Operator Spaces and Ideals in Fourier Algebras

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

In this thesis we study ideals in the Fourier algebra, $A(G)$, of a locally compact group G.

For a locally compact abelian group G, necessary conditions for a closed ideal in $A(G)$ to be weakly complemented are given, and a complete characterization of the complemented ideals in $A(G)$ is given when G is a discrete abelian group. The closed ideals in $A(G)$ with bounded approximate identities are also characterized for any locally compact abelian group G.

When G is an arbitrary locally compact group, we exploit the natural operator space structure that $A(G)$ inherits as the predual of the group von Neumann algebra, $VN(G)$, to study ideals in $A(G)$. Using operator space techniques, necessary conditions for an ideal in $A(G)$ to be weakly complemented by a completely bounded projection are given for amenable G, and the ideals in $A(G)$ possessing bounded approximate identities are completely characterized for amenable G. Ideas from homological algebra are then used to study the biprojectivity of $A(G)$ in the category of operator spaces. It is shown that $A(G)$ is operator biprojective if and only if G is a discrete group. This result is then used to show that every completely complemented ideal in $A(G)$ is invariantly completely complemented when G is discrete.

We conclude by proving that for certain discrete groups G, there are complemented ideals in $A(G)$ which fail to be complemented or weakly complemented by completely bounded projections.

Acknowledgements

Contents

Chapter 1

Introduction and Overview

Let G be a locally compact group and let $A(G)$ denote the Fourier algebra of G. In this thesis we will survey certain aspects of the ideal theory of $A(G)$. In particular, our goal is to study necessary and sufficient conditions for closed ideals in $A(G)$ to be complemented or weakly complemented in $A(G)$, and to possess bounded approximate identities. We will place a particular emphasis on exploring the role that the theory of operator spaces plays in providing insight into these problems for a noncommutative locally compact group G.

1.1 Historical Overview

Historically, the problem of classifying ideals in Fourier algebras originates from research conducted in the 1960s and 1970s on the classification of ideals in commutative group algebras.

Let G be a locally compact abelian group with Pontryagin dual group $\widehat{G}$, and let $L^1(G)$ denote the group algebra of G. Since G is abelian, the Fourier (Gelfand) transform $\mathcal{F} : L^1(G) \to C_0(\widehat{G})$ allows us to simultaneously view $L^1(G)$ as a convolution algebra of integrable functions on G and as a commutative Banach algebra of continuous functions on $\widehat{G}$ - namely the image algebra $A(\widehat{G}) := \mathcal{F}(L^1(G)) \subseteq C_0(\widehat{G})$.

By identifying $L^1(G)$ with $\mathcal{F}(L^1(G))$ as above, we are able to observe a correspondence between ideals in $L^1(G)$ and closed subsets of $\widehat{G}$. Namely, for each closed ideal I in $L^1(G)$ we can consider its *hull* hI defined by

$$hI = \{x \in \widehat{G} : \ (\mathcal{F}f)(x) = 0 \ \forall f \in I\},$$

and conversely, for any closed subset $X \subseteq \widehat{G}$ we can consider the closed ideal $I(X)$ in $L^1(G)$

defined by

$$I(X) := \{f \in L^1(G): \ (\mathcal{F}f)(x) = 0 \ \forall x \in X\}.$$

In 1962, W. Rudin ([45]) utilized this connection between ideals in $L^1(G)$ and closed subsets of $\widehat{G}$ to classify, for compact abelian G, the complemented ideals in $L^1(G)$ in terms of their hulls. He proved, using an averaging argument with the Haar measure on G, that a closed ideal I in $L^1(G)$ is complemented if and only if there exists a measure μ in $M(G)$ whose Fourier-Stieltjes transform $\hat{\mu}$ is the characteristic function of hI. In 1966, H. Rosenthal ([42]) continued along the same lines by using averaging techniques - this time with invariant means on $L^\infty(G)$ - to obtain, for arbitrary locally compact abelian groups G, necessary and sufficient conditions for an ideal I in $L^1(G)$ to be weakly complemented in terms of its hull hI. The work of Rudin and Rosenthal was extended even further in 1973 by T.-S. Liu, A. van Rooij, and J. Wang ([36]) who - by characterizing weakly complemented ideals in $L^1(G)$ as precisely those ideals with bounded approximate identities - obtained a classification of the ideals in $L^1(G)$ with bounded approximate identities in terms of their hulls.

The work of Rudin, Rosenthal, and Liu *et al.* was the starting point of an ongoing research program that has spanned several decades in an attempt to classify the complemented ideals in commutative group algebras in terms of their hulls. At the present time, this program appears to be far from completion. See [1], [2], [3], [31], and [32] for results in this direction.

Now let us turn our attention to the primary object of interest in this thesis: the Fourier algebra. For a general locally compact group G, the Fourier algebra, $A(G)$, is a certain commutative regular semisimple Banach algebra of continuous functions contained in $C_0(G)$, which is naturally identified as the predual of the group von Neumann algebra, $VN(G)$ ([13]). For abelian G, $A(G)$ turns out to be isometrically isomorphic to $L^1(\widehat{G})$ and is precisely the image of $L^1(\widehat{G})$ under the Fourier transform. For arbitrary locally compact groups, $A(G)$ serves as a generalization of the dual group algebra $L^1(\widehat{G})$.

Since for abelian G, the Fourier transform allows us to identify $A(G)$ with $L^1(\widehat{G})$, we may interpret the work of Rudin, Rosenthal and Liu *et al.* on ideals commutative group algebras in terms of ideals in Fourier algebras of commutative groups. With this observation we are led to ask: is it possible to extend the results of Rudin, Rosenthal, Liu *et al.* to Fourier algebras of arbitrary locally compact groups? Despite the naturality of such a question, it turns out that generalizing these results is a highly nontrivial matter.

To understand the principal difficulty that arises in trying to classify complemented and

weakly complemented ideals in $A(G)$ for general G, it is necessary to first understand why the approach taken in the commutative setting does not extend to the general setting. For commutative groups, the most important tool used by Rudin, Rosenthal and Liu *et al.* was the invariant mean on $L^\infty(\widehat{G})$. Using invariant means, crucial averaging techniques could be applied to projections onto ideals and their annihilators to obtain information about their hulls. However, for a noncommutative group G, there no longer exists a Pontryagin dual group $\widehat{G}$ on which this averaging with invariant means can be done. This unfortunate fact makes it necessary to search for novel ways to do these averaging techniques for $A(G)$.

An initial hope in this direction was provided by the work of A. Ya. Helemski ([24], [25]) and P. Curtis and R. Loy ([8]) on homological properties of Banach algebras in the mid 1980s. They showed that for an arbitrary Banach algebra $\mathcal{A}$, it is possible to use certain homological properties of $\mathcal{A}$ such as Banach algebra amenability and biprojectivity to abstractly perform averaging arguments like those done concretely for commutative group algebras using invariant means. Based on the known homological structure of the group algebra $L^1(G)$ and the intuition that $A(G)$ is a dual object to $L^1(G)$, it was conjectured that $A(G)$ should be Banach algebra amenable if and only if G is an amenable group, and that $A(G)$ should be biprojective if and only if G is a discrete group. If these conjectures were true, then the work of Helemski, Curtis and Loy could easily be applied to study ideals in Fourier algebras of a broad class of noncommutative groups.

Unfortunately, this homological algebra approach failed because $A(G)$ turns out to behave quite unexpectedly as a Banach algebra. It can be shown that $A(G)$ is amenable as a Banach algebra if and only if G has an abelian subgroup of finite index ([29], [19]), and $A(G)$ is known to be biprojective only when G is a discrete group with an abelian subgroup of finite index ([50])! These facts about $A(G)$ render the techniques of Helemski, Curtis and Loy essentially ineffective at studying ideals in the Fourier algebra of a noncommutative group.

However, with the advent of the theory of abstract operator spaces in the late 1980s, a new approach opened up for studying ideals in Fourier algebras. Since $A(G)$ is always the predual of a von Neumann algebra (the group von Neumann algebra $VN(G)$), it can be shown that $A(G)$ admits a natural operator space structure from this duality. Furthermore, the operator space structure and algebra structure on $A(G)$ are compatible in such a way that $A(G)$ becomes what is called a *completely contractive Banach algebra.* In 1995 Z.-J. Ruan introduced the notion of operator amenability for completely contractive Banach algebras - an analogue of Banach algebra amenability adapted to the operator space setting - and established that $A(G)$ is operator amenable if and only if G is an amenable group ([44]). Inspired by Ruan's work, P. Wood ([50]) and O. Yu. Aristov ([4]) in 2002 independently

defined and investigated the operator biprojectivity of completely contractive Banach algebras - an operator space analogue of Banach algebra biprojectivity. In particular, they showed that $A(G)$ is operator biprojective if and only if G is a discrete group ([50]). These two homological results for $A(G)$ are very interesting because they recover, in the category of operator spaces, what our intuition about $A(G)$ predicts should happen. These results (among many others) strongly suggested that $A(G)$ is best understood when viewed as a completely contractive Banach algebra rather than merely as a Banach algebra.

But more to the point of this thesis, these homological results are important tools for studying of ideals in $A(G)$. Just as classical amenability and biprojectivity are effective averaging tools for a Banach algebra, operator amenability and operator biprojectivity turn out to be effective averaging tools for completely contractive Banach algebras. In this way, the operator space category provides us with the right framework in which to significantly extend our understanding of the ideal structure of $A(G)$ beyond abelian groups (see [18], [49], [50], [51], [52]).

1.2   Organization

This thesis is organized as follows.

Chapter 2 is a brief overview of some of the functional analytic background that will be needed in this thesis. We also establish the notation that will be used.

In Chapter 3 we study, for locally compact abelian groups G, complemented and weakly complemented ideals in $L^1(G)$, as well as ideals in $L^1(G)$ possessing bounded approximate identities. We survey the major results obtained by Rudin, Rosenthal and Liu *et al.* in [45], [42] and [36]. By identifying $A(G)$ with the dual group algebra $L^1(\widehat{G})$, we are able to interpret the results of this chapter in terms of ideals in Fourier algebras of locally compact abelian groups.

In Chapter 4 we turn our attention to ideals in the Fourier algebra, $A(G)$, of a general locally compact group G. The main purpose of this chapter is to illustrate, through a series of examples and results, the similarities and differences between the type of phenomena that can occur in the commutative and noncommutative settings, and to set the stage for the use of operator space techniques in the following chapter.

In Chapter 5 we begin to study the Fourier algebra as a completely contractive Banach algebra. Using ideas from homology theory for completely contractive Banach algebras developed by Wood ([52]), we show how properties of a completely contractive Banach algebra $\mathcal{A}$ such as operator amenability and operator biprojectivity can be used to study ideals in $\mathcal{A}$. Applying these techniques to the Fourier algebra, we are able to significantly

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generalize many of the results we obtained in Chapter 3 for abelian groups.

In Chapter 6 we consider the following question: do there exist locally compact groups G for which there are ideals in $A(G)$ that are complemented (or weakly complemented) in $A(G)$, but not complemented (or weakly complemented) by completely bounded projections? We prove using operator space techniques and certain results obtained in Chapter 5, that such ideals do indeed exist when G is a noncommutative free group.

Chapter 2

Preliminaries and Notation

In this chapter we will briefly overview some of the notation and concepts from functional analysis and harmonic analysis that will be used throughout this thesis.

2.1 Banach Spaces

All vector spaces considered in this thesis will be over the field $\mathbb{C}$. If X is a Banach space and $M \geq 0$, we denote by $b_M(X)$ the closed ball

$$b_M(X) = \{x \in X : \ \|x\| \leq M\}.$$

If X is an normed linear space with respect to a norm $\|\cdot\| : X \to [0, \infty)$, its Banach space completion is denoted by $\overline{X}^{\|\cdot\|}$ or simply $\overline{X}$ if no confusion shall arise. If X is a Banach space, X^* will denote its **dual space** - that is, the Banach space of all norm-continuous linear functionals on X. If $x \in X$ and $\phi \in X^*$, we will typically denote the evaluation of ϕ at x by $\langle x, \phi \rangle$. If Y is a linear subspace of a Banach space X, its **annihilator** in X^* is the closed subspace $Y^\perp \subseteq X^*$ defined by

$$Y^\perp := \{\phi \in X^* : \ \langle y, \phi \rangle = 0, \ \forall y \in Y\}.$$

If Z is a linear space which acts as a family of continuous linear functionals on a Banach space X, the linear topology on X induced by the family of functionals

$$\mathcal{F} = \{x \mapsto \langle x, z \rangle : \ x \in X, \ z \in Z\},$$

is denoted by the $\sigma(X, Z)$-topology. In this context we define the closed subspace

$$Z_\perp := \{x \in X : \langle x, z \rangle = 0, \ \forall z \in Z\} \subseteq X$$

and call it the **pre-annihilator** of Z. In the special case where $Z = X^*$ we call the $\sigma(X, X^*)$-topology the **weak-topology** on X. Recall that for any Banach space Y, there is a natural embedding $Y \hookrightarrow Y^{**}$ given by

$$y \mapsto \hat{y}, \quad \langle \phi, \hat{y} \rangle := \langle y, \phi \rangle, \qquad \forall y \in Y, \ \phi \in Y^*.$$

We call the resultant $\sigma(Y^*, Y)$-topology on Y^* the **weak*-topology** on Y^*.

One of the most important concepts in this thesis will be the notion of a **complemented subspace** of a Banach space. Let X be a Banach space and $Y \subseteq X$ a closed linear subspace. Y is said to be **complemented in** X if there exists a closed subspace $Z \subseteq X$ such that $X = Y + Z$ and $Z \cap Y = \{0\}$. It is an easy consequence of the closed graph theorem that Y is complemented in X if and only if there exists a bounded linear idempotent $P : X \to X$ satisfying $PX = Y$.

2.2 Operator Spaces

Our standard reference for Operator Spaces is [11].

Let X be a complex vector space. For each $n \in \mathbb{N}$ let $M_n(X)$ denote the vector space of $n \times n$ matrices over X. An **operator space** is a complex vector space X equipped with a family $\{\|\cdot\|_n\}_{n \in \mathbb{N}}$ of norms

$$\|\cdot\|_n : M_n(X) \to [0, \infty)$$

which satisfy the following two axioms:

$$\left\| \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \right\|_{n+m} = \max\{\|x\|_n, \|y\|_m\}, \quad \forall x \in M_n(X), \ y \in M_m(X), \tag{2.1}$$

$$\|\alpha x \beta\|_n \leq \|\alpha\| \|x\|_n \|\beta\| \ \ \forall x \in M_n(X), \alpha, \beta \in M_n(\mathbb{C}). \tag{2.2}$$

In (2.2), $M_n(\mathbb{C})$ is given its canonical norm as the algebra of bounded linear operators on the n-dimensional Hilbert space $l^2(n)$ and $\alpha x \beta$ denotes the usual bimodule action of $M_n(\mathbb{C})$ on the matrix space $M_n(X)$. That is, we define $(\alpha x \beta)_{ij} := \sum_{l=1}^n \sum_{k=1}^n \alpha_{ik} x_{kl} \beta_{lj}$. Note that

if $X = M_1(X)$ is complete then $M_n(X)$ is complete for all $n \in \mathbb{N}$.

The prototypical operator space is the C*-algebra $\mathcal{B}(\mathcal{H})$ consisting of all bounded operators on a Hilbert space $\mathcal{H}$. The norm on $M_n(\mathcal{B}(\mathcal{H}))$ is obtained by identifying $M_n(\mathcal{B}(\mathcal{H}))$ with $\mathcal{B}(\mathcal{H}^n)$ via the map

$$M_n(\mathcal{B}(\mathcal{H})) \ni [a_{ij}] \mapsto \left\{ (\xi_i)_{i=1}^n \mapsto (\sum_{k=1}^n a_{ik}\xi_k)_{i=1}^n \right\} \in \mathcal{B}(\mathcal{H}^n).$$

Similarly, any subspace of $\mathcal{B}(\mathcal{H})$ is an operator space with respect to the relativized operator space structure. Also any C^*-algebra $\mathcal{A}$ is an operator space by assigning $M_n(\mathcal{A})$ the (unique) norm that makes it a C^*-algebra.

Given operator spaces X and Y and a linear map $T : X \to Y$, we define for each $n \in \mathbb{N}$ the n**th amplification of** T to be the linear map $T^{(n)} : M_n(X) \to M_n(Y)$ given by

$$T^{(n)}[x_{ij}] = [Tx_{ij}].$$

We say that the map $T : X \to Y$ is **completely bounded** if

$$\|T\|_{cb} = \sup\{\|T^{(n)}\| : n \in \mathbb{N}\} < \infty.$$

We will frequently use **c.b.** as an abbreviation for the words "completely bounded". We call T a **complete contraction** if $\|T\|_{cb} \le 1$, a **complete isometry** if $T^{(n)}$ is an isometry for all $n \in N$, and a **complete quotient map** if for every $n \in \mathbb{N}$, $T^{(n)}$ is a quotient map (i.e. $\forall y \in T^{(n)}(M_n(X))$ and $\forall \epsilon > 0$, there exists an $x \in M_n(X)$ such that $T^{(n)}x = y$ and $\|x\|_n < \|y\|_n + \epsilon$.

The linear space of all c.b. maps from X to Y is denoted by $\mathcal{CB}(X, Y)$. When equipped with the norm $\| \cdot \|_{cb}$, $\mathcal{CB}(X, Y)$ is complete if and only if Y is complete. $\mathcal{CB}(X, Y)$ has a natural operator space structure given by identifying $M_n(\mathcal{CB}(X, Y))$ with $\mathcal{CB}(X, M_n(Y))$ via the map

$$[T_{ij}] \mapsto (x \mapsto [T_{ij}x]).$$

If X is an operator space and $f \in X^*$, then $\|f\|_{cb} = \|f\|$. Therefore $X^* = \mathcal{CB}(X, \mathbb{C})$ is naturally an operator space. Furthermore, the norm of $[f_{ij}] \in M_n(X^*)$ can be computed by the duality formula

$$\|[f_{ij}]\|_n = \sup\{\|[\langle x_{kl}, f_{ij}\rangle]\|_{M_{n^2}} : [x_{kl}] \in b_1(M_n(X))\}.$$

Given $T \in \mathcal{CB}(X, Y)$, the linear adjoint map $T \mapsto T^*$ is a complete isometry from $\mathcal{CB}(X, Y)$

into $\mathcal{CB}(Y^*, X^*)$. We denote the range of this map by $\mathcal{CB}^\sigma(Y^*, X^*)$. Of particular importance in abstract harmonic analysis is the fact that if $\mathcal{A}$ is a von Neumann algebra then its predual $\mathcal{A}_*$ is naturally an operator space via the canonical completely isometric inclusion $\mathcal{A}_* \hookrightarrow \mathcal{A}^*$.

Given two operator spaces X and Y. We give the ℓ^∞-direct sum $X \oplus_\infty Y$ an operator space structure by identifying $M_n(X \oplus_\infty Y)$ with $M_n(X) \oplus_\infty M_n(Y)$. The ℓ^1-direct sum $X \oplus_1 Y$ is given an operator space structure via the natural imbedding $X \oplus_1 Y \hookrightarrow (X \oplus_\infty Y)^*$. If X is complete and $Z \subseteq X$ is a closed subspace. Then X/Z is naturally an operator space by identifying $M_n(X/Z)$ with $M_n(X)/M_n(Z)$.

Given any Banach space X, there are two natural operator space structures that can be put on X. The first is the **maximal operator space structure**, denoted by $\mathrm{MAX}(X)$, and the second is the **minimal operator space structure**, denoted by $\mathrm{MIN}(X)$. The matricial norms on $\mathrm{MAX}(X)$ and $\mathrm{MIN}(X)$ are defined as follows. For $[x_{ij}] \in M_n(X)$ set

$$\|[x_{ij}]\|_{n,\mathrm{MAX}} = \sup\{\|[Tx_{ij}]\|_{M_k} : T \in b_1(\mathcal{B}(X, M_k(\mathbb{C}))), \ k \in \mathbb{N}\},$$

and

$$\|[x_{ij}]\|_{n,\mathrm{MIN}} = \sup\{\|f^{(n)}[x_{ij}]\|_{M_n} : f \in b_1(X^*)\}.$$

It can be shown that for any operator space Y, any bounded linear map $T : \mathrm{MAX}(X) \to Y$ is automatically completely bounded with $\|T\|_{cb} = \|T\|$, and any bounded linear map $T : Y \to \mathrm{MIN}(X)$ is automatically completely bounded with $\|T\|_{cb} = \|T\|$.

A very important theorem due to Z.-J. Ruan says that for any operator space X, there exists a Hilbert space $\mathcal{H}$ and a complete isometry $\phi : X \to \mathcal{B}(\mathcal{H})$. Thus, up to complete isometric isomorphism, all operator spaces can be realized as subspaces of $\mathcal{B}(\mathcal{H})$ for an appropriate Hilbert space $\mathcal{H}$.

The following two theorems concerning completely bounded maps from a unital C*-algebra $\mathcal{A}$ into $\mathcal{B}(\mathcal{H})$ will be needed in this thesis. The first theorem, due to Wittstock ([11]), is a useful generalization of the classical Hahn-Banach extension theorem to c.b. maps:

Theorem 2.2.1. *Let $\mathcal{A}$ be a unital C*-algebra and $X \subseteq \mathcal{A}$ be a subspace. Let $\mathcal{H}$ be a Hilbert space, and let $\phi : X \to \mathcal{B}(\mathcal{H})$ be a completely bounded map. Then there exists a completely bounded map $\psi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ which extends ϕ with $\|\psi\|_{cb} = \|\phi\|_{cb}$.*

The second theorem we need is a structure theorem for completely bounded maps from $\mathcal{A}$ into $\mathcal{B}(\mathcal{H})$:

Theorem 2.2.2. *Let $\mathcal{A}$ be a unital C^*-algebra, and let $T : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a completely bounded map. Then there exists a Hilbert space $\mathcal{K}$, a $*$-representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{K})$, bounded operators $V_i : \mathcal{H} \to \mathcal{K}$, $i = 1, 2$, with $\|T\|_{cb} = \|V_1\|\|V_2\|$ such that*

$$Ta = V_2^* \pi(a) V_1, \qquad \forall a \in \mathcal{A}.$$

Moreover, if $\|T\|_{cb} = 1$, then V_1 and V_2 may be taken to be isometries.

A proof of this theorem can be founded in [40].

2.3 Tensor Products of Banach Spaces and Operator Spaces

If X and Y are Banach spaces, their algebraic tensor product will be denoted by $X \otimes Y$. If $\| \cdot \|_\alpha$ is a norm on $X \otimes Y$ then the completion of $X \otimes Y$ with respect to $\| \cdot \|_\alpha$ will be denoted by $X \otimes^\alpha Y$.

In this thesis there are two tensor products that we will be primarily concerned with. These are the **Banach space projective tensor product** and the **operator space projective tensor product**. Given two Banach spaces X and Y, the Banach space projective tensor product of X and Y is denoted by $X \otimes^\gamma Y$ and defined to be the completion of $X \otimes Y$ with respect to the norm

$$\|u\|_\gamma = \inf \Big\{ \sum_i \|x_i\|\|y_i\| :\ u = \sum_i x_i \otimes y_i \in X \otimes Y \Big\}.$$

For Banach spaces X, Y, and Z, we call a bilinear map $T : X \times Y \to Z$ **jointly bounded** if we have

$$\|T\|_{jb} = \sup\{\|T(x, y)\| :\ x \in b_1(X),\ y \in b_1(Y)\} < \infty.$$

The Banach space of all such jointly bounded bilinear maps is denoted by $\mathcal{JB}(X \times Y; Z)$. An important feature of the Banach space projective tensor product is the following linearization result for jointly bounded maps: If X, Y and Z are Banach spaces, then the canonical map $\Phi : \mathcal{B}(X \otimes^\gamma Y, Z) \to \mathcal{JB}(X \times Y; Z)$ given by

$$\Phi T(x, y) = T(x \otimes y), \qquad \forall x \in X,\ y \in Y,$$

induces an isometric isomorphism

$$\mathcal{JB}(X \times Y; Z) \cong \mathcal{B}(X \otimes^\gamma Y, Z).$$

Another important feature of the Banach space projective tensor product is fact for any Banach spaces X and Y, the linear map $T \mapsto \phi_T$ from $\mathcal{B}(X, Y^*)$ to $(X \otimes^\gamma Y)^*$ defined by the dual pairing

$$\langle x \otimes y, \phi_T \rangle := \langle y, T(x) \rangle, \qquad \forall T \in \mathcal{B}(X, Y^*), \ x \in X, \ y \in Y,$$

induces an isometric isomorphism

$$(X \otimes^\gamma Y)^* \cong \mathcal{B}(X, Y^*).$$

The operator space projective tensor product serves as an analogue for the Banach space projective tensor product in the category of operator spaces. If X and Y are operator spaces, then the operator space projective tensor product of X and Y is denoted by $X \widehat{\otimes} Y$ and defined by the family of norms $\{ \| \cdot \|_{n,\wedge} : M_n(X \otimes Y) \to \mathbb{R}_+ \}_{n \in \mathbb{N}}$ where

$$\|u\|_{n,\wedge} = \inf \{ \|\alpha\| \|x\|_p \|y\|_q \|\beta\| : \ u = \alpha(x \otimes y)\beta \in M_n(X \otimes Y) \},$$

where $x \in M_p(X)$, $y \in M_q(X)$, $\alpha \in M_{n, p \times q}$, and $\beta \in M_{p \times q, n}$.

For operator spaces X, Y, and Z, we call a bilinear map $T : X \times Y \to Z$ **jointly completely bounded** if we have

$$
\begin{aligned}
\|T\|_{jcb} &= \sup_{m,n \in \mathbb{N}} \{ \|[T(x_{ij}, y_{kl})]\|_{mn} : \ [x_{ij}] \in b_1(M_m(X)), \ [y_{kl}] \in b_1(M_n(Y)) \} \\
&< \ \infty.
\end{aligned}
$$

The Banach space of all such jointly completely bounded bilinear maps is denoted by $\mathcal{JCB}(X \times Y; Z)$. $\mathcal{JCB}(X \times Y; Z)$ also becomes an operator space by identifying $M_n(\mathcal{JCB}(X \times Y; Z))$ with $\mathcal{JCB}(X \times Y; M_n(Z))$ for each $n \in \mathbb{N}$. By analogy with the Banach space case, we have the following useful results: If X, Y and Z are operator spaces then

$$\mathcal{JCB}(X \times Y; Z) \cong \mathcal{CB}(X \widehat{\otimes} Y, Z),$$

and

$$(X \widehat{\otimes} Y)^* \cong \mathcal{CB}(X, Y^*),$$

completely isometrically. The maps defining these isomorphisms are defined the same way as we did above for the Banach space projective tensor product versions.

2.4 Banach algebras and Completely Contractive Banach Algebras

In this section we list some of the important facts from the theory of Banach algebras and completely contractive Banach algebras that will be needed later.

If $\mathcal{A}$ is an associative algebra which is also an operator space and is such that the multiplication map

$$m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$$

extends to a complete contraction $m : \mathcal{A} \widehat{\otimes} \mathcal{A} \to \mathcal{A}$, then we call $\mathcal{A}$ a **completely contractive Banach algebra**. As in the Banach algebra category, we have that if $\mathcal{A}$ is a completely contractive Banach algebra and $I \subseteq A$ is a closed two-sided ideal, then both I and $\mathcal{A}/I$ are completely contractive Banach algebras with respect to the operator space structures they inherit from $\mathcal{A}$.

If $\mathcal{A}$ is any Banach algebra, we say that $\mathcal{A}$ is **unital** if it is unital as an algebra. A net $\{e_\alpha\}_\alpha \subseteq \mathcal{A}$ is called a **left approximate identity** for $\mathcal{A}$ if for all $a \in \mathcal{A}$ we have

$$\lim_\alpha e_\alpha a = a.$$

We can similarly define right approximate identities and two-sided approximate identities for $\mathcal{A}$. A (left, right, or two-sided) approximate identity $\{e_\alpha\}_\alpha \subseteq \mathcal{A}$ is called a **bounded approximate identity** if $\{e_\alpha\}_\alpha$ is an approximate identity and there exists an $M > 0$ such that $\|e_\alpha\| \le M$ for all α. A routine calculation shows that a Banach algebra $\mathcal{A}$ with a left bounded approximate identity and a right bounded approximate identity also has a two sided bounded approximate identity (see for example [5]).

The following result characterizing Banach algebras with bounded approximate identities will be useful. For a proof see [5].

Proposition 2.4.1. *Let $\mathcal{A}$ be a Banach algebra. Then the following are equivalent:*
(i). $\mathcal{A}$ has a two-sided bounded approximate identity.
(ii). There exists a bounded subset $\mathcal{U} \subset \mathcal{A}$ such that for any $a \in \mathcal{A}$ and any $\epsilon > 0$, there is some $u \in \mathcal{U}$ so that

$$\|ua - a\| < \epsilon \quad and \quad \|au - a\| < \epsilon.$$

Obviously Proposition 2.4.1 is true for left and right bounded approximate identities as well.

2.4.1   Ideals and Spectral Synthesis in Commutative Semisimple Banach Algebras

Let $\mathcal{A}$ be a commutative semisimple Banach algebra and let Σ denote the character space of $\mathcal{A}$. Since $\mathcal{A}$ is **semisimple** (that is, the Gelfand transform $\Gamma : \mathcal{A} \to C_0(\Sigma)$ is injective) we may without loss of generality assume that $\mathcal{A}$ sits concretely as a subalgebra of $C_0(\Sigma)$.

Let $\mathcal{A} \subseteq C_0(\Sigma)$ be as above and let $I \subseteq \mathcal{A}$ be a closed ideal. The **hull** of I is defined to be the set

$$hI = \{x \in \Sigma : \ f(x) = 0, \ \forall f \in \mathcal{A}\} \subseteq \Sigma.$$

It is easy to see that hI is always a closed subset of Σ. Conversely, given any closed subset $X \subseteq \Sigma$, we can define two ideals

$$I(X) = \{f \in \mathcal{A} : \ f(x) = 0, \ \forall x \in X\},$$

and

$$I_0(X) = \{f \in \mathcal{A} : \mathrm{supp}(f) \text{ is compact and } \mathrm{supp}(f) \cap X = \emptyset\}.$$

It is easy to see that $I(X)$ is always closed in $\mathcal{A}$, that $h(I(X)) = X$ and that $I(X)$ is maximal in the sense that $I \subseteq I(X)$ for any ideal I with $hI = X$. If $\mathcal{A}$ is furthermore **regular** (that is, for any closed subset $E \subseteq \Sigma$ and any $x \in \Sigma \backslash E$ there exists $f \in \mathcal{A}$ such that $f(E) = \{0\}$ and $f(x) = 1$) it can be shown ([9], Proposition 4.1.20) that

$$I_0(X) \subseteq I \subseteq I(X),$$

for any ideal $I \subseteq \mathcal{A}$ with $hI = X$.

If $\mathcal{A} \subseteq C_0(\Sigma)$ is a commutative regular semisimple Banach algebra and $X \subseteq \Sigma$ is a closed subset, then we say that X is a **set of spectral synthesis** if $I_0(X)$ is dense in $I(X)$. In other words, X is a set of spectral synthesis if $I(X)$ is the only closed ideal whose hull is X. A commutative regular semisimple Banach algebra $\mathcal{A} \subseteq C_0(\Sigma)$ is said to **admit spectral synthesis** if every closed subset $X \subseteq \Sigma$ is a set of synthesis.

It is well known that $C_0(\Sigma)$ admits spectral synthesis for any locally compact Hausdorff space Σ (see [14]). It is also well known that the Fourier algebra $A(G)$ admits spectral synthesis for any discrete abelian group G (see [14]). It is currently unknown whether $A(G)$ admits spectral synthesis for all discrete groups G.

2.4.2 Banach $\mathcal{A}$-Modules and Operator $\mathcal{A}$-Modules

Let $\mathcal{A}$ be a Banach algebra. A **left Banach $\mathcal{A}$-module** is defined to be a left $\mathcal{A}$-module X such that X is a Banach space and such that the module map $\pi : \mathcal{A} \otimes X \to X$ extends to a contraction $\pi : \mathcal{A} \otimes^\gamma X \to X$. One can similarly define a **right Banach $\mathcal{A}$-module** and a **Banach $\mathcal{A}$-bimodule**. If X is a left Banach $\mathcal{A}$-module, then X^* becomes a right Banach $\mathcal{A}$-module by dualizing the module actions. That is, we define

$$\langle x, \phi \cdot a \rangle := \langle a \cdot x, \phi \rangle, \qquad \forall a \in \mathcal{A},\ x \in X,\ \phi \in X^*.$$

We call X^* a **right dual Banach $\mathcal{A}$-module** in this case.

If $\mathcal{A}$ is a completely contractive Banach algebra. A **left operator $\mathcal{A}$-module** is defined to be a left $\mathcal{A}$-module X such that X is an operator space and such that the module map $\pi : \mathcal{A} \otimes X \to X$ extends to a complete contraction $\pi : \mathcal{A} \widehat{\otimes} X \to X$. As above, we can similarly define **right operator $\mathcal{A}$-modules**, **operator $\mathcal{A}$-bimodules**, and **dual operator $\mathcal{A}$-modules**. An $\mathcal{A}$-bimodule X is called **symmetric** if $a \cdot x = x \cdot a$ for all $x \in X$ and all $a \in \mathcal{A}$.

A left Banach (or operator) $\mathcal{A}$-module X is called **neounital** if

$$X = \{a \cdot x :\ a \in \mathcal{A},\ x \in X\}.$$

X is **essential** if $\mathrm{span}\{a \cdot x : a \in \mathcal{A},\ x \in X\}$ is dense in X. There are obviously similar definitions for neounital and essential right $\mathcal{A}$-modules and $\mathcal{A}$-bimodules.

If X is a left Banach (or operator) $\mathcal{A}$-module and Y is any Banach (or operator) space, then $X \otimes^\gamma Y$ (or $X \widehat{\otimes} Y$) becomes a left Banach (or operator) $\mathcal{A}$-module via the left module action

$$(a, x \otimes y) \mapsto a \cdot x \otimes y, \qquad \forall a \in \mathcal{A},\ x \in X,\ y \in Y.$$

If Y is also a right Banach (or operator) $\mathcal{A}$-module then $X \otimes^\gamma Y$ (or $X \widehat{\otimes} Y$) becomes a Banach (or operator) $\mathcal{A}$-bimodule.

If $\mathcal{A}$ is a (completely contractive) Banach algebra and X and Y are both left Banach (operator) $\mathcal{A}$-modules, then $\mathcal{B}(X, Y)$ ($\mathcal{CB}(X, Y)$) is naturally a Banach (operator) $\mathcal{A}$-bimodule via the module actions

$$(T \cdot a)(x) = T(a \cdot x), \qquad (a \cdot T)(x) = a \cdot (Tx).$$

If X and Y are both right $\mathcal{A}$-modules, then $\mathcal{B}(X, Y)$ ($\mathcal{CB}(X, Y)$) become Banach (operator)

$\mathcal{A}$-bimodules as via

$$(T \cdot a)(x) = (Tx) \cdot a, \qquad (a \cdot T)(x) = T(x \cdot a).$$

Let $\mathcal{A}$ be a completely contractive Banach algebra and let $\mathcal{A}_+$ denote its unitization. That is, $\mathcal{A}_+ = \mathcal{A} \oplus \mathbb{C}$ equipped with the multiplication

$$(a, \alpha)(b, \beta) = (ab + \alpha b + \beta a, \alpha\beta), \qquad \forall \alpha, \beta \in \mathbb{C}, \ a, b \in \mathcal{A}.$$

Note that any type of $\mathcal{A}$-module becomes an $\mathcal{A}_+$-module via the module actions

$$(a, \alpha) \cdot x = a \cdot x + \alpha x, \qquad x \cdot (a, \alpha) = x \cdot a + \alpha x.$$

If we equip $\mathcal{A}_+ = \mathcal{A} \oplus \mathbb{C}$ with the operator space structure $\mathcal{A} \oplus_1 \mathbb{C}$ as defined in Section 2.2, then $\mathcal{A}_+$ becomes a completely contractive Banach algebra (see [12]). With this structure on $\mathcal{A}_+$ we also have the following useful result:

Lemma 2.4.2. *([12]) If X is a left (right, or two sided) operator $\mathcal{A}$-module, then X is a neounital left (right, or two sided) operator $\mathcal{A}_+$-module.*

If $\mathcal{A}$ is a (completely contractive) Banach algebra and X is a Banach (operator) $\mathcal{A}$-bimodule, a linear map $D \in \mathcal{B}(\mathcal{A}, X)$ ($D \in \mathcal{CB}(\mathcal{A}, X)$) is called a **(completely) bounded derivation** if D satisfies the following Leibniz rule:

$$D(ab) = a \cdot D(b) + D(a) \cdot b, \qquad \forall a, b \in \mathcal{A}.$$

We say that D is **inner** if there exists some $x \in X$ so that

$$D(a) = a \cdot x - x \cdot a, \qquad \forall a \in \mathcal{A}.$$

A Banach algebra $\mathcal{A}$ is called **amenable** if for any dual Banach $\mathcal{A}$-bimodule X^*, we have that every bounded derivation $D : \mathcal{A} \to X^*$ is inner. Similarly, a completely contractive Banach algebra $\mathcal{A}$ is called **operator amenable** if for any dual operator $\mathcal{A}$-bimodule X^*, we have that every *completely bounded* derivation $D : \mathcal{A} \to X^*$ is inner. We note the obvious fact that a completely contractive Banach algebra that is amenable as a Banach algebra is automatically operator amenable.

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2.5   Exact Sequences

In Chapter 5 we will need the notion of an exact sequence of Banach spaces and/or Banach $\mathcal{A}$-modules over some Banach algebra $\mathcal{A}$. We will briefly discuss these notions here.

Consider a fixed sequence

$$\Sigma : 0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

of Banach spaces X, Y and Z where f and g are bounded linear maps. We say that Σ is **short exact** if $\ker f = 0$, $g(Y) = Z$, and $\mathrm{ran} f = \ker g$. If Σ is short exact, we say that Σ is **admissible** if there exist bounded linear maps $F : Y \to X$ and $G : Z \to Y$ so that $F \circ f = id_X$ and $g \circ G = id_Z$. In other words, Σ is exact if there exists a bounded left inverse for f and a bounded right inverse for g.

If $\mathcal{A}$ is a Banach algebra, X, Y and Z are (left, right or two-sided) Banach $\mathcal{A}$-modules, and f and g are Banach $\mathcal{A}$-module maps, we say that an admissible short exact sequence of the form Σ **splits** if the left and right inverses F and G can be taken to be Banach $\mathcal{A}$-module maps as well. In [8], it is shown that the existence of a bounded left inverse F for f implies the existence of a bounded right inverse G for g and visa versa. Furthermore F is a Banach $\mathcal{A}$-module map if and only if G is. In particular, to check that a short exact sequence Σ is admissible, it suffices to find only one of the inverses for f or g.

2.6   Abstract Harmonic Analysis

Our main references for this section are [13] and [14].

Let G be a locally compact group with unit e. In abstract harmonic analysis we are interested in studying algebras of functions and linear operators associated to G. Of fundamental importantance is the action of G on spaces of functions on G by translation: If $f : G \to \mathbb{C}$ is a function on G and $x \in G$, we define the **left translate** of f, denoted by $\delta_x * f$, and the **right translate** of f, denoted by $f * \delta_x$, to be the functions

$$(\delta_x * f)(s) = f(x^{-1}s), \qquad (f * \delta_x)(s) = f(sx).$$

Inversion in G gives rise to two important involutions on spaces of functions on G: if $f : G \to \mathbb{C}$ is a function we define $\tilde{f}$ and $\check{f}$ by

$$\tilde{f}(s) = \overline{f(s^{-1})}, \qquad \check{f}(s) = f(s^{-1}).$$

If X is a space of functions on G, we say that X is **left (respectively right) translation invariant** if $\delta_g * f \in X$ (resp. $f * \delta_g \in X$) for all $f \in X$ and $g \in G$. We say that X is **translation invariant** if X is both left and right translation invariant.

We denote by $C_b(G)$ the commutative C*-algebra of bounded continuous complex-valued functions on G, equipped with the supremum norm $\|\cdot\|_\infty$. $C_c(G)$ will denote the subalgebra of $C_b(G)$ of compactly supported continuous functions on G. A function $\phi \in C_b(G)$ is called **left (right) uniformly continuous** if the map $x \mapsto \delta_x * \phi$ ($x \mapsto \phi * \delta_x$) is continuous from G into $C_b(G)$. There are several C*-subalgebras of $C_b(G)$ that will be of interest to us. We will list the most important ones here:

$C_0(G)$: The algebra of continuous complex-valued functions on G vanishing at infinity.

$LUC(G)$: The algebra of left uniformly continuous functions on G.

$RUC(G)$: The algebra of right uniformly continuous functions on G.

$UC(G) = LUC(G) \cap RUC(G)$: The algebra of uniformly continuous functions on G.

It is not difficult to see that a function $\phi \in C_b(G)$ belongs to $LUC(G)$ (respectively $RUC(G)$) if and only if for any net $\{x_\alpha\}_\alpha \subseteq G$ converging to $e \in G$, we have $\lim_\alpha \|\delta_{x_\alpha} * \phi - \phi\|_\infty = 0$ (respectively $\lim_\alpha \|\phi * \delta_{x_\alpha} - \phi\|_\infty = 0$).

Let $M(G)$ denote the Banach space of all complex regular Borel measures on G. By the Reisz representation theorem $M(G)$ is isometrically isomorphic to $C_0(G)^*$ via the duality

$$\langle f, \mu \rangle = \int_G f d\mu, \qquad \forall f \in C_0(G), \ \mu \in M(G).$$

$M(G)$ is a Banach $*$-algebra. Multiplication in $M(G)$ is given by **convolution**: if μ and $\nu \in M(G)$, we define $\mu * \nu \in M(G)$ by

$$\int_G f d(\mu * \nu) = \int_G \int_G f(xy) d\mu(x) d\nu(y), \qquad \forall f \in C_0(G).$$

The involution on $M(G)$ is given by the map $\mu \mapsto \mu^*$ where

$$\int_G f d\mu^* = \overline{\int_G \tilde{f}, d\mu}, \qquad \forall f \in C_0(G).$$

$M(G)$ is called the **measure algebra of** G.

Let m be a fixed left Haar measure on G. For $1 \leq p \leq \infty$ we let $L^p(G) = L^p(G, m)$. When $p = 1$, $L^1(G)$ is called the **group algebra** or **convolution algebra**. The dual space of $L^1(G)$ is the commutative von Neumann algebra $L^\infty(G)$, the duality being given by

$$\langle f, \phi \rangle := \int_G f(x)\phi(x)dx, \qquad f \in L^1(G), \ \phi \in L^\infty(G),$$

where $\int_G \cdots dx$ denotes integration over G with respect to the Haar measure m. $L^1(G)$ can be isometrically identified with the subspace of $M(G)$ consisting of those $\mu \in M(G)$ that are absolutely continuous with respect to the Haar measure m. With this identification, $L^1(G)$ becomes a closed self-adjoint ideal in $M(G)$. If $\mu \in M(G)$ and $f, g \in L^1(G)$, then $\mu * f$, $f * \mu$, $f * g$ and f^* are given for m-almost every $x \in G$ by the formulae

$$
\begin{aligned}
(\mu * f)(x) &= \int_G f(y^{-1}x)d\mu(y), \\
(f * \mu)(x) &= \int_G f(xy^{-1})\Delta(y^{-1})d\mu(y), \\
(f * g)(x) &= \int_G g(y^{-1}x)f(y)dy, \\
f^*(x) &= \tilde{f}(x)\Delta(x^{-1}),
\end{aligned}
$$

where $\Delta : G \to (0, \infty)$ is the Haar modular function for G. Note the very important fact that $M(G)$ is always unital with unit δ_e and $L^1(G)$ always admits a contractive two-sided bounded approximate identity.

The spaces $L^\infty(G)$, $C_b(G)$, $LUC(G)$, $RUC(G)$ and $UC(G)$ can all be viewed as Banach $M(G)$-modules via the following left and right actions:

$$
(\mu * \phi)(x) = \int_G \phi(y^{-1}x)d\mu(y), \qquad (\phi * \mu)(x) = \int_G \phi(xy^{-1})d\mu(y).
$$

The Banach spaces $L^p(G)$ for $1 < p < \infty$ are also Banach $M(G)$-modules with left and right actions given pointwise m-almost everywhere for $f \in L^p(G)$ and $\mu \in M(G)$ by

$$
(\mu * f)(x) = \int_G f(y^{-1}x)d\mu(y), \qquad (f * \mu)(x) = \int_G f(xy^{-1})\Delta(y^{-1})d\mu(y).
$$

A locally compact group G is called **amenable** if there exists a left-translation invariant state $m \in L^\infty(G)^*$. That is, a state m such that

$$
\langle \delta_x * \phi, m \rangle = \langle \phi, m \rangle, \qquad \forall x \in G, \ \phi \in L^\infty(G).
$$

All compact and abelian locally compact groups are amenable, as well as any nilpotent or solvable group. The free group on n generators $\mathbb{F}_n$ is not amenable for any $n \geq 2$. The class of amenable locally compact groups has nice stability properties with respect to quotients, subgroups, increasing unions and extensions by amenable groups.

A **continuous unitary representation** of a locally compact group G is a continuous homomorphism $\pi : G \to \mathcal{U}(\mathcal{H})$, where $\mathcal{U}(\mathcal{H})$ denotes the group of unitary operators on the

Hilbert space $\mathcal{H}$, equipped with the relative weak operator topology from $\mathcal{B}(\mathcal{H})$.

Given a continuous unitary representation $\pi : G \to \mathcal{U}(\mathcal{H})$, we can define a map $\pi_1 : L^1(G) \to \mathcal{B}(\mathcal{H})$ by forming the weak operator topology convergent integral

$$\pi_1(f) = \text{WOT} - \int_G f(x)\pi(x)dx.$$

That is, we define π_1 by demanding that

$$\langle \pi_1(f)\xi|\eta \rangle := \int_G f(x)\langle \pi(x)\xi|\eta \rangle dx, \qquad \forall f \in L^1(G), \ \xi, \eta \in \mathcal{H}.$$

For any such π, π_1 is a non-degenerate $*$-representation of $L^1(G)$ on $\mathcal{H}$. Furthermore $\|\pi_1(f)\| \leq \|f\|_1$ for all $f \in L^1(G)$.

Conversely, given a non-degenerate $*$-representation $\sigma : L^1(G) \to \mathcal{B}(\mathcal{H})$, there exists a unique continuous unitary representation $\pi : G \to \mathcal{U}(\mathcal{H})$ which is defined on the dense subspace $\sigma(L^1(G))\mathcal{H}$ of $\mathcal{H}$ by the formula

$$\pi(s)\sigma(f)\xi = \sigma(\delta_s * f)\xi, \qquad \forall s \in G, \ f \in L^1(G), \ \xi \in \mathcal{H}.$$

Furthermore, it can be shown that $\pi_1 = \sigma$. Thus there is a one-to-one correspondence between non-degenerate $*$-representations of $L^1(G)$ and continuous unitary representations of G.

Two continuous unitary representations of G that are of particular importance in abstract harmonic analysis are the **left regular representation** $\lambda : G \to \mathcal{U}(L^2(G))$ and the **right regular representation** $\rho : G \to \mathcal{U}(L^2(G))$. λ and ρ are given by

$$\lambda(x)\xi = \delta_x * \xi, \quad \rho(x)\xi = \Delta(x)^{-1/2}(\xi * \delta_x) \qquad \forall x \in G, \ \xi \in L^2(G).$$

Note that the associated $*$-representation $\lambda_1 : L^1(G) \to \mathcal{B}(L^2(G))$ is given by the convolution

$$\lambda_1(f)\xi = f * \xi, \qquad \forall f \in L^1(G), \ \xi \in L^2(G),$$

and is always injective.

Since $\|\pi_1\| \leq 1$ for all continuous unitary representations $\pi : G \to \mathcal{U}(\mathcal{H})$ and we know there exists at least one injective non-degenerate $*$-represenation of $L^1(G)$ (λ_1 for example), we can define a new norm $\|\cdot\|_*$ on $L^1(G)$ by setting

$$\|f\|_* = \sup \big\{\|\pi_1(f)\| : \ \pi \text{ is a continuous unitary representation of } G\big\}.$$

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This is a C*-norm on $L^1(G)$ and we define the **enveloping C*-algebra** of G to be completion of $L^1(G)$ with respect to this norm. We denote this C*-algebra by

$$C^*(G) = \overline{L^1(G)}^{\|\cdot\|_*}.$$

We also define the **reduced group C*-algebra of** G to be

$$C^*_\lambda(G) = \overline{\lambda_1(L^1(G))}^{\|\cdot\|_{\mathcal{B}(L^2(G))}} \subseteq \mathcal{B}(L^2(G)).$$

2.6.1 Fourier Algebras and Fourier-Stieltjes Algebras

Let Σ_G denote the collection of all continuous unitary representations of G and let

$$B(G) = \{u \in C_b(G) : \ u = \langle \pi(\cdot)\xi|\eta\rangle \text{ for some } \pi \in \Sigma_G, \ \xi, \ \eta \in \mathcal{H}_\pi\}.$$

Given $\pi \in \Sigma_G$, a function of the form $u = \langle \pi(\cdot)\xi|\eta\rangle$ for some ξ and $\eta \in \mathcal{H}_\pi$ is called a **coefficient function** of the representation π. It follows from the existence of tensor products and direct sums of unitary representations that $B(G)$ is in fact a pointwise algebra of continuous functions on G. Indeed, if $u_1 = \langle \pi_1(\cdot)\xi_1|\eta_1\rangle \in B(G)$, $u_2 = \langle \pi_2(\cdot)\xi_2|\eta_2\rangle \in B(G)$ and $\alpha \in \mathbb{C}$, then

$$\begin{aligned}
\alpha u_1 + u_2 &= \langle (\pi_1 \oplus \pi_2)(\cdot)(\alpha\xi_1 \oplus \xi_2)|\eta_1 \oplus \eta_2\rangle \in B(G), \\
u_1 u_2 &= \langle (\pi_1 \otimes \pi_2)(\cdot)(\xi_1 \otimes \xi_2)|\eta_1 \otimes \eta_2\rangle \in B(G).
\end{aligned}$$

$B(G)$ can be identified with the Banach space dual of $C^*(G)$ via the dual pairing

$$\langle f, u \rangle := \int_G f(x)u(x)dx = \langle \pi_1(f)\xi|\eta\rangle$$

for $f \in L^1(G)$ and $u = \langle \pi(\cdot)\xi|\eta\rangle \in B(G)$. Equipped with the norm $\| \cdot \|_{B(G)} = \| \cdot \|_{C^*(G)^*}$, $B(G)$ becomes a commutative regular semisimple Banach algebra under pointwise operations which we call the **Fourier-Stieltjes algebra** of G. $B(G)$ is also a Banach *-algebra with isometric involution given by pointwise complex conjugation.

Observe that if λ is the left regular represntation of G and $\xi, \eta \in L^2(G)$, then the coefficient function $u = \langle \lambda(\cdot)\xi|\eta\rangle$ is precisely the convolution $\overline{\eta} * \check{\xi}$. The **Fourier algebra** of G, denoted by $A(G)$, is defined to be the norm closed subspace of $B(G)$ generated by the coefficient functions of the left regular representation $\lambda : G \to \mathcal{U}(L^2(G))$. That is, we

put

$$A(G) = \overline{\mathrm{span}\{u = \langle\lambda(\cdot)\xi|\eta\rangle = \overline{\eta} * \check{\xi} : \ \xi, \eta \in L^2(G)\}}^{\|\cdot\|_{B(G)}}$$

It is a highly important and non-trivial fact ([13]) that $A(G)$ is a closed ideal in $B(G)$ and can be realized as exactly the collection of coefficient functions of λ. That is,

$$A(G) = \{u = \langle\lambda(\cdot)\xi|\eta\rangle = \overline{\eta} * \check{\xi} : \ \xi, \eta \in L^2(G)\}.$$

Furthermore, the norm of $u \in A(G)$ can be realized as

$$\|u\|_{A(G)} := \|u\|_{B(G)} = \inf\{\|\xi\|_2\|\eta\|_2 : u = \overline{\eta} * \check{\xi}\}.$$

$A(G)$ is a self-adjoint commutative regular semisimple Banach algebra contained in $C_0(G)$, and its spectrum is topologically isomorphic to G ([13]).

The **group von Neumann algebra** of G, denoted by $VN(G)$, is defined as the WOT-closure of $\lambda_1(L^1(G))$ in $\mathcal{B}(L^2(G))$. $VN(G)$ can equivalently be realized as

$$VN(G) = \lambda(G)'' = \rho(G)' = \overline{\mathrm{span}\{\lambda(x) : \ x \in G\}}^{\mathrm{WOT}} = \overline{C_\lambda^*(G)}^{WOT}.$$

$VN(G)$ can be be identified with $A(G)^*$, the duality being given by

$$\langle u, T\rangle = \langle T\xi|\eta\rangle, \qquad \forall T \in VN(G), \ u = \overline{\eta} * \check{\xi} \in A(G).$$

From the above duality, it follows that the weak*-topology that $VN(G)$ inherits as the dual of $A(G)$, the weak operator topology on $VN(G)$ and the σ-weak operator topology on $VN(G)$ all coincide.

Since $L^1(G) \cong L^\infty(G)_*$ and $A(G) \cong VN(G)_*$ are preduals of von Neumann algebras, and since $M(G) \cong C_0(G)^*$ and $B(G) \cong C^*(G)^*$ are duals of C*-algebras, the algebras $L^1(G)$, $M(G)$, $A(G)$ and $B(G)$ all admit natural operator space structures through these dualities. Furthermore, with these operator space structures, $L^1(G)$, $M(G)$, $A(G)$ and $B(G)$ turn out to be completely contractive Banach algebras and the inclusions $L^1(G) \hookrightarrow M(G)$ and $A(G) \hookrightarrow B(G)$ are complete isometries.

Note that $VN(G)$, being the operator space dual of the commutative completely contractive Banach algebra $A(G)$, is a natural symmetric dual operator $A(G)$-bimodule. The module action being given by

$$\langle v, u \cdot T\rangle := \langle uv, T\rangle, \qquad \forall T \in VN(G), \ u, v \in A(G).$$

The operator space structure of a group algebra is such that $L^1(G) \cong \text{MAX}(L^1(G))$ completely isometrically and $L^1(G \times H) \cong L^1(G) \otimes^\gamma L^1(H) \cong L^1(G) \widehat{\otimes} L^1(H)$ completely isometrically for all locally compact groups G and H. The operator space structure of a Fourier algebra is considerably more subtle. Given locally compact groups G and H, $A(G) \cong \text{MAX}(A(G))$ completely isomorphically, and $A(G) \otimes^\gamma A(H) \cong A(G \times H)$ isomorphically if and only if G has an abelian subgroup of finite index (see [52] and [38] respectively). Note however that $A(G \times H) \cong A(G) \widehat{\otimes} A(H)$ completely isometrically for all locally compact groups G and H ([11]).

2.6.2 Multipliers of The Fourier Algebra

A linear map $\Gamma : A(G) \to A(G)$ is called an $A(G)$-**multiplier** if $\Gamma(uv) = u\Gamma v$ for all $u, v \in A(G)$. The linear space of multipliers is denoted by $MA(G)$. An application of the closed graph theorem and the regularity of the function algebra $A(G)$ shows that $MA(G) \subseteq \mathcal{B}(A(G))$, and that to each $\Gamma \in MA(G)$ there corresponds a unique $\phi \in C_b(G)$ such that $\Gamma u = \phi u$ for all $u \in A(G)$. Equipped with the norm inherited from $\mathcal{B}(A(G))$, $MA(G)$ becomes a self-adjoint Banach algebra of bounded continuous functions on G. $\Gamma \in MA(G)$ is called a **completely bounded** $A(G)$-**multiplier** if $\Gamma \in \mathcal{CB}(A(G))$. The space of completely bounded $A(G)$-multipliers is denoted by $M_{cb}A(G)$. Equipped with the norm inherited from $\mathcal{CB}(A(G))$, $M_{cb}A(G)$ also becomes a self-adjoint Banach algebra of bounded continuous functions on G. Note that in general we have the following contractive inclusions:

$$A(G) \subseteq B(G) \subseteq M_{cb}A(G) \subseteq MA(G) \subseteq C_b(G),$$

and it is known ([37]) that the amenability of G is characterized by the equality

$$MA(G) = B(G).$$

2.6.3 The Coset Ring and Idempotents in $B(G)$

Let G be a locally compact group. The **coset ring** of G, denoted by $\Omega(G)$, is the ring of subsets of G generated by left cosets of open subgroups of G. The **closed coset ring** of G, denoted by $\Omega_c(G)$, is the collection of all the elements in $\Omega(G_d)$ which are closed in G. Here, G_d denotes the abstract group G equipped with the discrete topology. Not that since any open coset of locally compact group G is automatically closed, it follows that $\Omega(G) \subset \Omega_c(G)$.

The structure of $\Omega_c(G)$ for locally compact abelian groups has been intensively studied in [21]. A generalization of the results in [21] to arbitrary locally compact groups has been

obtained in [16] and [17]. We summarize these results in the following theorem.

Theorem 2.6.1. *([21], [16], [17]) Let G be a locally compact group. A set $X \subseteq G$ belongs to $\Omega_c(G)$ if an only if there exist closed subgroups $H_1, \ldots, H_n \leq G$, $g_1, \ldots, g_n \in G$ and $\Delta_i \in \Omega(H_i)$ for $1 \leq i \leq n$ such that*

$$X = \bigcup_{i=1}^{n} g_i(H_i \backslash \Delta_i).$$

Furthermore if G is amenable, then every set $X \in \Omega_c(G)$ is a set of spectral synthesis for the Fourier algebra $A(G)$.

The following theorem due to Cohen [7] (for locally compact abelian groups) and Host [27] (for general locally compact groups) demonstrates the importance of the coset ring $\Omega(G)$ in the study of idempotents in the Fourier-Stieltjes algebra $B(G)$:

Theorem 2.6.2. *(**Cohen-Host Idempotent Theorem**) Let G be a locally compact group and $X \subseteq G$. Then the characteristic function 1_X belongs to $B(G)$ if and only if $X \in \Omega(G)$.*

2.7   Amenability and the Fourier Algebra

The amenability of a locally compact group G manifests itself in several interesting ways through properties of the Fourier algebra and its multiplier algebras. We summarize some of these results in the following theorem:

Theorem 2.7.1. *Let G be a locally compact group. The the following are equivalent:*
(1) G is amenable.
(2) $L^1(G)$ is an (operator) amenable (completely contractive) Banach algebra.
(3) $A(G)$ is an operator amenable completely contractive Banach algebra.
(4) $A(G)$ has a bounded approximate identity.
(5) $B(G) = MA(G)$ isometrically as Banach algebras, and $M_{cb}A(G) = B(G)$ completely isometrically as completely contractive Banach algebras.

The above list of equivalences is due to several mathematicians over several decades. The equivalence $(1) \Leftrightarrow (2)$ was first proved by Johnson in [28], $(1) \Leftrightarrow (3)$ is due to Ruan in [44], $(1) \Leftrightarrow (4)$ is known as Leptin's theorem [47], and the equivalence $(1) \Leftrightarrow (5)$ is due to both Losert in [37] and some unpublshed work of Z.-J. Ruan.

2.8 Commutative Harmonic Analysis

Throughout this thesis we will use the abbreviation LCA for the words "locally compact abelian". Let G be a LCA group and let Γ denote the collection of equivalence classes of irreducible continuous unitary representations of G. Because G is abelian, Schur's Lemma guarantees that every irreducible continuous unitary representation of G is one dimensional and any two such representations are unitarily equivalent if and only if they are equal. Therefore Γ can be realized as the set of continuous homomorphisms $\gamma : G \to \mathbb{T}$. Elements of Γ are called the **characters** of G. Given $\gamma \in \Gamma$ and $x \in G$ we will frequently denote $\gamma(x)$ by $\langle x, \gamma \rangle$. Γ is an abelian group under pointwise multiplication of characters. We call Γ the **Pontryagin dual group** of G and denote it also by $\widehat{G}$.

The dual group Γ becomes an LCA group if we define a basis $\mathcal{O}$ of open subsets of Γ as follows: For every $r > 0$ and every compact set $K \subseteq G$ let $\mathcal{U}_{K,r} = \{ \gamma \in \Gamma : \ |1 - \langle x, \gamma \rangle| < r, \ \forall x \in K \}$. $\mathcal{O}$ is then defined to be the collection of all such $\mathcal{U}_{K,r}$'s and their translates.

There is a natural embedding $\Phi : G \hookrightarrow \widehat{\widehat{G}} = \widehat{\Gamma}$ given by $\langle \gamma, \Phi(x) \rangle = \langle x, \gamma \rangle$ for all $\gamma \in \Gamma$, $x \in G$. The following celebrated duality theorem of L.S. Pontryagin ([46], Theorem 1.7.2) says that Φ is in fact an isomorphism:

Theorem 2.8.1. *Let G be a LCA group with Pontryagin dual $\widehat{G}$. Then G is topologically isomorphic to $\widehat{\widehat{G}}$ via the canonical map Φ.*

Let $H \leq G$ be a closed subgroup of a locally compact abelian group G. Define

$$H^\perp := \{ \gamma \in \widehat{G} : \ \langle H, \gamma \rangle = \{1\} \}.$$

Then $H^\perp$ is a closed subgroup of $\widehat{G}$. A consequence of the Pontryagin duality theorem is the duality between compact abelian groups and discrete abelian groups, and the duality between closed subgroups and quotient groups of LCA groups. We summarize these results in the following theorem:

Theorem 2.8.2. *Let G be a LCA group. Then $\widehat{G}$ is discrete iff G is compact. If H is a closed subgroup of G, then $\widehat{H} \cong \widehat{G}/H^\perp$ and $H \cong H^{\perp\perp}$.*

When G is abelian, the algebras $M(G)$ and $L^1(G)$ are both commutative. Given $\mu \in M(G)$, the **Fourier-Stieltjes transform** of μ is the function $\widehat{\mu} \in C_b(\Gamma)$ defined by

$$\widehat{\mu}(\gamma) := \int_G \langle x, \gamma^{-1} \rangle d\mu(x).$$

The map $\mathcal{F} : M(G) \to C_b(\Gamma)$ given by $\mu \mapsto \widehat{\mu}$ is called the **Fourier-Stieltjes transform**. $\mathcal{F}$ is in fact an injective $*$-homomorphism whose range is precisely the Fourier-Stieltjes

algebra $B(\Gamma)$. Furthermore $\|\widehat{\mu}\|_{B(G)} = \|\mu\|_{M(G)}$, and hence the Fourier-Stieltjes transform $\mathcal{F} : M(G) \to B(\Gamma)$ yields a (completely) isometric $*$-isomorphism between $M(G)$ and $B(\Gamma)$.

Restricting the Fourier-Stieltjes transform to $L^1(G)$, we obtain the **Fourier transform**

$$\mathcal{F}f(\gamma) = \widehat{f}(\gamma) := \int_G f(x)\langle x, \gamma^{-1}\rangle dx, \qquad \forall f \in L^1(G), \ \gamma \in \Gamma.$$

The Fourier transform is in fact a completely isometric $*$-isomorphism between $L^1(G)$ and $A(\Gamma)$ and $\mathcal{F}$ can be identified with the Gelfand transform for the commutative Banach algebra $L^1(G)$.

In order to make the $L^1(G)$-$L^\infty(G)$ duality more compatible with the Fourier and Fourier-Steiltjes transforms, it is customary in the commutative harmonic analysis literature (see for example [42] and [46]) to use the following modified duality formula: for G abelian, $f \in L^1(G)$, $\phi \in L^\infty(G)$ we set

$$\langle f, \phi \rangle := \int_G f(x)\check{\phi}(x)dx = \int_G f(x)\phi(x^{-1})dx.$$

For any LCA group G with Pontryagin dual Γ, the Fourier-Stieltjes transform allows us to identify the algebras $VN(\Gamma)$, $C^*(\Gamma)$, $MA(\Gamma)$ and $M_{cb}A(\Gamma)$ with familiar spaces of functions and measures on G. In particular, the Fourier-Stieltjes transform yields the following completely isometric isomorphisms:

$$\begin{aligned}
VN(\Gamma) &\cong L^\infty(G), \\
C^*(\Gamma) &\cong C^*_\lambda(\Gamma) \cong C_0(G), \\
M_{cb}A(\Gamma) &= MA(\Gamma) = B(\Gamma) \cong M(G).
\end{aligned}$$

2.8.1 The Bohr Compactification

Let G be a LCA group with dual Γ and let Γ_d denote the group Γ endowed with the discrete topology. Let $\overline{G} := \widehat{\Gamma_d}$. By Theorem 2.8.2, $\overline{G}$ is a compact abelian group and is called the **Bohr compactification** of G. The map $\beta : G \to \overline{G}$ given by

$$\langle \gamma, \beta(x) \rangle := \langle x, \gamma \rangle, \qquad \forall x \in G, \ \gamma \in \Gamma,$$

is a continuous injection of G into $\overline{G}$ with dense range.

Let $AP(G) = \overline{\mathrm{span}\{\gamma : \ \gamma \in \Gamma\}}^{\|\cdot\|_\infty} \subseteq C_b(G)$. $AP(G)$ is called the **algebra of almost periodic functions** on G. $AP(G)$ is a commutative C*-subalgebra of $UC(G)$ and is canonically isomorphic to $C(\overline{G})$, the C*-algebra of continuous complex valued functions

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on the Bohr compactification of G.

Chapter 3

Ideals in Commutative Group Algebras

Let G be a locally compact abelian group with Pontryagin dual group $\Gamma := \widehat{G}$. In this chapter, we will investigate necessary and sufficient conditions for a closed ideal $I \subseteq L^1(G)$ to be complemented in $L^1(G)$, as well as necessary and sufficent conditions for a closed ideal $I \subseteq L^1(G)$ to have a bounded approximate identity.

Recall that when G is abelian, the Fourier transform gives an isomorphism of Banach algebras $A(\Gamma) \cong L^1(G)$. Therefore the following analysis of closed ideals in commutative group algebras can be equally regarded as an analysis of ideals in Fourier algebras over abelian locally compact groups. The results of this chapter will therefore provide us with motivation and intuition in later chapters when we study studying ideals in the Fourier algebra of an arbitrary locally compact group G.

Our primary references for the material of this chapter are [36], [42] and [45].

3.1 Complemented and Weakly Complemented Ideals in Commutative Group Algebras

Let G be a LCA group with dual group Γ and let $I \subseteq L^1(G)$ be a closed ideal. Recall that the *hull* of I is the closed subset of Γ defined by

$$hI = \{x \in \Gamma : \ \hat{f}(x) = 0 \ \forall f \in I\},$$

where the map $f \mapsto \hat{f}$ is the Fourier (Gelfand) transform $\mathcal{F} : L^1(G) \to A(\Gamma)$. Our purpose in this section is to study the relationship between the existence of a Banach space complement

for I in $L^1(G)$ and the structure of the closed set hI.

We will begin with some preliminaries. The following proposition shows that the closed ideals in $L^1(G)$ are in one-to-one correspondence with the closed translation-invariant subspaces of $L^1(G)$.

Proposition 3.1.1. *Let $I \subseteq L^1(G)$ be a closed subspace. Then I is an ideal if and only if I is translation-invariant.*

Proof. Suppose first that I is closed and translation-invariant. Let $f \in I$, $g \in L^1(G)$, and let $\phi \in I^\perp$. Then for any $x \in G$,

$$(f * \check{\phi})(x) = \int_G f(y)\check{\phi}(y^{-1}x)dy = \int_G f(y)\phi(x^{-1}y)dy = \int_G f(xy)\phi(y)dy = 0.$$

That is, $f * \check{\phi} = 0$. Therefore $(g * f) * \check{\phi} = 0$, and in particular

$$0 = (f * g * \check{\phi})(e) = \int_G (f * g)(y)\check{\phi}(y)dy = \langle f * g, \phi \rangle.$$

Since $\phi \in I^\perp$ was arbitary, it follows from the Hahn-Banach theorem that $f * g \in I$.

Conversely suppose that I is a closed ideal. Let $x \in G$, $f \in I$, and let $\{e_\alpha\}_\alpha$ be a bounded approximate identity for $L^1(G)$. We want to show that $\delta_x * f \in I$. Since I is an ideal, $(e_\alpha * \delta_x) * f \in I$ for all α. Since I is closed, $\delta_x * f = \lim_\alpha e_\alpha * (\delta_x * f) = \lim_\alpha (e_\alpha * \delta_x) * f \in I$. $\square$

In our study of ideals in commutative group algebras, there is a special class of complemented ideals which will be of particular importance to us. These are the so-called *invariantly complemented* ideals.

Definition 3.1.2. *Let $I \subseteq L^1(G)$ be a closed ideal. We say that I is **invariantly complemented** in $L^1(G)$ if there exists a bounded projection $P : L^1(G) \to I$ such that*

$$P(g * f) = g * Pf,$$

*for all $f, g \in L^1(G)$. We call such a projection P an **invariant projection** onto I.*

In module theoretic language, the invariantly complemented ideals in $L^1(G)$ are simply the complemented ideals $I \subseteq L^1(G)$ for which there exists a projection $P : L^1(G) \to I$ which is also a Banach $L^1(G)$-module map. It should be obvious that there is nothing special here about $L^1(G)$ and that the notion of an invariantly complemented (left, right, or two-sided) ideal makes sense in any Banach algebra $\mathcal{A}$.

The following lemma shows that Banach $L^1(G)$-module maps from $L^1(G)$ into $L^1(G)$ are in one-to-one correspondence with the bounded G-module maps from $L^1(G)$ into $L^1(G)$. Furthermore, all such maps arise as convolution operators coming from measures in $M(G)$.

Lemma 3.1.3. *Let G be a locally compact group and let $T \in \mathcal{B}(L^1(G))$. Then the following are equivalent:*

(1) T commutes with translations. That is $T(\delta_x * f) = \delta_x * Tf$ for all $x \in G$ and $f \in L^1(G)$.

(2) T is a Banach $L^1(G)$-module map. That is $T(g * f) = g * Tf$ for all $f, g \in L^1(G)$.

(3) There exists a measure $\mu \in M(G)$ such that $Tf = f * \mu$ for all $f \in L^1(G)$.

Proof. (1) $\Rightarrow$ (2). Let $T \in \mathcal{B}(L^1(G))$ satisfy (1) and let $\phi \in L^\infty(G)$ be arbitrary. Since T is bounded, the map $f \mapsto \int_G (Tf)(x)\phi(x)dx$ defines a bounded linear functional on $L^1(G)$. Therefore there exists some $\psi \in L^\infty(G)$ so that $\int_G (Tf)(x)\phi(x)dx = \int_G f(x)\psi(x)dx$ for all $f \in L^1(G)$. Then for any $f, g \in L^1(G)$ we have

$$
\begin{aligned}
\int_G (T(f*g))(x)\phi(x)dx &= \int_G (f*g)(x)\psi(x)dx \\
&= \int_G \int_G f(y)(\delta_y * g)(x)\psi(x)dydx \\
&= \int_G f(y) \int_G T(\delta_y * g)(x)\phi(x)dxdy \\
&= \int_G f(y) \int_G (\delta_y * Tg)(x)\phi(x)dxdy \\
&= \int_G \int_G f(y)(\delta_y * Tg)(x)\phi(x)dydx \\
&= \int_G (f * Tg)(x)\phi(x)dx.
\end{aligned}
$$

Since $\phi \in L^\infty(G)$ and $f, g \in L^1(G)$ were arbitary, this shows that $T(f*g) = f * Tg$ for all $f, g \in L^1(G)$.

(2) $\Rightarrow$ (3). Suppose that T satisfies (2). Let $\{e_\alpha\}_\alpha$ be a bounded approximate identity for $L^1(G)$ and let $\mu \in M(G)$ be any weak*-cluster point of the bounded net $\{Te_\alpha\}_\alpha$. Without loss of generality, we may assume that $\mu = w^* - \lim_\alpha T(e_\alpha)$. Now, since for any fixed $\nu \in M(G)$, the map $\sigma \mapsto \sigma * \nu$ is weak*-weak* continuous from $M(G)$ into $M(G)$, we get for any $f \in L^1(G)$

$$
\begin{aligned}
Tf &= w^* - \lim_\alpha T(f * e_\alpha) \quad \text{(in fact this is a norm limit)} \\
&= w^* - \lim_\alpha f * Te_\alpha \\
&= f * \mu.
\end{aligned}
$$

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$(3) \Rightarrow (1)$. If there exists a measure $\mu \in M(G)$ such that $Tf = f * \mu$ for all $f \in L^1(G)$, then for any $x \in G$, we get $T(\delta_x * f) = (\delta_x * f) * \mu = \delta_x * (f * \mu) = \delta_x * Tf$. $\square$

If we apply Lemma 3.1.3 to invariant projections onto ideals in $L^1(G)$, we get the following simple characterization of invariantly complemented ideals:

Theorem 3.1.4. *Let $I \subseteq L^1(G)$ be a closed ideal. Then I is invariantly complemented in $L^1(G)$ if and only if there exists an idempotent measure $\mu \in M(G)$ (i.e. $\mu * \mu = \mu$) such that the map $f \mapsto f * \mu$ is a projection onto I.*

Proof. Obviously if such an idempotent measure μ exists, then I is invariantly complemented by the projection $P_\mu f = f * \mu$.

Conversely, if I is invariantly complemented by an invariant projection P, then Lemma 3.1.3 implies that there exists a measure $\mu \in M(G)$ such that $Pf = f * \mu$ for all $f \in L^1(G)$. To see that μ is idempotent, first observe that for any $\nu \in M(G)$ and any $f \in C_c(G)$, $f * \nu \in L^1(G) \cap C_b(G)$. In particular, $(f * \nu)(e)$ is well defined for all $f \in C_c(G)$ and $\nu \in M(G)$. Now, since $P^2 = P$, we get, for any $f \in C_c(G)$

$$
\begin{aligned}
\int_G f(y)d\mu(y) &= (\check{f} * \mu)(e) \\
&= (P\check{f})(e) \\
&= (P^2 \check{f})(e) \\
&= (\check{f} * \mu * \mu)(e) \\
&= \int_G f(y)d(\mu * \mu)(y).
\end{aligned}
$$

Since any measure in $M(G)$ is uniquely determined by how it integrates $C_c(G)$, the above equation indicates that $\mu = \mu * \mu$. $\square$

Now that we have seen how the invariantly complemented ideals in $L^1(G)$ can easily be characterized in terms of idempotent measures, a natural question that we would like to answer is the following: For which LCA groups G are all complemented ideals invariantly complemented? As we will see, it turns out that it is necessary and sufficient for G to be compact:

Proposition 3.1.5. *Let G be a compact abelian group. A closed ideal $I \subseteq L^1(G)$ is complemented if and only if I is invariantly complemented.*

Proof. Let $Q : L^1(G) \to I$ be an arbitrary bounded projection onto I and let dg denote the Haar measure on G normalized so that $\int_G dg = 1$. We will construct from Q another

projection $P : L^1(G) \to I$ by averaging against the Haar measure dg. More precisely, define a linear map $P : L^1(G) \to L^1(G)$ by setting

$$Pf = \int_G \delta_g * (Q(\delta_{g^{-1}} * f))dg, \qquad \forall f \in L^1(G).$$

Note that the above vector valued integral exists as a consequence of the compactness of G and norm continuity of the function $g \mapsto \delta_g * (Q(\delta_{g^{-1}} * f))$ (see [14] Theorem A3.1). Furthermore $\|P\| \leq \|Q\|$, so P is bounded. We will now show that P is an invariant projection onto I. If $f \in I$, then since I is translation invariant and Q is a projection onto I, it is clear that $Pf = f$. Therefore $I \subseteq \mathrm{ran}P$ and $P\big|_I = id_I$. Conversely since I is closed and translation invariant, we have $\mathrm{ran}P \subseteq \overline{\mathrm{ran}Q} = \overline{I} = I$. Therefore $\mathrm{ran}P = I$ and P is a projection onto I. To see that P is an invariant projection, it suffices by Lemma 3.1.3 to check that $P(\delta_x * f) = \delta_x * Pf$ for all $x \in G$ and all $f \in L^1(G)$. So fix $x \in G$ and $f \in L^1(G)$. Then we have (using the invariance of Haar measure)

$$\begin{aligned}
P(\delta_x * f) &= \int_G \delta_g * (Q(\delta_{g^{-1}x} * f))dg \\
&= \int_G \delta_{xg} * (Q(\delta_{g^{-1}} * f))dg \\
&= \int_G \delta_x * (\delta_g * (Q(\delta_{g^{-1}} * f)))dg \\
&= \delta_x * \int_G \delta_g * (Q(\delta_{g^{-1}} * f))dg \\
&= \delta_x * Pf,
\end{aligned}$$

and the proof is complete. $\qquad\square$

Combining Proposition 3.1.5 and Theorem 3.1.4 yields the following characterization of the complemented ideals in $L^1(G)$ for compact abelian G:

Corollary 3.1.6. *Let G be a compact abelian group. A closed ideal $I \subseteq L^1(G)$ is complemented iff there exists an idempotent measure $\mu \in M(G)$ such that $I = L^1(G) * \mu$.*

We will now show that the converse to Proposition 3.1.5 is also true, and consequently automatic invariant complementation for complemented ideals characterizes compact abelian groups.

Proposition 3.1.7. *Let G be a LCA group. Then G is compact if and only if every complemented ideal in $L^1(G)$ is invariantly complemented.*

Proof. If G is compact, then every complemented ideal in $L^1(G)$ is invariantly complemented by Proposition 3.1.5. Conversely suppose that every complemented ideal in $L^1(G)$

is invariantly complemented. Let Γ denote the dual group of G, let e_Γ be its unit, and consider the cofinite dimensional (and hence complemented) ideal $I(\{e_\Gamma\}) = \{f \in L^1(G) : \hat{f}(e_\Gamma) = 0\}$. Since $I(\{e_\Gamma\})$ is (by assumption) invarintly complemented, Corollary 3.1.6 implies that there is an idempotent measure $\mu \in M(G)$ such that $I(\{e_\Gamma\}) = L^1(G) * \mu$. Taking Fourier transforms, we have $\widehat{I(\{e_\Gamma\})} = \{\hat{f} \in A(\Gamma) : \hat{f}(e_\Gamma) = 0\} = A(\Gamma) \cdot \hat{\mu}$ where $\hat{\mu}$ (being the Fourier-Steiltjes transform of an idempotent measure) is an indicator function in $B(\Gamma)$. Since $A(\Gamma)$ is a regular function algebra ([13]), it follows that $\hat{\mu} = 1_{\Gamma \setminus \{e_\Gamma\}}$. Since $\hat{\mu}$ is continuous on Γ, $\Gamma \setminus \{e\}$ must be open in Γ and so Γ is discrete. By the Pontryagin duality theorem, it follows that G is compact. $\square$

Remark: It is worthwile to note that Theorem 3.1.7 remains true when we drop the hypothesis that G is an abelian group. That is, it can be shown that any locally compact group G is compact if and only if every complemented (left, right, or two-sided) ideal in $L^1(G)$ is invariantly complemented. See [45] regarding this.

We have so far been able to characterize the complemented ideals in $L^1(G)$ for compact abelian groups G in terms of the idempotent measures in $M(G)$. However when G is not compact, the Haar measure on G is no longer finite and thus the crucial averaging argument used in Proposition 3.1.5 no longer works to enable us to obtain invariant projections from arbitrary projections. To get around this issue, we recall that any LCA group G is amenable and therefore the algebra $L^\infty(G)$ always admits a translation-invariant mean $m : L^\infty(G) \to \mathbb{C}$ ([47]). We can use this mean to average projections onto translation invariant subspaces of $L^\infty(G)$ in a weak*-sense to obtain translation invariant ones. This idea is the content of the following theorem, whose usefulness will become apparent very shortly.

Theorem 3.1.8. *Let $B \subseteq A \subseteq L^\infty(G)$, with B a weak*-closed translation-invariant subspace of $L^\infty(G)$ and let A be a norm-closed translation-invariant subspace of $L^\infty(G)$. If $P : A \to B$ is a continuous projection, then there exists a continuous projection $Q : A \to B$ such that $Q(\delta_x * \phi) = \delta_x * Q\phi$ for all $x \in G$ and all $\phi \in A$.*

Proof. Recall that for $\phi \in L^\infty(G)$ and $f \in L^1(G)$, $\langle f, \phi \rangle = \int_G f(x)\phi(x^{-1})dx$ denotes the dual pairing between $L^1(G)$ and $L^\infty(G)$. Let $m \in L^\infty(G)^*$ be a left invariant mean on $L^\infty(G)$. For notational convenience, we shall view m as a finitely additive measure on G. Now, for fixed $\phi \in L^\infty(G)$, the map

$$f \mapsto \int_G \langle f, \delta_x * P(\delta_{x^{-1}} * \phi) \rangle dm(x),$$

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defines a bounded linear functional on $L^1(G)$ with norm dominated by $\|P\|\|\phi\|$. Let $Q\phi$ denote the associated element in $L^\infty(G)$, as specified by the above dual pairing. This gives us a linear operator $Q : A \to L^\infty(G)$, with $\|Q\| \le \|P\|$. We now show that Q is our required map.

Let us first show that Q is a projection onto B. Since B is weak*-closed, it follows from the Hahn-Banach theorem that $B_\perp^\perp = B$. Let $f \in B_\perp$ and $\phi \in A$. Since A and B are translation-invariant and $\mathrm{ran}P = B$, we have $\langle f, \delta_x * P(\delta_{x^{-1}} * \phi)\rangle = 0$ for each $x \in G$. Therefore

$$0 = \int_G \langle f, \delta_x * P(\delta_{x^{-1}} * \phi)\rangle dm(x) = \langle f, Q\phi\rangle.$$

Since $f \in B_\perp$ was arbitrary it follows that $Q\phi \in B_\perp^\perp = B$. That is $\mathrm{ran}Q \subseteq B$. Next suppose that $\phi \in B$. Then $\delta_x * \phi = P(\delta_x * \phi)$ for all $x \in G$, hence

$$\begin{aligned}
\langle f, Q\phi\rangle = \int_G \langle f, \delta_x * P(\delta_{x^{-1}} * \phi)\rangle dm(x) &= \int_G \langle f, (\delta_x * \delta_{x^{-1}} * \phi)\rangle dm(x)\\
&= \int_G \langle f, \phi\rangle dm(x)\\
&= \langle f, \phi\rangle,
\end{aligned}$$

for all $f \in L^1(G)$. This means that $Q\phi = \phi$ for all $\phi \in B$ and therefore Q is indeed a projection onto B.

Now let us show that Q commutes with translations. Fix $\phi \in A$, $f \in L^1(G)$, and $y \in G$. Then since

$$\langle f, \delta_y * \psi\rangle = \langle \delta_y * f, \psi\rangle,$$

for any $\psi \in L^\infty(G)$, we get

$$\begin{aligned}
\langle f, \delta_y * Q\phi\rangle &= \int_G \langle \delta_y * f, \delta_x * P(\delta_{x^{-1}} * \phi)\rangle dm(x)\\
&= \int_G \langle f, \delta_y * \delta_x * P(\delta_{x^{-1}} * \phi)\rangle dm(x)\\
&= \int_G \langle f, \delta_{yx} * P(\delta_{(yx)^{-1}} * (\delta_y * \phi))\rangle dm(x)\\
&= \langle f, Q(\delta_y * \phi)\rangle.
\end{aligned}$$

(by the translation invariance of m and A.)

Thus $\delta_y * Q\phi = Q(\delta_y * \phi)$ for all $\phi \in A$ and the proof is complete. $\qquad\square$

If $I \subseteq \mathcal{A}$ is a closed ideal in a Banach algebra $\mathcal{A}$ with annihilator $I^\perp \subseteq \mathcal{A}^*$, we will say that I is **weakly complemented** in $\mathcal{A}$ if there exists a continuous projection $P : \mathcal{A}^* \to I^\perp$.

Note that if I is complemented in $\mathcal{A}$ by a projection Q, then $P := id_{\mathcal{A}^*} - Q^*$ is a projection from $\mathcal{A}^*$ onto $I^\perp$ and so I is also weakly complemented. The converse does not hold in general as can be seen by combining Example 3.1.15 with Theorem 3.3.9 from later in this chapter.

If $I \subseteq L^1(G)$ is a closed ideal and if we let $A = L^\infty(G)$ and $B = I^\perp$ (using the notation of Theorem 3.1.8), we get the following useful corollary concerning the weak complementation of I.

Corollary 3.1.9. *Let G be a LCA group and suppose I is a weakly complemented closed ideal in $L^1(G)$. Then there exists a projection $P : L^\infty(G) \to I^\perp$ such that $\delta_x * P\phi = P(\delta_x * \phi)$ for all $x \in G$ and $\phi \in L^\infty(G)$.*

If G is a LCA group and Γ is the Pontryagin dual for G, we let Γ_d denote the group Γ equipped with the discrete topology. The following lemma shows us that the hulls of weakly complemented ideals in $L^1(G)$ have a very special structure. Namely, if $I \subseteq L^1(G)$ is weakly complemented, then the indicator function 1_{hI} must belong to $B(\Gamma_d)$, the Fourier-Stieltjes algebra of Γ_d:

Lemma 3.1.10. *Let G be a LCA group and let $\overline{G}$ denote its Bohr compactification. Suppose $I \subseteq L^1(G)$ is a weakly complemented closed ideal. Then there exists a measure $\mu \in M(\overline{G})$ such that the Fourier-Stieltjes transform $\hat{\mu}$ of μ is the characteristic function of hI.*

Proof. Since I is weakly complemented, Corollary 3.1.9 implies that there exists a translation invariant projection $P : L^\infty(G) \to I^\perp$. We now claim that $P(UC(G)) \subseteq UC(G)$. Indeed, let $f \in UC(G)$ and observe that

$$\|\delta_x * Pf - Pf\|_\infty = \|P(\delta_x * f - f)\|_\infty \leq \|P\|\|\delta_x * f - f\|_\infty,$$

which tends to 0 as $x \to e$. This shows that $Pf \in C_b(G)$ and furthermore that the map $x \mapsto \delta_x * Pf$ is continuous from G into $C_b(G)$. Consequently, $Pf \in UC(G)$.

Since $P(UC(G)) \subseteq UC(G)$ and the space $AP(G)$ of almost periodic functions on G is a closed subspace of $UC(G)$, the map

$$f \mapsto (Pf)(e),$$

defines a bounded linear functional on $AP(G)$. Since the restriction map $f \mapsto f\big|_G$ from $C(\overline{G})$ to $C(G)$ induces a $*$-isomorphism between $C(\overline{G})$ and $AP(G)$, the Riesz-representation theorem for $C(\overline{G})$ gives a unique measure $\mu \in M(\overline{G})$ such that

$$(Pf)(e) = \int_{\overline{G}} f(y^{-1})d\mu(y),$$

for each $f \in C(\overline{G})$. Furthermore, for any $x \in G$ we have

$$(Pf)(x) = (\delta_{x^{-1}} * Pf)(e) = (P(\delta_{x^{-1}} * f))(e) = \int_{G} f(xy^{-1}) d\mu(y) = (f * \mu)(x).$$

That is, $Pf = f * \mu$ for all $f \in AP(G) \cong C(\overline{G})$. Since $P^2 = P$, $\mu * \mu = \mu$. Since μ is idempotent, $\hat{\mu}$ is an indicator function on Γ_d. It remains to show that $\hat{\mu} = 1_{hI}$.

Let $\chi \in \Gamma$. Since $\Gamma \subseteq AP(G)$, we have, for each $x \in G$

$$(P\chi)(x) = (\chi * \mu)(x) = \int_{G} \chi(y^{-1}x) d\mu(y) = \chi(x) \int_{G} \chi(y^{-1}) d\mu(y) = \chi(x)\hat{\mu}(\chi).$$

That is, $P\chi = \hat{\mu}(\chi)\chi$ for all $\chi \in \Gamma$. Now observe that

$$
\begin{aligned}
\chi \in hI \quad &\Leftrightarrow \quad \chi \in I^{\perp} \\
&\Leftrightarrow \quad P\chi = \chi \\
&\Leftrightarrow \quad \hat{\mu}(\chi)\chi = \chi \\
&\Leftrightarrow \quad \hat{\mu}(\chi) = 1.
\end{aligned}
$$

That is, $\hat{\mu} = 1_{hI}$. $\qquad\square$

A consequence of Lemma 3.1.10 is the fact that for compact abelian groups, weakly complemented closed ideals in $L^1(G)$ are always complemented.

Corollary 3.1.11. *Let G be a compact abelian group and let I be a closed ideal in $L^1(G)$. If I is weakly complemented in $L^1(G)$, then I is complemented in $L^1(G)$.*

Proof. Let $I \subseteq L^1(G)$ be a weakly complemented closed ideal. From Lemma 3.1.10 and its proof we know that there exists a projection $P : L^\infty(G) \to I^{\perp}$ and an idempotent measure $\mu \in M(\overline{G}) = M(G)$ such that $Pf = f * \mu$ for all $f \in AP(G) = C(G)$. Now define $Q : L^1(G) \to I(hI)$ by setting $Qf = f - f * \mu$. Then Q is a continuous invariant projection onto $I(hI)$ with norm dominated by $1 + \|\mu\|_{M(G)}$. Since $\widehat{G} = \Gamma$ is discrete and abelian, $hI \subseteq \Gamma$ is a set of spectral synthesis. Therefore $I = I(hI)$ and we are done. $\qquad\square$

Let us now recall the definition of the **coset ring** and **closed coset ring** for a locally compact group Γ. The coset ring of Γ, denoted by $\Omega(\Gamma)$, is the ring of subsets of Γ generated by left cosets of open subgroups of Γ. The closed coset ring of Γ, denoted by $\Omega_c(\Gamma)$, is the collection of all the elements in $\Omega(\Gamma_d)$ which are closed in Γ. Let us also recall that a set $X \subseteq \Gamma$ belongs to $\Omega_c(\Gamma)$ if an only if there exist closed subgroups $H_1, \ldots, H_n \leq \Gamma$,

$g_1, \ldots, g_n \in \Gamma$ and $\Delta_i \in \Omega(H_i)$ for $1 \le i \le n$ such that

$$X = \bigcup_{i=1}^{n} g_i(H_i \backslash \Delta_i).$$

Finally, recall that the Cohen-Host idempotent theorem says that a set $X \subseteq \Gamma$ belongs to $\Omega(\Gamma)$ if and only if the indicator function 1_X belongs to $B(\Gamma)$.

Using the above coset ring terminology, we can summarize our current results in the following two theorems:

Theorem 3.1.12. *Let G be an arbitrary LCA group, let I be a closed ideal in $L^1(G)$, and let $hI \subseteq \Gamma$ be its hull. If I is weakly complemented in $L^1(G)$, then hI belongs to the closed coset ring $\Omega_c(\Gamma)$. In particular, if I is complemented ideal, then $hI \in \Omega_c(\Gamma)$.*

Proof. By Lemma 3.1.10, there exists an idempotent $\mu \in M(\overline{G})$ such that $\hat{\mu} = 1_{hI} \in B(\Gamma_d)$. By the Cohen-Host idempotent theorem, it follows that $hI \in \Omega(\Gamma_d)$. However, since I is an ideal of continuous functions on Γ, then hI is closed in Γ. Thus $hI \in \Omega_c(\Gamma)$. $\square$

Theorem 3.1.13. *Let G be a compact abelian group and let $I \subseteq L^1(G)$ be a closed ideal. Then the following are equivalent:*

(1) I is complemented.

(2) I is weakly complemented.

(3) $hI \in \Omega(\Gamma)$.

(4) $I = L^1(G) * \mu$ for some idempotent measure $\mu \in M(G)$.

Proof. (1) $\Rightarrow$ (2) and (4) $\Rightarrow$ (1) are obvious. (2) $\Rightarrow$ (3) follows from Theorem 3.1.12 and the fact that $\Omega(\Gamma) = \Omega_c(\Gamma)$ because Γ is discrete. Finally, to prove (3) $\Rightarrow$ (4), let $hI \in \Omega(\Gamma)$. By the Cohen-Host idempotent theorem (Theorem 2.6.2), we have $1_{hI} \in B(\Gamma)$. Let $\mu \in M(G)$ be chosen so that $1 - \hat{\mu} = 1_{hI}$. Then it follows that $I(hI) = L^1(G) * \mu$. Since Γ is discrete and abelian, hI is a set of spectral synthesis, and so $I = I(hI) = L^1(G) * \mu$. $\square$

3.1.1   Some Examples

Let us now look at a few examples where we can use Theorem 3.1.12 to classify complemented ideals in $L^1(G)$ for certain groups in terms of their hulls.

Example 3.1.14. *Let $0 \ne I \ne \ell^1(\mathbb{Z})$ be a non-trivial ideal in $\ell^1(\mathbb{Z}) \cong A(\mathbb{T})$. Then I is complemented in $\ell^1(\mathbb{Z})$ if and only if its hull $hI \subseteq \mathbb{T}$ is finite.*

Proof. First suppose that hI is finite. Then I has finite codimension equal to $\dim(A(\mathbb{T})/I) = |hI|$. Since every subspace of a Banach space with finite codimension is complemented, I

is complemented in $\ell^1(\mathbb{Z})$. Conversely suppose that $0 \neq I \neq \ell^1(\mathbb{Z})$ is a non-trivial complemented ideal. Then since $\emptyset$ and $\mathbb{T}$ are both sets of spectral synthesis, we must have $\emptyset \subsetneq hI \subsetneq \mathbb{T}$. By Theorem 3.1.12, $hI \in \Omega_c(\mathbb{T})$, and so we may apply Theorem 2.6.1 to find closed subgroups $H_1, \ldots, H_n \leq \mathbb{T}$, $g_1, \ldots, g_n \in \mathbb{T}$ and $\Delta_i \in \Omega(H_i)$ for each $i \in \{1, \ldots, n\}$ such that

$$hI = \bigcup_{i=1}^{n} g_i(H_i \backslash \Delta_i).$$

Now suppose, to get a contradiction, that $|hI| = \infty$. Then for some $j \in \{1, \ldots n\}$ we must have $|H_j \backslash \Delta_j| = \infty$. Since $H_j \leq \mathbb{T}$ is a closed subgroup, we either have $H_j = \mathbb{T}$ or H_j is finite. Consequently, $H_j = \mathbb{T}$. Now, because $\Delta_j \in \Omega(H_j) = \Omega(\mathbb{T}) = \{\emptyset, \mathbb{T}\}$ (since $\mathbb{T}$ is connected) and $|H_j \backslash \Delta_j| = \infty$, we conclude that $H_j \backslash \Delta_j = \mathbb{T}$. This contradicts the fact that $hI \neq \mathbb{T}$, consequently hI is finite. □

We will conclude this section by giving an example (without proof) that in general the necessary condition in Theorem 3.1.12 for an ideal $I \subseteq L^1(G)$ to be complemented is not a sufficient condition:

Example 3.1.15. *Let $A = \mathbb{Z} \cup \sqrt{2}\mathbb{Z} \subseteq \mathbb{R}$. Then $I(A)$ is not complemented in $L^1(\mathbb{R})$ even though $A \in \Omega_c(\mathbb{R})$.*

Proof. Refer to [2] for the full details. The basic idea is to observe that $A = \mathbb{Z} \cup \sqrt{2}\mathbb{Z}$ is the union two subgroups $\mathbb{Z}$ and $\sqrt{2}\mathbb{Z}$. It can be shown that if H_1 and H_2 are two closed subgroups of a LCA group Γ for which $H_1 \cap H_2 = \{e_\Gamma\}$ and $I(H_1 \cup H_2) \subseteq A(\Gamma)$ is complemented, then $H_1 H_2$ must be topologically isomorphic, as a subgroup of Γ, to $H_1 \oplus H_2$ (See [2] Theorem 4.4). In particular $H_1 H_2$ must be closed in Γ. In our case we have $H_1 = \mathbb{Z}$ and $H_2 = \sqrt{2}\mathbb{Z}$. But $H_1 + H_2 = \mathbb{Z} + \sqrt{2}\mathbb{Z}$ is a proper dense subgroup of $\mathbb{R}$, and therefore cannot be topologically isomorphic to $\mathbb{Z} \oplus \sqrt{2}\mathbb{Z}$. Therefore $I(A)$ is not complemented in $L^1(\mathbb{R})$, despite the fact that $A \in \Omega_c(\mathbb{R})$. □

In Section 3.3, we will see that although not every ideal $I \subseteq L^1(G)$ with $hI \in \Omega_c(\Gamma)$ is complemented, such ideals are always *weakly* complemented in $L^1(G)$.

3.2 Sufficient Conditions for an Ideal to be Complemented

Over the last few decades, there has been a considerable amount of research directed towards fully classifying complemented ideals in commutative group algebras in terms of their hulls. Theorem 3.1.12 can be seen as the starting point for this classification program. See for example [1], [2], [3], [31], and [32]. Considerable progress has been made, however the full

classification appears to be far from completion. At the present time, there is no known body of techniques which is applicable to all LCA groups to determine if a given hull $X \in \Omega_c(\widehat{G})$ generates a complemented ideal $I(X)$. For example, the complemented ideals in $L^1(\mathbb{R})$ and $L^1(\mathbb{R}^2)$ have been fully classified in terms of their hulls, however the full classification remains open for $\mathbb{R}^n$ for $n \geq 3$ (See [1] and [3].)

Since our primary goal in this thesis is to study the connection between the operator space structure of the Fourier algebra and the structure of its ideals, an in depth study of sufficient conditions for an ideal in a commutative group algebra to be complemented would take us beyond our intended scope. We therefore will content ourselves with the above listed references for this line of research.

3.3 Bounded Approximate Identities in Ideals in Commutative Group Algebras

We will now shift our focus to the existence of bounded approximate identities in closed ideals in commutative group algebras. We shall see that there is a deep connection between the following two questions: (1) When is a closed ideal I in a commutative group algebra weakly complemented? And (2) when does a closed ideal I in a commutative group algebra have a bounded approximate identity? We shall see that for any LCA group G with dual Γ, an ideal in $L^1(G)$ has a bounded approximate identity if and only if it is weakly complemented, and furthermore we weill see that these ideals are all of the form $I = I(X)$ for some set $X \in \Omega_c(\Gamma)$.

We will first begin by recalling some general Banach algebraic notions. Let $\mathcal{A}$ be a Banach algebra and let I be any closed (left, right, or two-sided) ideal in $\mathcal{A}$. Recall that I is said to be **weakly complemented** in $\mathcal{A}$ if there exists a bounded projection $P : \mathcal{A}^* \to I^\perp$. Recall also that when I is a left ideal, $I^\perp$ has a natural dual right Banach $\mathcal{A}$-module structure given by

$$\langle b, \phi \cdot a \rangle := \langle ab, \phi \rangle, \qquad \forall a, b \in \mathcal{A}, \ \phi \in I^\perp.$$

Similarly, when I is a right (or two-sided) ideal, $I^\perp$ has a natural dual left Banach $\mathcal{A}$-module (or dual Banach $\mathcal{A}$-bimodule) structure. Finally, we say that a closed ideal $I \subseteq \mathcal{A}$ is **invariantly weakly complemented** in $\mathcal{A}$ if I is weakly complemented by a projection $P : \mathcal{A}^* \to I^\perp$ which is a Banach $\mathcal{A}$-module homomorphism between $\mathcal{A}^*$ and $I^\perp$.

The following useful proposition shows that for any Banach algebra $\mathcal{A}$ with a bounded two-sided approximate identity, the invariantly weakly complemented ideals in $\mathcal{A}$ are pre-

cisely those ideals with bounded approximate identities. More precisely:

Proposition 3.3.1. *Let $\mathcal{A}$ be a Banach algebra with a two-sided bounded approximate identity and let I be a closed left (or right) ideal in $\mathcal{A}$. Then I is invariantly weakly complemented in $\mathcal{A}$ if and only if I has a bounded right (or left) approximate identity.*

Proof. Let I be a closed left ideal in $\mathcal{A}$ with a bounded right approximate identity $\{e_\alpha\}_\alpha$. Without loss of generality we may assume that $\{e_\alpha\}_\alpha$ converges weak* to some $\Phi \in I^{**}$. Define $S \in \mathcal{B}(I^*, \mathcal{A}^*)$ by setting

$$\langle a, S\phi \rangle = \langle \phi \cdot a, \Phi \rangle, \qquad \forall \phi \in I^*, a \in \mathcal{A}.$$

Let $\iota : I \to \mathcal{A}$ be the canonical inclusion map. Note then that $\iota^* S = id_{I^*}$, since if $x \in I$ and $\phi \in I^*$

$$
\begin{aligned}
\langle x, \iota^* S\phi \rangle &= \langle x, S\phi \rangle \\
&= \langle \phi \cdot x, \Phi \rangle \\
&= \lim_\alpha \langle e_\alpha, \phi \cdot x \rangle \\
&= \lim_\alpha \langle xe_\alpha, \phi \rangle \\
&= \langle x, \phi \rangle.
\end{aligned}
$$

Now define a map $P \in \mathcal{B}(\mathcal{A}^*)$ by setting

$$P\phi = \phi - S\iota^*\phi, \qquad \forall \phi \in \mathcal{A}^*.$$

We claim that P is a projection from $\mathcal{A}^*$ onto $I^\perp$. Indeed for any $x \in I$ and $\phi \in \mathcal{A}$ we have

$$
\begin{aligned}
\langle x, P\phi \rangle &= \langle x, \phi \rangle - \langle x, S\iota^*\phi \rangle \\
&= \langle x, \phi \rangle - \langle x, \iota^* S\iota^*\phi \rangle \quad \text{(since } x \in I) \\
&= \langle x, \phi \rangle - \langle x, \iota^*\phi \rangle \quad \text{(since } \iota^* S = id_{I^*}) \\
&= \langle x, \phi \rangle - \langle x, \phi \rangle \quad \text{(since } x \in I) \\
&= 0.
\end{aligned}
$$

So $P(\mathcal{A}^*) \subseteq I^\perp$. Conversely, if $\phi \in I^\perp$, then $\iota^*\phi = 0$, giving $P\phi = \phi - S\iota^*\phi = \phi$. Thus P is indeed a projection onto $I^\perp$.

To see that $P : \mathcal{A}^* \to I^\perp$ is a right Banach $\mathcal{A}$-module map, let $a, x \in \mathcal{A}$, $\phi \in \mathcal{A}^*$, and

observe that

$$
\begin{aligned}
\langle x, Si^*(\phi \cdot a) \rangle &= \langle i^*(\phi \cdot a) \cdot x, \Phi \rangle \\
&= \langle (i^*\phi) \cdot (ax), \Phi \rangle \quad \text{(since } \iota^* \text{ is an } \mathcal{A}\text{-module map)} \\
&= \langle ax, Si^*\phi \rangle \\
&= \langle x, (Si^*\phi) \cdot a \rangle.
\end{aligned}
$$

Hence

$$
\begin{aligned}
P(\phi \cdot a) &= \phi \cdot a - Si^*(\phi \cdot a) \\
&= \phi \cdot a - (Si^*\phi) \cdot a \\
&= (P\phi) \cdot a,
\end{aligned}
$$

for all $a \in \mathcal{A}$, and $\phi \in \mathcal{A}^*$.

If I is a closed right ideal with a bounded left approximate identity $\{e_\alpha\}_\alpha$, we proceed almost exactly as above, except we now define a map $S' : I^* \to \mathcal{A}^*$ by setting

$$
\langle a, S'\phi \rangle = \langle a \cdot \phi, \Phi \rangle
$$

where $\Phi = w^* - \lim_\alpha e_\alpha \in I^{**}$, and we define our projection $P' : \mathcal{A}^* \to I^\perp$ by setting

$$
P'\phi = \phi - S'i^*\phi, \qquad \forall \phi \in \mathcal{A}^*.
$$

Now suppose that $I \subseteq \mathcal{A}$ is an invariantly weakly complemented closed left ideal, and let $P : \mathcal{A}^* \to I^\perp$ be such an invariant projection. Then $(id_{\mathcal{A}^{**}} - P^*) : \mathcal{A}^{**} \to I^{\perp\perp} = I^{**}$ is a weak*-weak* continuous left Banach $\mathcal{A}$-module map. Let $\{e_\alpha\}_\alpha$ be a two-sided bounded approximate identity for $\mathcal{A}$ and without loss of generality assume that $\Phi \in \mathcal{A}^{**}$ is a weak*-limit of the net $\{e_\alpha\}_\alpha$. Set $u := (id_{\mathcal{A}^{**}} - P^*)\Phi \in I^{**}$ and observe that for any $\phi \in I^*, a \in I$ we have

$$
\begin{aligned}
\langle \phi, a \cdot u \rangle &= \langle \phi \cdot a, (id_{\mathcal{A}^{**}} - P^*)\Phi \rangle \\
&= \langle \phi \cdot a - (P\phi) \cdot a, \Phi \rangle \\
&= \lim_\alpha \langle e_\alpha, \phi \cdot a - (P\phi) \cdot a \rangle \\
&= \lim_\alpha \langle ae_\alpha, \phi - P\phi \rangle \\
&= \langle a, \phi - P\phi \rangle \\
&= \langle a, \phi \rangle.
\end{aligned}
$$

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Therefore $a \cdot u = a$ for all $a \in I$. If we approximate $u \in I^{**}$ weak* by a bounded net of elements in I and then pass to convex combinations of these elements, we obtain a right bounded approximate identity for I.

If I is instead an invariantly weakly complemented right ideal, then the same argument as above applies to give an element $u \in I^{**}$ such that $u \cdot a = a$ for all $a \in I$. So I has a left bounded approximate identity. $\square$

We will now use the above proposition to show that for any LCA group G, every weakly complemented ideal in $L^1(G)$ is automatically invariantly weakly complemented.

Theorem 3.3.2. *Let G be a LCA group and let $I \subseteq L^1(G)$ be a closed ideal. If I is weakly complemented in $L^1(G)$, then I is invariantly weakly complemented in $L^1(G)$.*

Proof. Recall that for LCA groups G, the dual pairing between $L^1(G)$ and $L^\infty(G)$ is specified by

$$\langle f, \phi \rangle := \int_G f(x)\check{\phi}(x)dx, \qquad \forall f \in L^1(G), \ \phi \in L^\infty(G).$$

Observe that with this dual pairing, we have for all $f, g \in L^1(G)$ and $\phi \in L^\infty(G)$

$$
\begin{aligned}
\langle f * g, \phi \rangle &= \int_G \int_G f(y)g(y^{-1}z)\phi(z^{-1})dydz \\
&= \int_G \int_G f(y)g(z)\phi(z^{-1}y^{-1})dzdy \\
&= \int_G f(y)(g * \phi)(y^{-1})dy \\
&= \langle f, g * \phi \rangle.
\end{aligned}
$$

That is, the dual action of $L^1(G)$ on $L^\infty(G)$ is just the convolution product $(f, \phi) \mapsto f * \phi$ for $f \in L^1(G)$ and $\phi \in L^\infty(G)$.

Now suppose that I is a weakly complemented closed ideal in $L^1(G)$. By Corollary 3.1.9, we know that there exists a bounded projection $P : L^\infty(G) \to I^\perp$ that commutes with translations by elements of G, that is $P(\delta_x * \phi) = \delta_x * (P\phi)$ for all $x \in G$ and $\phi \in L^\infty(G)$. Furthermore, we know from the proof of Lemma 3.1.10 that such a projection P must map $UC(G)$ into itself.

Since $L^1(G) * L^\infty(G) \subseteq UC(G)$ (in fact they are equal - see [47]) and $P(UC(G)) \subseteq UC(G)$, it makes sense to define a new map $\tilde{P} : L^\infty(G) \to L^\infty(G)$ by setting

$$\langle f, \tilde{P}\phi \rangle = P(f * \phi)(e), \qquad \forall f \in L^1(G), \ \phi \in L^\infty(G).$$

It is clear that $\tilde{P} \in \mathcal{B}(L^\infty(G))$ with $\|\tilde{P}\| \leq \|P\|$. We will now show that $\tilde{P}$ is a projection onto $I^\perp$ which is also an $L^1(G)$-module map.

To do this, first observe that $P|_{UC(G)}$ commutes with the action of $L^1(G)$ on $UC(G)$. Indeed, since the map $y \mapsto \delta_y * \phi$ is continuous from G into $UC(G)$ for all $\phi \in UC(G)$, the vector-valued integral $\int_G f(y)\delta_y * \phi dy$ exists for all $f \in L^1(G)$ and $\phi \in UC(G)$ ([14] Theorem A3.3). Since continuous linear maps commute with vector-valued integrals ([14]), we have

$$
\begin{aligned}
P(f * \phi) &= P\left(\int_G f(y)\delta_y * \phi dy \right) \\
&= \int_G f(y)P(\delta_y * \phi)dy \\
&= \int_G f(y)\delta_y * (P\phi)dy \\
&= f * P\phi,
\end{aligned}
$$

for all $f \in L^1(G)$ and $\phi \in UC(G)$.

Now let $f \in I$ and let $\{e_\alpha\}_\alpha$ be a bounded approximate identity for $L^1(G)$, then for all $\phi \in L^\infty(G)$ we have

$$
\begin{aligned}
\langle f, \tilde{P}\phi \rangle &= \lim_\alpha \langle f * e_\alpha, \tilde{P}\phi \rangle \\
&= \lim_\alpha P(f * e_\alpha * \phi)(e) \\
&= \lim_\alpha (f * P(e_\alpha * \phi))(e) \quad \text{(since } L^1(G) * L^\infty(G) \subseteq UC(G) \text{ and} \\
&\qquad P|_{UC(G)} \text{ commutes with the action of } L^1(G) \text{ on } UC(G)) \\
&= \lim_\alpha \langle f, P(e_\alpha * \phi) \rangle \\
&= 0 \quad \text{(since } P \text{ is a projection onto } I^\perp \text{ and } f \in I).
\end{aligned}
$$

Thus $\tilde{P}(L^\infty(G)) \subseteq I^\perp$. Conversely, if $\phi \in I^\perp$, then $f * \phi \in I^\perp$ for all $f \in L^1(G)$ and hence

$$
\langle f, \tilde{P}\phi \rangle = P(f * \phi)(e) = (f * \phi)(e) = \langle f, \phi \rangle, \qquad \forall f \in L^1(G).
$$

Therefore $\tilde{P}$ is a projection onto $I^\perp$. Finally, to see that $\tilde{P}$ is an $L^1(G)$-module map, let

$f, g \in L^1(G)$ and $\phi \in L^\infty(G)$. Then we have

$$
\begin{aligned}
\langle f, g * \tilde{P}\phi \rangle &= \langle f * g, \tilde{P}\phi \rangle \\
&= P(f * g * \phi)(e) \\
&= (f * P(g * \phi))(e) \\
&= \langle f, \tilde{P}(g * \phi) \rangle.
\end{aligned}
$$

This completes the proof. $\qquad\square$

Putting Proposition 3.3.1 and Theorem 3.3.2 together, we obtain the following remarkable corollary:

Corollary 3.3.3. *Let G be a LCA group. Then a closed ideal $I \subseteq L^1(G)$ has a bounded approximate identity if and only if I is weakly complemented in $L^1(G)$.*

We are now ready to begin to relate the structure of the hull of a closed ideal $I \subseteq L^1(G)$ to the existence of a bounded approximate identity in I. With the help of Corollary 3.3.3, we will be able to show that a closed ideal I in $L^1(G)$ has a bounded approximate identity if and only if it is of the of the form $I = I(X)$ for some set $X \in \Omega_c(\Gamma)$.

We will see below that the difficult task in establishing this characterization of ideals with bounded approximate identities lies in proving that for any $X \in \Omega_c(\Gamma)$, the ideal $I(X) \subseteq L^1(G)$ always has a bounded approximate identity. The following approach to this problem is borrowed from [36].

Theorem 3.3.4. *Let $X, Y \subset \Gamma$ be closed sets. If the ideals $I(X)$ and $I(Y)$ both have bounded approximate identities, then so does the ideal $I(X \cup Y)$.*

Proof. Let $\{u_i\}_{i \in I}$ and $\{v_j\}_{j \in J}$ be bounded approximate identities for $I(X)$ and $I(Y)$ respectively. Note that for each i and each j, $w_{(i,j)} := u_i * v_j$ belongs to $I(X \cup Y)$. If we put the product order on the set $I \times J$, then it is easy to see that the net $\{w_{(i,j)}\}_{(i,j) \in I \times J}$ gives us a bounded approximate identity for the ideal $I(X \cup Y)$. $\qquad\square$

The above theorem can easily be extended the case of arbitrary finite unions of hulls.

Corollary 3.3.5. *Let $\{X_i\}_{i=1}^n$ be any finite collection of closed subsets of Γ such that each ideal $I(X_i)$ has a bounded approximate identity. Then the ideal $I(\bigcup_{i=1}^n X_i)$ also has a bounded approximate identity.*

Proof. Just apply Theorem 3.3.4 inductively. $\qquad\square$

Now let Γ be the dual of a LCA group G. Recall Theorem 2.6.1 which states that a set $X \subseteq \Gamma$ belongs to the closed coset ring $\Omega_c(\Gamma)$ if and only if X is a finite union of translates of sets of the form $H\backslash\Delta$, where $H \leq \Gamma$ is a closed subgroup of Γ and $\Delta \in \Omega(H)$, the coset ring of H. In a sense, we may regard the sets of the form $H\backslash\Delta$ as the "basic building blocks" of the closed coset ring $\Omega_c(\Gamma)$.

In the following theorem, we will show that for any closed subgroup $H \leq \Gamma$ and any $\Delta \in \Omega(H)$, the ideal $I(H\backslash\Delta)$ is always complemented in $L^1(G)$:

Theorem 3.3.6. *Let G be a LCA group with Pontryagin dual Γ. Let H be a closed subgroup of Γ and let $X \in \Omega(H)$. Then the ideal $I(X)$ is complemented in $L^1(G)$. In particular, since $H \in \Omega(H)$, the ideal $I(H)$ is always complemented.*

Proof. Let $H^\perp \leq G$ be the annihilator subgroup of H and let $G_1 := G/H^\perp$. By Theorem 2.8.2 we may identify $\widehat{G_1}$ topologically with the closed subgroup $H \leq G$. In particular the identification $\widehat{G_1} \cong H$ is given by the Pontryagin dual pairing

$$\langle xH^\perp, h\rangle := \langle x, h\rangle, \qquad \forall h \in H, \ xH^\perp \in G_1.$$

Now define a map $R : L^1(G) \to L^1(G_1)$ by setting

$$Rf(xH^\perp) = \int_{H^\perp} f(x\xi)d\xi, \qquad \forall xH^\perp \in G_1,$$

where $d\xi$ denotes a Haar measure on $H^\perp$. If we normalize the Haar measure $d(xH^\perp)$ on G_1 so that the Weil formula

$$\int_G f(x)dx = \int_{G_1} \int_{H^\perp} f(x\xi)d\xi d(xH^\perp), \qquad \forall f \in L^1(G), \tag{3.1}$$

always holds, then it is easy to see that R is a well defined linear contraction (refer to [41] for a proof of the Weil formula). Observe also that R, when viewed dually as an operator from $A(\Gamma)$ into $A(H)$, is just a restriction map. That is, $\widehat{Rf} = \hat{f}\big|_H$ for all $f \in L^1(G)$. Indeed, if $f \in L^1(G)$ and $h \in H$, then

$$
\begin{aligned}
\hat{f}(h) &:= \int_G f(x)\langle x, h^{-1}\rangle dx \\
&= \int_{G_1} \int_{H^\perp} f(x\xi)\langle x\xi, h^{-1}\rangle d\xi d(xH^\perp) \quad \text{(by formula (3.1))} \\
&= \int_{G_1} Rf(xH^\perp)\langle xH^\perp, h^{-1}\rangle d(xH^\perp) \\
&=: \widehat{Rf}(h).
\end{aligned}
$$

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We now need to use the concept of a **Bruhat function** for the subgroup $H^\perp$. A continuous function $\phi : G \to (0, \infty)$ is called a Bruhat function for $H^\perp$ if it satisfies the following two conditions:

(i) For each compact subset $K \subset G$, $\operatorname{supp}(\phi|_{KH^\perp})$ is compact, and
(ii)

$$\int_{H^\perp} \phi(x\xi)d\xi = 1, \qquad \forall x \in G.$$

It is well known that Bruhat functions always exist for any closed subgroup H of a locally compact group G. (See for example [47] Proposition 1.2.6).

Now let ϕ be any fixed Bruhat function for the closed subgroup $H^\perp \leq G$ and define a linear extension map $E : L^1(G_1) \to L^1(G)$ by setting

$$Ef(x) = \phi(x)f(xH^\perp), \qquad \forall f \in L^1(G_1), \ x \in G.$$

If $f \in L^1(G_1)$, then $\|Ef\|_1 = \|f\|_1$ as can be easily seen by applying formula (3.1). Therefore E is an isometry and well defined. Furthermore, E is a right inverse for R since $R(Ef(xH^\perp)) = \int_{H^\perp} \phi(x\xi)f(xH^\perp)d\xi = f(xH^\perp)\int_{H^\perp}\phi(x\xi)d\xi = f(xH^\perp)$ for a.e. $x \in G$.

Now suppose that $X \subseteq H$ belongs to $\Omega(H)$. By the Cohen-Host Idempotent Theorem (Theorem 2.6.2) there exists an idempotent $\mu \in M(G_1)$ such that $\hat\mu = 1_X \in B(H)$. Now define $P \in \mathcal{B}(L^1(G))$ by setting

$$Pf = f - E(\mu * Rf).$$

If $f \in L^1(G)$ and $h \in X \subseteq H$, then we have

$$
\begin{aligned}
\widehat{Pf}(h) &= \hat{f}(h) - [E(\mu * Rf)]\hat{}(h) \\
&= \hat{f}(h) - [R(E(\mu * Rf))]\hat{}(h) \\
&\quad \text{(since } h \in H \text{ and hence } [R(E(\mu * Rf))]\hat{}(h) = [E(\mu * Rf)]\hat{}(h)) \\
&= \hat{f}(h) - \widehat{\mu * Rf}(h) \quad \text{(since } R \circ E = id_{L^1(G_1)}) \\
&= \hat{f}(h) - 1_X(h)\widehat{Rf}(h) \\
&= \hat{f}(h) - \hat{f}(h) = 0.
\end{aligned}
$$

Therefore ran$P \subseteq I(X)$. Conversely if $f \in I(X)$ then

$$
\begin{aligned}
\widehat{\mu * Rf} &= 1_X \widehat{Rf} \\
&= 1_X \hat{f}\big|_H = 0.
\end{aligned}
$$

Therefore $Pf = f - 0 = f$ and P is a projection onto $I(X)$. $\qquad\square$

The above theorem has the following easy consequence in terms of the elementary building blocks of $\Omega_c(\Gamma)$:

Corollary 3.3.7. *Let G be a LCA group with dual Γ. Then for any closed subgroup $H \leq \Gamma$, $\Delta \in \Omega(H)$ and $g \in \Gamma$, the ideal $I(g(H \backslash \Delta))$ is complemented in $L^1(G)$.*

Proof. Letting $X = H \backslash \Delta$ in Theorem 3.3.6, we see that $I(H \backslash \Delta)$ is complemented in $L^1(G)$ by a projection P. Now define $Q : L^1(G) \to I(g(H \backslash \Delta))$ by $Qf = g \cdot P(g^{-1} \cdot f)$ (where $(\gamma \cdot f)(x) = \langle x, \gamma \rangle f(x)$ for all $x \in G$ and $\gamma \in \Gamma$). Then Q is easily seen to be the required projection. $\qquad\square$

It is worth pointing out that Corollary 3.3.7 provides us with a way of showing that not every complemented ideal in $L^1(G)$ is necessarily invariantly complemented:

Example 3.3.8. *Let Γ be a LCA group with a non-open closed subgroup H (for example $\Gamma = \mathbb{R}$, $H = \mathbb{Z}$) and let $G = \widehat{\Gamma}$. Then the ideal $I(H) \subset L^1(G)$ is a complemented ideal in $L^1(G)$ but is not invariantly complemented.*

Proof. $I(H)$ is complemented by Corollary 3.3.7. If $I(H)$ was invariantly complemented, Proposition 3.1.4 would imply that $I(H) = L^1(G) * \mu$ for some idempotent measure $\mu \in M(G)$. By taking Fourier transforms it follows that $\hat{\mu} = 1_{\Gamma \backslash H}$ and therefore $\Gamma \backslash H$ is clopen in Γ. This however contradicts the fact that H is not open in Γ. $\qquad\square$

We are now ready to exhibit the relationship between weak complementation of a closed ideal I in a commutative group algebra, the existence of a bounded approximate identity in I, and the structure of its hull hI:

Theorem 3.3.9. *Let G be a LCA group with Pontryagin dual Γ and let I be a closed ideal in $L^1(G)$. Then the following conditions are equivalent:*
 (a) I has a bounded approximate identity.
 (b) I is weakly complemented.
 (c) $hI \in \Omega_c(\Gamma)$.
 (d) $I = I(X)$ for some $X \in \Omega_c(\Gamma)$.

Proof. (a) $\Leftrightarrow$ (b). This is Corollary 3.3.3.

(b) $\Rightarrow$ (c). This is Theorem 3.1.12.

(c) $\Rightarrow$ (d). This is immediate because every element of $\Omega_c(\Gamma)$ is a set of spectral synthesis ([22]).

(d) $\Rightarrow$ (a). Let $X \in \Omega_c(\Gamma)$. Then by Theorem 2.6.1 there exist closed subgroups $H_1, \ldots, H_n \leq \Gamma$, $g_1, \ldots, g_n \in \Gamma$ and $\Delta_i \in \Omega(H_i)$ for $1 \leq i \leq n$ such that

$$X = \bigcup_{i=1}^{n} g_i(H_i \backslash \Delta_i).$$

For each $i \in \{1, \ldots, n\}$ the ideal $I(g_i(H_i \backslash \Delta_i))$ is complemented (and hence weakly complemented) in $L^1(G)$ by Corollary 3.3.7. By Corollary 3.3.3, each ideal $I(g_i(H_i \backslash \Delta_i))$ has a bounded approximate identity. By Corollary 3.3.5, $I(X)$ has a bounded appoximate identity. $\qquad\square$

3.4 Summary

In this chapter we have studied ideals in commutative group algebras. In particular we have focused on the following two questions: Given a LCA group G with dual Γ and a closed ideal $I \subseteq L^1(G)$, (1) When is I complemented in $L^1(G)$? and (2) When does I have a bounded approximate identity? We have discovered that the answers to these questions depend very much on the topological and geometric structure of the *hull* of I, $hI = \{x \in \Gamma : \hat{f}(x) = 0 \;\; \forall f \in I\} \subseteq \Gamma$. Our main results are summarized below:

Let G be a LCA group with Pontryagin dual Γ, then

(a). The ideal $I(H) \subseteq L^1(G)$ is complemented in $L^1(G)$ for any closed subgroup $H \leq \Gamma$. (This is a special case of Corollary 3.3.7).

(b). A closed ideal $I \subseteq L^1(G)$ is complemented in $L^1(G)$ only if $hI \in \Omega_c(\Gamma)$.

(c). G is compact if and only if every complemented ideal $I \subseteq L^1(G)$ is invariantly complemented. (In other words, G is compact if and only if every complemented ideal $I \subseteq L^1(G)$ is of the form $I = L^1(G) * \mu$ for some idempotent $\mu \in M(G)$).

(d). A closed ideal $I \subseteq L^1(G)$ has a bounded approximate identity $\Leftrightarrow$ I is weakly complemented $\Leftrightarrow$ $hI \in \Omega_c(\Gamma)$ (and in particular $I = I(hI)$).

In the next few chapters, we will study ideals in Fourier algebras of general locally compact groups and try to establish to what extent the results (a) - (d) generalize to the noncommutative setting.

Chapter 4

Ideals in Fourier Algebras

In this chapter we will begin to investigate ideals in $A(G)$ for general locally compact groups G. The main purpose of this chapter is to indicate that although it is possible to ask many of the same questions about ideals in $A(G)$ that we asked in Chapter 3 (regardless of whether G is commutative or not), these same questions become substantially more difficult to answer for noncommutative groups G. We will see that there are two major reasons why things become more difficult in the noncommutative setting, these are:

(1) When G is non-abelian, there is no longer a Pontryagin dual group $\widehat{G}$ allowing us to realize $A(G)$ as the group algebra $L^1(\widehat{G})$. This makes it impossible for us to use the averaging techniques developed in Theorems 3.1.4 and 3.1.8 from the last chapter to study ideals in $A(G)$.

(2) When G is non-abelian, G may fail to be amenable and the question of amenability turns out to be intimately related to the ideal theory of $A(G)$. We will see that the failure of the amenability of G can result in many pathologies arising in the study of complemented ideals and ideals with bounded approximate identities in $A(G)$.

4.1 Motivation

Let G be a LCA group and let $\widehat{G}$ denote its Pontryagin dual group. Recall that the Fourier-Stieltjes transform $\mathcal{F}$ defined on $M(\widehat{G})$ by

$$\mathcal{F}\mu(x) = \int_{\widehat{G}} \langle x^{-1}, \gamma \rangle d\mu(\gamma), \qquad \forall \mu \in M(\widehat{G}), \ x \in G,$$

lets us identify the "dual" algebras $M(\widehat{G})$ and $L^1(\widehat{G})$ with the Fourier-Stieltjes algebra $B(G)$ and the Fourier algebra $A(G)$, respectively. Using these identifications together with Pontryagin duality theory, it is easy to restate the major results obtained in Chapter 3 in

terms of the algebras $B(G)$ and $A(G)$. We will record them here for clarity.

Let G be a LCA group, then

(a). For any closed subgroup $H \leq G$, the ideal $I(H) = \{u \in A(G) : u(h) = 0 \ \forall h \in H\}$ is always complemented in $A(G)$. (This is a special case of Corollary 3.3.7).

(b). If a closed ideal $I \subseteq A(G)$ is weakly complemented in $A(G)$, then its hull hI belongs to $\Omega_c(G)$.

(c). G is discrete if and only if every complemented ideal $I \subseteq A(G)$ is invariantly complemented. (In other words, G is discrete if and only if every complemented ideal $I \subseteq A(G)$ is of the form $I = 1_X A(G)$ for some idempotent $1_X \in B(G)$).

(d). A closed ideal $I \subseteq A(G)$ has a bounded approximate identity if an only if I is weakly complemented, and this happens if and only if $I = I(X)$ for some $X \in \Omega_c(G)$.

It is important to note that in the above reformulation of our results from Chapter 3, we have effectively removed the "abelian" concept of Pontryagin duality from the picture and restated things in terms of objects that now make sense for all locally compact groups G - namely the closed coset ring $\Omega_c(G)$, and the algebras $B(G)$ and $A(G)$. This simple but important observation observation enables us to ask the following questions about ideals in $A(G)$ for arbitrary G.

Let G be a locally compact group:

(a$'$). Is the ideal $I(H) \subseteq A(G)$ always complemented in $A(G)$ for any closed subgroup $H \leq G$?

(b$'$). If I is a (weakly) complemented ideal in $A(G)$, does its hull hI necessarily belong to $\Omega_c(G)$?

(c$'$). Is G a discrete group if and only if every complemented ideal $I \subseteq A(G)$ is invariantly complemented?

(d$'$). Which ideals $I \subseteq A(G)$ have bounded approximate identities? Can they be char-

acterized in terms of their hulls or in terms of weak complementation as in the abelian case?

In this chapter we will explore these questions and show that the non-abelian situation is in general quite different from the abelian situation. Our primary references for the material of this chapter are [15], [17] and [18].

4.2   Ideals in $A(G)$ Vanishing on Closed Subgroups

Let G be a locally compact group and let $H \leq G$ be a closed subgroup. In this section, we will study the complementation properties of the closed ideal $I(H) = \{u \in A(G) : \ u\big|_H = 0\} \subset A(G)$.

At this point we know that when G is abelian, $I(H)$ is always complemented (by Corollary 3.3.7). When H is an open subgroup, we can also show that $I(H)$ is always complemented:

Proposition 4.2.1. *Let G be a locally compact group and let $H \leq G$ be an open subgroup. Then the ideal $I(H)$ is complemented $A(G)$. In particular, if G is discrete, then $I(H)$ is complemented in $A(G)$ for any subgroup $H \leq G$.*

Proof. If H is open in G, then $H \in \Omega(G)$. By Theorem 2.6.2, the characteristic function 1_H belongs to $B(G)$ and so $A(G) \cong 1_H A(G) \oplus I(H)$. If G is discrete, then every subgroup is open, and the result follows. $\qquad \square$

We will now endeavor to show that when G is not discrete and not abelian, the ideal $I(H)$ may fail to be complemented for some closed subgroups $H \leq G$. To do this, we will need a little preparatation.

The following proposition provides us with a useful way of characterizing the complemented ideals of the form $I(H)$.

Proposition 4.2.2. *Let H be a closed subgroup of a locally compact group G. Then the ideal $I(H)$ is complemented in $A(G)$ if and only if there exists a continuous linear map $\Gamma : A(H) \to A(G)$ satisfying $\Gamma u\big|_H = u$ for every $u \in A(H)$.*

Proof. First suppose that $P : A(G) \to I(H)$ is a continuous projection. It was shown by Herz ([26] Theorem 1) that the Fourier algebra of a locally compact group behaves very nicely with respect to restriction to subgroups. More precisely, Herz showed that the restriction map $v \mapsto v\big|_H$ from $A(G)$ to $A(H)$ is a continuous quotient map, and furthermore given any $u \in A(H)$, one can choose a $v \in A(G)$ such that $v\big|_H = u$ with $\|u\|_{A(H)} = \|v\|_{A(G)}$.

Now, given $u \in A(H)$, let $v \in A(G)$ be chosen so that $v\big|_H = u$, $\|v\|_{A(G)} = \|u\|_{A(H)}$ and define a map $\Gamma : A(H) \to A(G)$ by setting

$$\Gamma u = v - Pv.$$

Note that $\|\Gamma u\| \leq \|v\| + \|P\|\|v\| = (1 + \|P\|)\|u\|$. Also note that if v_1 is any other extension of $u \in A(H)$, then $v_1 - v \in I(H)$, and so $v_1 - Pv_1 - (v - Pv) = v_1 - v - (P(v_1 - v)) = 0$. Thus Γ is well defined. To see that Γ is linear, let $u_1, u_2 \in A(H)$, $\alpha \in \mathbb{C}$. Let v_1, v_2 be two extensions of u_1 and u_2 respectively. Then $\alpha v_1 + v_2$ is an extension of $\alpha u_1 + u_2$, so

$$\Gamma(\alpha u_1 + u_2) = \alpha v_1 + v_2 - P(\alpha v_1 + v_2) = \alpha \Gamma u_1 + \Gamma u_2.$$

Finally, since $\|\Gamma\| \leq (1 + \|P\|) < \infty$, Γ is bounded. Therefore Γ is the required extension map.

Conversely, suppose $\Gamma : A(H) \to A(G)$ is a continuous linear extension map. For $v \in A(G)$ define $Pv = v - \Gamma(v\big|_H)$. Then P is obviously linear, and $\|P\| \leq 1 + \|\Gamma\|$. Note that $Pv\big|_H = v\big|_H - \Gamma(v\big|_H)\big|_H = v\big|_H - v\big|_H = 0$, so $\mathrm{ran} P \subseteq I(H)$. Finally, if $u \in I(H)$, then $u\big|_H = 0$, so $Pu = u$. Therefore P is the required projection. $\square$

Remark: Note that if G is abelian and H is a closed subgroup of G, then Corollary 3.3.7 and Proposition 4.2.2 tell us that there always exists a continuous linear extension map $\Gamma : A(H) \to A(G)$ such that $\Gamma u\big|_H = u$ for all $u \in A(H)$. The construction is in fact quite explicit for abelian groups: Since $A(G) \cong L^1(\widehat{G})$ and $A(H) \cong L^1(\widehat{G}/H^\perp)$, we can define an extension map $\Gamma : L^1(\widehat{G}/H^\perp) \to L^1(\widehat{G})$ by setting $(\Gamma f)(x) = \phi(x)f(xH^\perp)$ where $\phi : \widehat{G} \to (0, \infty)$ is any Bruhat function for the subgroup $H^\perp \leq \widehat{G}$.

In what follows, we will see that for there to exist a continuous extension map $\Gamma : A(H) \to A(G)$ for a given closed subgroup $H \leq G$ (or equivalently a continuous projection $P : A(G) \to I(H)$), it is necessary for there to be a certain level of compatibility between the Banach space geometries of $A(H)$ and $A(G)$. In particular, we will see that if $A(G)$ satisfies the so-called *Radon-Nykodym property* and $I(H)$ is complemented, then $A(H)$ must also satisfy the Radon-Nykodym Property.

Definition 4.2.3. *A Banach space X is said to have the* **Radon-Nykodym property** **(RNP)** *if for any finite measure space (Ω, Σ, μ) and any μ-continuous vector measure $L : \Sigma \to X$ of bounded total variation, there exists a Bochner integrable function $g : \Omega \to X$*

such that

$$L(E) = \int_E g d\mu, \quad \forall E \in \Sigma.$$

We will also use the notion of complete reducibility of a unitary representation of a group.

Definition 4.2.4. *Let G be a locally compact group, let Σ_G denote the collection of all unitary equivalence classes of continuous unitary representations of G, and let $\widehat{G}$ denote the collection equivalence classes of irreducible continuous unitary representations of G. A representation $\pi \in \Sigma_G$ is said to be **completely reducible** if $\pi = \bigoplus_{\alpha \in A} \sigma_\alpha$, where each $\sigma_\alpha \in \widehat{G}$.*

Remarks: It follows directly from the very definition of the RNP that any closed subspace of a Banach space X with the RNP also has the RNP. It is also not too difficult to show that the RNP is preserved by Banach space isomorphisms. Indeed, suppose $\Phi : X \to Y$ is a Banach space isomorphism, X has the RNP, (Ω, Σ, μ) is a finite measure space, and $L : \Sigma \to Y$ is any μ-continuous vector measure of bounded total variation. Then the map $\Phi^{-1} \circ L : \Sigma \to X$ defines a μ-continuous vector measure of bounded total variation, and so by the RNP for X, there exists a Bochner integrable function $g : \Omega \to X$ such that

$$\Phi^{-1}L(E) = \int_E g d\mu, \quad \forall E \in \Sigma.$$

Setting $h = \Phi \circ g$, we obtain a Bochner integrable function $h : \Omega \to Y$ such that

$$L(E) = (\Phi(\Phi^{-1}L))(E) = \Phi\Big(\int_E g d\mu\Big) = \int_E (\Phi \circ g) d\mu = \int_E h d\mu, \quad \forall E \in \Sigma.$$

Consequently Y has the RNP.

Finally, we note that the RNP for the Fourier algebra of a locally compact group G and the complete reducibility of the left regular representation of G are intimately related. Indeed it was shown in [48] that as a Banach space, $A(G)$ satisfies the RNP if and only if the left regular representation $\lambda : G \to \mathcal{U}(L^2(G))$ is completely reducible.

Since the RNP behaves nicely with respect to closed subspaces and Banach space isomorphisms, we can use Proposition 4.2.2 to give us the following useful result concerning complemented ideals.

Proposition 4.2.5. *Let G be a locally compact group for which $A(G)$ has the RNP and let $H \leq G$ be a closed subgroup. If $I(H)$ is complemented in $A(G)$, then $A(H)$ has the RNP.*

Proof. By Proposition 4.2.2 there exists a continuous linear extension map $\Gamma : A(H) \rightarrow A(G)$ such that $\Gamma u\big|_H = u$ for all $u \in A(H)$. It is easy to see that $\Gamma(A(H))$ is a closed subspace of $A(G)$ and that $\Gamma : A(H) \rightarrow \Gamma(A(H))$ is a Banach space isomorphism. Since $A(G)$ has the RNP, so does the closed subspace $\Gamma(A(H))$. Since $\Gamma(A(H))$ is isomorphic to $A(H)$, then $A(H)$ has the RNP. $\qquad\square$

It turns out that Proposition 4.2.5 is precisely what we need to construct an example of a locally compact group G and a closed subgroup H such that the ideal $I(H)$ is not complemented in $A(G)$.

Let $G = \mathbb{R}_+ \ltimes \mathbb{R}$ be the semi-direct product of the multiplicative group $\mathbb{R}_+ = (0, \infty)$ and $\mathbb{R}$. G is commonly referred to as the "$ax + b$" group. This is because G is naturally identified as the group of affine transformations on $\mathbb{R}$. The identification is given by $G \ni (a, b) \mapsto (x \mapsto ax + b, \quad x \in \mathbb{R})$. Observe that G has a closed normal subgroup H which is isomorphic to $\mathbb{R}$. Namely H is the closed subgroup of G identified with the translations $x \mapsto x + b$ on $\mathbb{R}$.

Example 4.2.6. *Let G and H be as defined above. Then $I(H)$ is not complemented in $A(G)$.*

Proof. It is well known that the left regular representation λ_G of G is completely reducible ([33]), and so by our above remarks $A(G)$ has the RNP. If $I(H)$ was complemented in $A(G)$, then $A(H) \cong A(\mathbb{R})$ must have the RNP, and again by our above remarks, $\lambda_{\mathbb{R}}$ must be completely reducible. But it is well known that for any abelian group H, λ_H is completely reducible if and only if H is compact. Since $\mathbb{R}$ is not compact, $I(H)$ cannot be complemented. $\qquad\square$

The above example shows that there is a fundamental difference between the ideal structure of the Fourier algebra of a noncommutative group and the Fourier algebra of a commutative group. In the noncommutative setting, we have shown that even some of the most elementary ideals - those of the form $I(H)$ - may fail to be complemented in $A(G)$! The pricipal cause for this difference seems to be the fact that for a nonabelian noncompact group G, the left regular representation of G may still be completely reducible - a phenomenon that cannot occur if G is abelian.

4.3 Complemented Ideals in $A(G)$ whose Hulls do not Belong to $\Omega_c(G)$

In Chapter 3 (Theorem 3.1.12) we saw that if G is a LCA group, a necessary condition for an ideal $I \subseteq A(G)$ to be complemented in $A(G)$ is that its hull hI must belong to $\Omega_c(G)$, the closed coset ring of G. In this section we will show by example that this is no longer necessarily true when G is noncommutative. The example we will examine here is due to M. Leinert ([34], [35]) and involves the notion of a *Leinert set* in a discrete group G.

Definition 4.3.1. *Let G be a discrete group with unit e and let $E \subseteq G$ be a subset. E is called a **Leinert set** if for any $n \in \mathbb{N}$ and any $x_1, \ldots, x_{2n} \in E$ satisfying $x_i \neq x_{i+1}$ for $1 \leq i \leq 2n - 1$, we have*

$$x_1^{-1} x_2 x_3^{-1} x_4 \ldots x_{2n-1}^{-1} x_{2n} \neq e.$$

A typical example of a Leinert set in a discrete group G is a **free set** - i.e. a subset of G whose elements are in free relation. For example, the set of generators of any noncommutative free group forms a free set (and hence a Leinert set). Let us now collect some useful facts about Leinert sets, most of which we will present without proof:

Proposition 4.3.2. *Let G be a discrete group and let $E \subseteq G$ be a Leinert set. Then $\ell^2(E) \subseteq VN(G)$ and there exists a constant $C > 0$ so that $\|\lambda(f)\|_{VN(G)} \leq C\|f\|_2$ for all $f \in \ell^2(E)$.*

Proof. See [34]. $\qquad\square$

Recall that the multiplier algebra of $A(G)$ is defined as $MA(G) = \{\phi : G \to \mathbb{C} : \phi u \in A(G) \; \forall u \in A(G)\}$. If we dualize Proposition 4.3.2, we are able to see that for any Leinert set E, the indicator function 1_E belongs $MA(G)$.

Corollary 4.3.3. *Let G be a discrete group and let $E \subseteq G$ be a Leinert set. Then $1_E \in MA(G)$.*

Proof. Let $f \in \ell^1(G)$. By Proposition 4.3.2 there exists a constant $C > 0$ (independent of f) such that

$$\|\lambda(1_E f)\|_{VN(G)} \leq C\|1_E f\|_2 \leq C\|f\|_2 \leq C\|\lambda(f)\|_{VN(G)}.$$

Now let $\langle \cdot, \cdot \rangle$ denote the dual pairing between $A(G)$ and $VN(G)$, let $K(G) = \{v : G \to \mathbb{C} : \text{supp} v \text{ is finite}\} \subset A(G)$, and consider the linear map $u \mapsto 1_E u$ from $K(G)$ into $K(G)$.

Observe that for any $u \in K(G)$ and $f \in \ell^1(G)$, we have

$$|\langle 1_E u, \lambda(f) \rangle| = |\langle u, \lambda(1_E f) \rangle| \leq C \|u\|_{A(G)} \|\lambda(f)\|_{VN(G)}.$$

Consequently the map $u \mapsto 1_E u$ extends to a bounded linear map on $\overline{K(G)}^{\|\cdot\|_{A(G)}} = A(G)$. That is, $1_E \in MA(G)$. $\square$

In [35], Leinert goes even further and shows that for any infinite Leinert set E in a discrete group G, 1_E is a *completely bounded* multiplier of $A(G)$ (that is $1_E \in M_{cb}A(G)$), but that 1_E doeas not belong to $B(G)$. This now brings us to the main example of this section.

Example 4.3.4. *Let $\mathbb{F}_2$ denote the free group on two generators a and b. Let $E = \{a^n b a^{-n} : n \in \mathbb{N}\}$. Then the ideal $I(E) \subseteq A(\mathbb{F}_2)$ is complemented in $A(\mathbb{F}_2)$ even though $E \notin \Omega_c(\mathbb{F}_2)$.*

Proof. It is easy to see that the set E is an infinite free (hence Leinert) set in $\mathbb{F}_2$. Thus, by [35], $1_E \in M_{cb}A(\mathbb{F}_2)$ and $1_E \notin B(\mathbb{F}_2)$. Define $P : A(\mathbb{F}_2) \to I(E)$ by $Pu = (1 - 1_E)u$. Then P is a (completely) bounded projection onto $I(E)$. However, since $1_E \notin B(\mathbb{F}_2)$, we have $E \notin \Omega(\mathbb{F}_2) = \Omega_c(\mathbb{F}_2)$. $\square$

Example 4.3.4 simultaneously provides us with negative answers to both questions (b$'$) and (c$'$) which were posed in Section 4.1. That is, for nonabelian groups G it is not necessary for the hull of a complemented ideal $I \subseteq A(G)$ to belong to $\Omega_c(G)$, and it is not true that every complemented ideal I is of the form $1_X A(G)$ for some idempotent $1_X \in B(G)$ when G is discrete. It should also be noted that the nonamenability of $\mathbb{F}_2$ plays an important role in the construction of Example 4.3.4, since in order to construct sets E in a group G for which $1_E \in MA(G) \backslash B(G)$, Theorem 2.7.1 tells us that G must be apriori non-amenable.

4.4 Bounded Approximate Identities for Ideals in $A(G)$

Let us now consider the existence of bounded approximate identities in ideals in $A(G)$. Recall that for all LCA groups G, an ideal $I \subseteq A(G)$ posesses a bounded approximate identity if and only if its hull hI lies in $\Omega_c(G)$. In this section we will investigate to what extent this result can be generalized to $A(G)$ for all locally compact groups G. As we shall soon see, the amenability of G plays an important role in the existence of a bounded approximate identity in an ideal in $A(G)$.

Let us first begin by proving that $hI \in \Omega_c(G)$ is always a necessary condition for an ideal I in $A(G)$ to have a bounded approximate identity. This result is a partial generalization of Theorem 3.3.9 to arbitrary locally compact groups.

Proposition 4.4.1. *Let X be a closed subset of a locally compact group G. If $I \subseteq A(G)$ is a closed ideal with a bounded approximate identity and $hI = X$, then $X \in \Omega_c(G)$. If G is amenable, then $I = I(X)$.*

Proof. Let $\{u_\alpha\}_\alpha$ be a bounded approximate identity for I and let M be a norm bound for the net $\{u_\alpha\}_\alpha$. If $x \in X$, then $u_\alpha(x) = 0$ for all α. If $x \in G \backslash X$, there exists a function $u \in I$ such that $u(x) = 1$, and therefore

$$
\begin{aligned}
\lim_\alpha |u_\alpha(x) - 1| &= \lim_\alpha |(u_\alpha u)(x) - u(x)| \\
&\leq \limsup_\alpha \|u_\alpha u - u\|_{A(G)} \\
&= 0.
\end{aligned}
$$

That is $u_\alpha \to 1_{G \backslash X}$ pointwise on G. Now for each α, $u_\alpha \in A(G) \subseteq B(G_d)$ where G_d denotes the group G equipped with the discrete topology, and $\|u_\alpha\|_{B(G_d)} \leq \|u_\alpha\|_{B(G)} = \|u_\alpha\|_{A(G)} \leq M$ ([13]). Since bounded subsets of $B(G_d)$ are closed in the pointwise topology ([13] Corollary 2.25), we have $1_{G \backslash X} \in B(G_d)$ and $\|1_{G \backslash X}\|_{B(G_d)} \leq M$. By Theorem 2.6.2, $G \backslash X \in \Omega(G_d)$ which implies that $X \in \Omega(G_d)$. Since X is closed, $X \in \Omega_c(G)$. Finally, if G is amenable, then every element of $\Omega_c(G)$ is a set of spectral synthesis ([16]), giving $I = I(X)$. $\square$

Unfortunately, the converse to Proposition 4.4.1 is not true, at least when G is not an amenable group:

Theorem 4.4.2. *Let G be a locally compact group with unit e. Then the following are equivalent:*
(i). G is amenable.
(ii). The closed ideal $I(\{e\}) \subset A(G)$ has a bounded approximate identity.

Proof. $(ii) \Rightarrow (i)$. Suppose $I(\{e\})$ has a bounded approximate identity $\{u_\alpha\}_\alpha$ such that $\|u_\alpha\|_{A(G)} \leq M$ for each α. Pick $h \in A(G)$ such that $h(e) = 1 = \|h\|_{A(G)}$, and let $v_\alpha = u_\alpha - h u_\alpha + h$. Then $\{v_\alpha\}_\alpha$ is a net in $A(G)$ such that $\|v_\alpha\|_{A(G)} \leq 2M + 1$. Let $\psi \in A(G)$ be arbitrary. Since $\psi - h\psi \in I(\{e\})$ and $\{u_\alpha\}_\alpha$ is a bounded approximate identity for $I(\{e\})$, then $\psi - h\psi = \lim_\alpha u_\alpha \psi - h u_\alpha \psi$. But this means that

$$
\begin{aligned}
\lim_\alpha \|v_\alpha \psi - \psi\|_{A(G)} &= \lim_\alpha \|u_\alpha \psi - h u_\alpha \psi + h\psi - \psi\|_{A(G)} \\
&= \lim_\alpha \|u_\alpha \psi - h u_\alpha \psi - (\psi - h\psi)\|_{A(G)} \\
&= 0.
\end{aligned}
$$

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Therefore $\{v_\alpha\}_\alpha$ is a bounded approximate identity for $A(G)$. By Leptin's theorem (see Theorem 2.7.1), G is amenable.

$(i) \Rightarrow (ii)$. Let G be an amenable group. Then, again by Leptin's theorem, $A(G)$ has bounded approximate identity $\{v_\alpha\}_\alpha$. Now let $\{\mathcal{U}_\beta\}$ be a neighbourhood basis at the identity e (ordered by reverse inclusion). For each $\mathcal{U}_\beta$, choose $h_\beta \in A(G) \cap C_c(G)$ satisfying $\|h_\beta\|_{A(G)} = h_\beta(e) = 1$ and $\mathrm{supp}(h_\beta) \subseteq \mathcal{U}_\beta$. (Note that this can always be done: For each β choose a symmetric neighbourhood V_β of e with compact closure such that $\overline{V_\beta}^2 \subset \mathcal{U}_\beta$. Let $g_\beta = \frac{1_{V_\beta}}{\|1_{V_\beta}\|_2}$ and define $h_\beta = \overline{g}_\beta * \check{g}_\beta$.)

For each pair (α, β) define $u_{(\alpha,\beta)} \in I(\{e\})$ by setting

$$u_{(\alpha,\beta)} = v_\alpha - v_\alpha(e)h_\beta.$$

With the ususal product order on the set $\{(\alpha, \beta)\}$, $\{u_{(\alpha,\beta)}\}_{(\alpha,\beta)}$ is a bounded net in $I(\{e\})$. We now show that in fact this net is a bounded approximate identity for $I(\{e\})$.

Let $\psi \in I(\{e\})$ and let $\epsilon > 0$. By [13] Corollary 4.7, we can find a function $\psi_\epsilon \in A(G) \cap C_c(G)$ such that $\|\psi - \psi_\epsilon\|_{A(G)} < \frac{\epsilon}{4}$ and an open neighbourhood V of e such that $\mathrm{supp}(\psi_\epsilon) \cap V = \emptyset$. Choose α_0 large enough so that for all $\alpha \geq \alpha_0$, $\|v_\alpha \psi - \psi\|_{A(G)} < \frac{\epsilon}{4}$ and let $\mathcal{U}_{\beta_0} = V$. Note that if $\beta \geq \beta_0$, then $\mathrm{supp}(h_\beta) \subseteq \mathcal{U}_\beta \subseteq V$ and hence $h_\beta \psi_\epsilon = 0$. So, for all $(\alpha, \beta) \geq (\alpha_0, \beta_0)$ we have

$$
\begin{aligned}
\|u_{(\alpha,\beta)}\psi - \psi\|_{A(G)} &= \|u_{(\alpha,\beta)}(\psi - \psi_\epsilon) + u_{(\alpha,\beta)}\psi_\epsilon - \psi\|_{A(G)} \\
&\leq \|u_{(\alpha,\beta)}\|_{A(G)}\|\psi - \psi_\epsilon\|_{A(G)} + \|u_{(\alpha,\beta)}\psi_\epsilon - \psi\|_{A(G)} \\
&< 2\frac{\epsilon}{4} + \|u_{(\alpha,\beta)}\psi_\epsilon - \psi\|_{A(G)} \\
&= \frac{\epsilon}{2} + \|v_\alpha \psi_\epsilon - v_\alpha(e)h_\beta\psi_\epsilon - \psi\|_{A(G)} \\
&= \frac{\epsilon}{2} + \|v_\alpha \psi_\epsilon - \psi\|_{A(G)} \quad (\text{since } h_\beta \psi_\epsilon = 0) \\
&\leq \frac{\epsilon}{2} + \|v_\alpha(\psi_\epsilon - \psi)\|_{A(G)} + \|v_\alpha \psi - \psi\|_{A(G)} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \epsilon.
\end{aligned}
$$

Therefore $\{u_{(\alpha,\beta)}\}_{(\alpha,\beta)}$ is indeed a bounded approximate identity for $I(\{e\})$. $\qquad\square$

Theorem 4.4.2 provides us with many examples of closed ideals $I \subseteq A(G)$ with $hI \in \Omega_c(G)$ which do not posses bounded approximate identities: Just take any nonamenable group G with unit e and let $I = I(\{e\})$.

It should also be noted that for any nonamenable locally compact group G, the cofinite

dimensional ideal $I(\{e\})$ is always complemented in $A(G)$. Thus, at least for nonamenable groups, (weak) complementation of an ideal I does not necessarily imply the existence of a bounded approximate identity in I. This stands in contrast to the situation for abelian groups, where weak complementation of an ideal I is equivalent to the fact that $hI \in \Omega_c(G)$ and also to the fact that I has a bounded approximate identity.

4.5 Summary

In this chapter we began to study the structure of ideals in $A(G)$ for general locally compact groups G. Through various examples, we saw that the problem of determining when an ideal in $A(G)$ is complemented or when it has a bounded approximate identity is much more subtle in the non-abelian situation.

We first showed that for general locally compact groups G, ideals in $A(G)$ vanishing on closed subgroups of G may fail to be complemented. We then showed that for certain non-commutative discrete groups G, there exist invariantly complemented ideals in $A(G)$ which are not complemented by idempotents from the Fourier-Steiltjes algebra $B(G)$. Finally, we concluded by showing that for nonamenable groups G, the connection between weak complementation of ideals in $A(G)$, bounded approximate identities in ideals in $A(G)$, and the coset ring $\Omega_c(G)$ that exists for abelian G breaks down.

The point of this chapter was to indicate that if we are to try and understand when an ideal in the Fourier algebra of a general locally compact group is complemented or has a bounded approximate identity, a very different approach from the one we took in Chapter 3 is going to be required. In the next chapter, we will see that if we view the Fourier algebra as an operator space, we can make significant progress in our study of ideals in Fourier algebras for general locally compact groups.

Chapter 5

Operator Space Structure and Ideals in Fourier Algebras

In this chapter we will exploit the natural operator space structure of the Fourier algebra to study its ideals. We will see that by taking into account the operator space structure on $A(G)$, powerful tools from the theory of operator spaces can be used to generalize many results from Chapter 3 to $A(G)$ for large classes of locally compact groups G.

Our primary references for the material of this chapter are [16], [49], [50], [51] and [52].

5.1   Motivation and Overview

Much of what we have studied so far in this thesis has centred around the following two Banach algebraic questions: (1) If $\mathcal{A}$ is a Banach algebra and $I \subseteq \mathcal{A}$ is a closed ideal, then under what conditions is I complemented or weakly complemented in $\mathcal{A}$? (2) If I happens to be complemented or weakly complemented, then when is I complemented or weakly complemented by an invariant projection (i.e. a projection P that is also an $\mathcal{A}$-module map)?

Using the language of short exact sequences (introduced in Section 2.5), we can rephrase these two questions in terms of homological algebra as follows.

Let $I \subseteq \mathcal{A}$ be a closed ideal and consider the short exact sequence sequence of Banach $\mathcal{A}$-modules

$$\Sigma : 0 \longrightarrow I \overset{i}{\longrightarrow} \mathcal{A} \overset{q}{\longrightarrow} \mathcal{A}/I \longrightarrow 0$$

and its dual sequence

$$\Sigma^* : 0 \longrightarrow I^\perp \xrightarrow{\;q^*\;} \mathcal{A}^* \xrightarrow{\;i^*\;} I^* \longrightarrow 0$$

where $i : I \to \mathcal{A}$ is the inclusion map, $q : \mathcal{A} \to \mathcal{A}/I$ is the canonical quotient map and i^*, q^* are their linear adjoints. If we recall the notions of admissibility and splitting for exact sequences defined in Section 2.5, we see that the ideal I is complemented if and only if Σ is an admissible sequence, I is weakly complemented if and only if Σ^* is admissible, I is invariantly complemented if and only if Σ splits, and I is invariantly weakly complemented if and only if Σ^* splits.

It is therefore possible to regard our work so far as a problem in homological algebra concerning the admissibility and splitting of certain exact sequences of Banach $\mathcal{A}$-modules. In order to see why taking this homological perspective is useful, we note the following two results (from [8] and [25] respectively).

Theorem 5.1.1. *Let $\mathcal{A}$ be an amenable Banach algebra and let $I \subseteq \mathcal{A}$ be a closed (left, right, or two-sided) ideal. Then the exact sequence*

$$\Sigma^* : 0 \longrightarrow I^\perp \xrightarrow{\;q^*\;} \mathcal{A}^* \xrightarrow{\;i^*\;} I^* \longrightarrow 0$$

is admissible if and only if I has a (right, left, or two-sided) bounded approximate identity.

Theorem 5.1.2. *Let $\mathcal{A}$ be a Banach algebra, let $m : \mathcal{A} \otimes^\gamma \mathcal{A} \to \mathcal{A}$ denote the multiplication map and let $N = \ker m$. If the exact sequence*

$$\Delta : 0 \longrightarrow N \xrightarrow{\;i\;} \mathcal{A} \otimes^\gamma \mathcal{A} \xrightarrow{\;m\;} \mathcal{A} \longrightarrow 0$$

splits as a sequence of Banach $\mathcal{A}$-bimodules, then every complemented ideal in $\mathcal{A}$ is automatically invariantly complemented in $\mathcal{A}$.

If $\mathcal{A} = A(G)$ where G is a LCA group, then it is well known that $A(G) \cong L^1(\widehat{G})$ is amenable as a Banach algebra and it is also well known that the sequence Δ in Theorem 5.1.2 splits if and only if G is discrete ([47]). Therefore Theorems 5.1.1 and 5.1.2 provide an abstract means of obtaining results for ideals in $A(G)$ that were already obtained in Chapter 3 using averaging techniques with invariant means on $\widehat{G}$. The advantage of this abstract homological approach however is that these techniques, unlike those presented in Chapter 3, are potentially applicable to a wide variety of Banach algebras, including Fourier algebras of locally compact groups.

However, if G is an arbitrary locally compact group, we run into two major problems when trying to apply Theorems 5.1.1 and 5.1.2 to study ideals in $A(G)$. First, $A(G)$ turns out to be amenable as a Banach algebra for only a very small class of groups - those with an abelian subgroup of finite index ([19]). Second, the Banach algebra $A(G) \otimes^\gamma A(G)$ is quite poorly understood when G is not abelian, making it extremely difficult to determine when the sequence Δ in Theorem 5.1.2 splits. In fact, at the present time the sequence Δ is only known to split when G is discrete and the natural algebra homomorphism $\Phi :$ $A(G) \otimes A(G) \to A(G \times G)$ given by $\Phi(u \otimes v)(s,t) = u(s)v(t)$ extends continuously to a Banach algebra isomorphism $A(G) \otimes^\gamma A(G) \cong A(G \times G)$ ([50]). Unfortunately, this latter phenomenon occurs if and only if G has an abelian subgroup of finite index ([37]). So, despite the apparent generality of Theorems 5.1.1 and 5.1.2, when we apply them to the Fourier algebra $A(G)$, they can only be used when G is "almost" abelian!

However, if we choose to view the Fourier algebra as a completely contractive Banach algebra (replacing Banach algebra amenability with operator amenability and the Banach space projective tensor product "$\otimes^\gamma$" with its operator space counterpart "$\widehat{\otimes}$") two remarkable things happen: (1) $A(G)$ is operator amenable if and only if G is an amenable group ([44]), and (2) the homomorphism $\Phi : A(G) \otimes A(H) \to A(G \times H)$ defined above extends to a complete isometry $A(G \times H) \cong A(G)\widehat{\otimes}A(H)$ for any two locally compact groups G and H ([11]). That is, $A(G)$ is amenable as a completely contractive Banach algebra precisely when we should expect it to be, and the projective tensor product $A(G)\widehat{\otimes}A(G)$ is something we understand quite well - it is naturally the Fourier algebra of the product group $G \times G$.

These facts strongly suggest that we should attempt to develop completely contractive Banach algebra analogues of Theorems 5.1.1 and 5.1.2 which can be used to study ideals in $A(G)$ for noncommutative G. In this chapter, we develop the necessary homological tools to do precisely this. We will see in the following sections that these homological results will allow us to obtain several interesting consequences in terms of the ideal structure of $A(G)$.

5.2   Exact Sequences of Operator Modules

In this section we will introduce the basic concepts from homological algebra in the category of completely contractive Banach algebras that we will need. For a more detailed exposition of this theory, see [52].

The principlal objects of study in this chapter will be the so-called **short exact sequences** of operator spaces or operator modules. We define these as follows:

Consider a fixed sequence

$$\Sigma : 0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

of operator spaces X, Y and Z where f and g are completely bounded linear maps. As in the Banach space case discussed in Chapter 2, we say that Σ is **short exact** if $\ker f = 0$, $g(Y) = Z$, and $\operatorname{ran} f = \ker g$. We will call Σ **completely admissible** if there exist completely bounded left and right inverses for f and g respectively. If $\mathcal{A}$ is a completely contractive Banach algebra and X, Y and Z are in addition (left, right or two-sided) operator $\mathcal{A}$-modules, and f, g are $\mathcal{A}$-module maps, then we say that Σ **splits completely** if there exist completely bounded left and right inverses for f and g which are both operator $\mathcal{A}$-module maps. Finally, we call an arbitrary operator $\mathcal{A}$-module map $\phi : X \to Y$ **completely admissible** if there exists a c.b. linear map $\theta : Y \to X$ such that $\phi \circ \theta = id_{\operatorname{ran}\phi}$.

If we are given a short exact sequence of operator spaces

$$\Sigma : 0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0 \ ,$$

it is important to note that on the level of Banach spaces, the isomorphism $Z \cong Y/f(X)$ *always* holds. However as operator spaces, it may very well be that $Z \ncong Y/f(X)$. For example, if X is any Banach space, we can consider the short exact sequence

$$\Sigma : 0 \longrightarrow 0 \longrightarrow \operatorname{MAX}(X) \xrightarrow{id} \operatorname{MIN}(X) \longrightarrow 0 \ .$$

If X is infinite dimensional, then $\operatorname{MAX}(X)/0 = \operatorname{MAX}(X)$ is not completely isomorphic to $\operatorname{MIN}(X)$ ([11]). Thus we see that even on an elementary level, there are subtle differences between the study of exact sequences of Banach spaces and exact sequences of operator spaces.

In order to obtain homological results in the category of operator spaces that parallel the those from the Banach space category, we will need to eliminate the potential for pathologies like the one above. To do this, we will have to restrict the class of short exact sequences of operator spaces we deal with. With this in mind, we make the following definitions:

Definition 5.2.1. *Given two operator spaces X and Y, a c.b. map $T : X \to Y$ has the **complete isomorphism property** if $T(X)$ is closed in Y and the induced map $\tilde{T} : X/\ker T \to T(X)$ is a complete isomorphism.*

Definition 5.2.2. *A short exact sequence of operator spaces*

$$\Sigma : 0 \longrightarrow X \xrightarrow{\ f\ } Y \xrightarrow{\ g\ } Z \longrightarrow 0$$

*is called an **extension sequence** if the maps f and g both have the complete isomorphism property.*

The following elementary lemma shows that the class of extension sequences avoids the isomorphism dilemma for operator spaces discussed above.

Lemma 5.2.3. *Suppose that the short exact sequence*

$$\Sigma : 0 \longrightarrow X \xrightarrow{\ f\ } Y \xrightarrow{\ g\ } Z \longrightarrow 0$$

of operator spaces forms an extension sequence. Then $Y/f(X)$ is completely isomorphic to Z.

Proof. Since Σ is short exact, we have $\operatorname{ran} f = \ker g$. Since g has the complete isomorphism property, $Y/\ker g = Y/f(X)$ is c.b. ismorphic to Z. $\square$

Now let $\mathcal{A}$ be a Banach algebra. We remarked earlier in Chapter 2 that for a short exact sequence

$$\Sigma : 0 \longrightarrow X \xrightarrow{\ f\ } Y \xrightarrow{\ g\ } Z \longrightarrow 0$$

of Banach $\mathcal{A}$-modules, it suffices to find only *one* of either a bounded left inverse for f or a bounded right inverse for g to ensure that Σ is admissible. The following lemma shows that for extension sequences of operator modules over a completely contractive Banach algebra, the appropriate analogue holds.

Lemma 5.2.4. *Let $\mathcal{A}$ be a completely contractive Banach algebra and let*

$$\Sigma : 0 \longrightarrow X \xrightarrow{\ f\ } Y \xrightarrow{\ g\ } Z \longrightarrow 0$$

be an extension sequence of operator $\mathcal{A}$-modules. Then there exists a completely bounded map $F : Y \to X$ such that $Ff = id_X$ if and only if there exists a completely bounded map $G : Z \to Y$ such that $gG = id_Z$. Furthermore F is an $\mathcal{A}$-module map if and only if G is.

Proof. Suppose that F is a completely bounded left inverse for f. Then since $F \circ f = id_X$, the map $f \circ F : Y \to Y$ is a completely bounded projection onto $f(X) \subseteq Y$. Let Q denote the operator space complement of $f(X)$ in Y and consider the map $g\big|_Q : Q \to Z$. Since

$Q \cong Y/f(X)$ and the induced map $\tilde{g} : Y/f(X) \to Z$ is a completely bounded isomorphism, it follows that $g\big|_Q : Q \to Z$ is a c.b. isomorphism. Define $G := (g\big|_Q)^{-1} : Z \to Q \subseteq Y$. Then G is completely bounded and $gGz = g(g\big|_Q)^{-1}z = g\big|_Q(g\big|_Q)^{-1}z = z$ for all $z \in Z$. That is, G is a c.b. right inverse for g.

Now suppose that G is a completely bounded right inverse for g. Since $g \circ G = id_Z$, the map $G \circ g : Y \to Y$ is a completely bounded projection onto a subspace $P \subseteq Y$. Let Q be the operator space complement of P in Y. Then $id_Y - G \circ g$ is a completely bounded projection onto Q. Note that $g \circ (id_Y - G \circ g) = g - g \circ G \circ g = g - g = 0$, so that $Q \subseteq \ker g$, and conversely if $y \in \ker g$, then $Q \ni (id_Y - G \circ g)y = y$. That is, $\ker g = Q$. Since $Q = \ker g = f(X)$ and f has the complete isomorphism property, the map $f^{-1} : Q \to X$ is a completely bounded isomorphism. Now define

$$F : Y \to X,$$

by

$$F(y) = f^{-1}(y - Ggy).$$

By our above discussion F is well defined and completely bounded. Furthermore $Ff(x) = f^{-1}(f(x) - Ggf(x)) = f^{-1}(f(x) - 0) = x$ for all $x \in X$.

It remains to show that F is an $\mathcal{A}$-module map if and only if G is. Suppose first that F is an $\mathcal{A}$-module map. From the above proof of the existence of G, we know that $G = (g\big|_Q)^{-1}$ where $Q = (id_Y - fF)Y$ is the operator space complement of $f(X)$ in Y. Since F is a left $\mathcal{A}$-module map, Q is a submodule of Y. Since $g : Y \to Z$ is a module map, it follows that $G = (g\big|_Q)^{-1}$ is also a module map. Conversely suppose G is a left $\mathcal{A}$-module map (the right and two-sided cases are similar), then from the above formula for F,

$$F(a \cdot y) = f^{-1}(a \cdot y - Gg(a \cdot y)) = a \cdot f^{-1}(y - Ggy) = a \cdot F(y)$$

for all $a \in \mathcal{A}$, $y \in Y$. Therefore F is a left $\mathcal{A}$-module map. $\qquad \square$

From now on we will restrict our attention to extension sequences of operator spaces. In the following sections of this chapter, the two primary examples of extension sequences that we will be interested in are obviously the sequence

$$\Sigma : 0 \longrightarrow I \stackrel{i}{\longrightarrow} \mathcal{A} \stackrel{q}{\longrightarrow} \mathcal{A}/I \longrightarrow 0 \tag{5.1}$$

and its dual

$$\Sigma^* : 0 \longrightarrow I^\perp \xrightarrow{\ q^* \ } \mathcal{A}^* \xrightarrow{\ i^* \ } I^* \longrightarrow 0 \tag{5.2}$$

where $\mathcal{A}$ is is a completely contractive Banach algebra, $I \subseteq \mathcal{A}$ is a closed (left, right, or two-sided) ideal, $i : I \to \mathcal{A}$ is the inclusion map, $q : \mathcal{A} \to \mathcal{A}/I$ is the canonical quotient map and i^*, q^* are their linear adjoints respectively.

5.3 Operator Amenability and Weakly Completely Complemented Ideals in the Fourier Algebra

Let $\mathcal{A}$ be a completely contractive Banach algebra and let $I \subseteq \mathcal{A}$ be a (left, right or two-sided) closed ideal. By analogy with the definitions for a Banach algebra, we say that I is **completely complemented** in $\mathcal{A}$ if there exists a completely bounded projection $P : \mathcal{A} \to I$. We say that I is **weakly completely complemented** if there is a completely bounded projection $Q : \mathcal{A}^* \to I^\perp$. We say that I is **invariantly (weakly) completely complemented** if the c.b. projection $P : \mathcal{A} \to I$ ($Q : \mathcal{A}^* \to I^\perp$) can be chosen to be an $\mathcal{A}$-module map.

In this section we will use the homology tools introduced in the last section to develop an operator space analogue of Theorem 5.1.1. That is, we will prove that for an operator amenable completely contractive Banach algebra $\mathcal{A}$, an ideal $I \subseteq \mathcal{A}$ is weakly completely complemented if and only if I has a bounded approximate identity. Combining this result with Ruan's theorem on the operator amenability of the Fourier algebra, we will be able to prove that for any amenable locally compact group, a necessary condition for an ideal $I \subseteq A(G)$ to be completely complemented in $A(G)$ is that its hull hI belong to $\Omega_c(G)$.

Our main tool for this section is the following theorem, which says that for an operator amenable completely contractive Banach algebra $\mathcal{A}$, certain completely admissable extension sequences of operator $\mathcal{A}$-modules are always guaranteed to split completely.

Theorem 5.3.1. *Let $\mathcal{A}$ be an operator amenable completely contractive Banach algebra and let*

$$\Sigma : 0 \longrightarrow X^* \xrightarrow{\ f \ } Y \xrightarrow{\ g \ } Z \longrightarrow 0$$

be a completely admissible extension sequence of left or right operator $\mathcal{A}$-modules with X^* a dual operator $\mathcal{A}$-module. Then Σ splits completely as a sequence of left or right operator $\mathcal{A}$-modules.

Proof. Suppose that Σ is a completely admissible extension sequence of left operator $\mathcal{A}$-

modules. By complete admissibility, there exists a map $G \in \mathcal{CB}(Z, Y)$ satisfying $gG = id_Z$. Define a map $D : \mathcal{A} \to \mathcal{CB}(Z, Y)$ by the formula

$$Da = a \cdot G - G \cdot a$$

where $(a \cdot G)(z) = a \cdot (Gz)$ and $(G \cdot a)(z) = G(a \cdot z)$ are the natural operator $\mathcal{A}$-bimodule actions on $\mathcal{CB}(Z, Y)$. It is easy to see that $D : \mathcal{A} \to \mathcal{CB}(Z, Y)$ is a completely bounded derivation. Note however that since g is a left $\mathcal{A}$-module map, we have

$$
\begin{aligned}
g[(Da)(z)] &= g[(a \cdot G - G \cdot a)(z)] \\
&= g(a \cdot (Gz)) - gG(a \cdot z) \\
&= a \cdot gGz - gG(a \cdot z) \\
&= a \cdot z - a \cdot z = 0.
\end{aligned}
$$

Thus $D(\mathcal{A}) \subseteq \mathcal{CB}(Z, \ker g) = \mathcal{CB}(Z, \mathrm{im} f)$. Since $f : X^* \to \mathrm{im} f$ is a c.b. left module isomorphism, the map $f^{-1}D : \mathcal{A} \to \mathcal{CB}(Z, X^*) \cong (Z \widehat{\otimes} X)^*$ is a c.b. derivation into the dual operator $\mathcal{A}$-bimodule $(Z \widehat{\otimes} X)^*$. Since $\mathcal{A}$ is operator amenable, $f^{-1}D$ is an inner derivation and therefore there exists some $Q \in \mathcal{CB}(Z, X^*)$ so that

$$f^{-1}Da = a \cdot Q - Q \cdot a$$

Since f is a module map, we have

$$a \cdot G - G \cdot a = Da = f(f^{-1}Da) = f(a \cdot Q - Q \cdot a) = a \cdot fQ - fQ \cdot a.$$

Rearranging this equation gives $a \cdot G - a \cdot fQ = G \cdot a - fQ \cdot a$ for all $a \in \mathcal{A}$. Therefore the map $G' = G - fQ : Z \to Y$ is a left operator $\mathcal{A}$-module map. Furthermore, for all $z \in Z$ we have

$$gG'z = gGz - gfQz = gGz - 0 = z$$

since $gG = id_Z$ and $g \circ f = 0$. That is, $gG' = id_Z$. By Lemma 5.2.4, Σ splits completely.

Now suppose Σ is a completely admissible extension sequence of right operator $\mathcal{A}$-modules. We will reduce this case to the left module case by considering the so called **opposite algebra** of $\mathcal{A}$. The opposite algebra of $\mathcal{A}$, denoted by $\mathcal{A}^{op}$, is defined to be the operator space $\mathcal{A}$ equipped with a new multiplication map $\tilde{m} : \mathcal{A} \widehat{\otimes} \mathcal{A} \to \mathcal{A}$ defined by $\tilde{m}(a \otimes b) = ba$. It is easy to see from the commutativity of the operator space projective tensor product ([11]) that $\mathcal{A}^{op}$ is a completely contractive Banach algebra and that $\mathcal{A}^{op}$

is operator amenable if and only if $\mathcal{A}$ is. Furthermore there is an obvious correspondence between left operator $\mathcal{A}^{op}$-modules and right operator $\mathcal{A}$-modules. Now observe that our sequence Σ can be viewed as a completely admissible extension sequence of left operator $\mathcal{A}^{op}$-modules. From what we have already proven, Σ must split completely as a sequence of left $\mathcal{A}^{op}$-modules. But this just means that Σ splits completely a sequence of right operator $\mathcal{A}$-modules. $\qquad\square$

Applying Theorem 5.3.1 to the context of ideals yields the following corollary.

Corollary 5.3.2. *Let I be a closed left or right ideal in an operator amenable completely contractive Banach algebra $\mathcal{A}$. If I is weakly completely complemented, then I is invariantly weakly completely complemented.*

Proof. Suppose that I is a closed left (right) ideal in $\mathcal{A}$ and consider the extension sequence of right (left) operator $\mathcal{A}$-modules Σ^* defined in (5.2):

$$\Sigma^* : 0 \longrightarrow I^\perp \xrightarrow{q^*} \mathcal{A}^* \xrightarrow{i^*} I^* \longrightarrow 0 \ .$$

Since I is weakly completely complemented, Σ^* is completely admissible. Since $I^\perp \cong (\mathcal{A}/I)^*$ is always a dual operator $\mathcal{A}$-module and $\mathcal{A}$ is operator amenable, Σ^* splits completely as a sequence of right (left) operator $\mathcal{A}$-modules by Theorem 5.3.1. That is, I is invariantly weakly completely complemented. $\qquad\square$

Corollary 5.3.2 now yields our desired operator space version of Theorem 5.1.1.

Theorem 5.3.3. *Let $\mathcal{A}$ be an operator amenable completely contractive Banach algebra and $I \subseteq \mathcal{A}$ a closed (left, right, or two-sided) ideal. Then I is weakly completely complemented in $\mathcal{A}$ if and only if I has a (right, left, or two-sided) bounded approximate identity.*

Proof. Let I be a closed left ideal in $\mathcal{A}$ with a bounded right approximate identity $\{e_\alpha\}_\alpha$. We will proceed exactly as in the proof of Proposition 3.3.1. Without loss of generality we may assume that $\{e_\alpha\}_\alpha$ converges weak* to some $\Phi \in I^{**}$. Define $S \in \mathcal{B}(I^*, \mathcal{A}^*)$ by setting

$$\langle a, S\phi \rangle = \langle \phi \cdot a, \Phi \rangle, \qquad \forall \phi \in I^*, a \in \mathcal{A}.$$

Let $i : I \to \mathcal{A}$ be the canonical injection, $i^* : \mathcal{A}^* \to I^*$ its adjoint, and define $P : \mathcal{A}^* \to I^\perp$ by

$$\langle x, P\phi \rangle = \langle x, \phi \rangle - \langle x, Si^*\phi \rangle, \qquad \forall x \in \mathcal{A}, \phi \in \mathcal{A}^*.$$

It was already shown in Proposition 3.3.1 that P is a bounded projection from $\mathcal{A}$ onto $I^\perp$. It remains now to show that P is completely bounded, or equivalently, that S is completely bounded. To see this, let $[\phi_{ij}] \in b_1(M_n(I^*))$. Then

$$
\begin{aligned}
\|S^{(n)}[\phi_{ij}]\|_n &= \sup_{[a_{kl}]\in b_1(M_n(\mathcal{A}))} \|\langle\langle[S\phi_{ij}],[a_{kl}]\rangle\rangle\| \\
&= \sup_{[a_{kl}]\in b_1(M_n(\mathcal{A}))} \|\langle\Phi,\phi_{ij}\cdot a_{kl}\rangle\|_{M_{n^2}} \\
&= \sup_{[a_{kl}]\in b_1(M_n(\mathcal{A}))} \|\Phi^{(n^2)}[\phi_{ij}\cdot a_{kl}]\|_{M_{n^2}} \\
&\leq \sup_{[a_{kl}]\in b_1(M_n(\mathcal{A}))} \|\Phi^{(n^2)}\|\|[\phi_{ij}\cdot a_{kl}]\|_{M_{n^2}(I^*)} \\
&\leq \|\Phi^{(n^2)}\|\|[\phi_{ij}]\|_n \quad \text{(since I^* is a dual operator $\mathcal{A}$-bimodule)} \\
&= \|\Phi\| \quad \text{(since $\Phi \in I^{**}$ and therefore $\|\Phi\| = \|\Phi\|_{cb}$)},
\end{aligned}
$$

so S is completely bounded.

If I is a closed right ideal with a bounded left approximate identity $\{e_\alpha\}_\alpha$, the argument proceeds exactly as above, except the map $S : I^* \to \mathcal{A}^*$ is now defined by $\langle a, S\phi\rangle = \langle a\cdot\phi,\Phi\rangle$ where $\Phi = w^* - \lim_\alpha e_\alpha \in I^{**}$. If I is a two-sided ideal with a two-sided bounded approximate identity, then this is just a special case of either the left or right ideal case.

Now suppose that $I \subseteq \mathcal{A}$ is a weakly completely complemented closed left (or right) ideal. By Theroem 5.3.2, I is invariantly weakly completely complemented. Since $\mathcal{A}$ is operator amenabe, it has a bounded approximate identity. Proposition 3.3.1 then implies that I has a right (or left) bounded approximate identity.

Finally, if I is a two-sided weakly completely complemented ideal, then by the above arguments, I must have both a left and a right bounded approximate identity. It then follows from standard Banach algebra arguments that I has a two-sided bounded approximate identity. $\square$

We can use Theorem 5.3.3 to gain some new information on the ideal structure of the Fourier algebra of an amenable locally compact group.

Theorem 5.3.4. *Let G be an amenable locally compact group and let $I \subseteq A(G)$ be a closed ideal. If I is weakly completely complemented in $A(G)$, then $hI \in \Omega_c(G)$. In particular, a necessary condition for an ideal $I \subseteq A(G)$ to be completely complemented in $A(G)$ is that $hI \in \Omega_c(G)$.*

Proof. If I is weakly completely complemented, Theorem 5.3.3 implies that I has a bounded approximate identity (since G amenable $\Leftrightarrow A(G)$ is operator amenable). By Proposition

4.4.1, $hI \in \Omega_c(G)$. If I is a completely complemented ideal in $A(G)$, then in particular, I is weakly completely complemented, and the result follows. $\qquad\square$

When G is a discrete amenable group, we are able to use the Cohen-Host Idempotent Theorem (Theorem 2.6.2) to characterize the completely complemented ideals in $A(G)$.

Corollary 5.3.5. *Let G be a discrete amenable group and $I \subseteq A(G)$ a closed ideal. Then I is completely complemented in $A(G)$ if and only if $I = I(X) = 1_{G\backslash X} A(G)$ for some $X \in \Omega(G)$. In particular, every completely complemented ideal in $A(G)$ is invariantly completely complemented.*

Proof. If I is completely complemented, then $hI \in \Omega_c(G) = \Omega(G)$ by Theorem 5.3.4. Since G is discrete and amenable, hI is a set of spectral synthesis and therefore $I = I(hI) = 1_{G\backslash hI} A(G)$.

Conversely, if $X \in \Omega(G)$ the function 1_X belongs to $B(G) = M_{cb}A(G)$ and the map $u \mapsto u - 1_X u$ is a completely bounded projection onto $I(X)$. $\qquad\square$

Remarks: In this section we have, using operator space techniques, obtained a generalization of Theorem 3.1.12 and Proposition 3.1.5 from the class of LCA groups to all amenable locally compact groups. Note that even though the results of Chapter 3 make no reference to completely bounded projections, the results of this section are truly extensions of Theorem 3.1.12 and Proposition 3.1.7. The reason why this is the case is that when G is a LCA group with dual $\widehat{G}$, we have the identification $A(G) \cong L^1(\widehat{G}) = L^\infty(\widehat{G})_*$, and the natural operator space structure on $A(G) \cong L^1(\widehat{G})$ arising from this duality is therefore maximal ([11]). That is, $A(G) \cong \mathrm{MAX}(A(G))$ when G is abelian and thus every complemented ideal in $A(G)$ is automatically completely complemented.

It is also worth mentioning that if G is a nonamenable discrete group, it is known that $B(G) \neq M_{cb}A(G)$. This means that there may exist idempotents in $M_{cb}A(G)$ which are not characteristic functions of sets in $\Omega(G)$. This is exactly what happened in Example 4.3.4, where we exhibited a closed ideal $I \subseteq A(\mathbb{F}_2)$ that was invariantly completely complemented in $A(\mathbb{F}_2)$, even though its hull hI did not belong to $\Omega(\mathbb{F}_2)$. Example 4.3.4 shows that our hypothesis that G is amenable in Corollary 5.3.5 is necessary.

5.4 Ideals in $A(G)$ with Bounded Approximate Identities - Revisited

We will now revisit the problem of classifying ideals in Fourier algebras which have bounded approximate identities. Recall that in Chapter 4 (Proposition 4.4.1) we showed that for

any locally compact group G, a necessary condition for a closed ideal I in $A(G)$ to have a bounded approximate identity is that its hull hI belongs to $\Omega_c(G)$. In Chapter 3 (Theorem 3.3.9) we showed that when G is abelian, the converse to Proposition 4.4.1 is also true, and furthermore the closed ideals in $A(G)$ with bounded approximate identities are precisely those of the form $I(X)$ for some $X \in \Omega_c(G)$. Let us quickly review the approach we took to obtain this result. We first showed that if $H \leq G$ is any closed subgroup, $x \in G$ and $\Delta \in \Omega(H)$, then the ideal $I(x(H\backslash\Delta))$ is always complemented in $A(G)$. In particular, $I(x(H\backslash\Delta))$ is always weakly complemented. We then showed (Theorem 3.3.2) that any weakly complemented ideal in $L^1(G)$ is invariantly weakly complemented and then used Proposition 3.3.1 to deduce that $I(x(H\backslash\Delta))$ always has a bounded approximate identity. Finally, since any set $X \in \Omega_c(G)$ is a finite union of sets of the form $x(H\backslash\Delta)$, Corollary 3.3.5 gives a bounded approximate identity for $I(X)$ for any $X \in \Omega_c(G)$.

If we now shift our attention to general locally compact groups with the hope of generalizing Theorem 3.3.9, we immediately know from Theorem 4.4.2 that we must restrict our attention to amenable groups. Furthermore, Example 4.2.6 shows that ideals of the form $I(x(H\backslash\Delta))$ are not always complemented in $A(G)$ (even for amenable G!), so the approach taken above for abelian groups will require a substantial modification.

In this section we will show that by utilizing the connections between operator amenability, weakly completely complemented ideals, and ideals with bounded approximate identities that we developed in the last section, Theorem 3.3.9 can in fact be generalized to $A(G)$ for all amenable groups G.

5.4.1 Bounded Approximate Identities for Ideals Vanishing on Closed Subgroups

To characterize the ideals with bounded approximate identities in $A(G)$ for G amenable, we will first need to prove the following crucial result:

Let G be an amenable locally compact group and let $H \leq G$ be any closed subgroup. Then the ideal $I(H) \subseteq A(G)$ has a bounded approximate identity.

We will see that the above result is the key to eventually generalizing Theorem 3.3.9 to all amenable locally compact groups. Let us begin with a few preparatory results.

Lemma 5.4.1. *Let H be any amenable locally compact group. Then there exists a completely contractive projection $P : \mathcal{B}(L^2(H)) \to VN(H)$.*

Proof. Since H is amenable, there exists a left invariant mean $m \in L^\infty(H)^*$. Let $\rho : H \to$

$\mathcal{U}(L^2(H))$ denote the right regular representation of H and define for each $T \in \mathcal{B}(L^2(H))$ an operator $PT \in \mathcal{B}(L^2(H))$ by the formula

$$\langle PTf|g \rangle = \int_H \langle \rho(s)T\rho(s^{-1})f|g \rangle dm(s), \qquad \forall f,g, \in L^2(H).$$

In other words, we define a linear map $P : \mathcal{B}(L^2(H)) \to \mathcal{B}(L^2(H))$ given by the weak operator topology convergent integral

$$T \mapsto PT = \text{WOT} - \int_H \rho(s)T\rho(s^{-1})dm(s).$$

Note that if $T \in B(L^2(H))$ then the left-translation invariance of m implies that $\rho(t)PT\rho(t^{-1}) = PT$ for all $t \in H$. Since $VN(H) = \rho(H)'$, $PT \in VN(H)$. Furthermore, if $T \in VN(H)$ then

$$PT = \int_H \rho(s)T\rho(s^{-1})dm(s) = \int_H Tdm(s) = T.$$

Thus $P : B(L^2(H)) \to VN(H)$ is a projection, and it remains to show that P is completely contractive.

Observe that P is completely positive, since $P^{(n)} : M_n(\mathcal{B}(L^2(H))) \to M_n(VN(H))$ is given for all $n \in \mathbb{N}$ by

$$P^{(n)}([T_{ij}]) = \int_H \text{diag}(\rho(s))[T_{ij}]\text{diag}(\rho(s^{-1}))dm(s),$$

which is positive whenever $[T_{ij}] \in M_n(\mathcal{B}(L^2(H)))$ is positive. Observe also that $P(I) = I$. Since P is completely positive and unital, P is a complete contraction ([11]). $\square$

Lemma 5.4.2. *Let H be a closed subgroup of a locally compact group G. Then the quotient Banach algebra $A(G)/I(H)$ is isometrically isomorphic to $A(H)$.*

Proof. By the Herz restriction theorem ([26], Theorem 1), the restriction map $u \mapsto u\big|_H$ defines a quotient map from $A(G)$ onto $A(H)$. In other words, we have $A(H) = A(G)\big|_H$ as spaces of functions on H and

$$\|v\|_{A(H)} = \inf\{\|w\|_{A(G)} : w\big|_H = v\}, \qquad \forall v \in A(H).$$

Now define $\Phi : A(G)/I(H) \to A(H)$ by setting

$$\Phi(u + I(H)) = u\big|_H, \qquad \forall\, u + I(H) \in A(G)/I(H).$$

Note that $u+I(H) = v+I(H) \Leftrightarrow u-v \in I(H) \Leftrightarrow u\big|_H = u\big|_H \Leftrightarrow \Phi(u+I(H)) = \Phi(v+I(H))$,

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and hence Φ is well defined. It is clear that Φ is also an algebra homomorphism. Since $A(G)\big|_H = A(H)$, Φ is surjective. Finally, to see that Φ is isometric (and in particular injective), let $u \in A(G)$ and let $\|\cdot\|_Q$ denote the quotient norm on $A(G)/I(H)$. We then have

$$
\begin{aligned}
\|u + I(H)\|_Q & := \inf\{\|u + v\|_{A(G)} : \ v \in I(H)\} \\
& = \inf\{\|w\|_{A(G)} : \ w\big|_H = u\big|_H\} \\
& = \|u\big|_H\|_{A(H)} \\
& = \|\Phi(u + I(H))\|_{A(H)}.
\end{aligned}
$$

Thus $\Phi : A(G)/I(H) \to A(H)$ is an isometric isomorphism. $\qquad\square$

By dualizing Lemma 5.4.2, we obtain a very important isomorphism on the level of the group von Neumann algebras.

Proposition 5.4.3. *Let G be a locally compact group and let H be a closed subgroup of G. Then the annnihilator $I(H)^\perp$ is a von Neumann subalgebra of $VN(G)$ which is $*$-isomorphic to $VN(H)$.*

Proof. Let $\Phi : A(G)/I(H) \to A(H)$ be the isometric Banach algebra isomorphism obtained in Lemma 5.4.2. By taking the adjoint of Φ, we obtain a weak*-weak* continuous dual Banach space isomorphism

$$
\Psi := \Phi^* : VN(H) \to I(H)^\perp.
$$

To show that the annihilator $I(H)^\perp \subseteq VN(G)$ is a von Neumann subalgebra and that $I(H)^\perp$ is $*$- isomorphic to $VN(H)$, it suffices to show that Ψ is a $*$-homomorphism.

Let λ_G and λ_H denote the left regular representations of G and H respectively. If $x \in H$ and $u \in A(G)$, we have

$$
\langle u, \Psi \lambda_H(x) \rangle = \langle u\big|_H, \lambda_H(x) \rangle = \langle u, \lambda_G(x) \rangle.
$$

That is,

$$
\Psi \lambda_H(x) = \lambda_G(x), \qquad \forall x \in H.
$$

It follows from the above equation (together with the fact that λ_G and λ_H are unitary representations) that the restriction of Ψ to the $*$-subalgebra $\mathcal{D} = \mathrm{span}\{\lambda_H(x) : \ x \in H\} \subseteq VN(H)$ is a $*$-homomorphism from $\mathcal{D}$ into $I(H)^\perp$. However, since $\mathcal{D}$ is weak*-dense

in $VN(H)$ and Ψ is weak*-weak*continuous, Ψ must extend to a $*$-homomorphism from $VN(H) = \overline{\mathcal{D}}^{w^*}$ onto ran$\Psi = I(H)^\perp$. This completes the proof. $\qquad\square$

An interesting consequence of the above proposition is the following strengthening of Lemma 5.4.2.

Corollary 5.4.4. *Let G be a locally compact group and let H be a closed subgroup, then the isomorphism $\Phi : A(G)/I(H) \to A(H)$ is completely isometric.*

Proof. From Proposition 5.4.3 we know that Φ^* is a $*$-isomorphism and hence a complete isometry. This implies that Φ is also a complete isometry. $\qquad\square$

Using the above results, we can deduce that the ideal $I(H)$ is always weakly completely complemented in $A(G)$ for any closed subgroup H of an amenable locally compact group G.

Proposition 5.4.5. *Let G be an amenable locally compact group and let H be a closed subgroup. Then $I(H)$ is always weakly completely complemented in $A(G)$.*

Proof. Let $\Psi : VN(H) \to I(H)^\perp$ be the $*$-isomorphism from Proposition 5.4.3. Since $\Psi^{-1} : I(H)^\perp \to VN(H)$ is completely bounded, the Arveson-Wittstock Extension theorem (Theorem 2.2.1) implies the existence of a completely bounded extension of Ψ^{-1}, say $\Omega : \mathcal{B}(L^2(G)) \to \mathcal{B}(L^2(H))$. Let $P : \mathcal{B}(L^2(H)) \to VN(H)$ be the completely contractive projection constructed in Lemma 5.4.1 and define $\tilde{Q} := \Psi \circ P \circ \Omega : \mathcal{B}(L^2(G)) \to I(H)^\perp$. Since $\Omega\big|_{I(H)^\perp} = \Psi^{-1}$, it is easy to see that $\tilde{Q}$ is a completely contractive projection from $\mathcal{B}(L^2(G))$ onto $I(H)^\perp$. Now simply let $Q : VN(G) \to I(H)^\perp$ to be the restriction $\tilde{Q}\big|_{VN(G)}$. $\qquad\square$

We now have all of the tools we need to establish the existence of a bounded approximate identity in $I(H)$.

Theorem 5.4.6. *Let G be an amenable locally compact group and let $H \leq G$ be a closed subgroup. Then the ideal $I(H) \subseteq A(G)$ always has a bounded approximate identity.*

Proof. By Proposition 5.4.5, $I(H)$ is always weakly completely complemented in $A(G)$. Since G is amenable, $A(G)$ is an operator amenable completely contractive Banach algebra, and therefore Theorem 5.3.3 implies that $I(H)$ has a bounded approximate identity. $\qquad\square$

5.4.2 The Complete Characterization of Ideals with Bounded Approximate Identities

We will now use Theorem 5.4.6 to conclude more generally that for any amenable group G, a closed ideal $I \subseteq A(G)$ has a bounded approximate identity if and only if $I = I(X)$ for

some $X \in \Omega_c(G)$. The approach we take from here closely parallels the approach taken in Chapter 3 for LCA groups.

To begin, we recall a useful Banach algebraic result.

Lemma 5.4.7. *Let $\mathcal{A}$ be a Banach algebra and $I \subseteq \mathcal{A}$ a closed left (respectively right or two-sided) ideal. If $\mathcal{A}/I$ and I both have left (respectively right or two-sided) bounded approximate identities, then $\mathcal{A}$ has a left (respectively right or two-sided) bounded approximate identity.*

Proof. We will only consider the left ideal case, as the other cases are similar. Suppose $\mathcal{A}/I$ and I both have left bounded approximate identities, say $\{e_\alpha + I\}_{\alpha \in A} \subset b_M(\mathcal{A}/I)$ and $\{f_\beta\}_{\beta \in B} \subset b_N(I)$ respectively. Note that since $\{e_\alpha + I\}_{\alpha \in A} \subset b_M(\mathcal{A}/I)$, we may assume that a net of coset representatives $\{e_\alpha\}_{\alpha \in A} \subset \mathcal{A}$ has been chosen to be uniformly bounded by $M + 1$.

Consider the set $\mathcal{U} := \{e_\alpha + f_\beta - f_\beta e_\alpha : \alpha \in A, \beta \in B\}$. We will show that $\mathcal{U}$ is a left bounded approximate unit for $\mathcal{A}$.

First note that $\mathcal{U}$ is a bounded subset of $\mathcal{A}$. Next, let $a \in \mathcal{A}$ and let $\epsilon > 0$ be given. Choose $\alpha_0 \in A$ and $x \in I$ so that

$$\|e_{\alpha_0} a - a + x\| < \frac{\epsilon}{3N}.$$

Now choose $\beta_0 \in B$ so that

$$\|f_{\beta_0} x - x\| < \frac{\epsilon}{3N}.$$

Letting $u = e_{\alpha_0} + f_{\beta_0} - f_{\beta_0} e_{\alpha_0} \in \mathcal{U}$, we get

$$
\begin{aligned}
\|ua - a\| &= \|e_{\alpha_0} a + f_{\beta_0} a - f_{\beta_0} e_{\alpha_0} a - a\| \\
&= \|e_{\alpha_0} a - a + x - x + f_{\beta_0}(a - e_{\alpha_0} a - x) + f_{\beta_0} x\| \\
&\leq \|e_{\alpha_0} a - a + x\| + \|f_{\beta_0}(a - e_{\alpha_0} a - x)\| + \|f_{\beta_0} x - x\| \\
&< \frac{\epsilon}{3N} + N\frac{\epsilon}{3N} + \frac{\epsilon}{3N} \leq \epsilon.
\end{aligned}
$$

Therefore, $\mathcal{U}$ is a left bounded approximate unit for $\mathcal{A}$. By Proposition 2.4.1, $\mathcal{A}$ has a bounded left approximate identity. $\square$

Using Lemma 5.4.7 and Theorem 5.4.6, we can show that the ideals of the form "$I(g(H\backslash\Delta))$" always have a bounded approximate identity.

Lemma 5.4.8. *Let G be an amenable locally compact group, $H \leq G$ a closed subgroup, $\Delta \in \Omega(H)$, and $g \in G$. Then the closed ideal $I(g(H\backslash\Delta)) \subseteq A(G)$ has a bounded approximate identity.*

Proof. By translating by $g \in G$, it is necessary and sufficient to prove that $I(H\backslash\Delta)$ has a bounded approximate identity. First consider the closed ideal in $A(H)$ defined by

$$I_H(H\backslash\Delta) = \{u \in A(H) : \ u(x) = 0 \ \forall x \in H\backslash\Delta\}.$$

We claim that $I_H(H\backslash\Delta)$ has a bounded approximate identity. To see this, observe that H is an amenable group since it is a closed subgroup of G. By Leptin's theorem (see Theorem 2.7.1), $A(H)$ must have a bounded approximate identity, say $\{e_\alpha\}_\alpha$. Since $\Delta \in \Omega(H)$, $1_\Delta \in B(H)$ and the map $P : A(H) \to I_H(H\backslash\Delta)$ defined by $Pu = 1_\Delta u$ is a completely bounded projection. The net $\{1_\Delta e_\alpha\}_\alpha$ is easily seen to be a bounded approximate identity for $I_H(H\backslash\Delta)$.

Now recall from Proposition 5.4.2 that there is a completely isometric Banach algebra isomorphism $\Phi : A(G)/I(H) \to A(H)$. It follows from the definition of Φ that

$$\Phi^{-1}\big|_{I_H(H\backslash\Delta)} : I_H(H\backslash\Delta) \to I(H\backslash\Delta)/I(H)$$

is another isometric Banach algebra isomorphism. Since we know that $I_H(H\backslash\Delta)$ has a bounded approximate identity, it follows that $I(H\backslash\Delta)/I(H)$ must also have a bounded approximate identity. By Theorem 5.4.6, $I(H)$ has a bounded approximate identity. Since $I(H)$ and $I(H\backslash\Delta)/I(H)$ both have bounded approximate identities, Lemma 5.4.7 implies that $I(H\backslash\Delta)$ has a bounded approximate identity. $\square$

Now we arrive at our main theorem:

Theorem 5.4.9. *Let G be an amenable locally compact group. Then a closed ideal $I \subseteq A(G)$ has a bounded approximate identity if and only if $I = I(X)$ for some $X \in \Omega_c(G)$.*

Proof. Suppose that I has a bounded approximate identity. Then $hI \in \Omega_c(G)$ by Proposition 4.4.1. It is also shown in [16] that every element of $\Omega_c(G)$ is a set of spectral synthesis when G is amenable, hence $I = I(hI)$. Conversely, suppose $X \in \Omega_c(G)$. Then by Theorem 2.6.1 we can write $X = \bigcup_{i=1}^n g_i(H_i\backslash\Delta_i)$ where for $1 \leq i \leq n$, $g_i \in G$, $H_i \leq G$ is a closed subgroup and $\Delta_i \in \Omega(H_i)$. By Lemma 5.4.8, each ideal $I(g_i(H_i\backslash\Delta_i))$ has a bounded approximate identity. We can therefore apply Corollary 3.3.5 (or more precisely the obvious generalization of Corollary 3.3.5 to $A(G)$ for general locally compact groups) to obtain a bounded approximate identity for $I(\bigcup_{i=1}^n g_i(H_i\backslash\Delta_i)) = I(X)$. $\square$

Remark: It worth noting that although Theorem 5.4.9 is a theorem about the Banach algebraic structure of the Fourier algebra, its proof uses the operator space structure of the

Fourier algebra in a highly non-trivial way. Theorem 5.4.9 testifies to the value of operator space techniques in the study of the Fourier algebra of a locally compact group.

5.5 Operator Projectivity and Invariantly Complemented Ideals in $A(G)$

In Chapter 3 (Theorem 3.1.13) we observed that the complemented ideals in $A(G)$ for discrete abelian G have a particularly simple structure. Namely, if G is a discrete abelian group and $I \subseteq A(G)$ is a closed ideal with hull $X \subseteq G$, then I is complemented if and only if the characteristic function $1_{G \setminus X}$ belongs to $B(G)$ and consequently $I = 1_{G \setminus X} A(G)$.

In Section 5.3 (Corollary 5.3.5) we were able to use the operator space structure of $A(G)$ to generalize the above result to all amenable discrete groups in the following way: If G is an amenable discrete group and $I \subseteq A(G)$ is a closed ideal with hull $X \subseteq G$, then I is completely complemented if and only if the characteristic function $1_{G \setminus X}$ belongs to $B(G)$ and consequently $I = 1_{G \setminus X} A(G)$.

In this section we want to further generalize these results to all (possibly non-amenable) discrete groups G. In view of Example 4.3.4, we know that when G is not amenable, there may exist idempotents in $M_{cb} A(G)$ which do not belong to $B(G)$. Therefore we cannot, in general, expect the completely complemented ideals in $A(G)$ for discrete G to be fully characterized by the idempotents in $B(G)$. However, in this section we will be able to show that the next best thing happens: that is, if G is any discrete group and $I \subseteq A(G)$ is any closed ideal with hull $X \subseteq G$, then I is completely complemented if and only if $I = 1_{G \setminus X} A(G)$ where $1_{G \setminus X}$ is an idempotent in $M_{cb} A(G)$.

To prove the above statement, we need to show that whenever G is a discrete group and $I \subseteq A(G)$ is a completely complemented ideal, then I is automatically invariantly completely complemented. (This is because the idempotents in $M_{cb} A(G)$ coincide with the completely bounded $A(G)$-invariant projections onto ideals in $A(G)$).

Thus, using the homological algebra terminology developed in Section 5.2, our task is to show that for any discrete group G and any closed ideal $I \subseteq A(G)$, the extension sequence

$$\Sigma : 0 \longrightarrow I \overset{i}{\longrightarrow} A(G) \overset{q}{\longrightarrow} A(G)/I \longrightarrow 0 \qquad (5.3)$$

is completely admissible if and only if it splits completely.

We shall see that the answer to this problem is intimately related to a notion from homological algebra called *projectivity*. We will now define and explore this notion in the category of operator spaces.

Definition 5.5.1. *Let $\mathcal{A}$ be a completely contractive Banach algebra and let X be a left operator $\mathcal{A}$-module. We say that X is a **left operator projective** operator $\mathcal{A}$-module if given any left operator $\mathcal{A}$-modules Y and Z, any completely admissible surjection $\phi : Y \to Z$, and any c.b. module map $\theta : X \to Z$, there exists a c.b. module map $\psi : X \to Y$ such that $\phi \circ \psi = \theta$. In other words, the following diagram commutes (as a diagram of left operator $\mathcal{A}$-modules).*

$$\begin{array}{ccc} & & X \\ & \psi \swarrow & \downarrow \theta \\ Y & \xrightarrow{\phi} & Z \end{array}$$

(5.4)

Given the above definition, it should be clear how one can also define a right operator projective right operator $\mathcal{A}$-module and an operator biprojective operator $\mathcal{A}$-bimodule. For our purposes (i.e. to study the Fourier algebra), it will be sufficient for us to understand the notion of operator projectivity for left modules. We will therefore stick to the "left" case and refer the reader to [49], [50] and [52] for the other situations.

The notion of projectivity has its origins in module theory in abstract algebra (see for example [43]). This notion was studied in the Banach algebra category in [25]. The notion of projectivity in the category of operator spaces was introduced and studied in [49], [50], and [52]. We will see that operator projectivity is an extremely powerful concept when studying ideal theory in completely contractive Banach algebras.

To see why operator projectivity is a useful notion for us when considering invariant complementation for ideals in a completely contractive Banach algebra, we have the following two results relating projectivity to the splitting of exact sequences.

Proposition 5.5.2. *Let Z be a left operator projective $\mathcal{A}$-module and suppose that*

$$\Sigma : 0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

is a completely admissible sequence of left operator $\mathcal{A}$-modules. Then Σ splits completely.

Proof. Since $g : Y \to Z$ is a completely admissible surjection, then by projectivity there exists a c.b. module map $G : Z \to Y$ that extends the identity map $id_Z : Z \to Z$. That is, $g \circ G = id_Z$. By Theorem 5.2.4, Σ splits completely. $\square$

Theorem 5.5.3. *Let I be a completely complemented left ideal in a completely contractive Banach algebra $\mathcal{A}$. If $\mathcal{A}/I$ is left operator projective, then I is invariantly completely complemented in $\mathcal{A}$.*

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Proof. Consider the exact sequence

$$\Sigma : 0 \longrightarrow I \xrightarrow{\ i\ } \mathcal{A} \xrightarrow{\ q\ } \mathcal{A}/I \longrightarrow 0$$

where i is the canonical inclusion and q is the canonical quotient map. It is easy to see that Σ forms an extension sequence of left operator $\mathcal{A}$-modules. Since I is completely complemented, there exists a c.b. projection $P : \mathcal{A} \to I$ satisfying $P \circ i = id_I$. By Theorem 5.2.4, Σ is then completely admissible. Applying Proposition 5.5.2, we get that Σ splits completely. In particular, there exists c.b. projection $\tilde{P} : \mathcal{A} \to I$ which is also a left module homomorphism. $\qquad\square$

So the question of determining when a completely complemented left ideal $I \subseteq \mathcal{A}$ is invariantly completely complemented in $\mathcal{A}$ is very much related to the left operator projectivity of the quotient algebra $\mathcal{A}/I$. We will now try to find a condition on $\mathcal{A}$ to ensure that $\mathcal{A}/I$ is always left operator projective. This will require some further development of the notion of projectivity.

Let $\mathcal{A}$ be a completely contractive Banach algebra and let $\mathcal{A}_+$ denote its unitization as defined in Chapter 2. Recall that $\mathcal{A}_+$ is a completely contractive Banach algebra and that any operator $\mathcal{A}$-module automatically becomes a neounital operator $\mathcal{A}_+$-module. Recall also from Chapter 2 that for any operator space X, $\mathcal{A}_+ \widehat{\otimes} X$ is naturally a left operator $\mathcal{A}$-module when equipped with the module action

$$a \cdot ((b, \lambda) \otimes x)) = (ab + \lambda a) \otimes x, \qquad \forall (a, \lambda) \in \mathcal{A}_+,\ b \in \mathcal{A},\ x \in X.$$

The following technical lemma asserts that $\mathcal{A}_+ \widehat{\otimes} X$ is left operator projective for any operator space X.

Lemma 5.5.4. *Let X be any operator space, then $\mathcal{A}_+ \widehat{\otimes} X$ is a left operator projective operator $\mathcal{A}$-module.*

Proof. Let Y and Z be left operator $\mathcal{A}$-modules, let $\phi : Y \to Z$ be a completely admissible surjection and let $\theta : \mathcal{A}_+ \widehat{\otimes} X \to Z$ be a c.b. $\mathcal{A}$-module map. We need to show that there exists a c.b. module map $\psi : \mathcal{A}_+ \widehat{\otimes} X \to Y$ such that $\phi \circ \psi = \theta$.

Now, since $\phi : Y \to Z$ is completely admissible, there exists some $\alpha \in \mathcal{CB}(Z, Y)$ such that $\phi \circ \alpha = id_Z$. Define a bilinear map $\tilde{\psi}$ from $\mathcal{A}_+ \times X$ into Y by the equation

$$\tilde{\psi}((a, x)) := a \cdot \alpha(\theta(e \otimes x)),$$

where $a \in \mathcal{A}_+$, $x \in X$ and e denotes the unit in $\mathcal{A}_+$. We claim that $\tilde{\psi} \in \mathcal{JCB}(\mathcal{A}_+ \times X; Y)$.

To see this, let $[b_{ij}] \in b_1(M_p(\mathcal{A}_+))$ and $[x_{kl}] \in b_1(M_q(X))$ and observe that

$$
\begin{aligned}
\|\tilde{\psi}^{(p,q)}([b_{ij}],[x_{kl}])\|_{pq} &= \|[\tilde{\psi}(b_{ij},x_{kl})]\|_{pq} \\
&= \|[b_{ij} \cdot \alpha(\theta(e \otimes x_{kl}))]\|_{pq} \\
&\leq \|[b_{ij}]\|_p\|[\alpha(\theta(e \otimes x_{kl}))]\|_q \\
&\leq \|\alpha\|_{cb}\|[\theta(e \otimes x_{kl})]\|_q \\
&\leq \|\alpha\|_{cb}\|\theta\|_{cb} < \infty.
\end{aligned}
$$

Since $[b_{ij}] \in b_1(M_p(\mathcal{A}_+))$ and $[x_{kl}] \in b_1(M_q(X))$, and $p,q \in \mathbb{N}$ were arbitary, we conclude that $\tilde{\psi} \in \mathcal{JCB}(\mathcal{A}_+ \times X; Y)$. Also, since $\mathcal{JCB}(\mathcal{A}_+ \times X; Y) \cong \mathcal{CB}(\mathcal{A}_+\widehat{\otimes}X, Y)$ completely isometrically (see Chapter 2), the map $\psi : \mathcal{A}_+\widehat{\otimes}X \to Y$ given on elementary tensors by

$$
\psi(a \otimes x) = \tilde{\psi}(a, x)
$$

is completely bounded. We now claim that the map $\psi \in \mathcal{CB}(\mathcal{A}_+\widehat{\otimes}X, Y)$ is our desired map. First note that ψ is a left $\mathcal{A}_+$ (and hence $\mathcal{A}$)-module map since

$$
\begin{aligned}
\psi(a \cdot (b \otimes x)) &= \psi(ab \otimes x) \\
&= (ab) \cdot \alpha(\theta(e \otimes x)) \\
&= a \cdot (b \cdot \alpha(\theta(e \otimes x))) \\
&= a \cdot \psi(b \otimes x). \quad \forall a, b \in \mathcal{A}_+, \ x \in X.
\end{aligned}
$$

Since $\phi \circ \alpha = id_Z$ and ϕ is a module map, we get

$$
\phi(\psi(a \otimes x)) = \phi(a \cdot \alpha(\theta(e \otimes x))) = a \cdot \theta(e \otimes x) = \theta(a \otimes x),
$$

for all $a \in \mathcal{A}_+$ and $x \in X$. Extending by linearity and noting that $\mathcal{A}_+ \otimes X$ is dense in $\mathcal{A}_+\widehat{\otimes}X$, we obtain $\phi \circ \psi = \theta$. This completes the proof. $\qquad\square$

Using the above technical lemma, we can obtain a useful characterization of left operator projective $\mathcal{A}$-modules in terms of the unitization $\mathcal{A}_+$.

Lemma 5.5.5. *Let X be a left operator $\mathcal{A}$-module, let $\pi_L : \mathcal{A}_+\widehat{\otimes}X \to X$ be the natural left operator $\mathcal{A}_+$-module map and let $N = \ker \pi_L$. Then the short exact sequence*

$$
\Sigma_L : 0 \longrightarrow N \overset{i}{\longrightarrow} \mathcal{A}_+\widehat{\otimes}X \overset{\pi_L}{\longrightarrow} X \longrightarrow 0
$$

is always completely admissible. Moreover Σ_L splits completely as a sequence of left operator

$\mathcal{A}$-modules if and only if X is left operator projective.

Proof. First note that the short exact sequence Σ_L forms an extension sequence since i is a complete isometry onto its range and π_L is a complete quotient map. Next, note that the map $\tau : X \to \mathcal{A}_+ \widehat{\otimes} X$ given by $\tau(x) = e \otimes x$ is a c.b. right inverse for π_L. By Theorem 5.2.4, Σ_L is therefore completely admissible.

Suppose first that X is left operator projective. By Proposition 5.5.2, Σ_L splits completely as a sequence of left operator $\mathcal{A}$-modules.

Conversely suppose that Σ_L splits completely as a sequence of left operator $\mathcal{A}$-modules. That is, there exists a c.b. $\mathcal{A}$-module map $\rho : X \to \mathcal{A}_+ \widehat{\otimes} X$ which is a right inverse for π_L. Now let $\phi : Y \to Z$ be a completely admissible surjection and let $\theta : X \to Z$ be any c.b. $\mathcal{A}$-module map. We need to show that there exists a c.b. $\mathcal{A}$-module map $\psi : X \to Y$ so that $\phi \circ \psi = \theta$. To do this, define $\theta' : \mathcal{A}_+ \widehat{\otimes} X \to Z$ by setting $\theta' = \theta \circ \pi_L$. Then θ' is a c.b. $\mathcal{A}$-module homomorphism. By Lemma 5.5.4, $\mathcal{A}_+ \widehat{\otimes} X$ is left operator projective, hence there exists a c.b. $\mathcal{A}$-module map $\psi' : \mathcal{A}_+ \widehat{\otimes} X \to Y$ satisfying $\phi \circ \psi' = \theta'$. Now define $\psi : X \to Y$ by setting $\psi = \psi' \circ \rho$. Note that ψ is obviously a c.b. $\mathcal{A}$-module homomorphism. Finally observe that $\phi \circ \psi = \phi \circ \psi' \circ \rho = \theta' \circ \rho = \theta \circ \pi_L \circ \rho = \theta \circ id_X = \theta$, as required. $\qquad\square$

We now return to the study of ideals. Let $\mathcal{A}$ be a completely contractive Banach algebra with multiplication map $m : \mathcal{A} \widehat{\otimes} \mathcal{A} \to \mathcal{A}$ and let $I \subseteq \mathcal{A}$ be a completely complemented left ideal. Recall that Theorem 5.5.3 suggests that if we want to establish whether I is invariantly completely complemented in $\mathcal{A}$, we need to understand when the quotient algebra $\mathcal{A}/I$ is a left operator projective $\mathcal{A}$-module.

The following theorem shows that if there exists a c.b. module map $\rho : \mathcal{A} \to \mathcal{A} \widehat{\otimes} \mathcal{A}$ which is as a right inverse for the multiplication map m, then the left operator projectivity of $\mathcal{A}/I$ is guaranteed for any closed left ideal $I \subseteq \mathcal{A}$.

Theorem 5.5.6. *Let $\mathcal{A}$ be a completely contractive Banach algebra and let I be a closed left ideal in $\mathcal{A}$. Suppose there exists a c.b. bimodule map $\rho : \mathcal{A} \to \mathcal{A} \widehat{\otimes} \mathcal{A}$ which is a right inverse for the multiplication map $m : \mathcal{A} \widehat{\otimes} \mathcal{A} \to \mathcal{A}$. Then $\mathcal{A}/I$ is left operator projective.*

Proof. For the sake of brevity, we will prove this result with the additional assumption that the ideal I is essential as a left operator $\mathcal{A}$-module. The proof of the theorem for arbitrary (not necessarily essential) closed left ideals requires a lot more machinery from homological algebra. See [49] and [50] for the full details.

Let $\rho : \mathcal{A} \to \mathcal{A} \widehat{\otimes} \mathcal{A}$ be a c.b. bimodule map which is a right inverse for the multiplication map $m : \mathcal{A} \widehat{\otimes} \mathcal{A} \to \mathcal{A}$, let $q : \mathcal{A} \to \mathcal{A}/I$ be the canonical quotient map and let $\pi_L : \mathcal{A} \widehat{\otimes} \mathcal{A}/I \to$

$\mathcal{A}/I$ be the left module map. Consider the following composition of c.b. module maps:

$$(id_\mathcal{A} \otimes q) \circ \rho : \mathcal{A} \to \mathcal{A} \widehat{\otimes} \mathcal{A}/I.$$

Observe that if $a \in \mathcal{A}$ and $z \in I$, then $\rho(az) = \rho(a) \cdot z \in (\mathcal{A} \widehat{\otimes} \mathcal{A}) \cdot I \subseteq \mathcal{A} \widehat{\otimes} I$. Consequently

$$(id_\mathcal{A} \otimes q)(\rho(az)) \;\; = \;\; 0.$$

Since I is essential, it follows by linearity and density that the map $(id_\mathcal{A} \otimes q) \circ \rho$ vanishes on all of I. Therefore we can define a new c.b. module map $\tilde{\rho} : \mathcal{A}/I \to \mathcal{A} \widehat{\otimes} \mathcal{A}/I$ by setting

$$\tilde{\rho}(a + I) = (id_\mathcal{A} \otimes q)(\rho(a)), \qquad \forall\, a + I \in \mathcal{A}/I.$$

We now claim that $\tilde{\rho}$ is a right inverse for π_L. To see this, let $a, b \in \mathcal{A}$ and observe that

$$\pi_L(id_\mathcal{A} \otimes q)(a \otimes b) = \pi_L(a \otimes (b + I)) = ab + I = q(m(a \otimes b)).$$

Thus by linearity, $\pi_L \circ (id_\mathcal{A} \otimes q)\big|_{\mathcal{A} \otimes \mathcal{A}} = q \circ m\big|_{\mathcal{A} \otimes \mathcal{A}}$. Since $\mathcal{A} \otimes \mathcal{A}$ is dense in $\mathcal{A} \widehat{\otimes} \mathcal{A}$ and the above maps are continuous, it follows that $\pi_L \circ (id_\mathcal{A} \otimes q) = q \circ m$. But then

$$
\begin{aligned}
\pi_L(\tilde{\rho}(a + I)) \;\; &= \;\; \pi_L(id_\mathcal{A} \otimes q)(\rho(a)) \\
&= \;\; q(m(\rho(a))) \\
&= \;\; q(a) \\
&= \;\; a + I, \quad \forall\, a + I \in \mathcal{A}/I.
\end{aligned}
$$

This shows that $\tilde{\rho}$ is indeed a c.b. right inverse for π_L.

Now, since the inclusion $\mathcal{A} \widehat{\otimes} \mathcal{A}/I \hookrightarrow \mathcal{A}_+ \widehat{\otimes} \mathcal{A}/I$ is a completely isometric module map, the map $\tilde{\rho}$ is also c.b. right inverse to the module map $\pi_L : \mathcal{A}_+ \widehat{\otimes} \mathcal{A}/I \to \mathcal{A}/I$. But this means that the completely admissible short exact sequence

$$\Sigma_L : 0 \longrightarrow N \overset{i}{\longrightarrow} \mathcal{A}_+ \widehat{\otimes} \mathcal{A}/I \overset{\pi_L}{\longrightarrow} \mathcal{A}/I \longrightarrow 0$$

(where $N := \ker \pi_L$) splits completely. By Lemma 5.5.5, $\mathcal{A}/I$ is left operator projective. $\quad\square$

In view of the preceding theorem, we give the following definition.

Definition 5.5.7. *A completely contractive Banach algebra $\mathcal{A}$ is called **operator biprojective** if there exists an operator $\mathcal{A}$-bimodule map $\rho : \mathcal{A} \to \mathcal{A} \widehat{\otimes} \mathcal{A}$ which is a right inverse for the canonical multiplication map $m : \mathcal{A} \widehat{\otimes} \mathcal{A} \to \mathcal{A}$.*

Since any completely contractive Banach algebra $\mathcal{A}$ is canonically an operator $\mathcal{A}$-bimodule over itself, it is natural to ask: when is $\mathcal{A}$ is an operator biprojective operator $\mathcal{A}$-bimodule? It turns out that $\mathcal{A}$ is an operator biprojective operator $\mathcal{A}$-bimodule if and only if $\mathcal{A}$ is operator biprojective in the sense of Definition 5.5.7. Refer to [50] for details regarding this.

If we now put together Theorem 5.5.6 and Theorem 5.5.3, we arrive at the main theorem of this section.

Theorem 5.5.8. *Let $\mathcal{A}$ be an operator biprojective completely contractive Banach algebra and let $I \subseteq \mathcal{A}$ be a completely complemented left ideal. Then I is invariantly completely complemented.*

Observe that Theorem 5.5.8 is the completely contractive Banach algebra version of Theorem 5.1.2 that we promised at the beginning this chapter.

5.5.1 Invariantly Completely Complemented Ideals in the Fourier Algebra

Let us now consider the Fourier algebra $A(G)$ for a locally compact group G. We have seen in Theorem 5.5.8 that the operator biprojectivity of $A(G)$ is intimately related to the question of whether or not a completely complemented ideal in $A(G)$ is always invariantly completely complemented. Let us now settle the question of when $A(G)$ is operator biprojective:

Theorem 5.5.9. *Let G be a locally compact group. Then $A(G)$ is operator biprojective if and only if G is discrete.*

Proof. Suppose that $A(G)$ is operator biprojective. Consider the closed ideal $I(\{e\}) = \{u \in A(G) : u(e) = 0\}$ where $e \in G$ denotes the identity element of G. Since $I(\{e\})$ has codimension one, there exists a bounded projection $P : A(G) \to I(\{e\})$. Define $Q : A(G) \to A(G)$ by $Qu = u - P(u)$. Then Q is a bounded projection onto $\ker P \cong A(G)/I(\{e\}) \cong \mathbb{C}$. Since $\mathrm{ran}Q$ is one dimensional, Q is completely bounded ([11] Corollary 2.2.4) and hence P is completely bounded. Since $I(\{e\})$ is completely complemented in $A(G)$ and $A(G)$ is operator biprojective, Theorem 5.5.8 implies that there exists a c.b. projection $\tilde{P} : A(G) \to I(\{e\})$ such that $\tilde{P}(uv) = u\tilde{P}v$ for all $u, v \in A(G)$. Identifying $\tilde{P}$ with the unique function in $M_{cb}A(G)$ that it defines, we see that the function $1_{G\backslash\{e\}} \in M_{cb}A(G) \subseteq C_b(G)$. It follows that the singleton set $\{e\}$ is open in G and so G must be discrete.

Conversely suppose that G is discrete. Recall that for any locally compact group G there is a completely isometric $A(G)$-bimodule isomorphism $\Phi : A(G)\widehat{\otimes}A(G) \to A(G \times G)$

defined on elementary tensors by

$$\Phi(u \otimes v)(s,t) = u(s)v(t) \quad \forall (s,t) \in G \times G, \ u, v \in A(G).$$

The isomorphism Φ allows us to identify $A(G)\widehat{\otimes}A(G)$ with $A(G \times G)$. From this identification it follows that the operator biprojectivity of $A(G)$ is equivalent to the existence of a c.b. $A(G)$-bimodule map $\rho : A(G) \to A(G \times G)$ which is as a right inverse for the induced map $\tilde{m} = m \circ \Phi^{-1}$.

Now denote by $\Delta_G = \{(g,g) : \ g \in G\}$ the diagonal subgroup of $G \times G$. Since the map $g \mapsto (g,g)$ is a topological isomorphism between G and Δ_G, the map $\Psi : A(G) \to A(\Delta_G)$ given by $\Psi u(g,g) = u(g)$ is a completely isometric isomorphism between $A(G)$ and $A(\Delta_G)$. Furthermore, since G is discrete, Δ_G is open in $G \times G$ and therefore $A(\Delta_G)$ embeds completely isometrically into $A(G \times G)$ as the ideal of functions in $A(G \times G)$ supported on Δ_G. Let $\iota : A(\Delta_G) \hookrightarrow A(G \times G)$ denote this embedding. Now define a map $\rho : A(G) \to A(G \times G)$ by setting

$$\rho(u) = \iota \Psi u.$$

The map ρ is just the natural embedding of $A(G)$ into $\iota A(\Delta_G) \subset A(G \times G)$. Note that ρ is a composition of c.b. bimodule maps and therefore ρ is also a c.b. bimodule map. It remains to show that ρ is a right inverse for $\tilde{m} = m \circ \Phi^{-1}$. To see this, first let $g \in G$ and let δ_g be the function in $A(G)$ which takes the value 1 at g and the value 0 elswehere. Then we have

$$\rho(\delta_g) = \iota \Psi(\delta_g) = \delta_{(g,g)} = \Phi(\delta_g \otimes \delta_g),$$

which implies that

$$(\tilde{m} \circ \rho)(\delta_g) = m(\Phi^{-1}\Phi(\delta_g \otimes \delta_g)) = m(\delta_g \otimes \delta_g) = \delta_g.$$

By linearity, it follows that $\tilde{m}(\rho(u)) = u$ for all $u \in K(G) = \mathrm{span}\{\delta_g : g \in G\}$. Since $K(G)$ is dense in $A(G)$, $\tilde{m} \circ \rho = id_{A(G)}$, and $A(G)$ is therefore operator biprojective. $\square$

Using the operator biprojectivity of $A(G)$ for discrete G together with Theorem 5.5.8 yields our desired result concerning completely complemented ideals in $A(G)$ for discrete G.

Corollary 5.5.10. *Let G be a locally compact group. Then G is discrete if and only if every completely complemented ideal in $A(G)$ is invariantly completely complememented. In particular, for every discrete group G and every completely complemented ideal $I \subseteq A(G)$,*

the characteristic funcion $1_{G \backslash hI}$ *belongs to* $M_{cb}A(G)$ *and* $I = 1_{G \backslash hI}A(G)$.

Proof. If G is discrete, Theorem 5.5.9 and Theorem 5.5.8 imply that every completely complemented ideal in $A(G)$ is invariantly completely complemented. Conversely suppose that every completely complemented ideal in $A(G)$ is invariantly completely complemented. Then in particular the ideal $I(\{e\}) = \{u \in A(G) : u(e) = 0\}$ is invariantly completely complemented. But as we saw in the proof of Theorem 5.5.9, this forces G to be discrete.

Now suppose G is discrete and that I is any completely complemented ideal in $A(G)$. Since we know that I is invariantly completely complemented, there must exist a completely bounded projection $P : A(G) \to I$ such that $P(uv) = uPv$ for all $u, v \in A(G)$. It follows from the definition of $M_{cb}A(G)$ that there exists some $f \in M_{cb}A(G)$ so that $Pu = fu$ for all $u \in A(G)$. Since $P^2 = P$, $f^2 = f$ and therefore $f = 1_X$ for some subset $X \subseteq G$. Since $I = fA(G) = 1_X A(G) \subseteq I(h(I))$, the regularity of $A(G)$ forces us to have $hI \subseteq G \backslash X$. Conversely if $y \in G \backslash X$ then $1_X u(y) = 0$ for all $u \in A(G)$. Since $1_X A(G) = I$, we get $y \in hI$. Therefore $X = G \backslash hI$ and $I = 1_{G \backslash hI}A(G)$. $\square$

Chapter 6

Complemented Ideals in the Fourier Algebra that are not Completely Complemented

In Chapter 5 we saw that by utilizing the operator space structure of the Fourier algebra, it is possible to answer several interesting questions about the structure of the completely complemented and weakly completely complemented ideals in this algebra. The results of that chapter provide very compelling evidence that the Fourier algebra is best viewed as an object in the category of operator spaces. However, suppose that I is a closed ideal in the Fourier algebra of a locally compact group which is known to be complemented or weakly complemented. If we wish to study the structure of I using the techniques of Chapter 5, we are required to apriori know that I is not only (weakly) complemented, but in fact (weakly) *completely* complemented. Thus we are naturally led to ask the following two questions about I

(1) *Is it possible for I to be (weakly) complemented but not (weakly) completely complemented?*

(2) *If the answer to Question (1) is yes, then is it possible to generalize the results of Chapter 5 so that they can still apply to ideals like I that are not necessarily (weakly) completely complemented?*

In this chapter we will provide an affirmative answer to Question (1). More specifically, using operator space techniques we will show by example that for the noncommutative free

group $\mathbb{F}_N$ on N generators ($N \geq 2$, possibly infinite), the Fourier algebra $A(\mathbb{F}_N)$ always contains a complemented ideal I which is neither completely complemented nor weakly completely complemented. The key to constructing our example is to realize that, for a discrete group G, the algebra $M_{cb}A(G)$ acts naturally as Schur multipliers on $\mathcal{B}(\ell^2(G))$.

Unfortunately an answer to Question (2) is not known at this time, so that question will have to be left unanswered in this thesis.

Our primary reference for material on multipliers on free groups found in this chapter is [23]. There are some original results in this chapter, particularly Proposition 6.3.1 and Theorem 6.3.6.

6.1 Completely Bounded Multipliers and Schur Multipliers on Discrete Groups

In this section, we will introduce for a discrete group G, the space $V^\infty(G)$ of Schur multipliers on $\mathcal{B}(\ell^2(G))$ and show how every function $\phi \in M_{cb}A(G)$ can be represented naturally as an element of $V^\infty(G)$.

Let G be a discrete group and let $\{\delta_g : g \in G\}$ denote the canonical orthonormal basis for $\ell^2(G)$. Recall that any operator $T \in \mathcal{B}(\ell^2(G))$ can be written uniquely as the infinite matrix

$$T = [T(g,h)]_{(g,h)\in G\times G},$$

where

$$T(g,h) := \langle T\delta_h | \delta_g \rangle_{\ell^2(G)}.$$

Definition 6.1.1. *A function $\sigma : G \times G \to \mathbb{C}$ is called a **Schur multiplier** of $\mathcal{B}(\ell^2(G))$ if the infinite matrix*

$$S_\sigma T := [\sigma(g,h)T(g,h)]_{(g,h)\in G\times G},$$

belongs to $\mathcal{B}(\ell^2(G))$ for all $T = [T(g,h)]_{(g,h)\in G\times G} \in \mathcal{B}(\ell^2(G))$. We denote vector space of Schur multipliers of $\mathcal{B}(\ell^2(G))$ by $V^\infty(G)$.

It is an easy consequence of the closed graph theorem that for any $\sigma \in V^\infty(G)$, the map S_σ is automatically a bounded linear map on $\mathcal{B}(\ell^2(G))$. Thus $V^\infty(G)$ is naturally a pointwise Banach algebra of functions on $G \times G$ when equipped with the operator norm inherited from $\mathcal{B}(\mathcal{B}(\ell^2(G)))$.

The following proposition gives us a useful way to manufacture examples of functions belonging to $V^\infty(G)$.

Proposition 6.1.2. *Let G be a discrete group, let $\mathcal{K}$ be a Hilbert space, and let $\xi, \eta : G \to \mathcal{K}$ be bounded functions. Then the function $\sigma_{\xi,\eta} : G \times G \to \mathbb{C}$ defined by*

$$\sigma_{\xi,\eta}(g,h) = \langle \xi(h)|\eta(g)\rangle_{\mathcal{K}}, \qquad \forall (g,h) \in G \times G,$$

belongs to $V^\infty(G)$, and furthermore $\|\sigma_{\xi,\eta}\|_{V^\infty(G)} \leq \|\xi\|_\infty \|\eta\|_\infty$.

Proof. Let u and v be finitely supported functions on G such that $\|u\|_2 = \|v\|_2 = 1$. Then for any $T = [T(g,h)]_{(g,h) \in G \times G} \in \mathcal{B}(\ell^2(G))$ we have

$$
\begin{aligned}
\left| \langle S_{\sigma_{\xi,\eta}} T u | v \rangle_{\ell^2(G)} \right| &= \left| \left\langle [\langle \xi(h)|\eta(g)\rangle_{\mathcal{K}} T(g,h)]_{(g,h)} u \big| v \right\rangle_{\ell^2(G)} \right| \\
&= \left| \sum_{g \in G} \sum_{h \in G} u(h)\overline{v}(g) \langle \xi(h)|\eta(g)\rangle_{\mathcal{K}} T(g,h) \right| \\
&= \left| \sum_{g \in G} \sum_{h \in G} \langle u(h)\xi(h)|v(g)\eta(g)\rangle_{\mathcal{K}} T(g,h) \right| \\
&=: \left| \left\langle [T(g,h)id_{\mathcal{K}}]_{(g,h)}(u\xi) \big| (v\eta) \right\rangle_{\ell^2(G,\mathcal{K})} \right| \\
&\leq \left\| [T(g,h)id_{\mathcal{K}}]_{(g,h)} \right\|_{\mathcal{B}(\ell^2(G,\mathcal{K}))} \cdot \|u\xi\|_{\ell^2(G,\mathcal{K})} \cdot \|v\eta\|_{\ell^2(G,\mathcal{K})} \\
&\leq \|T\|\|u\|_2\|\xi\|_\infty \|v\|_2\|\eta\|_\infty \\
&\leq \|T\|\|\xi\|_\infty \|\eta\|_\infty.
\end{aligned}
$$

Since u, v and $T \in \mathcal{B}(\ell^2(G))$ were arbitrary, it follows that $\|\sigma_{\xi,\eta}\|_{V^\infty(G)} = \|S_{\sigma_{\xi,\eta}}\| \leq \|\eta\|_\infty \|\xi\|_\infty$ and we are done. $\qquad\square$

Remark: Although it is not important for our purposes, it is worthwile to note that the converse to Theorem 6.1.2 is also true. That is, a function $\sigma : G \times G \to \mathbb{C}$ belongs to $V^\infty(G)$ if and only if there exists a Hilbert space $\mathcal{K}$ and bounded functions $\xi, \eta : G \to \mathcal{K}$ such that

$$\sigma_{\xi,\eta}(g,h) = \langle \xi(g)|\eta(h)\rangle_{\mathcal{K}}, \qquad \forall (g,h) \in G \times G.$$

Furthermore, $\|\sigma\|_{V^\infty(G)} = \inf\{\|\xi\|_\infty\|\eta\|_\infty : \sigma(\cdot,\cdot) = \langle \xi(\cdot)|\eta(\cdot)\rangle_{\mathcal{K}}\}$. See [40] for a proof of this result.

In the next theorem we show that the completely bounded multipliers of $A(G)$ can be nicely realized as certain coefficient type functions on G. This will allow us to see that $M_{cb}A(G)$ acts naturally as Schur multipliers on $\mathcal{B}(\ell^2(G))$.

Theorem 6.1.3. *Let G be a discrete group and let $\phi \in M_{cb}A(G)$. Then there exists a Hilbert space $\mathcal{K}$ and bounded functions $\xi, \eta : G \to \mathcal{K}$ such that*

$$\phi(gh^{-1}) = \langle \xi(h) | \eta(g) \rangle_{\mathcal{K}}, \qquad \forall g, h \in G. \tag{6.1}$$

Proof. Suppose that $\phi \in M_{cb}A(G)$. Then by definition, the map $m_\phi : A(G) \to A(G)$ defined by $m_\phi u = \phi u$ is a completely bounded map on $A(G)$. By duality, it then follows that the map

$$M_\phi := m_\phi^* : VN(G) \to VN(G) \subseteq \mathcal{B}(\ell^2(G))$$

which is given by $M_\phi \lambda(f) = \lambda(\phi f)$ for $f \in \ell^1(G)$, is completely bounded. By Theorem 2.2.2, there exists some Hilbert space $\mathcal{K}$, a $*$-representation $\pi : VN(G) \to \mathcal{B}(\mathcal{K})$, and bounded operators $V_i : \ell^2(G) \to \mathcal{K}$ $(i = 1, 2)$, with $\|M_\phi\|_{cb} = \|V_1\|\|V_2\|$, such that

$$M_\phi a = V_2^* \pi(a) V_1, \qquad \forall a \in VN(G).$$

In particular we have $\phi(x)\lambda(x) = M_\phi(\lambda(x)) = V_2^* \pi(\lambda(x)) V_1$ for all $x \in G$. If we now let $\{\delta_g : g \in G\}$ denote the canonical orthonormal basis for $\ell^2(G)$, then for any $g, h \in G$ we get

$$
\begin{aligned}
\phi(gh^{-1}) &= \langle \phi(gh^{-1})\delta_g | \delta_g \rangle_{\ell^2(G)} \\
&= \langle \phi(gh^{-1})\lambda(gh^{-1})\delta_h | \delta_g \rangle_{\ell^2(G)} \\
&= \langle V_2^* \pi(\lambda(gh^{-1})) V_1 \delta_h | \delta_g \rangle_{\ell^2(G)} \\
&= \langle \pi(\lambda(h^{-1})) V_1 \delta_h | \pi(\lambda(g^{-1})) V_2 \delta_g \rangle_{\mathcal{K}}.
\end{aligned}
$$

Defining $\xi(x) = \pi(\lambda(x^{-1})) V_1 \delta_x$ and $\eta(x) = \pi(\lambda(x^{-1})) V_2 \delta_x$ for $x \in G$, we obtain

$$\phi(gh^{-1}) = \langle \xi(h) | \eta(g) \rangle_{\mathcal{K}}, \qquad \forall g, h \in G,$$

with $\|\xi\|_\infty \leq \|V_2\| < \infty$ and $\|\eta\|_\infty \leq \|V_1\| < \infty$. $\qquad \square$

Remark: The converse of Theorem 6.1.3 is also true. Refer to [30] for a proof of this.

If we now put together Proposition 6.1.2 and Theorem 6.1.3, we obtain the following useful corollary.

Corollary 6.1.4. *Let G be a discrete group and let $\phi \in M_{cb}A(G)$. Then the function $\sigma_\phi : G \times G \to \mathbb{C}$ defined by*

$$\sigma_\phi(g, h) = \phi(gh^{-1}), \qquad \forall(g, h) \in G \times G$$

belongs to $V^\infty(G)$.

Note that Corollary 6.1.4 provides a necessary condition for a function $\phi : G \to \mathbb{C}$ to belong to $M_{cb}A(G)$. (In fact it follows from the remarks which follow Proposition 6.1.2 and Theorem 6.1.3 that the condition given in Corollary 6.1.4 is also a sufficient.)

In Section 6.3, we will use Corollary 6.1.4 as a tool to distinguish between elements of $MA(G)$ and $M_{cb}A(G)$ when G is a noncommutative free group. This will enable us to find complemented ideals in $A(G)$ that are not completely complemented.

6.2 Multipliers on Free Groups

We will now restrict our attention to noncommutative free groups.

Let $N \geq 2$ be a fixed positive integer. Throughout this section, $\mathbb{F}_N$ will denote the noncommutative free group on N fixed generators $\{x_1, \ldots, x_n\}$.

Observe that any $g \in \mathbb{F}_N$ has a unique expression as a finite product of generators x_i and their inverses x_i^{-1} which does not contain any two adjacent inverse factors ($x_i x_i^{-1}$ or $x_i^{-1} x_i$). We will call this unique expression the **word** for g. The number of factors in the word for g is called the **length** of g, and is denoted by $|g|$. For example, if $g = x_1 x_2 x_1^{-2} x_3 x_2$, then $|g| = 6$. For the unit $e \in \mathbb{F}_N$, we have the convention $|e| = 0$. In this way, we obtain a function

$$|\cdot| : \mathbb{F}_N \to \mathbb{N} \cup \{0\}.$$

We will call the function $|\cdot|$ the **length function** for $\mathbb{F}_N$.

The length function and its connection to harmonic analysis on $\mathbb{F}_N$ has been extensively studied by Haagerup in [23]. In particular, the following result ([23] Lemma 1.7) will be useful to us.

Theorem 6.2.1. *Let $N \geq 2$ be a positive integer and let $\phi : \mathbb{F}_N \to \mathbb{C}$ be a function for which*

$$\sup_{g \in \mathbb{F}_N} |\phi(g)|(1 + |g|)^2 < \infty.$$

Then $\phi \in MA(\mathbb{F}_N)$ and

$$\|\phi\|_{MA(\mathbb{F}_N)} \leq 2 \sup_{g \in \mathbb{F}_N} |\phi(g)|(1 + |g|)^2.$$

We would like to extend Theorem 6.2.1 to noncommutative free groups with infinitely many generators. This can easily be done since the inequlities in Theorem 6.2.1 have no dependence on the number N of generators of $\mathbb{F}_N$.

Corollary 6.2.2. *Let $\mathbb{F}_I$ be a noncommutative free group on infinitely many generators $\{x_i\}_{i \in I}$ and let $\phi : \mathbb{F}_I \to \mathbb{C}$ be a function for which*

$$\sup_{g \in \mathbb{F}_I} |\phi(g)|(1+|g|)^2 < \infty.$$

Then $\phi \in MA(\mathbb{F}_I)$ and

$$\|\phi\|_{MA(\mathbb{F}_I)} \leq 2 \sup_{g \in \mathbb{F}_I} |\phi(g)|(1+|g|)^2.$$

Proof. Let $K(\mathbb{F}_I)$ denote the algebra of finitely supported complex valued functions on $\mathbb{F}_I$. Since $A(\mathbb{F}_I) = \overline{K(\mathbb{F}_I)}^{\|\cdot\|_{A(\mathbb{F}_I)}}$ and $\phi K(\mathbb{F}_I) \subseteq K(\mathbb{F}_I)$, it suffices to show that

$$\|\phi u\|_{A(\mathbb{F}_I)} \leq 2 \sup_{g \in \mathbb{F}_I} |\phi(g)|(1+|g|)^2 \|u\|_{A(\mathbb{F}_I)}.$$

for every $u \in K(\mathbb{F}_I)$.

To see this, let $u \in K(\mathbb{F}_I)$ be arbitrary. Since the support of u is finite, we can find a finite subset $J \subseteq I$ such that u is supported on the finitely generated subgroup $\mathbb{F}_J = \langle x_i : i \in J \rangle \leq \mathbb{F}_I$. That is, $u \in K(\mathbb{F}_J)$. Since the natural inclusion $K(\mathbb{F}_J) \hookrightarrow K(\mathbb{F}_I)$ extends to a complete isometry $A(\mathbb{F}_J) \hookrightarrow A(\mathbb{F}_I)$, we get

$$
\begin{aligned}
\|\phi u\|_{A(\mathbb{F}_I)} &= \|\phi u\|_{A(\mathbb{F}_J)} \\
&\leq 2 \sup_{g \in \mathbb{F}_J} |\phi(g)|(1+|g|)^2 \|u\|_{A(\mathbb{F}_J)} \\
&\qquad \text{(by Theorem 6.2.1)} \\
&\leq 2 \sup_{g \in \mathbb{F}_I} |\phi(g)|(1+|g|)^2 \|u\|_{A(\mathbb{F}_I)}.
\end{aligned}
$$

$\square$

Remark: It is perhaps worth noting an interesting consequence of Corollary 6.2.2. Let I be any index set, let $M \in \mathbb{N}$, and let $E_M = \{g \in \mathbb{F}_I : |g| \leq M\}$. It follows from Corollary 6.2.2 that $I_M = \{\psi \in MA(\mathbb{F}_I) : \text{supp}\psi \subseteq E_M\} = \ell^\infty(E_M)$. That is, the ideal $I_M \subseteq MA(\mathbb{F}_I)$ is always as big as it possibly can be!

We are now in a position to construct complemented ideals in Fourier algebras of noncommutative free groups that are neither completely complemented nor weakly completely complemented.

6.3 The Construction

Let $\mathbb{F}_\infty$ denote the noncommutative free group on countably many generators $\{x_i\}_{i\in\mathbb{N}}$ and consider the set

$$E := \{x_i x_j^{-1} : 1 \le i \le j < \infty\} \subset \mathbb{F}_\infty.$$

The following proposition shows that the characteristic function 1_E is a bounded (but not completely bounded) multiplier of $A(\mathbb{F}_\infty)$.

Proposition 6.3.1. *The function 1_E belongs to $MA(\mathbb{F}_\infty)$ but not $M_{cb}A(\mathbb{F}_\infty)$.*

Proof. Observe that

$$\sup_{g\in\mathbb{F}_\infty} |1_E(g)|(1+|g|)^2 = \sup_{g\in E}(1+|g|)^2 = (1+2)^2 = 9,$$

so $1_E \in MA(\mathbb{F}_\infty)$ by Corollary 6.2.2.

Now let $\phi := 1_E$ and suppose, to get a contradiction, that $\phi \in M_{cb}A(\mathbb{F}_\infty)$. It then follows from this assumption and Corollary 6.1.4 that the function $\sigma_\phi : \mathbb{F}_\infty \times \mathbb{F}_\infty \to \mathbb{C}$ given by

$$\sigma_\phi(g,h) = \phi(gh^{-1}), \qquad \forall (g,h) \in \mathbb{F}_\infty \times \mathbb{F}_\infty$$

belongs to $V^\infty(\mathbb{F}_\infty)$.

Let us now consider the associated Schur multiplier S_{σ_ϕ}. Let $\{\delta_g : g \in \mathbb{F}_\infty\}$ denote the canonical orthonormal basis for $\ell^2(\mathbb{F}_\infty)$ and let $S = \{x_i\}_{i\in\mathbb{N}}$. Then $\mathcal{B}(\ell^2(S))$ can be identified with the corner $P\mathcal{B}(\ell^2(\mathbb{F}_\infty))P \subset \mathcal{B}(\ell^2(\mathbb{F}_\infty))$ where P is the orthogonal projection from $\ell^2(\mathbb{F}_\infty)$ onto the subspace $\ell^2(S)$. If $T = [T(x_i, x_j)]_{(i,j)\in\mathbb{N}\times\mathbb{N}} \in \mathcal{B}(\ell^2(S))$, then $S_{\sigma_\phi}T$ is given by the infinite matrix

$$S_{\sigma_\phi}T = [1_E(x_i x_j^{-1})T(x_i, x_j)]_{(i,j)\in\mathbb{N}\times\mathbb{N}}$$

where

$$1_E(x_i x_j^{-1}) = \begin{cases} 1, & \text{if } i \le j \\ 0, & \text{if } i > j \end{cases}$$

Thus the map $T \mapsto S_{\sigma_\phi}T$ is just the upper-triangular truncation map on $\mathcal{B}(\ell^2(S))$. Since $\ell^2(S)$ is not finite dimensional, it follows that upper triangular truncation is not bounded on $\mathcal{B}(\ell^2(S))$ (see problems 8.15 and 8.16 in [40]). The unboundedness of S_{σ_ϕ} contradicts the fact that $\sigma_\phi \in V^\infty(\mathbb{F}_\infty)$, and therefore we must have $\phi = 1_E \in MA(\mathbb{F}_\infty)\backslash M_{cb}A(F_\infty)$. $\qquad\square$

Remark: In Proposition 6.3.1 we showed the strict inclusion $M_{cb}A(\mathbb{F}_\infty) \subsetneq MA(\mathbb{F}_\infty)$. This result was first proved in a paper by M. Bozejko in 1981 ([6]). It should however be noted that Bozejko's proof is non-constructive in the sense that he does not write down an explicit function $\phi \in MA(\mathbb{F}_\infty) \backslash M_{cb}A(\mathbb{F}_\infty)$. In fact, according to the author's knowledge, the example given in Proposition 6.3.1 appears to be the only explicit example ever given of a multiplier of the Fourier algebra that is not completely bounded.

We will now show that Proposition 6.3.1 provides us with an example of an ideal in $A(\mathbb{F}_\infty)$ that is complemented but not completely complemented.

Corollary 6.3.2. *The ideal* $I(\mathbb{F}_\infty \backslash E) = \{u \in A(\mathbb{F}_\infty) : u\big|_{\mathbb{F}_\infty \backslash E} = 0\}$ *is complemented in* $A(\mathbb{F}_\infty)$ *but not completely complemented.*

Proof. Since $1_E \in MA(\mathbb{F}_\infty)$, the map $u \mapsto 1_E u$ is a bounded projection from $A(\mathbb{F}_\infty)$ onto $I(\mathbb{F}_\infty \backslash E)$. Thus $I(\mathbb{F}_\infty \backslash E)$ is (invariantly) complemented. Now, if $I(\mathbb{F}_\infty \backslash E)$ was completely complemented, then Corollary 5.5.10 would imply that $I(\mathbb{F}_\infty \backslash E)$ is invariantly completely complemented, which means that $1_E \in M_{cb}A(\mathbb{F}_\infty)$. This contradicts Proposition 6.3.1, and therefore $I(\mathbb{F}_\infty \backslash E)$ is not completely complemented. $\square$

At this point we know that the ideal $I(\mathbb{F}_\infty \backslash E) \subseteq A(\mathbb{F}_\infty)$ is not completely complemented. It is, however, conceivable that $I(\mathbb{F}_\infty \backslash E)$ could still be *weakly* completely complemented. (After all, we have already seen that it is possible for an ideal I to be weakly completely complemented when it is not even Banach space complemented - see Example 4.2.6 together with Theorem 5.4.5.) We will now show that the ideal $I(\mathbb{F}_\infty \backslash E) \subseteq A(\mathbb{F}_\infty)$ is not even weakly completely complemented. To do this, we will need to consider the cb-multiplier closure of $A(\mathbb{F}_\infty)$. We will now briefly introduce this completely contractive Banach algebra.

For any locally compact group G, the Fourier algebra $A(G)$ embeds completely contractively into $M_{cb}A(G)$. It is therefore possible to construct a new completely contractive Banach algebra from $A(G)$ by endowing $A(G)$ with the $M_{cb}A(G)$ operator space structure and taking the closure in the $M_{cb}A(G)$-norm. We denote this new completely contractive Banach algebra by $A_{cb}(G)$ and call it the **cb-multiplier closure of** $A(G)$. Since G is amenable if and only if $B(G) = M_{cb}A(G)$ completely isometrically, it follows that $A_{cb}(G) \neq A(G)$ if and only if G is non-amenable. Thus $A_{cb}(G)$ is only really of interest when G is a non-amenable group (like $\mathbb{F}_\infty$). We will need the algebra $A_{cb}(G)$ for the following two results, each of which is found in [20].

Theorem 6.3.3. *Let G be a discrete group such that $A_{cb}(G)$ is operator amenable, and let I be a weakly completely complemented closed ideal of $A(G)$. Then there is a subset $F \subseteq G$ with $1_F \in M_{cb}A(G)$ such that $I = I(F)$. In particular, I is invariantly completely complemented.*

Proof. See Theorem 3.4 of [20]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Theorem 6.3.4. $A_{cb}(\mathbb{F}_\infty)$ *is operator amenable.*

Proof. The countably generated free group $\mathbb{F}_\infty$ is a so-called *weakly amenable* group. That is, $A(\mathbb{F}_\infty)$ has an approximate identity which is bounded in the $\|\cdot\|_{M_{cb}A(\mathbb{F}_\infty)}$-norm (see [10]). Furthermore the group C*-algebra $C^*(\mathbb{F}_\infty)$ is well known to have a point seperating family of finite dimensional irreducible *-representations (this can be seen by observing that $\mathbb{F}_\infty$ has "lots" of finite group quotients). It then follows from Theorem 2.7 of [20] that $A_{cb}(\mathbb{F}_\infty)$ is operator amenable. $\qquad$ $\square$

Applying Theorems 6.3.3 and 6.3.4 to the complemented ideal $I(\mathbb{F}_\infty \backslash E) \subseteq A(\mathbb{F}_\infty)$, we get

Corollary 6.3.5. *The ideal $I(\mathbb{F}_\infty \backslash E) \subseteq A(\mathbb{F}_\infty)$ is weakly complemented, but not weakly completely complemented.*

Proof. Since $I := I(\mathbb{F}_\infty \backslash E)$ is complemented, it is obviously weakly complemented. If I was weakly completely complemented, it would follow from Theorem 6.3.3 that $1_E \in M_{cb}A(\mathbb{F}_\infty)$. But this contradicts Proposition 6.3.1, so I is not weakly completely complemented. $\qquad$ $\square$

By noting that any noncommutative free group on $N \geq 2$ generators contains an isomorphic copy of $\mathbb{F}_\infty$, we can easily extend Corollaries 6.3.2 and 6.3.5 to include these groups. We thus arrive at the main theorem of this chapter.

Theorem 6.3.6. *Let $\mathbb{F}_N$ be a noncommutative free group on $N \geq 2$ generators. Then $A(\mathbb{F}_N)$ contains a complemented ideal that is neither completely complemented nor weakly completely complemented.*

Proof. Fix $N \geq 2$ and let $a, b \in F_N$ be any two elements in free relation. Let $S = \{a^i b a^{-i}\}_{i \in \mathbb{N}}$. It is easy to see that S is a countable free set in $\mathbb{F}_N$ and hence the subgroup $H = \langle S \rangle \leq \mathbb{F}_N$ is isomorphic to the countably generated noncommutative free group $\mathbb{F}_\infty$. By Corollaries 6.3.2 and 6.3.5, the Fourier algebra $A(H)$ contains a complemented ideal I that is neither completely complemented nor weakly completely complemented. Since the natural inclusions $A(H) \hookrightarrow A(\mathbb{F}_N)$ and $VN(H) \hookrightarrow VN(\mathbb{F}_N)$ are complete isometries, and since the maps $u \mapsto 1_H u$ from $A(\mathbb{F}_N)$ onto $A(H)$ and $T \mapsto 1_H \cdot T$ from $VN(\mathbb{F}_N)$ onto

$VN(H)$ are completely contractive projections, it follows that the ideal $I \subseteq A(H) \hookrightarrow A(\mathbb{F}_N)$ is complemented in $A(\mathbb{F}_N)$ but neither completely complemented nor weakly completely complemented in $A(\mathbb{F}_N)$. $\square$

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