Stabilizing the Psychological Dynamics of People in a Crowd

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

This thesis investigates the use of control theory as a means to study and ultimately control the psychological dynamics of people in a crowd. Gustav LeBon's suggestibility theory, a well-known account of collective behaviour, is used to develop a discrete-time nonlinear model of psychological crowd behaviour that, consistent with suggestibility theory, is open-loop unstable. As a first attempt to stabilize the dynamics, linear observer-based output-feedback techniques and a collection of simple nonlinear control strategies are pursued. The poor performance afforded by these schemes motivates an agent-oriented control strategy in which authoritative figures, termed control agents, are interspersed within the crowd and, similar to the technique of feedback linearization, use knowledge of the system dynamics to issue signals that propagate through the crowd to drive specific components of the state to zero. It is shown that if these states are chosen judiciously then it follows that a collection of other state signals are, themselves, zero. This realization is used to develop a stability result for a simple crowd structure and this result is, in turn, used as a template to develop similar results for crowds of greater complexity. Simulations are used to verify the functionality of the reported schemes and the advantages of using multiple control agents, instead of a single control agent, are emphasized. While the mathematical study of complex social phenomena, including crowds, is prefixed by an assortment of unique challenges, the main conclusion of this thesis is that control theory is a potentially powerful framework to study the underlying dynamics at play in such systems.

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Chapter 1

Introduction

The complex biological underpinnings and multifaceted nature of the human mind make quantifying a person's behaviour, even in a restricted domain of activity, a formidable pursuit. Not surprisingly, the study of social systems, in which the behaviour of multiple people collectively define a social entity, is especially challenging. That said, there are many instances where the behaviour of groups has considerable bearing on the economic, political, or social interests of both the comprising members and third parties alike. Consequently, if engineering methods could be used to study crowd behaviour, then it would seem natural that the results to emerge would have widespread appeal. As an analytical tool, control theory has proven a remarkably powerful construct for studying a broad array of systems. While many of these systems are modeled using well-defined physical laws, e.g., electro-mechanical systems, this impressive track record nevertheless begs the question, "Can control theoretic techniques be used to study social phenomena?" It is in this exploratory spirit that this thesis investigates the use of control theory as a means to study and ultimately control the psychological behaviour of people in a crowd.

Chapter 2 begins the aforementioned investigation by pursuing a dynamic model of psychological crowd behaviour. A brief overview of crowd psychology is presented before narrowing in on a well-known nineteenth-century formulation of group behaviour: Gustav LeBon's suggestibility theory. The key elements of this formulation, most notably the idea that members of a crowd become highly suggestible and readily assume the actions and attitudes of their neighbours, is captured by a discrete-time nonlinear dynamic model. This model is, to the best of the author's knowledge, the first of its kind to focus exclusively on the psychological dynamics that drive crowd behaviour, unlike the majority of existing crowd models, which focus instead on the physical motion of crowds throughout a space. Moreover, while these other formulations typically treat the crowd macroscopically as an aggregate mass, the model used in this work explicitly accounts for each crowd member's individual psychological state and each interaction between crowd members explicitly.

Chapter 3 develops the crowd control problem by providing a definition for crowd stability, formalizing the control objectives, and introducing a notational system for representing a crowd. The primary objective is to stabilize the crowd and ensure each member behaves in a calm and orderly manner, though other aspects of performance, including the scalability and sensing requirements of the control strategies reported, are also discussed. As a first attempt to stabilize the dynamics, a linear observer-based output-feedback controller and a collection of simple nonlinear control strategies are discussed. The poor performance afforded by these schemes, in addition to the fact they are highly impractical and affect change by overriding the dynamics rather than working through social channels that exist within the model, motivate an entirely new stabilization strategy based on the idea of a control agent, that is, an authoritative figure that uses knowledge of the crowd's state to issue control signals with the intent of stabilizing the crowd.

Having formulated the crowd control problem, Chapter 4 considers a control strategy in which a single control agent is used to stabilize a queue, that is a one-dimensional crowd in which each member has no more than two neighbours. The control strategy employed is similar to feedback linearization methods in that it relies on a priori knowledge of the crowd dynamics and the ability to sense the entire state in order to issue signals that propagate through the crowd to drive specific components of the state to zero. This approach is then extended to develop stability results for queues containing multiple control agents. It is shown that while the use of multiple control agents increases the complexity involved in stabilizing a queue, doing so allows for significant improvements with respect to several aspects of performance. In Chapter 5, these multiple-control-agent strategies to stabilize a queue are leveraged to develop stability results for a general crowd in which a member may have two or more neighbours and, consequently, the social network linking crowd members may be significantly more complex.

Finally, Chapter 6 summarizes the prominent ideas of the thesis and reflects on the feasibility of using control theory to study psychological crowd behaviour. Also mentioned is an assortment of open research directions that have arisen during the development of this thesis, but that on account of time constraints, or their tangential nature, or both, have received only modest deliberation.

Chapter 2

Crowd Behaviour

2.1 A Brief History of Crowd Psychology

Psychological crowd theory deals with the thought processes, emotions, and attitudes of crowds. The field rose to prominence in late nineteenth-century Europe as a consequence of the masses rising against oppressive political regimes. Seminal contributions were made by a number of intellectuals, including Gustav LeBon, Scipio Sighele, and Gabriel Tarde. Early treatments of the subject were characterized by their sensationalistic assertions and emotionally charged rhetoric. Despite (or perhaps partly due to) the impassioned treatment, crowd theory fell out of favour by the early 1900's, but by the mid-twentieth century, interest in the field was rekindled by social psychologists eager to treat the subject with an intellectual scrutiny and scientific rigor it had not received in the past. The field blossomed and today interest in crowd psychology continues within both academic circles and popular culture. The latter point is evidenced by a number of best-selling books (e.g., [4, 13]) that examine the pervasive nature of group behaviour in various aspects of society, and the former point is supported by a number of modern academic monographs, including [3, 5, 6, 7, 8, 9, 10, 11]. While these references present a wide-ranging account of modern theories to explain collective behaviour, the dynamic model that forms the basis of this work is rooted on a well-known account put forth by Gustav LeBon [2] in the latter part of the nineteenth century. LeBon's ideas have been exploited by politicians (famous examples include Hitler and Mussolini) and by the media for decades [2]. The decision to focus exclusively on this particular formulation was made on account of this model being relatively simple and having displayed enduring significance within the field; moreover, although the theory is imperfect, other formulations are not without their own limitations.

2.2 Gustav LeBon's Suggestibility Theory

LeBon founded his perspective of crowd behaviour on observations of group activity made during the turbulent events of the French Revolution. As with other early contributions, LeBon's account in [2] has a sensationalistic flair and his descriptions are entirely qualitative. He also adopted an expressive free-flowing style that leads to his commentary having, at least at times, a ranting and long-winded quality. These factors make the task of capturing the key elements of LeBon's assertions in a mathematical form a challenging assignment, and one inevitably involving a degree of personal judgement. To this end, the following is a list of those elements put forth by LeBon that, in the author's opinion, are obligatory requisites of any dynamic model striving to capture the tenets of suggestibility theory:

(a) The Law of Mental Unity of Psychological Crowds: LeBon states that people in a psychological crowd, who are not necessarily in close physical proximity, form an entity with its own personality, mannerisms, and way of thinking. LeBon stresses that these attributes are not an aggregate of the associated traits of the comprising members, but rather those of an entirely new entity and uses the analogy of chemical reactions producing substances with properties entirely different from those of the initial reactants. To describe this phenomenon, LeBon uses the term *the law of mental unity of psychological crowds* and relies on this idea to help explain his observation that a person in a crowd can act in manner that is in stark contrast to their typical conduct. LeBon makes a note that this tendency gives crowds an infantile quality and makes them prone to exaggerated reactions with tendencies similar to that of a person under hypnotic suggestion.

(b) Suggestibility: The most significant subconscious trait that drives crowd behaviour is, according to LeBon, that of *suggestibility*. He states that it is suggestibility that makes a crowd "hover in a state of expectant attention" and be "readily open to acting on any idea"[2]. LeBon argues that suggestibility stems from a subconscious tendency to conform and deep-rooted

desire to imitate. According to LeBon, people are especially suggestible to words, phrases, and imagery that evoke strong sentiment and emotionally-charged responses. For example, the ideas of freedom, democracy, and justice elicit impassioned responses and tend to make a person more suggestible to future propositions. Furthermore, LeBon mentions that these reactions can be made all the more intense by repetition and reinforcement.

(c) Prestige: LeBon uses the term *prestige* to refer to a person's ability to influence the behaviour of neighbouring crowd members. For example, a highly prestigious person will exert greater influence over the actions and attitudes of neighbouring members as compared to a less prestigious person. LeBon describes prestige as a "domination of the mind" that causes people to be filled with a sense of "inexplicable astonishment," and that "paralyzes the ability to think"[2]. In this sense, LeBon argues that both prestige and suggestibility are necessary for an idea to take root and spread among members of a crowd. Specifically, a suggestible person will adopt an idea only if the incentive to do so comes from a person having at least some prestige.

(d) Remote Factors: LeBon uses the term *remote factors* to account for those intangible attributes of a crowd that determine how beliefs and opinions propagate among members. Certainly, the list of remote factors is long; however, a collection of the more pronounced factors include the race, social customs, the historical period, past experiences, and the degree of education of the members comprising the crowd. Remote factors reflect the fact that while an idea may draw a particular strong reaction and spreads rampantly among members of one crowd, the same notion may elicit a much less pronounced response in a crowd made up of different members.

(e) Immediate Factors: According to LeBon, *immediate factors* are those events that cause an idea to take shape and send an otherwise unassuming group of people spiraling into the suggestibility-fueled mindset of a psychological crowd. As mentioned, repeated exposure to words, phrases, and imagery that appeal to a person's sentiment are especially effective at evoking the crowd mentality. LeBon is quick to point out that immediate factors may be positive or negative, indicating crowds are capable of both positive and negative action.

2.3 An Example of Psychological Crowd Behaviour

Psychological crowd behaviour can arise in a host of situations where people are exposed to a common set of ideas, attitudes, and events and for which there is some means of communication between members. These requirements are in place, for example, whenever people convene at a common venue, be it waiting for admission to a music concert, attending a social demonstration, or shopping at a local market. While gatherings of this sort are a ubiquitous occurrence of modern life, they nevertheless foster a rich assortment of intriguing social phenomena. Specifically, crowds permit the propagation of ideas, attitudes, and actions by using the social interactions between the comprising members as a conduit for information exchange [3, 6, 8].

LeBon provides various examples of crowd behaviour in [2]. To illustrate how dynamic social behaviour can develop in a crowd, in this section we consider the scenario in which avid music fans are patiently waiting outside a concert hall for admission to an event. United by their anticipation for the show, individuals engage one another in conversation. In doing so, they set up a social network in which ideas, attitudes, and the associated actions circulate among the group. Now suppose an individual is alerted, perhaps through a cellular phone call, that reports indicate the headline act is nowhere to be found. Troubled by the development, this individual shares the information with neighbouring fans, who in turn pass their views of the subject onto their neighbours. Among the impassioned fans, conflicting viewpoints, regarding whether the concert will proceed or have to be cancelled, evolve and transmute with time. In this example, the remote factors include the educational history, socioeconomic status, ethnic background, age, and personal histories of the ticket holders. The immediate factor is the event that initiates the psychological crowd mentality, in this case the unnerving phone call made to one of the concert goers. In the case where the crowd is composed of particularly demonstrative individuals, it is possible the original rumor could incite aggressive and even hostile behaviour as it is disseminated. While perhaps extreme, this scenario illustrates the notion that crowds support the propagation of ideas and attitudes and that in the right setting the resultant behaviour can be dramatic.

2.4 A Dynamic Model of Crowd Behaviour

The purely qualitative and open-ended nature of LeBon's formulation make it ill-suited to quantitative discussion and precludes mathematical analysis entirely. To profit from any insight that control theory may be able to offer to the study of crowd behaviour, we strive to capture the tenets of LeBon's suggestibility theory in a system theoretic form. To this end, the following is one attempt to transmogrify the qualitative traits of Section 2.2 to a state description of crowd behaviour. To begin this abstraction, the term *agent* is introduced to refer to a member of the psychological crowd and the psychological crowd itself is said to be composed of agents. This terminology stresses the ability of a person to act and influence their environment through social interactions with other crowd members. Additionally, the term is used when discussing multi-agent systems, a field in which the author believes this formulation of crowd behaviour fits naturally. It is worth reiterating that the model we pursue focuses entirely on the psychological dynamics of a crowd and we assume the position of each member, as determined by some ordering procedure, does not vary with time. Interpreting LeBon's commentary, we model the state of each agent using four signals, referred to as prestige, action, delayed-action, and suggestibility, with each component described as follows:

- *Prestige of agent in position i*: $p_i[k] > 0$ is a measure of the ability of the agent in position i to influence the behaviour of other agents. Prestige is a positive quantity, as the concept of negative prestige has no sensible connotation within LeBon's model, and our future analysis simplifies if we disallow the case where $p_i[k] = 0$.

- *Action of agent in position i*: $a_i[k]$ is a quantification of the behaviour of the agent in position i as it relates to acceptance of an idea ($a_i[k] > 0$) or its antitheses ($a_i[k] < 0$). Action values for which $|a_i[k]| \approx 0$ are indicative of mild acceptance of an idea or its opposite notion, and are associated with a calm and orderly agent. Larger values of $|a_i[k]|$ are indicative of more extreme degrees of acceptance, ranging from devoted to frenetic in accordance with $|a_i[k]|$. In the concert example of Section 2.3, positive action may be associated with the belief the headline act will perform, while negative action is therefore associated with the belief the show will be cancelled.

- *Delayed action of agent in position i*: $b_i[k]$ is the value of $a_i[k]$ one time instant in the past. It is introduced so the dynamics may be expressed in state form.

- *Suggestibility of agent in position i*: $s_i[k] > 0$ is a measure of the affinity of the agent in position i to incorporate the behaviour of neighbouring agents into their own behaviour, by mimicking the actions and conduct of others. As with prestige, suggestibility is modeled as a positive quantity.

We propose the following model, reported in [1, 12], to capture the psychological dynamics of the agent in position i[1]:

$$
\begin{aligned}
p_i[k+1] &= c_p p_i[k] + \mu_{pa,i}|a_i[k]| && (2.1)\\
a_i[k+1] &= c_a a_i[k] + \mu_{apa,i} s_i^2[k] \sum_{O_j \in \mathcal{N}(O_i)} d_{i,j} p_j[k] a_j[k] && (2.2)\\
b_i[k+1] &= a_i[k] && (2.3)\\
s_i[k+1] &= \mu_{s,i} S \alpha^{c_s \beta_i[k]} && (2.4)\\
\beta_i[k] &:= \mu_{sa,i} \left(a_i[k] - b_i[k]\right)^2 + \mu_{sap,i} \sum_{O_j \in \mathcal{N}(O_i)} d_{i,j} p_j[k] |a_j[k]| + \\
&\quad\ \mu_{ssp,i} \sum_{O_j \in \mathcal{N}(O_i)} d_{i,j} p_j[k] \left(s_j[k] - s_i[k]\right). && (2.5)
\end{aligned}
$$

In (2.1)–(2.5) the μ parameters are agent-specific positive gains used to scale the contributions of the various social effects. In (2.4), $\alpha > 1$ is a growth constant and $S > 0$ is a nominal suggestibility value. The constants

[1]It should be noted that the dynamics reported in [1] and listed for completeness in Appendix A are slightly different than those reported in [12], which correspond to (2.1)–(2.5). Namely, the dynamics used in [1] represent an earlier model that we later slightly modified to be, in our opinion, more in tune with LeBon's ideas. More specifically, the models differ only with respect a few terms in the respective suggestibility equations. The majority of our results will be based on (2.1)–(2.5), but consistent with the chronology of our work some of our early results are based on the model in [1]. We opted to include these results in Example 2.4.1 and Sections 3.2-3.3, despite the fact they are predicated on a different model, because they played a pivotal role in shaping the research and, once again, the models are much more similar than they are distinct. In Chapters 4 and 5 we deal exclusively with (2.1)–(2.5)

c_p, c_a, and c_s reside in the interval $(0,1)$ and capture the tendency of $p_i[k]$ and $a_i[k]$ to decay towards zero and $s_i[k]$ to approach $\mu_{s,i}S$ in the absence of adequate social excitation. The quantity $\mathcal{N}(O_i)$ is the *neighbour set* of the agent in position i and consists of the set of agents for which agent i has direct interaction with through conversation, gesturing, or other social exchanges. It is worth mentioning that this notation was adopted in favour of the more natural $\mathcal{N}_i$ as it proves more appropriate in the context of the labeling system that we introduce in a later section. The $d_{i,j}$ terms are constants used to weight the interactions between agents i and j and will typically be taken as one for all $d_{i,j}$ such that $O_j \in \mathcal{N}(O_i)$ and zero otherwise. In regard to the time set used, (2.1)–(2.5) was developed as a discrete-time model since in this application it is more natural to consider that sensors and actuators measure and act in a periodic rather than a continuous-time manner.

We do not claim that the specific relations in (2.1)–(2.5) precisely model the dynamics of people in a crowd. However, we believe the trends and cause-effect relationships are consistent with LeBon's observations. Specifically, note that:

- The leftmost term on the right side of (2.1) is a decay term: it captures the idea that a person's prestige diminishes if they take no action or have no opinion on a subject (i.e., $a_i[k] = 0$). The second term in (2.1) captures the idea that having greater action, be it positive or negative, makes a person more influential and, therefore, more prestigious.

- Second, the leftmost term on the right side of (2.2) is a decay term: it reflects the notion that if a person is not suggestible, or is in contact only with neighbours that have either no prestige or no action, then that person's attitude tends toward zero, i.e., $a_i[k] \to 0$ as $k \to \infty$. The summation in (2.2) accounts for the trend that suggestible members adjust their action in accord with the prestige and action of neighbouring agents. The suggestibility component of these terms, namely $s_i[k]$, is squared to reflect the heightened significance suggestibility has on the evolution of action. Whether using a power of 2 is correct or not is not of concern here; the intention is simply to account for the fact that, according to LeBon, suggestibility plays a key role in the development of action.

- Third, the suggestibility dynamics in (2.4) are an exponential function of $\beta_i[k]$, with the relation for $\beta_i[k]$ specified in (2.5). The first term on the right side of (2.5) reflects the idea that dramatic changes in action, from one time instant to the next, tend to make a person more suggestible. The first summation in (2.5) captures the fact that merely interacting with prestigious individuals also tends to make a person more suggestible. The second summation in (2.5) captures Lebon's idea that the suggestibility of person i rises if they interact with members who are more suggestible than they are and, conversely, declines if they interact with members who are less suggestible than they are.

- Fourth, the remote factors enter (2.1)–(2.5) in the form of the various μ parameters, the decay constants c_p, c_a, and c_s, the nominal suggestibility value S, and the exponential growth constant α. The immediate factors are modeled by the initial state of the crowd.

- As a last comment, observe that the interaction between suggestibility and attitude in (2.2) and (2.4) reflects the unstable nature of crowds so prominently asserted by LeBon.

Having proposed a dynamic model for crowd behaviour, it would be natural at this stage to validate (2.1)–(2.5) and ensure the dynamics are a faithful representation of not only LeBon's suggestibility theory, but documented cases of crowd behaviour as well. However, given the qualitative nature of LeBon's teachings, a lack of existing experimental results, and the host of ethical and logistic barriers one faces when attempting to gather new experimental data, model validation is not only poorly defined, but impractical and outside the scope of this thesis. In response, the following approach was adopted: a range of values was determined for each parameter such that any combination of parameter values residing in these regions provide what we deem to be reasonable simulation results. While by no means model validation in a rigorous sense, the concessions made were deemed appropriate given the inexplicit nature of the subject matter and the exploratory theme of this thesis.

Example 2.4.1 To provide a sense of the dynamic behaviour contained in the crowd dynamics reported in [1], Figure 2.1 illustrates the progression of the prestige, action, and suggestibility states of each agent in a 100-agent crowd arranged on a 10×10 grid. For simplicity, the values of the μ parameters were chosen to be the same for all agents in the crowd and are equal to (for all $1 \leq i \leq 100$)

$$\mu_{pa,i} = 0.02295,\ \ \mu_{apa,i} = 0.00553,\ \ \mu_{sa,i} = 0.04346,$$
$$\mu_{ssp,i} = 0.04064,\ \ \mu_{sap,i} = 0.04796,\ \ \mu_{s,i} = 1.07062.$$

The decay constants and nominal suggestibility values used are equal to

$$c_p = 0.95,\ \ c_a = 0.55,\ \ c_s = 0.75,\ \ S = 20.$$

For this simulation, it is assumed a person interacts only with those people situated immediately adjacent to them, such that $\mathcal{N}(O_i)$ contains eight agents for each person i except for those situated on either an edge or a corner of the grid. The $d_{i,j}$ values used are equal to the reciprocal of the distance between person i and person j, i.e., 1 or $1/\sqrt{2}$, for all $d_{i,j}$ such that $O_j \in \mathcal{N}(O_i)$ and zero otherwise. The initial conditions, shown in the first row of the figure (time instant $k = 0$), corresponds to a crowd with almost neutral attitude ($|a_i[0]| < 0.1$) except for two people in the top row, for which $a_i[0] \approx -2$. Consistent with the ideas emphasized by LeBon, and as one may have expected given the interplay between (2.2) and (2.4), open-loop simulations reveal that all components of the state quickly swell in magnitude as the negative attitudes of the two people spread to the whole crowd. From a crowd control perspective, this scenario is representative of an unruly horde and generally undesirable, since there is no assurance the crowd will ultimately return to a calm and orderly state.

□

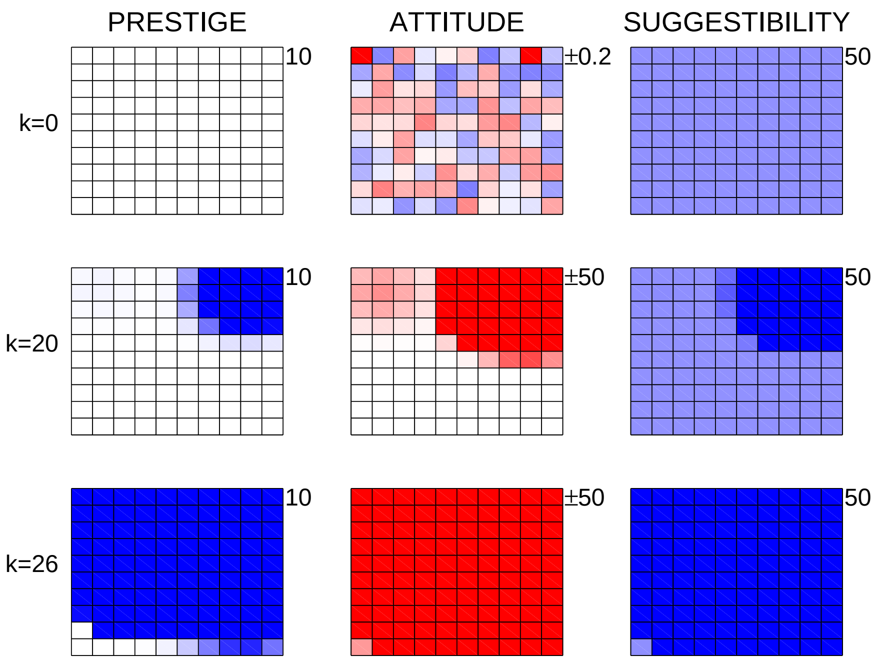

Figure 2.1: Open-loop response of the nonlinear plant dynamics in [1], showing prestige, attitude, and suggestibility at various time instants. Blue indicates positive numbers and red indicates negative numbers. The number at the top right of each plot indicates the colour saturation value, e.g., the attitude plot at $k = 0$ has maximum intensity for any agent with $|a_i[0]| \geq 0.2$.

Chapter 3

Formulating the Crowd Control Problem

This chapter is concerned with presenting the logistic details needed to define the crowd control problem. We describe the general class of crowds that are of interest and present a collection of notation and terminology to concisely describe a crowd and pinpoint its key topological features. Given the unstable nature of the crowd dynamics, special attention is allocated to expounding our definition of exactly what it means to stabilize a crowd. This notion is then used to guide the investigation of preliminary control strategies based on both linearization methods and simple nonlinear heuristic schemes. The poor performance and impractical nature of the associated controllers motivates an entirely new approach to control. To this end, we introduce the concept of a *control agent* as a human controller capable of effecting social change and use this paradigm to formally define the crowd control problem investigated in subsequent chapters.

3.1 Preliminaries

The crowds we consider are composed of $n \geq 2$ agents, identified by a means of ordering, to be presented shortly, as agent 1 through agent n. As we are interested in dynamic crowd behaviour, it is assumed all agents have entered the psychological crowd mindset and are, thus, subject to the dynamics in

(2.1)–(2.5). The overall crowd state is taken as the collection of all agent states and defined by

$$x[k] = [p_1[k], \ldots, p_n[k], a_1[k], \ldots, a_n[k], b_1[k], \ldots, b_n[k], s_1[k], \ldots, s_n[k]]^T, \tag{3.1}$$

where $p_i[k]$, $a_i[k]$, $b_i[k]$, and $s_i[k]$ are the prestige, the action, the delayed-action, and the suggestibility of agent i at time k, respectively.

3.2 Equilibrium Points

For a general crowd, (2.1)–(2.5) can have numerous equilibrium points and, on account of the complexity of the crowd dynamics, characterizing them is arduous. If, instead, we assume that all systems parameters are chosen uniformly, that all agents have equivalent state values in equilibrium (i.e., $p_{i,o} = p_o, \ldots, s_{i,o} = s_o$, for $i = 1 \ldots n$), that agents are positioned at each point of an infinitely-large rectangle grid, so as to avoid fringing effects near the edges of the grid, and, finally, that each agent interacts with only their immediate neighbours, such that each agents has precisely eight neighbours, then the calculation of equilibrium points is greatly simplified and reduces to finding the roots of a single third-order polynomial. Using this approach and the parameters specified in Example 2.4.1 gives three equilibrium points, listed in Table 3.1. The stability of each equilibrium point may be evaluated using linearization methods as detailed in Appendix A. Specifically, linearizing equations (2.1)–(2.5) about an equilibrium point, x_o, yields a system of the form $\Delta x[k+1] = A \cdot \Delta x[k]$ where $\Delta x := x - x_o$ and, assuming $n = 100$, where A is 400×400. For this selection of parameter values, the stability of each equilibrium point, as determined using linearization methods, is summarized in Table 3.1; the equilibrium point corresponding to $a_i[k] = 0$ is asymptotically stable, while the two remaining equilibria are unstable. Although our classification of equilibrium points relies on a number of simplifying assumptions, our approach makes it algebraically tractable to acquire at least a degree of insight into the equilibrium characteristics of (2.1)–(2.5) and, by determining equilibrium points about which to linearize, allow us to, on at least some level, evaluate the feasibility of control strategies based on a linear model of crowd behaviour.

p_o	a_o	b_o	s_o	Stability
0	0	0	21.4124	stable
0.0260	0.0566	0.0566	21.4125	unstable
0.0260	-0.0566	-0.0566	21.4125	unstable

Table 3.1: The three equilibrium points of the open-loop system.

3.3 A First Attempt to Control the Crowd

It has been noted that the interplay between action and suggestibility in (2.1)–(2.5) suggest the dynamics are inherently unstable, a suspicion supported by the explosive growth of crowd states witnessed in the simulations of Example 2.4.1. Indeed, the equilibrium points of the system, summarized in Table 3.1, confirm the crowd dynamics are unstable and the crowd states swell in magnitude for the majority of initial conditions. Given this realization, a natural recourse is to pursue stabilization schemes that ensures $x[k]$ assumes values representative of calm and orderly behaviour among all agents. At this time, we propose an objective consistent with this vision and try to force, for any initial condition and for all i, $a_i[k] \to 0$, $p_i[k] \to 0$, $b_i[k] \to 0$, and, $s_i[k] \to \mu_{s,i}S$. The system equations in (2.1)–(2.5) imply this objective is equivalent to regulating just the action states, $a_1[k], a_2[k], \ldots, a_n[k]$. Hence, in terms of Figure 3.1, the objective is to force $y[k] \to 0$ where $y[k] := [a_1[k], \ldots, a_n[k]]^T$. Given the social nature of the problem, the sensors and actuators used deserve special mention:

- *Sensing:* The sensor is a person who observes the psychological condition of the crowd. Of the state variables $p_i[k]$, $a_i[k]$, and $s_i[k]$, it is most realistic to assume that only $a_i[k]$, i.e., the action of agent i, can be measured.

- *Actuation:* There are several plausible ways in which a control signal can enter (2.1)–(2.5). Only one scheme is considered here, namely the scenario in which the controller (also a person) directly influences an agent's action. In this case, equation (2.2) assumes the form:

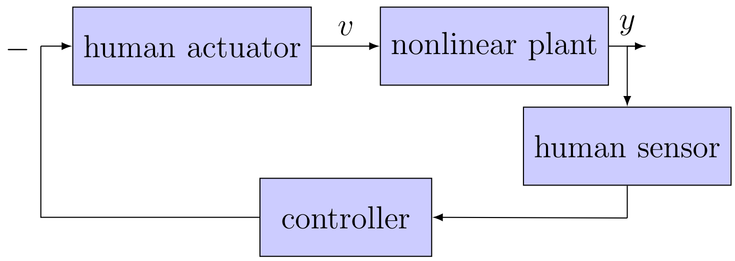

Figure 3.1: A block diagram showing the plant, with input $v := [v_1, \ldots, v_n]^T$ and output $y := [a_1, \ldots, a_n]^T$, connected with a controller. Sensing and actuation are performed by a human.

$$a_i[k+1] = c_a a_i[k] + \mu_{apa,i} s_i^2[k] \sum_{O_j \in \mathcal{N}(O_i)} \alpha_{i,j} p_j[k] a_j[k] + v_i[k], \; i = 1, \ldots, n, \tag{3.2}$$

where $v_i[k]$ is the control signal affecting the action of agent i.

3.3.1 A Control Scheme for a Linearized Crowd Model

As a first attempt to stabilize the crowd dynamics, we consider an observer-based pole-placement controller for a linearized plant model, as shown in Figure 3.2. With the goal of driving all action states to zero, it is natural to linearize about the first equilibrium point in Table 3.1. However, doing so results in an overly simplified model that fails to account for many of the nonlinear terms present in the crowd dynamics. As such, we elect, instead, to linearize about the second equilibrium point in Table 3.1, for which the resultant linear time-invariant plant has the form (see Appendix A):

$$\Delta x[k+1] = A \cdot \Delta x[k] + B \Delta v[k] \tag{3.3}$$

$$\Delta y[k] = C \Delta x[k]. \tag{3.4}$$

The controller used in Figure 3.2 is based on a traditional Luenberger ob-

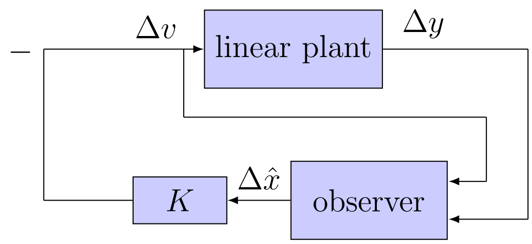

Figure 3.2: An observer-based state-feedback controller with $\Delta v := v - v_o$, $\Delta y := y - y_o$, and $\Delta x := x - x_o$.

server of the form:

$$\Delta\hat{x}[k+1] = A\hat{x}[k] + Bv[k] + H(\Delta y[k] - C\Delta\hat{x}[k]) \quad (3.5)$$

$$\Delta v = -K\Delta\hat{x}[k]. \quad (3.6)$$

To specify the controller gain matrices K and H, standard pole-placement algorithms were used to place the 800 closed-loop poles (i.e., the eigenvalues of $A - BK$ and $A - HC$) at random locations between 0.4 and 0.6 on the real axis. Snapshots from a typical simulation run are provided in Figure 3.3. Although the controller succeeds in regulating action, the transient performance is notably poor and, even more concerning, the linear controller relies on driving the perturbed components of various prestige signals to negative values, a mechanism which, given the small positive value of the second equilibrium point and the condition that prestige is positive, is feasible only for a very small region of attraction. Finally, simulations reveal the observer-based pole-placement scheme is highly sensitive to measurement noise. The lack of success afforded by linearization schemes suggest it may be worthwhile to explore control schemes that can be applied directly to the nonlinear plant dynamics.

3.3.2 Heuristic Nonlinear Control Schemes

In this subsection, we consider two nonlinear schemes inspired by their heuristic sensibility. Each method takes note of the fact that the control signal in (3.2) appears directly in the action state equation of each agent. To this end,

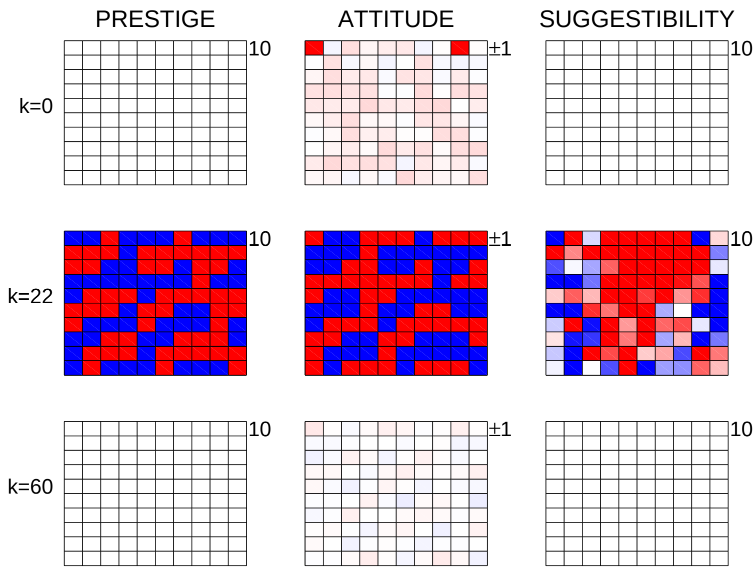

Figure 3.3: Closed-loop response of the (linearized) crowd using the observer-based state-feedback controller (3.5)–(3.6). The signals shown are perturbation signals, i.e., Δp, Δa, and Δs.

the following control law has intrinsic appeal:

$$v_i[k] = -a_i[k], \; i = 1, \ldots, n. \tag{3.7}$$

In this approach, the controller tries, at each time instant, to drive the action state of each agent to zero by canceling the action value of each agent with their action value from the previous time instant. The structure of (3.7) results in (3.2) assuming the form of (2.2) with c_a replaced by $c_a - 1$. Linearization arguments readily conclude the controller locally stabilizes the equilibrium point corresponding to $a_i[k] = 0$, for all i. Simulation results, such as those in Figure 3.4, indicate that the region of attraction is greatly increased compared to that of the open-loop system. From a practical standpoint, the control scheme is, however, rather resource intensive: assuming a controller is capable of issuing a single control signal at each time instant, this scheme requires n people to act as actuators, one for each agent in the crowd! This is an extravagant use of resources and one which can potentially be made more resource-effective by focusing the control action solely on the

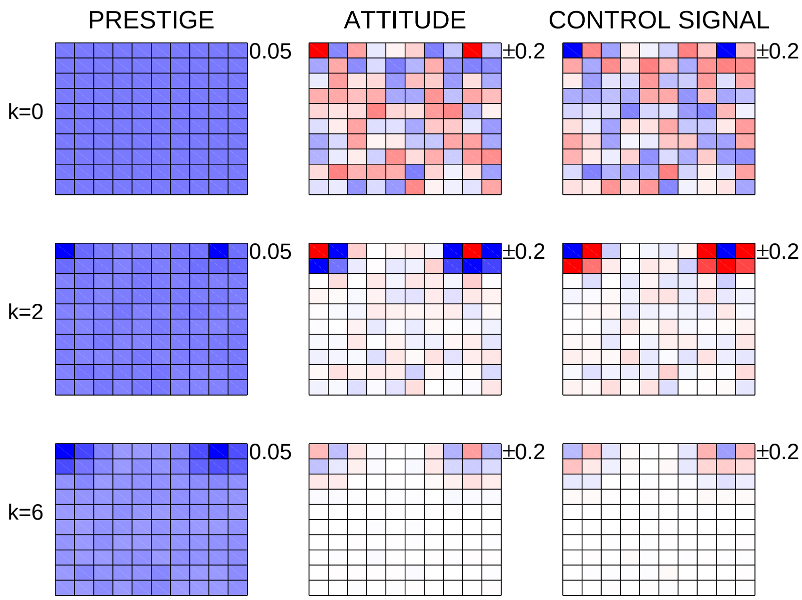

Figure 3.4: Closed-loop response of the (nonlinear) system using the controller (3.7). The third column shows the control signal, $v_i[k]$.

most outspoken agent, that is the agent with the largest action magnitude. With this vision in mind, introduce

$$m[k] := \min\{\arg\max_i |a_i[k]|\},$$

and the second heuristic control scheme:

$$v_i[k] = \begin{cases} 0 & \text{if} \quad i \neq m[k] \\ -a_i[k] & \text{if} \quad i = m[k]. \end{cases} \tag{3.8}$$

Simulations of (3.8), such as those in Figure 3.5, reveal that this scheme works, but with a smaller region of operation as compared to (3.7); proving that the result works is difficult on account of the switching nature of the controller.

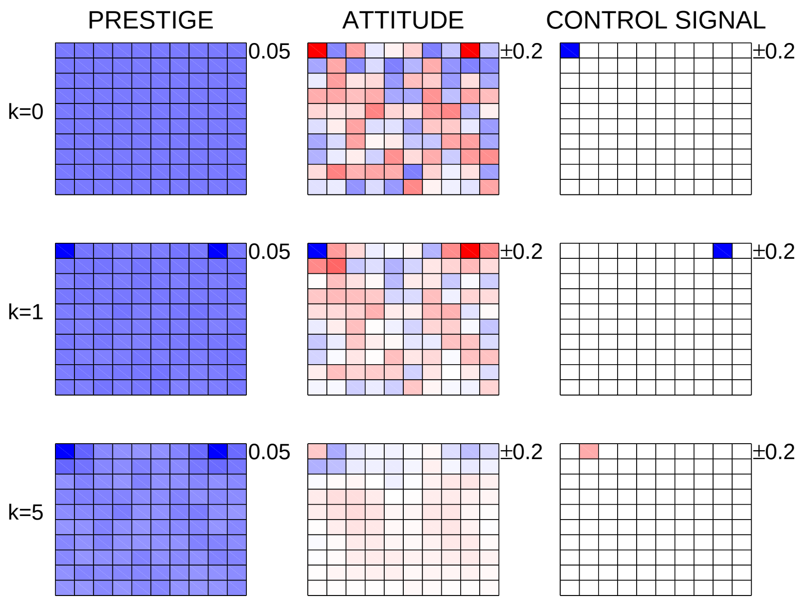

Figure 3.5: Closed-loop response of the (nonlinear) system using the controller (3.8). The third column shows the control signal, $v_i[k]$.

Despite the appeal of (3.8) as a simple control scheme, it is discouraging for two reasons: first, it relies on overriding the dynamics in (3.2), an assumption that stands in obstinate defiance of LeBon's ideas and the social nature of group behaviour emphasized in the development of (2.1)–(2.5); second, (3.8) requires a human controller to scamper about the crowd while targeting the most outspoken agent at each time instant. These attributes leave (3.8) afflicted by a strong lack of practicality and suggest a more appropriate control strategy would employ a radically different, and more practical, paradigm in keeping with the social nature of the crowd control problem.

3.4 Introducing the Control Agent and Problem Formulation

In response to the weaknesses of the control schemes in the previous section, we introduce the *control agent* as a human capable of effecting social change by working directly with the nonlinear crowd dynamics, a wholly more feasibly paradigm as it relates to both the social nature of crowd behaviour and controller implementation. Below we describe in more detail the characteristics of a control agent and formulate the problem addressed in this thesis.

The problem we pursue considers a crowd of $n \geq 2$ agents that have entered the psychological mindset and are thus subject to the dynamics in (2.1)–(2.5). As a control tool, we position $m \geq 1$ control agents throughout the crowd and task these individuals with maintaining orderly behaviour among the agents. We view a control agent as an authoritative figure, such as a security guard or police officer, who has a disposition towards preserving decorum, has keen judgment, and possesses the self-composure and retrospective psyche needed to resist succumbing to the crowd mentality. Given these traits, the control agent is modeled as being able to sense $x[k]$ and use this knowledge alongside an understanding of (2.1)–(2.5) to consciously affect the behaviour of agents in the crowd[1]. In a later section, we describe various schemes to refer to specific members of the crowd, including control agent i. For now, we reflect the aforementioned attributes by assigning the following dynamics to control agent i:

$$p_i[k] = \hat{p}_i \tag{3.9}$$

$$a_i[k] = u_i[k] = u_i(x[k]) \tag{3.10}$$

$$s_i[k] = 0. \tag{3.11}$$

These three equations define the functionality of each control agent. The prestige equalling the constant $\hat{p}_i > 0$ in (3.9) attests to a control agent maintaining a constant level of influence. In (3.10), it is assumed control agent i can set its action state, $u_i[k]$, to any desired value in order to affect the behaviour of agents in the crowd. The functional dependence of $u_i[k]$ on $x[k]$ indicates the control agent can sense the entire state and that the action of a control agent is based on the behaviour of agents in the crowd. Finally, $s_i[k]$ equalling zero in (3.11) signifies the control agent is impervious to suggestion and acts as an individual, rather than a member of the psychological crowd. From a control perspective, the control agent introduces the familiar concept of feedback by using knowledge of the system's state to modify the system output. Once again, the roles of both sensor and actuator

[1]We recognize that, as mentioned earlier, it is not realistic to measure the entire state; however, such an assumption is necessary at this stage.

are performed by people, specifically the control agents themselves, a more well-defined and practical controller compared to the preliminary renditions of Sections 3.3.1 and 3.3.2.

Previously, we had argued that given the unstable nature of (2.1)–(2.5), a natural objective is to stabilize the crowd dynamics and ensure each agent acts in a calm and orderly manner. However, we placed no restrictions on the controllers used to achieve this objective. Having introduced the notion of a control agent, we refine our earlier notion of stability and present the definition of stability used throughout the remainder of this thesis:

Definition 3.4.1 A crowd composed of n agents and m control agents is defined to be $\mathcal{C}(\lambda)$*–stabilizable*, for integer $\lambda > 0$, if there exist m causal control laws, each of the form (3.9)–(3.11), capable of driving, from any initial state $x[0]$, the action state of all agents to zero in no more than λ time instants and subsequently holding all action states at zero. If such a collection of control laws is implemented, the crowd is said to be $\mathcal{C}(\lambda)$*–stable* or $C(\lambda)$*–stabilized* and the control laws, $\mathcal{C}(\lambda)$*–stabilizing*. The integer λ is called the *stabilization time.* □

Notice once again that our definition of stability addresses only the action states of agents in the crowd. However, it follows from (2.1)–(2.5) that if the action of all agents is zero, $p_i[k]$ will tend to zero asymptotically and $s_i[k]$ will approach $\mu_{s,i}S$ for each agent. Hence, the notion of $\mathcal{C}(\lambda)$–stability is consistent with state signals remaining bounded and all agents acting in a calm and orderly manner. The control objective of this thesis may, therefore, be stated succinctly as developing a set of $\mathcal{C}(\lambda)$–stabilizing control laws for a crowd comprised of n agents and m control agents. Given the discrete nature of the psychological dynamics in (2.1)–(2.5), our control objective amounts to seeking a dead-beat response with respect to the collection of all agent action states.

While the definition of $\mathcal{C}$–stability involves driving the action state of all agents in the crowd to zero, the following definition is concerned with a specific agent having zero action:

Definition 3.4.2 Agent i is said to be *zeroed* for $k \geq \tilde{k}$ if the action state of agent i is zero for $k \geq \tilde{k}$ (i.e., $a_i[k] = 0, \forall k \geq \tilde{k}$). □

The concepts of $\mathcal{C}(\lambda)$–stability and a zeroed agent are clearly related by the fact that a crowd is $\mathcal{C}(\lambda)$–stabilized if and only if all agents in the crowd are zeroed for all times greater than or equal to λ.

3.5 Notation and Terminology

This section presents a collection of notation and terminology appropriate for describing a crowd's key topological features. We begin by using the symbols O and X to refer to a generic agent and a generic control agent, respectively. As a further classification aid, the members that comprise a crowd are categorized into two groups: the set of n agents and the set of m control agents, denoted by $\mathcal{O} = \{O^1, \ldots, O^n\}$ and $\mathcal{X} = \{X^1, \ldots, X^m\}$, respectively. The structure of the social network through which members interact is described by the aggregate collection of neighbour sets, denoted, henceforth, by $\mathcal{N} = \{\mathcal{N}(O^1), \ldots, \mathcal{N}(O^n), \mathcal{N}(X^1), \ldots, \mathcal{N}(X^m)\}$. Using just these quantities, a crowd $\mathcal{C}$ may be completely described by the triple $\mathcal{C} = (\mathcal{O}, \mathcal{X}, \mathcal{N})$.

As a simple illustration of how we use the newly-introduced notation and how we graphically display a crowd, we say the crowd in Figure 3.6 is composed of 15 members: 12 agents and 3 control agents. The social network through which members communicate is represented by the edges connecting the various members. That is, $O^j \in \mathcal{N}(O^i)$ if and only if there is an edge connecting O^j and O^i in the figure. For example, the neighbour set of agent O^6 in Figure 3.6 is $\mathcal{N}(O^6) = \{O^3, O^7, O^8\}$.

Queues, that is one-dimensional crowd structures, may be depicted in a similar manner, as illustrated in Figure 3.7. In this case, the queue consists of six agents, located, according to a left-to-right enumeration starting from zero, in positions $1, 2, 3, 5, 6$, and 8, and three control agents, located in positions $0, 4$, and 7. Given all members of a queue have exactly two neighbours, aside from members situated at either end of the queue which have only a single neighbour, queues possess considerable structure and we distill the information content of Figure 3.7 and represent the queue symbolically as

$$XOOOXOOXO. \tag{3.12}$$

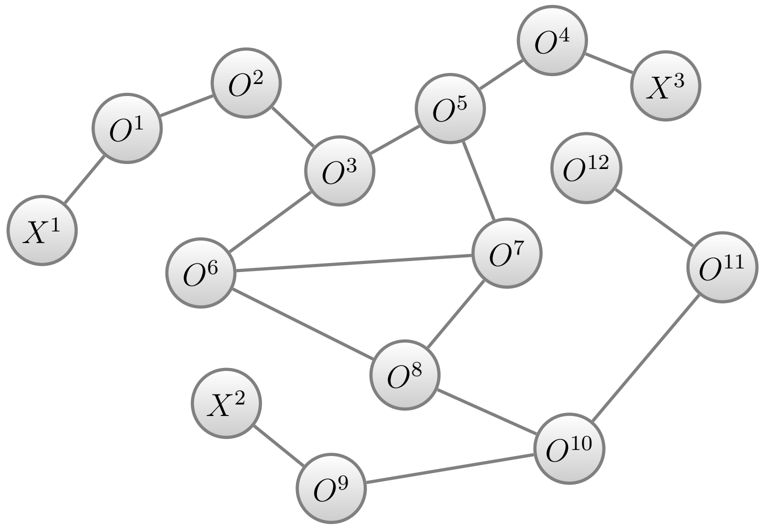

Figure 3.6: A simple crowd composed of 12 agents and 3 control agents

Within a queue, we use the notation O_i and X_j to denote the agent located in position i and the control agent located in position j, respectively. When using this subscript notation, the queue in (3.12) may be rewritten as

$$X_0O_1O_2O_3X_4O_5O_6X_7O_8. \tag{3.13}$$

Subscript notation is also useful for conveniently describing the neighbour set of agents in a queue. For example, the neighbour set of agent O_i in the queue $X_0O_1 \ldots O_n$ is given by

$$\mathcal{N}(O_i) = \begin{cases} \{X_0, O_2\} & \text{for} \quad i = 1 \\ \{O_{i-1}, O_{i+1}\} & \text{for} \quad i = 2, \ldots, n-1 \\ \{O_{n-1}\} & \text{for} \quad i = n, \end{cases} \tag{3.14}$$

while the neighbour set of the control agent in the queue is $\mathcal{N}(X_0) = \{O_1\}$. Occasionally, it is useful to explicitly refer to a particular agent or control agent in the queue relative to other agents or control agents, respectively. To this end, the notation O^i and X^j is used to refer to the i^{th} agent and j^{th} control agent, based on an enumeration starting from one at the left end of the queue of all agents and control agents, respectively. When using

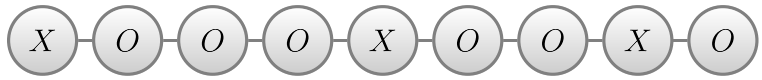

Figure 3.7: A simple queue composed of 6 agents and 3 control agents

superscript notation, the queue in (3.12) may be rewritten as

$$X^1 O^1 O^2 O^3 X^2 O^4 O^5 X^3 O^6. \tag{3.15}$$

To concisely represent groupings of $\bar{n}$ adjacent agents we use the notation $\bar{O}$ and, when necessary, indicate the number of members that comprise such a grouping using the notation $\bar{O}\,(\bar{n})$. Using this scheme and subscript notation, the queue in (3.12) may be written as

$$X_0 \bar{O}\,(3)\, X_4 \bar{O}\,(2)\, X_7 \bar{O}\,(1)\,. \tag{3.16}$$

Finally, the notational systems we have introduced may be used interchangeably, depending on the information that is to be conveyed. For example, two of the many possible representations, in addition to representations provided in (3.12)–(3.16), for the queue in (3.12) include $X_0 \bar{O}\,(3)\, X_4 O_5 O^5 X O^6$ and $X^1 O_1 O^2 O^3 X^2 \bar{O}\,(2)\, X^3 O^6$. While there are numerous ways to describe a queue, particularly as the number of members gets large, we make an asserted effort to use the aforementioned notational system sparingly, employing the most basic representation that suffices to communicate the intended information.

In two-dimensional crowds, the social network describing the interaction between crowd members may contain multiple social channels or *paths* through which ideas and actions propagate. This is a perception that has considerable bearing on stabilization strategies of Chapter 5 and one worth formalizing in greater depth:

Definition 3.5.1

(a) The *path* $\ell_{j,k}$ is a sequence of distinct crowd members consisting of a single control agent and one or more agents, such that the sequence begins with X^j, terminates with O^k, and adjacent members of the sequence are neighbours of one another.

(b) The *length of path* $\ell_{j,k}$ is denoted by $|\ell_{j,k}|$ and equals the number of agents in $\ell_{j,k}$, excluding the control agent.

(c) In general, there may be multiple paths, still finite in number, linking X^j with O^k. In this case, we define the *collection of all such paths*, arranged by path length, as $L_{j,k} = \{\ell^1_{j,k}, \ell^2_{j,k}, \ell^3_{j,k}, \ldots\}$, where $\ell^1_{j,k}, \ell^2_{j,k}, \ell^3_{j,k} \ldots$ are paths from X^j to O^k and $|\ell^1_{j,k}| \leq |\ell^2_{j,k}| \leq |\ell^3_{j,k}|$, etc.

(d) The *set of path lengths* corresponding to paths from X^j to O^k is $|L_{j,k}| = \{|\ell^1_{j,k}|, |\ell^2_{j,k}|, |\ell^3_{j,k}|, \ldots\}$.

(e) The set of paths from all of the m control agents to O^k is defined by $L_k := L_{1,k} \cup L_{2,k} \cup \cdots \cup L_{m,k}$ and the set of path lengths from all of the m control agents to O^k by $|L_k| := |L_{1,k}| \cup |L_{2,k}| \cup \cdots \cup |L_{m,k}|$.

□

Example 3.5.2 To illustrate the ideas of Definition 3.5.1, consider the crowd of Figure 3.6 and the set of paths from X^1 to O^6. In this case, there are three paths from X^1 to O^6, namely

$$\ell^1_{1,6} = \{X^1, O^1, O^2, O^3, O^6\}, \quad (3.17)$$
$$\ell^2_{1,6} = \{X^1, O^1, O^2, O^3, O^5, O^7, O^6\}, \quad (3.18)$$
$$\ell^3_{1,6} = \{X^1, O^1, O^2, O^3, O^5, O^7, O^8, O^6\}, \quad (3.19)$$

which have lengths of 4, 6, and 7, respectively. Therefore, $|L_{1,6}| = \{4, 6, 7\}$. Similarly,

$$\ell^1_{2,6} = \{X^2, O^9, O^{10}, O^8, O^6\}, \quad (3.20)$$
$$\ell^2_{2,6} = \{X^2, O^9, O^{10}, O^8, O^7, O^6\}, \quad (3.21)$$
$$\ell^3_{2,6} = \{X^2, O^9, O^{10}, O^8, O^7, O^5, O^3, O^6\}, \quad (3.22)$$
$$\ell^1_{3,6} = \{X^3, O^4, O^5, O^3, O^6\}, \quad (3.23)$$
$$\ell^2_{3,6} = \{X^3, O^4, O^5, O^7, O^6\}, \quad (3.24)$$
$$\ell^3_{3,6} = \{X^3, O^4, O^5, O^7, O^8, O^6\}, \quad (3.25)$$

so that $L_6 = \{\ell^1_{1,6}, \ell^2_{1,6}, \ell^3_{1,6}, \ell^1_{2,6}, \ell^2_{2,6}, \ell^3_{2,6}, \ell^1_{3,6}, \ell^2_{3,6}, \ell^3_{3,6}\}$ and $|L_6| = \{4, 5, 6, 7\}$.
□

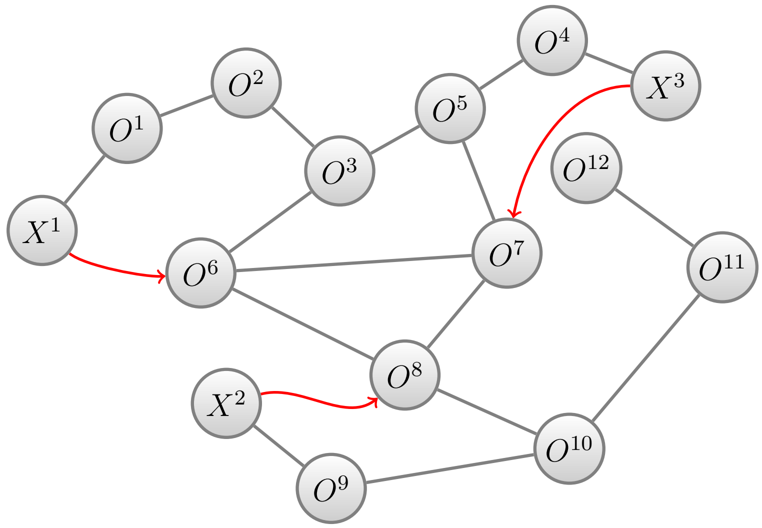

Figure 3.8: A set of allocation pairings for the crowd in Figure 3.6. The allocation pairings shown are $(X^1 \to O^6)$, $(X^2 \to O^8)$, and $(X^3 \to O^7)$.

The control strategies we employ rely on each control agent *targeting* a particular agent, as described by the following definition:

Definition 3.5.3 The notation $(X^j \to O^k)$ is used to denote that X^j is targeting O^k by attempting to drive its action state to zero. We say that $(X^j \to O^k)$ is an *allocation pairing* because it describes how X^j allocates its effort, i.e., by targeting O^k instead of another agent. $\square$

Figure 3.8 illustrates one possible set of allocation pairings for the simple crowd in Figure 3.6. Having introduced the concept of an allocation pairing, the definition of a *target set* follows naturally:

Definition 3.5.4 The *target set* is the subset of agents which are targeted by one or more control agents. $\square$

For example, the target set associated with the set of allocation pairings shown in Figure 3.8 is $\{O^6, O^7, O^8\}$.

Chapter 4

One-Dimensional Crowds: Stabilizing a Queue

This chapter describes a control law that $\mathcal{C}$–stabilizes a class of queues. We begin by considering a control law to $\mathcal{C}$–stabilize the queue $X_0 O_1 \cdots O_n$ and build upon this result to develop stability results for a broader class of queues. In turn, these results are extended in Chapter 5 to develop stability results for a general class of crowds. Referring to the queue $X_0 O_1 \cdots O_n$, we develop our results progressively through a series of three propositions: Proposition 4.2.1 establishes that the action state $a_i[k+i]$ has rich structure with respect to $u_0[k]$. Proposition 4.2.2 extends this result by certifying that $a_i[k+i]$ can be driven to zero by appropriate selection of $u_0[k]$. Proposition 4.2.3 investigates the implications of holding the action of the rightmost agent at zero indefinitely on other agents in the queue. Finally, Propositions 4.2.2 and 4.2.3 are leveraged in Theorem 4.2.4 to yield the main stability result. Central to the arguments that follow is an appreciation for the temporal structure with which $u_0[k]$ propagates through the queue. To gain insight into this paradigm it is useful to first consider the case where $n = 2$, for which enumeration of the agent states is algebraically tractable. Finally, to limit algebraic complexity, we assume throughout the remainder of this thesis that all nonzero $d_{i,j}$ values in (2.1)–(2.5) are equal to one, though the analysis is equally applicable for any viable $d_{i,j}$.

4.1 The Queue $X_0O_1O_2$

Consider the simple queue $X_0O_1O_2$ and the goal of driving the action state of the last agent to zero at time $k = 2$, i.e., we desire $a_2[2] = 0$. Given the queue structure, X_0 cannot affect the state of O_2 directly, but only indirectly by issuing control signals to O_1 that subsequently influence O_2 through the interaction between the two agents. Due to the small size of the queue being considered, it is possible to enumerate the relevant state signals and study the algebraic forms that emerge. To this end, the state of O_1 at time $k = 1$ may be calculated using (2.1)–(2.5) and is given by:

$$p_1[1] = c_p p_1[0] + \mu_{pa,1} |a_1[0]| \tag{4.1}$$

$$a_1[1] = c_a a_1[0] \ + \mu_{apa,1} s_1^2[0] \left(\hat{p}_0 u_0[0] + p_2[0] a_2[0]\right) \tag{4.2}$$

$$s_1[1] = \mu_{s,1} S \alpha^{c_s \beta_1[0]} \tag{4.3}$$

$$\begin{aligned} \beta_1[0] \ = \ & \mu_{sa,1} \left(a_1[0] - b_1[0]\right)^2 - \mu_{ssp,1} \hat{p}_0 s_1[0] + \mu_{ssp,1} p_2[0] \left(s_2[0] - s_1[0]\right) + \\ & \mu_{sap,1} \left(\hat{p} \left|u_0[0]\right| + p_2[0] \left|a_2[0]\right|\right). \end{aligned} \tag{4.4}$$

Similarly, the state of O_2 at time $k = 1$ is given by:

$$p_2[1] \ = \ c_p p_2[0] + \mu_{pa,2} |a_2[0]| \tag{4.5}$$

$$a_2[1] \ = \ c_a a_2[0] + \mu_{apa,2} s_2^2[0] p_1[0] a_1[0] \tag{4.6}$$

$$s_2[1] \ = \ \mu_{s,2} S \alpha^{c_s \beta_2[0]} \tag{4.7}$$

$$\begin{aligned} \beta_2[0] \ = \ & \mu_{sa,2} \left(a_2[0] - b_2[0]\right)^2 + \mu_{ssp,2} p_1[0] \left(s_1[0] - s_2[0]\right) + \\ & \mu_{sap,2} p_1[0] \left|a_1[0]\right|. \end{aligned} \tag{4.8}$$

At time $k = 1$, the term $u_0[0]$ has no effect on O_2, as illustrated by the absence of $u_0[0]$ in (4.5)–(4.8). Rather, the implications of $u_0[0]$ on O_2 are experienced at time $k = 2$, at which point

$$a_2[2] = c_a a_2[1] + \mu_{apa,2} s_2^2[1] p_1[1] a_1[1]. \tag{4.9}$$

Substituting (4.1), (4.2), (4.6), and (4.7) in (4.9) permits $a_2[2]$ to be written entirely in terms of system parameters and agent states at time $k = 0$ through the expression

$$\begin{aligned} a_2[2] \; &= c_a^2 a_2[0] + c_a \mu_{apa,2} s_2^2[0] p_1[0] a_1[0] + \mu_{apa,2} \mu_{s,2}^2 S^2 \alpha^{2c_s \beta_2[0]} \times \\ &\quad (c_p p_1[0] + \mu_{pa,1} \left| a_1[0] \right|) \times \big(c_a a_1[0] + \mu_{apa,1} s_1^2[0] \left(\hat{p}_0 u_0[0] + p_2[0] a_2[0] \right) \big) . \end{aligned}$$

The coefficient of $u_0[0]$ in the above relation, namely the already rather lengthy expression $\mu_{apa,2}\mu_{s,2}^2 S^2 \alpha^{2c_s\beta_2[0]} \left(c_p p_1[0] + \mu_{pa,1} \left|a_1[0]\right|\right) \mu_{apa,1} s_1^2[0]\hat{p}_0$, consists of system parameters and positive functions of the agent states at time $k = 0$. This implies $a_2[2]$ is a positively sloped linear function of $u_0[0]$ and may be driven to any real value by appropriate selection of $u_0[0]$. Specifically, to force $a_2[2]$ to zero we require

$$\begin{aligned} u_0[0] = &- \frac{c_a^2 a_2[0] + c_a \mu_{apa,2} s_2^2[0] p_1[0] a_1[0]}{\mu_{apa,2} \mu_{s,2}^2 S^2 \alpha^{2c_s\beta_2[0]} \left(c_p p_1[0] + \mu_{pa,1} \left|a_1[0]\right|\right) \mu_{apa,1} s_1^2[0] \hat{p}_0} - \\ &\frac{c_a a_1[0]}{\mu_{apa,1} s_1^2[0] \hat{p}_0} - \frac{p_2[0] a_2[0]}{\hat{p}_0} . \end{aligned}$$

Furthermore, by generalizing to time k, it is possible to determine the appropriate value of the control signal at time k needed to ensure $a_2[k+2]$ is zero. In the event the action state of the rightmost agent is held at zero indefinitely, we may use the crowd dynamics to draw an interesting conclusion about the action of O_1. Specifically, assume $a_2[k] = 0$ for all $k \geq 2$. In this case, the action state equation of O_2 assumes the form

$$a_2[k+1] = 0 = \mu_{apa,2} s_2^2[k] p_1[k] a_1[k], \;\; k \geq 2. \tag{4.10}$$

Given $s_2[k]$ and $p_1[k]$ are positive at all times, equality in the above relation necessitates $a_1[k] = 0$ for $k \geq 2$; hence, there exists a control law, (3.9)–(3.11), capable of $\mathcal{C}\,(2)$–stabilizing the queue $X_0 O_1 O_2$.

In summary, by explicitly propagating $u_0[0]$ through the queue dynamics, an equation suitable for determining the causal control law needed to drive $a_2[k]$ to zero for $k \geq 2$ may be developed and, moreover, $a_2[k] = 0$ for $k \geq 2$ also implies $a_1[k] = 0$ for $k \geq 2$, i.e., the queue is $\mathcal{C}\,(2)$–stabilized. However, in developing this result, for a relatively small queue, considerable algebraic complexity has emerged.

4.2 The Queue $X_0O_1\cdots O_n$

To extend the results of Section 4.1 to larger queues of the form $X_0O_1\cdots O_n$ it is necessary to make the most of the modest structure present in (2.1)–(2.5). To this end, the following proposition exploits the fact that the terms belonging to the summation in (2.2) are linear with respect to the action state of neighbouring agents:

Proposition 4.2.1 Consider the queue $X_0O_1\cdots O_n$. For $i = 1,\ldots,n$ and $k \geq 0$ the action state of agent O_i at time $k+i$ may be expressed in the form

$$a_i[k+i] = a_i^{\overline{u}_0}[k+i] + a_i^{u_0}[k+i]u_0[k], \tag{4.11}$$

where $a_i^{\overline{u}_0}[k+i]$ and $a_i^{u_0}[k+i]$ are both functions of only $x[k]$ and in particular are independent of $u_0[k]$. Furthermore, the term $a_i^{u_0}[k+i]$ is nonzero. □

Proof: From (2.2), the action state of agent O_1 at time $k+1$ is given by $c_a a_1[k] + \mu_{apa,1}s_1^2[k]\left(\hat{p}_0 u_0[k] + p_2[k]a_2[k]\right)$. Defining $\mu_{apa,1}s_1^2[k]\hat{p}_0$ and $c_a a_1[k] + \mu_{apa,1}s_1^2[k]p_2[k]a_2[k]$ as $a_1^{u_0}[k+1]$ and $a_1^{\overline{u}_0}[k+1]$ respectively and noting that $\hat{p}_0$ and $s_1[k]$ are positive, (4.11) is immediately confirmed for $i = 1$. Proceeding along inductive lines, for some $\ell \in [1, n-1]$, assume the proposition statement is true for $i = \ell$, which was verified for the base case $\ell = 1$ above. For $\ell \in [1, n-2]$ it follows from the action-state update equation of agent $O_{\ell+1}$ and the induction assumption that

$$a_{\ell+1}[k+\ell+1] = c_a a_{\ell+1}[k+\ell] + \mu_{apa,\ell+1}s_{\ell+1}^2[k+\ell] \times \left(p_\ell[k+\ell]\left(a_\ell^{\overline{u}_0}[k+\ell] + a_\ell^{u_0}[k+\ell]u_0[k]\right) + p_{\ell+2}[k+\ell]a_{\ell+2}[k+\ell]\right).$$

The above expression has the form in (4.11) for $i = \ell+1$ with:

$$a_{\ell+1}^{u_0}[k+\ell+1] := \mu_{apa,\ell+1}s_{\ell+1}^2[k+\ell]p_\ell[k+\ell]a_\ell^{u_0}[k+\ell] \tag{4.12}$$

$$a_{\ell+1}^{\overline{u}_0}[k+\ell+1] := c_a a_{\ell+1}[k+\ell] + \mu_{apa,\ell+1}s_{\ell+1}^2[k+\ell]\times \left(p_\ell[k+\ell]a_\ell^{\overline{u}_0}[k+\ell] + p_{\ell+2}[k+\ell]a_{\ell+2}[k+\ell]\right). \tag{4.13}$$

Repeating this procedure for $\ell = n-1$ yields an expression for $a_{\ell+1}^{u_0}[k+\ell+1]$ identical to that in (4.12), while the expression for $a_{\ell+1}^{\overline{u}_0}[k+\ell+1]$ is equal to the relation in (4.13) with the term $\mu_{apa,\ell+1}s^2_{\ell+1}[k+\ell]p_{\ell+2}[k+\ell]a_{\ell+2}[k+\ell]$ removed.

Since $u_0[k]$ propagates through the queue, in that it influences agent states, at a rate no faster than one agent per discrete time interval, all prestige, action, and suggestibility states of agents $O_{\ell+1}$ and, for $\ell < n-1$, $O_{\ell+2}$ are independent of $u_0[k]$ at time $k+\ell$. Furthermore, from (2.1), the prestige of agent O_ℓ is a function of this agent's prestige and action at the previous instant and therefore $p_\ell[k+\ell]$ is also independent of control signal $u_0[k]$. Hence, for $\ell \in [1, n-1]$, $a_{\ell+1}$, $a_{\ell+2}$, $s_{\ell+1}$, p_ℓ, and $p_{\ell+2}$ in the relations for $a^u_{\ell+1}[k+\ell+1]$ and $a^{\overline{u}}_{\ell+1}[k+\ell+1]$ are independent of $u_0[k]$. By the induction assumption, $a^{u_0}_\ell[k+\ell]$ and $a^{\overline{u}_0}_\ell[k+\ell]$ are independent of $u_0[k]$ and it follows that $a^{u_0}_{\ell+1}[k+\ell+1]$ and $a^{\overline{u}_0}_{\ell+1}[k+\ell+1]$ are independent of $u_0[k]$.

Next, by the induction assumption, $a^{u_0}_\ell[k+\ell]$ and $a^{\overline{u}_0}_\ell[k+\ell]$ are exclusive functions of $x[k]$. Other signals appearing in the relations for $a^{u_0}_{\ell+1}[k+\ell+1]$ and $a^{\overline{u}_0}_{\ell+1}[k+\ell+1]$ can be expressed in terms of $x[k]$ by propagating elements of the state at time k through the queue dynamics and it follows that $a^{u_0}_{\ell+1}[k+\ell+1]$ and $a^{\overline{u}_0}_{\ell+1}[k+\ell+1]$ can be viewed as exclusive functions of $x[k]$.

Finally, again by the induction assumption, $a^{u_0}_\ell[k+\ell]$ is nonzero. Since $s_{\ell+1}[k+\ell]$ and $p_\ell[k+\ell]$ are positive it follows that $a^{u_0}_{\ell+1}[k+\ell+1]$ in (4.12) is also nonzero.

This series of results confirm the proposition statement is true for $i=\ell+1$. It follows from induction that for $i = 1,\ldots,n$ and $k \geq 0$ the action state of agent O_i may be expressed as $a^{\overline{u}_0}_i[k+i] + a^{u_0}_i[k+i]u_0[k]$ where $a^{\overline{u}_0}_i[k+i]$ and $a^{u_0}_i[k+i]$ are both functions of only $x[k]$; moreover, $a^{u_0}_i[k+\ell]$ is nonzero.■

Proposition 4.2.1 may be used to conclude that $a_i[k+i]$ can be driven to zero by appropriate selection of $u_0[k]$:

Proposition 4.2.2 Consider the queue $X_0O_1\cdots O_n$. For $i = 1,\ldots,n$ and $k \geq 0$ there exists a causal, state-feedback control law, (3.9)–(3.11), capable of driving $a_i[k+i]$ to zero. □

Proof: It follows from (4.11) in Proposition 4.2.1 that for $i=1,\ldots,n$ and $k \geq 0$, $a_i[k+i]$ may be written in the form

$$a_i[k+i] = a^{\overline{u}_0}_i[k+i] + a^{u_0}_i[k+i]u_0[k], \tag{4.14}$$

where $a_i^{\overline{u}_0}[k+i]$ and $a_i^{u_0}[k+i]$ are both independent of $u_0[k]$ and $a_i^{u_0}[k+i]$ is nonzero. Since (4.14) is linear in $u_0[k]$, the action state of agent O_i at time $k+i$ may be driven to zero by selecting

$$u_0[k] = -\frac{a_i^{\overline{u}_0}[k+i]}{a_i^{u_0}[k+i]}. \tag{4.15}$$

From Proposition 4.2.1, $a_i^{\overline{u}_0}[k+i]$ and $a_i^{u_0}[k+i]$ are functions of only $x[k]$; hence, $u_0[k]$ in (4.15) depends only on $x[k]$ and (3.9)–(3.11) may be regarded as a causal, state-feedback controller.■

The following proposition investigates the implications of holding the action state of the rightmost agent in the queue, namely agent O_n, at zero indefinitely. Specifically, holding $a_n[k]$ at zero for $k \geq \tilde{k} \geq 0$ is shown to imply $a_{n-1}[k], \ldots, a_1[k]$ are also zero for $k \geq \tilde{k}$:

Proposition 4.2.3 Consider the queue $X_0 O_1 \cdots O_n$. Let $\tilde{k} \geq 0$ denote an instant in time. If the action state of O_n is sustained at zero for all $k \geq \tilde{k}$ then the action state of each agent in the queue is zero for all $k \geq \tilde{k}$. □

Proof: As indicated in the proposition statement, consider the scenario in which the action state of the rightmost agent in the queue is zero for $k \geq \tilde{k}$. The implications on other agents in the queue may be inferred through an inductive argument. To this end, for some $\ell \in [2, n]$, assume $a_i[k] = 0$ for all $i \geq \ell$ and $k \geq \tilde{k}$. This condition is true for the base case in which $\ell = n$. The action-state-update equation of agent O_ℓ gives

$$a_\ell[k+1] = c_a a_\ell[k] + \mu_{apa,\ell} s_\ell^2[k] \sum_{O_j \in \mathcal{N}(O_\ell)} p_j[k] a_j[k].$$

From the induction assumption, $a_\ell[k] = 0$ for $k \geq \tilde{k}$, and

$$0 = \mu_{apa,\ell} s_\ell^2[k] \sum_{O_j \in \mathcal{N}(O_\ell)} p_j[k] a_j[k], \;\; k \geq \tilde{k}. \tag{4.16}$$

In the event $\ell = n$, the neighbour set $\mathcal{N}(O_\ell)$ is simply $\{O_{\ell-1}\}$ and (4.16) reduces to

$$0 = \mu_{apa,\ell} s_\ell^2[k] p_{\ell-1}[k] a_{\ell-1}[k], \;\; k \geq \tilde{k}. \tag{4.17}$$

Alternatively, if $\ell \neq n$, $\mathcal{N}(O_\ell) = \{O_{\ell-1}, O_{\ell+1}\}$; however, it follows from the induction assumption that $a_{\ell+1}[k] = 0$ for $k \geq \tilde{k}$, and (4.16) once again reduces to (4.17). Since $s_\ell[k]$ and $p_{\ell-1}[k]$ are positive, equality in (4.17) necessitates $a_{\ell-1}[k] = 0$ for $k \geq \tilde{k}$. This result establishes that $a_i[k] = 0$ for $i \geq \ell - 1$ and $k \geq \tilde{k}$. It follows from induction that $a_i[k] = 0$ for $i = 1, \ldots, n$ and $k \geq \tilde{k}$, implying that if $a_n[k]$ can be held at zero for $k \geq \tilde{k}$ then the action states $a_{n-1}[k], \ldots, a_1[k]$ are also zero for $k \geq \tilde{k}$.■

Propositions 4.2.2 and 4.2.3 can be used to yield a stability result for queues under the supervision of a single control agent:

Theorem 4.2.4 There exists a control law, (3.9)–(3.11), that $\mathcal{C}(n)$–stabilizes the queue $X_0O_1 \cdots O_n$. □

Proof: From Proposition 4.2.2 the causal, state-feedback controller of the form (3.9)–(3.11) with (3.10) given by (4.15) for $i = n$ and $k \geq 0$, specifically

$$u_0[k] = -\frac{a_n^{\overline{u}_0}[k+n]}{a_n^{u_0}[k+n]}, \; k \geq 0, \tag{4.18}$$

results in $a_n[k] = 0$ for $k \geq n$. For this control law, Proposition 4.2.3 mandates $a_i[k] = 0$ for $i = 1, \ldots, n$ and $k \geq n$. Therefore, the queue $X_0O_1 \cdots O_n$ is $\mathcal{C}(n)$–stabilized using (3.9)–(3.11) with (3.10) given by (4.18).■

4.3 Computational Issues for the Queue $X_0O_1 \cdots O_n$

While Theorem 4.2.4 guarantees the existence of a $\mathcal{C}(n)$–stabilizing control law, it does not explicitly address a practical approach to calculate $u_0[k]$. Namely, Theorem 4.2.4 establishes that the queue $X_0O_1 \cdots O_n$ may be $\mathcal{C}(n)$–stabilized using the result in (4.18); however, evaluation of (4.18) directly amounts to an enumerative approach that, as discussed in Section 4.1, is unwieldy for all but the smallest of queues. Alternatively, here we consider a recursive method to compute $u_0[k]$ that is well-suited to numeric evaluation. Development of this approach requires no more than an added degree of bookkeeping throughout Propositions 4.2.1 and 4.2.2 to keep track of the relationships that emerge in relating $a_i^{u_0}[k+i]$ and $a_i^{\overline{u}_0}[k+i]$ to signals at time $k+i-1$ for $i=1,\ldots,n$. For example, at time $k+1$ we may write, as in Section 4.1, $a_1^{\overline{u}_0}[k+1]$ and $a_1^{u_0}[k+1]$ in terms of state signals and the control agent's prestige $\hat{p}_0$. Similarly, at time $k+2$ we may write $a_2^{\bar{u}_0}[k+2]$ and $a_2^{u_0}[k+2]$ in terms of state signals at time k_o+1, $a_i^{u_0}[k+i]$, and $a_i^{\overline{u}_0}[k+i]$ and so on until $a_n^{u_0}[k+n]$ and $a_n^{\overline{u}_n}[k+n]$ are written in terms of state signals at time $k+n-1$, $a_{n-1}^{u_0}[k+n-1]$ and $a_{n-1}^{\overline{u}_0}[k+n-1]$. The recursive equations that result from such an analysis are given in (4.19) and (4.20) for $n \geq 2$:

$$a_i^{u_0}[k+i] = \begin{cases} \mu_{apa,1} s_1^2[k]\hat{p}_0 & , i=1 \\ \mu_{apa,i} s_i^2[k+i-1] p_{i-1}[k+i-1] a_{i-1}^{u_0}[k+i-1] & , i=2\ldots n \end{cases} \tag{4.19}$$

$$a_i^{\overline{u}_0}[k+i] = \begin{cases} c_a a_1[k] + \mu_{apa,1} s_1^2[k] p_2[k] a_2[k] & , i=1 \\ c_a a_i[k+i-1] + \mu_{apa,i} s_i^2[k+i-1]\left(p_{i-1}[k+i-1] \times \right. \\ \left. a_{i-1}^{\overline{u}_0}[k+i-1] + p_{i+1}[k+i-1] a_{i+1}[k+i-1]\right) \\ & , 2 \leq i \leq n-1 \\ c_a a_i[k+i-1] + \\ \mu_{apa,n} s_n^2[k+n-1] p_{n-1}[k+n-1] a_{n-1}^{\overline{u}_0}[k+n-1] & , i=n. \end{cases} \tag{4.20}$$

Evaluation of (4.19) and (4.20) requires knowledge of specific agent states over the time interval $[k, k+n-1]$. These requisite signals may be determined using a computer routine that iteratively computes the state, using (2.1)–(2.5), over the appropriate time interval. Having garnered all of the necessary state values, it is straightforward to calculate $u_0[k]$ using (4.18), (4.19), and (4.20). It is worth reinforcing that, given an understanding of the social dynamics, all signals defined at time instants on the interval $[k, k+n-1]$ may be expressed, using (2.1)–(2.5), as functions of the components of $x[k]$. Therefore, the control law is indeed causal as implied by the existence of a $\mathcal{C}(n)$–stabilizing control law in Theorem 4.2.4.

4.4 Simulation Results for the Queue $X_0 O_1 \cdots O_n$

This section substantiates the stability result in Theorem 4.2.4 through a number of simulations. We also consider the performance of the control law in cases where the control agent is unable to definitively sense $x[k]$ or issue $u_0[k]$ with absolute precision. In all simulations, agents are assumed to have the same set of μ parameters with each parameter value randomly selected within a predefined range, as described in Example 2.4.1. Also, the control agent's prestige is set at $\hat{p}_0 = 0.5$.

The functionality of the $\mathcal{C}(n)$–stabilizing control law in (3.9)–(3.11) with (3.10) given by (4.18) for the cases in which $n = 3$ and $n = 6$ is illustrated in Figures 4.1 and 4.2, respectively. Notice that the control law does indeed drive the action states of all agents to zero. Also noteworthy, from the plots, is that many of the elements of the control signal used to $\mathcal{C}(6)$–stabilize the queue $X_0O_1O_2O_3O_4O_5O_6$ are considerably larger than their counterparts in the control signal used to $\mathcal{C}(3)$–stabilize the queue $X_0O_1O_2O_3$. This observation is explored in the second column of Table 4.1, which lists the first element of the control signal needed to $\mathcal{C}(n)$–stabilize the queue $X_0O_1 \cdots O_n$ for various values of n. These entries suggest a trend in which $|u_0[0]|$ increases as n is increased. Intuitively, given the unstable dynamics and limited rate at which $u_0[k]$ propagates through the queue, the value of $|u_0[0]|$ rises as n increases since $u_0[0]$ must offset states at time n that have had more time to swell and grow in magnitude.

The last two columns of Table 4.1 provide two measures by which to assess the viability of the control scheme in non-ideal conditions. To investigate the effects of a human control agent being unable to sense $x[k]$ with complete accuracy, column 4 of Table 4.1 provides a measure of the action of the rightmost agent in the queue resulting from a sensed $x[0]$ that is randomly skewed by $\pm 1\%$ from its true value. To investigate this scenario in more detail, Figure 4.3 traces the progression of the relevant state signals of agents in the queue $X_0O_1O_2O_3O_4O_5O_6$ for the case in which the sensed value of $x[k]$ is randomly skewed in the manner described above. Note that the limitations in sensing prevent the control law from driving all action states to zero at time $k = 6$ and that even at time $k = 20$ the the majority of state signals are still nonzero. However, as a consolatory remark, the simulation does indicate the control law succeeds in preventing state signals from experiencing dramatic swells in magnitude. Finally, to study the ramifications of a human actuator being unable to issue signals with absolute precision, column 5 of Table 4.1 lists the value of $a_n[n]$ when $u_0[0]$ is randomly altered by $\pm$ 1 percent from its true value.

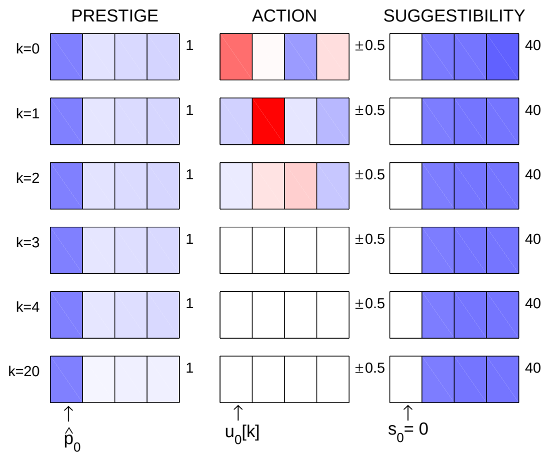

Figure 4.1: Progression of prestige, action, and suggestibility states among members of the queue $X_0O_1O_2O_3$ subject to the $\mathcal{C}\,(3)$–stabilizing control law. The components of the control agent's state, including control signal $u_0[k]$, are displayed in the leftmost column of each plot.

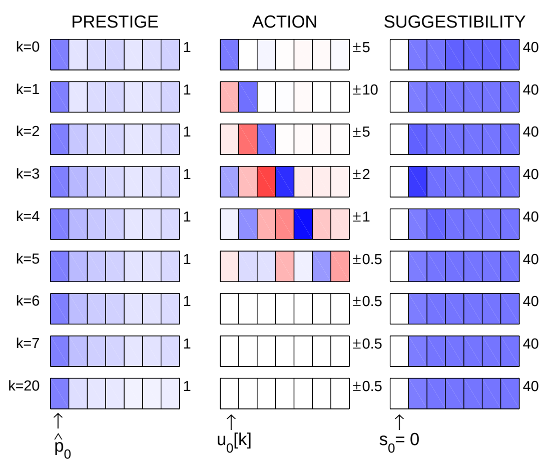

Figure 4.2: Progression of the prestige, action, and suggestibility states among members of the queue $X_0O_1O_2O_3O_4O_5O_6$ subject to the $\mathcal{C}\,(6)$–stabilizing control law. The colour saturation value is varied with time to provide greater contrast among the colours used to denote state values in each plot.

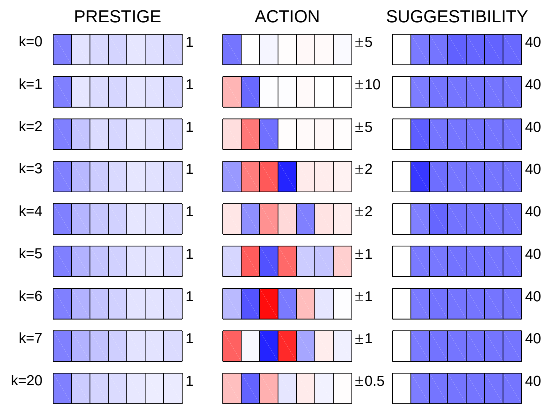

Figure 4.3: Progression of the prestige, action, and suggestibility states among members of the queue $X_0O_1O_2O_3O_4O_5O_6$ subject to the $\mathcal{C}$–stabilizing control law (3.9)–(3.11) with (3.10) given by (4.18) for a random ±1% error in the sensed value of $x[k]$ used to construct $u_0[k]$.

No. of agents n	$u_0[0]$	Ideal value of $a_n[n]$	$a_n[n]$ for ±1% error in sensed $x[0]$	$a_n[n]$ for ±1% error in $u[0]$
2	-0.112	0.0	0.000	-0.001
3	-0.285	0.0	-0.003	0.002
4	-0.099	0.0	-0.010	0.000
5	1.046	0.0	0.012	-0.002
6	2.563	0.0	0.021	0.002
7	-9.290	0.0	0.012	0.006
8	-46.565	0.0	-0.130	0.027
9	-150.320	0.0	-0.293	-0.173
10	-256.583	0.0	-4.968	1.004
11	-302.837	0.0	-54.765	7.291

Table 4.1: Data illustrating that, as the queue length increases, $u_0[0]$ tends to grow in magnitude and the control law becomes dramatically more sensitive to noise.

In summary, it is possible to use a single control agent to $\mathcal{C}(n)$–stabilize n agents, as in the queue $X_0O_1 \ldots O_n$; however, as queue length increases the control strategy discussed has four limitations:

1. The stabilization time increases,
2. The sensing burden of the control agent increases,
3. Components of the stabilizing control law grow in magnitude, and
4. The control law becomes dramatically more sensitive to system noise.

In the next section, we highlight the significance of the first two shortcomings and consider the use of multiple control agents as a mechanism to reduce the stabilization time and sensing burden of each control agent.

4.5 The Queues $X\bar{O}X\bar{O}\cdots X\bar{O}$ and $X\bar{O}X\bar{O}\cdots X\bar{O}X$

In the hope of alleviating the shortcomings highlighted at the end of the preceding section, we now consider the use of multiple control agents to $\mathcal{C}$–stabilize a queue. However, when using approaches of this nature, care must be taken to ensure the collection of control agents operate in a coordinated fashion to avoid one control agent inadvertently mitigating the intended effect of another. To a large extent, the stability results presented here build directly on the results developed for the queue $X_0O_1 \cdots O_n$ in the preceding sections.

Using combinatorial arguments, the number of unique queue arrangements consisting of n agents and m control agents is $(n+m)!/(n!m!)$. However, it is not necessary to come up with a separate control scheme for each of these arrangements. Indeed, since control agents are unaffected by the state of neighbouring members, they act as barriers to the transmission of psychological phenomena within the queue. Consequently, from a control perspective, a grouping of two or more control agents positioned at either end of the queue offers no added functionality beyond what can be achieved using only a single control agent. Similarly, collections of three or more control agents within the interior of the queue offer no added functionality beyond what can be achieved using a grouping of just two control agents. For example, in attempting to $\mathcal{C}$–stabilize a queue of three agents, the performance that can be achieved using five control agents in the arrangement $XXOXXXOO$ is no better than what is possible when using only three control agents in the arrangement $XOXXOO$. Furthermore, given the shielding effect of control agents, placing two control agents next to one another within the queue isolates the psychological behaviour of agents on either side of the grouping from those on the other. Hence, for equivalent initial conditions and control action, the propagation of agent states in $XOXXOO$ is identical to that in the two disjoint queues XOX and XOO.

Using the above observations, it follows that any arrangement involving $m > 1$ control agents (with a control agent on at least one end of the queue) can be decomposed to a collection of disjoint queues assuming either the form $X\bar{O}X\bar{O}\cdots X\bar{O}$ or the form $X\bar{O}X\bar{O}\cdots X\bar{O}X$. Therefore, if $\mathcal{C}$–stabilization schemes can be developed for these two topologies, it follows that any queue comprised n agents and m control agents (with a control agent on at least one end of the queue) can be $\mathcal{C}$–stabilized. To this end, the following subsections examine the C–stabilizability of these two general topologies.

4.5.1 The Queue $X\bar{O}\cdots X\bar{O}$

Queues of the form $X\bar{O}X\bar{O}\cdots X\bar{O}$ may be viewed as the composite of queues of the form $X\bar{O}$. To aid analysis, we apply the notation introduced at the beginning of Section 3.5 and denote the family of queues under consideration by $X^1\bar{O}\,(n_1)\,X^2\bar{O}\,(n_2)\cdots X^m\bar{O}\,(n_m)$ with $n_1 + \cdots + n_m = n$. Now consider the segment $X^m\bar{O}\,(n_m)$ at the right end of the queue. By sensing only the state signals of agents in $\bar{O}\,(n_m)$, it follows from Theorem 4.2.4 that there exists a causal control law for X^m, of the form (3.9)–(3.11), such that the action state of all agents in $\bar{O}\,(n_m)$ are driven to, and subsequently held at, zero for $k \geq n_m$. In using such a control law, it follows from the action-state-update equation of the leftmost agent in $\bar{O}\,(n_m)$ that $u^m[k] = 0$ for $k \geq n_m$. Tracing the effect of this condition from right-to-left within the queue, it follows that the rightmost agent in $\bar{O}\,(n_{m-1})$ receives no direct excitation from X^m for $k \geq n_m$. Therefore, from the perspective of the action state of this agent, it is as if the segment $X^m\bar{O}\,(n_m)$ is entirely absent for $k \geq n_m$. Hence, by reapplying the preceding argument, the action state of all agents in $\bar{O}(n_{m-1})$ can be driven to, and subsequently held at, zero for $k \geq n_m + n_{m-1}$ by appropriate selection of a causal control law of the form (3.9)–(3.11) for X^{m-1}. By working from right-to-left throughout the queue and repeatedly applying these arguments it follows that the action state of each agent in $X^1\bar{O}(n_1)\,X^2\bar{O}(n_2)\cdots X^m\bar{O}(n_m)$ may be zeroed for all times $k \geq n_m + \cdots + n_1 = n$ using causal control laws of the form (3.9)–(3.11) for X^m through X^1. Therefore, the queue $X^1\bar{O}(n_1)\,X^2\bar{O}(n_2)\cdots X^m\bar{O}(n_m)$ is $C(n)$–stabilizable.

While the stabilization time of this sequential control strategy is no better than that of the single-agent approach of Section 4.2, it is noteworthy that this multi-control-agent approach requires that X^i sense only the state of the n_i agents in $\bar{O}(n_i)$. Therefore, by evenly positioning control agents throughout the queue the number of state variables that need to be sensed by any given control agent can be reduced by a factor of approximately m over the single-control-agent approach of Section 4.2. However, given the sequential nature of this multi-control-agent scheme, it is necessary that each control agent X^i for $i = 1, \ldots, m-1$ be aware of the time instant at which control agent X^{i+1} has zeroed all agents in $\bar{O}(n_{i+1})$, so that X^i can begin the process of zeroing all agents in $\bar{O}(n_i)$. This requirement for communication between control agents is characteristic of multi-control-agent approaches to stabilization. The sequential nature of this stabilization approach is illustrated in Figure 4.4 for the case in which two control agents, positioned according to the arrangement $X_0O_1O_2O_3X_4O_5O_6O_7$, are tasked with $\mathcal{C}(6)$–stabilizing a queue containing six agents.

4.5.2 The Queue $X\bar{O}X\bar{O}\cdots X\bar{O}X$

The queue $X\bar{O}X\bar{O}\cdots X\bar{O}X$ is distinguished from the general class of queues considered in Section 4.5.1 only by the presence of an additional control agent at the right end of the queue. In fact, if the newly-added-right-most control agent issues a control signal that is at all times zero, the arguments of the preceding subsection may be applied to establish the existence of a set of $\mathcal{C}$–stabilizing control laws for the queue $X\bar{O}X\bar{O}\cdots X\bar{O}X$. However, intuition suggests there is a more advantageous set of control laws, capable of improving at least some aspect of performance, that mandate the rightmost control agent take a more active role in $\mathcal{C}$–stabilizing the queue. To this end, we consider a control scheme to stabilize the queue $X\bar{O}X$ in the following

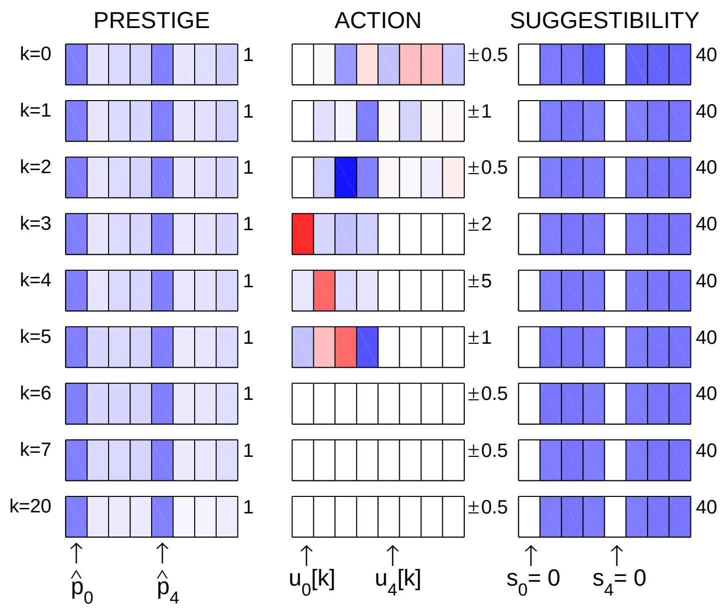

Figure 4.4: Progression of the prestige, action, and suggestibility states among members of the queue $X_0O_1O_2O_3X_4O_5O_6O_7$ subject to the $\mathcal{C}$–stabilizing control laws discussed in Section 4.5.1. Note that X_0 does not attempt to drive the action state of agents in $O_1O_2O_3$ to zero until X_4 has driven the action state of agents in $O_5O_6O_7$ to zero.

subsection; once control laws have been established to $\mathcal{C}$–stabilize the queue $X\bar{O}X$, the existence of control laws to $\mathcal{C}$–stabilize the queue $X\bar{O}X\bar{O}\cdots X\bar{O}X$ follows directly from the discussion in Section 4.5.1.

4.5.3 The Queue $X\bar{O}X$

This subsection discusses the existence of control laws for X^1 and X^2 that $\mathcal{C}$–stabilize the queue $X^1O_1\cdots O_nX^2$. In developing these results we leverage the similarities with respect to the queue $X_0O_1\cdots O_n$ from Section 4.2, but reduce the stabilization time by sagaciously coordinating the attitude adjustments administered by X^1 and X^2. Moreover, by using the two control agents in this way, the sensing burden on each control agent is approximately half of that faced by the single control agent used in Section 4.2. The cost of realizing these benefits is once again the requirement that X^1 and X^2 be able to readily communicate with one another.

We begin with Proposition 4.5.1 that extends Proposition 4.2.2 to the case in which a single control agent is positioned at each end of the queue.

Proposition 4.5.1 Consider the queue $X^1 O_1 \cdots O_n X^2$. Let $n^* = \lceil n/2 \rceil$. For $1 \leq i \leq n^*$, $n^*+1 \leq j \leq n$, and $k \geq 0$ there exists a set of two causal, state-feedback control laws, each of the form (3.9)–(3.11), for X^1 and X^2 capable of driving the action state of agents O_i and O_j to zero at times $k+i$ and $k+(n-j+1)$, respectively. $\square$

Proof: For simplicity we consider the cases in which n is even and odd separately. We begin with the case in which n is even. Given that control signals $u^1[k]$ and $u^2[k]$ propagate through the queue, in that they influence agent action states, at a rate no faster than one control agent per discrete-time interval, it follows that the state of agents $O_1 \cdots O_{n^*}$ are independent of $u^2[k]$ and the prestige of X^2 for all time instants on the interval $[k, k+n^*]$. Consequently, on this time interval, the evolution of state components belonging to agents O_1 through O_{n^*} in the queue $X^1 O_1 \cdots O_n X^2$ are equivalent to the associated progression of state components in the queue $X^1 O_1 \cdots O_n$. It follows from Proposition 4.2.1 that over the time interval $[k, k+n^*]$ the action state of agent O_i for $1 \leq i \leq n^*$ at time $k+i$ may be written as

$$a_i[k+i] = a_i^{\overline{u^1}}[k+i] + a_i^{u^1}[k+i]u^1[k], \tag{4.21}$$

where $a_i^{\overline{u^1}}[k+i]$ and $a_i^{u^1}[k+i]$ are functions of only the state $x[k]$ and $a_i^{u^1}[k+i]$ is different from zero. Then for $1 < i < n^*$ it follows from Proposition 4.2.1 that $a_i[k+i]$ for $1 \leq i \leq n^*$ may be driven to zero by a causal, state-feedback control law of the form (3.9)–(3.11) with (3.10) given by

$$u^1[k] = -\frac{a_i^{\overline{u^1}}[k+i]}{a_i^{u^1}[k+i]},\ 1 \leq i \leq n^*. \tag{4.22}$$

With regard to agents $O_{n^*+1} \cdots O_n$, an entirely similar argument may be applied to conclude that the action state of agent O_j for $n^*+1 \leq j \leq n$ at time $k+(n-j+1)$ may be driven to zero by appropriate selection of $u^2[k]$. Therefore, for even n, it follows that there exists a pair of causal control laws, each of the form (3.9)–(3.11), capable of driving the action state of agents O_i and O_j to zero at times $k+i$ and $k+(n-j+1)$ respectively, for $1 \leq i \leq n^*$ and $n^*+1 \leq j \leq n$.

In the event n is odd, the preceding arguments may be readily applied to verify the proposition statement for agents $O_1 \cdots O_{n^*-1}$ as well as agents $O_{n^*+1} \cdots O_n$. Therefore, it remain only to establish the result for O_{n^*}. In this case, $a_{n^*}[k+n^*]$ will depend on both $u^1[k]$ and $u^2[k]$. However, the action state of O_{n^*} still has rich structure with respect to $u^1[k]$. Specifically, the preceding arguments indicate $a_{n^*-1}[k+n^*-1]$ may be written as $a^{\overline{u^1}}_{n^*-1}[k+n^*-1] + a^{u^1}_{n^*-1}[k+n^*-1]u^1[k]$ where each term is a function of only the state $x[k]$ and $a^{u^1}_{n^*-1}[k+n^*-1]$ is nonzero. Then from the action-state-update equation of O_{n^*}, the action state $a_{n^*}[k+n^*]$ may be expressed in the form $a^{\overline{u^1}}_{n^*}[k+n^*] + a^{u^1}_{n^*}[k+n^*]u^1[k]$, where

$$a^{u^1}_{n^*}[k+n^*] = \mu_{apa,n^*} s^2_{n^*}[k+n^*-1] \times p_{n^*-1}[k+n^*-1] a^{u^1}_{n^*-1}[k+n^*-1] \tag{4.23}$$

$$\begin{aligned} a^{\overline{u^1}}_{n^*}[k+n^*] = c_a a_{n^*}[k+n^*-1] + \mu_{apa,n^*} s^2_{n^*}[k+n^*-1] \times \\ (p_{n^*-1}[k+n^*-1] a^{\overline{u^1}}_{n^*-1}[k+n^*-1] + p_{n^*+1}[k+n^*-1] \\ a_{n^*+1}[k+n^*-1]). \end{aligned} \tag{4.24}$$

Using similar arguments as in Proposition 4.2.1, it follows that $a^{\overline{u^1}}_{n^*}[k+n^*]$ and $a^{u^1}_{n^*}[k+n^*]$ are functions of the state $x[k]$ as well as $u^2[k]$ and the prestige of X^2; moreover, $a^{u^1}_{n^*}[k+n^*]$ is nonzero and $a_{n^*}[k+n^*]$ may be driven to zero by (4.22) for $i = n^*$. Therefore, if X^1 and X^2 are able to communicate, such that X^1 is conscious of the value of $u^2[k]$ and the prestige of X^2, then there exists a causal control law of the form (3.9)–(3.11) for X^1 capable of driving $a_{n^*}[k+n^*]$ to zero; thereby confirming the proposition statement holds in the case of both even and odd n.■

Proposition 4.5.2 builds upon Proposition 4.2.3 by considering the implications of holding the action state of two adjacent agents, in the queue $X^1 O_1 \cdots O_n X^2$, at zero indefinitely. Specifically, holding $a_j[k]$ and $a_{j+1}[k]$ at zero for $k \geq \tilde{k} \geq 0$ is shown to imply $a_1[k], \ldots, a_n[k]$ are all zero for $k \geq \tilde{k}$:

Proposition 4.5.2 Consider the queue $X^1 O_1 \cdots O_n X^2$. Let $\tilde{k} \geq 0$ denote an instant in time. If the action state of two adjacent agents, O_j and O_{j+1}, in the queue are sustained at zero for all $k \geq \tilde{k}$ then the action state of each agent in the queue is identically zero for all $k \geq \tilde{k}$. □

Proof: As indicated in the proposition statement, consider the scenario in which the action state of agents O_j and O_{j+1} are both zero for $k \geq \tilde{k}$. Furthermore, consider the queue segments $X^1 O_1 \cdots O_j O_{j+1}$ and $O_j O_{j+1} \cdots O_n X^2$. The implications of $a_j[k]$ and $a_{j+1}[k]$ being zero for all $k \geq \tilde{k}$ may be inferred through an inductive approach. To this end, consider the segment $X^1 O_1 \cdots O_j O_{j+1}$ and for some $\ell \in [2, j]$, assume $a_i[k] = 0$ for all $\ell \leq i \leq j$ and $k \geq \tilde{k}$. This condition is true for the base case $\ell = j$. The action-state-update equation of O_ℓ gives

$$a_\ell[k+1] = c_a a_\ell[k] + \mu_{apa,\ell} s_\ell^2[k] \sum_{O_j \in \mathcal{N}(O_\ell)} p_j[k] a_j[k].$$

From the induction assumption, $a_\ell[k] = a_{\ell+1}[k] = 0$ for $k \geq \tilde{k}$, and with $\mathcal{N}(O_\ell) = \{O_{\ell-1}, O_{\ell+1}\}$ the above equation simplifies to

$$0 = \mu_{apa,\ell} s_\ell^2[k] p_{\ell-1}[k] a_{\ell-1}[k], \; k \geq \tilde{k}. \tag{4.25}$$

Since $s_\ell[k]$ and $p_{\ell-1}[k]$ are both positive, equality in (4.25) necessitates that $a_{\ell-1}[k] = 0$ for $k \geq \tilde{k}$. This result establishes that $a_i[k] = 0$ for $\ell - 1 \leq i \leq j$ and $k \geq \tilde{k}$. It follows from induction that $a_i[k] = 0$ for $i = 1, \ldots, j$ and $k \geq \tilde{k}$, implying that if $a_j[k]$ and $a_{j+1}[k]$ are held at zero for $k \geq \tilde{k}$ then the action states $a_1[k], \ldots, a_{j-1}[k]$ are also zero for $k \geq \tilde{k}$.

A similar argument may be applied to the queue segment $O_j O_{j+1} \cdots O_n X^2$ to conclude that $a_{j+2}[k], \ldots, a_n[k]$ are also zero for $k \geq \tilde{k}$. Therefore, if the action state of two adjacent agents in the queue are held at zero for $k \geq \tilde{k}$, it follows the action states of all agents in the queue are zero for $k \geq \tilde{k}$.■

Propositions 4.5.1 and 4.5.2 are used in tandem in Theorem 4.5.3 to develop a stability result for the queue $X^1 O_1 \cdots O_n X^2$:

Theorem 4.5.3 There exist two control laws, each of the form (3.9)–(3.11), that $\mathcal{C}(n^*)$–stabilizes the queue $X^1 O_1 \cdots O_n X^2$, with $n^* = \lceil n/2 \rceil$. □

Proof: To provide $\mathcal{C}(n^*)$–stability, the combined effect of $u^1[k]$ and $u^2[k]$ must ensure the action state of all agents in the queue are identically zero at time n^* and subsequently held at zero for all future times. From Proposition 4.5.1 it follows that the action state of agents O_{n^*} and O_{n^*+1} may be driven to zero and subsequently held at zero by repeated application of causal control laws of the form (3.9)–(3.11), where some degree of communication between X^1 and X^2 is necessary in the case where n is odd. That is, $a_{n^*}[k]$ and $a_{n^*+1}[k]$ may be driven to zero and held at zero for $k \geq n^*$ by selecting $u^1[k]$ and $u^2[k]$ according to

$$u^1[k] = -\frac{a_{n^*}^{\overline{u^1}}[k+n^*]}{a_{n^*}^{u^1}[k+n^*]}, \quad u^2[k] = -\frac{a_{n^*+1}^{\overline{u^2}}[k+n^*]}{a_{n^*+1}^{u^2}[k+n^*]}, \quad k \geq 0. \tag{4.26}$$

By selecting $u^1[k]$ and $u^2[k]$ to hold $a_{n^*}[k]$ and $a_{n^*+1}[k]$ at zero for $k \geq n^*$, respectively, it follows from Proposition 4.5.2 that $a_i[k] = 0$ for $i = 1, \ldots, n$ and $k \geq n^*$. Therefore, the queue $X^1 O_1 \cdots O_n X^2$ is $\mathcal{C}(n^*)$–stabilized by two controllers of the form (3.9)–(3.11), with $u^1[k]$ and $u^2[k]$ given by (4.26).■

Theorem 4.5.3 indicates that, by adding a second control agent to the right end of the queue, the stabilization time for a queue of n agents may be reduced by a factor of approximately two as compared to the single-control-agent approach of Section 4.2. This reduction in stabilization time is illustrated in Figure 4.5 for the queue $X_0 O_1 O_2 O_3 O_4 O_5 O_6 X_7$. Note that the sensing requirements are also reduced compared to the single-control-agent case because each of X^1 and X^2 need only sense the state of half the agents in the queue. However, it is critical that the two control agents communicate with one another for several purposes: to coordinate how control action will be directed (i.e., X^1 and X^2 must agree to focus on driving the action states of O_{n^*} and O_{n^*+1} to zero, respectively), to share the components of the state $x[k]$ measured by each control agent with the other control agent, and, in the case where n is odd, to allow one of the control agents to know the control signal and prestige value of the other control agent. Ultimately, there are both benefits (namely, improved performance) and costs (namely, the need for communication) associated with using multiple control agents to $\mathcal{C}$–stabilize a queue.

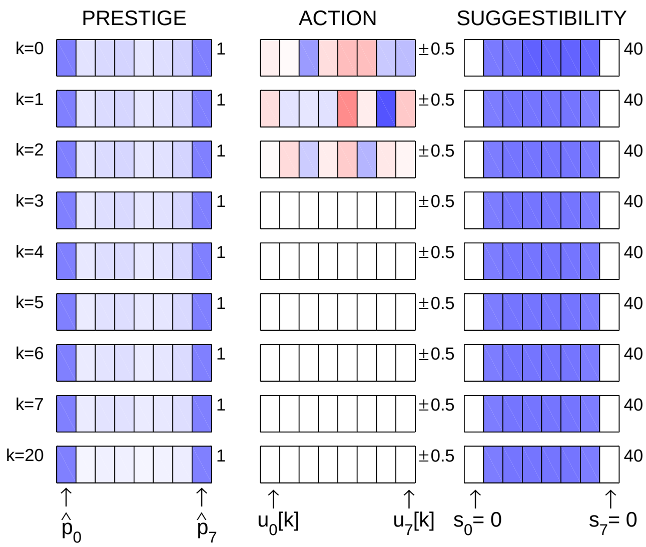

Figure 4.5: Progression of the prestige, action, and suggestibility states among members of the queue $X_0O_1O_2O_3O_4O_5O_6X_7$ subject to the $\mathcal{C}$–stabilizing control laws discussed in Section 4.5.3.

4.5.4 The Queues $\bar{O}X\bar{O}X\bar{O}\cdots\bar{O}X$ and $\bar{O}X\bar{O}X\cdots X\bar{O}$

The control strategies of the preceding sections have required that a control agent be positioned on at least one end of the queue. We consistently assumed this end to be the left end of the queue and proceeded to consider the queues $X\bar{O}$, $X\bar{O}X\bar{O}\cdots X\bar{O}$, and $X\bar{O}X\cdots X\bar{O}X$. Keeping in mind the discussion at the beginning of Section 4.5, the only queue topologies we have yet to consider are the queues $\bar{O}X\bar{O}X\cdots\bar{O}X$ and $\bar{O}X\bar{O}X\cdots\bar{O}X\bar{O}$. The queue $\bar{O}X\bar{O}X\cdots\bar{O}X$ is just the mirror image of the queue $X\bar{O}X\bar{O}\cdots X\bar{O}$ discussed in Section 4.5.1. Consequently, earlier stability results readily apply to the queue $\bar{O}X\bar{O}X\cdots\bar{O}X$ and it remains only to consider the queue $\bar{O}X\bar{O}X\cdots X\bar{O}$.

To aid analysis, the queue $\bar{O}X\bar{O}X\cdots\bar{O}X\bar{O}$ can be thought of as consisting of two segments: the left segment $\bar{O}X\bar{O}$ and a right segment of the form $X\bar{O}X\bar{O}\cdots X\bar{O}$. We have discussed a technique to $\mathcal{C}$–stabilize the right segment in Section 4.5.1. Therefore, if we can $\mathcal{C}$–stabilize the left segment then the existence of a set of $\mathcal{C}$–stabilizing control laws for the queue $\bar{O}X\bar{O}X\cdots\bar{O}X\bar{O}$ would follow from earlier results. However, $\mathcal{C}$–stabilizing

$\bar{O}X\bar{O}$ is difficult because doing so requires the control agent to simultaneously zero agents on both its left and right sides and it is not apparent if the crowd dynamics are sufficiently pliable to accommodate such an approach. Interestingly enough, for the simple queue O^1XO^2, it turns out the control agent is able to provide $\mathcal{C}(2)$–stability. The control agent achieves this feat by issuing a control signal at time $k = 1$ that appropriately shuffles the states of each agent so that appropriate selection of the the control signal at time $k = 2$ is able to zero both agents – a somewhat surprising result. Determining if similar results apply to larger queues of the form $\bar{O}X\bar{O}$ is, however, difficult on account of the algebraic complexity that emerges in tracing the propagation of the control signal through various agent states at successive time instants. Since the requirement of positioning one control agent at the end of the queue is quite reasonable, we elect to state the following conjecture for the queue $\bar{O}X\bar{O}$ and, in the next chapter, proceed to consider the problem of $\mathcal{C}$–stabilizing a two-dimensional crowd.

Conjecture 4.5.4 There exists a control law of the form (3.9)–(3.11) that $\mathcal{C}$–stabilizes the queue $\bar{O}X\bar{O}$. □

4.6 A Summary of Stabilization Strategies for a Queue

In this chapter, we have established the existence of a set of $\mathcal{C}(\lambda)$–stabilizing control laws, each of the form (3.9)–(3.11), for queues with a control agent positioned on at least one end. By enumerating the relevant state signals of agents in the queue $X_0O_1O_2$ at successive time instants, we were able to determine an expression, in terms of the initial state, for the value of $u_0[k]$ needed to drive the action of agent O_2 to zero. To develop similar results for longer queues of the form $X_0O_1 \ldots O_n$, we introduced three propositions that capitalize on the basic structure of both the social network present in queues and the dynamics in (2.1)–(2.5). These results were, in turn, used to establish a causal state-feedback $\mathcal{C}(n)$–stabilizing control law for the queue $X_0O_1 \ldots O_n$. Simulations reveal the control strategy performs admirably in ideal conditions, but that, in the presence of system noise, performance deteriorates dramatically as queue length increases. Furthermore, the sensing requirements and the stabilization time both scale poorly as a function of queue length. Recognizing that the performance of the single-control-agent scheme degrades rapidly as n increases, we extended our results to the case in which $m > 1$ control agents are positioned throughout the queue (again, with a control agent positioned on at least one end). Benefits of the use of multiple control agents include a reduction in stabilization time (for example, placing control agents at both ends of the queue affords a reduction in the stabilization time from n samples to approximately $n/2$ samples) and a reduction in the sensing burden placed on each control agent. A cost of having multiple control agents is, however, the need for the control agents to be able to communicate with one another.

Chapter 5

Two-Dimensional Crowds: Stability in the General Case

In this chapter, we exploit the ideas of Chapter 4 to develop stability results that apply in the case of two-dimensional crowds. In the case of two-dimensional crowds, members are no longer restricted to having at most two neighbours and, consequently, the social networks to emerge have the potential to be much more complex than those encountered in the previous chapter. As a result, determining how the m control agents should allocate their control action, with the goal of $\mathcal{C}$–stabilizing the whole crowd, is not immediately obvious. Our high-level strategy is to have each control agent target a different agent in the crowd; if these pairings are chosen wisely, then the stability results developed for the queue $X_0O_1 \cdots O_n$ percolate through to the two-dimensional crowd control problem. We also exploit the ability of control agents to communicate with each other to surmount any difficulties the extra interactions between agents may introduce.

We present our results in the form of algorithms. Algorithm 5.1.1 may be used to test for conditions that are sufficient to guarantee all agents in a particular target set can be zeroed in finite time. In turn, given a set of zeroed agents, Algorithm 5.2.2 can be used to determine if there are other agents which, given the structure of the social network, are themselves necessarily zeroed. Finally, in Algorithm 5.3.1, we describe how Algorithms 5.1.1 and 5.2.2 may be used in succession as part of a design procedure to identify at which positions in a crowd control agents should be placed to guarantee $\mathcal{C}$–stabilizability.

5.1 Algorithm 5.1.1: Determining if All Agents in a Target Set Can be Zeroed

In a general two-dimensional crowd, there are no restrictions on the number of neighbours an agent may have. As a result, there are typically multiple paths through which ideas and attitudes may propagate. For this reason, identifying whether or not a particular control agent can drive the action state of a particular agent to zero is, in general, a more involved undertaking than was the case when we considered queues. To pursue the matter further, we assume a fixed set of allocation pairs (i.e., a predetermined allocation of control effort) and ask the question, "Do there exist m control laws, each of the form (3.9)–(3.11), capable of zeroing all agents in the target set?" Algorithm 5.1.1 provides a means to test for conditions that are sufficient to guarantee that the answer to this question is yes.

To aid in presentation, we denote the allocation pairs, without loss of generality, by $\{(X^i \to O^i), i = 1, \ldots, m\}$, where O^i and X^i belong to the set of agents and control agents, respectively, for $i = 1 \ldots, m$. Such an assignment is always possible by simply reassigning the superscript indices of agents in the crowd. Algorithm 5.1.1 constructs a finite sequence, $\Upsilon_0, \Upsilon_1, \ldots, \Upsilon_\infty$, of allocation pairing sets. Inclusion of the allocation pairing $(X^i \to O^i)$ in Υ_∞ guarantees the target agent O^i can be zeroed by X^i using a control law of the form (3.9)–(3.11).

Algorithm 5.1.1 The algorithm consists of three steps:

Step 1 Initialize α to zero and initialize Υ_0 to the empty set $\emptyset$.

Step 2 Determine if there exists an allocation pairing, $(X^i \to O^i)$, for which $|\ell^1_{i,i}|$ is the unique minimum of $|L_i|$. In the event no such pairing exists, the algorithm terminates with $\Upsilon_\infty = \emptyset$. Otherwise, define $\Upsilon_1 := \{(X^i \to O^i)\}$ and proceed to Step 3 with $\alpha = 1$.

Step 3 Determine if there exists a pair, $(X^i \to O^i)$, not already in Υ_α, for which:

(a) $|\ell^1_{i,i}|$ is the unique minimum of $|L_i|$, or

(b) for each j such that $|\ell^1_{j,i}| \leq |\ell^1_{i,i}|$ it follows that $(X^j \to O^i) \in \Upsilon_\alpha$.

In the event no such pairing exists, the algorithm terminates with $\Upsilon_\infty = \Upsilon_\alpha$. Otherwise, define $\Upsilon_{\alpha+1} = \Upsilon_\alpha \cup \{(X^i \to O^i)\}$ and repeat Step 3 for $\alpha \to \alpha + 1$.

If Υ_∞ contains all allocation pairings then the allocation of control effort in question ensures that each agent in the target set can be zeroed by appropriate selection of a control law of the form (3.9)–(3.11) for $X^1, \ldots, X^m$. Algorithm 5.1.1 can also be used to determine whether or not information exchange between a pair of control agents is needed in order to zero all agents in the target set. Specifically, given paths $\ell^1_{j,i}$ and $\ell^1_{i,i}$ with $|\ell^1_{j,i}| \leq |\ell^1_{i,i}|$, zeroing O^i using control laws of the form (3.9)–(3.11) for each control agent requires that X^i be informed in advance of the control action taken by X^j.

□

The following examples illustrate how Algorithm 5.1.1 may be applied to determine if all agents in a predetermined target set can be zeroed.

Example 5.1.2 Consider the crowd of Figure 3.6 and assume the predetermined allocation pairing of interest is the same as that illustrated in Figure 3.8. In this case, the allocation pairings are $\{(X^1 \to O^6), (X^2 \to O^8), (X^3 \to O^7)\}$ and the target set is $\{O^6, O^7, O^8\}$. Applying Algorithm 5.1.1 gives:

1. $\Upsilon_0 = \emptyset$, $\alpha = 0$ (Step 1).

2. $|\ell^1_{2,8}|$ is the unique minimum of $|L_8|$.

3. $\Upsilon_1 = \{(X^2 \to O^8)\}$, $\alpha = 1$ (Step 2).

4. $|\ell^1_{3,7}|$ is the unique minimum of $|L_7|$.

5. $\Upsilon_2 = \{(X^2 \to O^8), (X^3 \to O^7)\}$, $\alpha = 2$ (Step 3a).

6. $|\ell^1_{1,6}|$ is not the unique minimum of $|L_6|$, since there are paths equal in length to $\ell^1_{1,6}$ from both X^2 to O^6 and from X^3 to O^6. However, the allocation pairings $(X^2 \to O^8)$ and $(X^3 \to O^7)$ both belong to Υ_2.

7. $\Upsilon_3 = \{(X^2 \to O^8), (X^3 \to O^7), (X^1 \to O^6)\}$, $\alpha = 3$ (Step 3b).

8. All allocation pairings belong to Υ_3 and Algorithm 5.1.1 terminates with $\Upsilon_\infty = \{(X^2 \to O^8), (X^3 \to O^7), (X^1 \to O^6)\}$.

In this example, for the control strategy to work, at each time instant X^1 must be apprised of the control action taken by both X^2 and X^3.

□

Example 5.1.3 To illustrate how Algorithm 5.1.1 generalizes the control ideas discussed in Chapter 4, consider applying it to the queue $X_0 O_1 \ldots O_n$. In Chapter 4, we successfully used the allocation pairing $(X_0 \to O_n)$ to zero O_n. There is only one path from X_0 to O_n, namely the path through all agents, and we conclude, this time by Algorithm 5.1.1, that X_0 is able to zero O_n. □

It is readily appreciated from the above examples that Algorithm 5.1.1 terminates in finite time. Moreover, regarding the issue of uniqueness, we conjecture the following:

Conjecture 5.1.4 The Υ_∞ to emerge from Algorithm 5.1.1 is unique. □

5.2 Algorithm 5.2.2: Determining if Agents Outside the Target Set Are Zeroed When the Target Set is Zeroed

In earlier discussions pertaining to the queue $X_0 O_1 \ldots O_n$, we had argued that if all agents to the right of O_i were zeroed for $k \geq \tilde{k}$ then O_i is necessarily zeroed for $k \geq \tilde{k}$. In the following proposition, we show that this concept has a natural analogue in the world of two-dimensional crowds.

Proposition 5.2.1 Consider a crowd $\mathcal{C}$, a group of agents Ω_α for which all members are zeroed for $k \geq \tilde{k}$, and agent $O_i \in \Omega_\alpha$ that has exactly one neighbour, O_h, not belonging to Ω_α. Then O_h is zeroed for $k \geq \tilde{k}$. □

Proof: The result follows directly from the action-state update equation of O_i. Specifically, (2) gives

$$a_i[k+1] = c_a a_i[k] + \mu_{apa,i} s_i^2[k] \times \left(p_h[k] a_h[k] + \sum_{O_j \in \mathcal{N}(O_i), j \neq h} p_j[k] a_j[k] \right). \tag{5.1}$$

Since O_i belongs to Ω_α, both $a_i[k+1]$ and $a_i[k]$ are zero for $k \geq \tilde{k}$. Furthermore, aside from O_h, all of O_i's other neighbours belong to Ω_α and, thus, are zeroed for $k \geq \tilde{k}$, from which it follows the summation in (5.1) is zero. Given prestige and suggestibility are positive, equality in (5.1) necessitates $a_h[k] = 0$ for $k \geq \tilde{k}$, and, hence, O_h is zeroed for $k \geq \tilde{k}$.■

Given a set of agents that are zeroed for $k \geq \tilde{k}$, Proposition 5.2.1 can be used iteratively to determine a larger set of agents, denoted Ω_∞, that are also necessarily zeroed for $k \geq \tilde{k}$, as described in the following algorithm:

Algorithm 5.2.2 The algorithm consists of three steps:

Step 1 Initialize α to zero and initialize Ω_0 to the set of agents that are known to be zeroed for $k \geq \tilde{k}$.

Step 2 Determine if there exists an agent, say O_i, in Ω_α that has exactly one neighbour, say O_h, that does not belong to Ω_α.

Case 1 In the event there is no such agent, the algorithm terminates with $\Omega_\infty := \Omega_\alpha$.

Case 2 Conversely, if such an agent does exist, proceed to Step 3.

Step 3 From Proposition 5.2.1, O_h is zeroed for $k \geq \tilde{k}$. Define

$$\Omega_{\alpha+1} := \Omega_\alpha \cup \{O_h\} \tag{5.2}$$

and repeat Step 2 for $\alpha \to \alpha + 1$.

□

The following examples illustrate how Algorithm 5.2.2 may be applied to a crowd with a known set of zeroed agents.

Example 5.2.3 Consider once more the two-dimensional crowd of Figure 3.6 and assume the predetermined allocation of control effort is the same as that illustrated in Figure 3.8. Finally, assume all agents in the target set have been zeroed, a feat we know is possible from Example 5.1.2. Algorithm 5.2.2 can be used to identify a superset of agents which are also zeroed.

To begin, the zeroed set is $\Omega_0 = \{O^6, O^7, O^8\}$. Applying Algorithm 5.2.2, observe that O^5 is the only neighbour of O^7 not in Ω_0. Hence, O^5 is zeroed and, as per Step 3 of Algorithm 5.2.2, we define $\Omega_1 := \{O^6, O^7, O^8, O^5\}$. Next, we note that O^3 is the only neighbour of O^6 not in Ω_1 and, as before, O^3 must be zeroed, and we define $\Omega_2 := \{O^6, O^7, O^8, O^5, O^3\}$. This procedure of identifying an agent who has one neighbour not already in the set of zeroed agents can be repeated until Algorithm 5.2.2 eventually terminates with $\Omega_\infty := \{O^6, O^7, O^8, O^5, O^3, O^{10}, O^4, O^2, O^1\}$. At this stage, we cannot conclude that any other members of the crowd are necessarily zeroed. While this example discusses a specific crowd and a specific allocation of control effort, it is readily appreciated that Algorithm 5.2.2 always terminates in finite time. □

Example 5.2.4 To illustrate that Algorithm 5.2.2 is also a generalization of earlier results, consider the familiar queue $X_0 O_1 \ldots O_n$ and assume O_n is zeroed. Applying Algorithm 5.2.2 and noting that O_n has exactly on neighbour, we can conclude that O_{n-1} is zeroed. Similarly, noting that O_{n-1}'s neighbours are O_{n-2} and O_n and that O_n is zeroed, we conclude from Algorithm 5.2.2 that O_{n-2} is zeroed. Repeatedly applying steps 2 and 3 of Algorithm 5.2.2 confirms all agents in the queue are zeroed, a reaffirmation of Proposition 4.2.3. □

It is worth noting that the sequence of zeroed agents, $\Omega_0, \Omega_1, \ldots$, to emerge from Algorithm 5.2.2 need not be unique. For instance, in Example 5.2.3, we could have defined $\Omega_1 := \{O^6, O^7, O^8, O^{10}\}$, as opposed to the set actually used. However, it is not the sequence $\Omega_0, \Omega_1, \ldots$ that is of particular concern, but rather the terminating value of the sequence, Ω_∞. The following proposition states that while the feasible sequences to emerge from Algorithm 5.2.2 may not be distinct, the corresponding terminal value is unique:

Proposition 5.2.5 Consider a crowd $\mathcal{C}$ and a set of agents, Ω_0, that are zeroed for $k \geq \tilde{k}$. The terminating value of any sequence that emerges from application of Algorithm 5.2.2 is unique. □

Proof (by contradiction): Consider two distinct sequences of zeroed crowd members, say $\{\Omega_0, \Omega_1, \ldots, \Omega_\infty\}$ and $\{\Omega'_0, \Omega'_1, \ldots, \Omega'_\infty\}$, that emerge from application of Algorithm 5.2.2. Assume $\Omega_\infty \neq \Omega'_\infty$. It follows that there must be at least one agent, say O_h, that belongs to one of Ω_∞ or Ω'_∞, but not the other. Without loss of generality, assume O_h belongs to Ω_∞ but not Ω'_∞. To aid in discussion, we introduce the following quantities:

$$\alpha := \max\{i : \Omega_i = \Omega'_i\} \text{ and } \beta := \min\{i : O_h \in \Omega_i\}. \tag{5.3}$$

It is noted that each of these quantities is well defined: in the case of α, since $\Omega_o = \Omega'_o$ and, in the case of β, since O_h belongs to Ω_∞. Additionally, the definitions of α and β necessitate $\alpha < \beta$.

Algorithm 5.2.2 proceeds to construct $\Omega_{\alpha+1}, \ldots, \Omega_\beta$ by successively adding one or more agents, say $\hat{O}_1, \hat{O}_2, \ldots, \hat{O}_d = O_h$, one agent at a time, to Ω_α. The fact that $\hat{O}_1$ belongs to $\Omega_{\alpha+1}$ implies there must be at least one agent in Ω_α, say O^*_α, for which $\hat{O}_1$ is the only neighbour not itself in Ω_α. However, since $\Omega'_\alpha = \Omega_\alpha$, this argument is equally applicable to Ω'_α and, hence, $\hat{O}_1$ must belong to Ω'_∞.

To show that $\hat{O}_i \in \Omega'_\infty$ for all $i \in [1, d]$ and, hence, that $O_h \in \Omega'_\infty$ we employ an inductive approach. To this end, assume that for a particular $\ell \in [1, d]$ that agent $\hat{O}_j \in \Omega'_\infty$, for all $1 \leq j \leq \ell$; a result that is apparent for the base case $\ell = 1$. The fact that $\hat{O}_{\ell+1} \in \Omega_{\alpha+\ell+1}$ implies there exists at least one agent, say $O^*_{\alpha+\ell}$, in $\Omega_{\alpha+\ell}$ whose only neighbour not in $\Omega_{\alpha+\ell}$ is $\hat{O}_{\ell+1}$. However, given the assumption that $\hat{O}_j \in \Omega'_\infty$ for $1 \leq j \leq \ell$, this argument is equally applicable to the set $\Omega'_\alpha \cup \hat{O}_1 \cup \cdots \cup \hat{O}_\ell \subseteq \Omega'_\infty$. It follows from induction that $\hat{O}_i \in \Omega'_\infty$ for $i \in [1, d]$, implying O_h belongs to Ω'_∞ and thereby refuting the initial assumption that O_h does not belong to Ω'_∞ and, hence, that $\Omega_\infty \neq \Omega'_\infty$. Rather, any agent in Ω_∞ necessarily also belongs to Ω'_∞ and vice versa.■

The question arises as to whether or not Algorithm 5.2.2 generates all agents that are zeroed. We conjecture that the answer is yes:

Conjecture 5.2.6 For any crowd $\mathcal{C}$ and set of zeroed agents Ω_o, the set Ω_∞, generated by Algorithm 5.2.2, is the largest necessarily-zeroed set. □

5.3 Algorithm 5.3.1: A Design Procedure For Stabilizing Any Crowd

It would be useful to have a mechanism to position control agents within the crowd which, given the appropriate control action, provide $\mathcal{C}$–stability. The information that can be gained upon successive application of Algorithms 5.1.1 and 5.2.2 suggests one possible means to this end. Given there are only a finite number of ways to position m control agents and target m agents in an n-agent crowd, all possible combinations may be readily enumerated. Algorithms 5.1.1 and 5.2.2 may then be applied to each of the possible combinations to determine if the associated target set can indeed be zeroed and, in cases where this is true, if the associated Ω_∞, that is a superset of necessarily zeroed agents, is equal to $\mathcal{O} = \{O^1, \ldots, O^n\}$. Algorithm 5.3.1 uses this approach to determine where control agents should be located to $\mathcal{C}$–stabilize a crowd, while limiting the number of control agents required.

Algorithm 5.3.1 The algorithm consists of three steps:

Step 1 Initialize m, the number of control agents that will be used to $\mathcal{C}$–stabilize the crowd, to n. Assign to each agent a control agent and task the control agents with ensuring their respective agents are zeroed by some finite time. That such an approach is always possible may be confirmed by noting that the path from each control agent to the associated agent is the unique minimum of all such paths and, therefore, from Algorithm 5.1.1 all agents in the target set may be zeroed. Moreover, the fact that the target set is the set of all agents implies the crowd is $\mathcal{C}$–stabilizable

Step 2 For each crowd arrangement involving $m - 1$ control agents identify all allocation pairings.

Step 3 For each of the combinations identified in Step 2, use Algorithms 5.1.1 and 5.2.2 to determine, first, if sufficient conditions hold to guarantee that all agents in the target set can be zeroed and, if so, second, if sufficient conditions hold to guarantee that all agents in the crowd are necessarily zeroed given the target set is zeroed.

Case 1 If, for every arrangement from Step 2, either of the above conditions do not hold, then we can guarantee the crowd is $\mathcal{C}$–stabilizable only if we use at least m control agents. The algorithm terminates and the crowd may be $\mathcal{C}$–stabilized using any of the $\mathcal{C}$–stabilizing m-control-agent schemes.

Case 2 Conversely, there is at least one $\mathcal{C}$–stabilizing control scheme involving $m-1$ control agents. Repeat Step 2 for $m \to m-1$.

□

Example 5.3.2 In this example, we illustrate how Algorithm 5.3.1 may be applied to $\mathcal{C}$–stabilize a crowd. To this end, Figure 5.1 shows the crowd of Figure 3.6 with the three control agents from the original figure removed. Before proceeding, recall from Example 5.2.3 that because $\Omega_\infty \neq \mathcal{O}$ we cannot guarantee the arrangement in Figure 3.6 is $\mathcal{C}$–stabilizing. By applying Algorithm 5.3.1 we seek to position control agents throughout the crowd to guarantee $\mathcal{C}$–stability, while limiting the number of control agents used.

We begin by assigning to each agent in the crowd a control agent that is tasked with zeroing their respective agent. This scenario is illustrated in Figure 5.2 where the crowd of Figure 5.1 has been augmented with 12 control agents. The red arrows in the figure illustrate the allocation pairings, such that $(X^1 \to O^1)$, $(X^4 \to O^2)$, and so on. Using 12 control agents to $\mathcal{C}$–stabilize the crowd is likely overkill and, as per Algorithm 5.3.1, we attempt to reduce this number by removing control agents and using Algorithms 5.1.1 and 5.2.2 to verify the resulting crowd is still $\mathcal{C}$–stabilizable. To expedite the process, we remove multiple control agents at a time. For example, upon removing control agents X^4, X^6, X^8, and X^{10} and using the allocation pairings shown in Figure 5.3, Algorithms 5.1.1 and 5.2.2 confirm the crowd is $\mathcal{C}$–stabilizable. In an attempt to further reduce the number of control agents, we remove control agents X^5 and X^{11} from the crowd and reallocate the control effort as per Figure 5.4. Once again, Algorithms 5.1.1 and 5.2.2 confirm this arrangement is $\mathcal{C}$–stabilizable. Likewise, upon removing control agents X^9 and X^{12}, and reallocating the control effort as per Figure 5.5 the crowd is still $\mathcal{C}$–stabilizable. In this example, we have not enumerated every possible combination of control agent locations and allocation pairings; however, by applying Algorithm 5.3.1 we were able to quickly $\mathcal{C}$–stabilize the crowd while limiting the number of control agent required.

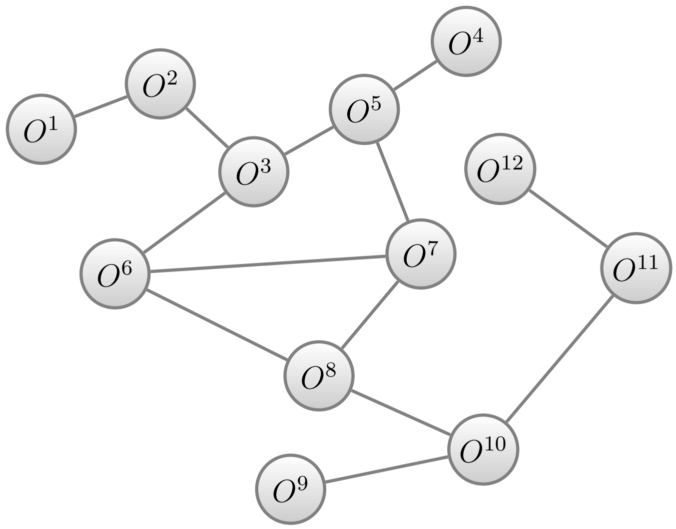

Figure 5.1: A crowd consisting of 12 agents that has no control action.

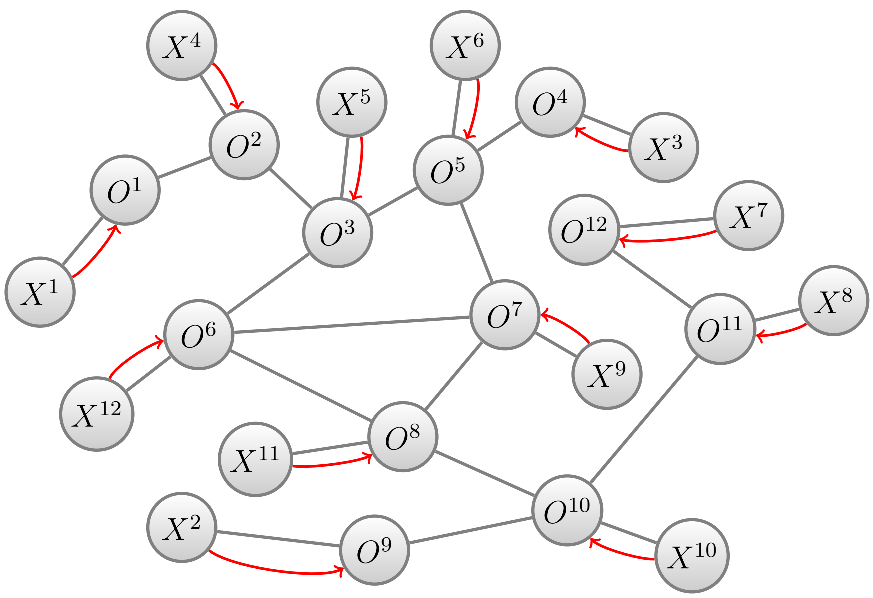

Figure 5.2: A scheme that uses 12 control agents to $\mathcal{C}$–stabilize the crowd.

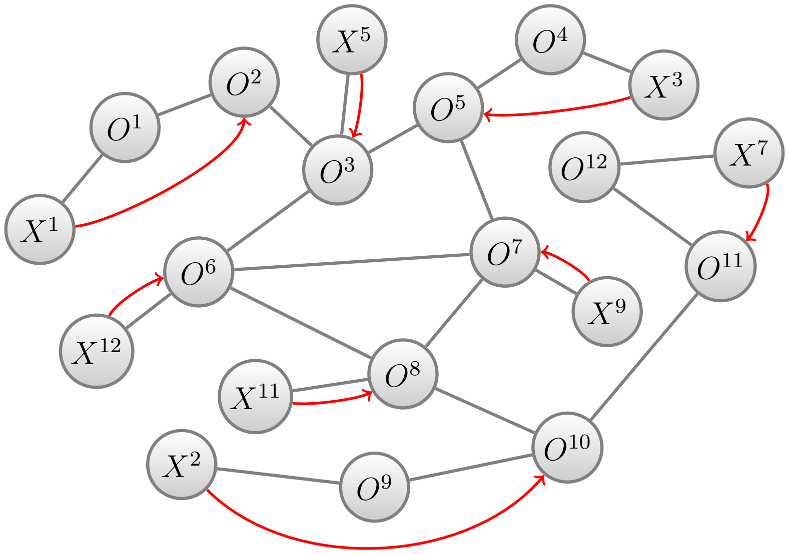

Figure 5.3: A scheme that uses 8 control agents to $\mathcal{C}$–stabilize the crowd.

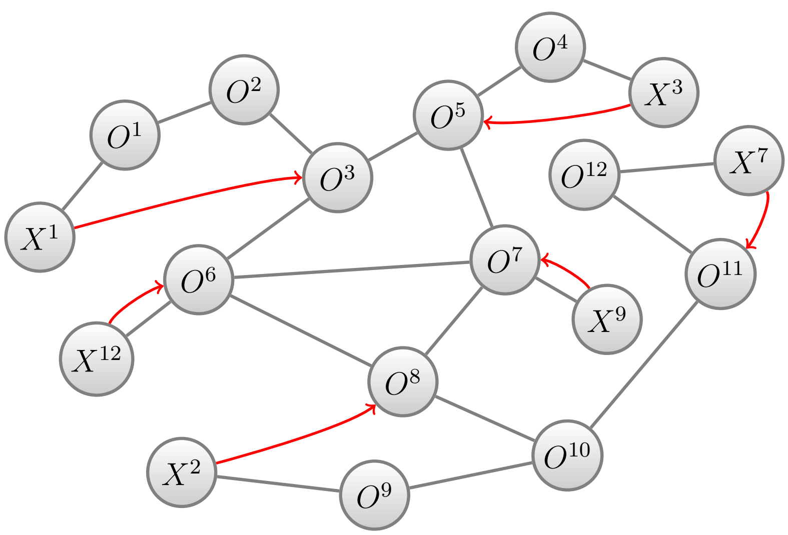

Figure 5.4: A scheme that uses 6 control agents to $\mathcal{C}$–stabilize the crowd.

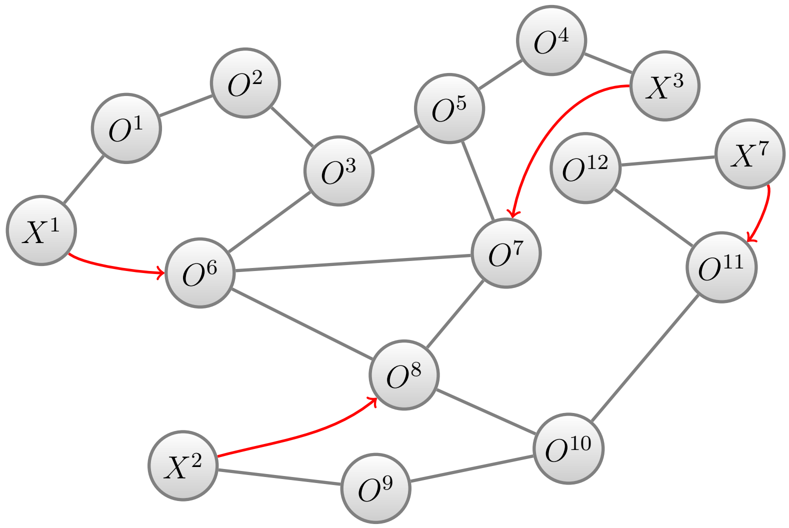

Figure 5.5: A scheme that uses just 4 control agents to $\mathcal{C}$–stabilize the crowd.

$\square$

Chapter 6

Summary and Future Directions

This thesis was motivated by the question, “Is it possible to use techniques from control theory to study and ultimately control the psychological behaviour of people in a crowd?” To investigate the matter, we developed a dynamic model of crowd behaviour based on the notion of suggestibility put forth in the social commentary of Gustav LeBon. The discrete-time nonlinear dynamic model we employed uses information about the crowd’s current state to describe the effect of the social interactions between neighbours on the psychological state of each member. Simulations and linearization arguments revealed that the crowd dynamics are unstable.

With the goal of ensuring all agents act in a calm and orderly manner, we presented a definition for what it means to stabilize a crowd and used linearization techniques as well as simple nonlinear methods to provide stability. These schemes provided varying degrees of success, but even the best methods had serious limitations and motivated an entirely new approach to control. In response, we introduced the control agent as a means for affecting psychological change among crowd members and first employed this paradigm by using a single control agent to stabilize a queue. We showed that given a priori knowledge of the crowd dynamics and the ability to sense the entire state, the control agent can drive the action state of any agent in the queue to zero. Moreover, we showed that if this target agent is chosen judiciously and its action state is held at zero, which may be achieved by repeatedly applying the aforementioned control law, then it necessarily follows the action state of all agents in the queue are zero and, hence, the queue can be stabilized. Simulations confirm the functionality of the control law but reveal that, as queue length increases, the control law becomes dramatically more sensitive to system noise, while the stabilization time increases and the sensing burden on the lone control agent becomes onerous.

To address some of the limitations of using a single control agent to stabilize a queue we positioned multiple control agents throughout the queue and had them operate in a cooperative manner. We showed that, in addition to providing stability, such schemes offer the benefits of decreasing the stabilization time and reducing the sensing burden placed on any particular control agent. By introducing the appropriate machinery, we exploited the results developed for queues to establish similar results that apply to the more general case of two-dimensional crowds. More specifically, our results culminated in an algorithm that can be used to stabilize any two-dimensional crowd and paves the way for a number of exciting and potentially fruitful directions in which to steer the research discussed in this thesis.

From a modeling perspective, we have hedged our research efforts on a model of crowd behaviour predicated on suggestibility theory. However, over the past century, a number of alternative theories to explain group behaviour have emerged and many of these formulations stand in stark contrast to LeBon's assertions. For example, the ideas of conformity, cognitive dissonance, persuasion, social facilitation, de-individualization, and group polarization, to name a few, all capture social-psychological ideas that are unaccounted for in our model. Also on a modeling note, we may account for instances where an agent's neighbours have the potential to evolve dynamically (e.g., due to the physical movement of crowd members) by including an explicit dependence on time when defining the neighbour set of each agent. By augmenting our model to include one or more of these factors we may not only make the model more realistic, but the new dynamics may also inspire novel control strategies or be more receptive to experimental validation.

There are also a number of open research directions that are likely to be of interest to the control community. Here we provide a sampling of those ideas that, in the author's opinion, show considerable promise or are of particular personal interest. To begin, it would be highly desirable, given its practical significance, to develop a control strategy whereby the control agents, collectively, need only sense a subset of the entire state (for example, perhaps only the action states). This capability would dramatically reduce the overall sensing burden as well as promote the idea of sensing only those signals that are manifested in tangible forms suitable for human perception. In terms of two-dimensional crowds, there is an onus to describe the minimum sensing and communication requirements needed to compute stabilizing control signals, as well as to develop a systematic procedure for calculating these signals. Understanding the structure of the crowd's social network will inevitably be key to realizing these ambitions. Given these networks are effectively graphs, it may be advantageous to incorporate ideas from graph theory into the crowd control problem and leverage results that serve to streamline or extend our existing findings. It would also be interesting to examine the sensitivity of control schemes involving multiple control agents in the case of both queues and two-dimensional crowds. Our single-control-agent stabilization strategy had poor sensitivity characteristics and using multiple control agents may offer improvements in this regard. Another potentially fruitful direction is to cast the problem of stabilizing a crowd as an optimal control problem. Namely, minimizing the appropriate cost function, subject to the constraint that the crowd is stabilized, is a natural recourse to identify control schemes that achieve a favourable balance between the number of control agents used, the stabilization time, and the sensing requirements of the control strategy.

There are also incentives for reexamining the control objectives and philosophical approach towards stabilization at the grassroots level. For example, in this work, we focused on driving the action state of each agent in the crowd to zero and benefited from the implications of this condition on other crowd states. It may, however, be more appropriate to consider containment strategies that only guarantee state components do not exceed certain predetermined values. This alternate formulation captures the idea that a crowd need not have zero action in order to be deemed well-behaved. Another research direction notes that while we have reported on a number of the benefits afforded by coordinating the behaviour of multiple control agents, our results pertain to a specific stabilization strategy. Pursuing new control strategies and cooperative information-sharing protocols among the control agents may afford greater improvements with respect to the metrics we have discussed or prove advantageous from perspectives we have not yet considered. Finally, at select points in this thesis we have relied on conjecture to encapsulate our suspicions about results that have yet to be formally verified. It is natural to attempt to resolve these proclamations by either affirming or refuting their validity.

The open research directions we have listed address key issues with respect to both the modeling and control of psychological crowd behaviour. Moreover, these initiatives have been formulated in a system-theoretic framework for which control theory is both an insightful and powerful analysis tool. Consequently, while this work has focused predominantly on stabilizing the psychological dynamics of people in a crowd, the panoply of open research directions suggest there are many exciting and untapped social-psychological research initiatives that hinge critically on the control-theoretic framework. Therefore, in regard to the question that motivated this thesis, we state the following conclusion: while the mathematical study of complex social phenomena, including crowds, is prefixed by an assortment of unique challenges, control theory is, nevertheless, a potentially powerful framework to study the underlying dynamics at play in such systems.

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Appendix A

Developing a Linear Dynamic Model of Psychological Crowd Behaviour

The linear model of crowd behaviour considered in Section 3.3.1 was obtained by linearizing the nonlinear model of crowd behaviour reported in [1], for which the dynamics of the agent in position i are given by:

$$
\begin{aligned}
p_i[k+1] &= c_p p_i[k] + \mu_{pa,i}|a_i[k]| && \text{(A.1)}\\
a_i[k+1] &= c_a a_i[k] + \mu_{apa,i} s_i^2[k] \sum_{O_j \in \mathcal{N}(O_i)} d_{i,j} p_j[k] a_j[k] + v_i[k] && \text{(A.2)}\\
b_i[k+1] &= a_i[k] && \text{(A.3)}\\
s_i[k+1] &= c_s s_i[k] + \mu_{sa,i} \left(a_i[k] - b_i[k]\right)^2 + \mu_{sap,i} \sum_{O_j \in \mathcal{N}(O_i)} d_{i,j} p_j[k] |a_j[k]| + \\
& \quad \mu_{ssp,i} \sum_{O_j \in \mathcal{N}(O_i)} d_{i,j} p_j[k] \left(s_j[k] - s_i[k]\right) + \mu_{s,i} S \left(2 - c_s\right) - s_i[k]. && \text{(A.4)}
\end{aligned}
$$

The meaning of the state signals and system parameters in (A.1)–(A.4) are the same as in (2.1)–(2.5) and $v_i[k]$ the control signal affecting agent i. In this appendix, we linearize the nonlinear dynamics in (A.1)–(A.4) about the generic operating point $p_i = p_0$, $a_i = a_0 > 0$, $b_i = b_0$, and $s_i = s_0$

for $i = 1, \ldots, n$. Taking the crowd's state and input signal as $x[k] = [p_1[k], \ldots, p_n[k], a_1[k], \ldots, a_n[k],\ b_1[k], \ldots, b_n[k],\ s_1[k], \ldots, s_n[k]]^T$ and $v[k] = [v_1[k], \ldots, v_n[k]]^T$, respectively, linearizing yields a linear time-invariant system of the form

$$\Delta x[k+1] = A\Delta x[k] + B\Delta v[k], \tag{A.5}$$

where

$$A = \begin{pmatrix} A_{p,p} & A_{p,a} & A_{p,b} & A_{p,s} \\ A_{a,p} & A_{a,a} & A_{a,b} & A_{a,s} \\ A_{b,p} & A_{b,a} & A_{b,b} & A_{b,s} \\ A_{s,p} & A_{s,a} & A_{s,b} & A_{s,s} \end{pmatrix} \text{ and } B = \begin{pmatrix} 0_n \\ I_n \\ 0_n \\ 0_n \end{pmatrix}.$$

The sub-matices of the A matrix can be written as

$$A_{p,p} = c_p I_n,\ A_{p,b} = 0_n,\ A_{p,s} = 0_n,\ A_{a,b} = 0_n,$$

$$A_{b,p} = O_n,\ A_{b,a} = I_n,\ A_{b,b} = O_n,\ A_{b,s} = O_n, A_{s,b} = 0_n,$$

$$A_{p,a} = \begin{pmatrix} \mu_{pa,1} & & \\ & \ddots & \\ & & \mu_{pa,n} \end{pmatrix}$$

$$A_{a,p} = \begin{pmatrix} \epsilon_{1,1} & \ldots & \epsilon_{1,n} \\ \vdots & \ddots & \vdots \\ \epsilon_{n,1} & \ldots & \epsilon_{n,n} \end{pmatrix},\ A_{a,s} = \begin{pmatrix} \delta_1 & & \\ & \ddots & \\ & & \delta_n \end{pmatrix},$$

$$A_{s,p} = \begin{pmatrix} \pi_{1,1} & \ldots & \pi_{1,n} \\ \vdots & \ddots & \vdots \\ \pi_{n,1} & \ldots & \pi_{n,n} \end{pmatrix},\ A_{s,a} = \begin{pmatrix} \tau_{1,1} & \ldots & \tau_{1,n} \\ \vdots & \ddots & \vdots \\ \tau_{n,1} & \ldots & \tau_{n,n} \end{pmatrix},$$

$$A_{a,a} = \begin{pmatrix} c_a & \eta_{1,2} & \cdots & \eta_{1,n} \\ \eta_{2,1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \eta_{n-1,n} \\ \eta_{n,1} & \cdots & \eta_{n,n-1} & c_a \end{pmatrix}, \ A_{s,s} = \begin{pmatrix} \zeta_1 & \rho_{1,2} & \cdots & \rho_{1,n} \\ \rho_{2,1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \rho_{n-1,n} \\ \rho_{n,1} & \cdots & \rho_{n,n-1} & \zeta_n \end{pmatrix},$$

where

$$\delta_i = 2\mu_{apa,i} s_o p_o a_o \sum_{O_j \in \mathcal{N}(O_i)} d_{i,j}, \qquad \epsilon_{i,j} = \mu_{apa,i} s_o^2 a_o d_{i,j}, \ \tau_{i,j} = \mu_{sap,i} p_o d_{i,j},$$

$$\zeta_i = c_s - 1 - \mu_{ssp,i} p_o \sum_{O_j \in \mathcal{N}(O_i)} d_{i,j}, \ \ \eta_{i,j} = \mu_{apa,i} s_o^2 p_o d_{i,j}, \ \rho_{i,j} = \mu_{ssp,i} p_o d_{i,j}, \text{ and}$$

$$\pi_{i,j} = \mu_{sap,i} |a_o| d_{i,j}.$$