Black Hole Thermodynamics and the Tunnelling Method for Particle Emission

by

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Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners. I understand that my thesis may be made electronically available to the public.

Abstract

The semi-classical black hole tunnelling method is a useful technique to calculate black hole temperature and understand black hole thermodynamics. I will investigate the black hole tunnelling method in detail. I will compare two different approaches used to calculate black hole tunnelling. The tunnelling method can be applied to a broad range of spacetimes and I will show this explicitly in order to demonstrate the robustness of the tunnelling technique. In particular, I will apply the tunnelling method to spacetimes including: Rindler (the method can recover the Unruh temperature), and more general spacetimes (such as Kerr-Newman and Taub-NUT). I will also discuss the 5d Kerr-Gödel spacetimes in detail (while showing a previous unobserved property of these spaces). Once the parameter space of Kerr-Gödel is understood in detail, I will show how the tunnelling method can also be successfully applied to the Kerr-Gödel black hole.

Finally, the key result of my thesis involves extending the tunnelling method to model fermion emission. The previous tunnelling calculations all involved the emission of scalar particles. I will model the emission of spin-1/2 fermions from various spacetimes including the Rindler spacetime and general non-rotating black holes. I will also model the emission of charged spin-1/2 fermions from the Kerr-Newman spacetime to show that the method is also applicable to rotating spacetimes. In all these cases I show that the correct Hawking temperature (Unruh temperature in the case of Rindler) is recovered for spin-1/2 fermion emission. Although this final result is not surprising, it is an important result because it confirms that Dirac particles will radiate from the black hole at the same temperature as scalar particles. It has always been assumed that this is the case but there is very little literature involving fermion radiation of black holes. So the results of my calculations are twofold, I demonstrate that Dirac particles are emitted at the same temperature as scalar particles from a black hole and it shows how robust the semi-classical tunnelling technique is.

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1 Introduction

A black hole is an object for which gravity is so strong that (classically) no matter or radiation can escape from it. If only a classical system is considered, it would be impossible to define a temperature for a black hole because it would be impossible for anything to be in thermal equilibrium with a black hole. This is because everything would go into the black hole but nothing will come out. The topic of black hole thermodynamics has been a subject of interest since the 1970's when Bekenstein first conjectured that there was a fundamental relationship between the properties of black holes and the laws of thermodynamics [1]. This conjecture was strengthened by Hawking, who was able to show that black holes can radiate when quantum effects are taken in account [2]-[4]. He was able to show that this black hole radiation was purely thermal. This meant that black holes have a well defined temperature and can truly be thought of as thermodynamic objects. This was an important discovery because classically nothing could escape from a black hole. So black hole radiation was discovered as by-product of early quantum gravity calculations [2]-[4] and this emphasizes the importance of trying to find a full quantum theory of gravity. This is because new physics should be found once a complete quantum theory of gravity is formulated and any discoveries could be as important as black hole radiation. The discovery of black hole radiation also opened up new mysteries such as the information loss problem. The information loss problem results from the argument of whether the black hole radiation should be purely thermal or not. If the radiation is purely that of a black body then it should not contain any information with it and after the black hole evaporates the information of what made up the black hole will be gone forever. The information loss problem is of a particular concern to quantum gravity; it is controversial whether information will actually be lost or if the radiation should have a modified emission that is not truly thermal. This is controversial because different physicists disagree over whether or not black holes should lose information and the most famous example of this is the Thorne-Hawking-Preskill bet. In 1997 Kip Thorne and Stephen Hawking made a public bet with John Preskill; Hawking and Thorne bet that information would be lost in a black hole and Preskill bet that information must not be lost. In 2004 Hawking publicly conceded the bet but Thorne has not conceded and the issue still remains an open problem. Because a full quantum theory of gravity does not yet exist, I will use semi-classical techniques to investigate black hole radiation and black hole thermodynamics. I will show that despite their somewhat limited nature these semi-classical techniques are surprisingly effective for calculating black hole radiation for a wide range of cases.

It is very important to understand black hole radiation because it has implications in the near future since it has been suggested that mini black holes might be created in the Large Hadron Collider (LHC) [5]. So it is important be certain that black holes will evaporate away if any black holes are actually created in the LHC (however unlikely that may be). A goal of my research will be to investigate the emission of fermions from black holes to see if any new physics can be found for fermion radiation in comparison to scalar particle radiation. I will study the semi-classical tunnelling methods for scalar particle emission and then extend these methods to model fermion emission. To the lowest order in WKB approximation, I will recover the same temperature for the emission of spin-1/2 fermions as for scalar particles. I will investigate the validity of these semi-classical techniques when applied to a wide range of horizons.

The laws of black hole thermodynamics first came about from the an analogy between the laws of black hole physics and the laws of thermodynamics. Once Hawking showed that the black hole had a well defined temperature [2] this meant that the laws of black hole physics are in fact the actual laws of black hole thermodynamics and not just an analogy.

An important property of a black hole is its surface gravity, the surface gravity is essentially a measurement of the force, as exerted at infinity, needed to keep a unit test mass at the horizon. The surface gravity is calculated by using the Killing field χ^a which is normal to the horizon of a black hole. The Killing field χ^a is defined by the equation $\chi^a = \xi^a + \Omega_H \psi^a$ where ξ^a is the stationary Killing field (i.e. ∂_t in the usual coordinates), ψ^a is the axial killing field (∂_ϕ in the usual coordinates), and Ω_H is the angular velocity of a black hole (note: the angular velocity of a locally nonrotating observer is $\Omega = \frac{dt}{d\phi} = -\frac{g_{t\phi}}{g_{\phi\phi}}$; Ω_H is simply Ω evaluated at the black hole horizon). The horizon of a stationary black hole has surface gravity κ where κ is defined by $\nabla^a(\chi^b\chi_b) = -2\kappa\chi^a$. The surface gravity has the property that $\mathcal{L}_\chi \kappa = 0$ (where $\mathcal{L}_\chi$ is the lie derivative with respect to χ^a). It is also possible to write an explicit formula for the surface gravity as $\kappa^2 = -1/2(\nabla^a\chi^b)(\nabla_a\chi_b)$. The surface gravity is constant over the horizon of a black hole. This is analogous to the zeroth law of thermodynamics which states that the temperature is constant throughout a body in thermal equilibrium. So in this sense the surface gravity is analogous to the temperature.

Another important property of black hole physics is how the mass of a black hole changes in terms of changes of its area, angular momentum and charge. These are all related by the equation:

$$dM = \frac{\kappa}{8\pi} dA + \Omega_H dJ + \Phi dQ \tag{1.1}$$

where M is the mass of the black hole, κ is the surface gravity, A is the area of the horizon, Ω_H is the angular velocity, J is the angular momentum of the black hole (note: $J = \frac{1}{16\pi}\int\limits_S \epsilon_{abcd}\nabla^c\psi^d$, where S is a 2-sphere in the asymptotic region), Q is the electric charge, and Φ is the electrostatic potential (i.e. $\Phi = \frac{Q}{r}$ for a point charge) . This is analogous to the first law of thermodynamics:

$$dE = TdS + \text{work terms} \tag{1.2}$$

where E is the energy, T is the temperature, and S is the entropy. Notice that a change in mass is a change in energy (i.e. $E = Mc^2$ and I am using $c = 1$ units so $E = M$). The terms $\Omega_H dJ + \Phi dQ$ are work terms. This implies that

$\frac{\kappa}{8\pi}dA$ is analogous to TdS. So the temperature is analogous to surface gravity and black hole area is analogous to the entropy.

Another important property of black holes is that their area cannot decrease by any (classical) process (i.e. $dA \geq 0$). This is analogous to the second law of thermodynamics which is the fact that the entropy of the universe must increase ($dS \geq 0$). So once again area is seen to be analogous to entropy.

Finally it is impossible to construct a black hole that has vanishing surface gravity by any (finite number of) physical processes. The only black holes that have zero surface gravity are extremal black holes. A Kerr-Newman black hole (black hole with rotation and electric charge) is extremal when $M^2 = a^2 + Q^2$ (where $a = J/M$ is the rotation parameter of the black hole). A normal Kerr-Newman black hole has $M^2 > a^2 + Q^2$. The closer you get to an extremal black hole (say by adding more angular momentum) the harder it becomes to move any closer [6] (i.e. it becomes even more difficult to add angular momentum). One formulation of the third law of thermodynamics states that it is impossible to achieve zero temperature in a finite number of steps. So once again the relation between surface gravity and temperature can be seen (with this formulation). Note that the analogy doesn't work for an alternate form of the third law of thermodynamics which states that entropy approaches zero as the temperature approaches zero, since extremal black holes have zero surface gravity but non-zero area.

It was Bekenstein that first claimed that these similarities were more than an analogy [1]. He claimed that $TdS = \frac{\kappa}{8\pi}dA$, so that the temperature of the black hole was proportional to the surface gravity and that the entropy was proportional to the area. This was later shown by Hawking [2] who calculated the temperature of a black hole to be explicitly

$$T_H = \frac{\kappa}{2\pi} \tag{1.3}$$

So from this, the entropy of the black hole can be inferred to be $S = \frac{A}{4}$. So it was possible to conclude that the laws of black hole physics are in reality the laws of black hole thermodynamics.

There are now several methods used for deriving Hawking radiation [2]-[59] and calculating the black hole temperature. The original method considered the creation of a black hole in the context of a collapse geometry, calculating the Bogoliubov transformations between the initial and final states of incoming and outgoing radiation [2],[9]. The more popular method of analytic continuation to a Euclidean section (the Wick Rotation method) emerged soon after [3]. Relying on the methods of finite-temperature quantum field theory, an analytic continuation $t \to i\tau$ of the black hole metric is performed and the periodicity of τ (denoted by β) is chosen in order to remove a conical singularity that would otherwise be present at fixed points of the $U(1)$ isometry generated by $\partial/\partial\tau$ (the event horizon in the original Lorentzian section). The black hole is then considered to be in equilibrium with a scalar field that has inverse temperature β at infinity. Recently, some other methods of calculating black hole temperature have been created such as the black hole tunnelling methods [10]-[58] and the anomaly method [59].

The tunnelling method is a particularly interesting method for calculating black hole temperature since it provides a dynamical model of the black hole radiation. My goal is to study this tunnelling method in detail. I plan to extend the method beyond its initial application to Schwarzschild black holes. I also plan to broaden the method beyond modelling scalar particles. In this thesis I will demonstrate the robustness of the tunnelling method by showing that it can be applied to a broad range of spacetimes and can be extended to model fermion emission. In the original calculations the tunnelling method was only applied to a Schwarzschild black hole [10]-[12]. Due to the semi-classical nature of the model, it was not expected to be as powerful as it has turned out to be. Therefore the robustness of the method needs to be checked explicitly to verify that the method is applicable to a broad range of spacetimes. I will also test the validity of extending the method to the tunnelling of Dirac particles.

In the 1990's, a semi-classical method of modeling Hawking radiation as a tunnelling effect was proposed [10] and has garnered a lot of interest [10]-[58]. This method involves calculating the imaginary part of the action for the (classically forbidden) process of s-wave emission across the horizon (first considered by Kraus and Wilczek [10]-[12]), which in turn is related to the Boltzmann factor for emission at the Hawking temperature. Using the WKB approximation the tunnelling probability for the classically forbidden trajectory of the s-wave coming from inside to outside the horizon is given by:

$$\Gamma \propto \exp(-2\mathrm{Im}I) \tag{1.4}$$

where I is the classical action of the trajectory to leading order in $\hbar$ (here set equal to unity). Expanding the action in terms of the particle energy, the Hawking temperature is recovered at linear order. In other words for $2I = \beta E + O(E^2)$ this gives

$$\Gamma \sim \exp(-2I) \simeq \exp(-\beta E) \tag{1.5}$$

which is the regular Boltzmann factor for a particle of energy E where β is the inverse temperature of the horizon. The higher order terms are a self-interaction effect resulting from energy conservation [11],[14]; however, for calculating the temperature, expansion to linear order is all that is required. There are two different approaches that are used to calculate the imaginary part of the action for the emitted particle. The first black hole tunnelling method developed was the null geodesic method used by Parikh and Wilczek [14], which followed from the work of Kraus and Wilczek [10]-[12]. The other approach to black hole tunnelling is the Hamilton-Jacobi Ansatz used by Angheben et al, which is an extension of the complex path analysis of Padmanabhan et al [19]-[22].

The null geodesic method considers a null s-wave emitted from the black hole. Based on the previous work analyzing the full action in detail [10]-[12], the only part of the action that contributes an imaginary term is $\int_{r_{in}}^{r_{out}} p_r dr$, where $p_r$is the momentum of the emitted null s-wave. Then by using Hamilton's equation and knowledge of the null geodesics it is possible to calculate the imaginary part of the action.

The Hamilton-Jacobi ansatz involves consideration of an emitted scalar particle, ignoring its self-gravitation, and assumes that its action satisfies the relativistic Hamilton-Jacobi equation. From the symmetries of the metric, one picks an appropriate ansatz for the form of the action. This method is motivated by applying the WKB approximation to the Klein-Gordon equation. This method can be extended to other types of particles (i.e. other than scalar particles) by applying the WKB approximation to other wave equations such as the Dirac equation to model spin-1/2 fermions.

The black hole tunnelling method has a lot of strengths when compared to other methods for calculating the temperature. The calculations are straightforward and relatively simple. The tunnelling method is robust in the sense that it can be applied to a wide variety of exotic spacetimes; I will demonstrate this property as a major theme of my thesis. Related research has shown that it can be successfully applied to spacetimes such as Kerr and Kerr-Newman cases [30, 32, 37], black rings [35], the 3-dimensional BTZ black hole [28, 36], Vaidya [43], other dynamical black holes [47], Taub-NUT spacetimes [37], and Gödel spacetimes [48]. The tunnelling method can even been applied to horizons that are not black hole horizons, such as Rindler Spacetimes [19],[37] and it has been shown that the Unruh temperature [7] is in fact recovered. The tunnelling method can even be applied to the cosmological horizons of de Sitter spacetimes [16],[24],[25],[28], [38],[51],[58]. The applications to de Sitter spacetimes demonstrate a particular advantage of the tunnelling method over the Wick rotation method. This is because the Wick rotation method cannot be applied when a Schwarzschild black hole is embedded in a de Sitter spacetime but the tunnelling method can be applied. I will review these results as part of my thesis. Another strength of the tunnelling method is that it can be extended beyond the emission of scalar particles and can model particles that have spin [50],[54]-[58]. I was the first to extend the tunnelling method to model Dirac particles [50] and I will demonstrate this as part of this thesis. Finally the tunnelling method is important because it gives an intuitive picture of black hole radiation. An s-wave particle follows a trajectory from the inside of the black hole to the outside, a classically forbidden process. Because of energy conservation, the radius of the black hole shrinks as a function of the energy of the outgoing particle; in this sense the particle creates its own tunnelling barrier. This also provides a dynamical model of black hole radiation since the mass of the black hole decreases (albeit slow dynamics since the mass cannot be changing rapidly for this model).

In the second chapter of this thesis, I will review key properties of the tunnelling method. Since this is a review chapter, it also contains a review of work done by others in addition to extensions of the tunnelling methods that I have done. The later chapters will consist entirely of original calculations that I have done in the papers [37],[48],[50],[57]. I will start by comparing the two approaches used to calculate the black hole temperature (which are the null geodesic method and the Hamilton-Jacobi ansatz) and this is a recap of the comparison of the methods that I did in [37]. I will review the extension of these methods to model charged particle emission [31],[45]. These charged particle results will be useful when I discuss extremal black holes in the third chapter and when I extend fermion tunnelling to include electric charge in the second last chapter. I will also review the application of the tunnelling approaches to cosmological horizons [16],[24],[25],[28],[38],[51],[58] and describe some of the controversy surrounding tunnelling from de Sitter spacetimes. This has implications for the universe since measurements of type Ia supernovae [60] and measurements from the Wilkinson Microwave Anisotropy Probe (WMAP) [61] suggest that the universe has a positive cosmological constant. The de Sitter spacetimes will also provide a useful point of comparison when I discuss the Kerr-Gödel spacetime in a later chapter. I will finish the chapter by outlining the factor of 2 issue that can occur with the tunnelling calculation.

In the third chapter, I will examine the tunnelling methods in the context of a broader class of spacetimes. One of the prime motivations is to understand the applicability of the method to other spacetimes such as Rindler, Kerr-Newman, Taub-NUT-AdS and extremal black holes. The Rindler metric is interesting because it demonstrates that the tunnelling method works with other types of horizons. An early attempt to apply a tunnelling method to Rindler space was carried out by Padmanabhan [19] using the Hamilton-Jacobi approach and later I independently arrived at the same result by applying the null geodesic approach to Rindler space [37]. It is useful to extend the method to the Kerr-Newman metric because this spacetime no longer has spherical symmetry. This gives the calculation some fundamentally new properties beyond a general spherically symmetric black hole. It becomes necessary to transform to a corotating metric so that the ergosphere can be safely ignored. Applying tunnelling to the Kerr-Newman spacetime demonstrates how to apply the method to rotating spacetimes in general. Since spherical symmetry is lost for a rotating black hole it becomes necessary to break the emission up into emitted rings instead of a full s-wave. It is also found that the angular momentum of the emitted particle is very important in the rotating case since there is a term in the tunnelling probability that depends on the angular momentum of the emitted wave and the angular velocity of the black hole. The Taub-NUT metric is a generalization of the Schwarzschild metric and has played an important role in the conceptual development of general relativity and in the construction of brane solutions in string theory and M-theory. [62]. They have interesting thermodynamic properties; their entropy is not proportional to the area of the event horizon and their free energy can sometimes be negative [63],[64],[65],[66]. I will obtain a general expression for the temperature for a subclass of Taub-NUT spacetimes without closed timelike curves (CTCs) that can be compared to those obtained via Wick rotation methods. I will show agreement in all relevant cases. I will finish the third chapter with a discussion of issues that occur when applying the method to extremal black holes, concentrating on the specific case of the extremal Reissner-Nordström spacetime.

In the fourth chapter I will investigate the application of the tunnelling method to Gödel black holes [67]-[80], specifically I will apply the method to

Schwarzschild-Gödel and Kerr-Gödel black holes [70]. In general, a Gödel-type solution can be described as spacetime for which the spacetime itself has rotation. Gödel spacetimes also have the property that they contain closed timelike curves. Various black holes embedded in Gödel universe backgrounds have been obtained as exact solutions [68, 70, 76]. The string-theoretic implications of these spacetimes make them a lively subject of interest; in particular, since closed-timelike curves (CTCs) exist in Gödel spacetimes, these solutions have be used to investigate the implications of CTCs for string theory [72, 73, 75, 77]. It has also been shown that Gödel type solutions are T-dual to pp-waves [69]-[71]. The black hole solutions I will use are of the Schwarzschild-Kerr type embedded in a Gödel universe [70]. A study of their thermodynamic behaviour [78, 80] has indicated that the expected relations of black hole thermodynamics are satisfied. Making use of standard Wick-rotation methods, their temperature has been shown to equal $\kappa/2\pi$ (where κ is the surface gravity) their entropy to equal $A/4$ (where A is the surface area of the black hole) and the first law of thermodynamics has been shown to be satisfied. I will analyze the thermodynamic properties of the Kerr-Gödel spacetime with the tunnelling method. The presence of CTCs merits consideration of the applicability of the tunnelling method to Kerr-Gödel spacetimes. Due to the presence of a CTC "horizon" (in addition to the usual black-hole horizons) some qualitatively new features appear. My investigation of these spacetimes is in large part motivated by the fact that these new features provide additional tests as to the robustness of the tunnelling approach. This shows that the tunnelling method can work in higher dimensions, since Kerr-Gödel is 5D. It also provides another type of "horizon" that can be tested due to the presence of the CTC "horizon". In this case it is found that no tunnelling occurs across the CTC "horizon". I will begin the chapter by reviewing the Kerr-Gödel spacetime and some of its properties. I will then describe properties of its parameter space and show that either the CTC horizon is outside both black hole horizons, inside both black hole horizons, or in coincidence with one of the horizons. I claim that it is not possible for the CTC horizon to be strictly in between the two black hole horizons, a property previously overlooked in discussions of this spacetime [80]. I will extend the investigation further insofar as I will include a brief discussion of the issues that occur when the CTC horizon is inside the black hole horizons. I will finish the chapter by applying the tunnelling method to calculate the temperature of Kerr-Gödel spacetimes, showing consistency with previous results.

In the fifth chapter I will extend the tunnelling method to model the emission of spin-1/2 fermions. For fermion emission it becomes necessary to consider the effect that spin will have on the black hole. Due to the fact that there are statistically as many particles with the spin in one direction as in the other, the effect of the spin of each type of fermion will cancel themselves out. So to the lowest WKB order of approximation the rotation of the black hole will not change. (There may still be the possibility that higher order correction may cause an imbalance between spin emission in which case the rotation of the black hole might change). Since a black hole has a well defined temperature it should radiate all types of particles like a black body at that temperature (ignoring grey body effects). The emission spectrum therefore is expected to contain particles of all spins; the implications of this expectation were studied 30 years ago [8]. The first application of the tunnelling methods only involved scalar particles. In fact comparatively little has been done for fermion radiation for black holes. The Hawking temperature for fermion radiation has been calculated for 2d black holes [81] using the Bogoliubov transformation and more recently was calculated for evaporating black holes using a technique called the generalized tortoise coordinate transformation (GTCT) [82]-[84]. The latter result [84] is interesting because there is a contribution to the fermion emission probability due to a coupling effect between the spin of the emitted fermion and the acceleration of the Kinnersley black hole (note, a Kinnersley black hole is a spacetime which contains a black hole that is accelerating). From this one may infer that when fermions are emitted from rotating black holes there might be a coupling between the spin of the fermion and angular momentum of the rotating black hole present in the tunnelling probability. Unfortunately the tunnelling method to lowest order of WKB will not recover such an effect. This is not to say that such an effect does not exist, just that the lowest order of approximation is not able to calculate it. This is an interesting topic for further research In order to model fermion emission, I will follow an analogous approach to the original approach used by Padmanabhan et al [19]. The Hamilton-Jacobi ansatz emerged from an application of the WKB approximation to the Klein-Gordon equation. I will apply a WKB approximation to the Dirac Equation to model Dirac particle emission. I will start with the Rindler spacetime first and confirm that the Unruh temperature is recovered. Insofar as fermionic vacua are distinct from bosonic vacua and can lead to distinct physical results [85], this result is non-trivial. I will then extend this technique to a general non rotating 4-D black hole metric and show that the Hawking temperature is recovered. I will illustrate this result in several coordinate systems (Schwarzschild, Painlevé, and Kruskal) to demonstrate that the result is independent of this choice. One of the assumptions of this semi-classical calculation will be to neglect any change of angular momentum of the black hole due to the spin of the emitted particle. For zero angular momentum black holes, with mass much larger than the Planck mass, this is a good approximation. Furthermore, statistically particles of opposite spin will be emitted in equal numbers, yielding no net change in the angular momentum of the black hole (although higher-order statistical fluctuations could be present). I will finish the chapter by extending the tunnelling method to model charged spin 1/2 particle emission from rotating black holes. To this end I will apply the fermion tunnelling method to the Kerr-Newman black hole for both massless and massive charged particle emission. This extension introduces some non-trivial technical features associated with the choice of γ matrices. So in the end I will confirm that spin 1/2 fermions are emitted at the expected Hawking Temperature from a variety of black holes, providing further evidence for the universality of black hole radiation.

2 Review of Tunnelling Method

In this chapter I will review the basic calculations for black hole tunnelling and give an overview of some of the conceptual issues related to the calculations. In the first section of this chapter I will compare the two approaches used to model black hole tunnelling. I will start by reviewing the null geodesic method [14], which I will follow by reviewing the Hamilton-Jacobi ansatz [28]. The calculations will model the tunnelling of uncharged scalar particles from general non-rotating black holes. In the following section of this chapter, I will review the calculations for the emission of charged scalar particles from Reissner-Nordström black holes [12],[29],[31],[32],[34],[45]. I am reviewing charged particle emission because the results will prove useful when I discuss extremal black holes at the end of the next chapter. The results for charged scalar particle emission will also provide a useful framework for setting up charged fermion emission in the second last chapter. I will then demonstrate that the tunnelling method can also be applied to cosmological horizons by reviewing the calculations that have been done for both de Sitter spacetime and Schwarzschild-dS. In this section I will also discuss the controversy surrounding these calculations for particles tunnelling from the cosmological horizon and the resulting sign ambiguity. In the final section of the chapter I will briefly discuss the factor of 2 issue of the tunnelling method.

2.1 Null Geodesic method compared to Hamilton-Jacobi Ansatz

2.1.1 Null Geodesic Method

I will begin by reviewing the null geodesic method used by Parikh and Wilczek [14] that followed from the work of Kraus and Wilczek [10],[11],[12]. The basic idea behind this approach is to regard Hawking radiation as a quantum tunnelling process. However, unlike other tunnelling processes in which two separated classical turning points are joined by a trajectory in imaginary time, the tunnelling barrier is created by the outgoing particle itself, whose trajectory is from the inside of the black hole to the outside, a classically forbidden process. The probability of tunnelling is proportional to the exponential of (negative) two times the imaginary part of the classical action for this tunnelling process in the WKB limit. Because of energy conservation, the radius of the black hole shrinks as a function of the energy of the outgoing particle. Since the horizon shrinks in response to the motion of the particle, in this sense the particle creates its own tunnelling barrier.

Applying the WKB approximation to the Schrödinger equation will give a wave equation of the form $\phi \propto \exp(\frac{iI}{\hbar})$, where I is solved along the classically forbidden trajectory (therefore I will be complex as a result). By multiplying the wave function by its complex conjugate the result is a semi-classical tunnelling probability for the emitted particle of the form:

$$\Gamma \propto \exp(-2\text{Im}I) \tag{2.1}$$

where $\hbar$ has been set equal to unity. The Hamilton-Jacobi ansatz (which will be discussed in the next subsection) also uses this as a starting point of its calculation. The Hamilton-Jacobi method applies the WKB approximation to the Klein-Gordon equation instead of the Schrödinger equation. So these two methods end up differing in how the action is calculated.

For the null geodesic method the only part of the action that contributes an imaginary term to the final tunnelling probability is $\int_{r_{in}}^{r_{out}} p_r dr$, where p_r is the (radial) momentum of the emitted null s-wave. Other contributions to the action I will in general be terms of the form $-\displaystyle\int Edt$, $\displaystyle\int p_\phi d\phi$, and $\displaystyle\int p_\theta d\theta$ (This is known because of Hamilton's principle) and will be ignored since they do not contribute to the final tunnelling rate. For a stationary spacetime, it can be seen that the energy integral will simply correspond to $-Et$ which is entirely real and does not contribute to the tunnelling probability (2.1). The angular terms will also be real and therefore do not contribute. It is also possible to simply ignore any effects of the angular terms by assuming that the emitted s-wave is only moving radially, in which case the angular terms will automatically be zero. It should be noted that Kraus and Wilczek solved the most general action for the full system of the shell and the background completely in their papers [10],[11],[12] which provides a more explicit proof that only $\int_{r_{in}}^{r_{out}} p_r dr$ contributes to the tunnelling rate as claimed. See the appendix for a recap of this derivation of the action from [10],[11],[12] (this also shows the importance of using the Painlevé form of the metric).

I will now demonstrate a generalization of the null geodesic calculation for a general non-rotating black hole. (The original Parikh and Wilczek paper [14] only applied the method specifically to Schwarzschild and Reissner-Nordström black holes). A general static spherically metric can be written in the form:

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{g(r)} + r^2 d\Omega^2 \tag{2.2}$$

which covers a broad range of black hole metrics. For this generic black hole metric both $f(r)$ and $g(r)$ vanish at the black hole horizon r_0 (i.e. $f(r_0) = g(r_0) = 0$). I am also assuming that the black hole is non-extremal which means that the two functions $f(r)$ and $g(r)$ only have first order zeros at the horizon. In other words the first derivatives of these functions exist at the horizon and are non-zero (i.e. $f'(r_0) \neq 0, g'(r_0) \neq 0$).

The first step for the null geodesic method is to convert the metric into Painlevé form [86] so that there will no longer be a singularity at the horizon. This is easily accomplished via the transformation:

$$t \to t - \int \sqrt{\frac{1-g\,(r)}{f\,(r)\,g\,(r)}} dr \tag{2.3a}$$

yielding

$$ds^2 = -f(r)dt^2 + 2\sqrt{f(r)}\sqrt{\frac{1}{g(r)} - 1}drdt + dr^2 + r^2 d\Omega^2 \tag{2.4}$$

Using the Painlevé form of the metric is a vital part of the null geodesic calculation. This coordinate system also has a number of interesting features beyond removing the singularity at the horizon. A Painlevé metric has the properties that at any fixed time the spatial geometry is flat and at any fixed radius the boundary geometry is the same as that of the of the unaltered metric (2.2).

The radial null geodesics for this metric correspond to:

$$\dot{r} = \sqrt{\frac{f(r)}{g(r)}}\left(\pm 1 - \sqrt{1-g(r)}\right) \tag{2.5}$$

where the plus/minus signs correspond to outgoing/ingoing null geodesics.

In the spherically symmetric case, the emitted particle is taken to be in an outgoing s-wave mode and this corresponds to the plus sign in (2.5). At the horizon, $\dot{r}=0$ since $\frac{f(r)}{g(r)}$ is well defined there (which follows from the fact that $f'$and g' are both non zero at the horizon). The imaginary part of the action for an outgoing s-wave from r_{in} to r_{out} is expressed as

$$I = \int_{r_{in}}^{r_{out}} p_r dr = \int_{r_{in}}^{r_{out}} \int_0^{p_r} dp_r' dr \tag{2.6}$$

where r_{in} and r_{out} are the respective initial and final radii of the black hole. The trajectory between these two radii is the barrier the particle must tunnel through.

I will assume that the emitted s-wave has energy $\omega' << M$ and that the total energy of the space-time was originally M. Invoking conservation of energy to this approximation, the s-wave moves in a background spacetime of energy $M \to M-\omega'$. In order to evaluate the integral, I employ Hamilton's equation $\dot{r} = \frac{dH}{dp_r}|_r$ to switch the integration variable from momentum to energy ($dp_r = \frac{dH}{\dot{r}}$), giving

$$I = \int_{r_{in}}^{r_{out}} \int_M^{M-\omega} \frac{dr}{\dot{r}} dH = \int_0^{\omega} \int_{r_{in}}^{r_{out}} \frac{dr}{\dot{r}}(-d\omega') \tag{2.7}$$

where $dH = -d\omega'$ because total energy $H = M-\omega'$ with M constant. Note that $\dot{r}$ is implicitly a function of $M-\omega'$. For the special cases where this function is known (e.g. Schwarzschild) the integral in (2.7) can be solved exactly in terms of ω [14]. I will review this result explicitly for the Schwarzschild black hole and then return to solving the general case. For a Schwarzschild black hole $f(r) = g(r) = (1-\frac{2M}{r})$ and the radial geodesic when the black hole mass is $M-\omega'$ (i.e. when background spacetime is reduced in mass by ω') is:

$$\dot{r} = \left(1 - \sqrt{\frac{2(M-\omega')}{r}}\right)$$

$$\therefore I = \int_0^{\omega} \int_{r_{in}}^{r_{out}} \frac{dr}{1-\sqrt{\frac{2(M-\omega')}{r}}}(-d\omega')$$

$$\Longrightarrow \operatorname{Im} I = \operatorname{Im} \int_0^{\omega} +4\pi i(M - \omega')d\omega'$$
$$\operatorname{Im} I = +4\pi\omega(M - \frac{\omega}{2}) \tag{2.8}$$

The sign is positive because $r_{in} > r_{out}$ since the black hole horizon before emission is located at $r_{in} = 2M$ and the black hole horizon after emission is $r_{out} = 2(M - \omega)$. This was demonstrated by Parikh and Wilczek in their paper by switching the order of integration [14]. Plugging this into the expression for the semi-classical emission rate (2.1) gives:

$$\Gamma \sim \exp(-2 \operatorname{Im} I) = \exp(-8\pi\omega(M - \frac{\omega}{2}) \tag{2.9}$$
$$= \exp(+\Delta S_{BH}) \tag{2.10}$$

where ΔS_{BH} is the change in the black hole's Bekenstein-Hawking entropy S_{BH}. When only the lowest order of ω is considered, the expression reduces to $\exp(-8\pi M\omega)$ which is the same as the Boltzmann factor (i.e. $\exp(-\frac{E}{k_b T})$) for a particle of energy ω at the Hawking Temperature $T_H = \frac{1}{8\pi M}$. (Note I am using the unit convention that $k_b = 1$).

A generalization of the null geodesic method is to consider spacetimes with a well defined ADM mass but where it may not be possible to solve the above integral exactly. Even in the most general case it is still possible to obtain self gravitation effects as a perturbative expansion of ω (this was first considered in [26]). From this point on, I will ignore the higher order terms of ω and just concern myself with solving the expression to the lowest order. It should also be noted that there has been some question to the validity of the higher order terms of the tunnelling rate since it has been claimed that the semi-classical tunnelling probability is not invariant under canonical transformations in general [39]. It should be noted that when the only the lowest order of ω is used, the resulting Boltzmann factor is invariant under such canonical transformations.

I will now return to the general expression for the action (2.7); in general it is always possible to perform a series expansion in ω in order to find the temperature. To first order this gives:

$$I = \int_0^{\omega} \int_{r_{in}}^{r_{out}} \frac{dr}{\dot{r}(r, M - \omega')}(-d\omega') = -\omega \int_{r_{in}}^{r_{out}} \frac{dr}{\dot{r}(r, M)} + O(\omega^2)$$
$$\simeq \omega \int_{r_{out}}^{r_{in}} \frac{dr}{\dot{r}(r, M)} \tag{2.11}$$

To proceed any further, this integral needs to be estimated. First notice that $r_{in} > r_{out}$ because the black hole decreases in mass as the s-wave is emitted; consequently the radius of the event horizon decreases. Therefore, I will write $r_{in} = r_0(M) - \epsilon$ and $r_{out} = r_0(M - \omega) + \epsilon$ where $r_0(M)$ denotes the location of the event horizon of the original background space-time before the emission of particles. Henceforth the notation r_0 will be used to denote $r_0(M)$. Note that with this generalization, no explicit knowledge of the total energy or mass is required since r_0 is simply the radius of the event horizon before any particles are emitted.

There is a pole at the horizon where $\dot{r} = 0$. Remember for a non-extremal black hole that $f'(r_0)$ and $g'(r_0)$ are both non-zero at the horizon; so $\frac{1}{\dot{r}}$ only has a simple pole at the horizon with a residue of $\frac{2}{\sqrt{f'(r_0)g'(r_0)}}$. Hence the imaginary part of the action will be

$$\operatorname{Im} I = \frac{2\pi\omega}{\sqrt{f'(r_0)g'(r_0)}} + O(\omega^2) \tag{2.12}$$

Therefore the tunnelling probability is

$$\Gamma = \exp\left(-2\mathrm{Im}I\right) = \exp\left(-\beta\omega\right) \tag{2.13}$$

and so the resulting Hawking temperature $T_H = \beta^{-1}$is

$$T_H = \frac{\sqrt{f'(r_0)g'(r_0)}}{4\pi} \tag{2.14}$$

It is easy to see that for Schwarzschild black hole the correct result of $T_H = \frac{1}{8\pi M}$ follows once again. Situations in which the horizons do not have a simple pole correspond to extremal black holes and need to be handled separately. In fact extremal black holes push the limits of the tunnelling method. The tunnelling method itself may not even be valid for extremal black holes since there are multiple conceptual issues involved. One such conceptual issue that arises when applying tunnelling methods to the extremal black holes involves the fact that the tunnelling model is dynamic; so emission of a neutral particle from the black hole implies a naked singularity, in violation of cosmic censorship. This will be discussed in the more detail in a later chapter.

2.1.2 Hamilton-Jacobi Ansatz

I will now review an alternate method for calculating black hole tunnelling that makes use of the Hamilton-Jacobi equation as an ansatz [28]. This method ignores the effects of the particle self-gravitation and is based on the work of Padmanabhan and his collaborators [19],[20],[21]. In general the method involves using the WKB approximation to solve a wave equation. The simplest case to model is scalar particles, which therefore involves applying the WKB approximation to the Klein-Gordon equation. The result, to the lowest order of WKB approximation, is a differential equation that can be solved by plugging in a suitable ansatz. The ansatz is chosen by using the symmetries of the spacetime to assume separability. After plugging in a suitable ansatz, the resulting equation can be solved by integrating along the classically forbidden trajectory, which starts inside the horizon and finishes at the outside observer (usually at infinity). Since this trajectory is classically forbidden the equation will have a simple pole located at the horizon. So it is necessary to apply the method of complex path analysis and deflect the path around the pole. Since

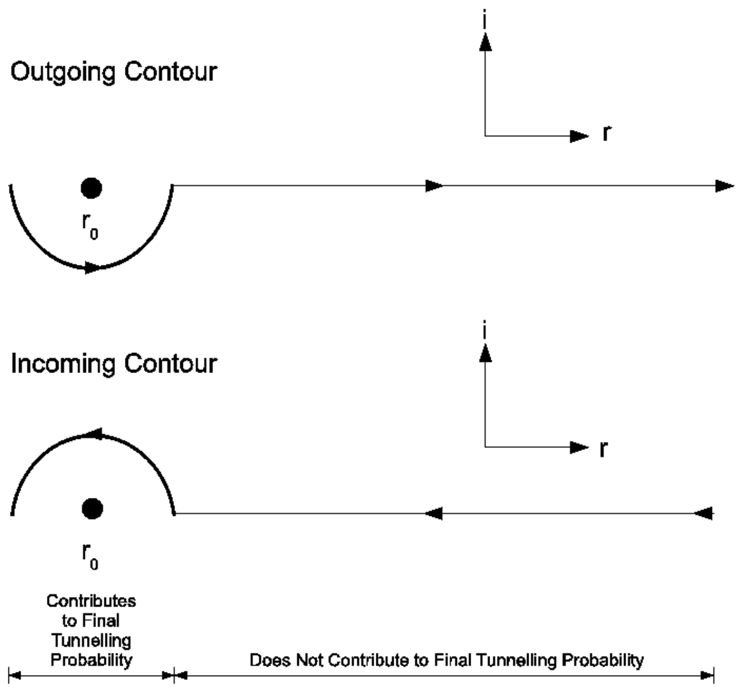

Figure 1: Diagram of contours between black hole and observer for both outgoing and incoming trajectories

I am only concerned with calculating the semi-classical tunnelling probability, I will need to multiply the resulting wave equation by its complex conjugate. So the portion of the trajectory that starts outside the black hole and continues to the observer will not contribute to the final tunnelling probability and can be safely ignored. Therefore, the only part of the wave equation that contributes to the tunnelling probability is the contour around the black hole horizon (for a visual representation of the deformation of the contour see Figure 1).

I will consider a general (non-extremal) black hole metric of the form

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{g(r)} + C(r)h_{ij}dx^i dx^j \tag{2.15}$$

The Klein Gordon equation for a scalar field ϕ is:

$$g^{\mu\nu}\partial_\mu\partial_\nu\phi - \frac{m^2}{\hbar^2}\phi = 0 \tag{2.16}$$

Applying the WKB approximation by assuming an ansatz of the form

$$\phi(t, r, x^i) = \exp[\frac{i}{\hbar}I(t, r, x^i) + I_1(t, r, x^i) + O(\hbar)]$$

and then inserting this back into the Klein Gordon equation will result in the Hamilton-Jacobi equation to the lowest order in $\hbar$:

$$-\left[g^{\mu\nu}\partial_\mu I \partial_\nu I + m^2\right] + O(\hbar) = 0 \tag{2.17}$$

(obtained after dividing by the exponential term and multiplying by $\hbar^2$).

Also notice that:

$$\Gamma \propto |\phi|^2 = \exp(-\frac{2\,\mathrm{Im}\,I}{\hbar}) \tag{2.18}$$

For the Hamilton-Jacobi ansatz it is common [28] to skip these early steps and simply start a calculation by assuming that the classically forbidden trajectory from inside to outside the horizon is given by:

$$\Gamma \propto \exp(-2\mathrm{Im}I) \tag{2.19}$$

where $\hbar$ has been set to unity and the classical action I satisfies the relativistic Hamilton-Jacobi equation

$$g^{\mu\nu}\partial_\mu I \partial_\nu I + m^2 = 0 \tag{2.20}$$

For the black hole metric the Hamilton-Jacobi equation is explicitly

$$-\frac{(\partial_t I)^2}{f(r)} + g(r)(\partial_r I)^2 + \frac{h^{ij}}{C(r)}\partial_i I \partial_j I + m^2 = 0 \tag{2.21}$$

There exists a solution of the form

$$I = -Et + W(r) + J(x^i) + K \tag{2.22}$$

where

$$\partial_t I = -E, \qquad \partial_r I = W'(r), \qquad \partial_i I = J_i$$

and K and the J_i's are constant (K can be complex). Since ∂_t is the timelike killing vector for this coordinate system, E is the energy of the particle as detected by an observer at infinity. This is because at infinity the norm of the timelike killing vector ∂_t is (minus) unity. Solving for $W(r)$ yields

$$W_\pm(r) = \pm\int \frac{dr}{\sqrt{f(r)g(r)}}\sqrt{E^2 - f(r)(m^2 + \frac{h^{ij}J_iJ_j}{C(r)})} \tag{2.23}$$

since the equation was quadratic in terms of $W(r)$. One solution corresponds to scalar particles moving away from the black hole (i.e. + outgoing) and the other solution corresponds to particles moving toward the black hole (i.e. $-$ incoming). Imaginary parts of the action can only come about due to the pole at the horizon or from the imaginary part of K. The probabilities of crossing the horizon each way are proportional to

$$\mathrm{Prob}[out] \propto \exp[-\frac{2}{\hbar}\,\mathrm{Im}\,I] = \exp[-\frac{2}{\hbar}(\mathrm{Im}\,W_+ + \mathrm{Im}\,K)] \tag{2.24}$$

$$\mathrm{Prob}[in] \propto \exp[-\frac{2}{\hbar}\,\mathrm{Im}\,I] = \exp[-\frac{2}{\hbar}(\mathrm{Im}\,W_- + \mathrm{Im}\,K)] \tag{2.25}$$

To ensure that the probability is normalized, so that any incoming particles crossing the horizon have a 100% chance of entering the black hole, it is necessary to set $\operatorname{Im} K = -\operatorname{Im} W_-$ and since $W_+ = -W_-$ this implies that the probability of a particle tunnelling from inside to outside the horizon is:

$$\Gamma \propto \exp[-\frac{4}{\hbar} \operatorname{Im} W_+] \tag{2.26}$$

It is also possible to start with an ansatz for the action that does not contain the constant K. In such a case it is necessary to take a ratio of (2.24) and (2.25) to get the correct tunnelling rate (2.26).

Henceforth I will set $\hbar$ to unity and also drop the "+" subscript from W. Integrating around the pole at the horizon leads to the result

$$W = \frac{\pi i E}{\sqrt{g'(r_0) f'(r_0)}} \tag{2.27}$$

where the imaginary part of W is now manifest. This leads to a tunnelling probability of:

$$\Gamma = \exp[-\frac{4\pi}{\sqrt{f'(r_0)g'(r_0)}} E] \tag{2.28}$$

and implies the usual Hawking temperature of:

$$T_H = \frac{\sqrt{f'(r_0)g'(r_0)}}{4\pi} \tag{2.29}$$

It is also possible to get the correct Hawking temperature by parameterizing the outgoing probability in terms of the proper radial distance and ignoring the incoming probability. This was shown in [28] and I also used this method in order to get the correct Hawking Temperature in [37]. I have reproduced the calculation in the appendix. It is conceptually stronger to either normalize the incoming probability or to take the ratio of Prob[*out*] over Prob[*in*]. This is the way that Padmanabhan originally calculated the tunnelling rate [19] and using this technique is what Mitra [44] proposed as a possible solution to the factor of 2 issue. (Note: the factor of 2 issue will be discussed briefly at the end of this chapter).

2.1.3 Hamilton-Jacobi Ansatz applied to Kruskal-Szekeres Metric

By applying the Hamilton-Jacobi method to spacetimes that do not have coordinate singularities at the horizon, the validity of the method can be checked for these cases. It has be shown that the correct Hawking temperature is recovered when the Hamilton-Jacobi tunnelling method is applied to a wide range of transformed Schwarzschild spacetimes In a paper of Padmanabhan's, he showed that the method could be applied to Painlevé and Lemaître coordinates [21]. Mitra also applied the Hamilton-Jacobi tunnelling method to wide range of coordinates such as the Eddington-Finkelstein coordinates, Painlevé coordinates and the Kruskal spacetime [46]. A non-singular spacetime that the Hamilton-Jacobi method has not specifically been applied to is the Kruskal-Szekeres metric. I modelled fermion tunnelling of spin-1/2 particles for Kruskal-Szekeres metric in [50]; before I tried Dirac particle calculation, I had modelled scalar particle emission in the Kruskal-Szekeres metric as a preliminary exercise. The scalar particle calculation was not included in [50], although it could be seen as a simplification of the Dirac particle calculation. I will apply the Hamilton-Jacobi method to Kruskal-Szekeres metric to explicitly demonstrate the fact that the Hamilton-Jacobi method works when it is applied to metrics that do not have any coordinate singularities at the horizon. This calculation will be useful guide for the tunnelling calculation of Rindler observers in Minkowski space that will be done in the next chapter. The calculation will also useful when I model the tunnelling of Dirac particles from a Kruskal-Szekeres metric in the last chapter before the conclusions. The Kruskal-Szekeres metric is:

$$ds^2 = f(r)\left(-dT^2 + dX^2\right) + r^2 d\Omega^2 \tag{2.30}$$

where:

$$f(r) = \frac{32M^3 e^{-\frac{r}{2M}}}{r} \qquad\qquad (\frac{r}{2M} - 1)e^{r/2M} = X^2 - T^2 \tag{2.31}$$

The metric (2.30) is well behaved at both the future and past horizons $X = \pm T$ (corresponding to $r = 2M$). Note that the metric has a timelike Killing vector $X\partial_T + T\partial_X$ (and not ∂_T since ∂_T is only timelike and not a Killing vector for this metric).

$$I = I(X,T) + J(\theta,\phi) \tag{2.32}$$

For convenience, assume the simplest case $J = 0$ and $m = 0$. The calculation is more straightforward in this case. Plugging this action into the Hamilton-Jacobi equation yields:

$$\frac{1}{f(r)}\left(-(\partial_T I)^2 + (\partial_X I)^2\right) = 0$$

$$\implies I = h(X-T) \text{ or } I = f(X+T) \tag{2.33}$$

In order to solve the equations, a definition for the energy of the wave (as detected by the observer) is required. I will define the energy by using the timelike killing vector

$$\partial_\chi = N(X\partial_T + T\partial_X)$$

where N is a normalization constant chosen so that the norm of the Killing vector is equal to -1 at infinity. This yields:

$$\partial_\chi = \frac{1}{4M}(X\partial_T + T\partial_X) \tag{2.34}$$

and therefore the energy is defined by:

$$\partial_\chi I = -E \tag{2.35}$$

Using (2.35) with the two solutions (2.33) it is now possible to solve for I in each case. For the outgoing solution (i.e. $h(X-T)$):

$$\frac{1}{4M}(X\partial_T I + T\partial_X I) = -E$$

$$\begin{aligned} 4ME &= (X-T)h'(X-T) \\ h'(X-T) &= \frac{4ME}{(X-T)} \end{aligned}$$

which has a simple pole at the black hole horizon $X = T$. Setting $\eta = X - T$ gives the expression

$$h'(\eta) = \frac{4ME}{\eta} \tag{2.36}$$

Integrating (2.36) around the pole at the horizon (doing a half circle contour) implies

$$\operatorname{Im} I_{out} = 4\pi ME$$

For the outgoing solution:

$$\frac{1}{4M}(X\partial_T I + T\partial_X I) = -E$$

$$\begin{aligned} -4ME &= (X+T)k'(X+T) \\ k'(X+T) &= \frac{-4ME}{(X+T)} \end{aligned}$$

Notice that this equation does not have a pole at the black hole horizon $X = T$. Hence for incoming particles

$$\operatorname{Im} I_{in} = 0$$

and so Prob[in] = 1 already. The final result for the tunnelling probability is

$$\Gamma = \frac{\text{Prob}[out]}{\text{Prob}[in]} = \exp[-2\operatorname{Im} I_{out}] = \exp[-8\pi ME]$$

and this means the Hawking Temperature $T_H = \frac{1}{8\pi M}$ is recovered for the Kruskal-Szekeres metric.

2.2 Charged Particle Emission from Reissner-Nordström Black Holes

I will now review how charged particle emission works for each of the tunnelling approaches. For scalar particle emission, this is a topic that has been already been covered in great detail by others [12],[29],[31],[32],[34],[45]. I am reviewing the charged tunnelling calculations because I will use the expressions when I discuss extremal black holes and I will also use the charged scalar particle calculations to motivate my charged fermion calculations at the end of the thesis. I will start with the charged null geodesic calculation first and then follow with the Hamilton-Jacobi calculation. I will start by reviewing the Reissner-Nordström spacetime.

The Reissner-Nordström metric and vector potential are:

$$ds^2 = -(1 - \frac{2M}{r} + \frac{Q^2}{r^2})dt^2 + \frac{dr^2}{(1 - \frac{2M}{r} + \frac{Q^2}{r^2})} + r^2 d\Omega^2 \tag{2.37}$$

$$A_a = -\frac{Q}{r}(dt)_a \tag{2.38}$$

Note: that I will only be working with a non-extremal black hole so I am assuming $M^2 > Q^2$. The horizons for this spacetime are located at:

$$r_{\pm} = M \pm \sqrt{M^2 - Q^2} \tag{2.39}$$

2.2.1 Null Geodesic Method Applied to Charged Emission from Reissner-Nordström Black Hole

In this subsection I will review the calculation done in [31]. In order to apply the null geodesic method the Reissner-Nordström metric (2.37) needs to be converted into Painlevé form (2.4) which is explicitly:

$$ds^2 = -(1 - \frac{2M}{r} + \frac{Q^2}{r^2})dt^2 + 2\sqrt{\frac{2M}{r} - \frac{Q^2}{r^2}}dtdr + dr^2 + r^2 d\Omega^2 \tag{2.40}$$

Notice that after a charged particle q is emitted, the electromagnetic potential of the background spacetime will become

$$A_t = -\frac{Q - q}{r}$$

The radial null geodesics for this metric correspond to:

$$\dot{r} = \left(\pm 1 - \sqrt{\frac{2M}{r} - \frac{Q^2}{r^2}}\right) \tag{2.41}$$

where the plus/minus signs correspond to outgoing/ingoing null geodesics.

In order to model charged particle emission the effect of the electromagnetic field needs to be taken into account. The background system consists of a black hole with an electromagnetic field outside of the black hole. The classical action that will be solved in this case will be of the form $\int L dt$. Where the Lagrangian of the matter-gravity system can be written as:

$$L = L_m + L_e$$

where:

$$L_e = \frac{-1}{4} F_{\mu\nu} F^{\mu\nu}$$

is the Lagrangian function of the electromagnetic field corresponding to the generalized coordinates $A_\mu = (A_t, 0, 0, 0)$. When the charged particle tunnels out the system transitions from one state to another (i.e. $A_t = -\frac{Q}{r} \rightarrow A_t = -\frac{Q-q}{r}$). But from the expression for L_e it has been found that $A_\mu = (A_t, 0, 0, 0)$ is an ignorable coordinate [31]. So the freedom corresponding to A_t is removed by writing the action as:

$$I = \int_{t_i}^{t_f} (L - P_{A_t} \dot{A}_t) dt \tag{2.42}$$

The action is then rewritten to drop any terms that do not contribute to the final tunnelling probability (remember only the imaginary part of the action matters for the tunnelling methods):

$$\operatorname{Im} I = \operatorname{Im} \int_{r_{in}}^{r_{out}} \left[p_r - \frac{P_{A_t} \dot{A}_t}{\dot{r}} \right] dr = \operatorname{Im} \int_{r_{in}}^{r_{out}} \left[\int_{(0.0)}^{(p_r, P_{A_t})} dp'_r - \frac{\dot{A}_t}{\dot{r}} dP'_{A_t} \right] dr \tag{2.43}$$

where P_{A_t} is the electromagnetic field's canonical momentum conjugate to A_t.

Using Hamilton's equations:

$$\dot{r} = \frac{dH}{dp_r}|_{(r; A_t, P_{A_t})}$$
$$\dot{A}_t = \frac{dH}{dP_{A_t}}|_{(A_t; r, p_r)}$$

yields:

$$\operatorname{Im} I = \operatorname{Im} \int_{r_{in}}^{r_{out}} \left[\int_{(M, E_Q)}^{(M-\omega, E_{Q-q})} \frac{1}{\dot{r}} (dH)_{r; A_t, P_{A_t}} - \frac{1}{\dot{r}} (dH)_{A_t; r, pr} \right] dr$$

where E_Q represents the energy of the electromagnetic field and:

$$(dH)_{r; A_t, P_{A_t}} = d(M - \omega') = -d\omega'$$
$$(dH)_{A_t; r, pr} = -\frac{Q - q'}{r} dq'$$

(This result has assumed that the black hole can be treated as a charged conducting sphere.)

The action can now be rewritten as:

$$\operatorname{Im} I = -\operatorname{Im} \int_{r_{in}}^{r_{out}} \left[\int_{(0,0)}^{(\omega,q)} \frac{r(r+\sqrt{2(M-\omega')r-(Q-q')^2})}{(r-r'_+)(r-r'_-)} d\omega' \right.$$
$$\left. - \frac{(Q-q')(r+\sqrt{2(M-\omega')r-(Q-q')^2})}{(r-r'_+)(r-r'_-)} dq' \right] dr \tag{2.44}$$

where:

$$r'_\pm = M-\omega' \pm \sqrt{(M-\omega')^2-(Q-q')^2}$$

The authors of [31] solve this integral (2.44) to show that:

$$\operatorname{Im} I = -\frac{1}{2}\Delta S_{BH}$$

where $\Delta S_{BH} = S_{BH}(M-\omega, Q-q) - S_{BH}(M,Q)$ is the change of the entropy after the emission of the particle. (Note: $S_{BH}(M,Q) = \pi[M+\sqrt{M^2-Q^2}]^2$). Giving a full tunnelling probability:

$$\Gamma \sim \exp(\Delta S_{BH}) \tag{2.45}$$

I will assume that the charge of the s-wave is much smaller than the total charge of the black hole ($q' << Q$) along with my usual assumption that the energy of the s-wave is much smaller than the mass of the black hole ($\omega' << M$). So I will once again ignore the self-gravitation (and self-interaction) effects of the s-wave shell. So I will rewrite the action to lowest order in energy and charge:

$$\begin{aligned}
\operatorname{Im} I &= -\operatorname{Im}\left[\int_0^{\omega}\int_{r_{in}}^{r_{out}} \frac{r(r+\sqrt{2Mr-Q^2})}{(r-r_+)(r-r_-)} dr d\omega' - \int_0^{q}\int_{r_{in}}^{r_{out}} \frac{Q(r+\sqrt{2Mr-Q^2})}{(r-r_+)(r-r_-)} dr dq' \right] \\
&= \operatorname{Im} \pi i \frac{r_+(r_+ + \sqrt{2Mr_+ - Q^2})}{r_+ - r_-} \left[\omega - \frac{qQ}{r_+}\right] \\
&= \pi \frac{(M+\sqrt{M^2-Q^2})^2}{\sqrt{M^2-Q^2}} \left[\omega - \frac{qQ}{M+\sqrt{M^2-Q^2}}\right]
\end{aligned}$$

This gives a tunnelling rate (2.1) of:

$$\Gamma \propto \exp\left(-2\pi \frac{(M+\sqrt{M^2-Q^2})^2}{\sqrt{M^2-Q^2}} \left[\omega - \frac{qQ}{M+\sqrt{M^2-Q^2}}\right]\right) \tag{2.46}$$

and this gives the Hawking temperature for a Reissner-Nordström black hole:

$$T_H = \frac{1}{2\pi}\frac{\sqrt{M^2-Q^2}}{(M+\sqrt{(M^2-Q^2)})^2} \tag{2.47}$$

2.2.2 Hamilton-Jacobi Ansatz Applied to Charged Emission from Reissner-Nordström Black Hole

In this section I will review the calculation preformed in [45]. In order to model a charged particle tunnelling from a Reissner-Nordström black hole using the Hamilton-Jacobi method it is necessary to use the charged relativistic Hamilton-Jacobi equation:

$$g^{\mu\nu}(\partial_\mu I - qA_\mu)(\partial_\nu I - qA_\nu) + m^2 = 0 \tag{2.48}$$

for a particle of charge q. For the Reissner-Nordström metric (2.37), the Hamilton-Jacobi equation is explicitly:

$$-\frac{(\partial_t I - qA_t)^2}{f(r)} + f(r)(\partial_r I)^2 + \frac{1}{r^2}(\partial_\theta I)^2 + \frac{1}{r^2\sin^2\theta}(\partial_\phi I)^2 + m^2 = 0$$

where: $f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$

There exists a separable solution of the form:

$$I = -Et + W(r) + Y(\theta, \phi) \tag{2.49}$$

where:

$$\partial_t I = -E, \qquad \partial_r I = W'(r), \qquad \partial_\theta Y = Y_\theta, \qquad \partial_\phi Y = Y_\phi \tag{2.50}$$

Since ∂_t is the timelike killing vector for this coordinate system, E is the energy of the particle as detected by an observer at infinity.

Solving the Hamilton-Jacobi equation for $W(r)$ yields:

$$W_\pm(r) = \pm\int \frac{dr}{f(r)}\sqrt{(E + qA_t)^2 - f(r)\left(m^2 + \frac{1}{r^2}(Y_\theta)^2 + \frac{1}{r^2\sin^2\theta}(Y_\phi)^2\right)} \tag{2.51}$$

since the equation was quadratic in terms of $W(r)$. One solution corresponds to scalar particles moving away from the black hole (i.e. + outgoing) and the other solution corresponds to particles moving toward the black hole (i.e. - incoming). Imaginary parts of the action can only come due the pole at the horizon. The probabilities of crossing the horizon each direction are proportional to

$$\mathrm{Prob}[out] \propto \exp[-2\,\mathrm{Im}\, I] = \exp[-2\,\mathrm{Im}\, W_+] \tag{2.52}$$

$$\mathrm{Prob}[in] \propto \exp[-2\,\mathrm{Im}\, I] = \exp[-2\,\mathrm{Im}\, W_-] \tag{2.53}$$

To get the correct tunnelling rate it is necessary to take a ratio of the outgoing and incoming rates:

$$\Gamma \propto \frac{\mathrm{Prob}[out]}{\mathrm{Prob}[in]}$$

$$\Gamma \propto \exp[-4\,\mathrm{Im}\, W_+] \tag{2.54}$$

I will now drop the "+" subscript from W. Integrating around the pole at the horizon leads to the result:

$$W = \frac{\pi i(E - q\frac{Q}{r_+})}{f'(r_+)}$$
$$W = \pi i \frac{(M + \sqrt{M^2 - Q^2})^2}{2\sqrt{M^2 - Q^2}} \left[E - \frac{qQ}{M + \sqrt{M^2 - Q^2}} \right] \tag{2.55}$$

where the imaginary part of W is now manifest. This leads to a tunnelling probability of:

$$\Gamma = \exp\left(-2\pi \frac{(M + \sqrt{M^2 - Q^2})^2}{\sqrt{M^2 - Q^2}} \left[\omega - \frac{qQ}{M + \sqrt{M^2 - Q^2}} \right] \right) \tag{2.56}$$

and recovers the correct Hawking temperature:

$$T_H = \frac{1}{2\pi} \frac{\sqrt{M^2 - Q^2}}{(M + \sqrt{(M^2 - Q^2)})^2} \tag{2.57}$$

2.3 Review of Emission Through a Cosmological Horizon

One strength of semi-classical tunnelling methods is the ability to apply it to a wide variety of horizons beyond ordinary black hole horizons. For example the tunnelling method has been applied the cosmological horizons of various de Sitter space times [16],[24],[25],[28],[38],[51],[58]. This method has an advantage over Wick rotation methods, which cannot be applied in general when black holes are embedded in a de Sitter space. This is because when a black hole is embedded in de Sitter, multiple horizons are present and in general they have different temperatures. This means it will not be possible to have a heat bath that is in equilibrium with both horizons and the Wick rotation calculation will not work as a result. There are special cases in which the black hole can have an electric charge and the parameters are picked in such a way that "lukewarm" solutions [87] (for which the black hole and cosmological horizon are in equilibrium at the same temperature) can be found and the Wick rotation method can be applied. Discussing the thermodynamics of de Sitter space is therefore a tricky topic due to the presence of the two horizons that in general aren't in thermodynamic equilibrium.

So the addition of a cosmological constant to black hole spacetime makes the topic of gravitational thermodynamics more complicated. A further level of complexity results if, instead of a cosmological constant, dark energy is assumed. An alternative view of our universe is that cosmological constant should be regarded as a thermally fluctuating parameter [51],[88],[89] but there are multiple models for the behaviour of dark energy (Padmanabhan provides a review of approaches to dark energy in [90]). The tunnelling method has been applied to a particular dark energy model [51]. What is found is that a sign ambiguity can occur between calculations depending on if it is assumed that the cosmological constant is changing or not. So the multiple approaches to dark energy further confuse the concept of gravitational thermodynamics. As a result there does exist some controversy regarding conceptual interpretation of the tunnelling model across a cosmological horizon [51]; in general the differing views are that:

- A tunnelling model must mean that the cosmological constant will change [51]
- Trying to claim that the tunnelling model forces a variable cosmological constant goes beyond what a tunnelling model can tell you [52]
- The tunnelling picture does not even necessarily lead to Hawking radiation from the de Sitter horizon [53]

For the earliest null geodesic calculations, the cosmological constant was assumed to remain a constant after a tunnelling process [16],[24]. For my calculations I am concerned only with the emission of particles that have energy much lower than the energy of the de Sitter space time and therefore I will not consider a changing cosmological constant. As a result I will be using the sign conventions of the early papers [16],[19],[24],[28] in order to review the tunnelling calculations for de Sitter spacetimes.

I will start by showing the null geodesic tunnelling calculations from [16],[24] for the de Sitter and Schwarzschild-dS spacetimes. These calculations will be relevant for when I talk about Kerr-Gödel black holes in a later chapter and will allow me to compare and contrast the these cases. The de Sitter and Schwarzschild-dS metrics are given by:

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega^2 \tag{2.58}$$

where:

$$f(r) = 1 - \frac{r^2}{l^2} \text{ ;for de Sitter}$$
$$f(r) = 1 - \frac{2m}{r} - \frac{r^2}{l^2} \text{ ;for Schwarzschild-dS}$$

The cosmological constant for each of these spacetimes is $\Lambda = \frac{3}{l^2}$ For the de Sitter spacetime the cosmological horizon is located a $r_c = l$. For the Schwarzschild-dS black hole, when $\frac{27m^2}{l^2} < 1$ (i.e. the Nariai bound) is satisfied then the spacetime will have two horizons and the metric will be non-extremal. The two horizons correspond to a Schwarzschild horizon r_s and a cosmological horizon r_c. For the case when $m << l$ the horizons will be located at: $r_s \cong 2m$ and $r_c \cong l - m$.

In order to apply the null geodesic method, I will once again need to convert the metric into Painlevé form but in order to model particles tunnelling across the cosmological horizon it is necessary to choose:

$$t \to t + \int \sqrt{\frac{1 - f(r)}{(f\,(r))^2}}\,dr \tag{2.59a}$$

Notice that the sign of the transformation (2.59a) is opposite to the sign in (2.3a). This is due to the fact that the particles are tunnelling the opposite direction. The trajectory travels from outside the cosmological horizon in towards an observer within the cosmological horizon. Note, if the space time is just de Sitter (without a black hole) the observer will be located at the origin. If the spacetime is Schwarzschild-dS the observer will be located between the black hole horizon and the cosmological horizon. (For a more detailed discussion of the location of the observer in de Sitter spacetimes see the appendix section on applying the Hamilton-Jacobi method to de Sitter spaces). The resulting Painlevé metric has the opposite sign on the $dtdr$ term when compared to (2.4):

$$ds^2 = -f(r)dt^2 - 2\sqrt{1 - f(r)}drdt + dr^2 + r^2 d\Omega^2 \tag{2.60}$$

The radial null geodesics for this metric correspond to:

$$\dot{r} = \left(\pm 1 + \sqrt{1 - f(r)}\right) \tag{2.61}$$

where the plus/minus signs correspond to outgoing/ingoing null geodesics. Notice that it is the ingoing solution is the solution that vanishes at the cosmological horizon. (This is because the tunnelling is from outside to inside).

The imaginary part of the action for an s-wave crossing the horizon is:

$$I = \int_{r_i}^{r_f} p_r dr = \int_{r_i}^{r_f} \int_0^{p_r} dp'_r dr \tag{2.62}$$

where r_i and r_f are the respective initial and final radii of the cosmological horizon. The trajectory between these two radii is the barrier the particle must tunnel through.

In order to evaluate the integral, I once again employ Hamilton's equation $\dot{r} = \frac{dH}{dp_r}|_r$ to switch the integration variable from momentum to energy ($dp_r = \frac{dH}{\dot{r}}$), giving

$$I = \int_{r_i}^{r_f} \int_0^{E} \frac{dr}{\dot{r}} dH \tag{2.63}$$

The energy of the emitted particle should be smaller than the energy of the de Sitter spacetime. Note that the total energy (mass) of the de Sitter spacetime dS_3 is $M = \frac{1}{8G}$ [91] (where G is Newtons constant). So it is possible to solve to the lowest order in energy by assuming that $E << \frac{1}{8G}$. For de Sitter:

$$\begin{aligned} I &= E \int_{r_i}^{r_f} \frac{l}{r - l} dr \\ &= \pi i l E \end{aligned}$$

This gives a semiclassical tunnelling rate (2.1)

$$\Gamma_c \propto e^{-2\pi l E} \tag{2.64}$$

and a temperature for de Sitter space of:

$$T_{dS} = \frac{1}{2\pi l}$$

Parikh was also able to go further by using the following expression to model how the radial null geodesics are effected by the shell energy E'[16]:

$$\dot{r} = \left(\sqrt{\frac{r^2}{l^2} - 8GE'} - 1\right)$$

and he found a tunnelling rate of:

$$\Gamma_c \propto \exp\left[\frac{\pi l}{2G}\left(\sqrt{1-8GE} - 1\right)\right] \tag{2.65}$$

Notice that (2.65) reduces to (2.64) when $8GE << 1$ (as expected).

For Schwarzschild-dS (which is covered in [24]), the action (across the cosmological horizon) will be:

$$I = E\int_{r_i}^{r_f} \frac{1}{\sqrt{\frac{2m}{r} + \frac{r^2}{l^2}} - 1} dr$$

$$\implies \operatorname{Im} I = i\pi E \frac{l^2 r_c^2}{r_c^3 - ml^2}$$

$$\therefore \Gamma_c \propto \exp\left[-2\pi E \frac{l^2 r_c^2}{r_c^3 - ml^2}\right] \tag{2.66}$$

which gives a temperature for the cosmological horizon of:

$$T_{dS} = \frac{r_c^3 - ml^2}{2\pi l^2 r_c^2} \tag{2.67}$$

In order to calculate the tunnelling rate across the black hole horizon it is possible to use the standard null geodesic equations (2.3a),(2.5),(2.4) (notice the sign difference between those equations and the respective equations used for cosmological horizon tunnelling (2.59a),(2.61),(2.60)).

$$\implies \operatorname{Im} I = i\pi E \frac{l^2 r_s^2}{ml^2 - r_s^3}$$

$$\therefore \Gamma_s \propto \exp\left[-2\pi E \frac{l^2 r_s^2}{ml^2 - r_s^3}\right]$$

This gives a Hawking temperature of:

$$T_H = \frac{ml^2 - r_s^3}{2\pi l^2 r_s^2} \tag{2.68}$$

Both of the expressions for temperature are positive due to the fact that $r_c^3 > ml^2$, $r_s^3 < ml^2$. This can be seen explicitly in the case when the parameters are far from the Nariai bound ($\frac{27m^2}{l^2} << 1$) when $r_s \cong 2m$ and $r_c \cong l$ and the usual Hawking and de Sitter temperatures are recovered (i.e. $T_H \cong \frac{1}{8\pi m}$, $T_{dS} \cong \frac{1}{2\pi l}$). Notice that T_H is greater than T_{dS}. So the black hole will be radiating at a faster rate than the cosmological horizon and the system will eventually become empty de Sitter space. Equality of the two temperatures (in this case) only occurs at the Nariai bound where the system becomes extremal and the temperatures disappear. The property that $T_H \geq T_{dS}$ was shown explicitly in [24] for Schwarzschild-dS and I am making use of that result. This inequality may not necessarily be true for all types of de Sitter black holes since there is a special "lukewarm" solution for Reissner-Nordström-dS [87] where the black hole and de Sitter horizons have the same non-zero temperature and are not extremal. This doesn't contradict the result (for Schwarzschild-dS) that $T_H \geq T_{dS}$ but it does suggest that it would be worth investigating Reissner-Nordström-dS in more detail to see if there are any choices of parameters for which the de Sitter horizon will have a greater temperature than the Reissner-Nordström black hole. While this topic would be interesting to investigate in more detail, it is beyond the scope of my thesis.

It should also be noted, there is a sign ambiguity that exists between various papers when the tunnelling method is applied to cosmological horizons (i.e. compare the results in [51] to the earlier papers [16],[19],[24],[28]). The paper [51] gets a tunnelling rate across the cosmological horizon of $e^{2\pi l E}$ and some other authors have also taken up this "alternate" sign convention [58].

The controversy seems to boil down to argument of whether the cosmological horizon will increase or decrease in size. As a result this ends up being dependent on how the dark energy is assumed to behave. The horizons of the Schwarzschild-dS spacetime (well away from the Nariai bound) are approximately $r_s \cong 2m, r_c \cong l - m$. The usual argument is that, when spherical shell of energy ω is emitted by the cosmological horizon, the metric outside of the shell will be modified to that of Schwarzschild-dS with a mass parameter equal to $m + \omega$. Using the fact that a Schwarzschild-dS space with a larger mass parameter will have less energy overall [91], then this result is consistent with the cosmological horizon losing energy due to tunnelling. After the emitted s-wave eventually enters the black hole, the final locations of horizons will be $r_s \cong 2(m+\omega), r_c \cong l-(m+\omega) = l-m-\omega$. So the radius of the cosmological horizon will decrease in size with these assumptions. Also notice that under this argument, l does not change but the spacetime itself is changed by the particles tunnelling across the cosmological horizon and as a result the radius of the cosmological horizon is modified. The cosmological constant still remains constant in this case.

The author of [51] conversely takes the view that the cosmological constant itself should be changing after tunnelling across the cosmological horizon following the model used in [88],[89]. He assumes that the parameter l should increase by ω (after a particle of energy ω is emitted) and therefore he assumes that the radius of the cosmological horizon should increase. This explains why he gets a different sign on the final tunnelling probability. The claims of his paper have sparked controversy (see [52],[53]). In [51] the author assumes that after the tunnelling the space transforms from a straight de Sitter to a straight de Sitter spacetime, so he seems to ignore the fact that the resulting spacetime (outside the s-wave) is Schwarzschild-dS. He uses the first law of thermodynamics $dE = TdS$ with $T = \frac{1}{2\pi l}$ and $S = \pi l^2$ and then assumes that he can write $dS = 2\pi l dl$ but I think there should be an important distinction made between the radius of the cosmological horizon and the cosmological parameter l itself. Just because it can be written in terms of a changing l doesn't mean it is correct; it is still possible to take $r_i = l$ initially and have r change but l stay fixed. So I think there needs to be a stronger argument as to why l has to change. This model of dark energy being used is only one of many possible dark energy models (Padmanabhan reviewed multiple models of dark energy in [90] and Medved [52] provides some references to papers that oppose the thermally fluctuating parameter viewpoint). With this dark energy model the resulting expression for the energy is $dE = dl$ and from this he argues the energy of the de Sitter horizon is $E = l + const$ (this does not seem consistent with the results in [91] which gives the energy of a de Sitter space as $\frac{1}{8G}$ regardless of l). In his paper, he uses results from [91] (that shows that the the energy of Schwarzschild-dS has less energy than de Sitter) to claim "if an emitted particle has positive "frequency" ω it will be measured as negative "energy" $E = -\omega$" [51]. He uses this claim to justify his argument that $dE = dl$ which would mean that the energy of the de Sitter horizon (and therefore the parameter l itself) is actually increasing. To recap his argument, a positive 'frequency' particle is emitted from the horizon with negative 'energy' (this is used to justify $e^{2\pi l E}$ as a normal Boltzmann factor) so the energy of the de Sitter horizon increases and therefore the parameter l has increased. It is not apparent to me that such an argument is correct. Another intuitive problem I have with this model is the question of how this (negative) energy is extracted from the cosmological constant. In the early de Sitter tunnelling calculations [16],[19],[24],[28] the cosmological constant does not change and particles simply tunnel across the cosmological horizon (as a result the cosmological horizon does change even though l does not); these particles originally exist beyond the horizon, so intuitively the mechanism is clear. In [51] somehow (negative) energy is extracted from the cosmological constant (or dark energy quintessence) itself which is not localized to any particular location. So the mechanism is not conceptually clear to me how particles tunnelling across the cosmological horizon somehow changes the dark energy everywhere in the universe. So as a result, I prefer the assumptions of the earlier papers [16],[19],[24],[28] (with regard to the sign ambiguity) and it is this choice of sign convention that I have used for my calculations in this section.

2.4 Factor of 2 Issue

As a review of the current state of the tunnelling method it is also useful to mention the controversies involving the technique in order to gain a full picture of the method. I discussed in the last section how sign ambiguities can arise in the case of tunnelling across the cosmological horizon. Another issue that occurs involving the tunnelling method involves an ambiguous factor of 2. For a metric in standard Schwarzschild form if the tunnelling calculation is preformed without taking the ratios of the outgoing and incoming probabilities the resulting tunnelling rate will be off by factor of 2 from when the Painlevé form of the metric is used (refer to Hamilton-Jacobi section). As a result this will give a temperature that will be twice as big as the Hawking temperature. This in essence is the factor of 2 issue. An alternative remedy is to ignore the incoming path and integrate in terms of the proper spatial distance (see appendix). There is a group of researchers that were not satisfied with either of these prescriptions to get the usual result and for a time argued that this factor of 2 result was the actually the correct result [40],[41]. In other words they claimed that the black hole temperature should really be a factor of 2 different from the usual Hawking temperature. Recently, they have changed their viewpoint and no longer claim that the black hole temperature should be a factor of 2 different from the Hawking temperature [42]. They have developed a modification of the tunnelling method that will no longer give answers that are a factor of 2 different. In their method they have shown a "temporal contribution" will fix the factor of 2 issue. So their method provides another alternative prescription for getting the standard Hawking temperature. I will continue to stick with the prescription of using the ratio of outgoing and ingoing probabilities throughout the thesis.

3 Survey of Scalar Particle Emission from various spacetimes

The tunnelling method was only originally applied Schwarzschild black holes [10]. This method was not originally created to be a robust temperature calculation that could rival the Wick rotation method in both simplicity and range of application. The technique was simply intended to be a straightforward semi-classical model that would give intuitive insight into Hawking radiation. Since the model is only semi-classical, it is not immediately apparent that it would be particularly robust when applied to a large variety of spacetimes. Surprisingly, this is what is actually found [10]-[58]. The tunnelling method can even be applied in some cases where the Wick rotation method cannot (i.e. Schwarzschild-dS calculations were reviewed in the last chapter). So as a result the tunnelling method can also be applied to cosmological horizons in de Sitter spaces and also the apparent horizons of Rindler observers. In order to demonstrate the effectiveness of the tunnelling method I will apply the technique to a range of spacetimes. I will start by applying the tunnelling method to Rindler space to show that the method will recover the Unruh temperature. An important extension of the method is the application to rotating spacetimes. I will show how the tunnelling can be extended to rotating spacetimes by applying the method to the Kerr-Newman spacetimes. I will then show that the method can also be extended to spacetimes with NUT charge (a type of magnetic mass). So in particular, I will apply the tunnelling method to Taub-NUT-AdS. I will finish by discussing the application of tunnelling to extremal black holes which pushes the realm of applicability of the method. The results for the tunnelling rate (for extremal black holes) tend to contain a diverging real component in the action and while the real component of the action is normally ignored in tunnelling calculations, the fact that it is diverging may be an indication that the tunnelling method breaks down in the case of extremal black holes. I will also show that it is possible to tunnel from extremal black hole to extremal black hole when charged emission is considered. In this case there is no diverging real component but it does end up requiring $\omega = |q|$ for the emitted wave.

3.1 Rindler Space

I will now illustrate how these methods can also be applied to the horizon as seen by an accelerated observer. I shall employ multiple different coordinate systems for Rindler spaces to show that correct Unruh temperature results are recovered from applying the tunnelling methods to such a range of coordinate choices. Padmanabhan preformed a similar calculation in [19] for Rindler spaces but only with the Hamilton-Jacobi method (and not the null geodesic method). Padmanabhan also preformed Hamilton-Jacobi tunnelling calculations where he discussed “local Rindler frames” as part of his calculations [22] but here he was still calculating temperatures for horizons with a set surface gravity. I was the first to use the null geodesic method to calculate tunnelling across the Rindler horizon in [37]. It should be noted that in order to apply the tunnelling method to a Rindler spacetime the energy of the emitted particle should be smaller than the energy of a Rindler spacetime. This means an estimate for the energy of a Rindler spacetime is required and I use the results from [92] to define an energy for the Rindler spacetime.

I will use the following two forms of the Rindler metric in my calculations:

$$ds^2 = -a^2x^2dt^2 + dx^2 \tag{3.1}$$

$$ds^2 = -(a^2\tilde{x}^2 - 1)dt^2 + \frac{a^2\tilde{x}^2}{a^2\tilde{x}^2 - 1}d\tilde{x}^2 \tag{3.2}$$

where a is the proper acceleration of the hyperbolic trajectory. Here there is no well defined total mass or energy of the spacetime, but there are well defined horizons. These two metrics are related by the equation $x^2 = \frac{a^2\tilde{x}^2-1}{a^2}$. For the metric (3.1) the horizon is located at $x = 0$ and the timelike-killing vector ∂_t will have norm equal to (minus) unity at $x = \frac{1}{a}$ (this is where the Rindler observer is located with acceleration a). The metric (3.2) has the horizon at $\tilde{x} = \frac{1}{a}$, whereas the observer is located at $\tilde{x} = \frac{\sqrt{2}}{a}$.

I will consider a null particle emitted from the Rindler horizon, and it is reasonable to assume the emitted particle will have a Hamiltonian associated with it. However providing an explicit definition for the total energy of the space-time is less than clear, though it has been claimed [92] that one can associate a surface energy density $\sigma = \frac{a}{4\pi}$ with a Rindler horizon and a total energy $E = \frac{1}{4a}$ with the spacetime. In the context of the null geodesic method, I will expect that the Hamiltonian of the space-time will correspond to the total energy E (perhaps with respect to some reference energy via a limiting procedure) so as long as the emitted particles have energy much smaller than $\frac{1}{4a}$ ($E << \frac{1}{4a}$) then the method is applicable. I will proceed under this assumption that it is possible to use Hamilton's equation and follow the derivation for the null geodesic method as before. These assumptions will be justified a-posteriori.

The null geodesics for (3.1) in the x-direction are given by

$$\dot{x} = \pm ax$$

and so

$$I = \omega \int_{x_{in}}^{x_{out}} \frac{dx}{ax} = \frac{\pi\omega}{a}$$

yielding a the Unruh temperature:

$$T_H = \frac{a}{2\pi} = \frac{a}{2\pi}$$

I will now employ the Hamilton-Jacobi ansatz for the Rindler metric (3.2).

Here $f = a^2\tilde{x}^2 - 1, g = \frac{a^2\tilde{x}^2-1}{a^2\tilde{x}^2}$ and at the horizon $f'(1) = g'(1) = 2a$ so using (2.28)

$$\Gamma = \exp[-\frac{2\pi}{a}E]$$

again giving a temperature of $T_H = \frac{a}{2\pi}$.

It can be seen that the expected value for the temperature of Rindler space is recovered given these assumptions. This could perhaps be regarded further evidence that a total energy $E = \frac{1}{4a}$ can be associated with Rindler space.

3.1.1 Tunnelling for a Rindler observer in Minkowski Spacetime

I will also demonstrate that the tunnelling method can be applied to a Rindler observer following a hyperbolic path in Minkowski spacetime. The Minkowski metric does not contain a horizon by itself but a hyperbolic observer does see a Rindler horizon. I will show that even when the Minkowski metric is used the tunnelling method can still be applied. The Minkowski metric is given by:

$$ds^2 = \left(-dT^2 + dX^2\right) + dy^2 + dz^2 \tag{3.3}$$

where the Rindler observer follows a path of constant x_0, where x_0 is defined by:

$$x_0^2 = X^2 - T^2 \tag{3.4}$$

The metric (3.3) is well behaved everywhere including both the future and past horizons $X = \pm T$ seen by the Rindler observer. Note that there is freedom in choosing the timelike killing vector. The most obvious timelike killing vector is ∂_T but this is the timelike killing vector of a stationary (or inertial) observer. If the method was attempted by trying use ∂_T to define the energy you would not find any tunnelling at all. This is due to the fact that a stationary (or inertial) observer does not see a Rindler Horizon. Motivated by the calculations that I preformed for the Kruskal-Szekeres metric in the last chapter, it is possible to notice that the timelike killing vector $X\partial_T + T\partial_X$ also exists in Minkowski space. I claim that this is the killing vector of the Rindler observer but it still needs to be normalized for this observer. For convenience assume a solution for the Hamilton-Jacobi equation of the form:

$$I = I(X, T) \tag{3.5}$$

I am assuming the simplest case of the 2D with $m = 0$ for the emitted particles. Plugging this into the Hamilton-Jacobi equation:

$$\left(-(\partial_T I)^2 + (\partial_X I)^2\right) = 0$$

$$\implies I = h(X - T) \text{ or } I = f(X + T) \tag{3.6}$$

In order to solve the equations, I will need a definition of the energy of the wave. I will use the same timelike killing vector that I used for the Kruskal-Szekeres calculation in the last chapter due to the similarity of the two metrics. So I will define energy via the timelike killing vector:

$$\partial_\chi = N(X\partial_T + T\partial_X)$$

where N is a normalization constant chosen so that the norm of the Killing vector is equal to -1 at the location of Rindler observer. This yields

$$\partial_\chi = \frac{1}{x_0}(X\partial_T + T\partial_X) \tag{3.7}$$

and so

$$\partial_\chi I = -E \tag{3.8}$$

Using (3.8) with the two possible solutions (3.6) I will solve the equations. For the outgoing solution:

$$\frac{1}{x_0}(X\partial_T I + T\partial_X I) = -E$$

$$x_0 = (X-T)h'(X-T)$$
$$h'(X-T) = \frac{x_0}{(X-T)}$$

where prime denotes the derivative of the function h. Notice, h' has a simple pole at the black hole horizon $X = T$. Setting $\eta = X - T$:

$$h'(\eta) = \frac{x_0}{\eta} \tag{3.9}$$

Integrating (3.9) around the pole at the horizon (doing a half circle contour) implies

$$\mathrm{Im}\, I_{out} = \pi x_0 E$$

For the outgoing solution:

$$\frac{1}{x_0}(X\partial_T I + T\partial_X I) = -E$$

$$-x_0 = (X+T)k'(X+T)$$
$$k'(X+T) = \frac{-x_0}{(X+T)}$$

Where prime now denotes the derivative of the function k. Notice that this equation does not have a pole at the horizon $X = T$. Hence for incoming particles

$$\mathrm{Im}\, I_{in} = 0$$

and so $\mathrm{Prob}[in] = 1$ already. The final result for the tunnelling probability is

$$\Gamma = \frac{\mathrm{Prob}[out]}{\mathrm{Prob}[in]} = \exp[-2\,\mathrm{Im}\, I_{out}] = \exp[-2\pi E x_0]$$

Remembering that the acceleration of Rindler observer at x_0 is $a = \frac{1}{x_0}$ means that the Unruh Temperature $T_U = \frac{a}{2\pi}$ is recovered for a Rindler observer in Minkowski space.

3.2 Kerr-Newman Black Hole

The tunnelling method was only originally applied to the most basic non rotating black holes (i.e. Schwarzschild and Reissner-Nordström). As a result there was some question to how robust the method was and how far it could be extended. A particularly important extension of the method to try was rotating spacetimes. M. Angheben, M. Nadalini, L Vanzo, and S. Zerbini [28] were able to calculate tunnelling from rotating black holes by extending the Hamilton-Jacobi method to the BTZ and Kerr spaces. I was able to use this as a starting point for extending the null geodesic method to rotating spacetimes. In particular, I applied both the null geodesic method and Hamilton-Jacobi method to Kerr-Newman black hole [37]. One reason rotating spacetimes are an interesting extension of the tunnelling method because spherical symmetry is no longer present (although they are still axial-symmetric). As a result, it is possible to extend the null geodesic method incorrectly to rotating spacetimes and a naive first attempt of extending the method will not work (see appendix). Another reason that rotating spacetimes are important is that the resulting tunnelling rate contains another term in its exponential that is proportional to the angular velocity of the black hole horizon. This is not a property that can be seen for non-rotating black holes and it increases the understanding of the particle emission beyond just calculating the temperature. Since spherical symmetry is lost it becomes mathematically necessary to consider the emission of rings of particles instead of s-waves. Once it is found that all of these rings have the same emission probability it is possible to consider the whole as an emitted s-wave again The tunnelling calculation can also be affected by the presence of the ergosphere. The way to resolve this problem is to transform into a corotating coordinate system where the energy of the emitted particle becomes $E - \Omega_H J$ where Ω_H is the angular velocity of the black hole and J is the angular momentum of the particle.

I will now calculate the tunnelling emission for a Kerr-Newman black hole.

The Kerr-Newman metric and vector potential are given by:

$$ds^2 = -f(r,\theta)dt^2 + \frac{dr^2}{g(r,\theta)} - 2H(r,\theta)dtd\phi + K(r,\theta)d\phi^2 + \Sigma(r,\theta)d\theta^2 \tag{3.10}$$

$$A_a = -\frac{er}{\Sigma(r)}[(dt)_a - a^2\sin^2\theta(d\phi)_a] \tag{3.11}$$

$$\begin{aligned}
f(r,\theta) &= \frac{\Delta(r) - a^2\sin^2\theta}{\Sigma(r,\theta)}, \\
g(r,\theta) &= \frac{\Delta(r)}{\Sigma(r,\theta)}, \\
H(r,\theta) &= \frac{a\sin^2\theta(r^2+a^2-\Delta(r))}{\Sigma(r,\theta)} \\
K(r,\theta) &= \frac{(r^2+a^2)^2 - \Delta(r)a^2\sin^2\theta}{\Sigma(r,\theta)}\sin^2(\theta) \\
\Sigma(r,\theta) &= r^2 + a^2\cos^2\theta \\
\Delta(r) &= r^2 + a^2 + e^2 - 2Mr
\end{aligned}$$

I will assume a non-extremal black hole so that $M^2 > a^2 + e^2$ so that there are two horizons at $r_\pm = M \pm \sqrt{M^2 - a^2 - e^2}$.

There is a minor technical issue involved in applying the tunnelling method to rotating spacetimes, since the metric functions depend on the angle θ in general. In order to account for this it is no longer possible to just look a generic spherical wave; instead consider rings of emitted photons for arbitrary fixed $\theta = \theta_0$. In the end it will be discovered that the temperature is independent of θ_0 (as it should be).

A naive first attempt at utilizing the null geodesic method would be to just consider the transformation

$$dt = dT - \sqrt{\frac{1-g(r,\theta_0)}{g(r,\theta_0)f(r,\theta_0)}}dr$$

Such an attempt will not work due to other technical issues involving rotating spacetimes. (See the appendix for the calculation and a discussion of technical issues involved).

In order to apply the tunnelling method in this case, note that the Kerr-Newman metric can be rewritten as

$$ds^2 = -F(r,\theta)dt^2 + \frac{dr^2}{g(r,\theta)} + K(r,\theta)(d\phi - \frac{H(r,\theta)}{K(r,\theta)}dt)^2 + \Sigma(r)d\theta^2 \tag{3.12}$$

$$F(r,\theta) = f(r,\theta) + \frac{H^2(r,\theta)}{K(r,\theta)} = \frac{\Delta(r)\Sigma(r,\theta)}{(r^2+a^2)^2 - \Delta(r)a^2\sin^2\theta}$$

where at the horizon

$$\frac{H(r_+,\theta)}{K(r_+,\theta)} = \frac{a}{r_+^2 + a^2} = \Omega_H$$

So the metric near the horizon for fixed $\theta = \theta_0$ is

$$ds^2 = -F_r(r_+, \theta_0)(r-r_+)dt^2 + \frac{dr^2}{g_r(r_+, \theta_0)(r-r_+)} + K(r_+, \theta_0)(d\phi - \frac{H(r_+, \theta_0)}{K(r_+, \theta_0)}dt)^2 \tag{3.13}$$

and by defining $d\chi = d\phi - \frac{H(r_+, \theta_0)}{I(r_+, \theta_0)}dt$.

$$ds^2 = -F_r(r_+, \theta_0)(r - r_+)dt^2 + \frac{dr^2}{g_r(r_+, \theta_0)(r - r_+)} + K(r_+, \theta_0)(d\chi)^2 \tag{3.14}$$

The metric (3.14) is well-behaved for all θ_0 and is of the same form as (2.2) with $f(r) = F_r(r_+, \theta_0)(r - r_+)$ and $g(r) = g_r(r_+, \theta_0)(r - r_+)$. Hence it is easy to obtain the final result (2.14)

$$T_H = \frac{\sqrt{F_r(r_+, \theta_0)g_r(r_+, \theta_0)}}{4\pi}$$

Explicit calculation of $F_r(r_+, \theta_0)$ and $g_r(r_+, \theta_0)$ yields

$$g_r(r_+, \theta_0) = \frac{\Delta_r(r_+)}{\Sigma(r_+, \theta_0)} = \frac{2r_+ - 2M}{r_+^2 + a^2\cos^2(\theta_0)}$$
$$F_r(r_+, \theta_0) = \frac{\Delta_r(r_+)\Sigma(r_+, \theta_0)}{(r_+^2 + a^2)^2} = \frac{(2r_+ - 2M)(r_+^2 + a^2\cos^2(\theta_0))}{(r_+^2 + a^2)^2}$$

Although $F_r(r_+, \theta_0)$ and $g_r(r_+, \theta_0)$ each depend on θ_0, their product

$$F_r(r_+, \theta_0)g_r(r_+, \theta_0) = \frac{(2r_+ - 2M)^2}{(r_+^2 + a^2)^2}$$

is independent of this quantity. Hence the temperature is

$$T_H = \frac{1}{2\pi}\frac{r_+ - M}{r_+^2 + a^2} = \frac{1}{2\pi}\frac{(M^2 - a^2 - e^2)^{\frac{1}{2}}}{2M(M + (M^2 - a^2 - e^2)^{\frac{1}{2}}) - e^2}$$

for any angle.

I will now turn to the Hamilton-Jacobi method to find the temperature. The action is assumed to be of the form

$$I = -Et + J\phi + W(r, \theta_0)$$

and rewriting this in terms of $\chi(r_+) = \phi - \Omega_H t$

$$I = -(E - \Omega_H J)t + J\chi + W(r, \theta_0)$$

where it is assumed that $E - \Omega_H J > 0$. This demonstrates a nuance overlooked in the null geodesic method; the transformation to χ implies that E should be replaced by $E - \Omega_H J$ for the emitted particle. The reason for this is the presence of the ergosphere. The Killing field that is timelike everywhere is

$\chi = \partial_t + \Omega_H \partial_\phi$. A particle can escape to infinity only if $p_a \chi^a < 0$, and so $-E + \Omega_H J < 0$ where E and J are the energy and angular momentum of the particle.

Employing the metric in the near horizon form (3.13), the final result for $W(r, \theta_0)$ is the same as the standard Hamilton-Jacobi result for W (2.27) with E replaced by $E - \Omega_H J$:

$$W(r, \theta_0) = \frac{\pi i (E - \Omega_H J)}{\sqrt{F_r(r_+, \theta_0) g_r(r_+, \theta_0)}} = (E - \Omega_H J) \frac{\pi i (r_+^2 + a^2)}{2(r_+ - M)}$$

Giving a tunnelling rate of:

$$\Gamma \propto \exp(-\frac{2\pi(r_+^2 + a^2)}{(r_+ - M)}(E - \Omega_H J)$$

again yielding the temperature over the full surface of the Black Hole

$$T_H = \frac{1}{2\pi} \frac{r_+ - M}{r_+^2 + a^2} = \frac{1}{2\pi} \frac{(M^2 - a^2 - e^2)^{\frac{1}{2}}}{2M(M + (M^2 - a^2 - e^2)^{\frac{1}{2}}) - e^2}$$

in full agreement with the previous calculation and with Euclidean space techniques. The Hamilton-Jacobi calculation can also be done without converting to the corotating frame in the beginning but the calculation becomes messier (this calculation has been added to the appendix).

3.3 Taub-NUT-AdS

I will continue the strategy of checking the robustness of the tunnelling method by applying it to more exotic spacetimes. A particularly interesting exotic spacetime is the Taub-NUT-AdS spacetime. The Taub-NUT metric is a generalization of the Schwarzschild metric and has played an important role in the conceptual development of general relativity and in the construction of brane solutions in string theory and M-theory [62]. The Taub-NUT spacetime is an exact solution to Einstein's equations and contains a special conserved quantity known as the NUT charge. The NUT charge plays the role of a magnetic mass in the spacetime. The NUT charges induces a topology in the Euclidean section of the metric at infinity that is a Hopf fibration of a circle over a 2-sphere. "A counter example to almost anything" [93], Taub-NUT spaces have been of particular interest in recent years because of the role they play in furthering the understanding of the AdS-CFT correspondence [63],[64],[94]. Along these lines, the thermodynamics of various Taub-NUT solutions has been a subject of intense study in recent years. Their entropy is not proportional to the area of the event horizon and their free energy can sometimes be negative [63],[64],[65],[66]. Solutions of Einstein equations with a negative cosmological constant Λ and a nonvanishing NUT charge have a boundary metric that has closed timelike curves. The behaviour of quantum field theory is significantly different in such spaces, and it is of interest to understand how ADS/CFT works in these sorts of cases [95].

The previous thermodynamic calculations for Taub-NUT spaces have been carried out in the Euclidean section, using Wick rotation methods. For most Taub-NUT spaces the Lorentzian section has closed timelike curves (CTCs). As a consequence, determination of the temperature via the original method of Hawking – while mathematically clear – is somewhat problematic in terms of its physical interpretation. It is straightforward enough to analytically continue the time coordinate and various metric parameters to render the metric Euclidean. Regularity arguments then yield a periodicity for the time coordinate that can then be interpreted as a temperature. However the Lorentzian analogue of this procedure is less than clear, though it has been established that a relationship between distinct analytic continuation methods exists [96]. An independent method of computing the temperature associated with event horizons in NUT-charged spacetimes is certainly desirable. So the tunnelling method is a particularly interesting candidate to apply to Taub-NUT spaces. In order to avoid the problems involving CTCs I will calculate the temperature for a subclass of Taub-NUT spacetimes that don't have CTCs. This way the results can be compared to those obtained via Wick rotation methods to check the validity of the result. I will then discuss the tunnelling result in the presence of CTCs.

The general Taub-NUT-AdS solutions with cosmological constant $\Lambda = -3/\ell^2$ are given by [95]:

$$ds^2 = -F(r)(dt + 4n^2 f_k^2(\frac{\theta}{2})d\varphi)^2 + \frac{dr^2}{F(r)} + (r^2+n^2)(d\theta^2 + f_k^2(\theta)d\varphi^2) \quad (3.15)$$

where

$$F(r) = k\frac{r^2-n^2}{r^2+n^2} + \frac{-2Mr + \frac{1}{\ell^2}(r^4+6n^2r^2-3n^4)}{r^2+n^2} \quad (3.16)$$

and k is a discrete parameter that takes the values $1, 0, -1$ and defines the form of the function $f_k(\theta)$

$$f_k(\theta) = \begin{cases} \sin\theta & \text{for k} = 1 \\ \theta & \text{for k} = 0 \\ \sinh\theta & \text{for k} = -1 \end{cases} \quad (3.17)$$

One of the interesting properties of Taub-NUT spaces is the existence of closed timelike curves (CTCs) [93]. For these cases it is not clear how to apply the null geodesic method, since the emission of an s-wave particle would have to recur in a manner consistent with the presence of CTCs.

However there exists a special subclass of Hyperbolic Taub-NUT solutions (for $4n^2/\ell^2 \leq 1$) that do not contain CTCs. A discussion of Taub-NUT space and the special cases without CTCs appears in the appendix. I will consider these cases in what follows.

The temperature can be successfully calculated using the metric in the following form:

$$ds^2 = -Hdt^2 + \frac{dr^2}{F} + G(d\varphi - \frac{F4nf_k^2(\frac{\theta}{2})}{G}dt)^2 + (r^2+n^2)d\theta^2 \quad (3.18)$$

where:

$$H(r,\theta) = (F + F^2 \frac{16n^2 f_k^4(\frac{\theta}{2})}{G}) \tag{3.19}$$

$$G(r,\theta) = 4f_k^2(\frac{\theta}{2}) \left(r^2 + n^2 - f_k^2(\frac{\theta}{2})(4n^2F + k(r^2+n^2)) \right) \tag{3.20}$$

As with the Kerr-Newman black hole, I will consider rings at constant θ_0 and use the near horizon approximation.

At the horizon

$$\frac{G(r_+,\theta_0)}{f_k^2(\frac{\theta_0}{2})} = \begin{cases} 4\left((r_+^2+n^2)\cosh^2(\frac{\theta_0}{2})\right), & k=-1 \\ 4(r_+^2+n^2), & k=0 \\ 4\left((r_+^2+n^2)\cos^2(\frac{\theta_0}{2})\right), & k=1 \end{cases}$$

Only when $k=1$ (for which CTCs are present) and $\theta_0 = \pi$ (i.e. when $\cos(\frac{\theta_0}{2}) = 0$) are there any potential divergences at the horizon. Since

$$H_r(r_+,\theta_0) = F_r(r_+)$$

the metric near the horizon for fixed $\theta = \theta_0$ is:

$$\begin{aligned} ds^2 &= -F_r(r_+)(r-r_+)dt^2 + \frac{dr^2}{F_r(r_+)(r-r_+)} \\ &\quad + G(r_+,\theta_0)(d\varphi - \frac{F_r(r_+)4nf_k^2(\frac{\theta}{2})}{G(r_+,\theta_0)}(r-r_+)dt)^2 && (3.21) \\ &= -F_r(r_+)(r-r_+)dt^2 + \frac{dr^2}{F_r(r_+)(r-r_+)} + G(r_+,\theta)d\varphi^2 && (3.22) \end{aligned}$$

Notice that defining $\chi = \varphi - \Omega_H t$ as with the charged Kerr case is pointless since $\Omega_H = 0$. From this point the steps are the same as for the general procedures outlined for either the null-geodesic method or the Hamilton-Jacobi ansatz. Inserting this into the final result for temperature (either (2.14) or (2.29)) yields

$$T_H = \frac{F_r(r_+)}{4\pi} \tag{3.23}$$

which is the same form found using the Wick rotation method [95, 96].

To demonstrate this is straightforward. Consider the hyperbolic case ($k = -1$). The mass parameter can be written in terms of the other metric parameters upon recognition that $F(r_+) = 0$ yielding

$$M = \frac{r_+^4 + (6n^2-\ell^2)r_+^2 - n^2(3n^2-\ell^2)}{2\ell^2 r_+}$$

Using this mass in (3.23) yields an expression for the hyperbolic Taub-NUT temperature of

$$T_H = \frac{4\pi\ell^2 r_+}{3(r_+^2+n^2)-\ell^2} \tag{3.24}$$

Comparing this to the result [95] for the hyperbolic Taub-NUT temperature obtained from Wick rotation methods

$$T_H = \frac{4\pi\ell^2 r_+}{3(r_+^2 - N^2) - \ell^2} = \frac{F_r(r_+)}{4\pi} \tag{3.25}$$

(where N is the Wick rotated NUT charge) agreement is obtained upon recognizing that $n^2 = -N^2$ due to analytic continuation. Note however that there is an implicit analytic continuation in the definition of r_+, since $F(r_+, n) \to F(r_+, iN)$ [96]. Note this temperature result also assumes that the observer is at some location where $H(r, \theta) \approx 1$. For an arbitrary choice of parameters it may not be possible to have any location where $g_{00} \approx -1$. This means it will be necessary to pick a location for the observer and modify the temperature above by dividing by $\sqrt{-g_{00}}$ at the observer (i.e. Tolman redshift factor [97]). The reasoning for this is the same as for de Sitter spacetimes as discussed in the appendix.

I will close by commenting that although I considered only the $k = -1$ case to avoid problems with CTCs, both the $k = 0, 1$ cases can be formally carried through, yielding the result (3.23). In the context of the null geodesic method this situation could perhaps be interpreted by noting that Hawking radiation yields a thermal bath of particles, whose existence can statistically be reconciled with the presence of CTCs. In the context of the Hamilton-Jacobi ansatz the physical interpretation is less problematic provided the classical action for the particle can be considered to obey the Hamilton-Jacobi equation in the presence of CTCs. These results suggest a-posteriori the answer is yes, but the matter merits further study. In this context I will note recent work [98] demonstrating that there are no SU(2)-invariant (time-dependent) tensorial perturbations of asymptotically flat Lorentzian Taub-NUT space, calling into question the possibility that a physically sensible thermodynamics can be associated to Lorentzian Taub-NUT spaces without cosmological constant. Related work involves the investigating the thermodynamics of Kerr-Taub-NUT [99] that questions the applicability of the first law of thermodynamics to these spacetime. Whether or not such results extend to Taub-NUT spaces without CTCs is an interesting question.

3.4 Extremal Black Holes

Extremal black holes need to be treated separately from the other generalizations, since the integrand no longer has a single pole. The general results derived earlier are no longer valid and even the self gravitating terms may play a very important role. One of the properties that occurs in extremal case is the presence of a divergent real component in the action. Although such a term does not contribute to the imaginary part of the action, this may be an indication that the tunnelling approach is breaking down and the calculation is becoming pathological. Unlike the Wick-rotation method, which involves finding an equilibrium temperature, the tunnelling approach describes a dynamical system. In this latter context when a black hole is extremal the possibility exists that an emitted neutral particle may cause the creation of a naked singularity, in violation of cosmic censorship.

Such a pathological situation would be prevented if the tunnelling barrier had infinite height. However this is not found to be the case, and an evaluation of the imaginary part of the action yields a finite temperature. This is consistent with the proposal that extremal black holes can be in thermal equilibrium at any temperature [100].

For concreteness, I will consider the particular case of the Reissner-Nordström metric, please note that a diverging real component has also been seen to occur with the extremal GHS solution [28].

3.4.1 Extremal Reissner-Nordström

The Reissner-Nordström space-time is described by the metric

$$ds^2 = -(1 - \frac{2M}{r} + \frac{Q^2}{r^2})dt^2 + \frac{dr^2}{(1 - \frac{2M}{r} + \frac{Q^2}{r^2})} + r^2 d\Omega^2 \tag{3.26}$$

The black hole is non-extremal when $M^2 > Q^2$ and extremal when $Q = M$. For the non-extremal case the tunnelling approach yields a temperature of

$$T_H = \frac{1}{2\pi} \frac{\sqrt{M^2 - Q^2}}{(M + \sqrt{(M^2 - Q^2)})^2} \tag{3.27}$$

using either the null geodesic or Hamilton-Jacob methods. Notice that the limit $Q \to M$ gives a temperature of zero.

For the Reissner-Nordström case self gravitating effects have been calculated exactly [14] and the full emission rate (without charge) is

$$\Gamma \sim e^{-2I} = e^{-2\pi\left(2\omega(M - \frac{\omega}{2}) - (M-\omega)\sqrt{(M-\omega)^2 - Q^2} + M\sqrt{M^2 - Q^2}\right)} \tag{3.28}$$

Expanding this emission rate in powers of ω yields the temperature (3.27) to leading order. Note that setting $Q = M$ yields a contradictory result, since the second term in the exponent becomes imaginary. This unphysical situation corresponds to an extremal black hole emitting a particle, a situation in violation of cosmic censorship.

Consider a nearly extremal black hole that emits a particle so that the resulting black hole is extremal. This corresponds to substitution of $Q = (M - \omega)$ where the black hole emits a null particle of energy ω. Insertion of this value of Q into (3.27) yields

$$T_H = \frac{1}{2\pi} \frac{\sqrt{M^2 - (M-\omega)^2}}{(M + \sqrt{(M^2 - (M-\omega)^2)})^2} = \frac{1}{2\pi} \frac{\sqrt{2M\omega}}{M^2} + O(\omega) \tag{3.29}$$

Comparing this to the temperature obtained from the emission rate using (3.28) gives

$$\Gamma = e^{-2\pi\left(2\omega(M - \frac{\omega}{2}) + M\sqrt{M^2 - (M-\omega)^2}\right)} = e^{-2\pi\left(M\sqrt{2M\omega} + 2\omega M + O(\omega^{\frac{3}{2}})\right)}$$

and it is found that the temperature is $O(\sqrt{\omega})$ and again approaches zero the closer the original black hole is to extremality. Explicitly

$$T = \frac{1}{2\pi}\frac{\omega}{M\sqrt{2M\omega}} = \frac{1}{4\pi}\frac{\sqrt{2M\omega}}{M^2} \tag{3.30}$$

which differs from the value given in (3.29) by a factor of 1/2. This discrepancy arises due to an inappropriate expansion implicitly used in obtaining (3.29), which assumes that $\omega << \frac{M^2-Q^2}{2M}$, an invalid assumption for $Q = (M - \omega)$. In this context it should be noted that earlier work demonstrating that the transition probability of emitting a particle that will make the black hole extremal is zero [12].

The result is also odd because I obtain a temperature that depends on the energy of the emitted particle. I will pursue the extremal case further by considering a direct attempt to find the temperature by starting with the metric in its extremal form:

$$ds^2 = -(1-\frac{M}{r})^2 dt^2 + \frac{dr^2}{(1-\frac{M}{r})^2} + r^2 d\Omega^2 \tag{3.31}$$

Using the Hamilton-Jacobi ansatz (with proper spatial distance) as a first attempt only yields a diverging real component. i.e.

$$f(r) = g(r) = \frac{1}{M^2}(r-M)^2 + O((r-M)^3)$$

I will use proper (radial) spatial distance σ where:

$$\sigma = \int \frac{dr}{\sqrt{g(r)}} = M\int \frac{dr}{(r-M)} \simeq M\ln(r-M)$$
$$r - M = e^{\frac{\sigma}{M}}$$

where $M < r < \infty$ implies that bounds on σ are $-\infty < \sigma < \infty$. Rather than considering an observer at infinity in this case, I will consider an observer well outside the horizon at some r_1 corresponding to $\sigma(r_1)$. From the usual equations for the Hamilton Jacobi method (2.23)

$$\begin{aligned} W(r) &= \int \frac{dr}{\frac{1}{M^2}(r-M)^2}\sqrt{E^2 - \frac{1}{M^2}(r-M)^2(m^2 + \frac{h^{ij}J_iJ_j}{C(r)})} \\ &= M\int_{-\infty}^{\sigma(r_1)} \frac{d\sigma}{e^{\frac{\sigma}{M}}}\sqrt{E^2 - \frac{1}{M^2}e^{2\frac{\sigma}{M}}(m^2 + \frac{h^{ij}J_iJ_j}{C(r_0)})} \\ &= M\int_{-\infty}^{\sigma(r_1)} d\sigma\sqrt{E^2e^{-2\frac{\sigma}{M}} - \frac{1}{M^2}(m^2 + \frac{h^{ij}J_iJ_j}{C(r_0)})} \end{aligned} \tag{3.32}$$

For convenience I will choose a $\sigma(r_1)$ so that the term under the root is never negative. This integral is diverging and real, suggesting that no particles are emitted [28]. However this result is suspect in that it may be contingent on employing the near horizon approximation in the early stages of this method.

I will now turn to the null geodesic method. The outward radial null geodesic is given by

$$\dot{r} = 1 - \sqrt{1 - (1 - \frac{M}{r})^2} \tag{3.33}$$

$$= \frac{1}{2M^2}(r-M)^2 - \frac{1}{M^3}(r-M)^3 + O((r-M)^4) \tag{3.34}$$

Using the null geodesic method

$$\begin{aligned}
\operatorname{Im} I &\simeq \operatorname{Im}\left[-\omega \int_0^\pi \frac{\epsilon i e^{i\theta} d\theta}{\frac{1}{2M^2}\epsilon^2 e^{2i\theta}(1 + \frac{2}{M}\epsilon e^{i\theta})}\right] \\
&= -2\omega M^2 \operatorname{Im}\left[\int_0^\pi (\frac{i}{\epsilon e^{i\theta}} + \frac{2i}{M(1+\frac{2}{M}\epsilon e^{i\theta})})d\theta\right] \\
&= \operatorname{Im}\left[O(\frac{1}{\epsilon}) + 4M\omega[\ln(e^{-i\theta} + \frac{2\epsilon}{M})]|_0^\pi\right] \\
&= 4M\omega \operatorname{Im}\left[\ln\left(\frac{-1+\frac{2\epsilon}{M}}{1-\frac{2\epsilon}{M}}\right)\right] \\
&= (2n+1)\, 4\pi M\omega
\end{aligned} \tag{3.35}$$

where I have written $(r-M) = -\epsilon e^{i\theta}$ and n is an integer. The first part of the integral is a real contribution of $O(\frac{1}{\epsilon})$ that diverges as $\epsilon \to 0$ once again. It does not contribute to the imaginary part of the action. The imaginary part of the action implies a non-zero finite temperature

$$T_H = \frac{1}{8\pi M(2n+1)} \tag{3.36}$$

for any integer n. The extremal temperature is quantized in units of the temperature of a Schwarzschild black hole!

Note that this result depends crucially on the inclusion of the third order term, whose evaluation depends upon assumptions of the choice of Riemannian sheet. Had I expanded the integral for small ϵ, I would have obtained a value for the temperature given by $n = -1$ in eq. (3.36), i.e. a negative temperature for the extremal black hole.

Obtaining many (finite-valued) results for the temperature is reminiscent of the proposal that an extremal black hole can be in thermal equilibrium at any finite temperature [100]. However I can see that these strange results arise due to an inappropriate use of the WKB approximation in the null geodesic method. Although writing $(r-M) = -\epsilon e^{i\theta}$ is consistent with the the assumptions $r_{in} = r_0(M) - \epsilon$ and $r_{out} = r_0(M-\omega) + \epsilon$ (where $r_0(M)$ denotes the location of the event horizon of the original background space-time) for a non-extremal black hole, in fact the quantity r_{out} does not exist, since the extremal black hole cannot retain an event horizon upon emitting any neutral quantum of energy.

Its only option for future evolution would appear to be that of evolving into a naked singularity, which cosmic censorship forbids.

These results seem to imply that for black holes near extremality one must consider the full self-gravitating results, where the emitted particle drives the hole toward extremality. For an already extremal spacetime both methods yield a diverging real component in the action. This could be taken to imply that no particle can be emitted (since the alternative is creation of a naked singularity).

All of the previous discussion involved uncharged particles being emitted from a black hole. It is now interesting to consider the emission of charged particles from an extremal Reissner-Nordström black hole. In the last chapter I reviewed (from [31]) that the full emission rate (with charge) is:

$$\Gamma \sim \exp(\Delta S_{BH}) \tag{3.37}$$
$$= \exp\left(\pi\left[\left((M-\omega)+\sqrt{(M-\omega)^2-(Q-q)^2}\right)^2-\left(M+\sqrt{M^2-Q^2}\right)^2\right]\right) \tag{3.38}$$

With this expression it is possible for tunnelling to occur from extremal black hole to extremal black hole. Assume the black hole was originally extremal (i.e. $M^2 = Q^2$) and the resulting black hole after tunnelling is also extremal (i.e. $(M-\omega)^2 = (Q-q)^2 \implies \omega^2 = q^2$). Then 3.38 gives a resulting tunnelling probability of:

$$\Gamma = \exp(-2\pi\omega(M-\frac{\omega}{2})) \tag{3.39}$$

This would naively imply a temperature of:

$$T = \frac{1}{2\pi M}$$

or four times the Hawking temperature. The same result can actually be obtained in this case by using the low energy (and charge) approximation (i.e. $\omega << M$, $q << Q$) (2.46):

$$\Gamma = \exp\left(-2\pi\frac{(M+\sqrt{M^2-Q^2})^2}{\sqrt{M^2-Q^2}}\left[\omega-\frac{qQ}{M+\sqrt{M^2-Q^2}}\right]\right) \tag{3.40}$$

If you plug $q=\omega$ into this expression and take the limit as $Q \to M$:

$$\Gamma = \exp(-2\pi\omega M) \tag{3.41}$$

Which is consistent with the result above. This means that tunnelling from an extremal black hole to an extremal black hole is consistent with the WKB approximation in this case. In fact it is possible to start with the extremal metric, follow through the tunnelling method and again get the same result. This happens because even though the denominator of the integral will have a second order zero (i.e. $f(r) = (1-\frac{M}{r})^2$) the numerator will also vanish at the horizon (i.e. $\omega\left(1-\frac{Q}{r}\right)$ using $q=\omega$) so the resulting integral will have a simple pole at the horizon in this case. Because the tunnelling probability is non-zero doesn't necessarily mean that the temperature is non-zero though. In general the tunnelling method gives a tunnelling rate that is consistent with the first law of thermodynamics i.e.:

$$\begin{aligned}\Gamma &= \exp(\Delta S_{BH}) \\ &= \exp(\beta(\Delta E - \Phi\Delta Q)\end{aligned}$$

A particle emitted of energy ω corresponds to a change in black hole energy of $\Delta E = -\omega$ and a charge q corresponds to a change in black hole charge $\Delta Q = -q$. Giving:

$$\Gamma = \exp(-\beta(\omega - \Phi q)$$

This is the same form the tunnelling rate is commonly written as. So the fact that an extremal black hole can have a non-zero tunnelling rate corresponds to the change in entropy being non-zero but the temperature and the energy/work term (i.e. $\omega - \Phi q$) are both zero. (As shown explicitly above for the Reissner-Nordström black hole). The major caveat in order to have such tunnelling from extremal to extremal, is that an s-wave with $|q| = \omega$ is required. This seems unphysical and therefore unlikely to actually occur in our universe. It is useful to note, if such particles could exist then such emission would be consistent with the tunnelling model.

4 Analysis of Gödel Black holes

There has been a fair amount of activity in recent years studying Gödel-type solutions to 5d supergravity [67]-[80]. Various black holes embedded in Gödel universe backgrounds have been obtained as exact solutions [68, 70, 76] and their string-theoretic implications make them a lively subject of interest. For example Gödel type solutions have been shown to be T-dual to pp-waves [69]-[71]. Since closed-timelike curves (CTCs) exist in Gödel spacetimes these solutions can be used to investigate the implications of CTCs for string theory [72, 73, 75, 77]. These Gödel Black holes provide another useful testing ground to check the validity of the tunnelling method.

The particular black hole solutions of interest to me are of the Schwarzschild-Kerr type embedded in a Gödel universe [70]. A study of their thermodynamic behaviour [78, 80] has indicated that the expected relations of black hole thermodynamics are satisfied. Making use of standard Wick-rotation methods, their temperature has been shown to equal $\kappa/2\pi$ (where κ is the surface gravity) and confirmed that their entropy is equal to $A/4$ (where A is the surface area of the black hole). So the first law of thermodynamics has been shown to be satisfied. The presence of CTCs merits consideration of the applicability of the tunnelling method to Kerr-Gödel spacetimes. Due to the presence of a CTC "horizon" (I am using the term "horizon" loosely here). some qualitatively new features appear. The investigation of these spacetimes is in large part motivated by the fact that these new features provide additional tests as to the robustness of the tunnelling approach.

I will start this chapter with a thorough analysis of the Kerr-Gödel spacetime. I will begin by reviewing the Kerr-Gödel spacetime and some of its properties. I will then investigate properties of its parameter space and show that the CTC horizon is either outside both black hole horizons, inside both black hole horizons, or in coincidence with one of the horizons. I claim that it is not possible for the CTC horizon to be strictly in between the two black hole horizons, a property previously overlooked in discussions of this spacetime [80]. I will extend the investigation further insofar as include a brief discussion of the issues that occur when the CTC horizon is inside the black hole horizons. Once the Kerr-Gödel spacetime is understood I will then quickly review the tunnelling method and apply it to calculate the temperature of Kerr-Gödel spacetimes, showing consistency with previous thermodynamics results.

4.1 Summary of 5d Kerr-Gödel Spacetimes

The 5d Kerr-Gödel spacetime has the metric [70]

$$ds^2 = -f(r)(dt + \frac{a(r)}{f(r)}\sigma_3)^2 + \frac{dr^2}{V(r)} + \frac{r^2}{4}(\sigma_1^2 + \sigma_2^2) + \frac{r^2 V(r)}{4f(r)}\sigma_3^2 \tag{4.1}$$

$$A = \frac{\sqrt{3}}{2} j r^2 \sigma_3 \tag{4.2}$$

where

$$
\begin{aligned}
f(r) &= 1 - \frac{2m}{r^2} \\
a(r) &= jr^2 + \frac{ml}{r^2} \\
V(r) &= 1 - \frac{2m}{r^2} + \frac{16j^2m^2}{r^2} + \frac{8jml}{r^2} + \frac{2ml^2}{r^4}
\end{aligned}
$$

and the σ's are the right-invariant one-forms on SU(2), with Euler angles (θ, ϕ, ψ):

$$
\begin{aligned}
\sigma_1 &= \sin\phi d\theta - \cos\phi \sin\theta d\psi \\
\sigma_2 &= \cos\phi d\theta + \sin\phi \sin\theta d\psi \\
\sigma_3 &= d\phi + \cos\theta d\psi
\end{aligned}
$$

This metric may be obtained by embedding the Kerr black hole metric (with the two possible rotation parameters set to the same value i.e. $l_1 = l_2 = l$) in a 5-d Gödel universe.

This metric and gauge field satisfy the following 4+1 dimensional equations of motion:

$$
R_{\mu\nu} = 2(F_{\mu\alpha}F_{\nu}{}^{\alpha} - \frac{1}{6}g_{\mu\nu}F^2), \;\; D_\mu F^{\mu\nu} = \frac{1}{2\sqrt{3}}\tilde{\epsilon}^{\alpha\beta\gamma\mu\nu}F_{\alpha\beta}F_{\gamma\mu}
$$

where

$$
\tilde{\epsilon}_{\alpha\beta\gamma\mu\nu} = \sqrt{-g}\epsilon_{\alpha\beta\gamma\mu\nu}
$$

Some other useful ways to write the metric (4.1) are the expanded form:

$$
ds^2 = -f(r)dt^2 - 2a(r)dt\sigma_3 + g(r)\sigma_3^2 + \frac{dr^2}{V(r)} + \frac{r^2}{4}(\sigma_1^2 + \sigma_2^2)
$$

where

$$
g(r) = \frac{r^2V(r) - 4a^2(r)}{4f(r)} = -j^2r^4 + \frac{1 - 8mj^2}{4}r^2 + \frac{ml^2}{2r^2} \tag{4.3}
$$

and the lapse-shift form:

$$
ds^2 = -N^2dt^2 + g(r)(\sigma_3 - \frac{a(r)}{g(r)}dt)^2 + \frac{dr^2}{V(r)} + \frac{r^2}{4}(\sigma_1^2 + \sigma_2^2) \tag{4.4}
$$

where

$$
N^2 = f(r) + \frac{a^2(r)}{g(r)} = \frac{r^2V(r)}{4g(r)}
$$

When the parameters j and l are set to zero the metric simply reduces to the 5d Schwarzschild black hole, whose mass is proportional to the parameter m. The parameter j is the Gödel parameter and is responsible for the rotation of the spacetime; when $m = l = 0$ the metric reduces to that of the 5-d Gödel universe [67]. The parameter l is related to the rotation of the black hole. When $j = 0$

this reduces to the 5d Kerr black hole with the two possible rotation parameters (l_1, l_2) of the general 5-d Kerr spacetime set equal to l. When $l = 0$ the solution becomes the Schwarzschild-Gödel black hole. The metric is well behaved at the horizons and the scalars only become singular at the origin. It has been noted that the gauge field is not well behaved at the horizons [80] although it is possible to pass to a new gauge potential that is well behaved. When $g(r) < 0$ then ∂_ϕ will be timelike, indicating the presence of closed timelike curves since ϕ is periodic. The point at which $g(r) = 0$ is where the lapse (N^2) becomes infinite, implying that nothing can cross over to the CTC region from the region without CTC's. This property is implied by the geodesic solutions for Schwarzschild-Gödel found in [70] but I will argue in a later section of this chapter that this is a general property of Kerr-Gödel. The lapse vanishes when $V(r) = 0$; these points correspond to the black hole horizons.

The function $f(r)$ is equal to zero when $r = \sqrt{2m}$, corresponding to an ergosphere. The angular velocity of locally non-rotating observers is given by $\Omega = \frac{d\phi}{dt} = \frac{a(r)}{g(r)}$ with $\Omega_H = \frac{a(r_H)}{g(r_H)}$ denoting the angular velocity of the horizon. There is a special choice of parameters that will cause the angular velocity at the horizon to vanish (besides the trivial $l = j = 0$). When $l = -4jm$ then $V(r) = 0$ has solutions at $r^2 = 2m$ and $r^2 = 16j^2m^2$. The function $a(r)$ will be equal to zero for $r^2 = 2m$. Consequently Ω_H will vanish for the choice $l = -4jm$ at the horizon $r = \sqrt{2m}$.

For the case $l = 0$ there is only one black hole horizon located at $r_H = \sqrt{2m(1-8j^2m)}$. Clearly $1 > 8j^2m$ for the horizon to be well defined. A standard Wick-rotation approach yields a temperature $T_H = \frac{1}{2\pi\sqrt{2m(1-8j^2m)^3}}$ for the Schwarzschild-Gödel black hole [78], where the horizon has angular velocity $\Omega_H = \frac{4j}{(1-8j^2m)^2}$. There will be no CTC's for $r < r_{CTC} = \frac{\sqrt{(1-8mj^2)}}{2j}$ (the region where $g(r) > 0$), and the condition $r_{CTC} > r_H$ corresponds to $1 > 8j^2m$. Hence for $l = 0$ the CTC horizon is always outside of the black hole horizon. This property is not true for $l \neq 0$ and in the next section I will investigate the conditions under which the CTC horizon is no longer outside of the black hole horizons.

4.2 Analysis of Parameter Space of the 5d Kerr-Gödel

I will start by examining the parameter space of the 5d Kerr-Gödel spacetimes. The functions of interest are $g(r) = 0$, which determines the location of the CTC horizon and $V(r) = 0$ which determines the black hole horizons. I wish to find out how the horizons behave in terms of the parameters l and j. To simplify the analysis I will reparameterize as follows

$$J = j\sqrt{8m} \qquad L = \frac{l}{\sqrt{2m}} \qquad x = \frac{r^2}{2m} \tag{4.5}$$

so $x = 1$ at the ergosphere $(r^2 = 2m)$, $J^2 = 1$ when $8mj^2 = 1$, and the special case $l = -4jm$ corresponds to the choice $L = -J$.

The equations $V(r) = 0$ and $g(r) = 0$ now correspond respectively to the equations:

$$\frac{1}{x^2}(x^2 - (1 - J^2 - 2LJ)x + L^2) = 0 \tag{4.6}$$

$$-\frac{m}{2x}(J^2x^3 - (1 - J^2)x^2 - L^2) = 0 \tag{4.7}$$

There are two solutions to the quadratic equation (4.6) and there is only one real solution to (4.7) when L is non-zero (it can be shown that when $L = 0$ only the single non-zero solution of (4.7) is relevant). The solutions of (4.6) and (4.7) are respectively:

$$x_\pm = \frac{1}{2}\left((1 - J^2 - 2LJ) \pm \sqrt{(1 - J^2 - 2LJ)^2 - 4L^2}\right) \tag{4.8}$$

$$x_{ctc} = \frac{C(J, L)}{6J^2} + \frac{2(1 - J^2)^2}{3J^2C(J, L)} + \frac{1 - J^2}{3J^2} \tag{4.9}$$

where

$$C(J, L) = \left[108L^2J^4 + 8(1 - J^2)^3 + 12J^2\sqrt{3L^2(27L^2J^4 + 4(1 - J^2)^3)}\right]^{\frac{1}{3}}$$

The black hole is extremal when $x_+ = x_-$ and will occur when $J = \pm 1, J = -2L + 1, J = -2L - 1$. All three horizons will coincide when $J = -L = \pm 1$. Note that black hole horizons only exist when $x_\pm > 0$ since the horizon radii $r_\pm = \sqrt{2mx_\pm}$.

In Figure 2 I show a 3d plot of $\sqrt{x_{ctc}} - \sqrt{x_+}$ in terms of L and J. Note that when $J^2 > 1$ the value of $\sqrt{x_{ctc}} - \sqrt{x_+}$ is negative so the CTC horizon is inside the black hole horizon. In order to get a feel for how the horizons behave it is useful to plot all three horizons (inner, outer, and CTC) together for special values of J. The choices of J that are interesting are $J = -L$ (which is when $\Omega_H = 0$ at the horizon located at $x = 1$), and the extremal cases $J = 1$ and $J = -2L - 1$. These plots are shown in Figures 3, 4 a) and b) respectively. Notice that for figure 3 the CTC horizon is either outside both of r_+ and r_- (i.e. the black hole horizons) or inside both r_+ and r_- (this is also trivially true for the other two plots since they are extremal black holes). In all three plots the change from CTCs outside the black hole horizons to inside the horizons occurs when you go beyond the points $J = -L = \pm 1$.

I claim that is not possible to have the CTC horizon located in between the two black hole horizons. Assuming the contrary, consider the problem of finding values of J and L when the CTC horizon is in between the two black hole horizons. First look for solutions when the CTC horizon is in coincidence with one of the black hole horizons. So $x_{ctc} = x_\pm$ when the equation

$$(3J^2 + 2JL - 2)^2 + 4J^2 - 5J^4 = 0 \tag{4.10}$$

holds. Notice that $J = -L = \pm 1$ are solutions to (4.10).

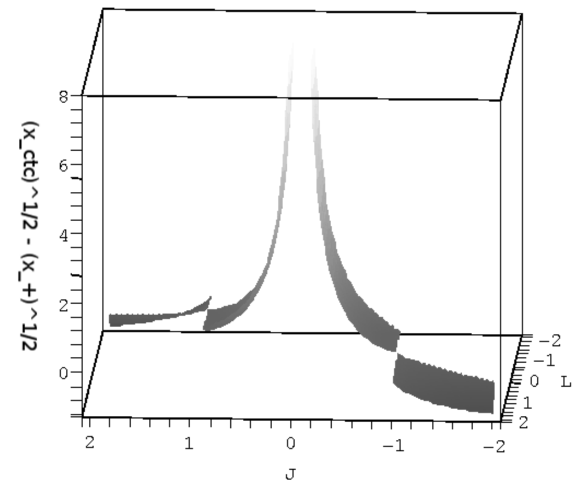

Figure 2: 3d plot of $\sqrt{x_{ctc}} - \sqrt{x_+}$ in terms of L and J.(i.e. compares location of the CTC horizon to largest black hole horizon) Note: Regions when $L^2 > 1$ are negative which means the CTC horizon is inside the black hole horizon. The region when $L^2 < 1$ is positive so the CTC horizon is outside the black hole horizon. The two points $J = -L = \pm 1$ are both zero and all horizons coincide. The peak at $J = 0$ corresponds to the infinite CTC horizon and indicates regular 5d Kerr.

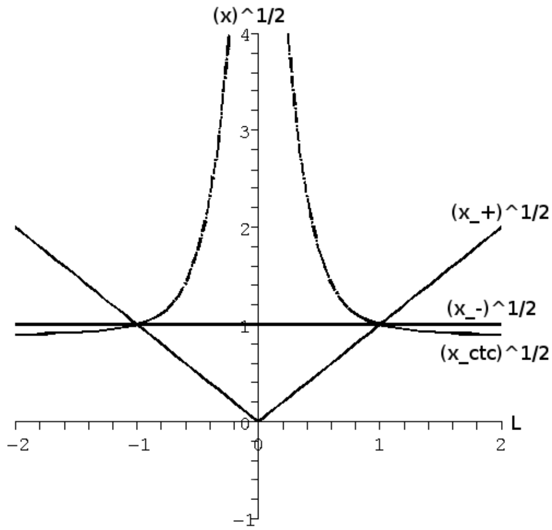

Figure 3: Plots of $\sqrt{x_{ctc}}, \sqrt{x_-}, \sqrt{x_+}$ in terms of L and when $J = -L$, Note: dashed line corresponds to $\sqrt{x_{ctc}}$ and solid lines are $\sqrt{x_-}, \sqrt{x_+}$. Notice the CTC is horizon is either outside both horizons or inside both horizons but never in between.

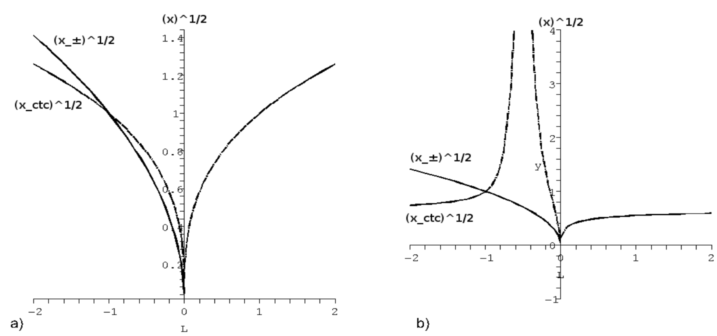

Figure 4: Plots of $\sqrt{x_{ctc}}, \sqrt{x_-}, \sqrt{x_+}$ it terms of a) L when $J = 1$ b) L when $J = -2L - 1$ Note: the dashed line corresponds to $\sqrt{x_{ctc}}$ and solid lines are $\sqrt{x_-}, \sqrt{x_+}$ (These are extremal cases so $\sqrt{x_-} = \sqrt{x_+}$) Notice, black hole horizons do not exist for a) when $L > 0$ and b) when $L > 0$. Also in b) $\sqrt{x_{ctc}}$ is infinite at $L = -\frac{1}{2}$ because. $J = 0$ which is 5d Kerr.

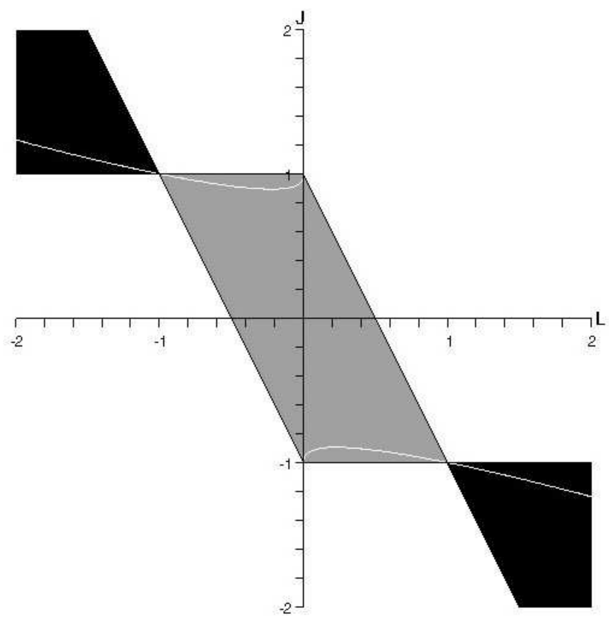

Figure 5: Plot of the horizon behavior in terms of J and L. In the grey region the CTC horizon is outside both black hole horizons. In the black region the CTC horizon is inside both black hole horizons. The white line corresponds to the special case when the CTC horizon is in coincidence with a black hole horizon (outer horizon in grey region, inner horizon in black region, and both at the special points $J = -L = \pm 1$). The white (uncoloured) region corresponds to naked singularities (no black hole horizons).

An analysis of the curve resulting from the left-hand-side of (4.10) indicates that when both $J^2 < 1$ and $L^2 < 1$ then the CTC horizon is coincident with the outer horizon; on either side of this curve the CTC horizon is outside both black hole horizons. When both $J^2 > 1$ and $L^2 > 1$ then the CTC horizon is coincident with the inner horizon, and on either side of this curve the CTC horizon is inside both r_+ and r_-. In all other regions of parameter space the metric (4.4) has naked singularities. Figure 5 illustrates this behaviour in terms of J and L. In the grey region the CTC horizon is outside both black hole horizons. In the black region the CTC horizon is inside both r_+ and r_-. The white line corresponds to the curves resulting from (4.10). In the white region the metric has no black hole horizons and naked singularities are present.

An alternate verification for the fact that the CTC horizon is never in between the black hole horizons may be obtained by substituting $x_\pm$ into $g(x)$, which shows that when $J^2 < 1$ then $g(x_-) > 0, g(x_+) \geq 0$ and for $J^2 > 1$ then $g(x_-) \leq 0, g(x_+) < 0$ (plots not shown). Conceptually it is easy to see why this property must be true by looking at the definition (4.3) of the function $g(r)$, which defines where the CTC horizon must be located. If $r = r_{ctc}$ then $g(r_{ctc}) = 0$ which implies $r_{ctc}^2 V(r_{ctc}) = 4a^2(r_{ctc})$. For this equality to be true then $V(r_{ctc})$ must be positive since every other term in the equation is positive. Since $V(r_{ctc})$ cannot be negative then r_{ctc} cannot be in between r_- and r_+.

Another property worth mentioning is the location of the black hole horizons in relation to $x = 1$ (i.e. the ergosphere $r = \sqrt{2m}$). When $J^2 < 1$ then $x_\pm \leq 1$ so the horizons are inside the ergosphere. When $J^2 > 1$ then $x_\pm \geq 1$ so the "horizons" are outside the ergosphere. Indeed when $J^2 > 1$ the surfaces $x_\pm = 1$ are not actually horizons, though I have been using this term as a counterpart to the $J^2 < 1$ case. Henceforth I shall refer to this as the "other region" of parameter space.

The finding that the CTC horizon can never be in between the two black hole horizons is contrary to assumptions made in previous work [80]. However the resultant thermodynamics is not significantly altered, as all main results consider only the situation when the CTC horizon is outside the black hole anyway. In the next two sections I will discuss the properties of the black hole region and the other region of parameter space.

4.2.1 Black Hole region of parameter space ($J^2 < 1$)

This is the region that is well understood and can be simply regarded as a Kerr Black Hole embedded in a Gödel space time, with the CTC horizon outside of the black hole horizons. To better understand this case I will take a look at the geodesics in the (t, r, ϕ) plane (with θ and ψ fixed). The metric becomes

$$ds^2 = -\frac{r^2 V(r)}{4g(r)} dt^2 + g(r)(d\phi - \frac{a(r)}{g(r)} dt)^2 + \frac{dr^2}{V(r)} \tag{4.11}$$

Note that $g(r_H) \geq 0$ for the choice of parameters ($-1 \leq J \leq 1, -\frac{1}{2} - \frac{J}{2} \leq L \leq \frac{1}{2} - \frac{J}{2}$) that are being considering. For convenience I impose the further restriction that $L \neq \frac{-3J^2+2+\sqrt{5J^4-4J^2}}{2J}$ and $L \neq \frac{-3J^2+2-\sqrt{5J^4-4J^2}}{2J}$ so that $g(r_H) > 0$ and the CTC horizon is strictly outside the outer black hole horizon.

The tangent vector to a geodesic is given by:

$$u^\alpha = [\dot{t}, \dot{r}, \dot{\phi}]$$

where dot denotes the derivative with respect to the affine parameter λ. For this metric ∂_t and ∂_ϕ are Killing vectors so in general the energy and angular momentum for these geodesics are respectively

$$\begin{aligned} E &= \frac{r^2V(r) - 4a^2(r)}{4g(r)}\dot{t} + a(r)\dot{\phi} \\ \ell &= -a(r)\dot{t} + g(r)\dot{\phi} \end{aligned}$$

I am interested in geodesics with $\ell = 0$. Note that for constant r the quantity $d\chi = d\phi - \frac{a(r)}{g(r)}dt$ is constant (i.e. $\frac{d\chi}{d\lambda} = 0$); for $r = r_H$ these correspond to geodesics for which $\chi = \phi - \Omega_H t$ is constant on the horizon (recall $\Omega_H = \frac{a(r_H)}{g(r_H)}$).

Setting $\ell = 0$ yields $\frac{a(r)}{g(r)}\ \dot{t} = \dot{\phi}$ and $E = \frac{r^2V(r)}{4g(r)}\dot{t}$., so for null geodesics I find $\dot{r} = \pm\frac{rV(r)}{2\sqrt{g(r)}}\dot{t}$, therefore:

$$u^\alpha_\pm = K_\pm\left[\frac{g(r)}{V(r)r^2}, \pm\frac{\sqrt{g(r)}}{2r}, \frac{a(r)}{V(r)r^2}\right] \tag{4.12}$$

where the plus/minus signs refer to outgoing/ingoing geodesics. When $l = 0$ this can be solved explicitly, and I recover the results for geodesic motion examined in ref. [70]. The $K_\pm$'s are constants related to the energy $E = \frac{K_\pm}{4}$. Choosing the normalization:

$$g_{\alpha\beta}u^\alpha_+ u^\beta_- = -1$$

at some point $r = r_0$, I obtain

$$K_+K_- = \frac{2r_0^2V(r_0)}{g(r_0)}$$

and for convenience I will pick $K_- = 1, K_+ = \frac{2r_0^2V(r_0)}{g(r_0)}$. The expansion scalar for null geodesics is

$$\Theta(u_\pm) = \pm\frac{K_\pm g'(r)}{4r\sqrt{g(r)}}$$

and it can be seen that for outgoing null rays there is a sign difference between geodesics starting inside the horizon ($r_0 < r_H$) and geodesics starting outside ($r_0 > r_H$), with no such change for ingoing geodesics, as expected for a trapped surface at $r = r_H$.

I can also say useful things about the CTC boundary. It occurs when $g(r_{ctc}) = 0$, so the expansions are infinite there. Furthermore $\frac{dr}{dt}$ is infinite and $\frac{dr}{d\lambda} = 0$ there, implying that null geodesics cannot cross the CTC boundary. These results are consistent with the observations for the Schwarzschild-Gödel ($l = 0$) case [70]: null geodesics will take infinite coordinate time t to go between the black hole horizon and CTC boundary. The CTC boundary is reached in finite affine parameter λ, although once the null ray reaches the CTC horizon it spirals back toward the black hole.

4.2.2 The Other Region of Parameter Space ($J^2 > 1$)

When $x_{ctc} < x_-$, i.e. the CTC boundary is the innermost surface, it is unclear what sort of object metric now represents. For convenience, I shall continue to use the term horizon to signify $x_\pm$, and the term ergosphere to denote the surface $x = 1$ ($r = \sqrt{2m}$), mindful of potential abuses of language. Both horizons are now outside of the ergosphere, but the CTC boundary can either be inside or outside of this surface, depending on the choice of parameters. For example for $J = 1.5, L = -2$, $x_{ctc} > 1$ (and $x_{ctc} < x_- < x_+$) but $x_{ctc} < 1$ for $J = 2, L = -2$.

To understand the causal properties of this spacetime I shall consider the metric for fixed θ and ψ for convenience. Consider the behaviour of

$$\begin{aligned} ds^2 &= -\frac{r^2 V(r)}{4g(r)}dt^2 + g(r)(d\phi - \frac{a(r)}{g(r)}dt)^2 + \frac{dr^2}{V(r)} \\ &= \frac{r^2 V(r)}{4\left|g(r)\right|}dt^2 - \left|g(r)\right|(d\phi + \frac{a(r)}{\left|g(r)\right|}dt)^2 + \frac{dr^2}{V(r)} \end{aligned} \tag{4.13}$$

where in the outer region $r > r_+$ it can be seen that $g(r) < 0$ and $V(r) > 0$, and I have rewritten the metric for fixed (θ, ψ) in the 2nd line above. For $r_- < r < r_+$ then $g(r) < 0$ and $V(r) < 0$.

I will now define a new coordinate $\chi = \phi - \frac{a(r_0)}{g(r_0)}t$ for some $r_0 > r_+$ and the metric is now

$$\begin{aligned} ds^2 = &(-f(r) + g(r)\Omega^2 - 2a(r)\Omega)dt^2 + 2(g(r)\Omega - a(r))dtd\chi \\ &+ g(r)d\chi^2 + \frac{dr^2}{V(r)} \end{aligned} \tag{4.14}$$

where $\Omega = \frac{a(r_0)}{g(r_0)}$. For a fixed value of $r = r_0 > r_+$ this metric simplifies to:

$$ds^2 = +\frac{r_0^2 V(r_0)}{4\left|g(r_0)\right|}dt^2 - \left|g(r_0)\right| d\chi^2 \tag{4.15}$$

Since $g_{tt} > 0, g_{\chi\chi} < 0$ notice that χ functions as the time coordinate, but only near $r = r_0$. For any given $r_0 > r_+$ it is possible to choose such a time coordinate χ in a neighbourhood of r_0 and the metric always has signature $(- + + + +)$.

When $r_- < r < r_+$ the signature of the metric becomes $(- - - + +)$ and so this region is not a physical spacetime. There is no choice of coordinate

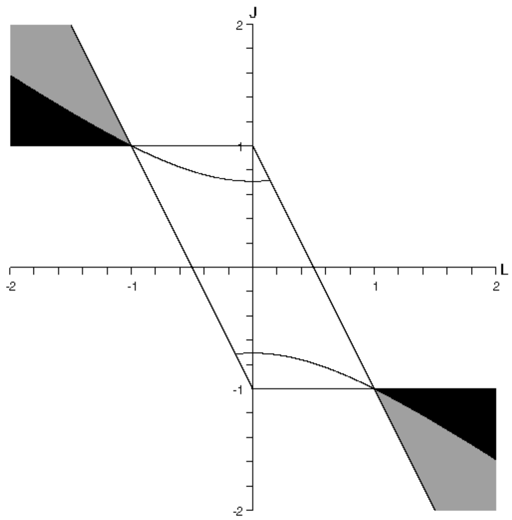

Figure 6: Plot of of horizon behavior in terms of J and L. Curve corresponds to $r_{ctc} = \sqrt{2m}$. In the black region $r_{ctc} > \sqrt{2m}$ and in the grey region $r_{ctc} < \sqrt{2m}$.

transformation that will allow the metric to have correct signature. This is easily seen by expanding the metric near r_+

$$\begin{aligned} ds^2 = &+ \frac{r_+^2 V'(r_+)}{4\,|g(r_+)|}(r-r_+)dt^2 + 2\left(g'(r_+)\Omega_+ - a'(r_+)\right)(r-r_+)dtd\chi \\ &- |g(r_+)|\, d\chi^2 + \frac{dr^2}{V'(r_+)(r-r_+)} \end{aligned}$$

indicating that the metric changes signature as r passes through r_+ from above. There is a conical singularity at $r = r_+$ that is removed by imposing a periodicity on t of $\frac{r_+ V'(r_+)}{8\pi\sqrt{|g(r_+)|}}$. Hence the region $r > r_+$ is a regular spacetime everywhere permeated by closed timelike curves due to the periodicity of ϕ and t. Setting $r < r_-$, I need to consider two distinct cases depending on where the "ergosphere" ($r_{ergo} = \sqrt{2m}$) is located with respect to the CTC horizon. These are $r_{ergo} < r_{ctc} < r_-$ (represented by the black region in figure 6 in terms of L and J parameters) and $r_{ctc} < r_{ergo} < r_-$ (represented by the grey region in figure 6 in terms of L and J parameters).

I will start with the $r_{ergo} < r_{ctc} < r_-$ case and I will restrict myself to CTC region $r_{ctc} < r < r_-$ so that $g(r) < 0$ and $V(r) > 0$. So for this case the metric is once again in the same form as (4.13). It is possible for an arbitrary r_0 ($r_{ctc} < r_0 < r_-$) to choose a coordinate $\chi = \phi - \frac{a(r_0)}{g(r_0)}t$ and the metrics (4.14) and (4.15) will be valid in this region. Expanding the metric near r_- gives

$$
\begin{aligned}
ds^2 = &+\frac{r_-^2|V'(r_-)|}{4\,|g(r_-)|}(r_- - r)dt^2 - 2\left(g'(r_-)\Omega_- - a'(r_-)\right)(r_- - r)dtd\chi \\
&- |g(r_-)|\,d\chi^2 + \frac{dr^2}{|V'(r_-)|(r_- - r)}
\end{aligned}
$$

again showing that when $r_- < r < r_+$ the signature of the metric becomes $(- - - + +)$. Removal of the conical singularity at $r = r_-$ is achieved by imposing a periodicity on t of $\frac{r_-|V'(r_-)|}{8\pi\sqrt{|g(r_-)|}}$. Notice that this differs from that imposed in the $r > r_+$ region, as expected for two regions that are disconnected spacetimes. Referring back to (4.14), for an arbitrary choice of χ it can be seen that $g_{\chi\chi} \to 0$ as the CTC horizon is approached. At the CTC horizon g_{tt} can be either positive or negative depending on the Ω that defined χ. Examining (4.14) at $r = r_{ctc}$, it is clear that g_{tt} will be positive if (a) Ω has an opposite sign to $a(r_{ctc})$ (in general this will be true since $\Omega = \frac{a(r_0)}{g(r_0)}$ and $g(r_0)$ is negative) and (b) $|\Omega| > \frac{f(r_{ctc})}{2|a(r_{ctc})|}$ (i.e. r_0 must be chosen to be close enough to r_{ctc} so that this inequality will be satisfied); otherwise g_{tt} will be negative at the CTC horizon. So for an arbitrary choice of parameters L and J it will not be possible to choose a single coordinate χ for which one can write the metric in a form in which ∇t is space-like everywhere between r_{ctc} and r_-. However for an arbitrary r_0 I can choose a coordinate χ so that in a neighbourhood of r_0 the metric can be written with ∇t space-like.

The $r_{ctc} < r_{ergo} < r_-$ case is a little more interesting due to the presence of the "ergosphere". Outside of the ergosphere the analysis remains the same as the previous $r < r_-$ case. Inside the ergosphere at any given r_0 one can still choose $\chi = \phi - \frac{a(r_0)}{g(r_0)}t$. However when $r_{ctc} < r < r_{ergo}$ it is sufficient to choose $\Omega = 0$ because $f(r)$ is negative and the metric

$$
\begin{aligned}
ds^2 &= -f(r)dt^2 - a(r)dtd\phi + g(r)d\phi^2 + \frac{dr^2}{V(r)} \\
&= |f(r)|dt^2 - a(r)dtd\phi + g(r)d\phi^2 + \frac{dr^2}{V(r)}
\end{aligned}
$$

is such that ϕ is the time coordinate inside the ergosphere.

4.3 Temperature Calculation

Turning now to the calculation of the black hole temperature, recall that the full metric in lapse shift form is (4.4). To employ the null geodesic method it is necessary to write the metric in a Painlevé form so that the null geodesic equations convey the semi-permeable nature of the black hole horizon (i.e. that it is easy to cross into the black hole but classically they cannot escape). The calculation will be treated in the came manner as the Kerr-Newman calculation of the last chapter. So I will convert to a corotating frame by defining $\chi =$

$\phi - \Omega_H t$. (for convenience I will set $d\chi = d\theta = d\psi = 0$). The emitted s-wave will carry angular momentum ℓ. I will do the null geodesic calculation for an s-wave of energy ω where $\omega = E - \Omega_H \ell$. (i.e. I will sub in $E - \Omega_H \ell$ into the last step to get the correct tunnelling probability).

$$\Gamma \simeq \exp(-\beta(E - \Omega_H \ell))$$

where Ω_H is the angular velocity of the black hole horizon. For this tunnelling probability to make sense I must require $E - \Omega_H \ell > 0$ as with the Kerr-Newman spacetime. This inequality corresponds to the s-wave being able to escape from the ergosphere. For calculating the temperature the metric I will use is:

$$ds^2 = -\frac{r^2 V(r)}{4g(r)} dt^2 + \frac{dr^2}{V(r)}$$

I can easily rewrite this in Painlevé form via:

$$t \to t - \int \frac{2\sqrt{g(r)}}{rV(r)} \sqrt{1 - V(r)} dr$$

giving (for constant χ, θ,and ψ) the following Painlevé metric

$$ds^2 = -\frac{r^2 V(r)}{4g(r)} dt^2 + \frac{r}{\sqrt{g(r)}} \sqrt{1 - V(r)} dr dt + dr^2$$

The radial null geodesic equation is given by:

$$\frac{dr}{dt} = \frac{r}{2\sqrt{g(r)}} (\pm 1 - \sqrt{1 - V(r)}) \tag{4.16}$$

where $+$ denotes outgoing and $-$ denotes ingoing geodesics (notice that $\frac{dr}{dt} = 0$ at the horizon for outgoing geodesics and $\frac{dr}{dt}$ is nonzero for ingoing geodesics). Inserting (4.16) into the usual expression for action:

$$I = \int_{r_{in}}^{r_{out}} \int_M^{M-\omega} \frac{dr}{\dot{r}} dH = \int_0^\omega \int_{r_{in}}^{r_{out}} \frac{dr}{\dot{r}} (-d\omega') \tag{4.17}$$

$1/\dot{r}$ has a first order pole at the horizon with residue $\frac{4\sqrt{g(r_H)}}{r_H V'(r_H)}$. So solving the integral:

$$\begin{aligned}
\operatorname{Im} I &= \frac{4\pi\omega\sqrt{g(r_H)}}{r_H V'(r_H)} + O(\omega^2) \\
\Gamma &\sim \exp(-2 \operatorname{Im} I) \simeq \exp\left(-\frac{8\pi\sqrt{g(r_H)}}{r_H V'(r_H)}\omega\right) \\
\Gamma &= \exp\left(-\frac{8\pi\sqrt{g(r_H)}}{r_H V'(r_H)}(E - \Omega_H \ell)\right)
\end{aligned}$$

This corresponds to a temperature:

$$T = \frac{r_H V'(r_H)}{8\pi\sqrt{g(r_H)}} \tag{4.18}$$

$$T = \frac{m(r_H^2(1 - 8j^2m - 4ml) - 2l^2)}{\pi r_H^3\sqrt{-4j^2r_H^6 + (1 - 8j^2m)r_H^4 + 2ml^2}} \tag{4.19}$$

This temperature is the same as that obtained using Wick-rotation methods [79, 80]; when $l = 0$ it reduces down to the Schwarzschild-Gödel temperature found in [78]. Note that the expression for the temperature diverges when $g(r_H) = 0$, which occurs when the CTC horizon is coincident with the outer horizon. The temperature is not defined when $g(r_H) < 0$, an unsurprising result considering the analysis of the other region of parameter space and the fact the when the CTC horizon is inside the r_- and r_+ horizons the derivation used is not valid. Not only is t not the correct time coordinate, but it is unclear how to even define tunnelling from inside r_+ because the region $r_- < r < r_+$ is not a spacetime.

Motivated by the de Sitter calculation consider what happens if the tunnelling method is applied the to the CTC horizon. From (4.16) it is known that $\frac{dr}{dt} \to \infty$ as $r \to r_{ctc}$.. This means that $1/\frac{dr}{dt}$ is simply zero at the CTC horizon. Since $1/\frac{dr}{dt}$ has no poles at the CTC horizon it means there is no tunnelling at the CTC horizon. So unlike a cosmological horizon nothing will tunnel out of a CTC horizon. Similarly to de Sitter spaces the observer will be inside the CTC horizon and outside the black hole horizon. In general when the parameters are chosen so that the CTC horizon is far away from the black hole horizon it should be possible to pick an observation location so that $\frac{r_{obs}^2 V(r_{obs})}{4g(r_{obs})} \approx 1$. In which case (4.19) will be the temperature seen by the observer. For a specific choice of parameters it may not be possible to have any location where $g_{00} \approx -1$. This means it will be necessary to pick a location for the observer and modify the temperature by dividing by the value of $\sqrt{-g_{00}}$ at the observer (i.e. Tolman redshift factor). The reasoning for this is the same as for de Sitter spacetimes as discussed in the appendix.

5 Fermion Emission

Since a black hole has a well defined temperature it should radiate all types of particles like a black body at that temperature. The emission spectrum therefore is expected to contain particles of all spins; the implications of this expectation were studied 30 years ago [8]. A strength of the tunnelling method is the ability to extend the model to other types of particles. Specifically, I extended the tunnelling method to model spin-1/2 fermions tunnelling from the black hole [50]. Prior to this no other fermion tunnelling model existed. In fact comparatively little has been done for fermion radiation for black holes at all. The Hawking temperature for fermion radiation has been calculated for 2d black holes [81] using the Bogoliubov transformation and more recently was calculated for evaporating black holes using a technique called the generalized tortoise coordinate transformation (GTCT) [82]-[84]. The latter result [84] is interesting because there is a contribution to the fermion emission probability due to a coupling effect between the spin of the emitted fermion and the acceleration of the Kinnersley black hole. From this one may infer that when fermions are emitted from rotating black holes there might be a coupling between the spin of the fermion and angular momentum of the rotating black hole present in the tunnelling probability. Unfortunately such a coupling for rotating black holes is not seen for the spin-1/2 tunnelling calculation. This is probably due to the fact that the method is a lowest order WKB approximation and such a coupling would probably need a higher order approximation to calculate.

In this chapter I will demonstrate how to extend the tunnelling method to model spin 1/2 particle emission from black holes. In order to do this I will follow an analogous approach to Hamilton-Jacobi method. Instead of applying a WKB approximation to the Klein Gordon equation, I will apply a WKB approximation to the Dirac Equation. I will consider Rindler spacetime first and confirm that the Unruh temperature is recovered. Insofar as fermionic vacua are distinct from bosonic vacua and can lead to distinct physical results [85], this result is non-trivial. I will then extend this technique to a general non-rotating 4-D black hole metric and show the Hawking temperature is recovered. I will illustrate this result in several coordinate systems – Schwarzschild, Painlevé, and Kruskal-Szekeres – to demonstrate that the result is independent of this choice. I will then extend the tunnelling of spin-1/2 particles to rotating black holes. From these calculations I will confirm that spin 1/2 fermions are also emitted at the Hawking Temperature. This final result, while not surprising, furnishes an important confirmation of the robustness of the tunnelling approach. This is one of the few calculations that can actually calculate the spin-1/2 fermion radiation. This demonstrates a strength of the tunnelling method over the Wick rotation method which can only model a black hole at equilibrium with a scalar particle heat bath..

It should be noted that one of the assumptions of this semi-classical calculation is to neglect any change of angular momentum of the black hole due to the spin of the emitted particle. For black holes with a mass much larger than the Planck mass this is a good approximation. Furthermore, statistically particles of opposite spin will be emitted in equal numbers, yielding no net change in the angular momentum of the black hole (although second-order statistical fluctuations will be present).

5.1 Spin 1/2 particles and Rindler Space

Remember that the tunnelling method for scalar particles can be applied to the apparent horizons of Rindler observers and will recover the Unruh temperature (this is shown in the third chapter). Motivated by this fact, I will start by applying fermion tunnelling to the Rindler space. A reason to start with Rindler space it is a little simpler to start with a spacetime that does not have angular coordinates. I will start with the Rindler calculation and then generalize the calculation to black holes.

I will only show the calculation explicitly for spin up case; the final result is also the same for the spin down case as can be easily shown using the methods described below. Due to the statistical nature of the heat bath I will assume that no angular momentum is imparted to the accelerating detector (i.e. on average there are as many spin up particles as spin down particles detected). The fermionic heat bath as seen by accelerated observers has many applications, such as understanding the effects of acceleration on entanglement [85].

I will use the following metric for Rindler spacetime

$$
\begin{aligned}
ds^2 &= -f(z)dt^2 + dx^2 + dy^2 + \frac{dz^2}{g(z)} \\
f(z) &= a^2z^2 - 1 \\
g(z) &= \frac{a^2z^2 - 1}{a^2z^2}
\end{aligned}
$$

so chosen for its convenience in extending the technique to non-rotating black holes. The Dirac equation is:

$$
i\gamma^{\mu} D_{\mu}\psi + \frac{m}{\hbar}\psi = 0 \tag{5.1}
$$

where:

$$
\begin{aligned}
D_{\mu} &= \partial_{\mu} + \Omega_{\mu} \\
\Omega_{\mu} &= \frac{1}{2} i \Gamma^{\alpha}{}_{\mu}{}^{\beta} \Sigma_{\alpha\beta} \\
\Sigma_{\alpha\beta} &= \frac{1}{4} i [\gamma^{\alpha}, \gamma^{\beta}]
\end{aligned}
$$

The γ^{μ} matrices satisfy $\{\gamma^{\mu}, \gamma^{\nu}\} = 2g^{\mu\nu} \times 1$. There are many different ways to choose the γ^μ matrices and I will use the following chiral form:

$$\gamma^t = \frac{1}{\sqrt{f(z)}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
$$\gamma^x = \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix}$$
$$\gamma^y = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}$$
$$\gamma^z = \sqrt{g(z)} \begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix}$$

The vierbeins used here are: $e_0^t = \frac{1}{\sqrt{f(z)}}, e_1^x = 1, e_2^y = 1, e_3^z = \sqrt{g(z)}$ and the $\sigma's$ are simply the Pauli Sigma Matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$
$$\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and $\xi_{\uparrow/\downarrow}$ are the eigenvectors of σ^3. Note that

$$\gamma^5 = i\gamma^t\gamma^x\gamma^y\gamma^z = \sqrt{\frac{g(z)}{f(z)}} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

is the resulting γ^5 matrix.

Measuring spin in the z-direction (i.e. the direction of the accelerating observer) I will employ the following ansatz for the Dirac field, respectively corresponding to the spin up and spin down cases:

$$\begin{aligned} \psi_\uparrow(t,x,y,z) &= \begin{bmatrix} A(t,x,y,z)\xi_\uparrow \\ B(t,x,y,z)\xi_\uparrow \end{bmatrix} \exp\left[\frac{i}{\hbar} I_\uparrow(t,x,y,z)\right] \\ &= \begin{bmatrix} A(t,x,y,z) \\ 0 \\ B(t,x,y,z) \\ 0 \end{bmatrix} \exp\left[\frac{i}{\hbar} I_\uparrow(t,x,y,z)\right] \end{aligned} \tag{5.2}$$

$$\begin{aligned} \psi_\downarrow(t,x,y,z) &= \begin{bmatrix} C(t,x,y,z)\xi_\downarrow \\ D(t,x,y,z)\xi_\downarrow \end{bmatrix} \exp\left[\frac{i}{\hbar} I_\downarrow(t,x,y,z)\right] \\ &= \begin{bmatrix} 0 \\ C(t,x,y,z) \\ 0 \\ D(t,x,y,z) \end{bmatrix} \exp\left[\frac{i}{\hbar} I_\downarrow(t,x,y,z)\right] \end{aligned} \tag{5.3}$$

Notice that the ansatz of spin up and spin down are eigenvectors of the spin operator (in the z-direction) defined by:

$$S^z = e^z_i S^i$$

where

$$S^i = \frac{\hbar}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}$$

(this follows from the chiral form of the gamma matrices I have chosen) and therefore from the vierbeins for this system

$$S^z = \sqrt{g(z)}\frac{\hbar}{2}\begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix} \tag{5.4}$$

Another important operator to define for fermions is the helicity

$$\begin{aligned} h &\equiv \hat{p}\cdot\mathbf{S} = \hat{p}_z e^z_3 S^3 \\ h &= \hat{p}_z\sqrt{g(z)}\frac{\hbar}{2}\begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix} \end{aligned} \tag{5.5}$$

Remember that at the Rindler observer $g(z) = 1$ so the spin in the z-direction and helicity will be measured as $\pm\hbar/2$ by the observer. Where $\pm$ depends on which spin ansatz is used and also the direction of motion in the case of helicity.

In order to apply the WKB approximation I will insert the ansatz (5.2) for spin up particles into the Dirac Equation. Dividing by the exponential term and multiplying by $\hbar$ the resulting equations to leading order in $\hbar$ are:

$$-B\left(\frac{1}{\sqrt{f(z)}}\partial_t I_\uparrow + \sqrt{g(z)}\partial_z I_\uparrow\right) + Am = 0 \tag{5.6}$$

$$-B\left(\partial_x I_\uparrow + i\partial_y I_\uparrow\right) = 0 \tag{5.7}$$

$$A\left(\frac{1}{\sqrt{f(z)}}\partial_t I_\uparrow - \sqrt{g(z)}\partial_z I_\uparrow\right) + Bm = 0 \tag{5.8}$$

$$-A\left(\partial_x I_\uparrow + i\partial_y I_\uparrow\right) = 0 \tag{5.9}$$

Note that although A, B are not constant, their derivatives – and the components Ω_μ – are all of a higher order in $\hbar$ and so can be neglected to lowest order in WKB.

When $m \neq 0$ equations (5.6) and (5.8) couple whereas when $m = 0$ they decouple. I will employ the ansatz

$$I_\uparrow = -Et + W(z) + P(x, y) \tag{5.10}$$

and insert it into equations (5.6-5.9)

$$-B\left(\frac{-E}{\sqrt{f(z)}}+\sqrt{g(z)}W'(z)\right)+mA=0 \tag{5.11}$$

$$-B\left(P_x+iP_y\right)=0 \tag{5.12}$$

$$-A\left(\frac{E}{\sqrt{f(z)}}+\sqrt{g(z)}W'(z)\right)+mB=0 \tag{5.13}$$

$$-A\left(P_x+iP_y\right)=0 \tag{5.14}$$

where I only consider the positive frequency contributions without loss of generality. Equations (5.12) and (5.14) both yield $(P_x+iP_y)=0$ regardless of A or B, implying

$$P(x,y)=h(x+iy) \tag{5.15}$$

where h is some arbitrary function.

Consider first $m=0$. Equations (5.11) and (5.13) then have two possible solutions

$$A=0 \text{ and } W'(z)=W'_+(z)=\frac{E}{\sqrt{f(z)g(z)}}$$
$$B=0 \text{ and } W'(z)=W'_-(z)=\frac{-E}{\sqrt{f(z)g(z)}}$$

corresponding to motion away from (+) and toward (-) the horizon, chosen to be at $z=1/a$.

Since the solution $[A,0,0,0]$ is an eigenvector of γ^5 and has a negative eigenvalue; its spin and momentum vectors are opposite, which is consistent with the fact that the particle is moving down toward the horizon and the spin is up. This will also have negative helicity since $\hat{p}_r$ is in the negative r-direction which is consistent with the left handed chirality. The solution $[0,0,B,0]$ is also an eigenvector of γ^5 with positive eigenvalue; its spin and momentum vectors are therefore in the same direction, consistent with the particle being spin up and moving away from the horizon. This also has positive helicity consistent with having right handed chirality.

Hence with the Rindler horizon at $z=1/a$ the $(\pm)$ cases correspond to outgoing/incoming solutions of the same spin. Note that neither of these cases is an antiparticle solution since I have assumed positive frequency modes as a part of the ansatz. In computing the imaginary part of the action take note that $P(x,y)$ must be complex (other than the trivial solution of $P=0$), and so will yield a contribution. However it is the same for both incoming and outgoing solutions, and so will cancel out in computing the emission probability

$$\Gamma\propto\frac{\mathrm{Prob}[out]}{\mathrm{Prob}[in]}=\frac{\exp[-2(\mathrm{Im}\,W_++\mathrm{Im}\,h)]}{\exp[-2(\mathrm{Im}\,W_-+\mathrm{Im}\,h)]} \tag{5.16}$$

$$=\exp[-2(\mathrm{Im}\,W_+-\mathrm{Im}\,W_-)=\exp[-4\,\mathrm{Im}\,W_+] \tag{5.17}$$

using reasoning similar to the scalar case.

Solving for W:

$$W_+(z) = \int \frac{Edz}{\sqrt{f(z)g(z)}}$$

and after integrating around the pole (and dropping the + subscript)

$$W = \frac{\pi i E}{\sqrt{g'(z_0)f'(z_0)}} = \frac{\pi i E}{2a} \tag{5.18}$$

The resulting tunnelling probability is

$$\Gamma = \exp[-\frac{2\pi}{a}E]$$

recovering

$$T_H = \frac{a}{2\pi} \tag{5.19}$$

which is the Unruh temperature.

In the massive case equations (5.11) and (5.13) no longer decouple. I will start by eliminating the function $W'(z)$ from the two equations so I can find an equation relating A and B in terms of known values. Subtracting $B\times$(5.13) from $A\times$ (5.11) gives

$$\frac{2ABE}{\sqrt{f(z)}} + mA^2 - mB^2 = 0$$

$$m\sqrt{f(z)}(\frac{A}{B})^2 + 2E(\frac{A}{B}) - m\sqrt{f(z)} = 0$$

and so

$$\frac{A}{B} = \frac{-E \pm \sqrt{E^2 + m^2 f(z)}}{m\sqrt{f(z)}}$$

where

$$\lim_{z\to z_0}\left(\frac{-E + \sqrt{E^2 + m^2 f(z)}}{m\sqrt{f(z)}}\right) = 0$$

$$\lim_{z\to z_0}\left(\frac{-E - \sqrt{E^2 + m^2 f(z)}}{m\sqrt{f(z)}}\right) = -\infty$$

Consequently at the Rindler horizon either $\frac{A}{B} \to 0$ or $\frac{A}{B} \to -\infty$, i.e. either $A \to 0$ or $B \to 0$. For $A \to 0$ at the horizon, I solve (5.13) in terms of m and insert into (5.11)

$$-B\left(\frac{-E}{\sqrt{f(z)}} + \sqrt{g(z)}W'(z)\right) + \frac{A^2}{B}\left(\frac{E}{\sqrt{f(z)}} + \sqrt{g(z)}W'(z)\right) = 0$$

$$\frac{EB}{\sqrt{f(z)}}\left(1 + \frac{A^2}{B^2}\right) - B\sqrt{g(z)}W'(z)\left(1 - \frac{A^2}{B^2}\right) = 0$$

$$W'(z) = W'_+(z) = \frac{E}{\sqrt{f(z)g(z)}} \frac{\left(1 + \frac{A^2}{B^2}\right)}{\left(1 - \frac{A^2}{B^2}\right)}$$

whereas for $B \to 0$ at the horizon, I solve (5.11) in terms of m and insert into (5.13) to get

$$-A\left(\frac{E}{\sqrt{f(z)}} + \sqrt{g(z)}W'(z)\right) + \frac{B^2}{A}\left(\frac{-E}{\sqrt{f(z)}} + \sqrt{g(z)}W'(z)\right) = 0$$

$$-\frac{EA}{\sqrt{f(z)}}\left(1 + \frac{B^2}{A^2}\right) - A\sqrt{g(z)}W'(z)\left(1 - \frac{B^2}{A^2}\right) = 0$$

$$W'(z) = W'_-(z) = \frac{-E}{\sqrt{f(z)g(z)}} \frac{\left(1 + \frac{B^2}{A^2}\right)}{\left(1 - \frac{B^2}{A^2}\right)}$$

Since the extra contributions vanish at the horizon, the result of integrating around the pole for W in the massive case is the same as the massless case and the Unruh temperature is recovered for the fermionic Rindler vacuum.

The spin-down case proceeds in a manner fully analogous to the spin-up case discussed above. Other than some changes of sign the equations are of the same form as the spin up case. For both the massive and massless cases the Unruh temperature (5.19) is obtained, implying that both spin up and spin down particles are emitted at the same rate.

5.2 Fermion Emission of Non-Rotating Black Holes

I will now turn to a non-rotating black hole. I will ignore any change in the angular momentum of the black hole due to the spin of the emitted particle. This is a good approximation for black holes of sufficient large mass. The zero angular momentum state is maintained because statistically as many particles with spin in one direction will be emitted as particles with spin in the opposite direction. In order to check the robustness of the fermion tunnelling calculation, I will also check that the method works when coordinates that do not have singularities at the horizon are used. So I will explicitly demonstrate that the fermion calculation works with the Painlevé and Kruskal-Szekeres metrics.

I will now extend the fermion tunnelling approach to a general black hole with spherical symmetry. The metric is

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{g(r)} + r^2(d\theta^2 + \sin^2(\theta)d\phi^2) \tag{5.20}$$

where, for this case, I will pick for the γ matrices

$$
\begin{aligned}
\gamma^t &= \frac{1}{\sqrt{f(r)}}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \\
\gamma^r &= \sqrt{g(r)}\begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix} \\
\gamma^\theta &= \frac{1}{r}\begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix} \\
\gamma^\phi &= \frac{1}{r\sin\theta}\begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}
\end{aligned}
$$

and I measure spin in terms of the r-direction. The matrix for γ^5 is

$$
\gamma^5 = i\gamma^t\gamma^r\gamma^\theta\gamma^\phi = i\sqrt{\frac{g(r)}{f(r)}}\frac{1}{r^2\sin\theta}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
$$

The spin up (i.e. +ve r-direction) and spin down (i.e. -ve r-direction) solutions have the form

$$
\begin{aligned}
\psi_\uparrow(t,r,\theta,\phi) &= \begin{bmatrix} A(t,r,\theta,\phi)\xi_\uparrow \\ B(t,r,\theta,\phi)\xi_\uparrow \end{bmatrix}\exp\left[\frac{i}{\hbar}I_\uparrow(t,r,\theta,\phi)\right] \\
&= \begin{bmatrix} A(t,r,\theta,\phi) \\ 0 \\ B(t,r,\theta,\phi) \\ 0 \end{bmatrix}\exp\left[\frac{i}{\hbar}I_\uparrow(t,r,\theta,\phi)\right] \qquad (5.21) \\
\psi_\downarrow(t,x,y,z) &= \begin{bmatrix} C(t,r,\theta,\phi)\xi_\downarrow \\ D(t,r,\theta,\phi)\xi_\downarrow \end{bmatrix}\exp\left[\frac{i}{\hbar}I_\downarrow(t,r,\theta,\phi)\right] \\
&= \begin{bmatrix} 0 \\ C(t,r,\theta,\phi) \\ 0 \\ D(t,r,\theta,\phi) \end{bmatrix}\exp\left[\frac{i}{\hbar}I_\downarrow(t,r,\theta,\phi)\right] \qquad (5.22)
\end{aligned}
$$

I will only solve the spin up case explicitly since the spin-down case is fully analogous. Employing the ansatz (5.21) into the Dirac equation results in

$$
-\left(\frac{iA}{\sqrt{f(r)}}\partial_t I_\uparrow + B\sqrt{g(r)}\partial_r I_\uparrow\right) + Am = 0 \qquad (5.23)
$$

$$
-\frac{B}{r}\left(\partial_\theta I_\uparrow + \frac{1}{\sin\theta}i\partial_\phi I_\uparrow\right) = 0 \qquad (5.24)
$$

$$
\left(\frac{iB}{\sqrt{f(r)}}\partial_t I_\uparrow - A\sqrt{g(r)}\partial_r I_\uparrow\right) + Bm = 0 \qquad (5.25)
$$

$$
-\frac{A}{r}\left(\partial_\theta I_\uparrow + \frac{1}{\sin\theta}i\partial_\phi I_\uparrow\right) = 0 \qquad (5.26)
$$

to leading order in $\hbar$. I assume the action takes the form

$$I_{\uparrow} = -Et + W(r) + J(\theta, \phi) \tag{5.27}$$

where I only concern myself with positive frequency contributions as before. This yields

$$\left(\frac{iAE}{\sqrt{f(r)}} - B\sqrt{g(r)}W'(r)\right) + mA = 0 \tag{5.28}$$

$$-\frac{B}{r}\left(J_\theta + \frac{1}{\sin\theta} iJ_\phi\right) = 0 \tag{5.29}$$

$$-\left(\frac{iBE}{\sqrt{f(r)}} + A\sqrt{g(r)}W'(r)\right) + Bm = 0 \tag{5.30}$$

$$-\frac{A}{r}\left(J_\theta + \frac{1}{\sin\theta} iJ_\phi\right) = 0 \tag{5.31}$$

Notice that (5.29) and (5.31) result in the same equation regardless of A or B (i.e. $\left(J_\theta + \frac{1}{\sin\theta} iJ_\phi\right) = 0$ must be true), implying that $J(\theta, \phi)$ must be a complex function. As with the Rindler case, the same solution for J is obtained for both the outgoing and incoming cases. Consequently the contribution from J cancels out upon dividing the outgoing probability by the incoming probability as in (5.17). I can therefore ignore J from this point (or else pick the trivial $J = 0$ solution).

Equations (5.28) and (5.30) (for $m = 0$) have two possible solutions:

$$A = -iB \text{ and } W'(r) = W'_+(r) = \frac{E}{\sqrt{f(r)g(r)}}$$

$$A = iB \text{ and } W'(r) = W'_-(r) = \frac{-E}{\sqrt{f(r)g(r)}}$$

where W_+ corresponds to outward solutions and W_- correspond to the incoming solutions. The overall tunnelling probability is

$$\Gamma = \frac{\mathrm{Prob}[out]}{\mathrm{Prob}[in]} = \frac{\exp[-2(\mathrm{Im}\, W_+)]}{\exp[-2(\mathrm{Im}\, W_-)]} = \exp[-4\, \mathrm{Im}\, W_+] \tag{5.32}$$

with

$$W_+(r) = \int \frac{E dr}{\sqrt{f(r)g(r)}}$$

After integrating around the pole (and dropping the + subscript) I find

$$W = \frac{\pi i E}{\sqrt{g'(r_0) f'(r_0)}} \tag{5.33}$$

giving

$$\Gamma = \exp[-\frac{4\pi}{\sqrt{g'(r_0) f'(r_0)}} E] \tag{5.34}$$

for the resultant tunnelling probability to leading order in $\hbar$.

So I recover the expected Hawking Temperature

$$T_H = \frac{\sqrt{f'(r_0)g'(r_0)}}{4\pi} \tag{5.35}$$

in the massless case.

Solving equations (5.28) and (5.30) for A and B in the case that $m \neq 0$ leads to the result:

$$\left(\frac{A}{B}\right)^2 = \frac{-iE + \sqrt{f(r)}m}{iE + \sqrt{f(r)}m}$$

and approaching the horizon it can be seen that $\lim_{r\to r_0}\left(\frac{A}{B}\right)^2 = -1$. Following a procedure similar to what was done above, I will obtain the same result for the Hawking Temperature as in the massless case.

The spin-down calculation is very similar to the spin-up case discussed above. Other than some changes of sign, the equations are of the same form as the spin up case. For both the massive and massless spin down cases the Hawking temperature (5.35) is obtained, implying that both spin up and spin down particles are emitted at the same rate. This is consistent with the initial assumption that there are as many spin up as spin down fermions emitted.

5.2.1 Painlevé Coordinates

The Painlevé coordinate system is an important coordinate for scalar particle emission because of its use with the null geodesic method. It has also been shown that the Painlevé coordinates work with the Hamilton-Jacobi method [21],[46]. It is important to confirm that the tunnelling of Dirac particles also works with the Painlevé coordinates. In this section I will demonstrate that Painlevé coordinates can be used to recover the results of the preceding section, albeit by a somewhat different computational route.

Using the transformation

$$t \to t - \int \sqrt{\frac{1-g\,(r)}{f\,(r)\,g\,(r)}}dr \tag{5.36}$$

with the metric (5.20) will give:

$$ds^2 = -f(r)dt^2 + 2\sqrt{f(r)}\sqrt{\frac{1}{g(r)} - 1}drdt + dr^2 + r^2 d\Omega^2 \tag{5.37}$$

which is the Painlevé form of a spherically symmetric metric.

This coordinate system has a number of interesting features. At any fixed time the spatial geometry is flat. At any fixed radius the boundary geometry for the Painlevé metric is exactly the same as that of the unaltered black hole metric. Also, this form of the Painlevé metric is a very convenient metric to use for black hole tunnelling since the imaginary part of the action for the incoming solution is zero which means Prob$[in] = 1$ as has been seen in the scalar particle case for the null geodesic method. This property also holds for fermion tunnelling.

I will choose the representation for the γ matrices to be

$$\begin{aligned}
\gamma^t &= \frac{1}{\sqrt{f(r)}}\begin{pmatrix} 0 & 1+\sqrt{1-g(r)}\sigma^3 \\ -1+\sqrt{1-g(r)}\sigma^3 & 0 \end{pmatrix} \\
\gamma^r &= \sqrt{g(r)}\begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix} \\
\gamma^\theta &= \frac{1}{r}\begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix} \\
\gamma^\phi &= \frac{1}{r\sin\theta}\begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}
\end{aligned}$$

The matrix for γ^5 for this case is:

$$\gamma^5 = i\gamma^t\gamma^r\gamma^\theta\gamma^\phi = \sqrt{\frac{g(r)}{f(r)}}\frac{1}{r^2\sin\theta}\begin{pmatrix} -1-\sqrt{1-g(r)}\sigma^3 & 0 \\ 0 & +1-\sqrt{1-g(r)}\sigma^3 \end{pmatrix}$$

Measuring spin in the r-direction as before, I will have the two following ansatz for the spin 1/2 Dirac field which correspond to the spin up (i.e. +ve r-direction) and spin down (i.e. -ve r-direction) cases respectively:

$$\begin{aligned}
\psi_\uparrow(t,r,\theta,\phi) &= \begin{bmatrix} A(t,r,\theta,\phi)\xi_\uparrow \\ B(t,r,\theta,\phi)\xi_\uparrow \end{bmatrix}\exp\left[\frac{i}{\hbar}I_\uparrow(t,r,\theta,\phi)\right] \\
&= \begin{bmatrix} A(t,r,\theta,\phi) \\ 0 \\ B(t,r,\theta,\phi) \\ 0 \end{bmatrix}\exp\left[\frac{i}{\hbar}I_\uparrow(t,r,\theta,\phi)\right] \qquad (5.38) \\
\psi_\downarrow(t,x,y,z) &= \begin{bmatrix} C(t,r,\theta,\phi)\xi_\downarrow \\ D(t,r,\theta,\phi)\xi_\downarrow \end{bmatrix}\exp\left[\frac{i}{\hbar}I_\downarrow(t,r,\theta,\phi)\right] \\
&= \begin{bmatrix} 0 \\ C(t,r,\theta,\phi) \\ 0 \\ D(t,r,\theta,\phi) \end{bmatrix}\exp\left[\frac{i}{\hbar}I_\downarrow(t,r,\theta,\phi)\right] \qquad (5.39)
\end{aligned}$$

Once again I will only solve the spin up case explicitly. Insertion of the ansatz into the Dirac equation results in the following equations to the leading order in $\hbar$.

$$-B\left(\frac{1}{\sqrt{f(r)}}\left(1+\sqrt{1-g(r)}\right)\partial_t I_\uparrow+\sqrt{g(r)}\partial_r I_\uparrow\right)+Am=0 \tag{5.40}$$

$$-\frac{B}{r}\left(\partial_\theta I_\uparrow+\frac{1}{\sin\theta}i\partial_\phi I_\uparrow\right)=0 \tag{5.41}$$

$$A\left(\frac{1}{\sqrt{f(r)}}\left(1-\sqrt{1-g(r)}\right)\partial_t I_\uparrow-\sqrt{g(r)}\partial_r I_\uparrow\right)+Bm=0 \tag{5.42}$$

$$-\frac{A}{r}\left(\partial_\theta I_\uparrow+\frac{1}{\sin\theta}i\partial_\phi I_\uparrow\right)=0 \tag{5.43}$$

To solve these equations I pick the ansatz (5.27) for the action, again working only with positive frequency contributions. The equations for J are the same as in the last section, and I can dispense with this function for the same reasons as before. I will obtain

$$B\left(\frac{1}{\sqrt{f(r)}}\left(1+\sqrt{1-g(r)}\right)E-\sqrt{g(r)}W'(r)\right)+Am=0 \tag{5.44}$$

$$-A\left(\frac{1}{\sqrt{f(r)}}\left(1-\sqrt{1-g(r)}\right)E+\sqrt{g(r)}W'(r)\right)+Bm=0 \tag{5.45}$$

Equations (5.44) and (5.45) (for $m=0$) have two possible solutions:

$$A=0 \text{ and } W'(r)=W'_+(r)=\frac{E\left(1+\sqrt{1-g(r)}\right)}{\sqrt{f(r)g(r)}}$$

$$B=0 \text{ and } W'(r)=W'_-(r)=\frac{-E\left(1-\sqrt{1-g(r)}\right)}{\sqrt{f(r)g(r)}}$$

W_+ corresponds to outward solutions and W_- correspond to the incoming solutions. Notice that W'_+ have a pole at the horizon but W'_- has a well defined limit at the horizon and does not have a pole (i.e. $\lim_{r\to r_0} W'_-(r) = \frac{-E}{2}\sqrt{\frac{g'(r_0)}{f'(r_0)}}$). This implies that the the imaginary part W_- is zero and confirms that Prob$[in]=1$. So the overall tunnelling probability is:

$$\Gamma \propto \text{Prob}[out]$$
$$\Gamma \propto \exp[-2\,\text{Im}\,W_+]$$

$\therefore$

$$W_+(r)=\int\frac{E\left(1+\sqrt{1-g(r)}\right)dr}{\sqrt{f(r)g(r)}}$$

and after integrating around the pole (and dropping the + subscript):

$$W=\frac{2\pi iE}{\sqrt{g'(r_0)f'(r_0)}} \tag{5.46}$$

So the resulting tunnelling probability is once again:

$$\Gamma = \exp[-\frac{4\pi}{\sqrt{g'(r_0)f'(r_0)}}E]$$

and the normal Hawking Temperature is also recovered for the Painlevé massless case

$$T_H = \frac{\sqrt{f'(r_0)g'(r_0)}}{4\pi} \tag{5.47}$$

Solving equations (5.44) and (5.45) for A and B in the case that $m \neq 0$ leads to the results that $A \to 0$ as $r \to r_0$ or $B \to 0$ as $r \to r_0$. So the same final result will be recovered in the massive case.

5.2.2 Kruskal-Szekeres Metric

It is useful to check that the tunnelling calculations can be applied to multiple coordinate systems to check the validity. Another useful metric to check the tunnelling method with is the Kruskal-Szekeres metric. In the second chapter I showed how scalar particle tunnelling could be successfully applied to the Kruskal-Szekeres metric. Motivated by the scalar particle calculation, I will extend fermion tunnelling to the Kruskal-Szekeres metric:

$$ds^2 = f(r)\left(-dT^2 + dX^2\right) + r^2 d\Omega^2 \tag{5.48}$$

where:

$$f(r) = \frac{32M^3 e^{-\frac{r}{2M}}}{r} \qquad\qquad (\frac{r}{2M} - 1)e^{r/2M} = X^2 - T^2$$

The metric (5.48) is well behaved at both the future and past horizons $X = \pm T$ (corresponding to $r = 2M$). Note that the metric has a timelike Killing vector $X\partial_T + T\partial_X$ (and not ∂_T).

For this calculation I will employ the following representation for the γ matrices

$$\gamma^T = \frac{1}{\sqrt{f(r)}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
$$\gamma^X = \frac{1}{\sqrt{f(r)}}\begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix}$$
$$\gamma^\theta = \frac{1}{r}\begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix}$$
$$\gamma^\phi = \frac{1}{r\sin\theta}\begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}$$

where spin is measured referenced to the X-direction. The matrix for γ^5 is

$$\gamma^5 = i\gamma^t\gamma^r\gamma^\theta\gamma^\phi = \frac{1}{f(r)}\frac{1}{r^2\sin\theta}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

The spin up (i.e. +ve X-direction) and spin down (i.e. -ve X-direction) solutions have the form

$$\begin{aligned}
\psi_\uparrow(T,X,\theta,\phi) &= \begin{bmatrix} A(T,X,\theta,\phi)\xi_\uparrow \\ B(T,X,\theta,\phi)\xi_\uparrow \end{bmatrix} \exp\left[\frac{i}{\hbar} I_\uparrow(T,X,\theta,\phi)\right] \\
&= \begin{bmatrix} A(T,X,\theta,\phi) \\ 0 \\ B(T,X,\theta,\phi) \\ 0 \end{bmatrix} \exp\left[\frac{i}{\hbar} I_\uparrow(T,X,\theta,\phi)\right] \qquad (5.49) \\
\psi_\downarrow(T,X,y,z) &= \begin{bmatrix} C(T,X,\theta,\phi)\xi_\downarrow \\ D(T,X,\theta,\phi)\xi_\downarrow \end{bmatrix} \exp\left[\frac{i}{\hbar} I_\downarrow(T,X,\theta,\phi)\right] \\
&= \begin{bmatrix} 0 \\ C(T,X,\theta,\phi) \\ 0 \\ D(T,X,\theta,\phi) \end{bmatrix} \exp\left[\frac{i}{\hbar} I_\downarrow(T,X,\theta,\phi)\right] \qquad (5.50)
\end{aligned}$$

Once again inserting the spin-up ansatz (5.49) (the spin-down case being similar) into the Dirac equation yields the following equations

$$-\frac{B}{\sqrt{f(r)}}\left(\partial_T I_\uparrow + \partial_X I_\uparrow\right) + Am = 0 \qquad (5.51)$$

$$-\frac{B}{r}\left(\partial_\theta I_\uparrow + \frac{1}{\sin\theta} i\partial_\phi I_\uparrow\right) = 0 \qquad (5.52)$$

$$\frac{A}{\sqrt{f(r)}}\left(\partial_T I_\uparrow - \partial_X I_\uparrow\right) + Bm = 0 \qquad (5.53)$$

$$-\frac{A}{r}\left(\partial_\theta I_\uparrow + \frac{1}{\sin\theta} i\partial_\phi I_\uparrow\right) = 0 \qquad (5.54)$$

to leading order in $\hbar$. This time it is only possible to infer that the action takes the form

$$I_\uparrow = -I(X,T) + J(\theta,\phi) \qquad (5.55)$$

The equations for J are unchanged from previous calculations. I will ignore these equations since they do not affect the final result and only concern myself with solving for $I(X,T)$.

In order to solve the equations a definition for the energy of the wave is required. I will define energy via the timelike killing vector

$$\partial_\chi = N(X\partial_T + T\partial_X)$$

where N is a normalization constant chosen so that the norm of the Killing vector is equal to -1 at infinity. This yields

$$\partial_\chi = \frac{1}{4M}(X\partial_T + T\partial_X) \qquad (5.56)$$

and so

$$\partial_\chi I = -E \qquad (5.57)$$

Using (5.57) with (5.51) and (5.53) it is possible to solve the equations.

Consider first the massless case. Here either $A = 0$ or $B = 0$. For $A = 0$ (outgoing case):

$$\partial_T I + \partial_X I = 0$$
$$\frac{1}{4M}(X\partial_T I + T\partial_X I) = -E$$

The first equation implies the general solution of $I = h(X - T)$ and the second in turn leads to

$$4ME = (X - T)h'(X - T)$$
$$h'(X - T) = \frac{4ME}{(X - T)}$$

which has a simple pole at the black hole horizon $X = T$. Setting $\eta = X - T$ yields

$$h'(\eta) = \frac{4ME}{\eta} \tag{5.58}$$

Integrating (5.58) around the pole at the horizon (doing a half circle contour) implies

$$\operatorname{Im} I_{out} = 4\pi ME$$

for outgoing particles.

For the incoming case $B = 0$ and so

$$\partial_T I - \partial_X I = 0$$
$$\frac{1}{4M}(X\partial_T I + T\partial_X I) = -E$$

The first equation implies the general solution $I = k(X + T)$ and so the second leads to

$$-4ME = (X + T)k'(X + T)$$
$$k'(X + T) = \frac{-4ME}{(X + T)}$$

Note that this equation does not have a pole at the black hole horizon $X = T$. Hence for incoming particles

$$\operatorname{Im} I_{in} = 0$$

and so $\text{Prob}[in] = 1$ like in the Painlevé case. The final result for the tunnelling probability is

$$\Gamma = \frac{\text{Prob}[out]}{\text{Prob}[in]} = \exp[-2\operatorname{Im} I_{out}] = \exp[-8\pi ME]$$

and the Hawking Temperature $T_H = \frac{1}{8\pi M}$ is recovered in the massless case.

In the massive case, use equations (5.57), (5.51) and (5.53) to solve for $\frac{A}{B}$. A straightforward calculation yields

$$\frac{A}{B} = \frac{-4ME \pm \sqrt{16M^2E^2 + m^2 f(r)(X^2 - T^2)}}{\sqrt{f(r)}m(X+T)} \tag{5.59}$$

notice, as the black hole horizon ($X = T$) is approached that either $\frac{A}{B} \to 0$ or $\frac{A}{B} \to \frac{-4ME}{\sqrt{f(2M)}mT} = \frac{-4ME}{\sqrt{f(2M)}mX}$. Subtracting (5.51)/$A$ from (5.53)/B leads to

$$\partial_T I = -\partial_X I \frac{(1 - (\frac{A}{B})^2)}{(1 + (\frac{A}{B})^2)}$$

and so from (5.57):

$$\partial_X I = \frac{4ME(1 + (\frac{A}{B})^2)}{\left[X(1 - (\frac{A}{B})^2) - T(1 + (\frac{A}{B})^2)\right]} \tag{5.60}$$

where $\frac{A}{B} \to 0$ at $X = T$.

From (5.59) it follows:

$$\lim_{X \to T} \left[X(1 - (\frac{A}{B})^2) - T(1 + (\frac{A}{B})^2)\right] = 0$$

and

$$\lim_{X \to T} \frac{\partial}{\partial X} \left[X(1 - (\frac{A}{B})^2) - T(1 + (\frac{A}{B})^2)\right]$$
$$= \lim_{X \to T} \left[(1 - (\frac{A}{B})^2) - 2(X+T)(\frac{A}{B})\frac{\partial}{\partial X}(\frac{A}{B})\right] = 1$$

Consequently $\partial_X I$ has a simple pole at the black hole horizon implying $\mathrm{Im}\, I_{out} = 4\pi ME$ in the massive case. Note that when $\frac{A}{B} \to \frac{-4ME}{\sqrt{f(2M)}mT}$ then $\partial_X I$ does not have a pole at the horizon, implying that $\mathrm{Im}\, I_{in} = 0$. The rest of the calculation proceeds as before, and the Hawking temperature is recovered in the massive case.

5.3 Charged Spin 1/2 Particle Emission From Kerr-Newman Black Holes

It is non-trivial to extend the tunnelling method from non-rotating spacetimes to rotating spacetimes. Issues with scalar particle tunnelling in rotating spacetimes were discussed in the third chapter and in the appendix. Similar issues also occur for fermion tunnelling from rotating spacetimes. It is very important to choose an appropriate ansatz for the gamma matrices in rotating spacetimes (see appendix). In order to extend fermion tunnelling to rotating spacetimes I will consider particle emission from the Kerr-Newman solution. Since the

Kerr-Newman metric is being considered I will also consider charged particle emission from this spacetime (remember charged emission of scalar particles was reviewed in the second chapter). The Kerr-Newman metric and vector potential are given by:

$$
\begin{aligned}
ds^2 &= -f(r,\theta)dt^2 + \frac{dr^2}{g(r,\theta)} - 2H(r,\theta)dtd\phi + K(r,\theta)d\phi^2 + \Sigma(r,\theta)d\theta^2 \\
A_a &= -\frac{er}{\Sigma(r)}[(dt)_a - a\sin^2\theta(d\phi)_a] \qquad (5.61)\\
f(r,\theta) &= \frac{\Delta(r) - a^2\sin^2\theta}{\Sigma(r,\theta)}, \\
g(r,\theta) &= \frac{\Delta(r)}{\Sigma(r,\theta)}, \\
H(r,\theta) &= \frac{a\sin^2\theta(r^2+a^2-\Delta(r))}{\Sigma(r,\theta)} \\
K(r,\theta) &= \frac{(r^2+a^2)^2 - \Delta(r)a^2\sin^2\theta}{\Sigma(r,\theta)}\sin^2(\theta) \\
\Sigma(r,\theta) &= r^2 + a^2\cos^2\theta \\
\Delta(r) &= r^2 + a^2 + e^2 - 2Mr
\end{aligned}
$$

Since the tunnelling method is not immediately applicable to extremal black holes without further technical considerations (see the third chapter), I will only assume a non-extremal black hole so that $M^2 > a^2 + e^2$. In fact fermion tunnelling from an extremal black holes will have another technical problem beyond even the scalar particle case; the assumption that the effect the spin of the particle on the black hole can be safely ignored will not be valid for an extremal black hole.

Since the black hole is not extremal, there are two horizons located at $r_\pm = M \pm \sqrt{M^2 - a^2 - e^2}$. It is convenient for these calculations to work with the function $F(r,\theta) = -\left(g^{tt}\right)^{-1}$ where

$$F(r,\theta) = f(r,\theta) + \frac{H^2(r,\theta)}{K(r,\theta)} = \frac{\Delta(r)\Sigma(r,\theta)}{(r^2+a^2)^2 - \Delta(r)a^2\sin^2\theta} \qquad (5.62)$$

and where the angular velocity at the black hole horizon is

$$\Omega_H = \frac{H(r_+,\theta)}{K(r_+,\theta)} = \frac{a}{r_+^2 + a^2} \qquad (5.63)$$

I will only show the calculation explicitly for the spin up case; the final result is also the same for the spin down case as can be easily shown using the methods described below. In the non-rotating case a statistical argument is used to justify the assumption that overall a zero angular momentum state is maintained for fermion emission, because as many particles with spin pointing radially outward (spin up) would be emitted as particles with spin pointed radially inward (spin down). This argument is still valid in the rotating case; the statistical distribution of spins in the fermion emission spectrum should not alter the angular momentum of the black hole.

The Dirac equation with electric charge is:

$$i\gamma^{\mu}(D_{\mu} - \frac{iq}{\hbar}A_{\mu})\psi + \frac{m}{\hbar}\psi = 0 \tag{5.64}$$

where:

$$D_{\mu} = \partial_{\mu} + \Omega_{\mu} \tag{5.65}$$

$$\Omega_{\mu} = \frac{1}{2}i\Gamma^{\alpha}{}_{\mu}{}^{\beta}\Sigma_{\alpha\beta} \tag{5.66}$$

$$\Sigma_{\alpha\beta} = \frac{1}{4}i[\gamma^{\alpha}, \gamma^{\beta}] \tag{5.67}$$

The γ^{μ} matrices satisfy $\{\gamma^{\mu}, \gamma^{\nu}\} = 2g^{\mu\nu} \times 1$. I will choose a representation for them in the form:

$$\begin{aligned} \gamma^{t} &= \frac{1}{\sqrt{F(r,\theta)}}\gamma^{0} \quad \gamma^{r} = \sqrt{g(r,\theta)}\gamma^{3} \qquad \gamma^{\theta} = \frac{1}{\sqrt{\Sigma(r,\theta)}}\gamma^{1} \\ \gamma^{\phi} &= \frac{1}{\sqrt{K(r,\theta)}}\left(\gamma^{2} + \frac{H(r,\theta)}{\sqrt{F(r,\theta)K(r,\theta)}}\gamma^{0}\right) \end{aligned} \tag{5.68}$$

where the γ^{a}'s are simply the following chiral γ's for Minkowski space

$$\begin{aligned} \gamma^{0} &= \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \quad \gamma^{1} = \begin{pmatrix} 0 & \sigma^{1} \\ \sigma^{1} & 0 \end{pmatrix} \\ \gamma^{2} &= \begin{pmatrix} 0 & \sigma^{2} \\ \sigma^{2} & 0 \end{pmatrix} \quad \gamma^{3} = \begin{pmatrix} 0 & \sigma^{3} \\ \sigma^{3} & 0 \end{pmatrix} \end{aligned} \tag{5.69}$$

and the $\sigma's$ are the Pauli Matrices

$$\sigma^{1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^{2} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^{3} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5.70}$$

and I denote $\xi_{\uparrow/\downarrow}$ for the eigenvectors of σ^{3}. Note that

$$\gamma^{5} = i\gamma^{t}\gamma^{r}\gamma^{\theta}\gamma^{\phi} = \sqrt{\frac{g}{FK\Sigma}}\begin{pmatrix} -I + \frac{H}{\sqrt{FK}}\sigma^{2} & 0 \\ 0 & I + \frac{H}{\sqrt{FK}}\sigma^{2} \end{pmatrix} \tag{5.71}$$

is the resulting γ^{5} matrix.

The spin up (i.e. +ve r-direction) ansatz for the Dirac field, has the form (motivated by the nonrotating cases):

$$\begin{aligned}\psi_{\uparrow}(t,r,\theta,\phi) &= \begin{bmatrix} A(t,r,\theta,\phi)\xi_{\uparrow} \\ B((t,r,\theta,\phi)\xi_{\uparrow} \end{bmatrix} \exp\left[\frac{i}{\hbar} I_{\uparrow}(t,r,\theta,\phi)\right] \\ &= \begin{bmatrix} A(t,r,\theta,\phi) \\ 0 \\ B(t,r,\theta,\phi) \\ 0 \end{bmatrix} \exp\left[\frac{i}{\hbar} I_{\uparrow}(t,r,\theta,\phi)\right] \end{aligned} \tag{5.72}$$

In order to apply the WKB approximation I will insert the ansatz (5.72) for spin up particles into the Dirac equation. Dividing by the exponential term and multiplying by $\hbar$ the resulting equations to leading order in $\hbar$ are

$$\begin{aligned} 0 = -B \Bigg[& \frac{1}{\sqrt{F(r,\theta)}}\partial_t I_{\uparrow} + \sqrt{g(r,\theta)}\partial_r I_{\uparrow} + \frac{H(r,\theta)}{K(r,\theta)\sqrt{F(r,\theta)}}\partial_\phi I_{\uparrow} \\ & + \frac{qer}{\Sigma(r,\theta)\sqrt{F(r,\theta)}}\left(1 - \frac{H(r,\theta)}{K(r,\theta)} a\sin^2(\theta)\right)\Bigg] + Am \end{aligned} \tag{5.73}$$

$$0 = -B\left[\frac{i}{\sqrt{K(r,\theta)}}(\partial_\phi I_{\uparrow} - \frac{qer}{\Sigma(r,\theta)} a\sin^2\theta) + \frac{1}{\sqrt{\Sigma(r,\theta)}}\partial_\theta I_{\uparrow}\right] \tag{5.74}$$

$$\begin{aligned} 0 = A \Bigg[& \frac{1}{\sqrt{F(r,\theta)}}\partial_t I_{\uparrow} - \sqrt{g(r,\theta)}\partial_r I_{\uparrow} + \frac{H(r,\theta)}{K(r,\theta)\sqrt{F(r,\theta)}}\partial_\phi I_{\uparrow} \\ & + \frac{qer}{\Sigma(r,\theta)\sqrt{F(r,\theta)}}\left(1 - \frac{H(r,\theta)}{K(r,\theta)} a\sin^2(\theta)\right)\Bigg] + Bm \end{aligned} \tag{5.75}$$

$$0 = -A\left[\frac{i}{\sqrt{K(r,\theta)}}(\partial_\phi I_{\uparrow} - \frac{qer}{\Sigma(r,\theta)} a\sin^2\theta) + \frac{1}{\sqrt{\Sigma(r,\theta)}}\partial_\theta I_{\uparrow}\right] \tag{5.76}$$

Note that although A, B are not constant, their derivatives – and the components Ω_μ – are all of a higher order in $\hbar$ and so can be neglected to lowest order in WKB.

When $m \neq 0$ equations (5.73) and (5.75) couple whereas when $m = 0$ they decouple. I will employ the standard ansatz

$$I_{\uparrow} = -Et + J\phi + W(r,\theta) \tag{5.77}$$

and insert it into equations (5.73-5.76) (where I will only consider the positive frequency contributions without loss of generality). To simplify the expressions, I will expand the equations near the horizon and find:

$$0 = -B\left(\frac{(-E+\Omega_H J+\frac{qer_+}{r_+^2+a^2})}{\sqrt{F_r(r_+,\theta)(r-r_+)}}+\sqrt{g_r(r_+,\theta)(r-r_+)}W_r(r,\theta)\right)+Am \quad (5.78)$$

$$0 = -B\left(\frac{i}{\sqrt{K(r_+,\theta)}}(J-\frac{qer_+}{\Sigma(r_+,\theta)}a\sin^2\theta)+\frac{1}{\sqrt{\Sigma(r_+,\theta)}}W_\theta(r,\theta)\right) \quad (5.79)$$

$$0 = A\left(\frac{(-E+\Omega_H J+\frac{qer_+}{r_+^2+a^2})}{\sqrt{F_r(r_+,\theta)(r-r_+)}}-\sqrt{g_r(r_+,\theta)(r-r_+)}W_r(r,\theta)\right)+Bm \quad (5.80)$$

$$0 = -A\left(\frac{i}{\sqrt{K(r_+,\theta)}}(J-\frac{qer_+}{\Sigma(r_+,\theta)}a\sin^2\theta)+\frac{1}{\sqrt{\Sigma(r_+,\theta)}}W_\theta(r,\theta)\right) \quad (5.81)$$

where

$$g_r(r_+,\theta) = \frac{\Delta_r(r_+)}{\Sigma(r_+,\theta)} = \frac{2r_+-2M}{r_+^2+a^2\cos^2(\theta)}$$
$$F_r(r_+,\theta) = \frac{\Delta_r(r_+)\Sigma(r_+,\theta)}{(r_+^2+a^2)^2} = \frac{(2r_+-2M)(r_+^2+a^2\cos^2(\theta))}{(r_+^2+a^2)^2}$$

In the massless case it is possible to pull $\frac{1}{\sqrt{\Sigma(r_+,\theta)}}$ out of equations (5.78) and (5.80), making these equations independent of θ. Furthermore, equations (5.79) and (5.81) have no explicit r dependence. From this, it is possible to conclude that near the black horizon it is possible to further separate the function W

$$W(r,\theta) = W(r)+\Theta(\theta)$$

and the resulting equations (5.79) and (5.81) both yield the same equation for Θ regardless of A or B.

Equations (5.78) and (5.80) then have two possible solutions

$$A = 0 \text{ and } W'(r) = W'_+(r) = \frac{(E-\Omega_H J-\frac{qer_+}{r_+^2+a^2})(r_+^2+a^2)}{\Delta_r(r_+)(r-r_+)}$$
$$B = 0 \text{ and } W'(r) = W'_-(r) = \frac{-(E-\Omega_H J-\frac{qer_+}{r_+^2+a^2})(r_+^2+a^2)}{\Delta_r(r_+)(r-r_+)}$$

where the prime denotes a derivative with respect to r and $W_{+/-}$ corresponds to outgoing/incoming solutions.

The probabilities of crossing the horizon in each direction are proportional to

$$\text{Prob}[out] \propto \exp[-2\,\text{Im}\,I] = \exp[-2(\text{Im}\,W_+ + \text{Im}\,\Theta)] \quad (5.82)$$
$$\text{Prob}[in] \propto \exp[-2\,\text{Im}\,I] = \exp[-2(\text{Im}\,W_- + \text{Im}\,\Theta)] \quad (5.83)$$

To ensure that the probabilities are correctly normalized so that any incoming particles crossing the horizon have a 100% chance of entering the black hole I will need to divide each equation by (5.83). From this the probability of going from outside to inside the horizon will be equal to 1 and this implies that the probability of a particle tunnelling from inside to outside the horizon is:

$$\Gamma \propto \frac{\mathrm{Prob}[out]}{\mathrm{Prob}[in]} = \frac{\exp[-2(\operatorname{Im} W_+ + \operatorname{Im}\Theta)]}{\exp[-2(\operatorname{Im} W_- + \operatorname{Im}\Theta)]} = \exp[-4 \operatorname{Im} W_+] \tag{5.84}$$

Solving for W_+ yields

$$W_+(r) = \int \frac{(E - \Omega_H J - \frac{qer_+}{r_+^2+a^2})(r_+^2 + a^2)}{\Delta_r(r_+)(r - r_+)}$$

and after integrating around the pole (and dropping the + subscript) I obtain

$$\begin{aligned} W &= \frac{\pi i(E - \Omega_H J - \frac{qer_+}{r_+^2+a^2})(r_+^2 + a^2)}{2r_+ - 2M} \\ \operatorname{Im} W &= (E - \Omega_H J - \frac{qer_+}{r_+^2 + a^2})\frac{\pi}{2}\frac{r_+^2 + a^2}{(r_+ - M)} \end{aligned} \tag{5.85}$$

The resulting tunnelling probability is

$$\Gamma = \exp\left[-2\pi\frac{r_+^2 + a^2}{(r_+ - M)}\left(E - \Omega_H J - \frac{qer_+}{r_+^2 + a^2}\right)\right]$$

giving the expected Hawking temperature

$$T_H = \frac{1}{2\pi}\frac{r_+ - M}{r_+^2 + a^2} = \frac{1}{2\pi}\frac{(M^2 - a^2 - e^2)^{\frac{1}{2}}}{2M(M + (M^2 - a^2 - e^2)^{\frac{1}{2}}) - e^2} \tag{5.86}$$

for a charged rotating black hole.

In the massive case equations (5.78) and (5.80) no longer decouple and analysis of the tunnelling is more subtle. I will begin by eliminating the function $W_r(r, \theta)$ from these two equations and will find an equation relating A and B in terms of known quantities. Subtracting $B\times$(5.80) from $A\times$ (5.78) gives

$$0 = \frac{2AB(E - \Omega_H J - \frac{qer_+}{r_+^2+a^2})}{\sqrt{F_r(r_+, \theta)(r - r_+)}} + mA^2 - mB^2 = 0 \tag{5.87}$$

$$\begin{aligned} 0 = {} & m\sqrt{F_r(r_+, \theta)(r - r_+)}(\frac{A}{B})^2 + 2(E - \Omega_H J - \frac{qer_+}{r_+^2 + a^2})(\frac{A}{B}) \\ & - m\sqrt{F_r(r_+, \theta)(r - r_+)} \end{aligned} \tag{5.88}$$

and so

$$\frac{A}{B} = \frac{-(E - \Omega_H J - \frac{qer_+}{r_+^2+a^2}) \pm \sqrt{(E - \Omega_H J - \frac{qer_+}{r_+^2+a^2})^2 + m^2 F_r(r_+, \theta)(r - r_+)}}{m\sqrt{F_r(r_+, \theta)(r - r_+)}}$$

where

$$\lim_{r\to r_+}\left(\frac{-(E-\Omega_H J-\frac{qer_+}{r_+^2+a^2})\pm\sqrt{(E-\Omega_H J-\frac{qer_+}{r_+^2+a^2})^2+m^2F_r(r_+,\theta)(r-r_+)}}{m\sqrt{F_r(r_+,\theta)(r-r_+)}}\right) \tag{5.89}$$

$$=\left\{\begin{array}{c} 0 \\ -\infty \end{array}\right. \tag{5.90}$$

for the upper/lower sign respectively.

Consequently at the horizon either $\frac{A}{B}\to 0$ or $\frac{A}{B}\to -\infty$, i.e. either $A\to 0$ or $B\to 0$. For $A\to 0$ at the horizon, I will solve (5.80) in terms of m and insert into (5.78), obtaining

$$W_r(r,\theta)=\frac{(E-\Omega_H J-\frac{qer_+}{r_+^2+a^2}{}_+)}{\sqrt{F_r(r_+,\theta)g_r(r_+,\theta)}(r-r_+)}\frac{\left(1+\frac{A^2}{B^2}\right)}{\left(1-\frac{A^2}{B^2}\right)} \tag{5.91}$$

Note that the θ-dependence drops out of this expression, i.e.

$$W_r(r,\theta)\equiv W'_+(r)=\frac{(E-\Omega_H J-\frac{qer_+}{r_+^2+a^2})}{\sqrt{F_r(r_+,\theta)g_r(r_+,\theta)}(r-r_+)}\frac{\left(1+\frac{A^2}{B^2}\right)}{\left(1-\frac{A^2}{B^2}\right)}$$

since $\frac{A}{B}$ is zero at the horizon the result of integrating around the pole is the same as in the massless case. For $B\to 0$ I can simply rewrite the expression (5.91) in terms of $\frac{B}{A}$ to get

$$W_r(r,\theta)\equiv W'_-(r)=\frac{-(E-\Omega_H J-\frac{qer_+}{r_+^2+a^2})}{\sqrt{F_r(r_+,\theta)g_r(r_+,\theta)}(r-r_+)}\frac{\left(1+\frac{B^2}{A^2}\right)}{\left(1-\frac{B^2}{A^2}\right)}$$

Again, since the extra contributions vanish at the horizon, the result of integrating around the pole for W in the massive case is the same as the massless case and I recover the Hawking temperature (5.86) for the Kerr-Newman black hole.

The spin-down case proceeds in a manner fully analogous to the spin-up case discussed above. Other than some changes of sign, the equations are of the same form as the spin up case. For both the massive and massless cases the temperature (5.86) is obtained, implying that both spin up and spin down particles are emitted at the same rate.

6 Conclusions

In this thesis I have investigated the black hole tunnelling method and black hole thermodynamics. I have shown that the method can successfully be applied to a wide range of spacetimes. I have extended the method to model Dirac particle emission and verified its validity in multiple spacetimes. These results indicate that the tunnelling method can be seen to be robust in the sense that it works with an extensive range of horizons. Since the tunnelling method works so effectively this implies that black hole radiation can be understood as a physical phenomenon. The areas of future research would be to consider higher order calculations in WKB in both the scalar field and fermionic case. In particular, it would be worth investigating fermion emission from rotating spacetimes beyond lowest order to see if a coupling term between angular velocity of the black hole and the spin of the fermions can be found. This is interesting because if such a coupling term could be found than it would be a discovery of new physics and would show how fermion emission varies from scalar particle emission. The method can also be extended to model other types of particles by using various wave equations.

I have examined and compared the two different approaches to the tunnelling method for finding the black hole temperatures. The two approaches to black hole tunnelling are the null geodesic method and the Hamilton-Jacobi ansatz. In the null geodesic method the s-wave is massless and follows null geodesics of the Painlevé form of the black hole metric. With the null geodesic method it is possible to calculate the self interaction effect resulting from energy conservation of the system. It is also possible to ignore the self interaction by doing a perturbative expansion in terms of the particle's energy as long as this energy is much smaller than the energy of the system (i.e. ADM mass). The Hamilton-Jacobi ansatz is a result of complex path analysis and it ignores self gravitation effects but it can model massive particle emission. The complex path analysis techniques can also be extended to model fermion emission.

I have shown that the tunnelling methods are extremely robust for non-extremal black holes, yielding results commensurate with other methods for general non-rotating black holes, de Sitter spaces, Rindler space, rotating black holes, Taub-NUT black holes, and 5D Kerr-Gödel black holes. The method can also be extended beyond regular scalar particle emission to charged scalar particle emission, fermion emission, and charged fermion emission. The tunnelling method is a straightforward calculation that has a breadth of applicability to rival any other black hole temperature calculation. In fact the tunnelling method can even be applied to cases for which the usual Euclideanization techniques cannot work. For example the tunnelling method has been applied to de Sitter spaces. There is a sign ambiguity for de Sitter spaces depending on a-priori assumptions of how de Sitter space should behave but it is still possible to do more with the tunnelling method than the Wick rotation method in this case. The Wick rotation method only models a scalar particle heat bath while the tunnelling method can also model fermion emission. The tunnelling method is also a dynamical model as opposed to the static nature of scalar particle heat bath of the Wick rotation method. Admittedly the tunnelling method is only applicable for slowly changing dynamics (i.e. when the mass changes from $M \rightarrow M - \omega$ for $\omega << M$) but it is still a step beyond a static model. Another strength of the method is the fact that the observer does not have to be at infinity. While the calculation often assumes an observer at infinity for simplicity it is not a prerequisite. When the observer is not at infinity (or near it i.e. $r_{obs} >> r_h$) it is found that the temperature is adjusted by the Tolman redshift factor at the observer (see appendix section discussing de Sitter). The tunnelling method is also useful because it gives a intuitive picture of black hole radiation. In this thesis I have shown explicitly how robust the method is by applying it to a wide range of spacetimes and extending the method to model fermion emission.

I reviewed key properties of the tunnelling method in the second chapter. I showed the calculations for the null geodesic method and the Hamilton-Jacobi ansatz and reviewed the extension of these methods to model charged particle emission from Reissner-Nordström black holes. I showed how the method could be applied to cosmological horizons and described some of the issues related to applying the tunnelling method to de Sitter spacetimes. The calculations I showed assumed that the cosmological constant remains constant after emission and the cosmological horizon will shrink. I also mentioned the sign ambiguity that exists due to the fact that some calculations assume that the cosmological constant should be thermally fluctuating parameter and claim resulting cosmological horizon should grow [51].

In the third chapter I started applying the tunnelling methods to a wider range of spacetimes. In particular I applied the method to Rindler, Kerr-Newman, Taub-NUT-AdS and extremal black holes. The Rindler metric calculation demonstrates that the tunnelling method it applicable to other types of horizons. The Kerr-Newman calculation showed how the calculation could be extended when there is no longer spherical symmetry. It becomes necessary to transform to a corotating metric so that the ergosphere can be ignored. I have provided independent verification of the temperatures obtained for Taub-NUT spaces without CTCs via analytic continuation methods. Indeed it is not too difficult to show that the temperatures even match when CTCs are present, though in this case an a-priori justification for the method is unclear. Finally, I investigated extremal black holes, for which the tunnelling method is somewhat more problematic due to its dynamic nature. I found that the temperature is proportional to the energy of the emitted particles for black holes close to extremality. I also found that both methods yield a divergent real part to the action for extremal black holes, which is suggestive of a full suppression of particle emission. However the null geodesic method has a nonzero finite imaginary parts, whose values yields a countably infinite number of possible finite temperatures for an extremal Reissner-Nordström black hole. This rather strange result arises because of a breakdown of the WKB method in the null geodesic approximation. This suggests limitations on the method. Another calculation with extremal black holes was to model charged particle emission that go from an extremal black hole to an extremal black hole. This calculation did not give a diverging real component as with the other extremal cases. The calculation gave an emission probability of $\Gamma = \exp(-2\pi\omega M)$ which would naively indicate a temperature four times the Hawking temperature. Unfortunately this calculation is entirely dependant on being able to emit radiation with ($\omega = |q|$) which may not be physically possible.

In the fourth chapter, I discussed Kerr-Gödel black holes in relation to the tunnelling method. In this chapter I reviewed some of the general properties of the Kerr-Gödel spacetime and performed detailed analysis of its parameter space. There are two distinct classes of parameter space (three if naked singularities are also counted). One is the class $J^2 < 1$, corresponding to black holes for which the CTC horizon r_{CTC} is exterior to the black hole horizons at r_+ and r_- . When $J^2 > 1$, I obtain the other class (the "other" region of parameter space), for which the CTC horizon is inside both of the other surfaces r_+ and r_-. I find that these are the only two possibilities (apart from naked singularities); there is no "in-between" region where $r_+ > r_{CTC} > r_-$, contrary to previous expectations [80]. Despite the presence of CTCs, I find that the tunnelling method applied to the black hole region of parameter space yields a temperature consistent with previous calculations made via Wick rotation methods. I also find (when $r_{ctc} > r_+$) that there is no tunnelling through the CTC horizon. In a sense a Kerr-Gödel black hole is analogous to a black hole embedded in de Sitter space with the major difference being the CTC horizon will not radiate. I have discussed technical problems that occur in trying to apply the tunnelling method to the "other" region of parameter space due the fact that the region $r_- < r < r_+$ does not have the correct signature. Higher-order corrections from self interaction remains an interesting problem to explore in this case.

In the fifth chapter I extend the tunnelling method to model fermion emission. I was the first to show that computing the Unruh and Hawking temperatures using the tunnelling method holds for Dirac fermions. The fermion tunnelling calculation also provides evidence that fermions will radiate at the same temperature as scalar particles due to the presence of these horizons. Comparatively few demonstrations that fermionic radiation has the same temperature as scalar radiation appears in the literature [81]-[84]. These all involve either lower dimensional calculations of the Bogoliubov transformation [81] or use of the GTCT [82]-[84] to calculate fermion radiation from evaporating black holes. For accelerated observers using Rindler coordinates I have found the expected Unruh temperature. I also applied fermion tunnelling to a general static spherically symmetric black hole metric in both Schwarzschild and Painlevé form, and found that the usual Hawking temperature is recovered. The fact that the results do not depend on coordinate singularities was demonstrated by showing the same results hold for the Kruskal-Szekeres metric. Extending fermion tunnelling to rotating spacetimes in which the emitted particles have orbital angular momentum and charge was natural next step. I have successfully extended fermion tunnelling to model the emission of charged fermions from a rotating charged black hole at the end of the chapter. The analysis yields the expected Hawking temperature consistent with black hole universality. However there are subtle technical issues involved with choosing an appropriate ansatz for the Dirac field consistent with the choice of γ matrices, and failure to make such a choice leads to a breakdown in the method.

I have successfully been able to model fermion emission (to the lowest order of WKB). This thesis has investigated the lowest order of the tunnelling method in detail, so the next step is to investigate higher order effects. Future work would involve computing corrections to the tunnelling probability by fully taking into account conservation of energy to yield corrections to the fermion emission temperature. In various scalar field cases this is inherent in the null geodesic method and can be incorporated into the Hamilton-Jacobi tunnelling approach [26]. Another avenue of future research is to perform tunnelling calculations to higher order in WKB (in both the scalar field and fermionic cases) in order to calculate grey body effects. It was hoped that when applying fermion tunnelling to rotating black holes that the emission would have been of the form $\exp(-\frac{1}{T_H}(E-\Omega_H J_\phi+C))$, where C parametrizes the coupling between the spin of the field and the angular momentum of the black hole. Unfortunately such a coupling term was not seen to the lowest order of WKB. The possibility still exists of finding a coupling term once the calculation is taken beyond the lowest order of WKB. It is also worth investigating the possibility of calculating a density matrix for the emitted particles from a tunnelling approach in order to calculate correlations between particles. Another interesting extension of the tunnelling method is to model other types of fermions. For example tunnelling of spin-3/2 fermions could be modelled by starting with the Rarita-Schwinger equation.

Appendix

Derivation of Action for Null Geodesic Method

The first tunnelling papers [10],[11],[12] started with a spherically symmetric metric in ADM form and then derived the full action for black hole plus the emitted shell system. The action is ultimately written in a Hamiltonian form in terms of all the canonically conjugate momenta. Since the system will only have one effective degree of freedom, the constraints of the theory are solved in order to eliminate the dependence of all momenta from the action, except the one conjugate to the radius of the shell. In order to simplify the action, a gauge choice is picked which corresponds to choosing the Painlevé metric. The result at the end of this derivation is that the action is of the standard form used for the null geodesic method. What follows is a detailed recap of Kraus and Wilczek's derivation from [10],[11],[12].

The ADM form of the metric is:

$$ds^2 = -N^t(t,r)^2 dt^2 + L(t,r)^2[dr + N^r(t,r)dt]^2 + R(t,r)^2[d\theta^2 + \sin^2\theta d\phi^2] \quad (6.1)$$

The action for the spherical shell is:

$$S^s = -m\int\sqrt{-\hat{g}_{\mu\nu}d\hat{x}^\mu d\hat{x}^\nu} = -m\int dt\sqrt{\hat{N}^{t2} - \hat{L}^2(\frac{d\hat{r}}{dt} + \hat{N}^r)^2} \quad (6.2)$$

where m is the rest mass of the shell and the quantities with "ˆ" are evaluated at the shell ($\hat{g}_{\mu\nu} = g_{\mu\nu}(\hat{t},\hat{r})$). The action for the black hole and emitted shell system is:

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}\mathcal{R} - m\int dt\sqrt{\hat{N}^{t2} - \hat{L}^2(\frac{d\hat{r}}{dt} + \hat{N}^r)^2} + \text{boundary terms} \quad (6.3)$$

This is written in canonical form as:

$$S = \int p\frac{d\hat{r}}{dt}dt + \int dtdr[\pi_R\dot{R} + \pi_L\dot{L} - N^t(\mathcal{H}_t^s + \mathcal{H}_t^G) - N^r(\mathcal{H}_r^s + \mathcal{H}_r^G)] - \int dtM_{ADM} \quad (6.4)$$

with

$$\mathcal{H}_t^s = \sqrt{(p/\hat{L})^2 + m^2}\delta(r-\hat{r}) \quad ; \quad \mathcal{H}_r^s = -p\delta(r-\hat{r})$$

$$\mathcal{H}_t^G = \frac{L\pi_L^2}{2R^2} - \frac{\pi_L\pi_R}{R} + \left(\frac{RR'}{L}\right)' - \frac{R'^2}{2L} - \frac{L}{2} \quad ; \quad \mathcal{H}_r^G = R'\pi_R - L\pi_L'$$

where: $\cdot$ represents $\frac{d}{dt}$, $'$ represents $\frac{d}{dr}$, M_{ADM} is the ADM mass of the system. It should be noted that in this approach the total ADM mass is allowed the vary while the mass of the black hole is fixed. In later null geodesic calculations the mass of the black hole is changing and the total ADM mass is fixed.

In order to eliminate the gravitational degrees of freedom the constraints need to be identified. The first constraints are obtained by varying with respect to N^t and N^r:

$$\mathcal{H}_t = \mathcal{H}_t^s + \mathcal{H}_t^G = 0 \quad ; \quad \mathcal{H}_r = \mathcal{H}_r^s + \mathcal{H}_r^G = 0 \tag{6.5}$$

By solving these constraints and inserting the solutions back into the action (6.4), the dependence on π_R and π_L can be eliminated. From the following linear combination of the constraints:

$$0 = \frac{R'}{L}\mathcal{H}_t + \frac{\pi_L}{RL}\mathcal{H}_r = -\mathcal{M}' + \frac{\hat{R}'}{\hat{L}}\mathcal{H}_t^s + \frac{\hat{\pi}_L}{\hat{R}\hat{L}}\mathcal{H}_r^s \tag{6.6}$$

where

$$\mathcal{M} = \frac{\pi_L^2}{2R} + \frac{R}{2} - \frac{RR'^2}{2L^2} \tag{6.7}$$

Away from the shell the solution to this constraint is $\mathcal{M}$ =constant. When a static slice ($\pi_L = \pi_R = 0$) is considered, it can be seen the solution is a static slice of Schwarzschild geometry with $\mathcal{M}$ as the mass parameter. The shell causes $\mathcal{M}$ to be discontinuous at $\hat{r}$, therefore:

$$\begin{aligned} \mathcal{M} &= M \quad r < \hat{r} \\ \mathcal{M} &= M_+ \quad r > \hat{r} \end{aligned}$$

Since there is no other matter outside the shell $M_{ADM} = M_+$. Using (6.6) and (6.7) to solve the constraints to find for π_R and π_L

$$\begin{aligned} \pi_L &= R\sqrt{(R'/L)^2 - 1 + \frac{2M}{R}} \quad ; \quad \pi_R = \frac{L}{R'}\pi_L' \quad r < \hat{r} \\ \pi_L &= R\sqrt{(R'/L)^2 - 1 + \frac{2M_+}{R}} \quad ; \quad \pi_R = \frac{L}{R'}\pi_L' \quad r > \hat{r} \end{aligned} \tag{6.8}$$

The relation between M_+ and M is found by solving the constraints at the position of the shell. This is done by choosing coordinates such that L and R are continuous across the shell. Also π_L and π_R both need to be free of singularities at the shell. Integration of the constraints across the shell gives:

$$\begin{aligned} \pi_L(\hat{r} + \epsilon) - \pi_L(\hat{r} - \epsilon) &= -\frac{p}{\hat{L}} \\ R'(\hat{r} + \epsilon) - R'(\hat{r} - \epsilon) &= -\frac{1}{\hat{R}}\sqrt{p^2 + m^2\hat{L}^2} \end{aligned} \tag{6.9}$$

When the constraints are satisfied a variation of the action takes the form:

$$dS = p d\hat{r} + \int dr(\pi_R \delta R + \pi_L \delta L) - M_+ dt \tag{6.10}$$

The geometry inside the shell is taken to be fixed (M is held constant) and the geometry outside the shell will vary in order to satisfy the constraints. The action will be easier to integrate by initially varying the geometry away from the shell To begin with, start from an arbitrary geometry and vary L until $\pi_L = \pi_R = 0$, while holding $\hat{r}, p, R, \hat{L}$ fixed:

$$\begin{aligned}
\int dS &= \int_{r_{\min}}^{\infty} dr \int_{\pi=0}^{L} \delta L(\pi_L) \\
&= \int_{r_{\min}}^{\hat{r}-\epsilon} dr \int_{\pi=0}^{L} \delta L \left(R\sqrt{(R'/L)^2 - 1 + \frac{2M}{R}} \right) + \int_{\hat{r}+\epsilon}^{\infty} dr \int_{\pi=0}^{L} \delta L \left(R\sqrt{(R'/L)^2 - 1 + \frac{2M_+}{R}} \right) \\
&= \int_{r_{\min}}^{\hat{r}-\epsilon} dr \left[RL\sqrt{(R'/L)^2 - 1 + \frac{2M}{R}} + RR' \ln \left| \frac{R'/L - \sqrt{(R'/L)^2 - 1 + \frac{2M}{R}}}{\sqrt{|1 - \frac{2M}{R}|}} \right| \right] \\
&+ \int_{\hat{r}+\epsilon}^{\infty} dr \left[RL\sqrt{(R'/L)^2 - 1 + \frac{2M_+}{R}} + RR' \ln \left| \frac{R'/L - \sqrt{(R'/L)^2 - 1 + \frac{2M_+}{R}}}{\sqrt{|1 - \frac{2M_+}{R}|}} \right| \right]
\end{aligned} \tag{6.11}$$

The constant resulting from the lower limit of L integration has been discarded. It is also possible to vary L and R, while keeping $\pi_L = \pi_R = 0$, to some set geometry. In this case all the momenta vanish so there is no contribution to the action from such a variation.

There are still nonzero variations at the shell to be considered. Inserting an arbitrary variation of L and R into the action:

$$dS = \int_{r_{\min}}^{\infty} dr(\pi_R \delta R + \pi_L \delta L) - \left[\frac{\partial S}{\partial \hat{R}'}(\hat{r}+\epsilon) - \frac{\partial S}{\partial \hat{R}'}(\hat{r}-\epsilon) \right] d\hat{R} + \frac{\partial S}{\partial M_+} dM_+$$

Since R' is discontinuous at the shell,

$$\frac{\partial S}{\partial \hat{R}'}(\hat{r}+\epsilon) - \frac{\partial S}{\partial \hat{R}'}(\hat{r}-\epsilon)$$

does not vanish and needs to be subtracted so that the relations

$$\frac{\delta S}{\delta R} = \pi_R \quad ; \qquad \frac{\delta S}{\delta L} = \pi_L$$

will hold. Using (6.11) the term to be subtracted is

$$
\begin{aligned}
&-\left[\frac{\partial S}{\partial \hat{R}'}(\hat{r}+\epsilon)-\frac{\partial S}{\partial \hat{R}'}(\hat{r}-\epsilon)\right]d\hat{R}\\
&= d\hat{R}\left(-\hat{R}\ln\left|\frac{R'(\hat{r}-\epsilon)/\hat{L}-\sqrt{(R'(\hat{r}-\epsilon)/\hat{L})^2-1+\frac{2M}{\hat{R}}}}{\sqrt{|1-\frac{2M}{\hat{R}}|}}\right|\right.\\
&\left.+\hat{R}\ln\left|\frac{R'(\hat{r}+\epsilon)/\hat{L}-\sqrt{(R'(\hat{r}-\epsilon)/\hat{L})^2-1+\frac{2M_+}{\hat{R}}}}{\sqrt{|1-\frac{2M_+}{\hat{R}}|}}\right|\right)
\end{aligned}
\tag{6.12}
$$

It is also necessary to subtract a term:

$$
\frac{\partial S}{\partial M_+}dM_+ = -\int_{\hat{r}+\epsilon}^{\infty} dr L\frac{\sqrt{(R'/L)^2-1+\frac{2M_+}{R}}}{1-\frac{2M_+}{\hat{R}}}dM_+
$$

The remaining variations are $p, \hat{r}, t$. Variations in t simply give $dS = -M_+dt$. The variations in p and $\hat{r}$ do not need to be considered separately. This is because when the constraints are satisfied their variations are already accounted for in the expression for S. Collecting all the terms the full expression for the action is:

$$
\begin{aligned}
S = &\int_{r_{\min}}^{\hat{r}-\epsilon} dr\left[RL\sqrt{(R'/L)^2-1+\frac{2M}{R}}+RR'\ln\left|\frac{R'/L-\sqrt{(R'/L)^2-1+\frac{2M}{R}}}{\sqrt{|1-\frac{2M}{R}|}}\right|\right]\\
&+\int_{\hat{r}+\epsilon}^{\infty} dr\left[RL\sqrt{(R'/L)^2-1+\frac{2M_+}{R}}+RR'\ln\left|\frac{R'/L-\sqrt{(R'/L)^2-1+\frac{2M_+}{R}}}{\sqrt{|1-\frac{2M_+}{R}|}}\right|\right]\\
&-\int dt\frac{d\hat{R}}{dt}\hat{R}\left[\ln\left|\frac{R'(\hat{r}-\epsilon)/\hat{L}-\sqrt{(R'(\hat{r}-\epsilon)/\hat{L})^2-1+\frac{2M}{\hat{R}}}}{\sqrt{|1-\frac{2M}{\hat{R}}|}}\right|\right.\\
&\left.-\ln\left|\frac{R'(\hat{r}+\epsilon)/\hat{L}-\sqrt{(R'(\hat{r}-\epsilon)/\hat{L})^2-1+\frac{2M_+}{\hat{R}}}}{\sqrt{|1-\frac{2M_+}{\hat{R}}|}}\right|\right]\\
&+\int dt\int_{\hat{r}+\epsilon}^{\infty} dr L\frac{\sqrt{(R'/L)^2-1+\frac{2M_+}{R}}}{1-\frac{2M_+}{\hat{R}}}\frac{dM_+}{dt}-\int M_+dt
\end{aligned}
\tag{6.13}
$$

This can be shown to be the correct expression by differentiating it. It is possible to rewrite this action in a more conventional form as the time integral of a Lagrangian. The action (6.13) is given for an arbitrary choice of L and R that is consistent with the constraints. There are many L's and R's that are equivalent to each other by a change of coordinates, so there is a large amount redundant information present in this model. To obtain a form of the action which only depends on the physical variables $p, \hat{r}$ it is possible to make a specific choice for L and R (choose a gauge). In making the choice

$$R'(\hat{r}+\epsilon) - R'(\hat{r}-\epsilon) = -\frac{1}{\hat{R}}\sqrt{p^2 + m^2\hat{L}^2}$$

must be satisfied. If R is chosen for all $r > \hat{r}$ then $R'(\hat{r}-\epsilon)$ is fixed by this constraint. It will still be possible to choose R is for $r < \hat{r}-\epsilon$ (away from the shell). The notation is chosen so that $R'_<$ denotes the value of R' that is near the shell but far enough away so that R is still freely specifiable. There is an analogous definition for $R'_>$, in this case there is freedom to choose $R'_> = R'(\hat{r}+\epsilon)$. Using this notation the time derivative of S is:

$$\begin{aligned}
\mathcal{L} = \frac{dS}{dt} &= \frac{d\hat{r}}{dt}\hat{R}\hat{L}\left[\sqrt{(R'_</\hat{L})^2 - 1 + \frac{2M}{\hat{R}}} - \sqrt{(R'_>/\hat{L})^2 - 1 + \frac{2M_+}{\hat{R}}}\right] \\
&- \frac{d\hat{R}}{dt}\hat{R}\ln\left|\frac{R'(\hat{r}-\epsilon)/\hat{L} - \sqrt{(R'(\hat{r}-\epsilon)/\hat{L})^2 - 1 + \frac{2M}{\hat{R}}}}{(R'_</\hat{L}) - \sqrt{(R'_</\hat{L})^2 - 1 + \frac{2M}{\hat{R}}}}\right| \\
&+ \int\limits_{r_{\min}}^{\hat{r}-\epsilon} dr(\pi_R\dot{R} + \pi_L\dot{L}) + \int\limits_{\hat{r}+\epsilon}^{\infty} dr(\pi_R\dot{R} + \pi_L\dot{L}) - M_+
\end{aligned} \tag{6.14}$$

The calculations are simplified by considering a massless particle ($m = 0$) and defining $\eta = \pm 1 = \mathrm{sgn}(p)$. The constraints (6.9) become:

$$\begin{aligned}
R'(\hat{r}+\epsilon) - R'(\hat{r}-\epsilon) &= -\frac{\eta p}{\hat{R}} \\
\sqrt{(R'(\hat{r}-\epsilon)/L)^2 - 1 + \frac{2M}{\hat{R}}} &= \sqrt{(R'(\hat{r}+\epsilon)/L)^2 - 1 + \frac{2M_+}{\hat{R}}} + \frac{p}{\hat{L}\hat{R}}
\end{aligned} \tag{6.15}$$

and inserting into the Lagrangian yields:

$$\begin{aligned}
\mathcal{L} &= \frac{d\hat{r}}{dt}\hat{R}\hat{L}\left[\sqrt{(R'_</\hat{L})^2 - 1 + \frac{2M}{\hat{R}}} - \sqrt{(R'_>/\hat{L})^2 - 1 + \frac{2M_+}{\hat{R}}}\right] \\
&- \eta\frac{d\hat{R}}{dt}\hat{R}\ln\left|\frac{R'_>/\hat{L} - \eta\sqrt{(R'_>/\hat{L})^2 - 1 + \frac{2M_+}{\hat{R}}}}{R'_</\hat{L} - \eta\sqrt{(R'_</\hat{L})^2 - 1 + \frac{2M}{\hat{R}}}}\right| \\
&+ \int\limits_{r_{\min}}^{\hat{r}-\epsilon} dr(\pi_R\dot{R} + \pi_L\dot{L}) + \int\limits_{\hat{r}+\epsilon}^{\infty} dr(\pi_R\dot{R} + \pi_L\dot{L}) - M_+
\end{aligned} \tag{6.16}$$

It is possible to choose the gauge to make the Lagrangian as simple as possible. So by choosing L and R to be time independent then $\pi_R\dot{R} + \pi_L\dot{L} = 0$. The expressions are also simplified when $R' = L$. The resulting metric should also not have any coordinate singularities. All of these properties are satisfied when Painlevé coordinates are chosen (i.e. $L = 1$, $R = r$). Hence the importance of writing black holes in Painlevé form for the null geodesic method. By choosing Painlevé coordinates, the Lagrangian reduces to:

$$\mathcal{L} = \frac{d\hat{r}}{dt}[\sqrt{2M\hat{r}} - \sqrt{2M_+\hat{r}}] - \eta\frac{d\hat{r}}{dt}\hat{r}\ln\left|\frac{\sqrt{\hat{r}} - \eta\sqrt{2M_+}}{\sqrt{\hat{r}} - \eta\sqrt{2M}}\right| - M_+ \tag{6.17}$$

The canonical momentum conjugate to $\hat{r}$ obtained from the Lagrangian is simply:

$$p_r = \frac{\partial L}{\partial(\frac{d\hat{r}}{dt})} = [\sqrt{2M\hat{r}} - \sqrt{2M_+\hat{r}}] - \eta\hat{r}\ln\left|\frac{\sqrt{\hat{r}} - \eta\sqrt{2M_+}}{\sqrt{\hat{r}} - \eta\sqrt{2M}}\right|$$

So the action can be written in canonical form as:

$$S = \int dt(p_r\frac{d\hat{r}}{dt} - M_+) \tag{6.18}$$

This is the same action that is assumed for the null geodesic method (ignoring the M_+ contribution which is real).

Hamilton-Jacobi Calculation Using Proper Radial Distance

As was shown in the second chapter the Hamilton-Jacobi ansatz can start with the assumption that the tunnelling probability for the classically forbidden trajectory from inside to outside the horizon is given by:

$$\Gamma \propto \exp(-2\text{Im}I) \tag{6.19}$$

where $\hbar$ has been set to unity. I assume that the action of the outgoing particle is given by the classical action I that satisfies the relativistic Hamilton-Jacobi equation

$$g^{\mu\nu}\partial_\mu I\partial_\nu I + m^2 = 0 \tag{6.20}$$

For a metric of the form

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{g(r)} + C(r)h_{ij}dx^i dx^j \tag{6.21}$$

the resulting Hamilton-Jacobi equation for this metric is

$$-\frac{(\partial_t I)^2}{f(r)} + g(r)(\partial_r I)^2 + \frac{h^{ij}}{C(r)}\partial_i I\partial_j I + m^2 = 0 \tag{6.22}$$

There exists a solution of the form

$$I = -Et + W(r) + J(x^i) \tag{6.23}$$

where

$$\partial_t I = -E, \qquad \partial_r I = W'(r), \qquad \partial_i I = J_i$$

and that the J_i's are constant. Solving for $W(r)$ yields

$$W(r) = \int \frac{dr}{\sqrt{f(r)g(r)}} \sqrt{E^2 - f(r)(m^2 + \frac{h^{ij}J_iJ_j}{C(r)})} \tag{6.24}$$

(for an outgoing particle) and the imaginary part of the action can only come from the pole at the horizon. One way to get the correct Hawking temperature is to parameterize in terms of the proper spatial distance [28].

With the null-geodesic method the Painlevé coordinate r was the proper spatial distance. In this case the proper spatial distance between any two points at some fixed t is given by

$$d\sigma^2 = \frac{dr^2}{g(r)} + C(r)h_{ij}dx^i dx^j \tag{6.25}$$

Since I am only concerned with radial rays, the resulting proper radial spatial distance is:

$$d\sigma^2 = \frac{dr^2}{g(r)}$$

Employing the near horizon approximation

$$\begin{aligned} f(r) &= f'(r_0)(r - r_0) + ... \\ g(r) &= g'(r_0)(r - r_0) + ... \end{aligned} \tag{6.26}$$

it is found that

$$\sigma = \int \frac{dr}{\sqrt{g(r)}} \simeq 2\frac{\sqrt{r - r_0}}{\sqrt{g'(r_0)}} \tag{6.27}$$

is the proper radial distance. So for particles emitted radially

$$\begin{aligned} W(\sigma) &= \frac{2}{\sqrt{g'(r_0)f'(r_0)}} \int \frac{d\sigma}{\sigma} \sqrt{E^2 - \frac{\sigma^2}{4} g'(r_0)f'(r_0)\left(m^2 + \frac{h^{ij}J_iJ_j}{C(r_0)}\right)} \\ &= \frac{2\pi i E}{\sqrt{g'(r_0)f'(r_0)}} \end{aligned} \tag{6.28}$$

and from this point the computation is the same as for the previous method, yielding

$$T_H = \frac{\sqrt{f'(r_0)g'(r_0)}}{4\pi} \tag{6.29}$$

for the temperature. So this calculation gives the correct temperature without considering the incoming rays.

Hamilton-Jacobi Calculation for de Sitter Spacetimes

Remember the de Sitter and Schwarzschild-dS metrics are:

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega^2 \tag{6.30}$$

where:

$$f(r) = 1 - \frac{r^2}{l^2} \text{ ;for de Sitter}$$
$$f(r) = 1 - \frac{2m}{r} - \frac{r^2}{l^2} \text{ ;for Schwarzschild-dS}$$

For the de Sitter spacetime the cosmological horizon is located a $r_c = l$. For the Schwarzschild-dS black hole, when $\frac{27m^2}{l^2} < 1$ (i.e. the Nariai bound) is satisfied then the spacetime will have two horizons and will be non-extremal. The two horizons correspond to a Schwarzschild horizon r_s and a cosmological horizon r_c. For the case when $m << l$ the horizons will be located at: $r_s \cong 2m$ and $r_c \cong l - m$.

For these de Sitter metrics the Hamilton-Jacobi equation is explicitly

$$-\frac{(\partial_t I)^2}{f(r)} + f(r)(\partial_r I)^2 + \frac{1}{r^2}(\partial_\theta I)^2 + \frac{1}{r^2 \sin^2\theta}(\partial_\phi I)^2 + m^2 = 0 \tag{6.31}$$

In order to apply the Hamilton-Jacobi method, an ansatz for the action I is required:

$$I = -Et + W(r) + J(\theta, \phi) \tag{6.32}$$

For an asymptotically flat spacetime, E is the energy of the s-wave as detected by the observer at infinity. This is because for an asymptotically flat spacetime the timelike killing ∂_t has a norm of (minus) unity at infinity. For a de Sitter spacetime the observer is located inside the cosmological horizon. For regular de Sitter space (i.e. no Schwarzschild black hole) ∂_t will be normalized (to minus 1) at the origin (i.e. $f(0) = 1$). So it is straightforward to apply the Hamilton-Jacobi tunnelling method to de Sitter space and get the usual tunnelling rate:

Solving for $W(r)$ yields

$$W_\mp(r) = \mp \int \frac{dr}{f(r)} \sqrt{E^2 + f(r)\left(m^2 + \frac{1}{r^2}(J_\theta)^2 + \frac{1}{r^2 \sin^2\theta}(J_\phi)^2\right)} \tag{6.33}$$

$$= \mp \int \frac{l^2 dr}{l^2 - r^2} \sqrt{E^2 + f(r)\left(m^2 + \frac{1}{r^2}(J_\theta)^2 + \frac{1}{r^2 \sin^2\theta}(J_\phi)^2\right)} \tag{6.34}$$

$$W_\mp(r) = \pm \pi i \frac{El}{2} \tag{6.35}$$

Notice that for a cosmological horizon W_- corresponds to tunnelling out of the cosmological horizon and heading towards the observer (at the origin) and W_+

corresponds to entering the cosmological horizon. Therefore

$$\mathrm{Prob}[out] \propto \exp[-2\,\mathrm{Im}\,W_-] \tag{6.36}$$

$$\mathrm{Prob}[in] \propto \exp[-2\,\mathrm{Im}\,W_+] \tag{6.37}$$

The resulting semi-classical probability of a particle tunnelling from inside to outside the horizon is:

$$\Gamma \propto \frac{\mathrm{Prob}[out]}{\mathrm{Prob}[in]} = \frac{\exp[-2(\mathrm{Im}\,W_-)]}{\exp[-2(\mathrm{Im}\,W_+)]} = \exp[-4\,\mathrm{Im}\,W_-] \tag{6.38}$$

$$\Gamma \propto \exp(-2\pi l E) \tag{6.39}$$

Unfortunately, when working with the Schwarzschild-dS metric, there is no location where $f(r_{obs}) = 1$ will be satisfied (i.e. $f(r) = 1$ only for $r = -\sqrt[3]{2ml^2}$ but $r > 0$). One solution to this is to consider cases that are far away from the Nariai bound ($\frac{27m^2}{l^2} << 1$). In this case it is possible to choose an r_{obs} for which $f(r_{obs}) \approx 1$. This will be true for any r_{obs} where $m << r_{obs} << l$ and it will always be possible to choose such an r_{obs} when the spacetime is well away from the Nariai bound.

Another way to resolve this issue is to manually normalize the timelike killing vector ∂_t so that it will norm of (minus) unity at the observer located at r_{obs}. This can be done by dividing the killing vector ∂_t by $\sqrt{f(r_{obs})}$. From this, if the energy detected by the observer at r_{obs} is E then the action should satisfy $\frac{1}{\sqrt{f(r_{obs})}}\partial_t I = -E$. This means the ansatz for the action should be modified to:

$$I = -\sqrt{f(r_{obs})}Et + W(r) + J(\theta, \phi) \tag{6.40}$$

in order to satisfy the equation for the energy. It should also be noted that r_{obs} is a constant and is determined by where the observer is located. I will replace $\sqrt{f(r_{obs})}E$ with $\tilde{E}$ so that the old Hamilton-Jacobi expressions can be used with E replaced with $\tilde{E}$. Now I will treat the particles crossing the horizons separately. In other words, I will consider paths that come from beyond one of the horizons and finish at the observer. This corresponds to an observer that is looking towards one of the horizons and is somehow blocking any particles coming from the other horizon behind him from being observed. This blocking would correspond to a perfect reflector being inserted between the black hole and the cosmological horizons. The effect would be to isolate the two regions and the thermodynamics becomes clear. The result would be two isolated heat baths with temperatures determined individually by each horizon. This would also require observers on each side of the reflector. This particular situation is unrealistic but this is only one way to approach the tunnelling method. It is also possible to consider both horizons at the same time which wouldn't require emission from one of the horizons to be blocked.

With these multiple horizon systems it is also possible to consider particle trajectories that start beyond one horizon, tunnels out, heads toward the other horizon and crosses the other horizon. The results for trajectories that follow this path have already been calculated by Shankaranarayanan in [25] and in the end the resulting temperature that he finds is:

$$T = \frac{1}{2\pi\alpha}\frac{\kappa_s\kappa_c}{\kappa_s + \kappa_c} \tag{6.41}$$

where:

$$\kappa_s = \alpha\left|\frac{m}{r_s^2} - \frac{r_s}{l^2}\right|$$
$$\kappa_c = \alpha\left|\frac{m}{r_c^2} - \frac{r_c}{l^2}\right|$$
$$\alpha = \frac{1}{\sqrt{1 - \sqrt[3]{27(\frac{m^2}{l^2})}}}$$

and the $\kappa's$ are the two surface gravities of the horizons. So for this case, there is radiation propagating between the two horizons and a static observer will be in a thermal bath at this temperature.

Now I will return back to the case of observing the horizons independently. Applying the Hamilton-Jacobi method to these horizons will give the following results:

For the cosmological horizon:

$$\Gamma_c \propto \exp\left[-2\pi\tilde{E}\frac{l^2 r_c^2}{r_c^3 - ml^2}\right]$$

which gives a temperature, in the case $\tilde{E} \cong E$ (i.e. $f(r_{obs}) \cong 1$) of:

$$T_{dS} = \frac{r_c^3 - ml^2}{2\pi l^2 r_c^2}$$

For the black hole horizon:

$$\therefore \Gamma_s \propto \exp\left[-2\pi\tilde{E}\frac{l^2 r_s^2}{ml^2 - r_s^3}\right]$$

Giving a Hawking temperature, in the case $\tilde{E} \cong E$ (i.e. $f(r_{obs}) \cong 1$) of:

$$T_H = \frac{ml^2 - r_s^3}{2\pi l^2 r_s^2} \tag{6.42}$$

If it is not possible to choose r_{obs} so that $f(r_{obs}) \cong 1$ than all of the above temperatures will be modified by a factor of $\sqrt{f(r_{obs})}$. i.e.

$$T_{obs} = \frac{T}{\sqrt{f(r_{obs})}}$$
$$T_{obs} = \frac{T}{\sqrt{-g_{00}}} \tag{6.43}$$

So the temperature has simply been adjusted by the Tolman redshift factor [97] which is not that surprising.

Technical Issues With Rotating Black holes

Technical Issues with Scalar Particle Emission From Rotating Black Holes

Remember the Kerr-Newman metric and vector potential are given by

$$ds^2 = -f(r,\theta)dt^2 + \frac{dr^2}{g(r,\theta)} - 2H(r,\theta)dtd\phi + K(r,\theta)d\phi^2 + \Sigma(r,\theta)d\theta^2 \tag{6.44}$$

$$A_a = -\frac{er}{\Sigma(r)}[(dt)_a - a^2\sin^2\theta(d\phi)_a] \tag{6.45}$$

$$\begin{aligned}
f(r,\theta) &= \frac{\Delta(r) - a^2\sin^2\theta}{\Sigma(r,\theta)}, \\
g(r,\theta) &= \frac{\Delta(r)}{\Sigma(r,\theta)}, \\
H(r,\theta) &= \frac{a\sin^2\theta(r^2+a^2-\Delta(r))}{\Sigma(r,\theta)} \\
K(r,\theta) &= \frac{(r^2+a^2)^2 - \Delta(r)a^2\sin^2\theta}{\Sigma(r,\theta)}\sin^2(\theta) \\
\Sigma(r,\theta) &= r^2 + a^2\cos^2\theta \\
\Delta(r) &= r^2 + a^2 + e^2 - 2Mr
\end{aligned}$$

I assume a non-extremal black hole with $M^2 > a^2 + e^2$ so that there are two horizons at $r_\pm = M \pm \sqrt{M^2 - a^2 - e^2}$. I will also assume rings of constant θ_0

A naive first attempt utilizing the null geodesic method would be to consider the transformation

$$dt = dT - \sqrt{\frac{1-g(r,\theta_0)}{g(r,\theta_0)f(r,\theta_0)}}dr$$

This gives the equation

$$\begin{aligned}
ds^2 = {} & -f(r,\theta_0)dT^2 + 2\sqrt{f(r,\theta_0)}\sqrt{\frac{1}{g(r,\theta_0)} - 1}drdT + dr^2 \\
& - 2Hd\phi(dT - \frac{\sqrt{\frac{1}{g(r,\theta_0)} - 1}}{\sqrt{f(r,\theta_0)}}dr) + Kd\phi^2
\end{aligned} \tag{6.46}$$

whose radial null geodesics correspond to

$$\dot{r} = \sqrt{\frac{f(r,\theta_0)}{g(r,\theta_0)}}\left(\pm 1 - \sqrt{1-g(r,\theta_0)}\right) \tag{6.47}$$

There remain divergences in the $dTdr$ and $drd\phi$ terms at the horizon, and $\frac{f(r,\theta_0)}{g(r,\theta_0)}$ is not well behaved there. Only for $\sin\theta_0 = 0$ are these eliminated. Restricting further the calculation to $\theta_0 = 0$ or π (in which case $\frac{f}{g} = 1$), the outgoing radial null geodesics along the z axis are

$$\dot{r} = 1 - \sqrt{1 - g(r,\theta_0)|_{\sin\theta_0 = 0}} \tag{6.48}$$

which yields

$$I = \omega \int_{r_{out}}^{r_{in}} \frac{dr}{\dot{r}} = \frac{2\pi\omega}{g'(r_+,\theta_0)|_{\sin\theta = 0}} = 2\pi\omega \frac{r_+^2 + a^2}{2(r_+ - M)}$$

for the imaginary part of the action. This in turn results in the temperature

$$T_H = \frac{1}{2\pi}\frac{r_+ - M}{r_+^2 + a^2} = \frac{1}{2\pi}\frac{(M^2 - a^2 - e^2)^{\frac{1}{2}}}{2M(M + (M^2 - a^2 - e^2)^{\frac{1}{2}}) - e^2} \tag{6.49}$$

which is the same as the found for the Kerr-Newman black hole by other means.

The restriction to two specific values of θ_0 is because of the presence of the ergosphere. The calculation breaks down because $f(r,\theta)$ is actually negative everywhere else at the horizon (i.e. inside the ergosphere) and ∂_T is not properly timelike there . The two values $\theta_0 = 0$ or π correspond to where the event horizon and ergosphere coincide.

Hamilton-Jacobi Method Applied to Rotating Black Holes Without Using the Corotating Frame

It is also possible to apply the Hamilton-Jacobi method directly to rotating black holes without converting the metric to the corotating frame Such a calculation is a little messier than when the corotating frame is used and mathematically they amount to the same thing at the end:

$$g^{\mu\nu}\partial_\mu I \partial_\nu I + m^2 = 0 \tag{6.50}$$

The resulting Hamilton-Jacobi equation for this Kerr-Newman metric is

$$0 = -\frac{(\partial_t I)^2}{F(r,\theta)} + g(r,\theta)(\partial_r I)^2 + \frac{-2H(r,\theta)}{F(r,\theta)K(r,\theta)}(\partial_t I)(\partial_\phi I) \tag{6.51}$$

$$+ \frac{f(r,\theta)}{F(r,\theta)K(r,\theta)}(\partial_\phi I)^2 + \frac{(\partial_\theta I)^2}{\Sigma(r,\theta)} + m^2 \tag{6.52}$$

Pick the solution of the form:

$$I = -Et + W(r) + J\phi$$

Plug into the Hamilton-Jacobi equation and manipulate the equation:

$$\begin{aligned}
0 &= \frac{E^2}{F(r,\theta)} + \frac{2H(r,\theta)}{F(r,\theta)K(r,\theta)}EJ + \frac{f(r,\theta)}{F(r,\theta)K(r,\theta)}J^2 + g(r,\theta)W'(r) + m^2 \\
0 &= -\frac{1}{F(r,\theta)}\left[E^2 - \frac{2H(r,\theta)}{K(r,\theta)}EJ + \frac{H^2(r,\theta)}{K^2(r,\theta)}J^2\right] \\
&+ \frac{H^2(r,\theta)}{F(r,\theta)K^2(r,\theta)}J^2 + \frac{f(r,\theta)}{F(r,\theta)K(r,\theta)}J^2 + g(r,\theta)W'(r) + m^2 \\
0 &= \frac{1}{F(r,\theta)}\left(E - \frac{H(r,\theta)}{K(r,\theta)}J\right)^2 + \frac{1}{K(r,\theta)}J^2 + g(r,\theta)W'(r) + m^2
\end{aligned}$$

Notice that near the horizon r_+:

$$\begin{aligned}
F(r,\theta) &= \frac{\Delta'(r_+)(r-r_+)}{(r_+^2+a^2)^2}\Sigma(r_+,\theta) \\
K(r,\theta) &= \frac{(r_+^2+a^2)^2}{\Sigma(r_+,\theta)}\sin^2\theta = K(r_+,\theta) \\
H(r,\theta) &= \frac{a\sin^2\theta(r_+^2+a^2)}{\Sigma(r_+,\theta)} \\
g(r,\theta) &= \frac{\Delta'(r_+)(r-r_+)}{\Sigma(r_+,\theta)} \\
\frac{H(r,\theta)}{K(r,\theta)} &= \frac{a}{(r_+^2+a^2)} = \Omega_H
\end{aligned}$$

This implies that near the horizon:

$$0 = \frac{(r_+^2+a^2)^2}{\Delta'(r_+)(r-r_+)\Sigma(r_+,\theta)}\left(E-\Omega_H J\right)^2 + \frac{1}{K(r_+,\theta)}J^2 + \frac{\Delta'(r_+)(r-r_+)}{\Sigma(r_+,\theta)}W'(r) + m^2$$

Imposing fixed $\theta = \theta_0$ in order to solve for W:

$$\begin{aligned}
W_\pm(r) = \pm\int &\frac{(r_+^2+a^2)dr}{\Delta'(r_+)(r-r_+)} \\
&\times \sqrt{\left(E-\Omega_H J\right)^2 - \Delta'(r_+)(r-r_+)\Sigma(r_+,\theta_0)(m^2 + \frac{1}{K(r_+,\theta_0)}J^2)}
\end{aligned}$$

So the final result for $W(r)$:

$$W(r) = \frac{\pi i(E-\Omega_H J)(r_+^2+a^2)}{\Delta'(r_+)} = (E-\Omega_H J)\frac{\pi i(r_+^2+a^2)}{2(r_+-M)}$$

Giving a tunnelling rate of:

$$\Gamma \propto \exp(-\frac{2\pi(r_+^2+a^2)}{(r_+-M)}(E-\Omega_H J)$$

and the correct temperature is recovered:

$$T_H = \frac{1}{2\pi}\frac{r_+ - M}{r_+^2 + a^2} = \frac{1}{2\pi}\frac{(M^2 - a^2 - e^2)^{\frac{1}{2}}}{2M(M + (M^2 - a^2 - e^2)^{\frac{1}{2}}) - e^2}$$

Fermion Emission From Rotating Black Holes

With rotating spacetimes the choice γ matrices is quite relevant, not only for ease of calculation but also for tractability. In order to demonstrate this I will repeat the calculation for a different (and less convenient) choice of γ matrices. I will also set the charge q of the emitted particles to zero for simplicity.

Consider the choice

$$\tilde{\gamma}^t = \frac{1}{\sqrt{f(r,\theta)}}\left(\gamma^0 - \frac{H(r,\theta)}{\sqrt{F(r,\theta)K(r,\theta)}}\gamma^2\right) \qquad \tilde{\gamma}^r = \sqrt{g(r,\theta)}\gamma^3$$
$$\tilde{\gamma}^\theta = \frac{1}{\sqrt{\Sigma(r,\theta)}}\gamma^1 \qquad \tilde{\gamma}^\phi = \sqrt{\frac{f(r,\theta)}{F(r,\theta)K(r,\theta)}}\gamma^2 \tag{6.53}$$

where I am using the same chiral γ^a matrices (5.69) as before. This choice satisfies the correct anti-commutation relations $\{\tilde{\gamma}^\mu, \tilde{\gamma}^\nu\} = 2g^{\mu\nu}$, and corresponds to a different choice of tetrad basis for the metric.

Naively choosing the same ansatz as before

$$\psi_\uparrow(t, r, \theta, \phi) = \begin{bmatrix} A \\ 0 \\ B \\ 0 \end{bmatrix} \exp\left[\frac{i}{\hbar}I\right]$$

it is found upon insertion into the (chargeless) Dirac equation (5.1) the following is obtained

$$-B\left(\frac{1}{\sqrt{f}}\partial_t I + \sqrt{g}\partial_r I\right) + Am = 0 \tag{6.54}$$
$$-B\left(i\sqrt{\frac{f}{FK}}\partial_\phi I - i\frac{H}{\sqrt{FK}}\partial_t I + \frac{1}{\sqrt{\Sigma}}\partial_\theta I\right) = 0 \tag{6.55}$$
$$A\left(\frac{1}{\sqrt{f}}\partial_t I - \sqrt{g}\partial_r I\right) + Bm = 0 \tag{6.56}$$
$$-A\left(i\sqrt{\frac{f}{FK}}\partial_\phi I - i\frac{H}{\sqrt{FK}}\partial_t I + \frac{1}{\sqrt{\Sigma}}\partial_\theta I\right) = 0 \tag{6.57}$$

Repeating the same kind of analysis as before (using the ansatz $I = -Et + J\phi + W$) it is found from (6.54) and (6.56) that

$$W_{r\pm} = \frac{\pm E}{\sqrt{fg}}$$

Since f does not vanish at the horizon (except when $\sin\theta = 0$), this expression does not have a simple pole at the horizon. It is not possible to solve the expression for arbitrary θ, and the calculation becomes intractable. This situation is analogous to what happens in the scalar field case if one naively applies the null geodesic method to a rotating black hole by trying to force ϕ to be constant (i.e. $d\phi = 0$), as shown in the previous section.

In order to understand this issue in more detail it is useful to examine the similarity transformation between γ^μ and $\tilde{\gamma}^\mu$. It is found that:

$$\tilde{\gamma}^\mu = S\gamma^\mu S^{-1}, \text{ for all } \mu$$

when:

$$S = \begin{pmatrix} aI - b\sigma^2 & 0 \\ 0 & aI + b\sigma^2 \end{pmatrix}$$

where

$$a = \sqrt{\frac{1}{2}\left(\sqrt{\frac{F}{f}} + 1\right)} \qquad b = \sqrt{\frac{1}{2}\left(\sqrt{\frac{F}{f}} - 1\right)} \tag{6.58}$$

The transformation S is similar to a Lorentz boost in the ϕ direction. Applying it to the spin up ansatz used previously I find

$$\tilde{\psi}_\uparrow(t, r, \theta, \phi) = \begin{bmatrix} Aa \\ -Aib \\ Ba \\ Bib \end{bmatrix} \exp\left[\frac{i}{\hbar}I\right] \tag{6.59}$$

As $r \to \infty$ it can be seen that $a \to 1$ and $b \to 0$, yielding the same spin up ansatz in this limit. Inserting (6.59) into the (chargeless) Dirac equation (5.1) and following the same procedure as before results in the same expression (5.86) for the temperature. This is not surprising since all has been done is to apply a similarity transformation to the Dirac equation, and I shall not repeat the (somewhat more tedious) calculations here. My point is to emphasize the importance of choosing an appropriate ansatz for a given choice of γ matrices.

CTC's and Taub-NUT space

The presence of closed timelike curves in Taub-NUT space can be seen by considering the curve generated by the Killing vector ∂_φ and by examining $g_{\varphi\varphi}$

$$g_{\varphi\varphi} = 4f_k^2(\frac{\theta}{2})\left(r^2 + n^2 - f_k^2(\frac{\theta}{2})(4n^2F + k(r^2 + n^2))\right)$$

So for $k = 1, 0$, and $k = -1$ with $4n^2/\ell^2 > 1$ the quantity $g_{\varphi\varphi} < 0$, yielding a timelike ∂_φ; the curve $r = r_0$, $t = t_0$, and $\theta = \theta_0$ becomes a CTC.

However there is a range of hyperbolic Taub-NUT solutions that occur when $4n^2/\ell^2 \leq 1$ that don't contain CTC's. Now it is possible for $g_{\varphi\varphi}$ to be negative

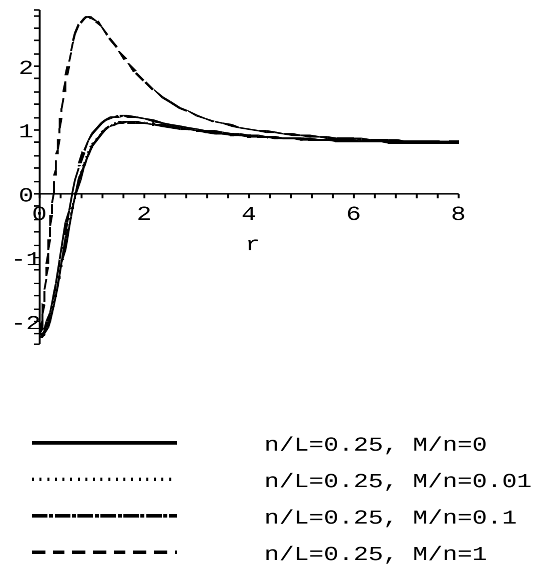

Figure 7: Plots when $4\mathrm{n}^2/\mathrm{L}^2 < 1$ for a range of masses

when $4n^2/\ell^2 < 1$but this occurs for small values of r_0 and happens inside the horizon. Explicitly when $k = -1$ then $g_{\varphi\varphi}$ is given by

$$g_{\varphi\varphi} = 4\sinh^2(\frac{\theta}{2})(r^2+n^2)\left(\cosh^2(\frac{\theta}{2}) - \frac{4n^2F}{r^2+n^2}\sinh^2(\frac{\theta}{2})\right)$$

So $g_{\varphi\varphi} \geq 0$ will always be true as long as $\frac{4n^2F}{r^2+n^2} \leq 1$. Figures 7-9 are plots of $1 - \frac{4n^2F}{r^2+n^2}$ for a range of mass and NUT-charge . On the plots the x-axis is r/n. The $k = -1$ case corresponds to hyperbolic solutions whose event horizon has radius $r_b > n$. Since $g_{\varphi\varphi}$ only becomes negative when $r < n$ (within $4n^2/\ell^2 \leq 1$) any CTCs are contained within the horizon (provided the mass is positive). So no CTC's are present outside of the horizon for the hyperbolic case when $4n^2/\ell^2 \leq 1$.

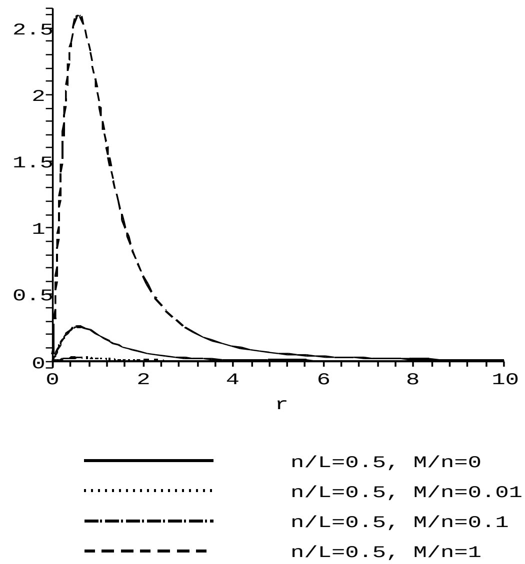

Figure 8: Plots for when $4\mathrm{n}^2/\mathrm{L}^2 = 1$ for range of masses.

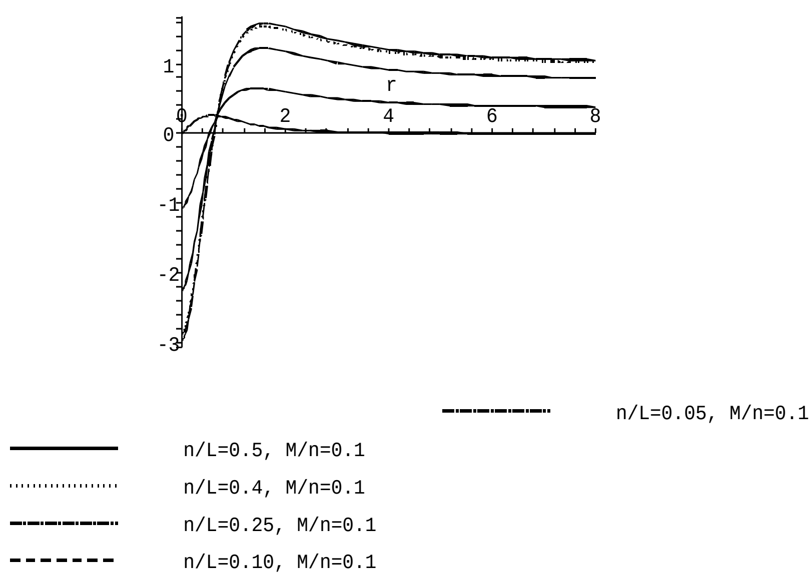

Figure 9: Plots for fixed mass and a range of $\mathrm{n}^2/\mathrm{L}^2$

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