Optimal Reinsurance Retentions under Ruin-Related Optimization Criteria

by

Zhi Li

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Zhi Li

Abstract

Quota-share and stop-loss/excess-of-loss reinsurances are two important reinsurance strategies. An important question, both in theory and in application, is to determine optimal retentions for these reinsurances. In this thesis, we study the optimal retentions of quota-share and stop-loss/excess-of-loss reinsurances under ruin-related optimization criteria.

We attempt to balance the interest for a ceding company and a reinsurance company and employ an optimization criterion that considers the interests of both a cedent and a reinsurer. We also examine the influence of interest, dividend, commission, expense, and diffusion on reinsurance retentions.

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Chapter 1

Introduction

Quota-share and stop-loss/excess-of-loss reinsurances are two significant reinsurance strategies. An imperative question both in theory and in application is determining optimal retentions for each. Earlier studies involving the optimality of reinsurance contracts include Gerber (1979), Waters (1979, 1983), Goovaerts et al. (1989, 1990), Daykin et al. (1994), Buhlmann (1996), Bowers et al. (1997), Rolski et al. (1999), Schmitter (2001), Gollier (2003), Verlaak and Beirlant (2003), and references therein. To a large extent, early literature focused on the position of a ceding company. Further, few early models accounted for both the interests of a cedent and a reinsurer. Also, existing studies regarding optimal retentions in the collective risk model are based mainly on the classical compound Poisson risk model.

The thesis further develops risk models given the reinsurance precondition, with special consideration to the optimal reinsurance treaty under various criteria. The attempt to balance the interests of a ceding company and a reinsurance company, and to employ an optimization criterion that considers the interests of both a cedent and a reinsurer are undertaken. Additionally, the impact of economic and financial factors on the optimal retentions are examined. The factors included in this examination are the influence of interest, dividend, commission, expense, and diffusion. First, a brief historical background of reinsurance is helpful.

1.1 Reinsurance

Reinsurance is best thought of as “insurance for insurance companies,” a protection for primary insurers against unforeseen or extraordinary losses. Reinsurance serves to limit liability on specific risks, to increase individual insurer’s capacity to share liability when losses overwhelm the primary insurer’s resources, and to aid insurers in stabilizing financial concerns due to wide swings in profit and loss margins inherent to the insurance business. The company transferring the risk is called the ceding company or direct writer while the company adopting the risk is called the assuming company or reinsurer. Reinsurance contracts may cover a specific risk or a broad class of business. The reinsurer charges the reinsurance premium as the adequate compensation for assuming transferred risk from the cedent. There are two basic forms of reinsurance: proportional (pro-rata) and non-proportional (excess).

Proportional is a form of reinsurance where the amount ceded is defined at the point at which the risk is transferred, not at the point of claim. The amount of risk may vary with time. Proportional reinsurance contains two sub-forms, quota-share and surplus-share. For the purpose of the thesis, quota-share reinsurance is the sole focus. Under quota-share, premiums and losses are shared proportionately between the ceding company and the reinsurer, and the same percentage applies to all reinsurance policies in a given area of business.

Non-proportional is a form of reinsurance where the reinsurer's liability is not fixed in advance, but rather is dependent on the number or amount of claims incurred in a given period. Non-proportional reinsurance includes three sub-forms: catastrophe, stop-loss, and spread-loss. The main examination here with regards to non-proportional reinsurance is stop-loss reinsurance. Stop-loss dictates that the reinsurer pays some or all of the aggregate retained losses of a ceding company in excess of a predetermined dollar amount, or in excess of a percentage of the premium.

In most practical cases, the reinsurance protection of an insurance portfolio is not limited to one reinsurance type, rather it is organized through a combination of several methods of protection, or a so-called reinsurance program.

To achieve a balance between practical application and theoretical neatness, the thesis focuses on a quota-share and stop-loss/excess-of-loss combination. This form of reinsurance has been discussed by Centeno (1985, 1986, 2002(a), 2002(b)), Kaluszka (2001), Schmitter (2001), Cai and Tan (2007), Cai et al. (2008), and many others. Simply speaking, with the reinsurance, there exists a retention limit M and a quota-share level a. When a claim of size X arises, the cedent pays the amount of aX or M, whichever is less, and the reinsurer pays the remaining sum.

The premium principle attaches a premium to a risk for purposes of insurance, for example, Kaluszka (2005). Here, the expected value principle is employed as a default. The expected value principle determines that the premium paid is charged at a certain percentage of expected payout. Both direct writing insurance companies and reinsurers may use reinsurance to enter new markets, to try out new products, and to gain valuable underwriting experience. As previously stated, the ceding company benefits from reinsurance because it helps to manage financial risk, increase capacity, and achieve marketing goals. The obvious disadvantage is that the premiums paid reduce the amount of money available for other purposes and the cedent insurer's chance of earning unexpected profits is removed. Conversely, although reinsurers have the opportunity to invest income from premiums, and thus increase profitability, reinsurance also emphasizes certain problems, such as surplus strains on financial resources.

For insurance customers, and the insurance industry in general, reinsurance presents genuine advantages. Reinsurance aids customers by making certain that necessary coverage is available and affordable. Due to reinsurance, insurers are able to provide the amount of coverage requested, even if the amount is beyond the single insurer's retention limit.

For industry, reinsurance provides a greater spread of risk and the wider the spread of risk, the less the likelihood that any single insurer will suffer catastrophic financial loss from unexpectedly high claims. If each company has less exposure to catastrophic loss, the industry as a whole is better protected. This reinsurance makes the insurance industry stronger financially and provides a more stable and reliable marketplace for customers, investors and insurance companies. Balancing the interests between the cedent company and the reinsurer, and considerations of the effects of interest, dividends, commissions, expenses, and diffusion on reinsurance retentions are discussed in the thesis.

1.2 Outline of Thesis

As previously stated, a reinsurance contract involves two parties, an insurer and a reinsurer. In most existing reinsurance literature, optimal retentions only consider the interest of one party, what is optimal only for an insurer or for a reinsurer. However, it is worth questioning and determining optimal retentions that are, in some sense, fair to both parties. Chapters 2 and 3 derive the optimal retentions, considering the interests of both parties, to maximize the joint survival probability of an insurer and a reinsurer.

In particular, Chapter 2 presents the explicit solution for optimal retentions for quota-share and stop-loss in combined reinsurance treaties, which is derived based on the idea proposed by Ignatov et al. (2004) and Kaishev et al. (2006) for the single period claim model. It is very difficult to attain the explicit expression for the joint survival probability in a collective risk model with excess-of-loss reinsurance. Even for the exponential claim case, the explicit expression is not available. It is impossible to determine the optimal retentions by maximizing the joint survival probability directly. Hence, in Chapter 3, two methods are developed to manage the optimal retentions in the aggregate claim model. First, we prove a lower bound for the joint survival probability by using the association property of the aggregate claims of an insurer and a reinsurer, and then optimize the treaty by maximizing the lower bound. The second method develops a bivariate gamma approximation for the joint distribution of the surpluses of an insurer and a reinsurer. This bivariate gamma approximation itself is significant and is also utilized to approximate the joint distribution of two dependent nonnegative random variables. Using the bivariate gamma approximation in multi-period claim models, optimal retentions for quota-share and stop-loss in a combined reinsurance treaty can be determined. Note, for equity on both sides, the initial surplus is not considered in Chapters 2 and 3.

Conversely, most existing literature regarding optimal retentions in a collective risk model has basis in the classical compound Poisson risk model. These optimal retentions do not consider the effect of economic or financial factors on optimal retentions. Chapters 4 through 6 consider optimal retentions in risk models with interest rates, dividends, commissions, expenses, and diffusions. Determining optimal retentions in risk models, which include interest and dividends, presents quite a complex inquiry. Chapter 4 attempts to generalize De Vylder's approximation and presents the development of a De Vylder-type approximation for an interest-included compound Poisson risk model. The De Vylder-type approximation is also itself noteworthy and is useful in studying additional inquiries into the interest-included compound Poisson risk model. This approximation is utilized in determining optimal retentions for quota-share and excess-of-loss in combined reinsurance treaties. The influence of interest rates on the optimal retentions is illustrated numerically.

Chapter 5 explores the criterion of maximizing the expected total discounted dividends disbursed up to ruin to derive the optimal retention of a quota-share reinsurance in the compound Poisson risk model with dividends. Based on Gerber and Shiu (2006), the model is generated to include the reinsurance factor for the Erlang claim and is compared to the exponential claim. This chapter illustrates the effects of dividends on the optimal retentions by studying both exponential and Erlang (2) claims.

It is difficult to determine the explicit formulas for the infinite-time and finite-time ruin probabilities, which means it is not possible to obtain the optimal retention levels by minimizing the ruin probabilities directly. However, when certain conditions are applied, the upper bounds of the ruin probabilities exist. By discovering the minimum upper bounds, we can determine the optimal retention levels and the ruin probabilities are limited so that the risk does exceed a certain limit.

Considering the uncertain economic events in the surplus process, Chapter 6 extends Centeno's (1985-2002) work to a jump-diffusion risk model; it also includes the commission and expenses in determining the net premium for a ceding company.

Using the criterion of maximizing the adjustment coefficient, there exists a simple explicit formula to determine the reinsurance retention level. First, Chapter 6 derives the optimal retention limit by minimizing the Lundberg upper bound for the infinite-time ruin probability. Second, we derive the upper bound for the finite-time ruin probability in the jump-diffusion risk model by the martingale approach. We then study the excess-of-loss reinsurance and the optimal retentions of reinsurance by minimizing the upper bound.

Finally, concluding remarks and questions for further research are offered in Chapter 7. Some other literature we used in this thesis are Grandell (1991), Klugman et al. (1998), Willmot and Lin (2001).

1.3 Notations and Definitions

The following notations and definitions are used throughout the thesis.

Shown first is the notation set for the claim frequency.

- $\{Y_i\}_{i=1}^{\infty}$ are independent and identically distributed non-negative random variables with the common distribution as Y and Y_i is the time between the $(i-1)$th claim and the ith claim. The common distribution function is $H(y) = \Pr\{Y \leq y\}$ with mean of $E(Y) = \frac{1}{\lambda}$.

- T_n denotes the time of the nth claim, which equals to $T_n = Y_1 + Y_2 + \cdots + Y_n$ with $T_0 = 0$.

- $\{N(t)\}_{t \geq 0}$ is an ordinary renewal process and $N(t)$ denotes the number of claims, which occur in a certain time interval $(0, t]$, it can be written as $N(t) = \sup\{n :$

$T_n \leq t\}$.

The foundational assumption of the work set forth here is the reinsurance treaty. The assumption is made that an insurer first sets a quota-share retention level of a, then sets a stop-loss retention limit of M such that the insurer retains $X_I = \min(aX, M) = (aX \wedge M)$ when a claim of size X occurs. The remaining claim $X_R = X - X_I$ is designated for the reinsurance company. The expressed objective is to seek the quota-share level a and the retention limit M to minimize the insurer's risk in different risk models.

When the retention limit of M is infinite, the treaty then becomes a pure quota-share reinsurance. When the quota-share level a is one, the treaty becomes a pure stop-loss reinsurance. These two scenarios, as special cases, will receive further discussion in the following chapters. When retention limit M or quota-share level entire insurance business to the reinsurer. Because of the well-established fact that the insurance company does not profit from this action, this situation is not considered.

Hereafter, the subscript "I" represents the aspect from a ceding insurance company and the subscript "R" represents the aspect from a reinsurer.

Secondly, the notation set for the claim size is as noted below.

- $\{X_i\}_{i=1}^{\infty}$ is a sequence of independent and identically distributed non-negative random variables, which are independent of $\{Y_i\}_{i=1}^{\infty}$. The random variable X_i is the claim amount of the ith claim with a common distribution function of $F(x) = \Pr\{X \leq x\}$ and an average claim size of $E(X) = \mu$.
- $X_{I_i} = \min(aX_i, M)$ is the amount of the ith claim paid by the insurance company, and $X_{R_i} = X_i - X_{I_i}$ is the amount of the ith claim paid by the reinsurance

company.

In the single period case, for simplicity, X_I is used instead of X_{I_i} and X_R is used instead of X_{R_i}.

In the multi-period case, the sum of all claims up to a time of t is considered. The total amounts paid by the cedent and reinsurer are represented as follows $S_I(t) = \sum_{i=1}^{N(t)} X_{I_i}$ and $S_R(t) = \sum_{i=1}^{N(t)} X_{R_i}$, respectively.

Thirdly, the notation set for the insurance company and the corresponding reinsurance company is as follows:

- θ_I and θ_R are the security loading factors for the insurer and reinsurer, respectively. Here $\theta_I < \theta_R$, which means the insurer cannot reinsure the whole risk with a certain profit.

- P_I and P_R are the premiums received by the insurer and reinsurer, respectively. They follow the expected value principle. This principle discussed further in later chapters.

- u_I and u_R are the non-negative initial surpluses for both companies respectively.

- The company surplus is equal to the initial surplus plus the net premium received, minus the net claim paid out. The respective surplus processes $\{U_I(t)\}$ and $\{U_R(t)\}$ are written as $U_I(t) = u_I + P_I t - S_I(t)$ and $U_R(t) = u_R + P_R t - S_R(t)$.

Chapter 2

A Fair Optimal Retention for Insurers and Reinsurers: Explicit Solutions

Let us first consider single contract, one period insurance. In the reinsurance contract, the two parties, cedent and reinsurer, have conflicting interests. Each party strives to minimize risk for a higher proportion of originally occurring premium income. The optimal contract must appear as a reasonable compromise between the two interests. Conversely, because both companies have common objectives in managing their shared risk, the insurer and the reinsurer have common interests and can be considered as partners.

One vital condition of achieving solvency and financial stability lies in maximizing survival probability; in reality maximizing the likelihood of survival is the top priority for each partner. As natural approach, considering total premium income and aggregated claims are shared between the ceding company and the reinsurer, as a method of maximizing the joint survival probability, this chapter will concentrate on defining the conditions of such a contract, optimal with respect to the interests of both parties.

Similar to previous assumptions, for a claim with size X, the insurance company pays

$$X_I = aX \wedge M,$$

and the reinsurance pays the remaining, which is

$$X_R = X - (aX \wedge M) \, .$$

This chapter will attempt to determine the maximum of joint survival or solvency probability under different reinsurance treaties, namely quota-share, stop-loss, and combination reinsurance.

Let us consider the premiums agreement between an insurance company and a reinsurance company. According to the expected value principle, the net premium received by the reinsurance company should be

$$P_R = (1+\theta_R) E\left(X_R\right),$$

and the premium received by the insurance company should be

$$P_I = (1+\theta_I) E\left(X\right) - P_R.$$

The joint survival or solvency probability is the probability that both companies will survive, i.e., the claims paid out are less than or equal to the premium received for both parties. It is written as $\Pr\{X_I \leq P_I, X_R \leq P_R\}$.

2.1 Quota-share Reinsurance

When we consider quota-share reinsurance, the retention level M goes to infinity. The quota-share level a is the only variable to be considered. This implies that the reinsurance amount of a claim with size X is $(1-a)X$, and the expected loss of the reinsurer is $E(X_R) = (1-a)E(X)$, where a is any percentage between 0% to 100%.

Hence, the premium received by the reinsurance company should be

$$P_R = (1+\theta_R)(1-a)E\left(X\right).$$

Recall that the insurance premium after reinsurance should be greater than zero, which means

$$P_I = \left(1+\theta_I\right)E(X) - \left(1+\theta_R\right)\left(1-a\right)E(X) > 0,$$

or

$$a > \frac{\theta_R - \theta_I}{1+\theta_R}.$$

Hence the joint solvency probability of the insurance company and the reinsurance company, $\Pr_{joint}(a)$, is expressed as

$$\begin{aligned}
& \Pr_{joint}(a) \\
= \; & \Pr\{X_I \leq P_I, X_R \leq P_R\} \\
= \; & \Pr\Big\{aX \leq (1+\theta_I)E\left(X\right) - (1+\theta_R)(1-a)E\left(X\right), \\
& (1-a)X \leq (1+\theta_R)(1-a)E\left(X\right)\Big\} \\
= \; & \Pr\{X \leq \frac{\theta_I - \theta_R}{a}E\left(X\right) + (1+\theta_R)E\left(X\right),\ X \leq (1+\theta_R)E\left(X\right)\} \\
= \; & \Pr\{X \leq (1+\theta_R)E\left(X\right) - \frac{\theta_R - \theta_I}{a}E\left(X\right)\} \\
= \; & F\Big((1+\theta_R)E\left(X\right) - \frac{\theta_R - \theta_I}{a}E\left(X\right)\Big). \qquad (2.1)
\end{aligned}$$

Since $\theta_R > \theta_I$, the joint solvency probability $\Pr_{joint}(a)$ is increasing in a. Thus, the maximum of the joint solvency probability is

$$F((1+\theta_I)E(X)),$$

which means the optimal quota-share level a is one and the ceding company retains all business and does not use any reinsurer. In other words, optimal reinsurance does not exist if an insurer only uses quota-share reinsurance.

2.2 Stop-loss Reinsurance

Under the stop-loss reinsurance treaty the quota-share a is equal to one. The retention limit M ($M > 0$) is the only variable that considered. The problem becomes finding the optimal M necessary to maximize the joint survival probability for both companies.

The expected reinsurance claim amount is equal to

$$\begin{aligned}
& E(X_R) \\
= \; & E(X) - E(X \wedge M) \\
= \; & \int_0^\infty x dF(x) - \int_0^M x dF(x) - \int_M^\infty M dF(x) \\
= \; & \int_M^\infty x dF(x) - \int_M^\infty M dF(x) \\
= \; & \int_M^\infty (x - M) \, dF(x).
\end{aligned}$$

Hence, the premium received by the reinsurance company is

$$P_R = (1+\theta_R) \int_M^\infty (x-M) dF(x),$$

and the net premium received by the ceding company is

$$P_I = (1+\theta_I)E(X) - (1+\theta_R)\int_M^\infty (x-M)dF(x).$$

Under this treaty, the joint solvency probability of the insurance company and the reinsurance company $\Pr_{joint}(M)$ is expressed as

$$\begin{aligned}
& \Pr_{joint}(M) \\
= \; & \Pr\{X_I \le P_I, X_R \le P_R\} \\
= \; & \Pr\{(X\wedge M) \le P_I, X-(X\wedge M) \le P_R\} \\
= \; & \Pr\{X \le M\}\Pr\{X\wedge M \le P_I, X-(X\wedge M) \le P_R | X \le M\} \\
& + \Pr\{X > M\}\Pr\{X\wedge M \le P_I, X-(X\wedge M) \le P_R | X > M\} \\
= \; & \Pr\{X \le M\}\Pr\{X \le P_I | X \le M\} \\
& + \Pr\{X > M\}\Pr\{M \le P_I, X - \; M \le P_R | X > M\} \\
= \; & \Pr\{X \le P_I, X \le M\} + \Pr\{M \le P_I\}\Pr\{X \le M+P_R, X > M\} \\
= \; & I_{\{M\le P_I\}}\left(\Pr\{X \le M\} + \Pr\{M < X \le M+P_R\}\right) + I_{\{M>P_I\}}\Pr\{X \le P_I\} \\
= \; & I_{\{M\le P_I\}}\Pr\{X \le M+P_R\} + I_{\{M>P_I\}}\Pr\{X \le (1+\theta_I)E(X) - P_R\} \\
= \; & \begin{cases} F(M+P_R) & \text{when } M+P_R \le (1+\theta_I)E(X), \\ F((1+\theta_I)E(X) - P_R) & \text{when } M+P_R > (1+\theta_I)E(X). \end{cases}
\end{aligned} \tag{2.2}$$

Here $I_{\{\cdot\}}$ is the indicator function, i.e., $I_{\{A\}} = 1$ if A is true and 0 otherwise.

Since

$$\frac{\partial(M+P_R)}{\partial M} = 1 + (1+\theta_R)\frac{\partial}{\partial M}\int_M^\infty (x-M)\,dF(x) = \; (1+\theta_R)F(M) - \theta_R,$$

and

$$\frac{\partial^2(M+P_R)}{\partial M^2} = (1+\theta_R)f(M) > 0,$$

we can conclude that $M + P_R$ is a convex function with respect to M and attains its minimum at

$$M = F^{-1}\left(\frac{\theta_R}{1+\theta_R}\right).$$

On the other hand, because

$$\frac{\partial((1+\theta_I)E(X) - P_R)}{\partial M} = (1+\theta_R)\,(1 - F(M)) > 0,$$

and

$$\frac{\partial^2((1+\theta_I)E(X) - P_R)}{\partial M^2} = -(1+\theta_R)f(M) < 0,$$

we can conclude that $((1+\theta_I)E(X) - P_R)$ is a monotone increasing concave function with respect to M.

Therefore, the relationships among lines $M + P_R$, $(1+\theta_I)E(X) - P_R$, and $(1+\theta_I)E(X)$ are expressed as identified in Figure 2.1 and Figure 2.2, based on different cases and parameters.

Case I: In this scenario, as shown in Figure 2.1, the joint survival function is defined on the dark brown line. It implies that the maximum survival probability is

$$F((1+\theta_I)E(X))$$

and it has two qualified retention levels, which should satisfy:

$$M + (1+\theta_R)\int_M^{\infty}(x - M)\,dF(x) = (1+\theta_I)E(X).$$

Case II: In this scenario, as shown in Figure 2.2, the joint survival function is defined on the line

$$(1+\theta_I)E(X) - P_R.$$

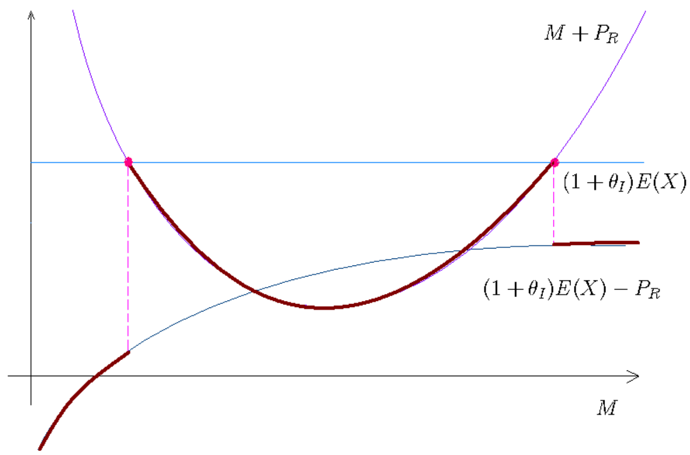

Figure 2.1: Stop-loss Reinsurance: Case I

To maximize the joint survival probability, the retention limit should be infinite. This implies that the insurance company will not use reinsurance, or the optimal reinsurance does not exist in this case.

Obviously, as M goes to infinity, the criteria $M > (1+\theta_I)E(X) - P_R$ will be satisfied, and the maximum joint solvency probability will be $F((1+\theta_I)E(X))$.

The above discussion establishes that when the claim amount distribution and the insurance loading satisfy the criteria presented in Case I, the optimal M to maximize the joint survival probability is achievable and the optimal M is the solution to the equation

$$M + (1+\theta_R)\int_M^\infty (x - M)\, dF(x) = (1+\theta_I)E(X).$$

Otherwise, the ceding company simply does not acquire any reinsurance and the optimal stop-loss reinsurance does not exist.

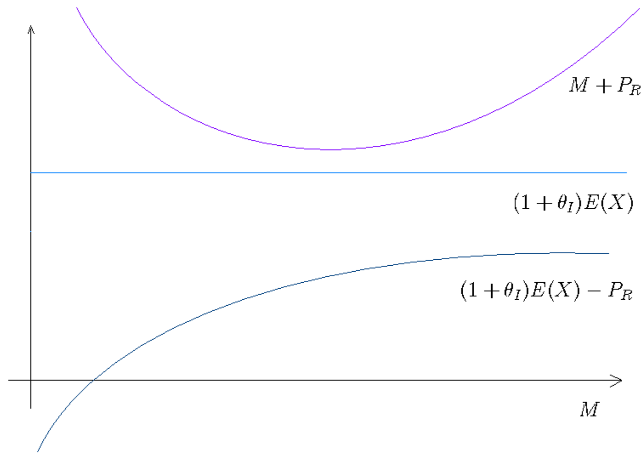

Figure 2.2: Stop-loss Reinsurance: Case II

2.3 Combination of Quota-share and Stop-loss Reinsurances

2.3.1 Joint Survival Probability

With the two special scenarios in Sections 2.1 and 2.2 receiving note, the focus shifts to the general scenario with both the retention limit M and the quota-share level a in the treaty. Here, $0 < a \leq 1$ and $M > 0$.

Lemma 2.3.1 The joint solvency probability of the insurance company and the reinsurance company $\Pr_{joint}(a, M)$ is expressed as

$$\begin{aligned} & \Pr_{joint}(a, M) \\ = \; & \Pr\{X_I \le P_I, X_R \le P_R\} \\ = \; & I_{\{M \le P_I\}} \Pr\{X \le M + P_R\} + I_{\{M > P_I\}} \Pr\{X \le \frac{P_I}{a}\}, \end{aligned} \tag{2.3}$$

where the premium received for the cedent and reinsurer are respectively

$$P_R = (1 + \theta_R) \left((1 - a)E(X) + \int_{\frac{M}{a}}^{\infty} (ax - M) \, dF(x) \right), \tag{2.4}$$

and

$$P_I = (1 + \theta_I)E\,(X) - P_R. \tag{2.5}$$

Here M is the retention limit, a is the quota-share level, and $F(x)$ is the claim distribution function. The parameters θ_I and θ_R represent the security loading factors for the insurance company and reinsurance company respectively with $\theta_I < \theta_R$.

Proof: First, note that

$$\begin{aligned} & E(X_R) \\ = \; & E\,[X - (aX \wedge M)] \\ = \; & E\,(X) - E\,(aX \wedge M) \\ = \; & \int_0^{\infty} x dF(x) - \int_0^{\frac{M}{a}} ax dF(x) - \int_{\frac{M}{a}}^{\infty} M dF(x) \\ = \; & (1 - a) \int_0^{\infty} x dF(x) + a \int_0^{\infty} x dF(x) - \int_0^{\frac{M}{a}} ax dF(x) - \int_{\frac{M}{a}}^{\infty} M dF(x) \\ = \; & (1 - a) \int_0^{\infty} x dF(x) + \int_{\frac{M}{a}}^{\infty} ax dF(x) - \int_{\frac{M}{a}}^{\infty} M dF(x) \\ = \; & (1 - a)E(X) + \int_{\frac{M}{a}}^{\infty} (ax - M) \, dF(x). \end{aligned} \tag{2.6}$$

Hence, according to the expected value principle, we can conclude that the reinsurance premium is

$$P_R = (1+\theta_R)E\left(X_R\right) = (1+\theta_R)\left((1-a)E(X) + \int_{\frac{M}{a}}^{\infty} (ax - M)\, dF(x)\right),$$

and the insurance premium is equal to

$$P_I = (1+\theta_I)E\left(X\right) - P_R.$$

The underlying insurance concept is that the total premium from both the ceding company and the reinsurance company should equal the total expected loss with a certain margin. This is also known as the expected value principle.

Additionally, note that

$$\begin{aligned} & \frac{P_I}{a} \\ = & \frac{(1+\theta_I)E\left(X\right) - (1+\theta_R)\left((1-a)E(X) + \int_{\frac{M}{a}}^{\infty} (ax - M)\, dF(x)\right)}{a} \\ = & (1+\theta_R)E(X) - \frac{\left(\theta_R - \theta_I\right)E(X) + (1+\theta_R)\int_{\frac{M}{a}}^{\infty} (ax - M)\, dF(x)}{a}, \end{aligned}$$

and

$$\begin{aligned} & \frac{P_R}{1-a} \\ = & \frac{(1+\theta_R)\left((1-a)E(X) + \int_{\frac{M}{a}}^{\infty} (ax - M)\, dF(x)\right)}{1-a} \\ = & (1+\theta_R)E(X) + \frac{(1+\theta_R)\int_{\frac{M}{a}}^{\infty} (ax - M)\, dF(x)}{1-a}. \end{aligned}$$

Because the reinsurance loading is greater than the insurance loading, $\theta_R > \theta_I$, we conclude that

$$\frac{P_I}{a} < \frac{P_R}{1-a}.$$

Hence, the joint survival probably of both parties, $\Pr_{joint}(a, M)$, is expressed as

$$\begin{aligned}
& \Pr_{joint}(a, M) \\
= & \Pr\{X_I \le P_I, X_R \le P_R\} \\
= & \Pr\{aX \wedge M \le P_I, X - (aX \wedge M) \le P_R\} \\
= & \Pr\{aX \le M\} \Pr\{aX \wedge M \le P_I, X - (aX \wedge M) \le P_R | aX \le M\} \\
& \quad + \Pr\{aX > M\} \Pr\{aX \wedge M \le P_I, X - (aX \wedge M) \le P_R | aX > M\} \\
= & \Pr\{aX \le M\} \Pr\{aX \le P_I, (1-a)X \le P_R | aX \le M\} \\
& \quad + \Pr\{aX > M\} \Pr\{M \le P_I, X - \ M \le P_R | aX > M\} \\
= & \Pr\{aX \le M\} \Pr\{X \le \frac{P_I}{a}, X \le \frac{P_R}{1-a} | aX \le M\} \\
& \quad + \Pr\{aX > M\} \Pr\{M \le P_I, X \le \ M + P_R | aX > M\} \\
= & \Pr\{aX \le M\} \Pr\{X \le \frac{P_I}{a} | aX \le M\} \\
& \quad + \Pr\{aX > M\} \Pr\{M \le P_I\} \Pr\{X \le \ M + P_R | aX > M\} \\
= & \Pr\{X \le \frac{P_I}{a}, aX \le M\} + \Pr\{M \le P_I\} \Pr\{X \le M + P_R, aX > M\} \\
= & \Pr\{X \le \frac{P_I}{a}, X \le \frac{M}{a}\} + \Pr\{M \le P_I\} \Pr\{\frac{M}{a} < X \le M + P_R\} \\
= & I_{\{M \le P_I\}} \left(\Pr\{X \le \frac{M}{a}\} + \Pr\{\frac{M}{a} < X \le M + P_R\} \right) + I_{\{M > P_I\}} \Pr\{X \le \frac{P_I}{a}\} \\
= & I_{\{M \le P_I\}} \Pr\{X \le M + P_R\} + I_{\{M > P_I\}} \Pr\{X \le \frac{P_I}{a}\}. \qquad (2.7)
\end{aligned}$$

It can be rewritten as

$$\Pr_{joint}(a, M) = \begin{cases} F\left(M + P_R\right) & \text{when } M \le P_I, \\ F\left(\frac{P_I}{a}\right) & \text{when } M > P_I. \end{cases}$$

■

2.3.2 Optimal Quota-share and Retention Limits

Prior to the determination of the optimal quota-share a and retention limit M to maximize the joint survival probability of both companies, various properties of the extremum are worthy of review.

Lemma 2.3.2 If $f(x, y)$ is a two-dimensional function with a relative extremum at a point (x_0, y_0), and has continuous partial derivatives at this point, then $f'_x(x_0, y_0) = 0$ and $f'_y(x_0, y_0) = 0$. The second partial derivatives test classifies the point as a local maximum or relative minimum.

Let $A = f''_{xx}(x_0, y_0)$, $B = f''_{xy}(x_0, y_0)$, $C = f''_{yy}(x_0, y_0)$ and $\Delta = B^2 - AC$. Then,

(1) If $\Delta < 0$ and $A > 0$, the point is a relative minimum.

(2) If $\Delta < 0$ and $A < 0$, the point is a relative maximum.

(3) If $\Delta > 0$, the point is a saddle point.

(4) If $\Delta = 0$, higher order tests must be used. ■

Lemma 2.3.3 The formula set for the partial derivatives of the integral part in the premium is

$$
\begin{aligned}
\frac{\partial \int_{\frac{M}{a}}^{\infty} (ax-M)\, dF(x)}{\partial M} &= \int_{\frac{M}{a}}^{\infty} -1 dF(x) - \frac{1}{a} (ax-M)\, f(x)|_{x=\frac{M}{a}} \\
&= -\left(1 - F(\frac{M}{a})\right), \\
\frac{\partial^2 \int_{\frac{M}{a}}^{\infty} (ax-M)\, dF(x)}{\partial M^2} &= \frac{\partial F(\frac{M}{a})}{\partial M} = \frac{1}{a} f(\frac{M}{a}), \\
\frac{\partial^2 \int_{\frac{M}{a}}^{\infty} (ax-M)\, dF(x)}{\partial a \partial M} &= \frac{\partial F(\frac{M}{a})}{\partial a} = -\frac{M}{a^2} f(\frac{M}{a}), \\
\frac{\partial \int_{\frac{M}{a}}^{\infty} (ax-M)\, dF(x)}{\partial a} &= \int_{\frac{M}{a}}^{\infty} x dF(X) + \frac{M}{a^2} (ax-M)\, f(x)|_{x=\frac{M}{a}} = \int_{\frac{M}{a}}^{\infty} x dF(X), \\
\frac{\partial^2 \int_{\frac{M}{a}}^{\infty} (ax-M)\, dF(x)}{\partial a^2} &= \frac{\partial}{\partial a} \int_{\frac{M}{a}}^{\infty} x dF(X) = \frac{M}{a^2} x f(x)|_{x=\frac{M}{a}} = \frac{M^2}{a^3} f(\frac{M}{a}).
\end{aligned}
$$

■

These formulas will be used in subsequent chapters.

Lemma 2.3.4 Let $f(a, M)$ be a general continuous bivariate function with $a \in R^+$ and $M \in R^+$. If for any fixed M, $f(a, M)$ is a unimodal function with respect to a and $f(\hat{a}, M) = \max_a f(a, M)$; and for any fixed a, $f(a, M)$ is a unimodal function with respect to M and $f(a, \hat{M}) = \max_M f(a, M)$; then $f(\hat{a}, \hat{M}) = \max_{a,M} f(a, M)$.

Proof: For $\forall a, \forall M$, there is

$$f(a, M) \le f(a, \hat{M}) \le f(\hat{a}, \hat{M}) \text{ and } f(a, M) \le f(\hat{a}, M) \le f(\hat{a}, \hat{M}),$$

hence $f(a, M) \le f(\hat{a}, \hat{M})$.

■

Given the above three lemmas, the optimization of the retention level and quota-share by derivation to get the extremum follows. However, due to the specialty of premium relationship, there exists a resourceful way to achieve the goals.

Theorem 2.3.1 If the condition

$$\min_a G_1\left(a, aF^{-1}\left(\frac{\theta_R}{1+\theta_R}\right)\right) < (1+\theta_I)E\left(X\right) \tag{2.8}$$

holds, then the maximum of the joint survival probability of the ceding company and the reinsurance company can be achieved when the quota-share level a and retention limit M satisfy

$$M + P_R = (1+\theta_I)E\left(X\right), \tag{2.9}$$

where

$$G_1(a, M) = M + P_R.$$

If condition (2.8) does not hold, the maximum of the joint survival probability is achieved for the largest possible M satisfying

$$(\theta_R - \theta_I)E\left(X\right) = (1+\theta_R)M\left(1 - F(\frac{M}{a})\right). \tag{2.10}$$

This set of $(\hat{a}, \hat{M})$ also satisfies

$$1 - F(\frac{M}{a}) - 2\frac{M}{a}f(\frac{M}{a}) < 0,$$

and

$$0 < P_I < M.$$

Proof: From Lemma 2.3.1, recalling the expected value principle,

$$P_I + P_R = (1+\theta_I)E(X),$$

the following exists

$$
\begin{aligned}
\Pr_{joint}(a, M) &= \begin{cases} F\left(M + P_R\right) & \text{when } M \leq P_I \\ F\left(\frac{P_I}{a}\right) & \text{when } M > P_I \end{cases} \\
&= \begin{cases} F\left(M + P_R\right) & \text{when } M \leq (1+\theta_I)E\left(X\right) - P_R \\ F\left(\frac{(1+\theta_I)E(X) - P_R}{a}\right) & \text{when } M > (1+\theta_I)E\left(X\right) - P_R \end{cases} \\
&= \begin{cases} F\left(M + P_R\right) & \text{when } M + P_R \leq (1+\theta_I)E\left(X\right) \\ F\left(\frac{(1+\theta_I)E(X) - P_R}{a}\right) & \text{when } (1+\theta_I)E\left(X\right) - P_R < M. \end{cases}
\end{aligned}
$$

Since the cumulative distribution function is monotone-increasing and right-continuous, we can conclude that the maximum of joint survival probability exists. The maximum value is less than or equal to $F\left((1+\theta_I)E\left(X\right)\right)$ or $F\left(\frac{M}{a}\right)$, whichever is greater. It depends on the relationship of $M + P_R$ and $(1+\theta_I)E\left(X\right)$. A further detailed discussion follows.

Let $G_1(a, M) = M + P_R$. From Lemma 2.3.3, recalling lemma 2.3.2, we have the following equations

$$
\begin{aligned}
\frac{\partial G_1(a, M)}{\partial M} &= 1 - (1+\theta_R)\left(1 - F(\frac{M}{a})\right) = (1+\theta_R)F(\frac{M}{a}) - \theta_R, \\
\frac{\partial G_1(a, M)}{\partial a} &= (1+\theta_R)\left(\int_{\frac{M}{a}}^{\infty} x dF(X) - E(X)\right) = -(1+\theta_R)\int_0^{\frac{M}{a}} x dF(X) < 0, \\
A &= \frac{\partial^2 G_1(a, M)}{\partial M^2} = (1+\theta_R)\frac{1}{a}f(\frac{M}{a}) > 0, \\
B &= \frac{\partial^2 G_1(a, M)}{\partial a \partial M} = -(1+\theta_R)\frac{M}{a^2}f(\frac{M}{a}) < 0, \\
C &= \frac{\partial^2 G_1(a, M)}{\partial a^2} = (1+\theta_R)\frac{M^2}{a^3}f(\frac{M}{a}) > 0,
\end{aligned}
$$

and

$$\Delta = B^2 - AC = 0.$$

Hence, with respect to a, the function $G_1(a, M)$ is a monotonically-decreasing convex function. When a has its smallest possible value, the function $G_1(a, M)$ attains its maximum. With respect to M, $G_1(a, M)$ is a convex function, which attains its minimum when

$$F(\frac{M}{a}) = \frac{\theta_R}{1+\theta_R}.$$

The global maximum requires further discussion.

Given that $(1+\theta_I)E(X)$ is not related to a and M, the relationship of $G_1(a, M)$ and $(1+\theta_I)E(X)$ has two cases, which can be expressed as provided in Figure 2.3.

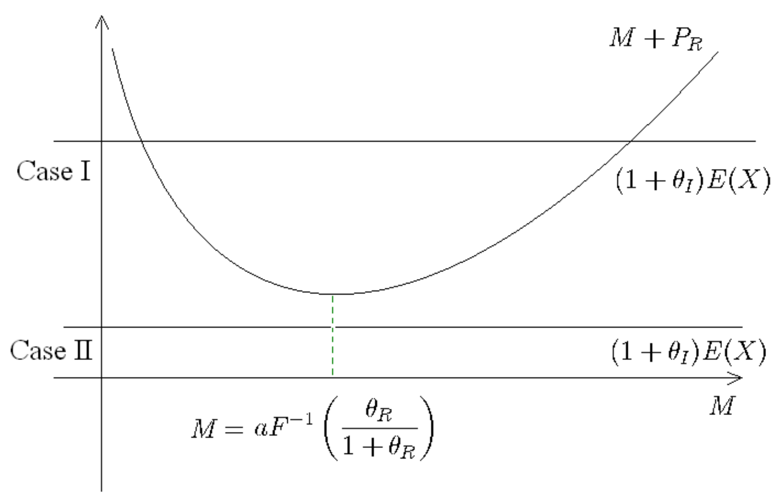

Figure 2.3: Function $G_1(a, M)$

Let

$$\begin{aligned} G_2(a, M) &= \frac{(1+\theta_I)E(X) - P_R}{a} \\ &= \frac{(\theta_I - \theta_R)E(X) - (1+\theta_R)\int_{\frac{M}{a}}^{\infty}(ax - M)\,dF(x)}{a} + (1+\theta_R)E(X). \end{aligned}$$

Similarly, from Lemma 2.3.3, recalling Lemma 2.3.2, the following equations are presented

$$\begin{aligned}
\frac{\partial G_2(a,M)}{\partial M} &= \frac{(1+\theta_R)}{a}\left(1-F(\frac{M}{a})\right) > 0, \\
\frac{\partial G_2(a,M)}{\partial a} &= \frac{-(1+\theta_R)\int_{\frac{M}{a}}^{\infty} x dF(X)}{a} \\
&\quad -\frac{(\theta_I-\theta_R)E\left(X\right)-(1+\theta_R)\int_{\frac{M}{a}}^{\infty}\left(ax-M\right)dF(x)}{a^2} \\
&= \frac{(\theta_R-\theta_I)E\left(X\right)-(1+\theta_R)M\left(1-F(\frac{M}{a})\right)}{a^2}, \\
A &= \frac{\partial^2 G_2(a,M)}{\partial M^2} = -\frac{(1+\theta_R)}{a^2}f(\frac{M}{a}) < 0, \\
B &= \frac{\partial^2 G_2(a,M)}{\partial a \partial M} = \frac{(1+\theta_R)M}{a^3}f(\frac{M}{a}) - \frac{(1+\theta_R)}{a^2}\left(1-F(\frac{M}{a})\right), \\
C &= \frac{\partial^2 G_2(a,M)}{\partial a^2} \\
&= 2\frac{(1+\theta_R)M\left(1-F(\frac{M}{a})\right)-(\theta_R-\theta_I)E\left(X\right)}{a^3} - \frac{(1+\theta_R)M^2 f(\frac{M}{a})}{a^4},
\end{aligned}$$

and the extremum indicator is

$$
\begin{aligned}
\Delta &= B^2 - AC \\
&= \left(\frac{(1+\theta_R)M}{a^3} f(\frac{M}{a}) - \frac{(1+\theta_R)}{a^2}\left(1 - F(\frac{M}{a})\right)\right)^2 \\
&\quad +2\frac{(1+\theta_R)}{a^2} f(\frac{M}{a}) \frac{(1+\theta_R)M\left(1 - F(\frac{M}{a})\right) - (\theta_R - \theta_I)E(X)}{a^3} \\
&\quad -\frac{(1+\theta_R)}{a^2} f(\frac{M}{a}) \frac{(1+\theta_R)M^2 f(\frac{M}{a})}{a^4} \\
&= \frac{(1+\theta_R)^2 M^2}{a^6} f^2(\frac{M}{a}) - 2\frac{(1+\theta_R)^2 M}{a^5} f(\frac{M}{a})\left(1 - F(\frac{M}{a})\right) \\
&\quad +\frac{(1+\theta_R)^2}{a^4}\left(1 - F(\frac{M}{a})\right)^2 \\
&\quad +2\frac{(1+\theta_R)^2 M\left(1 - F(\frac{M}{a})\right)}{a^5} f(\frac{M}{a}) - 2\frac{(1+\theta_R)}{a^2} f(\frac{M}{a}) \frac{(\theta_R - \theta_I)E(X)}{a^3} \\
&\quad -\frac{(1+\theta_R)^2 M^2}{a^6} f^2(\frac{M}{a}) \\
&= \frac{(1+\theta_R)^2}{a^4}\left(1 - F(\frac{M}{a})\right)^2 - 2\frac{(1+\theta_R)(\theta_R - \theta_I)E(X)}{a^5} f(\frac{M}{a}).
\end{aligned}
$$

Hence, with respect to M, the function $G_2(a, M)$ is a monotonically-increasing concave function. The maximum $G_2(a, M)$ attains when M has the largest possible value. With respect to a, the function $G_2(a, M)$ is a concave function, which attains its maximum when

$$
(\theta_R - \theta_I)E(X) = (1+\theta_R)M\left(1 - F(\frac{M}{a})\right).
$$

When the above condition is satisfied, the second derivative with respect to a is negative because

$$
C = -\frac{(1+\theta_R)M^2 f(\frac{M}{a})}{a^4} < 0.
$$

The premium after reinsurance is

$$\begin{aligned}
P_I &= (1+\theta_I)E\left(X\right) - (1+\theta_R)\left((1-a)E(X) + \int_{\frac{M}{a}}^{\infty}\left(ax - M\right)dF(x)\right) \\
&= \left(\theta_I - \theta_R + a(1+\theta_R)\right)E(X) - (1+\theta_R)a\int_{\frac{M}{a}}^{\infty} xdF(x) \\
&\quad + M(1+\theta_R)\left(1 - F(\frac{M}{a})\right) \\
&= a(1+\theta_R)E(X) - (1+\theta_R)a\int_{\frac{M}{a}}^{\infty} xdF(x) \\
&= a(1+\theta_R)\int_0^{\frac{M}{a}} xdF(x) \\
&> 0,
\end{aligned}$$

which indicates that the net premium for the ceding company is always positive.

The extremum indicator is

$$\begin{aligned}
\Delta &= \frac{(1+\theta_R)^2}{a^4}\left(1 - F(\frac{M}{a})\right)^2 - 2\frac{(1+\theta_R)(1+\theta_R)M\left(1 - F(\frac{M}{a})\right)}{a^5}f(\frac{M}{a}) \\
&= \frac{(1+\theta_R)^2}{a^4}\left(1 - F(\frac{M}{a})\right)\left(1 - F(\frac{M}{a}) - 2\frac{M}{a}f(\frac{M}{a})\right),
\end{aligned}$$

and

$$\begin{aligned}
G_2(a, M) &= \frac{(\theta_I - \theta_R)E\left(X\right) - (1+\theta_R)\int_{\frac{M}{a}}^{\infty}\left(ax - M\right)dF(x)}{a} + (1+\theta_R)E(X) \\
&= \frac{(\theta_I - \theta_R)E\left(X\right) - (1+\theta_R)a\int_{\frac{M}{a}}^{\infty} xdF(x) + M(1+\theta_R)\int_{\frac{M}{a}}^{\infty} dF(x)}{a} \\
&\quad + (1+\theta_R)E(X) \\
&= (1+\theta_R)E(X) - (1+\theta_R)\int_{\frac{M}{a}}^{\infty} xdF(x) \\
&= (1+\theta_R)\int_0^{\frac{M}{a}} xdF(x).
\end{aligned}$$

Now a discussion of the two situations is given case-by-case.

Case I: If the distribution of the specific claim amount satisfies Case I, which implies that

$$\min G_1\left(a, aF^{-1}\left(\frac{\theta_R}{1+\theta_R}\right)\right) < (1+\theta_I)E\left(X\right),$$

the answer to the maximum of the joint survival probability is fairly simple. It is necessary to find a and M that satisfy

$$M + P_R = (1+\theta_I)E\left(X\right).$$

Under this condition, $G_1(a, M) = (1+\theta_I)E(X)$. Also, recall that the cedent does not profit from ceding all the losses, which means $\theta_R > \theta_I$, and thus $G_2(a, M)$ is always less than $G_1(a, M)$.

The maximum of joint survival probability for both companies is

$$F\left(G_1(a, M)\right) = F\left((1+\theta_I)E\left(X\right)\right).$$

This condition is equal to $P_I = M$, i.e., the insurance premium after reinsurance will always be positive and equal to the retention limit.

Case II: In this situation, to achieve the maximum of the joint survival probability of two companies, the largest possible M must satisfy

$$(\theta_R - \theta_I)E\left(X\right) = (1+\theta_R)M\left(1 - F(\frac{M}{a})\right)$$

according to the specified claim distribution.

Since $0 < P_I = (1+\theta_I)E(X) - P_R < M$, the retention limit M and the quota-share a should satisfy

$$M > a(1+\theta_R)\int_0^{\frac{M}{a}} x dF(x).$$

On the other hand, since $\Delta < 0$, the retention limit M and the quota-share a should also satisfy

$$1 - F(\frac{M}{a}) - 2\frac{M}{a} f(\frac{M}{a}) < 0.$$

When the above-referenced criteria can be satisfied, the maximum joint survival probability is $F\left(G_2(a, M)\right).$ ■

2.4 Examples

As the optimal quota-share level a and retention limit M are dependent of the specific claim size distributions, exponential and Pareto claims are utilized here as examples. This section first discusses the properties of each claim size distribution. Because these distributions are employed throughout later chapters, this endeavor is significant. Using these properties, the special models are constructed according to the claim distribution, and finally the numbers are inserted into the model to indicate the numerical results. The results are illustrated to display the intuitive view.

2.4.1 Optimal Retention with Exponential Claims

Let us begin with the exponential distribution. Supposing the average claim size is β, we have the following expressions:

$$\begin{aligned}
f(x) &= \frac{1}{\beta} e^{-\frac{x}{\beta}}, \\
1 - F(\frac{M}{a}) &= e^{-\frac{M}{a\beta}}, \\
1 - F(\frac{M}{a}) - 2\frac{M}{a} f(\frac{M}{a}) &= e^{-\frac{M}{a\beta}}(1 - \frac{2M}{a\beta}), \\
\int_{\frac{M}{a}}^{\infty} (ax - M) \frac{1}{\beta} e^{-\frac{x}{\beta}} dx &= a\beta e^{-\frac{M}{a\beta}}, \\
E(X) &= \int_0^{\infty} x \frac{1}{\beta} e^{-\frac{x}{\beta}} dx = \beta, \\
F\left((1+\theta_I)E\left(X\right)\right) &= 1 - e^{-(1+\theta_I)}, \\
P_R &= (1+\theta_R)\beta \left(1 - a(1 - e^{-\frac{M}{a\beta}})\right), \\
P_I &= \beta \left(\theta_I - \theta_R + a \left(1 - e^{-\frac{M}{a\beta}}\right)(1+\theta_R)\right), \\
G_1(a, M) &= M + (1+\theta_R)\beta \left(1 - a(1 - e^{-\frac{M}{a\beta}})\right), \\
G_2(a, M) &= \beta\left(1+\theta_R\right)\left(1 - e^{-\frac{M}{a\beta}}\right) - \frac{\beta\left(\theta_R - \theta_I\right)}{a}.
\end{aligned}$$

From Figure 2.3, the minimum $G_1(a, M)$ is achieved when

$$1 - F(\frac{M}{a}) = \frac{1}{1+\theta_R},$$

which is

$$e^{-\frac{M}{a\beta}} = \frac{1}{1+\theta_R}.$$

The function $G_1(a, M)$ can be rewritten as

$$\begin{aligned}
& G_1(a, M) \\
= {} & M + (1+\theta_R)\beta \left(1 - a(1 - e^{-\frac{M}{a\beta}})\right) \\
= {} & a\beta \ln(1+\theta_R) + (1+\theta_R)\beta \left(1 - a\frac{\theta_R}{1+\theta_R}\right) \\
= {} & \beta\left(\ln(1+\theta_R) - \theta_R\right) a + (1+\theta_R)\beta.
\end{aligned}$$

For Case I to be true, it implies

$$\min(G_1(a)) < (1+\theta_I)E(X).$$

Note that in this case

$$\ln(1+\theta_R) - \theta_R < 0,$$

and also recall that when the quota-share a falls between 0 and 1, we should take $a = 1$. With future calculations, the loading factors of cedent and reinsurer should satisfy that

$$\ln(1+\theta_R) < \theta_I < \theta_R.$$

Here, the quota-share a and retention limit M should satisfy $P_I = M$, which is

$$a\beta\left(1 - e^{-\frac{M}{a\beta}}\right)(1+\theta_R) - M = (\theta_R - \theta_I)\,\beta,$$

to maximize the joint survival probability for both companies.

Let claims have average sizes of 100 and let the reinsurance loading be 20%. Since $\ln(1.2) = 0.182$, for Case I, we take insurance security loading as 19%. The combinations of all qualified quota-share a and retention level M are shown in Figure 2.4.

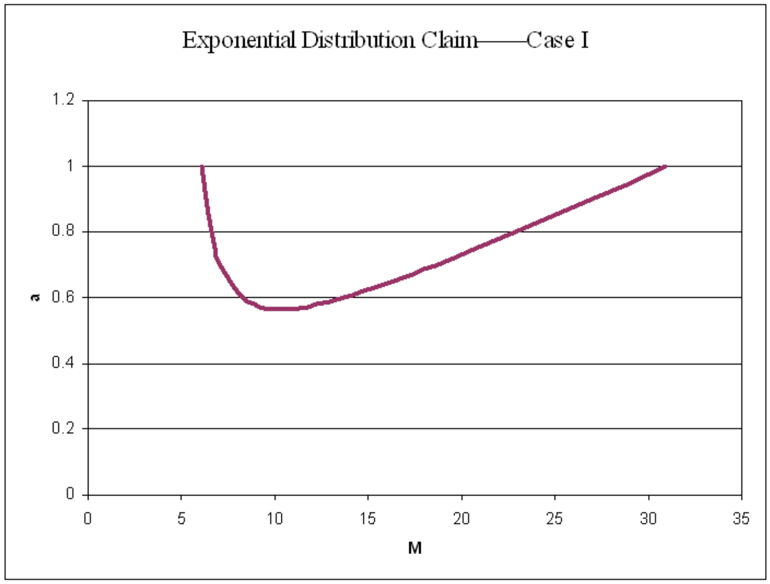

Figure 2.4: Combination of Qualified a and M for Exponential Claims

Any combination of the quota-share a and stop-loss M in the curve yields the same maximized joint survival probability. This line is identified as the "indifference line". The maximum joint survival probability for both companies is

$$F\left((1+\theta_I)E\left(X\right)\right) = 1 - e^{-(1+\theta_I)} = 69.58\%.$$

From Figure 2.4, note that, for a given quota-share level a, there exists two retention limits of M. We might ask: "which one is better?" Both Ms are indifferent to the counterparts as a whole. However, the ceding company tends to choose a smaller M to avoid potential losses, while the reinsurer tends to choose a higher M to avoid potential losses.

Take $a = 0.8$ for example. The corresponding retention level M can be 22.87 or 6.58 as noted in Table 2.1.

Table 2.1: "Indifference Line" Comparison for Exponential Claims

M	P_I	P_R	$\Pr(X_I < P_I)$	$\Pr(X_R < P_R)$
22.87	22.87	96.13	81.32%	76.94%
6.58	6.58	112.42	92.73%	71.95%

Case II provides the scenario of $\ln(1+\theta_R) > \theta_I$. In the following equation, M is the largest value that satisfies

$$(\theta_R - \theta_I)E(X) = (1+\theta_R)M(1 - F(\frac{M}{a})),$$

which can be rewritten as

$$(\theta_R - \theta_I)\beta = (1+\theta_R)Me^{-\frac{M}{a\beta}}.$$

Also to make $1 - F(\frac{M}{a}) - 2\frac{M}{a}f(\frac{M}{a}) < 0$, it requires that $a < \frac{2M}{\beta}$; and

$$0 < P_I = (1+\theta_I)E(X) - P_R < M$$

can be rewritten as

$$0 < \frac{(\theta_R - \theta_I)\,M}{(1+\theta_R)\,M - (\theta_R - \theta_I)\beta} < a < \frac{M^2 + (\theta_R - \theta_I)\,M\beta}{(1+\theta_R)\,M\beta - (\theta_R - \theta_I)\beta^2} < 1.$$

See below Figure 2.5 with $\theta_I = 18\%$.

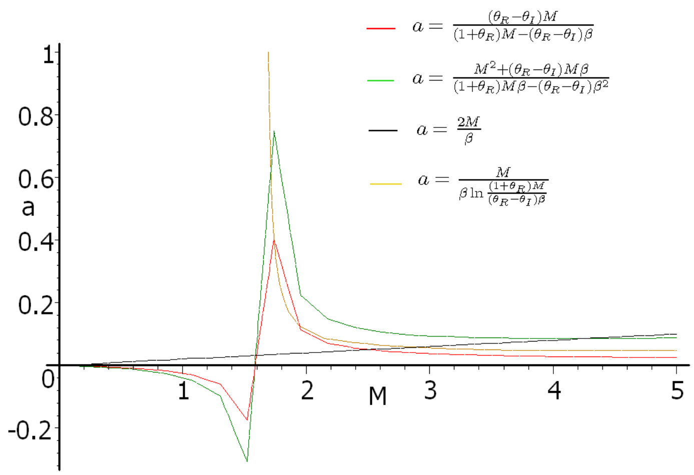

Figure 2.5: Exponential Distribution - Case II

The red line is

$$a = \frac{(\theta_R - \theta_I)\, M}{(1+\theta_R)\, M - (\theta_R - \theta_I)\beta};$$

the green line is

$$a = \frac{M^2 + (\theta_R - \theta_I)\, M\beta}{(1+\theta_R)\, M\beta - (\theta_R - \theta_I)\beta^2};$$

the black line is

$$a = \frac{2M}{\beta};$$

and the yellow line is

$$a = \frac{M}{\beta \ln \frac{(1+\theta_R)M}{(\theta_R - \theta_I)\beta}}.$$

Hence, the optimal quota-share level is $a = 0.054957$ and the optimal retention level is $M = 2.74787$. ■

2.4.2 Optimal Retention with Pareto Claims

A heavy-tailed claim is next for consideration — a Pareto distribution with parameters k and β. The assumption is that the claim amount distribution has a finite mean, variance, skewness, and excess kurtosis, which results in $k > 4$.

The following is the results for the Pareto distribution.

$$
\begin{aligned}
f(x) &= \frac{k\beta^k}{(x+\beta)^{k+1}}, \\
1 - F(\frac{M}{a}) &= \left(\frac{a\beta}{M+a\beta}\right)^k, \\
1 - F(\frac{M}{a}) - 2\frac{M}{a} f(\frac{M}{a}) &= \left(\frac{a\beta}{M+a\beta}\right)^k - 2\left(\frac{a\beta}{M+a\beta}\right)^k \frac{kM}{(M+a\beta)}, \\
\int_{\frac{M}{a}}^{\infty} (ax - M)\frac{k\beta^k}{(x+\beta)^{k+1}} dx &= \frac{a\beta}{k-1}\left(\frac{a\beta}{M+a\beta}\right)^{k-1}, \\
E(X) &= \frac{\beta}{k-1}, \\
F((1+\theta_I)E(X)) &= 1 - \left(\frac{k-1}{\theta_I + k}\right)^k,
\end{aligned}
$$

and

$$
\begin{aligned}
P_R &= (1+\theta_R)\frac{\beta}{k-1} - (1+\theta_R)\frac{a\beta}{k-1}\left(1 - \left(\frac{a\beta}{M+a\beta}\right)^{k-1}\right), \\
P_I &= (\theta_I - \theta_R)\frac{\beta}{k-1} + (1+\theta_R)\frac{a\beta}{k-1}\left(1 - \left(\frac{a\beta}{M+a\beta}\right)^{k-1}\right), \\
G_1(a, M) &= M + (1+\theta_R)\frac{\beta}{k-1} - (1+\theta_R)\frac{a\beta}{k-1}\left(1 - \left(\frac{a\beta}{M+a\beta}\right)^{k-1}\right), \\
G_2(a, M) &= (\theta_I - \theta_R)\frac{\beta}{a\,(k-1)} + (1+\theta_R)\frac{\beta}{k-1}\left(1 - \left(\frac{a\beta}{M+a\beta}\right)^{k-1}\right).
\end{aligned}
$$

From Figure 2.3, the minimum of $G_1(a, M)$ is attained when

$$1 - F(\frac{M}{a}) = \frac{1}{1+\theta_R},$$

or

$$\left(\frac{a\beta}{M+a\beta}\right)^k = \frac{1}{1+\theta_R}.$$

$G_1(a, M)$ is rewritten as

$$\begin{aligned}
& G_1(a, M) \\
= \; & M + (1+\theta_R)\frac{\beta}{k-1} - (1+\theta_R)\frac{a\beta}{k-1}\left(1 - \left(\frac{a\beta}{M+a\beta}\right)^{k-1}\right) \\
= \; & a\beta\,(1+\theta_R)^{\frac{1}{k}} - a\beta + (1+\theta_R)\frac{\beta}{k-1} - (1+\theta_R)\frac{a\beta}{k-1} + (1+\theta_R)\frac{a\beta}{k-1}\,(1+\theta_R)^{-1+\frac{1}{k}} \\
= \; & a\beta\,(1+\theta_R)^{\frac{1}{k}}\,(1+\frac{1}{k-1}) - a\beta + (1+\theta_R)\frac{\beta}{k-1} - (1+\theta_R)\frac{a\beta}{k-1} \\
= \; & \frac{\beta}{k-1}ka\,(1+\theta_R)^{\frac{1}{k}} + \frac{\beta}{k-1}(1-ak) + \frac{\beta}{k-1}\,(1-a)\,\theta_R \\
= \; & \beta\frac{1+\theta_R}{k-1} + \frac{\beta}{k-1}\left(k\,(1+\theta_R)^{\frac{1}{k}} - k - \theta_R\right)a.
\end{aligned}$$

Because the quota-share a is between 0 and 1, recall that $k > 1$, we have

$$k\,(1+\theta_R)^{\frac{1}{k}} - k - \theta_R < 0.$$

In order to qualify for Case I criteria which implies $\min(G_1(a)) < (1+\theta_I)E(X)$, the loading factors of the cedent and reinsurer should satisfy

$$k\,(1+\theta_R)^{\frac{1}{k}} - k < \theta_I < \theta_R.$$

In this case, the quota-share level a and retention limit M should satisfy $P_I = M$, which is

$$(\theta_I - \theta_R)\frac{\beta}{k-1} + (1+\theta_R)\frac{a\beta}{k-1}\left(1 - \left(\frac{a\beta}{M+a\beta}\right)^{k-1}\right) = M,$$

to maximize the joint survival probability for both companies.

To further study this Pareto distribution, a numerical example is helpful. To compare it to the above-referenced, exponentially distributed claim, requires the assumption that both share an identical mean, i.e., the average of the claim size is equal under both distributions. Additionally, the security loadings are assumed to be equal.

Let the claim have an average size of 100, $k = 5$, $\beta = 400$, and the reinsurance loading be 20%. Since $5\,(1+0.2)^{\frac{1}{5}} - 5 = 0.186$, for Case I, we also take insurance security loading to be 19%. The maximum joint survival probability for both companies is 72.8%. Figure 2.6 illustrates the combinations of all qualified quota-share a and retention level M for both Pareto and exponential claim distributions.

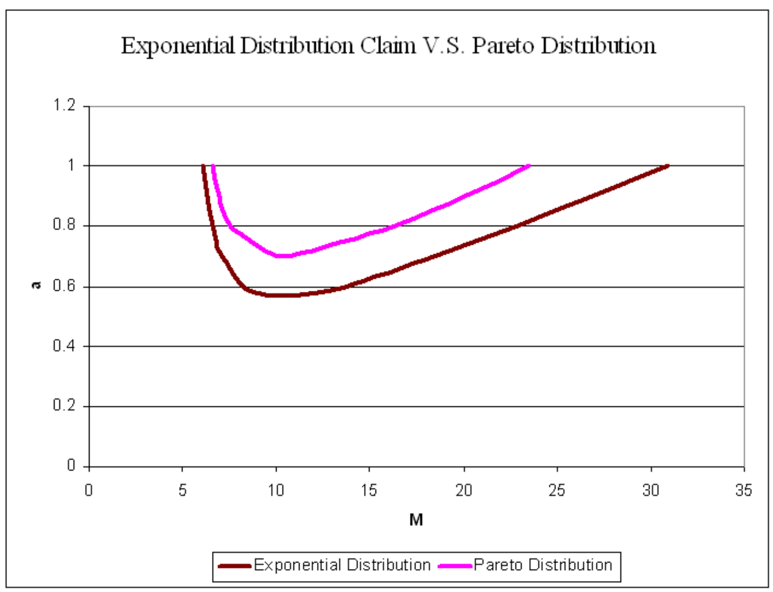

Figure 2.6: Exponential Distribution Vs. Pareto Distribution

From Figure 2.6, we can see that for any given retention level M, exponential distribution tends to select the smaller quota share a, i.e., it uses less reinsurance than Pareto distribution. It is given that both claim distributions have the same expected value. Consider a mixed exponential distribution and the 2-exponential distribution to explain this situation further.

Here, the density function of the mixed exponential distribution is

$$f(x) = q\mu_1 e^{-\frac{x}{\mu_1}} + (1-q)\mu_2 e^{-\frac{x}{\mu_2}}$$

where q is the weight. For a mixed exponential claim with the mean as 100, the equation (2.9) becomes

$$(1+\theta_I)100 - (1+\theta_R)\left((1-a)100 + qa\mu_1 e^{-\frac{M}{a\mu_1}} + (1-q)a\mu_2 e^{-\frac{M}{a\mu_2}}\right) = M,$$

where $q\mu_1 + (1-q)\mu_2 = 100$.

The hypoexponential distribution is the sum of two independent exponential distributed random variables with different parameters ($X = X_1 + X_2$). The density function is

$$f(x) = \int_0^x f_1(x-y)f_2(y)dy = \int_0^x \frac{1}{\mu_1}e^{-(x-y)/\mu_1}\frac{1}{\mu_2}e^{-y/\mu_2}dy = \frac{e^{-\frac{x}{\mu_1}} - e^{-\frac{1}{\mu_2}x}}{\mu_1 - \mu_2},$$

where μ_1 and μ_2 are the average sizes of the two exponential distributions X_1 and X_2 respectively, and $\mu_1 \neq \mu_2$. This distribution has the mean of $\mu_1 + \mu_2$, and the variance is $\mu_1^2 + \mu_2^2$. Note that if $\mu_1 = \mu_2$, this distribution is a gamma distribution with density function $f(x) = \frac{x}{\mu_1^2}e^{-\frac{x}{\mu_1}}$. For a mixed exponential claim with the mean as 100, the equation (2.9) becomes

$$(1+\theta_I)100 = M + (1+\theta_R)\left((1-a)100 + \frac{a}{\mu_1-\mu_2}\left(\mu_1^2 e^{-\frac{M}{a\mu_1}} - \mu_2^2 e^{-\frac{M}{a\mu_2}}\right)\right),$$

where $\mu_1 + \mu_2 = 100$.

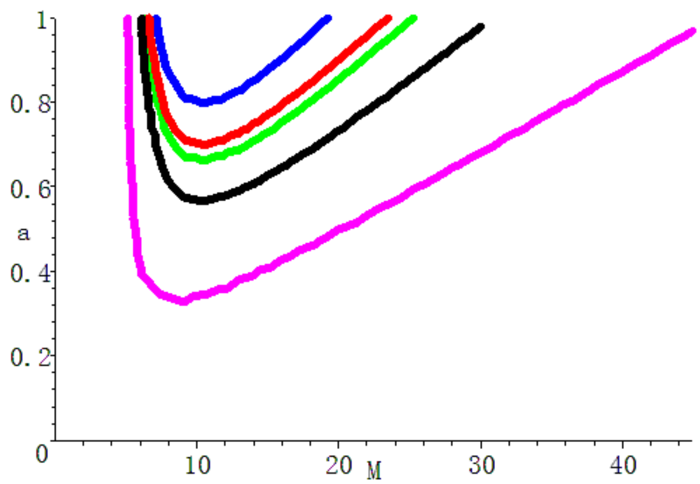

Figure 2.7: Mixed Exponential Distribution Vs. Pareto Distribution

As the standard deviation increases, the claim volatility increases, and there is a higher probability of incurring the higher amount claim, which affects the reinsurance company. To reduce the liability of the reinsurer and to ensure both companies survive, the optimal treaty should shift the claim to the ceding company to lower the larger claims that have to be absorbed by the reinsurer. This is indicated in Figure 2.7 and Table 2.2.

Table 2.2: Means and Standard Deviations of the Distributions

Color	Distribution	Mean	Stand Deviation
purple	hypoexponential	$90 + 10 = 100$	91
black	Exponential	$1 \times 100 = 100$	100
green	Mixed Exp 1	$0.3 \times 170 + 0.7 \times 70 = 100$	119
red	Pareto	100	129
blue	Mixed Exp 2	$0.4 \times 175 + 0.6 \times 50 = 100$	132

If $k\,(1+\theta_R)^{\frac{1}{k}} - k > \theta_I$, we will consider this scenario as Case II.

Now, M is the largest one that satisfies

$$(\theta_R - \theta_I)E(X) = (1+\theta_R)M(1 - F(\frac{M}{a})),$$

which can be rewritten as

$$(\theta_R - \theta_I)\frac{\beta}{k-1} = (1+\theta_R)M\left(\frac{a\beta}{M+a\beta}\right)^k.$$

To ensure $1 - F(\frac{M}{a}) - 2\frac{M}{a}f(\frac{M}{a}) < 0$, we need to have $M + a\beta < 2kM$ and

$$0 < P_I = (1+\theta_I)E(X) - P_R < M$$

can be rewritten as

$$\frac{(\theta_R - \theta_I)}{(1+\theta_R)} < a\left(1 - \left(\frac{a\beta}{M+a\beta}\right)^{k-1}\right) < \frac{k-1}{(1+\theta_R)\beta}M + \frac{(\theta_R - \theta_I)}{(1+\theta_R)}.$$

Figure 2.8 employs $\theta_I = 18\%$ as the exponential distribution discussed previously

.

The red line is

$$\frac{(\theta_R - \theta_I)}{(1+\theta_R)} = a\left(1 - \left(\frac{a\beta}{M+a\beta}\right)^{k-1}\right);$$

the black line is

$$M + a\beta = 2kM;$$

and the yellow line is

$$(\theta_R - \theta_I)\frac{\beta}{k-1} = (1+\theta_R)M\left(\frac{a\beta}{M+a\beta}\right)^k.$$

Here, P_I is strictly less than M, so not present is line

$$a\left(1 - \left(\frac{a\beta}{M+a\beta}\right)^{k-1}\right) = \frac{k-1}{(1+\theta_R)\beta}M + \frac{(\theta_R - \theta_I)}{(1+\theta_R)}.$$

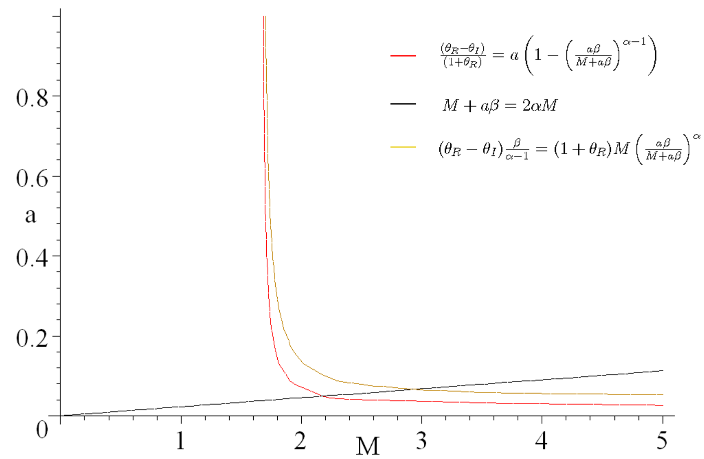

Figure 2.8: Pareto Distribution - Case II

The optimal quota-share level is $a = 0.063507$ and the retention level is $M = 2.8225$. ∎

Chapter 3

A Fair Optimal Retention for Insurers and Reinsurers by Maximizing the Lower Bound of the Joint Survival Probability and the Bivariate Translated Gamma Approximation

3.1 Motivations

Chapter 2 discusses properties of the reinsurance treaty for the ceding company and the reinsurance company under the single period case. Chapter 3 expands the research to the multi-period aggregate claims. Here we consider

$$S_I = \sum_{i=1}^{N} X_{I_i} = \sum_{i=1}^{N} \min(aX_i, M),$$

as the claims retained by the ceding company, and

$$S_R = \sum_{j=1}^{N} X_{R_j} = \sum_{j=1}^{N} \left(X_j - \min(aX_j, M)\right),$$

as the claims ceded to the reinsurer.

Using the previous notations, the premiums P_I and P_R in the joint survival probability

$$\Pr\left\{S_I \leq P_I, S_R \leq P_R\right\},$$

are expressed as

$$\begin{aligned} P_R &= E(N)(1+\theta_R)E\left[X - (aX \wedge M)\right] \\ &= E(N)(1+\theta_R)\left((1-a)E(X) + \int_{\frac{M}{a}}^{\infty} (ax - M)\, dF(x)\right) \end{aligned}$$

which is the reinsurance premium, and

$$\begin{aligned} P_I &= E(N)\left((1+\theta_I)E\left(X\right) - (1+\theta_R)E\left[X - (aX \wedge M)\right]\right) \\ &= E(N)(1+\theta_I)E\left(X\right) - E(N)(1+\theta_R)\left((1-a)E(X) + \int_{\frac{M}{a}}^{\infty} (ax - M)\, dF(x)\right) \end{aligned}$$

which is the premium after reinsurance.

It is a challenge to calculate the joint survival probability of S_I and S_R. Thus, it is not feasible to determine the optimal a and M by maximizing the joint survival probability directly. However, using the properties of associated random variables, a lower bound $L(a, M)$ is derived for the joint survival probability, namely

$$\Pr\left\{S_I \leq P_I, S_R \leq P_R\right\} \geq L(a, M).$$

Then we can determine the optimal a and M by maximizing the lower bound. This idea is equivalent to minimizing the upper bound of the corresponding ruin probability. This method is considered in sections 3.2 and 3.3 of this chapter.

Another commonly used method for the study of distribution functions is approximation. As for the distribution of S_I or S_R, a simple method is to use the normal approximation to the distribution of S_I or S_R. However, because the normal distribution is symmetric and the distribution of aggregate claims is often skewed, it is well known in actuarial literature that an efficient approximation to the distribution of S_I or S_R is the translated gamma distribution, which has a positive third central moment, as do the compound Poisson distributions with positive claim amounts. A more difficult question than calculating the distribution of S_I or S_R is to calculate the joint distribution of S_I and S_R. We may use a simple bivariate normal distribution approximation to the joint distribution of S_I and S_R. However, similar to the univariate case, the bivariate normal distribution approximation is not sound for the joint distribution of S_I and S_R.

In sections 3.4 through to 3.6 of this chapter, we will develop a translated bivariate gamma approximation to the joint distribution of S_I and S_R. The distribution parameters are selected by equating the covariance, first moment, second and third central moments of (S_I, S_R) with the corresponding characteristics of the translated bivariate gamma distributions. Then, we use the translated bivariate gamma approximation to determine the optimal retention levels a and M.

3.2 Association of Aggregate Claims of Cedent and Reinsurer

To derive a lower bound for the joint survival probability, we first recall the definitions and properties of associated random variables. In this chapter, increasing is defined as non-decreasing and a multivariate function is said to be increasing if the function is increasing in each argument.

Definition: Random variables $T_1, ..., T_n$ are said to be associated if

$$Cov[f(T_1, ..., T_n),\ g(T_1, ..., T_n)] \geq 0$$

for all increasing functions f and g for which $E[f(T_1, ..., T_n)]$, $E[g(T_1, ..., T_n)]$, and $E[f(T_1, ..., T_n)g(T_1, ..., T_n)]$ exist.

This concept of association is introduced in Esary, et al (1967). The concept is executed in many applied probability and statistics studies. In particular, many interesting probability inequalities or bounds for distribution functions are derived for associated random variables.

The following Lemma identifies several important properties of associated random variables, which will be used to derive the lower bound for the joint survival probability.

Lemma 3.2.1 Association has the following properties:

1. Any subset of associated random variables is associated.
2. If two sets of associated random variables are independent of one another, then their union is a set of associated random variables.

3. The set consisting of a single random variable is associated.

4. Increasing functions of associated random variables are associated.

5. Independent random variables are associated.

Proof: The proofs of (1)-(4) are given in (P1)-(P4) of Esary, et al (1967), and the proof of (5) is proved in Theorem 2.1 of Esary, et al (1967). ■

The following lemma identifies lower bounds for the joint distribution function and joint survival function of associated random variables.

Lemma 3.2.2 Let $T_1, ..., T_n$ be associated random variables and f_i, $i = 1, ..., k$, be increasing functions. Then,

$$\Pr\{f_1(T_1, ..., T_n) \leq t_1, ..., f_k(T_1, ..., T_n) \leq t_k\} \geq \prod_{i=1}^{k} \Pr\{f_i(T_1, ..., T_n) \leq t_i\}$$

and

$$\Pr\{f_1(T_1, ..., T_n) > t_1, ..., f_k(T_1, ..., T_n) > t_k\} \geq \prod_{i=1}^{k} \Pr\{f_i(T_1, ..., T_n) > t_i\}$$

for all $t_1, ..., t_k$.

Proof: The proof is given in Theorem 5.1 of Esary, et al (1967). ■

At this point, we are ready to derive the lower bound for the joint survival probability $\Pr\{S_I \leq P_I, S_R \leq P_R\}$.

Theorem 3.1.1 For any $0 < a \leq 1$ and $M > 0$, the random sums S_I and S_R are associated. Thus, the joint survival probability $\Pr\{S_I \leq P_I, S_R \leq P_R\}$ satisfies

$$\Pr\{S_I \leq P_I,\ S_R \leq P_R\} \geq \Pr\{S_I \leq P_I\} \Pr\{S_R \leq P_R\}.$$

Proof: First, we notice that for any $n = 1, 2, ...,$ the functions $h(x_1, ..., x_n) = \sum_{i=1}^{n} \min(ax_i, M)$ and $l(x_1, ..., x_n) = \sum_{i=1}^{n}(x_i - \min(ax_i, M))$ are nondecreasing. Thus, $\sum_{i=1}^{n} \min(aX_i, M)$ and $\sum_{j=1}^{n} (X_j - \min(aX_j, M))$ are associated since $X_1, ..., X_n$ are independent and Lemma 3.2.1 (4) and (5). Then, for any nondecreasing function f and g, we have

$$
\begin{aligned}
& E[f(S_I)g(S_R)] \\
= & \sum_{n=0}^{\infty} E\Big[f\Big(\sum_{i=1}^{n} \min(aX_i, M)\Big)\, g\Big(\sum_{j=1}^{n} (X_j - \min(aX_j, M))\Big)\Big] \Pr\{N = n\} \\
\geq & \sum_{n=0}^{\infty} E\Big[f\Big(\sum_{i=1}^{n} \min(aX_i, M)\Big)\Big]\, E\Big[g\Big(\sum_{j=1}^{n} (X_j - \min(aX_j, M))\Big)\Big] \Pr\{N = n\} \\
= & E\big[f_1(N)g_1(N)\big],
\end{aligned}
$$

where the inequality follows from the association of $\sum_{i=1}^{n} \min(aX_i, M)$ and $\sum_{j=1}^{n} (X_j - \min(aX_j, M))$, the sequences f_1 and g_1 are defined as

$$f_1(n) = E\Big[f\Big(\sum_{i=1}^{n} \min(aX_i, M)\Big)\Big],$$

and

$$g_1(n) = E\Big[g\Big(\sum_{j=1}^{n} (X_j - \min(aX_j, M))\Big)\Big].$$

Clearly, both $f_1(n)$ and $g_1(n)$ are nondecreasing in n. Thus, by Lemma 3.2.1(3), we get

$$E\big[f_1(N)g_1(N)\big] \geq E\big[f_1(N)\big]E\big[g_1(N)\big].$$

Note that

$$
\begin{aligned}
& E\big[f_1(N)\big] \\
= \ & \sum_{n=0}^{\infty} E\Big[f\Big(\sum_{i=1}^{n} \min(aX_i, M)\Big)\Big] \Pr\{N=n\} \\
= \ & E\Big[f\Big(\sum_{i=1}^{N} \min(aX_i, M)\Big)\Big] \\
= \ & E[f(S_I)],
\end{aligned}
$$

and

$$
\begin{aligned}
& E\big[g_1(N)\big] \\
= \ & E\Big[g\Big(\sum_{j=1}^{n} (X_j - \min(aX_j, M))\Big)\Big] \Pr\{N=n\} \\
= \ & E\Big[g\Big(\sum_{j=1}^{N} (X_j - \min(aX_j, M))\Big)\Big] \\
= \ & E[g(S_R)].
\end{aligned}
$$

Hence,

$$E[f(S_I)g(S_R)] \geq E[f(S_I)]E[g(S_R)],$$

which implies that S_I and S_R are associated. Thus, the lower bound for the joint survival probability follows from Lemma 3.2.2. ■

3.3 The Fair Optimal Retention by Maximizing Lower Bound of Joint Survival Probability

In the previous section, we derived the lower bound for the joint survival probability. We denote the lower bound by $L(a, M)$, namely,

$$\begin{aligned} & L(a, M) \\ = \; & \Pr\{S_I \leq P_I\} \Pr\{S_R \leq P_R\} \\ = \; & \Pr\left\{\sum_{i=1}^{N} \min(aX_i, M) \leq P_I\right\} \Pr\left\{\sum_{j=1}^{N} (X_j - \min(aX_j, M)) \leq P_R\right\}. \end{aligned}$$

Thus, we can determine the optimal a and M by maximizing the lower bound, which is to find $\hat{a}$ and $\hat{M}$ so that

$$L(\hat{a}, \hat{M}) = \max_{a,M} L(a, M).$$

One of the advantages of using this optimization criterion is that we can use the available computational methods to compute the distributions of S_I and S_R, both of which are compound distributions. In particular, when the claim sizes are integer-valued and the distribution of the claim number N belongs to the $(a, b, 0)$ class, noting that the parameter a here is different from the retention level a, Panjer recursion formula is used to calculate the distributions of S_I and S_R for all possible retention levels of a and M; the optimal a and M is then selected from the calculations. This procedure is illustrated in the following subsections by considering the excess-of-loss reinsurance, namely the retention level $a = 1$.

In doing so, we assume that the claim size distribution F is defined on $0, 1, 2, ..., m$ representing a multiple of a convenient monetary unit, namely, X, which has the following probability function

$$f(k) = \Pr\{X = k\}, \quad k = 0, 1, 2, ..., m$$

with $\sum_{k=0}^{m} f(k) = 1$.

We are interested in the retentions levels M satisfying $0 < M < m$, meaning an excess-of-loss reinsurance is employed and the insurer can not cede all the loss to the reinsurer. Note that $X \wedge M$ is defined on $0, 1, 2, ..., M$ and its probability function is

$$p(k) = \Pr\{X \wedge M = k\} = \Pr\{X = k\} = f(k), \quad k = 0, 1, ..., M - 1,$$

and

$$p(M) = \Pr\{X \wedge M = M\} = \Pr\{X \geq M\} = f(M) + \cdots + f(m).$$

Furthermore, $X - X \wedge M = (X - M)_+$ is defined on $0, 1, ..., m - M$ and its probability function is

$$q(0) = \Pr\{(X - M)_+ = 0\} = \Pr\{X \leq M\} = f(0) + \cdots + f(M),$$

and

$$q(k) = \Pr\{(X - M)_+ = k\} = \Pr\{X = M + k\} = f(M + k), \quad k = 1, ..., m - M.$$

Thus, when the distribution of N belongs to the $(a, b, 0)$ class, we can apply the Panjer recursion formula to the probability functions of

$$\Pr\Big\{\sum_{i=1}^{N} \min(X_i, M) \leq P_I\Big\},$$

and

$$\Pr\Big\{\sum_{j=1}^{N} (X_j - \min(X_j, M)) \leq P_R\Big\} = \Pr\Big\{\sum_{j=1}^{N} (X_j - M)_+ \leq P_R\Big\},$$

respectively, for all $0 < M < m$ and then find the optimal M which maximizes the lower bound

$$L(1, M) = \Pr\left\{\sum_{i=1}^{N} \min(X_i, M) \le P_I\right\} \Pr\left\{\sum_{j=1}^{N} (X_j - M)_+ \le P_R\right\}.$$

As illustrations, we consider compound Poisson, compound binomial, and compound negative binomial cases, respectively. Let us review the Panjer recursion formula first.

Lemma 3.3.1 Consider a random sum of

$$S = X_1 + X_2 + \cdots + X_N,$$

where $X_1, X_2, ...$ are i.i.d. integer-valued nonnegative random variables with the same probability function as X, and N is a counting random variable independent of $\{X_1, X_2, ...\}$. Denote the probability functions of S, N, and X by

$$\begin{aligned} g(k) &= \Pr\{S = k\}, \quad k = 0, 1, 2, \cdots, \\ h(k) &= \Pr\{N = k\}, \quad k = 0, 1, 2, \cdots, \\ f(k) &= \Pr\{X = k\}, \quad k = 0, 1, 2, \cdots. \end{aligned}$$

If the distribution of N is in the $(a, b, 0)$ class, namely

$$\frac{h(k)}{h(k-1)} = a + b\frac{1}{k},$$

holds for $k = 1, 2, ...$, then the Panjer recursion formula is expressed as

$$g(k) = \frac{1}{1 - af(0)} \sum_{j=1}^{k} \left(a + \frac{bj}{k}\right) f(j) g(k-j), \quad k = 1, 2, 3, \cdots, \tag{3.1}$$

with

$$g(0) = \sum_{n=0}^{\infty} f^n(0) \Pr(N = n) = P(f(0)), \tag{3.2}$$

where $P(z)$ is the probability generating function of N. ■

Thus, using Lemma 3.3.1, we can apply the Panjer recursive formula to calculate $\Pr\{S_I \leq P_I\}$ and $\Pr\{S_R \leq P_R\}$, respectively, and hence to obtain the value $L(1, M) = \Pr\{S_I \leq P_I\} \Pr\{S_R \leq P_R\}$ for all possible M. Finally, the optimal M can be determined from these values.

Let us review several statistical properties of compound Poisson (CP), compound binomial (CB), and compound negative binomial (CNB) distributions. The probability function of a Poisson random variable is

$$h(k) = \frac{e^{-\lambda}\lambda^k}{k!},$$

where $\lambda > 0$ and $k = 0, 1, 2, ...$

The probability function of a binomial random variable is

$$h(k) = \binom{n}{k}\eta^k(1-\eta)^{n-k},$$

where $0 < \eta < 1$ and $k = 0, 1, ..., n$.

The probability function of a negative binomial random variable is

$$h(k) = \frac{\Gamma(r+k)}{k!\Gamma(r)}\eta^r(1-\eta)^k,$$

where $k = 0, 1, 2, \cdots$; $r = 1, 2, 3, \cdots$; and $0 < \eta < 1$.

Table 3.1 further summarizes other statistical properties for the three compound models.

Table 3.1: Summary of Compound Poisson (CP), Compound Binomial (CB) and Compound Negative Binomial (CNB) Models

	CP	CB	CNB
$h(k)$	$\frac{e^{-\lambda}\lambda^k}{k!}$	$\binom{n}{k}\eta^k(1-\eta)^{n-k}$	$\frac{\Gamma(r+k)}{k!\Gamma(r)}\eta^r(1-\eta)^k$
$\frac{h(k)}{h(k-1)}$	$0+\lambda\frac{1}{k}$	$-\frac{\eta}{1-\eta}+\frac{(n+1)\eta}{1-\eta}\frac{1}{k}$	$(1-\eta)+(r-1)(1-\eta)\frac{1}{k}$
a	0	$-\frac{\eta}{1-\eta}$	$(1-\eta)$
b	λ	$\frac{(n+1)\eta}{1-\eta}$	$b=(r-1)(1-\eta)$
$P(z)$	$e^{\lambda(z-1)}$	$(1-\eta+\eta z)^n$	$\left(\frac{\eta}{1-(1-\eta)z}\right)^r$
$E(N)$	λ	$n\eta$	$\frac{r(1-\eta)}{\eta}$
$var(N)$	λ	$n\eta(1-\eta)$	$\frac{r(1-\eta)}{\eta^2}$

3.3.1 Bounded Claims

The first example considers a discrete uniform distribution claim. Let the probability function of X satisfy,

$$f(k)=\frac{1}{m+1},\quad k=0,1,...,m.$$

Thus, the probability functions of $X\wedge M$ and $(X-M)_+$ can be expressed as

$$p(k) = \frac{1}{m+1},\quad k=0,1,\cdots,M-1,$$

with

$$p(M) = \frac{m+1-M}{m+1};$$

and

$$q(k) = \frac{1}{m+1},\quad k=1,\cdots,m-M,$$

with

$$q(0) = \frac{M+1}{m+1}.$$

In conclusion,

$$E\left[X\right] = \frac{m}{2},$$

and

$$E\left[X - (X \wedge M)\right] = \frac{(m+1-M)(m-M)}{2(m+1)}.$$

Further, the premiums for the cedent and reinsurer are

$$P_R = E[N](1+\theta_R)E\left[X - (X \wedge M)\right],$$

and

$$P_I = E[N](1+\theta_I)E\left[X\right] - P_R.$$

The lower bound for the joint survival probability for the excess-of-loss reinsurance is

$$L(1, M) = \Pr\{S_I \le P_I\}\Pr\{S_R \le P_R\} = \sum_{k=0}^{int(P_I)} g_I(k) \sum_{k=0}^{int(P_R)} g_R(k),$$

where $int()$ rounds a number down to the nearest integer.

With respect to the numerical examples, select the same security loadings of $\theta_I = 0.1$ for the ceding company and $\theta_R = 0.2$ for the reinsurer as in the previous examples. Moreover, let the claim frequency be 100, i.e., $\lambda = 100$ for the compound Poisson model; $n = 200, \eta = 0.5$ for the compound binomial model; and $r = 100, \eta = 0.5$ for the compound negative binomial model.

Table 3.2 provides the maximum of the lower bound of joint survival probability with respect to different values of m. From the table, we can see that the compound negative binomial model has the largest optimal retention level, the compound binomial model has the smallest retention level, and the retention level for the compound Poisson model is between the compound negative binomial model and the compound binomial model. This suggests that for heavier tails, to maximize the lower bound

Table 3.2: Optimal Excess-of-loss Retentions with Discrete Uniform Distribution Claims

	CB	CP	CNB
$m = 99$	$E[X] = 49.5$		
$\max L$	72.967%	66.413%	59.636%
Optimal M	59	60	64
$m = 149$	$E[X] = 74.5$		
$\max L$	72.982%	66.414%	59.641%
Optimal M	89	90	94
$m = 199$	$E[X] = 99.5$		
$\max L$	72.990%	66.415%	59.646%
Optimal M	115	120	125

of the joint survival probability, the optimal retention levels should be higher. The ceding company assumes more responsibility and the reinsurer accepts less responsibility. The claim volatility increases as the standard deviation increases, and there is a higher probability of incurring the high amount claim affecting the reinsurance company. To reduce the liability of the reinsurer and to ensure both companies survive, the optimal treaty should shift the claim to the ceding company to lower the large claims that must be absorbed by the reinsurer. The same reason is found is the examples provided in Chapter 2. Alternatively, compound negative binomial distribution yields the severest claim, because it has the largest volatility. For this reason it has the smallest lower bound for the joint survival probability.

3.3.2 Unbounded Claims

Geometric Distribution

Unbounded claims are examined in this section. The first example we consider is geometric distribution with parameter g. The probability function of claim size X satisfies

$$f(k) = g(1-g)^k, \quad k = 0, 1, 2, \cdots .$$

Thus, the probability functions of $X \wedge M$ and $(X - M)_+$ is expressed as

$$p(k) \;=\; g(1-g)^k, \quad k = 0, 1, \cdots, M-1,$$

with

$$p(M) \;=\; (1-g)^M;$$

and

$$q(k) \;=\; g(1-g)^{M+k}, \quad k = 1, 2, 3, \cdots$$

with

$$q(0) \;=\; 1 - (1-g)^{M+1}.$$

We can conclude that

$$E[X] = \frac{1-g}{g},$$

and

$$E[X - (X \wedge M)] = \frac{(1-g)^{M+1}}{g}.$$

Note that the claim sizes are unbounded in this case. The retention limit M can be any level, namely $M = 1, 2, \ldots$.. When we search for the optimal limit, we have to calculate the lower bound of the joint survival probability for all possible limits of M. However, it is not feasible to do such computations with an infinite number of limits of M. Furthermore, if the retention limit M goes to infinity, the premium received by the reinsurer is going to be zero, which is not interesting to the reinsurer. Hence, for the unbounded claims, we are interested in all the possible limits of M so that the reinsurance premium $P_R(M)$ is at least greater than a certain level. Thus, the possible limits of M are finite and the computations are feasible.

To do so, we assume that the reinsurance premium satisfies $P_R(M) \geq \alpha P_R(0)$ for some $0 < \alpha < 1$, which is equivalent to assume $E[(X - M)_+] \geq \alpha E[X]$. This assumption means that the reinsurance premium should be greater than a certain percentage of the expected total claims. In the following examples and computations, we set $\alpha = 10\%$. In a geometric claim case, it implies $M \leq \ln 0.1 / \ln(1 - g)$. Thus, using the same method for the discrete uniform distribution, we obtain the optimized treaties under different scenarios in table 3.3. If the same constraint is applied on the retention limit to the discrete uniform distribution, the information in table 3.2 is still valid.

Discrete Pareto Distribution

Krishna and Pundira (2008) introduced a discrete Pareto distribution and assumed that the probability function of the discrete Pareto random variable X is defined as

$$\Pr(X = x) = \left(\frac{\beta}{x + \beta}\right)^k - \left(\frac{\beta}{x + 1 + \beta}\right)^k, \quad x = 0, 1, 2, \cdots .$$

Table 3.3: Optimal Excess-of-loss Retentions with Geometric Distribution Claims

	CB	CP	CNB
$g = 2/101$	$E[X] = 49.5$		
$\max L$	58.631%	55.180%	51.494%
Optimal M	72	74	76
$g = 2/151$	$E[X] = 74.5$		
$\max L$	58.697%	55.225%	51.518%
Optimal M	108	112	114
$g = 2/201$	$E[X] = 99.5$		
$\max L$	58.716%	55.235%	51.522%
Optimal M	144	150	155

Thus, the probability functions of $X \wedge M$ and $(X - M)_+$ are expressed as

$$p(x) = \left(\frac{\beta}{x+\beta}\right)^k - \left(\frac{\beta}{x+1+\beta}\right)^k, \quad x = 0, 1, \cdots, M-1,$$

with

$$p(M) = \left(\frac{\beta}{M+\beta}\right)^k;$$

and

$$q(x) = \left(\frac{\beta}{M+x+\beta}\right)^k - \left(\frac{\beta}{M+1+x+\beta}\right)^k, \quad x = 1, 2, 3, \cdots$$

with

$$q(0) = 1 - \left(\frac{\beta}{M+1+\beta}\right)^k.$$

We can conclude that

$$\begin{aligned}
E(X) &= \left(\frac{\beta}{1+\beta}\right)^k - \left(\frac{\beta}{1+1+\beta}\right)^k + 2\left(\frac{\beta}{2+\beta}\right)^k - 2\left(\frac{\beta}{2+1+\beta}\right)^k + 3\left(\frac{\beta}{3+\beta}\right)^k - \cdots \\
&= \left(\frac{\beta}{1+\beta}\right)^k + \left(\frac{\beta}{2+\beta}\right)^k + \left(\frac{\beta}{3+\beta}\right)^k + \cdot \\
&= \sum_{n=1}^{\infty} \left(\frac{\beta}{n+\beta}\right)^k \\
&= (-1)^k \frac{1}{(k-1)!} \beta^k \mathrm{Psi}\left(k-1, 1+\beta\right),
\end{aligned}$$

and

$$E\left[X-(X \wedge M)\right] = (-1)^k \frac{1}{(k-1)!} \beta^k \mathrm{Psi}\left(k-1, M+1+\beta\right).$$

Here $\mathrm{Psi}(n,x)$ is nth derivative of Psi function $\psi\left(x\right) = \frac{d}{dx}\ln\Gamma\left(x\right)$, and gamma function is $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt$.

Using the pervious method, and let $k=5$, we list the optimized treaties under different scenarios in table 3.4.

Table 3.4: Optimal Excess-of-loss Retentions with Discrete Pareto Distribution Claims

	CB	CP	CNB
$\beta = 200$	$E[X] = 49.5028$		
$\max L$	54.413%	54.266%	53.478%
Optimal M	94	136	226
$\beta = 300$	$E[X] = 74.5014$		
$\max L$	54.266%	51.761%	48.967%
Optimal M	136	157	161
$\beta = 400$	$E[X] = 99.5010$		
$\max L$	53.478%	51.232%	48.723%
Optimal M	226	305	307

Table 3.4 indicates that the ceding company must assume more responsibility for heavy-tailed claims and that the lower bound of the joint survival probability decreases as the claim severity increases.

3.4 Bivariate Translated Gamma Distributions

From this section, we will develop a translated bivariate gamma approximation to the joint distribution of (S_I, S_R).

Several forms of bivariate gamma distributions and their properties have received extensive study. See Nadarajah et al. (2006(a), 2006(b)), Chou et al. (2005), Zhou et al. (2005), and references therein.

This thesis uses the form promoted by Mathai and Moschopoulos (1991).

Definition: Let $V_i \sim G(\alpha_i, \beta_i)$, $i = 0, 1, 2$, $\alpha_i > 0$, $\beta_i > 0$, where V_i's are mutually independent, with density function being

$$g(x; \alpha_i, \beta_i) = \frac{x^{\alpha_i - 1} e^{-x/\beta_i}}{\beta_i^{\alpha_i} \Gamma(\alpha_i)}.$$

Let $Z_i = \frac{\beta_i}{\beta_0} V_0 + V_i$, $i = 1, 2$. The density of (Z_1, Z_2) is a bivariate gamma density.

Note here, if Z_1 represents the ceding company and Z_2 represents the reinsurer, V_0 can be thought of as the common interest part between the partners.

Properties: The joint moment generating function of (Z_1, Z_2) is

$$M(t_1, t_2) = (1 - \beta_1 t_1 - \beta_2 t_2)^{-\alpha_0} (1 - \beta_1 t_1)^{-\alpha_1} (1 - \beta_2 t_2)^{-\alpha_2}.$$

The following properties can be obtained directly from the moment generating function:

$$
\begin{aligned}
E(Z_1) &= \frac{\partial M(t_1,t_2)}{\partial t_1}|_{t_1=t_2=0} = \beta_1\left(\alpha_0+\alpha_1\right);\\
E(Z_1^2) &= \frac{\partial^2 M(t_1,t_2)}{\partial t_1^2}|_{t_1=t_2=0} = \beta_1^2\left(\alpha_0+\alpha_1\right)\left(\alpha_0+\alpha_1+1\right);\\
E(Z_1Z_2) &= \frac{\partial^2 M(t_1,t_2)}{\partial t_2\partial t_1}|_{t_1=t_2=0} = \beta_1\beta_2\left(\alpha_0+\alpha_0^2+\alpha_0\alpha_2+\alpha_0\alpha_1+\alpha_1\alpha_2\right);\\
E(Z_1^3) &= \frac{\partial^3 M(t_1,t_2)}{\partial^3 t_1}|_{t_1=t_2=0} = \beta_1^3\left(\alpha_0+\alpha_1\right)\left(\alpha_0+\alpha_1+1\right)\left(\alpha_0+\alpha_1+2\right).
\end{aligned}
$$

Hence for $i=1,2$, we have the following equations

$$
\begin{aligned}
E(Z_i) &= \beta_i\left(\alpha_0+\alpha_i\right);\\
Var(Z_i) &= E(Z_i^2)-E^2(Z_i) = \beta_i^2\left(\alpha_0+\alpha_i\right);\\
Cov(Z_1,Z_2) &= E(Z_1,Z_2)-E(Z_1)E(Z_2) = \alpha_0\beta_1\beta_2;\\
\mu_3(Z_i) &= E(Z_i^3)-3E(Z_i^2)E(Z_i)+2E^3(Z_i) = 2\beta_i^3\left(\alpha_0+\alpha_i\right).
\end{aligned}
$$

Here, $\mu_3(Z_i)$ is the third central moment of Z_i. Since $V_i > 0$ and $V_i = Z_i - \frac{\beta_i}{\beta_0}V_0$, we have $V_0 < \frac{\beta_0}{\beta_i}Z_i$.

Density: From the definitions and above discussions, the joint density of V_0, Z_1, and Z_2 can be expressed as

$$
\begin{aligned}
g(v_0,z_1,z_2) &= \frac{(v_0)^{\alpha_0-1}\exp(-v_0/\beta_0)}{\beta_0^{\alpha_0}\Gamma(\alpha_0)}\\
&\quad\times\frac{\left(z_1-\frac{\beta_1}{\beta_0}v_0\right)^{\alpha_1-1}\exp(-\left(z_1-\frac{\beta_1}{\beta_0}v_0\right)/\beta_1)}{\beta_1^{\alpha_1}\Gamma(\alpha_1)}\\
&\quad\times\frac{\left(z_2-\frac{\beta_2}{\beta_0}v_0\right)^{\alpha_2-1}\exp(-\left(z_2-\frac{\beta_2}{\beta_0}v_0\right)/\beta_2)}{\beta_2^{\alpha_2}\Gamma(\alpha_2)}.
\end{aligned}
$$

Hence, the joint density of Z_1 and Z_2 is

$$
\begin{aligned}
& g(z_1, z_2) \\
= & \int_{v_0} g(v_0, z_1, z_2) dv_0 \\
= & \int_{v_0} \frac{(v_0)^{\alpha_0 - 1} \exp(-v_0/\beta_0)}{\beta_0^{\alpha_0}\Gamma(\alpha_0)} \frac{\left(z_1 - \frac{\beta_1}{\beta_0} v_0\right)^{\alpha_1 - 1} \exp(-\left(z_1 - \frac{\beta_1}{\beta_0} v_0\right)/\beta_1)}{\beta_1^{\alpha_1}\Gamma(\alpha_1)} \\
& \quad \times \frac{\left(z_2 - \frac{\beta_2}{\beta_0} v_0\right)^{\alpha_2 - 1} \exp(-\left(z_2 - \frac{\beta_2}{\beta_0} v_0\right)/\beta_2)}{\beta_2^{\alpha_2}\Gamma(\alpha_2)} dv_0 \\
= & \frac{e^{-\frac{z_1}{\beta_1}} e^{-\frac{z_2}{\beta_2}}}{\Gamma(\alpha_0)\beta_1^{\alpha_1}\Gamma(\alpha_1)\beta_2^{\alpha_2}\Gamma(\alpha_2)} \\
& \quad \times \int_{v_0 = 0}^{\min(\frac{\beta_0}{\beta_1} z_1, \frac{\beta_0}{\beta_2} z_2)} \left(\frac{v_0}{\beta_0}\right)^{\alpha_0 - 1} \left(z_1 - \beta_1 \frac{v_0}{\beta_0}\right)^{\alpha_1 - 1} \left(z_2 - \beta_2 \frac{v_0}{\beta_0}\right)^{\alpha_2 - 1} \exp(\frac{v_0}{\beta_0}) d\frac{v_0}{\beta_0} \\
& \qquad \left(\text{let } v = \frac{v_0}{\beta_0}\right) \\
= & \frac{e^{-\frac{z_1}{\beta_1}} e^{-\frac{z_2}{\beta_2}}}{\Gamma(\alpha_0)\beta_1^{\alpha_1}\Gamma(\alpha_1)\beta_2^{\alpha_2}\Gamma(\alpha_2)} \int_{v=0}^{\min(\frac{z_1}{\beta_1}, \frac{z_2}{\beta_2})} (v)^{\alpha_0 - 1} (z_1 - \beta_1 v)^{\alpha_1 - 1} (z_2 - \beta_2 v)^{\alpha_2 - 1} e^v dv.
\end{aligned}
$$

Lemma 3.4.1 ***(Joint Distribution Function)*** : The cumulative distribution function of the bivariate gamma distribution is

$$
G(y_1, y_2) = \begin{cases} G_1(y_1, y_2) & \text{when } \frac{y_2}{\beta_2} < \frac{y_1}{\beta_1} \\ & \\ G_2(y_1, y_2) & \text{when } \frac{y_1}{\beta_1} < \frac{y_2}{\beta_2} \end{cases},
$$

where

$$
\begin{aligned}
& G_1(y_1, y_2) \\
= \; & \int_0^{\frac{y_2}{\beta_2}} \int_{x_2}^{\frac{y_1}{\beta_1}} \frac{e^{-x_1} e^{-x_2} \, (x_1)^{\alpha_1 - 1} \, \, (x_2)^{\alpha_2 - 1}}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^{x_2} (v)^{\alpha_0 - 1} \left(1 - \frac{v}{x_1}\right)^{\alpha_1 - 1} \\
& \times \left(1 - \frac{v}{x_2}\right)^{\alpha_2 - 1} e^v dv dx_1 dx_2 \\
& + \int_0^{\frac{y_2}{\beta_2}} \int_0^{x_2} \frac{e^{-x_1} e^{-x_2} \, (x_1)^{\alpha_1 - 1} \, \, (x_2)^{\alpha_2 - 1}}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^{=x_1} (v)^{\alpha_0 - 1} \left(1 - \frac{v}{x_1}\right)^{\alpha_1 - 1} \\
& \times \left(1 - \frac{v}{x_2}\right)^{\alpha_2 - 1} e^v dv dx_1 dx_2 \qquad (3.3)
\end{aligned}
$$

and

$$
\begin{aligned}
& G_2(y_1, y_2) \\
= \; & \int_{\frac{y_1}{\beta_1}}^{\frac{y_2}{\beta_2}} \int_0^{\frac{y_1}{\beta_1}} \frac{e^{-x_1} e^{-x_2} \, (x_1)^{\alpha_1 - 1} \, \, (x_2)^{\alpha_2 - 1}}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^{x_1} (v)^{\alpha_0 - 1} \left(1 - \frac{v}{x_1}\right)^{\alpha_1 - 1} \\
& \times \left(1 - \frac{v}{x_2}\right)^{\alpha_2 - 1} e^v dv dx_1 dx_2 \\
& + \int_0^{\frac{y_1}{\beta_1}} \int_0^{x_2} \frac{e^{-x_1} e^{-x_2} \, (x_1)^{\alpha_1 - 1} \, \, (x_2)^{\alpha_2 - 1}}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^{x_1} (v)^{\alpha_0 - 1} \left(1 - \frac{v}{x_1}\right)^{\alpha_1 - 1} \\
& \times \left(1 - \frac{v}{x_2}\right)^{\alpha_2 - 1} e^v dv dx_1 dx_2 \\
& + \int_0^{\frac{y_1}{\beta_1}} \int_{x_2}^{\frac{y_1}{\beta_1}} \frac{e^{-x_1} e^{-x_2} \, (x_1)^{\alpha_1 - 1} \, \, (x_2)^{\alpha_2 - 1}}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^{x_2} (v)^{\alpha_0 - 1} \\
& \times \left(1 - \frac{v}{x_1}\right)^{\alpha_1 - 1} \left(1 - \frac{v}{x_2}\right)^{\alpha_2 - 1} e^v dv dx_1 dx_2. \qquad (3.4)
\end{aligned}
$$

Here, parameters $\alpha_i > 0$ and $\beta_i > 0$, for $i = 0, 1, 2$.

Proof: The cumulative distribution function is

$$
\begin{aligned}
& G(y_1, y_2) \\
= & \int_0^{y_2}\int_0^{y_1} g(z_1, z_2) dz_1 dz_2 \\
= & \int_0^{y_2}\int_0^{y_1} \frac{e^{-\frac{z_1}{\beta_1}} e^{-\frac{z_2}{\beta_2}} z_1^{\alpha_1 - 1} z_2^{\alpha_2 - 1}}{\Gamma(\alpha_0)\beta_1^{\alpha_1}\Gamma(\alpha_1)\beta_2^{\alpha_2}\Gamma(\alpha_2)} \int_{v=0}^{\min(\frac{z_1}{\beta_1},\frac{z_2}{\beta_2})} (v)^{\alpha_0 - 1}\left(1 - \frac{\beta_1}{z_1}v\right)^{\alpha_1 - 1} \\
& \times \left(1 - \frac{\beta_2}{z_2}v\right)^{\alpha_2 - 1} e^v dv dz_1 dz_2 \\
& \qquad \text{let } \frac{z_1}{\beta_1} = x_1 \text{ and } \frac{z_2}{\beta_2} = x_2 \\
= & \int_0^{\frac{y_2}{\beta_2}}\int_0^{\frac{y_1}{\beta_1}} \frac{e^{-x_1}e^{-x_2}\left(\beta_1 x_1\right)^{\alpha_1 - 1}\ \left(\beta_2 x_2\right)^{\alpha_2 - 1}}{\Gamma(\alpha_0)\beta_1^{\alpha_1}\Gamma(\alpha_1)\beta_2^{\alpha_2}\Gamma(\alpha_2)} \\
& \qquad \times \int_{v=0}^{\min(x_1, x_2)} (v)^{\alpha_0 - 1}\left(1 - \frac{v}{x_1}\right)^{\alpha_1 - 1}\left(1 - \frac{v}{x_2}\right)^{\alpha_2 - 1} e^v dv d\beta_1 x_1 d\beta_2 x_2 \\
= & \int_0^{\frac{y_2}{\beta_2}}\int_0^{\frac{y_1}{\beta_1}} \frac{e^{-x_1}e^{-x_2}\left(x_1\right)^{\alpha_1 - 1}\ \left(x_2\right)^{\alpha_2 - 1}}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_{v=0}^{\min(x_1, x_2)} (v)^{\alpha_0 - 1} \\
& \times \left(1 - \frac{v}{x_1}\right)^{\alpha_1 - 1}\left(1 - \frac{v}{x_2}\right)^{\alpha_2 - 1} e^v dv dx_1 dx_2.
\end{aligned}
$$

Note for any $f(x) > 0$, if $x_1 < x_2$, we have $\int_0^{x_1} f(v)dv < \int_0^{x_2} f(v)dv$.

Let $\Xi(\alpha_0, y) = \int_0^{y} (v)^{\alpha_0 - 1}\left(1 - \frac{v}{x_1}\right)^{\alpha_1 - 1}\left(1 - \frac{v}{x_2}\right)^{\alpha_2 - 1} e^v dv$.

1. When $\frac{y_2}{\beta_2} < \frac{y_1}{\beta_1}$, where $0 < x_1 < \frac{y_1}{\beta_1}$ and $0 < x_2 < \frac{y_2}{\beta_2}$, there are three cases:

1.1. $x_2 < \frac{y_2}{\beta_2} < x_1 < \frac{y_1}{\beta_1}$, here $\Xi(\alpha_0, x_2) < \Xi(\alpha_0, x_1)$;

1.2. $x_2 < x_1 < \frac{y_2}{\beta_2} < \frac{y_1}{\beta_1}$, here $\Xi(\alpha_0, x_2) < \Xi(\alpha_0, x_1)$;

1.3. $x_1 < x_2 < \frac{y_2}{\beta_2} < \frac{y_1}{\beta_1}$, here $\Xi(\alpha_0, x_1) < \Xi(\alpha_0, x_2)$.

Hence, the joint cumulative distribution function is

$$
\begin{aligned}
& G(y_1, y_2) \\
= \quad & \int_0^{\frac{y_2}{\beta_2}} \int_{x_2}^{\frac{y_1}{\beta_1}} \frac{e^{-x_1} e^{-x_2} (x_1)^{\alpha_1 - 1} \ (x_2)^{\alpha_2 - 1}}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^{x_2} (v)^{\alpha_0 - 1} \left(1 - \frac{v}{x_1}\right)^{\alpha_1 - 1} \\
& \times \left(1 - \frac{v}{x_2}\right)^{\alpha_2 - 1} e^v dv dx_1 dx_2 \\
& + \int_0^{\frac{y_2}{\beta_2}} \int_0^{x_2} \frac{e^{-x_1} e^{-x_2} (x_1)^{\alpha_1 - 1} \ (x_2)^{\alpha_2 - 1}}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^{x_1} (v)^{\alpha_0 - 1} \left(1 - \frac{v}{x_1}\right)^{\alpha_1 - 1} \\
& \times \left(1 - \frac{v}{x_2}\right)^{\alpha_2 - 1} e^v dv dx_1 dx_2.
\end{aligned}
$$

2. When $\frac{y_1}{\beta_1} < \frac{y_2}{\beta_2}$, where $0 < x_1 < \frac{y_1}{\beta_1}$ and $0 < x_2 < \frac{y_2}{\beta_2}$, there are three cases:

2.1. $x_1 < \frac{y_1}{\beta_1} < x_2 < \frac{y_2}{\beta_2}$, here $\Xi(\alpha_0, x_1) < \Xi(\alpha_0, x_2)$;

2.2. $x_1 < x_2 < \frac{y_1}{\beta_1} < \frac{y_2}{\beta_2}$, here $\Xi(\alpha_0, x_1) < \Xi(\alpha_0, x_2)$;

2.3. $x_2 < x_1 < \frac{y_1}{\beta_1} < \frac{y_2}{\beta_2}$, here $\Xi(\alpha_0, x_2) < \Xi(\alpha_0, x_1)$.

Hence, the joint cumulative distribution function is

$$
\begin{aligned}
& G(y_1, y_2) \\
= \; & \int_{\frac{y_1}{\beta_1}}^{\frac{y_2}{\beta_2}} \int_0^{\frac{y_1}{\beta_1}} \frac{e^{-x_1} e^{-x_2} \left(x_1\right)^{\alpha_1 - 1} \;\left(x_2\right)^{\alpha_2 - 1}}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^{x_1} (v)^{\alpha_0 - 1} \\
& \times \left(1 - \frac{v}{x_1}\right)^{\alpha_1 - 1} \left(1 - \frac{v}{x_2}\right)^{\alpha_2 - 1} e^v dv dx_1 dx_2 \\
& + \int_0^{\frac{y_1}{\beta_1}} \int_0^{x_2} \frac{e^{-x_1} e^{-x_2} \left(x_1\right)^{\alpha_1 - 1} \;\left(x_2\right)^{\alpha_2 - 1}}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^{x_1} (v)^{\alpha_0 - 1} \left(1 - \frac{v}{x_1}\right)^{\alpha_1 - 1} \\
& \times \left(1 - \frac{v}{x_2}\right)^{\alpha_2 - 1} e^v dv dx_1 dx_2 \\
& + \int_0^{\frac{y_1}{\beta_1}} \int_{x_2}^{\frac{y_1}{\beta_1}} \frac{e^{-x_1} e^{-x_2} \left(x_1\right)^{\alpha_1 - 1} \;\left(x_2\right)^{\alpha_2 - 1}}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^{x_2} (v)^{\alpha_0 - 1} \\
& \times \left(1 - \frac{v}{x_1}\right)^{\alpha_1 - 1} \left(1 - \frac{v}{x_2}\right)^{\alpha_2 - 1} e^v dv dx_1 dx_2.
\end{aligned}
$$

■

3.5 Approximation to Aggregate Claims for Insurers and Reinsurers

Let us consider an aggregate claim model where $S = \sum_{i=1}^{N} X_i$ and where X_i's are i.i.d. random variables, and independent of random variable N.

The law of total variance provides

$$
\begin{aligned}
E(S) &= E(N)E(X); \\
Var(S) &= E(N)Var(X) + Var(N)E^2(X); \\
\mu_3(S) &= E(\mu_3(S|N)) + \mu_3(E(S|N)) + 3Cov(E(S|N), var(S|N)) \\
&= E(N)\mu_3(X_i) + \mu_3(NE(X_i)) + 3Cov(NE(X_i), NVar(X_i)) \\
&= E(N)\mu_3(X) + E^3(X)\mu_3(N) + 3E(X)Var(X)Var(N).
\end{aligned}
$$

Similarly, use the conditional expectation,

$$
\begin{aligned}
E(S_I S_R) &= E(\sum_{i=1}^{N} X_{I_i} \sum_{j=1}^{N} X_{R_j}) \\
&= E(E(\sum_{i=1}^{N} X_{I_i} \sum_{j=1}^{N} X_{R_j} | N)) \\
&= E(NE(X_{I_i} X_{R_i}) + \left(N^2 - N\right) E(X_{I_i} X_{R_j})) \\
&= E(X_I X_R)E(N) + E\left(N^2 - N\right) E(X_I)E(X_R).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
Cov(S_I, S_R) &= E(S_I S_R) - E(S_I)E(S_R) \\
&= E(X_I X_R)E(N) + E(N^2)E(X_I)E(X_R) \\
&\quad -E(N)E(X_I)E(X_R) - E(N)E(X_I)E(N)E(X_R) \\
&= E(X_I X_R)E(N) - E(N)E(X_I)E(X_R) \\
&\quad +E(N^2)E(X_I)E(X_R) - E(N)E(X_I)E(N)E(X_R) \\
&= E(N)Cov(X_I, X_R) + Var(N)E(X_I)E(X_R).
\end{aligned}
$$

The translated Gamma approximation for the joint survival probabilities of both parties under the aggregate claim is the next procedural stage. In order to obtain the approximation distribution parameters, we need to match the moments of the original claim to the corresponding characteristics of the translated bivariate gamma distribution.

Because there are seven parameters in the model, namely, $\alpha_0, \alpha_1, \alpha_2, \beta_1, \beta_2, \omega_1, \omega_2$, the intuitive approach is to use the first three moments and the covariance of S_I and S_R. Theorem 3.5.1 provides the approximation parameters.

Theorem 3.5.1 Under the bivariate translated gamma approximation, the joint survival probability of the cedent and reinsurer can be approximated as

$$\Pr\{S_I \leq P_I, S_R \leq P_R\} \approx G(P_I - \omega_1, P_R - \omega_2). \tag{3.5}$$

Here $G(y_1, y_2)$ is the bivariate gamma CDF from Lemma (3.4.1), and the corresponding bivariate gamma parameters are

$$\begin{aligned}
\beta_1 &= \frac{\mu_3(S_I)}{2Var(S_I)}; \\
\beta_2 &= \frac{\mu_3(S_R)}{2Var(S_R)}; \\
\alpha_0 &= \frac{4Var(S_I)Var(S_R)cov(S_I, S_R)}{\mu_3(S_I)\mu_3(S_R)}; \\
\alpha_1 &= \frac{4Var^3(S_I)}{\mu_3^2(S_I)} - \frac{4Var(S_I)Var(S_R)cov(S_I, S_R)}{\mu_3(S_I)\mu_3(S_R)}; \\
\alpha_2 &= \frac{4Var^3(S_R)}{\mu_3^2(S_R)} - \frac{4Var(S_I)Var(S_R)cov(S_I, S_R)}{\mu_3(S_I)\mu_3(S_R)}; \\
\omega_1 &= E(S_I) - \frac{2Var^2(S_I)}{\mu_3(S_I)}; \\
\omega_2 &= E(S_R) - \frac{2Var^2(S_R)}{\mu_3(S_R)}.
\end{aligned} \tag{3.6}$$

Proof: Using a similar idea from Bowers et al. (1997), it is possible to approximate the joint distribution of aggregate claims (S_I, S_R) using a joint translated gamma distribution that matches the moments.

Let $S_I = Z_1 + \omega_1$ and $S_R = Z_2 + \omega_2$, where $\omega_1 > 0$ and $\omega_2 > 0$.

The joint survival probability of both companies is

$$\begin{aligned} & \Pr\{S_I \leq P_I, S_R \leq P_R\} \\ = & \Pr\{Z_1 \leq P_I - \omega_1, Z_2 \leq P_R - \omega_2\} \\ = & G(P_I - \omega_1, P_R - \omega_2), \end{aligned}$$

if the first three moments, as well as the covariance of the insurance and reinsurance aggregate claim amount can be matched by the translated gamma.

Because central moments of the translated gamma are the same as ones of the gamma distribution, this procedure imposes the following requirements

$$\begin{aligned} Cov(S_I, S_R) &= \alpha_0 \beta_1 \beta_2; \\ E(S_I) &= (\alpha_0 + \alpha_1)\,\beta_1 + \omega_1; \\ E(S_R) &= (\alpha_0 + \alpha_2)\,\beta_2 + \omega_2; \\ Var(S_I) &= (\alpha_0 + \alpha_1)\,\beta_1^2; \\ Var(S_R) &= (\alpha_0 + \alpha_2)\,\beta_2^2; \\ \mu_3(S_I) &= 2\,(\alpha_0 + \alpha_1)\,\beta_1^3; \\ \mu_3(S_R) &= 2\,(\alpha_0 + \alpha_2)\,\beta_2^3. \end{aligned} \tag{3.7}$$

From the above discussion, the bivariate gamma parameter set of (3.6) can be matched and the theorem is proven.

■

Recall that in the bivariate gamma parameter, all α_i's and β_i's should be greater than zero. However, for α_1 and α_2, these criteria may not be satisfied under certain distributions. In this case, we will force $\alpha_1 = \alpha_2 = 1$ and use the covariance with the first two moments to do the five parameter match, which is stated as Theorem 3.5.2.

Theorem 3.5.2 Under the bivariate translated gamma approximation, the joint survival probability of the cedent and reinsurer is expressed as

$$\Pr\{S_I \leq P_I, S_R \leq P_R\} \approx G(P_I - \omega_1, P_R - \omega_2). \tag{3.8}$$

Here $G(y_1, y_2)$ is the bivariate gamma CDF from Lemma 3.4.1, and the corresponding bivariate gamma parameters are

$$\begin{aligned}
\alpha_0 &= \frac{cov(S_I, S_R)}{\sqrt{var(S_R)var(S_I)} - cov(S_I, S_R)}; \\
\beta_1 &= \sqrt{var(S_I)\left(1 - \frac{cov(S_I, S_R)}{\sqrt{var(S_R)var(S_I)}}\right)}; \\
\beta_2 &= \sqrt{var(S_R)\left(1 - \frac{cov(S_I, S_R)}{\sqrt{var(S_R)var(S_I)}}\right)}; \\
\omega_1 &= E(S_I) - \sqrt{var(S_I)\frac{\sqrt{var(S_R)var(S_I)}}{\sqrt{var(S_R)var(S_I)} - cov(S_I, S_R)}}; \\
\omega_2 &= E(S_R) - \sqrt{var(S_R)\frac{\sqrt{var(S_R)var(S_I)}}{\sqrt{var(S_R)var(S_I)} - cov(S_I, S_R)}}.
\end{aligned} \tag{3.9}$$

Proof: Similar to Theorem 3.5.1, only the first two central moments and the covariance are matched, i.e. only the first five equations in the equation array (3.7) are matched. Here, $\alpha_1 = \alpha_2 = 1$. ■

As stated above, the goal is to find the optimal retention level M and quota-share a required to maximize the joint survival probability of the ceding company and the reinsurer. The question can be reformulated as:

$$\max_{a,M} \Pr\{S_I \leq P_I, S_R \leq P_R\} = \max_{a,M} G(P_I - \omega_1, P_R - \omega_2). \tag{3.10}$$

Because there is no explicit form for the integral,

$$\int_{v=0}^{\min(x_1,x_2)} (v)^{\alpha_0-1} \left(1 - \frac{v}{x_1}\right)^{\alpha_1-1} \left(1 - \frac{v}{x_2}\right)^{\alpha_2-1} e^v dv,$$

equation (3.10) must be solved numerically.

3.6 Approximation of Fair Optimal Retention and Numerical Examples

Here, we will consider three compound distributions, compound Poisson, compound binomial and compound negative binomial as in section 3.3. The claim we consider will be exponential claim and Pareto claim. As stated previously, the ceding company cannot profit from reinsurance, i.e., for insurance loads $\theta_I < \theta_R$. Also, the premium after reinsurance P_I is positive.

For an exponential distribution with p.d.f. $f(x) = \frac{1}{\mu}e^{-\frac{x}{\mu}}$, we have the following expressions

$$
\begin{aligned}
E(X) &= \mu; \\
P_R &= (1+\theta_R)(1-a)\mu + (1+\theta_R)a\mu e^{-\frac{M}{a\mu}}; \\
P_I &= (1+\theta_I)\mu - (1+\theta_R)(1-a)\mu - (1+\theta_R)a\mu e^{-\frac{M}{a\mu}}; \\
E(X_I) &= -e^{-\frac{M}{a\mu}}a\mu + a\mu; \\
E(X_I^2) &= -2Me^{-\frac{M}{a\mu}}a\mu - 2e^{-\frac{M}{a\mu}}a^2\mu^2 + 2a^2\mu^2; \\
E(X_I^3) &= -3e^{-\frac{M}{a\mu}}M^2a\mu - 6e^{-\frac{M}{a\mu}}Ma^2\mu^2 - 6e^{-\frac{M}{a\mu}}a^3\mu^3 + 6a^3\mu^3; \\
E(X_R) &= \mu\left(1 + e^{-\frac{M}{a\mu}}a - a\right); \\
E(X_R^2) &= 2\mu e^{-\frac{M}{a\mu}}M(1-a) - 2a\mu^2\,(2-a)\left(1 - e^{-\frac{M}{a\mu}}\right) + 2\mu^2; \\
E(X_R^3) &= 3\mu e^{-\frac{M}{a\mu}}M^2\frac{(1-a)^2}{a} \\
&\quad + 6\mu^3\,(1-a)^3\,(1 - e^{-\frac{M}{a\mu}}) + 6\mu^2 e^{-\frac{M}{a\mu}}\left(\mu + 2M - 3Ma + Ma^2\right); \\
E(X_I\ X_R\ ) &= \mu\left(e^{-\frac{M}{a\mu}}M\,(2a-1) + 2a\mu\,(1-a)\left(1 - e^{-\frac{M}{a\mu}}\right)\right).
\end{aligned}
$$

Consider the compound Poisson model first. When N is Poisson distributed with the expected value λ, we have $E(N) = Var(N) = \mu_3(N) = \lambda$.

It implies

$$\begin{aligned}
Var(S) &= E(N)Var(X) + Var(N)E^2(X) = \lambda E(X^2); \\
\mu_3(S) &= E(N)\mu_3(X) + E^3(X)\mu_3(N) + 3E(X)Var(X)Var(N) \\
&= \lambda\left(\mu_3(X) + E^3(X) + 3E(X)Var(X)\right) \\
&= \lambda\left(\mu_3(X) + E^3(X) + 3E(X)E(X^2) - 3E^3(X)\right) \\
&= \lambda E(X^3); \\
Cov(S_I, S_R) &= E(N)Cov(X_I, X_R) + Var(N)E(X_I)E(X_R) \\
&= \lambda Cov(X_I, X_R) + \lambda E(X_I)E(X_R) \\
&= \lambda E(X_I, X_R).
\end{aligned}$$

In the seven parameter model, equation (3.6) becomes

$$\begin{aligned}
\beta_1 &= \frac{E(X_I^3)}{2E(X_I^2)}; \\
\beta_2 &= \frac{E(X_R^3)}{2E(X_R^2)}; \\
\alpha_0 &= \frac{4\lambda E(X_I, X_R)E(X_R^2)E(X_I^2)}{E(X_R^3)E(X_I^3)}; \\
\alpha_1 &= 4\lambda\frac{E(X_I^2)}{E(X_I^3)}\left(\frac{E(X_I^2)E(X_I^2)}{E(X_I^3)} - \frac{E(X_I, X_R)E(X_R^2)}{E(X_R^3)}\right); \\
\alpha_2 &= \frac{4\lambda E(X_R^2)}{E(X_R^3)}\left(\frac{E(X_R^2)E(X_R^2)}{E(X_R^3)} - \frac{E(X_I, X_R)E(X_I^2)}{E(X_I^3)}\right); \\
\omega_1 &= \lambda E(X_I) - \frac{2\lambda E^2(X_I^2)}{E(X_I^3)}; \\
\omega_2 &= \lambda E(X_R) - \frac{2\lambda E^2(X_R^2)}{E(X_R^3)}.
\end{aligned} \tag{3.11}$$

Recall the assumption in the model is that β_1, β_2, α_0, α_1, and α_2 should all be greater than zero. In the above equation array, β_1, β_2 and α_0 will always be greater than zero. However, for α_1 and α_2 this certainty is not presented, especially for α_2 in the exponential case, as the maximum value of α_2 is zero.

The results indicates that the seven-parameter translated bivariate gamma distribution may not be a good approximation for the compound Poisson distribution when the distributed claim size is exponential. Hence, a five-parameter translated bivariate gamma distribution is used to approximate the distribution of joint survival probability.

We have $\alpha_1 = \alpha_2 = 1$, and the other five parameters are

$$\begin{aligned}
\alpha_0 &= \frac{E(X_I X_R)}{\sqrt{E(X_I^2)E(X_R^2)} - E(X_I X_R)}; \\
\beta_1 &= \sqrt{\lambda E(X_I^2)\left(\frac{\sqrt{E(X_I^2)E(X_R^2)} - E(X_I X_R)}{\sqrt{E(X_I^2)E(X_R^2)}}\right)}; \\
\beta_2 &= \sqrt{\lambda E(X_R^2)\left(\frac{\sqrt{E(X_I^2)E(X_R^2)} - E(X_I X_R)}{\sqrt{E(X_I^2)E(X_R^2)}}\right)}; \\
\omega_1 &= \lambda E(X_I) - \sqrt{\lambda E(X_I^2)\frac{\sqrt{E(X_I^2)E(X_R^2)}}{\sqrt{E(X_I^2)E(X_R^2)} - E(X_I X_R)}}; \\
\omega_2 &= \lambda E(X_R) - \sqrt{\lambda E(X_R^2)\frac{\sqrt{E(X_I^2)E(X_R^2)}}{\sqrt{E(X_I^2)E(X_R^2)} - E(X_I X_R)}}.
\end{aligned} \tag{3.12}$$

For Pareto distribution claim with probability density function

$$f(x) = \frac{k\beta^k}{(x+\beta)^{k+1}},$$

to compare it to the exponential claim on the same page, we need to use the five-parameter bivariate gamma model instead of the seven-parameter bivariate gamma model. Several statistical properties are listed as the follows:

$$
\begin{aligned}
E(X) &= \frac{\beta}{k-1}; \\
P_R &= (1+\theta_R)\frac{\beta}{k-1} - (1+\theta_R)\frac{a\beta}{k-1}\left(1-\left(\frac{a\beta}{M+a\beta}\right)^{k-1}\right); \\
P_I &= (\theta_I-\theta_R)\frac{\beta}{k-1} + (1+\theta_R)\frac{a\beta}{k-1}\left(1-\left(\frac{a\beta}{M+a\beta}\right)^{k-1}\right); \\
E(X_I) &= a\frac{\beta}{k-1} - \frac{M+a\beta}{k-1}\left(\frac{a\beta}{M+a\beta}\right)^{k}; \\
E(X_I^2) &= \frac{2a^2\beta^2}{(k-1)(k-2)} - \frac{2\,(a\beta+Mk-M)}{(k-2)}\frac{(M+a\beta)}{(k-1)}\left(\frac{a\beta}{M+a\beta}\right)^{k}; \\
E(X_R) &= \frac{a\beta+M}{k-1}\left(\frac{a\beta}{M+a\beta}\right)^{k} + \beta\frac{1-a}{k-1}; \\
E(X_R^2) &= \frac{2\,(1-a)^2\,\beta^2}{(k-1)\,(k-2)} \\
&\quad + 2\frac{2a\beta-a^2\beta-Mka+Ma+Mk}{a\,(k-2)}\frac{(M+a\beta)}{(k-1)}\left(\frac{a\beta}{M+a\beta}\right)^{k}; \\
E(X_I\ X_R\ ) &= \frac{2a\,(1-a)\,\beta^2}{(k-1)\,(k-2)} \\
&\quad + \left(\frac{a\beta}{M+a\beta}\right)^{k}\frac{(M+a\beta)}{(k-1)}\frac{(2a^2\beta-2a\beta+2aMk-2aM-Mk)}{a\,(k-2)}.
\end{aligned}
$$

We are now ready to address the numerical examples. The security loading factors are $\theta_I = 0.1$ and $\theta_R = 0.2$ for the cedent and reinsurer respectively. Let the mean of the inter claim time be 1 ($E(N) = 1$), i.e., $\lambda = 100$ for the compound Poisson model; $n = 200, \eta = 0.5$ for the compound binomial model; and $r = 100, \eta = 0.5$ for the compound negative binomial model. Let the original claim size has a mean of 100 ($E(X) = 100$), i.e., $\mu = 100$ for the exponential claim, and $k = 5$, $\beta = 400$ for the Pareto claim.

Table 3.5: Optimal Retentions via Bivariate Gamma Approximation

Claim Size	Claim Frequency	a	M	$G(P_I - \omega_1, P_R - \omega_2)$
Exponential	CB	0.93	900	54%
	CP	0.96	970	53%
	CNB	1	1000	52%
Pareto	CB	0.46	980	50%
	CP	0.49	980	49%
	CNB	0.52	1000	48%

Using MATLAB to code the program, the optimal reinsurance treaties are summarized in Table 3.5 for the six scenarios. In programming, we use researching method. The parameter a changes from 0 to 100% by incremental of 1%, and M changes from 1 to 1000 by incremental 10.

Table 3.5 indicates that the claim size distribution has a greater influence on the optimal treaties than does the claim frequency distribution. The Pareto distribution has a lower quota-share a and a higher retention level M compared to the exponential claims with the same mean. This implies that for the heavier tail distributed claim, the ceding company should seek greater reinsurance to maximize joint survival probability for both companies.

3.6.1 Comparison of Bivariate Translated Gamma Approximation with the Method of Maximizing Lower Bound

We can discretize a continuous distribution function and use the corresponding discretization distribution function to approximate the continuous distribution function.

Thus, we can apply the method of maximizing the lower bound of the joint survival probability discussed in Section 3.3 to find the optimal retentions for the continuous distribution function. We will compare the results from the discretization distribution function with those from the bivariate translated gamma approximation.

To do so, let us consider the excess-loss treaty with $a = 1$. We are going to compare the bivariate translated approximation with the lower bound method. Note that for the exponential distribution in Section 3.6, the corresponding discretization exponential distribution is a geometrical distribution with the probability function

$$\Pr(X = x) = e^{-\frac{x}{\mu}} - e^{-\frac{x+1}{\mu}} = \left(1 - e^{-\frac{1}{\mu}}\right) e^{-\frac{x}{\mu}},\ x = 0, 1, 2, ...,$$

where $g = 1 - e^{-\frac{1}{\mu}}$. For the Pareto distribution in Section 3.6, the corresponding discretization Pareto distribution is a discrete Pareto with the probability function

$$\Pr(X = x) = \left(\frac{\beta}{x+\beta}\right)^k - \left(\frac{\beta}{x+1+\beta}\right)^k,\ x = 0, 1, 2, ...$$

Using the method in Section 3.3, we obtain the optimal retentions based on the discretization exponential and Pareto distributions, which are presented in Table 3.6. The table indicates that the optimal retentions using the bivariate gamma approximation are slightly larger than those from maximizing the lower bound. However, the joint survival probabilities estimated from the bivariate gamma approximation are less than the lower bound of the joint survival probabilities. This means that the bivariate gamma approximation underestimates the joint survival probabilities. In this sense, the approach of maximizing the lower bound in Section 3.3 is better than the bivariate gamma approximation.

Table 3.6: Approach Comparison

	Discretization Exponential $g = 1 - e^{-\frac{1}{\beta}}$		Discretization Pareto $\beta = 400,\ k = 5$	
	Optimal M	$\max L$	Optimal M	$\max L$
CB	142	59%	226	53%
CP	148	55%	305	51%
CNB	156	52%	307	49%
	Bivariate Gamma Approximation with $a = 1$			
	Optimal M	$\max L$	Optimal M	$\max L$
CB	146	54%	230	49%
CP	165	53%	310	48%
CNB	194	52%	311	48%

Chapter 4

Optimal Retentions with Interest: A De Vylder-Type Approximation

In insurance business, many interest included models have been suggested and studied. See, for example, Cai and Dickson (2003, 2004), Cai (2004)and Capasso and Bakstein (2005). This chapter considers the compound Poisson risk model modified by the inclusion of interest, as well as reinsurance. The assumption is that the insurer receives interest on its surplus at a constant continuously compounding interest $\delta > 0$.

Let $U_\delta(t)$ be the surplus of the cedent company at time t with $U_\delta(0) = u_I$ after reinsurance. Then, $U_\delta(t)$ can be expressed as

$$U_\delta(t) = u_I e^{\delta t} + \frac{P_I(e^{\delta t} - 1)}{\delta} - \sum_{k=1}^{N(t)} X_{I_k} e^{\delta(t-T_k)}, \tag{4.1}$$

where T_k is the time of the kth claim.

The time of ruin is expressed as $\tau_\delta = \inf\{t : U_\delta(t) < 0\}$, and the ultimate ruin probability with the inclusion of interest δ is

$$\psi_\delta(u_I) = \Pr\{\tau_\delta < \infty\} = \Pr\{\cup_{t\geq 0}\,(U_\delta(t) < 0)\}.$$

4.1 Objectives

In this chapter, we want to find the optimal retention levels a and M by minimizing the ruin probability $\psi_\delta(u_I)$. However, the explicit formula for $\psi_\delta(u_I)$ is available only for a few special cases. It is very difficult to minimize the ruin probability $\psi_\delta(u_I)$ directly for general cases.

De Vylder (1978) utilized a simple, yet ingenious method to replace a compound Poisson risk process with general claims by a compound Poisson risk process with exponential claims, which makes the first three moments of the risk process with general claims equal to the ones of the risk process with exponential claims. Thus, the ruin probability in the risk process with general claims is approximated by the ruin probability in the risk process with exponential claims, which has a closed and explicit form.

In this chapter, we extend De Vylder's approximation to the compound Poisson risk model with interest. With the De Vylder-type approximation, we can determine the optimal retention levels a and M and consider the effect of the interest on the retentions.

The idea is as follows. We replace the risk process $U_\delta(t)$ in (4.1) by $\widetilde{U}_\delta(t)$, a new compound Poisson risk process with the same initial surplus u_I and the same interest force δ, but a new Poisson process $\tilde{N}(t)$ and exponential claims, which is defined as

$$\tilde{U}_\delta(t) = u_I e^{\delta t} + \frac{\tilde{P}_I(e^{\delta t}-1)}{\delta} - \sum_{k=1}^{\tilde{N}(t)} \tilde{X}_k e^{\delta(t-\tilde{T}_k)}, \tag{4.2}$$

where $\tilde{N}(t)$ is a Poisson process with rate $\tilde{\lambda}$, $\tilde{T}_k$ is the kth claim time, $\tilde{P}_I$ is the premium rate, $\tilde{X}_k$ has an exponential distribution with mean $\tilde{\mu}$.

The parameters $\tilde{\lambda}$, $\tilde{P}_I$ and $\tilde{\mu}_I$ are chosen, so that

$$E\left[U_\delta^k(t)\right] = E\left[\tilde{U_\delta^k}(t)\right] \text{ for } k = 1, 2, 3; t \geq 0.$$

Then, the probability of ultimate ruin in the initial process is approximated by the probability of ruin in the new process.

Denote the ruin probability in the risk process $\tilde{U}_\delta$ by $\tilde{\psi}_\delta(u_I)$. The closed and explicit formula is given by

$$\tilde{\psi}_\delta(u_I) = \frac{\Gamma(\frac{\tilde{\lambda}}{\tilde{\delta}}, \frac{\tilde{P}_I}{\tilde{\delta}\tilde{\mu}} + \frac{u_I}{\tilde{\mu}})}{\Gamma(\frac{\tilde{\lambda}}{\tilde{\delta}}, \frac{\tilde{P}_I}{\tilde{\delta}\tilde{\mu}}) + \frac{\tilde{\delta}}{\tilde{\lambda}}\left(\frac{\tilde{P}_I}{\tilde{\lambda}\tilde{\mu}}\right)^{\tilde{\lambda}/\tilde{\delta}} e^{-\frac{\tilde{P}_I}{\tilde{\delta}\tilde{\mu}}}}, \tag{4.3}$$

where $\Gamma(a, b) = \int_b^\infty x^{a-1}e^{-x}dx$, $a > 0$, $b \geq 0$, is the incomplete gamma function. See, for example, Segerdahl (1942) or Gerber (1979).

Therefore, the ruin probability $\psi_\delta(u_I)$ can be approximated by $\tilde{\psi}_\delta(u_I)$, which is employed here to determine the optimal reinsurance quota-share a and stop-loss limit M in this multi-period claim process. To determine the De Vylder approximation, we need to calculate the first three moments of $U_\delta(u_I)$, which are given in the following section.

4.2 Moments of Surplus Process with Interest

Lemma 4.2.1 For the surplus process defined in equation (4.1), the first three moments of $U_\delta(t)$ are provided by

$$\begin{aligned} E\left[U_\delta(t)\right] &= u_I e^{\delta t} + \left(P_I - \lambda\mu_1\right)\frac{e^{\delta t}-1}{\delta}, && (4.4)\\ E\left[U_\delta^2(t)\right] &= \left(u_I e^{\delta t}\right)^2 + 2u_I\left(P_I - \lambda\mu_1\right)e^{\delta t}\frac{e^{\delta t}-1}{\delta} \\ &\quad + \left(P_I - \lambda\mu_1\right)^2\left(\frac{e^{\delta t}-1}{\delta}\right)^2 + \lambda\mu_2\frac{e^{2\delta t}-1}{2\delta}, && (4.5)\\ E\left[U_\delta^3(t)\right] &= \left(u_I e^{\delta t}\right)^3 + 3u_I^2\left(P_I - \lambda\mu_1\right)e^{2\delta t}\frac{e^{\delta t}-1}{\delta} + 3u_I\lambda\mu_2 e^{\delta t}\frac{e^{2\delta t}-1}{2\delta} \\ &\quad + 3u_I\left(P_I - \lambda\mu_1\right)^2 e^{\delta t}\left(\frac{e^{\delta t}-1}{\delta}\right)^2 + \left(P_I - \lambda\mu_1\right)^3\left(\frac{e^{\delta t}-1}{\delta}\right)^3 \\ &\quad + 3\lambda\mu_2\left(P_I - \lambda\mu_1\right)\frac{e^{\delta t}-1}{\delta}\frac{e^{2\delta t}-1}{2\delta} - \lambda\mu_3\frac{e^{3\delta t}-1}{3\delta}, && (4.6) \end{aligned}$$

where μ_1, μ_2, and μ_3 are the first three moments of the random variable X_I.

Proof: Let

$$X_\delta(t) = \sum_{k=1}^{N(t)} X_{I_k} e^{-\delta T_k} = \sum_{k=1}^{N(t)} h(T_k, X_{I_k}),$$

where $h(t,x) = xe^{-\delta t}$.

Recall that $N(t)$ is a Poisson process with rate $\lambda > 0$; $\{X_{I_k}, k \geq 1\}$ are i.i.d. jump size independent of $N(t)$ and T_k is the time of jump k with $k \geq 1$. Further, assume that the random variable X_{I_k}, the claim paid by the ceding company, has a distribution function of $G(x)$.

According to Lemma 2.2(ii) of Rosinski (1990),

$$E\left[e^{izX_\delta(t)}\right] = e^{\lambda\int_0^t\int_0^\infty\left(e^{izxe^{-\delta v}}-1\right)dG(x)dv}.$$

Letting $s = -zi$ results in the Laplace transform of $X_\delta(t)$

$$E\left[e^{-sX_\delta(t)}\right] = e^{\lambda \int_0^t \int_0^\infty (e^{-sxe^{-\delta v}} - 1) dG(x) dv} = e^{\lambda \int_0^t (\widehat{g}(se^{-\delta v}) - 1) dv},$$

where $\widehat{g}(s) = \int_0^\infty e^{-sx} dG(x)$.

Thus, the Laplace transform of $\sum_{k=1}^{N(t)} X_{I_k} e^{\delta(t-T_k)}$ is given as

$$\begin{aligned}
\varphi(s) &= E\left[e^{-se^{\delta t} X_\delta(t)}\right] \\
&= e^{\lambda \int_0^t (\widehat{g}(se^{\delta t} e^{-\delta v}) - 1) dv} = e^{\lambda \int_0^t (\widehat{g}(se^{\delta(t-v)}) - 1) dv} \\
&= e^{\lambda \int_0^t (\widehat{g}(se^{\delta x}) - 1) dx}. \qquad (4.7)
\end{aligned}$$

Note that $\varphi(s) = E\left[e^{-sX}\right]$ and $\varphi^{(k)}(s) = E\left[(-X)^k e^{-sX}\right]$, which indicates that $\varphi(0) = 1$ and $\varphi^{(k)}(0) = (-1)^k \mu_k$, for $k = 1, 2, 3, ...$

Hence, for $\varphi(s)$ defined as (4.7),

$$\begin{aligned}
\varphi'(s) &= \varphi(s) \lambda \int_0^t \widehat{g}'(se^{\delta x}) e^{\delta x} dx; \\
\varphi''(s) &= \varphi'(s) \lambda \int_0^t \widehat{g}'(se^{\delta x}) e^{\delta x} dx + \varphi(s) \lambda \int_0^t \widehat{g}''(se^{\delta x}) e^{2\delta x} dx; \\
\varphi'''(s) &= \varphi''(s) \lambda \int_0^t \widehat{g}'(se^{\delta x}) e^{\delta x} dx + \varphi'(s) \lambda \int_0^t \widehat{g}''(se^{\delta x}) e^{2\delta x} dx \\
&\quad + \varphi'(s) \lambda \int_0^t \widehat{g}''(se^{\delta x}) e^{2\delta x} dx + \varphi(s) \lambda \int_0^t \widehat{g}'''(se^{\delta x}) e^{3\delta x} dx.
\end{aligned}$$

It implies

$$
\begin{aligned}
\varphi'(0) &= \lambda \int_0^t \widehat{g}'(0) e^{\delta x} dx = -\lambda\mu_1 \int_0^t e^{\delta x} dx = -\lambda\mu_1 \frac{e^{\delta t}-1}{\delta} \\
&= -E\left[\sum_{k=1}^{N(t)} X_{I_k} e^{\delta(t-T_k)}\right]; \\
\varphi''(0) &= \varphi'(0)\lambda \int_0^t \widehat{g}'(0) e^{\delta x} dx + \varphi(0)\lambda \int_0^t \widehat{g}''(0) e^{2\delta x} dx \\
&= -\lambda\mu_1 \frac{e^{\delta t}-1}{\delta}\lambda\left(-\mu_1\right)\frac{e^{\delta t}-1}{\delta} + \lambda\mu_2 \frac{e^{2\delta t}-1}{2\delta} \\
&= \left(\lambda\mu_1\right)^2 \left(\frac{e^{\delta t}-1}{\delta}\right)^2 + \lambda\mu_2 \frac{e^{2\delta t}-1}{2\delta} \\
&= E\left[\left(\sum_{k=1}^{N(t)} X_{I_k} e^{\delta(t-T_k)}\right)^2\right]; \\
\varphi'''(0) &= \left(\left(\lambda\mu_1\right)^2 \left(\frac{e^{\delta t}-1}{\delta}\right)^2 + \lambda\mu_2 \frac{e^{2\delta t}-1}{2\delta}\right)\lambda \int_0^t \widehat{g}'(0) e^{\delta x} dx \\
&\quad + \left(-\lambda\mu_1 \frac{e^{\delta t}-1}{\delta}\right)\lambda \int_0^t \widehat{g}''(0) e^{2\delta x} dx \\
&\quad + \left(-\lambda\mu_1 \frac{e^{\delta t}-1}{\delta}\right)\lambda \int_0^t \widehat{g}''(0) e^{2\delta x} dx + \lambda \int_0^t \widehat{g}'''(0) e^{3\delta x} dx \\
&= \left(\left(\lambda\mu_1\right)^2 \left(\frac{e^{\delta t}-1}{\delta}\right)^2 + \lambda\mu_2 \frac{e^{2\delta t}-1}{2\delta}\right)\lambda\left(-\mu_1\right)\frac{e^{\delta t}-1}{\delta} \\
&\quad + \left(-\lambda\mu_1 \frac{e^{\delta t}-1}{\delta}\right)\lambda\mu_2 \frac{e^{2\delta t}-1}{2\delta} \\
&\quad + \left(-\lambda\mu_1 \frac{e^{\delta t}-1}{\delta}\right)\lambda\mu_2 \frac{e^{2\delta t}-1}{2\delta} + \lambda\left(-\mu_3\right)\frac{e^{3\delta t}-1}{3\delta} \\
&= -\left(\lambda\mu_1\right)^3 \left(\frac{e^{\delta t}-1}{\delta}\right)^3 - 3\lambda^2\mu_1\mu_2 \frac{e^{2\delta t}-1}{2\delta}\frac{e^{\delta t}-1}{\delta} - \lambda\mu_3 \frac{e^{3\delta t}-1}{3\delta} \\
&= -E\left[\left(\sum_{k=1}^{N(t)} X_{I_k} e^{\delta(t-T_k)}\right)^3\right].
\end{aligned}
$$

Hence, the first three moments of $U_\delta(t)$, defined as (4.1), are as follows

$$\begin{aligned}
E\left[U_\delta(t)\right] &= u_I e^{\delta t} + P_I \frac{e^{\delta t}-1}{\delta} - \lambda\mu_1 \frac{e^{\delta t}-1}{\delta} \\
&= u_I e^{\delta t} + \left(P_I - \lambda\mu_1\right) \frac{e^{\delta t}-1}{\delta}; \\
E\left[U_\delta^2(t)\right] &= \left(u_I e^{\delta t} + P_I \frac{e^{\delta t}-1}{\delta}\right)^2 - 2\left(u_I e^{\delta t} + P_I \frac{e^{\delta t}-1}{\delta}\right)\lambda\mu_1 \frac{e^{\delta t}-1}{\delta} \\
&\quad + \left(\lambda\mu_1\right)^2 \left(\frac{e^{\delta t}-1}{\delta}\right)^2 + \lambda\mu_2 \frac{e^{2\delta t}-1}{2\delta} \\
&= \left(u_I e^{\delta t}\right)^2 + 2u_I\left(P_I - \lambda\mu_1\right) e^{\delta t} \frac{e^{\delta t}-1}{\delta} \\
&\quad + \left(P_I - \lambda\mu_1\right)^2 \left(\frac{e^{\delta t}-1}{\delta}\right)^2 + \lambda\mu_2 \frac{e^{2\delta t}-1}{2\delta}; \\
E\left[U_\delta^3(t)\right] &= \left(u_I e^{\delta t} + P_I \frac{e^{\delta t}-1}{\delta}\right)^3 - 3\left(u_I e^{\delta t} + P_I \frac{e^{\delta t}-1}{\delta}\right)^2 \lambda\mu_1 \frac{e^{\delta t}-1}{\delta} \\
&\quad +3\left(u_I e^{\delta t} + P_I \frac{e^{\delta t}-1}{\delta}\right)\left(\left(\lambda\mu_1\right)^2 \left(\frac{e^{\delta t}-1}{\delta}\right)^2 + \lambda\mu_2 \frac{e^{2\delta t}-1}{2\delta}\right) \\
&\quad - \left(\lambda\mu_1\right)^3 \left(\frac{e^{\delta t}-1}{\delta}\right)^3 - 3\lambda^2\mu_1\mu_2 \frac{e^{2\delta t}-1}{2\delta}\frac{e^{\delta t}-1}{\delta} - \lambda\mu_3 \frac{e^{3\delta t}-1}{3\delta} \\
&= \left(u_I e^{\delta t}\right)^3 + 3u_I^2\left(P_I - \lambda\mu_1\right) e^{2\delta t} \frac{e^{\delta t}-1}{\delta} + 3u_I\lambda\mu_2 e^{\delta t} \frac{e^{2\delta t}-1}{2\delta} \\
&\quad +3u_I\left(P_I - \lambda\mu_1\right)^2 e^{\delta t} \left(\frac{e^{\delta t}-1}{\delta}\right)^2 + \left(P_I - \lambda\mu_1\right)^3 \left(\frac{e^{\delta t}-1}{\delta}\right)^3 \\
&\quad +3\lambda\mu_2\left(P_I - \lambda\mu_1\right) \frac{e^{\delta t}-1}{\delta}\frac{e^{2\delta t}-1}{2\delta} - \lambda\mu_3 \frac{e^{3\delta t}-1}{3\delta}.
\end{aligned}$$

■

4.3 A De Vylder-Type Approximation to Surplus Process with Interest

The idea of the De Vylder approximation is to replace a general claim surplus process $U_\delta(t)$ with a new exponential claim surplus process $\widetilde{U_\delta}(t)$ in a compound Poisson model. These two surplus processes hold the same initial values and interest rates. By matching the first three moments, the new premium $\widetilde{P_I}$, the new inter-claim time Poisson parameter $\widetilde{\lambda}$, and the claim size exponential parameter $\widetilde{\mu}$ may be determined.

Theorem 4.3.1 Assume that the first three moments of the retained claim X_I are μ_1, μ_2, and μ_3, respectively. To approximate the risk process $U_\delta(t)$ by the risk process $\tilde{U_\delta}(t)$, the parameters $\tilde{\mu}$, $\tilde{\lambda}$, and $\tilde{P_I}$ must satisfy

$$\begin{aligned} \tilde{\mu} &= \frac{\mu_3}{2\mu_2}, && (4.8)\\ \tilde{\lambda} &= \frac{4\mu_2^3}{\mu_3^2}\lambda, && (4.9)\\ \tilde{P_I} &= P_I - \lambda\mu_1 + \lambda\frac{2\mu_2^2}{\mu_3}. && (4.10) \end{aligned}$$

Proof: We use Lemma 4.2.1 to match the first three moments of $U_\delta(t)$ and $\tilde{U_\delta}(t)$.

For the first moment match of $E\left[U_\delta(t)\right] = E\left[\tilde{U_\delta}(t)\right]$, we have

$$u_I e^{\delta t} + (P_I - \lambda\mu_1)\frac{e^{\delta t}-1}{\delta} = u_I e^{\delta t} + \left(\tilde{P_I} - \widetilde{\lambda}\widetilde{\mu}\right)\frac{e^{\delta t}-1}{\delta},$$

which results in

$$P_I - \lambda\mu_1 = \widetilde{P_I} - \widetilde{\lambda}\widetilde{\mu}. \tag{4.11}$$

For the second moment match or $E\left[U_\delta^2(t)\right] = E\left[\tilde{U_\delta^2}(t)\right]$, we have

$$
\begin{aligned}
& \left(u_I e^{\delta t}\right)^2 + 2u_I\left(P_I - \lambda\mu_1\right)e^{\delta t}\frac{e^{\delta t}-1}{\delta} + \left(P_I - \lambda\mu_1\right)^2\left(\frac{e^{\delta t}-1}{\delta}\right)^2 + \lambda\mu_2\frac{e^{2\delta t}-1}{2\delta} \\
= \ & \left(u_I e^{\delta t}\right)^2 + 2u_I\left(\tilde{P}_I - \widetilde{\lambda}\widetilde{\mu}\right)e^{\delta t}\frac{e^{\delta t}-1}{\delta} + \left(\tilde{P}_I - \widetilde{\lambda}\widetilde{\mu}\right)^2\left(\frac{e^{\delta t}-1}{\delta}\right)^2 + \widetilde{\lambda}\widetilde{\mu}^2\frac{e^{2\delta t}-1}{2\delta},
\end{aligned}
$$

which results in

$$
\lambda\mu_2 = \widetilde{\lambda}\widetilde{\mu}^2. \tag{4.12}
$$

For the third moment match or $E\left[U_\delta^3(t)\right] = E\left[\tilde{U_\delta^3}(t)\right]$, we have

$$
\begin{aligned}
& \left(u_I e^{\delta t}\right)^3 + 3u_I^2\left(P_I - \lambda\mu_1\right)e^{2\delta t}\frac{e^{\delta t}-1}{\delta} + 3u_I\lambda\mu_2 e^{\delta t}\frac{e^{2\delta t}-1}{2\delta} \\
& + 3u_I\left(P_I - \lambda\mu_1\right)^2 e^{\delta t}\left(\frac{e^{\delta t}-1}{\delta}\right)^2 + \left(P_I - \lambda\mu_1\right)^3\left(\frac{e^{\delta t}-1}{\delta}\right)^3 \\
& + 3\lambda\mu_2\left(P_I - \lambda\mu_1\right)\frac{e^{\delta t}-1}{\delta}\frac{e^{2\delta t}-1}{2\delta} - \lambda\mu_3\frac{e^{3\delta t}-1}{3\delta} \\
= \ & \left(u_I e^{\delta t}\right)^3 + 3u_I^2\left(\tilde{P}_I - \tilde{\lambda}\widetilde{\mu}\right)e^{2\delta t}\frac{e^{\delta t}-1}{\delta} + 3u_I\tilde{\lambda}\widetilde{\mu}^2 e^{\delta t}\frac{e^{2\delta t}-1}{2\delta} \\
& + 3u_I\left(\tilde{P}_I - \widetilde{\lambda}\widetilde{\mu}\right)^2 e^{\delta t}\left(\frac{e^{\delta t}-1}{\delta}\right)^2 + \left(\tilde{P}_I - \widetilde{\lambda}\widetilde{\mu}\right)^3\left(\frac{e^{\delta t}-1}{\delta}\right)^3 \\
& + 3\tilde{\lambda}\widetilde{\mu}^2\left(\tilde{P}_I - \widetilde{\lambda}\widetilde{\mu}\right)\frac{e^{\delta t}-1}{\delta}\frac{e^{2\delta t}-1}{2\delta} - 2\tilde{\lambda}\widetilde{\mu}^3\frac{e^{3\delta t}-1}{3\delta},
\end{aligned}
$$

which results in

$$
\lambda\mu_3 = 2\tilde{\lambda}\widetilde{\mu}^3. \tag{4.13}
$$

Therefore, equations (4.12) and (4.13) imply

$$
\widetilde{\mu} = \frac{\mu_3}{2\mu_2},
$$

equation (4.12) suggests

$$\widetilde{\lambda} = \frac{\lambda\mu_2}{\widetilde{\mu}^2} = \frac{\lambda\mu_2}{\frac{\mu_3}{2\mu_2}\frac{\mu_3}{2\mu_2}} = \frac{4\mu_2^3}{\mu_3^2}\lambda,$$

and equation (4.11) implies

$$\widetilde{P_I} = P_I - \lambda\mu_1 + \widetilde{\lambda}\widetilde{\mu} = P_I - \lambda\mu_1 + \lambda\frac{2\mu_2^2}{\mu_3}.$$

■

4.4 Approximation of the Optimal Retentions with Interest

The goal is to determine the optimal retention level M and the quota-share a to minimize the ruin probability (4.3)

$$\tilde{\psi}_\delta(u_I) = \frac{\Gamma(\frac{\widetilde{\lambda}}{\delta}, \frac{\widetilde{P_I}}{\delta\widetilde{\mu}} + \frac{u_I}{\widetilde{\mu}})}{\Gamma(\frac{\widetilde{\lambda}}{\delta}, \frac{\widetilde{P_I}}{\delta\widetilde{\mu}}) + \frac{\delta}{\widetilde{\lambda}}\left(\frac{\widetilde{P_I}}{\widetilde{\lambda}\widetilde{\mu}}\right)^{\widetilde{\lambda}/\delta} e^{-\frac{\widetilde{P_I}}{\delta\widetilde{\mu}}}},$$

where $\tilde{\mu}$, $\tilde{\lambda}$, and $\tilde{P}_I$ are as indicated in Theorem 4.3.1, i.e.,

$$\begin{aligned}
\tilde{\mu} &= \frac{\mu_3}{2\mu_2}; \\
\tilde{\lambda} &= \frac{4\mu_2^3}{\mu_3^2}\lambda; \\
\tilde{P}_I &= P_I - \lambda\mu_1 + \lambda\frac{2\mu_2^2}{\mu_3};
\end{aligned}$$

and

$$\Gamma(a, b) = \int_b^\infty x^{a-1}e^{-x}dx.$$

Here, the exponential claim and the Pareto claim for numerical illustrations are used again. As in previous chapters, both of them have the unchanged average claim sizes of 100 and average inter-claim times of 1. The security loadings for the cedent and reinsurer are $\theta_I = 0.1$ and $\theta_R = 0.2$, respectively.

In particular, for the exponential claim, we have

$$\begin{aligned}
P_I &= \lambda\mu\left((1+\theta_I) - (1+\theta_R)(1-a) - (1+\theta_R)ae^{-\frac{M}{a\mu}}\right);\\
\mu_1 &= -e^{-\frac{M}{a\mu}}a\mu + a\mu;\\
\mu_2 &= -2Me^{-\frac{M}{a\mu}}a\mu - 2e^{-\frac{M}{a\mu}}a^2\mu^2 + 2a^2\mu^2;\\
\mu_3 &= -3e^{-\frac{M}{a\mu}}M^2a\mu - 6e^{-\frac{M}{a\mu}}Ma^2\mu^2 - 6e^{-\frac{M}{a\mu}}a^3\mu^3 + 6a^3\mu^3;\\
\mu &= 100;\\
\theta_I &= 0.1;\\
\theta_R &= 0.2;
\end{aligned}$$

For the Pareto claim, we have

$$
\begin{aligned}
P_I &= \lambda(\theta_I - \theta_R)\frac{\beta}{k-1} + \lambda(1+\theta_R)\frac{a\beta}{k-1}\left(1 - \left(\frac{a\beta}{M+a\beta}\right)^{k-1}\right);\\
\mu_1 &= \frac{a\beta}{k-1} - \frac{M+a\beta}{k-1}\left(\frac{a\beta}{M+a\beta}\right)^{k};\\
\mu_2 &= \frac{2a^2\beta^2}{(k-1)(k-2)} - \frac{2\left(a\beta + Mk - M\right)}{(k-2)}\frac{(M+a\beta)}{(k-1)}\left(\frac{a\beta}{M+a\beta}\right)^{k};\\
\mu_3 &= \frac{6a^3\beta^3}{(k-1)\,(k-2)\,(k-3)}\\
&\quad -3\frac{(2a^2\beta^2 + 2\beta a M k - 2\beta a M - 3kM^2 + k^2M^2 + 2M^2)}{(k-2)\,(k-3)}\frac{(M+a\beta)}{(k-1)}\left(\frac{a\beta}{M+a\beta}\right)^{k};\\
\beta &= 400;\\
k &= 5;\\
\theta_I &= 0.1;\\
\theta_R &= 0.2;
\end{aligned}
$$

For interest rate δ, we consider three scenarios $\delta = 1\%$, $\delta = 5\%$, and $\delta = 10\%$. For the initial surplus u_I, we consider seven scenarios from $u_I = 100$ up to $u_I = 400$. Hence, there are 21 scenario combinations for each claim distribution, which are listed as follows. Here, for each scenario, the optimal reinsurance treaties and the corresponding ruin probability were derived with MATLAB. We also compared the ruin probability when there is no reinsurance.

Table 4.1: Exponential Claim with Interest Rate $\delta = 1\%$

u_I	a	M	$\psi_\delta^*(u_I)$	$\psi_\delta(u_I)$ without Reinsurance
100	58.8%	125	87.31%	91.41%
150	57.4%	122	81.14%	87.30%
200	55.5%	118	75.09%	83.31%
250	54.1%	115	69.18%	79.44%
300	52.7%	112	63.42%	75.69%
350	50.8%	108	57.83%	72.06%
400	49.4%	105	52.42%	68.56%

Table 4.2: Exponential Claim with Interest Rate $\delta = 5\%$

u_I	a	M	$\psi_\delta^*(u_I)$	$\psi_\delta(u_I)$ without Reinsurance
100	100%	30	60.36%	84.74%
150	100%	24	39.55%	77.65%
200	100%	9	18.64%	70.94%
250	57.8%	9	1.98%	64.62%
300	57.8%	9	$8.40E-04$	58.69%
350	57.8%	9	$1.64E-05$	53.16%
400	57.8%	9	$1.68E-07$	48.01%

Table 4.3: Exponential Claim with Interest Rate $\delta = 10\%$

u_I	a	M	$\psi_\delta^*(u_I)$	$\psi_\delta(u_I)$ without Reinsurance
100	100%	9	25.57%	79.69%
150	57.8%	9	1.23%	70.59%
200	57.8%	9	$1.40E-04$	62.23%
250	57.8%	9	$5.58E-07$	54.60%
300	57.8%	9	$1.03E-09$	47.68%
350	57.8%	9	$1.04E-12$	41.47%
400	57.8%	9	$6.66E-16$	35.91%

Table 4.4: Pareto Claim with Interest Rate $\delta = 1\%$

u_I	a	M	$\psi_\delta^*(u_I)$	$\psi_\delta(u_I)$ without Reinsurance
100	65.5%	120	87.41%	93.01%
150	64.4%	118	81.28%	89.64%
200	61.7%	113	75.27%	86.34%
250	60.6%	111	69.39%	83.13%
300	58.4%	107	63.66%	79.99%
350	57.3%	105	58.09%	76.94%
400	54.6%	100	52.69%	73.97%

Table 4.5: Pareto Claim with Interest Rate $\delta = 5\%$

u_I	a	M	$\psi^*_\delta(u_I)$	$\psi_\delta(u_I)$ without Reinsurance
100	100%	31	60.58%	87.01%
150	100%	24	39.77%	80.95%
200	100%	9	18.72%	75.2%
250	71.5%	9	1.98%	69.75%
300	71.5%	9	$8.40E-04$	64.6%
350	71.5%	9	$1.64E-05$	59.74%
400	71.5%	9	$1.68E-07$	55.16%

Table 4.6: Pareto Claim with Interest Rate $\delta = 10\%$

u_I	a	M	$\psi^*_\delta(u_I)$	$\psi_\delta(u_I)$ without Reinsurance
100	100%	9	25.63%	80.39%
150	71.5%	9	1.23%	72.73%
200	71.5%	9	$1.40E-04$	65.67%
250	71.5%	9	$5.59E-07$	59.17%
300	71.5%	9	$1.03E-09$	53.22%
350	71.5%	9	$1.05E-12$	47.78%
400	71.5%	9	$6.66E-16$	42.82%

Without reinsurance, because of the heavier tail, the Pareto claim possesses the higher ruin probability given the same conditions.

The tables above demonstrate that the ruin probability decreases dramatically when both the initial value and interest rate increased. The ruin probability difference between the two claim distributions is significantly smaller due to the reinsurance. The Pareto claim demonstrates a higher quota-share level, which is consistent with the conclusion in Chapter 2. With the optimal reinsurance treaty, the ruin probability is effectively controlled, given the proper initial value and interest rate. Figure 4.1 gives the demonstration.

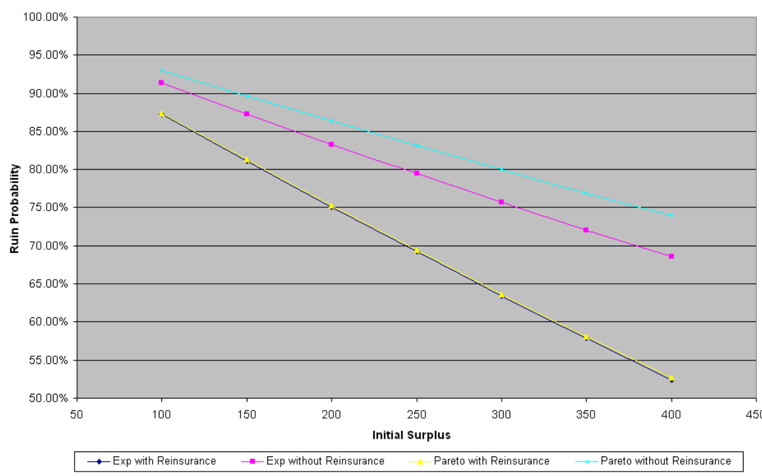

Figure 4.1: Ruin Probabilities with Reinsurance v.s. without Reinsurance

Chapter 5

Optimal Retentions with Dividends

In this chapter, the influence of dividends on optimal reinsurance retentions is considered. The assumption is that the ceding insurance company pays dividends to shareholders according to a dividend strategy.

Bruno De Finetti (1957) first suggested that a company would seek to maximize the expectation of the present value of all dividends before possible ruin. When the surplus of the company is a discrete process with steps of size plus or minus only one, the optimal dividend payment strategy is a barrier strategy. This means, any surplus above a certain level would be paid as dividends to the shareholders of the company.

Jeanblanc-Picqu and Shiryaev (1995) and Asmussen and Taksar (1997) further proved that in the Brownian motion model with a dividend ceiling the optimal dividend strategy is a threshold strategy, which is, dividends should be paid out at the highest admissible rate as soon as the surplus exceeds a certain threshold.

Gerber and Shiu (2006) generalized the model to a compound Poisson process and provided the explicit formula of the expected dividend value for exponential claim amount distribution.

Recent research includes the works by Dickson and Drekic (2006); and Cheung, Dickson and Drekic (2008). The first paper demonstrates a method to approximate the expected dividend before the possible ruin and the latter paper presents the general formula for Erlang family when an initial surplus is less than a certain threshold level.

Based on the aforementioned theories, this chapter extends the model to the reinsurance aspect. The objective is to maximize the expectation of the present value of all dividends prior to possible ruin by determining the optimal reinsurance treaties.

Here, only quota-share reinsurance receives consideration, with excess-of-loss reinsurance reserved for future research. The optimal quota-share reinsurance treaty for the exponential claim amount case is first discussed, followed by Erlang(2) claim with numerical examples provided.

5.1 Assumptions

Using the previous notations, let a be the quota-share retention level and M be the stop-loss limit. u_I is the initial surplus, μ is the expected claim size and λ is the claim frequency in a compound Poisson process. The definition sets for this chapter are as follows.

- $X_I = \min(aX, M)$: the ceding company claim payment with original claim size of X.

- P_I: net premium rate of the ceding company after reinsurance.

- $\{S_{a,M}(t)\}$: aggregate claim process after the reinsurance, i.e.,

$$S_{a,M}(t) = \sum_{i=1}^{N(t)} \min(aX_i, M).$$

- $\{U_{a,M}(t)\}$: surplus process after the reinsurance, i.e.,

$$U_{a,M}(t) = u_I + P_I t - S_{a,M}(t).$$

- $D(t)$: the discounted aggregate dividends paid between time 0 and t.

- $Z_{a,M}(t)$: company's net surplus at time t, after reinsurance and dividend payment, i.e.,

$$Z_{a,M}(t) = U_{a,M}(t) - D(t).$$

- T: the time of ruin, i.e.,

$$T = \inf\{t \geq 0 | Z_{a,M}(t) < 0\}.$$

- δ: the force of interest for valuation. Here, we only consider the dividends accumulated from the interest and we do not consider the interest effectiveness for received premium, nor the paid out claim.

- D: present value of all dividends until ruin, i.e.,

$$D = \int_0^T e^{-\delta t} dD(t).$$

  Note that, under the threshold model, we further have

$$D = D_c \int_0^T e^{-\delta t} I_{\{U_{a,M}(t) \geq b\}} dt.$$

- $D_c \in (0, P_I)$: the dividend-rate ceiling. Here, the problem is considered under the constraint that only dividend strategies with the dividend rate bounded by a ceiling are admissible and the ceiling is less than the premium rate, i.e., $dD(t) \leq D_c dt$.

For the threshold strategy, threshold level b is used and the dividend payments comply with the following rules:

- $U_{a,M}(t) < b$: no dividends are paid;
- $U_{a,M}(t) > b$: dividends are paid at the maximal rate D_c.

To denote the expectation of the present value of all dividends until ruin, $V_{a,M}(u_I; b)$ is used where the initial surplus is u_I and threshold level is b. The objective is to maximize $V_{a,M}(u; b)$ by determining the optimal a and M. Here, let M go to infinity and consider quota-share reinsurance only.

5.2 Expected Discounted Dividends until Ruin with Quota-share Reinsurance and Exponential Claims

Under the quota-share reinsurance, similar to the discussions in Chapter 2, the insurance premium after the reinsurance is

$$P_I = (a(1 + \theta_R) - (\theta_R - \theta_I))\lambda\mu.$$

The claim after reinsurance is $X_I = aX$, with the c.d.f.

$$F_{X_I}(x) = P(X_I < x) = P(X < \frac{x}{a}) = F(\frac{x}{a}),$$

and the p.d.f. is

$$f_{X_I}(x) = \frac{1}{a} f(\frac{x}{a}).$$

For the original exponential claim distribution with p.d.f.

$$f(x) = \beta e^{-\beta x},$$

the claim payment of the ceding company is

$$f_{X_I}(x) = \frac{\beta}{a} e^{-\frac{\beta}{a} x},$$

which is an exponential distribution with the new parameter $\frac{\beta}{a}$. Hence, the formula (6.14) and (6.15) from Hans U Gerber and Shiu (2006) may be directly applied to calculate $V_a(u_I; b)$, which is the expectation of the present value of all dividends before possible ruin.

Case I: Initial surplus is less than the constant threshold, i.e., $0 \leq u_I \leq b$

$$V_a(u_I; b) = \frac{-\xi_3}{\beta/a} \frac{D_c}{\delta} \frac{(\beta/a + \xi_1) e^{\xi_1 u_I} - (\beta/a + \xi_2) e^{\xi_2 u_I}}{(\xi_1 - \xi_3) e^{\xi_1 b} - (\xi_2 - \xi_3) e^{\xi_2 b}}. \tag{5.1}$$

Case II: Initial surplus is greater than the constant threshold, i.e., $u_I \geq b$

$$V_a(u_I; b) = \frac{D_c}{\delta} \left[1 - e^{\xi_3 (u_I - b)}\right] + V_a(b; b) e^{\xi_3 (u_I - b)}, \tag{5.2}$$

and we have

$$V_a(0; 0) = \frac{-\xi_3}{\beta/a} \frac{D_c}{\delta}. \tag{5.3}$$

Here $\xi_1 > 0$, $\xi_2 < 0$ are the roots of the characteristic equation

$$P_I \xi^2 + (\frac{\beta}{a} P_I - \lambda - \delta) \xi - \frac{\beta}{a} \delta = 0,$$

and $\xi_3 < 0$ is the negative solution to

$$(P_I - D_c)\,\xi^2 + (\frac{\beta}{a}\,(P_I - D_c) - \lambda - \delta)\xi - \frac{\beta}{a}\delta = 0.$$

It implies

$$\xi_1 = \frac{-(\frac{\beta}{a}P_I - \lambda - \delta) + \sqrt{(\frac{\beta}{a}P_I - \lambda - \delta)^2 + 4P_I\frac{\beta}{a}\delta}}{2P_I}, \tag{5.4}$$

$$\xi_2 = \frac{-(\frac{\beta}{a}P_I - \lambda - \delta) - \sqrt{(\frac{\beta}{a}P_I - \lambda - \delta)^2 + 4P_I\frac{\beta}{a}\delta}}{2P_I}, \tag{5.5}$$

$$\xi_3 = \frac{-(\frac{\beta}{a}\,(P_I - D_c) - \lambda - \delta)}{2\,(P_I - D_c)} - \frac{\sqrt{(\frac{\beta}{a}\,(P_I - D_c) - \lambda - \delta)^2 + 4\,(P_I - D_c)\,\frac{\beta}{a}\delta}}{2\,(P_I - D_c)}, \tag{5.6}$$

The optimal quota-share level a should satisfy $\frac{\partial}{\partial a}V_a(u_I, b) = 0$.

Now, our task is to seek the optimal quota-share level a to maximize the expected dividend before the possible ruin. The initial method is to set the partial derivation of (5.1) and (5.2) with respect to a to 0. However, there is no such simple explicit formula for the desired quota-share. Thus, MATLAB programming is chosen as an alternative solution.

The base scenario has the following parameters: let interest rate be $\delta = 5\%$, claim size be 100 ($\beta = 0.01$), reinsurance security loading be $\theta_R = 0.2$, insurance security loading be $\theta_I = 0.1$, claim frequency $\lambda = 1$, initial value $u_I = 100$, and dividend rate ceiling $D_c = 5$. Table 5.1 offers the results.

Table 5.1: Optimal Quota-Share Level for Exponential Claims – Baseline

Threshold Level b	Optimal Quota-share a	Largest Expected Dividend $V_a(u_I; b)$
10	41%	40.45
20	43%	40.27
30	45%	39.98
40	47%	39.62
50	50%	39.19
60	53%	38.72
70	56%	38.22
80	60%	37.71
90	64%	37.19
100	68%	37.67
110	76%	35.68
120	83%	34.82
130	90%	34.08
140	98%	33.42
150	100%	32.81
160	100%	32.22
170	100%	31.64
180	100%	31.07
190	100%	30.51
200	100%	29.95

Table 5.1 notes the optimal quota-share level, a, increased, and the maximum of the expected dividend before ruin, $V_a(u_I; b)$, decreased as the threshold level b increases. The insurance meaning is that when the start point of dividend payment increases, the ceding company tends to use less reinsurance to generate a larger expected dividend before the possible ruin.

For practical study, the following six scenarios are examined:

Scenario 1: increase dividend ceiling D_c from 5 to 10. Both the optimal quota-share level a, and the largest expected dividend before ruin, $V_a(u_I; b)$ increases when the dividend ceiling D_c increases. Because more dividends are paid out, the ceding company tends to use less reinsurance to obtain the larger expected dividend before the possible ruin.

Scenario 2: increase the ceding company security loading θ_I from 10% to 19%. The optimal quota-share level, a, decreases and the largest expected dividend before ruin, $V_a(u_I; b)$, increases. When additional security loading is added to the ceding company, it uses more reinsurance. Because the difference between both companies is less, the maximum expected dividend for the cedent is more.

Scenario 3: increase interest rate δ from 5% to 10%. The largest expected dividend before ruin, $V_a(u_I; b)$, decreases when the interest rate δ increases. The optimal quota-share level, a, increased much faster along with the threshold level also increasing. Because a higher interest rate will generate higher dividends, the ceding company chooses the optimal reinsurance strategy according to the relationship between the threshold and initial surplus.

Scenario 4: increase claim frequency λ from 1% to 5%.The optimal quota-share level, a, increases while the largest expected dividend before ruin, $V_a(u_I; b)$, decreases.

When more claims occur, the ceding company chooses less reinsurance to maximize the expected dividend.

Scenario 5: decrease claim size from $100(= 1/0.01)$ to $50(= 1/0.02)$. Both the optimal quota-share level, a, and the largest expected dividend before ruin, $V_a(u_I; b)$, increase. When there is a smaller claim, the ceding company chooses reduced reinsurance to maximize the expected dividend.

Scenario 6: increase initial surplus u_I from 100 to 200. The optimal quota-share level, a, decreases while the largest expected dividend before ruin, $V_a(u_I; b)$, increases. When the initial surplus is higher, the ceding company chooses more reinsurance to maximize the expected dividend.

The following figures indicate the relationship between the optimal quota-share level and the corresponding largest expected dividend before ruin, with respect to the different parameters. The exact numerical results can be found on the attached Excel spreadsheet.

Note, all the quota-share levels, a, should be greater than $\frac{\theta_R-\theta_I}{1+\theta_R}$ to ensure the ceding premium after reinsurance, P_I, is greater than zero. Additionally, the dividend ceiling, D_C, is less than the premium received.

■

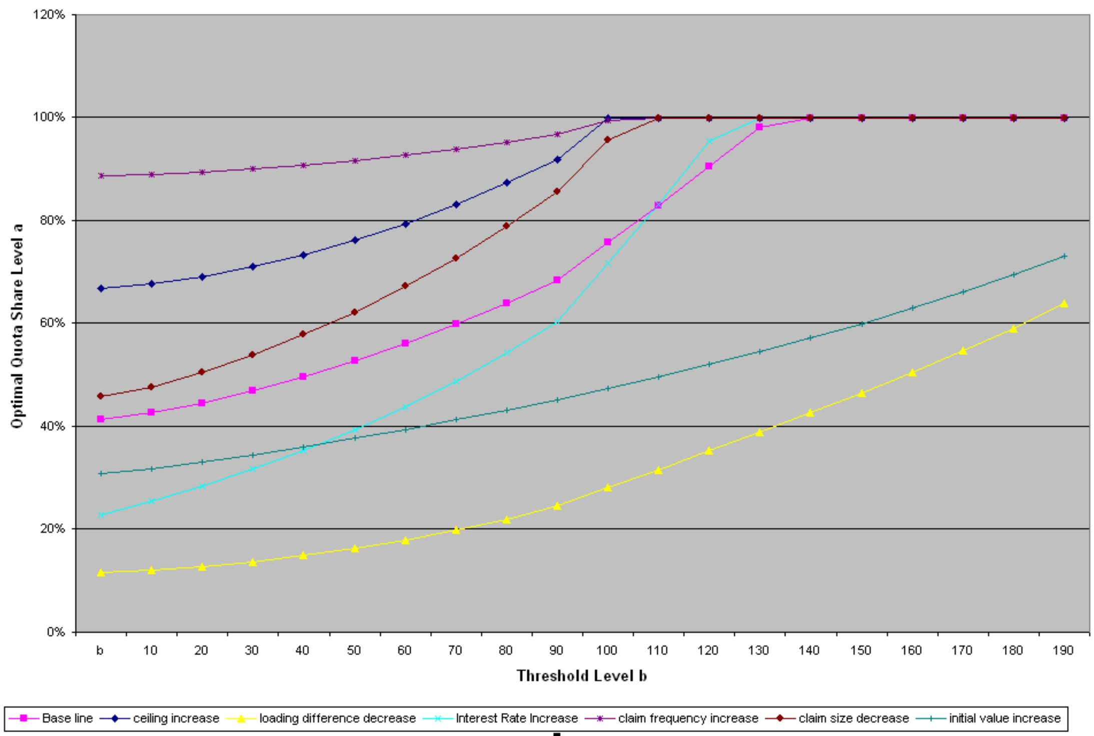

Figure 5.1: Optimal Quota-share Level in the Compound Poisson Process with Exponential Claims

5.3 Expected Discounted Dividends until Ruin with the Quota-share Reinsurance and Erlang (2) Claims

For an exponential claim, one of the special Erlang distributed claims, the formula provided by Gerber and Shiu (2006), is used to derive the optimal quota-share level. However, for a standard Erlang distributed claim, no such shortcut exists to calculate the expected dividend value after reinsurance. Hence, this generalized model must be constructed from scratch.

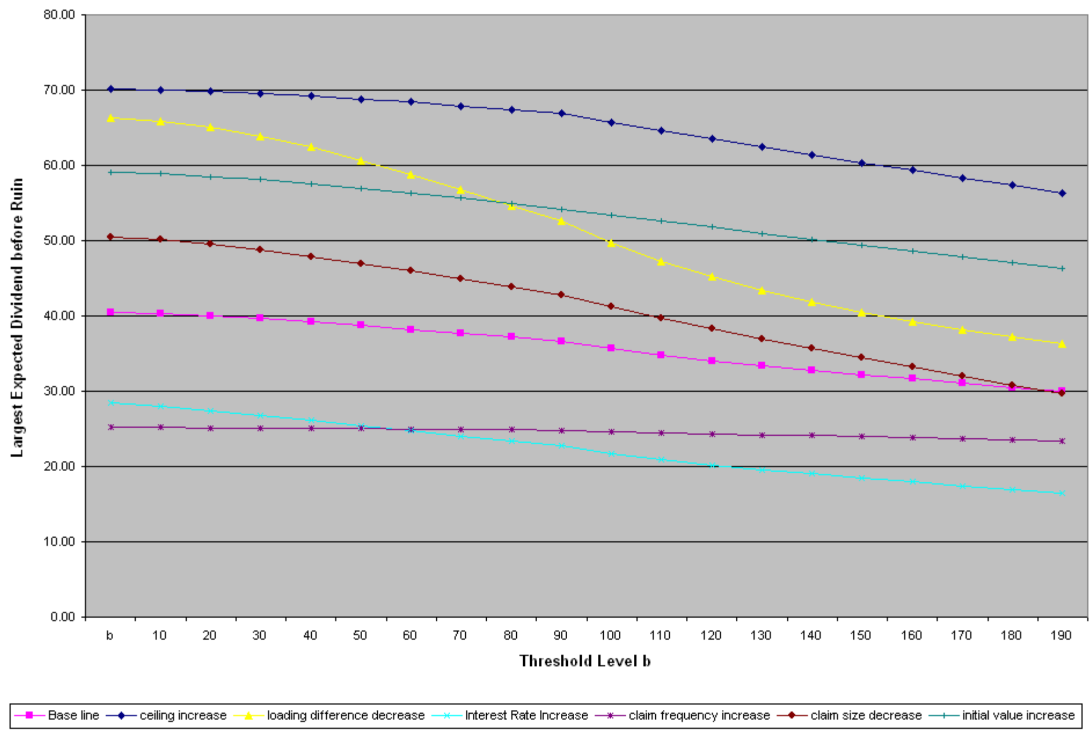

Figure 5.2: Maximum of Expected Discounted Dividend before Ruin in the Compound Poisson Process with Exponential Claims

The Erlang distribution we considered can be written as

$$f(x;k,\beta) = \frac{\beta^k x^{k-1} e^{-\beta x}}{(k-1)!}, \qquad k \geq 1.$$

Note, when $k = 1$, it becomes an exponential distribution.

To be more specific, only the Erlang(2) distribution receives consideration, i.e., $k = 2$, and

$$f(x;2,\beta) = \beta^2 x e^{-\beta x}.$$

The average claim is $\mu = \frac{2}{\beta}$. The claim after reinsurance can be written as

$$f_{X_a}(x) = \left(\frac{\beta}{a}\right)^2 x e^{-\frac{\beta}{a}x},$$

which is also an Erlang distribution where the parameter is $\frac{\beta}{a}$.

5.3.1 Characteristics of a Standard Cubic Function

Before we begin our discussion, a review of some characteristics of a standard cubic function

$$ax^3 + bx^2 + cx + d = 0, \tag{5.7}$$

is necessary, where a, b, c, d are any real numbers.

Lemma 5.3.1 Let u_1, u_2 and u_3 be the three roots of the cubic function (5.7), and let

$$A = \frac{1}{u_1\left(3au_1^2 + 2bu_1 + c\right)} + \frac{1}{u_2\left(3au_2^2 + 2bu_2 + c\right)} + \frac{1}{u_3\left(3au_3^2 + 2bu_3 + c\right)}.$$

We have

$$A = -\frac{1}{d}.$$

Proof: First note that, if u_1, u_2, and u_3 are the three roots of the cubic function (5.7), equation (5.7) can be re-written as

$$\begin{aligned} 0 &= (x - u_1)(x - u_2)(x - u_3) \\ &= x^3 - x^2(u_1 + u_2 + u_3) + x(u_1u_2 + u_1u_3 + u_2u_3)x - u_1u_2u_3. \end{aligned}$$

This indicates

$$u_1 + u_2 + u_3 = -\frac{b}{a},$$

and

$$u_1u_2 + u_1u_3 + u_2u_3 = \frac{c}{a}.$$

Hence, u_2 and u_3 is expressed as

$$
\begin{aligned}
u_2 &= \frac{-\left(\frac{b}{a}+u_1\right)+\sqrt{\left(\frac{b}{a}+u_1\right)^2-4u_1\left(\frac{b}{a}+u_1\right)-4\frac{c}{a}}}{2};\\
u_3 &= \frac{-\left(\frac{b}{a}+u_1\right)-\sqrt{\left(\frac{b}{a}+u_1\right)^2-4u_1\left(\frac{b}{a}+u_1\right)-4\frac{c}{a}}}{2}.
\end{aligned}
$$

Thus, A can also be rewritten as

$$
\begin{aligned}
& A \\
={}& \frac{1}{u_1\left(3au_1^2+2bu_1+c\right)}+\frac{1}{u_2\left(3au_2^2+2bu_2+c\right)}+\frac{1}{u_3\left(3au_3^2+2bu_3+c\right)}\\
={}& \frac{1}{-bu_1^2-2cu_1-3d}+\frac{1}{-bu_2^2-2cu_2-3d}+\frac{1}{-bu_3^2-2cu_3-3d}\\
={}& -\left(\frac{1}{bu_1^2+2cu_1+3d}+\frac{1}{bu+2cu_2+3d}+\frac{1}{bu_3^2+2cu_3+3d}\right).
\end{aligned}
$$

Substituting u_2 and u_3 to the above equation, after some tedious yet simple algebra, we have

$$
\frac{1}{bu_2^2+2cu_2+3d}+\frac{1}{bu_3^2+2cu_3+3d}=\frac{b^3-bu_1^2a^2-4bca-2cu_1a^2+6da^2}{B}
$$

and

$$
\begin{aligned}
& \frac{1}{bu_1^2+2cu_1+3d}+\frac{1}{bu+2cu_2+3d}+\frac{1}{bu_3^2+2cu_3+3d}\\
={}& -\frac{-27a^2d^2+18bcda-4ac^3-4b^3d+b^2c^2}{d\left(27a^2d^2-18abcd+4ac^3+4b^3d-b^2c^2\right)}\\
={}& \frac{1}{d},
\end{aligned}
$$

where

$$
\begin{aligned}
B ={}& -ab^2cu_1^2+4a^2c^2u_1^2-3a^2bdu_1^2-ab^2du_1-b^3cu_1\\
&+4abc^2u_1-6a^2cdu_1+9a^2d^2+4ac^3-b^2c^2+2b^3d-10abcd.
\end{aligned}
$$

Hence, $A=-\frac{1}{d}$. ■

Lemma 5.3.2 For the cubic function (5.7), all the roots are real when the discriminant of the cubic polynomial

$$\Delta = 4ac^3 - b^2c^2 + 4b^3d + 27a^2d^2 - 18abcd$$

is non-negative. All the roots are different when Δ is strictly positive.

■

5.3.2 Integro-differential Equations for Erlang(2) Claims

As with the exponential case for an Erlang(2) distributed claim with parameter β, let $V(u_I;b)$ denote the expectation of the present value of all dividends until ruin, where u_I is the initial surplus and b is the threshold. The function $V(u_I;b)$ satisfies the following integro-differential equations:

For $0 < u < b$,

$$P_I V'(u_I;b) - (\lambda+\delta)V(u_I;b) + \lambda \int_0^{u_I} V(u_I - x;b) f(x)dx = 0; \tag{5.8}$$

and for $u > b$,

$$D_c + (P_I - D_c)\, V'(u_I;b) - (\lambda+\delta)V(u_I;b) + \lambda \int_0^{u_I} V(u_I - x;b) f(x)dx = 0. \tag{5.9}$$

Following is an examination of some characteristics for integro-differential equations of an Erlang(2) distribution.

Lemma 5.3.3 In an Erlang(2) distribution $f(x)$ with parameter β, the integro-differential equation (5.8) is converted into a linear differential equation with constant coefficients,

$$\begin{aligned} & P_I V^{(3)}(u_I;b) + (2\beta P_I - \lambda - \delta)V''(u_I;b) \\ + & \left(P_I\beta^2 - 2\beta(\lambda+\delta)\right)V'(u_I;b) - \delta\beta^2 V(u_I;b) = 0; \end{aligned} \tag{5.10}$$

and equation (5.9) is converted to

$$\begin{aligned} & (P_I - D_c)V^{(3)}(u_I;b) + \left(2\beta(P_I - D_c) - (\lambda+\delta)\right)V''(u_I;b) \\ + & \left((P_I - D_c)\beta^2 - 2\beta(\lambda+\delta)\right)V'(u_I;b) - \beta^2\delta V(u_I;b) + \beta^2 D_c = 0. \end{aligned} \tag{5.11}$$

Here, P_I is the net received premium, λ is claim frequency, δ is dividend accumulation interest rate, and D_c is dividend ceiling. All of these are constants with respect to the initial surplus u_I.

Proof: First, note that $f(0;2,\beta) = 0$,

$$f'(x;2,\beta) = \beta^2 e^{-\beta x} - \beta^3 x e^{-\beta x} = \beta^2 e^{-\beta x} - \beta f(x;2,\beta),$$

and

$$\int_0^{u_I} V(u_I - x;b)f(x)dx = \int_0^{u_I} V(x;b)f(u_I - x)dx.$$

Hence, for $\int_0^{u_I} V(x;b)f(u_I - x)dx$, with respect to u_I, the first partial derivatives of the equations are

$$
\begin{aligned}
&\frac{\partial}{\partial u_I}\int_0^{u_I} V(x;b)f(u_I-x)dx = V(u_I;b)f(0)+\int_0^{u_I} V(x;b)f'(u_I-x)dx \\
= \;& \beta^2\int_0^{u_I} V(x;b)e^{-\beta(u_I-x)}dx - \beta\int_0^{u_I} V(x;b)f(u_I-x)dx;
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial}{\partial u_I}\int_0^{u_I} V(x;b)\beta^2 e^{-\beta(u_I-x)}dx = V(u_I;b)\beta^2 e^{-\beta(u_I-u_I)} - \beta\int_0^{u_I} V(x;b)\beta^2 e^{-\beta(u_I-x)}dx \\
= \;& \beta^2 V(u_I;b) - \beta^3\int_0^{u_I} V(x;b)e^{-\beta(u_I-x)}dx;
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial}{\partial u_I}\int_0^{u_I} V(x;b)\beta f(u_I-x)dx \\
= \;& \beta^3\int_0^{u_I} V(x;b)e^{-\beta(u_I-x)}dx - \beta^2\int_0^{u_I} V(x;b)f(u_I-x)dx;
\end{aligned}
$$

and the second partial derivative of the equation is

$$
\begin{aligned}
&\frac{\partial^2}{\partial u_I^2}\int_0^{u_I} V(u_I-x;b)f(x)dx \\
= \;& \beta^2 V(u_I;b) - 2\beta^3\int_0^{u_I} V(x;b)e^{-\beta(u_I-x)}dx + \beta^2\int_0^{u_I} V(x;b)f(u_I-x)dx.
\end{aligned}
$$

Applying operator β^2 to equation (5.8), we have

$$
\beta^2 P_I V'(u_I;b) - \beta^2(\lambda+\delta)V(u_I;b) + \lambda\beta^2\int_0^{u_I} V(u_I-x;b)f(x)dx = 0.
$$

Applying $2\beta\frac{\partial}{\partial u_I}$, we have

$$
\begin{aligned}
&2\beta P_I V''(u_I;b) - 2\beta(\lambda+\delta)V'(u_I;b) \\
+ \;& 2\lambda\beta^3\int_0^{u_I} V(x;b)e^{-\beta(u_I-x)}dx - 2\lambda\beta^2\int_0^{u_I} V(x;b)f(u_I-x)dx = 0.
\end{aligned}
$$

Applying operator $\frac{\partial^2}{\partial u_I^2}$, we have

$$
\begin{aligned}
&P_I V^{(3)}(u_I;b) - (\lambda+\delta)V''(u_I;b) + \lambda\beta^2 V(u_I;b) \\
- \;& 2\lambda\beta^3\int_0^{u_I} V(x;b)e^{-\beta(u_I-x)}dx + \lambda\beta^2\int_0^{u_I} V(x;b)f(u_I-x)dx = 0.
\end{aligned}
$$

Hence, equation (5.8) can be written as (5.10).

Similarly, equation (5.9) can be written as (5.11) after applying operator

$$\frac{\partial^2}{\partial u_I^2} + 2\beta\frac{\partial}{\partial u_I} + \beta^2.$$

$\blacksquare$

5.3.3 Expected Discounted Dividend until Ruin for Erlang(2) Claims

Theorem 5.3.1 For an Erlang(2) claim with parameter β, threshold level b, and initial surplus u_I, the formula for the expected dividend before the possible ruin is as follows:

for $0 \le u_I \le b$,

$$V(u_I; b) = K\left(e^{\xi_1 u_I} + \frac{(\beta+\xi_2)^2(\xi_3-\xi_1)}{(\beta+\xi_1)^2(\xi_2-\xi_3)}e^{\xi_2 u_I} + \frac{(\beta+\xi_3)^2(\xi_1-\xi_2)}{(\beta+\xi_1)^2(\xi_2-\xi_3)}e^{\xi_3 u_I}\right); \quad (5.12)$$

and for $u_I \ge b$,

$$V(u_I; b) = C_4 e^{\xi_4 u_I} + C_5 e^{\xi_5 u_I} + \frac{D_c}{\delta}. \quad (5.13)$$

Here, $\xi_3 < \xi_2 < 0 < \xi_1$ are the roots of the characteristic equation

$$P_I x^3 + (2\beta P_I - \lambda - \delta)x^2 + (P_I\beta^2 - 2\beta(\lambda+\delta))x - \beta^2\delta = 0; \quad (5.14)$$

and $\xi_5 < \xi_4 < 0$ are the negative roots of the characteristic equation

$$\begin{aligned} & (P_I - D_c)x^3 + (2\beta(P_I - D_c) - (\lambda+\delta))\,x^2 \\ + \quad & \big((P_I - D_c)\beta^2 - 2\beta(\lambda+\delta)\big)\,x - \beta^2\delta = 0, \end{aligned} \quad (5.15)$$

Here

$$K = \frac{D_c \xi_4 \xi_5 (\beta + \xi_1)^2 (\xi_2 - \xi_3)}{\delta \beta^2 \Upsilon}, \tag{5.16}$$

$$C_4 = \frac{D_c \xi_5 \left(\xi_4 + \beta\right)^2 e^{-\xi_4 b} \tilde{C}_4}{\delta \beta^2 \left(\xi_5 - \xi_4\right) \Upsilon}, \tag{5.17}$$

$$C_5 = \frac{D_c \xi_4 \left(\xi_5 + \beta\right)^2 e^{-\xi_5 b} \tilde{C}_5}{\delta \beta^2 \left(\xi_4 - \xi_5\right) \Upsilon}, \tag{5.18}$$

where

$$\begin{aligned}
\tilde{C}_4 &= \xi_1 \left(\xi_5 - \xi_1\right) \left(\xi_2 - \xi_3\right) e^{\xi_1 b} + \xi_2 \left(\xi_5 - \xi_2\right) \left(\xi_3 - \xi_1\right) e^{\xi_2 b} + \xi_3 \left(\xi_5 - \xi_3\right) \left(\xi_1 - \xi_2\right) e^{\xi_3 b}, \\
\tilde{C}_5 &= \xi_1 \left(\xi_4 - \xi_1\right) \left(\xi_2 - \xi_3\right) e^{\xi_1 b} + \xi_2 \left(\xi_4 - \xi_2\right) \left(\xi_3 - \xi_1\right) e^{\xi_2 b} + \xi_3 \left(\xi_4 - \xi_3\right) \left(\xi_1 - \xi_2\right) e^{\xi_3 b}, \\
\Upsilon &= \left(\xi_5 - \xi_1\right) \left(\xi_4 - \xi_1\right) \left(\xi_2 - \xi_3\right) e^{\xi_1 b} + \left(\xi_5 - \xi_2\right) \left(\xi_4 - \xi_2\right) \left(\xi_3 - \xi_1\right) e^{\xi_2 b} \\
&\quad + \left(\xi_5 - \xi_3\right) \left(\xi_4 - \xi_3\right) \left(\xi_1 - \xi_2\right) e^{\xi_3 b}.
\end{aligned}$$

Here, D_c is the dividend ceiling, δ is the interest rate at which the dividend accumulated, λ is claim frequency, and P_I is the net premium received.

Proof:

Case I: the initial surplus is less than the threshold.

For the linear differential equation (5.10), the general solution is

$$h(u_I) = C_1 e^{\xi_1 u_I} + C_2 e^{\xi_2 u_I} + C_3 e^{\xi_3 u_I}, \tag{5.19}$$

where ξ_1, ξ_2, ξ_3 are the roots of the characteristic equation (5.14), and C_1, C_2, C_3 are constants. $V(u_I; b) = K h(u_I)$, where K does not depend on u_I.

For equation (5.14), the discriminant

$$\begin{aligned}\Delta &= 15\beta^4 P_I^2\lambda^2 - 4\lambda^4\beta^2 - 12\beta^3 P_I\lambda^3 - 4\beta^5 P_I^3\lambda \\ &\quad -12\beta^3 P_I\lambda\delta^2 - 24\beta^3 P_I\lambda^2\delta - 2\beta^4 P_I^2\lambda\delta - 12\lambda^3\beta^2\delta - 12\lambda^2\beta^2\delta^2 - 4\lambda\delta^3\beta^2 \\ &= -\beta^2\lambda\left(4\beta P_I + \lambda\right)\left(-2\lambda + \beta P_I\right)^2 \\ &\quad -4\beta^2\lambda\delta\left(3\beta P_I\delta + 6\beta P_I\lambda + 3\beta^2 P_I^2 + 3\lambda^2 + 3\lambda\delta + \delta^2\right),\end{aligned}$$

is always less than 0. From Lemma 5.3.2, the equation has three different real roots.

Moreover, for this equation, the roots have two situations, either all roots are positive or there are two negative roots and one positive root. Under the first scenario, there should be: $2\beta P_I < \lambda + \delta$ and $\beta P_I > \beta(\lambda + \delta)$, which is impossible. Hence, the equation (5.14) must have three real roots, two of them are negative and the remaining one is positive. Here assume that $\xi_3 < \xi_2 < 0 < \xi_1$.

Putting equation (5.19) back to the equation (5.8), we have

$$\begin{aligned}& P_I\left(\xi_1 C_1 e^{\xi_1 u_I} + \xi_2 C_2 e^{\xi_2 u_I} + \xi_3 C_3 e^{\xi_3 u_I}\right) - (\lambda + \delta)\left(C_1 e^{\xi_1 u_I} + C_2 e^{\xi_2 u_I} + C_3 e^{\xi_3 u_I}\right) \\ + & \frac{\lambda C_1\beta^2}{(\beta + \xi_1)^2}(1 - \beta u_I e^{-\beta u_I} - \xi_1 u_I e^{-\beta u_I} - e^{-\beta u_I}) \\ + & \frac{\lambda C_2\beta^2}{(\beta + \xi_2)^2}(1 - \beta u_I e^{-\beta u_I} - \xi_2 u_I e^{-\beta u_I} - e^{-\beta u_I}) \\ + & \frac{\lambda C_3\beta^2}{(\beta + \xi_3)^2}(1 - \beta u_I e^{-\beta u_I} - \xi_3 u_I e^{-\beta u_I} - e^{-\beta u_I}) = 0.\end{aligned}$$

Equating the coefficient of $e^{-\beta u_I}$ with 0, we have

$$\frac{C_1}{(\beta + \xi_1)^2} + \frac{C_2}{(\beta + \xi_2)^2} + \frac{C_3}{(\beta + \xi_3)^2} = 0;$$

and equating the coefficient of $u_I e^{-\beta u_I}$ with 0, we have

$$\frac{C_1}{\beta + \xi_1} + \frac{C_2}{\beta + \xi_2} + \frac{C_3}{\beta + \xi_3} = 0.$$

After calculation, there is

$$C_2 = -\frac{(\beta+\xi_2)^2(\xi_1-\xi_3)}{(\beta+\xi_1)^2(\xi_2-\xi_3)}C_1,$$

and

$$C_3 = \frac{(\beta+\xi_3)^2(\xi_1-\xi_2)}{(\beta+\xi_1)^2(\xi_2-\xi_3)}C_1.$$

Hence, for $u \le b$,

$$V(u_I;b) = K\left(e^{\xi_1 u_I} + \frac{(\beta+\xi_2)^2(\xi_3-\xi_1)}{(\beta+\xi_1)^2(\xi_2-\xi_3)}e^{\xi_2 u_I} + \frac{(\beta+\xi_3)^2(\xi_1-\xi_2)}{(\beta+\xi_1)^2(\xi_2-\xi_3)}e^{\xi_3 u_I}\right).$$

Note, in the limiting case $D_c = P_I$, we have $V'(b^-;b) = 1$ and the result is

$$V(u_I;b) = \frac{\overline{V}}{\underline{V}}, \tag{5.20}$$

where

$$\begin{aligned}
\overline{V} &= (\beta+\xi_1)^2(\xi_2-\xi_3)e^{\xi_1 u_I} + (\beta+\xi_2)^2(\xi_3-\xi_1)e^{\xi_2 u_I} + (\beta+\xi_3)^2(\xi_1-\xi_2)\,e^{\xi_3 u_I},\\
\underline{V} &= \xi_1(\beta+\xi_1)^2(\xi_2-\xi_3)e^{\xi_1 u_I} + \xi_2(\beta+\xi_2)^2(\xi_3-\xi_1)e^{\xi_2 u_I} + \xi_3(\beta+\xi_3)^2(\xi_1-\xi_2)\,e^{\xi_3 u_I}.
\end{aligned}$$

Case II: the initial surplus is greater than the threshold.

The general solution for (5.11) is

$$V(u_I;b) = \sum_{i=4}^{6} C_i e^{\xi_i u_I} + \sum_{i=4}^{6} \frac{e^{\xi_i u_I}}{P'(\xi_i)} \int \beta^2 D_c e^{-\xi_i u_I} du_I,$$

where

$$P(x) = (P_I - D_c)x^3 + (2\beta(P_I - D_c) - (\lambda+\delta))\,x^2 + \big((P_I - D_c)\beta^2 - 2\beta(\lambda+\delta)\big)\,x - \beta^2\delta.$$

ξ_4, ξ_5, and ξ_6 are the real roots of the character equation $P(x) = 0$. Let $\xi_5 < \xi_4 < 0$ be the two negative roots and ξ_6 be the positive root.

Lemma 5.3.1 provides

$$\sum_{i=4}^{6} \frac{e^{\xi_i u_I}}{P'(\xi_i)} \int \beta^2 D_c e^{-\xi_i u_I} du = \frac{\beta^2 D_c}{\beta^2 \delta} = \frac{D_c}{\delta}.$$

Since $\lim_{u_I \to \infty} V(u_I; b) = \frac{D_c}{\delta}$, the coefficient for ξ_6, can not be greater than zero, which means $C_6 = 0$.

Hence, for $u_I > b$,

$$V(u_I; b) = C_4 e^{\xi_4 u_I} + C_5 e^{\xi_5 u_I} + \frac{D_c}{\delta}.$$

From the continuity condition, $V(b-; b) = V(b+; b)$, we have

$$\begin{aligned} & K\left(e^{\xi_1 b} + \frac{(\beta+\xi_2)^2(\xi_3-\xi_1)}{(\beta+\xi_1)^2(\xi_2-\xi_3)} e^{\xi_2 b} + \frac{(\beta+\xi_3)^2(\xi_1-\xi_2)}{(\beta+\xi_1)^2(\xi_2-\xi_3)} e^{\xi_3 b}\right) \\ = \;& C_4 e^{\xi_4 b} + C_5 e^{\xi_5 b} + \frac{D_c}{\delta}. \end{aligned} \tag{5.21}$$

For the convolution integral in equation (5.9), i.e., the integration part can be expressed as

$$\begin{aligned} & \int_0^{u_I} V(x; b) f(u_I - x) dx \\ = \;& K \int_0^b \left(e^{\xi_1 x} + \frac{(\beta+\xi_2)^2(\xi_3-\xi_1)}{(\beta+\xi_1)^2(\xi_2-\xi_3)} e^{\xi_2 x} + \frac{(\beta+\xi_3)^2(\xi_1-\xi_2)}{(\beta+\xi_1)^2(\xi_2-\xi_3)} e^{\xi_3 x}\right) \\ & \times \beta^2 (u_I - x) e^{-\beta(u_I - x)} dx \\ & + \int_b^{u_I} \left(C_4 e^{\xi_4 x} + C_5 e^{\xi_5 x} + \frac{D_c}{\delta}\right) \beta^2 (u_I - x) e^{-\beta(u_I - x)} dx. \end{aligned}$$

By setting the coefficients of $u_I e^{-\beta u_I}$ and $e^{-\beta u_I}$ to zero, we have

$$\begin{aligned} & K\left(\frac{\beta}{\beta+\xi_1} e^{\xi_1 b} + \frac{\beta(\beta+\xi_2)(\xi_3-\xi_1)}{(\beta+\xi_1)^2(\xi_2-\xi_3)} e^{\xi_2 b} + \frac{\beta(\beta+\xi_3)(\xi_1-\xi_2)}{(\beta+\xi_1)^2(\xi_2-\xi_3)} e^{\xi_3 b}\right) \\ = \;& C_4 \beta \frac{e^{\xi_4 b}}{\xi_4 + \beta} + C_5 \beta \frac{e^{\xi_5 b}}{\xi_5 + \beta} + \frac{D_c}{\delta}, \end{aligned} \tag{5.22}$$

and

$$
\begin{aligned}
& K\Big(\frac{\beta^2\left(b\beta+b\xi_1-1\right)e^{\xi_1 b}}{\left(\beta+\xi_1\right)^2}+\frac{\beta^2(\xi_3-\xi_1)\left(b\beta+b\xi_2-1\right)e^{\xi_2 b}}{(\beta+\xi_1)^2(\xi_2-\xi_3)} \\
& +\frac{\beta^2\left(\xi_1-\xi_2\right)\left(b\beta+b\xi_3-1\right)e^{\xi_3 b}}{(\beta+\xi_1)^2(\xi_2-\xi_3)}\Big) \\
= \; & C_4\frac{\beta^2\left(b\xi_4+b\beta-1\right)e^{\xi_4 b}}{\left(\xi_4+\beta\right)^2}+C_5\frac{\beta^2\left(b\xi_5+b\beta-1\right)e^{\xi_5 b}}{\left(\xi_5+\beta\right)^2}+\frac{D_c}{\delta}\left(b\beta-1\right); \quad (5.23)
\end{aligned}
$$

since

$$
\begin{aligned}
& \int_0^b e^{\xi_1 x}\beta^2(u_I-x)e^{-\beta(u_I-x)}dx \\
= \; & \beta^2\Big(\frac{u_I(\beta+\xi_1)e^{b\beta+\xi_1 b-\beta u_I}-u_I\left(\beta+\xi_1\right)e^{-\beta u_I}}{\left(\beta+\xi_1\right)^2} \\
& +\frac{-b(\beta+\xi_1)e^{b\beta+\xi_1 b-\beta u_I}+e^{b\beta+\xi_1 b-\beta u_I}-e^{-\beta u_I}}{\left(\beta+\xi_1\right)^2}\Big);
\end{aligned}
$$

$$
\begin{aligned}
& \int_b^{u_I} e^{\xi_4 x}\beta^2(u_I-x)e^{-\beta(u_I-x)}dx \\
= \; & \beta^2\Big(\frac{-u_I e^{\xi_4 b+b\beta-\beta u_I}\left(\xi_4+\beta\right)+be^{\xi_4 b+b\beta-\beta u_I}\left(\xi_4+\beta\right)}{\left(\xi_4+\beta\right)^2} \\
& +\frac{-e^{\xi_4 b+b\beta-\beta u_I}+e^{u_I\xi_4}}{\left(\xi_4+\beta\right)^2}\Big);
\end{aligned}
$$

and

$$
\int_b^{u_I}\beta^2(u_I-x)e^{-\beta(u_I-x)}dx=1-\beta u_I e^{b\beta-u_I\beta}+b\beta e^{b\beta-u_I\beta}-e^{b\beta-u_I\beta}.
$$

Note that the relationship

$$
1+\frac{(\xi_3-\xi_1)}{(\xi_2-\xi_3)}+\frac{(\xi_1-\xi_2)}{(\xi_2-\xi_3)}=0,
$$

and

$$1+\frac{(\beta+\xi_2)(\xi_3-\xi_1)}{(\beta+\xi_1)(\xi_2-\xi_3)}+\frac{(\beta+\xi_3)(\xi_1-\xi_2)}{(\beta+\xi_1)(\xi_2-\xi_3)}=0,$$

is used in the above calculations.

From (5.21), (5.22) and (5.23), K, C_4, C_5 are the solutions of $A^{-1}B$, where A is a 3×3 matrix with

$$\begin{aligned}
A_{11} &= \left(e^{b\xi_1}+\frac{(\beta+\xi_2)^2(\xi_3-\xi_1)}{(\beta+\xi_1)^2(\xi_2-\xi_3)}e^{b\xi_2}+\frac{(\beta+\xi_3)^2(\xi_1-\xi_2)}{(\beta+\xi_1)^2(\xi_2-\xi_3)}e^{b\xi_3}\right),\\
A_{21} &= \frac{\beta}{\beta+\xi_1}\left(e^{b\xi_1}+\frac{(\beta+\xi_2)(\xi_3-\xi_1)}{(\beta+\xi_1)(\xi_2-\xi_3)}e^{b\xi_2}+\frac{(\beta+\xi_3)(\xi_1-\xi_2)}{(\beta+\xi_1)(\xi_2-\xi_3)}e^{b\xi_3}\right),\\
A_{31} &= \frac{\beta^2}{(\beta+\xi_1)^2}\\
&\times \left((b\beta+b\xi_1-1)e^{b\xi_1}+\frac{(\xi_3-\xi_1)(b\beta+b\xi_2-1)}{(\xi_2-\xi_3)}e^{b\xi_2}+\frac{(\xi_1-\xi_2)(b\beta+b\xi_3-1)}{(\xi_2-\xi_3)}e^{b\xi_3}\right),
\end{aligned}$$

$$A_{*2}=\begin{pmatrix} -e^{b\xi_4}\\ -\frac{\beta}{\beta+\xi_4}e^{b\xi_4}\\ -\frac{\beta^2(b\beta+b\xi_4-1)}{(\beta+\xi_4)^2}e^{b\xi_4}\end{pmatrix},$$

$$A_{*3}=\begin{pmatrix} -e^{\xi_5 b}\\ -\frac{\beta}{\beta+\xi_5}e^{b\xi_5}\\ -\frac{\beta^2(b\beta+b\xi_5-1)}{(\beta+\xi_5)^2}e^{b\xi_5}\end{pmatrix},$$

and

$$B=\begin{pmatrix} \frac{D_c}{\delta}\\ \frac{D_c}{\delta}\\ \frac{D_c(\beta b-1)}{\delta}\end{pmatrix}.$$

After tedious calculation, the results for K, C_4, and C_5 as (5.16), (5.17), and (5.18) are produced.

When $u_I = b = 0$, we have

$$V(0;0) = \frac{D_c \xi_4 \xi_5}{\delta \beta^2}.$$

Hence, the theory set forth here is proven. Note that, if the formula 4.3 of Cheung et al. (2008) is simplified, the identical results occur through these two different approaches. ■

With the Erlang(2) distribution, the claim after quota-share reinsurance is still an Erlang(2) distribution. To calculate the expected claim after reinsurance, we can simply use $V_a(u_I;b)$ to replace $V(u_I;b)$; and $\frac{\beta}{a}$ to replace β—all the results follow.

5.3.4 Optimal Quota-Share Limit for Erlang (2) Claims

Assume Erlang (2) has the same average claim as the exponential example that is used in the previous section, using the average claim size 100, which denotes $\beta = 0.02$. In the base scenario, all parameters are the same: interest rate $\delta = 5\%$, reinsurance loading $\theta_R = 0.2$, insurance loading $\theta_I = 0.1$, claim frequency $\lambda = 1$, initial surplus $u_I = 100$, and dividend ceiling $D_c = 5$.

Using MATLAB to program the problem for the base scenario as well as the other six scenarios described for an exponential distribution, it is not surprising that both distributions have the same pattern. An exponential distribution is considered as an Erlang (1) distribution, and hence the two claim distributions belong to the same family. All the explanations used for the exponential claim are applied to Erlang (2), and they have the same underlying reinsurance meaning.

Table 5.2 provides the base line scenario, and Figure 5.3 and Figure 5.4 demonstrate the optimal quota-share level a and the largest expected dividend under different scenarios respectively. See attached Excel sheets for complete numerical results.

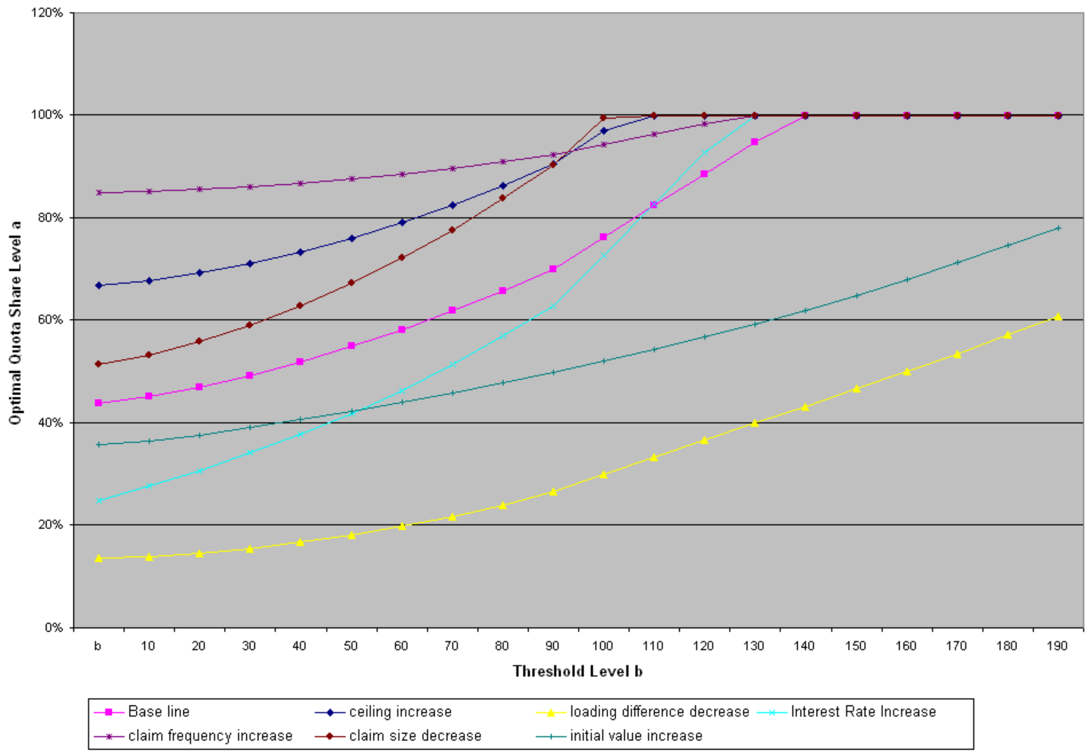

Figure 5.3: Optimal Quota-share Level in the Compound Poisson Process with Erlang(2) Claim

Table 5.2: Optimal Quota-Share Level for Erlang (2) Claims – Baseline

Threshold Level b	Optimal Quota-share a	Largest Expected Dividend $V_a(u_I; b)$
10	44%	40.32
20	45%	40.11
30	47%	39.78
40	49%	39.36
50	52%	38.87
60	55%	38.32
70	58%	37.74
80	62%	37.14
90	66%	36.53
100	70%	35.94
110	76%	34.90
120	82%	33.99
130	89%	33.19
140	95%	32.47
150	100%	31.82
160	100%	31.20
170	100%	30.59
180	100%	29.99
190	100%	29.40
200	100%	28.82

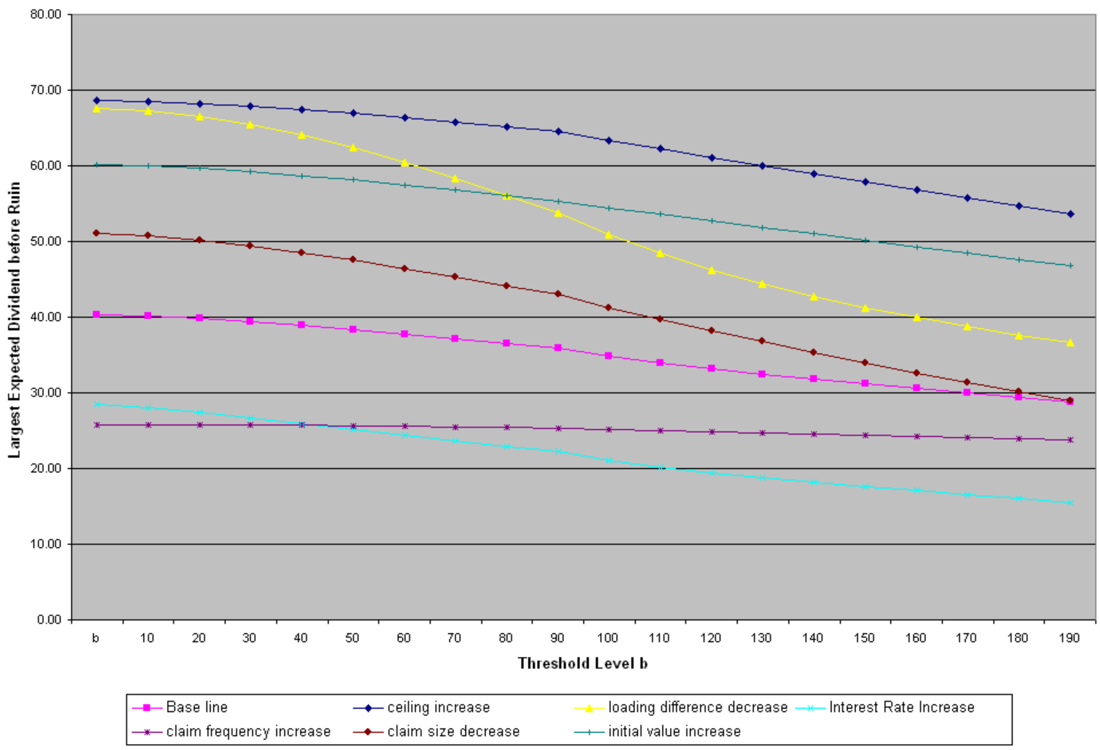

Figure 5.4: Maximum of Expected Discounted Dividend before Ruin in the Compound Poisson Process with Erlang(2) Claims

Chapter 6

Optimal Retentions with Commissions, Expenses, and Diffusion

The last part of this thesis focuses on a compound Poisson model with a diffusion process included, which represents the effects of business uncertainty in the surplus process. This chapter also includes commissions and expenses in an effort to mirror the real business world.

6.1 Jump-diffusion Risk Model with Commissions and Expenses

The risk model considered in this chapter is described as follows.

1. The distribution function of claim amount, $F(X)$, satisfies the following:

    - $F(0) = 0$; $0 \leq F(x) < 1$ for $0 < x < +\infty$;
    - $\frac{dF(x)}{dx}$ exists and is continuous;
    - $M_X(r)$ (moment generating function of X) exists for $r \in (-\infty, \tau)$ for some $0 < \tau \leq +\infty$ and $\lim_{r\to\tau} M_X(r) = \lim_{r\to\tau} E[e^{rX}] = +\infty$.

2. The number of claims is given by the Poisson process with parameter λ, which indicates that the inter claim time T has an exponential distribution and the claim frequency is λ.

3. The reinsurance treaty can be written as follows:

    - P – the insurer's gross premium income per unit of time.
    - α – positive loading coefficient.
    - e – insurer's expense rate.
    - c – commission payment rate. The reinsurer will pay the commission back to the insurer according to the business volume, which is $c(1-a)P$. Here, we assume that the insurer cannot reinsure the whole risk with a certain profit, i.e.,

    $$e > c$$

    and

    $$(1-e)P - (1+\alpha)\lambda\mu < 0.$$

    Relative safety loading can be written as

    $$\rho = \frac{(1-e)P}{\lambda\mu} - 1,$$

    which is positive.

- The premium paid to the reinsurer is

$$P_R = (1-c)(1-a)P + (1+\alpha)\lambda \int_{M/a}^{\infty} (ax - M)dF(x).$$

- The net premium remaining for the ceding company is

$$P_I = (1-e)P - P_R.$$

4. The surplus process is affected by a diffusion process $\{W(t)\}$, which is a Wiener process with infinitesimal drift of 0 and infinitesimal variance of $2D > 0$. It is independent of claim time and claim size. Further, $W(t) \sim N(0, 2Dt)$ for any $t > 0$. The surplus process in the reinsurance with the commissions, expenses, and diffusion is expressed as

$$U_{a,M}(t) = u + P_I t - \sum_{i=1}^{N(t)} \min(aX_i, M) + W(t), \tag{6.1}$$

where $u > 0$ is the initial surplus of the insurer.

Note, for $W \sim N(\mu, \sigma^2)$, the moment generating function is

$$M(r) = E[e^{rW}] = e^{\mu r + \sigma^2 r^2/2}.$$

5. Let $J(a, M)$ be the insurer's net profit after the reinsurance and recall

$$X_I = \min(aX, M).$$

The insurer's expected net profit per period of time (after reinsurance and expenses) is expressed as

$$E[J(a, M)] = P_I - \lambda E[X_I].$$

The infinite-time ruin probability in the model (6.1) is defined as

$$\psi_{a,M}(u) = \Pr\{U_{a,M}(t) < 0 \ \text{ for some } \ t > 0\},$$

which is the probability that the surplus of the insurer will be negative eventually.

The finite-time ruin probability in the model (6.1) is defined as

$$\psi_{a,M}(u,t) = \Pr\{U_{a,M}(s) < 0 \ \text{ for some } \ 0 < s \leq t\},$$

which is the probability that the surplus of the insurer will be negative in the time period $(0,\, t]$.

It is difficult to determine the explicit formulas for the infinite-time and finite-time ruin probabilities, which means that it is not possible to find the optimal retention levels a and M by minimizing the ruin probabilities directly. However, when certain conditions are applied, the upper bounds of the ruin probabilities exist. By discovering the minimum upper bounds, we can determine the optimal retention levels a and M and the ruin probabilities can be limited so that the risk does not exceed a certain limit.

Centeno (2002(a), 2002(b)) studied the optimal retention levels a and M in the compound Poisson risk model. In this chapter, we extend the model and results of Centeno (2002(a), 2002(b)) to the jump-diffusion risk model or the compound Poisson risk model with diffusion as described in (6.1).

6.2 Optimal Retentions by Minimizing Lundberg Upper Bound for Infinite-Time Ruin Probability

Motivated by Dufresne and Gerber's model (1991), in the risk model (6.1), for the given (a, M), the adjustment coefficient denoted by $R_{a,M}$ is the unique positive root of

$$\lambda E[e^{rX_I}] + Dr^2 = \lambda + P_I r,$$

when such a root exists, or zero otherwise.

Here, the goal is for M and a to maximize the $R_{a,M}$, so that the Lundberg upper bound for the infinite-time ruin probability, which is

$$\psi_{a,M}(u) \leq e^{-R_{a,M}u},$$

is minimized.

Because commission, expense and business uncertainty are included in the model, it is complicated to maximize $R_{a,M}$. We have to introduce a new set of definitions and preliminaries in this section about the properties of the adjustment coefficient $R_{a,M}$ and the other parameters.

Let us review some useful results before proceeding to the next step. To obtain the optimal quota-share and retention levels, the process involves numerous derivative calculations. The following formulas will be used in the work. Some are fairly tedious and for this reason, details are omitted.

6.2.1 Preliminaries

Lemma 6.2.1 The formula set for the partial derivatives of the reinsurance premium is

$$\begin{aligned}
\frac{\partial P_R}{\partial M} &= -(1+\alpha)\lambda(1-F(\frac{M}{a})) = -\frac{\partial P_I}{\partial M};\\
\frac{\partial^2 P_R}{\partial M^2} &= \frac{1}{a}(1+\alpha)\lambda f(\frac{M}{a}) = -\frac{\partial^2 P_I}{\partial M^2};\\
\frac{\partial P_R}{\partial a} &= -(1-c)P + (1+\alpha)\lambda\int_{M/a}^{\infty} x dF(x) = -\frac{\partial P_I}{\partial a};\\
\frac{\partial^2 P_R}{\partial a^2} &= (1+\alpha)\lambda\frac{M^2}{a^3} f(\frac{M}{a}) = -\frac{\partial^2 P_I}{\partial a^2};\\
\frac{\partial^2 P_R}{\partial a \partial M} &= -(1+\alpha)\lambda\frac{M}{a^2} f(\frac{M}{a}) = -\frac{\partial^2 P_I}{\partial a \partial M}.
\end{aligned}$$

Since the moment generating function of X_I can be written as

$$\begin{aligned}
E[e^{rX_I}] &= \int_0^{M/a} e^{rax} dF(x) + \int_{M/a}^{\infty} e^{rM} dF(x)\\
&= \int_0^{M/a} e^{rax} dF(x) + e^{rM}[1-F(\frac{M}{a})],
\end{aligned}$$

the partial derivative set for $E[e^{rX_I}]$ is

$$\begin{aligned}
\frac{\partial E[e^{rX_I}]}{\partial M} &= re^{rM}[1-F(\frac{M}{a})];\\
\frac{\partial^2 E[e^{rX_I}]}{\partial M^2} &= r^2 e^{rM}[1-F(\frac{M}{a})] - \frac{1}{a} re^{rM} f(\frac{M}{a});\\
\frac{\partial E[e^{rX_I}]}{\partial a} &= r\int_0^{M/a} x e^{rax} dF(x);\\
\frac{\partial^2 E[e^{rX_I}]}{\partial a^2} &= -r\frac{M^2}{a^3} e^{rM} f(\frac{M}{a}) + r^2\int_0^{M/a} x^2 e^{rax} dF(x);\\
\frac{\partial^2 E[e^{rX_I}]}{\partial a \partial M} &= \frac{rM}{a^2} e^{rM} f(\frac{M}{a}).
\end{aligned}$$

■

Centeno (1985) has proven the following lemma:

Lemma 6.2.2 Let $A = \{a : 0 < a \leq 1\ \}$ and suppose that there exists an M such that $E[J(a, M)] = 0\}$ and $a_0 = \frac{(e-c)P}{(1-c)P-\lambda E(X)}$. Then the following results hold:

1. $A = (a_0, 1]$;
2. For each $a \in A$, there exists a unique M such that $E[J(a, M)] = 0$, i.e. there is a function Φ mapping A into $(0, \infty)$, such that $M = \Phi(a)$ is equivalent to
$$E[J(a, M)] = 0;$$
3. $\Phi(a)$ is convex;
4. $\lim_{a \to a_0} \Phi(a) = +\infty$.

Lemma 6.2.3 $R_{a,M}$ is the one and the only one positive solution of

$$\lambda E[e^{rX_I}] + Dr^2 = \lambda + P_I r. \tag{6.2}$$

Proof: The function can be rewritten as

$$\lambda E[e^{rX_I}] = -Dr^2 + P_I r + \lambda.$$

Note, with respect to r, using the results of Lemma 1 from Centeno (2002), $E[e^{rX_I}]$ is a non-decreasing convex function, and the left side of the function is equal to λ at $r = 0$. On the other hand, the right side of the function is a concave function which is equal to λ at $r = 0$, which implies that the equation holds when $r = 0$. Figure 6.1 demonstrates that the two lines have one, and the only one, positive intersection point.

■

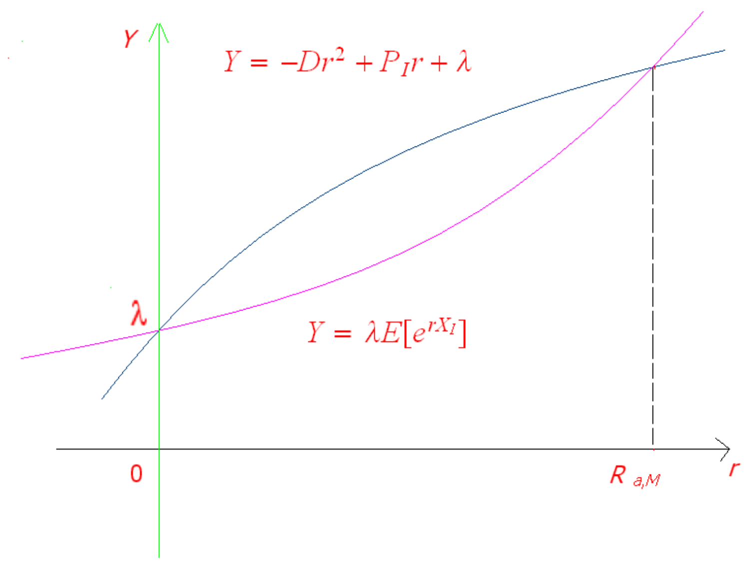

Figure 6.1: Unique Positive Solution of r

Lemma 6.2.4 The adjustment coefficient $R_{a,M}$ is positive if and only if $(a, M) \in L$, where L is the set of points for which the insurer's net expected profit is positive, i.e.,

$$L = \{(a, M) : 0 \leq a \leq 1, M \geq 0 \text{ and } E[J(a, M)] > 0\}.$$

And for any $(a, M) \in L$, $H'_{a,M}(r)$ is positive at $r = R_{a,M}$, where

$$H_{a,M}(r) = \lambda E[e^{rX_I}] + Dr^2 - P_I r - \lambda.$$

Proof: Let

$$\xi = \begin{cases} +\infty & \text{if } M < +\infty \\ \tau & \text{for } M = +\infty \end{cases},$$

where $M = +\infty$ means no excess-of-loss reinsurance.

From the discussion in Lemma 6.2.3, $H_{a,M}(r)$ is a convex function. Note,

$$\begin{cases} H_{a,M}(0) = 0, \\ \lim H_{a,M}(r)_{r\to\xi} = +\infty. \end{cases},$$

implies that the adjustment coefficient is positive if and only if $H'_{a,M}(0) < 0$.

Moreover, note that

$$\begin{aligned} \frac{\partial H_{a,M}(r)}{\partial r}|_{r=0} &= \lambda E[X_I e^{rX_I}] + 2Dr - P_I \,|_{r=0} \\ &= E[X_I] - P_I E[T] < 0, \end{aligned}$$

hence,

$$E[J(a,M)] = P_I - \lambda E[X_I] = -\frac{\partial H_{a,M}(r)}{\partial r}|_{r=0} > 0$$

is proved.

Furthermore, for any $(a, M) \in L$, $H'_{a,M}(r)$ is positive at $r = R_{a,M}$.

Figure 6.2 illustrates the proof.

■

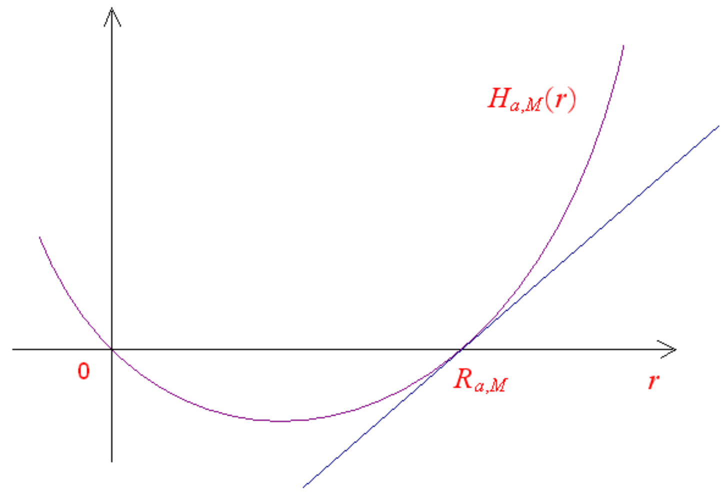

Figure 6.2: Illustration for Proof of Lemma 6.2.4

Given Lemma 6.2.2, it is equivalent to say that the adjustment coefficient is positive if and only if $a > a_0$ and $M > \Phi(a)$.

6.2.2 Adjustment Coefficient as a Function of Retention

Theorem 6.2.1 For a fixed value of $a \in (a_0, 1]$, $R_{a,M}$ is a unimodal function of M, attaining its maximum value at the only point satisfying

$$M = \frac{1}{R_{a,M}} \ln(1+\alpha), \tag{6.3}$$

where $R_{a,M}$ is the only positive solution of (6.2).

Proof: Recall, in the implicit function theorem, if y is an implicit function of x in the form $F(x,y) = 0$, the general formula holds:

$$\frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y}.$$

Hence, we have

$$\frac{\partial R_{a,M}}{\partial M} = -\frac{(\partial/\partial M)H_{a,M}(r)}{(\partial/\partial r)H_{a,M}(r)}|_{r=R_{a,M}}$$

and

$$\frac{\partial^2 R_{a,M}}{\partial M^2}|_{\frac{\partial R_{a,M}}{\partial M}=0} = -\frac{(\partial^2/\partial M^2)H_{a,M}(r)}{(\partial/\partial r)H_{a,M}(r)}|_{r=R_{a,M},\frac{\partial R_{a,M}}{\partial M}=0}.$$

Note, $\frac{\partial H_{a,M}(r)}{\partial r}|_{r=R_{a,M}} > 0$, from Lemma 6.2.4. We want to find out M such that

$$\frac{\partial R_{a,M}}{\partial M} = 0,$$

where

$$\frac{\partial^2 R_{a,M}}{\partial M^2}|_{\frac{\partial R_{a,M}}{\partial M}=0} < 0.$$

It is equivalent to ascertain M satisfying

$$\frac{\partial H_{a,M}(r)}{\partial M}|_{r=R_{a,M}} = 0$$

and

$$\frac{\partial^2 H_{a,M}(r)}{\partial M^2}|_{r=R_{a,M},\frac{\partial R_{a,M}}{\partial M}=0} > 0.$$

For the first partial derivative with respect to M, we have

$$\begin{aligned}
&\frac{\partial H_{a,M}(r)}{\partial M} \\
= \;& \lambda\frac{\partial E\left[e^{rX_I}\right]}{\partial M} - r\frac{\partial P_I}{\partial M} \\
= \;& \lambda r e^{rM}\left(1 - F(\frac{M}{a})\right) - (1+\alpha)\lambda r\left(1 - F(\frac{M}{a})\right) \\
= \;& \lambda r\left(1 - F(\frac{M}{a})\right)\left(e^{rM} - (1+\alpha)\right).
\end{aligned}$$

If

$$\frac{\partial R_{a,M}}{\partial M} = -\frac{(\partial/\partial M)H_M(r)}{(\partial/\partial r)H_M(r)}|_{r=R_M} = 0,$$

then M should satisfy

$$M = \frac{1}{R_{a,M}} \ln(1+\alpha),$$

which is (6.3). There must be such a point, because:

(i) When $r \to 0$, i.e., when the net profit is zero,

$$\lim_{r \to 0} \left\{ e^{rM} - (1+\alpha) \right\} = -\alpha.$$

(ii) When $r \to \infty$, i.e., the adjustment coefficient before the excess-of-loss reinsurance takes place

$$\lim_{r \to \infty} \left\{ e^{rM} - (1+\alpha) \right\} = \infty.$$

Also, for the second derivative with respect to M,

$$\begin{aligned}
& \frac{\partial^2 H_{a,M}(r)}{\partial M^2} \\
= \; & \lambda \frac{\partial^2 E\left[e^{rX_I}\right]}{\partial M^2} - r \frac{\partial^2 P_I}{\partial M^2} \\
= \; & \lambda \left(r^2 e^{rM} \left(1 - F(\frac{M}{a}) \right) - \frac{1}{a} r e^{rM} f(\frac{M}{a}) \right) + \frac{r}{a}(1+\alpha)\lambda f(\frac{M}{a}) \\
= \; & \lambda r^2 e^{rM} \left(1 - F(\frac{M}{a}) \right) + \frac{\lambda r}{a} f(\frac{M}{a}) \left((1+\alpha) - e^{rM} \right).
\end{aligned}$$

When M satisfies (6.3), we have

$$\frac{\partial^2 H_{a,M}(r)}{\partial M^2}\Big|_{r=R_{a,M}, \frac{\partial R_{a,M}}{\partial M}=0} = \lambda r^2 e^{rM} \left(1 - F(\frac{M}{a}) \right) > 0.$$

This indicates

$$\frac{\partial^2 R_{a,M}}{\partial M^2}\Big|_{\frac{\partial R_M}{\partial M}=0} = -\frac{(\partial^2/\partial M^2) H_{a,M}(r)}{(\partial/\partial r) H_{a,M}(r)}\Big|_{r=R_{a,M}, \frac{\partial R_{a,M}}{\partial M}=0} < 0.$$

Hence, the second derivative with respect to M of $R_{a,M}$ is negative when the first derivative is zero, which implies that for fixed $a \in (a_0, 1]$, $R_{a,M}$ has at most one turning point, and when such a point exists, it is a maximum. The maximum will exist and be finite at the only point satisfying

$$M = \frac{1}{R_{a,M}} \ln(1+\alpha).$$

■

Theorem 6.2.2 When the equation $\frac{\partial H_{a,M}(r)}{\partial M} = 0$ holds, M can be defined as a function of a, let it be $\Upsilon(a)$. Let

$$\hat{R}_a = \max_M(R_{a,M}) = R_{a,\Upsilon(a)},$$

then $\hat{R}_a$ is a unimodal function of a for $a \in (a_0, 1]$, and it attains its maximum at $a = 1$, if and only if $\lim_{a\to 1^-} \frac{d}{da}\hat{R}_a \geq 0$.

Proof: First we have

$$\frac{d\hat{R}_a}{da} = -\frac{(\partial/\partial a)H_{a,M}(r)}{(\partial/\partial r)H_{a,M}(r)}|_{M=\Upsilon(a),r=\hat{R}_a},$$

and

$$\begin{aligned} &\frac{d^2\hat{R}_a}{da^2}|_{\frac{d\hat{R}_a}{da}=0} \\ = \quad & -\frac{(\partial^2/\partial a^2)H_{a,M}(r)\times(\partial^2/\partial M^2)H_{a,M}(r) - [(\partial^2/\partial a\partial M)H_{a,M}(r)]^2}{(\partial/\partial r)H_{a,M}(r)\times(\partial^2/\partial M^2)H_{a,M}(r)}|_{M=\Upsilon(a),r=\hat{R}_a,\frac{d\hat{R}_a}{da}=0}. \end{aligned}$$

It has been previously proven that

$$\frac{\partial^2 H_{a,M}(r)}{\partial M^2}|_{M=\Upsilon(a),r=\widehat{R}_a} > 0,$$

and

$$\frac{\partial H_{a,M}(r)}{\partial r}|_{M=\Upsilon(a),r=\widehat{R}_a} > 0.$$

Similarly, the partial derivation with respect to a produces

$$
\begin{aligned}
\frac{\partial H_{a,M}(r)}{\partial a} &= \lambda r \int_0^{M/a} x e^{rax} dF(x) + (1+\alpha)\lambda r \int_{M/a}^{\infty} x dF(x) - (1-c)rP > 0; \\
\frac{\partial^2 H_{a,M}(r)}{\partial a^2} &= \lambda r^2 \int_0^{M/a} x^2 e^{rax} dF(x) + \frac{\lambda r M^2}{a^3} f(\frac{M}{a}) \left((1+\alpha) - e^{rM}\right); \\
\frac{\partial^2 H_{a,M}(r)}{\partial a \partial M} &= \frac{\lambda r M}{a^2} f(\frac{M}{a}) \left(e^{rM} - (1+\alpha)\right).
\end{aligned}
$$

Since M satisfies (6.3), this results in

$$
\begin{aligned}
& (\partial^2/\partial a^2) H_{a,M}(r) \times (\partial^2/\partial M^2) H_{a,M}(r)|_{r=R_{a,M};\frac{\partial}{\partial a}H_{a,M}(r)=0;\frac{\partial}{\partial a}H_{a,M}(r)=0} \\
- & [(\partial^2/\partial a \partial M) H_{a,M}(r)]^2|_{r=R_{a,M};\frac{\partial}{\partial a}H_{a,M}(r)=0;\frac{\partial}{\partial a}H_{a,M}(r)=0} \\
= & \lambda^2 r^4 e^{rM} \left(1 - F(\frac{M}{a})\right) \int_0^{M/a} x^2 e^{rax} dF(x) > 0,
\end{aligned}
$$

which indicates

$$\frac{d^2 \widehat{R}_a}{da^2}|_{\frac{d\widehat{R}_a}{da}=0} < 0.$$

Conversely, when $a \to a_0$, $\hat{R}_a$ goes to zero and we can demonstrate that the maximum of $\hat{R}_a$ is 1, if and only if $\lim_{a \to 1^-} \frac{d}{da}\hat{R}_a \geq 0$. Hence, the result is proven. ■

6.2.3 Optimal Retentions with Exponential Claims

Let the individual claim amount distribution be exponential with mean $1/\beta$, i.e.,

$$F(x) = 1 - e^{-\beta x}$$

and

$$f(x) = \beta e^{-\beta x}.$$

Here $\mu = 1/\beta$.

Let $\alpha = 0.8$, $c = 0.2$, $e = 0.3$, $P = 1.6$, $\beta = 1$, $\lambda = 1$, and $D = 0.02$. The parameters satisfy $e > c$ and

$$\rho = \frac{(1-e)P}{\lambda\mu} - 1 = 0.12 < \alpha.$$

In this case, we have

$$\begin{aligned}
E[e^{rX_I}] &= \int_0^{M/a} e^{rax} dF(x) + e^{rM}[1 - F(\frac{M}{a})] \\
&= \int_0^{M/a} \beta e^{rax} e^{-\beta x} dx + e^{rM} e^{-\beta \frac{M}{a}} \\
&= \frac{ra}{ra-\beta} e^{rM - \beta\frac{M}{a}} - \frac{\beta}{ra-\beta},
\end{aligned}$$

and

$$\begin{aligned}
P_I &= (1-e)P - P_R \\
&= (1-e)P - (1-c)(1-a)P - (1+\alpha)\lambda \int_{M/a}^{\infty} (ax - M)\beta e^{-\beta x} dx \\
&= (a + c - ca - e)P - (1+\alpha)\lambda \frac{a}{\beta} e^{-\beta\frac{M}{a}}.
\end{aligned}$$

Hence (6.2) becomes

$$\lambda(\frac{ra}{ra-\beta} e^{rM-\beta\frac{M}{a}} - \frac{\beta}{ra-\beta}) + Dr^2 = \lambda + [(a+c-ca-e)P - (1+\alpha)\lambda\frac{a}{\beta}e^{-\beta\frac{M}{a}}]r.$$

For

$$M = \frac{1}{R_{a,M}} \ln(1+\alpha),$$

the above equation after simplification becomes

$$\frac{\lambda r a^2}{(ra-\beta)\,\beta}(1+\alpha)^{1-\frac{\beta}{ar}} = \frac{\lambda a}{ra-\beta} + (a+c-ca-e)P - Dr,$$

When $a = 1$, the adjustment coefficient $R_{a,M}$ attains its maximum value when it satisfies

$$\left\{ \frac{\lambda r}{(r-\beta)\,\beta}(1+\alpha)^{1-\frac{\beta}{r}} = \frac{\lambda}{r-\beta} + (1-e)P - Dr \right\} |_{r=R_{a,M}} .$$

Using MATLAB, the optimal retention level is $M = 5.54$ and the maximum of the adjustment coefficient is $R_{a,M} = 0.10612$. With the initial surplus $u = 2$, the Lundberg's upper bound is 80.88%.

Compare this to the classical model without diffusion process, i.e., $D = 0$, the optimal retention level is $M = 5.45$ and the maximum of the adjustment coefficient is $R_{a,M} = 0.10789$. With the initial surplus $u = 2$, Lundberg's upper bound is 80.59%. It means that the upper bound of the ruin probability increases due to business uncertainty. ■

6.3 Optimal Retentions by Minimizing the Upper Bound for Finite-Time Ruin Probability

In this section, we consider the excess-of-loss reinsurance in the model (6.1) or when $a = 1$ in the model (6.1). We denote the finite-time ruin probability in this case by $\psi_M(u,t)$, namely $\psi_M(u,t) = \psi_{1,M}(u,t)$.

Let us review the definition of martingales first.

Definition Let $(X_t)_{t\in\mathfrak{R}_+}$ be a real-values family of random variables defined on the probability space $(\Omega, \mathfrak{F}, P)$ and let $(\mathfrak{F}_t)_{t\in\mathfrak{R}_+}$ be a filtration. The stochastic process $(X_t)_{t\in\mathfrak{R}_+}$ is said to be adapted to the family $(\mathfrak{F}_t)_{t\in\mathfrak{R}_+}$ if, for all $t \in \mathfrak{R}_+$, X_t is $\mathfrak{F}_t$-measurable. The stochastic process $(X_t)_{t\in\mathfrak{R}_+}$, adapted to the filtration $(\mathfrak{F}_t)_{t\in\mathfrak{R}_+}$, is a martingale with respect to this filtration, provided the following conditions hold:

1. X_t is P-integrable, for all $t \in \mathfrak{R}_+$;

2. for all $(s,t) \in \mathfrak{R}_+ \times \mathfrak{R}_+$, $s < t$: $E[X_t|\mathfrak{F}_s] = X_s$ almost surely.

Gerber (1979) derived the upper bound for the finite-time ruin probability in the compound Poisson risk model. Using the similar idea, we derive the upper bound for the finite-time ruin probability in the jump-diffusion risk model in this section.

Lemma 6.3.1 In the diffusion-included ruin process, the upper bound for the finite-time ruin probability after reinsurance is

$$\psi_M(u,t) \le \exp\left(\min_{r \ge R_M} f_M(r;u,t)\right), \tag{6.4}$$

where

$$f_M(r;u,t) = -ur + t\theta_M(r), \tag{6.5}$$

with

$$\theta_M(r) = \lambda E[e^{rX_I}] + Dr^2 - P_I r - \lambda. \tag{6.6}$$

Proof: Using the similar argument from Gerber (1973) or Gerber (1979), first note that

$$
\begin{aligned}
& E\left[\exp\left(-r\left(-\sum_{i=1}^{N(t)}\min(X_i,M)+W(t)\right)\right)|\mathfrak{F}(s)\right] \\
= \; & E\Bigg[\exp\Bigg(-r\left(-\sum_{i=1}^{N(t)}\min(X_i,M)+W(t)\right)+r\left(-\sum_{i=1}^{N(s)}\min(X_i,M)+W(s)\right) \\
& -r\left(-\sum_{i=1}^{N(s)}\min(X_i,M)+W(s)\right)\Bigg)|\mathfrak{F}(s)\Bigg] \\
= \; & E\left[\exp\left(-r\left(-\sum_{i=1}^{N(t-s)}\min(X_i,M)+W(t-s)\right)-r\left(-\sum_{i=1}^{N(s)}\min(X_i,M)+W(s)\right)\right)|\mathfrak{F}(s)\right] \\
= \; & E\left[\exp\left(-r\left(-\sum_{i=1}^{N(t-s)}\min(X_i,M)+W(t-s)\right)\right)\right] \\
& \times E\left[\exp\left(-r\left(-\sum_{i=1}^{N(s)}\min(X_i,M)+W(s)\right)\right)|\mathfrak{F}(s)\right] \\
= \; & \exp\left(\lambda(t-s)\left(E[e^{rX_I}]-1\right)\right)e^{(t-s)Dr^2}\exp\left(-r\left(-\sum_{i=1}^{N(s)}\min(X_i,M)+W(s)\right)\right).
\end{aligned}
$$

Here $\mathfrak{F}(s)$ is a filtration, which contains all information up to time s.

Hence it is known that

$$
\left\{\exp\left(-t\left(-rP_I+\lambda\left(E[e^{rX_I}]-1\right)+Dr^2\right)\right)\exp\left(-rU_M(t)\right)\right\} \tag{6.7}
$$

is a martingale because

$$
\begin{aligned}
& E\left[\left\{\exp\left(-t\left(-rP_I+\lambda\left(E[e^{rX_I}]-1\right)+Dr^2\right)\right)\exp\left(-rU_M(t)\right)\right\}|\mathfrak{F}(s)\right] \\
= \;& E\Bigg[\exp\Bigg(-t\left(-rP_I+\lambda\left(E[e^{rX_I}]-1\right)+Dr^2\right) \\
& -r\left(u+P_It-\sum_{i=1}^{N(t)}\min(X_i,M)+W(t)\right)\Bigg)|\mathfrak{F}(s)\Bigg] \\
= \;& \exp\left(-t\left(\lambda\left(E\left[e^{rX_I}\right]-1\right)+Dr^2\right)-ru\right)E\left[\exp\left(-r\left(-\sum_{i=1}^{N(t)}\min(X_i,M)+W(t)\right)\right)|\mathfrak{F}(s)\right] \\
= \;& \exp\left(-t\left(\lambda\left(E[e^{rX_I}]-1\right)+Dr^2\right)-ru\right)\exp\left(\lambda(t-s)\left(E[e^{rX_I}]-1\right)\right)e^{(t-s)Dr^2} \\
& \times\exp\left(-r\left(-\sum_{i=1}^{N(s)}\min(X_i,M)+W(s)\right)\right) \\
= \;& \exp\left(-s\left(\lambda\left(E[e^{rX_I}]-1\right)+Dr^2\right)\right)\exp\left(-ru-r\left(-\sum_{i=1}^{N(s)}\min(X_i,M)+W(s)\right)\right) \\
= \;& \exp\left(-s\left(-rP_I+\lambda\left(E[e^{rX_I}]-1\right)+Dr^2\right)\right) \\
& \times\exp\left(-ru-srP_I-r\left(-\sum_{i=1}^{N(s)}\min(X_i,M)+W(s)\right)\right) \\
= \;& \exp\left(-s\left(-rP_I+\lambda\left(E[e^{rX_I}]-1\right)+Dr^2\right)\right)\exp\left(-rU_M(s)\right).
\end{aligned}
$$

We then have

$$
E\left[\exp\left(-T\left(-rP_I+\lambda\left(E[e^{rX_I}]-1\right)+Dr^2\right)\right)\exp\left(-rU_M(T)\right)\right]=e^{-ru},
$$

where T is the time of ruin.

We can further conclude that, for $r\geq R_M$,

$$
\begin{aligned}
e^{-ru} &\geq E\left[\exp\left(-T\left(-rP_I+\lambda\left(E[e^{rX_I}]-1\right)+Dr^2\right)\right)|T\leq t\right]\psi_M(u,t) \\
&\geq \exp\left(-t\left(-rP_I+\lambda\left(E[e^{rX_I}]-1\right)+Dr^2\right)\right)\psi_M(u,t).
\end{aligned}
$$

Thus

$$\psi_M(u,t) \leq \min_{r \geq R_M} \exp\left(-ur + t\left(-P_I r + \lambda\left(E[e^{rX_I}] - 1\right) + Dr^2\right)\right),$$

and the lemma is proved. ■

Note here,

- For a given M, the adjustment coefficient R_M is the unique positive root of

$$\lambda E[e^{rX_I}] + Dr^2 = \lambda + P_I r,$$

  when such a root exists, or zero otherwise.

- Equation (6.6) implies that $\theta_M(r)$ is the only root of

$$\lambda E[e^{rX_I}] + Dr^2 = \lambda + P_I r + \theta_M(r). \tag{6.8}$$

- Because it is an excess-of-loss only reinsurance, the net premium received by the ceding company after reinsurance is

$$P_I = (1-e)P - (1+\alpha)\lambda\mu + (1+\alpha)\lambda E(X_I),$$

  where $X_I = \min(X, M)$.

6.3.1 Preliminaries

After studying the insurer's adjustment coefficient as a function of retention levels in the compound Poisson model with diffusion process, the research is confined to an excess-of-loss reinsurance. We can minimize the upper bound with reinsurance by proper retention level M.

Let us study some properties of $\theta_M(r)$ first

Lemma 6.3.2 For any $M > 0$,

(i) $\theta_M(0) = \theta_M(R_M) = 0$;

(ii) $\lim_{r\to\infty} \theta_M(r) = +\infty$ and $\lim_{r\to\infty} \frac{\theta_M(r)}{r} = +\infty$;

(iii) $P_I + \theta_M(r) \geq 0$ when $r \geq 0$;

(iv) $\frac{\partial \theta_M(0)}{\partial r} = \lambda E[X_M] - P_I$, which is negative;

(v) $\theta_M(r)$ is a convex function of r.

Proof:

(i), (ii) and (iii) come from equation (6.6) directly.

(iv) Differentiating (6.6) with respect to r, we have

$$\frac{\partial}{\partial r}\theta_M(r) = \lambda E[X_I e^{rX_I}] \ + 2Dr - P_I.$$

It implies for $r = 0$, $\frac{\partial \theta_M(0)}{\partial r} = \lambda E[X_I] - P_I$, which is negative.

(v) The second derivative with respect to r is

$$\frac{\partial^2}{\partial r^2}\theta_M(r) = \lambda E[X_I^2 e^{rX_I}] + 2D,$$

which is greater than zero. Hence, $\theta_M(r)$ is a convex function of r.

The relationship between $\theta_M(r)$ and r is illustrated in Figure 6.3.

■

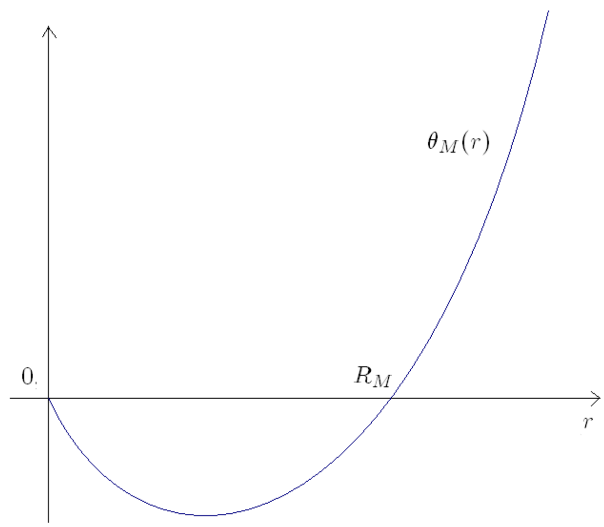

Figure 6.3: Relationship between $\theta_M(r)$ and r

Lemma 6.3.3 For each $M > 0$ and $r > 0$, $f_M(r; u, t) = -ur + t\theta_M(r)$ has a local minimum if and only if the expected surplus at time t is positive. In this case the minimizer is unique. Let it be $\hat{r}_M$.

Proof: Because

$$f_M(r; u, t) = -ur + t\theta_M(r),$$

from Lemma 6.3.2, we have

$$\lim_{r \to 0} f_M(r; u, t) = 0,$$

$$\lim_{r \to \infty} f_M(r; u, t) = +\infty,$$

$$\lim_{r \to 0} \frac{\partial}{\partial r} f_M(r; u, t) = -u + t \frac{\partial}{\partial r} \theta_M(0) < 0,$$

and

$$\frac{\partial^2}{\partial r^2} f_M(r; u, t) = t \frac{\partial^2}{\partial r^2} \theta_M(r) > 0.$$

First it may be concluded that with respect to r, $f_M(r;u,t)$ is a convex function for $r > 0$. Note, the expected surplus at time t can be rewritten as

$$u + (P_I - \lambda E[X_I])\, t = -\lim_{r \to 0} \frac{\partial}{\partial r} f_M(r;u,t).$$

Because $f_M(r;u,t)$ has a minimum in r, if and only if,

$$\lim_{r \to 0} \left[\frac{\partial}{\partial r} f_M(r;u,t) \right] < 0,$$

which means the expected surplus at time t is positive. Let it be $\hat{r}_M$. ■

Because the insurer cannot reinsure the whole risk with a certain profit, there exists a positive M_0 such that $M \in L$ if and only if $M > M_0$.

Lemma 6.3.4 Suppose that the expected surplus at time t is positive. Then $\hat{r}_M > R_M$, if and only if

$$\frac{u}{t} > \lambda E[X_I e^{R_M X_I}] \; + 2DR_M - P_I.$$

Here R_M is the unique positive root of equation (6.2) if $M > M_0$ or zero otherwise.

Proof: From the proof of Lemma 6.3.2 (iv), we have

$$\frac{\partial}{\partial r} f_M(r;u,t) = -u + \left(\lambda E[X_I e^{r X_I}] \; + 2Dr - P_I \; \right) t.$$

Since $\hat{r}_M$ is the solution of $\frac{\partial}{\partial r} f_M(r;u,t) = 0$, it is clear from Figure 6.4 that

$$\hat{r}_M > R_M$$

if and only if

$$\frac{\partial}{\partial r} f_M(r;u,t)|_{r=R_M} < 0.$$

Hence, the results follow. ■

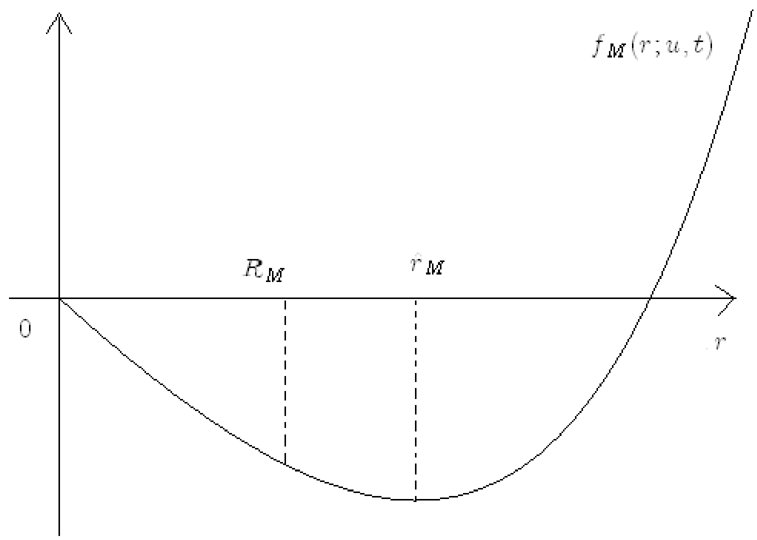

Figure 6.4: Function $f_M(r; u, t)$

Let M_1 be the minimum value of M for which the expected surplus at time t is non-negative, i.e.,

$$M_1 = \min\{M : M \geq 0 \text{ and } u + (P_I - \lambda E\,[X_I])\,t \geq 0\}.$$

Note that

$$\begin{aligned} & \lambda E[X_I] - P_I \\ = \;& \lambda E[X_I] - ((1-e)P - (1+\alpha)\lambda\mu + (1+\alpha)\lambda E(X_I)) \\ = \;& (1+\alpha)\lambda\mu - (1-e)P - \alpha\lambda E(X_I) \\ = \;& (1+\alpha)\lambda\mu - \lambda\mu\,(1+\rho) - \alpha\lambda E(X_I) \\ = \;& (\alpha-\rho)\lambda\mu - \alpha\lambda E(X_I), \end{aligned}$$

which means that $M_1 = 0$ if and only if $\frac{u}{t} \geq \lambda\mu(\alpha - \rho)$.

Note, when retention level M equals to 0, it implies that the ceding company surrenders all the business to the reinsurer, with $P_I = (\rho - \alpha)\lambda\mu$ and $X_I = 0$.

The corollary below follows from the previous proof.

Corollary 6.3.1 For each $M > M_1$, if

$$\frac{u}{t} > \lambda E[X_I e^{R_M X_I}] + 2DR_M - P_I,$$

then

$$\psi_M(u,t) \leq e^{f_M(u,t,\hat{r}_M)};$$

and if

$$\frac{u}{t} \leq \lambda E[X_I e^{R_M X_I}] + 2DR_M - P_I,$$

then

$$\psi_M(u,t) \leq e^{f_M(u,t,R_M)}.$$

Here R_M is the unique positive root of the equation (6.2) if $M > M_0$ or zero otherwise, and $\hat{r}_M$ is the solution to

$$\lambda E[X_I e^{rX_I}] + 2Dr - P_I = \frac{u}{t}.$$

Hence, we can conclude that for certain values of M, it is possible to improve Lundberg's inequality (infinite time). In some cases, the value of M that minimizes the upper bound provided by the inequality of Lemma 6.3.1 (finite time) is different from the value of M that maximizes the Lundberg's adjustment coefficient. That will be the case if

$$\frac{u}{t} > \lambda E[X_I e^{R_M X_I}] + 2DR_M - P_I,$$

where $\left(\hat{M}, \hat{R}_{\hat{M}}\right)$ is the solution of

$$\lambda E[e^{rX_I}] + Dr^2 = \lambda + P_I r,$$

and

$$e^{rM} = (1 + \alpha).$$

■

6.3.2 Upper Bound as a Function of Retention

With the previous proof, the upper bound can be expressed as a function of the retention.

Theorem 6.3.1 In the compound Poisson model with a diffusion process, to minimize the upper bound for the probability of ruin before time t, the stop-loss reinsurance retention level M satisfies the following.

(i) If $\frac{u}{t} \geq \lambda\mu(\alpha - \rho)$, then the upper bound for the probability of ruin before time t attains its minimum at $M = 0$.

(ii) If $\frac{u}{t} < \lambda\mu(\alpha - \rho)$, then the upper bound, considered as a function of M, has an absolute minimum which is attained at the unique point satisfying

$$M = \frac{1}{r^*} \ln(1 + \alpha),$$

with $r^* = \max(\hat{r}, \hat{R})$.

Here, $\hat{r}$ is the solution to

$$\frac{u}{t} = \lambda E[X_I e^{\hat{r}X_I}] \ + 2D\hat{r} - P_I,$$

and $\hat{R}$ is the adjustment coefficient which satisfies

$$\lambda E[e^{\hat{R}X_I}] + D\hat{R}^2 = \lambda + P_I\hat{R}.$$

Here, u is the initial surplus, λ is the average claim frequency, μ is the average claim size and $F(x)$ is the claim size distribution. The net premium for the ceding company after reinsurance, P_I, is greater than zero. The relative safety loading

$$\rho = \frac{(1-e)P}{\lambda\mu} - 1$$

is less than the positive loading coefficient α.

Proof: Note

$$\min_{M \geq M_1} \psi_M(u,t) \leq \exp\left(\min_{M \geq M_1} \min_{r \geq R(M)} f_M(r;u,t)\right) = \exp\left(\min_{M \geq M_1} \min_{r \geq R(M)} f_M(r;u,t)\right),$$

with

$$\begin{aligned} f_M(r;u,t) &= -ur + t\theta_M(r) \\ &= -ur + t\left(\lambda E[e^{rX_I}] + Dr^2 - P_I r - \lambda\right). \end{aligned}$$

Calculating the derivative with respect to M with the results from Lemma 6.2.1, we get

$$\begin{aligned} \frac{\partial}{\partial M} f_M(r;u,t) &= t\frac{\partial}{\partial M}\theta_M(r) \\ &= \left(\lambda \frac{\partial}{\partial M} E[e^{rX_I}] - r\frac{\partial P_I}{\partial M}\right) t \\ &= \left(\lambda r e^{rM}(1-F(M)) - r(1+\alpha)\lambda(1-F(M))\right) t \\ &= \left(e^{rM} - (1+\alpha)\right) \lambda r (1-F(M))\, t; \end{aligned}$$

and

$$
\begin{aligned}
\frac{\partial^2}{\partial M^2} f_M(r;u,t) &= t\frac{\partial^2}{\partial M^2}\theta_M(r) \\
&= \left(\lambda \frac{\partial^2}{\partial M^2} E[e^{rX_I}] - r\frac{\partial^2 P_I}{\partial M^2}\right) t \\
&= \left(\lambda \left(r^2 e^{rM} \left(1 - F(M)\right) - re^{rM} f(M)\right) + r(1+\alpha)\lambda f(M)\right) t \\
&= \lambda r^2 e^{rM} \left(1 - F(M)\right) t + \left((1+\alpha) - e^{rM}\right) \lambda r f(M) t.
\end{aligned}
$$

We can conclude that $\frac{\partial}{\partial M} f_M(r;u,t) = 0$ if and only if

$$
M = \frac{1}{r}\ln(1+\alpha), \tag{6.9}
$$

and

$$
\frac{\partial^2}{\partial M^2} f_M(r;u,t)|_{\frac{\partial}{\partial M} f_M(r;u,t)=0} = \lambda t r^2 e^{rM} \left(1 - F(M)\right)
$$

is always positive. Hence, for fixed r, u and t, $f_M(r;u,t)$ has a local minimum which is uniquely attained at the point $\widehat{M}(r)$ such that $\widehat{M}(r)$ is the solution to (6.9).

Recall Lemma 6.3.2, we have

$$
\begin{aligned}
\frac{\partial}{\partial r}\theta_M(r) &= \lambda E[X_I e^{rX_I}] + 2Dr - P_I, \\
\frac{\partial^2}{\partial r^2}\theta_M(r) &= \lambda E[X_I^2 e^{rX_I}] + 2D;
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial^2 \theta_M(r)}{\partial r \partial M} &= \frac{\partial \left(e^{rM} - (1+\alpha)\right) \lambda r \left(1 - F(M)\right)}{\partial r} \\
&= M e^{rM} \lambda r \left(1 - F(M)\right) + \left(e^{rM} - (1+\alpha)\right) \lambda \left(1 - F(M)\right).
\end{aligned}
$$

Furthermore, recall that

$$
E\left[X_I^2 e^{rX_I}\right] = \int_0^M x^2 e^{rx} dF(x) + M^2 e^{rM} \left(1 - F(M)\right),
$$

resulting in

$$
\begin{aligned}
& \frac{\partial^2\theta_M(r)}{\partial r^2}\frac{\partial^2\theta_M(r)}{\partial M^2} - \left(\frac{\partial^2\theta_M(r)}{\partial r\partial M}\right)^2 |_{\frac{\partial}{\partial M}f_M(r;u,t)=0} \\
= & \left(\lambda E[X_I^2 e^{rX_I}] + 2D\right)\lambda r^2 e^{rM}\left(1-F(M)\right) - M^2\lambda^2 r^2 e^{2rM}\left(1-F(M)\right)^2 \\
= & \left(\lambda \int_0^M x^2 e^{rx} dF(x) + 2D\right)\lambda r^2 e^{rM}\left(1-F(M)\right),
\end{aligned}
$$

which is always positive.

Let us now study the function $f_{\hat{M}(r)}(r;u,t)$, using the implicit function theorem, we can see that

$$\frac{d}{dr}f_{\hat{M}(r)}(r;u,t) = -u + t\frac{\partial\theta_M(r)}{\partial r}|_{\frac{\partial}{\partial M}f_M(r;u,t)=0},$$

and

$$\frac{d^2}{dr^2}f_{\hat{M}(r)}(r;u,t) = t\frac{\frac{\partial^2\theta_M(r)}{\partial r^2}\frac{\partial^2\theta_M(r)}{\partial M^2} - \left(\frac{\partial^2\theta_M(r)}{\partial r\partial M}\right)^2}{\frac{\partial^2\theta_M(r)}{\partial M^2}}|_{\frac{\partial}{\partial M}f_M(r;u,t)=0}.$$

This implies that $f_{\hat{M}(r)}(r;u,t)$ is a convex function of r, since $\frac{d^2}{dr^2}f_{\hat{M}(r)}(r;u,t)$ is always positive. Hence, we can conclude that there is at most one solution to $\frac{d}{dr}f_{\hat{M}(r)}(r;u,t) = 0$ and when it exists, it is the global minimum of $f_{\hat{M}(r)}(r;u,t)$. From definitions and Lemma 6.3.3, we have

$$\lim_{r\to 0} f_{\widehat{M}(r)}(r;u,t) = 0,$$

and

$$\lim_{r\to 0}\frac{d}{dr}f_{\widehat{M}(r)}(r;u,t) < 0.$$

If $\frac{u}{t} \geq \lambda\mu(\alpha-\rho)$, then M_1 is zero. This implies that the upper bound for the probability of ruin before time t attains its minimum at $M = 0$. Hence, part (i) is proven.

(ii) For $\frac{d}{dr} f_{\hat{M}(r)}(r; u, t) = 0$,

$$\frac{u}{t} = \lambda E[X_I e^{rX_I}] + 2Dr - P_I.$$

For $\frac{u}{t} < \lambda\mu(\alpha - \rho)$, let the solution to (6.9) be r_1 for $M = M_1$, which is finite. Also recall Jensen's inequality

$$E[X_{M_1} e^{r_1 X_{M_1}}] \geq E(X_{M_1}) E[e^{r_1 X_{M_1}}],$$

we have

$$\begin{aligned}
& \lim_{r \to r_1} \frac{d}{dr} f_{\widehat{M}(r)}(r; u, t) \\
= \; & -u + t \left(\lambda E[X_I e^{rX_I}] + 2Dr - P_I \right) |_{M=M_1} \\
= \; & -\lambda E[X_I] t + t\lambda E[X_I e^{rX_I}] + 2Drt|_{M=M_1} \\
\geq \; & 0.
\end{aligned}$$

Hence $\widehat{r}$ exists and it is smaller than r_1. The proof is complete.

■

6.3.3 Optimal Retentions with Exponential Claims

Using the same exponential claim example, and further assume that $u = 2$ and $t = 10$, we have

$$\frac{u}{t} = 0.2 < 0.66 = \lambda\mu(\alpha - \rho).$$

$M = \frac{1}{r^*} \ln(1 + \alpha)$, where $r^* = \max(\hat{r}, \hat{R})$.

Here, $\hat{r}$ is the solution to

$$\frac{u}{t} = \lambda E[X_I e^{\hat{r} X_I}] \; + 2D\hat{r} - P_I,$$

Table 6.1: Optimal Retentions with Diffusion

D		Upper Bound	Max r	Optimal M
0	Infinite	80.59%	0.10789	5.45
	Finite	79.75%	0.13815	4.25
0.02	Infinite	80.88%	0.10612	5.54
	Finite	80.05%	0.13571	4.33
0.2	Infinite	83.12%	0.09242	6.36
	Finite	82.38%	0.11767	5.00

and $\hat{R}$ is the adjustment coefficient satisfies

$$\lambda E[e^{\hat{R}X_I}] + D\hat{R}^2 = \lambda + P_I\hat{R}.$$

Note that

$$\begin{aligned}
P_I &= (1-e)P - (1+\alpha)\lambda\frac{1}{\beta}e^{-M\beta};\\
E[e^{rX_I}] &= \frac{\beta - re^{-M(\beta-r)}}{\beta - r};\\
E[X_I e^{rX_I}] &= \frac{\beta}{(\beta-r)^2} - \frac{\beta rM + \beta - Mr^2}{(\beta-r)^2}e^{-(\beta-r)M}.
\end{aligned}$$

Using MATLAB, we optimal retention level is $M = 4.33$ with $r^* = 0.1357$, and the upper bound for finite time is 80.05% with $f_M(r; u, t) = -0.2225$.

From Table 6.1, we can conclude that the upper bound for finite time is lower than the upper bound for infinite time, which is an improvement. The ceding company decreases the retention level for the lower upper bound of ruin probability. As the business uncertainty increases, the upper bound of ruin probability increases along with the optimal retention level M.

Chapter 7

Conclusions

7.1 Summary

The main purpose of the thesis is to derive a "fair", optimal reinsurance retention between a ceding company and a reinsurer and to study the effects of financial and economic factors including interest, dividend, commissions, expenses, and diffusion on optimal reinsurance retentions.

Chapter 2 illustrates the optimal retentions for a single period claim. The explicit expression for the probability of the joint survival of the cedent and the reinsurer is modeled. The relationships among the retention level, quota-share and maximums of the joint survival probability are derived. An optimal split of the total premium income, maximizing the joint survival is obtained. We illustrate the results using the exponential distribution claim and compare it to the Pareto claim case. The extreme cases, the quota-share only reinsurance and stop-loss only reinsurance, are discussed as well.

Following the discussion of the single period claim, the multi-period aggregate claim is examined. It is parallel to the case found in Chapter 2, but far more complex. It is very difficult to calculate the joint survival probability in the multi-period case, and thus it is not feasible to determine the optimal treaty directly. First, Chapter 3 uses the properties of associated random variables to derive a lower bound for the joint survival probability. Then, we can determine the optimal a and M by maximizing the lower bound. Second, Chapter 3 uses bivariate gamma distribution to approximate the joint survival probability of the cedent and the reinsurer. We derive the joint survival probability under the aggregate claim. However because there is no explicit analytical form of the optimal retention level, even for the simple compound Poisson model, the numerical results are used to compare the compound Poisson, compound binomial, and compound negative binomial for both the exponential and Pareto claims.

For the continuous time risk model, which is studied in Chapter 4 through 6, we focus on the ceding company's interests and consequently add more realistic aspects of the reinsurance business into the model.

The first element examined is the effect of interest rate. Chapter 4 advocates for a new exponential claim compound Poisson process by De Vylder's approximation, while including the interest rate in the surplus process. The explicit formulas are provided for the new parameter set and the exponential and Pareto distributions are analyzed as examples. The optimal reinsurance quota-share level and retention limit are derived, along with the minimum ultimate ruin probability. Note, the ruin probability decreases dramatically when the initial surplus and the interest rate increase. In order to minimize ruin probability, both quota-share level a and retention level M need to decrease as the initial surplus increases.

The second real insurance element used is dividend. Chapter 5 considers a compound Poisson process with the dividends accumulated at a constant rate. It uncovers the optimal reinsurance treaty necessary to maximize the expectation of the present value of all dividends before possible ruin. The discussion focuses on quota-share reinsurance treaties and derives optimal quota-share levels by varying the dividend ceiling, insurance loadings, claim frequency, claim size and initial surplus for both exponential claim and Erlang (2) claim. When the threshold level increases, the quota-share level also increases to achieve the maximal possible accumulated dividends.

In Chapter 6, special attention is given to a compound Poisson model with diffusion included. We add a diffusion process to the surplus process in the classical ruin model to present the uncertain events that affect the insurance industry on a day-to-day basis. Also, commissions and expenses are included. This chapter first focuses on the optimal reinsurance treaty necessary to maximize adjustment coefficient in the Lundberg upper bound for the infinite-time ruin probability. We give a simple explicit formula to determine the reinsurance retention level. Further, Chapter 6 uncovers the optimal treaty which is necessary to minimize the upper bound for the finite-time ruin probability. We derive such an upper bound by using the martingale approach.

7.2 Future Research

The thesis presents the optimal reinsurance treaty under different criteria. It considers the compromise interests of both parties in the treaty and expands the classical models to include additional business factors in an effort to mirror the real business world. We can extend the study to a much larger scale analysis of optimal criteria.

Take a single period contract as an example; ensuring maximization of joint survival probability is not the sole measurement to judge the optimal reinsurance treaty. There are numerous other quantitative metrics to consider in optimizing the interests for both the ceding company and the reinsurer.

In Chapter 2, we choose initial surpluses as zero to place both parties at a comparable starting level. Considered from another point of view, as the cedent and reinsurer usually have different initial surpluses, it is an interesting question to obtain the optimal treaty while the nonzero initial surpluses are included in the ruin process. We can ascertain the joint survival probability of both companies using a similar method, described in Chapter 2. We do possess the optimal treaties for any pure stop-loss reinsurance or pure quota-share reinsurance. However, when the treaty becomes a combination case, there is no explicit formula to obtain the answer.

In a stop-loss reinsurance scenario, the ceding company, in the interests of avoiding bankruptcy, can always set the retention levels such that the total claim pays out less than the initial surplus. From the cedent's perspective, it is important to maximize the expected return, while the reinsurer's obvious goal is to maximize the survival probability. This idea inspires the introduction of another measurement for the optimal reinsurance treaty. Minimize the ruin probability for the reinsurer while maximizing the expected return for the ceding company is desired. This, however, is outside of the scope of the research presented here.

In addition to the ruin probability, Value at Risk (VaR) or Conditional Tail Expectation (CTE) is used as a risk measure as well. Some studies are Wang et al. (2005) and Jorion (2001). VaR represents the loss amount under a given probability while integrating diversification effects and risk properties of a particular portfolio; hence, risk constraints at all levels of a hierarchical organization can be utilized coherently. VaR is a simple tool for the selection of strategic risk and provides a common language for risk management. CTE, also called Expected Shortfall or Tail-VaR, is defined as the average outcome that exceeds a specified percentile. CTE is calculated as the weighted average of the worst results of the stochastic simulation. From a prudent risk management perspective, the risk measure associated with loss must be as minimal as possible. The optimal retention level M and quota-share a is determined to minimize the corresponding VaR or CTE. It is another meaningful optimal criterion.

Simply stated, a set of measurements exist to define optimal conditions. Discussing several other optimal criteria from different perspectives aids the understanding of optimal quota-share and retention limit. Several optimal results do not exist in the current research. Discovering a close formula may be a significant challenge. The meaningful economic underlying factor must be balanced with maintaining mathematical solvability. This riveting topic has implications in the field of actuarial science and for the world-wide business community. Further research will, and no doubt, prove prosperous.

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