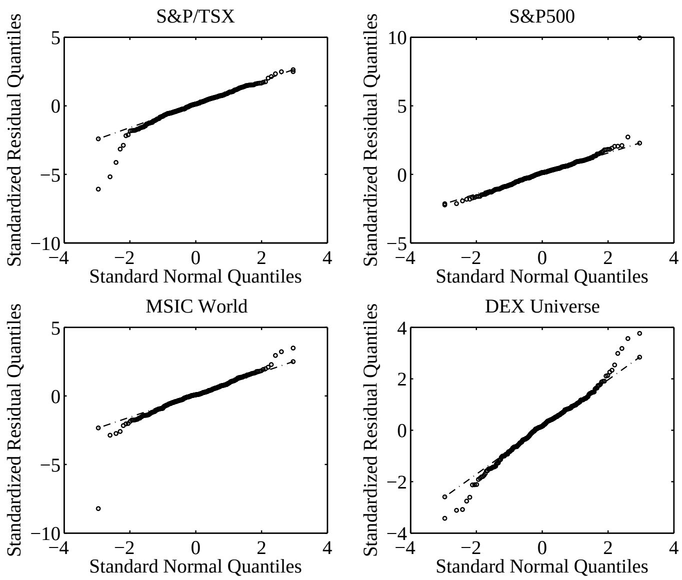


Figure 1: QQ-plots of the fitted residuals of the DVEC(1,1)-MVN model

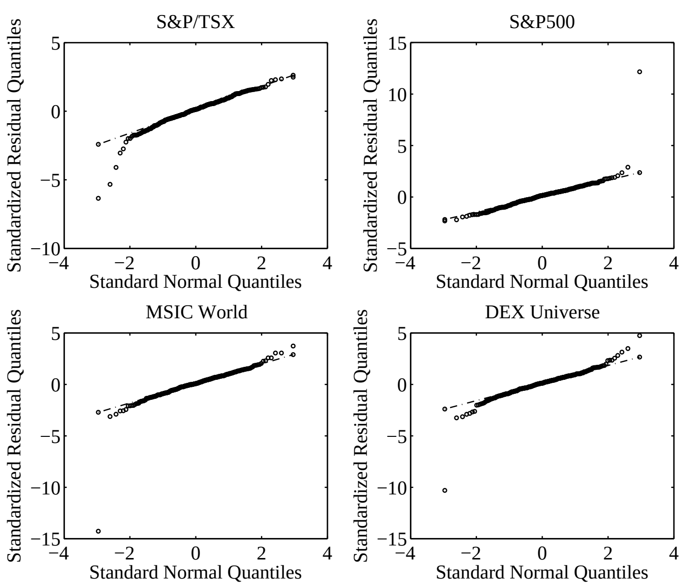


Figure 2: QQ-plots of the fitted residuals of the DVEC(1,1)-MVT model

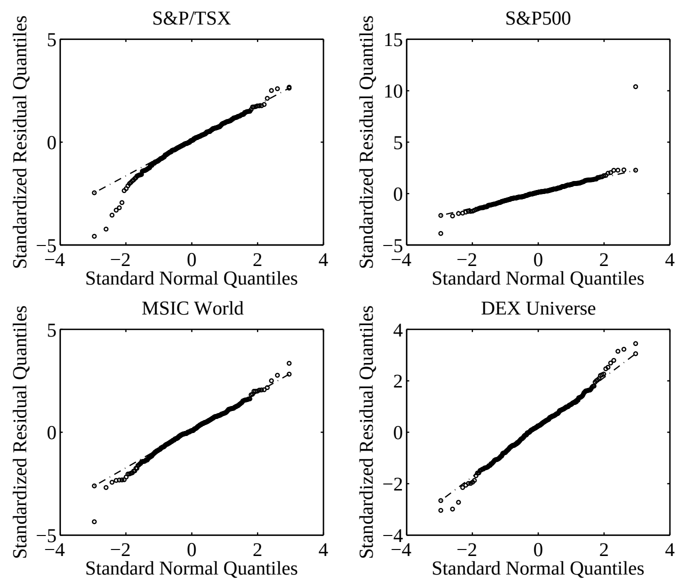


Figure 3: QQ-plots of the fitted residuals of the BEKK(1,1)-MVN model

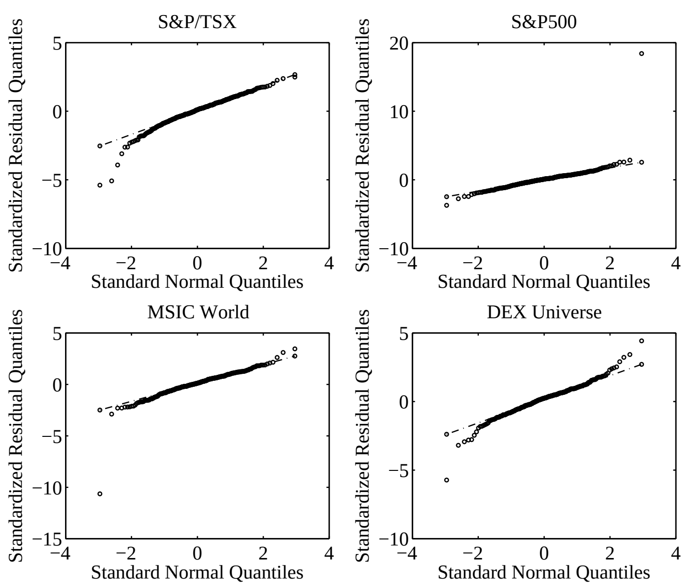


Figure 4: QQ-plots of the fitted residuals of the BEKK(1,1)-MVT model

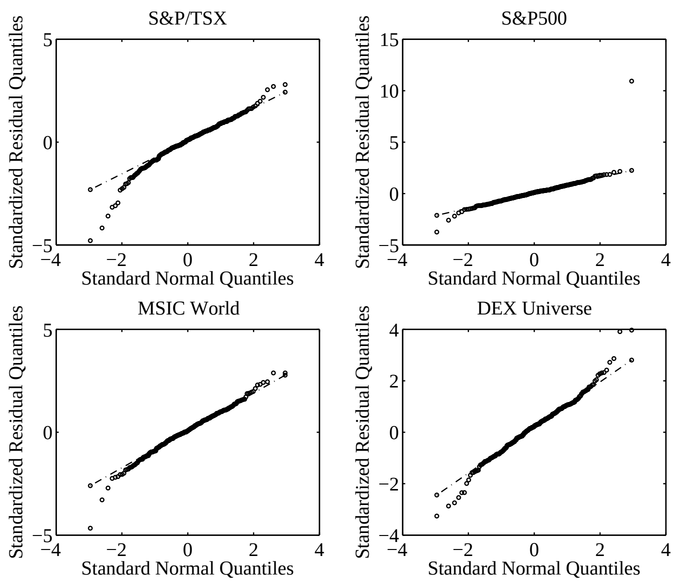


Figure 5: QQ-plots of the fitted residuals of the CCC-GARCH(1,1)-N model

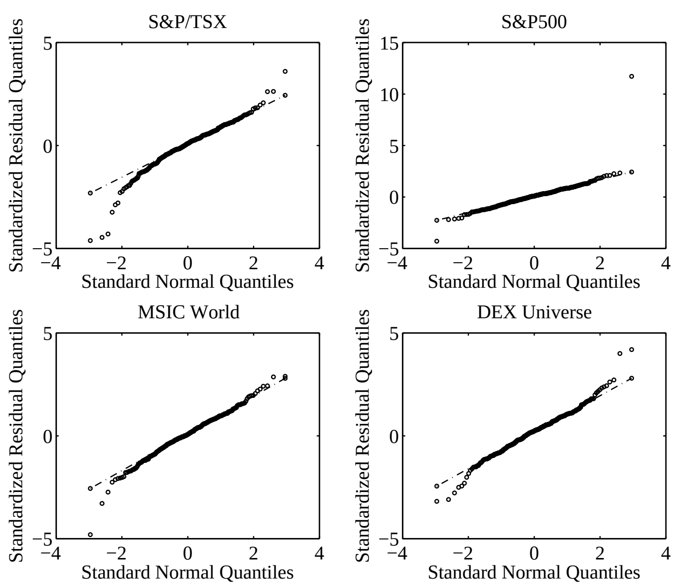


Figure 6: QQ-plots of the fitted residuals of the CCC-GARCH(1,1)-T model

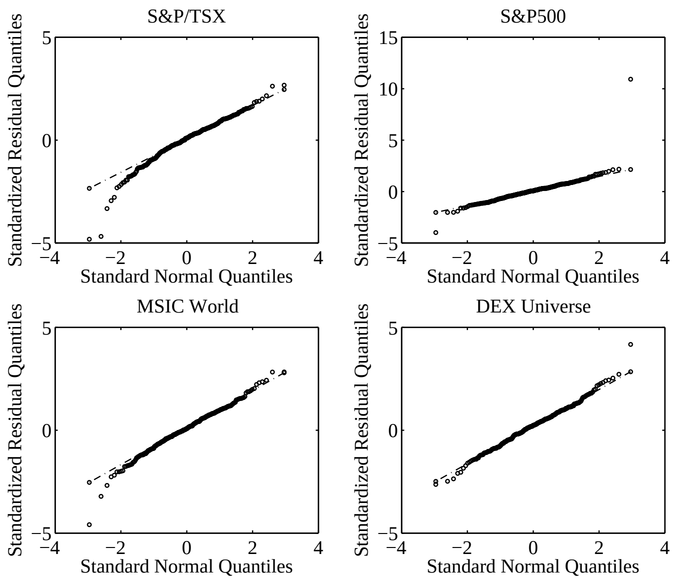


Figure 7: QQ-plots of the fitted residuals of the CCC-GARCH(1,1)-SKT model

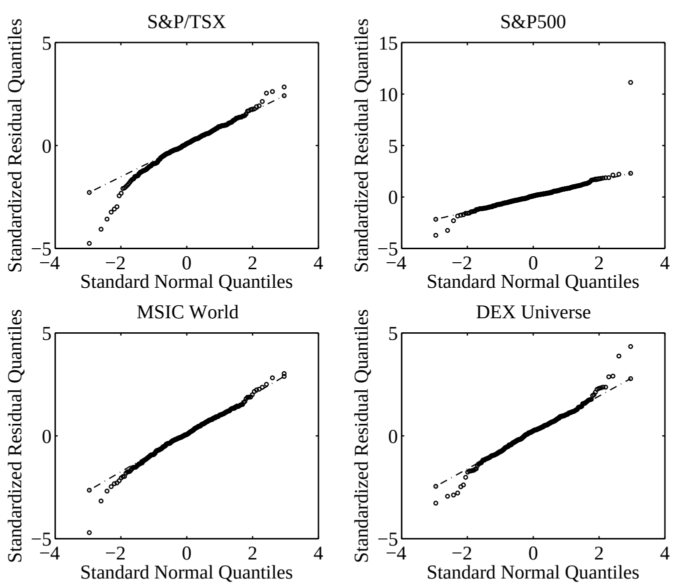


Figure 8: QQ-plots of the fitted residuals of the DCC(1,1)-GARCH(1,1)-N model

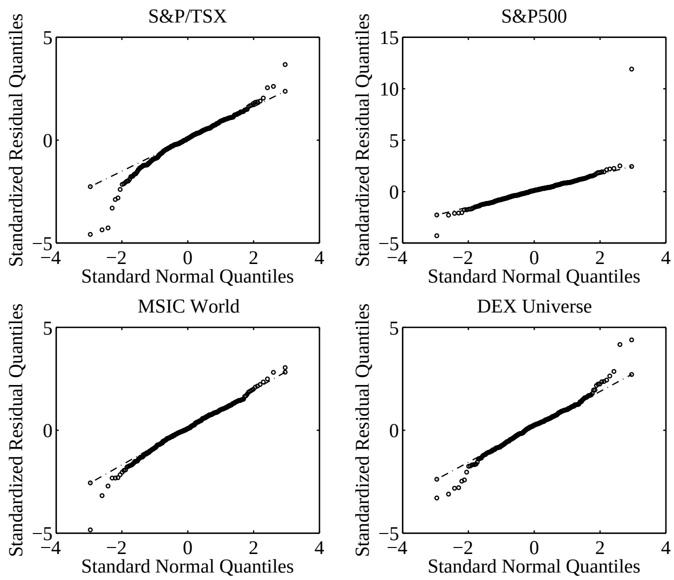


Figure 9: QQ-plots of the fitted residuals of the DCC(1,1)-GARCH(1,1)-T model

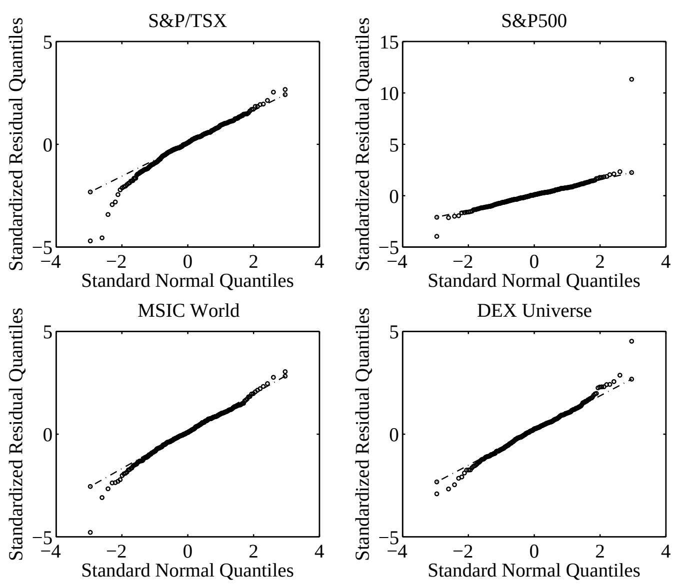


Figure 10: QQ-plots of the fitted residuals of the DCC(1,1)-GARCH(1,1)-SKT model

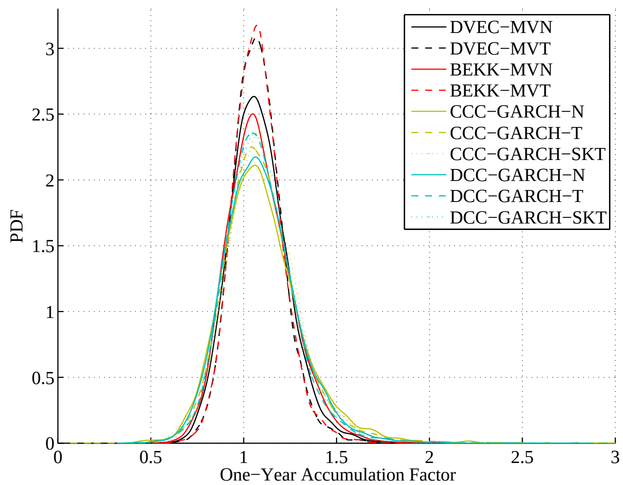


Figure 11: Smoothed distributions of the 1-year accumulation factor of Canadian equity balanced fund

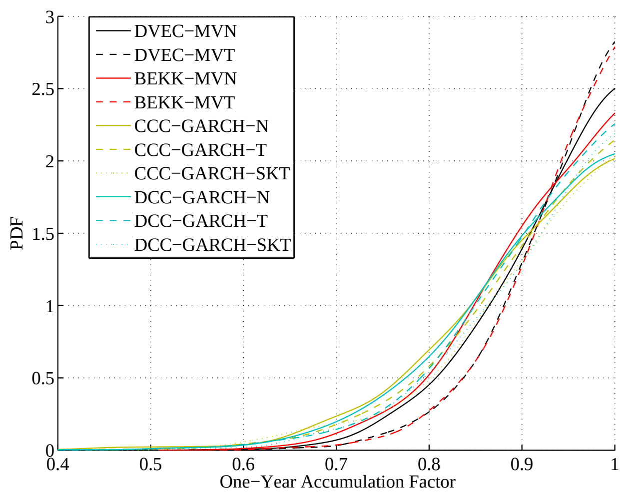


Figure 12: Left tail of the distributions of the 1-year accumulation factor of Canadian equity balanced fund

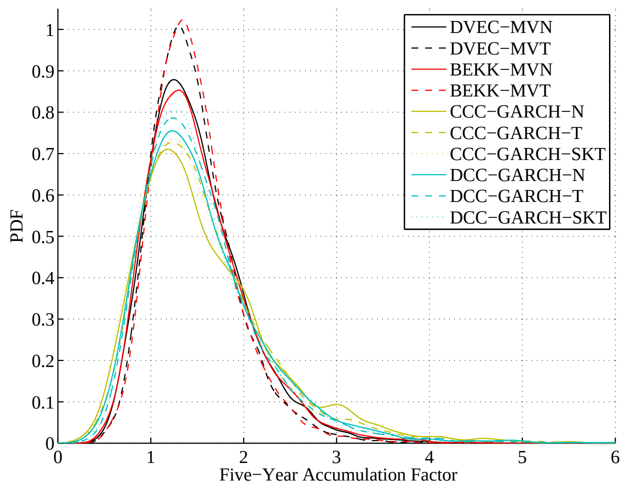


Figure 13: Smoothed distributions of the 5-year accumulation factor of Canadian equity balanced fund

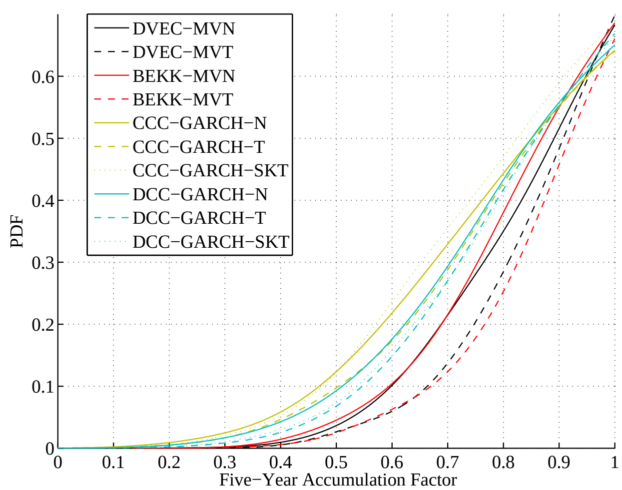


Figure 14: Left tail of the distributions of the 5-year accumulation factor of Canadian equity balanced fund

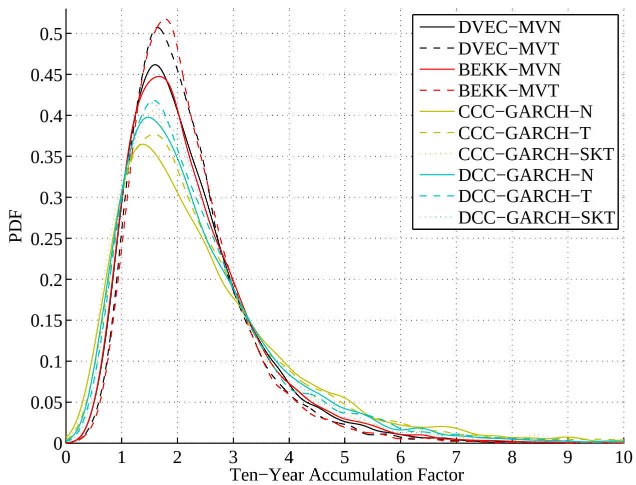


Figure 15: Smoothed distributions of the 10-year accumulation factor of Canadian equity balanced fund

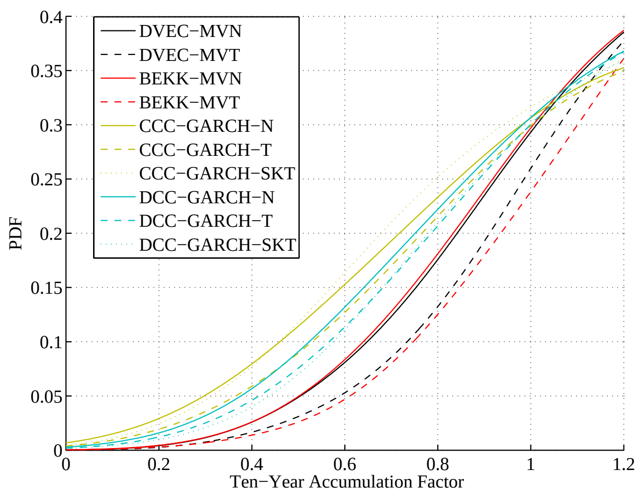


Figure 16: Left tail of the distributions of the 10-year accumulation factor of Canadian equity balanced fund

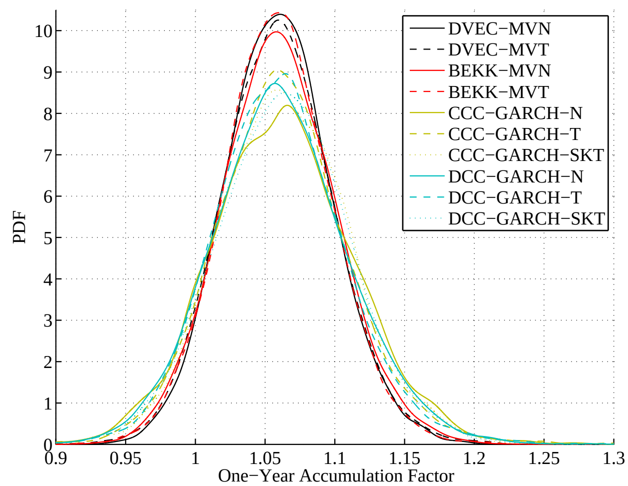


Figure 17: Smoothed distributions of the 1-year accumulation factor of Canadian fixed income balanced fund

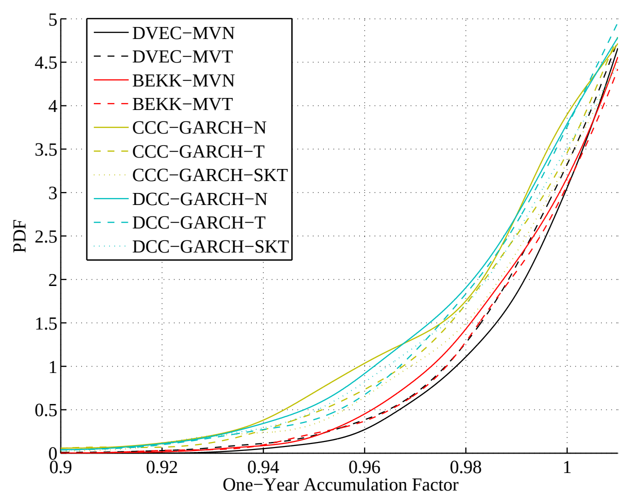


Figure 18: Left tail of the distributions of the 1-year accumulation factor of Canadian fixed income balanced fund

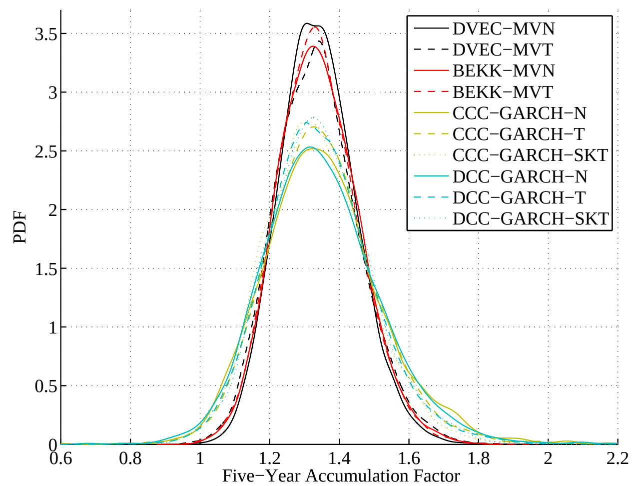


Figure 19: Smoothed distributions of the 5-year accumulation factor of Canadian fixed income balanced fund

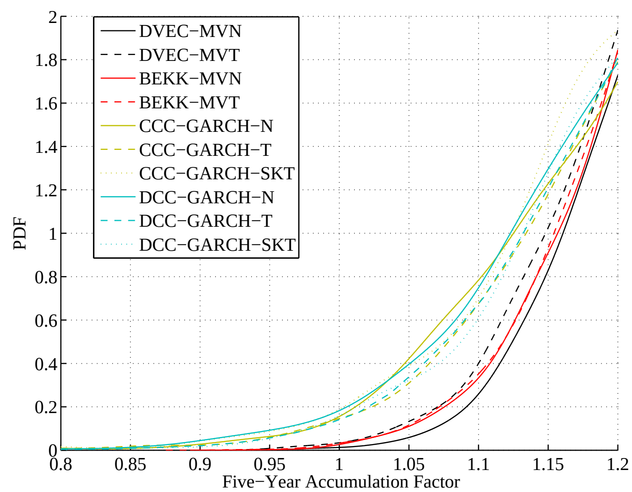


Figure 20: Left tail of the distributions of the 5-year accumulation factor of Canadian fixed income balanced fund

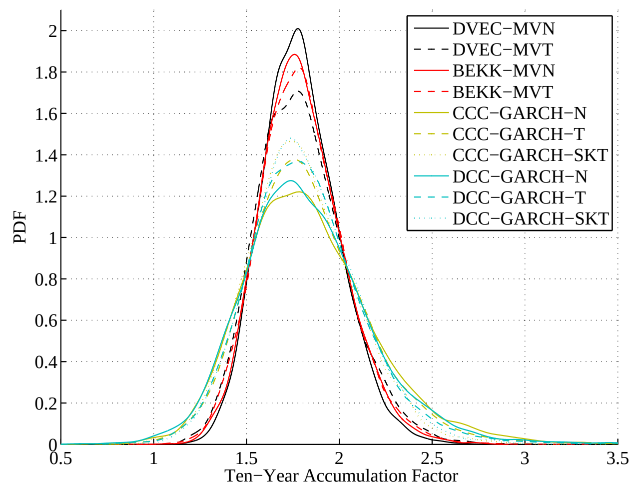


Figure 21: Smoothed distributions of the 10-year accumulation factor of Canadian fixed income balanced fund

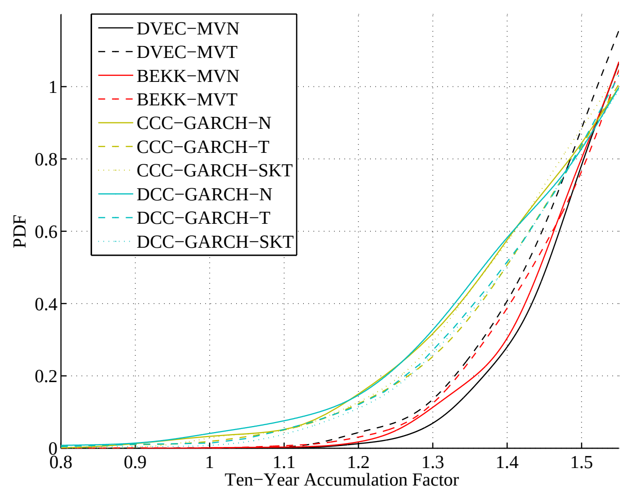


Figure 22: Left tail of the distributions of the 10-year accumulation factor of Canadian fixed income balanced fund

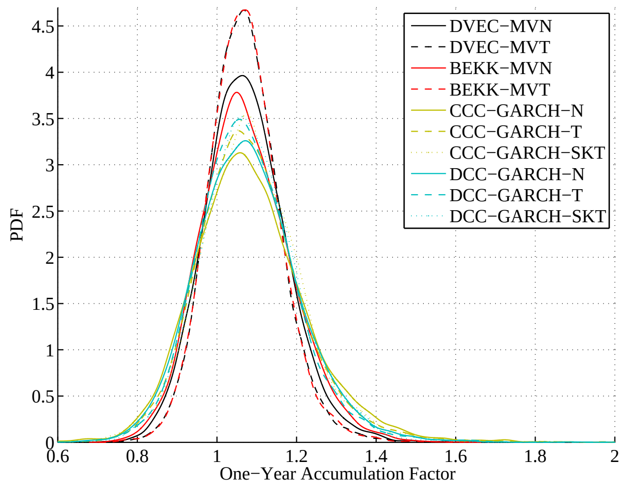


Figure 23: Smoothed distributions of the 1-year accumulation factor of Canadian neutral balanced fund

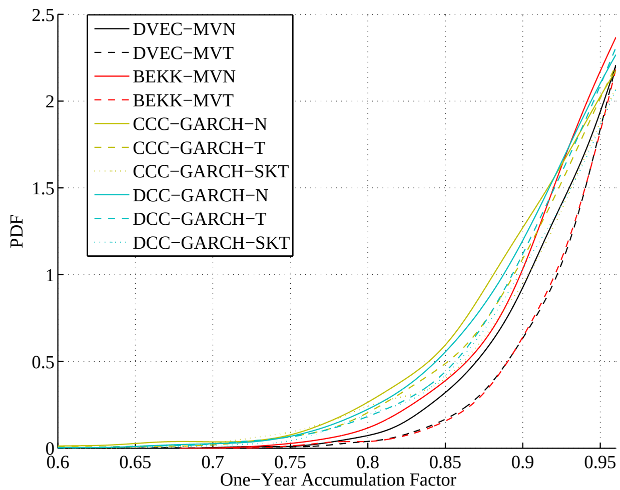


Figure 24: Left tail of the distributions of the 1-year accumulation factor of Canadian neutral balanced fund

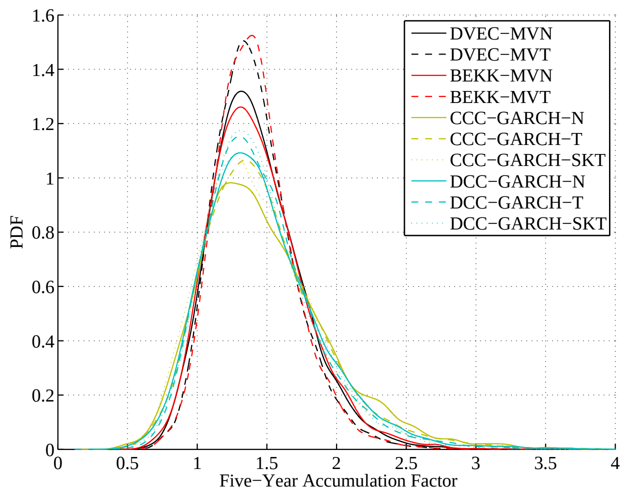


Figure 25: Smoothed distributions of the 5-year accumulation factor of Canadian neutral balanced fund

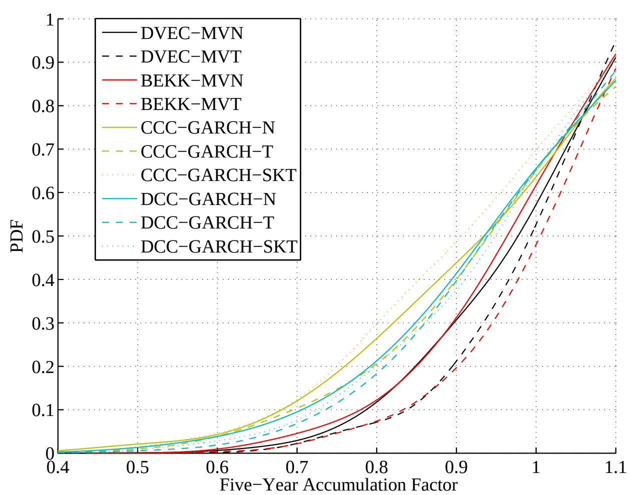


Figure 26: Left tail of the distributions of the 5-year accumulation factor of Canadian neutral balanced fund

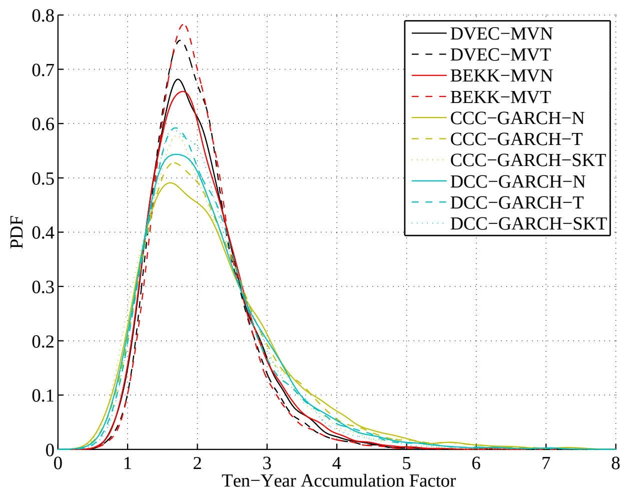


Figure 27: Smoothed distributions of the 10-year accumulation factor of Canadian neutral balanced fund

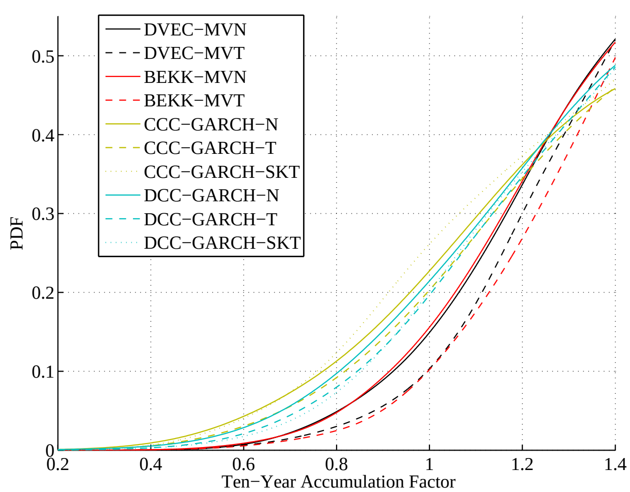


Figure 28: Left tail of the distributions of the 10-year accumulation factor of Canadian neutral balanced fund

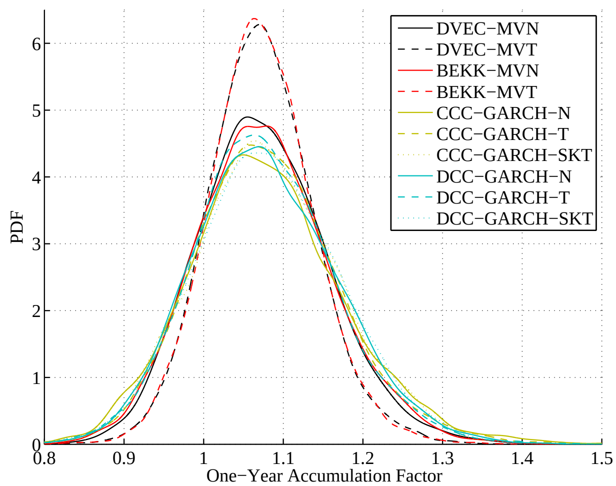


Figure 29: Smoothed distributions of the 1-year accumulation factor of global equity balanced fund

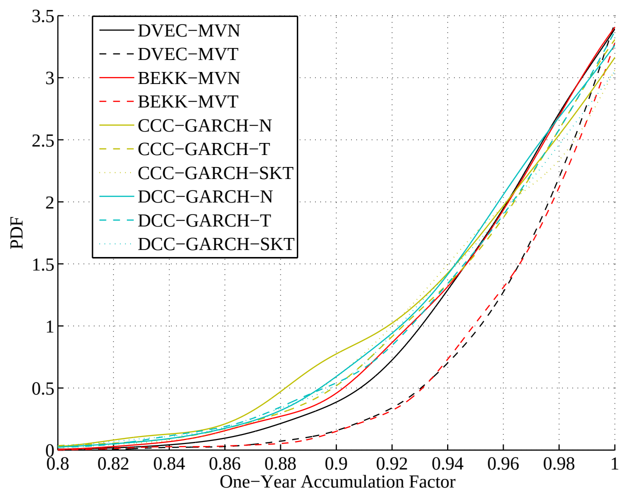


Figure 30: Left tail of the distributions of the 1-year accumulation factor of global equity balanced fund

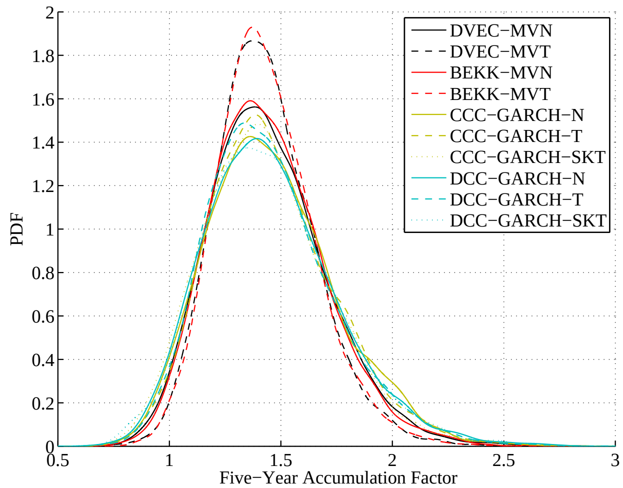


Figure 31: Smoothed distributions of the 5-year accumulation factor of global equity balanced fund

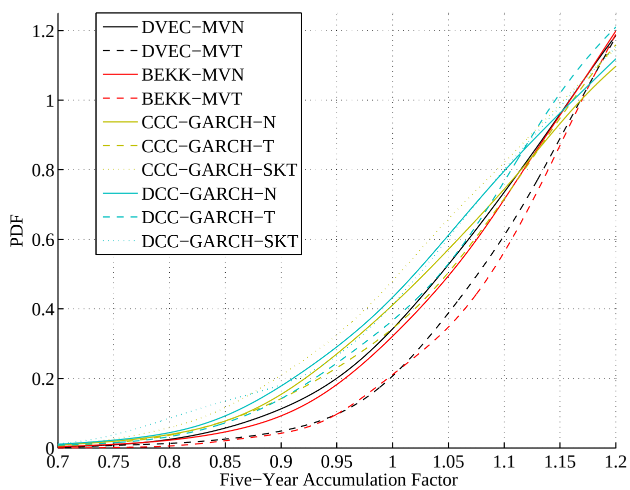


Figure 32: Left tail of the distributions of the 5-year accumulation factor of global equity balanced fund

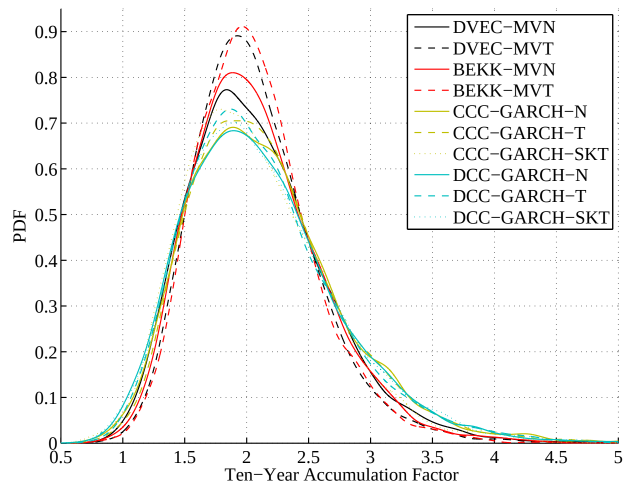


Figure 33: Smoothed distributions of the 10-year accumulation factor of global equity balanced fund

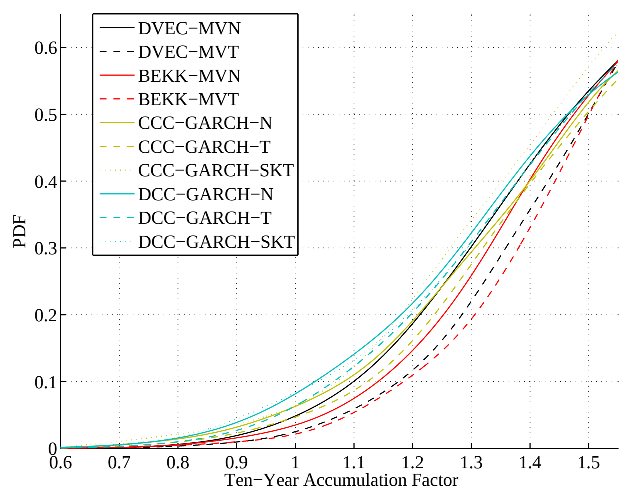


Figure 34: Left tail of the distributions of the 10-year accumulation factor of global equity balanced fund

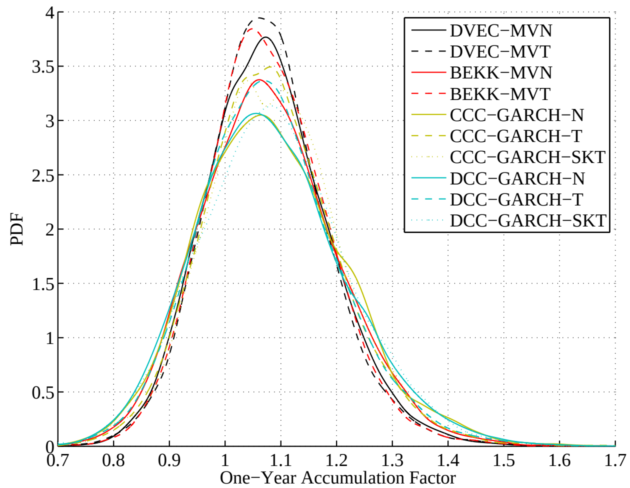


Figure 35: Smoothed distributions of the 1-year accumulation factor of US equity balanced fund

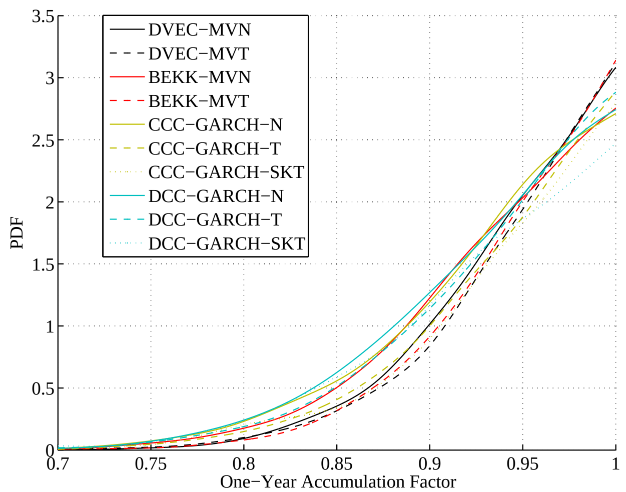


Figure 36: Left tail of the distributions of the 1-year accumulation factor of US equity balanced fund

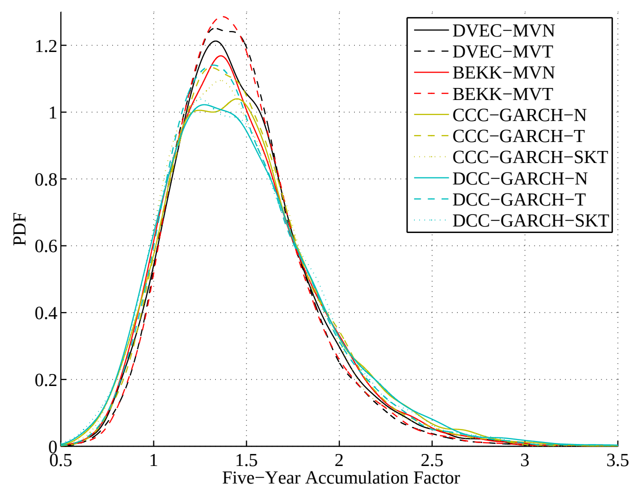


Figure 37: Smoothed distributions of the 5-year accumulation factor of US equity balanced fund

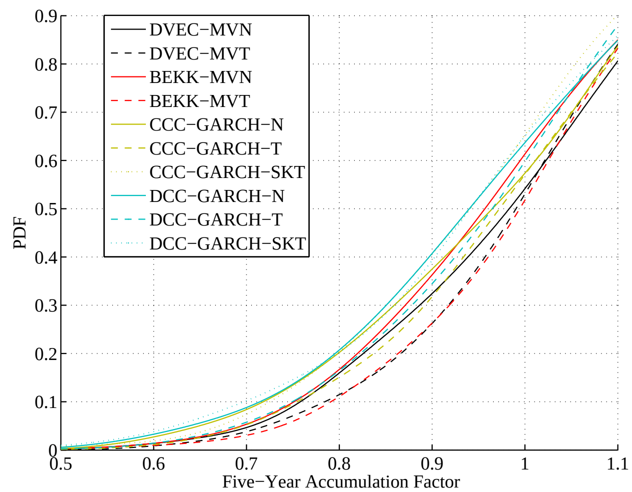


Figure 38: Left tail of the distributions of the 5-year accumulation factor of US equity balanced fund

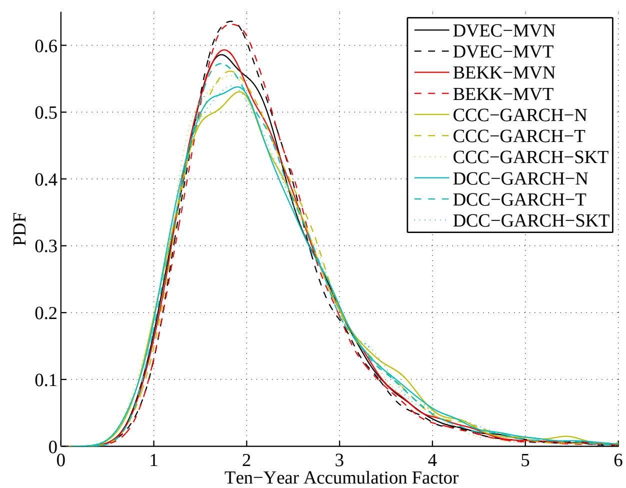


Figure 39: Smoothed distributions of the 10-year accumulation factor of US equity balanced fund

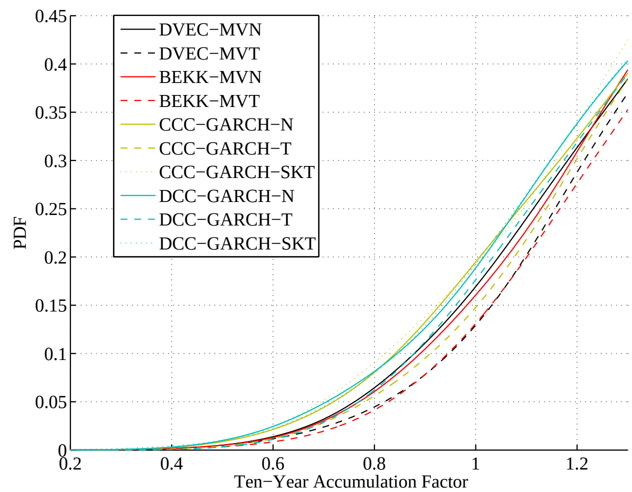


Figure 40: Left tail of the distributions of the 10-year accumulation factor of US equity balanced fund

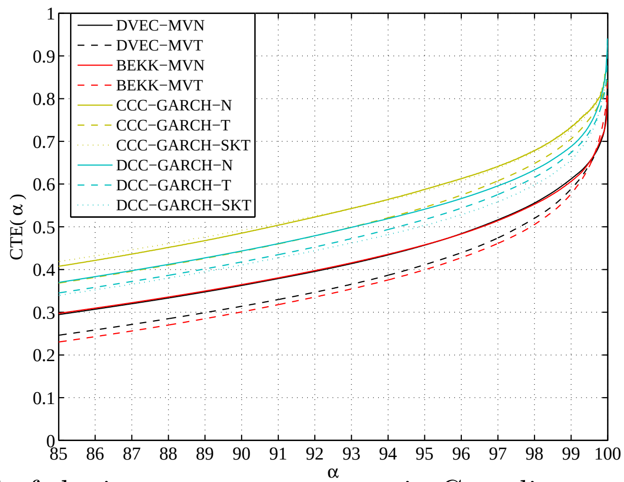


Figure 41: CTE of the investment guarantee in Canadian equity balanced fund

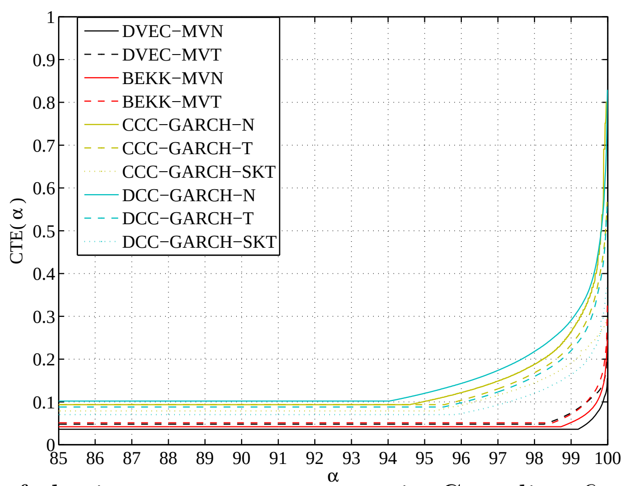


Figure 42: CTE of the investment guarantee in Canadian fixed income balanced fund

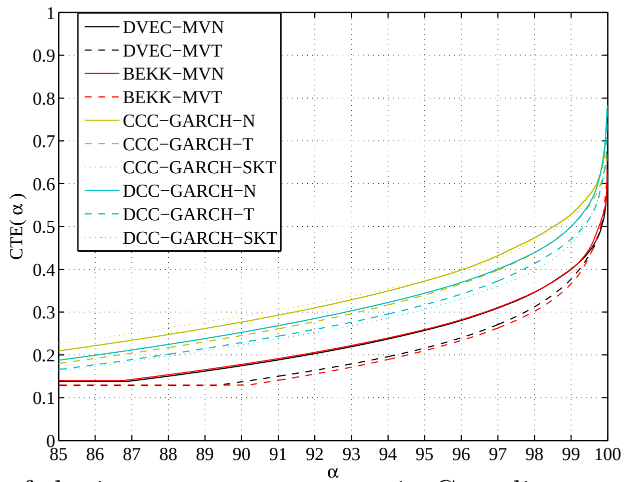


Figure 43: CTE of the investment guarantee in Canadian neutral balanced fund

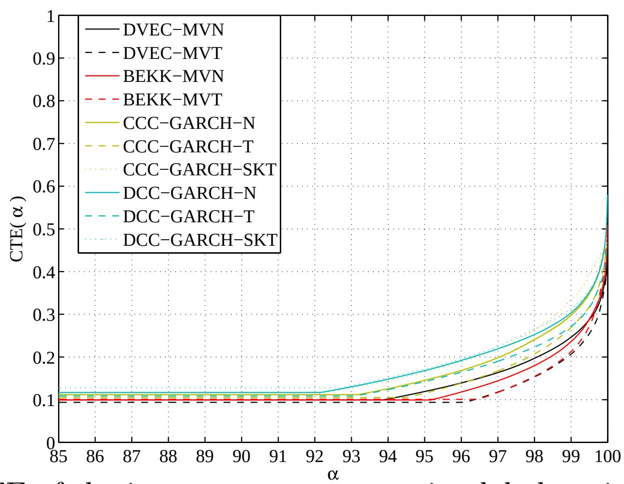


Figure 44: CTE of the investment guarantee in global equity balanced fund

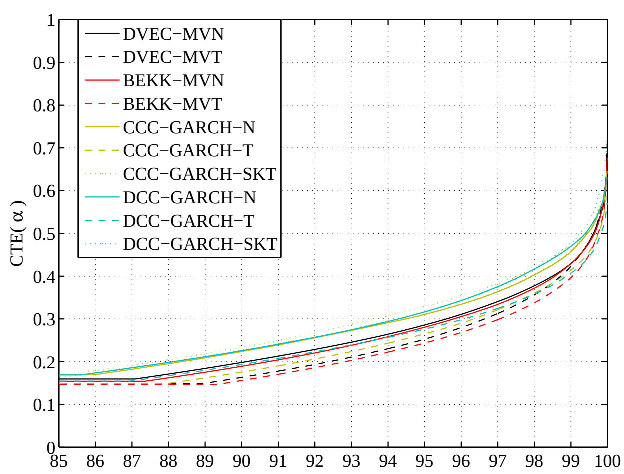


Figure 45: CTE of the investment guarantee in US equity balanced fund