On A New Approach to Model Reference Adaptive Control

by

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A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Doctor of Philosophy
in
Electrical and Computer Engineering

Waterloo, Ontario, Canada, 2008

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

The objective of adaptive control is to design a controller that can adjust its behaviour to tolerate uncertain or time-varying parameters. An adaptive controller typically consists of a linear time-invariant (LTI) compensator together with a tuning mechanism which adjusts the compensator parameters and yields a nonlinear controller. Because of the nonlinearity, the transient closed-loop behaviour is often poor and the control signal may become unduly large. Although the initial objective of adaptive control was to deal with time-varying plant parameters, most classical adaptive controllers cannot handle rapidly changing parameters.

Recently, the use of a linear periodic (LP) controller has been proposed as a new approach in the field of model reference adaptive control [20]. In this new approach, instead of estimating plant parameters, the *ideal control signal* (what the control signal would be if the plant parameters and states were measurable) is estimated. The resulting controller has a number of desirable features:

- it handles rapid changes in the plant parameters,
- it provides nice transient behaviour of the closed-loop system,
- it guarantees that the effect of the initial conditions declines to zero exponentially, and
- it generates control signals which are modest in size.

Although the linear periodic controller (LPC) has the above advantages, it has some imperfections. In order to achieve the desirable features, a rapidly varying control signal and a small sampling period are used. The rapidly time-varying control signal requires fast actuators which may not be practical. The second weakness of the LPC [20] is poor noise rejection behaviour. The small sampling period results in large controller gains and correspondingly poor noise sensitivity, since there is a clear trade-off between tracking and noise tolerance. As the last drawback, this controller requires knowledge of the exact plant relative degree.

Here we extend this work in several directions:

i In [20], the ∞-norm is used to measure the signal size. Here we redesign the controller to yield a new version which provides comparable results when the more common 2-norm is used to measure signal size.

ii A key drawback of the controller of [20] is that the control signal moves rapidly. Here we redesign the control law to significantly alleviate this problem.

iii The redesigned controller can handle large parameter variation and in the case that the sign of high frequency gain is known, the closed-loop system is remarkably noise-tolerant.

iv We prove that in an important special case, we can replace the requirement of knowledge of the exact relative degree with that of an upper bound on the relative degree, at least from the point of view of providing stability.

v A number of approaches to improve the noise behaviour of the controller are presented.

Acknowledgements

This dissertation is a report of the research work that has been done since I started my studies at the University of Waterloo in 2004. During this time I have worked with a number of people whose contribution made this work possible. It is a great pleasure to convey my gratitude at the beginning of this report.

I gratefully acknowledge my supervisor Professor Daniel Miller for his instructions, guidance, and support and spending huge amounts of time and energy in every single step of my PhD studies. His outstanding method to approach, analyse, and solve scientific problems has made a unique researcher of him, for this I am grateful. It has been a great privilege and experience for me to work under his supervision. As a role model, Dr Miller taught me how to think and this resulted in an exceptional growth in my whole life and of course I will be indebted to him as long as I live.

My special thanks to the Natural Sciences and Engineering Council of Canada and the University of Waterloo to provide financial support for this work. This research has also given me the opportunity of coming to Canada and the privilege of living in a free and peaceful country.

I would acknowledge my committee members Professors Daniel Davison, John Thistle, and Kirsten Morris for their constructive advice, guidance, and feedback. It is also a pleasure to thank Dr Amir Aghdam, my external committee member from Concordia University.

I am grateful to Professors Dwight Aplevich, Andrew Heunis, and David Wang for their help and advice. I have always appreciated Professor Sherman Shen for his supervision as the associate chair for graduate studies and Wendy Boles, Karen Schooley, Annette Dietrich, and Katja Czarnecki for their assistance as the graduate studies administrative staff. Many thanks to the technical staff Philip Regier, Paul Ludwig, and Fernando Rivero Hernandez for their help.

My exceptional gratitude to my dear friends Mehri Mehrjoo, Masoud Barakati, and Shahrzad Towfighian. I would like to thank my companions Mary Helen Stroeder and Hugh Stroeder for their fulfilling love, generous care, opening their home to me, and sharing Canadian culture with me.

I gratefully appreciate Wilfred Kwok and Julie Vale for several technical discussions, giving advice, and spending time with me. My thanks to Daniel Stroeder for reviewing my materials.

In closing, I gratefully acknowledge my dearly-loved family for all they have done for me and thank them for supporting me in my studies in Canada.

Dedication

To my adorable parents for their unfailing love and support

and

my beloved sister, Mitra, for her true friendship

Contents

List of Figures

List of Acronyms

LP	Linear Periodic
LPC	Linear Periodic Controller
LPTV	Linear Periodic Time-Varying
LTI	Linear Time-Invariant
LTV	Linear Time-Varying
MMSE	Minimum Mean Squared Error
MRAC	Model Reference Adaptive Control
MRAS	Model Reference Adaptive System
PPAC	Pole Placement Adaptive Control
SIBO	Single Input Bounded Output
SISO	Single Input Single Output

List of Symbols

C	the set of complex numbers
D	the set of complex numbers which lie in the open unit disk
$\mathbf{C}^-$	the set of complex numbers with negative real part
R	the set of real numbers
$\mathbf{R}^+$	the set of nonnegative real numbers
Z	the set of integer numbers
$\mathbf{Z}^+$	the set of nonnegative integer numbers
N	the set of natural numbers
$\mathbf{a} >> \mathbf{b}$	real number a is much bigger than real number b

Chapter 1

Introduction

1.1 The Control Problem

A significant part of control engineering is the application of the mathematical study of systems. These applications span a range of complexities from the nanoscopic to the macroscopic and a range of scales from operational amplifiers to hydroelectric power plants. The central concept in control theory is to regulate the dynamic behaviour of a system (*the plant*) by using a device (*the controller*) such that the response or output of the system satisfies a set of specified constraints. Since controllers are described by mathematical equations, it is necessary to have a mathematical *model* of the plant to achieve the control objectives; the more accurate the mathematical model, the closer its responses to those of the actual plant. A good model is simple enough to be practical for the control design and complicated enough to accurately describe the behaviour of the actual plant. This trade-off introduces an elementary error into the system. The error is compounded by incomplete or inexact data from identification experiments, introducing a discrepancy between the mathematical model and the actual plant. Collectively, these inaccuracies are termed *uncertainty*. Uncertainties may be classified as being one of two types: structured and unstructured [1, 4] . A structured uncertainty in the plant model may be expressed in terms of parameters such as gains, pole and zero locations and also in terms of the transfer function parameters [2]. Unstructured uncertainties are associated with unmodeled dynamics, truncation of high frequency modes, nonlinearities, and the effects of linearization and even time-variation and randomness in the system [2]. In addition to uncertainty, in some cases the plant parameters are time-varying.

For a controller to keep the plant within specified constraints, the controller must satisfy the desired properties of both the mathematical model (e.g., asymptotic behaviour) and the physical plant (e.g., adapting to the discrepancies between the model and reality). Two common techniques for designing controllers to deal with uncertainty and variable parameters are *robust control* and *adaptive control*. The goal of robust control is to design a fixed (time-invariant) controller which provides both stability and acceptable performance, even in the presence of uncertain parameters.

Adaptive control is another design method for controlling uncertain plants. An adaptive controller typically consists of an LTI compensator together with a tuning mechanism, which adjusts the compensator gains to match the plant. Because of the modification law, a typical adaptive controller is nonlinear. *Model reference adaptive control* (MRAC) is one of the most important schemes in adaptive control. This approach uses a pre-designed stable *reference model* describing the desired input-output relationship; the objective is to modify the controller parameters such that the output of the plant with unknown parameters asymptotically tracks the reference model output. Because of the modification law, an MRAC is nonlinear.

Although MRAC was investigated as far back as Whitaker et al. [34, 41] in the mid 1950's and early 1960's, the proof of global stability was completed almost two decades later [6, 8, 24, 31, 32]. The solution is achievable under certain assumptions:

(i) the plant is minimum phase, and

(ii) an upper bound of the plant order,

(iii) the plant relative degree, and

(iv) the sign of high-frequency gain

are known. Later attempts showed that in assumption (iii) an upper bound of the plant relative degree is sufficient [38], and assumption (iv) is unnecessary [27].

Further work pointed out that the adaptive controllers of the 1970's cannot tolerate unmodeled dynamics and/or bounded disturbances and yield instability [36]. *Robust adaptive control* [9, 11, 12, 13, 28, 29] uses controller redesign to help reduce this problem. An adaptive controller is robust if the global stability of the closed-loop is guaranteed in the presence of *reasonable* classes of unmodeled dynamics and bounded disturbances. By the mid 1990's several efforts were carried out on performance and transients of the MRAC; one is based on multiple models, switching, and selection or tuning [10, 25, 26, 30].

Although the initial motivation of adaptive control was that of handling time-varying plant parameters, the initial schemes were able to control plants with unknown but fixed parameters. The first efforts were made in the mid 1980's; these control designs were based on applying a persistent exciting disturbance to the control signal or restriction of the parameter variations to be slow or have a known form [7, 15, 16, 33, 39, 40]. Another drawback of most classical adaptive controllers is that they are nonlinear, which may make it difficult to predict the transient closed-loop behaviour, and there may be large transients, especially if the initial estimate of parameters is poor. Furthermore, this undesired feature may cause a large control signal, which might lead to saturation.

To sum up, most adaptive control methods have the following disadvantages:

- they may not handle time-varying plant parameters well,
- the transient closed-loop behaviour may be poor,

- because of the nonlinearity, it is not proved that the effect of initial conditions decays exponentially to zero, and
- the control signal may be large.

So the question is: can we design an adaptive controller that does not have these imperfections? Miller [20] proposed a new approach to the model reference adaptive control problem which does not have the mentioned drawbacks. The early versions of this work discuss the problem for the first-order plants [19, 18]. This approach uses a *linear sampled-data periodic* controller; unlike classical adaptive control methods, which are based on parameter estimation, this method directly estimates the control signal. This point of view has been applied on LQR [23], model matching [22], and H_∞ problems [21] as well. This new approach, when applied to the MRAC problem, has the following advantages:

- it can handle time-varying plant parameters,
- the transient behaviour of the system is improved; immediate tracking can be achieved if the initial conditions of the plant and controller are the same, and if the initial conditions are not the same, then the deviation goes exponentially to zero, and
- the control signal is not large and it can be as close as desired to the *ideal control signal* (the ideal control signal is the one obtained if the plant parameters and state were known and the ideal LTI compensator would be applied; this signal is modest in size).

The LPC has its own undesired properties as well:

- during each controller period the control signal takes different values; since the controller period is small, this requires fast actuators,
- in order to achieve the desired tracking, a small sampling period is used which results in large controller gains; this may lead the system to poor noise tolerance; in fact, there is a trade-off between the desired tracking and noise tolerance, and
- the plant relative degree must be known.

Here we extend this work in several directions.

First of all, in [20] the time-domain ∞-norm is used to measure signal size. However, the time-domain 2-norm is an equally (if not more) common way to measure signal size. Here we extend the approach so that results can be obtained in the 2-norm setting which are comparable to those of [20]. This new setting requires a structural change in the control law together with new proofs of key steps.

Second of all, in [20] the control signal moves vigorously, in particular during the so-called Estimation Phase, when probing takes place; the control signal jumps rapidly from one value to another. Here the controller is redesigned to reduce the size and number of the jumps. The approach is as follows. In the signal control law [20], the controller is periodic with each period consisting of two phases: in Estimation Phase the *ideal control law* is estimated, while in Control Phase a naturally scaled version of this estimate is applied. Here we estimate the *change* in the ideal control signal law between periods, which typically reduces the size of the probing signal. We also carry out the probing more efficiently, to reduce the number of jumps in the control signal as well. While proving this approach requires a good deal of effort, the results are very nice: the control signal is smoother, the controller can handle large parameter variation, the performance is better (for a given sampling period), and in the case that the sign of the high frequency gain is fixed the closed-loop system is remarkably noise tolerant.

The third part of this PhD thesis is focussed on the plant relative degree assumption. One of the undesired aspects of the LPC is that the plant relative degree should be known. In this work, we show that under some restrictive assumptions, this requirement can be relaxed to that of requiring an upper bound of the plant relative degree, at least from the point of view of providing stability.

Finally, to improve the noise rejection, four different approaches are presented. In the first approach, a low pass filter is used at the plant output. In the second approach, we modify the LPC in order to obtain a smaller estimation error, which allows us to choose larger sampling times and consequently better noise tolerance. While the idea of MMSE estimation is used in the third approach to minimize the effect of noise, the fourth approach is based on applying a probing signal with a larger size in order to obtain better noise rejection. Finally, we combine the fourth approach with the redesigned controller (in the case that the sign of high frequency gain can change) to achieve the advantages of both methods. The *noise tolerant redesigned LPC* attains better noise rejection and performance as well as a better behaved control signal.

The outline of this work is as follows. In the next section we present some preliminary mathematics. Chapter 2 gives an overview of classical adaptive control. A high level explanation of the LPC proposed in [20] is presented in Chapter 3. In Chapter 4, a controller redesign is carried out so that the LPC [20] works when using the 2-norm. To achieve a smoother control signal and a larger sampling period, we redesign the LPC in Chapter 5. The robustness of the LPC, i.e. providing stability for plants with lower relative degree, is discussed in Chapter 6. Four different approaches to obtain better noise rejection are proposed in Chapter 7; in this chapter, we also present the noise tolerant redesigned LPC. Last of all, in Chapter 8, we provide a summary, concluding remarks and discuss future work.

1.2 Preliminary Mathematics

The Euclidean norm of the vector $x \in \mathbf{R}^n$, used to measure its size, is

$$\|x\| = \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}} = (xx^T)^{\frac{1}{2}},$$

and the corresponding induced norm of the matrix $M \in \mathbf{R}^{m\times n}$ is

$$\|M\| = \sup_{\|x\|=1} \frac{\|Mx\|}{\|x\|}.$$

The 2-norm of the continuous vector signal $y(t)$ is defined as

$$\|y\|_2 = \left(\int_0^{\infty} y^T(t)y(t)dt\right)^{\frac{1}{2}},$$

while the 2-norm of the discrete vector signal $y(kT)$ is defined as:

$$\|y\|_2 = (\sum_{k=0}^{\infty} y^T(kT)y(kT))^{\frac{1}{2}}.$$

For vector signals $y_1, y_2, \cdots, y_n$ we have

$$\|y_1 + y_2 + \cdots + y_n\|_2^2 \le n(\|y_1\|_2^2 + \|y_2\|_2^2 + \cdots + \|y_n\|_2^2),$$

so

$$\|y_1 + y_2 + \cdots + y_n\|_2 \le \sqrt{n}(\|y_1\|_2 + \|y_2\|_2 + \cdots + \|y_n\|_2).$$

We also define

$$\|y\|_{2,\,[t_1,t_2)} = \left(\int_{t_1}^{t_2} y^T(t)y(t)dt\right)^{\frac{1}{2}},$$

so if $0 \le t_1 \le t_1 \le \cdots \le t_n$ then

$$\sum_{i=1}^{n-1} \|y\|_{2,\,[t_i,t_{i+1})} \;\le\; \sqrt{n}\;\|y\|_{2,\,[t_1,t_n)}. \tag{1.1}$$

We will be using several Cauchy-Schwartz inequalities:

(i) For real numbers $x_i,\ y_j,\ \ i,j = 1,2,\cdots,n$ we have

$$|x_1y_1 + x_2y_2 + \cdots + x_ny_n| \le \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}\sqrt{y_1^2 + y_2^2 + \cdots + y_n^2}.$$

(ii) For integrable functions $f_1(x)$ and $f_2(x)$ on $[a,b]$ we have

$$\int_a^b f_1(x)f_2(x)dx \le \sqrt{\int_a^b f_1^2(x)dx}\sqrt{\int_a^b f_2^2(x)dx}.$$

We denote the set of piecewise continuous functions from $\mathbf{R}^+$ to $\mathbf{R}^{n\times m}$ as $PC(\mathbf{R}^{n\times m})$. For $f \in PC(\mathbf{R}^{n\times m})$, define

$$\|f\|_\infty := \mathrm{esssup}_{t\in\mathbf{R}^+}\|f(t)\|.$$

We let $PC_\infty(\mathbf{R}^{n\times m})$ denote the set of $f \in PC(\mathbf{R}^{n\times m})$ for which $\|f\|_\infty < \infty$. A function $f \in PC(\mathbf{R}^{n\times m})$ is *piecewise smooth* on $[a,b] \subset \mathbf{R}$ if there exists a finite set of points $\{x_i\}$,

$$a = x_1 < x_2 < \cdots < x_k = b,$$

such that, on each interval (x_i, x_{i+1}), $i = 1, \cdots, k$, the functions f and $\dot{f}$ are continuous and bounded, and both have finite limits as $x \to x_i$ and $x \to x_{i+1}$. If $f \in PC(\mathbf{R}^{n\times m})$ is piecewise smooth on every finite interval $[a,b] \subset \mathbf{R}^+$, then it is *piecewise smooth* and we write $f \in PS(\mathbf{R}^{n\times m})$. If for $f \in PS(\mathbf{R}^{n\times m})$, we have $\|f\|_\infty < \infty$ and $\|\dot{f}\|_\infty < \infty$, $\quad f \in PC(\mathbf{R}^{n\times m})$, then we write $PS_\infty(\mathbf{R}^{n\times m})$. For $m \geq 1$, we say that $f \in PS^m(\mathbf{R}^{n\times m})$ if $f, f^{(1)}, \cdots, f^{(m-1)}$ are absolutely continuous and $f^{(m)} \in PS(\mathbf{R}^{n\times m})$; we also define $PS^m_\infty(\mathbf{R}^{n\times m})$ to be the set of $f \in PS^m(\mathbf{R}^{n\times m})$ for which $\|f\|_\infty, \|\dot{f}\|_\infty, \cdots, \|f^{(m+1)}\|_\infty$ are all bounded. With $T > 0$, $PS(\mathbf{R}^{n\times m}, T)$ is the set of functions f such that every discontinuity of f and $\dot{f}$ are at most T time units apart; in a similar way $PS_\infty(\mathbf{R}^{n\times m}, T)$, $PS^m(\mathbf{R}^{n\times m}, T)$, and $PS^m_\infty(\mathbf{R}^{n\times m}, T)$ are defined. For simplicity, instead of writing $PC(\mathbf{R}^{n\times m})$ we omit $(\mathbf{R}^{n\times m})$ and write PC; in a similar way we simply write PC_∞, PS, PS_∞, PS^m, PS^m_∞, $PS(T)$, $PS_\infty(T)$, $PS^m(T)$, and $PS^m_\infty(T)$.

We say that $f : \mathbf{R}^+ \to \mathbf{R}^{n\times m}$ is of order T^j, and write $f = \mathcal{O}(T^j)$, if there exists constants $c_1 > 0$ and $T_1 > 0$ so that

$$||f(T)|| \leq c_1 T^j, \quad T \in (0, T_1).$$

On occasion we have a function f which depends not only on $T > 0$ but also depends implicitly on a variable θ restricted to a set $\mathcal{P} \subset PS$; we say that $f = \mathcal{O}(T^j)$ if there exist constants $c_1 > 0$ and $T_1 > 0$ so that

$$||f(T)|| \leq c_1 T^j, \quad T \in (0, T_1), \quad \theta \in \mathcal{P}.$$

Chapter 2

Classical Adaptive Control

2.1 Introduction

The objective of this chapter is to introduce classical identifier-based adaptive control methods together with their strengths and weaknesses. In this approach, an adaptive controller typically consists of an LTI compensator together with a tuning mechanism that adjusts the compensator parameters to match the plant, as illustrated in Figure 2.1. Because of tuning mechanism, the resulting controller is nonlinear.

Classical adaptive control has two different principal approaches: the *indirect* and *direct methods*. The indirect method recursively estimates plant parameters, and updates the controller parameters accordingly. The direct method adjusts controller parameters directly with no plant parameter identification [8]. Adaptive control has two important branches based on two different control objectives: *model reference control* and *pole placement*. In model reference control the objective is asymptotic tracking of a large class of signals as it is restricted to minimum phase plants; in pole placement the objective is stability and possibly asymptotic tracking of a small class of signals such as steps, and the minimum phase property is not required.

This chapter is organized in the following manner. Section 2.2 presents an

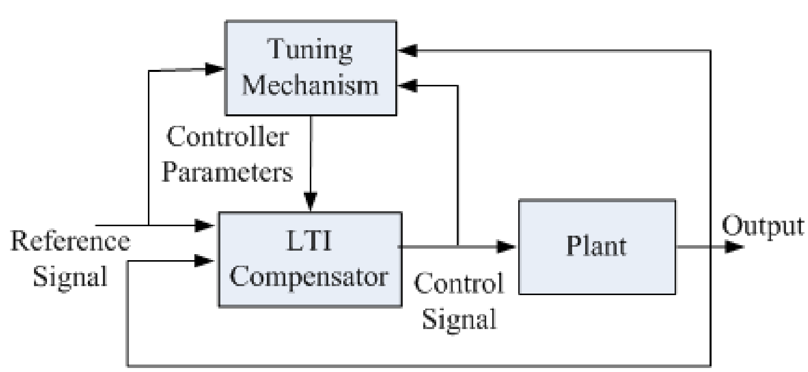

Figure 2.1: A typical adaptive system diagram

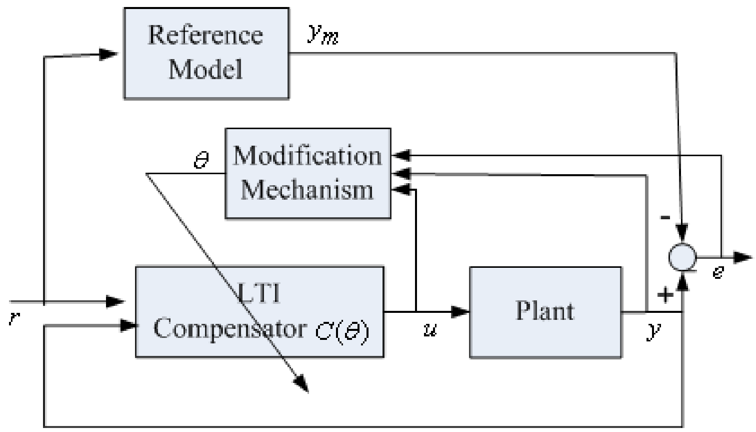

Figure 2.2: Block diagram of an MRAS [13]

overview of model reference adaptive control. The pole placement adaptive control problem is presented in Section 2.3. In each section, the structure of the controller is explained by an example.

2.2 Model Reference Adaptive Control (MRAC)

One of the most important schemes in adaptive control is MRAC. Figure 2.2 shows the block diagram of a classical MRAC. In this approach, there is a stable *reference model*, which describes the desired input-output properties of the closed-loop system. The control system consists of an LTI compensator $C(\theta)$ with modifiable parameters θ together with a modification mechanism (as tuning mechanism), which estimates the controller parameters $\theta(t)$ online. Here r is exogenous signal, which $e = y - y_m$ is the tracking error between the reference model output and the plant output.

The goal is to design the control law such that signals in the closed-loop system are bounded and for every bounded reference signal r all the tracking error e converges asymptotically to zero with time. The classical assumptions [8, 24, 32] are

- the plant is minimum phase,
- an upper bound on the plant order is known,
- the plant relative degree is known, and
- the sign of the plant high frequency gain is known.

2.2.1 Example

To illustrate MRAC, a simple example is presented. Consider the first-order, discrete-time plant

$$y(t+1) = ay(t) + bu(t), \quad y(0) = y_0, \quad t \geq 0, \tag{2.1}$$

where y and u are the plant output and input signals, respectively, and a and b are fixed but unknown plant parameters. The stable scalar reference model is described as follows:

$$y_m(t+1) = a_m y_m(t) + b_m r(t), \quad y_m(0) = y_{m_0}, \quad t \geq 0,$$

where y_m and r are the reference model output and input signals, respectively, and $|a_m| < 1$. The control objective is to design the control signal such that u and y are bounded and the tracking error $e(t) = y(t) - y_m(t)$ decays to zero. Here the unknown plant parameters are $\theta^* = \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbf{R}^2$, so if we choose the control law to be

$$\begin{aligned} u(t) &= \frac{1}{b}(b_m r(t) + (a_m - a)y(t)) \\ &= \underbrace{\begin{bmatrix} \frac{-(a-a_m)}{b} & \frac{b_m}{b} \end{bmatrix}}_{k^*(\theta^*)} \underbrace{\begin{bmatrix} y(t) \\ r(t) \end{bmatrix}}_{\phi(t)}, \end{aligned} \tag{2.2}$$

then the closed-loop system is as follows:

$$y(t+1) = a_m y(t) + b_m r(t), \quad y(0) = y_0.$$

Since a and b are unknown, k^* cannot be calculated directly, so it will be estimated. The idea is to recursively estimate θ^*, and then use this estimate in (2.2). Since we have a division by b in (2.2) , we must take care that the estimate of b is never zero. To this end, we adopt the Projection Algorithm (see [9]) for estimating the plant parameters.

With $c > 0$, $\hat{\theta}(0) = \begin{bmatrix} \hat{a}(0) \\ \hat{b}(0) \end{bmatrix} \in \mathbf{R}^2$ satisfying $\hat{b}(0) \neq 0$, and $\delta \in (0,1) \cup (1,2)$, we define the prediction error by

$$e_M(t) = y(t) - a_m\, y(t-1) - \phi^T(t-1)\, \hat{\theta}(t-1),$$

together with the parameter estimate update law:

$$\begin{bmatrix} \hat{a}(t) \\ \hat{b}(t) \end{bmatrix} = \hat{\theta}(t) = \hat{\theta}(t-1) + \gamma(t) \frac{\phi(t-1)}{c + \phi(t-1)^T \phi(t-1)} e_M(t), \quad \hat{\theta}(0) = \hat{\theta}_0;$$

the gain $\gamma(t)$ is given by

$$\gamma(t) = \begin{cases} 1 & if \ \ [\hat{\theta}(t-1) + \frac{\phi(t-1)}{c+\phi(t-1)^T\phi(t-1)} e_M(t)] \neq 0, \\ \delta & \text{otherwise,} \end{cases}$$

which ensures that $\hat{b}(t) \neq 0$ for $t \geq 0$. We now set

$$u(t) = \frac{1}{\hat{b}(t)}[-(\hat{a} - a_m)y(t) + b_m r(t)].$$

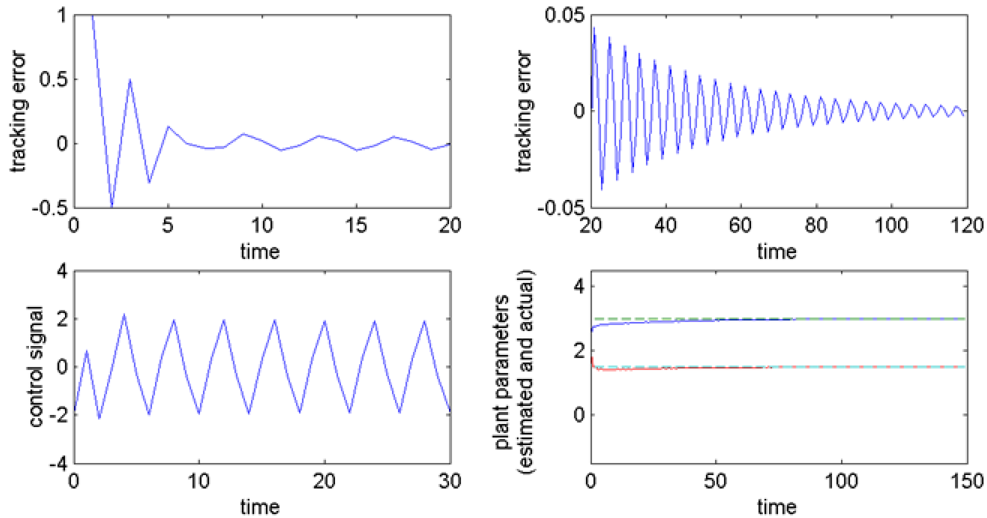

Figure 2.3: The simulation results for the first-order system and the MRAC applied with good initial estimates.

The control law is clearly highly nonlinear.

A simulation was carried out (see Figure 2.3) with $a = 3$, $b = 1.5$, $a_m = -0.5$, $b_m = -1$, $c = 1$, $\delta = 1.5$, the reference input $r(t) = cos(\pi t/2)$, the initial conditions $y(0) = 1$, $y_m(0) = 0$, and $\hat{\theta}(0) = \theta = [2.5 \quad 2]^T$. We see that the tracking error decays to zero and the control signal is bounded.

Remark 2.1: If the initial parameter estimate error is small, then the control signal is modest in size, the tracking error converges to zero, and the transient behaviour is guaranteed.

In the next simulation (see Figure 2.4) the initial estimates are poor, $\hat{\theta}(0) = [-1 \quad -1]^T$. Figure 2.4 shows that although the tracking error converges to zero and the estimation of parameters converges to the actual ones, but because the initial estimates are poor, the transient behaviour is poor, and the control signal is large.

Remark 2.2: In the face of poor initial estimates, the MRAC may yield a large control signal, and the transient behaviour is not guaranteed.

In the next simulation, the effect of time-varying parameters are studied. First we consider the case that the rate of parameters change is low. In this simulation,

$$\begin{bmatrix} a(t) \\ b(t) \end{bmatrix} = \begin{bmatrix} -2 + cos(\lambda t) \\ 1.5 + cos(\lambda t) \end{bmatrix}, \; a_m = -0.5,\; b_m = -1,\; c = 1,\; \delta = 1.5,$$

the reference input $r(t) = cos(\pi t/2)$, the initial condition $u(0) = 0$, $y(0) = -1$, $y_m(0) = 0$, and $\hat{\theta}(0) = [1 \quad 1]^T$. In Figure 2.5 we set $\lambda = 0.01$, so that the plant parameters are slowly time-varying; we see that the tracking error is small, the control and output signals are bounded, and the estimated parameters are close to the actual ones. In Figure 2.6, we set $\lambda = 0.1$, so the rate of parameters change is high; both the tracking error and the parameter estimation error are large, which

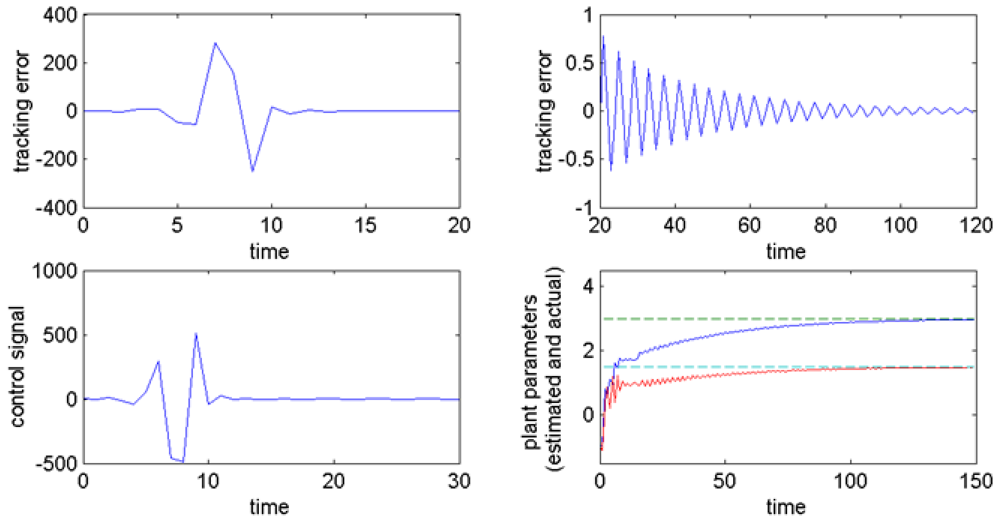

Figure 2.4: The simulation results for the first-order system and the MRAC applied with poor initial estimates.

illustrates that this controller cannot handle rapidly changing parameters.

2.3 Pole Placement Adaptive Control (PPAC)

In MRAC it is assumed that the plant is minimum phase, which allows a guarantee of exact asymptotic tracking. PPAC is an approach that does not require the plant to be minimum phase. However, exact asymptotic tracking is now guaranteed for only a small class of reference signals [9]. The other difference of MRAC and PPAC is the knowledge of plant order. While in MRAC, the designer needs to know an upper bound of the plant order, in PPAC the requirement is the exact plant order. In this method, a pole placement algorithm and parameter estimation are combined to yield a controller that attempts to place the closed-loop poles at desired locations; if designed properly, asymptotic tracking can be guaranteed for sinusoids of prescribed frequencies.

2.3.1 Example

In this section the structure of a PPAC is explained using a simple example. Consider the second-order system

$$y(t+2) = a_1 y(t+1) + a_2 y(t) + b_1 u(t+1) + b_2 u(t),$$

which can be a non-minimum phase system (i.e. $|\frac{b2}{b1}| > 1$) with unknown parameters a_1, a_2, b_1, and b_2. The desired closed-loop characteristic polynomial is as follows:

$$a^*(q^{-1}) = 1 - a_{m_1}\, q^{-1} - a_{m_2}\, q^{-2} - a_{m_3}\, q^{-3}, \tag{2.3}$$

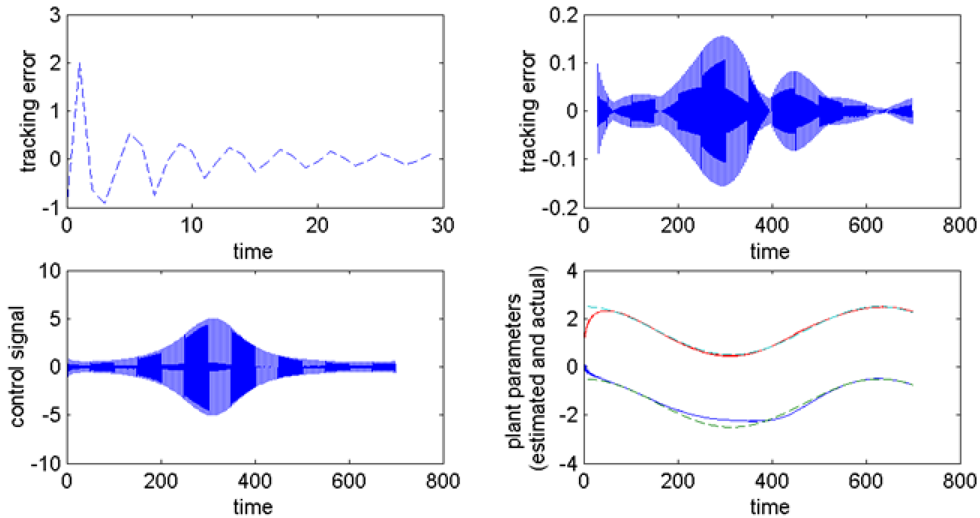

Figure 2.5: The simulation results for the first-order system with slowly time-varying parameters and the MRAC applied with $\lambda = 0.01$.

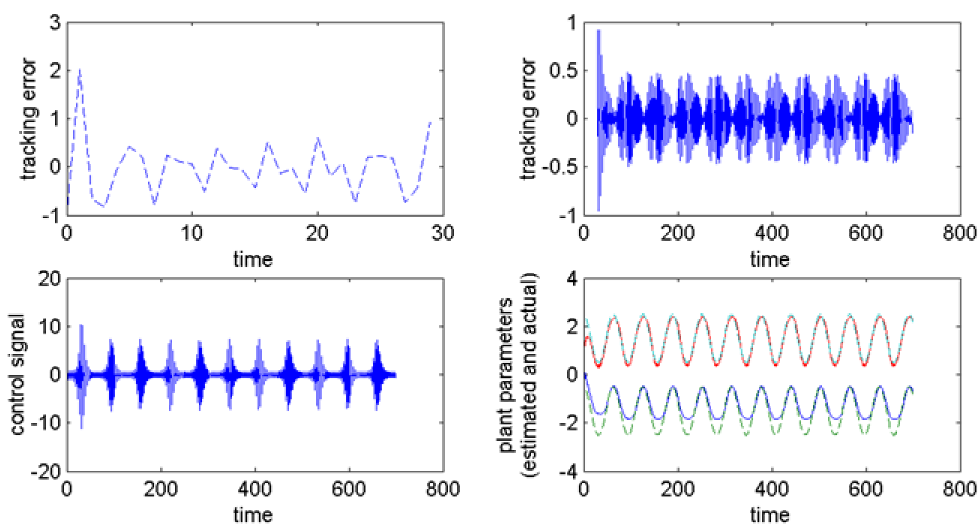

Figure 2.6: The simulation results for the first-order system with rapidly time-varying parameters and the MRAC applied with $\lambda = 0.1$.

where the operator q^{-1} is a unit delay [9]. The control objective is to design the controller such that u and y are bounded, and such that, in certain sense, the closed-loop characteristic polynomial asymptotically reaches (2.3); we also have a reference input $y^*(t)$. The PPAC requires that

$$a(q^{-1}) = 1 - a_1 q^{-1} - a_2 q^{-2}, \quad b(q^{-1}) = b_1 q^{-1} + b_2 q^{-2}$$

be co-prime. Define the vector of unknown parameters as

$$\theta^* = \begin{bmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{bmatrix} \in \mathbf{R}^4.$$

Since polynomials $a(q^{-1})$ and $b(q^{-1})$ are co-prime, we can use the Sylvester matrix to find polynomials

$$l(q^{-1}) = 1 + l_1 q^{-1}, \quad p(q^{-1}) = p_0 + p_1 q^{-1},$$

such that

$$a(q^{-1})l(q^{-1}) + b(q^{-1})p(q^{-1}) = a^*(q^{-1}),$$

If we choose the control law as

$$u(t) = [-l_1 u(t-1) + p_0(y^*(t) - y(t)) + p_1(y^*(t-1) - y(t-1))],$$

then the closed-loop system is as follows:

$$[a^*(q^{-1})y](t) = [b(q^{-1})p(q^{-1})y^*](t).$$

Since θ^* is unknown, the ideal control law cannot be calculated. Instead, the elements of θ are estimated; estimates of l and p are computed accordingly, and these estimates are used in the control law. Since we need the estimates of a and b to be co-prime, we assume prior knowledge of a *convex closed* set S in the parameter space $\mathbf{R}^4$ for which the corresponding polynomials are co-prime. Because of the properties of S, we can project any estimate that is located outside S uniquely to the closest one in S; we label this projection Π.

With $c > 0$, $\hat{\theta}(0) = \hat{\theta}_0 \in \mathbf{R}^4$, and $\beta \in (0, 2)$, we define the control law as follows:

$$\begin{aligned}
e_P(t) &= y(t) - \phi^T(t-1)\,\hat{\theta}_p(t-1), \\
\hat{\theta}(t) &= \Pi[\hat{\theta}(t-1) + \beta\frac{\phi(t-1)}{c + \phi(t-1)^T\phi(t-1)}e_P(t)], \;\; \hat{\theta}(0) = \hat{\theta}_0, \\
\hat{a}(t, z^{-1}) &= 1 + \hat{\theta}_{(1)}(t)z^{-1} + \hat{\theta}_{(2)}(t)z^{-2}, \\
\hat{b}(t, z^{-1}) &= \hat{\theta}_{(3)}(t)z^{-1} + \hat{\theta}_{(4)}(t)z^{-2},
\end{aligned}$$

solve

$$\hat{a}(t, z^{-1})\hat{l}(t, z^{-1}) + \hat{b}(t, z^{-1})\hat{p}(t, z^{-1}) = a^*(z^{-1}),$$

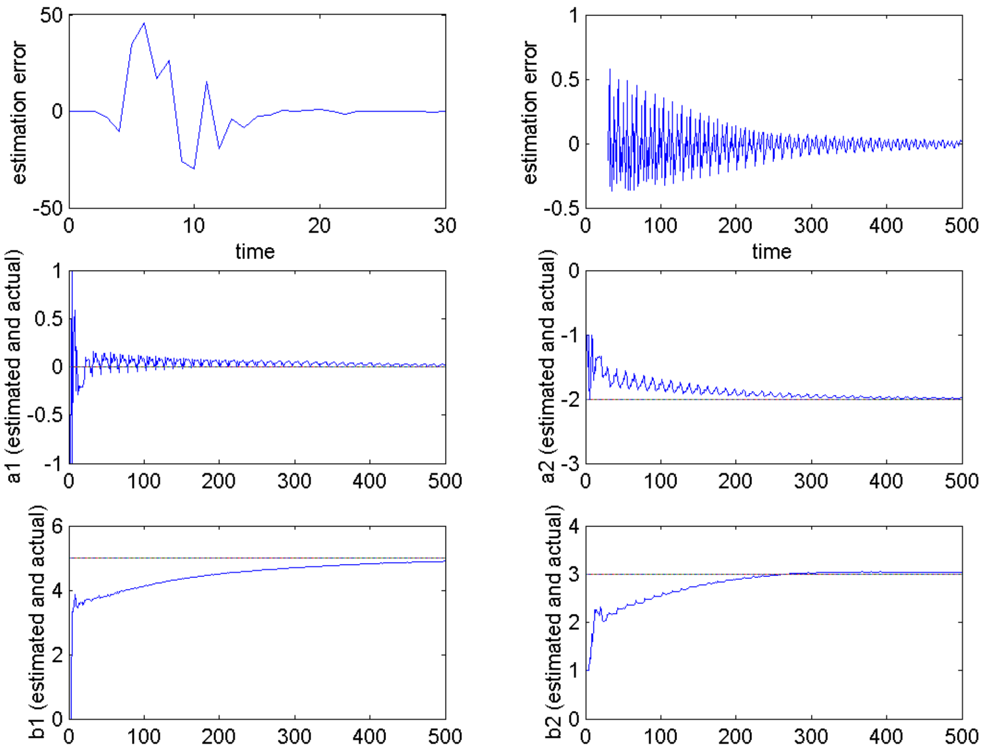

Figure 2.7: The simulation results for the second-order system when the PPAC is applied

uniquely for $\hat{l}$, $\hat{b}$, and then set

$$u(t) = -\hat{l}_1(t)u(t-1) + \hat{p}_0(t)(y^*(t) - y(t)) + \hat{p}_1(t)(y^*(t-1) - y(t-1))$$

Similar to the case of MRAC, it is clear that this control law is nonlinear. A simulation was carried out (see Figure 2.7) with $a(z^{-1}) = 1 + 2z^{-2}$, $b(z^{-1}) = 5z^{-1} + 3z^{-2}$, $a^*(z^{-1}) = 1 + z^{-1} + 0.25z^{-2}$, $c = 1$, $\beta = 1$, the reference input $y^*(t) = cos(\pi t/2) + sin(\pi t/3)$, the initial condition $y(0) = y(1) = 1$, $u(0) = u(1) = u(2) = 0$, and $\hat{\theta}_0 = [-1 \quad -1 \quad -1 \quad 1]^T$. We see the control and output signals are bounded and because the estimated parameters converge to the actual ones, the closed-loop polynomial converges to the desired one. As Figure 2.7 shows, the estimation error e_p declines to zero. However, in this case as well, the transients are large.

2.4 Summary and Concluding Remarks

In this chapter we introduced classical adaptive control methods, and demonstrated their strengths and weaknesses. In these methods an identifier is used to estimate unknown plant (or LTI compensator) parameters, and these are used to update compensator parameters accordingly. The approach can tolerate parameter uncertainty and slowly time-varying parameters. While the asymptotic behaviour is always guaranteed, the transient behaviour may be poor if the parameter uncertainty is too large and the initial estimate is poor. Furthermore, the nonlinearity embedded in the controller makes the behaviour hard to predict. In the next chapter we discuss an alternative approach to adaptive control which alleviates these drawbacks and tolerates rapid parameter variation.

Chapter 3

A Linear Periodic Approach to MRAC

3.1 Introduction

As discussed in Chapter 2, classical identifier-based adaptive control methodologies have several negative features:

- they cannot handle rapidly time-varying plant parameters,
- the transient closed-loop behaviour may be poor,
- they are nonlinear, which may make the control signal unduly large, the effect of the initial conditions and the input cannot be separated out, and it is not proven that the effect of initial conditions declines to zero.

In response to this, a new approach to the MRAC was proposed in [20]. The approach yields a linear periodic controller (LPC), and it has the following desirable features:

- it allows for rapidly time-varying parameters,
- it provides smooth transient behaviour (immediate tracking rather than asymptotic),
- the effect of initial conditions decays exponentially to zero, and
- the control signal is modest in size.

This chapter presents an overview to the idea together with its strengths and weaknesses. In this approach, instead of estimating controller parameters (direct method) or plant parameters (indirect method), the *ideal control signal* (what the control signal would be if the plant states and parameters were known) is estimated. The system is periodically probed to estimate the value of the ideal control signal, and then a suitably weighted estimate is applied. The control period is divided into two phases: an Estimation Phase and a Control Phase; during the Estimation

Phase the ideal control signal is estimated, while in the Control Phase a suitably weighted estimate of the ideal control signal is applied to the plant.

The outline of this chapter is as follows. The problem setup is presented in Section 3.2. Section 3.3 deals with modeling of the uncertain plant. A high level explanation of the LPC is presented in Section 3.4. While Section 3.5 gives the controller construction, in Section 3.6 everything is pulled together to prove the main result. Finally we carry out two simulations in Section 3.7.

3.2 Problem Formulation

In this section the structure of the set of uncertain plants, reference model, and the controller are presented. The SISO linear, time-varying plant P is described by

$$\begin{array}{rcl} \dot{x}(t) & = & A(t)x(t) + B(t)u(t),\ x(t_0) = x_0, \\ y(t) & = & C(t)x(t), \end{array} \tag{3.1}$$

with $x(t) \in \mathbf{R}^n$ the plant state, $u(t) \in \mathbf{R}$ the plant input, and $y(t) \in \mathbf{R}$ the plant output; we associate the plant with the triple $(A(t), B(t), C(t))$. We allow a good deal of model uncertainty, to be described in detail later on; the set of possible models is labelled $\mathcal{P}$.

The stable SISO LTI reference model P_m is given by

$$\begin{array}{rcl} \dot{x}_m(t) & = & A_m x_m(t) + B_m u_m(t),\ x_m(t_0) = x_{m_0}, \\ y_m(t) & = & C_m x_m(t), \end{array}$$

with $x_m(t) \in \mathbf{R}^{n_m}$ the reference model state, $u_m(t) \in \mathbf{R}$ the reference model input, and $y_m(t) \in \mathbf{R}$ the reference model output. The reference model describes the desired behaviour of the closed-loop system, and is chosen to be stable. The goal is to design a controller that provides stability and makes the plant output track the model reference output. To this end, we define the tracking error as

$$e(t) := y_m(t) - y(t).$$

The controller is a sampled data controller, which periodically samples u_m and y. We use an anti-aliasing filter before sampling u_m: with $\sigma > 0$ we choose an anti-aliasing filter of the form

$$\dot{\bar{u}}_m = -\sigma \bar{u}_m + \sigma u_m, \quad \bar{u}_m(t_0) = \bar{u}_{m_0}, \tag{3.2}$$

whose input-output map (with zero initial conditions) is labelled F_σ. Accordingly, we define a new version of the reference model with $\bar{u}_m$ as the input:

$$\begin{array}{rcl} \dot{\bar{x}}_m & = & A_m \bar{x}_m + B_m \bar{u}_m, \quad \bar{x}_m(t_0) = x_{m_0}, \\ \bar{y}_m & = & C_m \bar{x}_m. \end{array} \tag{3.3}$$

If we choose $\gamma_m > 0$ and $\lambda_m < 0$ so that

$$||e^{A_m t}|| \;\; \leq \;\; \gamma_m e^{\lambda_m t}, \quad t \geq 0,$$

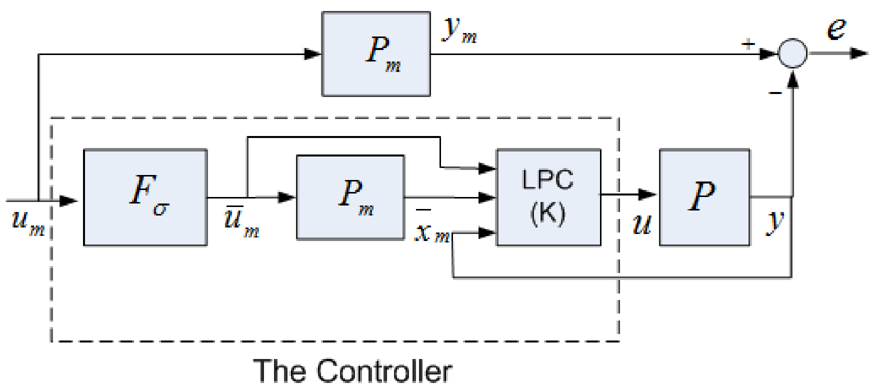

Figure 3.1: The feedback diagram

it follows that

$$\begin{aligned} |\bar{y}_m(t) - y_m(t)| \;\; &\leq \;\; ||B_m|| \times ||C_m|| \frac{\gamma_m + 1}{\sigma - ||A_m||} e^{\lambda_m t} |\bar{u}_{m_0}| + \\ &\quad \frac{||B_m|| \times ||C_m||}{\sigma - ||A_m||} [\gamma_m \frac{||A_m||}{|\lambda_m|} + 1] \; ||u_m||_\infty, \;\; t \geq 0. \end{aligned} \tag{3.4}$$

Last of all, let us define the linear periodic controller by

$$\begin{aligned} z[k+1] &= F(k)z[k] + G(k)y(kh) + H(k)\bar{x}_m(kh) + J(k)\bar{u}_m(kh), \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad z[k_0] = z_0 \in \mathbf{R}^l, \\ u(kh+\tau) &= L(k)z[k] + M(k)y(kh), \qquad\qquad\qquad \tau \in [0, h), \end{aligned} \tag{3.5}$$

whose gains F, G, H, J, and L are periodic of period $p \in \mathbf{N}$; the sampling time is h and the period of the controller is $T := ph$, and we associate this system with the 7-tuple (F, G, H, J, L, h, p). Observe that (3.5) can be implemented with a sampler, a zero-order-hold, and an l^{th} order periodically time-varying, discrete-time system of period p .

Remark 3.1: *The reference model is chosen by the control system designer to embody the desired closed-loop behaviour. Hence, we can consider our controller to be a combination of the anti-aliasing filter (3.2), the reference model (3.3), and the discrete-time, periodic compensator (3.5), as indicated in Figure 3.1.*

The feedback configuration is given in Figure 3.1. As the figure shows, the controller is a mixture of discrete and continuous subsystems, so the closed-loop state is a combination of discrete and continuous states, defined as

$$x_{sd}(t) = \begin{bmatrix} x(t) \\ \bar{x}_m(t) \\ \bar{u}_m(t) \\ z[k] \end{bmatrix}, \;\; t \in [kh, (k+1)h).$$

Definition 3.1 [20]: *The controller (3.2), (3.3) and (3.5)* exponentially stabilizes *$\mathcal{P}$ if there exist constants $\gamma > 0$ and $\lambda < 0$ so that, for every $P \in \mathcal{P}$, set of initial* *conditions* x_0*,* $\bar{x}_{m_0}$*,* $\bar{u}_{m_0}$*, and* z_0*, set of initial times* $k_0 \in \mathbf{Z}^+$ *and* $t_0 = k_0 h$*, with* $u_m(t) = 0$ *for* $t \geq t_0$ *we have*

$$\|x_{sd}(t)\| \leq \gamma e^{\lambda(t-t_0)} \|x_{sd}(t_0)\|, \quad t \geq t_0.$$

3.3 Modeling of the Uncertain Plant

At this point we must explain in more detail the class of model uncertainty which will be tolerated here. We adopt the setup of [20], which we repeat here for the reader's convenience. The classical assumptions adopted in MRAC are: (i) the plant is minimum phase, and (ii) an upper bound n on the plant order is known, (iii) the plant relative degree m is known, and (iv) the sign of the high-frequency gain g is known. We make natural time-varying generalizations of Assumptions (i), (ii), and (iii), and replace (iv) by a lower bound on its magnitude. Given that we wish to prove *uniform tracking*, independent of the plant parameters, we will require that the parameters lie in a compact set. We also make some technical assumptions on the smoothness of certain parameters.

We start with a general LTV differential of an input-output form, defined in terms of the auxiliary variable η:

$$\begin{aligned} \textstyle\sum_{i=0}^{n} a_i(t) D^i \eta &= g(t) u \\ y &= \textstyle\sum_{i=0}^{n-m} b_i(t) D^i \eta; \end{aligned} \tag{3.6}$$

we assume that the differential equation is normalized such that $a_n = b_{n-m} = 1$. Using the definition $(D^i f)(t) := \frac{d^i f}{dt^i}(t)$, we define the input and output operators by

$$a(D,t) := \sum_{i=0}^{n} a_i(t) D^i, \quad b(D,t) := \sum_{i=0}^{n-m} b_i(t) D^i.$$

Clearly if the parameters are time-invariant, then the differential equation yields a transfer function of

$$g \frac{b(s)}{a(s)}.$$

Following [20], the goal is to isolate the zero dynamics and then use the feedback linearization technique. With the definition of the state variables as

$$w := \begin{bmatrix} \eta \\ D\eta \\ \vdots \\ D^{n-m-1}\eta \end{bmatrix}, \quad v := \begin{bmatrix} y \\ Dy \\ \vdots \\ D^{m-1}y \end{bmatrix},$$

we can write the zero dynamics as follows:

$$\dot{w}(t) = \underbrace{\begin{bmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ -b_0(t) & -b_1(t) & \cdots & -b_{n-m-1}(t) \end{bmatrix}}_{=:A_1(t)} w(t) + \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}}_{=:b_1} y(t). \tag{3.7}$$

Now we need to express $\dot{v}$ in terms of v and w. It is enough to find polynomials

$$\beta(D,t) = \sum_{i=0}^{m} \beta_i(t) D^i, \quad \alpha(D,t) = \sum_{i=0}^{n-m-1} \alpha_i(t) D^i$$

with $\beta_m(t) = 1$, which satisfy

$$\begin{aligned} \dot{v}(t) &= \underbrace{\begin{bmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ -\beta_0(t) & -\beta_1(t) & \cdots & -\beta_{m-1}(t) \end{bmatrix}}_{=:A_2(t)} v(t) + \\ & \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}}_{=:b_2} \underbrace{\begin{bmatrix} \alpha_0(t) & \cdots & \alpha_{n-m-1}(t) \end{bmatrix}}_{=:c_1(t)} w(t) + \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix} g(t)u(t). \end{aligned} \tag{3.8}$$

With

$$c_2 = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}, \tag{3.9}$$

we would thereby end up with a state-space model of

$$\begin{aligned} \begin{bmatrix} \dot{w}(t) \\ \dot{v}(t) \end{bmatrix} &= \underbrace{\begin{bmatrix} A_1(t) & b_1 c_2 \\ b_2 c_1(t) & A_2(t) \end{bmatrix}}_{:=A(t)} \underbrace{\begin{bmatrix} w(t) \\ v(t) \end{bmatrix}}_{:=x(t)} + g(t) \underbrace{\begin{bmatrix} 0 \\ b_2 \end{bmatrix}}_{:=B} u(t) \\ y &= \underbrace{\begin{bmatrix} 0 & c_2 \end{bmatrix}}_{:=C} \begin{bmatrix} w(t) \\ v(t) \end{bmatrix}. \end{aligned} \tag{3.10}$$

In the case that the differential equation is time-invariant (a and b are fixed), α and β can be obtained by long division:

$$\frac{a(s)}{b(s)} = \beta(s) - \frac{\alpha(s)}{b(s)}.$$

Now the question is: under what conditions on a, b, and g are sufficient to guarantee that we can find the polynomials of α and β satisfying (3.8)?

It turns out that it is sufficient that there be bounds on the derivative(s) of these parameters, although jumps are also tolerated. To proceed we collect all of the parameters of the plant (3.6) into a single vector:

$$\theta(t) := \left[\; a_0(t) \; \cdots \; a_{n-1}(t) \;\; b_0(t) \; \cdots \; b_{n-m-1}(t) \;\; g(t) \; \right]^T \in \mathbf{R}^{2n-m+1}.$$

The parameters of the transformed system (3.10) can be collected into another vector:

$$\bar{\theta}(t) := \left[\; \alpha_0(t) \; \cdots \; \alpha_{n-m-1}(t) \;\; \beta_0(t) \; \cdots \; \beta_{m-1}(t) \;\; b_0 \; \cdots \; b_{n-m-1} \;\; g(t) \; \right]^T.$$

Now we list some assumptions on the original parameter θ which are sufficient for the existence of $\bar{\theta}$.

Assumptions on the vector θ [20].

Assumption 1: (Compact set) There exists a compact set $\Gamma \subset \mathbf{R}^{2n-m+1}$ so that $\theta(t) \in \Gamma$ for all $t \geq 0$.

Assumption 2: (Infrequent jumps) There exists a $T_0 > 0$ so that $\theta \in PS_\infty(T_0)$.

Assumption 3: (Bounded derivative) There exists a constant μ_1 so that $\|\dot{\theta}\|_\infty \leq \mu_1$.

Assumption 4: (Regularity of g) There exists a positive constant $\underline{g}$ so that $|g(t)| \geq \underline{g}$, $t \geq 0$.

Assumption 5: (Smoothness of b) The parameters $b_i \in PS^m_\infty(T_0)$, $i = 0, 1, \cdots, n-m-1$; also, there are constants $\delta_j > 0$, $j = 1, \cdots, m+1$, so that

$$\|D^j(b_i)\|_\infty \leq \delta_j, \;\; i = 0, 1, \ldots, n-m-1, \;\;\; j = 1, \cdots, m+1.$$

Assumption 6: (Uniformly exponentially stable zero dynamics) There exist constants $\gamma_0 > 0$ and $\lambda_0 < 0$ so that the transition matrix Φ_{A_1} corresponding to A_1 satisfies

$$\|\Phi_{A_1}(t, t_0)\| \leq \gamma_0 e^{\lambda_0 (t - t_0)}, \;\; t \geq t_0 \geq 0.$$

Remark 3.2: *Observe that because of the use of "essential supremum" in the definition of the norm, Assumption 3 does not rule out jumps in the parameters. Also, note that Assumption 6 is a generalization of the classical minimum phase requirement. Assumption 5 is a technical condition needed to ensure that we can carry out the desired transformation.*

Hence, for every $n \geq m \geq 1$, compact set $\Gamma \subset \mathbf{R}^{2n-m+1}$, set of positive constants μ_1, T_0, $\delta_1, \cdots, \delta_{m+1}$, $\underline{g}$, and γ_0, and negative constant λ_0, there exists a natural class of models of the form (3.6) which satisfy Assumptions 1-6; we label this class

$$\mathcal{P}(n, m, \Gamma, \mu_1, T_0, \delta_1, \cdots, \delta_{m+1}, \underline{g}, \gamma_0, \lambda_0).$$

It turns out that Assumptions 1-6 are sufficient to guarantee the existence of α, β, and hence $\bar{\theta}$. Before showing that, we list some key properties that we will require of the transformed vector $\bar{\theta}$.

Assumptions on the vector $\bar{\theta}$ [20].

Assumption 1′: (compact set) There exists a compact set $\bar{\Gamma}$ so that $\bar{\theta}(t) \in \bar{\Gamma}$ for all $t \geq 0$.

Assumption 2′: (Infrequent jumps) There exists a $\bar{T}_0 > 0$ so that $\bar{\theta} \in PS_\infty(\bar{T}_0)$.

Assumption 3′: (Bounded derivative) There exists a constant $\bar{\mu}_1$ so that $\|\dot{\bar{\theta}}\|_\infty \leq \bar{\mu}_1$.

Assumptions 4, 6

For every $n \geq m \geq 1$, compact set $\bar{\Gamma} \subset \mathbf{R}^{2n-m+1}$, set of positive constants $\bar{\mu}_1$, $\bar{T}_0$, $\underline{g}$, and γ_0, and negative constant λ_0, there exists a natural class of models of the form (3.7) and (3.8) which satisfy Assumptions $1' - 3'$ and Assumptions 4 and 6; we label this class

$$\bar{\mathcal{P}}(n, m, \bar{\Gamma}, \bar{\mu}_1, \bar{T}_0, \underline{g}, \gamma_0, \lambda_0).$$

We will not distinguish between the vector $\bar{\theta}$ and the corresponding model of the form (3.7) and (3.8).

Proposition 3.1 [20]: *For every set of uncertainty of the form*

$$\mathcal{P}(n, m, \Gamma, \mu_1, T_0, \delta_1, \cdots, \delta_{m+1}, \underline{g}, \gamma_0, \lambda_0)$$

there exist constants $\bar{\mu}_0$ *and* $\bar{T}_0$ *as well as a compact set* $\bar{\Gamma} \subset \mathbf{R}^{2n-m+1}$ *so that for every*

$$\theta \in \mathcal{P}(n, m, \Gamma, \mu_1, T_0, \delta_1, \cdots, \delta_{m+1}, \underline{g}, \gamma_0, \lambda_0)$$

there exist polynomials $\alpha(D, t)$ *and* $\beta(D, t)$ *satisfying (3.8) so that the corresponding new parameter vector* $\bar{\theta}$ *satisfies*

$$\bar{\theta} \in \bar{\mathcal{P}}(n, m, \bar{\Gamma}, \bar{\mu}_1, \bar{T}_0, \underline{g}, \gamma_0, \lambda_0).$$

From now on we will assume that our set of plant uncertainty is of the form

$$\mathcal{P}(n, m, \Gamma, \mu_1, T_0, \delta_1, \cdots, \delta_{m+1}, \underline{g}, \gamma_0, \lambda_0)$$

which, in turn, induces a class of uncertainty in our new parameter vector $\bar{\theta}$ of the form

$$\bar{\mathcal{P}}(n, m, \bar{\Gamma}, \bar{\mu}_1, \bar{T}_0, \underline{g}, \gamma_0, \lambda_0)$$

(or simply $\bar{\mathcal{P}}$) with corresponding state space models of the form (3.7) and (3.8). At this point we combine the plant, reference model and anti-aliasing filter to obtain the *generalized plant*:

$$\begin{bmatrix} \dot{w} \\ \dot{v} \\ \dot{\bar{x}}_m \\ \dot{\bar{u}}_m \end{bmatrix} = \underbrace{\begin{bmatrix} A_1(t) & b_1 c_2 & 0 & 0 \\ b_2 c_1(t) & A_2(t) & 0 & 0 \\ 0 & 0 & A_m & b_m \\ 0 & 0 & 0 & -\sigma \end{bmatrix}}_{=:\bar{A}(t)} \underbrace{\begin{bmatrix} w \\ v \\ \bar{x}_m \\ \bar{u}_m \end{bmatrix}}_{=:\bar{x}} + g(t) \underbrace{\begin{bmatrix} 0 \\ b_2 \\ 0 \\ 0 \end{bmatrix}}_{=:\bar{B}} u(t) + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ \sigma \end{bmatrix}}_{=:\bar{E}} u_m(t), \qquad \bar{x}(t_0) = \bar{x}_0, \tag{3.11}$$

$$y = \begin{bmatrix} 0 & c_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} w \\ v \\ \bar{x}_m \\ \bar{u}_m \end{bmatrix}, \tag{3.12}$$

$$\bar{e} = \underbrace{\begin{bmatrix} 0 & -c_2 & C_m & 0 \end{bmatrix}}_{=:\bar{C}} \begin{bmatrix} w \\ v \\ \bar{x}_m \\ \bar{u}_m \end{bmatrix}. \tag{3.13}$$

The goal is to construct the control law $u(t)$ such that the closed-loop stability is guaranteed and the map of $u_m \to \bar{e}$ is small. At this point we make a natural assumption on the relative degree of the reference model.

Assumption 7: The relative degree of the reference model is at least m.

3.4 The Approach

In this section, the proposed LPC is presented. We start with a high level description and then turn to a more concrete one. We start with a description of the *ideal LTI control law* which we would use in the case of complete state and parameter information. We then explain a sequence of three approximations to this control law, with the last one being of the form (3.5).

3.4.1 The Ideal Controller

To motivate the ideal control law, we start with the time-invariant, first-order case (so $y = x$ and $y_m = x_m$):

$$\begin{aligned} \dot{y} &= ay + gu, \\ \dot{\bar{y}}_m &= a_m \bar{y}_m + B_m \bar{u}_m. \end{aligned} \tag{3.14}$$

The control objective is to make the tracking error of $\bar{e} = \bar{y}_m - y$ small. Start with the tracking error dynamics:

$$(\dot{\bar{y}}_m - \dot{y}) = a_m(\bar{y}_m - y) + [b_m \bar{u}_m - gu + (a_m - a)y].$$

If we set

$$b_m \bar{u}_m - gu + (a_m - a)y = 0,$$

then the mismatch decays to zero like $e^{a_m t}$, which leads us to the following definition of the **ideal control law**:

$$u = \frac{1}{g}(b_m \bar{u}_m + (a_m - a)y) = \frac{1}{g} \underbrace{\begin{bmatrix} a_m - a & 0 & b_m \end{bmatrix}}_{=:K} \begin{bmatrix} y \\ \bar{y}_m \\ \bar{u}_m \end{bmatrix}.$$

This controller provides both stability and model matching.

The ideal control law in the general case uses the same basic idea. We first define

$$\Lambda_2 := \begin{bmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad f_2(t) := \begin{bmatrix} \beta_0(t) & \cdots & \beta_{m-1}(t) \end{bmatrix}. \tag{3.15}$$

Since (Λ_2, b_2) is controllable, we can choose a vector of $\bar{f}_2$ so that $\bar{A}_2 := \Lambda_2 + b_2\bar{f}_2$ is stable with eigenvalues with real parts less than λ_m, which means that there exists a constant $\bar{\gamma}_m > \gamma_m$ so that

$$\|e^{\bar{A}_2 t}\| \leq \bar{\gamma}_m e^{\lambda_m t}, \;\; t \geq 0.$$

If we now set

$$u(t) = \frac{1}{g(t)}[-c_1(t)w(t) + (\bar{f}_2 - f_2(t))v(t)] + u_n(t),$$

with u_n representing a feed-forward term to be chosen below, then we will decouple v from w and stabilize v, yielding

$$\begin{aligned} \dot{v} &= \bar{A}_2 v + b_2[g(t)u_n(t)], \\ y &= c_2 v. \end{aligned}$$

Because of Assumption 7 we can choose $k_1 \in \mathbf{R}^{1\times n_m}$ and $k_2 \in \mathbf{R}$ so that

$$\frac{P_m(s)}{c_2(sI - \bar{A}_2)^{-1}b_2} = k_2 + k_1(sI - A_m)^{-1}B_m. \tag{3.16}$$

Now we choose the feed-forward term u_n by

$$u_n(t) = \frac{1}{g(t)}[k_2\bar{u}_m(t) + k_1\bar{x}_m(t)],$$

which yields the **ideal control** law of

$$u(t) = \frac{1}{g(t)}\underbrace{\left[\begin{array}{cccc} -c_1(t) & \bar{f}_2 - f_2(t) & k_1 & k_2 \end{array}\right]}_{=:K(t)}\bar{x}(t). \tag{3.17}$$

By labelling this control signal $u^0(t)$ and the corresponding generalized plant response $\bar{x}^0$, the ideal closed-loop system is as follows:

$$\begin{aligned} \dot{\bar{x}}^0(t) &= \underbrace{\left[\begin{array}{cccc} A_1(t) & b_1c_2 & 0 & 0 \\ 0 & \bar{A}_2 & b_2k_1 & b_2k_2 \\ 0 & 0 & A_m & B_m \\ 0 & 0 & 0 & -\sigma \end{array}\right]}_{=:\bar{A}_{cl}(t)}\bar{x}^0(t) + \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ \sigma \end{array}\right]u_m(t), \\ \bar{e}^0(t) &= \left[\begin{array}{cccc} 0 & -c_2 & C_m & 0 \end{array}\right]\bar{x}^0(t) \end{aligned}$$

Since the plant is minimum phase, A_1 is exponentially stable, so it follows that this closed-loop system is exponentially stable and the map from $u_m \longmapsto \bar{e}^0$ is 0.

3.4.2 The First Approximation

Since the plant parameters and system states are not completely known, the ideal control law (3.17) cannot be implemented exactly. Instead, this quantity will be periodically estimated and applied. The $\frac{1}{g}$ term will be problematic in carrying out the estimation, so we will approximate it by a polynomial, which will be easier to deal with. To this end, from Assumptions 1 and 4, it follows that the set $\mathcal{G}$ defined by

$$\mathcal{G} := \{g \in \mathbf{R} : \text{there exists a } \psi \in \mathbf{R}^{2n-m} \text{ so that} \begin{bmatrix} \psi \\ g \end{bmatrix} \in \bar{\Gamma}\}$$

is a compact set, which does not include zero; indeed, there exists a constant $\bar{g} > 0$ so that $\mathcal{G} \subset [-\bar{g}, -\underline{g}] \cup [\underline{g}, \bar{g}]$. From the Stone-Weierstrass Approximation Theorem [37], we know that we can approximate $1/g$ arbitrarily well over $\mathcal{G}$ via a polynomial.

> **Proposition 3.2 [20]:** *The summation* $\sum_{i=0}^{\infty} \frac{(-1)^i}{(\bar{g}^2)^{i+1}} g(g^2 - \bar{g}^2)^i$ *converges uniformly to* $\frac{1}{g}$ *on* $\mathcal{G}$.

Proof: Carry out a Taylor Series expansion of $\frac{1}{g^2}$ about $\bar{g}^2$ and then multiply both expressions by g.

□

Unfortunately, Taylor Series expansions tend to converge slowly. There have been investigations into optimal approximation, e.g. [5] (pp. 126-138). In any event for every $\varepsilon > 0$ there exist $q \in \mathbf{N}$ and $c_i \in \mathbf{R}$ such that the polynomial $\hat{f}_\varepsilon(g) = \sum_{i=0}^{q} c_i g^i$ satisfies

$$\left|1 - g\hat{f}_\varepsilon(g)\right| < \varepsilon, \quad g \in \mathcal{G}. \tag{3.18}$$

This brings us to the first approximation of (3.17): we define the **approximate ideal control law** to be

$$u(t) = \hat{f}_\varepsilon(g)K(t)\bar{x}(t). \tag{3.19}$$

When (3.19) is applied to the generalized plant (3.11)-(3.13), we let u^ε denote the corresponding control signal, $\hat{x}^\varepsilon$ denote the corresponding state, and $\hat{e}^\varepsilon$ denote the corresponding error; the equations describing this closed-loop system are

$$\begin{aligned} \dot{\bar{x}}^\varepsilon(t) &= \bar{A}_{cl}(t)\bar{x}^\varepsilon(t) + \bar{E}u_m(t) + [g(t)\hat{f}_\varepsilon(g(t)) - 1]\bar{B}K(t)\bar{x}^\varepsilon(t) \\ &=: \bar{A}^\varepsilon_{cl}(t)\bar{x}^\varepsilon(t) + \bar{E}u_m(t), \\ \bar{e}^\varepsilon(t) &= \bar{C}\bar{x}^\varepsilon(t). \end{aligned}$$

We label the transition matrices corresponding to $\overline{A}_{cl}(t)$ and $\overline{A}^\varepsilon_{cl}(t)$ by $\Phi_{cl}(t, \tau)$ and $\Phi^\varepsilon_{cl}(t, \tau)$, respectively.

Proposition 3.3 [20]: *Suppose that $\lambda \in (max\{\lambda_0, \lambda_m\}, 0)$. Then there exist constants $\overline{\varepsilon} > 0$ and $\gamma > 0$ so that for every $\varepsilon \in (0, \overline{\varepsilon})$ and $\bar{\theta} \in \bar{\mathcal{P}}$, the closed-loop system (with $t_0 = 0$) satisfies*

$$\begin{aligned} \left\| \bar{x}^{\varepsilon}(t) - \overline{x}^{0}(t) \right\| &\leq \gamma\varepsilon(e^{\lambda t}\left\|\overline{x}_0\right\| + \left\|u_m\right\|_\infty) \\ \left\|\bar{e}^{\varepsilon}(t) - \bar{C}\Phi_{cl}(t,0)\overline{x}_0\right\| &\leq \gamma\varepsilon(e^{\lambda t}\left\|\overline{x}_0\right\| + \left\|u_m\right\|_\infty) \\ \left\|\bar{x}^{\varepsilon}(t)\right\| &\leq \gamma e^{\lambda t}\left\|\overline{x}_0\right\| + \gamma\left\|u_m\right\|_\infty \quad t \geq 0; \end{aligned}$$

furthermore, the transition matrix satisfies

$$\|\Phi^{\varepsilon}_{cl}(t,\tau)\| \leq \gamma e^{\lambda(t-\tau)}, \quad t \geq \tau \geq 0.$$

Remark 3.3: *This result states, in particular, that when $\bar{x}_0 = 0$, the deviations between the tracking error and the state of the closed-loop system when the approximated control law (3.19) is applied, namely $\bar{x}^{\varepsilon}(t)$ and $\bar{e}^{\varepsilon}(t)$, and the tracking error and states of the closed-loop system when the ideal control law (3.17) is applied, namely $\bar{x}^0(t)$ and $\bar{e}^0$, can be made as small as desired by selecting the parameter $\varepsilon > 0$.*

So, as expected, as the approximation of $\frac{1}{g}$ improves, the steady-state tracking error improves. At this point we freeze $\lambda \in (\max\{\lambda_0, \lambda_m\}, 0)$ and $\lambda_1 \in (\max \lambda_0, \lambda_m\}, \lambda)$. From Proposition 3.2, we can choose $\varepsilon > 0$, a polynomial $\hat{f}_{\varepsilon}(g) = \sum_{i=0}^{q} c_i g^i$ satisfying (3.18), and a constant $\gamma > 0$ so that for all $\bar{\theta} \in \bar{\mathcal{P}}$, we have

$$\|\Phi^{\varepsilon}_{cl}(t,\tau)\| \leq \gamma e^{\lambda_1(t-\tau)}, \quad t \geq \tau \geq 0.$$

3.4.3 The Second Approximation

Since neither g nor $\bar{x}$ is measurable, (3.19) cannot be implemented, so the goal is to approximate (3.19) in a linear fashion. We use h small and with q the order of the polynomial approximation to $\frac{1}{g}$, we choose $p > (2q+1)m$; recall that the controller period is $T = ph$. The second approximation is as follows:

$$u(t) = \begin{cases} 0, & t \in [kT, kT + (2q+1)mh), \\ \frac{p}{p-(2q+1)m}\hat{f}_{\varepsilon}(g(kT))K(kT)\bar{x}^{\varepsilon}(kT), & t \in [kT + (2q+1)mh, (k+1)T). \end{cases} \tag{3.20}$$

Clearly if h and T are both small, this is a close approximation of

$$u(t) = \hat{f}_{\varepsilon}(g(kT))K(kT)\bar{x}^{\varepsilon}(kT), \quad t \in [kT, (k+1)T),$$

which, in turn, is a good approximation of (3.19). We cannot construct (3.20) for the same reason that we cannot construct (3.19), but the former is amenable to

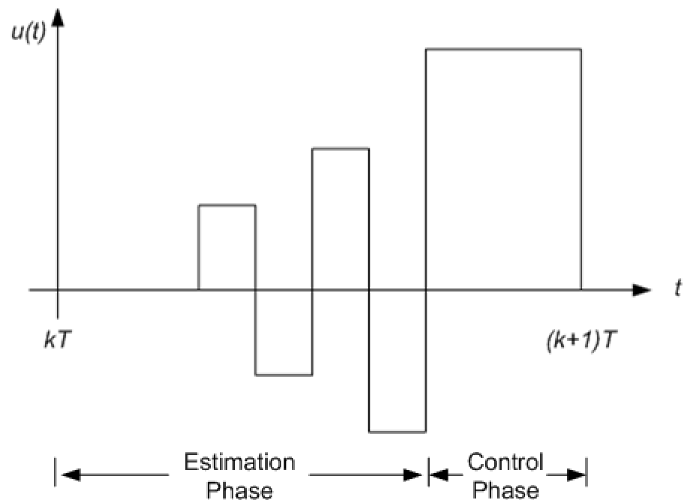

Figure 3.2: A control period consisting of the Estimation and Control Phases [20]

estimation. Every period is divided into two phases: an Estimation Phase and a Control Phase, as illustrated in Figure 3.2. In the Estimation Phase, the quantity $\hat{f}_\varepsilon(g(kT))K(kT)\hat{x}^\varepsilon(kT)$ is estimated; although a degree of probing is used, it is carried out in such a way that its effect at the end of the Estimation Phase is very small. In the Control Phase, the above estimate, scaled by $\frac{p}{p-(2q+1)m}$, is applied to the plant; this scaling is chosen to reflect the fact that the Control Phase is exactly $\frac{p-(2q+1)m}{p}$ of the whole period. In the next two subsections we will explain how the Estimation Phase is implemented.

3.5 Controller Construction

In this subsection, the implementation of the above LPC is presented. We start by explaining how to carry out the Estimation Phase. Define two $(m+1)\times(m+1)$ matrices:

$$S_m = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2^2 & \cdots & 2^m \\ & & \vdots & & \\ 1 & m & m^2 & \cdots & m^m \end{bmatrix}, \; H_m(h) = \begin{bmatrix} 1 & & & & \\ & h & & & \\ & & h^2/(2!) & & \\ & & & \ddots & \\ & & & & h^m/(m!) \end{bmatrix} \tag{3.21}$$

First consider the simple system with transfer function $\frac{g}{s^m}$ with a state-space representation of

$$\dot{x} = \begin{bmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ g \end{bmatrix} u,$$

$$y = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} x.$$

By setting

$$u(t) = \bar{u}, \quad t \in [0, mh),$$

it follows that

$$\begin{aligned} y(t) &= \begin{bmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{m-1}}{(m-1)!} \end{bmatrix} x(0) + \frac{t^m}{m!}\bar{u} \\ &= \begin{bmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{m-1}}{(m-1)!} & \frac{t^m}{m!} \end{bmatrix} \begin{bmatrix} x(0) \\ g\bar{u} \end{bmatrix} \quad , t \in [0, mh]. \end{aligned}$$

If we define the sequence of output samples by

$$\mathcal{Y}(t) := \begin{bmatrix} y(t) \\ y(t+h) \\ \vdots \\ y(t+mh) \end{bmatrix},$$

it follows that

$$H_m(h)^{-1} S_m^{-1} \mathcal{Y}(0) = \begin{bmatrix} x(0) \\ g\bar{u} \end{bmatrix}.$$

The following lemma illustrates how this form of estimation can be carried out in the general case.

Lemma 3.1 (Key Estimation Lemma (KEL-3)) [20]: *There exist constants $\gamma > 0$ and $\bar{h} > 0$ so that for all $t_0 \in \mathbf{R}^+$, $x_0 \in \mathbf{R}^{n+n_m+1}$, $u_m \in PC_\infty$, $h \in (0, \bar{h})$, and $\bar{\theta} \in \bar{\mathcal{P}}$, the solution of (3.11)-(3.13) with*

$$u(t) = \bar{u}, \quad t \in [t_0, t_0 + mh)$$

has the following two properties :

(i) In all cases

$$\|\bar{x}(t) - \bar{x}(t_0)\| \leq \gamma h(\|\bar{x}(t_0)\| + \|\bar{u}\| + \|u_m\|_\infty), \quad t \in [t_0, t_0 + mh),$$

$$\|H_m(h)^{-1} S_m^{-1} \mathcal{Y}(t_0)\| \leq \gamma(\|\bar{x}(t_0)\| + \|\bar{u}\| + \|u_m\|_\infty).$$

(ii) If $\bar{\theta}(t)$ is absolutely continuous on $[t_0, t_0 + mh]$, then we can obtain a tighter approximation:

$$\begin{aligned} \left\| H_m(h)^{-1} S_m^{-1} \mathcal{Y}(t_0) - \begin{bmatrix} v(t_0) \\ f_2(t_0)v(t_0) + c_1(t_0)w(t_0) + g(t_0)\bar{u} \end{bmatrix} \right\| &\leq \\ \gamma h(\|\bar{x}(t_0)\| + \|\bar{u}\| + \|u_m\|_\infty). \end{aligned}$$

Now let us see how KEL-3 can be applied. First, for simplicity, we assume that $\bar{\theta} \in \bar{\mathcal{P}}$ is absolutely continuous. By setting

$$u(t) = 0, \quad t \in [kT, kT + mh),$$

and considering the definition of $K(t)$ in (3.17), it follows that

$$\left[\begin{array}{cc} \bar{f}_2 & -1 \end{array} \right] H_m(h)^{-1} S_m^{-1} \mathcal{Y}(kT) + \left[\begin{array}{cc} k_1 & k_2 \end{array} \right] \left[\begin{array}{c} \bar{x}_m(kT) \\ \bar{u}_m(kT) \end{array} \right] \approx K(kT)\bar{x}(kT).$$

Choose a scaling factor $\rho > 0$, and set

$$u(t) = \rho \times \text{ estimate of } K(kT)\bar{x}(kT), \quad t \in [kT + mh, kT + 2mh);$$

it follows that

$$\frac{1}{\rho} \left[\begin{array}{cccc} 0 & \cdots & 0 & 1 \end{array} \right] H_m(h)^{-1} S_m^{-1} [\mathcal{Y}(kT + mh) - \mathcal{Y}(kT)] \approx g(kT)K(kT)\bar{x}(kT).$$

If we repeat this procedure then a close estimate of

$$\frac{1}{g(kT)} K(kT)\bar{x}(kT) \approx \sum_{i=0}^{q} c_i \underbrace{g(kT)^i K(kT)\bar{x}(kT)}_{=:\phi_i(kT)}$$

can be obtained by the end of the Estimation Phase (the interval $[kT, kT + (2q + 1)mh]$); in the Control Phase a suitably weighted of this estimate is applied. If the sampling time T is small enough, then any discontinuity will be infrequent and the controller will work. With the above motivation, we end up with the controller structure as:

THE PROPOSED CONTROLLER [20] ($t_0 = 0$)

Estimation Phase: $[kT, kT + (2q+1)mh)$

$$\hat{\phi}_i(kT) = \begin{cases} \begin{bmatrix} \bar{f}_2 & -1 \end{bmatrix} H_m(h)^{-1} S_m^{-1} \mathcal{Y}(kT) + \begin{bmatrix} k_1 & k_2 \end{bmatrix} \begin{bmatrix} \bar{x}_m(kT) \\ \bar{u}_m(kT) \end{bmatrix}, & \text{if } i = 0, \\ \frac{1}{\rho} \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} H_m(h)^{-1} S_m^{-1} \begin{bmatrix} \mathcal{Y}(kT + (2i-1)mh) - \mathcal{Y}(kT) \end{bmatrix}, & \\ & \text{if } i = 1, \cdots, q, \end{cases} \tag{3.22}$$

$$u(t) = \begin{cases} 0 & t \in [kT, kT + mh), \\ \rho\hat{\phi}_{i-1}(kT) & t \in [kT + (2i-1)mh, kT + 2imh), \; i = 1, \cdots, q \\ -\rho\hat{\phi}_{i-1}(kT) & t \in [kT + 2imh, kT + (2i+1)mh), \; i = 1, \cdots, q. \end{cases} \tag{3.23}$$

Control Phase: $[kT + (2q+1)mh, (k+1)T)$

$$u(t) = \frac{p}{p - (2q+1)m} \sum_{i=0}^{q} c_i \hat{\phi}_i(kT), \quad t \in [kT + (2q+1)mh, (k+1)T). \tag{3.24}$$

Lemma 3.2 [20]: *The control law (3.22)-(3.24) has a representation of the form (3.5) (with parameters (F, G, H, J, L, M, h, p)) which is deadbeat; in fact, $F(0) = 0$.*

Now we present a proposition which consider the closed-loop system over one period. It is shown that the closed-loop system with control law (3.22)-(3.24) behaves very much the same as the closed-loop system with the approximated ideal control law (3.19).

Lemma 3.3 (Key Control Lemma (KCL-3)) [20]: *There exist constants $\gamma > 0$ and $\overline{T} > 0$ so that for all $k \in \mathbf{Z}^+, T \in (0, \overline{T})$, and $\bar{\theta} \in \bar{\mathcal{P}}$, the solution of (3.11) with u given by (3.22)-(3.24) has the following properties:*

(i) In all cases

$$\left\| \bar{x}(t) - \Phi_{cl}^{\varepsilon}(t, kT)\bar{x}(kT) - \int_{kT}^{t} \Phi_{cl}^{\varepsilon}(t,\tau) E u_m(\tau) d\tau \right\| \;\; \leq \;\; \gamma T(\|\bar{x}(kT)\| + \|u_m\|_{\infty}),$$

$$\|u(t)\| \leq \gamma(\|\bar{x}(kT)\| + \|u_m\|_{\infty}), t \in [kT, (k+1)T),$$

(ii) If $\bar{\theta}(t)$ is absolutely continuous on $[kT, (k+1)T]$, then

$$\begin{aligned} \|\bar{x}((k+1)T) - \Phi_{cl}^{\varepsilon}((k+1)T, kT)\bar{x}(kT) - \int_{kT}^{(k+1)T} \Phi_{cl}^{\varepsilon}((k+1)T, \tau)\bar{E} u_m(\tau) d\tau\| \\ \leq \gamma T^2(\|\bar{x}(kT)\| + \|u_m\|_{\infty}). \end{aligned}$$

KCL-3 describes the closed-loop system behaviour during each period. By the following proposition, it is proven that the proposed controller (3.22)-(3.24) acts like the approximated ideal control law (3.19) for all time. We adopt the following notation: we let $\hat{u}^{\varepsilon}$ denote the control signal, $\hat{\bar{x}}^{\varepsilon}$ denote the state, $\hat{\bar{e}}^{\varepsilon}$ denote the error, and $\hat{\bar{y}}^{\varepsilon}$ denote the plant output when (3.22)-(3.24) is applied. Recall that we use $\bar{x}^{\varepsilon}$ and $\bar{e}^{\varepsilon}$ to denote the corresponding signals when (3.19) is applied.

Proposition 3.4 [20]: *There exist constants $\bar{T} > 0$, $\lambda < 0$ and $\gamma > 0$ so that for every $\bar{\theta} \in \bar{\mathcal{P}}$, $\bar{x}_0 \in \mathbf{R}^{n+n_m+1}$, $u_m \in PC_{\infty}$ and $T \in (0, \bar{T})$, we have*

$$\|\bar{x}^{\varepsilon}(t) - \hat{\bar{x}}^{\varepsilon}(t)\| \leq \gamma T e^{\lambda t}\|\bar{x}_0\| + \gamma T \|u_m\|_{\infty}, \;\; t \geq 0.$$

3.6 The Main Result

Now we can put together all results of the last few sections to prove that the LPC provides both stability and good tracking for all admissible models.

Theorem 3.1 [20]: *For every* $\delta > 0$ *and* $\lambda \in (\max\{\lambda_0, \lambda_m\}, 0)$ *there exists a controller of the form (3.2), (3.3), and (3.5) so that for every* $\bar{\theta} \in \bar{\mathcal{P}}$, $\bar{x}_0 \in \mathbf{R}^{n+n_m+1}$, *and* $u_m \in PC_\infty$, *we have that the closed-loop system is exponentially stable, and when* $t_0 = k_0 = 0$, *we have*

$$|\hat{\bar{y}}^{\varepsilon}(t) - y_m(t) - \bar{C}\Phi_{cl}(t,0)\bar{x}(0)| \leq \delta e^{\lambda t}\|\bar{x}_0\| + \delta\|u_m\|_\infty, \quad t \geq 0.$$

Remark 3.4: *This result states that the deviation between the output of the closed-loop system when the proposed control law (3.22)-(3.24) is applied, namely* $\hat{\bar{y}}^{\varepsilon}(t)$, *and the output of the closed-loop system when the ideal control law (3.17) is applied, namely*

$$y_m(t) + \bar{C}\Phi_{cl}(t,0)\bar{x}_0,$$

can be made as small as desired by selecting the controller parameters $\varepsilon > 0$ *and* $T > 0$.

Remark 3.5: *Theorem 3.1 and the control law (3.22)-(3.24) demonstrate that, unlike classical adaptive control methods, the proposed controller has these desirable features:*

- *the controller is linear,*
- *there is no tuning phase and near exact tracking is achieved immediately,*
- *the* $e^{\lambda t}\|\bar{x}_0\|$ *term shows that the effect of non-zero initial conditions decays exponentially to zero,*
- *since the ideal control law (3.17) is modest in size and the applied control signal is similar to it, the control signal is not large,*
- *parameters can be time-varying without pre-specified structure.*

Remark 3.6: *It is clear that to achieve good tracking, we need to have a good estimate of* $\frac{1}{g}$ *which needs a small* $\varepsilon > 0$*; this good estimate yields a high order polynomial* $\hat{f}_\varepsilon(g) = \sum_{i=0}^{q} c_i g^i$*. We also need to have the sampled-data controller period* T *be small. These two facts result in the sampling period* $h = \frac{T}{p} < \frac{T}{(2q+1)m}$ *(recall that* m *is the plant relative degree) must be very small which means that some of the controller gains should be large (looking at the control law (3.22)-(3.24) shows that some are proportional to* $\frac{1}{h^m}$*). Large controller gains lead to poor noise tolerance, which is one of the imperfections of the proposed controller.*

3.7 Examples

Here we present two examples to illustrate the proposed design methodology and its positive and negative features.

3.7.1 Example 1

The plant model is first order:

$$\dot{y}(t) = a(t)y(t) + g(t)u(t).$$

The set of plant uncertainty is given by

$$\Gamma = \{ \begin{bmatrix} a \\ g \end{bmatrix} \in \mathbf{R}^2 : a \in [-1, 1],\ g^2 \in [1, 1.4] \},$$

$$\mathcal{P} = \mathcal{P}(n = 1,\ m = 1,\ \Gamma, \mu_1 = 1, T_0 = 5, \gamma_0 = 1, \lambda_0 = -5).$$

While the reference model is

$$\dot{\bar{x}}_m = -\bar{x}_m + \bar{u}_m,$$

the anti-aliasing filter is

$$\dot{\bar{u}}_m = -50\bar{u}_m + 50u_m.$$

We set $\rho = 1$.

For approximating the polynomial $\frac{1}{g}$, we use the approximation of:

$$\hat{f}_{0.01}(g) = 2.1647g - 1.5153g^3 + 0.3433g^5,$$

so $q = 5$, and we choose $p = 25 > 2(2q + 1)$, i.e. the control phase is 66% of the time. We set $T = 0.25$ (so that $h = 0.01$). Figure 3.3 shows the simulation results with $y_0 = 3$, $\bar{u}_{m_0} = 0$, u_m a square wave given by

$$u_m = sign(\cos(\frac{2\pi t}{15})),$$

and

$$a(t) = \cos(t/2) \text{ and } g(t) = [1.2 + 0.2cos(t/2)] * sign[cos(t/4)];$$

as shown in Figure 3.3, the effect of the initial conditions declines exponentially to zero, the tracking is quite close, and the control signal is modest in size, even with time-varying plant parameters. We add a noise signal of the form

h * random sequence uniformly distributed between ± 1

to the output measurement at the time $t = 66$ sec; we see that it degrades the tracking slightly. In Figure 3.4 we provide a close-up of the control signal during the first three periods.

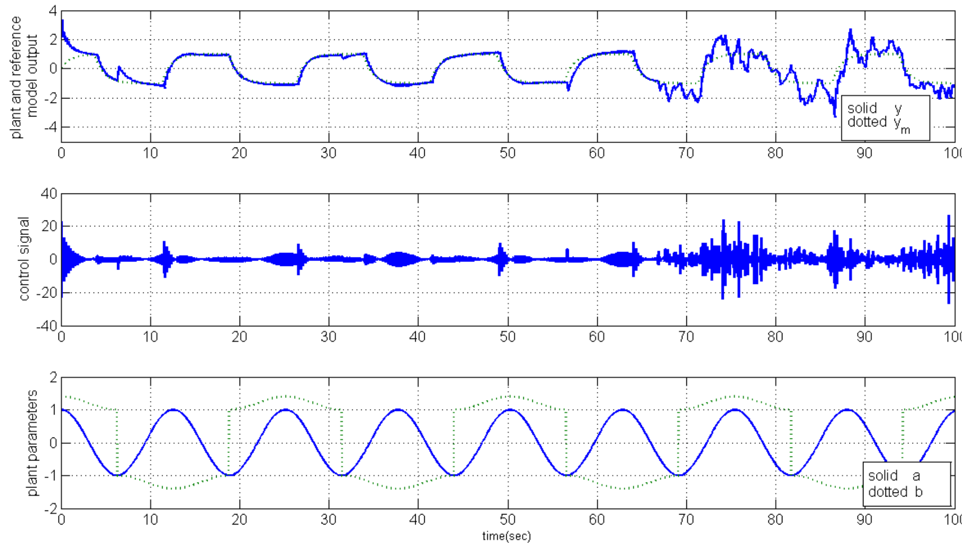

Figure 3.3: Example with a and g varying with time

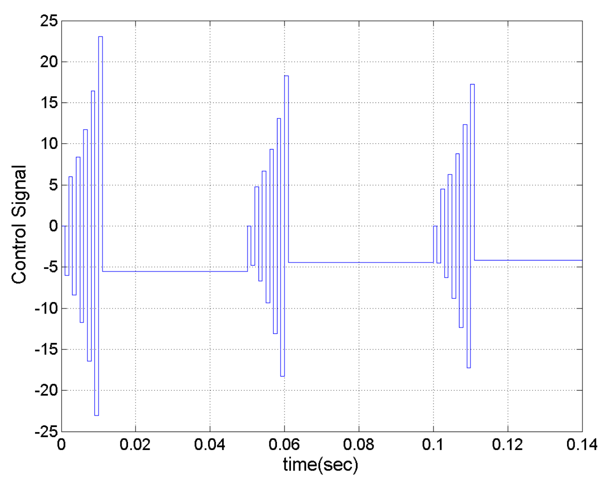

Figure 3.4: A close-up of the control signal

3.7.2 Example 2

In this example a minimum phase relative degree two model of a DC motor is considered [17]:

$$\begin{aligned} \dot{x} &= \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & \beta_1(t) \end{bmatrix}}_{=:A(t)} x + \underbrace{\begin{bmatrix} 0 \\ g(t) \end{bmatrix}}_{=:B(t)} u, \\ y &= \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{=:C} x. \end{aligned}$$

The set of plant uncertainty is given by

$$\bar{\Gamma} = \{ \begin{bmatrix} \beta_0 \\ \beta_1 \\ g \end{bmatrix} \in \mathbf{R}^3 : \beta_0 = 0,\ \beta_1 \in [-5,-1],\ \ g \in [1,2] \},$$

$$\bar{\mathcal{P}} = \bar{\mathcal{P}}(n = 2, m = 2, \bar{\Gamma}, \bar{\mu}_1 = 1, \bar{T}_0 = 10, \gamma_0 = 1, \lambda_0 = -5\}.$$

The reference model is relative degree two with the state space representation:

$$\begin{aligned} \dot{x}_m &= \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x_m + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_m, \\ y_m &= \begin{bmatrix} 1 & 0 \end{bmatrix} x_m. \end{aligned}$$

The anti-aliasing filter is chosen as

$$\dot{\bar{u}}_m = -50\bar{u}_m + 50u_m.$$

We set $\rho = 1$.

For the approximation of $\frac{1}{g}$, we use the polynomial

$$\hat{f}_{0.01}(g) = 2.1647 - 1.5153g + 0.3433g^2,$$

so $q = 2$ and we choose $p = 25 > 2(2q + 1)m$ and set $T = 0.25$ (so that $h = 0.01$). A simulation was carried out (see Figure 3.5) with $x_0 = [1 \ \ 1]^T$, $y_{m_0} = \bar{u}_{m_0} = 0$, u_m a square wave given by

$$u_m = sign(\cos(\frac{2\pi t}{15})),$$

and

$$\beta_1(t) = -3 + 2 * \cos(t/2) \text{ and } g(t) = 1.5 + 0.5 * \sin(t/4).$$

We see that the tracking is quite good and the control signal is modest in size, even in the face of significant time variations. We add a noise signal of the form

h^2 * random sequence uniformly distributed between ± 1

to the output measurement at the time $t = 66$ sec; we see that it degrades the tracking slightly. In Figure 3.6 we provide a close-up of the control signal during the first three periods.

As these two examples show, although the noise signals are much smaller (proportional to the sampling time in Example 1, and to the square of the sampling time in Example 2) than the reference signals, they have noticeable effects on the output signals. In other words, the closed-loop system has poor noise tolerance, which is a drawback of the LPC. To see another undesired feature of the LPC, we provided a close-up of the control signals during the first three periods (see Figure 3.4 and Figure 3.6); observe that during the Estimation Phase, the control signals take a range of values, which results in a rapidly time-varying control signal, which requires a fast actuator.

3.8 Summary and Concluding Remarks

The underlying motivation of the LPC [20] is to design a model reference adaptive control scheme for time-varying minimum phase systems, which, unlike most classical MRACs, is linear and periodic. Rather than estimating parameters, the LPC directly estimates what the control signal would be if the plant parameters and states were known and the ideal LTI compensator were applied. This control method has the following positive features:

i it handles rapidly time-varying parameters,

ii it provides smooth transient behaviour,

iii the effect of initial conditions decays exponentially to zero, and

iv the control signal is modest in size.

In contrast to the desirable features, the LPC [20] has its own drawbacks. In order to achieve the aforementioned advantages, a small sampling period is required, which results in a rapidly time-varying control signal. This control signal requires fast actuators, which may not be practical. The second disadvantage of the LPC [20] is poor noise tolerance. A small sampling period results in large controller gains and consequently poor noise tolerance behaviour. The last drawback is that this controller requires knowledge of the exact plant relative degree. While in the next chapter we redesign the controller so that it works in the context of the 2-norm, Chapters 5, 6, and 7 are devoted to redesigning the approach to alleviate these drawbacks.

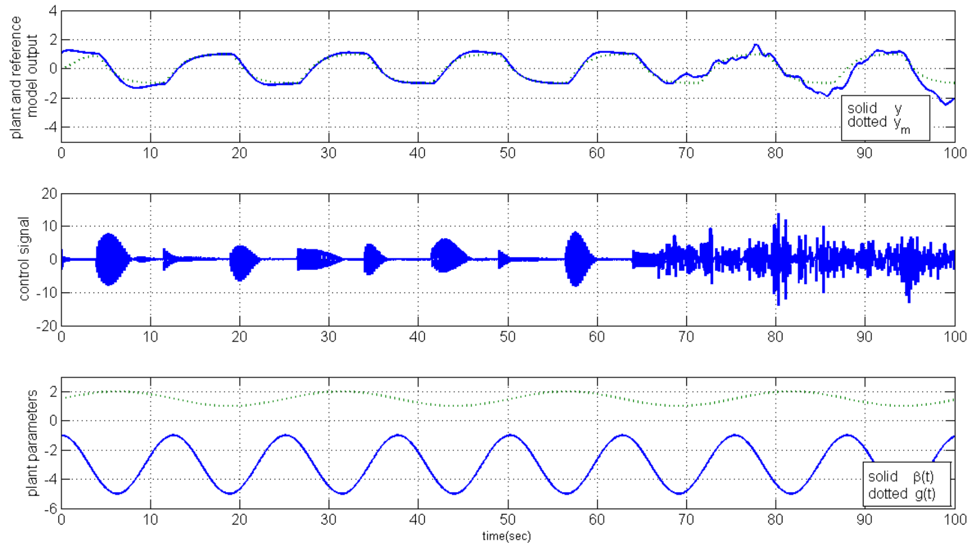

Figure 3.5: Example with β_1 and g varying with time

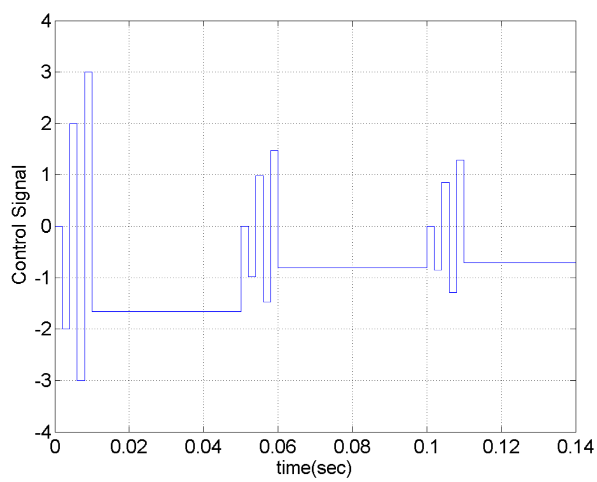

Figure 3.6: A close-up of the control signal

Chapter 4

The Linear Periodic MRAC in the Context of the 2-Norm

4.1 Introduction

As discussed in Chapter 3, to overcome the undesirable features of most adaptive control methodologies, a new approach to the model reference adaptive control problem was proposed in [20]. In this approach, rather than estimating the plant or controller parameters, as is often done in the classical approach, the quantity being estimated is the *ideal control signal*, which could be used if the plant parameters and states were known. The approach yields an LPC, with nice transient and steady-state tracking properties, as well as an ability to handle rapid time-variations.

In the LPC [20], the time-domain ∞-norm is used to measure signal size. However, the time-domain 2-norm is an equally common way to measure signal size and is the one used in the popular H_∞ paradigm. Here we extend the approach so that results can be obtained in the 2-norm settling which are comparable to those of [20]. This requires structural change in the control law together with new proofs of key steps. Since we will be using a sampled-data controller, we will need to add an anti-aliasing filter before sampling the plant output[1]. The filter can be incorporated into the plant and the reference model at the expense of increasing the complexity and worsening the performance and noise tolerance of the closed-loop system.

The outline of this work is as follows. The problem setup is presented in Section 4.2. A high level explanation of the LPC is presented in Section 4.3. In Section 4.4, a controller redesign is carried out so that the idea works when using the 2-norm. To show how the proposed controller works, two examples are presented in Section 4.5. Last of all, Section 4.6 provides a summary and concluding remarks.

[1]This is not required when using the ∞-norm to measure the size of a time-domain signal, since the discrete-time ∞-norm of a sampled signal is bounded above by the continuous-time ∞-norm of this signal.

4.2 Problem Formulation

The problem setup is similar to that of Chapter 3, except we have added an anti-aliasing filter before sampling the plant output. The SISO linear time-varying plant P is described by

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t), \; x(t_0) = x_0, \\ y(t) &= C(t)x(t), \end{aligned} \tag{4.1}$$

with $x(t) \in \mathbf{R}^n$ the plant state, $u(t) \in \mathbf{R}$ the plant input, and $y(t) \in \mathbf{R}$ the plant output; we associate the plant with the triple $(A(t), B(t), C(t))$. We allow a good deal of model uncertainty, to be described in detail later on; the set of possible models is labelled P.

The stable SISO LTI reference model is given by

$$\begin{aligned} \dot{x}_m(t) &= A_m x_m(t) + B_m u_m(t), \; x_m(t_0) = x_{m_0}, \\ y_m(t) &= C_m x_m(t), \end{aligned}$$

with $x_m(t) \in \mathbf{R}^{n_m}$ the reference model state, $u_m(t) \in \mathbf{R}$ the reference model input, and $y_m(t) \in \mathbf{R}$ the reference model output. The reference model describes the desired behaviour of the closed-loop system, and is chosen to be stable. The goal is to design a controller that provides stability, and makes the plant output track the model reference output. To this end, we define the tracking error as

$$e(t) := y_m(t) - y(t).$$

The controller is a sampled data controller, which periodically samples u_m and y. We use an anti-aliasing filter before sampling u_m: with $\sigma > 0$ we choose an anti-aliasing filter of the form

$$\dot{\bar{u}}_m = -\sigma \bar{u}_m + \sigma u_m, \quad \bar{u}_m(t_0) = \bar{u}_{m_0}, \tag{4.2}$$

whose input-output map is labelled F_σ. Accordingly, we define a new version of the reference model with $\bar{u}_m$ as the input:

$$\begin{aligned} \dot{\bar{x}}_m &= A_m \bar{x}_m + B_m \bar{u}_m, \quad \bar{x}_m(t_0) = x_{m_0}, \\ \bar{y}_m &= C_m \bar{x}_m. \end{aligned} \tag{4.3}$$

If we choose $\gamma_m > 0$ and $\lambda_m < 0$ so that

$$||e^{A_m t}|| \;\leq\; \gamma_m e^{\lambda_m t}, \quad t \geq 0,$$

it follows that

$$\begin{aligned} ||\bar{y}_m - y_m||_2 \;\leq\; & \frac{1}{\sigma}\left[||C_m||.||B_m|| + ||C_m||.||A_m||.||B_m|| \frac{\gamma_m}{|\lambda_m|} \right] ||u_m||_2 + \\ & \frac{\gamma_m}{|\lambda_m|} ||C_m||.||B_m|| \times \frac{1}{\sqrt{2\sigma}} ||\bar{u}_{m_0}||. \end{aligned} \tag{4.4}$$

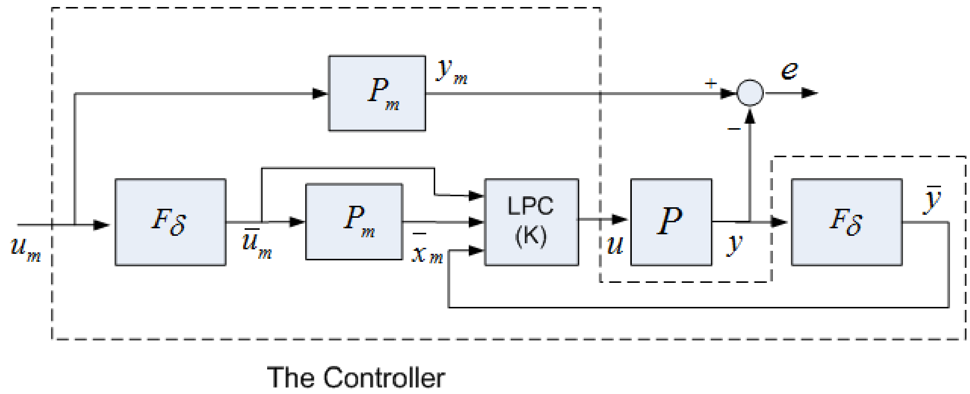

Figure 4.1: The feedback diagram in the context of the 2-norm

Since sampling a continuous time signal with finite energy may yield a discrete-time signal with infinite energy[2], we need to make changes to the controller of [20] for it to work in this new setting. Clearly, an anti-aliasing filter is in order, which we place at the plant output. Hence, with $\sigma > 0$ we choose the anti-aliasing filter at the plant output as follows:

$$\dot{\bar{y}} = -\sigma\bar{y} + \sigma y, \quad \bar{y}(t_0) = \bar{y}_0. \tag{4.5}$$

Finally let us define the linear periodic controller by

$$\begin{aligned} z[k+1] &= F(k)z[k] + G(k)y(kh) + H(k)\bar{x}_m(kh) + J(k)\bar{u}_m(kh), \quad z[k_0] = z_0 \in \mathbf{R}^l, \\ u(kh+\tau) &= L(k)z[k] + M(k)y(kh), \qquad \tau \in [0,h), \end{aligned} \tag{4.6}$$

whose gains F, G, H, J, and L are periodic of period $p \in \mathbf{N}$; the sampling time is h, and the period of the controller is $T := ph$. We associate this system with the 7-tuple (F, G, H, J, L, h, p). Observe that (4.6) can be implemented with a sampler, a zero-order-hold, and an l^{th} order periodically time-varying discrete-time system of period p.

Remark 4.1: *The reference model is chosen by the control system designer to embody the desired closed-loop behaviour. Hence, we can consider our controller to be a combination of the anti-aliasing filter (4.2), the reference model (4.3), the anti-aliasing filter (4.5), and the discrete-time periodic compensator (4.6).*

---

[2] An example of such signal is

$$x(t) = \begin{cases} a & kT \le t < kT + \frac{b}{2^k} \\ 0 & kT + \frac{b}{2^{|k|}} \le t < (k+1)T \end{cases} \quad a \in \mathbf{R}, \quad 0 < b \le T, \quad k \in \mathbf{Z}, \quad t \in [kT, (k+1)T).$$

The feedback configuration is given in Figure 4.1. As shown in the figure, the controller is a mixture of discrete and continuous subsystems, so the closed-loop state is a combination of discrete and continuous states, defined as

$$x_{sd}(t) = \begin{bmatrix} x(t) \\ \bar{y}(t) \\ \bar{x}_m(t) \\ \bar{u}_m(t) \\ z[k] \end{bmatrix}, \qquad t \in [kh, (k+1)h).$$

Definition 4.1: *The controller (4.2), (4.3), (4.5), and (4.6)* exponentially stabilizes $\mathcal{P}$ *if there exist constants* $\gamma > 0$ *and* $\lambda < 0$ *so that, for every* $P \in \mathcal{P}$*, set of initial conditions* x_0*,* $\bar{y}_0$*,* $\bar{x}_{m_0}$*,* $\bar{u}_{m_0}$*, and* z_0*, and set of initial times* $k_0 \in \mathbf{Z}^+$ *and* $t_0 = k_0 h$*, with* $u_m(t) = 0$ *for* $t \geq t_0$ *we have*

$$\|x_{sd}(t)\| \leq \gamma e^{\lambda(t-t_0)} \|x_{sd}(t_0)\|, \quad t \geq t_0.$$

Accordingly, we have the new version of the generalized-plant:

$$\begin{bmatrix} \dot{w} \\ \dot{v} \\ \dot{\bar{y}} \\ \dot{\bar{x}}_m \\ \dot{\bar{u}}_m \end{bmatrix} = \underbrace{\begin{bmatrix} A_1(t) & b_1 c_2 & 0 & 0 & 0 \\ b_2 c_1(t) & A_2(t) & 0 & 0 & 0 \\ 0 & \sigma c_2 & -\sigma & 0 & 0 \\ 0 & 0 & 0 & A_m & B_m \\ 0 & 0 & 0 & 0 & -\sigma \end{bmatrix}}_{=:\bar{A}(t)} \underbrace{\begin{bmatrix} w \\ v \\ \bar{y} \\ \bar{x}_m \\ \bar{u}_m \end{bmatrix}}_{=:\bar{x}} + g(t) \underbrace{\begin{bmatrix} 0 \\ b_2 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{=:\bar{B}} u(t) + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \sigma \end{bmatrix}}_{=:\bar{E}} u_m(t), \qquad \bar{x}(t_0) = \bar{x}_0, \tag{4.7}$$

$$\bar{y} = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix}}_{=:\bar{C}_1} \begin{bmatrix} w \\ v \\ \bar{y} \\ \bar{x}_m \\ \bar{u}_m \end{bmatrix}. \tag{4.8}$$

The plant output and the corresponding tracking error are given by

$$y = \underbrace{\begin{bmatrix} 0 & c_2 & 0 & 0 & 0 \end{bmatrix}}_{=:\bar{C}} \begin{bmatrix} w \\ v \\ \bar{y} \\ \bar{x}_m \\ \bar{u}_m \end{bmatrix}, \tag{4.9}$$

$$\bar{e} = \underbrace{\begin{bmatrix} 0 & -c_2 & 0 & C_m & 0 \end{bmatrix}}_{=:\bar{C}_2} \begin{bmatrix} w \\ v \\ \bar{y} \\ \bar{x}_m \\ \bar{u}_m \end{bmatrix}. \tag{4.10}$$

The goal is to construct the control law $u(t)$ such that the closed-loop stability is guaranteed and the map of $u_m \rightarrow \bar{e}$ is small.

Remark 4.2: *In the new setup, the measured and controlled outputs are $\bar{y}$ and y, respectively.*

4.3 The Approach

In this section, the proposed LPC is presented; the design is based on that of [20], with a modification arising from the anti-aliasing filter applied to y. We start with a high level description and then turn to a more concrete one. We start with a description of the *ideal LTI control law* which we would use in the case of complete state and parameter information. We then explain a sequence of these approximations to this control law, with the last one being of the form (4.6).

4.3.1 The Ideal Control Law

To motivate the ideal control law, we start with a time-invariant, first-order case (so $y = x$ and $y_m = x_m$):

$$\left.\begin{aligned} \dot{y} &= ay + gu, \\ \dot{\bar{y}} &= -\sigma\bar{y} + \sigma y, \\ \dot{\bar{y}}_m &= a_m\bar{y}_m + b_m\bar{u}_m. \end{aligned}\right\} \tag{4.11}$$

The control objective is to make the plant act like the reference model, so we would like the tracking error of $\bar{e} = \bar{y}_m - y$ to be small. Start with the tracking error dynamics:

$$(\dot{\bar{y}}_m - \dot{y}) = a_m(\bar{y}_m - y) + [b_m\bar{u}_m - gu + (a_m - a)y].$$

If we set

$$b_m\bar{u}_m - gu + (a_m - a)y = 0,$$

then the mismatch decays to zero like $e^{a_m t}$, which leads us to the following definition of the **ideal control law**:

$$u = \frac{1}{g}(b_m\bar{u}_m + (a_m - a)y) = \frac{1}{g}\underbrace{\begin{bmatrix} a_m - a & 0 & 0 & b_m \end{bmatrix}}_{=:K} \begin{bmatrix} y \\ \bar{y} \\ \bar{y}_m \\ \bar{u}_m \end{bmatrix}.$$

The controller provides both stability and model matching.

The ideal control law in the general case requires the same idea as the LPC [20] explained in Chapter 3. With (Λ_2, b_2) defined in equations (3.8) and (3.15) of Chapter 3, Sections 3.3 and 3.4, since (Λ_2, b_2) is controllable, we can choose a vector of $\bar{f}_2$ so that $\bar{A}_2 := \Lambda_2 + b_2\bar{f}_2$ is stable with eigenvalues with real parts less than λ_m, which means that there exist $\bar{\gamma}_m > \gamma_m$ so that

$$\|e^{\bar{A}_2 t}\| \leq \bar{\gamma}_m e^{\lambda_m t}, \quad t \geq 0.$$

If we now set

$$u(t) = \frac{1}{g(t)}[-c_1(t)w(t) + (\bar{f}_2 - f_2(t))v(t)] + u_n(t),$$

with u_n representing a feed-forward term to be chosen below, then we will decouple v from w and stabilize v, yielding

$$\begin{aligned} \dot{v} &= \bar{A}_2 v + b_2[g(t)u_n(t)] \\ y &= c_2 v. \end{aligned}$$

Because of Assumption 7 of Chapter 3 we can choose $k_1 \in \mathbf{R}^{1\times n_m}$ and $k_2 \in \mathbf{R}$ so that

$$\frac{P_m(s)}{c_2(sI - \bar{A}_2)^{-1}b_2} = k_2 + k_1(sI - A_m)^{-1}B_m. \tag{4.12}$$

Now we choose the feed-forward term u_n by

$$u_n(t) = \frac{1}{g(t)}[k_2\bar{u}_m(t) + k_1\bar{x}_m(t)],$$

which yields the **ideal control** law of

$$u(t) = \frac{1}{g(t)}\underbrace{\begin{bmatrix} -c_1(t) & \bar{f}_2 - f_2(t) & 0 & k_1 & k_2 \end{bmatrix}}_{=:K(t)}\bar{x}(t). \tag{4.13}$$

By labelling this control signal $u^0(t)$ and the corresponding generalized plant response $\bar{x}^0$, the ideal closed-loop system is as follows:

$$\begin{aligned} \dot{\bar{x}}^0(t) &= \underbrace{\begin{bmatrix} A_1(t) & b_1c_2 & 0 & 0 & 0 \\ 0 & \bar{A}_2 & 0 & b_2k_1 & b_2k_2 \\ 0 & \sigma c_2 & -\sigma & 0 & 0 \\ 0 & 0 & 0 & A_m & B_m \\ 0 & 0 & 0 & 0 & -\sigma \end{bmatrix}}_{=:\bar{A}_{cl}(t)} \bar{x}^0(t) + \bar{E}u_m(t) \\ \bar{y}^0(t) &= \bar{C}_1\bar{x}^0(t), \\ \bar{e}^0(t) &= \bar{C}_2\bar{x}^0(t), \end{aligned}$$

Since the plant is minimum phase, A_1 is exponentially stable, so it follows that this closed-loop system is exponentially stable and the map from $u_m \longmapsto \bar{e}^0$ is 0.

4.3.2 The First Approximation

Since the plant parameters and system states are not completely known, the ideal control law (4.13) cannot be implemented exactly. Instead, this quantity will be periodically estimated and applied. The $\frac{1}{g}$ term will be problematic in carrying out the estimation, so we will approximate it by a polynomial, which will be easier to deal with. To this end, from Assumptions 1 and 4 of Chapter 3, it follows that the set $\mathcal{G}$ defined by

$$\mathcal{G} := \{g \in \mathbf{R} : \text{there exists a } \psi \in \mathbf{R}^{2n-m} \text{ so that} \begin{bmatrix} \psi \\ g \end{bmatrix} \in \Gamma\}$$

is a compact set, which does not include zero; indeed, there exists a constant $\bar{g} > 0$ so that $\mathcal{G} \subset [-\bar{g}, -\underline{g}] \cup [\underline{g}, \bar{g}]$. From Proposition 3.2 of Chapter 3 we can approximate $1/g$ arbitrarily well over $\mathcal{G}$ via a polynomial, i.e. for every $\varepsilon > 0$ there exist $q \in \mathbf{N}$ and $c_i \in \mathbf{R}$ such that the polynomial $\hat{f}_\varepsilon(g) = \sum_{i=0}^{q} c_i g^i$ satisfies

$$\left|1 - g\hat{f}_\varepsilon(g)\right| < \varepsilon, \quad g \in \mathcal{G}. \tag{4.14}$$

This brings us to the first approximation of (4.13): we define the **approximate ideal control law** to be

$$u(t) = \hat{f}_\varepsilon(g(t))K(t)\bar{x}(t). \tag{4.15}$$

When (4.15) is applied to the generalized plant (4.7)-(4.10), we let u^ε denote the corresponding control signal, $\hat{x}^\varepsilon$ denote the corresponding state, and $\hat{e}^\varepsilon$ denote the corresponding error; the equations describing this closed-loop system are

$$\begin{aligned} \dot{\bar{x}}^\varepsilon(t) &= \bar{A}_{cl}(t)\bar{x}^\varepsilon(t) + \bar{E}u_m(t) + [g(t)\hat{f}_\varepsilon(g(t)) - 1]\bar{B}K(t)\bar{x}^\varepsilon(t) \\ &=: \bar{A}^\varepsilon_{cl}(t)\bar{x}^\varepsilon(t) + \bar{E}u_m(t), \\ \bar{e}^\varepsilon(t) &= \bar{C}\bar{x}^\varepsilon(t). \end{aligned}$$

We label the transition matrices corresponding to $\overline{A}_{cl}(t)$ and $\overline{A}^\varepsilon_{cl}(t)$ by $\Phi_{cl}(t,\tau)$ and $\Phi^\varepsilon_{cl}(t,\tau)$, respectively.

Proposition 4.1: *Suppose that* $\lambda \in (max\left\{\lambda_0, \lambda_m\right\}, 0)$. *Then there exist constants* $\bar{\varepsilon} > 0$ *and* $\gamma > 0$ *so that for every* $\varepsilon \in (0, \bar{\varepsilon})$ *and* $\bar{\theta} \in \bar{\mathcal{P}}$, *the closed-loop system (with* $t_0 = 0$*) satisfies*

$$\begin{aligned} \left\|\bar{x}^{\varepsilon} - \bar{x}^{0}\right\|_2 &\leq \gamma\varepsilon(\|\bar{x}_0\| + \|u_m\|_2), \\ (\int_0^{\infty} |\bar{e}^{\varepsilon}(t) - \bar{C}\Phi_{cl}(t,0)\bar{x}_0|^2 dt)^{\frac{1}{2}} &\leq \gamma\varepsilon(\|\bar{x}_0\| + \|u_m\|_2), \\ \|\bar{x}^{\varepsilon}\|_2 &\leq \gamma(\|\bar{x}_0\| + \|u_m\|_2); \end{aligned}$$

furtheremore, the transition matrix satisfies

$$\|\Phi_{cl}^{\varepsilon}(t,\tau)\| \leq \gamma e^{\lambda(t-\tau)}, \;\; t \geq \tau \geq 0.$$

Proof: See Appendix A.

Thus, as expected, as the approximation of $\frac{1}{g}$ improves, the steady-state tracking error improves. At this point we freeze $\lambda \in (\max\{\lambda_0, \lambda_m\}, 0)$ and $\lambda_1 \in (\max \lambda_0, \lambda_m\}, \lambda)$. From Proposition 4.1, we can choose $\varepsilon > 0$, a polynomial $\hat{f}_{\varepsilon}(g) = \sum_{i=0}^{q} c_i g^i$ satisfying (4.14), and a constant $\gamma > 0$ so that for all $\bar{\theta} \in \bar{\mathcal{P}}$, we have

$$\|\Phi_{cl}^{\varepsilon}(t,\tau)\| \leq \gamma e^{\lambda_1(t-\tau)}, \;\; t \geq \tau \geq 0.$$

4.3.3 The Second Approximation

Observe that (4.15) cannot be implemented, since neither g nor $\bar{x}$ is measurable, so the goal is to approximate (4.15) in a linear fashion. We adopt the technique of Chapter 3. We use h to be small and with q the order of the polynomial approximation set to $\frac{1}{g}$, we choose

$$p > (2q+1)(m+1);$$

recall that the controller period is $T = ph$. The second approximation is as follows:

$$u(t) = \begin{cases} 0 & t \in [kT, kT + (2q+1)(m+1)h), \\ \frac{p}{p-(2q+1)(m+1)} \hat{f}_{\varepsilon}(g(kT))K(kT)\bar{x}^{\varepsilon}(kT) & t \in [kT + (2q+1)(m+1)h, (k+1)T); \end{cases} \tag{4.16}$$

clearly if h and T are both small, (4.16) is a close approximation of

$$u(t) = \hat{f}_{\varepsilon}(g(kT))K(kT)\bar{x}^{\varepsilon}(kT), \quad t \in [kT, (k+1)T),$$

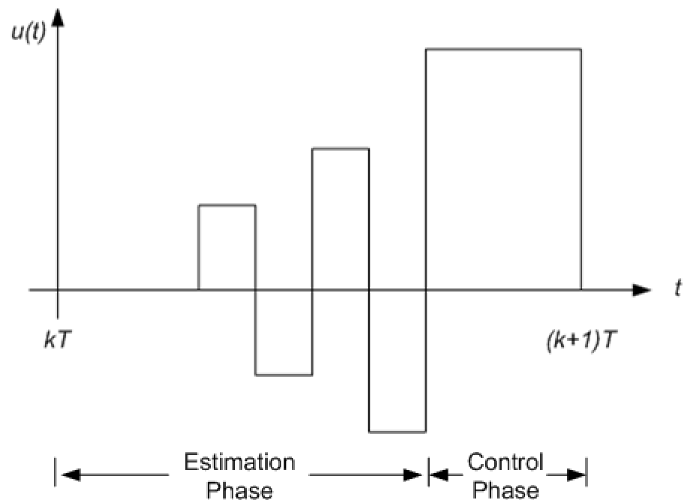

Figure 4.2: A control period consisting of the Estimation and Control Phases [20]

which, in turn, is a good approximation of (4.15). We cannot construct (4.16) for the same reason that we cannot construct (4.15), but the former is amenable to estimation. Every period is divided into two phases: an estimation phase and a control phase, as illustrated in Figure 4.2. In the Estimation Phase, the quantity

$$\hat{f}_\varepsilon(g(kT))K(kT)\hat{x}^\varepsilon(kT)$$

is estimated; although a degree of probing is used, it is carried out in such a way that its effect at the end of the Estimation Phase is very small. In the Control Phase, the above estimate, scaled by

$$\frac{p}{p-(2q+1)(m+1)},$$

is applied; this scaling is chosen to reflect the fact that the Control Phase is exactly

$$\frac{p-(2q+1)(m+1)}{p}$$

of the whole period. In the next two subsections we will explain how the Estimation Phase is implemented.

4.3.4 The First-Order Case (constant parameters and $q = 1$)

Here we consider the simplest possible case, that of first order with constant parameters, as displayed in equation (4.11). The error satisfies

$$(\dot{\bar{y}}_m - \dot{y}) = a_m(\bar{y}_m - y) + [b_m\bar{u}_m - gu + (a_m - a)y],$$

and ideally we would like to set

$$u(t) = \hat{f}_\varepsilon(g)K\bar{x}(t) = \sum_{i=0}^{q} c_i g^i [b_m \bar{u}_m(t) + (a_m - a)y(t)].$$

(Recall that we've chosen $p > 2(2q+1) = 6$ and $h > 0$, and have set $T = ph$.) First we look at the first period $[0, T)$. For simplicity, we consider the case of $q = 1$, so the first step is to construct an approximation of

$$\sum_{i=0}^{1} c_i \underbrace{g^i [b_m \bar{u}_m(0) + (a_m - a)y(0)]}_{=:\phi_i(0)}.$$

Since the controller does not directly measure the output signal y, first we need to estimate the terms containing $y(0)$. From the filter equation (4.5) we have

$$y(t) = \frac{1}{\sigma}\dot{\bar{y}}(t) + \bar{y}(t). \tag{4.17}$$

Assuming that $\bar{y}$ is smooth, we have

$$\begin{aligned} \dot{\bar{y}}(0) &\approx \frac{1}{h}[\bar{y}(h) - \bar{y}(0)] \\ &\approx \frac{1}{h}[\bar{y}(2h) - \bar{y}(h)] \\ &\approx \frac{1}{2h}[-\bar{y}(2h) + 4\bar{y}(h) - 3\bar{y}(0)]. \end{aligned} \tag{4.18}$$

Combining (4.17) and (4.18) results in

$$a_m y(0) \approx \frac{a_m}{2\sigma h}[-\bar{y}(2h) + 4\bar{y}(h) - 3\bar{y}(0)] + a_m \bar{y}(0).$$

The assumption of smoothness of $\bar{y}$ also yields

$$\ddot{\bar{y}}(0) \approx \frac{1}{h^2}[\bar{y}(0) - 2\bar{y}(h) + \bar{y}(2h)]. \tag{4.19}$$

To obtain an estimate of ay, we combine the filter equation (4.5) with the plant equation:

$$\begin{aligned} \dot{\bar{y}} &= -\sigma\bar{y} + \sigma y \\ \Rightarrow \ddot{\bar{y}} &= -\sigma\dot{\bar{y}} + \sigma\dot{y} \\ &= -\sigma\dot{\bar{y}} + \sigma a y + \sigma b u. \end{aligned} \tag{4.20}$$

Suppose that we initially set

$$u(t) = 0, \quad t \in [0, 2h),$$

so regarding (4.20), we obtain

$$\ddot{\bar{y}}(t) = -\sigma\dot{\bar{y}}(t) + \sigma a y(t), \quad t \in [0, 2h).$$

Thus, assuming once again that $\bar{y}$ is smooth, we have

$$\begin{aligned} ay(0) &= \frac{1}{\sigma}\ddot{\bar{y}}(0) + \dot{\bar{y}}(0) \\ &\approx \frac{1}{\sigma h^2}[\bar{y}(0) - 2\bar{y}(h) + \bar{y}(2h)] + \frac{1}{2h}[-\bar{y}(2h) + 4\bar{y}(h) - 3\bar{y}(0)]. \end{aligned}$$

Hence, we can make a good estimate of

$$\phi_0(0) := b_m\bar{u}_m(0) + (a_m - a)y(0),$$

namely

$$\begin{aligned} \hat{\phi}_0(0) &:= b_m\bar{u}_m(0) + a_m\bar{y}(0) + \frac{a_m}{2\sigma h}[-\bar{y}(2h) + 4\bar{y}(h) - 3\bar{y}(0)] - \\ &\quad \frac{1}{2h}[-\bar{y}(2h) + 4\bar{y}(h) - 3\bar{y}(0)] - \frac{1}{\sigma h^2}[\bar{y}(0) - 2\bar{y}(h) + \bar{y}(2h)] \\ &\approx \phi_0(0). \end{aligned}$$

To form an estimate of $\phi_1(0) = g\phi_0(0)$, we will carry out some experiments. With $\rho > 0$ a scaling factor (we make this factor small so that it does not disturb the system very much), set

$$u(t) = \rho\hat{\phi}_0(0), \quad t \in [2h, 4h).$$

Of course, in completing this experiment, we have excited the state. This can be largely undone by applying

$$u(t) = -\rho\hat{\phi}_0(0), \quad t \in [4h, 6h).$$

To obtain an estimate of $bu(2h) = \rho g\hat{\phi}_0(0)$, we return to (4.20) which we rewrite as

$$\begin{aligned} bu(2h) &= \frac{1}{\sigma}[\ddot{\bar{y}}(2h) + \sigma\dot{\bar{y}}(2h) - \sigma a y(2h)] \\ &= \frac{1}{\sigma}\ddot{\bar{y}}(2h) + \dot{\bar{y}}(2h) - ay(2h) \\ &\approx \frac{1}{\sigma}\ddot{\bar{y}}(2h) + \dot{\bar{y}}(2h) - ay(0) \\ &\approx \frac{1}{\sigma}\ddot{\bar{y}}(2h) + \dot{\bar{y}}(2h) - \frac{1}{\sigma}\ddot{\bar{y}}(0) - \dot{\bar{y}}(0); \end{aligned}$$

combining this equation with (4.18) and (4.19) results in

$$\begin{aligned} bu(2h) \approx\ & \frac{1}{\sigma h^2}[\bar{y}(2h) - 2\bar{y}(3h) + \bar{y}(4h)] + \frac{1}{2h}[-\bar{y}(4h) + 4\bar{y}(3h) - 3\bar{y}(2h)] - \\ & \frac{1}{\sigma h^2}[\bar{y}(0) - 2\bar{y}(h) + \bar{y}(2h)] - \frac{1}{2h}[-\bar{y}(2h) + 4\bar{y}(h) - 3\bar{y}(0)]. \end{aligned}$$

Hence,

$$\begin{aligned} \hat{\phi}_1(0) &:= \frac{1}{2\rho h}[-\bar{y}(4h) + 4\bar{y}(3h) - 3\bar{y}(2h) + \bar{y}(2h) - 4\bar{y}(h) + 3\bar{y}(0)] + \\ &\quad \frac{1}{\sigma\rho h^2}[-\bar{y}(0) + 2\bar{y}(h) - \bar{y}(2h) + \bar{y}(2h) - \bar{y}(3h) + \bar{y}(4h)] \\ &\approx \phi_1(0). \end{aligned} \tag{4.21}$$

At the end of the Estimation Phase, we are at $t = 6h$, and we have an estimate of $\phi_1(0)$ to form our control signal to be applied during the Control Phase. We now set u during the Control Phase to be

$$u(t) = \frac{p}{p-6}[c_0\hat{\phi}_0(0) + c_1\hat{\phi}_1(0)], \quad t \in [6h, ph).$$

It follows that

$$\begin{aligned} y(ph) &= y(T) \\ &= e^{pah}y(0) + \int_0^T e^{a(T-\tau)}gu(\tau)d\tau \\ &\approx e^{aT}y(0) + Tg\hat{f}(g)[b_m\bar{u}_m(0) + (a_m - a)y(0)] \\ &\approx e^{a_mT}y(0) + \int_0^T e^{a_m(T-\tau)}b_m\bar{u}_m(\tau)d\tau, \end{aligned}$$

as desired.

The above analysis was quite informal although reasonably intuitive. In the next section we extend this to the general case and rigorously prove a result on estimation.

4.3.5 The General Case

Now we will show the controller construction for the general n^{th} order case with time-varying parameters. First, we need some notation. Let us define $1 \times (m+2)$ vector

$$\tilde{c}_2 := \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix}, \tag{4.22}$$

and with two $(m+2) \times (m+2)$ matrices S_{m+1} and $H_{m+1}(h)$ defined by equation (3.21) of Section 3.5, b_2 and c_2 defined by equations (3.8) (3.9), respectively, of Section 3.3, and Λ_2 defined by equation (3.15) of Section 3.4, we define two $(m+2) \times (m+2)$ matrices

$$\tilde{\Lambda}_2 := \begin{bmatrix} \Lambda_2 & 0 & b_2 \\ \sigma c_2 & -\sigma & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \bar{F}_{m+1}(\sigma) := \begin{bmatrix} \tilde{c}_2 \\ \tilde{c}_2\tilde{\Lambda}_2 \\ \vdots \\ \tilde{c}_2\tilde{\Lambda}_2^m \\ \tilde{c}_2\tilde{\Lambda}_2^{m+1} \end{bmatrix}. \tag{4.23}$$

To motivate the upcoming result, first consider the special case in which the plant is LTI with transfer function $\frac{g}{s^m}$, which means that w has zero dimension:

$$\begin{aligned}\dot{v} &= \underbrace{\begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & 0 \end{bmatrix}}_{=\Lambda_2} v + \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{=b_2} gu, \\ \bar{y}(t) &= \underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}}_{=c_2} v,\end{aligned}$$

so combining the plant with the output filter yields

$$\begin{aligned}\begin{bmatrix} \dot{v}(t) \\ \dot{\bar{y}}(t) \end{bmatrix} &= \underbrace{\begin{bmatrix} \Lambda_2 & 0 \\ \sigma c_2 & -\sigma \end{bmatrix}}_{=:\bar{\Lambda}_2} \begin{bmatrix} v(t) \\ \bar{y}(t) \end{bmatrix} + \underbrace{\begin{bmatrix} b_2 \\ 0 \end{bmatrix}}_{=:\bar{b}_2} gu(t), \\ \bar{y}(t) &= \underbrace{[0 \quad 1]}_{=:\bar{c}_2} \begin{bmatrix} v(t) \\ \bar{y}(t) \end{bmatrix}. \end{aligned} \tag{4.24}$$

Now suppose that u is a constant:

$$u(t) = \bar{u}, \quad t \geq 0.$$

Then we combine this equation with (4.24):

$$\begin{aligned}\begin{bmatrix} \begin{bmatrix} \dot{v}(t) \\ \dot{\bar{y}}(t) \end{bmatrix} \\ g\dot{u}(t) \end{bmatrix} &= \underbrace{\begin{bmatrix} \bar{\Lambda}_2 & \bar{b}_2 \\ 0 & 0 \end{bmatrix}}_{=\tilde{\Lambda}_2} \begin{bmatrix} \begin{bmatrix} v(t) \\ \bar{y}(t) \end{bmatrix} \\ gu(t) \end{bmatrix}, \\ \bar{y}(t) &= \underbrace{[\bar{c}_2 \quad 0]}_{=\tilde{c}_2} \begin{bmatrix} \begin{bmatrix} v(t) \\ \bar{y}(t) \end{bmatrix} \\ gu(t) \end{bmatrix}, \quad t \geq 0;\end{aligned}$$

notice that $(\tilde{c}_2, \tilde{\Lambda}_2)$ is observable. It is easy to see that

$$\bar{y}(t) = \begin{bmatrix} 1 & t & \cdots & \frac{t^{m+1}}{(m+1)!} \end{bmatrix} \begin{bmatrix} \tilde{c}_2 \\ \tilde{c}_2\tilde{\Lambda}_2 \\ \vdots \\ \tilde{c}_2\tilde{\Lambda}_2^{m+1} \end{bmatrix} \begin{bmatrix} v(0) \\ \bar{y}(0) \\ g\bar{u} \end{bmatrix} + \mathcal{O}(t^{m+2}) \begin{bmatrix} v(0) \\ \bar{y}(0) \\ g\bar{u} \end{bmatrix}.$$

If we sample $\bar{y}(t)$ at multiples of h, we obtain

$$\begin{bmatrix} \bar{y}(0) \\ \bar{y}(h) \\ \vdots \\ \bar{y}((m+1)h) \end{bmatrix} = S_{m+1} H_{m+1}(h) \bar{F}_{m+1}(\sigma) \begin{bmatrix} v(0) \\ \bar{y}(0) \\ g\bar{u} \end{bmatrix} + \mathcal{O}(h^{m+2}) \begin{bmatrix} v(0) \\ \bar{y}(0) \\ g\bar{u} \end{bmatrix},$$

so

$$\bar{F}_{m+1}^{-1}(\sigma)H_{m+1}^{-1}(h)S_{m+1}^{-1}\begin{bmatrix}\bar{y}(0)\\ \bar{y}(h)\\ \vdots\\ \bar{y}((m+1)h)\end{bmatrix}=\begin{bmatrix}v(0)\\ \bar{y}(0)\\ g\bar{u}\end{bmatrix}+\mathcal{O}(h)\begin{bmatrix}v(0)\\ \bar{y}(0)\\ g\bar{u}\end{bmatrix}.$$

Indeed, as long as $u(t)$ is constant (equal to $\bar{u}$) on $t \in [0,(m+1)h)$, then we can estimate $v(0)$ and $g\bar{u}$ (we do not need to estimate $\bar{y}(0)$ since we can measure it). Since we are using a sequence of $m+2$ samples of $\bar{y}(t)$ we define

$$\bar{\mathcal{Y}}(t) := \begin{bmatrix} \bar{y}(t) & \bar{y}(t+h) & \cdots & \bar{y}(t+(m+1)h) \end{bmatrix}^T.$$

The following result shows how to do estimation in the general case, with a precise bound on the estimation error.

Lemma 4.1: *(Key Estimation Lemma (KEL-4)) There exist constants $\gamma > 0$ and $\bar{h} > 0$ so that for all $t_0 \in \mathbf{R}^+$, $\bar{x}_0 \in \mathbf{R}^{n+n_m+2}$, $u_m \in PC_\infty$, $h \in (0,\bar{h})$, and $\bar{\theta} \in \bar{\mathcal{P}}$, the solution of (4.7)-(4.10) with*

$$u(t) = \bar{u}, \quad t \in [t_0, t_0+(m+1)h)$$

has the following two properties:

(i) In all cases

$$\|\bar{x}(t)-\bar{x}(t_0)\| \le \gamma(h\|\bar{x}(t_0)\| + h\|\bar{u}\| + h^{\frac{1}{2}}\|u_m\|_{2\,[t_0,t)}), \quad t \in [t_0, t_0+(m+1)h),$$

$$\|\bar{F}_{m+1}^{-1}(\sigma)H_{m+1}^{-1}(h)S_{m+1}^{-1}\bar{\mathcal{Y}}(t_0)\| \le \gamma(\|\bar{x}(t_0)\| + \|\bar{u}\| + h^{\frac{1}{2}}\|u_m\|_{2\,[t_0,t_0+(m+1)h)}).$$

(ii) If $\bar{\theta}(t)$ is absolutely continuous on $[t_0, t_0+(m+1)h)$, then we can obtain a tighter approximation:

$$\left\|\bar{F}_{m+1}^{-1}(\sigma)H_{m+1}^{-1}(h)S_{m+1}^{-1}\bar{\mathcal{Y}}(t_0) - \begin{bmatrix}v(t_0)\\ f_2(t_0)v(t_0)+c_1(t_0)w(t_0)+g(t_0)\bar{u}\\ \bar{y}(t_0)\end{bmatrix}\right\| \le \gamma h(\|\bar{x}(t_0)\| + \|\bar{u}\| + h^{\frac{1}{2}}\|u_m\|_{2\,[t_0,t_0+(m+1)h)}).$$

Proof: See Appendix A.

To see the application of KEL-4, suppose that $\bar{\theta} \in \bar{\mathcal{P}}$ is absolutely continuous. If we set

$$u(t) = 0, \quad t \in [kT, kT + (m+1)h),$$

with $K(t)$ defined by (4.13), it follows that

$$\begin{bmatrix} \bar{f}_2 & -1 & 0 \end{bmatrix} \bar{F}_{m+1}^{-1}(\sigma) H_{m+1}^{-1}(h) S_{m+1}^{-1} \bar{\mathcal{Y}}(kT) + \begin{bmatrix} k_1 & k_2 \end{bmatrix} \begin{bmatrix} \bar{x}_m(kT) \\ \bar{u}_m(kT) \end{bmatrix}$$
$$\approx K(kT)\bar{x}(kT).$$

Similar to Chapter 3, with $\rho > 0$ a scaling factor, now set

$$u(t) = \rho \times \text{ estimate of } K(kT)\bar{x}(kT), \quad t \in [kT + (m+1)h, kT + 2(m+1)h);$$

then it follows

$$\frac{1}{\rho} \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 \end{bmatrix} \quad \bar{F}_{m+1}^{-1}(\sigma) H_{m+1}^{-1}(h) S_{m+1}^{-1} [\bar{\mathcal{Y}}(kT + (m+1)h) - \bar{\mathcal{Y}}(kT)] \approx$$
$$g(kT)K(kT)\bar{x}(kT).$$

Similar to the Chapter 3 approach, repeating this procedure, a close estimate of

$$\frac{1}{g(kT)} K(kT)\bar{x}(kT) \approx \sum_{i=0}^{q} c_i \underbrace{g(kT)^i K(kT)\bar{x}(kT)}_{=:\phi_i(kT)}$$

can be obtained by the end of the Estimation Phase (the interval $[kT, kT + (2q + 1)(m+1)h)$); in the Control Phase a suitably weighted of this estimate is applied. If the sampling time T is small enough, then any discontinuity will be infrequent, and the controller will work. Finally, we end up with the controller structure as:

THE PROPOSED CONTROLLER ($t_0 = 0$)

Estimation Phase: $[kT, kT + (2q+1)(m+1)h)$

$$\hat{\phi}_i(kT) = \begin{cases} \begin{bmatrix} \bar{f}_2 & -1 & 0 \end{bmatrix} \bar{F}_{m+1}^{-1}(\sigma) H_{m+1}^{-1}(h) S_{m+1}^{-1} \bar{\mathcal{Y}}(kT) + \begin{bmatrix} k_1 & k_2 \end{bmatrix} \begin{bmatrix} \bar{x}_m(kT) \\ \bar{u}_m(kT) \end{bmatrix} & \text{if } i = 0, \\ \frac{1}{\rho} \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 \end{bmatrix} \bar{F}_{m+1}^{-1}(\sigma) H_{m+1}^{-1}(h) S_{m+1}^{-1} \times & \\ \quad \begin{bmatrix} \bar{\mathcal{Y}}(kT + (2i-1)(m+1)h) - \bar{\mathcal{Y}}(kT) \end{bmatrix} & \text{if } i = 1, \cdots, q, \end{cases} \tag{4.25}$$

$$u(t) = \begin{cases} 0 & t \in [kT, kT + (m+1)h), \\ \rho \hat{\phi}_{i-1}(kT) & t \in [kT + (2i-1)(m+1)h, kT + 2i(m+1)h), \\ & i = 1, \cdots, q, \\ -\rho \hat{\phi}_{i-1}(kT) & t \in [kT + 2i(m+1)h, kT + (2i+1)(m+1)h), \\ & i = 1, \cdots, q. \end{cases} \tag{4.26}$$

Control Phase: $[kT + (2q+1)(m+1)h, (k+1)T)$

$$u(t) = \frac{p}{p - (2q+1)(m+1)} \sum_{i=0}^{q} c_i \hat{\phi}_i(kT), \quad t \in [kT + (2q+1)(m+1)h, (k+1)T). \tag{4.27}$$

Lemma 4.2: *The control law (4.25)-(4.27) has a representation of the form (4.6) (with parameters (F, G, H, J, L, M, h, p)) and is deadbeat; in fact, $F(0) = 0$.*

Proof: See Appendix A.

Now we present a proposition which considers the closed-loop system over one period. It is shown that the closed-loop system with control law (4.25)-(4.27) behaves very much the same as the closed-loop system with the approximate ideal control law (4.15).

Lemma 4.3:*(Key Control Lemma (KCL-4)) There exist constants $\gamma > 0$ and $\overline{T} > 0$ so that for all $k \in \mathbf{Z}^+, T \in (0, \overline{T})$, and $\bar{\theta} \in \bar{\mathcal{P}}$, the solution of (4.7)-(4.10) with u given by (4.25)-(4.27) has the following properties:*

(i) In all cases

$$\left\| \bar{x}(t) - \Phi^{\varepsilon}_{cl}(t, kT)\bar{x}(kT) - \int_{kT}^{t} \Phi^{\varepsilon}_{cl}(t,\tau)\bar{E}u_m(\tau)d\tau \right\| \leq \gamma(T\left\|\bar{x}(kT)\right\| + T^{\frac{1}{2}}\left\|u_m\right\|_{2,[kT,t)}),$$

$$\|u(t)\| \leq \gamma(\|\bar{x}(kT)\| + T^{\frac{1}{2}}\|u_m\|_{2,[kT,t)}), t \in [kT, (k+1)T).$$

(ii) If $\bar{\theta}(t)$ is absolutely continuous on $[kT, (k+1)T)$, then

$$\|\bar{x}((k+1)T) - \Phi^{\varepsilon}_{cl}((k+1)T, kT)\bar{x}(kT) - \int_{kT}^{(k+1)T} \Phi^{\varepsilon}_{cl}((k+1)T,\tau)\bar{E}u_m(\tau)d\tau\| \leq \gamma(T^2\|\bar{x}(kT)\| + T^{\frac{3}{2}}\|u_m\|_{2,\ [kT,(k+1)T)}).$$

Proof: See Appendix A.

The KEL-4 and KCL-4 describe the closed-loop system behaviour during each period. In the following result, it is proven that the proposed controller (4.25)-(4.27) acts like the approximated ideal control law (4.15) for all time. Before proceeding, we need some notation: we let $\hat{u}^{\varepsilon}$ denote the control signal, $\hat{\bar{x}}^{\varepsilon}$ denote the state, $\hat{\bar{e}}^{\varepsilon}$ denote the error, and $\hat{y}^{\varepsilon}$ denote the plant output when (4.25)-(4.27) is applied. Recall that we use $\bar{x}^{\varepsilon}$, $\bar{e}^{\varepsilon}$, and y^{ε} to denote the corresponding signals when (4.15) is applied.

Proposition 4.2: *There exist constants* $\bar{T} > 0$, $\lambda < 0$, *and* $\gamma > 0$ *so that for every* $\bar{\theta} \in \bar{\mathcal{P}}$, $u_m \in PC_\infty$, *and* $T \in (0, \bar{T})$, *we have*

$$\begin{aligned} \|\hat{\bar{x}}^\varepsilon(t)\| &\leq \gamma(e^{\lambda t}\|\bar{x}_0\| + \|u_m\|_{2\,[0,t)}), \quad t \geq 0, \\ \|\bar{x}^\varepsilon - \hat{\bar{x}}^\varepsilon\|_2 &\leq \gamma(T\|\bar{x}_0\| + T\|u_m\|_2). \end{aligned}$$

Proof: See Appendix A.

4.4 The Main Result

Now we can put together all of the results of this chapter to prove that the proposed LPC provides both stability and good tracking in the face of plant uncertainty.

Theorem 4.1: *For every* $\delta > 0$ *and* $\lambda \in (\max\{\lambda_0, \lambda_m\}, 0)$ *there exists a controller of the form (4.2),(4.3), (4.5), and (4.6) so that for every* $\bar{\theta} \in \bar{\mathcal{P}}$, $\bar{x}_0 \in \mathbf{R}^{n+n_m+1}$, *and* $u_m \in PC_\infty$, *we have that the closed-loop system is exponentially stable, and when* $t_0 = k_0 = 0$, *we have*

$$\left(\int_0^\infty |\hat{y}^\varepsilon(t) - y_m(t) - \bar{C}\Phi_{cl}(t,0)\bar{x}(0)|^2 dt\right)^{\frac{1}{2}} \leq \delta(\|\bar{x}_0\| + \|u_m\|_2), \;\; t \geq 0.$$

Remark 4.3: *If* $\bar{x}(0) = 0$, *then this means that*

$$\|\hat{y}^\varepsilon - y_m\|_2 \leq \delta\|u_m\|_2.$$

Remark 4.4: *Theorem 4.1 states that the difference between the output* $\hat{y}^\varepsilon$, *of the closed-loop system when the proposed controller is applied, and the output of the closed-loop system when the ideal control law (4.13) is applied, namely,*

$$y_m(t) + \bar{C}\Phi_{cl}(t,0)\bar{x}_0,$$

can be made as small as desired by selecting the controller parameters $\varepsilon > 0$ *and* $T > 0$ *properly.*

Remark 4.5: *Often we may have reference signals which have infinite 2-norm, such as a sinusoidal. Because of causality, Theorem 2 also means that, for every* $\tau > 0$

$$\left(\int_0^\tau |\hat{y}^\epsilon(t) - y_m(t) - \bar{C}\Phi_{cl}(t,0)\bar{x}(0)|^2 dt\right)^{\frac{1}{2}} \leq \delta\|\bar{x}_0\| + \delta\left(\int_0^\tau |u_m(t)|^2 dt\right)^{\frac{1}{2}}, \;\; \tau \geq 0.$$

Proof:

First suppose that $t_0 = k_0 = 0$. We start with the bound on the transients. If we choose $\sigma > 0$ sufficiently large that

$$\frac{1}{\sigma}\left[||C_m||.||B_m|| + ||C_m||.||A_m||.||B_m||\frac{\gamma_m}{|\lambda_m|}\right] + \frac{\gamma_m}{|\lambda_m|}||C_m||.||B_m|| \times \frac{1}{\sqrt{2\sigma}} \leq \frac{\delta}{3},$$

it follows from (4.4) that

$$||\bar{y}_m - y_m||_2 \;\leq\; (\delta/3)\,(||\bar{x}_0|| + ||u_m||_2). \tag{4.28}$$

Let $\lambda \in (\max\{\lambda_0, \lambda_m\}, 0)$ and $\lambda_1 \in (\max\{\lambda_0, \lambda_m\}, \lambda)$; from Proposition 4.1 there exist constants $\bar{\varepsilon} > 0$ and $\gamma > 1$ so that for every $\varepsilon \in (0, \bar{\varepsilon})$ and $\bar{\theta} \in \bar{\mathcal{P}}$, the closed-loop system satisfies

$$\begin{aligned} \|\Phi^{\varepsilon}_{cl}(t,\tau)\| &\leq \gamma_1 e^{\lambda_1(t-\tau)}, \quad t \geq \tau \geq 0, \\ \|\bar{x}^{\varepsilon}\|_2 &\leq \gamma_1(\|\bar{x}_0\| + \|u_m\|_2), \\ \Big(\int_0^{\infty} |\bar{e}^{\varepsilon}(t) - \bar{C}\Phi_{cl}(t,0)\bar{x}_0|^2 dt\Big)^{\frac{1}{2}} &\leq \gamma_1\varepsilon(\|\bar{x}_0\| + \|u_m\|_2). \end{aligned} \tag{4.29}$$

Thus, now choose $\varepsilon \in (0, \bar{\varepsilon})$ so that

$$\gamma_1\varepsilon < \delta/3.$$

It follows that

$$\Big(\int_0^{\infty} |\bar{e}^{\varepsilon}(t) - \bar{C}\Phi_{cl}(t,0)\bar{x}_0|^2 dt\Big)^{\frac{1}{2}} \;\leq\; (\delta/3)(\|\bar{x}_0\| + \|u_m\|_2), \;\; t \geq 0. \tag{4.30}$$

From Proposition 4.2 there exist $\bar{T} > 0$ and $\gamma_2 > 0$ so that for every $\bar{\theta} \in \bar{\mathcal{P}}$, $u_m \in PC_\infty$, and $T \in (0, \bar{T})$ we have

$$\begin{aligned} \|\hat{\bar{x}}^{\varepsilon}(t)\| &\leq \gamma_2(e^{\lambda t}\|\bar{x}_0\| + \|u_m\|_{2,[0,t)}), \quad t \geq 0, \\ \|\hat{\bar{e}}^{\varepsilon} - \bar{e}^{\varepsilon}\|_2 &\leq \gamma_2(T\|\bar{x}_0\| + T\|u_m\|_2), \\ \|\hat{\bar{x}}^{\varepsilon} - \bar{x}^{\varepsilon}\|_2 &\leq \gamma_2(T\|\bar{x}_0\| + T\|u_m\|_2). \end{aligned} \tag{4.31}$$

Choose $T \in (0, \bar{T})$ so that

$$\gamma_2 T < \delta/3;$$

it follows that

$$\|\hat{\bar{e}}^{\varepsilon} - \bar{e}^{\varepsilon}\|_2 \;\leq\; (\delta/3)(\|\bar{x}_0\| + \|u_m\|_2).$$

If we combine this equation with (4.28) and (4.30) we end up with

$$\begin{aligned}
&(\int_0^\infty |\hat{y}^\varepsilon(t) - y_m(t) - \bar{C}\Phi_{cl}(t,0)\bar{x}_0|^2 dt)^{\frac{1}{2}} \\
\leq\ &(\int_0^\infty |\hat{y}^\varepsilon(t) - y^\varepsilon(t) + y^\varepsilon(t) - \bar{y}_m(t) + \bar{y}_m(t) - y_m(t) - \bar{C}\Phi_{cl}(t,0)\bar{x}_0|^2 dt)^{\frac{1}{2}} \\
\leq\ &\|\hat{\bar{e}}^\varepsilon - \bar{e}^\varepsilon\|_2 + ||\bar{y}_m - y_m||_2 + (\int_0^\infty |\bar{e}^\varepsilon(t) - \bar{C}\Phi_{cl}(t,0)\bar{x}_0|^2 dt)^{\frac{1}{2}} \\
\leq\ &(\delta/3)(\|\bar{x}_0\| + \|u_m\|_2) + (\delta/3)(\|\bar{x}_0\| + \|u_m\|_2) + (\delta/3)(\|\bar{x}_0\| + \|u_m\|_2) \\
\leq\ &\delta(\|\bar{x}_0\| + \|u_m\|_2), \quad t \geq 0.
\end{aligned}$$

Hence, the controller given by (4.25)-(4.27) (with a representation of the form (4.6)) together with (4.2),(4.3), and (4.5) satisfies the constraints on the error.

It remains to prove that this controller provides exponentially stability. So first suppose that $t_0 = k_0 = 0$ and that $u_m = 0$. From Proposition 4.2, there exists a constant $\gamma_3 > 0$ such that

$$\|\hat{\bar{x}}^\varepsilon(t)\| \leq \gamma_3 e^{\lambda t}\|\bar{x}_0\|, \quad t \geq 0.$$

From Lemma 4.2 we know that the representation (4.6) of (4.25)-(4.27) is deadbeat; indeed, $F(0) = 0$. Hence, if we solve (4.6), it follows that there exists a constant $\gamma_4 > 0$ so that

$$\begin{aligned}
\|z[k]\| &\leq \gamma_4 \max\{\|\hat{\bar{x}}^\varepsilon(k-j)\| : 0 \leq j \leq p,\ k-j \geq 0\} \\
&\leq \gamma_3\gamma_4 e^{\lambda(k-p)T}\|\bar{x}_0\| \\
&\leq \underbrace{\gamma_3\gamma_4 e^{-\lambda p\bar{T}}}_{=:\gamma_5} e^{\lambda kT}\|\bar{x}_0\|, \quad k \geq 1.
\end{aligned}$$

Recall that the state of the closed loop system is defined by

$$x_{sd}(t) := \begin{bmatrix} \hat{\bar{x}}^\varepsilon(t) \\ z[k] \end{bmatrix}, \quad t \in [kT, (k+1)T),$$

so it follows that

$$\|x_{sd}(t)\| \leq \underbrace{(\gamma_3 + \gamma_5 e^{-\lambda\bar{T}})}_{=:\gamma_6} e^{\lambda t}\|x_{sd}(0)\|, \quad t \geq 0.$$

Now we turn to the case when $t_0 = k_0 h \geq 0$. Since the controller (4.6) is periodic of period T, it follows that if our starting point is $t_0 = jT$, then

$$\|x_{sd}(t)\| \leq \gamma_6 e^{\lambda(t-jT)}\|x_{sd}(jT)\|, \quad t \geq jT.$$

Since the plant parameters are uniformly bounded, it is straight-forward to prove that nothing un-toward happens if we start between samples: there exists a constant γ_7 so that for every $t_0 = k_0 h \geq 0$, we have

$$\|x_{sd}(t)\| \leq \gamma_7 e^{\lambda(t-t_0)}\|x_{sd}(t_0)\|, \quad t \geq t_0,$$

i.e., we have exponential stability.

□

4.5 Examples

In this section two examples as Chapter 3 are presented, one with relative degree one and one with relative degree two.

4.5.1 Example 1

The plant model is first order:

$$\dot{y}(t) = a(t)y(t) + g(t)u(t).$$

The set of plant uncertainty is given by

$$\Gamma = \{ \begin{bmatrix} a \\ g \end{bmatrix} \in \mathbf{R}^2 : a \in [-1, 1],\ g^2 \in [1, 1.4]\},$$

$$\mathcal{P} = \mathcal{P}(\Gamma, \mu_1 = 1, T_0 = 5, \gamma_0 = 1, \lambda_0 = -5).$$

The reference model is

$$\dot{\bar{x}}_m = -\bar{x}_m + \bar{u}_m,$$

while reference model anti-aliasing filter is chosen to be

$$\dot{\bar{u}}_m = -50\bar{u}_m + 50u_m,$$

as is the anti-aliasing output filter:

$$\dot{\bar{y}} = -50\bar{y} + 50y.$$

For approximating the polynomial $\frac{1}{g}$, we use

$$\hat{f}_{0.01}(g) = 2.1647g - 1.5153g^3 + 0.3433g^5,$$

so $q = 5$, and we choose $p = 50 > 2(2q + 1)$, i.e. the Control Phase is 66% of the time. We set $T = 0.1$ (so that $h = 0.002$) and $\rho = 1$. Figure 4.6 shows the simulation results with $y_0 = 3$, $\bar{u}_{m_0} = 0$, u_m a square wave given by

$$u_m = sign(\cos(\frac{2\pi t}{15})),$$

and

$$a(t) = \cos(t/2) \text{ and } g(t) = [1.2 + 0.2cos(t/2)] * sign[cos(t/4)];$$

we add a noise signal of the form

h * random sequence uniformly distributed between ± 1

to the output measurement at the time $t = 66$ sec. We see that the effect of the initial condition decays to zero and finally good tracking is attained, even in the presence of rapid time-variations and noisy measurements, although it is clear that the effect of the tiny amount of noise is quite significant.

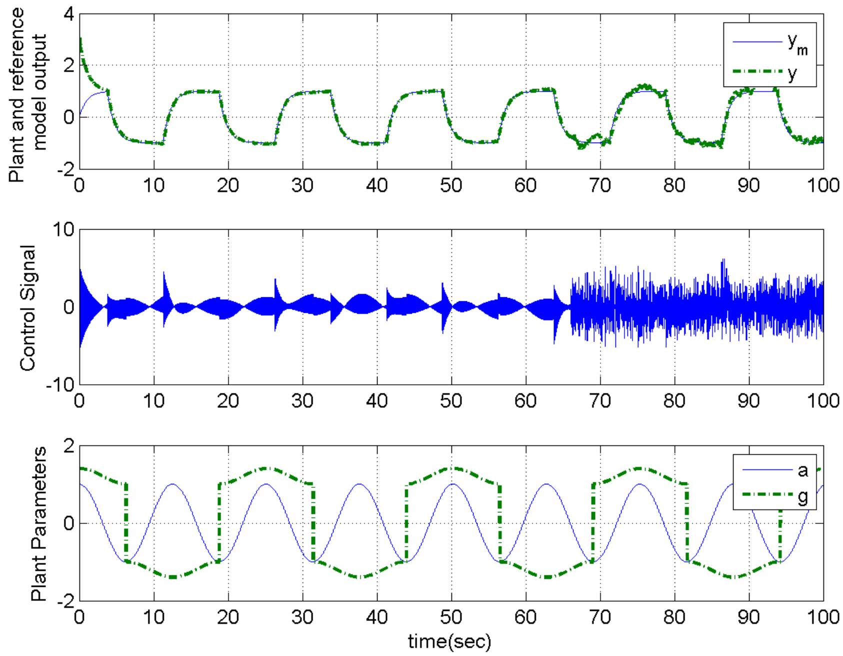

Figure 4.3: Example with a and g varying with time and the output anti-aliasing filter

4.5.2 Example 2

In this example a minimum phase relative degree two model of a DC motor is considered [17]:

$$\begin{aligned}\dot{x} &= \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & \beta_1(t) \end{bmatrix}}_{=:A(t)} x + \underbrace{\begin{bmatrix} 0 \\ g(t) \end{bmatrix}}_{=:B(t)} u, \\ y &= \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{=:C} x.\end{aligned}$$

The set of plant uncertainty is given by

$$\bar{\Gamma} = \{ \begin{bmatrix} \beta_0 \\ \beta_1 \\ g \end{bmatrix} \in \mathbf{R}^3 : \beta_0 = 0,\ \beta_1 \in [-5, -1],\ \ g \in [1, 2] \},$$

$$\bar{\mathcal{P}} = \bar{\mathcal{P}}(n = 2, m = 2, \bar{\Gamma}, \bar{\mu}_1 = 1, \bar{T}_0 = 10, \gamma_0 = 1, \lambda_0 = -5\}.$$

The reference model is relative degree two with the state space representation:

$$\begin{aligned}\dot{x}_m &= \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x_m + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_m, \\ y_m &= \begin{bmatrix} 1 & 0 \end{bmatrix} x_m.\end{aligned}$$

The anti-aliasing filter is chosen as

$$\dot{\bar{u}}_m = -50\bar{u}_m + 50u_m,$$

and the anti-aliasing output filter is given by

$$\dot{\bar{y}} = -50\bar{y} + 50y.$$

The range of g is smaller than in Example 1, so for the approximation of $\frac{1}{g}$, we use the polynomial

$$\hat{f}_{0.01}(g) = 2.1647 - 1.5153g + 0.3433g^2,$$

so $q = 2$, we choose $p = 40 > 2(2q+1)m$, and set $T = 0.04$ (so that $h = 0.001$), and $\rho = 1$. A simulation was carried out (see Figure 4.4) with $x_0 = [1 \quad 1]^T$, $y_{m_0} = \bar{u}_{m_0} = 0$, u_m a square wave given by

$$u_m = sign(\cos(\frac{2\pi t}{15})),$$

and

$$\beta_1(t) = -3 + 2 * \cos(t/2) \text{ and } g(t) = 1.5 + 0.5 * \sin(t/4).$$

We add a noise signal of the form
h^2 * random sequence uniformly distributed between ± 1
to the output measurement at the time $t = 66$ sec. While Figure 4.4 illustrates the simulation results, Figure 4.5 provides a close-up of the control signal during the first three periods. As shown in these two examples, the tracking is quite close, the control signal is modest in size, and the effect of the initial conditions declines exponentially to zero even with time-varying plant parameters.

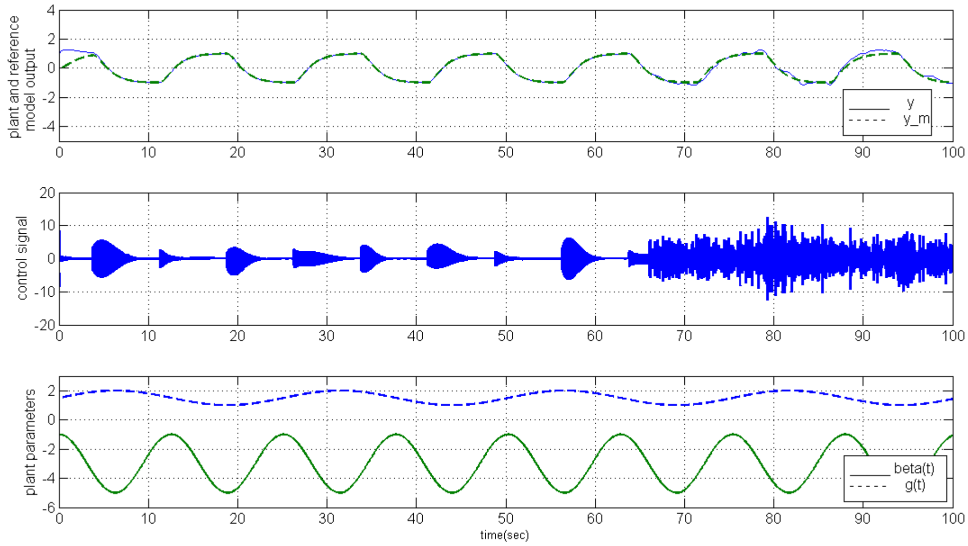

Figure 4.4: Example with β_1 and g varying with time

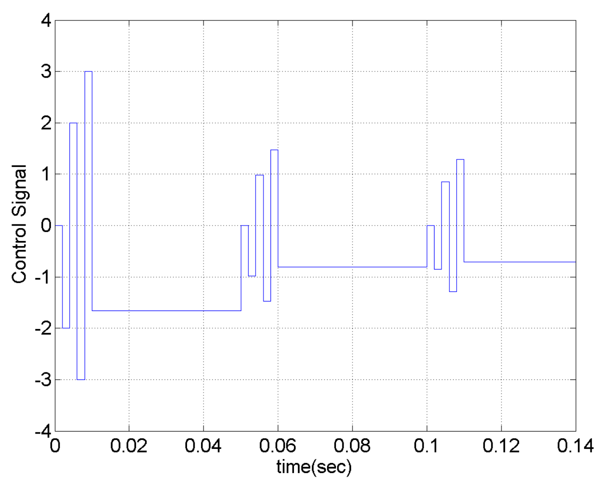

Figure 4.5: A close-up of the control signal

4.5.3 Example 3

In this example we analyze the performance and the noise behaviour of the closed-loop system. We compute the $\mathcal{L}_\infty$ induced norm when the plant parameters are fixed and investigate how the performance and the noise behaviour of the sampled-data system depends on the sampling period. The first computation of the $\mathcal{L}_\infty$ induced norm of the LPC [20] was carried out by Xiaosong [43]. The tool that is used is the *lifting* technique (see Chen and Francis [3] which establishes a strong correspondence between periodic systems and time invariant infinite-dimensional systems.

We adopt one admissible plant model of the set of plant uncertainty of Example 1 of this chapter as

$$\dot{y}(t) = y(t) + u(t).$$

In the LPC [20], for approximating the polynomial $\frac{1}{g}$, we use the estimation polynomial

$$\hat{f}(g) = g.$$

For the LPC both in L_1 framework (∞-norm) and H^∞ framework (2-norm). With the controller designed as above but with h free, in Figure 4.6, we plot $||T_{u_m}||$ and $||T_n||$ as a function of h. As expected, for both controllers, while the former goes to zero, the latter goes to infinity as $h \longrightarrow 0$. As this figure illustrates, for the new framework H^∞ (2-norm) the closed-loop performance and noise-tolerance deteriorate.

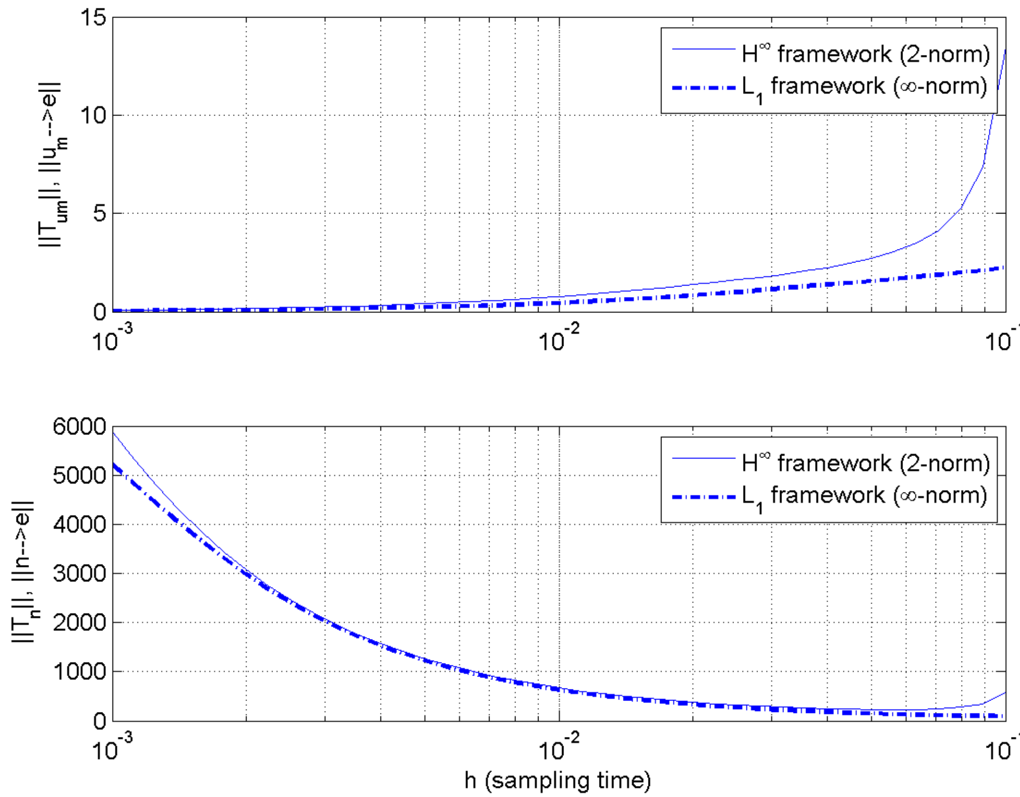

Figure 4.6: The $||u_m \to e||$ and $||n \to e||$ map for two controllers; the high frequency gain sign is fixed

4.6 Summary and Conclusion

The performance of many controllers is determined by the size of specified signals. The dependence of the 2-norm on the entire signal as well as its measurement of energy both in time and frequency domains make it more commonly used, e.g. in the H^∞ framework, than the ∞-norm for measuring signals in control designs.

Miller [18, 17, 20] proposed a model reference adaptive control for time varying, minimum phase systems, which unlike most MRACs, is linear and periodic. Rather than estimating parameters, the linear periodic time-varying controller directly estimates the control signal (what the control signal would be if the plant parameters and states were known, and the ideal linear time-invariant compensator was applied). The obtained results are in the ∞-norm.

In this chapter, we modified the controller [20] by adding an anti-aliasing filter to the plant output measurement, and produced results which hold when using the 2-norm to measure the signal size. Adding this filter resulted in increasing the complexity and deteriorating the performance and noise tolerance of the closed-loop system.

Chapter 5

The Redesigned LPC

5.1 Introduction

In the adaptive control technique of [20], discussed in Chapter 3 and extended in Chapter 4, the adaptive controller operates periodically, first estimating the *ideal control signal* using a probing signal, and then applying this (suitably scaled) quantity. Carrying out estimation and control in series is different than that used in classical adaptive control, where it is done in parallel. That, in itself, is not a major concern, although doing estimation and control in parallel is more aesthetically pleasing than in series. The real problem arises from the fact that the probing signal is quite large and must be of short duration for the approach to work, which is hard on the actuator. Since the goal is to estimate the *ideal control signal* and this quantity does not change rapidly, it is reasonable to expect that consecutive estimates are quite similar, so it would make more sense to estimate the *difference* in the value of the ideal control signal between consecutive periods, for that should be roughly proportional in size to the size of the period. While this idea is intuitively reasonable, the details are involved and the proof is intricate. Since the probing is now a second-order effect rather than a first-order effect as in [20], it has additional advantage that we need not cancel out the effect of the probing, which reduces the number of experiments (or probes) during a given period. Because of the fact that the control signal is always close to the *ideal control signal*, we might expect the required period T to be larger than in the similar design for the same level of performance.

Remark 5.1: *If the sole concern is an active actuator, an obvious solution would be to put a low-pass-filter at the plant input, and proceed as before. Using the original technique of [20], the input to the low-pass-filter would still be discontinuous and rapidly moving, but the control signal would be continuous. Of course, the relative degree of the augmented plant (the plant together with the low-pass-filter) would increase by one, which would increase the size of the gains, and therefore decrease the noise tolerance, which is a major drawback. If this is not a major concern then this could be considered with the new idea of simultaneous estimation and control* *discussed above.*

5.2 The Problem Formulation

In the following we adopt the same setup as in [20], which was discussed in Chapter 3. For completeness we collect relevant notation. We start with the linear time-varying plant:

$$\begin{array}{rcl} \dot{x}(t) & = & A(t)x(t) + B(t)u(t),\ x(t_0) = x_0, \\ y(t) & = & C(t)x(t). \end{array} \tag{5.1}$$

We will allow a good deal of model uncertainty: the set of possible models is labelled $\bar{\mathcal{P}}$; roughly speaking, the set of models corresponding to a time-varying counterpart of the classical set with a compactness requirement added. We adopt the approach illustrated in Figure 5.1, which we borrow from Chapter 3. It includes an anti-aliasing filter F_σ applied to u_m before it is sampled

$$\dot{\bar{u}}_m = -\sigma \bar{u}_m + \sigma u_m, \ \ \bar{u}_m(t_0) = \bar{u}_{m_0}, \tag{5.2}$$

a reference model P_m which embodies the desired closed loop behaviour with the state-space representation:

$$\begin{array}{rcl} \dot{\bar{x}}_m & = & A_m \bar{x}_m + B_m \bar{u}_m, \quad \bar{x}_m(t_0) = x_{m_0}, \\ \bar{y}_m & = & C_m \bar{x}_m, \end{array} \tag{5.3}$$

and last, but most important, a sampled-data periodic controller, listed below:

$$\begin{array}{ll} z[k+1] = F(k)z[k] + G(k)y(kh) + H(k)\bar{x}_m(kh) + J(k)\bar{u}_m(kh), & z[k_0] = z_0 \in \mathbf{R}^l, \\ u(kh+\tau) = L(k)z[k] + M(k)y(kh), & \tau \in [0, h) \end{array} \tag{5.4}$$

with z representing the state. The input-output map is labelled K and the gains F, G, H, J, L and M are periodic of period $p \in \mathbf{N}$; the period of the controller is $T := ph$, and we associate this system with the 8-tuple (F, G, H, J, L, M, h, p). As noted before, but worth emphasizing, observe that (5.4) can be implemented with a sampler, a zero-order-hold, and an l^{th} order periodically time-varying discrete-time system of period p .

Following Chapter 3, we can characterize our time-varying plant by the vector $\bar{\theta}$ which lies in the set

$$\bar{\mathcal{P}}(n, m, \bar{\Gamma}, \bar{\mu}_1, \bar{T}_0, \underline{g}, \gamma_0, \lambda_0).$$

Each admissible $\bar{\theta}$ corresponds to a plant model of the form

$$\begin{array}{rcl} \begin{bmatrix} \dot{w}(t) \\ \dot{v}(t) \end{bmatrix} & = & \underbrace{\begin{bmatrix} A_1(t) & b_1 c_2 \\ b_2 c_1(t) & A_2(t) \end{bmatrix}}_{=:A(t)} \underbrace{\begin{bmatrix} w(t) \\ v(t) \end{bmatrix}}_{=:x(t)} + \underbrace{g(t) \begin{bmatrix} 0 \\ b_2 \end{bmatrix}}_{=:B} u(t), \\ y(t) & = & \underbrace{\begin{bmatrix} 0 & c_2 \end{bmatrix}}_{=:C} \begin{bmatrix} w(t) \\ v(t) \end{bmatrix}; \end{array} \tag{5.5}$$

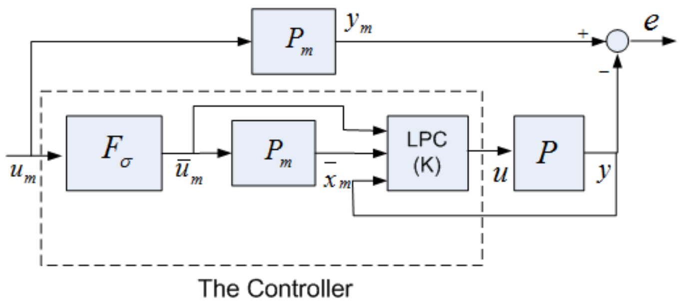

Figure 5.1: The feedback diagram. (Here $\bar{P}_m(s) = (sI - A_m)^{-1}B_m$.)

here the system parameters $A_1,\ A_2,\ b_1,\ b_2,\ c_1,\ c_2$ have rich structure (for details see Chapter 3) and are implicitly functions of $\bar{\theta}$.

It is convenient at this point to combine the plant, the reference model and the pre-filter together to form a **generalized plant**:

$$\begin{bmatrix} \dot{w} \\ \dot{v} \\ \dot{\bar{x}}_m \\ \dot{\bar{u}}_m \end{bmatrix} = \underbrace{\begin{bmatrix} A_1(t) & b_1c_2 & 0 & 0 \\ b_2c_1(t) & A_2(t) & 0 & 0 \\ 0 & 0 & A_m & B_m \\ 0 & 0 & 0 & -\sigma \end{bmatrix}}_{=:\bar{A}(t)} \underbrace{\begin{bmatrix} w \\ v \\ x_m(t) \\ \bar{u}_m(t) \end{bmatrix}}_{=:\bar{x}} + g(t) \underbrace{\begin{bmatrix} 0 \\ b_2 \\ 0 \\ 0 \end{bmatrix}}_{=:\bar{B}} u(t) +$$

$$\underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ \sigma \end{bmatrix}}_{=:\bar{E}} u_m, \quad \bar{x}(t_0) = \bar{x}_0, \tag{5.6}$$

$$y = \underbrace{\begin{bmatrix} 0 & c_2 & 0 & 0 \end{bmatrix}}_{=:\bar{\bar{C}}} \begin{bmatrix} w \\ v \\ \bar{x}_m \\ \bar{u}_m \end{bmatrix} \tag{5.7}$$

$$\bar{e} = \underbrace{\begin{bmatrix} 0 & -c_2 & C_m & 0 \end{bmatrix}}_{=:\bar{C}} \begin{bmatrix} w \\ v \\ \bar{x}_m \\ \bar{u}_m \end{bmatrix}; \tag{5.8}$$

here $\bar{e}$ represents the difference between y and the reference model P_m driven by the filtered reference model input $\bar{u}_m$, and can be proven to be close to the actual tracking error e (see equation (3.4) of Chapter 3). Hence, for each model in our class of models $\bar{\mathcal{P}}$, we have a well-defined generalized plant. *Our control objective will be to make the closed loop map from $u_m \longmapsto \bar{e}$ small while guaranteeing closed loop stability.*

To find our 'ideal control law' we need to define several gains. Arguing as in Chapter 3, we can choose $k_1 \in \mathbf{R}^{1\times n_m}$, $k_2 \in \mathbf{R}$ and $\bar{f}_2 \in \mathbf{R}^{1\times n_m}$ (independent of

$\bar{\theta} \in \bar{\mathcal{P}}$) so that the **ideal control law** given by

$$u(t) = \frac{1}{g(t)} \underbrace{\begin{bmatrix} -c_1(t) & \bar{f}_2 - f_2(t) & k_1 & k_2 \end{bmatrix}}_{=:K(t)} \bar{x}(t) \tag{5.9}$$

both stabilizes the system and yields a transfer function from $\bar{u}_m \to \bar{e}$ to be exactly zero, the time-varying components of the closed loop system have been removed). Of course, to implement this control law requires knowledge of both the generalized plant state $\bar{x}$ and the plant parameters $g(t)$, $f_2(t)$ and $c_1(t)$, none of which are accessible. Hence, we require a sequence of approximations.

In the first approximation, we observe that $g(t)$ takes values in a compact set $\mathcal{G}$ not containing zero; indeed, there exists a constant $\bar{g} > 0$ so that $\mathcal{G} \subset [-\bar{g}, -\underline{g}] \cup [\underline{g}, \bar{g}]$. From Proposition 3.2 of Chapter 3 we can approximate $1/g$ arbitrarily well over $\mathcal{G}$ via a polynomial, i.e. for every $\varepsilon > 0$ there exist $q \in \mathbf{N}$ and $c_i \in \mathbf{R}$ such that the polynomial $\hat{f}_\varepsilon(g) = \sum_{i=0}^{q} c_i g^i$ satisfies

$$|1 - g\hat{f}_\varepsilon(g)| < \varepsilon, \quad g \in \mathcal{G}. \tag{5.10}$$

Hence, the **approximate ideal control law** is

$$u(t) = \hat{f}_\varepsilon(g(t))K(t)\bar{x}(t). \tag{5.11}$$

As shown in [20] and Proposition 3.2 of Chapter 3, if ε is chosen small enough, then the **approximate ideal control law** works almost as well as the **ideal control law**.

With $\varepsilon > 0$ chosen to be small enough that the **approximate ideal control law** is near optimal in a sense made precise in [20] and Chapter 3, clearly the control law

$$\hat{f}_\varepsilon(g(kT))K(kT)\bar{x}(kT), \quad t \in [kT, (k+1)T),$$

should work almost as well as the approximate ideal control law. Since this cannot be implemented either, we now turn to the control law implemented in [20] and discussed in Chapter 3: with

$$\hat{\phi}_i(kT) := \text{estimate of } g(kT)^i K(kT)\bar{x}(kT), \quad i = 0, 1, \cdots, q,$$

and $\rho > 0$, the control law is

$$u(t) = \begin{cases} 0 & t \in [kT, kT + mh), \\ \rho\hat{\phi}_{i-1}(kT) & t \in [kT + (2i-1)mh, kT + 2imh), \\ & i = 1, \cdots, q \\ -\rho\hat{\phi}_{i-1}(kT) & t \in [kT + 2imh, kT + (2i+1)mh), \\ & i = 1, \cdots, q \\ \frac{p}{p-(2q+1)m} \sum_{i=0}^{q} c_i \hat{\phi}_i(kT) & t \in [kT + (2q+1)mh, (k+1)T). \end{cases} \tag{5.12}$$

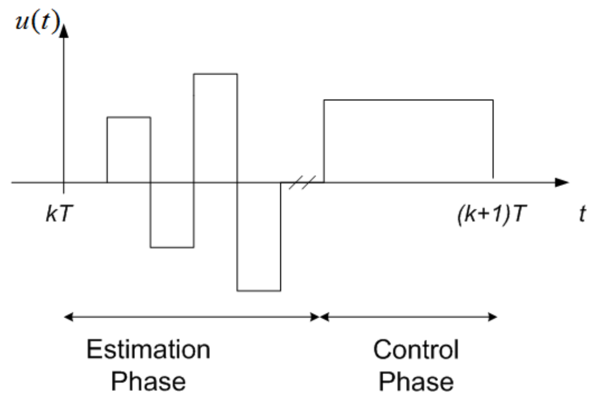

Figure 5.2: A typical control signal over a period for the controller of [20].

Hence, the controller operates by first estimating $g(kT)^i K(kT)\bar{x}(kT)$ for $i = 0, 1, \cdots, q$, combining them to form $\hat{f}_\varepsilon(g(kT))K(kT)\bar{x}(kT)$, and then applying this suitably scaled quantity. It turns out that the forming of the estimates $\hat{\phi}_i(kT)$ can be carried out linearly, resulting in a linear periodic controller, and the approach can be proven to be as close to the optimal control law as desired by choosing ε and T sufficiently small. This control methodology is illustrated in Figure 5.2.

As mentioned in the previous section, and clearly visible in Figure 5.2, the estimation and control are done in series, and results in a control signal which switches rapidly between different values. It is intuitively reasonable that

$$\hat{f}_\varepsilon(g(kT))K(kT)\bar{x}(kT)$$

should be quite close to

$$\hat{f}_\varepsilon(g(kT-T))K(kT-T)\bar{x}(kT-T),$$

so it would clearly make sense to apply

$$\hat{f}_\varepsilon(g(kT-T))K(kT-T)\bar{x}(kT-T)$$

as the nominal control law during $[kT, (k+1)T)$ and try to estimate

$$\hat{f}_\varepsilon(g(kT))K(kT)\bar{x}(kT) - \hat{f}_\varepsilon(g(kT-T))K(kT-T)\bar{x}(kT-T)$$

by suitable experiments, hopefully without disturbing the plant very much, from which we can obtain an estimate of

$$\hat{f}_\varepsilon(g(kT))K(kT)\bar{x}(kT);$$

this idea is illustrated pictorially in Figure 5.3. While the approach is intuitively reasonable, it requires a careful design and completely new proofs to show that it works, which is the goal of this chapter. It turns out that as long as $\varepsilon < 1$, this approach works: this is completely unexpected, but the mechanism at work will be explained in the next section.

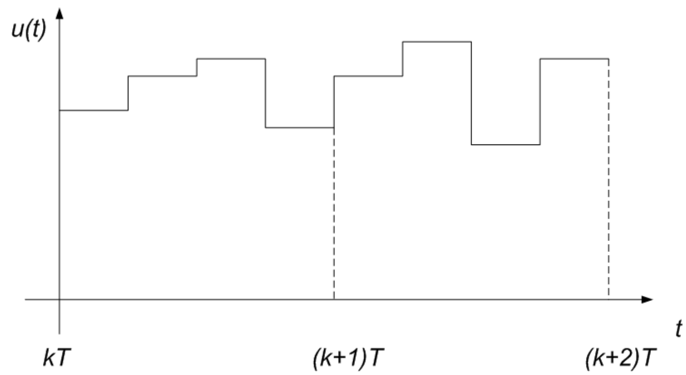

Figure 5.3: A typical control signal over a period for the new controller.

5.3 Controller Construction

In this section we present the redesigned controller. We start with the high level motivation of the approach. We then move on to the controller design for the first-order case, followed by a reasonably straight-forward extension to the general case. Next we present the formal controller definition, followed by a proof that it satisfies our objective.

5.3.1 The High Level Analysis

In the LPC of [20], the term $\hat{\phi}_0(kT)$ provides an estimate of $K(kT)\bar{x}^\varepsilon(kT)$ and $\hat{\phi}_1(kT)$ provides an estimate of $g(kT)^i K(kT)\bar{x}^\varepsilon(kT)$; we use these estimations to form a good estimate of the **approximated ideal control law** $\hat{f}_\varepsilon(g(t))K(t)\bar{x}(t)$: constants c_i, $i = 0, 1, \cdots, q$, are chosen so that

$$\hat{f}_\varepsilon(g(kT))K(kT)\bar{x}^\varepsilon(kT) \approx \sum_{i=0}^{q} c_i \hat{\phi}_i(kT).$$

Here the goal is to estimate the **ideal control law** rather than **approximate the ideal control law**. As in [20], with $\varepsilon > 0$ we first choose a polynomial $\hat{f}_\varepsilon(g) = \sum_{i=0}^{q} c_i g^i$ so that

$$|1 - g\hat{f}_\varepsilon(g)| < \varepsilon, \quad g \in \mathcal{G},$$

although now we will not require ε to be small. Here the variable $\hat{u}^o(kT)$ is used to represent an estimate of the ideal control signal at time kT. We apply this control signal on $[kT, (k+1)T)$, and apply an additional probing signal to create a new estimate $\hat{u}^o[(k+1)T]$ to be used during the following period. It turns out that if we apply the same formula for $\hat{\phi}_i$ as in [20] then the presence of $\hat{u}^o(kT)$ will affect the estimates $\hat{\phi}_i(kT)$: if all plant parameters are continuous on $[kT, (k+1)T)$ then

(i)

$$\hat{\phi}_0(kT) \approx [K(kT)\bar{x}(kT) - g(kT)\hat{u}^o(kT)]$$

rather than

$$\hat{\phi}_0(kT) \approx K(kT)\bar{x}^\varepsilon(kT),$$

and

(ii)

$$\begin{aligned}\hat{\phi}_i(kT) &\approx g(kT)^i\hat{\phi}_0(kT)\\ &\approx g(kT)^i\left[K(kT)\bar{x}(kT) - g(kT)\hat{u}^o(kT)\right]\end{aligned}$$

rather than

$$\begin{aligned}\hat{\phi}_i(kT) &\approx g(kT)^i\hat{\phi}_0(kT)\\ &\approx g(kT)^i K(kT)\bar{x}^\varepsilon(kT).\end{aligned}$$

Notice that now $\frac{1}{g(kT)}\hat{\phi}^o(kT)$ gives a good approximation to how much $\hat{u}_0(kT)$ differs from $\frac{1}{g(kT)}K(kT)\bar{x}(kT)$, so it is natural to define

$$\begin{aligned}\hat{u}^o[(k+1)T] &= \hat{u}^o(kT) + \sum_{i=0}^{q} c_i\hat{\phi}_i(kT)\\ &\approx \hat{u}^o(kT) + \sum_{i=0}^{q} c_i g(kT)^i\hat{\phi}_0(kT)\\ &\approx \hat{u}^o(kT) + \hat{f}_\varepsilon(g(kT))[K(kT)\bar{x}(kT) - g(kT)\hat{u}^o(kT)]\\ &= [1 - g(kT)\hat{f}_\varepsilon(g(kT))]\hat{u}^o(kT) + \hat{f}_\varepsilon(g(kT))K(kT)\bar{x}(kT).\end{aligned} \tag{5.13}$$

Now

$$|1 - g\hat{f}_\varepsilon(g)| \leq \varepsilon, \quad g \in \mathcal{G};$$

we impose the reasonable condition that $\varepsilon < 1$. If g, K, and $\bar{x}$ are slowly moving, it is clear that if T is small, then after some initial transients

$$\hat{u}^o[(k+1)T] \approx \frac{1}{g(kT)}K(kT)\bar{x}(kT),$$

which is quite close to the ideal value

$$\frac{1}{g((k+1)T)}K((k+1)T)\ \bar{x}((k+1)T).$$

Hence, this mechanism should do a good job of estimating the ideal control law

$$\frac{1}{g(t)}K(t)\bar{x}(t)$$

rather than a good job of estimating the approximate ideal control law

$$\hat{f}_\varepsilon(g(t))K(t)\bar{x}^\varepsilon(t).$$

While a small ε was required in [20] to ensure good performance, here all we need is $\varepsilon \in [0, 1)$; this is totally unexpected and an extremely desirable feature.

Remark 5.2: *Because of Assumption 1, Chpater 3, page 21, on $\bar{\mathcal{P}}$, the set $\mathcal{G}$ is compact and does not include zero.*

(i) *If every element of $\mathcal{G}$ has the same sign (all positive or all negative), then there exist $\underline{g}$, $\bar{g} \in \mathbf{R}$ satisfying*

$$\begin{array}{rcl} 0 & < & \underline{g} \; < \; \bar{g} \quad \text{or} \quad -\bar{g} \; < \; -\underline{g} \; < \; 0, \\ \mathcal{G} & \subset & [\underline{g}, \bar{g}] \quad \text{or} \quad \mathcal{G} \; \subset \; [-\bar{g}, -\underline{g}]. \end{array}$$

If we set

$$\hat{f}_\varepsilon(g) = \frac{1}{2\bar{g}}$$

if every element of $\mathcal{G}$ is positive and

$$\hat{f}_\varepsilon(g) = \frac{-1}{2\bar{g}}$$

when every element of $\mathcal{G}$ is negative, then it is easy to confirm that

$$|1 - g\hat{f}_\varepsilon(g)| \;\leq\; \underbrace{1 - \frac{\underline{g}}{2\bar{g}}}_{=:\varepsilon} \;<\; 1, \quad g \in \mathcal{G}.$$

This means that we can use a zero-th order polynomial if we so desire, which means that probing is not required at all!

(ii) *Now suppose $\mathcal{G}$ contains both positive and negative values. Then there exist $\underline{g}$, $\bar{g} \in \mathbf{R}$ satisfying*

$$\begin{array}{rcl} 0 & < & \underline{g} \; < \; \bar{g}, \\ \mathcal{G} & \subset & [-\bar{g}, -\underline{g}] \cup [\underline{g}, \bar{g}]. \end{array}$$

If we set

$$\hat{f}_\varepsilon(g) = \left(\frac{1}{2\bar{g}^2}\right) g,$$

then it is easy to confirm that

$$|1 - g\hat{f}_\varepsilon(g)| \;\leq\; \underbrace{1 - \frac{1}{2}\left(\frac{\underline{g}}{\bar{g}}\right)^2}_{=:\varepsilon} \;<\; 1, \quad g \in \mathcal{G}.$$

This means that we can use a first-order polynomial in this case. Hence, in this new design approach we only need to probe once!

5.3.2 The First-Order Case (constant parameters)

Here we consider the simplest possible case, that of first-order with constant parameters borrowed from Chapter 3:

$$\begin{aligned} \dot{y} &= ay + gu, \\ \dot{\bar{y}}_m &= a_m \bar{y}_m + b_m \bar{u}_m. \end{aligned}$$

The error satisfies

$$(\dot{\bar{y}}_m - \dot{y}) = a_m(\bar{y}_m - y) + [b_m \bar{u}_m - gu + (a_m - a)y],$$

and ideally we would like to set

$$u(t) = \frac{1}{g} \underbrace{\begin{bmatrix} a_m - a & 0 & b_m \end{bmatrix}}_{=:K} \underbrace{\begin{bmatrix} y \\ \bar{x}_m \\ \bar{u}_m \end{bmatrix}}_{=:\bar{x}},$$

or its sampled-data approximation:

$$u(t) = \frac{1}{g} K \bar{x}(kT), \quad t \in [kT, (k+1)T).$$

We proceed as follows. We start with an estimate of the ideal control signal, which we label $\hat{u}^o(0)$; the goal is to apply this estimate while at the same time probing the system to obtain a good estimate at time $t = T$, namely $\hat{u}^o(T)$. Motivated by the last subsection, we would like to form an approximation of

$$\sum_{i=0}^{q} c_i \underbrace{g^i [b_m \bar{u}_m(0) + (a_m - a)y(0) - g\hat{u}^o(0)]}_{:=\phi_i(0)}.$$

Suppose we initially set

$$\bar{u}(t) = \hat{u}^o(0), \quad t \in [0, h).$$

Since a and g are constrained to compact sets, it follows that

$$\begin{aligned} y(h) &= e^{ah} y(0) + g \int_0^h e^{a\tau} \hat{u}^o(0) d\tau \\ &= [1 + ah + \mathcal{O}(h^2)] y(0) + [gh + \mathcal{O}(h^2)] \hat{u}^o(0). \end{aligned}$$

Hence

$$\frac{1}{h}[y(h) - y(0)] = ay(0) + g\hat{u}^o(0) + \mathcal{O}(h) y(0) + \mathcal{O}(h) \hat{u}^o(0).$$

Thus, at this point we have a good estimate of $ay(0) + g\hat{u}^o(0)$, with the quality of the estimate improving as $h \to 0$. Hence, we can form an accurate estimate of

$$\phi_0(0) := b_m \bar{u}_m(0) + (a_m - a)y(0) - g\hat{u}^o(0),$$

namely

$$\begin{aligned}\hat{\phi}_0(0) &:= b_m\bar{u}_m(0) + a_m y(0) - \frac{1}{h}[y(h) - y(0)] \\ &= b_m\bar{u}_m(0) + (a_m - a)y(0) - g\hat{u}^o(0) + \mathcal{O}(h)y(0) + \mathcal{O}(h)\hat{u}^o(0) \\ &= \phi_0(0) + \mathcal{O}(h)y(0) + \mathcal{O}(h)\hat{u}^o(0).\end{aligned}$$

To form estimates of $\phi_i(0) = g^i\phi_0(0)$, we will carry out some experiments. With $\rho > 0$ a scaling factor (we make this factor small so that it does not disturb the system very much), set

$$u(t) = \rho\hat{\phi}_0(0) + \hat{u}^o(T), \quad t \in [h, 2h).$$

Then

$$\begin{aligned}y(2h) &= e^{2ah}y(0) + g\int_0^h e^{a(2h-\tau)}\hat{u}^o(0)d\tau + g\int_h^{2h} e^{a(2h-\tau)}[\rho\hat{\phi}_0(0) + \hat{u}^o(0)]d\tau \\ &= [1 + 2ah + \mathcal{O}(h^2)]y(0) + [2gh + \mathcal{O}(h^2)]\hat{u}^o(0) + [gh + \mathcal{O}(h^2)]\rho\hat{\phi}_0(0) \\ &= [1 + 2ah]y(0) + 2gh\hat{u}^o(0) + \rho gh\hat{\phi}_0(0) + \mathcal{O}(h^2)y(0) + \mathcal{O}(h^2)\hat{u}^o(0) + \\ &\quad \mathcal{O}(h^2)\bar{u}_m(0).\end{aligned}$$

Hence, define

$$\begin{aligned}\hat{\phi}_1(T) &:= \frac{1}{\rho h}[y(2h) - 2y(h) + y(0)] \\ &= g\phi_0(0) + \mathcal{O}(h)y(0) + \mathcal{O}(h)\hat{u}^o(0) + \mathcal{O}(h)\bar{u}_m(0) \\ &= \phi_1(0) + \mathcal{O}(h)y(0) + \mathcal{O}(h)\hat{u}^o(0) + \mathcal{O}(h)\bar{u}_m(0).\end{aligned}$$

Of course, in completing this experiment, we have excited the state. This can be largely undone by applying

$$u(t) = -\rho\hat{\phi}_0(0) + \hat{u}^o(0), \quad t \in [2h, 3h);$$

however, to save time we may as well do two steps at once by (almost) cancelling out the effect of this experiment at the same time as we do a new experiment: set

$$u(t) = \rho\hat{\phi}_1(0) - \rho\hat{\phi}_0(0) + \hat{u}^o(0), \quad t \in [2h, 3h).$$

This results in

$$\begin{aligned}y(3h) &= e^{3ah}y(0) + g\int_0^h e^{a(3h-\tau)}\hat{u}^o(0)d\tau + g\int_h^{2h} e^{a(3h-\tau)}[\rho\hat{\phi}_0(0) + \hat{u}^o(0)]d\tau + \\ &\quad g\int_{2h}^{3h} e^{a(3h-\tau)}[\rho\hat{\phi}_1(0) - \rho\hat{\phi}_0(0) + \hat{u}^o(0)]d\tau \\ &= [1 + 3ah + \mathcal{O}(h^2)]y(0) + [3gh + \mathcal{O}(h^2)]\hat{u}^o(0) + [gh + \mathcal{O}(h^2)]\rho\hat{\phi}_1(0) + \\ &\quad \mathcal{O}(h^2)\hat{\phi}_0(0) \\ &= [1 + 3ah]y(0) + 3gh\hat{u}^o(0) + \rho gh\hat{\phi}_1(0) + \mathcal{O}(h^2)y(0) + \mathcal{O}(h^2)\hat{u}^o(0) + \\ &\quad \mathcal{O}(h^2)\bar{u}_m(0),\end{aligned}$$

which means that

$$\begin{aligned}\frac{1}{\rho h}[y(3h) - y(2h) - y(h) + y(0)] &= g\phi_1(0) - g\hat{\phi}_0(0) + \\ &\qquad \mathcal{O}(h)y(0) + \mathcal{O}(h)\hat{u}^o(0) + \mathcal{O}(h)\bar{u}_m(0).\end{aligned}$$

Using the fact that $\hat{\phi}_1(0) \approx g\hat{\phi}_0(0)$, a good estimate of $\phi_2(0)$ is

$$\begin{aligned}\hat{\phi}_2(0) &= \frac{1}{\rho h}[y(3h) - y(2h) - y(h) + y(0)] + \hat{\phi}_1(0) \\ &= g\phi_1(0) - g\hat{\phi}_0(0) + \hat{\phi}_1(0) + \mathcal{O}(h)y(0) + \mathcal{O}(h)\hat{u}^o(0) + \mathcal{O}(h)\bar{u}_m(0) \\ &= \phi_2(0) + \mathcal{O}(h)y(0) + \mathcal{O}(h)\hat{u}^o(0) + \mathcal{O}(h)\bar{u}_m(0).\end{aligned}$$

This can be repeated $q-2$ more times. At $t = (q+1)h$, we have estimates of $\phi_0(0), \cdots, \phi_q(0)$; the last step is to approximately cancel out the effect of the last probe:

$$u(t) = -\rho\hat{\phi}_{q-1}(0) + \hat{u}^o(0), \quad t \in [(q+1)h, (q+2)h).$$

So we set $T := (q+2)h$, update the estimate of the optimal control signal:

$$\hat{u}^o(T) := \hat{u}^o(0) + \sum_{i=0}^{q} c_i\hat{\phi}_0(0),$$

and repeat the procedure.

Remark 5.3: *Now we carry out estimation and control simultaneously. This is both intuitively appealing as well as more efficient. We shall soon see via a simulation study that this also has better performance for a sampling period h.*

5.3.3 The General Case

Now we will show how this approach works for the general n^{th} order case with time-varying parameters. Since the key ideas are the same and only the details are different, we only summarize the approach. We do so on the interval $[kT, (k+1)T)$.

We start with an estimate of $u^o(kT)$, which we denote by $\hat{u}^o(kT)$. If we set

$$u(t) = \hat{u}^o(kT), \quad t \in [kT, kT + mh),$$

and consider the definition of $K(t)$ in (5.9), it follows from KEL-3 that we should define

$$\begin{aligned}\hat{\phi}_0(kT) &:= \begin{bmatrix} \bar{f}_2 & -1 \end{bmatrix} H_m(h)^{-1} S_m^{-1} \mathcal{Y}(kT) + \begin{bmatrix} k_1 & k_2 \end{bmatrix} \begin{bmatrix} \bar{x}_m(kT) \\ \bar{u}_m(kT) \end{bmatrix} \\ &\approx K(kT)\bar{x}(kT) - g(kT)\hat{u}^o(kT).\end{aligned}$$

By choosing a scaling factor $\rho > 0$, we set

$$\begin{aligned} u(t) &= \rho \times \text{ estimate of } [K(kT)\bar{x}(kT) - g(kT)\hat{u}^o(kT)] + \hat{u}^o(kT) \\ &= \rho\hat{\phi}_0(kT) + \hat{u}^o(kT), \quad t \in [kT + mh, kT + 2mh); \end{aligned}$$

it follows that we should define

$$\begin{aligned} \hat{\phi}_1(kT) &= \frac{1}{\rho} \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} H_m(h)^{-1} S_m^{-1} [\mathcal{Y}(kT + mh) - \mathcal{Y}(kT)] \\ &\approx g(kT)[K(kT)\bar{x}(kT) - g(kT)\hat{u}^o(kT)] \\ &\approx g(kT)\hat{\phi}_0(kT). \end{aligned}$$

In the next stage in order to achieve a close estimate of $\phi_2(kT) = g(kT)\phi_1(kT)$, we apply $\rho\hat{\phi}_1(kT)$ as the probing signal and at the same time cancel out the effect of $\hat{\phi}_0(kT)$ by setting

$$u(t) = \rho\hat{\phi}_1(kT) - \rho\hat{\phi}_0(kT) + \hat{u}^o(kT), \quad t \in [kT + mh, kT + 2mh).$$

This results in

$$\begin{aligned} &\frac{1}{\rho} \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} H_m(h)^{-1} S_m^{-1} [\mathcal{Y}(kT + 2mh) - \mathcal{Y}(kT)] \\ &\quad \approx g(kT)[\hat{\phi}_1(kT) - \hat{\phi}_0(kT)] \\ &\quad \approx g(kT)\hat{\phi}_1(kT) - \hat{\phi}_1(kT), \end{aligned}$$

which means that we should define

$$\begin{aligned} \hat{\phi}_2(kT) &:= \frac{1}{\rho} \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} H_m(h)^{-1} S_m^{-1} [\mathcal{Y}(kT + 2mh) - \mathcal{Y}(kT)] + \hat{\phi}_1(kT) \\ &\approx g(kT)\hat{\phi}_1(kT). \end{aligned}$$

If we repeat this procedure then we can obtain a good estimate of $\phi_i(kT), \quad i = 3, 4, \cdots, q$, which we can use to update our estimate of the optimal control signal:

$$\begin{aligned} \hat{u}^o[(k+1)T] &= \hat{u}^o(kT) + \sum_{i=0}^{q} c_i \hat{\phi}_i(kT) \\ &\approx \hat{u}^o(kT) + \sum_{i=0}^{q} c_i g(kT)^i [K(kT)\bar{x}(kT) - g(kT)\hat{u}^o(kT)] \\ &= [1 - \hat{f}_\varepsilon(g(kT))g(kT)]\hat{u}^o(kT) + \hat{f}_\varepsilon(g(kT))K(kT)\bar{x}(kT); \end{aligned}$$

here, we set

$$T := \begin{cases} mh & \text{if } q = 0, \\ (q+2)mh & \text{else.} \end{cases}$$

If $\varepsilon < 1$, we would expect $\hat{u}^o(kT)$ to converge to a good estimate of $K(kT)\bar{x}(kT)$, assuming that T is small enough. Of course, the above high level analysis assumes that there are no discontinuities in the plant parameters; if the sampling period is small enough, then any discontinuity will be infrequent and we will be able to prove that the controller still works.

THE PROPOSED CONTROLLER ($t_0 = 0$)

For $k \in \mathbf{Z}^+$:

$$\hat{\phi}_i(kT) = \begin{cases} \begin{bmatrix} \bar{f}_2 & -1 \end{bmatrix} H_m(h)^{-1} S_m^{-1} \mathcal{Y}_m(kT) + \begin{bmatrix} k_1 & k_2 \end{bmatrix} \begin{bmatrix} \bar{x}_m(kT) \\ \bar{u}_m(kT) \end{bmatrix}, & \text{if } i = 0, \\ \frac{1}{\rho} \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} H_m(h)^{-1} S_m^{-1} \begin{bmatrix} \mathcal{Y}_m(kT + mh) - \mathcal{Y}_m(kT) \end{bmatrix}, & \text{if } i = 1, \\ \frac{1}{\rho} \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} H_m(h)^{-1} S_m^{-1} \begin{bmatrix} \mathcal{Y}_m(kT + imh) - \mathcal{Y}_m(kT) \end{bmatrix} + \hat{\phi}_{i-1}(kT), & \text{if } i = 2, \cdots, q, \end{cases} \tag{5.14}$$

If $q = 0$ then

$$u(t) = \hat{u}^o(kT), \quad t \in [kT, kT + mh) = [kT, (k+1)T),$$

and if $q > 0$

$$u(t) = \begin{cases} \hat{u}^o(kT) & t \in [kT, kT + mh), \\ \rho\hat{\phi}_0(kT) + \hat{u}^o(kT) & t \in [kT + mh, kT + 2mh), \ i = 1 \\ \rho\hat{\phi}_{i-1}(kT) - \rho\hat{\phi}_{i-2}(kT) + \hat{u}^o(kT) & t \in [kT + imh, kT + (i+1)mh), \\ & i = 2, \cdots, q, \\ -\rho\hat{\phi}_{q-1}(kT) + \hat{u}^o(kT) & t \in [kT + (q+1)mh, \\ & kT + (q+2)mh). \end{cases} \tag{5.15}$$

$$\hat{u}^o[(k+1)T] = \hat{u}^o(kT) + \sum_{i=0}^{q} c_i \hat{\phi}_i(kT), \quad t \in [kT + (p-1)h, (k+1)T). \tag{5.16}$$

At this stage, the goal is to prove that the state of the system can be made as close as desired to the approximate ideal control law. We first examine a single period.

Lemma 5.1 (Key Control Lemma (KCL-5)): *There exist constants $\gamma > 0$ and $\overline{T} > 0$ so that for all $k \in \mathbf{Z}^+, T \in (0, \overline{T})$, and $\bar{\theta} \in \bar{\mathcal{P}}$, the solution of (5.6)-(5.8) with u given by (5.14)-(5.16) has the following properties:*

(i) In all cases

$$\left\| \bar{x}(t) - \Phi_{cl}(t, kT)\bar{x}(kT) - \int_{kT}^{t} \Phi_{cl}(t,\tau)\bar{E}u_m(\tau)d\tau \right\| \\ \leq \gamma T(\|\bar{x}(kT)\| + |\hat{u}^o(kT)| + \|u_m\|_\infty),$$

$$\|u(t)\| \leq \gamma(\|\bar{x}(kT)\| + |\hat{u}^o(kT)| + \|u_m\|_\infty), \quad t \in [kT, (k+1)T),$$

$$\|\hat{u}^o[(k+1)T]\| \leq \gamma(\|\bar{x}(kT)\| + |\hat{u}^o(kT)| + \|u_m\|_\infty),$$

(ii) If $\bar{\theta}(t)$ is absolutely continuous on $[kT, (k+1)T]$, then

$$|\hat{u}^o[(k+1)T] - [1 - \hat{f}_\varepsilon(g(kT))g(kT)]\hat{u}^o(kT) - \hat{f}_\varepsilon(g(kT))K(kT)\bar{x}(kT)| \\ \leq \gamma T(\|\bar{x}(kT)\| + |\hat{u}^o(kT)| + \|u_m\|_\infty),$$

$$\|\bar{x}((k+1)T) - \Phi_{cl}((k+1)T, kT)\bar{x}(kT) - \int_{kT}^{(k+1)T} \Phi_{cl}((k+1)T,\tau)\bar{E}u_m(\tau)d\tau - \\ \int_{kT}^{(k+1)T} \Phi_{cl}((k+1)T,\tau)\bar{B}[g(kT)u(kT) - K(kT)\bar{x}(kT)]d\tau|| \\ \leq \gamma T^2(\|\bar{x}(kT)\| + |\hat{u}^o(kT)| + \|u_m\|_\infty).$$

Proof: See the Appendix B.

Now by the following proposition, it is proven that the proposed controller (5.14)-(5.16) acts like the ideal control law (5.9) for all time. To be precise, we let $\hat{u}^o$ denote the estimate of the ideal control signal at time kT, $\hat{\bar{x}}$ denote the state, $\hat{\bar{e}}$ denote the error, and $\hat{y}$ denote the plant output when (5.14)-(5.16) is applied. Recall that we use $\bar{x}$, $\bar{e}$ and y to denote the corresponding signals when ideal control law (5.9) is applied.

Proposition 5.1: *There exist constants* $\bar{T} > 0$, $\lambda < 0$ *and* $\gamma > 0$ *so that for every* $\bar{\theta} \in \bar{\mathcal{P}}$, $u_m \in PC_\infty$, *and* $T \in (0, \bar{T})$, *we have*

$$\begin{aligned} \|\bar{x}(t) - \hat{\bar{x}}(t)\| &\leq \gamma T^{\frac{1}{2}} e^{\lambda t}(\|\bar{x}_0\| + |\hat{u}_0^o|) + \gamma T^{\frac{1}{2}} \|u_m\|_\infty, \;\; t \geq 0, \\ |\hat{u}^o(kT)| &\leq \gamma e^{\lambda kT}(\|\bar{x}_0\| + |\hat{u}_0^o|) + \gamma \|u_m\|_\infty, \;\; k \geq 0. \end{aligned}$$

Proof: See the Appendix B.

In the next lemma, the controller structure and stability are discussed.

Lemma 5.2: *The control law (5.14)-(5.16) has a representation of the form (5.4) given by* (F, G, H, J, L, M, h, p) *with* F *periodic with period* p *and* F *and* z *partitioned as*

$$F(jp+i) \;=\; \begin{bmatrix} F_{11}(jp+i) & F_{12}(jp+i) \\ F_{21}(jp+i) & F_{22}(jp+i) \end{bmatrix}, \qquad z[jp+i] = \begin{bmatrix} z_1[jp+i] \\ z_2[jp+i] \end{bmatrix},$$
$$j \in \mathbf{Z}^+, \;\; i \in \{0, \cdots, p-1\},$$

with the following properties:

(i) $F_{22}(jp+i) = 1, \qquad j \in \mathbf{Z}^+, \;\; i \in \{0, \cdots, p-1\}.$

(ii) *The second state is associated with* $\hat{u}^o$ *in the following sense:*

$$z_2[jp+i] = \begin{cases} \hat{u}^o(jT), & i \in \{0, \cdots, p-2\}, \\ \hat{u}^o[(j+1)T], & i = p-1, \end{cases} \qquad j \in \mathbf{Z}^+.$$

(iii) *The first sub-system is deadbeat in the following sense:*

$$F_{11}(p-1)F_{11}(p-2)\cdots F_{11}(0) = 0.$$

Proof: See the Appendix B.

Now we need to prove that the new controller provides both stability and close tracking for all admissible models.

Theorem 5.1: *For every $\delta > 0$ and $\lambda \in (\max\{\lambda_0, \lambda_m\}, 0)$ there exists a controller of the form (5.2), (5.3), and (5.4) with the following properties:*

(i) *the controller exponentially stabilizes $\bar{\mathcal{P}}$, and*

(ii) *for every $\bar{\theta} \in \bar{\mathcal{P}}$, $\bar{x}_0 \in R^{n+n_m+1}$, and $u_m \in PC_\infty$, when $t_0 = k_0 = 0$ the closed-loop system satisfies*

$$|\hat{y}(t) - y_m(t) - \bar{C}\Phi_{cl}(t,0)\bar{x}(0)| \;\leq\; \delta e^{\lambda t}(\|\bar{x}_0\| + |\hat{u}_0^o|) + \delta\|u_m\|_\infty, \;\; t \geq 0.$$

Proof:

First suppose that $t_0 = k_0 = 0$. We start with the bound on the transients. To proceed, following the derivation of equation (3.4) of Chapter 3, Section 3.2, first we choose $\sigma > ||A_m||$ sufficiently large that

$$\frac{||B_m|| \times ||C_m||}{\sigma - ||A_m||}[\gamma_m + 1] \;+\; \frac{||B_m|| \times ||C_m||}{\sigma - ||A_m||}[\gamma_m \frac{||A_m||}{|\lambda_m|} + 1] \;\leq\; \delta/2,$$

it follows from (3.4) that

$$|\bar{y}_m(t) - y_m(t)| \;\leq\; (\delta/2)\;(e^{\lambda_m t}||\bar{x}_0|| + ||u_m||_\infty) \quad t \geq 0. \tag{5.17}$$

Recall that $\bar{x}$ and y are the closed-loop system state and output respectively when the ideal control law (5.9) is applied, so there exist $\lambda_1 < 0$ and $\gamma_1 > 0$ so that

$$\begin{aligned} \|\bar{x}(t)\| &\leq \gamma_1 e^{\lambda_1 t}\|\bar{x}_0\| + \gamma_1\|u_m\|_\infty, \\ \|y(t)\| &\leq \gamma_1 e^{\lambda_1 t}\|\bar{x}_0\| + \gamma_1\|u_m\|_\infty, \quad t \geq 0. \end{aligned} \tag{5.18}$$

From Proposition 5.1 there exist $\bar{T} > 0$, $\lambda_2 < 0$, and $\gamma_2 > 0$ so that for every $\bar{\theta} \in \bar{\mathcal{P}}$ and $T \in (0, \bar{T})$ we have

$$\begin{aligned} &\|\hat{\bar{x}}(t) - \bar{x}(t)\| \leq \gamma_2 T^{\frac{1}{2}} e^{\lambda_2 t}(\|\bar{x}_0\| + |\hat{u}_0^o|) + \gamma_2 T^{\frac{1}{2}}\|u_m\|_\infty, && t \geq 0, \\ &|\hat{u}^o(kT)| \leq \gamma_2 e^{\lambda_2 kT}(\|\bar{x}_0\| + |\hat{u}_0^o|) + \gamma_2\|u_m\|_\infty, && k \geq 0. \end{aligned} \tag{5.19}$$

Choose $T \in (0, \bar{T})$ so that

$$\gamma_2 T^{\frac{1}{2}}||\bar{\bar{C}}|| < \delta/2;$$

then it follows that

$$|\hat{y}(t) - y(t)| \leq (\delta/2)[e^{\lambda_1 t}(\|\bar{x}_0\| + |\hat{u}_0^o|) + \|u_m\|_\infty], \;\; t \geq 0.$$

With $\lambda := \max\{\lambda_1, \lambda_2, \lambda_m\}$, if we combine this with (5.17) and use the fact that the map $\bar{u}_m \to \bar{e}$ is zero, i.e. $\bar{e}(t) - \bar{C}\Phi_{cl}(t,0)\bar{x}(0) = 0$, we have

$$\begin{aligned}
|\hat{y}(t) \;\; - \;\; & y_m(t) - \bar{C}\Phi_{cl}(t,0)\bar{x}(0)| \;\le\; |\hat{y}(t) - y(t) + y(t) - \bar{y}_m(t) + \\
& \bar{y}_m(t) - y_m(t) - \bar{C}\Phi_{cl}(t,0)\bar{x}(0)| \\
\le \;\; & |\hat{y}(t) - y(t)| + |\underbrace{y(t) - \bar{y}_m(t) - \bar{C}\Phi_{cl}(t,0)\bar{x}(0)}_{=0}| + |\bar{y}_m(t) - y_m(t)| \\
\le \;\; & (\delta/2)[e^{\lambda_2 t}(\|\bar{x}_0\| + |\hat{u}_0^o|) + \|u_m\|_\infty)] + (\delta/2)(e^{\lambda_m t}\|\bar{x}_0\| + \|u_m\|_\infty) \\
\le \;\; & \delta[e^{\lambda t}(\|\bar{x}_0\| + |\hat{u}_0^o|) + \|u_m\|_\infty], \quad t \ge 0.
\end{aligned}$$

Hence, the controller given by (5.14)-(5.16) (with a representation of the form (5.4)) together with (5.2) and (5.3) guarantees property (ii).

It remains to prove that this controller provides exponentially stability. Thus, first suppose that $t_0 = k_0 = 0$ and that $u_m = 0$. From (5.18) and (5.19) we see that

$$\|\hat{\bar{x}}(t)\| \;\le\; \underbrace{(\gamma_1 + \delta/2)}_{=:\gamma_3} e^{\lambda t}(\|\bar{x}_0\| + |\hat{u}_0^o|), \quad t \ge 0. \tag{5.20}$$

Now let us look at the controller states. Recall that Lemma 5.2 states that with the controller states z partitioned as

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

we have

$$|z_2[jp+i]| \le \max\{|\hat{u}^o(jT)|, |\hat{u}^o[(j+1)T]|\}, \quad j \in \mathbf{Z}^+, \quad i \in \{0, \cdots, p-1\},$$

and the first subsystem is deadbeat, so there exists a constant $\gamma_4 > 0$ so that

$$\begin{aligned}
||z_1[jp+i]|| \;\;\le\;\; & \gamma_4 \big(\max\{||\hat{\bar{x}}((jp+i-l)h)|| : 0 \le i-l \le p-1\} + \\
& \max\{|\hat{u}^o(jT)|, |\hat{u}^o[(j+1)T]|\}\big), \quad j \in \mathbf{Z}^+, \quad i \in \{0, \cdots, p-1\}.
\end{aligned}$$

If we use the obtained bound (5.19) for $\hat{u}^o(jT)$, and then (5.20) to get a bound on $\max\{||\hat{\bar{x}}(jp+i-l)|| : 0 \le i-l \le p-1,\ j \in \mathbf{Z}^+,\ \ i \in \{0, \cdots, p-1\}$, we have

$$\begin{aligned}
||z_1[jp+i]|| \;\;\le\;\; & \gamma_3\gamma_4 e^{\lambda(jp+i-l)h}(\|\bar{x}_0\| + |\hat{u}_0^o|) + \gamma_2\gamma_3 e^{\lambda jT}(\|\bar{x}_0\| + |\hat{u}_0^o|) \\
\le\;\; & \underbrace{(\gamma_3\gamma_4 e^{-\lambda p\bar{T}} + \gamma_2\gamma_3 e^{-\lambda p\bar{T}})}_{=:\gamma_4} e^{\lambda(jp+i)h}(\|\bar{x}_0\| + |\hat{u}_0^o|), \\
& \qquad\qquad\qquad j \in \mathbf{Z}^+, \quad i \in \{0, \cdots, p-1\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
||z[jp+i]|| \;\;\le\;\; & ||z_1[jp+i]|| + ||z_2[jp+i]|| \\
\le\;\; & \underbrace{(\gamma_4 + \gamma_2 e^{-\lambda\bar{T}})}_{=:\gamma_5} e^{\lambda(jp+i)h}(\|\bar{x}_0\| + |\hat{u}_0^o|), \\
& \qquad\qquad\qquad j \in \mathbf{Z}^+, \;\; i \in \{0, \cdots, p-1\}.
\end{aligned}$$

Setting $k = jp + i$ yields

$$||z[k]|| \quad \leq \quad \gamma_5 e^{\lambda kh}(\|\bar{x}_0\| + |\hat{u}_0^o|), \quad k \in \mathbf{Z}^+.$$

Hence, with

$$x_{sd}(t) := \begin{bmatrix} \hat{\bar{x}}(t) \\ z[k] \end{bmatrix}, \quad t \in [kh, (k+1)h),$$

so it follows that

$$\|x_{sd}(t)\| \leq \underbrace{(\gamma_3 + \gamma_5 e^{-\lambda \bar{T}})}_{=:\gamma_6} e^{\lambda t} \|x_{sd}(0)\|, \quad t \geq 0.$$

Now we turn to the case when $t_0 = k_0 h > 0$ and $u_m(t) = 0$ for $t \geq 0$. Since the controller (5.4) is periodic of period T, it follows that if our starting time is an integer of multiple of T, i.e. of the form $t_0 = jT$, then from above

$$\|x_{sd}(t)\| \leq \gamma_6 e^{\lambda(t-t_0)} \|x_{sd}(t_0)\|, \quad t \geq t_0.$$

Since the plant and controller parameters are uniformly bounded, it is straightforward to prove that nothing un-toward happens if we start between periods: there exists a constant $\gamma_7 \geq \gamma_6$ so that for every $k_0 \geq 0$, with $t_0 := k_0 h$ we have

$$\|x_{sd}(t)\| \leq \gamma_7 e^{\lambda(t-t_0)} \|x_{sd}(t_0)\|, \quad t \geq t_0,$$

i.e. we have exponential stability.

□

5.4 Examples

In this section several examples with relative degree one and one with relative degree two are presented. In all cases, we compare the new controller to that of [20].

5.4.1 Example 1

The plant model is first order:

$$\dot{y}(t) = a(t)y(t) + b(t)u(t).$$

The set of plant uncertainty is given by

$$\Gamma = \{ \begin{bmatrix} a \\ g \end{bmatrix} \in \mathbf{R}^2 : a \in [-1, 1], \; g \in [0.5, 1.5] \},$$

$$\mathcal{P} = \mathcal{P}(\Gamma, \mu_1 = 1, T_0 = 5, \gamma_0 = 1, \lambda_0 = -5).$$

The reference model is

$$\dot{\bar{x}}_m = -\bar{x}_m + \bar{u}_m,$$

while reference model anti-aliasing filter is chosen to be

$$\dot{\bar{u}}_m = -50\bar{u}_m + 50u_m.$$

For the LPC of [20], we approximate the polynomial $\frac{1}{g}$, as in [20]:

$$\hat{f}_{0.01}(g) = 2.1647 - 1.5153g + 0.3433g^2,$$

so $q = 2$, and we choose $p = 10 > 2m(2q + 1) = 10$, i.e. the Control Phase is 50% of the controller period. For the new approach, since every element of the compact set $\mathcal{G} := [0.5, 1.5]$ is positive, we can apply Remark 5.2 and set $\hat{f}_\varepsilon(g) = \frac{1}{3}$ and $p = 1$. In both controllers, we set $\rho = 1$, and $h = 0.01$, so in the LPC [20] $T = 0.1$ and in the redesigned LPC, $T = 0.01$. Figure 5.4 shows the simulation results with $y_0 = 3$, $\bar{u}_{m_0} = 0$, u_m a square wave given by

$$u_m = sign(\cos(\frac{2\pi t}{15})),$$

and

$$a(t) = \cos(t/2) \text{ and } g(t) = [1 + 0.5sin(t/4)];$$

we add a noise signal of the form

h * random sequence uniformly distributed between ± 1

to the output measurement at the time $t = 66$ sec. In Figure 5.5 we provide a close-up of the control signal during the time interval $t \in [0, 0.4]$ sec. As shown in Figures 5.4 and 5.5, the redesigned LPC has significantly better noise rejection and performance and a smoother control signal.

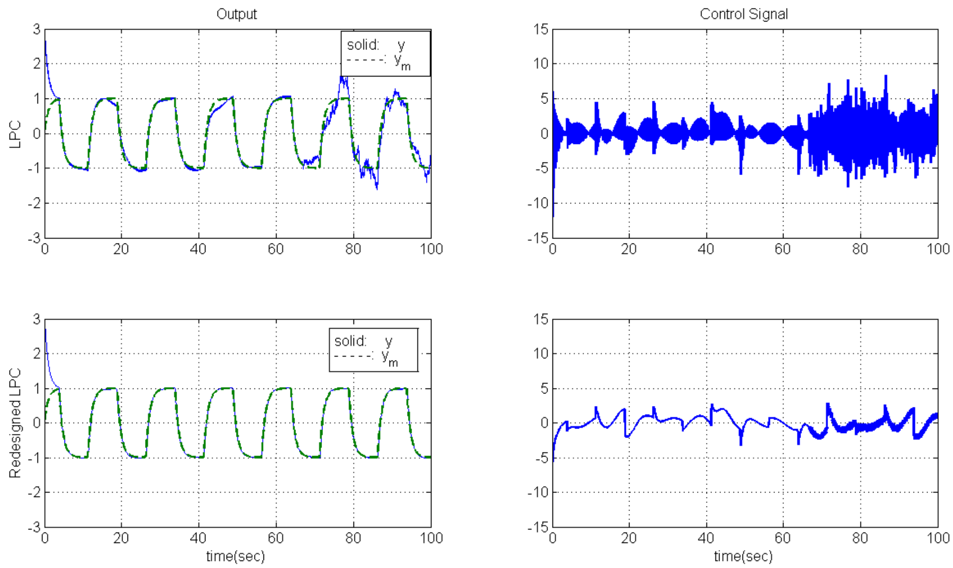

Figure 5.4: The plant output and control signal with a and g varying with time

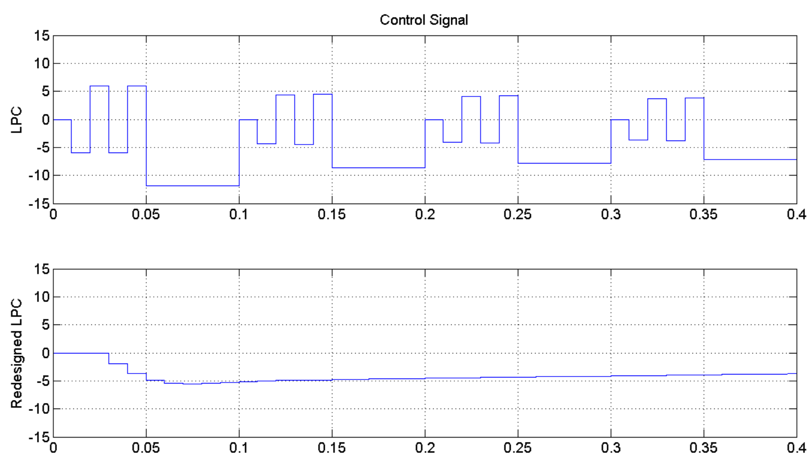

Figure 5.5: A close-up of the control signal

5.4.2 Example 2

The plant model is first order:

$$\dot{y}(t) = a(t)y(t) + b(t)u(t).$$

The set of plant uncertainty is given by

$$\Gamma = \{ \begin{bmatrix} a \\ g \end{bmatrix} \in \mathbf{R}^2 : a \in [-1, 1],\ g^2 \in [1, 1.4] \},$$

$$\mathcal{P} = \mathcal{P}(n = 1,\ m = 1,\ \Gamma, \mu_1 = 1, T_0 = 5, \gamma_0 = 1, \lambda_0 = -5).$$

The reference model is

$$\dot{\bar{x}}_m = -\bar{x}_m + \bar{u}_m,$$

and the anti-aliasing filter is

$$\dot{\bar{u}}_m = -50\bar{u}_m + 50u_m.$$

For the LPC of [20], we approximate the polynomial $\frac{1}{g}$, as in [20]:

$$\hat{f}_{0.01}(g) = 2.1647g - 1.5153g^3 + 0.3433g^5,$$

so $q = 5$, and we choose $p = 25 > 2m(2q + 1) = 22$, i.e. the Control Phase is 56% of the controller period. For the new approach, since the compact set $\mathcal{G} := [-1.4, -1] \cup [1, 1.4]$ contains both positive and negative values, we can apply Remark 5.2 and set $\hat{f}_\varepsilon(g) = \frac{1}{2.8}g$ and $p = 3$. In both controllers, we set $\rho = 1$, and $h = 0.01$, so in the LPC [20] $T = 0.25$ and in the redesigned LPC, $T = 0.03$.

Figure 5.6 shows the simulation results with $y_0 = 3$, $\bar{u}_{m_0} = 0$, u_m a square wave given by

$$u_m = sign(\cos(\frac{2\pi t}{15})),$$

and

$$a(t) = \cos(t/2) \text{ and } g(t) = [1.2 + 0.2cos(t/2)] * sign[cos(t/4)];$$

we add a noise signal of the form

h * random sequence uniformly distributed between ± 1

to the output measurement at the time $t = 66$ sec. As shown in Figure 5.6, the redesigned LPC has better noise rejection and performance. In Figure 5.7 we provide a close-up of the control signal during the time interval $t \in [0, 1]$ sec.

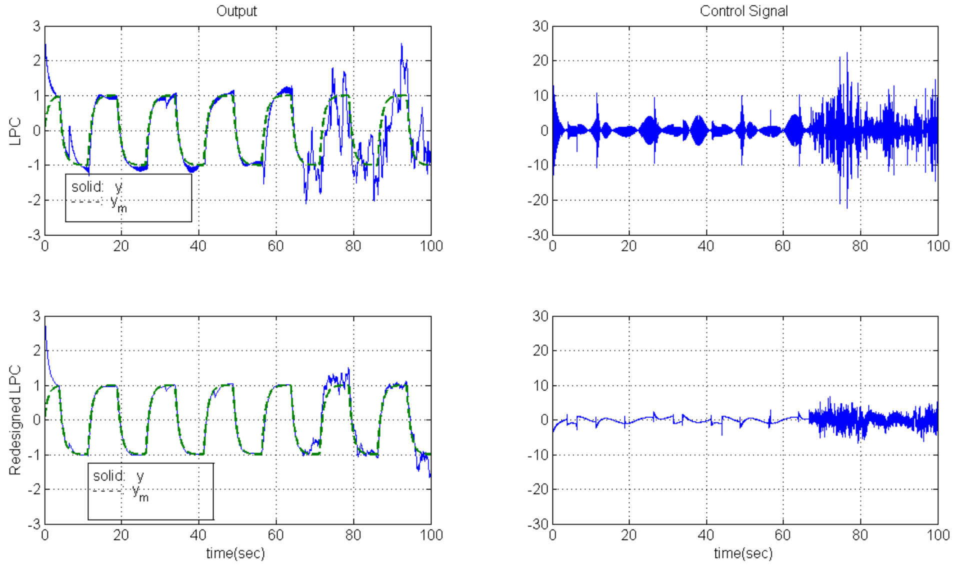

Figure 5.6: The plant output and control signal with a and g varying with time

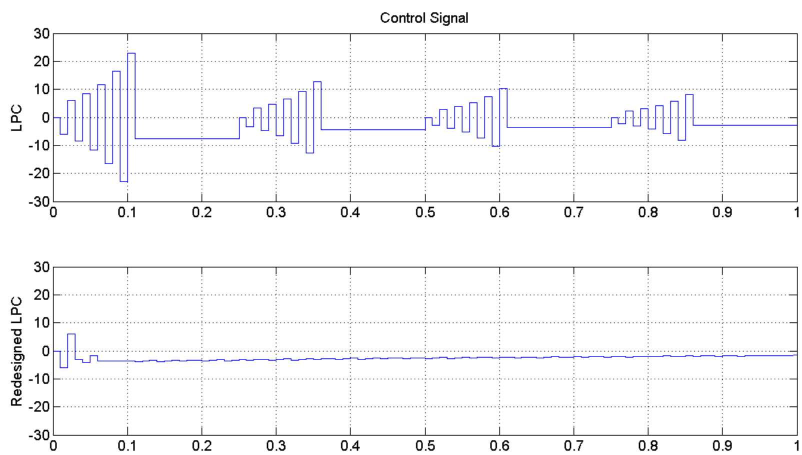

Figure 5.7: A close-up of the control signal

5.4.3 Example 3

In this example a minimum phase relative degree two model of a DC motor is considered [17]:

$$\begin{aligned} \dot{x} &= \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & \beta_1(t) \end{bmatrix}}_{=:A(t)} x + \underbrace{\begin{bmatrix} 0 \\ g(t) \end{bmatrix}}_{=:B(t)} u, \\ y &= \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{=:C} x. \end{aligned}$$

The set of plant uncertainty is given by

$$\bar{\Gamma} = \{ \begin{bmatrix} \beta_0 \\ \beta_1 \\ g \end{bmatrix} \in \mathbf{R}^3 : \beta_0 = 0,\ \beta_1 \in [-5,-1],\ \ g \in [1,2]\},$$

$$\bar{\mathcal{P}} = \bar{\mathcal{P}}(n = 2, m = 2, \bar{\Gamma}, \bar{\mu}_1 = 1, \bar{T}_0 = 10, \gamma_0 = 1, \lambda_0 = -5\}.$$

The reference model is relative degree two with the state space representation:

$$\begin{aligned} \dot{x}_m &= \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x_m + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_m, \\ y_m &= \begin{bmatrix} 1 & 0 \end{bmatrix} x_m. \end{aligned}$$

The anti-aliasing filter is chosen as

$$\dot{\bar{u}}_m = -50\bar{u}_m + 50u_m.$$

For the LPC of [20], we approximate the polynomial $\frac{1}{g}$, as in [20]:

$$\hat{f}_{0.01}(g) = 2.1647 - 1.5153g + 0.3433g^2,$$

so $q = 2$, and we choose $p = 25 > 2m(2q + 1) = 20$, i.e. the Control Phase is 60% of the controller period. For the redesigned LPC, since the compact set $\mathcal{G} := [1,2]$ contains only positive values, we can apply Remark 5.2 and set $\hat{f}_\varepsilon(g) = \frac{1}{4}$ and $p = m = 2$. In both controllers, we set $\rho = 1$, and $h = 0.01$, so in the LPC [20] $T = 0.25$ and in the redesigned LPC, $T = 0.02$.

A simulation was carried out (see Figure 5.8 with $x_0 = [1 \ \ 1]^T$, $y_{m_0} = \bar{u}_{m_0} = 0$, u_m a square wave given by

$$u_m = sign(\cos(\frac{2\pi t}{15})),$$

and

$$\beta_1(t) = -3 + 2 * \cos(t/2) \text{ and } g(t) = 1.5 + 0.5 * \sin(t/4).$$

We add a noise signal of the form

h^2 * random sequence uniformly distributed between ± 1

to the output measurement at the time $t = 66$ sec; we see that it degrades the tracking slightly. In Figure 5.9 we provide a close-up of the control signal during the time interval $t \in [0, 1]$ sec. As Figures 5.8 and 5.9 illustrate, the redesigned LPC has much better noise tolerance and performance and a smoother control signal.

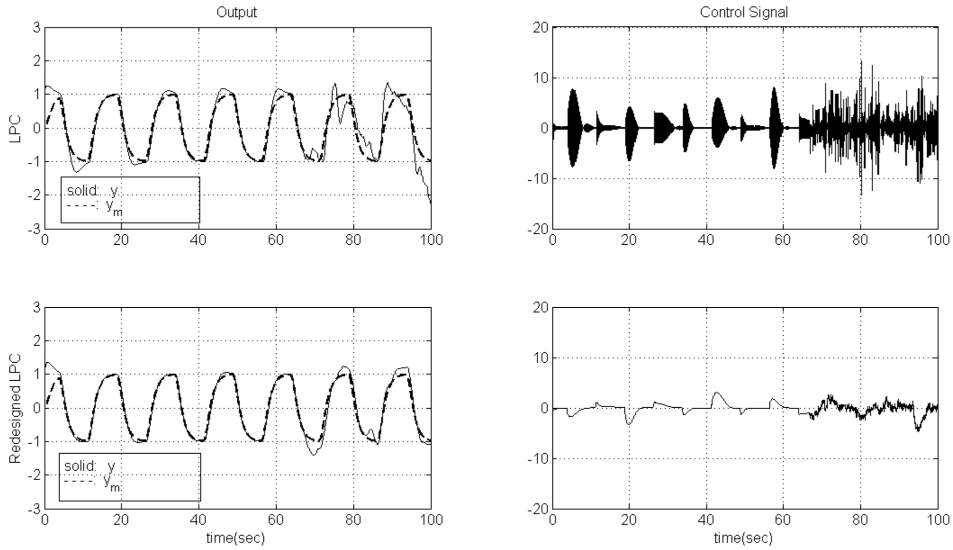

Figure 5.8: The plant output and control signal with β_1 and g varying with time

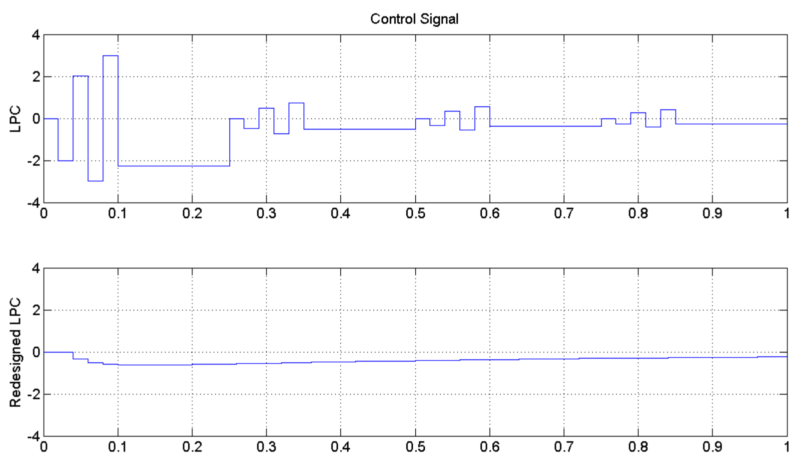

Figure 5.9: A close-up of the control signal

5.4.4 Example 4: Large Range of Parameter Variation and Noise Size

In the next three simulations, we show that the redesigned LPC handles large parameter variation and in the case that every element of $\mathcal{G}$ has the same sign (all positive or all negative) this controller is remarkably noise tolerant.

(a) Relative Degree One and the High Gain Frequency Sign is Fixed: In the first simulation we adopt the setup of Example 1 of this chapter but with larger parameter variation. The plant model is first order:

$$\dot{y}(t) = a(t)y(t) + g(t)u(t).$$

The set of plant uncertainty is given by

$$\Gamma = \{ \begin{bmatrix} a \\ g \end{bmatrix} \in \mathbf{R}^2 : a \in [-15, 15],\ g \in [-37, -1] \},$$

$$\mathcal{P} = \mathcal{P}(\Gamma, \mu_1 = 1, T_0 = 5, \gamma_0 = 1, \lambda_0 = -5).$$

The reference model is

$$\dot{\bar{x}}_m = -\bar{x}_m + \bar{u}_m,$$

and the anti-aliasing filter is

$$\dot{\bar{u}}_m = -50\bar{u}_m + 50u_m.$$

Since every element of the compact set $\mathcal{G} := [-37, -1]$ is negative, we apply Remark 5.2 and set $\hat{f}_\varepsilon(g) = \frac{-1}{100}$ and $p = 1$. We also set $\rho = 1$, and $h = 0.0004$, so $T = 0.0004$.

Figure 5.10 shows the simulation results and plant parameters with $y_0 = 3$, $\bar{u}_{m_0} = 0$, u_m a square wave given by

$$u_m = sign(\cos(\frac{2\pi t}{15})),$$

and

$$a(t) = 15\cos(t/2) \text{ and } g(t) = [-19 - 18sin(t/4)];$$

we add a noise signal of the form

$100h$ * random sequence uniformly distributed between ± 1

to the output measurement at the time $t = 20$ sec. As Figure 5.10 shows, although the size of noise is large (a hundred times of the sampling time), the plant output has not been disturbed largely. This illustrates the significant noise tolerance of the redesigned LPC in the case that the high frequency gain does not change sign.

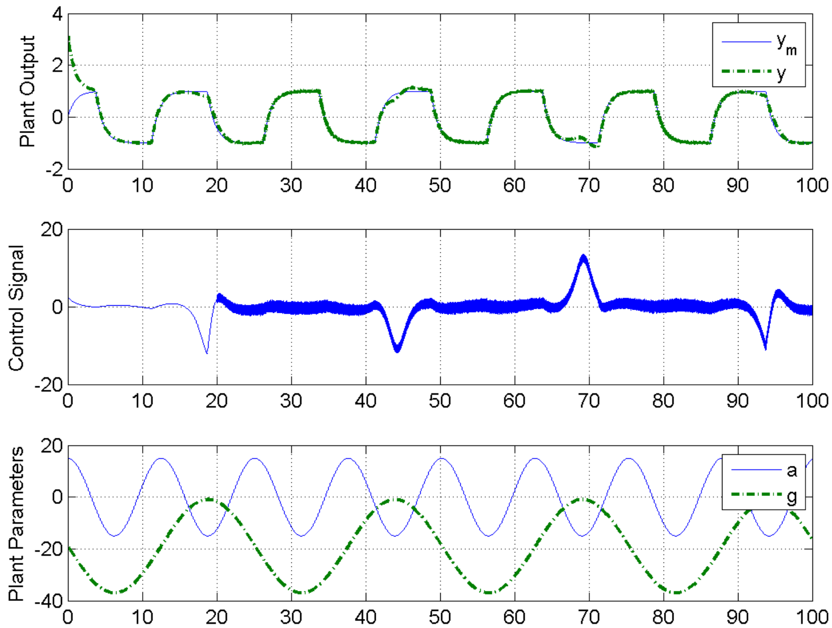

Figure 5.10: Example 4, part (a) with a relative degree one plant, large plant parameter variation and noise size, and the fixed high frequency gain sign

(b) Relative Degree Two and the High Gain Frequency Sign is Fixed: In the second simulation we adopt the setup of Example 3 of this chapter but with larger parameter variation. In this simulation a minimum phase relative degree two model of a DC motor is considered [17]:

$$\begin{aligned} \dot{x} &= \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & \beta_1(t) \end{bmatrix}}_{=:A(t)} x + \underbrace{\begin{bmatrix} 0 \\ g(t) \end{bmatrix}}_{=:B(t)} u, \\ y &= \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{=:C} x. \end{aligned}$$

The set of plant uncertainty is given by

$$\bar{\Gamma} = \{ \begin{bmatrix} \beta_0 \\ \beta_1 \\ g \end{bmatrix} \in \mathbf{R}^3 : \beta_0 = 0,\ \beta_1 \in [-9, 1],\ \ g \in [0.5, 11.5]\},$$

$$\bar{\mathcal{P}} = \bar{\mathcal{P}}(n = 2, m = 2, \bar{\Gamma}, \bar{\mu}_1 = 1, \bar{T}_0 = 10, \gamma_0 = 1, \lambda_0 = -5\}.$$

The reference model is relative degree two with the state space representation:

$$\begin{aligned} \dot{x}_m &= \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x_m + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_m, \\ y_m &= \begin{bmatrix} 1 & 0 \end{bmatrix} x_m. \end{aligned}$$

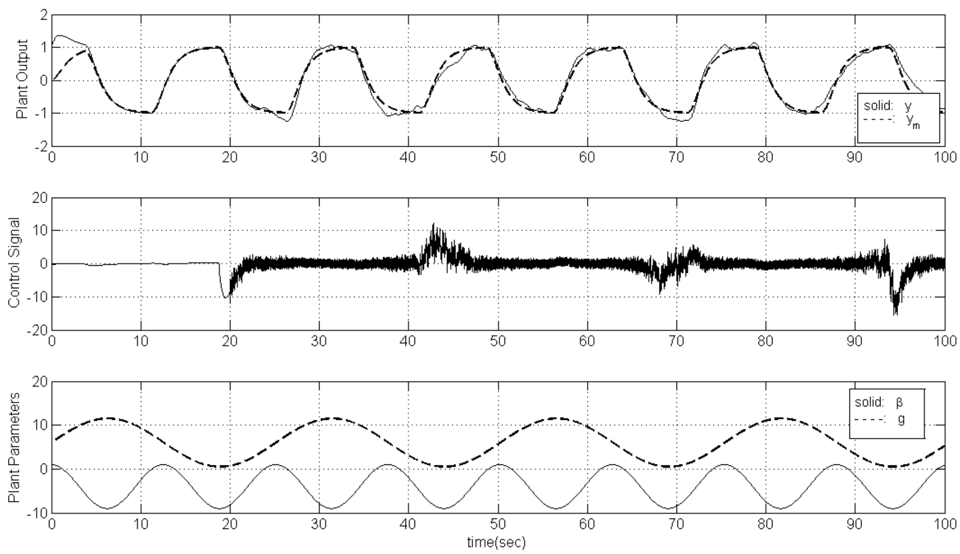

Figure 5.11: Example 4, part (b) with a relative degree two plant, large plant parameter variation and noise size, and the fixed high frequency gain sign

The anti-aliasing filter is chosen as

$$\dot{\bar{u}}_m = -50\bar{u}_m + 50u_m.$$

Since the compact set $\mathcal{G} := [0.5, 11.5]$ contains only positive values, we apply Remark 5.2 and set $\hat{f}_\varepsilon(g) = \frac{1}{12}$ and $p = m = 2$. We also set $\rho = 1$, and $h = 0.001$, so $T = 0.002$.

A simulation was carried out (see Figure 5.11 with $x_0 = [1 \;\; 1]^T$, $y_{m_0} = \bar{u}_{m_0} = 0$, u_m a square wave given by

$$u_m = sign(\cos(\frac{2\pi t}{15})),$$

and

$$\beta_1(t) = -4 + 5 * \cos(t/2) \text{ and } g(t) = 6 + 5.5 * \sin(t/4).$$

We add a noise signal of the form

$5h^2$ * random sequence uniformly distributed between ± 1

to the output measurement at the time $t = 20$ sec. As Figure 5.11 shows, with large noise size, the plant output has not been disturbed largely.

(c) Relative Degree One and the High Gain Frequency Sign is Varying: In this simulation we adopt the setup of Example 2 of this chapter but with larger parameter variation. The plant model is first order:

$$\dot{y}(t) = a(t)y(t) + b(t)u(t).$$

The set of plant uncertainty is given by

$$\Gamma = \{\begin{bmatrix} a \\ g \end{bmatrix} \in \mathbf{R}^2 : a \in [-3,3],\ g^2 \in [1,16]\},$$

$$\mathcal{P} = \mathcal{P}(\Gamma, \mu_1 = 1, T_0 = 5, \gamma_0 = 1, \lambda_0 = -5).$$

The reference model is

$$\dot{\bar{x}}_m = -\bar{x}_m + \bar{u}_m,$$

and the anti-aliasing filter is

$$\dot{\bar{u}}_m = -50\bar{u}_m + 50u_m.$$

Since the compact set $\mathcal{G} := [-4,-1] \cup [1,4]$ includes both positive and negative values, we apply Remark 5.2, page 69, and set $\hat{f}_\varepsilon(g) = \frac{g}{20}$ and $p = 3$. We also set $\rho = 1$, and $h = 0.0004$, so $T = 0.0012$.

Figure 5.12 shows the simulation results and plant parameters with $y_0 = 3$, $\bar{u}_{m_0} = 0$, u_m square a wave given by

$$u_m = sign(\cos(\frac{2\pi t}{15})),$$

and

$$a(t) = 3\cos(t/2) \text{ and } g(t) = [2.5 + 1.5\cos(t/2)] * sign[cos(t/4)].$$

We add a noise signal of the form

h * random sequence uniformly distributed between ± 1

to the output measurement at the time $t = 20$ sec. As Figure 5.12 shows, the controller handles large parameter variation.

As shown in Figures 5.10, 5.11, and 5.12 the redesigned LPC remarkably handles a large range of plant parameter variation. Also Figures 5.10 and 5.11 illustrate that in the case that the high frequency gain sign is fixed, the redesigned LPC is significantly noise tolerant.

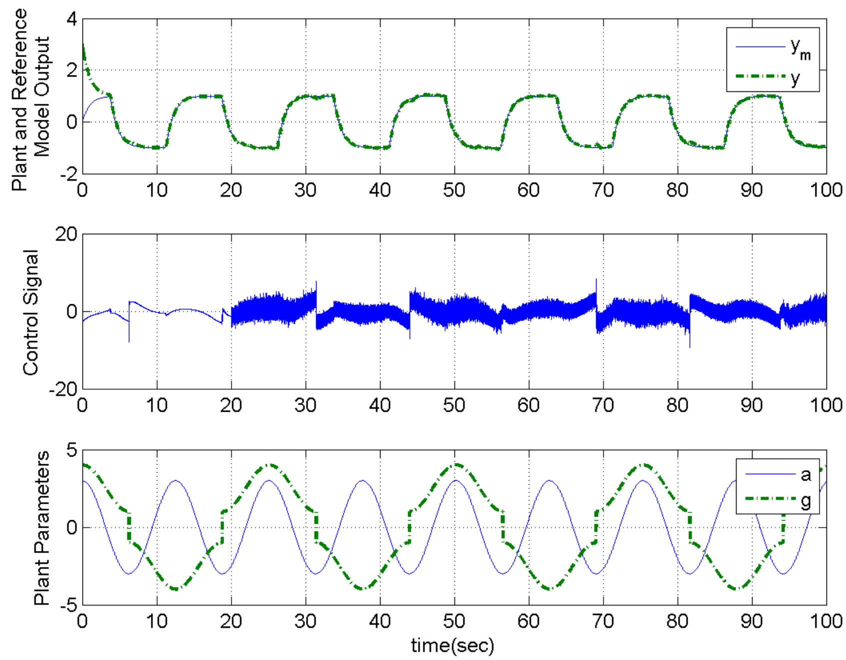

Figure 5.12: Example 4, part (c) with a relative degree one plant, large plant parameter variation, and high frequency gain sign changing

5.4.5 Example 5: Performance and Noise Behaviour

In this example we analyze the performance and the noise behaviour of the closed-loop system. We compute the $\mathcal{L}_\infty$ induced norm when the plant parameters are fixed and investigate how the performance and the noise behaviour of the sampled-data system depends on the sampling period. The first computation of the $\mathcal{L}_\infty$ induced norm of the LPC [20] was carried out by Xiaosong [43]. The tool that is used is the *lifting* technique (see Chen and Francis [3] which establishes a strong correspondence between periodic systems and time invariant infinite-dimensional systems.

We consider two cases: (i) the high frequency gain sign is fixed and (ii) the high frequency g can obtain both positive and negative values. We inject noise n at the plant output; we let T_{u_m} denote the map from $u_m \longrightarrow e$ and T_n denote the map from $n \longrightarrow e$.

(a) The high frequency gain sign is fixed: We adopt one admissible plant model of the set of plant uncertainty of Example 1 of this chapter as

$$\dot{y}(t) = y(t) + u(t).$$

In the LPC [20], for approximating the polynomial $\frac{1}{g}$, we use the estimation polynomial

$$\hat{f}(g) = 2.1647 - 1.5153g + 0.3433g^2.$$

For the redesigned LPC, the estimation polynomial $\hat{f}_\varepsilon(g) = \frac{1}{3}$ is used. With the controller designed as above but with h free, in Figure 5.13, we plot $||T_{u_m}||$ and $||T_n||$ as a function of h. As expected, for both controllers, the former goes to zero as $h \longrightarrow 0$, with that of the new controller goes to zero much faster; indeed, the gain of $||T_n||$ for the redesigned LPC converges to a constant as $h \longrightarrow 0$, while that for the LPC [20] blows up. This demonstrates that the noise behaviour is significantly improved.

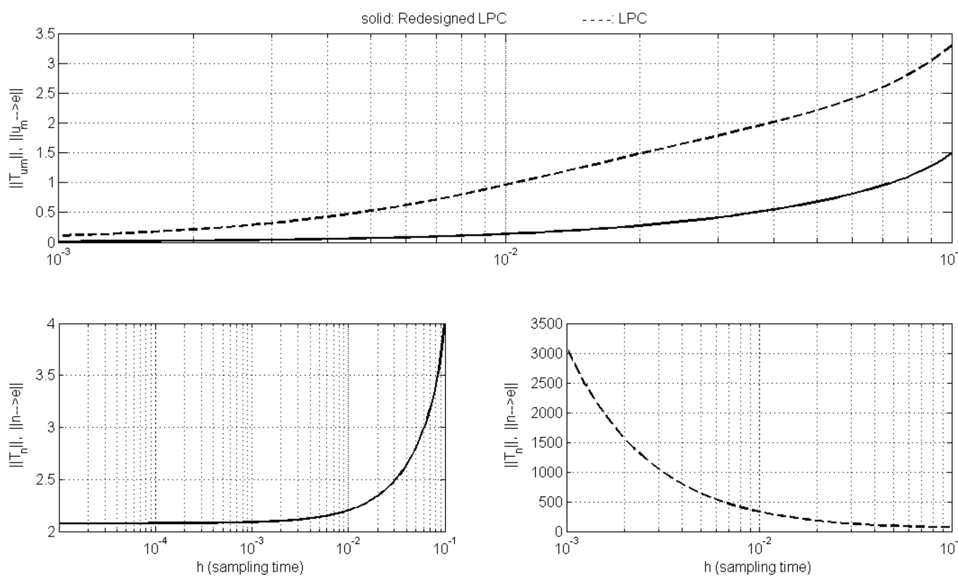

Figure 5.13: The $||u_m \rightarrow e||$ and $||n \rightarrow e||$ map for two controllers; the high frequency gain sign is fixed

(b) The high frequency gain can obtain positive and negative values: We adopt one admissible plant model of the set of plant uncertainty of Example 2 as

$$\dot{y}(t) \quad = \quad y(t) + u(t).$$

In the LPC [20], for approximating the polynomial $\frac{1}{g}$, we use the estimation polynomial

$$\hat{f}(g) = 2.1647g - 1.5153g^3 + 0.3433g^5.$$

For the redesigned LPC, the estimation polynomial $\hat{f}_\varepsilon(g) = \frac{g}{2.8}$ is used. With the controller designed as above but with h free, in Figure 5.14, we plot $||T_{u_m}||$ and $||T_n||$ as a function of h. As expected, for both controllers, while the former goes to zero, the latter goes to infinity as $h \longrightarrow 0$. As the figure shows the redesigned LPC remarkably improves both noise behaviour and performance.

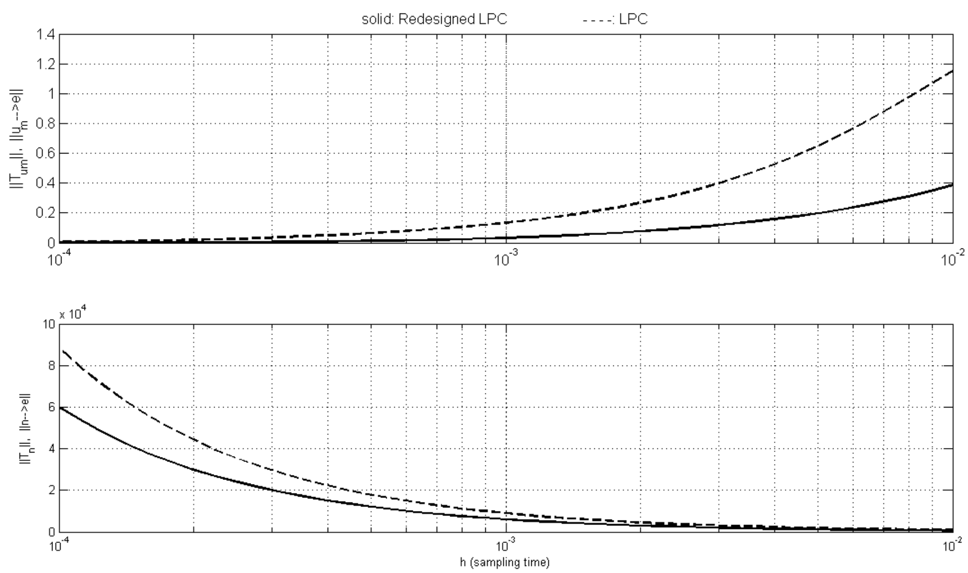

Figure 5.14: The $||u_m \rightarrow e||$ and $||n \rightarrow e||$ map for two controllers; the high frequency gain can obtain positive and negative values

5.4.6 Example 6: Effect of c_0

In the next simulation, the stability analysis of the redesigned LPC with respect to c_0 is carried out. We consider the case when the high frequency gain g is known, since in the next chapter we deal with this case and small values of c_0. We inject noise n at the plant output; we let T_{u_m} denote the map from $u_m \longrightarrow e$ and T_n denote the map from $n \longrightarrow e$.

We adopt one admissible plant model of the set of plant uncertainty of Example 1 of this chapter as

$$\dot{y}(t) = y(t) + u(t).$$

For the redesigned LPC, the estimation polynomial $\hat{f}_\epsilon(g) = c_0$ is used. With the controller designed as above but with h free, in Figure 5.15, we plot $||T_{u_m}||$ and $||T_n||$ as a function of h. As this figure shows, small values of c_0 result in instability of the closed-loop system in smaller sampling times. This result matches (5.13), since small values of c_0 make the term

$$1 - g(kT)\hat{f}_\epsilon(g(kT))$$

closer to the critical value one and consequently instability margin. In Chapter 6, we will show that with c_0 small (which demands small sampling time) and some other conditions, the redesigned LPC provides stability for plants with lower relative degree.

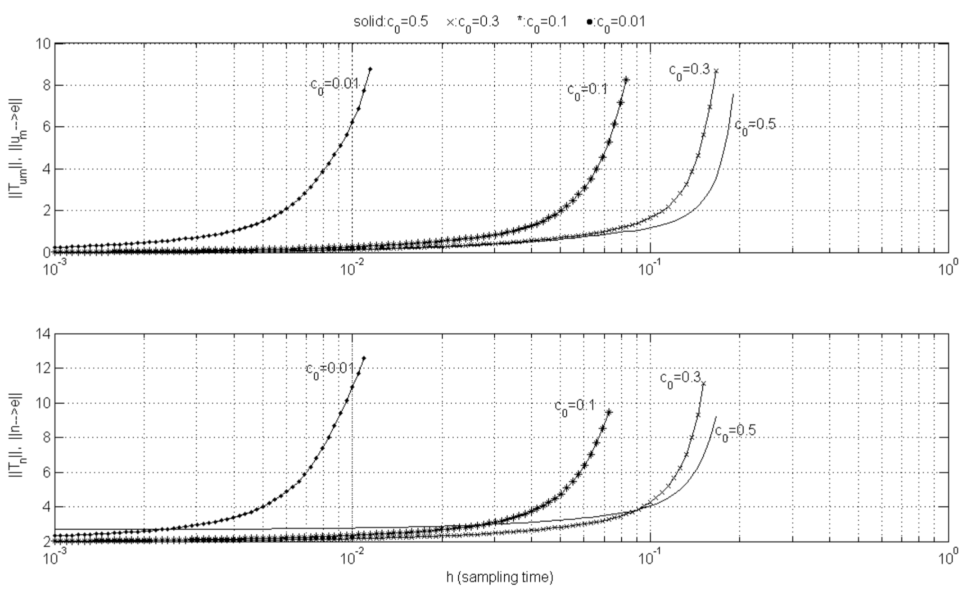

Figure 5.15: The $||u_m \rightarrow e||$ and $||n \rightarrow e||$ map for the redesigned LPC and different values of c_0; the high frequency gain is positive

5.5 Summary and Concluding Remarks

In [20] Miller presented a new adaptive control technique. This method was discussed in Chapter 3 and extended in Chapter 4. This adaptive control technique has a number of significant desirable features:

- it handles rapid changes in the plant parameters,
- it provides nice transient behaviour of the closed-loop system,
- it guarantees that the effect of the initial conditions declines to zero exponentially, and
- it generates control signals which are modest in size.

Having these advatages, we look at its drawbacks:

- during each controller period the control signal takes different values; since the controller period is small, this requires fast actuators,
- in order to achieve the desired tracking, a small sampling period is used which results in large controller gains; this may lead the system to poor noise tolerance; in fact, there is a trade-off between the desired tracking and noise tolerance, and
- the plant relative degree must be known.

In the LPC [20], the control signal moves vigorously, in particular during the Estimation Phase, when probing takes place; the control signal jumps rapidly from one value to another. Here the controller is redesigned to reduce the size and number of the jumps. The approach is as follows. In the signal control law [20], the controller is periodic with each period consisting of two phases: in Estimation Phase the *ideal control law* is estimated, while in Control Phase a naturally scaled version of this estimate is applied. In the redesigned controller, we estimate the *change* in the ideal control signal law between periods, which reduces the size of the probing during the Estimation Phase. We also carry out the probing more efficiently, to reduce the number of jumps in the control signal as well. While proving this approach required a good deal of effort, the results were good: the control signal is smoother, the performance and noise tolerance are considerably better (for a given period). While in this chapter the LPC was redesigned to improve the first two drawbacks, the next chapter is devoted to modifying the LPC to relax the last undesirable feature.

Chapter 6

Analyzing the Relative Degree

6.1 Introduction

Knowledge of the plant relative degree is typically assumed in model reference adaptive control. While in early works, knowledge of the exact relative degree was required, later attempts [38] relaxed this requirement and showed that an upper bound of the plant relative degree is sufficient. Unfortunately, the LPC [20] demands knowledge of the exact plant relative degree. In this chapter, we deal with this requirement in the context of the redesigned LPC introduced in Chapter 5. The goal is to analyze and possibly redesign the redesigned LPC to relax this demand; instead of the exact relative degree, only an upper bound is required, at least in an important special case. More specifically, we will show that the redesigned LPC, developed for relative degree $l \geq 1$, provides stability for plants with lower relative degree, i.e. an upper bound of the (instead of the exact) plant relative degree is sufficient, if

(i) the plant high frequency gain is known and always has the same sign,

(ii) to approximate the term $\frac{1}{g}$, we use the constant approximation polynomial $\hat{f}_\epsilon(g) = c_0 \neq 0$, and c_0 has the same sign of the high frequency gain and $|c_0|$ is very small, (which requires small sampling time).

(iii) the controller parameters vector $\bar{f}_2$ is chosen large in an appropriate sense, and

(iv) the plant is time-invariant.

Now for an outline of this chapter. In Section 6.2, we show that assumption (i) is necessary for the redesigned LPC to require only an upper bound, instead of the exact, relative degree of the plant; a counterpart-example proves that if the approximation polynomial order is higher than zero, the controller may result in an unstable closed-loop system for plants with lower relative degree. Finally Section 6.3 proves that if conditions (i)-(iv) are met, then in the redesigned LPC, an upper bound (instead of the exact) on the plant relative degree is sufficient.

6.2 The Requirement on the Sign of the High Frequency Gain

In this section we present a counter-example to show that, if the sign of the high frequency gain g is unknown, and the LPC is applied to a plant with a lower relative degree, then closed-loop stability is not guaranteed.

To proceed, we use a very simple example; specifically, suppose that the uncertainty used for designing the controller is the set of transfer functions

$$\{\frac{g}{s^2 + a_1 s + a_0} : \quad g = \pm 1, \ |a_i| \le \bar{a}, \quad i = 0, 1\}, \tag{6.1}$$

with a corresponding state-space model

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} x + \begin{bmatrix} 0 \\ g \end{bmatrix} u, \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x, \end{aligned}$$

while the actual plant lies in

$$\{\frac{g}{s+\alpha} : \quad g = \pm 1, \ |\alpha| \le \bar{\alpha}\},$$

with a corresponding state-space model of

$$\dot{y} = -\alpha y + g u. \tag{6.2}$$

(We can certainly define the corresponding $\mathcal{P}$ with the usual notation, but we omit this for brevity). Thus, we apply the approach of Chapter 5 for the relative degree two case: we choose a stable reference model

$$\begin{aligned} \dot{\bar{x}}_m &= \underbrace{\begin{bmatrix} 0 & 1 \\ -\bar{a}_0 & -\bar{a}_1 \end{bmatrix}}_{=Am} \bar{x}_m + \underbrace{\begin{bmatrix} 0 \\ \bar{g} \end{bmatrix}}_{=B_m} \bar{u}_m, \\ \bar{y}_m &= \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{=C_m} x_m. \end{aligned}$$

With

$$\bar{f}_2 = \begin{bmatrix} -f_1 & -f_2 \end{bmatrix},$$

chosen so that

$$\begin{bmatrix} 0 & 1 \\ -f_1 & -f_2 \end{bmatrix},$$

is stable[1], then we can design the ideal control law for the modelled plant, which is of the form

$$u = \frac{1}{g} \begin{bmatrix} a_0 - f_1 & a_1 - f_2 \end{bmatrix} x + \frac{1}{g} k_1 \bar{x}_m + \frac{1}{g} \bar{u}_m.$$

[1]This is clearly the case if and only if $f_1 > 0$ and $f_2 > 0$.

Now we apply the sampled-data controller introduced in Chapter 5. Given that $g = \pm 1$ in the nominal model class, it is natural to choose an estimate of $\frac{1}{g}$ to be g; indeed, we get an exact match by setting

$$\begin{aligned} \hat{f}_\varepsilon(g) &= \hat{f}_0(g) \\ &= g. \end{aligned}$$

Hence, we have $\rho = 1$, $m = 2$, and $T = 3mh$, so we define

$$\begin{aligned} \hat{\phi}_0(kT) &:= \begin{bmatrix} -f_1 & -f_2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & \frac{h^2}{2} \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}^{-1} \times \\ & \quad \begin{bmatrix} y(kT) \\ y(kT+h) \\ y(kT+2h) \end{bmatrix} + k_2 \bar{u}_m(kT) + k_1 \bar{x}_m(kT), \end{aligned}$$

$$\begin{aligned} \hat{\phi}_1(kT) &:= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & \frac{h^2}{2} \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}^{-1} \times \\ & \quad \begin{bmatrix} y(kT+2h) - y(kT) \\ y(kT+3h) - y(kT+h) \\ y(kT+4h) - y(kT+2h) \end{bmatrix}. \end{aligned}$$

Since $\hat{f}_\epsilon(g) = g$, the control signal is given by

$$u(t) = \begin{cases} \hat{u}^o(kT), & t \in [kT, kT+2h), \\ \hat{u}^o(kT) + \hat{\phi}_0(kT), & t \in [kT+2h, kT+4h), \\ \hat{u}^o(kT) - \hat{\phi}_0(kT), & t \in [kT+4h, kT+6h), \end{cases} \tag{6.3}$$

and

$$\hat{u}^o[(k+1)T] = \hat{u}^o(kT) + \hat{\phi}_1(kT). \tag{6.4}$$

When this control law is applied to the nominal model class (6.1), we know from Theorem 5.1 that we get closer and closer to optimal behaviour as T (or h) $\to 0$.

Now let us apply the control law (6.3) to the system (6.2). Here we have

$$u(t) = \hat{u}^o(kT), \quad t \in [kT, kT+2h).$$

Combining this with the first-order system (6.2), yields

$$\begin{aligned} y(kT+h) &= e^{-\alpha h} y(kT) + \int_0^h g e^{-\alpha\tau} \hat{u}^o(kT) d\tau \\ &\approx (1 - \alpha h + \frac{(\alpha h)^2}{2}) y(kT) + g(h - \alpha \frac{h^2}{2}) \hat{u}^o(kT) \\ y(kT+2h) &= e^{-\alpha 2h} y(kT) + \int_0^{2h} g e^{-\alpha\tau} \hat{u}^o(kT) d\tau \\ &\approx (1 - 2\alpha h + \frac{(\alpha 2h)^2}{2}) y(kT) + g(2h - \alpha \frac{(2h)^2}{2}) \hat{u}^o(kT), \end{aligned}$$

for small h, which results in

$$\hat{\phi}_0(kT) \approx (-f_1 + \alpha f_2 - \alpha^2)y(kT) + g(\alpha - f_2)\hat{u}^o(kT) + k_1\bar{x}_m(kT) + k_2\bar{u}_m(kT). \tag{6.5}$$

Using the obtained value of $\hat{\phi}_0(kT)$ in (6.5), if we apply (6.3) to the system (6.2), we will have

$$\begin{aligned} y(kT+3h) &= e^{-\alpha 3h}y(kT) + \int_0^{3h} e^{-\alpha\tau} g\hat{u}^o(kT)d\tau + \int_{2h}^{3h} e^{-\alpha(3h-\tau)} g\hat{\phi}_0(kT)d\tau \\ &\approx (1 - \alpha 3h + \frac{(\alpha 3h)^2}{2})y(kT) + g(3h - \alpha\frac{(3h)^2}{2})\hat{u}^o(kT) + g(h - \alpha\frac{h^2}{2})\hat{\phi}_0(kT), \end{aligned}$$

and

$$\begin{aligned} y(kT+4h) &= e^{-\alpha 4h}y(kT) + \int_0^{4h} e^{-\alpha\tau} g\hat{u}^o(kT)d\tau + \int_{2h}^{4h} e^{-\alpha(4h-\tau)} g\hat{\phi}_0(kT)d\tau \\ &\approx (1 - \alpha 4h + \frac{(\alpha 4h)^2}{2})y(kT) + g(4h - \alpha\frac{(4h)^2}{2})\hat{u}^o(kT) + g(2h - \alpha\frac{(2h)^2}{2})\hat{\phi}_0(kT), \end{aligned}$$

which leads to

$$\begin{aligned} \hat{\phi}_1(kT) &\approx -\alpha g\hat{\phi}_0(kT) \\ &\approx -\alpha g\left[(-f_1 + \alpha f_2 - \alpha^2)y(kT) + g(\alpha - f_2)\hat{u}^o(kT) + \right. \\ &\quad \left. k_1\bar{x}_m(kT) + k_2\bar{u}_m(kT)\right]. \end{aligned}$$

For T small, if we apply the corresponding control signal (6.3) to the plant (6.2) we obtain

$$\begin{aligned} y[(k+1)T] &= e^{-\alpha T}y(kT) + g\int_0^T e^{-\alpha(T-\tau)}u(kT+\tau)d\tau \\ &\approx e^{-\alpha T}y(kT) + gT\hat{u}^o(kT). \end{aligned}$$

The update on the control signal is

$$\begin{aligned} \hat{u}^o[(k+1)T] &= \hat{u}^o(kT) + \hat{\phi}_1(kT) \\ &= \hat{u}^o(kT) - \alpha g[(-f_1 + \alpha f_2 - \alpha^2)y(kT) + g(\alpha - f_2)\hat{u}^o(kT) + \\ &\quad k_1\bar{x}_m(kT) + k_2\bar{u}_m(kT)]. \end{aligned}$$

Combining the last two equations results in the following closed-loop system:

$$\begin{aligned} \begin{bmatrix} y[(k+1)T] \\ \hat{u}^o[(k+1)T] \end{bmatrix} &\approx \begin{bmatrix} e^{-\alpha T} & gT \\ -\alpha g(-f_1 + \alpha f_2 - \alpha^2) & 1 - g^2\alpha(\alpha - f_2) \end{bmatrix} \begin{bmatrix} y(kT) \\ \hat{u}^o(kT) \end{bmatrix} + \\ &\quad \begin{bmatrix} 0 \\ -\alpha g \end{bmatrix} [k_1\bar{x}_m(kT) + k_2\bar{u}_m(kT)], \end{aligned}$$

with the characteristic polynomial

$$\eta(\lambda) \;:=\; \lambda^2 + \lambda \underbrace{\left[-e^{-\alpha T} - 1 + g^2\alpha(\alpha - f_2)\right]}_{=:S_1} + \underbrace{e^{-\alpha T} - e^{-\alpha T} g^2 \alpha(\alpha - f_2) + \alpha g^2 T(-f_1 + \alpha f_2 - \alpha^2)}_{=:S_2},$$

which is Hurwitz if the roots stay in the open-unit disc. It is easy to show that this polynomial is not always Hurwitz. For example if $\alpha > 0$ is small, then

$$S_1 \to -2 + g^2\alpha(\alpha - f_2) < -2,$$

as $T \to 0$, so $\eta(\lambda)$ has roots outside the open-unit disc, which results in closed-loop instability. Thus, the control law (6.3) does not gurantee the closed-loop stability.

We conclude that, even in a very simple case, if the relative degree is unknown, then more information about the system parameters are required for our approach to work. In the next section we will demonstrate that knowledge of the sign of the high frequency gain is sufficient.

6.3 Stability of the LPC for Plants with Lower Relative Degree

In this section we prove that if the sign of the high frequency gain is known, then the redesigned LPC, designed for plants with relative degree l, provides closed-loop stability for the LTI plants for lower/equal relative degrees $m = 1, 2, \cdots, l$, if certain controller parameters are designed properly.

6.3.1 The Set of Uncertainty

At this point we explain the class of model uncertainty which will be handled here. We adopt the setup [20], which was repeated in Section 3.3 of Chapter 3:

$$\bar{\mathcal{P}}_m(n, m, \bar{\Gamma}, \bar{\mu}_1, \bar{T}_0, \delta_1, \cdots, \delta_{m+1}, \underline{g}, \gamma_0, \lambda_0).$$

Now with $n_i \geq i \geq 1, \;\; i = 1, \cdots, l$ we define the following sets of plant uncertainty with different relative degrees:

$$\begin{aligned}
&\bar{\mathcal{P}}_1(n_1, m = 1, \bar{\Gamma}_1, \bar{\mu}_1, \bar{T}_0, \delta_1, \delta_2, \underline{g}, \gamma_0, \lambda_0),\\
&\bar{\mathcal{P}}_2(n_2, m = 2, \bar{\Gamma}_2, \bar{\mu}_1, \bar{T}_0, \delta_1, \cdots, \delta_3, \underline{g}, \gamma_0, \lambda_0),\\
&\vdots\\
&\bar{\mathcal{P}}_l(n_l, m = l, \bar{\Gamma}_l, \bar{\mu}_1, \bar{T}_0, \delta_1, \cdots, \delta_{l+1}, \underline{g}, \gamma_0, \lambda_0),
\end{aligned}$$

so that the high frequency gains $g_1, \cdots, g_l$ have the same sign. For this section the class of plant uncertainty is of the form

$$\bar{\mathcal{P}} = \cup_{i=1}^{l} \bar{\mathcal{P}}_i.$$

6.3.2 The Controller Structure

In this section we present the redesigned LPC of Chapter 5, designed for a system with the relative degree l which stabilizes plants with lower relative degree. First, let us recall two $(l+1)\times(l+1)$ matrices S_l and $H_l(h)$, which are defined in (3.21) of Section 3.5 of Chapter 3, and the sequence of output samples by

$$\mathcal{Y}(t) := \begin{bmatrix} y(t) & y(t+h) & \cdots & y(t+lh) \end{bmatrix}^T .$$

Recall that we choose $\bar{f}_2 \in \mathbf{R}^{1\times l}$ so that the eigenvalues of

$$\underbrace{\begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & 0 \end{bmatrix}}_{\Lambda_2} + \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{b_2} \bar{f}_2$$

lie to the left of those of A_m.To achieve closed-loop stability we will see that we have to define $r_0, r_1, \ldots, r_l$ so that all polynomials

$$s^m + \frac{r_{m-1}}{r_m}\lambda^* s^{m-1} + \frac{r_{m-2}}{r_m}{\lambda^*}^2 s^{m-2} + \cdots + \frac{r_1}{r_m}{\lambda^*}^{m-1} s + \frac{r_0}{r_m}{\lambda^*}^m,$$
$$m = 1, 2, \cdots, l, \quad r_l := 1,$$

are Hurwitz. Now if we choose λ^* large enough and define the vector $\bar{f}_2$ as

$$\bar{f}_2(\lambda^*) := \begin{bmatrix} -r_0{\lambda^*}^l & -r_1{\lambda^*}^{l-1} & \cdots & -r_{l-1}\lambda^* \end{bmatrix},$$

then $\Lambda_2 + b_2\bar{f}_2$ is Hurwitz with eigenvalues on the left of those of A_m.

Because we have assumed that the sign of the high frequency gain is known, according to Remark 5.2 we can choose $\hat{f}_\varepsilon$ to be a constant, i.e. the constant approximation polynomial $\hat{f}_\epsilon(g) = c_0$. Hence, $q = 0$, so the control law is as follows:

THE PROPOSED CONTROLLER ($t_0 = 0$)

$$u(t) = \hat{u}^o(kT), \qquad t \in [kT, kT + lh),$$
$$\hat{\phi}_0(kT) = \begin{bmatrix} \bar{f}_2(\lambda^*) & -1 \end{bmatrix} H_l(h)^{-1} S_l^{-1} \mathcal{Y}_l(kT) + \begin{bmatrix} k_1 & k_2 \end{bmatrix} \begin{bmatrix} \bar{x}_m(kT) \\ \bar{u}_m(kT) \end{bmatrix} \qquad (6.6)$$
$$\hat{u}^o[(k+1)T] = \hat{u}^o(kT) + c_0\hat{\phi}_0(kT).$$

6.3.3 Applying the Control Law, Designed for Plants with Relative Degree l, to Plants with Relative Degree $m \leq l$

In this section we apply the control law (6.6), designed for plants with relative degree l to a plant with relative degree $m \leq l$ and analyze the closed-loop stability and performance. In this analysis, we assume that the plant is time-invariant and the rest of analysis is based on this assumption. Before proceeding, observe that with $x(t)$, A, B, and C defined in (5.5) and $\bar{x}(t)$, $\bar{A}$, $\bar{B}$, and $\bar{\bar{C}}$ defined in equations (5.6) and (5.7) of Chapter 5, Section 5.2, we have

$$\begin{array}{l}\bar{\bar{C}}\bar{A}^i\bar{x}(kT) = CA^ix(kT), \\ \bar{\bar{C}}\bar{A}^j\bar{B} = CA^jB,\end{array} \quad i \in \{0,1,\cdots,l\}, \quad i \in \{0,1,\cdots,l-1\}, \quad k \in \mathbf{Z}^+.$$

Recall that

$$u(t) = \hat{u}^o(kT), \quad t \in [kT, kT+lh),$$

so from Lemma 3.1 (KEL-3) [2] of Chapter 3, for every $l \in \mathbf{N}$ we have:

$$\begin{aligned}
&H_l^{-1}(h)S_l^{-1}\mathcal{Y}_l(kT) - \begin{bmatrix} C \\ CA \\ \vdots \\ CA^l \end{bmatrix} x(kT) + \begin{bmatrix} 0 \\ CB \\ \vdots \\ CA^{l-1}B \end{bmatrix} \hat{u}^o(kT) \\
&= H_l^{-1}(h)S_l^{-1}\mathcal{Y}_l(kT) - \underbrace{\begin{bmatrix} \bar{\bar{C}} \\ \bar{\bar{C}}\bar{A} \\ \vdots \\ \bar{\bar{C}}\bar{A}^l \end{bmatrix}}_{=:\mathcal{O}_l} \hat{\bar{x}}(kT) + \underbrace{\begin{bmatrix} 0 \\ \bar{\bar{C}}\bar{B} \\ \vdots \\ \bar{\bar{C}}\bar{A}^{l-1}\bar{B} \end{bmatrix}}_{=:p_l} \hat{u}^o(kT) \\
&= \mathcal{O}(h)(||\hat{\bar{x}}(kT)|| + |\hat{u}^o(kT)|).
\end{aligned}$$

Thus, with

$$\hat{\phi}_0(kT) = \begin{bmatrix} \bar{f}_2 & -1 \end{bmatrix} H_l^{-1}(h)S_l^{-1}\mathcal{Y}(kT) + \begin{bmatrix} k_1 & k_2 \end{bmatrix} \begin{bmatrix} \bar{x}_m(kT) \\ \bar{u}_m(kT) \end{bmatrix},$$

we have

$$\begin{aligned}
\hat{\phi}_0(kT) = &\begin{bmatrix} \bar{f}_2 & -1 \end{bmatrix} \mathcal{O}_l\hat{\bar{x}}(kT) + \begin{bmatrix} \bar{f}_2 & -1 \end{bmatrix} p_l\hat{u}^o(kT) + \\
&\begin{bmatrix} k_1 & k_2 \end{bmatrix} \begin{bmatrix} \bar{x}_m(kT) \\ \bar{u}_m(kT) \end{bmatrix} + \mathcal{O}(T)||\hat{\bar{x}}(kT)|| + \mathcal{O}(T)|\hat{u}^o(kT)|.
\end{aligned}$$

[2] Technically speaking, a bound is proven in this theorem. Since we need an equality, we use the above equality form.

If we combine this equation with the update rule for $\hat{u}^o$ in (6.6) then it follows

$$\begin{aligned}\hat{u}^o[(k+1)T) &= c_0 \begin{bmatrix} \bar{f}_2 & -1 \end{bmatrix} \mathcal{O}_l \hat{\bar{x}}(kT) + \left(1 + c_0 \begin{bmatrix} \bar{f}_2 & -1 \end{bmatrix} p_l\right) \hat{u}^o(kT) + \\ & \quad c_0 \begin{bmatrix} k_1 & k_2 \end{bmatrix} \begin{bmatrix} \bar{x}_m(kT) \\ \bar{u}_m(kT) \end{bmatrix} + \\ & \quad \mathcal{O}(T)||\hat{\bar{x}}(kT)|| + \mathcal{O}(T)|\hat{u}^o(kT)|. \end{aligned} \tag{6.7}$$

Now observe that we can choose perturbations Δ_1 and Δ_2, satisfying

$$\begin{aligned} ||\Delta_1(kT)|| &\leq T^2||\hat{\bar{x}}(kT)||, \\ ||\Delta_2(kT)|| &\leq T^2||\hat{u}^o(kT)||, \end{aligned}$$

so that applying the control law (6.6) yields

$$\begin{aligned} \hat{\bar{x}}[(k+1)T] &= e^{\bar{A}T}\hat{\bar{x}}(kT) + \int_0^T e^{\bar{A}(T-\tau)}\bar{B}\hat{u}^o(kT)d\tau \\ &= (I + \bar{A}T)\hat{\bar{x}}(kT) + T\bar{B}\hat{u}^o(kT) + \Delta_1(kT) + \Delta_2(kT), \end{aligned}$$

for small T. If we combine this equation, (6.6), and (6.7) together, we have

$$\begin{aligned} \begin{bmatrix} \hat{\bar{x}}[(k+1)T] \\ \hat{u}^o[(k+1)T] \end{bmatrix} &= \begin{bmatrix} I + \bar{A}T + \Delta_1(kT) & \bar{B}T + \Delta_2(kT) \\ c_0 \begin{bmatrix} \bar{f}_2 & -1 \end{bmatrix} \mathcal{O}_l + \mathcal{O}(T) & 1 + c_0 \begin{bmatrix} \bar{f}_2 & -1 \end{bmatrix} p_l + \mathcal{O}(T) \end{bmatrix} \times \\ & \quad \begin{bmatrix} \hat{\bar{x}}(kT) \\ \hat{u}^o(kT) \end{bmatrix} + c_0 \begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix} \begin{bmatrix} \bar{x}_m(kT) \\ \bar{u}_m(kT) \end{bmatrix}. \end{aligned} \tag{6.8}$$

The following result shows that under certain conditions this closed-loop system is stable.

Lemma 6.1: *There exists a constant* $\overline{T} > 0$ *so that for all* $T \in (0, \overline{T})$*, the closed-loop system (6.8) is stable for all* $\bar{\theta} \in \bar{\mathcal{P}}$ *if*

(i) $\sup_{\bar{\theta}\in\bar{\mathcal{P}}} \left|1 + c_0 \begin{bmatrix} \bar{f}_2 & -1 \end{bmatrix} p_l\right| < 1$, *and*

(ii) *there exists a compact set* $\Upsilon \subset \mathbf{C}^-$ *so that*

$$sp\left[\bar{A} - \bar{B}\frac{\begin{bmatrix} \bar{f}_2 & -1 \end{bmatrix} \mathcal{O}_l}{\begin{bmatrix} \bar{f}_2 & -1 \end{bmatrix} p_l}\right] \subset \Upsilon, \quad \bar{\theta} \in \bar{\mathcal{P}}.$$

Proof: See Appendix C.

To prove closed-loop stability, we need to show that conditions (i) and (ii) of Lemma 6.1 are satisfied if $\bar{f}_2$ is chosen properly. Lemma 6.2 states under certain conditions parts (i) and (ii) of Lemma 6.1 are met.

Lemma 6.2: *There exist constants $\lambda^{**} >> 1$ large enough and $1 >> \bar{c}_0 > 0$ small enough so that for all $\bar{\theta} \in \bar{\mathcal{P}}$ and $\lambda^* \geq \lambda^{**}$ if*

(a) *the high frequency gain g is known,*

(b) $0 < |c_0| \leq \bar{c}_0$ *and c_0 has the same sign as the high frequency gain g, and*

(c) *the coefficients $r_0, r_1, \ldots, r_l$ are defined so that the polynomials*

$$s^m + \frac{r_{m-1}}{r_m}\lambda^* s^{m-1} + \cdots + \frac{r_1}{r_m}{\lambda^*}^{l-1} s + r_0 {\lambda^*}^l, \quad m = 1, 2, \cdots, l, \quad r_l := 1,$$

are Hurwitz and $\bar{f}_2$ is given by

$$\bar{f}_2(\lambda^*) := \begin{bmatrix} -r_0{\lambda^*}^l & -r_1{\lambda^*}^{l-1} & \cdots & -r_{l-1}\lambda^* \end{bmatrix},$$

and

(d) *the plant is time-invariant, then*

(i) $\sup_{\bar{\theta}\in\bar{\mathcal{P}}} \left|1 + c_0 \begin{bmatrix} \bar{f}_2 & -1 \end{bmatrix} p_l\right| < 1$, *and*

(ii) *there exists a compact set $\Upsilon \subset \mathbf{C}^-$ so that* $sp\left[\bar{A} - \bar{B}\frac{\begin{bmatrix} \bar{f}_2 & -1 \end{bmatrix}\mathcal{O}_l}{\begin{bmatrix} \bar{f}_2 & -1 \end{bmatrix}p_l}\right] \subset \Upsilon, \quad \bar{\theta} \in \bar{\mathcal{P}}.$

Proof: See Appendix C.

Now we need to prove that the new controller provides stability for plants with lower relative degree.

Theorem 6.1: *There exist constants $\overline{T} > 0$, $\lambda^{**} >> 1$ large enough, and $1 >> \bar{c}_0 > 0$ small enough so that for all $\bar{\theta} \in \bar{\mathcal{P}}$, $T \in (0, \overline{T})$, and $\lambda^* \geq \lambda^{**}$ if conditions (a)-(d) of Lemma 6.2 are satisfied then the control law (6.6) designed for plants with relative degree l provides stability for all plants with lower relative degree.*

Proof:

Recall that if we apply the control law (6.6) designed for plants with relative degree l to plants with relative degree $m \leq l$, then the closed-loop system is given by (6.8). Regarding Lemma 6.2, there exist constants $\lambda^{**} >> 1$ large enough and

$\bar{c}_0$ small enough so that if conditions (a)-(c) of this lemma, which are conditions (a)-(c) of Theorem 6.1, are met then

(i) $\sup_{\bar{\theta}\in\bar{\mathcal{P}}} \left|1 + c_0 \begin{bmatrix} \bar{f}_2 & -1 \end{bmatrix} p_l\right| < 1$, and

(ii) there exists a compact set $\Upsilon \subset \mathbf{C}^-$ so that $sp\left[\bar{A} - \bar{B}\frac{\begin{bmatrix} \bar{f}_2 & -1 \end{bmatrix}\mathcal{O}_l}{\begin{bmatrix} \bar{f}_2 & -1 \end{bmatrix}p_l}\right] \subset \Upsilon, \quad \bar{\theta} \in \bar{\mathcal{P}}$,

which are the conditions (i) and (ii) of Lemma 6.1. Thus, based on Lemma 6.1 the closed-loop system (6.8) is stable for all $\bar{\theta} \in \bar{\mathcal{P}}$ and relative degree $m \leq l$.

□

6.4 Example

In this section one example is presented. In this simulation the plant relative degree is one and the controller, designed for a plant with relative degree two, provides stability for the first-order plant. The time-invariant plant model is first order:

$$\dot{y}(t) = ay(t) + gu(t),$$

with $a = 3$ and $g = -4$. The set of plant uncertainty is given by

$$\Gamma_1 = \{\begin{bmatrix} a \\ g_1 \end{bmatrix} \in \mathbf{R}^2 : a = 3,\ g_1 = -4\},$$

$$\Gamma_2 = \{\begin{bmatrix} a_1 \\ a_2 \\ g_2 \end{bmatrix} \in \mathbf{R}^3 : g_2 \in [-4, -1]\},$$

$$\begin{aligned} \mathcal{P} &= \mathcal{P}_1(n = 1, m = 1, \Gamma_1, \mu_1 = 1, T_0 = 5, \gamma_0 = 1, \lambda_0 = -5) \cup \\ &\quad \mathcal{P}_2(n = 2, m = 2, \Gamma_2, \mu_1 = 1, T_0 = 5, \gamma_0 = 1, \lambda_0 = -5), \end{aligned}$$

Since the high frequency gains g_1 and g_2 are both negative, we can apply Remark 5.2 and set $\hat{f}_\varepsilon(g) = \frac{-1}{200}$ and $p = 2$. We set $\rho = 1$, and $h = 0.01$, so $T = 0.02$.

Figure 6.1 shows the simulation results with $y_0 = 1$, $\bar{u}_{m_0} = 0$, u_m a square wave given by

$$u_m = sign(\cos(\frac{2\pi t}{15}));$$

as shown in Figure 6.1, the redesigned LPC designed for relative degree two, provides stability for the first-order plant, but unfortunately the tracking provided is very poor.

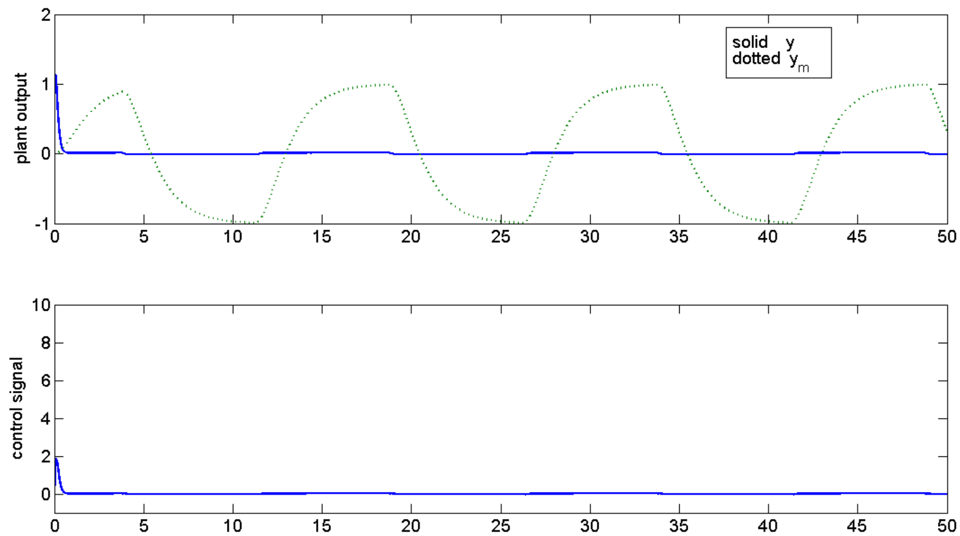

Figure 6.1: The plant output and control signal of the first-order plant and the redesigned LPC designed for relative degree two

6.5 Summary and Concluding Remarks

One of the undesirable features of the LPC [20] is that knowledge of the exact plant relative degree is required. This chapter is devoted to this requirement of the redesigned LPC, introduced in Chapter 5. We showed that in the redesigned controller an upper bound of the (instead of the exact) plant relative degree is sufficient, if

(i) the plant high frequency gain is known and always has the same sign,

(ii) the approximation polynomial is chosen a constant with a small size and has the same sign of the high frequency gain,

(iii) the controller gains are chosen large, and

(iv) the plant is time-invariant.

Chapter 7

Improving the Noise behaviour

7.1 Introduction

In many control designs, noise tolerance is an important goal. Two main sources of noise effect are actuators and sensors. When applying the control signal, there can be noise acting on the actuator at the plant input. Noise can also affect measurement sensors when determining the plant output.

Although the LPC [20] has significant desirable features, it suffers from poor noise tolerance. The reason is that in order to achieve good tracking performance, a small sampling period is used. This results in large controller gains and consequently poor noise tolerance. There is a clear trade-off between noise rejection and performance. This chapter is devoted to improving the noise tolerance of the LPC [20] as well as examining and improving the noise tolerance of the redesigned controller of Chapter 5. Figure 7.1 illustrates the LPC [20] feedback configuration with noises. In this diagram, n_1 and n_2 are the noise effects on the actuators and measurement sensors, respectively. This chapter deals with the most problematic term, namely the measurement noise n_2, and presents several methods to achieve better noise tolerance behaviour.

The outline of this chapter is organized as follows. In Section 6.2, we present four different ideas of noise reduction of the measurement noise n_2. In the first method, we apply a low-pass filter after the output. In the second approach, we redesign the LPC to obtain a smaller estimation error, which allows us to choose larger sampling time which hopefully will produce better noise rejection. In the third approach, the MMSE estimators are used to minimize the effect of noise. In the last approach, we adopt a more vigorous probing signal in order to achieve better noise tolerance. Finally, in Section 6.3, we combine the last approach with the redesigned controller of Chapter 5 to take advantage of both methods. The *noise tolerant redesigned LPC* is more tolerant to noise, has better performance, and requires slower actuators.

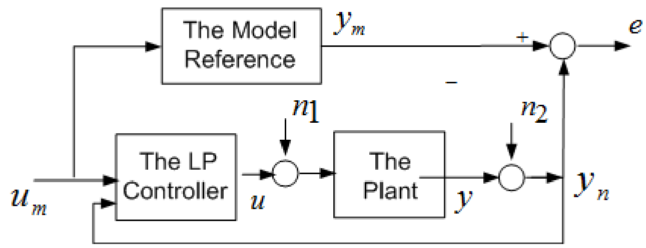

Figure 7.1: The feedback diagram with noises.

7.2 Noise Rejection Ideas

In this section, we present four different controller redesigns to improve noise tolerance of the controller of [20]. We carry out a time-domain simulation to produce a glimpse of the noise behaviour and we also compute the noise gain and tracking error gain (in the induced norm sense) as a function of the sampling period.

In the time-domain simulations, we simulate the plant and the controllers and then plot the plant output, the reference model output, and the control signal. In all time-domain simulations of Section 7.2, we adopt the following example:
The plant model is first order:

$$\dot{y}(t) = a(t)y(t) + g(t)u(t).$$

The set of plant uncertainty is given by

$$\Gamma = \{\begin{bmatrix} a \\ g \end{bmatrix} \in \mathbf{R}^2 : a \in [-1, 1],\ g \in [0.8, 1.2]\},$$

$$\mathcal{P} = \mathcal{P}(\Gamma, \mu_1 = 1, T_0 = 5, \gamma_0 = 1, \lambda_0 = -5).$$

For approximating the polynomial $\frac{1}{g}$, we use

$$\hat{f}_{0.01}(g) = g,$$

so $q = 1$. In all controllers, we set $\rho = 1$, and $h = 0.005$. The simulations are carried out with $y_0 = 3$, $\bar{u}_{m_0} = 0$, u_m a square wave:

$$u_m = sign(\cos(\frac{2\pi t}{15})),$$

and

$$a(t) = \cos(t/2) \text{ and } g(t) = [1 + 0.2sin(t/4)];$$

we add a noise signal of the form
h * random sequence uniformly distributed between ± 1
to the output measurement at the time $t = 30$ sec.

An analysis of performance and the noise rejection is also carried out. We inject noise n at the plant output[1]; we let T_{u_m} denote the map from $u_m \longrightarrow e$ and T_n

[1]We use n to denote the noise signal rather than n_2 to minimize notation.

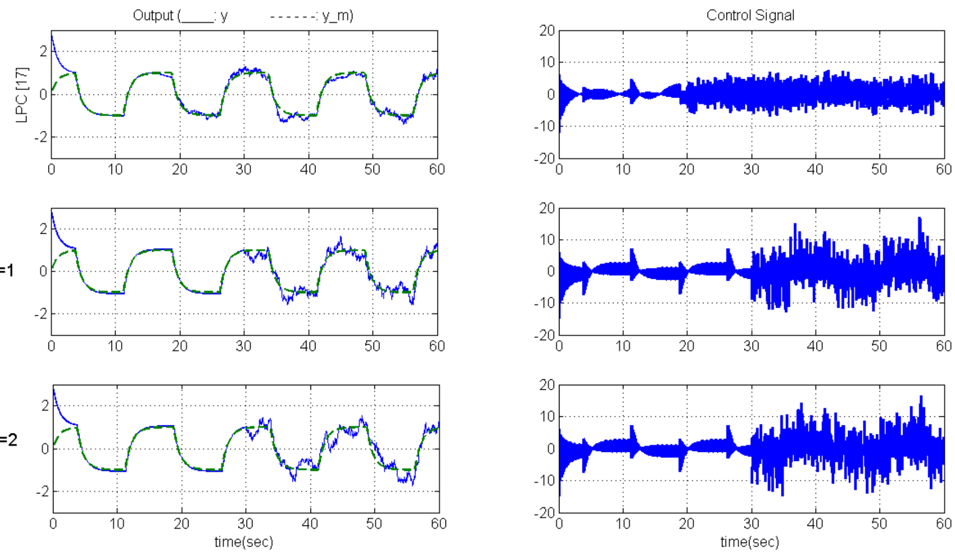

Figure 7.2: The time-domain simulation results for the LPC [20] and the controller with a low pass filter for different values of σ.

denote the map from $n \longrightarrow e$. With the designed controllers but with h free, we plot $||T_r||$ and $||T_n||$ as a function of h. As we will see, in all approaches, the former goes to zero as $h \longrightarrow 0$, while the latter goes to infinity as $h \longrightarrow 0$.
In all cases, the plant is an admissible plant model of the set of plant uncertainty of the time-domain simulation example, namely

$$\dot{y}(t) = y(t) + u(t), \quad t \geq 0.$$

7.2.1 Using a Low-Pass Filter

In the first approach, to reduce the effect of the measurement noise, we add a first-order low pass filter $F_\sigma(s) = \frac{\sigma}{s+\sigma}$ at the plant output. To redesign the LPC [20] in Chapter 3 to deal with the 2-norm case, we added an anti-aliasing filter at the plant output. Here we do the same: we add this filter at the plant output, which results in increasing the relative degree by one[2].

Figure 7.2 shows the time-domain simulation results for different values of σ. As the figure illustrates, this method does not improve the noise rejection. In Figure 7.3, we provide the operator gains. This method worsens the performance as well as the noise rejection, and larger values of σ make the performance even worse.

The reason of this deterioration is that using a low-pass filter increases the relative degree by one which results in the longer estimation phase.

[2]Since the high frequency gain of the resulting plant is the high frequency of the original plant times σ, large σ results in instability.

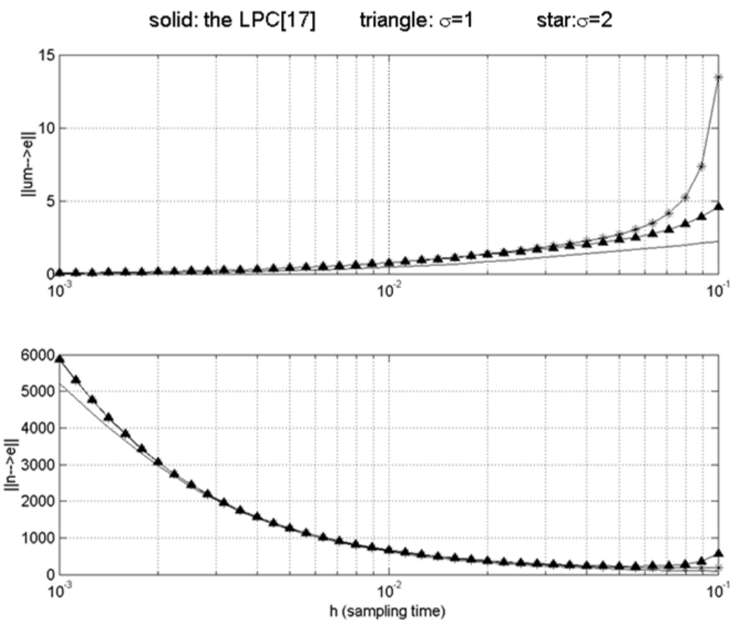

Figure 7.3: The performance, $||u_m \rightarrow e||$, and the noise rejection behaviour, $||n \rightarrow e||$, of the LPC [20] and the controller with an output low pass filter and different values of σ.

7.2.2 Controller With the Smaller Estimation Error

In the design of [20], the control signal is chosen to make sure that the error is $\mathcal{O}(h)$. In this approach, we redesign the controller [20] such that the error is $\mathcal{O}(h^2)$. So, to reach good tracking, we can choose larger values of the sampling time, which should lead to better noise rejection.

For simplicity, let us examine the first-order case:

$$(\dot{\bar{y}}_m - \dot{y}) = a_m(\bar{y}_m - y) + [b_m\bar{u}_m - gu + (a_m - a)y]$$

and we would like to set

$$u(t) = \hat{f}_\epsilon(g)K\bar{x}(t) = \sum_{i=0}^{q} c_i g^i [b_m\bar{u}_m(t) + (a_m - a)y(t)].$$

Recall that we've chosen $p > 2(2q+1) = 6$ and $h > 0$, and have set $T = ph$. First we look at the first period $[0, T)$. In the first step we would like to construct an approximation of

$$\sum_{i=0}^{1} c_i \underbrace{g^i[b_m\bar{u}_m(0) + (a_m - a)y(0)]}_{=:\phi_i(0)}.$$

Now we need to approximate $ay(0)$. Suppose that we initially set

$$\bar{u}(t) = 0, \quad t \in [0, 2h),$$

so it follows that

$$\begin{aligned} y(h) &= e^{ah}y(0) = [1 + ah + \frac{a^2h^2}{2} + \mathcal{O}(h^3)]y(0), \\ y(2h) &= e^{2ah}y(0) = [1 + 2ah + \frac{a^2(2h)^2}{2} + \mathcal{O}(h^3)]y(0). \end{aligned}$$

Hence

$$\frac{1}{2h}[-y(2h) + 4y(h) - 3y(0)] = ay(0) + \mathcal{O}(h^2)y(0).$$

So at this point we have a good estimate of $a_m y(0)$ and $ay(0)$, with the quality of the estimate improving as $h \to 0$. Hence, we can make a good estimate of

$$\phi_0(0) := b_m \bar{u}_m(0) + (a_m - a)y(0),$$

namely

$$\begin{aligned} \hat{\phi}_0(0) &= b_m \bar{u}_m(0) + a_m y(0) - \frac{1}{2h}[-y(2h) + 4y(h) - 3y(0)] \\ &= \phi_0(0) + \mathcal{O}(h^2)y(0), \end{aligned}$$

so

$$\phi_0(0) = \hat{\phi}_0(0) + \mathcal{O}(h^2)y(0).$$

To form an estimate of $\phi_1(0) = g\phi_0(0)$, we will carry out some experiments. With $\rho > 0$ a scaling factor (we make this small so that it does not disturb the system very much), set

$$u(t) = \rho\hat{\phi}_0(0), \quad t \in [2h, 4h).$$

Of course, in completing this experiment we have given the state a boost. This can be largely undone by applying

$$u(t) = -\rho\hat{\phi}_0(0), \quad t \in [4h, 6h),$$

Then

$$\begin{aligned} y(4h) &= e^{4ah}y(0) + (\int_0^{2h} e^{a\tau}d\tau)gu(2h) \\ &= [1 + 4ah + \frac{(4ah)^2}{2} + \mathcal{O}(h^3)]y(0) + [2h + a\frac{(2h)^2}{2} + \mathcal{O}(h^3)]g\hat{\phi}_0(0), \\ y(3h) &= e^{3ah}y(0) + (\int_0^{h} e^{a\tau}d\tau)gu(2h) \\ &= [1 + 3ah + \frac{(3ah)^2}{2} + \mathcal{O}(h^3)]y(0) + [h + a\frac{h^2}{2} + \mathcal{O}(h^3)]g\hat{\phi}_0(0). \end{aligned}$$

Hence

$$g\hat{\phi}_0(0) = \frac{1}{-2h}[y(4h) - 4y(3h) + 6y(2h) - 4y(h) + y(0)] + \mathcal{O}(h^2)y(0),$$

so define

$$\begin{aligned}\hat{\phi}_1(0) &= \frac{1}{-2h}[y(4h) - 4y(3h) + 6y(2h) - 4y(h) + y(0)] \\ &= \phi_1(0) + \mathcal{O}(h^2)y(0).\end{aligned}$$

which is a good estimate of $\phi_1(0)$. At the end of the estimation phase, we are at $t = 6h$, and we have estimate of $\phi_1(0)$ to form our control signal to be applied during the control phase. We now set u during the control phase to be

$$u(t) = \frac{p}{p-6}\hat{\phi}_1(0), \quad t \in [6h, ph).$$

It follows that

$$\begin{aligned}y(ph) = y(T) &= e^{pah}y(0) + Tg\hat{\phi}_1(0) + \mathcal{O}(h)y(0) + \mathcal{O}(h^2)\bar{y}(0) \\ &= e^{aT}y(0) + Tg\hat{f}(g)[b_m\bar{u}_m(0) + (a_m - a)y(0)] + \\ &\quad \mathcal{O}(h)y(0) + \mathcal{O}(h^2)\bar{y}(0) \\ &\approx e^{a_mT}y(0) + \int_0^T e^{a_m(T-\tau)}b_m\bar{u}_m(\tau)d\tau,\end{aligned}$$

as desired. Figure 7.4 shows the time-domain simulation results for the controller [20] and the proposed controller. In Figure 7.5 the operator gains are illustrated. As these figures show, this method worsens both the performance and the noise rejection.

Although the new approach has a smaller estimation error, the response has not been improved. The reason is that, although the smaller estimation error improves performance, it requires a longer estimation phase which worsens the performance. In fact, there is a trade-off between the estimation error and the length of the estimation phase.

7.2.3 Using MMSE Estimation

In all previous approaches, estimation is carried out in only one way. The following approach is based on getting different samples in order to approximate. Let $n \in N[0, \delta^2]$ define a realization of a normal random vector with mean value 0 and the variance δ^2, representing the noise signal. Now we begin with a first-order system and the controller [20]:

$$u(t) = 0, \;\; t \in [kT, kT + 2h),$$

which gives

$$\begin{aligned}y(kT + h) &= (1 + ah)y(kT) + \mathcal{O}(h^2), \\ y(kT + 2h) &= (1 + 2ah)y(kT) + \mathcal{O}(h^2).\end{aligned}$$

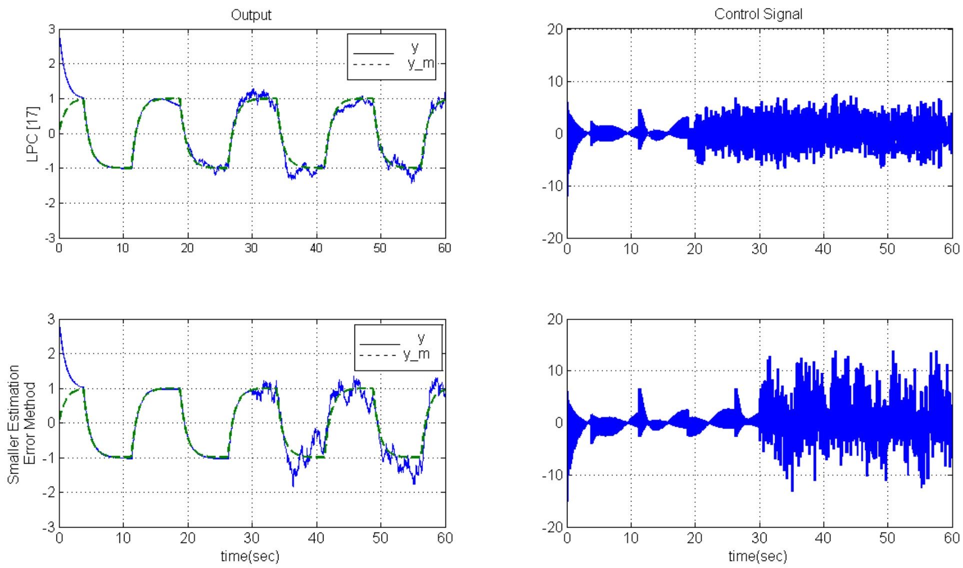

Figure 7.4: The time-domain simulation results for the LPC [20] and the controller with smaller estimation error.

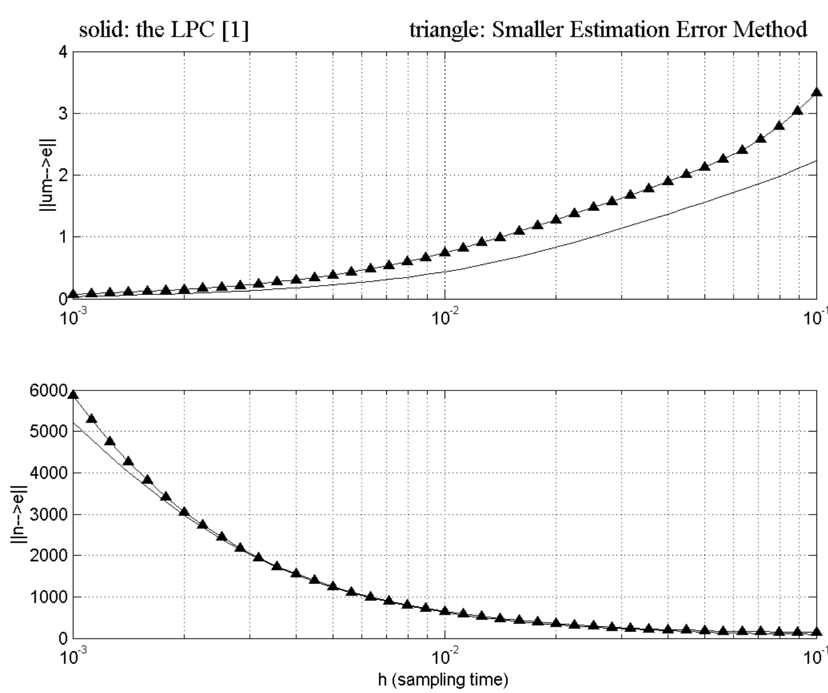

Figure 7.5: The performance, $||u_m \to e||$, and the noise rejection behaviour, $||n \to e||$, of the LPC [20] and the controller with smaller estimation error.

Let y_n denote the measured output. The measured output is a combination of the real output and the noise; in fact, what we get as the measured output is as follows

$$\begin{aligned} y_n(kT) &= y(kT) + n(kT), \\ y_n(kT+h) &= y(kT+h) + n(kT+h), \\ y_n(kT+2h) &= y(kT+2h) + n(kT+2h). \end{aligned}$$

We can approximate $ay(kT)$ in three different ways as follows:

$$\begin{aligned} S_1(kT) &= \frac{y_n(kT+2h) - y_n(kT+h)}{h} = \frac{y(kT+2h) + n(kT+2h) - y(kT+h) - n(kT+h)}{h} \\ &= ay(0) + \underbrace{\frac{n(kT+2h) - n(kT+h)}{h}}_{:=\bar{n}_1(kT)} + \mathcal{O}(h), \end{aligned}$$

$$\begin{aligned} S_2(kT) &= \frac{y_n(kT+h) - y_n(kT)}{h} = \frac{y(kT+h) + n(kT+h) - y(kT) - n(kT)}{h} \\ &= ay(kT) + \underbrace{\frac{n(kT+h) - n(kT)}{h}}_{:=\bar{n}_2(kT)} + \mathcal{O}(h), \end{aligned}$$

$$\begin{aligned} S_3(kT) &= \frac{y_n(kT+2h) - y_n(kT)}{2h} = \frac{y(kT+2h) + n(kT+2h) - y(kT) - n(kT)}{2h} \\ &= ay(kT) + \underbrace{\frac{n(kT+2h) - n(kT)}{2h}}_{:=\bar{n}_3(kT)} + \mathcal{O}(h), \end{aligned}$$

which can be rewritten as

$$\underbrace{\left[\begin{array}{ccc} S_1(kT) & S_2(kT) & S_3(kT) \end{array}\right]^T}_{:S^T(kT)} \approx \underbrace{\left[\begin{array}{ccc} 1 & 1 & 1 \end{array}\right]^T}_{:=H^T} ay(kT) + \\ \left[\begin{array}{ccc} \bar{n}_1(kT) & \bar{n}_2(kT) & \bar{n}_3(kT) \end{array}\right]^T,$$

with $\bar{n}_1 \in N[0, 2\frac{\delta^2}{h^2}]$, $\bar{n}_2 \in N[0, 2\frac{\delta^2}{h^2}]$, $\bar{n}_3 \in N[0, \frac{\delta^2}{2h^2}]$, random vectors with mean value 0. Now let us define the weighting matrix

$$W = \left[\begin{array}{ccc} 2\frac{\delta^2}{h^2} & 0 & 0 \\ 0 & 2\frac{\delta^2}{h^2} & 0 \\ 0 & 0 & \frac{\delta^2}{2h^2} \end{array}\right].$$

If we use the method of MMSE [35] estimation, the best estimation of $ay(0)$ is as follows:

$$\begin{aligned} Est[a\hat{y(kT)}] &= (H^T W^{-1} H)^{-1} H^T W^{-1} S(kT) \\ &= 0.25 S_1(kT) + 0.25 S_2(kT) + 0.5 S_3(kT). \end{aligned}$$

In order to construct the term ϕ_0, we need the term $y(kT)$. Although we have this term, we use the same method to find the best estimation. The term $\phi_1 = b\phi_0$ is estimated by this method as well.

To compare the performance of this controller, two simulations are carried out. The results of the time-domain simulation are shown in Figure 7.6. As the figure illustrates, the MMSE has better noise rejection. The operator gains are illustrated in Figure 7.7. As the figure shows, the new method improves the noise rejection but worsens the performance.

In summary, since the MMSE uses different samples, which requires longer estimation phase, the performance drops down. However, this method improves the noise rejection.

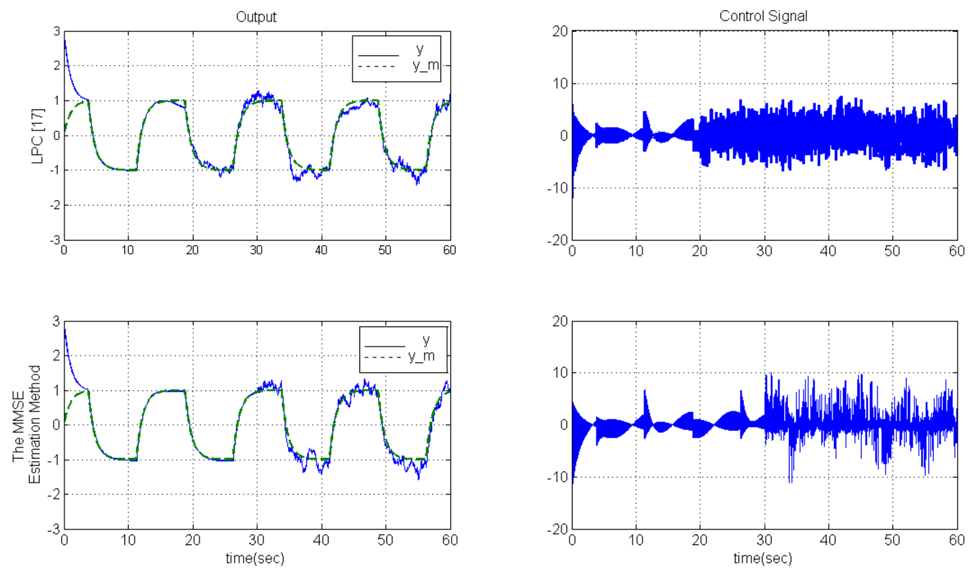

Figure 7.6: The response and the control signal of the closed-loop system with the controller [20] and the MMSE estimating method.

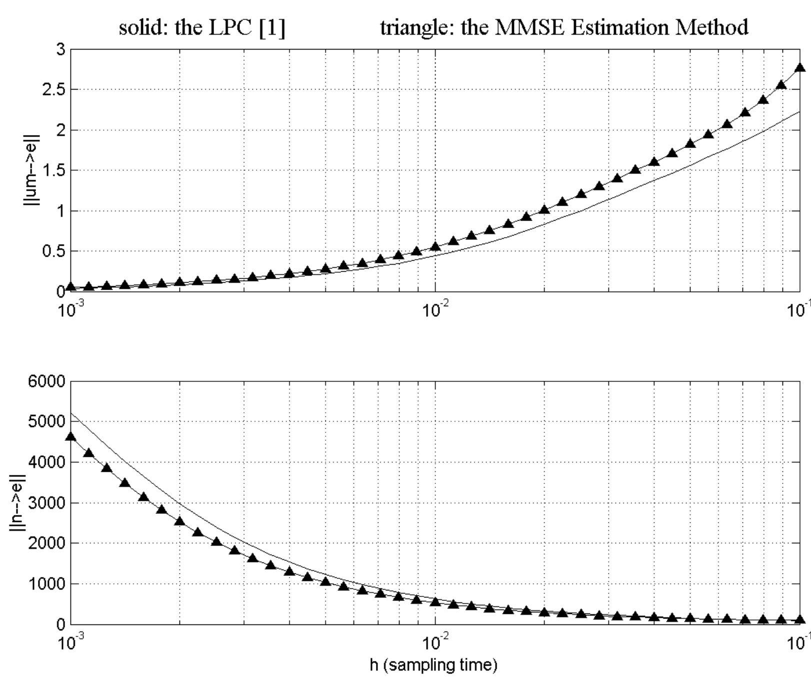

Figure 7.7: The performance, $||u_m \rightarrow e||$, and the noise rejection behaviour, $||n \rightarrow e||$, of the controller [20] and the MMSE estimating method.

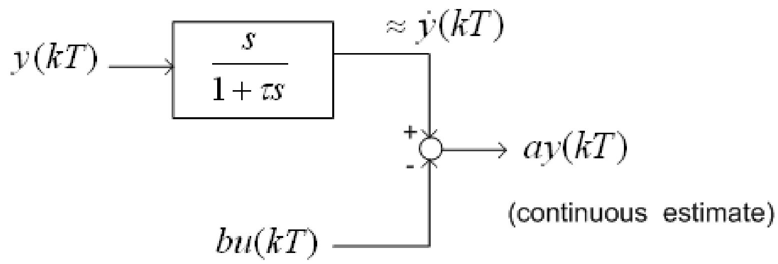

Figure 7.8: The procedure of the continuous estimate of the unknown term $ay(kT)$.

7.2.4 The Vigorous Probing Controller

In this approach, we use two techniques: (i) obtaining a continuous approximation, instead of a discrete one, of the term $\hat{\phi}_0(kT)$, and (ii) applying larger values of the free parameter for the control signal.

For simplicity, consider a first-order plant. First, we need to obtain a continuous estimate of the term $\phi_0(kT)$. To do this we have

$$\begin{aligned} \phi_0(kT) &= (a_m - a)y(kT) + b_m u_m(kT) \\ &= \underbrace{-ay(kT)}_{\text{continuous estimate}} + \underbrace{a_m y(kT) + b_m u_m(kT)}_{\text{measureable}}. \end{aligned}$$

Now, the term $ay(t)$ can be written as:

$$\dot{y}(t) = ay(t) + bu(t) \tag{7.1}$$
$$\Rightarrow ay(t) = \dot{y}(t) - bu(t). \tag{7.2}$$

So to have a continuous estimate of $ay(kT)$, we can simply estimate the two terms $\dot{y}(kT)$ and $bu(kT)$. Figure 7.8 shows how the continuous estimate can be carried out by a filter, which is a differentiator rolled off at high frequencies; in this filter, τ should be small.

Now, we need to have an estimation of $bu(kT)$. Let us define the term

$$b_m \bar{u}_m(kT) + (a_m - a)y(kT) := u^*(kT).$$

Recall that in the proposed controller [20], first we estimate $\phi_0(kT) = u^*(kT)$ and then we define $\phi_1(kT)$ by

$$\phi_1(kT) = bu^*(kT).$$

In (7.2), we need the term $bu(kT)$ so we also define $\phi_1(kT) = bu^*(KT)$.

In the next step, we probe the plant more vigorously when trying to estimate $b^i\hat{\phi}_0(kT)$. With $\sigma \in (0,1)$, we set

$$u(t) = \begin{cases} 0 & t \in [kT, kT + mh), \\ \rho h^{-\sigma}\hat{\phi}_{i-1}(kT) & t \in [kT + (2i-1)mh, kT + 2imh),\ i = 1, \cdots, q, \\ -\rho h^{-\sigma}\hat{\phi}_{i-1}(kT) & t \in [kT + 2imh, kT + (2i+1)mh),\ i = 1, \cdots, q. \end{cases} \tag{7.3}$$

Finally, we combine two the ideas together: the continuous estimate of $ay(kT)$ and the vigorous control signal (7.3).

Figure 7.9 presents the time-domain simulation results for the controller [20] and the proposed vigorous probing controller with different values of σ. In this simulation, we choose $\tau = 0.04$ in the estimator. As the figure shows, the vigorous probing controller has good noise rejection, especially for higher values of σ, but the control signal is big, which is a negative aspect of this method. In Figure 7.10, we plot operator gains; as expected, vigorous probing controller has better noise rejection, especially for bigger values of σ, but poorer performance.

To sum up, probing the control signal with larger values improves the noise rejection, worsens the performance, and makes a larger control signal.

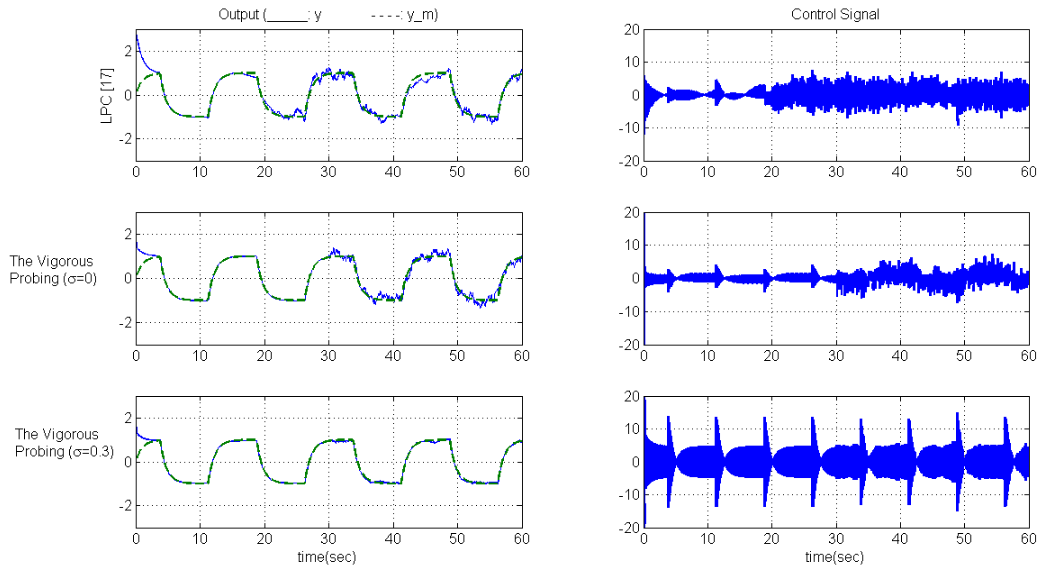

Figure 7.9: The output and control signal of the closed-loop system with the controller [20], and the vigorous probing controller for different values of σ.

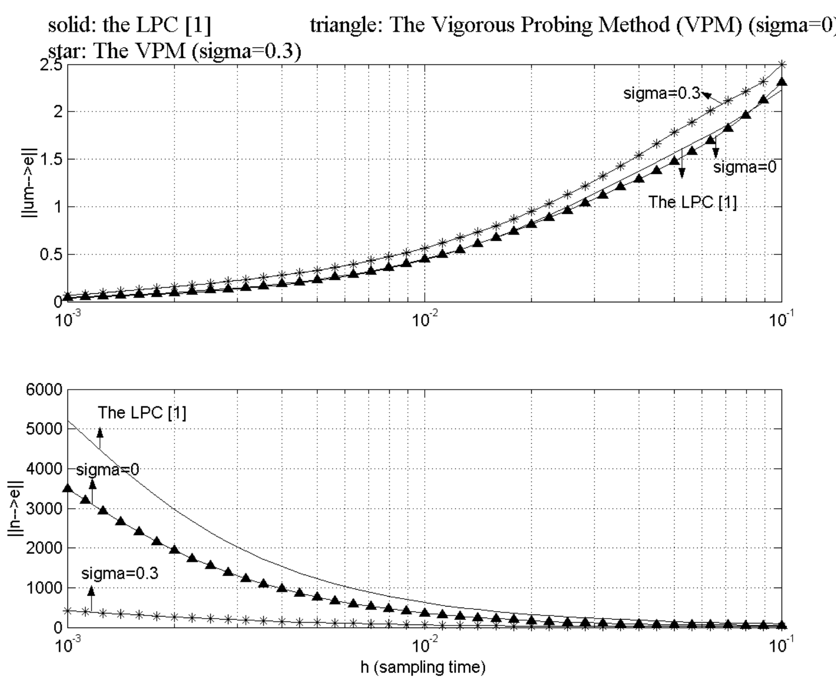

Figure 7.10: The performance, $||u_m \to e||$, and the noise rejection behaviour, $||n \to e||$, of the controller [20] and the vigorous probing controller with different values of σ.

7.3 The Noise Tolerant Redesigned LPC

In Chapter 5 we saw that in the case that the high frequency gain does not change sign, the redesigned LPC significantly improves the noise tolerance, but when the sign change is allowed, the noise rejection is not improved too much. The objective of this subsection is to show that if we increase the value of the scaling factor ρ, but not too much, then the new controller of Chapter 5 yields a larger control signal and poorer performance but improves the noise tolerance.

7.3.1 Examples

In this subsection, two examples are presented to illustrate the proposed methodology.

Time Domain Example

We adopt the setup of Example 2 of Chapter 5, Section 5.4.2; the plant model is first order:

$$\dot{y}(t) = a(t)y(t) + g(t)u(t).$$

The set of plant uncertainty is given by

$$\Gamma = \{\begin{bmatrix} a \\ g \end{bmatrix} \in \mathbf{R}^2 : a \in [-1, 1],\ g^2 \in [1, 1.4]\},$$

$$\mathcal{P} = \mathcal{P}(n = 1,\ m = 1,\ \Gamma, \mu_1 = 1, T_0 = 5, \gamma_0 = 1, \lambda_0 = -5).$$

While the reference model is

$$\dot{\bar{x}}_m = -\bar{x}_m + \bar{u}_m,$$

the anti-aliasing filter is

$$\dot{\bar{u}}_m = -50\bar{u}_m + 50u_m.$$

As the LPC [20], for approximating the polynomial $\frac{1}{g}$, we use the approximation of:

$$\hat{f}_{0.01}(g) = 2.1647g - 1.5153g^3 + 0.3433g^5,$$

so $q = 5$, and we choose $p = 25 > 2m(2q + 1) = 22$, i.e. the Control Phase is 56% of the controller period. For the redesigned LPC, since the compact set $\mathcal{G} := [-1.4, -1] \cup [1, 1.4]$ contains both positive and negative values, we can apply Remark 5.2 and set $\hat{f}_\varepsilon(g) = \frac{1}{3}g$ and $p = 3$. We also set $h = 0.01$, so in the LPC [20] $T = 0.25$ and in the redesigned LPC, $T = 0.03$. Moreover, in the LPC [20], we set $\rho = 1$, and for the redesigned LPC we use different values of ρ.

Figure 7.11 shows the simulation results with $y_0 = 3$, $\bar{u}_{m_0} = 0$, u_m a square wave given by

$$u_m = sign(\cos(\frac{2\pi t}{15})),$$

and

$$a(t) = \cos(t/2) \text{ and } g(t) = [1.2 + 0.2cos(t/2)] * sign[cos(t/4)].$$

We add a noise signal of the form

h * random sequence uniformly distributed between ± 1

to the output measurement at the time $t = 30$ sec. As shown in Figure 7.11, the noise tolerant redesigned LPC, designed for the case that the plant high frequency gain sign can have both positive and negative values, has much better noise rejection and higher performance when ρ is larger than 1, but not too large.

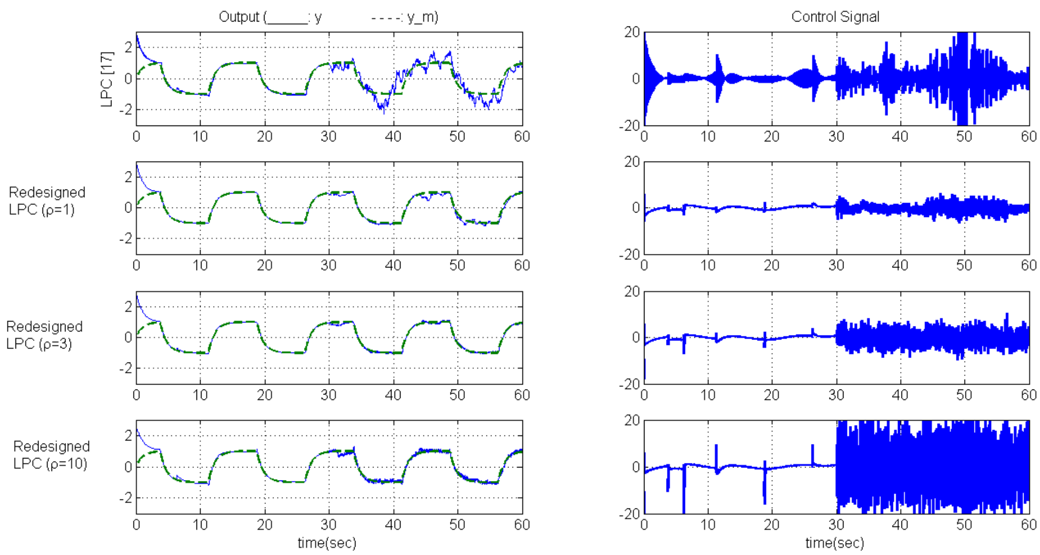

Figure 7.11: The output and control signal of the closed-loop system with the controller [20], and the noise tolerant redesigned LPC for different values of ρ.

Performance and Noise Behaviour

In this simulation we analyze the performance and the noise behaviour of the closed-loop system. We compute the $\mathcal{L}_\infty$ induced norm when the plant parameters are fixed and investigate how the performance and the noise behaviour of the sampled-data system depends on the sampling period. The first computation of the $\mathcal{L}_\infty$ induced norm of the LPC [20] was carried out by Xiaosong [43]. The tool that is used is the *lifting* technique (see Chen and Francis [3] which establishes a strong correspondence between periodic systems and time invariant infinite-dimensional systems.

We inject noise n at the plant output; we let T_{u_m} denote the map from $u_m \longrightarrow e$ and T_n denote the map from $n \longrightarrow e$. We adopt one admissible plant model of the set of plant uncertainty of the time domain example of this subsection as

$$\dot{y}(t) = y(t) + u(t).$$

In the LPC [20], for approximating the polynomial $\frac{1}{g}$, we use the estimation polynomial

$$\hat{f}(g) = 2.1647g - 1.5153g^3 + 0.3433g^5.$$

For the redesigned LPC, the estimation polynomial $\hat{f}_\varepsilon(g) = \frac{g}{3}$ is used. With the controller designed as above but with h free, in Figure 7.12, we plot $||T_{u_m}||$ and $||T_n||$ as a function of h. As expected, for both controllers, while the former goes to zero, the latter goes to infinity as $h \longrightarrow 0$.

As the figure shows, increasing ρ results in better noise tolerance, but deteriorates the performance. We see that while increasing ρ from 1 to $\rho = 3$ or $\rho = 5$ remarkably improves noise tolerance, it slightly worsens the performance. Meanwhile increasing ρ from 5 to $\rho = 10$ results in minor noise tolerance improvement and performance deterioration. These results show that the noise tolerant redesigned LPC with ρ larger than 1 but not too large values significantly improves performance and noise tolerance compared with the LPC [20].

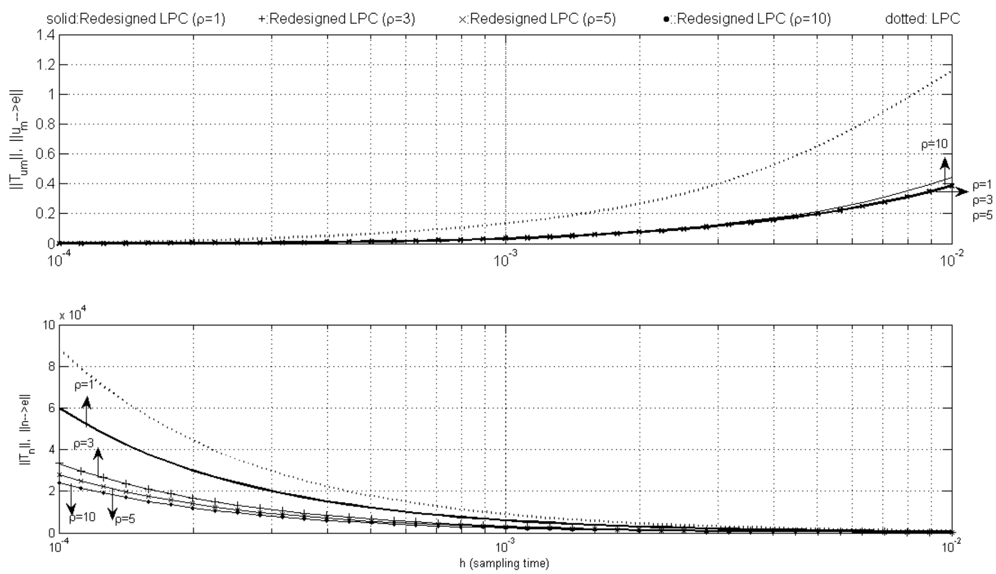

Figure 7.12: The $||u_m \to e||$ and $||n \to e||$ for the LPC [20] and the noise tolerant redesigned LPC with different values of ρ

7.4 Summary and Concluding Remarks

Although the LPC [20] has several advantages, it suffers from poor noise tolerance. In this controller, in order to achieve the desirable features, a small sampling period is used. This results in large controller gains and consequently poor noise tolerance. There is a clear trade-off between noise rejection and performance. In this chapter, we presented four different ideas of noise reduction of the measurement noise and examined a way to improve the noise tolerance of the redesigned controller of Chapter 5.

In the first method, we applied a low-pass filter after the output. Using a low pass filter to reduce the noise effect of a signal works only when the signal has enough time to settle down. In this approach, we take a few samples of the output in a short time. Thus, using the filter does not improve the noise rejection. This approach also leads to poorer performance, since using a low-pass filter increases the relative degree by one and consequently results in a longer Estimation Phase.

In the second approach, we redesigned the LPC to obtain a smaller estimation error, allowing us to choose larger sampling time and achieving better noise rejection. Although the new approach has a smaller estimation error, the response has not been improved. The reason is that, although the smaller estimation error improves performance, it requires a longer Estimation Phase, which worsens the performance. In fact, there is a trade-off between the estimation error and the length of Estimation Phase.

In the third approach, the MMSE estimators are used to minimize the effect of noise. The MMSE approach improves the noise rejection but worsens the performance, since this method uses different samples, requiring a longer Estimation Phase.

In the last approach, we probed the control signal with larger values in order to achieve better noise tolerance. Probing the control signal with larger values, improves the noise rejection but worsens the performance and makes a larger control signal.

In short, none of the attempts at modifying the LPC [20] were very successful at improving the noise tolerance-tracking performance trade-off. Thus, we examined the redesigned controller of Chapter 5. In the case that the high frequency gain does not change sign, this controller is significantly noise tolerant. In the case that the high frequency gain sign can have both positive and negative values, we considered the free parameter ρ and determined that for large (but not too large) values we could improve the noise tolerance with minimal change in the tracking performance.

Chapter 8

Conclusions

8.1 A Summary

Adaptive control is a widely used approach which deals with plants with uncertain or time-varying parameters. An adaptive controller typically consists of an LTI compensator together with a tuning mechanism which adjusts the compensator gains to match the plant. Because of the adjustment law, a typical adaptive controller is nonlinear. This nonlinearity often results in poor transient closed-loop behaviour and large control signal. Although the initial motivation for adaptive control was to cope with time-varying plant parameters, most classical adaptive controllers cannot handle rapidly changing parameters.

Recently Miller [20] proposed an LPC as a new approach in the field of model reference adaptive control. This approach uses a linear sampled-data periodic controller; unlike classical adaptive control schemes, which are based on parameter estimation, this method directly estimates the control signal. The achieved controller has the following advantages:

- it can handle time-varying plant parameters,
- the transient behaviour of the system is improved; immediate tracking can be achieved if the initial conditions of the plant and controller are the same, and if the initial conditions are not the same, then the deviation goes exponentially to zero,
- the control signal is not large and it can be as close as desired to the *ideal control signal* (the ideal control signal is the one obtained if the plant parameters and state were known and the ideal LTI compensator would be applied; this signal is modest in size).

Although the LPC [20] has the above desirable features, it has some imperfections:

- during each controller period the control signal takes different values; since the controller period is small, this requires fast actuators,

- in order to achieve the desired tracking, a small sampling period is used which results in large controller gains; this may lead the system to poor noise tolerance; in fact, there is a trade-off between the desired tracking and noise tolerance,
- the plant relative degree must be known.

In this PhD research, we extended this work in several directions.

First of all, in [20] the time-domain ∞-norm is used to measure signal size. However, the time-domain 2-norm is an equally (if not more) common way to measure signal size. Here we extended the approach so that results can be obtained in the 2-norm setting which are comparable to those of [20]. This new setting requires a structural change in the control law together with new proofs of key steps.

Second of all, in [20] the control signal moves vigorously, in particular during the so-called Estimation Phase, when probing takes place; the control signal jumps rapidly from one value to another. In this research the controller has been redesigned to reduce the size and number of the jumps. The approach is as follows. In the signal control law [20], the controller is periodic with each period consisting of two phases: in Estimation Phase the *ideal control law* is estimated, while in Control Phase a naturally scaled version of this estimate is applied. In this work, we estimated the *change* in the ideal control signal law between periods, which reduces the size of the probing. We also carried out the probing more efficiently, to reduce the number of jumps in the control signal as well. While proving this approach requires a good deal of effort, the results are very nice: the control signal is smoother, the controller can handle large parameter variation, the performance is better (for a given sampling period), and in the case that the sign of high frequency gain is fixed the closd-loop system is remarkably noise tolerant.

The third part of this PhD thesis has been focussed on the plant relative degree assumption. One of the undesired aspects of the LPC is that the plant relative degree should be known. In this work, we show that under some restrictive assumptions, this requirement can be relaxed to that of requiring an upper bound of the plant relative degree, at least from the point of view of providing stability.

Finally, to improve the noise rejection, four different approaches were presented. In the first approach, a low pass filter has been used at the plant output. In the second approach, we modified the LPC in order to obtain a smaller estimation error, which allows us to choose larger sampling times and consequently better noise tolerance. While the idea of MMSE estimation has been used in the third approach to minimize the effect of noise, the fourth approach has been based on applying a probing signal with a larger size in order to obtain better noise rejection. Finally, we demonstrated that the degree of freedom present in the redesigned controller, mentioned above, can be used to achieve the advantages of both methods. The *noise tolerant redesigned LPC* attains better noise rejection and performance as well as a better behaved control signal, as compared to the original controller of [20].

8.2 Future Research

In Chapter 5, we proposed the redesigned LPC with considerable desirable features, such as a smoother control signal, higher noise tolerance, and better performance. These advantages make an appropriate controller of the redesigned LPC for practical adaptive control applications: a smoother control signal alleviates the requirement of fast actuators, higher noise tolerance is of great benefit in real-life control problems which demand noisy environments, and finally better performance eases the necessity of fast sampling which is an advantageous aspect in practical control purposes. Implementation of the redesigned LPC for control of time-varying systems, such as fuel consumption, would be an extension of this research work.

The LPC [20] uses a *linear sampled-data periodic* controller; unlike classical adaptive control methods, which are based on parameter estimation, this method directly estimates the control signal. We modified this controller to achieve the redesigned LPC of Chapter 5. The modification is based on the idea of combining the Estimation and Control Phases. This idea might be used on [21, 22, 23] which apply the same point of view of that of the LPC [20].

In Chapter 6 we modified the redesigned LPC of Chapter 5 and showed that with the new setting, the controller provides stability for plants with the same/lower relative degrees. We could perhaps redesign/modify this controller to achieve both stability and performance.

In the redesigned LPC of Chapter 5, in the case that the high frequency sign can change, the controller period is three sampling times. We would adopt more effective approach to carry out the control in two sampling times, instead of three. This modification could result in higher noise-tolerance and better performance.

Appendices

.1 APPENDIX A

Before starting the proofs, four technical results are presented, which are used in other proofs. In these results, $B(t)$ is a time-varying vector and $A(t)$ is a time-varying square matrix; we let Φ_A denote the transition matrix corresponding to A.

Lemma A.1 of [17]: Consider the differential equation

$$\dot{\eta}(t) = A(t)\eta(t),$$

with $A \in PC_\infty$ and $c := \|A\|_\infty$. Then

$$\begin{aligned} \|\Phi_A(t, t_0)\| &\leq e^{c(t-t_0)}, \\ \|\Phi_A(t, t_0) - I\| &\leq (e^{c(t-t_0)} - 1), \quad t \geq t_0. \end{aligned}$$

Lemma A.2 of [17]: Consider the differential equation

$$\dot{\eta}(t) = [A(t) + \Delta(t)]\eta(t),$$

and suppose that there exist $c > 0$ and $\lambda < 0$ satisfying

$$\|\Phi_A(t, t_0)\| \leq ce^{\lambda(t-t_0)}, \quad t \geq t_0.$$

With $\lambda_1 \in (\lambda, 0)$ and $\|\Delta\|_\infty \leq \frac{|\lambda - \lambda_1|}{2c}$, it follows that

$$\|\Phi_{A+\Delta}(t, t_0)\| \leq 2ce^{\lambda_1(t-t_0)}, \quad t \geq t_0.$$

Lemma A.3: Consider the time-varying system

$$\dot{\eta}(t) = A(t)\eta(t) + B(t)u(t), \quad \eta(t_0) = \eta_0.$$

Suppose that $b := \sup_{t \geq 0} ||B(t)|| < \infty$, and that there exist $c > 0$ and $\lambda < 0$ satisfying

$$\|\Phi_A(t, t_0)\| \leq ce^{\lambda(t-t_0)}, \qquad t \geq t_0.$$

Then

$$\|\eta(t)\| \leq ce^{\lambda(t-t_0)} \|\eta(t_0)\| + \frac{bc}{\sqrt{2|\lambda|}} \|u(t)\|_{2\,[t_0,t)}, \quad t \geq t_0.$$

Proof:

Since the system is linear, we can partition $\eta(t)$ into its zero-input-response and zero-state-response

$$\eta(t) = \eta_{ZIR}(t) + \eta_{ZSR}(t).$$

The zero-input-response satisfies

$$\|\eta_{ZIR}(t)\| \leq ce^{\lambda(t-t_0)} \|\eta(t_0)\|, \quad t \geq t_0. \tag{1}$$

While the zero-state-response satisfies

$$\|\eta_{ZSR}(t)\| \leq \int_{t_0}^{t} ce^{\lambda(t-\tau)} b \ \|u(\tau)\| \, d\tau, \quad t \geq t_0.$$

If we apply the Cauchy Swartz inequality to this integral, we obtain

$$\|\eta_{ZSR}(t)\| \leq \frac{bc}{\sqrt{2|\lambda|}} \|u\|_{2\ [t_0,t)}, \quad t \geq t_0. \tag{2}$$

Combining this equation with (1) yields

$$\|\eta(t)\| \leq ce^{\lambda(t-t_0)} \|\eta(t_0)\| + \frac{bc}{\sqrt{2|\lambda|}} \|u\|_{2\ [t_0,t)}, \quad t \geq t_0.$$

□

Lemma A.4: Consider the time-varying system

$$\dot{\eta}(t) = A(t)\eta(t) + B(t)u(t).$$

Suppose that $b := \sup_{t\geq 0} ||B(t)|| < \infty$, and that there exist $c > 0$ and $\lambda < 0$ satisfying

$$\|\Phi_A(t, t_0)\| \leq ce^{\lambda(t-t_0)}, \qquad t \geq t_0.$$

Then

$$\|\eta(t)\|_{2\ [t_0,\infty)} \leq \frac{c}{\sqrt{2|\lambda|}} \|\eta(t_0)\| + \frac{bc}{|\lambda|} \|u\|_{2\ [t_0,\infty)}.$$

Proof:

We adopt the method of the previous proof. $\eta_{ZIR}(t)$ and $\eta_{ZSR}(t)$ are the zero-input-response and zero-state-response, respectively. For the zero-input-response we have

$$\|\eta_{ZIR}(t)\| \leq ce^{\lambda(t-t_0)} \|\eta(t_0)\|, \quad t \geq t_0,$$

so

$$\|\eta_{ZIR}(t)\|_{2\ [t_0,\infty)} \leq \frac{c}{\sqrt{2|\lambda|}} \|\eta(t_0)\|. \tag{3}$$

For the zero-state-response we have

$$\|\eta_{ZSR}(t)\| \le \int_{t_0}^{t} ce^{\lambda(t-\tau)} b\, ||u(\tau)|| d\tau.$$

We can regard the RHS output of an LTI system with transfer function $\frac{bc}{s-\lambda}$, driven by $||u(t)||$, so using Parseval's Theorem it follows that

$$\|\eta_{ZSR}\|_{2\ [t_0,\infty)} \le \frac{bc}{|\lambda|} \|u\|_{2,\ [t_0,\infty)}\,.$$

If we combine this equation with (3), then it follows that

$$\|\eta(t)\|_{2\ [t_0,\infty)} \le \frac{c}{\sqrt{2\,|\lambda|}} \|\eta(t_0)\| + \frac{bc}{|\lambda|} \|u\|_{2\ [t_0,\infty)}\,.$$

□

Proof of Proposition 4.1:

First we examine Φ_{cl}. Let Φ_1 denote the transition matrix associated with A_1; it follows from Assumption 6 of Chapter 3 that there exist constants γ_0 and $\lambda_0 < 0$ so that, for every $\bar{\theta} \in \bar{\mathcal{P}}$, we have

$$||\Phi_1(t,\tau)|| \le \gamma_0 e^{\lambda_0(t-\tau)}, \quad t \ge \tau.$$

Observe that $-\sigma < \lambda_m$ by hypothesis and that A_m and $\bar{A}_2$ have all eigenvalues with a real part less than λ_m. Now let $\lambda \in (\max\{\lambda_0, \lambda_m\}, 0)$; with $\lambda_1 \in (\max\{\lambda_0, \lambda_m\}, \lambda)$, it follows from the structure of A_{cl} that there exists a $\gamma_1 > 0$ so that for every $\bar{\theta} \in \bar{\mathcal{P}}$, we have

$$||\Phi_{cl}(t,\tau)|| \le \gamma_1 e^{\lambda_1(t-\tau)}, \quad t \ge \tau \ge 0. \tag{4}$$

$$\frac{\gamma_1}{\sqrt{2|\lambda_1|}} \|\overline{x}_0\| + \frac{\gamma_1||\bar{E}||}{|\lambda_1|} \|u_m\|_2 \le \underbrace{\max\{\frac{\gamma_1}{\sqrt{2|\lambda_1|}}, \frac{\gamma_1||\bar{E}||}{|\lambda_1|}\}}_{:=\gamma_2}(\|\overline{x}_0\| + \|u_m\|_2). \tag{5}$$

Now define

$$\Delta_\varepsilon(t) := \overline{B}[g(t)\hat{f}_\varepsilon(g(t)) - 1]K(t);$$

which means that

$$A^\varepsilon_{cl}(t) := \overline{A}_{cl} + \Delta_\varepsilon(t).$$

By definition of $\hat{f}_\varepsilon$ and the boundedness of the plant parameters, there exists a constant γ_3 so that for every $\varepsilon > 0$ we have

$$\|\Delta_\varepsilon(t)\| \le \gamma_3\varepsilon, \quad \bar{\theta} \in \bar{\mathcal{P}}, \tag{6}$$

so it follows from Lemma A.2 and (4) that there exist constants $\gamma_4 > 0$ and $\varepsilon_1 > 0$ so that for every $\varepsilon \in (0, \varepsilon_1)$ and $\bar{\theta} \in \bar{\mathcal{P}}$, we have

$$\|\Phi_{cl}^{\varepsilon}(t,\tau)\| \leq \gamma_4 e^{\lambda(t-\tau)}, \;\; t \geq \tau \geq 0. \tag{7}$$

Now let us define

$$\bar{\bar{x}}^{\varepsilon}(t) := \overline{x}^{0}(t) - \bar{x}^{\varepsilon}(t).$$

Then

$$\dot{\bar{\bar{x}}}^{\varepsilon}(t) = \overline{A}_{cl}^{\varepsilon}(t)\bar{\bar{x}}^{\varepsilon}(t) - \Delta_{\varepsilon}(t)\overline{x}^{0}(t).$$

Using this equation, Lemma A.4, (5), (6), (7), and the fact that $\overline{x}_0 = 0$, it follows that for all $\varepsilon \in (0, \varepsilon_1)$ and $\bar{\theta} \in \bar{\mathcal{P}}$, we have

$$\|\bar{\bar{x}}^{\varepsilon}\|_2 \leq \underbrace{\frac{\gamma_3\gamma_4}{\sqrt{2\,|\lambda|}}}_{:=\gamma_5} \varepsilon \,\|\overline{x}_0\|_2 \,.$$

If we combine this equation with (5) we have that, for all $\varepsilon \in (0, \varepsilon_1)$ and $\bar{\theta} \in \bar{\mathcal{P}}$:

$$\|\bar{\bar{x}}^{\varepsilon}\|_2 \leq \gamma_2\gamma_5\varepsilon(\|\overline{x}_0\| + \|u_m\|_2). \tag{8}$$

Since the map of $u_m \to \bar{e}$ is zero, if we multiply both sides of this inequality by $\|\bar{C}\|$, the bound on the error follows immediately. Last of all, using (5) and (8) it follows that for all $\varepsilon \in (0, \varepsilon_1)$ and $\bar{\theta} \in \bar{\mathcal{P}}$, we have

$$\|\bar{x}^{\varepsilon}\|_2 \leq (\gamma_2 + \gamma_2\gamma_5\varepsilon_1)(\|\overline{x}_0\| + \|u_m\|_2).$$

□

Proof of Lemma 4.1 (KEL-4):

We have

$$\dot{\bar{x}}(t) = \bar{A}(t)\bar{x}(t) + g(t)\bar{B}u + \bar{E}u_m(t), \quad t \in [t_0, t_0 + (m+1)h). \tag{9}$$

Define

$$\begin{aligned}\gamma_1 \;\; := \;\; & \sup_{\bar{\theta}\in\bar{\mathcal{P}}}(\|\bar{A}\|_\infty + ||g||_\infty\|\bar{B}\| + \|\bar{E}\| + \\ & \quad ||f_2||_\infty + ||c_1||_\infty + ||g||_\infty + ||\dot{f}_2||_\infty + ||\dot{c}_1||_\infty + ||\dot{g}||);\end{aligned}$$

it follows from the standing assumptions that γ_1 is finite. (Notice that the ∞-norm contains an essential supremum, so that infinite derivatives at the discontinuities do not show up when computing γ_1.)

Let us first prove the first bound. Let $\Phi(t,\tau)$ denote the transition matrix associated with $\bar{A}(t)$. From Lemma A.1 we have that

$$\|\Phi(t,\tau)\| \leq e^{\gamma_1(t-\tau)}, \;\; \|\Phi(t,\tau) - I\| \leq e^{\gamma_1(t-\tau)} - 1, \;\; t \geq \tau \geq 0.$$

If we solve (9) and apply the Cauchy-Schwartz inequality, we see that for small h we have

$$\begin{aligned}
\|\bar{x}(t)-\bar{x}(t_0)\| &\leq (e^{\gamma_1(t-t_0)}-1)\|\bar{x}(t_0)\| + \int_{t_0}^{t} e^{\gamma_1(t-\tau)}\gamma_1\|\bar{u}\|\,d\tau + \\
&\quad \int_{t_0}^{t} e^{\gamma_1(t-\tau)}\gamma_1\|u_m\|\,d\tau \\
&\leq (e^{\gamma_1(t-t_0)}-1)(\|\bar{x}(t_0)\|+\|\bar{u}\|) + \frac{\gamma_1\sqrt{e^{2\gamma_1(t-t_0)}-1}}{\sqrt{2\gamma_1}}\|u_m\|_{2\,[t_0,t)} \\
&\leq 2\gamma_1 mh(\|\bar{x}(t_0)\|+\|\bar{u}\|) + \gamma_1\sqrt{2m}\sqrt{h}\|u_m\|_{2\,[t_0,t)} \\
&\leq \gamma_2(h\|\bar{x}(t_0)\| + h\|\bar{u}\| + h^{\frac{1}{2}}\|u_m\|_{2\,[t_0,t)}),\quad t\in[t_0,t_0+(m+1)h),
\end{aligned} \tag{10}$$

where $\gamma_2 := 2\gamma_1 m$. This yields the first part of (i) of Lemma 4.1.

Now let us look at v in detail. Observe that

$$\begin{aligned}
\dot{v}(t) &= (\Lambda_2 + b_2 f_2(t))v(t) + b_2 c_1(t)w(t) + g(t)b_2\bar{u} \\
&= \Lambda_2 v(t) + b_2 \underbrace{[f_2(t)v(t) + c_1(t)w(t) + g(t)\bar{u}]}_{=:\psi(t)}.
\end{aligned}$$

Using the bound on v and w from equation (10) and the fact that f_2, c_1, and g are bounded by γ_1, it follows that there exists a constant γ_3 so that for small h we have

$$|\psi(t)| \leq \gamma_3(\|\bar{x}(t_0)\| + \|\bar{u}\| + h^{\frac{1}{2}}\|u_m\|_{2\,[t_0,t)}),\quad t\in[t_0,t_0+(m+1)h). \tag{11}$$

Combining the differential equations for v an $\bar{y}$ yields

$$\begin{bmatrix}\dot{v}(t)\\ \dot{\bar{y}}(t)\end{bmatrix} = \underbrace{\begin{bmatrix}\Lambda_2 & 0\\ \sigma c_2 & -\sigma\end{bmatrix}}_{=\bar{\Lambda}_2}\underbrace{\begin{bmatrix}v(t)\\ \bar{y}(t)\end{bmatrix}}_{=:\bar{v}(t)} + \underbrace{\begin{bmatrix}b_2\\ 0\end{bmatrix}}_{=\bar{b}_2}\psi(t), \tag{12}$$

$$\bar{y}(t) = \underbrace{[0\quad 1]}_{=\bar{c}_2}\begin{bmatrix}v(t)\\ \bar{y}(t)\end{bmatrix}. \tag{13}$$

Solving this differential equation yields

$$\bar{v}(t) = e^{\bar{\Lambda}_2(t-t_0)}\bar{v}(t_0) + \int_{t_0}^{t} e^{\bar{\Lambda}_2(t-\tau)}\bar{b}_2\psi(\tau)\,d\tau,$$

which results in

$$\bar{y}(t) = \bar{c}_2 e^{\bar{\Lambda}_2(t-t_0)}\bar{v}(t_0) + \underbrace{\int_{t_0}^{t} \bar{c}_2 e^{\bar{\Lambda}_2(t-\tau)}\bar{b}_2\psi(\tau)\,d\tau}_{=:\bar{\Delta}(t)}.$$

From the form of $\bar{\Lambda}_2$, $\bar{b}_2$, and $\bar{c}_2$ it follows that

$$\begin{aligned}\bar{c}_2 e^{\bar{\Lambda}_2 t} &= \sum_{i=0}^{m+1} \bar{c}_2 \,\frac{\bar{\Lambda}_2^i t^i}{i!} + \mathcal{O}(t^{m+2}) \\ &= \begin{bmatrix} 1 & t & \cdots & \frac{t^{m+1}}{(m+1)!} \end{bmatrix} \begin{bmatrix} \bar{c}_2 \\ \bar{c}_2\bar{\Lambda}_2 \\ \vdots \\ \bar{c}_2(\bar{\Lambda}_2)^{m+1} \end{bmatrix} + \mathcal{O}(t^{m+2}),\end{aligned}$$

and

$$\begin{aligned}\bar{c}_2 e^{\bar{\Lambda}_2 t}\bar{b}_2 &= \sum_{i=0}^{m} \bar{c}_2 \,\frac{\bar{\Lambda}_2^i \bar{b}_2 t^i}{i!} + \mathcal{O}(t^{m+1}) \\ &= \bar{c}_2 \,\frac{\bar{\Lambda}_2^m \bar{b}_2 t^m}{m!} + \mathcal{O}(t^{m+1}).\end{aligned}$$

Hence, solving the differential equation for $\bar{y}$ results in

$$\bar{y}(t) = \begin{bmatrix} 1 & t-t_0 & \cdots & \frac{(t-t_0)^{m+1}}{(m+1)!} \end{bmatrix} \begin{bmatrix} \bar{c}_2 \\ \bar{c}_2\bar{\Lambda}_2 \\ \vdots \\ \bar{c}_2\bar{\Lambda}_2^{m+1} \end{bmatrix} \bar{v}(t_0) + \mathcal{O}(t^{m+2})\bar{v}(t_0) + \bar{\Delta}(t),$$

From the form of $\bar{\Lambda}_2$, $\bar{b}_2$, and $\bar{c}_2$ it follows that there exists a constant $\gamma_4 > 0$ so that for small h we have

$$\begin{aligned}||\underbrace{\bar{c}_2 e^{\bar{\Lambda}_2 t} - \sum_{i=0}^{m+1} \bar{c}_2 \,\frac{\bar{\Lambda}_2^i t^i}{i!}}_{=:\Delta_1(t)}|| &= ||\bar{c}_2 e^{\bar{\Lambda}_2 t} - \begin{bmatrix} 1 & t & \cdots & \frac{t^{m+1}}{(m+1)!} \end{bmatrix} \begin{bmatrix} \bar{c}_2 \\ \bar{c}_2\bar{\Lambda}_2 \\ \vdots \\ \bar{c}_2(\bar{\Lambda}_2)^{m+1} \end{bmatrix} || \\ &\le \gamma_4 t^{m+2}, \quad t \in [0, (m+1)h),\end{aligned}$$

and

$$\begin{aligned}||\underbrace{\bar{c}_2 e^{\bar{\Lambda}_2 t}\bar{b}_2 - \sum_{i=0}^{m} \bar{c}_2 \,\frac{\bar{\Lambda}_2^i t^i}{i!}\bar{b}_2}_{=:\Delta_2(t)}|| &= ||\bar{c}_2 e^{\bar{\Lambda}_2 t}\bar{b}_2 - \bar{c}_2 \,\frac{\bar{\Lambda}_2^m t^m}{m!}\bar{b}_2|| \\ &\le \gamma_4 t^{m+1}, \quad t \in [0, (m+1)h).\end{aligned}$$

Hence, solving the equation (12) for $\bar{y}$ yields, for small h:

$$\begin{aligned}\bar{y}(t) &= \begin{bmatrix} 1 & (t-t_0) & \cdots & \frac{(t-t_0)^{m+1}}{(m+1)!} \end{bmatrix} \begin{bmatrix} \bar{c}_2 \\ \bar{c}_2\bar{\Lambda}_2 \\ \vdots \\ \bar{c}_2\bar{\Lambda}_2^{m+1} \end{bmatrix} \bar{v}(t_0) + \\ &\underbrace{\Delta_1(t-t_0)\bar{v}(t_0) + \bar{c}_2\frac{\bar{\Lambda}_2^m}{m!}\bar{b}_2\int_{t_0}^t (t-\tau)^m\psi(\tau)d\tau + \int_{t_0}^t \Delta_2(t-\tau)\psi(\tau)d\tau}_{=:\Delta_3(t)}, \\ &\qquad t\in[t_0, t_0+(m+1)h). \end{aligned} \tag{14}$$

Now observe that

$$\begin{aligned}\tilde{c}_2\tilde{\Lambda}^i &= \begin{bmatrix} \bar{c}_2 & 0 \end{bmatrix} \begin{bmatrix} \bar{\Lambda}_2 & \bar{b}_2 \\ 0 & 0 \end{bmatrix}^i \\ &= \begin{bmatrix} \bar{c}_2 & 0 \end{bmatrix} \begin{bmatrix} \bar{\Lambda}_2^i & \bar{\Lambda}_2^{i-1}\bar{b}_2 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \bar{c}_2\bar{\Lambda}_2^i & \bar{c}_2\bar{\Lambda}_2^{i-1}\bar{b}_2 \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

which means that

$$\bar{c}_2\bar{\Lambda}^i\bar{v}(t_0) = \tilde{c}_2\tilde{\Lambda}^i \begin{bmatrix} \bar{v}(t_0) \\ 0 \end{bmatrix}.$$

Hence, (14) can be rewritten as

$$\begin{aligned}\bar{y}(t) &= \begin{bmatrix} 1 & (t-t_0) & \cdots & \frac{(t-t_0)^{m+1}}{(m+1)!} \end{bmatrix} \bar{F}_{m+1}(\sigma) \begin{bmatrix} \bar{v}(t_0) \\ 0 \end{bmatrix} + \Delta_3(t), \\ &\qquad t\in[t_0, t_0+(m+1)h). \end{aligned} \tag{15}$$

Using the above bounds on $\Delta_1(t)$ and $\Delta_2(t)$ given above together with the bound on ψ given in (11), it follows that there exists a constant γ_5 so that for small h:

$$\begin{aligned} ||\Delta_3(t)|| &\le \gamma_4(t-t_0)^{m+2}||\bar{v}(t_0)|| + \\ &\quad |\bar{c}_2\frac{\bar{\Lambda}_2^m}{m!}\bar{b}_2| \int_{t_0}^t (t-\tau)^m d\tau \times \gamma_3(||\bar{x}(t_0)|| + ||\bar{u}|| + h^{\frac{1}{2}}||u_m||_{2,\,[t_0,t)}) + \\ &\quad \gamma_4\int_{t_0}^t (t-\tau)^{m+1}d\tau \times \gamma_3(||\bar{x}(t_0)|| + ||\bar{u}|| + h^{\frac{1}{2}}||u_m||_{2,\,[t_0,t)}) \\ &= \gamma_4(t-t_0)^{m+2}||\bar{v}(t_0)|| + \\ &\quad \gamma_3|\bar{c}_2\frac{\bar{\Lambda}_2^m}{(m+1)!}\bar{b}_2|\,(t-t_0)^{m+1}(||\bar{x}(t_0)|| + ||\bar{u}|| + h^{\frac{1}{2}}||u_m||_{2,\,[t_0,t)}) + \\ &\quad \frac{\gamma_3\gamma_4}{m+2}(t-t_0)^{m+2}(||\bar{x}(t_0)|| + ||\bar{u}|| + \\ &\qquad\qquad h^{\frac{1}{2}}||u_m||_{2,\,[t_0,t)}) \\ &\le \gamma_5(t-t_0)^{m+1}(||\bar{x}(t_0)|| + ||\bar{u}|| + h^{\frac{1}{2}}||u_m||_{2,\,[t_0,t_0+(m+1)h)}), \quad t\in[t_0, t_0+(m+1)h). \end{aligned}$$

Hence, if we substitute this equation into (15) and evaluate it at $t = t_0,\ t_0 + h,\ \cdots,\ t_0 + (m+1)h$ we obtain, for small h:

$$||\bar{y}(t_0) - S_{m+1}H_{m+1}(h)\bar{F}_{m+1}(\sigma)\begin{bmatrix} \bar{v}(t_0) \\ 0 \end{bmatrix}|| = \begin{bmatrix} \Delta_3(t_0) \\ \Delta_3(t_0+h) \\ \vdots \\ \Delta_3(t_0+(m+1)h) \end{bmatrix}$$

$$\le \gamma_5 \left(\sum_{j=1}^{m+1} j^{m+1} \right)^{\frac{1}{2}} h^{m+1}(||\bar{x}(t_0)|| + ||\bar{u}|| + h^{\frac{1}{2}}||u_m||_{2,\ [t_0,t_0+(m+1)h)}),$$
$$t \in [t_0, t_0+(m+1)h);$$

using the fact that $||H_{m+1}^{-1}(h)|| = \frac{(m+1)!}{h^{m+1}}$, it follows that for small h:

$$||\bar{F}_{m+1}^{-1}(\sigma)H_{m+1}^{-1}(h)S_{m+1}^{-1}\bar{y}(t_0) - \begin{bmatrix} \bar{v}(t_0) \\ 0 \end{bmatrix}|| \ \le$$
$$||\bar{F}_{m+1}^{-1}(\sigma)||\ ||S_{m+1}^{-1}||\ \gamma_5 h^{m+1} \left(\sum_{j=1}^{m+1} j^{m+1} \right)^{\frac{1}{2}} (||\bar{x}(t_0)|| + ||\bar{u}|| +$$
$$h^{\frac{1}{2}}||u_m||_{2,\ [t_0,t_0+(m+1)h)}),$$

which is the second part of (i).

Now let us assume that g and $\bar{A}$ are absolutely continuous on $[t_0, t_0+(m+1)h)$. Now we re-examine (12) and take this into account. To proceed, using the definition of ψ observe that

$$\dot{\psi}(t) = \dot{f}_2(t)v(t) + f_2(t)\dot{v}(t) + c_1(t)\dot{w}(t) + \dot{c}_1(t)w(t) + \dot{g}(t)\bar{u}; \tag{16}$$

using the bound on $\bar{x}$ given in (10), we can obtain a bound on $\dot{v}$ and $\dot{w}$, while γ_1 provides a bound on $||\dot{f}_2(t)||$, $||\dot{c}_1(t)||$, and $|\dot{g}(t)|$, so there exists a constant γ_6 so that for small h,

$$|\dot{\psi}(t)| \le \gamma_6(\|\bar{x}(t_0)\| + \|\bar{u}\| + h^{\frac{1}{2}}\|u_m\|_{2\ [t_0,t)}),\quad t \in [t_0, t_0+(m+1)h).$$

Now we can rewrite (12), (13), and (16) :

$$\begin{aligned} \begin{bmatrix} \dot{\bar{v}}(t) \\ \dot{\psi}(t) \end{bmatrix} &= \underbrace{\begin{bmatrix} \bar{\Lambda}_2 & \bar{b}_2 \\ 0 & 0 \end{bmatrix}}_{=\tilde{\Lambda}_2} \underbrace{\begin{bmatrix} \bar{v}(t) \\ \psi(t) \end{bmatrix}}_{=:\tilde{v}(t)} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{=:\tilde{b}_2} \dot{\psi}(t), \\ \bar{y}(t) &= \underbrace{\begin{bmatrix} \bar{c}_2 & 0 \end{bmatrix}}_{=\tilde{c}_2} \begin{bmatrix} \bar{v}(t) \\ \psi(t) \end{bmatrix}. \end{aligned} \tag{17}$$

From the form of $\tilde{\Lambda}_2$, $\tilde{b}_2$, and $\tilde{c}_2$ it follows that there exists a constant γ_7 so that for small h we have

$$\underbrace{||\,\tilde{c}_2 e^{\tilde{\Lambda}_2 t} - \sum_{i=0}^{m+1} \tilde{c}_2 \frac{\tilde{\Lambda}_2^i t^i}{i!}\,||}_{=:\Delta_4(t)} \ \le\ \gamma_7 t^{m+2},\quad t \in [0, (m+1)h),$$

and

$$\|\underbrace{\tilde{c}_2 e^{\tilde{\Lambda}_2 t}\tilde{b}_2}_{=:\Delta_5(t)}\| = \|\sum_{i=m+1}^{\infty} \tilde{c}_2 \frac{\tilde{\Lambda}_2^i t^i}{i!}\tilde{b}_2\| \leq \gamma_7 t^{m+1}, \quad t \in [0,(m+1)h).$$

Hence, solving equation (17) we obtain

$$\bar{y}(t) = \begin{bmatrix} 1 & (t-t_0) & \cdots & \frac{(t-t_0)^{m+1}}{(m+1)!} \end{bmatrix} \begin{bmatrix} \tilde{c}_2 \\ \tilde{c}_2\tilde{\Lambda}_2 \\ \vdots \\ \tilde{c}_2\tilde{\Lambda}_2^{m+1} \end{bmatrix} \tilde{v}(t_0) + \underbrace{\Delta_4(t-t_0)\tilde{v}(t_0) + \int_{t_0}^{t} \Delta_5(t-\tau)\dot{\psi}(\tau)d\tau}_{=:\Delta_6(t)}, \quad t \in [t_0, t_0+(m+1)h). \tag{18}$$

Now it follows that there exists a constant γ_8 so that for small h:

$$\begin{aligned} \|\Delta_6(t)\| &\leq \gamma_7(t-t_0)^{m+2}\|\tilde{v}(t_0)\| + \int_{t_0}^{t} \gamma_7(\tau-t_0)^{m+1}d\tau \times \gamma_6(\|\bar{x}(t_0)\| + \|\bar{u}\| + h^{\frac{1}{2}}\|u_m\|_{2,\,[t_0,t_0+(m+1)h)}) \\ &\leq \gamma_7(t-t_0)^{m+2}\|\tilde{v}(t_0)\| + \frac{\gamma_6\gamma_7}{m+2}(t-t_0)^{m+2}(\|\bar{x}(t_0)\| + \|\bar{u}\| + h^{\frac{1}{2}}\|u_m\|_{2,\,[t_0,t_0+(m+1)h)}) \\ &\leq \gamma_8(t-t_0)^{m+2}(\|\bar{x}(t_0)\| + \|\bar{u}\| + h^{\frac{1}{2}}\|u_m\|_{2,\,[t_0,t_0+(m+1)h)}), \quad t \in [t_0, t_0+(m+1)h). \end{aligned}$$

Hence, if we substitute this into (18) and evaluate it at $t = t_0,\ t_0+h,\ \cdots,\ t_0+(m+1)h$, we obtain, for small h:

$$\|\bar{y}(t_0) - S_{m+1}H_{m+1}(h)\bar{F}_{m+1}(\sigma)\tilde{v}(t_0)\| = \begin{bmatrix} \Delta_6(t_0) \\ \Delta_6(t_0+h) \\ \vdots \\ \Delta_6(t_0+(m+1)h) \end{bmatrix}$$

$$\leq \gamma_8 \left(\sum_{j=1}^{m+1} j^{m+2}\right)^{\frac{1}{2}} h^{m+2}(\|\bar{x}(t_0)\| + \|\bar{u}\| + h^{\frac{1}{2}}\|u_m\|_{2,\,[t_0,t_0+(m+1)h)});$$

using the fact that

$$\|H_{m+1}^{-1}(h)\| = \frac{(m+1)!}{h^{m+1}},$$

and

$$\tilde{v}(t_0) = \begin{bmatrix} v(t_0) \\ \bar{y}(t_0) \\ f_2(t_0)v(t_0) + c_1(t_0)w(t_0) + g(t_0)\bar{u} \end{bmatrix},$$

it follows that for small h:

$$||\bar{F}_{m+1}^{-1}(\sigma)H_{m+1}^{-1}(h)S_{m+1}^{-1}\bar{y}(t_0) - \begin{bmatrix} v(t_0) \\ \bar{y}(t_0) \\ f_2(t_0)v(t_0) + c_1(t_0)w(t_0) + g(t_0)\bar{u} \end{bmatrix}|| \leq$$

$$||\bar{F}_{m+1}^{-1}(\sigma)||\ ||S_{m+1}^{-1}||\ \gamma_8 h \left(\sum_{j=1}^{m+1} j^{m+2}\right)^{\frac{1}{2}} (m+1)!\ (||\bar{x}(t_0)|| + ||\bar{u}|| +$$

$$h^{\frac{1}{2}}||u_m||_{2,\ [t_0,t_0+(m+1)h)}),$$

as desired.

□

Proof of Lemma 4.2:

To prove this, the interval of $[jT,(j+1)T)$ will be analyzed. In this regard, the state z has dimension $2(m+1)+2 = 2m+4$:

- z_1 keeps track of

$$\frac{1}{\rho}\underbrace{\begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \bar{F}_{m+1}^{-1} H_{m+1}(h)^{-1} S_{m+1}^{-1}}_{=:\begin{bmatrix} \delta_0 & \delta_1 & \cdots & \delta_{m+1} \end{bmatrix}} \begin{bmatrix} \bar{y}(jT) \\ \bar{y}(jT+h) \\ \vdots \\ \bar{y}(jT+(m+1)h) \end{bmatrix}.$$

- z_2 keeps track of the current $\hat{\phi}_i(jT)$, $i = 1,\cdots,p$, being constructed. We define

$$\begin{bmatrix} \bar{\delta}_0 & \bar{\delta}_2 & \cdots & \bar{\delta}_{m+1} \end{bmatrix} := \begin{bmatrix} \bar{f}_2 & -1 & 0 \end{bmatrix} \bar{F}_{m+1}^{-1} H_{m+1}^{-1}(h) S_{m+1}^{-1},$$

  to be used in the construction of $\hat{\phi}_0(jT)$.

- z_3 is used to construct the control signal to be used during the Estimation Phase.

- z_4 is used to construct the control signal to be applied during the Control Phase.

The periodically time-varying gains F, G, H, J, and L (*with period* p) are partitioned as

$$F(k) = \begin{bmatrix} F_1(k) \\ F_2(k) \\ F_3(k) \\ F_4(k) \end{bmatrix},\ G(k) = \begin{bmatrix} G_1(k) \\ G_2(k) \\ G_3(k) \\ G_4(k) \end{bmatrix},\ H(k) = \begin{bmatrix} H_1(k) \\ H_2(k) \\ H_3(k) \\ H_4(k) \end{bmatrix},\ J(k) = \begin{bmatrix} J_1(k) \\ J_2(k) \\ J_3(k) \\ J_4(k) \end{bmatrix},$$

$$L(k) = \begin{bmatrix} L_1(k) & L_2(k) & L_3(k) & L_4(k) \end{bmatrix}.$$

Since the controller is periodic of period T, for simplicity we will assume that $j = 0$ which means that we are analyzing the first period $[0, T)$. We would like z_1 to keep track of a weighted version of $\bar{y}(0)$:

$$z_1(k) = \underbrace{\begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \bar{F}_{m+1}^{-1} H_{m+1}^{-1}(h) S_{m+1}^{-1}}_{\begin{bmatrix} \delta_0 & \delta_1 & \cdots & \delta_{m+1} \end{bmatrix}} \begin{bmatrix} \bar{y}(0) \\ \bar{y}(h) \\ \vdots \\ \bar{y}((m+1)h) \end{bmatrix}, \quad k = m+2, \cdots, p,$$

so we set

$$(F_1, G_1, H_1, J_1)(k) = \begin{cases} (\begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}, \delta_0, 0, 0) & k = 0, \\ (\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \delta_i, 0, 0,) & k = 1, \cdots, m+1, \\ (\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, 0, 0, 0) & k = m+2, \cdots, p-1. \end{cases}$$

The objective of z_2 is to construct the current $\hat{\phi}_i(0)$, $i = 1, \cdots, q$. To this end, we set

$$(F_2, G_2, H_2, J_2)(k) = \begin{cases} (0, 0, 0, 0) & k = 0, \cdots, m, \\ (\begin{bmatrix} -1 & 0 & 0 & 0 \end{bmatrix}, \delta_0, 0, 0) & k = (2i-1)(m+1), \\ & (i = 1, \cdots, q), \\ (\begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}, \delta_j, 0, 0) & k = (2i-1)(m+1) + j, \\ & \quad (i = 1, \cdots, q), \\ & (j = 1, \cdots, m+1), \\ (\begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}, 0, 0, 0) & \text{all remaining } k \in \{0, \cdots, p-1\}. \end{cases}$$

This choice of parameters results in

$$z_2(2i(m+1) + j) = \hat{\phi}_i(0), \quad j = 1, \cdots, m+1, \quad i = 1, \cdots, q.$$

The state z_3 is used to construct the control signal to be used during the Estimation Phase. While z_2 captures $\hat{\phi}_1(0), \cdots, \hat{\phi}_{q-1}(0)$, we can obtain $\hat{\phi}_0(0)$ from samples of $\bar{y}$, $\bar{x}_m$, and $\bar{u}_m$. To this end, set

$$(F_3, G_3, H_3, J_3)(k) = \begin{cases} (0, \bar{\delta}_0, k_1, k_2) & k = 0, \\ (\begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}, \bar{\delta}_k, 0, 0) & k = 1, \cdots, m+1, \\ (\begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}, 0, 0, 0) & k = (2i+1)(m+1) - 1, \\ & (i = 1, \cdots, q), \\ (\begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}, 0, 0, 0) & \text{all remaining } k \in \{0, \cdots, p-1\}. \end{cases}$$

Hence,

$$\begin{aligned} z_3((m+1) + j) &= z_3(m+1) + \bar{\delta}_{m+1}\bar{y}((m+1)h) = \hat{\phi}_0(0), \quad j = 1, \cdots, 2(m+1) - 1, \\ z_3((2i+1)(m+1) + j) &= \hat{\phi}_i(0), \quad i = 1, \cdots, q, \quad j = 0, \cdots, m. \end{aligned}$$

The state z_4 is used to form the control signal to be used during the Control Phase. To this end, we set

$$(F_4, G_4, H_4, J_4)(k) = \begin{cases} (0, 0, 0, 0) & k = 0, \\ (\begin{bmatrix} 0 & 0 & c_0 & 1 \end{bmatrix}, 0, 0, 0) & k = m+2, \\ (\begin{bmatrix} 0 & c_i & 0 & 1 \end{bmatrix}, 0, 0, 0) & k = 2i(m+1) + 1, \\ & (i = 1, \cdots, q), \\ (\begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}, 0, 0, 0) & \text{all remaining } k \in \{0, \cdots, p-1\}. \end{cases}$$

So

$$z_4(k) = \sum_{i=0}^{q} c_i \hat{\phi}_i(0), \quad k = 2q(m+1)+2, \cdots, p.$$

The construction of the control signal is as follows

$$(L,M)(k) = \begin{cases} (0,0) & k = 0, \cdots, m, \\ (\begin{bmatrix} 0 & \rho & 0 & 0 \end{bmatrix}, \rho\bar{\delta}_{m+1}) & k = m+1, \\ (\begin{bmatrix} 0 & 0 & \rho & 0 \end{bmatrix}, 0) & k = (2i+1)(m+1)+j, \\ & \quad (i = 0, \cdots, q-1)\ (j = 0, \cdots, m), \\ (\begin{bmatrix} 0 & 0 & -\rho & 0 \end{bmatrix}, 0) & k = (2i+2)(m+1)+j, \\ & \quad (i = 0, \cdots, q-1)\ (j = 0, \cdots, m), \\ (\begin{bmatrix} 0 & 0 & 0 & \frac{p}{p-2q-1} \end{bmatrix}, 0) & k = (2q+1)(m+1), \cdots, p-1. \end{cases}$$

By examining the definitions of $F_1,\ F_2,\ F_3$, and F_4 it follows that $F(0) = 0$, so

$$F(p-1) \times \cdots \times F(1) \times F(0) = 0,$$

which means that the controller is deadbeat.

□

Proof of Lemma 4.3 (KCL-4):

First define

$$\begin{aligned} \gamma_1 &= esssup_{t \geq 0}(\|\bar{A}(t)\| + ||g(t)||.||\bar{B}|| + ||\bar{E}|| + ||c_1(t)|| + ||f_2(t)||), \\ \gamma_2 &= esssup_{t \geq 0}(\|\dot{\bar{A}}(t)\| + ||\dot{g}(t)|| + ||\dot{c}_1(t)|| + ||\dot{f}_2(t)||); \end{aligned}$$

it follows from the standing assumptions that γ_1 and γ_2 are finite. Now let us examine the equation given in (4.8), namely

$$\dot{\bar{x}}(t) = \bar{A}(t)\bar{x}(t) + g(t)\overline{B}u(t) + \overline{E}u_m(t). \tag{19}$$

Thus, using Lemma A.1 and the Cauchy-Schwartz inequality it follows that

$$\|\bar{x}(t)\| \leq e^{\gamma_1 T}\,\|\bar{x}(kT)\| + (e^{2\gamma_1 T}-1)^{\frac{1}{2}}\gamma_1(\|u(t)\|_{2,[kT,t)} + \|u_m\|_{2,[kT,t)}), \quad t \in [kT,(k+1)T).$$

From KEL-4 there exists a constant γ_3 so that for small T

$$\begin{aligned} \|\hat{\phi}_0(kT)\| &\leq \gamma_3(\|\bar{x}(kT)\| + T^{\frac{1}{2}}\,\|u_m\|_{2,[kT,kT+(m+1)h)}), \\ \|\hat{\phi}_i(kT)\| &\leq \gamma_3(\|\bar{x}(kT)\| + \|\bar{x}(kT+(2i-1)(m+1)h)\| + \rho\|\hat{\phi}_{i-1}(kT)\| + \\ & \qquad T^{\frac{1}{2}}\,\|u_m\|_{2,[kT,kT+2i(m+1)h)}) \quad i = 1,2,\cdots,q. \end{aligned} \tag{20}$$

From KEL-4 we also have that there exists a constant γ_4 so that for small T

$$\begin{aligned} \|\bar{x}(kT+(m+1)h)\| &\leq \gamma_4(\|\bar{x}(kT)\| + T^{\frac{1}{2}}\,\|u_m(t)\|_{2,[kT,kT+(m+1)h)}), \\ \|\bar{x}(kT+2(m+1)h)\| &\leq \gamma_4(\|\bar{x}(kT+(m+1)h)\| + \rho T\|\hat{\phi}_0(kT)\| + \\ & \qquad T^{\frac{1}{2}}\,\|u_m\|_{2,[kT+(m+1)h,kT+2(m+1)h)}) \\ &\leq \gamma_4^2\,\|\bar{x}(kT)\| + \rho\gamma_4 T\|\hat{\phi}_0(kT)\| + \\ & \qquad \underbrace{\gamma_4(\gamma_4+1)}_{=:\gamma_5} T^{\frac{1}{2}}\,\|u_m\|_{2,[kT,kT+2(m+1)h)}), \end{aligned}$$

If we repeat this procedure, then there exists a constant γ_6 such that

$$\begin{aligned}\|\bar{x}(kT+(2i-1)(m+1)h)\| &\leq \gamma_6(\|\bar{x}(kT)\| + \rho T \sum_{j=0}^{i-2} \|\hat{\phi}_j(kT)\| + \\ & T^{\frac{1}{2}} \|u_m\|_{2,[kT,kT+(2i-1)(m+1)h)}), \quad i = 1, 2, \cdots, q.\end{aligned}$$

Combining this with (20) we see that there exists a constant γ_7 so that for small T

$$\begin{aligned}| \hat{\phi}_i(kT)\| &\leq \gamma_7(\|\bar{x}(kT)\| + \rho T \sum_{j=0}^{i-2} \|\hat{\phi}_j(kT)\| + \rho\|\hat{\phi}_{i-1}(kT)\| + \\ & T^{\frac{1}{2}} \|u_m\|_{2,[kT,kT+2i(m+1)h)}), \quad i = 1, \cdots, q.\end{aligned}$$

Solving iteratively, we see that there exists a constant γ_8 so that for small T

$$\|\hat{\phi}_i(kT)\| \leq \gamma_8(\|\bar{x}(kT)\| + T^{\frac{1}{2}} \|u_m\|_{2,[kT,kT+2i(m+1)h)}), \qquad i = 1, \cdots, q. \tag{21}$$

From the definition of the controller we have

$$\begin{aligned}u(t) &= 0, \quad t \in [kT, kT+(m+1)h), \\ \|u(t)\| &= \rho\|\hat{\phi}_{i-1}(kT)\|, \quad i = 1, \cdots, q, \\ & \quad t \in [kT+(2i-1)(m+1)h, kT+(2i+1)(m+1)h),\end{aligned}$$

and

$$\begin{aligned}\|u(t)\| &\leq \frac{p}{p-(2q+1)(m+1)} \sum_{i=0}^{q} |c_i| \, \|\hat{\phi}_i(kT)\|, \\ & t \in [kT+(2q+1)(m+1)h, (k+1)T),\end{aligned}$$

so

$$\begin{aligned}\|u(t)\| &\leq \underbrace{\frac{p(q+1)}{p-(2q+1)(m+1)}\bar{c}}_{\gamma_9} \max_i \left\{\|\hat{\phi}_i(kT)\|\right\}, \\ & i = 0, 1, \cdots, q, \quad t \in [kT+(2q+1)(m+1)h, (k+1)T),\end{aligned}$$

where

$$\bar{c} = max\{1, max_i\,\{c_i\}\}.$$

Combining this with (21) we see that for small T

$$\|u(t)\| \leq \gamma_8\gamma_9(\|\bar{x}(kT)\| + T^{\frac{1}{2}} \|u_m\|_{2,[kT,t)}), \quad t \in [kT, (k+1)T). \tag{22}$$

If we now solve (19) once again and use Lemma A.1,

$$\begin{aligned}\|\bar{x}(t) - \bar{x}(kT)\| &\leq (e^{\gamma_1 T} - 1)\, \|\bar{x}(kT)\| + \gamma_1 \int_0^T e^{\gamma_1(T-\tau)} \|u(kT+\tau)\| d\tau + \\ & \frac{1}{\sqrt{2\gamma_1}}(e^{2\gamma_1 T} - 1)\gamma_1 \, \|u_m\|_{2,[kT,t)}, \quad t \in [kT, (k+1)T).\end{aligned}$$

Combining this with (22), there exists a constant $\gamma_{10} > 0$ such that for small T:

$$\|\bar{x}(t) - \bar{x}(kT)\| \leq \gamma_{10}(T\|\bar{x}(kT)\| + T^{\frac{1}{2}}\|u_m\|_{2,[kT,t)}), \quad t \in [kT, (k+1)T). \qquad (23)$$

From Lemma A.1 we have that there exists a constant γ_{11} so that for all $\bar{\theta} \in \bar{\mathcal{P}}$, we have

$$\|\Phi^{\varepsilon}_{cl}(t, kT) - I\| \leq (e^{\|\overline{A}^{\varepsilon}_{cl}\|_{\infty}(t-kT)} - 1) \leq \gamma_{11}T,$$

$$\begin{aligned}\left\|\int_{kT}^{t} \Phi^{\varepsilon}_{cl}(t,\tau)\bar{E}u_m(\tau)d\tau\right\| &\leq \left(\int_{kT}^{t} e^{2\|\overline{A}^{\varepsilon}_{cl}\|_{\infty}(t-\tau)}\left\|\bar{E}\right\|^2 d\tau\right)^{\frac{1}{2}} \|u_m\|_{2,[kT,t)}, \\ &\leq \gamma_{11}T^{\frac{1}{2}}\|\overline{u}_m\|_{2,[kT,t)}, \quad t \in [kT, (k+1)T).\end{aligned}$$

If we combine these inequalities with (23) we have

$$\left\|\bar{x}(t) - \Phi^{\varepsilon}_{cl}(t,kT)\bar{x}(kT) - \int_{kT}^{t}\Phi^{\varepsilon}_{cl}(t,\tau)\bar{E}u_m(\tau)d\tau\right\| \leq \gamma_{10}(T\|\bar{x}(kT)\| +$$
$$T^{\frac{1}{2}}\|u_m\|_{2,[kT,t)}) + \gamma_{11}T\|\bar{x}(kT)\| + \gamma_{11}T^{\frac{1}{2}}\|u_m\|^2_{2,[kT,t)}, \quad t \in [kT, (k+1)T),$$

so there exists a constant γ_{12} such that for small T:

$$\begin{aligned}&\left\|\bar{x}(t) - \Phi^{\varepsilon}_{cl}(t,kT)\bar{x}(kT) - \int_{kT}^{t}\Phi^{\varepsilon}_{cl}(t,\tau)\bar{E}u_m(\tau)d\tau\right\| \leq \\ &\qquad \gamma_{12}(T\|\bar{x}(kT)\| + T^{\frac{1}{2}}\|u_m\|_{2,[kT,t)}), \quad t \in [kT, (k+1)T).\end{aligned}$$

Now we look at the more interesting case when $\bar{\theta} \in \bar{\mathcal{P}}$ is absolutely continuous on $[kT, (k+1)T)$. Define

$$L(t) = [c_1(t) \;\; f_2(t) \;\; 0 \;\; 0 \;\; 0].$$

From KEL-4 we know that there exists a constant γ_{13} so that for small T we have

$$\|\hat{\phi}_0(kT) - K(kT)\bar{x}(kT)\| \leq \gamma_{13}(T\|\bar{x}(kT)\| + T^{\frac{3}{2}}\|u_m\|_{2,\,[kT,kT+(m+1)h)}) \qquad (24)$$

and

$$\begin{aligned}\|\hat{\phi}_i(kT) \;\; - \;\; &\frac{1}{\rho}L(kT + (2i-1)(m+1)h)\,\bar{x}(kT + (2i-1)(m+1)h) + \\ &\frac{1}{\rho}L(kT)\bar{x}(kT) - g(kT + (2i-1)(m+1)h)\hat{\phi}_{i-1}(kT)\| \leq \\ &\gamma_{13}(T\|\bar{x}(kT)\| + T\|\bar{x}(kT + (2i-1)(m+1)h)\| + T\|\phi_{i-1}(kT)\| + \\ &T^{\frac{3}{2}}\|u_m\|_{2,\,[kT,kT+(m+1)h)} + T^{\frac{3}{2}}\|u_m\|_{2,\,[kT+(2i-1)(m+1)h,kT+2i(m+1)h)}), \\ &\qquad\qquad i = 1, 2, \cdots, q.\end{aligned}$$

Using (21) to obtain a bound on $||\hat{\phi}_{i-1}(kT)||$, and (23) to obtain a bound on $||\bar{x}(t)||$ for $t \in [kT,(k+1)T)$, we see that there exists a constant γ_{14} so that

$$\begin{aligned}\|\hat{\phi}_i(kT) \quad - \quad & \frac{1}{\rho}L(kT+(2i-1)(m+1)h)\ \bar{x}(kT+(2i-1)(m+1)h)+ \\ & \quad \frac{1}{\rho}L(kT)\bar{x}(kT) - g(kT+(2i-1)(m+1)h)\ \hat{\phi}_{i-1}(kT)\| \\ \leq \quad & \gamma_{14}(T\|\bar{x}(kT)\| + T^{\frac{3}{2}}\|u_m\|_{2,\ [kT,kT+2i(m+1)h)}).\end{aligned}$$

Now we combine this with (23) and the fact that L is a sub-matrix of $\bar{A}$ and is absolutely continuous on $[kT,(k+1)T)$ to obtain a nice relationship between $\hat{\phi}_i(kT)$ and $\hat{\phi}_{i-1}(kT)$:

$$\begin{aligned}|\hat{\phi}_i(kT) \quad - \quad & g(kT)\hat{\phi}_{i-1}(kT)| \leq \\ & \frac{1}{\rho}|(L(\bar{x}(kT+(2i-1)(m+1)h) - L(kT))\bar{x}(kT+(2i-1)(m+1)h)| + \\ & \frac{1}{\rho}|L(kT)(\bar{x}(kT+(2i-1)(m+1)h) - \bar{x}(kT))| + \\ & |[g(kT+(2i-1)(m+1)h) - g(kT)]\hat{\phi}_{i-1}(kT)| + \\ & \gamma_{14}(T\|\bar{x}(kT)\| + T^{\frac{3}{2}}\|u_m\|_{2,\ [kT,kT+2i(m+1)h)}) \\ \leq \quad & \frac{1}{\rho}\gamma_2 T\left[(1+\gamma_{10}T)\|\bar{x}(kT)\| + \gamma_{10}T^{\frac{1}{2}}\|u_m\|_{2,\ [kT,kT+(2i-1)(m+1)h)}\right] + \\ & \frac{1}{\rho}\gamma_1\gamma_{10}(T\|\bar{x}(kT)\| + T^{\frac{1}{2}}\|u_m\|_{2,\ [kT,kT+(2i-1)(m+1)h)}) + \\ & \gamma_2\gamma_{14}(T\|\bar{x}(kT)\| + T^{\frac{3}{2}}\|u_m\|_{2,\ [kT,kT+(2i-1)(m+1)h)}) + \\ & \gamma_{14}(T\|\bar{x}(kT)\| + T^{\frac{3}{2}}\|u_m\|_{2,\ [kT,kT+2i(m+1)h)}),\end{aligned}$$

so that there exists a constant γ_{15} so that for small T we have

$$\begin{aligned}|\hat{\phi}_i(kT) - g(kT)\hat{\phi}_{i-1}(kT)| \quad \leq \quad & \gamma_{15}(T\|\bar{x}(kT)\| + T^{\frac{1}{2}}\|u_m\|_{2,\ [kT,kT+2i(m+1)h)}), \\ & i = 1,2,\cdots,q.\end{aligned}$$

If we combine this with our bound on $\hat{\phi}_0(kT)$ given in (24), it follows that there exists a constant γ_{16} so that for small T

$$\begin{aligned}|\hat{\phi}_i(kT) - g(kT)^i K(kT)\bar{x}(kT)| \quad \leq \quad & \gamma_{16}(T\|\bar{x}(kT)\| + T^{\frac{1}{2}}\|u_m\|_{2,\ [kT,kT+2i(m+1)h)}), \\ & i = 0,1,\cdots,q.\end{aligned}$$

This means, in turn, that there exists a constant γ_{17} so that for small T

$$|\sum_{i=0}^{q} c_i\hat{\phi}_i(kT) - \hat{f}^{\varepsilon}(g(kT))K(kT)\bar{x}(kT)| \leq \gamma_{17}(T\|\bar{x}(kT)\| + T^{\frac{1}{2}}\|u_m\|_{2,\ [kT,(k+1)T)}). \tag{25}$$

Now let us look at x. We have

$$\begin{aligned}\dot{\bar{x}}(t) &= \bar{A}(t)\bar{x}(t) + \bar{E}u_m(t) + g(t)\bar{B}u(t)\\ &= \bar{A}^{\varepsilon}_{cl}(t)\bar{x}(t) + \bar{E}u_m(t) + (\bar{A}(t) - \bar{A}^{\varepsilon}_{cl}(t))\bar{x}(t) + g(t)\bar{B}u(t).\end{aligned}$$

Roughly speaking, we need to prove that

$$\psi(t) := \int_{kT}^{t} \Phi^{\varepsilon}_{cl}(t,\tau)[(\bar{A}(\tau) - \bar{A}^{\varepsilon}_{cl})\bar{x}(\tau) + g(\tau)\bar{B}u(\tau)], \ \ t \in [kT,(k+1)T),$$

when evaluated at $T = (k+1)T$, goes to zero as $T^{\frac{3}{2}}$ as $T \to 0$. But

$$(\bar{A}(t) - \bar{A}^{\varepsilon}_{cl}(t))\bar{x}(t) = -g(t)\hat{f}^{\varepsilon}(g(t))\bar{B}K(t)\bar{x}(t),$$

so it follows that

$$\dot{\psi}(t) = \bar{A}^{\varepsilon}_{cl}(t)\psi(t) + g(t)\bar{B}\underbrace{[-\hat{f}^{\varepsilon}(g(t))K(t)\bar{x}(t) + u(t)]}_{=:\eta(t)}, \ \ \psi(kT) = 0.$$

Using (23) and the bounds on the plant parameters and their derivatives, it follows that there exists a constant γ_{18} so that for small T

$$\begin{aligned}&|\eta(t) - u(t) + \hat{f}^{\varepsilon}(g(kT))K(kT)\bar{x}(kT)|\\ &\le \|\hat{f}^{\varepsilon}(g(kT))K(kT) - \hat{f}^{\varepsilon}(g(t))K(t)\| \times \|\bar{x}(kT)\| + \|\hat{f}_{\varepsilon}(g(t))K(t)\| \times\\ &\quad \|\bar{x}(t) - \bar{x}(kT)\|\\ &\le [|\hat{f}^{\varepsilon}(g(kT)) - \hat{f}^{\varepsilon}(g(t))| \times \sup_{\tau\ge 0}||K(\tau)|| \|\bar{x}(kT)\| + |\hat{f}^{\varepsilon}(g(t))| \times\\ &\quad \|K(t) - K(kT)\|] \times \|\bar{x}(kT)\| + (\sup_{\tau\ge 0}||g(\tau)|| + \varepsilon)\sup_{\tau\ge 0}||K(\tau)|| \times \|\bar{x}(t) - \bar{x}(kT)\|\\ &\le \gamma_{18}(T\|\bar{x}(kT)\| + T^{\frac{1}{2}}\|u_m\|_{2,\,[kT,t)}), \ \ t \in [kT,(k+1)T).\end{aligned}$$

Arguing in a similar way, it follows that there exists a constant γ_{19} so that

$$\begin{aligned}\|g(t)\eta(t) - g(kT)u(t) \ &+ \ g(kT)\hat{f}^{\varepsilon}(g(kT))K(kT)\bar{x}(kT)\|\\ &\le \ \gamma_{19}(T\|\bar{x}(kT)\| + T^{\frac{1}{2}}\|u_m\|_{2,\,[kT,t)}), \ \ t \in [kT,(k+1)T).\end{aligned} \tag{26}$$

From Lemma A.2 we have that for small T

$$\|\Phi^{\varepsilon}_{cl}(t,\tau) - I\| \le 2\gamma_1(t-\tau), \ \ t \ge \tau \ge 0, \ \ |t-\tau| \le T.$$

Using this equation and (22) and (23) to obtain a bound on η, it follows that there exists a constant γ_{20} so that

$$\begin{aligned}\|\psi((k+1)T) \ &- \ \int_{kT}^{(k+1)T} \bar{B}g(\tau)\eta(\tau)d\tau\| = \|\int_{kT}^{(k+1)T}(\Phi^{\varepsilon}_{cl}(t,\tau) - I)\bar{B}g(\tau)\eta(\tau)d\tau\|\\ &\le \ 2\gamma_1\frac{T^2}{2}\|\bar{B}\| \times \sup_{\tau\ge 0}\|g(\tau)\| \max_{\tau\in[kT,(k+1)T]}|\eta(\tau)|\\ &\le \ \gamma_{20}T^2(\|\bar{x}(kT)\| + \|u_m\|_{2,\,[kT,(k+1)T)}).\end{aligned}$$

Combining this equation with (26), it follows that there exists a constant γ_{21} so that for small T

$$\begin{aligned}\|\psi((k+1)T) - g(kT)\bar{B} \int_{kT}^{(k+1)T} & [u(\tau) - \hat{f}^{\varepsilon}(g(kT))K(kT)\bar{x}(kT)]d\tau\| \\ \leq & \gamma_{21}(T^2\|\bar{x}(kT)\| + T^{\frac{3}{2}}\|u_m\|_{2,\,[kT,(k+1)T)}).\end{aligned} \tag{27}$$

Using the definition of u, it follows that for small T

$$\int_{kT}^{(k+1)T} [u(\tau) - \hat{f}^{\varepsilon}(g(kT))K(kT)\bar{x}(kT)]\, d\tau = T\sum_{i=0}^{q} c_i[\hat{\phi}_i(kT) - g(kT)^i K(kT)\bar{x}(kT)].$$

If we now use (25) to provide an estimate of $\hat{\phi}_i(kT)$, it follows that for small T

$$\|\int_{kT}^{(k+1)T} [u(\tau) - \hat{f}^{\varepsilon}(g(kT))K(kT)\bar{x}(kT)]d\tau\| \leq \gamma_{17}(T^2\|\bar{x}(kT)\| + T^{\frac{3}{2}}\|u_m\|_{[kT,(k+1)T)}).$$

If we now combine this with (27), we see that for small T

$$\|\psi((k+1)T)\| \leq (\gamma_1\gamma_{17} + \gamma_{21})(T^2\|\bar{x}(kT)\| + T^{\frac{3}{2}}\|u_m\|_{2,\,[kT,(k+1)T)}),$$

as required.

□

Proof of Proposition 4.2:

The system equation for $\bar{x}^{\varepsilon}$ is

$$\bar{x}^{\varepsilon}(t) = \Phi_{cl}^{\varepsilon}(t,kT)\bar{x}^{\varepsilon}(kT) + \int_{kT}^{t} \Phi_{cl}^{\varepsilon}(t,\tau)\bar{E}u_m(\tau)d\tau, \quad t \geq kT. \tag{28}$$

By hypothesis ε has been chosen so that there exists a constant γ_1 so that

$$\|\Phi_{cl}^{\varepsilon}(t,\tau)\| \leq \gamma_1 e^{\lambda_1(t-\tau)}, \quad t \geq \tau,$$

so

$$\|\bar{x}^{\varepsilon}(t)\| \leq \gamma_1 e^{\lambda_1 t}\|\bar{x}_0\| + \frac{\gamma_1}{\sqrt{2|\lambda_1|}}\|\bar{E}\|\sqrt{|e^{2\lambda_1 t} - 1|} \times \|u_m\|_{2,\,[0,t)}, \quad t \geq 0, \tag{29}$$

and

$$\|\bar{x}^{\varepsilon}\|_2 \leq \frac{\gamma_1}{\sqrt{2|\lambda_1|}}\|\bar{x}_0\| + \frac{\gamma_1}{\sqrt{2|\lambda_1|}}\|\bar{E}\| \times \|u_m\|_2, \quad t \geq 0.$$

We can use KCL-4 to get bounds on $\hat{\bar{x}}^\varepsilon$ and $\hat{u}^\varepsilon$. There exists a constant γ_2 so that for small T:

$$\begin{aligned}\|\hat{\bar{x}}^\varepsilon(t) - \Phi^\varepsilon_{cl}(t,kT)\hat{\bar{x}}^\varepsilon(kT) \quad &- \quad \int_{kT}^{t} \Phi^\varepsilon_{cl}(t,\tau)\bar{E}u_m(\tau)d\tau\| \ \leq \\ \gamma_2(T\|\hat{\bar{x}}^\varepsilon(kT)\| \quad &+ \quad T^{\frac{1}{2}}\|u_m\|_{2,\,[kT,t)}), \end{aligned} \tag{30}$$

$$\|\hat{u}^\varepsilon(t)\| \leq \gamma_2(\|\hat{\bar{x}}^\varepsilon(kT)\| + T^{\frac{1}{2}}\|u_m\|_{2,[kT,t)}), \quad t \in [kT,(k+1)T), \tag{31}$$

and when $\bar{\theta}(t)$ is absolutely continuous on $[kT,(k+1)T)$, for small T we have:

$$\begin{aligned}\|\hat{\bar{x}}^\varepsilon[(k+1)T] \quad &- \quad \Phi^\varepsilon_{cl}((k+1)T,kT)\hat{\bar{x}}^\varepsilon[kT] - \int_{kT}^{(k+1)T} \Phi^\varepsilon_{cl}((k+1)T,\tau)\bar{E}u_m(\tau)d\tau\| \\ &\leq \quad \gamma_2(T^2(\|\hat{\bar{x}}^\varepsilon(kT)\| + T^{\frac{3}{2}}\|u_m\|_{2,\,[kT,(k+1)T)}). \end{aligned} \tag{32}$$

Now we will prove that $\hat{\bar{x}}^\varepsilon$ is well-behaved at the sample points when T is small. Since we have to allow for the occasional parameter jump, our proof is a little tricky: we will require that our period T be much smaller than the lower bound T_0 on the time between the parameter jumps. To this end, we will introduce a new integer valued variable

$$r(T) = \text{integer part of } (\frac{T_0}{T});$$

it follows that

$$r(T)T \leq T_0,$$

and $r(T)T \to T_0$ as $T \to 0$.

Now let us proceed with our analysis of $\hat{\bar{x}}^\varepsilon$. Using the differential equation for $\hat{\bar{x}}^\varepsilon$ and (30) and (32), we see that we can choose perturbations $\Delta_1, \cdots, \Delta_4$, satisfying

$$\begin{aligned}\|\Delta_1(kT)\| \quad &\leq \quad \gamma_2 T^2\|\hat{\bar{x}}^\varepsilon(kT)\|, \\ \|\Delta_2(kT)\| \quad &\leq \quad \begin{cases} 0 & \text{if } \bar{\theta}(t) \text{ is a.c. on } [kT,(k+1)T) \\ \gamma_2 T\|\hat{\bar{x}}^\varepsilon(kT)\| & \text{else,} \end{cases} \\ \|\Delta_3(kT)\| \quad &\leq \quad \gamma_2 T^{\frac{3}{2}}\|u_m\|_{2,\,[kT,(k+1)T)}, \\ \|\Delta_4(kT)\| \quad &\leq \quad \begin{cases} 0 & \text{if } \bar{\theta}(t) \text{ is a.c. on } [kT,(k+1)T) \\ \gamma_2 T^{\frac{1}{2}}\|u_m\|_{2,\,[kT,(k+1)T)} & \text{else,} \end{cases}\end{aligned}$$

so that

$$\begin{aligned}\hat{\bar{x}}^\varepsilon[(k+1)T] \quad &= \quad \Phi^\varepsilon_{cl}((k+1)T,kT)\hat{\bar{x}}^\varepsilon[kT] + \int_{kT}^{(k+1)T} \Phi^\varepsilon_{cl}((k+1)T,\tau)\bar{E}u_m(\tau)d\tau + \\ &\quad\quad \Delta_1(kT) + \Delta_2(kT) + \Delta_3(kT) + \Delta_4(kT). \end{aligned} \tag{33}$$

Using our bound on $\|\Phi^\varepsilon_{cl}(t,\tau)\|$, we end up with

$$\begin{aligned}\|\hat{\bar{x}}^\varepsilon(kT)\| \quad &\leq \quad \gamma_1 e^{\lambda_1 kT}\|\bar{x}_0\| + \sum_{i=0}^{k-1} \gamma_1 e^{\lambda_1(k-1-j)T}[\gamma_1 \frac{\|\bar{E}\|}{\sqrt{2\,|\lambda_1|}}\sqrt{1-e^{\lambda_1 T}} \times \\ &\quad\quad \|u_m\|_{2,\,[kT,(k+1)T)} + \|\Delta_1(kT)\| + \|\Delta_2(kT)\| + \|\Delta_3(kT)\| + \|\Delta_4(kT)\|]. \end{aligned}$$

Let $\psi(kT)$ denote the RHS of this equation. Then

$$\begin{aligned}\psi[(k+1)T] &= e^{\lambda_1 T}\psi(kT)+\gamma_1[\gamma_1\frac{\|\bar{E}\|}{\sqrt{2\,|\lambda_1|}}\sqrt{1-e^{\lambda_1 T}}\times\|u_m\|_{2,\,[kT,(k+1)T)}+\\ &\quad \|\Delta_1(kT)\|+\|\Delta_2(kT)\|+\|\Delta_3(kT)\|+\|\Delta_4(kT)\|]. \qquad (34)\end{aligned}$$

Since

$$\|\hat{\bar{x}}^{\varepsilon}(kT)\|\leq\psi(kT),$$
$$\psi(0)=\gamma_1\|\hat{\bar{x}}^{\varepsilon}(0)\|=\gamma_1\|\bar{x}_0\|,$$

it is enough to get a bound on ψ.

The goal is to obtain a tight bound on ψ. Before proceeding, observe that for small $T>0$ we have

$$e^{\lambda_1 T}+\gamma_2 T^2\leq e^{(\lambda_1+\lambda)T/2}=:e^{\lambda_2 T}.\quad [\lambda_2:=(\lambda_1+\lambda)/2]$$

Hence, from (34) it follows that there exists a constant $\gamma_3>0$ so that for small T we have

$$\psi[(k+1)T]\leq\begin{cases} e^{\lambda_2 T}\psi(kT)+\gamma_3 T^{\frac{1}{2}}\|u_m\|_{2,\,[kT,(k+1)T)} & \text{if } \bar{\theta}(t) \text{ is a.c. on } [kT,(k+1)T) \\ (1+\gamma_2 T)\psi(kT)+\gamma_3 T^{\frac{1}{2}}\|u_m\|_{2,\,[kT,(k+1)T)} & \text{else.}\end{cases} \qquad (35)$$

Now we analyze the system every $r(T)$ steps apart, and use the fact that $\bar{\theta}(t)$ will be absolutely continuous except for at most one of these steps to obtain a crude bound on ψ:

$$\begin{aligned}\psi[kT &+ r(T)T]\leq e^{\lambda_2 T[r(T)-1]}(1+\gamma_2 T)\psi[kT]+\\ &\gamma_3 T^{\frac{1}{2}}\sum_{j=k}^{k+r(T)-1}(1+\gamma_2 T)^{(k+r(T)-1-j)}\|u_m\|_{2,\,[jT,(j+1)T)},\\ &\leq e^{\lambda_2 T[r(T)-1]}(1+\gamma_2 T)\psi[kT]\;+\;\gamma_3 T^{\frac{1}{2}}\left[\frac{(1+\gamma_2 T)^{2r(T)}-1}{(1+\gamma_2 T)^2-1}\right]^{\frac{1}{2}}\times\\ &\quad \|u_m\|_{2,\,[kT,(k+r(T))T)},\end{aligned}$$

which for small T:

$$\begin{aligned}\psi[kT+r(T)T] &\leq e^{\lambda_2 T[r(T)-1]}e^{\gamma_2 T}\psi[kT]\;+\;\underbrace{\left(\gamma_3\sqrt{\frac{e^{2\gamma_2 T_0}-1}{2\gamma_2}}+1\right)}_{=:\gamma_4}\times\\ &\quad \|u_m\|_{2,\,[kT,(k+r(T))T)}.\end{aligned}$$

For small T we have

$$\lambda_2 T[r(T)-1]+\gamma_2 T\leq\lambda r(T)T,$$
$$\lambda r(T)T\leq\frac{1}{2}\lambda T_0;$$

then

$$\begin{aligned}\psi[kT+r(T)T] &\leq e^{\lambda r(T)T}\psi[kT]+\gamma_4\|u_m\|_{2,\,[kT,kT+r(T)T)}\\ \Rightarrow\ \psi[ir(T)T] &\leq (e^{\lambda T})^{ir(T)}\psi(0)+\gamma_4\sum_{l=0}^{i-1}e^{\lambda T\,(i-1-l)r(T)}\|u_m\|_{2,\,[lr(T)T,(l+1)r(T)T)}.\end{aligned} \tag{36}$$

It follows that for small T:

$$\begin{aligned}\psi[ir(T)T] &\leq (e^{\lambda T})^{ir(T)}\psi(0)+\gamma_4\sqrt{1+e^{2\lambda Tr(T)}+e^{4\lambda Tr(T)}+\cdots}\|u_m\|_{2,\,[0,ir(T)T)}\\ &\leq (e^{\lambda T})^{ir(T)}\psi(0)+\gamma_4\frac{1}{\sqrt{1-e^{\lambda T_0/2}}}\|u_m\|_{2,\,[0,ir(T)T)}.\end{aligned}$$

To see what happens at the sample points between $ir(T)T$ and $(i+1)r(T)T$, for $j\in\{0,1,\cdots,r(T)-1\}$ we can use the crude bound provided in (35) and argue as above:

$$\begin{aligned}&\psi[ir(T)T+jT]\leq(e^{\gamma_2 T})^{r(T)}\psi[ir(T)T]+\gamma_4\ \ \|u_m\|_{[ir(T)T,ir(T)T+jT)}\\ \leq\ &(e^{\gamma_2 T})^{r(T)}\ \{(e^{\lambda T})^{ir(T)}\psi(0)+\gamma_4\frac{1}{\sqrt{1-e^{\lambda T_0/2}}}\|u_m\|_{2,\,[0,ir(T)T)}\ \}+\\ &\gamma_4\ \|u_m\|_{[ir(T)T,ir(T)T+jT)}\\ \leq\ &e^{\gamma_2 T_0}e^{-\lambda T_0}(e^{\lambda T})^{ir(T)+j}\psi(0)+\underbrace{\gamma_4(1+\frac{1}{\sqrt{1-e^{\lambda T_0/2}}})}_{=:\gamma_5}\times\|u_m\|_{2,\,[0,ir(T)T+jT)},\end{aligned} \tag{37}$$

for small T. Hence, for small T

$$\|\psi(kT)\|\leq e^{(\gamma_2-\lambda)T_0}(e^{\lambda T})^k\psi(0)+\gamma_5\|u_m\|_{2,\,[0,kT)},$$

so using the definition of ψ, it follows that

$$\begin{aligned}\|\hat{\bar{x}}^{\varepsilon}(kT)\| &\leq \psi(kT)\\ &\leq \underbrace{\gamma_1 e^{(\gamma_2-\lambda)T_0}}_{=:\gamma_6}(e^{\lambda T})^k\|\bar{x}_0\|+\gamma_5\|u_m\|_{2,\,[0,kT)}.\end{aligned} \tag{38}$$

We now examine the deviation between $\hat{\bar{x}}^{\varepsilon}$ and $\bar{x}^{\varepsilon}(kT)$. To this end, define

$$\tilde{\bar{x}}^{\varepsilon}(t):=\hat{\bar{x}}^{\varepsilon}(t)-\bar{x}^{\varepsilon}(t).$$

Using (33) and (38) it follows immediately that there exists a constant $\gamma_7>0$ so that for small $T>0$ we have that the following crude bound holds at the sample points:

$$\|\tilde{\bar{x}}^{\varepsilon}(kT)\|\leq\gamma_7(e^{\lambda kT}\|\bar{x}_0\|+\|u_m\|_{2,\,[0,kT)}),\quad k\geq 0.$$

Now we use (30) to prove that nothing untoward happens between sample points. So for $t \in [kT, (k+1)T)$:

$$\begin{aligned}\|\hat{\bar{x}}^\varepsilon(t) - \hat{\bar{x}}^\varepsilon(kT)\| &\leq \|(\Phi_{cl}^\varepsilon(t, kT) - I)\hat{\bar{x}}^\varepsilon(kT)\| + \int_{kT}^t \|\Phi_{cl}^\varepsilon(t,\tau)\| \, \|\bar{E}\| \, \|u_m(\tau)\| d\tau + \\ &\qquad \gamma_2(T\|\hat{\bar{x}}^\varepsilon(kT)\| + T^{\frac{1}{2}}\|u_m\|_{2\ [kT,t)}).\end{aligned}$$

It follows that there exists a constant $\gamma_8 > 0$ such that

$$\|\hat{\bar{x}}^\varepsilon(t) - \hat{\bar{x}}^\varepsilon(kT)\| \leq \gamma_8\ (T\|\hat{\bar{x}}^\varepsilon(kT)\| + T^{\frac{1}{2}}\|u_m\|_{2\ [kT,t)}), \quad t \in [kT, (k+1)T). \tag{39}$$

Combining this with (38), for small T:

$$\begin{aligned}\|\hat{\bar{x}}^\varepsilon(t)\| &\leq (1+\gamma_8 T)\|\hat{\bar{x}}^\varepsilon(kT)\| + \gamma_8 T^{\frac{1}{2}}\|u_m\|_{2\ [kT,t)} \\ &\leq (1+\gamma_8 T)\gamma_6((e^{\lambda T})^k\|\bar{x}_0\| + \gamma_5\|u_m\|_{2,\ [0,kT)}) + \gamma_8 T^{\frac{1}{2}}\|\bar{u}_m\|_{2\ [kT,t)} \\ &\leq (1+\gamma_8 T)\gamma_6 e^{\lambda t}e^{-\lambda T}\|\bar{x}_0\| + ((1+\gamma_8 T)\gamma_6\gamma_5 + \gamma_8 T^{\frac{1}{2}})\|u_m\|_{2\ [kT,t)}, \\ &\qquad t \in [kT, (k+1)T).\end{aligned}$$

Thus there exists a constant $\gamma_9 > 0$ such that for small T

$$\|\hat{\bar{x}}^\varepsilon(t)\| \leq \gamma_9(e^{\lambda t}\|\bar{x}_0\| + \|u_m\|_{2\ [0,t)}), \quad t \geq 0, \tag{40}$$

as desired.

Now we will examine $\tilde{\bar{x}}^\varepsilon$ to get a bound on $\sum_{k=0}^{\infty}\|\tilde{x}^\varepsilon(kT)\|^2$. Before proceeding, we need a technical result, which is handy for the rest of the proof of this proposition:

Claim A.1:

$$\sum_{j=0}^{\infty}\|\hat{\bar{x}}^\varepsilon[jT]\|^2 = \mathcal{O}(\frac{1}{T})\|\bar{x}_0\|^2 + \mathcal{O}(\frac{1}{T})\|u_m\|_2^2.$$

Proof: The proof of Claim A.1 is given below.

First examine the behaviour of $\tilde{x}^\varepsilon$ at the sample points. If we combine (28) evaluated at $t = (k+1)T$ with (33), we end up with

$$\tilde{\bar{x}}^\varepsilon[(k+1)T] = \Phi_{cl}^\varepsilon((k+1)T, kT)\tilde{\bar{x}}^\varepsilon[kT] + \Delta_1(kT) + \Delta_2(kT) + \Delta_3(kT) + \Delta_4(kT),$$

so

$$\begin{aligned}
\tilde{\bar{x}}^{\varepsilon}[(k+1)T] &\leq \gamma_1 \sum_{j=0}^{k-1} e^{\lambda_1(k-1-j)T}(\|\Delta_1(kT)\| + \|\Delta_2(kT)\| + \|\Delta_3(kT)\| + \|\Delta_4(kT)\|) \\
&= \underbrace{\sum_{j=0}^{k-1} \gamma_1 e^{\lambda_1(k-1-j)T}\|\Delta_1(kT)\|}_{=:S_1[k]} + \underbrace{\sum_{j=0}^{k-1} \gamma_1 e^{\lambda_1(k-1-j)T}\|\Delta_2(kT)\|}_{=:S_2[k]} + \\
&\quad \underbrace{\sum_{j=0}^{k-1} \gamma_1 e^{\lambda_1(k-1-j)T}\|\Delta_3(kT)\|}_{=:S_3[k]} + \underbrace{\sum_{j=0}^{k-1} \gamma_1 e^{\lambda_1(k-1-j)T}\|\Delta_4(kT)\|}_{=:S_4[k]}.
\end{aligned}$$

Now from the definition of Δ_1 and Δ_2 together with Claim A.1 it follows that

$$\begin{aligned}
\|S_1\|_2 &\leq \left(\frac{\gamma_1^2}{1 - e^{2\lambda_1 T}}\right)^{\frac{1}{2}} \left(\sum_{j=0}^{\infty} \|\Delta_1(jT)\|^2\right)^{\frac{1}{2}} \\
&\leq \gamma_2 T^2 \left(\frac{\gamma_1^2}{1 - e^{2\lambda_1 T}}\right)^{\frac{1}{2}} \|\hat{\bar{x}}^{\varepsilon}\|_2 \\
&= \mathcal{O}(T^2)\mathcal{O}(\frac{1}{T})^{\frac{1}{2}}(\mathcal{O}(\frac{1}{T})^{\frac{1}{2}}\|\bar{x}_0\| + \mathcal{O}(\frac{1}{T})^{\frac{1}{2}}\|u_m\|_2) \\
&= \mathcal{O}(T)\|\bar{x}_0\| + \mathcal{O}(T)\|u_m\|_2,
\end{aligned}$$

and for small T:

$$\begin{aligned}
\|S_2\|_2 &\leq \left(\frac{\gamma_1^2}{1 - e^{2\lambda r(T)T}}\right)^{\frac{1}{2}} \left(\sum_{j=0}^{\infty} \|\Delta_2(jT)\|^2\right)^{\frac{1}{2}}, \\
&\leq \gamma_2 T \left(\frac{2\gamma_1^2}{1 - e^{\lambda T_0}}\right)^{\frac{1}{2}} \|\hat{\bar{x}}^{\varepsilon}\|_2, \\
&= \mathcal{O}(T)\ \mathcal{O}(1)\ (\mathcal{O}(\frac{1}{T})^{\frac{1}{2}}\|\bar{x}_0\| + \mathcal{O}(\frac{1}{T})^{\frac{1}{2}}\|u_m\|_2), \\
&= \mathcal{O}(T^{\frac{1}{2}})\|\bar{x}_0\| + \mathcal{O}(T^{\frac{1}{2}})\|u_m\|_2.
\end{aligned}$$

Using the definition of Δ_3 we can write

$$\begin{aligned}
\|S_3\|_2 &\leq \left(\frac{\gamma_1^2}{1 - e^{2\lambda_1 T}}\right)^{\frac{1}{2}} \left(\sum_{j=0}^{\infty} \|\Delta_3(jT)\|^2\right)^{\frac{1}{2}} \\
&\leq \gamma_2 T^{\frac{3}{2}} \left(\frac{\gamma_1^2}{1 - e^{2\lambda_1 T}}\right)^{\frac{1}{2}} \|u_m\|_2 \\
&= \mathcal{O}(T^{\frac{3}{2}})\mathcal{O}(T^{\frac{-1}{2}})\|u_m\|_2 \\
&= \mathcal{O}(T)\|u_m\|_2.
\end{aligned}$$

Using the definition of Δ_4, for small T we have

$$\begin{aligned}\|S_4\|_2 &\leq (\frac{\gamma_1^2}{1-e^{2\lambda_1 r(T)T}})^{\frac{1}{2}}(\sum_{k=0}^{\infty}\|\Delta_4(kT)\|^2)^{\frac{1}{2}}\\ &\leq \gamma_2 T^{\frac{1}{2}}\ (\frac{2\gamma_1^2}{1-e^{\lambda_1 T_0}})^{\frac{1}{2}}\|u_m\|_2\\ &= \mathcal{O}(T^{\frac{1}{2}})\|u_m\|_2.\end{aligned}$$

Combining S_1, S_2, S_3 and S_4 together gives

$$(\sum_{k=0}^{\infty}\|\tilde{\bar{x}}^{\varepsilon}(kT)\|^2)^{\frac{1}{2}} = \mathcal{O}(T^{\frac{1}{2}})\|\bar{x}_0\| + \mathcal{O}(T^{\frac{1}{2}})\|u_m\|_2. \tag{41}$$

Now we use the bound on $\sum_{k=0}^{\infty}\|\tilde{\bar{x}}^{\varepsilon}(kT)\|^2$ given in (41) and the bound on $\sum_{k=0}^{\infty}\|\hat{\bar{x}}^{\varepsilon}(kT)\|^2$ given in Claim A.1 to obtain a bound on $\|\hat{\bar{x}}^{\varepsilon} - \bar{x}^{\varepsilon}\|_2$. By the definition of 2-norm, it follows that

$$\|\hat{\bar{x}}^{\varepsilon} - \bar{x}^{\varepsilon}\|_2^2 = \int_0^{\infty}\|\hat{\bar{x}}^{\varepsilon}(t) - \bar{x}^{\varepsilon}(t)\|^2 dt = \sum_{k=0}^{\infty}\int_{kT}^{(k+1)T}\|\hat{\bar{x}}^{\varepsilon}(t) - \bar{x}^{\varepsilon}(t)\|^2 dt.$$

If we use the formula for $\bar{x}^{\varepsilon}$ from (28) it follows that

$$\begin{aligned}\|\hat{\bar{x}}^{\varepsilon} - \bar{x}^{\varepsilon}\|_2^2 &= \sum_{k=0}^{\infty}\Bigg(\int_{kT}^{(k+1)T}\|\hat{\bar{x}}^{\varepsilon}(t) - \Phi_{cl}^{\varepsilon}(t,kT)\bar{x}^{\varepsilon}(kT) -\\ &\qquad \int_{kT}^{t}\Phi_{cl}^{\varepsilon}(t,\tau)\bar{E}u_m(\tau)d\tau\|^2 dt\Bigg)\\ &\leq 2\sum_{k=0}^{\infty}\Bigg[\int_{kT}^{(k+1)T}\|\hat{\bar{x}}^{\varepsilon}(t) - \Phi_{cl}^{\varepsilon}(t,kT)\hat{\bar{x}}^{\varepsilon}(kT) +\\ &\qquad \int_{kT}^{t}\Phi_{cl}^{\varepsilon}(t,\tau)\bar{E}u_m(\tau)d\tau\|^2 dt +\\ &\qquad \int_{kT}^{(k+1)T}\underbrace{\|\Phi_{cl}^{\varepsilon}(t,kT)\ (\hat{\bar{x}}^{\varepsilon}(kT) - \bar{x}^{\varepsilon}(kT))\|^2}_{\leq\|\Phi_{cl}^{\varepsilon}(t,kT)\|^2\ \|\hat{\bar{x}}^{\varepsilon}(kT)-\bar{x}^{\varepsilon}(kT)\|^2} dt\Bigg].\end{aligned}$$

Using (30) it follows that

$$\begin{aligned}\|\hat{\bar{x}}^{\varepsilon} - \bar{x}^{\varepsilon}\|_2^2 &\leq 2\sum_{k=0}^{\infty}[\int_{kT}^{(k+1)T} 2\gamma_2^2(T^2\|\hat{\bar{x}}^{\varepsilon}(kT)\|^2 + T\|u_m\|_{2,\ [kT,t)}^2)dt +\\ &\qquad \gamma_1^2\int_{kT}^{(k+1)T}\|\hat{\bar{x}}^{\varepsilon}(kT) - \bar{x}^{\varepsilon}(kT)\|^2 dt].\end{aligned}$$

This equation yields

$$\begin{aligned}\|\hat{\bar{x}}^{\varepsilon} - \bar{x}^{\varepsilon}\|_2^2 &\leq 2T\sum_{k=0}^{\infty}\left[2\gamma_2^2(T^2\|\hat{\bar{x}}^{\varepsilon}(kT)\|^2 + T\|u_m\|_{2,\ [kT,(k+1)T)}^2) +\right.\\ &\qquad \left.\gamma_1^2\|\hat{\bar{x}}^{\varepsilon}(kT) - \bar{x}^{\varepsilon}(kT)\|^2\right].\end{aligned}$$

Using Claim A.1 and (41), it follows that

$$\begin{aligned}\|\hat{\bar{x}}^{\varepsilon}-\bar{x}^{\varepsilon}\|_2^2 &= \mathcal{O}(T^3)(\mathcal{O}(\frac{1}{T})\|\bar{x}_0\|^2+\mathcal{O}(\frac{1}{T})\|u_m\|_2^2)+\mathcal{O}(T^2)\|u_m\|_2^2+\\ &\quad \mathcal{O}(T)(\mathcal{O}(T)\|\bar{x}_0\|^2+\mathcal{O}(T)\|u_m\|_2)\\ &= \mathcal{O}(T^2)\|\bar{x}_0\|^2+\mathcal{O}(T^2)\|u_m\|_2^2.\end{aligned}$$

It follows that there exists a $\gamma_{10}>0$ such that for small T

$$\|\hat{\bar{\bar{x}}}^{\varepsilon}-\bar{x}^{\varepsilon}\|_2 \leq \gamma_{10}(T\|\bar{x}_0\|+T\|u_m\|_2),$$

as desired.

□

Proof of Claim A.1:

We need to prove:

$$\sum_{j=0}^{\infty}\|\hat{\bar{x}}^{\varepsilon}[jT]\|^2=\mathcal{O}(\frac{1}{T})\|\bar{x}_0\|^2+\mathcal{O}(\frac{1}{T})\|u_m\|_2^2.$$

From (36) we have

$$\psi[ir(T)T] \leq \underbrace{(e^{\lambda T})^{ir(T)}\psi(0)}_{t_1[i]}+\underbrace{\gamma_4\sum_{l=0}^{i-1}e^{\lambda T\,(i-1-l)r(T)}\|u_m\|_{2,\,[lr(T)T,(l+1)r(T)T)}}_{t_2[i]},$$

so

$$\begin{aligned}\sum_{i=0}^{\infty}|\psi[ir(T)T]|^2 &= \sum_{i=0}^{\infty}|t_1[i]+t_2[i]|^2\\ &\leq 2\sum_{i=0}^{\infty}|t_1[i]|^2+2\sum_{i=0}^{\infty}|t_2[i]|^2.\end{aligned}$$

But for small T

$$\begin{aligned}\sum_{i=0}^{\infty}|t_1[i]|^2 &= \sum_{i=0}^{\infty}(e^{2\lambda Tr(T)})^i\,\psi^2(0)\\ &= \frac{1}{1-e^{2\lambda Tr(T)}}\psi^2(0)\\ &\leq \frac{2}{1-e^{2\lambda T_0}}\psi^2(0)\\ &= \mathcal{O}(1)\psi^2(0). \qquad (42)\end{aligned}$$

With the definition of $g[i]:=\|u_m\|_{2,\,[ir(T)T,(i+1)r(T)T)}$, we can obtain $t_2[i]$ as the output of a digital filter as shown in Figure 1. Parseval's theorem states that for

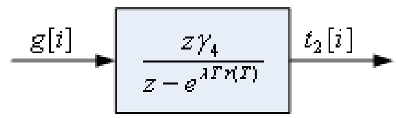

Figure 1: Block diagram of $t_2[i]$ and $g[i]$

small T

$$\begin{aligned}\sum_{i=0}^{\infty}|t_2[i]|^2 &\leq \sup_{|z|=1}|\frac{\gamma_4 z}{z-e^{\lambda\, Tr(T)}}|^2 \sum_{i=0}^{\infty}|g[i]|^2 \\ &= |\frac{\gamma_4}{1-e^{\lambda\, Tr(T)}}|^2 \, \|u_m\|_2^2 \\ &\leq \frac{2\gamma_4^2}{(1-e^{\lambda\, T_0/2})^2}\|u_m\|_2^2 \\ &= \mathcal{O}(1)\|u_m\|_2^2. \end{aligned} \tag{43}$$

Combining this equation with (42), it follows that

$$\sum_{i=0}^{\infty}|\psi[ir(T)T]|^2 \;=\; \mathcal{O}(1)\psi^2(0)+\mathcal{O}(1)\|u_m\|_2^2;$$

this means that there exists a constant γ_{11} so that for small T:

$$\sum_{i=0}^{\infty}|\psi[ir(T)T]|^2 \;\leq\; \gamma_{11}\psi^2(0)+\gamma_{11}\|u_m\|_2^2. \tag{44}$$

Now let us look between multipliers of $r(T)T$. To this end, using the fact that the controller is periodic of period T, it follows from (36) that for every $i \in \mathbf{Z}^+$ and $j \in \{0,1,\cdots,r(T)\}$:

$$\begin{aligned}\psi[ir(T)T+jT] \;\leq\; & (e^{\lambda T})^{ir(T)}\psi(jT)+ \\ & \gamma_4\sum_{l=0}^{i-1}e^{\lambda T\;(i-1-l)r(T)}\|u_m\|_{2,\,[lr(T)T+jT,(l+1)r(T)T+jT)}.\end{aligned}$$

If we argue in the same way as that used to obtain (44), we have

$$\sum_{i=0}^{\infty}|\psi[ir(T)T+jT]|^2 \;\leq\; \gamma_{11}\psi^2(jT)+\gamma_{11}\|u_m\|_2^2, \quad j=0,1,\cdots,r(T)-1.$$

Hence

$$\begin{aligned}\sum_{k=0}^{\infty}|\psi[kT]|^2 &= \sum_{j=0}^{r(T)-1}\sum_{i=0}^{\infty}|\psi[ir(T)T+jT]|^2 \\ &\leq \sum_{j=0}^{r(T)-1}(\gamma_{11}|\psi(jT)|^2+\gamma_{11}\|u_m\|_2^2) \\ &\leq \left(\sum_{j=0}^{r(T)-1}\gamma_{11}|\psi(jT)|^2\right)+\gamma_{11}r(T)\|u_m\|_2^2. \end{aligned} \tag{45}$$

Using (35) it follows that

$$\begin{aligned} \psi[(k+1)T] &\leq (1+\gamma_2 T)\psi[kT] + \gamma_3 T^{\frac{1}{2}} \|u_m\|_{2,\ [kT,(k+1)T)} \\ \Rightarrow \psi[kT] &\leq (1+\gamma_2 T)^k \psi[0] + \gamma_3 T^{\frac{1}{2}} \sum_{j=0}^{k-1} \|(1+\gamma_2 T)^{k-1-j} \|u_m\|_{2,\ [jT,(j+1)T)}. \end{aligned}$$

Now for $k \in 0, 1, \cdots, r(T)-1$ we have

$$(1+\gamma_2 T)^k \;\leq\; (e^{\gamma_2 T})^{r(T)} \;\leq\; e^{\gamma_2 T_0},$$

so

$$\psi[kT] \;\leq\; \underbrace{e^{\gamma_2 T_0}\psi[0]}_{=:\psi_1[kT]} + \underbrace{\gamma_3 T^{\frac{1}{2}} \sum_{j=0}^{k-1} e^{\gamma_2 T_0} \|u_m\|_{2,\ [jT,(j+1)T)}}_{=:\psi_2[kT]},$$

which results in

$$\sum_{j=0}^{r(T)-1} \psi^2[jT] \;\leq\; 2 \sum_{j=0}^{r(T)-1} \psi_1^2[jT] + 2 \sum_{j=0}^{r(T)-1} \psi_2^2[jT]. \tag{46}$$

It is easy to see that

$$\sum_{j=0}^{r(T)-1} \psi_1^2[jT] \;\leq\; r(T) e^{2\,\gamma_2 T_0} \psi^2[0].$$

Since $r(T) = \mathcal{O}(\frac{1}{T})$, we have

$$\sum_{j=0}^{r(T)-1} \psi_1^2[jT] \;\leq\; \mathcal{O}(\frac{1}{T})\psi^2[0]. \tag{47}$$

Now we need to get a bound for $\sum_{j=0}^{r(T)-1} \psi_2^2[jT]$. First we rewrite $\psi_2[kT]$, $k \in 0, 1, \ldots, r(T)-1$, in matrix form:

$$\begin{bmatrix} \psi_2[0] \\ \psi_2[T] \\ \vdots \\ \psi_2[(r(T)-1)T] \end{bmatrix} = \gamma_3 T^{\frac{1}{2}} e^{\gamma_2 T_0} \underbrace{\begin{bmatrix} 0 & & & & \\ 1 & \ddots & & & \\ & \ddots & & & \\ 1 & 1 & \cdots & 1 & 0 \end{bmatrix}}_{W_{r(T)}} \begin{bmatrix} \|u_m\|_{2,\ [0,T)} \\ \|u_m\|_{2,\ [T,2T)} \\ \vdots \\ \|u_m\|_{2,\ [(r(T)-1)T, r(T)T)} \end{bmatrix}.$$

Using Greshgorin Circles [42], it can be easily proven that $\|W_k\|^2 \;\leq\; \frac{1}{2}k^2$, so we have

$$\begin{aligned}
\sum_{j=0}^{r(T)-1} \psi_2^2[jT] &\leq \gamma_3^2\ T\ e^{2\gamma_2 T_0} \|W_{r(T)}\|^2 \sum_{j=0}^{r(T)-1} \|u_m\|^2_{2,\ [jT,(j+1)T)} \\
&\leq \gamma_3^2\ T\ e^{2\gamma_2 T_0} \frac{1}{2}\ r^2(T) \sum_{j=0}^{r(T)-1} \|u_m\|^2_{2,\ [jT,(j+1)T)} \\
&= \mathcal{O}(\frac{1}{T}) \|u_m\|^2_{2,\ [0,r(T)T)} \\
&\leq \mathcal{O}(\frac{1}{T}) \|u_m\|^2_2.
\end{aligned}$$

Combining this equation with (45), (46), (47) and using the definition of ψ, it follows that:

$$\begin{aligned}
\sum_{k=0}^{\infty} \|\hat{\bar{x}}^{\varepsilon}[kT]\|^2 &\leq \gamma_{11}^2 \sum_{j=0}^{r(T)-1} |\psi(jT)|^2 + \gamma_{11} r(T) \|u_m\|_2^2 \\
&\leq 2\gamma_{11}^2 \left(\sum_{j=0}^{r(T)-1} |\psi_1(jT)|^2 + \sum_{j=0}^{r(T)-1} |\psi_2(jT)|^2 \right) + \gamma_{11} r(T) \|u_m\|_2^2 \\
&= 2\gamma_{11}^2 \left(\mathcal{O}(\frac{1}{T}) \psi^2(0) + \mathcal{O}(\frac{1}{T}) \|u_m\|_2^2, \right) + \gamma_{11} \mathcal{O}(\frac{1}{T}) \|u_m\|_2^2 \\
&= \mathcal{O}(\frac{1}{T}) \psi^2(0) + \mathcal{O}(\frac{1}{T}) \|u_m\|_2^2 \\
&= \mathcal{O}(\frac{1}{T}) \|\bar{x}_0\|^2 + \mathcal{O}(\frac{1}{T}) \|u_m\|_2^2,
\end{aligned}$$

as desired.

□

.2 APPENDIX B

Proof of Lemma 5.1 (KCL-5):

First define

$$\gamma_1 := \sup_{\bar\theta \in \bar{\mathcal{P}}} (\|\bar{A}\|_\infty + \|\bar{A}_{cl}\|_\infty + \|g\bar{B}\|_\infty + \|\bar{E}\| + ||g||_\infty + ||\bar{B}||_\infty),$$

$$\gamma_2 := \sup_{\bar\theta \in \bar{\mathcal{P}}} (\|\dot{\bar{A}}\|_\infty + \|\dot{g}\|_\infty);$$

it follows from the standing assumptions that both γ_1 and γ_2 are finite. (Notice that our ∞-norm contains an essential supremum, so that infinite derivatives at the discontinuities do not show up when computing γ_2.)

First we look at the worst case scenario, in which there may be a parameter jump during a period. From the KEL there exists a constant $\gamma_3 > 0$ so that for small T

$$\begin{aligned}
\|\hat{\phi}_0(kT)\| &\leq \gamma_3(\|\bar{x}(kT)\| + |\hat{u}^o(kT)| + \|u_m\|_\infty), \\
\|\hat{\phi}_1(kT)\| &\leq \gamma_3 \left(\|\bar{x}(kT)\| + \|\bar{x}(kT+mh)\| + |\hat{u}^o(kT)| + \|\hat{\phi}_0(kT)\| + \|u_m\|_\infty\right), \\
\|\hat{\phi}_i(kT)\| &\leq \gamma_3 \left(\|\bar{x}(kT)\| + \|\bar{x}(kT+imh)\| + |\hat{u}^o(kT)| + \right. \\
&\qquad \left. \|\hat{\phi}_{i-1}(kT)\| + \|\hat{\phi}_{i-2}(kT)\| + \|u_m\|_\infty\right), \qquad i = 2, \cdots, q.
\end{aligned}$$

Solving iteratively, it follows that there exists a constant $\tilde{\gamma}_3 > 0$ so that for small T:

$$\|\hat{\phi}_i(kT)\| \leq \tilde{\gamma}_3 \left(\max_{\tau \in [kT,(k+1)T]} \|\bar{x}(\tau)\| + |\hat{u}^o(kT)| + \|u_m\|_\infty \right), \quad i = 1, \cdots, q. \quad (48)$$

From the definition of the controller we have

$$||u(t)|| = \begin{cases}
||\hat{u}^o(kT)|| & t \in [kT, kT+mh), \\
||\rho\hat{\phi}_0(kT) + \hat{u}^o(kT)|| & \\
 & t \in [kT+mh, kT+2mh), \\
 & \\
||\rho\hat{\phi}_{i-1}(kT) - \rho\hat{\phi}_{i-2}(kT) + \hat{u}^o(kT)|| & \\
 & t \in [kT+imh, kT+(i+1)mh), \\
 & i = 2, \cdots, q, \\
 & \\
|| - \rho\hat{\phi}_{q-1}(kT) + \hat{u}^o(kT)|| & kT+(q+1)mh, kT+(q+2)mh),
\end{cases}$$

and

$$\hat{u}^o(kT+T) = \hat{u}^o(kT) + \sum_{i=0}^{q} c_i \hat{\phi}_i(kT).$$

Combining this with (48) and we see that there exists a constant γ_4 so that for small T

$$\|u(t)\| \leq \gamma_4(\max_{\tau\in[kT,(k+1)T]} \|\bar{x}(\tau)\| + |\hat{u}^o(kT)| + \|u_m\|_\infty), \quad t \in [kT,(k+1)T) \tag{49}$$

$$|\hat{u}^o(kT+T)| \leq \gamma_4(\max_{\tau\in[kT,(k+1)T]} \|\bar{x}(\tau)\| + |\hat{u}^o(kT)| + \|u_m\|_\infty). \tag{50}$$

Now let us examine the system equation

$$\dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + g(t)\bar{B}u(t) + \bar{E}u_m(t). \tag{51}$$

Solving this equation, using Lemma A.1, and the bound on u, it follows that

$$\begin{aligned}\|\bar{x}(t)\| \;\leq\; & e^{\gamma_1 T}\|\bar{x}(kT)\| + (e^{\gamma_1 T}-1)[\gamma_4 \max_{\tau\in[kT,(k+1)T]} \|\bar{x}(\tau)\| + \gamma_4|\hat{u}^o(kT)| + \\ & (1+\gamma_4)\|u_m\|_\infty)]\,, \qquad t\in[kT,(k+1)T].\end{aligned}$$

By simplifying this implicit inequality in $||\bar{x}(t)||$, it follows that for small T we have

$$\begin{aligned}\|\bar{x}(t)\| \;\leq\; & \frac{1+2\gamma_1 T}{1-2\gamma_1\gamma_4 T}\|\bar{x}(kT)\| + 2\gamma_1\frac{\gamma_4}{1-2\gamma_1\gamma_4 T}T|\hat{u}^o(kT)| + \\ & 2\gamma_1\frac{1+\gamma_4}{1-2\gamma_1\gamma_4 T}T\|u_m\|_\infty, \quad t\in[kT,(k+1)T].\end{aligned} \tag{52}$$

If we now solve (51) once again and use Lemma A.1, we end up with

$$\begin{aligned}\|\bar{x}(t)-\bar{x}(kT)\| \;\leq\; & (e^{\gamma_1 T}-1)\|\bar{x}(kT)\| + (e^{\gamma_1 T}-1)\;\times \\ & [\gamma_4 \max_{\tau\in[kT,(k+1)T]} \|\bar{x}(\tau)\| + \gamma_4|\hat{u}^o(kT)| + (1+\gamma_4)\|u_m\|_\infty], \\ & \qquad\qquad t\in[kT,(k+1)T].\end{aligned}$$

If we combine this with (49), (50), and (52) and the definition of u it follows that there exists a constant γ_5 so that for small T we have

$$\left.\begin{aligned}\|\bar{x}(t)-\bar{x}(kT)\| \;&\leq\; \gamma_5 T(\|\bar{x}(kT)\| + |\hat{u}^o(kT)| + \|u_m\|_\infty), \\ \|u(t)\| \;&\leq\; \gamma_5(\|\bar{x}(kT)\| + |\hat{u}^o(kT)| + \|u_m\|_\infty), \;\; t\in[kT,(k+1)T], \\ |\hat{u}^o(kT+T)| \;&\leq\; \gamma_5(\|\bar{x}(kT)\| + |\hat{u}^o(kT)| + \|u_m\|_\infty), \\ |\phi_i(kT)| \;&\leq\; \gamma_5(\|\bar{x}(kT)\| + |\hat{u}^o(kT)| + \|u_m\|_\infty), \quad i=0,1,\cdots,q.\end{aligned}\right\} \tag{53}$$

From Lemma A.1 applied to $\bar{A}_{cl}(t)$, it follows that there exists a constant γ_6 so that for all $\bar{\theta}\in\bar{\mathcal{P}}$, we have

$$\begin{aligned}\|\Phi_{cl}(t,kT)-I\| \;&\leq\; (e^{\|\bar{A}_{cl}\|_\infty(t-kT)}-1) \leq \gamma_6 T, \\ \|\int_{kT}^t \Phi_{cl}(t,\tau)\bar{E}u_m(\tau)d\tau\| \;&\leq\; \int_{kT}^t [e^{\|\bar{A}_{cl}\|_\infty(t-\tau)}\|\bar{E}\|\times\|u_m\|_\infty]d\tau \\ &\leq\; \gamma_6 T\|u_m\|_\infty, \;\; t\in[kT,(k+1)T],\end{aligned}$$

for small T. If we combine this with (53) we have

$$\begin{aligned}
\|\bar{x}(t) &- \Phi_{cl}(t,kT)\bar{x}(kT) - \int_{kT}^{t}\Phi_{cl}(t,\tau)\bar{E}u_m(\tau)d\tau\| \\
&\leq \|\bar{x}(t)-\bar{x}(kT)\| + \|(\Phi_{cl}(t,kT)-I)\bar{x}(kT)\| + \|\int_{kT}^{t}\Phi_{cl}(t,\tau)\bar{E}u_m(\tau)d\tau\| \\
&\leq \gamma_5 T(\|\bar{x}(kT)\| + |\hat{u}^o(kT)| + |u_m\|_\infty) + \gamma_6 T\|\bar{x}(kT)\| + \gamma_6 T\|u_m\|_\infty \\
&\leq (\gamma_5+\gamma_6)T(\|\bar{x}(kT)\| + |\hat{u}^o(kT)| + |u_m\|_\infty), \quad t\in[kT,(k+1)T],
\end{aligned}$$

for small T, as required. If we combine this with (53), the proof of part (i) is complete.

Now we look at the more interesting case of when $\bar{\theta}\in\bar{\mathcal{P}}$ is absolutely continuous on $[kT,(k+1)T)$. Define

$$L(t) = \left[\begin{array}{cccc} c_1(t) & f_2(t) & 0 & 0 \end{array}\right].$$

From the KEL we know that there exists a constant γ_7 so that for small T we have

$$\|\hat{\phi}_0(kT) - K(kT)\bar{x}(kT) + g(kT)\hat{u}^o(kT)\| \leq \gamma_7 T(\|\bar{x}(kT)\| + |\hat{u}^o(kT)| + \|u_m\|_\infty), \quad (54)$$

$$\begin{aligned}
&\|\hat{\phi}_1(KT) - \frac{1}{\rho}L(kT+mh)\bar{x}(kT+mh) + \frac{1}{\rho}L(kT)\bar{x}(kT) - \\
&\quad g(kT+mh)\hat{\phi}_0(kT) + \frac{1}{\rho}g(kT)\hat{u}^o(kT) - \frac{1}{\rho}g(kT+mh)\hat{u}^o(kT)\| \\
\leq\ & \gamma_7 T(\|\bar{x}(kT)\| + \|\bar{x}(kT+mh)\| + \|\phi_0(kT)\| + |\hat{u}^o(kT)| + \|u_m\|_\infty),
\end{aligned}$$

and

$$\begin{aligned}
&\|\hat{\phi}_i(KT) - \frac{1}{\rho}L(kT+imh)\bar{x}(kT+imh) + \frac{1}{\rho}L(kT)\bar{x}(kT) - \\
&\quad g(kT+imh)\hat{\phi}_{i-1}(kT) + \\
&\quad \frac{1}{\rho}g(kT)\hat{u}^o(kT) - \frac{1}{\rho}g(kT+imh)\hat{u}^o(kT) + g(kT+imh)\hat{\phi}_{i-2}(kT) - \hat{\phi}_{i-1}(kT)\| \\
\leq\ & \gamma_7 T(\|\bar{x}(kT)\| + \|\bar{x}(kT+imh)\| + \|\phi_{i-1}(kT)\| + \|\phi_{i-2}(kT)\| + \\
&\quad |\hat{u}^o(kT)| + \|u_m\|_\infty), \qquad i=2,\cdots,q;
\end{aligned}$$

using (53) we see that there exists a constant γ_8 so that

$$\begin{aligned}
&\|\hat{\phi}_1(KT) - \frac{1}{\rho}L(kT+mh)\bar{x}(kT+mh) + \frac{1}{\rho}L(kT)\bar{x}(kT) - \\
&\quad g(kT+mh)\hat{\phi}_0(kT) + \frac{1}{\rho}g(kT)\hat{u}^o(kT) - \frac{1}{\rho}g(kT+mh)\hat{u}^o(kT)\| \\
\leq\ & \gamma_8 T(\|\bar{x}(kT)\| + |\hat{u}^o(kT)| + \|u_m\|_\infty),
\end{aligned}$$

and for $i = 2, \ldots, q$, we have

$$\begin{aligned}
& \|\hat{\phi}_i(KT) - \frac{1}{\rho}L(kT+imh)\bar{x}(kT+imh) + \frac{1}{\rho}L(kT)\bar{x}(kT) - \\
& \quad g(kT+imh)\hat{\phi}_{i-1}(kT) + \\
& \quad \frac{1}{\rho}g(kT)\hat{u}^o(kT) - \frac{1}{\rho}g(kT+imh)\hat{u}^o(kT) + g(kT+imh)\hat{\phi}_{i-2}(kT) - \hat{\phi}_{i-1}(kT)\| \\
\leq \;& \gamma_8 T(\|\bar{x}(kT)\| + |\hat{u}^o(kT)| + \|u_m\|_\infty).
\end{aligned}$$

Now we use (53) together with the fact that L is a sub-matrix of $\bar{A}$ and is absolutely continuous on $[kT,(k+1)T)$ to show that for small T:

$$\begin{aligned}
||\hat{\phi}_1(kT) \;\; - \;\; & g(kT)\hat{\phi}_0(kT)|| \leq \frac{1}{\rho}||[L(kT+mh) - L(kT)]\bar{x}(kT+mh)|| + \\
& \frac{1}{\rho}||L(kT)[\bar{x}(kT+mh) - \bar{x}(kT)]|| + \\
& ||[g(kT+mh) - g(kT)]\hat{\phi}_0(kT)|| + \frac{1}{\rho}|[g(kT+mh) - g(kT)]\hat{u}^o(kT)| + \\
& \gamma_8 T(||\bar{x}(kT)|| + |\hat{u}^o(kT)| + ||u_m||_\infty) \\
\leq \;\; & \frac{1}{\rho}\gamma_2 T(1+\gamma_5 T)[||\bar{x}(kT)|| + |\hat{u}^o(kT)| + ||u_m||_\infty] + \\
& \frac{1}{\rho}\gamma_1\gamma_5 T[||\bar{x}(kT)|| + |\hat{u}^o(kT)| + ||u_m||_\infty] + \\
& \gamma_2\gamma_5 T[||\bar{x}(kT)|| + |\hat{u}^o(kT)| + ||u_m||_\infty] + \frac{1}{\rho}\gamma_2 T|\hat{u}^o(kT)| \\
& \gamma_8 T(||\bar{x}(kT)|| + |\hat{u}^o(kT)| + ||u_m||_\infty),
\end{aligned}$$

and for $i = 2, \cdots, q$:

$$
\begin{aligned}
||\hat{\phi}_i(kT) \quad - \quad & g(kT)\hat{\phi}_{i-1}(kT)|| \leq \frac{1}{\rho}||[L(kT+imh) - L(kT)]\bar{x}(kT+imh)|| + \\
& \frac{1}{\rho}||L(kT)[\bar{x}(kT+imh) - \bar{x}(kT)]|| + \\
& ||[g(kT+imh) - g(kT)]\hat{\phi}_{i-1}(kT)|| + \\
& \frac{1}{\rho}|[g(kT+imh) - g(kT)]\hat{u}^o(kT)| + \\
& |\hat{\phi}_{i-1}(kT) - g(kT)\hat{\phi}_{i-2}(kT)| + |[g(kT+imh) - g(kT)]\hat{\phi}_{i-2}(kT)| + \\
& \gamma_8 T(||\bar{x}(kT)|| + |\hat{u}^o(kT)| + ||u_m||_\infty) \\
\leq \quad & \frac{1}{\rho}\gamma_2 T(1+\gamma_5 T)[||\bar{x}(kT)|| + |\hat{u}^o(kT)| + \\
& ||u_m||_\infty] + \frac{1}{\rho}\gamma_1\gamma_5 T[||\bar{x}(kT)|| + |\hat{u}^o(kT)| + ||u_m||_\infty] + \\
& \gamma_2\gamma_5 T[||\bar{x}(kT)|| + |\hat{u}^o(kT)| + ||u_m||_\infty] + \frac{1}{\rho}\gamma_2 T|\hat{u}^o(kT)| + \\
& |\hat{\phi}_{i-1}(kT) - g(kT)\hat{\phi}_{i-2}(kT)| + \gamma_2\gamma_5 T[||\bar{x}(kT)|| + |\hat{u}^o(kT)| + ||u_m||_\infty] + \\
& \gamma_8 T(||\bar{x}(kT)|| + |\hat{u}^o(kT)| + ||u_m||_\infty),
\end{aligned}
$$

It follows by iteration that there exists a constant γ_9 so that for small T, if $\bar{\theta} \in \bar{\mathcal{P}}$ is absolutely continuous on $[kT, (k+1)T)$ then we have

$$
|\hat{\phi}_i(kT) - g(kT)\hat{\phi}_{i-1}(kT)| \leq \gamma_9 T(\|\bar{x}(kT)\| + |\hat{u}^o(kT)| + \|u_m\|_\infty), \quad i = 1, \cdots, q.
$$

If we combine this equation with the bound on $\hat{\phi}_0(kT)$ given in (54), it follows that there exists a constant γ_{10} so that for small T, if $\bar{\theta} \in \bar{\mathcal{P}}$ is a.c. on $[kT, (k+1)T)$ then

$$
\begin{aligned}
|\hat{\phi}_i(kT) \quad - \quad & g(kT)^i[K(kT)\bar{x}(kT) - g(kT)\hat{u}^o(kT)]| \\
\leq \quad & \gamma_{10} T(\|\bar{x}(kT)\| + |\hat{u}^o(kT)| + \|u_m\|_\infty), \qquad i = 0, 1, \cdots, q. \qquad (55)
\end{aligned}
$$

Recall that $\hat{u}^o(kT+T)$ is constructed by

$$
\hat{u}^o(kT+T) \quad = \quad \hat{u}^o(kT) + \sum_{i=0}^{q} c_i \hat{\phi}_i(kT).
$$

Combining this equation with (55) and defining $\bar{c} := \max\{1, \max_i\{c_i\}\}$ yields:

$$\begin{aligned}
& |\hat{u}^o(kT+T) - [1-\hat{f}^\varepsilon(g(kT))g(kT)]\hat{u}^o(kT) - \hat{f}^\varepsilon(g(kT))K(kT)\bar{x}(kT)| \\
= \; & |\hat{u}^o[(k+1)T] - \hat{u}^o(kT) - \sum_{i=0}^{q} c_i g(kT)^i [K(kT)\bar{x}(kT) - g(kT)\hat{u}^o(kT)]| \\
= \; & |\sum_{i=0}^{q} c_i \hat{\phi}_i(kT) - \sum_{i=0}^{q} c_i g(kT)^i [K(kT)\bar{x}(kT) - g(kT)\hat{u}^o(kT)]| \\
= \; & |\sum_{i=0}^{q} c_i \left[\hat{\phi}_i(kT) - g(kT)^i [K(kT)\bar{x}(kT) - g(kT)\hat{u}^o(kT)]\right]| \\
\leq \; & \underbrace{(q+1)\bar{c}\gamma_{10}}_{=:\gamma_{11}} T(\|\bar{x}(kT)\| + |\hat{u}^o(kT)| + \|u_m\|_\infty);
\end{aligned}$$

hence, for small T, if $\bar{\theta} \in \bar{\mathcal{P}}$ is a.c. on $[kT, (k+1)T)$ then

$$\begin{aligned}
|\hat{u}^o(kT+T) \; & - \; [1-\hat{f}^\varepsilon(g(kT))g(kT)]\hat{u}^o(kT) - \hat{f}^\varepsilon(g(kT))K(kT)\bar{x}(kT)| \\
& \leq \; \gamma_{11} T(\|\bar{x}(kT)\| + |\hat{u}^o(kT)| + \|u_m\|_\infty),
\end{aligned}$$

which is the first equation in (ii).

Now let us look at $\bar{x}$. We have

$$\begin{aligned}
\dot{\bar{x}}(t) &= \bar{A}(t)\bar{x}(t) + \bar{E}u_m(t) + g(t)\bar{B}u(t) \\
&= \bar{A}_{cl}(t)\bar{x}(t) + \bar{E}u_m(t) + \underbrace{(\bar{A}(t) - \bar{A}_{cl}(t))}_{=-\bar{B}K(t)}\bar{x}(t) + g(t)\bar{B}u(t) \\
&= \bar{A}_{cl}(t)\bar{x}(t) + \bar{E}u_m(t) + \bar{B}[g(t)u(t) - K(t)\bar{x}(t)].
\end{aligned}$$

So we need to prove that

$$\psi(t) := \int_{kT}^{t} \Phi_{cl}(t,\tau)\bar{B}[\underbrace{g(\tau)u(\tau) - K(\tau)\bar{x}(\tau) - g(kT)\hat{u}^o(kT) + K(kT)\bar{x}(kT)}_{=:\eta(\tau)}]d\tau,$$

when evaluated at $t = (k+1)T$ is of the form

$$\mathcal{O}(T^2)(||\bar{x}(kT)|| + |\hat{u}^o(kT)| + ||u_m||_\infty)$$

when T is small and $\bar{\theta} \in \bar{\mathcal{P}}$ is a.c. on $[kT, (k+1)T)$. Now rewrite ψ as

$$\psi[(k+1)T] = \int_{kT}^{(k+1)T} \bar{B}\eta(\tau)d\tau + \int_{kT}^{(k+1)T} [\Phi_{cl}((k+1)T,\tau) - I]\bar{B}\eta(\tau)d\tau$$

$$\Rightarrow \; ||\psi[(k+1)T]|| \leq ||\bar{B}|| \times ||\int_{kT}^{(k+1)T} \eta(\tau)d\tau|| + \int_{kT}^{(k+1)T} ||\Phi_{cl}(t,\tau) - I|| \times ||\bar{B}|| \times ||\eta(\tau)||d\tau.$$

Next we obtain a crude bound on η. From (53) and the fact that $||K||$ is uniformly bounded as a function of $\bar{\theta} \in \bar{\mathcal{P}}$, we see that there exists a constant γ_{12} so that when T is small and $\bar{\theta} \in \bar{\mathcal{P}}$ is a.c. on $[kT, (k+1)T)$:

$$\begin{aligned} ||\eta(t)|| &\leq \gamma_1(||u(t)|| + ||\hat{u}^o(kT)||) + ||K||_\infty(||\bar{x}(t)|| + ||\bar{x}(kT)||) \\ &\leq ||K||_\infty(2 + \gamma_5 + \gamma_5 T)(||\bar{x}(kT)|| + |\hat{u}^o(kT)| + ||u_m||_\infty) \\ &\leq \gamma_{12}(||\bar{x}(kT)|| + |\hat{u}^o(kT)| + ||u_m||_\infty), \end{aligned}$$

and from Lemma A.1 we have

$$||\Phi_{cl}(t,\tau) - I|| \leq 2\gamma_1(t-\tau), \quad t \geq \tau \geq 0, \quad |t-\tau| \leq T,$$

so

$$\begin{aligned} \int_{kT}^{(k+1)T} ||\Phi_{cl}(t,\tau) - I|| \times ||\bar{B}|| \times ||\eta(\tau)|| d\tau &\leq 2\gamma_1\gamma_{12}T^2(||\bar{x}(kT)|| + \\ &\quad |\hat{u}^o(kT)| + ||u_m||_\infty). \end{aligned} \tag{56}$$

Now we obtain a crisper bound on η. Using (53) and the facts that K, g, and $\dot{g}$ are uniformly bounded function of $\bar{\theta} \in \bar{\mathcal{P}}$, it follows that there exists a constant γ_{13} so that when T is small and $\bar{\theta} \in \bar{\mathcal{P}}$ is a. c. in $[kT, (k+1)T)$:

$$\begin{aligned} ||\eta(t) \; - \; & g(kT)[u(t) - \hat{u}^o(kT)]|| = ||[g(t) - g(kT)]u(t) + [K(kT) - K(t)]\bar{x}(kT) + \\ & K(t)[\bar{x}(kT) - \bar{x}(t)]|| \\ \leq \; & T||\dot{g}||_\infty \times ||u(t)|| + T||\dot{K}||_\infty \times ||\bar{x}(kT)|| + ||K||_\infty||\bar{x}(kT) - \bar{x}(t)|| \\ \leq \; & \gamma_{13}T(||\bar{x}(kT)|| + |\hat{u}^o(kT)| + ||u_m||_\infty), \quad t \in [kT, (k+1)T). \end{aligned}$$

Using this equation together with the definition of u means that when T is small and $\bar{\theta} \in \bar{\mathcal{P}}$ is a. c. in $[kT, (k+1)T)$:

$$\begin{aligned} ||\int_{kT}^{(k+1)T} \eta(\tau) d\tau|| &\leq ||\int_{kT}^{(k+1)T} g(kT)[u(\tau) - \hat{u}^o(kT)] d\tau|| + \\ &\quad ||\int_{kT}^{(k+1)T} [\eta(\tau) - g(kT)[u(\tau) - \hat{u}^o(kT)]] d\tau|| \\ &\leq 0 + \gamma_{13}T^2(||\bar{x}(kT)|| + |\hat{u}^o(kT)| + ||u_m||_\infty). \end{aligned}$$

If we combine this with (56), it follows that when T is small and $\bar{\theta} \in \bar{\mathcal{P}}$ is a. c. in $[kT, (k+1)T)$:

$$||\psi((k+1)T)|| \leq T^2[\gamma_1\gamma_{12} + \gamma_{13}||\bar{B}||] \; [||\bar{x}(kT)|| + |\hat{u}^o(kT)| + ||u_m||_\infty],$$

as desired.

□

Proof of Proposition 5.1:

Set $t_0 = 0$ and let $\lambda \in (\max\{\lambda_0, \lambda_m\}, 0)$. The system equation for $\bar{x}^0$ is

$$\bar{x}^0(t) \;=\; \Phi_{cl}(t, kT)\bar{x}^0(kT) + \int_{kT}^{t} \Phi_{cl}(t,\tau)\bar{E}u_m(\tau)d\tau, \quad t \geq kT. \tag{57}$$

By Assumption 5 and the choice of $K(t)$ it follows that there exist a $\lambda_1 < \lambda$ and a constant γ_1 so that

$$||\Phi_{cl}(t,\tau)|| \;\leq\; \gamma_1 e^{\lambda_1 t}, \quad t \geq \tau, \tag{58}$$

which means that

$$||\bar{x}^0(t)|| \;\leq\; \gamma_1 e^{\lambda_1 t}||\bar{x}_0|| + \frac{\gamma_1}{|\lambda_1|}||\bar{E}|| \times ||u_m||_\infty, \quad t \geq 0. \tag{59}$$

We now use KCL to obtain bounds on $\hat{\bar{x}}^0$ and $\hat{u}^o$. There exists a constant γ_2 so that for small T:

$$\begin{aligned} ||\hat{\bar{x}}(t) - \Phi_{cl}(t,kT)\hat{\bar{x}}(kT) \;&-\; \int_{kT}^{t}\Phi_{cl}(t,\tau)\bar{E}u_m(\tau)d\tau|| \\ &\leq\; \gamma_2 T(||\hat{\bar{x}}(kT)|| + |\hat{u}^o(kT)| + ||u_m||_\infty), \end{aligned} \tag{60}$$

$$||\hat{u}(t)|| \;\leq\; \gamma_2 T(||\hat{\bar{x}}(kT)|| + |\hat{u}^o(kT)| + ||u_m||_\infty), \quad t \in [kT, (k+1)T), \tag{61}$$

$$||\hat{u}^o[(k+1)T]|| \;\leq\; \gamma_2(||\hat{\bar{x}}(kT)|| + |\hat{u}^o(kT)| + ||u_m||_\infty), \tag{62}$$

and when $\bar{\theta}$ is absolutely continuous on $[kT, (k+1)T)$, for small T we have

$$\begin{aligned} &||\hat{\bar{x}}[(k+1)T] - \Phi_{cl}((k+1)T, kT)\hat{\bar{x}}(kT) - \int_{kT}^{(k+1)T}\Phi_{cl}((k+1)T,\tau)\bar{E}u_m(\tau)d\tau - \\ &\quad \int_{kT}^{(k+1)T}\Phi_{cl}((k+1)T,\tau)\bar{B}[g(kT)\hat{u}^o(kT) - K(kT)\bar{x}(kT)]d\tau|| \\ &\leq\; \gamma_2 T^2(||\hat{\bar{x}}(kT)|| + |\hat{u}^o(kT)| + ||u_m||_\infty), \quad t \geq kT, \end{aligned} \tag{63}$$

and

$$\begin{aligned} &|\hat{u}^o[(k+1)T] - [1 - \hat{f}_\varepsilon(g(kT))g(kT)]\hat{u}^o(kT) - \hat{f}_\varepsilon(g(kT))K(kT)\hat{\bar{x}}(kT)| \\ &\leq\; \gamma_2 T(||\hat{\bar{x}}(kT)|| + |\hat{u}^o(kT)| + ||u_m||_\infty). \end{aligned} \tag{64}$$

Now we will prove that $\hat{\bar{x}}$ is well-behaved at the sample points when T is small. Since we have to allow for an occasional parameter jump, the proof is a bit tricky. Recall that, by hypothesis, there is a lower bound T_0 on the time between parameter jumps. To this end, recall that $t_0 = 0$, and let $\{t_i : \; i \in \mathbf{N}\}$ denote a sequence of times satisfying

- $t_1 \geq 0$, and $t_1 = 0$ if $\bar{\theta}(t)$ is discontinuous at $t = 0$,

- $t_{i+1} - t_i \geq T_0, \quad i \in \mathbf{N}$, and
- $\bar{\theta}(t)$ is absolutely continuous on $(t_i, t_{i+1}), \quad i \in \mathbf{N}$;

these constant times are typically not unique. For a given sampling period T, we identify the problematic periods on which $\bar{\theta}(t)$ loses absolute continuity: define $n_0(T) := 0$ and

$$n_i(T) := \left\lfloor \frac{t_i}{T} \right\rfloor, \quad i \in \mathbf{N}.$$

This means that $\bar{\theta}$ is absolutely continuous on

$$\cup_{i=0}^{\infty}[(n_i(T)+1)T, n_{i+1}(T)T),$$

with all discontinuities contained in

$$\cup_{i=0}^{\infty}[n_i(T)T, n_i(T)T + T).$$

Notice as well that for $i \in \mathbf{N}$

$$\limsup_{T \to 0}[n_{i+1}(T)T - [n_i(T)+1]T] \geq T_0;$$

the case of $i = 0$ is special since there could be a discontinuity very soon after the starting time t_0.

To proceed, we need to define an *indicator function* which is active during problematic periods: we define

$$\chi : \mathbf{Z}^+ \times \mathbf{R}^+ \to \{0, 1\},$$

by

$$\chi(j, T) = \begin{cases} 1 & \text{if} \quad j \in \{n_i(T) : i \in \mathbf{Z}^+\} \\ 0 & \text{else.} \end{cases}$$

Hence, for a given T, $\chi(j, T)$ equals 1 at exactly those j for which $[jT, (j+1)T)$ contains an element of $\{t_i : i \in \mathbf{Z}^+\}$. With this notation in hand, we can rewrite equations (60)-(64): there exists a constant $\tilde{\gamma}_2 \geq \gamma_2$ so that for small T:

$$\begin{aligned} & ||\hat{\bar{x}}[(k+1)T] - \Phi_{cl}((k+1)T, kT)\hat{\bar{x}}(kT) - \int_{kT}^{(k+1)T} \Phi_{cl}((k+1)T, \tau)\bar{E}u_m(\tau)d\tau - \\ & \quad \int_{kT}^{(k+1)T} \Phi_{cl}((k+1)T, \tau)\bar{B}[g(kT)\hat{u}^o(kT) - K(kT)\hat{\bar{x}}(kT)]d\tau|| \\ \leq \; & [\gamma_2 T^2 + \tilde{\gamma}_2 T\chi(k, T)](||\hat{\bar{x}}(kT)|| + |\hat{u}^o(kT)| + ||u_m||_\infty), \quad k \geq 0, \end{aligned} \tag{65}$$

and

$$\begin{aligned} & |\hat{u}^o[(k+1)T] - [1 - \hat{f}_\varepsilon(g(kT))g(kT)]\hat{u}^o(kT) - \hat{f}_\varepsilon(g(kT))K(kT)\hat{\bar{x}}(kT)| \\ \leq \; & [\gamma_2 T + \tilde{\gamma}_2 \chi(k, T)](||\hat{\bar{x}}(kT)|| + |\hat{u}^o(kT)| + ||u_m||_\infty), \quad k \geq 0. \end{aligned} \tag{66}$$

At this point we recall that $\hat{u}^o(kT)$ is trying to capture a good estimate of $\frac{1}{g(kT)}K(kT)\hat{\bar{x}}(kT)$, so it is natural to introduce a new state variable:

$$\hat{\eta}[kT] := \hat{u}^o[kT] - \frac{1}{g(kT)}K(kT)\hat{\bar{x}}(kT). \tag{67}$$

It follows immediately that the generalized plant state equation becomes

$$\begin{aligned} &||\hat{\bar{x}}[(k+1)T] - \Phi_{cl}((k+1)T, kT)\hat{\bar{x}}(kT) - \int_{kT}^{(k+1)T} \Phi_{cl}((k+1)T,\tau)\bar{E}u_m(\tau)d\tau - \\ &\quad \int_{kT}^{(k+1)T} \Phi_{cl}((k+1)T,\tau)\bar{B}g(kT)\hat{\eta}(kT)d\tau|| \\ \leq\ & [\gamma_2 T^2 + \tilde{\gamma}_2 T\chi(k,T)](||\hat{\bar{x}}(kT)|| + ||\hat{\eta}(kT)|| + ||u_m||_\infty), \quad k \in \mathbf{Z}^+. \end{aligned} \tag{68}$$

Using the fact that

$$\frac{K(kT)}{g(kT)} - \frac{K((k+1)T)}{g((k+1)T)} = \begin{cases} \mathcal{O}(T), & \text{if } \bar{\theta} \text{ is a.c. on } [kT,(k+1)T) \\ \mathcal{O}(1), & \text{else,} \end{cases}$$

after more algebra it is easy to verify that there exists a constant $\gamma_3 \geq \max\{\gamma_2, \tilde{\gamma}_2\}$ so that for small T

$$\begin{aligned} ||\hat{\eta}[(k+1)T] \ &- \ [1 - \hat{f}_\varepsilon(g(kT))g(kT)]\hat{\eta}(kT)|| \\ &\leq \ [\gamma_3 T + \gamma_3\chi(k,T)](||\hat{\bar{x}}(kT)|| + ||\hat{\eta}(kT)|| + ||u_m||_\infty), \\ &\quad k \in \mathbf{Z}^+. \end{aligned} \tag{69}$$

Now let us examine the difference equation in more details. Combining (58) and (68) yields:

$$\begin{aligned} ||\hat{\bar{x}}[kT]|| \ &\leq \ \gamma_1 e^{\lambda_1(k-k_0)T}||\hat{\bar{x}}[k_0T]|| + \sum_{j=k_0}^{k-1}\gamma_1 e^{\lambda_1(k-1-j)T}\left[\gamma_1 T||\bar{E}|| \times ||u_m||_\infty + \right. \\ &\quad \gamma_1 T||\bar{B}|| \times ||g||_\infty ||\hat{\eta}(jT)|| + \\ &\quad \left. [\gamma_3 T^2 + \gamma_3 T\chi(k,T)](||\hat{\bar{x}}(jT)|| + ||\hat{\eta}(jT)|| + ||u_m||_\infty)\right], \quad k \geq k_0 \geq 0, \end{aligned}$$

and solving (69) yields:

$$\begin{aligned} ||\hat{\eta}[kT]|| \ &\leq \ \varepsilon^{k-k_0}||\hat{\eta}(k_0)|| + \\ &\quad \sum_{j=k_0}^{k-1}\varepsilon^{k-1-j}[\gamma_3 T + \gamma_3\chi(j,T)](||\hat{\bar{x}}(jT)|| + ||\hat{\eta}(jT)|| + ||u_m||_\infty), \\ &\quad k \geq k_0 \geq 0. \end{aligned}$$

These two inequalities are tedious to handle. To this end, with $\gamma_4 := ||\bar{B}||\sup_{\bar{\theta}\in\bar{\mathcal{P}}}||g||_\infty$ we consider two associated difference equations:

$$\begin{aligned} \psi[(k+1)T] \ &= \ e^{\lambda_1 T}\psi[kT] + \gamma_1 T||\bar{E}|| \times ||u_m||_\infty + \gamma_1\gamma_4 T\upsilon[kT] + \\ &\quad [\gamma_3 T^2 + \gamma_3 T\chi(k,T)](\psi[kT] + \upsilon[kT] + ||u_m||_\infty), \\ \psi[0] \ &= \ ||\hat{\bar{x}}(0)||, \quad k \geq 0, \\ \upsilon[(k+1)T] \ &= \ \varepsilon\upsilon[kT] + [\gamma_3 T + \gamma_3\chi(k,T)](\psi[kT] + \upsilon[kT] + ||u_m||_\infty), \\ \upsilon[0] \ &= \ |\hat{\eta}(0)|, \quad k \geq 0. \end{aligned}$$

Claim B.1:

$$\begin{aligned} ||\hat{\bar{x}}(kT)|| &\leq \gamma_1 \psi[kT], \\ ||\hat{\eta}(kT)|| &\leq \upsilon[kT], \quad k \geq 0. \end{aligned}$$

Proof: The proof of Claim B.1 is given below.

We can combine these two equations into a state space form:

$$\begin{aligned} \begin{bmatrix} \Psi[(k+1)T] \\ \upsilon[(k+1)T] \end{bmatrix} &= \begin{bmatrix} e^{\lambda_1 T} + \gamma_3 T^2 & \gamma_1\gamma_4 T + \gamma_3 T^2 \\ \gamma_3 T & \varepsilon + \gamma_3 T \end{bmatrix} \begin{bmatrix} \Psi[kT] \\ \upsilon[kT] \end{bmatrix} + \\ & \begin{bmatrix} \gamma_1 T||\bar{E}|| + \gamma_3 T^2 \\ \gamma_3 T \end{bmatrix} ||u_m||_\infty + \\ & \chi(k,T) \begin{bmatrix} \gamma_3 T \\ \gamma_3 \end{bmatrix} (\Psi(kT) + \upsilon(kT) + ||u_m||_\infty), \quad k \geq 0. \end{aligned} \tag{70}$$

Claim B.2: Define

$$\bar{n}_i(T) := \left\lceil \frac{t_i + \sqrt{T}}{T} \right\rceil, \quad i \in \mathbf{N}.$$

For every $\lambda_2 \in (\lambda_1, \lambda)$ there exists a $\bar{T} > 0$ and $\gamma_5 > 0$ so that for every $\bar{\theta} \in PC$, $\bar{x}_0 \in \mathbf{R}^{n+n_m+1}$, and $T \in (0, \bar{T})$, we have

$$\begin{aligned} \Psi[kT] &\leq \gamma_5 e^{\lambda_2 kT}(\Psi[0] + \upsilon[0]) + \gamma_5 ||u_m||_\infty, \\ \upsilon[kT] &\leq \gamma_5 e^{\lambda_2 kT}(\Psi[0] + \upsilon[0]) + \gamma_5 ||u_m||_\infty, \quad k \in \mathbf{Z}^+, \end{aligned}$$

and

$$\begin{aligned} \upsilon[kT] \leq\ & \gamma_5 \sqrt{T} e^{\lambda_2 kT}(\Psi[0] + \upsilon[0]) + \gamma_5 \sqrt{T} ||u_m||_\infty, \quad \bar{n}_i(T) \leq k \leq n_{i+1}(T), \\ & i \in \mathbf{Z}^+. \end{aligned}$$

Proof: The proof of Claim B.2 is given below.

We can use Claims B.1 and B.2 to immediately conclude that, with $\gamma_6 := \gamma_1\gamma_5$, for small T we have

$$\begin{aligned} ||\hat{\bar{x}}(kT)|| &\leq \gamma_6 e^{\lambda_2 kT}(||\hat{\bar{x}}(0)|| + ||\hat{\eta}(0)||) + \gamma_6 ||u_m||_\infty, \\ ||\hat{\eta}(kT)|| &\leq \gamma_6 e^{\lambda_2 kT}(||\hat{\bar{x}}(0)|| + ||\hat{\eta}(0)||) + \gamma_6 ||u_m||_\infty, \quad k \geq 0, \\ ||\hat{\eta}(kT)|| &\leq \gamma_6 \sqrt{T} e^{\lambda_2 kT}(||\hat{\bar{x}}(0)|| + ||\hat{\eta}(0)||) + \gamma_6 \sqrt{T} ||u_m||_\infty, \\ & \qquad \bar{n}_i(T) \leq k \leq n_{i+1}(T), \quad i \in \mathbf{Z}^+. \end{aligned}$$

Using (67) and the fact that

$$\sup_{\bar\theta\in\bar{\mathcal{P}}}\sup_{t\geq 0}||\frac{K(t)}{g(t)}||<\infty,$$

it follows that there exists a constant γ_7 so that for small T:

$$\left.\begin{array}{l} ||\hat{\bar{x}}(kT)|| \leq \gamma_7 e^{\lambda_2 kT}(||\hat{\bar{x}}(0)||+||\hat{u}^o(0)||)+\gamma_7||u_m||_\infty, \\ ||\hat{u}^o(kT)|| \leq \gamma_7 e^{\lambda_2 kT}(||\hat{\bar{x}}(0)||+||\hat{u}^o(0)||)+\gamma_7||u_m||_\infty, \\ ||\hat{\eta}(kT)|| \leq \gamma_7 e^{\lambda_2 kT}(||\hat{\bar{x}}(0)||+||\hat{u}^o(0)||)+\gamma_7||u_m||_\infty, \quad k\geq 0, \\ ||\hat{\eta}(kT)|| \leq \gamma_7\sqrt{T} e^{\lambda_2 kT}(||\hat{\bar{x}}(0)||+||\hat{u}^o(0)||)+\gamma_7\sqrt{T}||u_m||_\infty, \\ \qquad\qquad\qquad\qquad \bar{n}_i(T)\leq k\leq n_{i+1}(T), \quad i\in\mathbf{Z}^+. \end{array}\right\} \tag{71}$$

This yields the second inequality of the proposition.

Now that we have proven that $\hat{\bar{x}}$ is well behaved at integer multiples of T, we will proceed to show that it is very close to $\bar{x}^o$ there. To this end, define

$$\tilde{\bar{x}} := \bar{x}^o - \hat{\bar{x}}.$$

Combining (57) and (68) and the fact that $\gamma_3 \geq \max\{\gamma_2, \bar\gamma_2\}$ yields

$$||\tilde{\bar{x}}[(k+1)T]-\Phi_{cl}((k+1)T,kT)\tilde{\bar{x}}(kT)-\int_{kT}^{(k+1)T}\Phi_{cl}((k+1)T,\tau)\bar{B}g(kT)\hat{\eta}(kT)d\tau||$$
$$\leq [\gamma_3 T^2+\gamma_3 T\chi(k,T)](||\hat{\bar{x}}(kT)||+|\hat{\eta}(kT)|+||u_m||_\infty).$$

Now

$$||\int_{kT}^{(k+1)T}\Phi_{cl}((k+1)T,\tau)\bar{B}g(kT)\hat{\eta}(kT)d\tau|| \quad\leq\quad \gamma_1||\bar{B}||\times||g(kT)||\times T\times ||\hat{\eta}(kT)||;$$

since g is uniformly bounded over all $\bar\theta\in\bar{\mathcal{P}}$, it follows that there exists a constant γ_8 so that for T:

$$\begin{aligned} ||\tilde{\bar{x}}(kT)|| \quad\leq\quad & \underbrace{\sum_{j=0}^{k-1}\gamma_1 e^{\lambda_1(k-1-j)T}[\gamma_8 T||\hat{\eta}(jT)||]}_{=:f_1(kT)}+ \\ & \underbrace{\sum_{j=0}^{k-1}\gamma_1\gamma_3 e^{\lambda_1(k-1-j)T}T^2[||\hat{\bar{x}}(jT)||+||\hat{\eta}(jT)||+||u_m||_\infty]}_{=:f_2(kT)}+ \\ & \underbrace{\sum_{j=0}^{k-1}\gamma_1\gamma_3 e^{\lambda_1(k-1-j)T}T\chi(j,T)[||\hat{\bar{x}}(jT)||+||\hat{\eta}(jT)||+||u_m||_\infty]}_{=:f_3(kT)}. \end{aligned} \tag{72}$$

The second term $f_2(kT)$ is easy to handle; using (71) it follows that

$$\begin{aligned} f_2(kT) &\leq \gamma_1\gamma_3\frac{1+2\gamma_7}{1-e^{\lambda_1 T}}T^2||u_m||_\infty + \frac{2\gamma_2\gamma_3\gamma_7}{1-e^{(\lambda_1-\lambda_2)T}e^{-\lambda_2 T}}T^2 e^{\lambda_2 kT} \times \\ &\quad (||\hat{\bar{x}}(0)|| + ||\hat{u}^o(0)||) \\ &= \mathcal{O}(T)[e^{\lambda_2 kT}(||\hat{\bar{x}}(0)|| + ||\hat{u}^o(0)||) + ||u_m||_\infty]. \end{aligned} \tag{73}$$

The third term $f_3(kT)$ takes a bit more work. Using the fact that

$$\begin{aligned} t_{i+1} - t_i &\geq T_0, \\ n_{i+1}(T) - n_i(T) &\geq T_0 - 2T, \quad i \in \mathbf{N}, \end{aligned}$$

it follows that

$$\begin{aligned} \sum_{j=0}^{k-1} e^{\lambda_1(k-1-j)T}\chi(j,T) &= \sum_{j=0}^{k-1} e^{\lambda_1 jT}\chi(k-1-j,T) \\ &\leq 1 + (1 + e^{\lambda_1(T_0-2T)} + e^{2\lambda_1(T_0-2T)} + \cdots) \\ &= 1 + \frac{1}{1-e^{\lambda_1(T_0-2T)}} \quad \text{(for small } T) \\ &= \mathcal{O}(1), \end{aligned}$$

and

$$\begin{aligned} \sum_{j=0}^{k-1} e^{\lambda_1(k-1-j)T}\chi(j,T)e^{\lambda_2 jT} &= \sum_{j=0}^{k-1} e^{\lambda_1 jT}\chi(k-1-j,T)e^{\lambda_2(k-1-j)T} \\ &= e^{\lambda_2 kT}e^{-\lambda_2 T}\sum_{j=0}^{k-1} e^{(\lambda_1-\lambda_2)jT}\chi(k-1-j,T) \\ &\leq e^{\lambda_2 kT}e^{-\lambda_2 T}[1 + e^{(\lambda_1-\lambda_2)(T_0-2T)} + \\ &\quad e^{2(\lambda_1-\lambda_2)(T_0-2T)} + \cdots] \\ &= \mathcal{O}(1)e^{\lambda_2 kT}. \end{aligned}$$

Combining these observations with (71) yields

$$f_3(kT) = \mathcal{O}(T)[e^{\lambda_2 kT}(||\hat{\bar{x}}(0)|| + ||\hat{u}^o(0)||) + ||u_m||_\infty]. \tag{74}$$

The first term $f_1(kT)$ is the trickiest to deal with. Define the set of integers

$$N(T) := (\cup_{i=0}^{\infty}[n_i(T), \bar{n}_i(T)]) \cap \mathbf{Z}^+.$$

Using (71), it follows that

$$\begin{aligned} f_1(kT) &= \gamma_1\gamma_8 T \sum_{j=0}^{k-1} e^{\lambda_1(k-1-j)T} ||\hat{\eta}(jT)|| \\ &\leq \gamma_1\gamma_8 T \sum_{j=0,\, j\in N(T)}^{k-1} e^{\lambda_1 jT}[\gamma_7 e^{\lambda_2(k-j-1)T}(||\hat{\bar{x}}(0)|| + ||\hat{u}^o(0)||) + \gamma_7||u_m||_\infty] + \\ &\quad \gamma_1\gamma_8 T \sum_{j=0,\ j\notin N(T)}^{k-1} e^{\lambda_1 jT}[\gamma_7\sqrt{T} e^{\lambda_2(k-1-j)T}(||\hat{\bar{x}}(0)|| + ||\hat{u}^o(0)||) + \\ &\quad \gamma_7\sqrt{T}||u_m||_\infty]. \end{aligned}$$

But

$$\begin{aligned} \bar{n}_i(T) - n_i(T) &= \left\lceil \frac{t_i + \sqrt{T}}{T} \right\rceil - \left\lfloor \frac{t_i}{T} \right\rfloor \\ &\leq \left(\frac{t_i + \sqrt{T}}{T} \right) - \left(\frac{t_i}{T} - 1 \right) \\ &\leq \frac{t_i + \sqrt{T} - t_i}{T} + 1 \\ &= \mathcal{O}(\frac{1}{\sqrt{T}}), \end{aligned}$$

so with some careful analysis we see that the first summation looks like

$$\mathcal{O}(T^{\frac{1}{2}})[e^{\lambda_2 kT}(||\hat{\bar{x}}(0)|| + ||\hat{u}^o(0)||) + ||u_m||_\infty],$$

while with some straight forward analysis the second summation clearly is of the form

$$\mathcal{O}(T^{\frac{1}{2}})[e^{\lambda_2 kT}(||\hat{\bar{x}}(0)|| + ||\hat{u}^o(0)||) + ||u_m||_\infty].$$

Hence,

$$f_1(kT) = \mathcal{O}(T^{\frac{1}{2}})[e^{\lambda_2 kT}(||\hat{\bar{x}}(0)|| + ||\hat{u}^o(0)||) + ||u_m||_\infty].$$

If we combine this with (73) and (74), we have that there exists a constant γ_9 so that for small T:

$$||\tilde{x}(kT)|| \leq \gamma_9 T^{\frac{1}{2}}[e^{\lambda_2 kT}(||\hat{\bar{x}}(0)|| + ||\hat{u}^o(0)||) + ||u_m||_\infty], \quad k \in \mathbf{Z}^+. \qquad (75)$$

Now we prove that everything is smooth between sample points. From (57), (60), and Lemma A.1 it follows that there exists a constant $\gamma_{10} > 0$ so that for small T:

$$\|\bar{x}^0(t) - \bar{x}^0(kT)\| \leq \gamma_{10} T(\|\bar{x}(kT)\| + |\hat{u}^o(kT)| + \|u_m\|_\infty), \ \ t \in [kT, kT+T],$$

$$\|\hat{\bar{x}}(t) - \hat{\bar{x}}(kT)\| \leq \gamma_{10} T(\|\hat{\bar{x}}(kT)\| + |\hat{u}^o(kT)| + \|u_m\|_\infty), \ \ t \in [kT, kT+T].$$

Hence, using these two equations together with (75), it follows that for $\delta \in [0,T]$ we have

$$\begin{aligned} \|\tilde{\bar{x}}(kT+\delta)\| &\leq \|\tilde{\bar{x}}(kT)\| + \|\tilde{\bar{x}}(kT+\delta) - \tilde{\bar{x}}(kT)\| \\ &\leq \|\tilde{\bar{x}}(kT)\| + \|\bar{x}(kT+\delta) - \bar{x}(kT)\| + \|\hat{\bar{x}}(kT+\delta) - \hat{\bar{x}}(kT)\| \\ &\leq \gamma_9\sqrt{T}(e^{\lambda_2 kT}\|\bar{x}_0\| + e^{\lambda_2 kT}|\hat{u}_0^o| + \|u_m\|_\infty) + \\ &\quad \gamma_{10}T(\|\bar{x}(kT)\| + |\hat{u}^o(kT)| + \|u_m\|_\infty) + \\ &\quad \gamma_{10}T(\|\hat{\bar{x}}(kT)\| + |\hat{u}^o(kT)| + \|u_m\|_\infty). \end{aligned}$$

Using (59) for a bound on $||\bar{x}^0(kT)||$ and (71) for a bound on $||\hat{\bar{x}}(kT)||$, and observing that $\hat{\bar{x}}(0) = \bar{x}^0(0) = \bar{x}_0$, it follows that there exists a constant $\gamma_{11} > 0$ so that for small $T > 0$ we have

$$\|\tilde{\bar{x}}(t)\| \leq \gamma_{11}\sqrt{T}e^{\lambda_2 t}(\|\bar{x}_0\| + |\hat{u}_0^o|) + \gamma_{11}\sqrt{T}\|u_m\|_\infty, \;\; t \geq 0,$$

which proves the first inequality of the proposition.

□

Proof of Claim B.2:

The key idea is as follows. We see from (70) that

$$\begin{aligned} \begin{bmatrix} \Psi[(k+1)T] \\ \upsilon[(k+1)T] \end{bmatrix} &= \begin{bmatrix} e^{\lambda_1 T} + \gamma_3 T^2 & \gamma_1\gamma_4 T + \gamma_3 T^2 \\ \gamma_3 T & \varepsilon + \gamma_3 T \end{bmatrix} \begin{bmatrix} \Psi[kT] \\ \upsilon[kT] \end{bmatrix} + \\ &\quad \begin{bmatrix} \gamma_1 T||\bar{E}|| + \gamma_3 T^2 \\ \gamma_3 T \end{bmatrix} ||u_m||_\infty, \\ &\quad k \notin \{n_0(T), n_1(T), n_2(T), \cdots\}, \end{aligned} \tag{76}$$

and that there exists a constant $\bar{\gamma}_1$ so that for small T

$$\begin{aligned} &\psi[(n_i(T)+1)T] \leq (e^{\lambda_1 T} + \gamma_3 T^2 + \gamma_3 T)\psi[n_i(T)T] + \\ &\quad \bar{\gamma}_1 T[\upsilon(n_i(T)T) + ||u_m||_\infty], \\ &\upsilon[(n_i(T)+1)T] \leq \bar{\gamma}_1[\psi[n_i(T)T] + \upsilon(n_i(T)T) + ||u_m||_\infty]. \end{aligned} \tag{77}$$

Now, we apply a similarity transformation to (76) to diagonalize the difference equation, which greatly simplifies the analysis; we then use (77) to get a bound on the size of the jump in the state on the problematic intervals $[n_i(T)T, (n_i(T)+1)T)$. We then glue the intervals together to prove the desired result.

To proceed, we first diagonalize the state matrix

$$M := \begin{bmatrix} e^{\lambda_1 T} + \gamma_3 T^2 & \gamma_1\gamma_4 T + \gamma_3 T^2 \\ \gamma_3 T & \varepsilon + \gamma_3 T \end{bmatrix}$$

of (76); we do so in two steps. The first goal is to choose real-valued $L(T)$ so that the (2,1) element of

$$\begin{bmatrix} 1 & 0 \\ L & 1 \end{bmatrix} M \begin{bmatrix} 1 & 0 \\ -L & 1 \end{bmatrix} \tag{78}$$

is zero. Equivalently, we want

$$(e^{\lambda_1 T} + \gamma_3 T^2)L + \gamma_3 T - (\gamma_1\gamma_4 T + \gamma_3 T^2)L^2 - (\varepsilon + \gamma_3 T)L = 0, \tag{79}$$

which results in

$$L = \underbrace{\frac{\varepsilon + \gamma_3 T}{e^{\lambda_1 T} + \gamma_3 T^2} L + \frac{\gamma_1\gamma_4 T + \gamma_3 T^2}{e^{\lambda_1 T} + \gamma_3 T^2} L^2 - \frac{\gamma_3 T}{e^{\lambda_1 T} + \gamma_3 T^2}}_{f_1(L)};$$

notice that f_1 is a continuous function of its argument. Since $\varepsilon \in (0,1)$, for T small we have that

$$f_1 : [0,1] \to [0,1);$$

hence, by Brouwer's Fixed Point Theorem there exists a solution to the above equation; in fact, it can be obtained by iteration, and it is easy to confirm that it is $\mathcal{O}(T)$ [1], i.e. there exists a $\bar{\gamma}_2$ so that, for small T,

$$||L(T)|| \leq \bar{\gamma}_2 T.$$

Now we apply another similarity transformation to (78) to diagonalize it: more specifically, the goal is to choose real-valued $p(T)$ so that

$$\begin{aligned} &\begin{bmatrix} 1 & p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ L & 1 \end{bmatrix} M \begin{bmatrix} 1 & 0 \\ -L & 1 \end{bmatrix} \begin{bmatrix} 1 & -p \\ 0 & 1 \end{bmatrix} \\ = &\begin{bmatrix} 1 & p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{\lambda_1 T} + \gamma_3 T^2 - (\gamma_1\gamma_4 T + \gamma_3 T^2)L & \gamma_1\gamma_4 T + \gamma_3 T^2 \\ 0 & \varepsilon + \gamma_3 T + (\gamma_1\gamma_4 T + \gamma_3 T^2)L \end{bmatrix} \times \\ &\begin{bmatrix} 1 & -p \\ 0 & 1 \end{bmatrix}, \end{aligned}$$

is diagonal. Equivalently, we want

$$\begin{aligned} [\varepsilon + \gamma_3 T + (\gamma_1\gamma_4 T + \gamma_3 T^2)L \;&-\; e^{\lambda_1 T} - \gamma_3 T^2 + (\gamma_1\gamma_4 T + \gamma_3 T^2)L]p \\ &+\; \gamma_1\gamma_4 T + \gamma_3 T^2 = 0, \end{aligned}$$

so p is $\mathcal{O}(T)$: there exists a constant $\bar{\gamma}_3$ so that for small T:

$$||p(T)|| \leq \bar{\gamma}_3 T.$$

[1] One can obtain a closed form expression for L by solving the quadratic equation (79), but it is tedious to analyse.

Using these similarity transformations, we can now define a new state:

$$\begin{bmatrix} \bar{\psi}(kT) \\ \bar{\upsilon}(kT) \end{bmatrix} := \begin{bmatrix} 1 & p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ L & 1 \end{bmatrix} \begin{bmatrix} \psi(kT) \\ \upsilon(kT) \end{bmatrix}.$$

Equation (70) becomes:

$$\begin{aligned} &\begin{bmatrix} \bar{\psi}[(k+1)T] \\ \bar{\upsilon}[(k+1)T] \end{bmatrix} = \\ &\begin{bmatrix} e^{\lambda_1 T} + \gamma_3 T^2 - (\gamma_1\gamma_4 T + \gamma_3 T^2)L(T) & 0 \\ 0 & \varepsilon + \gamma_3 T + (\gamma_1\gamma_4 T + \gamma_3 T^2)L(T) \end{bmatrix} \times \\ &\begin{bmatrix} \bar{\psi}(kT) \\ \bar{\upsilon}(kT) \end{bmatrix} + \\ &\begin{bmatrix} [1 + p(T)L(T)](\gamma_1 T||\bar{E}|| + \gamma_3 T^2) + \gamma_3 T p(T) \\ \gamma_3 T + L(T)(\gamma_1 T||\bar{E}|| + \gamma_3 T^2) \end{bmatrix} ||u_m||_\infty + \\ &\chi(k,T) \begin{bmatrix} \gamma_3 T[1 + p(T)L(T)] + \gamma_3 p(T) \\ \gamma_3 + \gamma_3 T L(T) \end{bmatrix} \times \\ &\left[\ (1 - L(T))\bar{\psi}(kT) + (1 + L(T)p(T) - p(T))\bar{\upsilon}(kT) + ||u_m||_\infty \ \right] \\ =: &\begin{bmatrix} p_1(T) & 0 \\ 0 & p_2(T) \end{bmatrix} \begin{bmatrix} \bar{\psi}(kT) \\ \bar{\upsilon}(kT) \end{bmatrix} + \begin{bmatrix} \bar{E}_1(T) \\ \bar{E}_2(T) \end{bmatrix} ||u_m||_\infty + \\ &\chi(k,T) \begin{bmatrix} B_1(T) \\ B_2(T) \end{bmatrix} \times \\ &\left[\ (1 - L(T))\bar{\psi}(kT) + (1 + L(T)p(T) - p(T))\bar{\upsilon}(kT) + ||u_m||_\infty . \ \right] \end{aligned} \tag{80}$$

For small T

$$\begin{aligned} p_1(T), p_2(T) &\in [0, 1), \\ p_1(T) &\le e^{(\lambda_1+\lambda)T/2} =: e^{\lambda_2 T} \quad [\lambda_2 := (\lambda_1 + \lambda)/2], \\ p_2(T) &\le \frac{\varepsilon + 1}{2}. \end{aligned}$$

Solving this difference equation yields, for $i \in \mathbf{Z}^+$:

$$\begin{aligned} \bar{\psi}[kT] &\le p_1(T)^{k-n_i(T)-1}\bar{\psi}[n_i(T)T + T] + \sum_{j=0}^{k-n_i(T)-2} p_1(T)^{k-n_i(T)-2-j}\bar{E}_1(T)||u_m||_\infty \\ &\le p_1(T)^{k-n_i(T)-1}\bar{\psi}[n_i(T)T + T] + \frac{p_1(T)^{k-n_i(T)-1} - 1}{p_1(T) - 1}\bar{E}_1(T)||u_m||_\infty \\ &= p_1(T)^{k-n_i(T)-1}\bar{\psi}[n_i(T)T + T] + \mathcal{O}(1)||u_m||_\infty, \quad k = n_i(T)+1, \cdots, n_{i+1}(T), \end{aligned} \tag{81}$$

and

$$\begin{aligned}
\bar{v}[kT] &\leq p_2(T)^{k-n_i(T)-1}\bar{v}[n_i(T)T+T] + \sum_{j=0}^{k-n_i(T)-2} p_2(T)^{k-n_i(T)-2-j}\bar{E}_2(T)||u_m||_\infty \\
&\leq p_2(T)^{k-n_i(T)-1}\bar{v}[n_i(T)T+T] + \frac{p_2(T)^{k-n_i(T)-1}-1}{p_2(T)-1}\bar{E}_2(T)||u_m||_\infty \\
&= p_2(T)^{k-n_i(T)-1}\bar{v}[n_i(T)T+T] + \mathcal{O}(T)||u_m||_\infty, \\
&\quad k = n_i(T)+1, \cdots, n_{i+1}(T). \qquad (82)
\end{aligned}$$

Now from (80)

$$\begin{aligned}
\bar{\psi}[(n_i(T)+1)T] &\leq p_1(T)\bar{\psi}(n_i(T)T) + \bar{E}_1(T)||u_m||_\infty + \bar{B}_1(T) \times \\
&\quad \big[(1-L(T))\bar{\psi}(n_i(T)T) + \\
&\quad (1+L(T)p(T)-p(T))\bar{v}(n_i(T)T) + ||u_m||_\infty\big] \\
&\leq (1+\mathcal{O}(T))\bar{\psi}(n_i(T)T) + \mathcal{O}(T)\bar{v}(n_i(T)T) + \mathcal{O}(T)||u_m||_\infty, \qquad (83)
\end{aligned}$$

and

$$\bar{v}[(n_i(T)+1)T] \leq \mathcal{O}(1)\bar{\psi}(n_i(T)T) + \mathcal{O}(1)\bar{v}(n_i(T)T) + \mathcal{O}(1)||u_m||_\infty. \qquad (84)$$

Combining (81)-(84) and evaluating at $k = n_{i+1}(T)$ yields

$$\begin{aligned}
\bar{\psi}[n_{i+1}(T)T] &\leq p_1(T)^{(n_{i+1}(T)-n_i(T))}\big\{(1+\mathcal{O}(T))\bar{\psi}(n_i(T)T) + \mathcal{O}(T)\bar{v}[n_i(T)T]\big\} + \\
&\quad \mathcal{O}(1)||u_m||_\infty,
\end{aligned}$$

$$\begin{aligned}
\bar{v}[n_{i+1}(T)T] &\leq p_2(T)^{(n_{i+1}(T)-n_i(T))} \times \\
&\quad \big\{\mathcal{O}(1)\bar{\psi}(n_i(T)T) + \mathcal{O}(1)\bar{v}(n_i(T)T) + \mathcal{O}(1)||u_m||_\infty\big\} + \mathcal{O}(T)||u_m||_\infty.
\end{aligned}$$

Now

$$\begin{aligned}
\liminf_{T\to 0}(n_{i+1}(T)-n_i(T))T &\geq T_0, \\
\lim_{T\to 0} p_2(T) &= \varepsilon, \\
p_1(T) &= e^{(\lambda_1+\mathcal{O}(T))T},
\end{aligned}$$

so using the fact that $\lambda_2 \in (\lambda_1, 0)$, it follows that there exists a $\bar{\gamma}_4$ so that for small T we have

$$\begin{aligned}
\bar{\psi}[n_{i+1}(T)T] + \bar{v}[n_{i+1}(T)T] &\leq e^{\lambda_2(n_{i+1}(T)-n_i(T))T}[\bar{\psi}(n_i(T)T) + \bar{v}(n_i(T)T)] + \\
&\quad 2\bar{\gamma}_4||u_m||_\infty,
\end{aligned}$$

as well as

$$\begin{aligned} \bar{\psi}[kT] &\leq (1+\gamma_4 T)e^{\lambda_2(k-n_i(T))T}[\bar{\psi}(n_i(T)T)+\bar{v}(n_i(T)T)]+\bar{\gamma}_4||u_m||_\infty, \\ \bar{v}[kT] &\leq \gamma_4 e^{\lambda_2(k-n_i(T))T}[\bar{\psi}(n_i(T)T)+\bar{v}(n_i(T)T)]+\bar{\gamma}_4||u_m||_\infty, \\ & \qquad n_i(T)\leq k\leq n_{i+1}(T), \end{aligned}$$

so for small T this leads to

$$\begin{aligned} \bar{\psi}[n_i(T)T]+\bar{v}[n_i(T)T] &\leq e^{\lambda_2[n_i(T)-n_1(T)]T}\left\{\bar{\psi}[n_1(T)T]+\bar{v}[n_1(T)T]\right\}+ \\ & \qquad \frac{3\bar{\gamma}_4}{1-e^{\lambda_2 T_0}}||u_m||_\infty, \quad i\in\mathbf{N}. \end{aligned}$$

and

$$\begin{aligned} \bar{\psi}[kT]+\bar{v}[kT] &\leq (1+\gamma_4+\gamma_4 T)e^{\lambda_2(k-n_i(T))T}[\bar{\psi}(n_i(T)T)+\bar{v}(n_i(T)T)]+ \\ & \qquad 2\bar{\gamma}_4||u_m||_\infty, \quad n_i(T)\leq k\leq n_{i+1}(T). \end{aligned}$$

If we combine these it follows that for small T and $n_i(T)\leq k\leq n_{i+1}(T)$:

$$\begin{aligned} \bar{\psi}[kT]+\bar{v}[kT] &\leq (1+\bar{\gamma}_4+\bar{\gamma}_4 T)e^{\lambda_2[k-n_i(T)]T}\times\left[\bar{\psi}[n_i(T)T]+\bar{v}[n_i(T)T]\right]+ \\ & \qquad 2\bar{\gamma}_4||u_m||_\infty \\ &\leq (1+2\bar{\gamma}_4)e^{\lambda_2[k-n_1(T)]T}\times\left[\bar{\psi}[n_1(T)T]+\bar{v}[n_1(T)T]\right]+ \\ & \qquad \underbrace{\left[(1+2\bar{\gamma}_4)\frac{3\bar{\gamma}_4}{1-e^{\lambda_2 T_0}}+2\bar{\gamma}_4\right]}_{=:\bar{\gamma}_5}||u_m||_\infty, \end{aligned} \tag{85}$$

which means that

$$\begin{aligned} \bar{\psi}[kT]+\bar{v}[kT] &\leq \bar{\gamma}_5 e^{\lambda_2[k-n_1(T)]T}\times\left[\bar{\psi}[n_1(T)T]+\bar{v}[n_1(T)T]\right]+\bar{\gamma}_5||u_m||_\infty, \\ & \qquad k\geq n_1(T). \end{aligned} \tag{86}$$

Now we have to extend this bound back to $t=0$. We have two possibilities:

(i) If $t_1=0$, then $n_1(T)=0$, so we are done.

(ii) If $t_1>0$, then $\bar{\theta}(t)$ is continuous on $[0,n_1(T)T)$, so solving (80) clearly yields a slightly modified version of (81) and (82): there exists a $\bar{\gamma}_6$ so that for all small T

$$\begin{aligned} \bar{\psi}[kT] &\leq e^{\lambda_2 kT}\bar{\psi}(0)+\bar{\gamma}_6||u_m||_\infty, \\ \bar{v}[kT] &\leq e^{\lambda_2 kT}\bar{v}(0)+\bar{\gamma}_6||u_m||_\infty, \quad 0\leq k\leq n_1(T), \end{aligned}$$

so

$$\bar{\psi}[kT]+\bar{v}[kT] \leq e^{\lambda_2 kT}\left[\bar{\psi}[0]+\bar{v}[0]\right]+2\bar{\gamma}_6||u_m||_\infty, \quad 0\leq k\leq n_1(T).$$

If we combine this with (86), we have

$$\bar{\psi}[kT]+\bar{v}[kT] \leq \bar{\gamma}_5 e^{\lambda_2 kT}\left[\bar{\psi}[0]+\bar{v}[0]\right]+(\bar{\gamma}_5+2\bar{\gamma}_5\bar{\gamma}_6)||u_m||_\infty, \quad k\geq 0. \tag{87}$$

Hence, clearly (87) holds in all cases. If we now solve for $\begin{bmatrix} \psi \\ \upsilon \end{bmatrix}$ in terms of $\begin{bmatrix} \bar{\psi} \\ \bar{\upsilon} \end{bmatrix}$:

$$\begin{bmatrix} \psi(kT) \\ \upsilon(kT) \end{bmatrix} = \begin{bmatrix} 1 & -p(T) \\ -L(T) & 1 + L(T)p(T) \end{bmatrix} \begin{bmatrix} \bar{\psi}(kT) \\ \bar{\upsilon}(kT) \end{bmatrix}, \tag{88}$$

and use the fact that $L(T) = \mathcal{O}(T)$ and $p(T) = \mathcal{O}(T)$, it follows that there exists a constant $\bar{\gamma}_7$ so that, for small T:

$$\begin{array}{l} \psi[kT] \leq \bar{\gamma}_7 e^{\lambda_2 kT}(\psi(0) + \upsilon(0)) + \bar{\gamma}_7 ||u_m||_\infty, \\ \upsilon[kT] \leq \bar{\gamma}_7 e^{\lambda_2 kT}(\psi(0) + \upsilon(0)) + \bar{\gamma}_7 ||u_m||_\infty, \quad k \geq 0, \end{array} \tag{89}$$

as desired.

It remains to find a tighter bound which holds during the last part of each interval $[n_i(T)T, n_{i+1}(T)T)$. To this end, define

$$\bar{n}_i(T) = \left\lceil \frac{t_i + \sqrt{T}}{T} \right\rceil, \quad i \in \mathbf{N};$$

henceforth we assume that T is sufficiently small that

$$\bar{n}_i(T) < n_{i+1}(T).$$

Notice that

$$(\bar{n}_i(T) - n_i(T))T \geq \frac{\sqrt{T}}{T}.$$

From (82) and (84) there exists a $\bar{\gamma}_8 > 0$ so that for small T:

$$\begin{aligned} \bar{\upsilon}[kT] \quad \leq \quad & \bar{\gamma}_8 (\frac{\varepsilon + 1}{2})^{k - n_i(T)} [\bar{\upsilon}(n_i(T)T) + \bar{\psi}(n_i(T)T) + ||u_m||_\infty] + \\ & \bar{\gamma}_8 T ||u_m||_\infty, \quad n_i(T) \leq k \leq n_{i+1}(T), \end{aligned}$$

so

$$\begin{aligned} \bar{\upsilon}[kT] \quad \leq \quad & \bar{\gamma}_8 \left(\frac{\varepsilon + 1}{2} \right)^{\frac{\sqrt{T}}{T}} e^{\lambda_2 (k - \bar{n}_i(T))T} [\bar{\upsilon}(n_i(T)T) + \bar{\psi}(n_i(T)T) + ||u_m||_\infty] + \\ & \bar{\gamma}_8 T ||u_m||_\infty, \quad \bar{n}_i(T) \leq k \leq n_{i+1}(T). \end{aligned}$$

Now $\frac{\varepsilon+1}{2} < 1$, so it is easy to confirm that

$$\left(\frac{\varepsilon + 1}{2} \right)^{\frac{1}{\sqrt{T}}} = \mathcal{O}(\sqrt{T}),$$

so there exists a constant $\bar{\gamma}_9$ so that for small T:

$$\begin{aligned} \bar{\upsilon}[kT] &\leq \bar{\gamma}_9\sqrt{T}e^{\lambda_2(k-\bar{n}_i(T))T}[\bar{\upsilon}(n_i(T)T)+\bar{\psi}(n_i(T)T)]+\bar{\gamma}_9\sqrt{T}||u_m||_\infty, \\ & \qquad \bar{n}_i(T)\leq k\leq n_{i+1}(T). \end{aligned}$$

Combining this with (87) and using the fact that $e^{\lambda_2(n_i(T)-\bar{n}_i(T))T}\to 1$ as $T\to 0$, it follows that there exists $\bar{\gamma}_{10}$ so that for small T:

$$\begin{aligned} \bar{\upsilon}[kT] &\leq \bar{\gamma}_{10}\sqrt{T}e^{\lambda_2 kT}[\bar{\upsilon}(0)+\bar{\psi}(0)]+\bar{\gamma}_{10}\sqrt{T}||u_m||_\infty, \\ & \qquad \bar{n}_i(T)\leq k\leq n_{i+1}(T),\ \ i\in\mathbf{Z}^+. \end{aligned}$$

Now

$$\upsilon(kT)=\mathcal{O}(T)\bar{\psi}(kT)+(1+\mathcal{O}(T^2))\bar{\upsilon}(kT),$$

so combining the above equation with (87) and (88), it follows that there exists a constant $\bar{\gamma}_{11}$ so that for small T:

$$\begin{aligned} \upsilon[kT] &\leq \bar{\gamma}_{11}\sqrt{T}e^{\lambda_2 kT}[\upsilon(0)+\psi(0)]+\bar{\gamma}_{11}\sqrt{T}||u_m||_\infty, \\ & \qquad \bar{n}_i(T)\leq k\leq n_{i+1}(T),\quad i\in\mathbf{Z}^+, \end{aligned}$$

as desired.

□

Proof of Lemma 5.1:

The states of ψ and υ satisfy

$$\begin{aligned} \psi[kT] &= e^{\lambda_1 kT}\psi(0)+\sum_{j=0}^{k-1}e^{\lambda_1(k-1-j)T}\left[\gamma_1 T||\bar{E}||\times||u_m||_\infty+\gamma_1\gamma_4 T\upsilon[jT]+\right. \\ & \qquad \left.[\gamma_3 T^2+\gamma_3 T\chi(j,T)](\psi[jT]+\upsilon[jT]+||u_m||_\infty)\right], \qquad (90) \\ \upsilon[kT] &= \varepsilon^k\upsilon(0)+\sum_{j=0}^{k-1}\varepsilon^{k-1-j}[\gamma_3 T+\gamma_3\chi(j,T)](\psi[jT]+\upsilon[jT]+||u_m||_\infty). \qquad (91) \end{aligned}$$

Clearly

$$\begin{aligned} ||\hat{\bar{x}}(0)|| &= \psi(0)\leq\psi(0), \\ ||\hat{\eta}(0)|| &= \upsilon(0)\leq\upsilon(0). \end{aligned}$$

So suppose

$$\begin{cases} ||\hat{\bar{x}}(jT)||\leq\psi[jT], \\ ||\hat{\eta}(jT)||\leq\upsilon[jT], \end{cases}\qquad j=0,1,...,k-1;$$

we must prove that they hold for $j = k$. Let's examine the update laws for $\hat{\bar{x}}$ and $\hat{\eta}$. We have

$$\begin{aligned}
||\hat{\bar{x}}[kT]|| &\leq \gamma_1 e^{\lambda_1 kT}||\hat{\bar{x}}(0)|| + \sum_{j=0}^{k-1} \gamma_1 e^{\lambda_1(k-1-j)T}\left[\gamma_1 T||\bar{E}|| \times ||u_m||_\infty + \right. \\
&\qquad \left. \gamma_1\gamma_4 T||\hat{\eta}[jT]|| + [\gamma_3 T^2 + \gamma_3 T\chi(j,T)](||\hat{\bar{x}}[jT]|| + ||\hat{\eta}[jT]|| + ||u_m||_\infty)\right] \\
&\leq \gamma_1 \psi[kT], \\
||\hat{\eta}[kT]|| &\leq \varepsilon^k ||\hat{\eta}(0)|| + \sum_{j=0}^{k-1} \varepsilon^{k-1-j}[\gamma_3 T + \gamma_3\chi(j,T)](||\hat{\bar{x}}[jT]|| + ||\hat{\eta}[jT]|| + ||u_m||_\infty) \\
&\leq \upsilon[jT].
\end{aligned}$$

□

Proof of Lemma 5.2:
Proof of Lemma 5.2:

To prove this result, the interval of $[jT, (j+1)T)$ will be analyzed. Here we analyze two cases: (i) $q = 0$, and (ii) $q \geq 1$. While in the first case the state z has dimension 2, in the second case the state z has dimension 7. First we prove the lemma for the case $q = 0$:

- z_1 is used to construct $\hat{\phi}_0(jT)$:

$$\hat{\phi}_0(jT) = \frac{1}{\rho}\underbrace{\left[\begin{array}{cc} \bar{f}_2 & -1 \end{array}\right] H_m(h)^{-1} S_m^{-1}}_{=:\left[\begin{array}{cccc} \bar{\delta}_0 & \bar{\delta}_1 & \cdots & \bar{\delta}_m \end{array}\right]} \left[\begin{array}{c} y(jT) \\ y(jT+h) \\ \vdots \\ y(jT+mh) \end{array}\right] + \left[\begin{array}{cc} k_1 & k_2 \end{array}\right] \left[\begin{array}{c} \bar{x}_m(jT) \\ \bar{u}_m(jT) \end{array}\right].$$

- z_2 is used to construct and keep track of $\hat{u}^o(jT)$.

The periodically time-varying gains F, G, H, J, and L (*with period* p) are partitioned accordingly as

$$F(k) = \left[\begin{array}{c} F_1(k) \\ F_2(k) \end{array}\right], \quad G(k) = \left[\begin{array}{c} G_1(k) \\ G_2(k) \end{array}\right], \quad H(k) = \left[\begin{array}{c} H_1(k) \\ H_2(k) \end{array}\right], \quad J(k) = \left[\begin{array}{c} J_1(k) \\ J_2(k) \end{array}\right],$$

$$L(k) = \left[\begin{array}{cc} L_1(k) & L_2(k) \end{array}\right].$$

Since the controller is periodic of period T, for simplicity we will assume that $j = 0$ which means that we are analysing the first period $[0, T)$. The state z_1 is used to construct $\hat{\phi}_0(0)$. We have

$$(F_1, G_1, H_1, J_1)(k) = \begin{cases} (\left[\begin{array}{cc} 0 & 0 \end{array}\right], \bar{\delta}_0, k_1, k_2) & k = 0, \quad p = 1, \\ (\left[\begin{array}{cc} 0 & 0 \end{array}\right], \bar{\delta}_0, 0, 0) & k = 0, \quad p > 1 \\ (\left[\begin{array}{cc} 1 & 0 \end{array}\right], \bar{\delta}_k, 0, 0,) & k = 1, \cdots, m \\ (\left[\begin{array}{cc} 1 & 0 \end{array}\right], 0, 0, 0) & k = m+1, \cdots, p-1, \quad p > m+1. \end{cases}$$

The state z_2 is used to form the $\hat{u}^o(jT)$. To this end, we set

$$(F_2, G_2, H_2, J_2)(k) = \begin{cases} (\begin{bmatrix} c_0 & 1 \end{bmatrix}, c_0\bar{\delta}_1, 0, 0) & k=0, \quad p=1 \\ (\begin{bmatrix} 0 & 1 \end{bmatrix}, c_0\bar{\delta}_{k+1}, c_0k_1, c_0k_2) & k=0, \quad p>1, \quad p=m. \\ (\begin{bmatrix} 0 & 1 \end{bmatrix}, 0, 0, 0) & k=0, \quad p>1, \quad p>m \\ (\begin{bmatrix} c_0 & 1 \end{bmatrix}, c_0\bar{\delta}_k, 0, 0) & k=p-1, \quad p=m, \quad m\geq 1 \\ (\begin{bmatrix} c_0 & 1 \end{bmatrix}, 0, 0, 0) & k=p-1, \quad p>m \quad m\geq 1 \\ (\begin{bmatrix} 0 & 1 \end{bmatrix}, 0, 0, 0) & \text{all remaining} \\ & k\in\{1,\cdots,p-2\}. \end{cases}$$

In the case that $p=m, \quad m\geq 1$ the initial value of the states should be chosen as

$$\begin{bmatrix} z_1(0) \\ z_2(0) \end{bmatrix} = \begin{bmatrix} (0) \\ -c_0\bar{\delta}_{k+1}y(0). \end{bmatrix}$$

The construction of the control signal is as follows

$$(L, M)(k) = \begin{cases} (\begin{bmatrix} c_0 & 1 \end{bmatrix}, c_0\bar{\delta}_1, 0, 0) & k=0, \quad p=1 \\ (\begin{bmatrix} 0 & 1 \end{bmatrix}, c_0\bar{\delta}_{k+1}, 0, 0) & k=0, \quad p=m, \quad m>1 \\ (\begin{bmatrix} 0 & 1 \end{bmatrix}, 0, 0, 0) & k=0, \quad p>m, \quad m>1 \\ (\begin{bmatrix} 0 & 1 \end{bmatrix}, 0, 0, 0) & k=p-1, \quad p=m, \quad m\geq 1 \\ (\begin{bmatrix} 1 & 1 \end{bmatrix}, 0, 0, 0) & k=p-1, \quad p>m \quad m\geq 1 \\ (\begin{bmatrix} 0 & 1 \end{bmatrix}, 0, 0, 0) & \text{all remaining} \\ & k\in\{1,\cdots,p-2\}. \end{cases}$$

The difference equation for the states is as

$$\begin{aligned} \begin{bmatrix} z_1(pj+i+1) \\ z_2(pj+i+1) \end{bmatrix} &= \begin{bmatrix} F1(1)(pj+i) & F1(2)(pj+i) \\ F2(1)(pj+i) & F_2(2)(pj+i) \end{bmatrix} \begin{bmatrix} z_1(pj+i) \\ z_2(pj+i) \end{bmatrix} + \\ & \begin{bmatrix} G_1(pj+i) \\ G_2(pj+i) \end{bmatrix} y(jT+ih) + \begin{bmatrix} H_1(pj+i) \\ 0 \end{bmatrix} \bar{x}_m(jT+ih) + \\ & \begin{bmatrix} J_1(pj+i) \\ 0 \end{bmatrix} \bar{u}_m(jT+ih), \quad j\in\mathbf{Z}^+, \quad i\in\{0,\cdots,p-1\}. \end{aligned}$$

Recall that the matrices $F_1, F_2, G_1, G_2, H_1, H_2, J_1$ and J_2 are periodic with period p for $j\in\mathbf{N}$. From the controller construction, it is easy to show that $F2(2)(pj+i) = 1, \quad j\in\mathbf{N}, \quad i\in\{0,\cdots,p-1\}$ and

$$z_2 = \hat{u}^o(jT)$$

as desired in parts (i) and (ii) of Lemma 5.2. To prove part (iii), using the fact that $F1(0)$, it follows

$$F_1(p-1)F_1(p-2)\cdots F_1(0) = 0,$$

i.e. the first sub-system is deadbeat, as desired in part (iii) of Lemma 5.

Now we prove the lemma for the case $q\geq 1$ and the state z has dimension 7:

- z_1 keeps track of the following entry, used to construct the $\hat{\phi}_i(jT)$ terms:

$$\frac{1}{\rho}\underbrace{\left[\begin{array}{cc} 0 & 1 \end{array}\right] H_m(h)^{-1} S_m^{-1}}_{=:\left[\begin{array}{cccc} \delta_0 & \delta_1 & \cdots & \delta_m \end{array}\right]} \left[\begin{array}{c} y(jT) \\ y(jT+h) \\ \vdots \\ y(jT+mh) \end{array}\right].$$

Construction of the $\hat{\phi}_i(jT)$ terms is carried out using two states:

- z_2 keeps track of the current $\hat{\phi}_i(jT) - \delta_m y(jT+(i+1)mh), \quad i = 1, \cdots, q,$ which is being constructed.

$$\left[\begin{array}{cccc} \delta_0 & \delta_1 & \cdots & \delta_m \end{array}\right] \left[\begin{array}{c} y(jT+imh) - y(jT) \\ y(jT+imh+h) - y(jT+h) \\ \vdots \\ y(jT+(i+1)mh) - y(jT+mh) \end{array}\right] + \hat{\phi}_{i-1}(jT) - \delta_m y(jT+(i+1)mh), \qquad i = 1, \cdots, q.$$

- z_3 keeps track of the current $\hat{\phi}_i(jT), \quad i = 1, \cdots, q$, which is being constructed.

$$\left[\begin{array}{cccc} \delta_0 & \delta_1 & \cdots & \delta_m \end{array}\right] \left[\begin{array}{c} y(jT+imh) - y(jT) \\ y(jT+imh+h) - y(jT+h) \\ \vdots \\ y(jT+(i+1)mh) - y(jT+mh) \end{array}\right] + \hat{\phi}_{i-1}(jT), \qquad i = 1, \cdots, q.$$

- z_4 is used to construct $\hat{\phi}_0(jT)$:

$$\hat{\phi}_0(jT) = \frac{1}{\rho}\underbrace{\left[\begin{array}{cc} \bar{f}_2 & -1 \end{array}\right] H_m(h)^{-1} S_m^{-1}}_{=:\left[\begin{array}{cccc} \bar{\delta}_0 & \bar{\delta}_1 & \cdots & \bar{\delta}_m \end{array}\right]} \left[\begin{array}{c} y(jT) \\ y(jT+h) \\ \vdots \\ y(jT+mh) \end{array}\right] + \left[\begin{array}{cc} k_1 & k_2 \end{array}\right] \left[\begin{array}{c} \bar{x}_m(jT) \\ \bar{u}_m(jT) \end{array}\right].$$

- z_5 is used to keep track of the current $\hat{\phi}_{i-1}(jT), \qquad i = 1, \cdots, q$; this is essentially a delyed versionof z_3.
- z_6 is used to construct $\sum_{i=0}^{q} c_i \hat{\phi}_i(jT)$.
- z_7 is used to construct and keep track of $\hat{u}^o(jT)$.

The periodically time-varying gains F, G, H, J, and L (*with period* p) are partitioned accordingly as

$$F(k) = \begin{bmatrix} F_1(k) \\ F_2(k) \\ F_3(k) \\ F_4(k) \\ F_5(k) \\ F_6(k) \\ F_7(k) \end{bmatrix}, \quad G(k) = \begin{bmatrix} G_1(k) \\ G_2(k) \\ G_3(k) \\ G_4(k) \\ G_5(k) \\ G_6(k) \\ G_7(k) \end{bmatrix}, \quad H(k) = \begin{bmatrix} H_1(k) \\ H_2(k) \\ H_3(k) \\ H_4(k) \\ H_5(k) \\ H_6(k) \\ H_7(k) \end{bmatrix}, \quad J(k) = \begin{bmatrix} J_1(k) \\ J_2(k) \\ J_3(k) \\ J_4(k) \\ J_5(k) \\ J_6(k) \\ J_7(k) \end{bmatrix},$$

$$L(k) = \begin{bmatrix} L_1(k) & L_2(k) & L_3(k) & L_4(k) & L_5(k) & L_6(k) & L_7(k) \end{bmatrix}.$$

Since the controller is periodic, for simplicity we will assume that $j = 0$.

For z_1 we have

$$(F_1, G_1, H_1, J_1)(k) = \begin{cases} (\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \delta_0, 0, 0) & k = 0 \\ (\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \delta_k, 0, 0,) & k = 1, \cdots, m \\ (\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, 0, 0, 0) & k = m+1, \cdots, p-1. \end{cases}$$

This means that

$$z_1(k) = \begin{bmatrix} 0 & 1 \end{bmatrix} H_m(h)^{-1} S_m^{-1} \mathcal{Y}_m(0), \quad k = m+1, \cdots, p-1.$$

The objective of z_2 is to construct of the current $\hat{\phi}_i(0) - \delta_m y((i+1)mh), \;\; i = 1, \cdots, q$. To this end, we set

$$(F_2, G_2, H_2, J_2)(k) = \begin{cases} (\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, 0, 0, 0) & k = 0, \cdots, m-1 \\ (\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \delta_{k-m}, 0, 0) & k = m, \cdots, 2m-2 \\ (\begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \delta_{m-1}, 0, 0) & k = 2m-1 \\ (\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \delta_0, 0, 0) & i = 2, \cdots, q, \quad k = im \\ (\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \delta_{k-im}, 0, 0) & i = 2, \cdots, q, \\ \qquad k = im+1, \cdots, (i+1)m-2, & \\ (\begin{bmatrix} -1 & 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \delta_{m-1}, 0, 0) & i = 2, \cdots, q, \\ & k = (i+1)m-1, \\ (\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, 0, 0, 0) & \text{all remaining} \\ & \quad k \in \{0, \cdots, p-1\}, \end{cases}$$

which results in

$$z_2(im+m) = \hat{\phi}_i(0) - \delta_m y((i+1)mh), \;\; i = 1, \cdots, q.$$

The state z_3 is used to construct the current $\hat{\phi}_i(0), \;\; i = 1, \cdots, q$. We have

$$(F_3, G_3, H_3, J_3)(k) = \begin{cases} (\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, 0, 0, 0,) & k = 0, \cdots, 2m-1 \\ (\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \delta_m, 0, 0) & i = 1, \cdots, q, \\ & k = (i+1)m, \\ (\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, 0, 0, 0) & \text{all remaining} \\ & \quad k \in \{0, \cdots, p-1\}, \end{cases}$$

which results in

$$z_3(k) = \hat{\phi}_i(0), \quad i = 1, \cdots, q, \quad k = (i+1)m + 1, \cdots, (i+2)m.$$

The state z_4 is used to construct $\hat{\phi}_0(0)$. We have

$$(F_4, G_4, H_4, J_4)(k) = \begin{cases} (\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \bar{\delta}_0, k_1, k_2) & k = 0 \\ (\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \bar{\delta}_k, 0, 0,) & k = 1, \cdots, m \\ (\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, 0, 0, 0) & k = m+1, \cdots, p-1. \end{cases}$$

The state z_5 is used to keep track of current $\hat{\phi}_{i-1}(0), \quad i = 1, \ldots, q$. To this end, set

$$(F_5, G_5, H_5, J_5)(k) = \begin{cases} (0, 0, 0, 0) & k = 0, \cdots, 2m \\ (\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, 0, 0, 0) & k = 2m+1 \\ (\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, 0, 0, 0) & i = 2, \cdots, q, \quad k = (i+1)m \\ (\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, 0, 0, 0) & \text{all remaining} \\ & \quad k \in \{0, \cdots, p-1\}. \end{cases}$$

The state z_6 is used to form $\sum_{i=0}^{q} c_i \hat{\phi}_i(0)$. To this end, we set

$$(F_6, G_6, H_6, J_6)(k) = \begin{cases} (\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, 0, 0, 0) & k = 0, \cdots, m \\ (\begin{bmatrix} 0 & 0 & 0 & c_0 & 0 & 0 & 0 \end{bmatrix}, 0, 0, 0) & k = m+1 \\ (\begin{bmatrix} 0 & 0 & c_i & 0 & 0 & 1 & 0 \end{bmatrix}, 0, 0, 0) & k = (i+1)m+1 \\ & (i = 1, \cdots, q-1) \\ (\begin{bmatrix} 0 & 0 & c_q & 0 & 0 & 1 & 0 \end{bmatrix}, \delta_m, 0, 0) & k = (q+1)m \\ (\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, 0, 0, 0) & \text{all remaining} \\ & \quad k \in \{0, \cdots, p-1\}. \end{cases}$$

Thus,

$$z_6(k) = \sum_{i=0}^{q} c_i \hat{\phi}_i(0), \quad k = (q+1)m + 1, \cdots, p.$$

The state z_7 is used to form the $\hat{u}^o(0)$. To this end, we set

$$(F_7, G_7, H_7, J_7)(k) = \begin{cases} (\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, 0, 0, 0) & k = p-1 \\ (\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, 0, 0, 0) & \text{all remaining} \\ & \quad k \in \{0, \cdots, p-2\}. \end{cases}$$

The construction of the control signal is as follows

$$(L, M)(k) = \begin{cases} (\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, 0) & k = 0, \cdots, m-1 \\ (\begin{bmatrix} 0 & 0 & 0 & \rho & 0 & 0 & 1 \end{bmatrix}, \rho\bar{\delta}_m) & k = m \\ (\begin{bmatrix} 0 & 0 & 0 & \rho & 0 & 0 & 1 \end{bmatrix}, 0) & k = m+1, \cdots, 2m-1 \\ (\begin{bmatrix} 0 & 0 & 0 & \rho & -\rho & 0 & 1 \end{bmatrix}, \rho\delta_m) & k = (i+1)m \\ & \quad (i = 1, \cdots, q-1) \\ (\begin{bmatrix} 0 & 0 & 0 & \rho & -\rho & 0 & 1 \end{bmatrix}, 0) & k = (i+1)m+1, \cdots, (i+2)m-1 \\ & \quad (i = 1, \cdots, q-1) \\ (\begin{bmatrix} 0 & 0 & 0 & 0 & -\rho & 0 & 1 \end{bmatrix}, 0) & k = (q+1)m, \cdots, (q+2)m-1 \\ (\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{p}{p-2q-1} & 1 \end{bmatrix}, 0) & k = (q+2)m, \cdots, p-1. \end{cases}$$

Now let us divide the states into two parts: $\zeta := \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{bmatrix}$ and z_7. Before proceeding, with an arbitrary $(1 \times n)$ vector Q we define the vector $Q_{[i:j]}, \quad i \leq j \leq n$, equal to a $(1 \times (j-i+1))$ row vector consisting of elements i^{th} to j^{th} of Q. Regarding this partitioning and definition, we define the following matrices

$$F_{11} := \begin{bmatrix} F_{1[1:6]} \\ F_{2[1:6]} \\ F_{3[1:6]} \\ F_{4[1:6]} \\ F_{5[1:6]} \\ F_{6[1:6]} \end{bmatrix}, \quad F_{12} := \begin{bmatrix} F_{1[7:7]} \\ F_{2[7:7]} \\ F_{3[7:7]} \\ F_{4[7:7]} \\ F_{5[7:7]} \\ F_{6[7:7]} \end{bmatrix}, \quad F_{21} := F_{7[1:6]}, \quad F_{22} := F_{7[7:7]}$$

$$\bar{G}_1 := \begin{bmatrix} G_{1[1:1]} \\ G_{2[1:1]} \\ G_{3[1:1]} \\ G_{4[1:1]} \\ G_{5[1:1]} \\ G_{6[1:1]} \end{bmatrix}, \quad \bar{H}_1 := \begin{bmatrix} H_{1[1:1]} \\ H_{2[1:1]} \\ H_{3[1:1]} \\ H_{4[1:1]} \\ H_{5[1:1]} \\ H_{6[1:1]} \end{bmatrix}, \quad \bar{J}_1 := \begin{bmatrix} J_{1[1:1]} \\ J_{2[1:1]} \\ J_{3[1:1]} \\ J_{4[1:1]} \\ J_{5[1:1]} \\ J_{6[1:1]} \end{bmatrix};$$

thus, the difference equation for the states is as

$$\begin{aligned} \begin{bmatrix} \zeta(pj+i+1) \\ z_6(pj+i+1) \end{bmatrix} &= \begin{bmatrix} F_{11}(pj+i) & F_{12}(pj+i) \\ F_{21}(pj+i) & F_{22}(pj+i) \end{bmatrix} \begin{bmatrix} \zeta(pj+i) \\ z_6(pj+i) \end{bmatrix} + \\ &\begin{bmatrix} \bar{G}_1(pj+i) \\ 0 \end{bmatrix} y(jT+ih) + \begin{bmatrix} \bar{H}_1(pj+i) \\ 0 \end{bmatrix} \bar{x}_m(jT+ih) + \\ &\begin{bmatrix} \bar{J}_1(i) \\ 0 \end{bmatrix} \bar{u}_m(jT+ih), \quad j \in \mathbf{Z}^+, \quad i \in \{0, \cdots, p-1\}. \end{aligned}$$

Recall that the matrices $F_{11},\ F_{12},\ F_{21},\ F_{22},\ \bar{G}_1\ \bar{H}_1$, and $\bar{J}_1$ are periodic with period p. From the controller construction, it is easy to show that $F_{22}(k) = 1, \quad k \in \mathbf{Z}^+$, and z_6 is constructed as

$$z_6(pj+i) = \begin{cases} \hat{u}^o(jT), & i \in \{0, 1, \cdots, p-2\} \\ \hat{u}^o[(j+1)T], & i = p-1, \end{cases}, \quad j \in \mathbf{Z}^+,$$

as desired in parts (i) and (ii) of Lemma 5.2. To prove part (iii), using the fact that $F_1(0),\ F_2(0), \cdots, F_6(0)$ are all zero vectors, it follows that $F_{11}(0) = 0$ and so

$$F_{11}(p-1)F_{11}(p-2)\cdots F_{11}(0) = 0,$$

i.e. the first sub-system is deadbeat, as desired in part (iii) of Lemma 5.2.

□

.3 APPENDIX C

Before starting the proofs, a technical result is presented which is used in other proofs.

Lemma C.1 [14]:
Consider the square matrix

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where A_{22} is nonsingular. On defining $L_0 := A_{22}^{-1} A_{21}$ and $A_0 := A_{11} - A_{12} L_0$, if

$$||A_{22}^{-1}|| \leq \frac{1}{3}(||A_0|| + ||A_{12}||\ ||L_0||)^{-1},$$

then there exists a matrix M with the property that

$$||M|| \leq \frac{2||A_0||\ ||L_0||}{||A_0|| + ||A_{12}||\ ||L_0||},$$

such that if $L := L_0 + M$, then

$$\begin{bmatrix} I & 0 \\ L & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ -L & I \end{bmatrix} = \begin{bmatrix} A_{11} - A_{12}L & A_{12} \\ 0 & A_{22} + LA_{12} \end{bmatrix}.$$

Proof of Lemma 6.1:

Suppose that the two conditions (i) and (ii) hold. Examining (6.8), the closed-loop system is stable if and only if eigenvalues of

$$A_{closed} := \begin{bmatrix} I + \bar{A}T + \Delta_1(kT) & \bar{B}T + \Delta_2(kT) \\ c_0 \begin{bmatrix} \bar{f}_2 & -1 \end{bmatrix} \mathcal{O}_l + \mathcal{O}(T) & 1 + c_0 \begin{bmatrix} \bar{f}_2 & -1 \end{bmatrix} p_l + \mathcal{O}(T) \end{bmatrix},$$

stay in the open unit disk. To proceed, we first carry out a similarity transformation to make the (2,1) element small, while maintaining the (1,2) element small. First we need some notation:

$$J := \frac{\begin{bmatrix} \bar{f}_2 & -1 \end{bmatrix} \mathcal{O}_l}{\begin{bmatrix} \bar{f}_2 & -1 \end{bmatrix} p_l},$$

and we define the similarity transformation T_1 as

$$T_1 := \begin{bmatrix} I & 0 \\ J & 1 \end{bmatrix}.$$

Then

$$\bar{A}_{closed} := T_1 A_{closed} T_1^{-1} =$$

$$\left[\begin{array}{c} I + (\bar{A} - \bar{B}J)T + \underbrace{\Delta_1(kT) - \Delta_2(kT)J}_{=:\Delta_{11}(kT)} \\ \underbrace{T\bar{A}J + TJ\bar{A} + J\Delta_1(kT) + \mathcal{O}(T) - TJ\bar{B}J + J\Delta_2(kT)J - \mathcal{O}(T)J}_{=:\Delta_{21}(kT)} \end{array}\right.$$

$$\left.\begin{array}{c} \underbrace{\bar{B}T + \Delta_2(kT)}_{=:\Delta_{12}(kT)} \\ 1 + c_0 \left[\begin{array}{cc} \bar{f}_2 & -1 \end{array}\right] p_l + \underbrace{\mathcal{O}(T) + TJ\bar{B} + J\Delta_2(kT)}_{=:\Delta_{22}(kT)} \end{array}\right]$$

$$=: \left[\begin{array}{cc} a_{11}(T) & a_{12}(T) \\ a_{21}(T) & a_{22}(T) \end{array}\right];$$

recall that there is a $\gamma_1 > 0$ so that

$$\begin{aligned} ||\Delta_1(kT)|| &\leq \gamma_1 T^2, \\ ||\Delta_2(kT)|| &\leq \gamma_1 T^2, \end{aligned}$$

so there exists a constant $\gamma_2 > 0$ so that

$$\begin{aligned} ||\Delta_{11}(kT)|| &\leq \gamma_2 T^2, \\ ||\Delta_{12}(kT)|| &\leq \gamma_2 T, \\ ||\Delta_{21}(kT)|| &\leq \gamma_2 T, \\ ||\Delta_{22}(kT)|| &\leq \gamma_2 T. \end{aligned}$$

Now we carry out a second similarity transformation to put $\bar{A}_{closed}$ into upper triangular form, so that stability will be determined by the diagonal terms. Before proceeding, define

$$\alpha := \sup_{\bar{\theta} \in \bar{\mathcal{P}}} |1 + c_0 \left[\begin{array}{cc} \bar{f}_2 & -1 \end{array}\right] p_l|,$$

which is less than one. Now we define the second similarity transformation as

$$T_2 := \left[\begin{array}{cc} I & 0 \\ W(T) & 1 \end{array}\right],$$

$W(T)$ will be defined in such a way that $T_2\bar{A}_{closed}T_2^{-1}$ is upper triangular. We have

$$T_2\bar{A}_{closed}T_2^{-1} = \begin{bmatrix} a_{11}(T) - a_{12}(T)W(T) & a_{12}(T) \\ W(T)a_{11}(T) + a_{21}(T) - W(T)a_{12}(T)W(T) - a_{22}(T)W(T) & a_{22}(T) + W(T)a_{12}(T) \end{bmatrix}. \tag{92}$$

Now we would like the (2,1) element to be zero, which is true if and only if

$$W(T) = \underbrace{a_{21}(T)a_{11}(T)^{-1} - W(T)a_{12}(T)W(T)a_{11}(T)^{-1} - a_{22}(T)W(T)a_{11}(T)^{-1}}_{=:\eta(W(T))}.$$

Now we have

$$\begin{aligned} a_{11}(T) &= I + \mathcal{O}(T), \\ a_{12}(T) &= \mathcal{O}(T), \\ a_{21}(T) &= \mathcal{O}(T), \\ a_{22}(T) &= 1 + c_0 \begin{bmatrix} \bar{f}_2 & -1 \end{bmatrix} p_l + \mathcal{O}(T), \end{aligned}$$

so

$$\begin{aligned} \eta(W(T)) = & [I + \mathcal{O}(T)]\mathcal{O}(T) + [I + \mathcal{O}(T)]W(T)\mathcal{O}(T)W(T) + \\ & [I + \mathcal{O}(T)] \left\{1 + c_0 \begin{bmatrix} \bar{f}_2 & -1 \end{bmatrix} p_l + \mathcal{O}(T)\right\} W(T). \end{aligned}$$

Hence, if $||W(T)||_2 \leq 1$, then there exists a $\gamma_3 > 0$ so that for small T:

$$\begin{aligned} ||\eta(W(T))|| \leq & (1+\gamma_3 T)\gamma_3 T + (1+\gamma_3 T)\gamma_3 T||W(T)||^2 + \\ & (1+\gamma_3 T)(\alpha + \gamma T)||W(T)|| \\ \leq & \alpha + 7\gamma_3 T. \end{aligned} \tag{93}$$

Hence, for small T, say $T \in (0, \bar{\bar{T}})$:

$$||\eta(W(T))|| \leq \frac{\alpha+1}{2}, \qquad \bar{\theta} \in \bar{\mathcal{P}}.$$

According to Brouwer's Fixed Point Theorem, the equation

$$W(T) = \eta(W(T))$$

has a solution for $T \in (0, \bar{T})$ satisfying

$$||W(T)|| \leq 1.$$

A tighter bound on its size can be obtained by examining (93):

$$\begin{aligned} ||W(T)|| = & ||\eta(W(T))|| \\ \leq & (1+\gamma_3 T)\gamma_3 T + \\ & (1+\gamma_3 T)\gamma_3 T||W(T)|| + (1+\gamma_3 T)(\alpha + \gamma_3 T)||W(T)|| \\ \Rightarrow & [1 - (1+\gamma_3 T)(\alpha + 2T)]\, ||W(T)|| \leq (1+\gamma_3 T)\gamma_3 T; \end{aligned}$$

for small T we have

$$(1+\gamma_3 T)(\alpha+2T) \;\leq\; \frac{\alpha+1}{2},$$

so for such T:

$$\begin{aligned}(1-\frac{\alpha+1}{2})\;||W(T)|| &\leq (1+\gamma_3 T)\gamma_3 T\\ \Rightarrow ||W(T)|| &\leq \frac{(1+\gamma_3 T)\gamma_3 T}{1-\left(\frac{\alpha+1}{2}\right)}\\ &= \mathcal{O}(T).\end{aligned}$$

Now we return to (92). For small T the closed-loop system is stable if and only if

$$sp[a_{11}(T)-a_{12}(T)W(T)] \subset \mathbf{D}, \quad \bar{\theta}\in\bar{\mathcal{P}},$$

and

$$|a_{22}(T)+W(T)a_{12}(T)| < 1, \quad \bar{\theta}\in\bar{\mathcal{P}}.$$

Now we have

$$a_{22}(T)+W(T)a_{12}(T) \;=\; 1+c_0\begin{bmatrix}\bar{f}_2 & -1\end{bmatrix}p_l+\mathcal{O}(T),$$

so for small T

$$|a_{22}(T)+W(T)a_{12}(T)| \;\leq\; \frac{\alpha+1}{2}, \quad \bar{\theta}\in\bar{\mathcal{P}}.$$

Also

$$\begin{aligned}a_{11}(T)-a_{12}(T)W(T) &= I+(\bar{A}-\bar{B}J)T+\mathcal{O}(T^2)\\ &= I+T[(\bar{A}-\bar{B}J)T+\mathcal{O}(T)].\end{aligned}$$

From condition (ii) it follows that there exists a compact set $\hat{\Upsilon}\subset\mathbf{C}^-$ containing υ so that for small T

$$sp[\bar{A}-\bar{B}J+\mathcal{O}(T)] \subset \hat{\Upsilon}, \quad \bar{\theta}\in\bar{\mathcal{P}}.$$

Hence, for such a range of T

$$sp[I+T(\bar{A}-\bar{B}J+\mathcal{O}(T))] \subset \{1+T\upsilon:\;\; \upsilon\in\hat{\Upsilon}\};$$

but this lies in $\mathbf{D}$ for small T. Hence, we conclude that there exists a $\bar{T}>0$ so that we have closed-loop stability for all $\bar{\theta}\in\bar{\mathcal{P}}$ and $T\in(0,\bar{T})$.

To prove the closed-loop stability we need to show that the controller states z are well behaved. First suppose that $t_0=k_0=0$. From the stability of the A_{closed} it follows that there exist constants $\gamma_4>0$ and $\lambda<0$ so that

$$\begin{aligned}&||\hat{\bar{x}}(kT)|| \leq \gamma_4 e^{\lambda kT}(||\bar{x}_0||+|\hat{u}_0^o|)+\gamma_4||u_m||_\infty,\\ &|\hat{u}^o(kT)| \leq \gamma_4 e^{\lambda kT}(||\bar{x}_0||+|\hat{u}_0^o|)+\gamma_4||u_m||_\infty,\end{aligned} \qquad k\in\mathbf{Z}^+. \tag{94}$$

Using an identical argument to that used in Theorem 5.1 we can show that there exists a constant $\gamma_5 > 0$ so that

$$|z[k]| \;\leq\; \gamma_5 e^{\lambda kh}(||\bar{x}_0|| + |\hat{u}_0^o|) + \gamma_5||u_m||_\infty, \quad k \in \mathbf{Z}^+.$$

It remains to prove that $\hat{\bar{x}}$ is well behaved between sample points. However, it follows from solving the system equation that for $t \in [kT, (k+1)T)$ that

$$\hat{\bar{x}}(t) \;=\; (I + \mathcal{O}(T))\hat{\bar{x}}(kT) + \mathcal{O}(T)||u_m||_\infty + \mathcal{O}(T)||\hat{u}^o(kT)||,$$

so it follows that there exists a $\gamma_6 > 0$ so that

$$\hat{\bar{x}}(t) \;\leq\; \gamma_6 e^{\lambda t}||\bar{x}_0|| + \gamma_6||u_m||_\infty.$$

In the more general case of $t_0 = k_0 h > 0$, we simply argue in the same way as in the proof of Theorem 5.1.

Hence, with

$$x_{sd}(t) := \begin{bmatrix} \hat{\bar{x}}(t) \\ z[k] \end{bmatrix}, \;\; t \in [kh, (k+1)h),$$

so it follows that

$$\|x_{sd}(t)\| \leq \underbrace{(\gamma_5 e^{-\lambda \bar{T}} + \gamma_6)}_{=:\gamma_7}(e^{\lambda t}\|x_{sd}(0)\| + \|u_m\|_\infty), \;\; t \geq 0,$$

as desired.

□

Proof of Lemma 6.2:

Recall that regarding Condition (d), the plant is time-invariant, so we observe the following two properties

- **Property 1:**

$$\begin{aligned}
C &= \begin{bmatrix} \bar{0} & 1 & 0 & \cdots & 0 \end{bmatrix} \\
CA &= \begin{bmatrix} \bar{0} & 0 & 1 & \cdots & 0 \end{bmatrix} \\
&\vdots \\
CA^{m-1} &= \begin{bmatrix} \bar{0} & 0 & \cdots & 0 & 1 \end{bmatrix} \\
CA^m &= \begin{bmatrix} \alpha_0 & \cdots & \alpha_{n-m-1} & -\beta_0 & \cdots & -\beta_{m-1} \end{bmatrix} \\
CA^{m+1} &= \begin{bmatrix} -b_0\alpha_{n-m-1} - \alpha_0\beta_{m-1} & \cdots & \beta_{m-1} + \beta_{m-1}^2 \end{bmatrix}.
\end{aligned}$$

For simplicity let us define:

$$\begin{aligned}
CA^{m+1} &= \begin{bmatrix} -b_0\alpha_{n-m-1} - \alpha_0\beta_{m-1} & \cdots & \beta_{m-1} + \beta_{m-1}^2 \end{bmatrix} \\
&:= \begin{bmatrix} \zeta_{0,0} & \cdots & \zeta_{n-1,0} \end{bmatrix}.
\end{aligned}$$

If we do the same procedure we will have

$$CA^l \;=\; \begin{bmatrix} \zeta_{0,l-m} & \cdots & \zeta_{n-1,l-m} \end{bmatrix}.$$

- **Property 2:**

$$\begin{aligned} CB &= 0 \\ CAB &= 0 \\ &\vdots \\ CA^{m-2}B &= 0 \\ CA^{m-1}B &= g \\ CA^{m}B &= -g\beta_{m-1}. \end{aligned}$$

For simplicity let us define:

$$CA^{m}B = \vartheta_0.$$

If we do the same procedure we will have

$$CA^{l-1}B = \vartheta_{l-m-1}.$$

First we prove part (i) of Lemma 6.2. To proceed, we use these properties to obtain

$$1 + c_0 \left[\begin{array}{cc} \bar{f}_2 & -1 \end{array} \right] p_l,$$

and

$$\left[\bar{A} - \bar{B} \frac{\left[\begin{array}{cc} \bar{f}_2 & -1 \end{array} \right] \mathcal{O}_l}{\left[\begin{array}{cc} \bar{f}_2 & -1 \end{array} \right] p_l} \right].$$

It is easy to show that

$$\left[\begin{array}{cc} \bar{f}_2 & -1 \end{array} \right] p_l = -gr_m {\lambda^*}^{l-m} - \vartheta_0 r_{m+1} {\lambda^*}^{l-m-1} - \cdots - \vartheta_{l-m-2} r_{l-1} \lambda^* - \vartheta_{l-m-1},$$

and since λ^* is very large we have

$$\left[\begin{array}{cc} \bar{f}_2 & -1 \end{array} \right] p_l \approx -gr_m {\lambda^*}^{l-m},$$

and so

$$1 + c_0 \left[\begin{array}{cc} \bar{f}_2 & -1 \end{array} \right] p_l \approx 1 - c_0 g r_m {\lambda^*}^{l-m};$$

it is easy to show that

$$|1 - c_0 g r_m {\lambda^*}^{l-m}| < 1,$$

if and only if c_0 and g have the same sign and

$$0 < |c_0| < \frac{1}{r_m |g| {\lambda^*}^{l-m}},$$

is small enough; thus, with this two conditions

$$|1 + c_0 \left[\begin{array}{cc} \bar{f}_2 & -1 \end{array} \right] p_l| < 1,$$

as desired in part (i) of Lemma 6.2.

Now we prove part (ii) of Lemma 6.2. The objective is to design the controller gains appropriately so that for all $\bar{\theta} \in \bar{\mathcal{P}}$ there exists a compact set $\Upsilon \subset \mathbf{C}^-$ so that

$$sp\left[\bar{A} - \bar{B}\frac{\left[\ \bar{f}_2 \quad -1\ \right]\mathcal{O}_l}{\left[\ \bar{f}_2 \quad -1\ \right]p_l}\right] \subset \Upsilon, \quad \bar{\theta} \in \bar{\mathcal{P}}.$$

To proceed, let us define

$$\begin{aligned} \mu_i &:= -\zeta_{l-m,i} - \sum_{j=0}^{l-m-1} \zeta_{i,j} r_{m+j} {\lambda^*}^{l-m-j} \\ &\approx -\zeta_{0,i} r_m {\lambda^*}^{l-m}, \quad i \in \{0, \cdots, n-m-1\}, \end{aligned}$$

then

$$\begin{aligned} &\left[\ \bar{f}_2 \quad -1\ \right]\mathcal{O}_l = \\ &\quad \left[\ \mu_0 \quad \cdots \quad \mu_{n-m-1} \quad -r_0{\lambda^*}^l + \mu_{n-m} \quad \cdots \quad -r_{m-1}{\lambda^*}^{l-m+1} + \mu_{n-1}\ \right] \\ &\quad \approx \left[\ -\zeta_{0,0} r_m {\lambda^*}^{l-m} \quad \cdots \quad -\zeta_{0,n-m-1} r_m {\lambda^*}^{l-m} \quad -r_0{\lambda^*}^l \quad \cdots \quad -r_{m-1}{\lambda^*}^{l-m+1}\ \right]. \end{aligned}$$

Thus,

$$\bar{B}\left[\ \bar{f}_2 \quad -1\ \right]\mathcal{O}_l = $$

$$\approx g\begin{bmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 \\ & \ddots & & & \ddots & \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ -\zeta_{0,0} r_m {\lambda^*}^{l-m} & \cdots & -\zeta_{0,n-m-1} r_m {\lambda^*}^{l-m} & -r_0{\lambda^*}^l & \cdots & -r_{m-1}{\lambda^*}^{l-m+1} \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ & \ddots & & & \ddots & \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

This structure yields

$$\bar{A} - \bar{B}\frac{\left[\ \bar{f}_2 \quad -1\ \right]\mathcal{O}_l}{\left[\ \bar{f}_2 \quad -1\ \right]p_l} = \begin{bmatrix} A - B\frac{\left[\ \bar{f}_2 \quad -1\ \right]\mathcal{O}_l}{\left[\ \bar{f}_2 \quad -1\ \right]p_l} & 0 & 0 \\ 0 & A_m & B_m \\ 0 & 0 & -\sigma \end{bmatrix}, \tag{95}$$

which is a diagonal matrix. Observe that A_m is stable and $\sigma > 0$, so we only need to design the controller gains so that for all $\bar{\theta} \in \bar{\mathcal{P}}$, there exists a compact set $\Upsilon \subset \mathbf{C}^-$ so that

$$sp\left[A - B\frac{\left[\ \bar{f}_2 \quad -1\ \right]\mathcal{O}_l}{\left[\ \bar{f}_2 \quad -1\ \right]p_l}\right] \subset \Upsilon, \quad \bar{\theta} \in \bar{\mathcal{P}}.$$

Now let us analyze this matrix. It is easy to show that

$$A - B\frac{\begin{bmatrix}\bar{f}_2 & -1\end{bmatrix}\mathcal{O}_l}{\begin{bmatrix}\bar{f}_2 & -1\end{bmatrix}p_l} \approx \left[\begin{array}{cc} \begin{bmatrix} 0 & 1 & & \\ & & \ddots & \\ 0 & 0 & \dots & 1 \\ -b_0 & -b_1 & \dots & -b_{n-m-1} \end{bmatrix} & \begin{bmatrix} 0 & 0 & & \\ & & \ddots & \\ 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & & \\ & & \ddots & \\ 0 & 0 & \dots & 0 \\ \alpha_0+\zeta_{0,0} & \alpha_1+\zeta_{0,1} & \dots & \alpha_{n-m-1}+\zeta_{0,n-m-1} \end{bmatrix} & \begin{bmatrix} 0 & 1 & & \\ & & \ddots & \\ 0 & 0 & \dots & 1 \\ -\beta_0-\frac{r_0}{r_m}{\lambda^*}^m & -\beta_1-\frac{r_1}{r_m}{\lambda^*}^{m-1} & \dots & -\beta_{m-1}-\frac{r_{m-1}}{r_m}\lambda^* \end{bmatrix} \end{array}\right]$$

Now we consider that

$$-\beta_j - \frac{r_j}{r_m}{\lambda^*}^{m-j} \quad \approx \quad -\frac{r_j}{r_m}{\lambda^*}^{m-j}, \qquad j \in \{0, \cdots, m-1\},$$

and define $\kappa_i := \alpha_i + \zeta_{0,i}, \qquad i \in \{0, \cdots, n-m-1\}$, so

$$A - B\frac{\begin{bmatrix}\bar{f}_2 & -1\end{bmatrix}\mathcal{O}_l}{\begin{bmatrix}\bar{f}_2 & -1\end{bmatrix}p_l} =: \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$\left[\begin{array}{cc} \begin{bmatrix} 0 & 1 & & \\ & & \ddots & \\ 0 & 0 & \dots & 1 \\ -b_0 & -b_1 & \dots & -b_{n-m-1} \end{bmatrix} & \begin{bmatrix} 0 & 0 & & \\ & & \ddots & \\ 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & & \\ & & \ddots & \\ 0 & 0 & \dots & 0 \\ \kappa_0 & \kappa_1 & \dots & \kappa_{n-m-1} \end{bmatrix} & \begin{bmatrix} 0 & 1 & & \\ & & \ddots & \\ 0 & 0 & \dots & 1 \\ -\frac{r_0}{r_m}{\lambda^*}^m & -\frac{r_1}{r_m}{\lambda^*}^{m-1} & \dots & -\frac{r_{m-1}}{r_m}\lambda^* \end{bmatrix} \end{array}\right]$$

with

$$A_{11} := \begin{bmatrix} 0 & 1 & & \\ & & \ddots & \\ 0 & 0 & \dots & 1 \\ -b_0 & -b_1 & \dots & -b_{n-m-1} \end{bmatrix}, \quad A_{12} := \begin{bmatrix} 0 & 0 & & \\ & & \ddots & \\ 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \end{bmatrix},$$

$$A_{21} := \begin{bmatrix} 0 & 0 & & \\ & & \ddots & \\ 0 & 0 & \dots & 0 \\ \kappa_0 & \kappa_1 & \dots & \kappa_{n-m-1} \end{bmatrix}, \quad A_{22} := \begin{bmatrix} 0 & 1 & & \\ & & \ddots & \\ 0 & 0 & \dots & 1 \\ -\frac{r_0}{r_m}\lambda^{*m} & -\frac{r_1}{r_m}\lambda^{*m-1} & \dots & -\frac{r_{m-1}}{r_m}\lambda^* \end{bmatrix}. \tag{96}$$

To achieve the desired result, first we need to transform

$$A - B\frac{\left[\begin{array}{cc} \bar{f}_2 & -1 \end{array}\right]\mathcal{O}_l}{\left[\begin{array}{cc} \bar{f}_2 & -1 \end{array}\right]p_l}$$

into an upper triangular matrix using a similarity transformation:

Claim C.1: There exists a $\bar{\lambda}^* >> 1$ large enough so that for $\lambda^* \geq \bar{\lambda}^*$, there exists a nonsingular transformation $\tilde{T} := \begin{bmatrix} I & 0 \\ L & I \end{bmatrix}$, where $||L|| = \mathcal{O}(\frac{1}{\lambda^*})$ and we have

$$\begin{aligned} \tilde{T}\left[A - B\frac{\left[\begin{array}{cc} \bar{f}_2 & -1 \end{array}\right]\mathcal{O}_l}{\left[\begin{array}{cc} \bar{f}_2 & -1 \end{array}\right]p_l}\right]\tilde{T}^{-1} &= \tilde{T}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\tilde{T}^{-1} \\ &= \begin{bmatrix} A_{11} - A_{12}L & A_{12} \\ 0 & A_{22} + LA_{12} \end{bmatrix}. \end{aligned}$$

Proof: The proof of Claim C.1 is given below.

Now we have the diagonal transformed matrix

$$\begin{bmatrix} A_{11} - A_{12}L & A_{12} \\ 0 & A_{22} + LA_{12} \end{bmatrix},$$

and we analyze the matrices on the diagonal. First we work on $A_{11} - A_{12}L$. Recall that because the plant is minimum phase, A_{11} is Hurwitz. From Claim C.1 for $\lambda^* >> 1$ large enough, $||A_{12}L|| = \mathcal{O}(1/\lambda^*)$ is small, so there exist constants $\gamma_1 > 0$ and $\lambda_1 > 0$ so that

$$||e^{(A_{11}-A_{12}L)t}|| \leq \gamma_1 e^{-\lambda_1 t}. \tag{97}$$

Now we look at $A_{22} + LA_{12}$. First we have to prove that A_{22} is Hurwitz. From the structure of A_{22} in (96), it is easy to show that the characteristic polynomial of this matrix is as

$$s^m + \frac{r_{m-1}}{r_m}\lambda^* s^{m-1} + \cdots + \frac{r_1}{r_m}{\lambda^*}^{m-1} s + \frac{r_0}{r_m}{\lambda^*}^m.$$

Regarding the fact that $r_0, r_1, \ldots, r_m$ are defined so that the above polynomial is Hurwitz, so A_{22} is Hurwitz. Similar to the stability proof of $A_11 - A_{12}L$, for $\lambda^* >> 1$ large enough, $||LA_{12}|| = \mathcal{O}(1/\lambda^*)$ is small, so there exist constants $\gamma_2 > 0$ and $\lambda_2 > 0$ so that

$$||e^{(A_{22}+LA_{12})t}|| \leq \gamma_2 e^{-\lambda_2 t}.$$

If we combine this inequality with (97), and we set $\lambda^{**} = \max\{\bar{\lambda}^*, \bar{\bar{\lambda}}^*\}$, then for $\lambda^* \geq \lambda^{**}$, for all $\bar{\theta} \in \bar{\mathcal{P}}$, there exists a compact set $\Upsilon \subset \mathbf{C}^-$ so that

$$sp\left[A - B\frac{\left[\begin{array}{cc}\bar{f}_2 & -1\end{array}\right]\mathcal{O}_l}{\left[\begin{array}{cc}\bar{f}_2 & -1\end{array}\right]p_l}\right] \subset \Upsilon, \quad \bar{\theta} \in \bar{\mathcal{P}};$$

recall that in the diagonal matrix (95) A_m is Hurwitz and $\sigma > 0$. Thus

$$sp\left[\bar{A} - \bar{B}\frac{\left[\begin{array}{cc}\bar{f}_2 & -1\end{array}\right]\mathcal{O}_l}{\left[\begin{array}{cc}\bar{f}_2 & -1\end{array}\right]p_l}\right] \subset \Upsilon, \quad \bar{\theta} \in \bar{\mathcal{P}},$$

as desired in part (ii) of Lemma 6.2.

□

Proof of Claim C.1:

To transform

$$\left[A - B\frac{\left[\begin{array}{cc}\bar{f}_2 & -1\end{array}\right]\mathcal{O}_l}{\left[\begin{array}{cc}\bar{f}_2 & -1\end{array}\right]p_l}\right] = \underbrace{\left[\begin{array}{cc}A_{11} & A_{12}\\ A_{21} & A_{22}\end{array}\right]}_{=:A_{closed}}$$

into an upper triangular matrix, first we show that $A_{22} = \mathcal{O}(\lambda)$ and then use Lemma C.1. Let us define the transformation matrix

$$T_1 = \left[\begin{array}{cccc}1 & & & \\ & \frac{1}{\lambda^*} & & \\ & & \ddots & \\ & & & \frac{1}{{\lambda^*}^{m-1}}\end{array}\right],$$

then we have

$$\begin{aligned} T_1 A_{22} T_1^{-1} &= \lambda^* \begin{bmatrix} 0 & 1 & & \\ & & \ddots & \\ 0 & 0 & \dots & 1 \\ -g\frac{r_0}{r_m} & -g\frac{r_1}{r_m} & \dots & -g\frac{r_{m-1}}{r_m} \end{bmatrix} \\ &= \mathcal{O}(\lambda^*). \end{aligned}$$

Now we apply Lemma C.1 [14] to transform A_{closed} into an upper triangular matrix. Since $A_{22} = \mathcal{O}(\lambda)$, on defining $L_0(t) := A_{22}^{-1}A_{21}(t)$ and $A_0(t) := A_{11}(t) - A_{12}L_0(t)$, since $\lambda^* >> 1$ is large enough, $||L_0|| = ||A_{22}^{-1}A_{21}|| = \mathcal{O}(1/\lambda^*)$, and $||A_0|| = ||A_{11} - A_{12}L_0|| = \mathcal{O}(1)$, we have

$$||A_{22}^{-1}|| \leq \frac{1}{3}(||A_0|| + ||A_{12}||\ ||L_0||)^{-1}.$$

So there exists a matrix $M(t)$ with the property that

$$\begin{aligned} ||M|| &\leq \frac{2||A_0||\ ||L_0||}{||A_0|| + ||A_{12}||\ ||L_0||} \\ &= \mathcal{O}(1/\lambda^*), \end{aligned}$$

such that if $L(t) := L_0(t) + M(t) = \mathcal{O}(1/\lambda^*)$, then

$$\begin{bmatrix} I & 0 \\ L & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ -L & I \end{bmatrix} = \begin{bmatrix} A_{11} - A_{12}L & A_{12} \\ 0 & A_{22} + LA_{22} \end{bmatrix},$$

as desired.

□

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