Inertial and Electromagnetic Aspects of Matter Induced from Five-Dimensional General Relativity

by

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Abstract

In this thesis, we examine the inertial and electromagnetic properties of matter induced from a five-dimensional Kaluza-Klein-type extension of General Relativity (referred to as "Induced Matter" theory). The research presented here consists of six exact solutions of the 5D vacuum field equations, representing three different physical configurations, which are analyzed for their inertial and electromagnetic properties (using, for the first time from within the Induced Matter formalism, a *charged*, *imperfect* fluid model).

The first two solutions represent spherically-symmetric charge distributions, describing what, in the appropriate limit, would be charged 'particles'. The next two solutions represent axially-symmetric 'magnetized' distributions, describing 'wires' carrying currents with axially-symmetric magnetic fields. The final two solutions are *conformally flat* solutions (5D conformally flat and 4D conformally flat in a 5D manifold) representing cosmological distributions. (Specifically, their 4D interpretations are that of de Sitter space.)

We also correct a previous error made in the analysis of the Liu-Wesson class of 5D charged solutions, recently published in ref. [2]. Specifically, that class of solutions was thought to represent charged *radiation*, whereas it actually represents 'nonradiative' fluid. The inertial (and electromagnetic) properties of the Liu-Wesson class are calculated here for the first time.

Contents

Chapter 1

Introduction

It has been well-known, from the work of Kaluza, Klein and others, that the field equations of General Relativity in five dimensions can be shown to contain the usual four dimensional Einstein field equations plus Maxwell's equations of electromagnetism, thereby effecting a unification of Einsteinian gravitation with Maxwellian electromagnetism (refs. [3], [4], [5], [6], [7], [8]).

Specifically, Kaluza and Klein associated the extended off-diagonal components of the 5D metric with the vector potentials of electromagnetism, and were able to show that the 15 field equations of 5D General Relativity could be broken down into the 10 field equations of 4D General Relativity in the presence of an electromagnetic field[1], plus the 4 equations of electromagnetism (in vacuum), plus a scalar wave equation (refs. [3], [4], [5], [2], [9]).

This attempt of Kaluza and Klein to unify gravitation and electromagnetism, however, included the so-called "cylinder condition" in which the fifth dimension was presummed to be curled up very small (on the order of Planck length), and

[1] *i.e.*, the 4D Einstein tensor equated to the electromagnetic energy-momentum tensor.

which forced the metric to be independent of the fifth coordinate. This was, in part to explain the observed absence of a fifth dimension, and, in part to try to incorporate the effects of quantization into 'Kaluza-Klein' theory (specifically, the quantization of charge). However, confining particles to a cylindrical surface the size of Planck length forced their masses to be on the order of Planck mass. This and other problems with the traditional Kaluza-Klein theory resulted in its being eclipsed by research into other areas (such as nuclear and quantum physics) which had a better chance of yielding verifiable results (refs. [3], [4], [10], [9]).

In more recent years, however, newer versions of 'Kaluza-Klein' (5D and higher-D GR) theories have been advanced in which the 'cylinder condition' of Kaluza and Klein is abandoned, and the metric is allowed to depend on the fifth coordinate. In one of these, referred to as Induced Matter Theory (due to Wesson, Ponce de Leon, Mashhoon, Liu, *etc.*), the 5D field equations are presumed to exist in vacuum, from which the 4D Einstein tensor can be extracted and equated to a *general* (4D) energy-momentum tensor. The energy-momentum tensor is then usually identified with that of either a perfect fluid, or (as in refs. [11] and [12]), with an imperfect fluid. Since all properties of matter can be derived from this, this version of 5D gravity allows for the unification of matter (and energy) with spacetime as well as the unification of gravitation and electromagnetism (refs. [4], [8], [2], [6]).

The unification of matter and energy with space and time also addresses the additional concerns raised by Mach over classical (Newtonian) mechanics; that is, relating the inertia of a test object with the material distribution of the universe. Since there are no sources in the Induced Matter theory (it is vacuum in 5D), then all matter comes from curvature effects in five dimensions, and no 'particle' can be said to be truly 'isolated'. Instead, matter and energy are represented as nonlocal distributions which depend on the global descriptions of the 5D manifold. As a result, the mass ('inertia') of a 'local' particle will depend on the *global* matter/energy distributions (refs. [13], [6], [8]).

For this thesis, several exact solutions were found for the 5D vacuum field equations and then interpreted physically in accord with the Induced Matter formalism. First, in chapter 2, the Induced Matter theory is examined more formally (mathematically) along with the motivations (inertial and electromagnetic) for desiring to investigate such a theory. The differences between the Induced Matter formalism and the original Kaluza-Klein theory are also elucidated.

In chapter 3, two off-diagonal, spherically-symmetric metrics representing 'charged particles' (at least, in the limiting cases) are investigated and their effective charges, masses, densities, *etc.*, calculated and analyzed.

These 'charged particle' metrics also depend on the fifth coordinate, and, as such, represent the first off-diagonal ('charged') solutions dependent on the fifth coordinate. One of the significances of such solutions is that it can be shown (for *diagonal* solutions, at least) that metrics which are *in*dependent of the fifth coordinate can only be modeled to describe radiation (ref.[8]). The dependence on the fifth coordinate has then been seen to be important in describing 'nonradiative' states of matter.

However, it was not fully appreciated until this thesis was done that this theorem (demonstrating a connection between the fifth coordinate and nonradiative equations of state) was derived only for *diagonal* metrics. Off-diagonal solutions are *not* so constrained by this theorem. A recent off-diagonal solution published by H. Liu and P. Wesson (ref. [2]), which also represents 'charged particles', *incorrectly* assigns their solution as radiation. As such, its *correct* nature has been investigated at the end of chapter 3.

In chapter 4, two off-diagonal, axially-symmetric metrics representing 'wires' (in the limiting case) carrying magnetic field-generating currents are presented. The first such metric is a completely general, radial-dependent solution to the 5D field equations for such a 'wire' metric. The second solution, though less general, is also dependent on the fifth coordinate (as well as the radial coordinate). The effective current density, linear mass density, *etc.*, are then calculated and analyzed for each metric.

In modeling the charged and magnetized metrics presented in chapters 3 and 4, *imperfect* charged fluid models were utilized; the first time such models (charged *and* imperfect) have been used within the Induced Matter formalism. (This, and other details of the Induced Matter theory, are discussed in more detail in chapter 2.)

In chapter 5, two classes of diagonal, *conformally flat* metrics, representing (in the 4D limit) cosmological solutions, are presented and analyzed. The first class of solutions possess a 4D conformally flat portion with an extra fifth-component portion, while the second solution is 5D conformally flat. These solutions, when analyzed, can be shown to represent 4D cosmological models (notably, de Sitter space).

Both sets of metrics depend on the radial coordinate, the temporal coordinate, and the fifth coordinate. As such, they are complicated metrics, mixed[2] functions of all three of coordinates (the 4D conformally flat solution being the *most general* such solution).

Finally, in chapter 6, discussion is presented on all the solutions found in this thesis, and conclusions drawn based on their analysis from within the Induced Matter formalism.

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[2]*i.e.*, *non*separable

In appendix A, the Riemann tensors for the 'charge metrics' presented in chapter 3 are shown, along with the derivation of those metrics. In appendix B, the Riemann tensors and derivations of the 'wire metrics' presented in chapter 4 are shown, and in appendix C, Riemann tensors and derivation of the cosmological metrics presented in chapter 5 are shown.

All metrics derived here were verified by on computer by GRTensor II, computer software developed by P. Musgrave, D. Pollney and K. Lake at Queen's University (ref. [1]). GRTensor II allows rapid calculation (and, thus, verification) of all GR-type metrics in any number of dimensions. However, GRTensor II does *not* allow for calculation of Maxwellian-type equations, which (due to the presence of charges and currents in this thesis), are also important here. As a result, in appendix D, a computer subroutine (written by the author) yielding calculation of Maxwell's equations, for the analysis of the charged and currented solutions given here, is presented.

Finally, in this thesis, unless otherwise stated, Latin super/subscripts run over all five dimensions, $a, b, ... = 0, 123, 5 = t, r\theta\phi, \psi$ (or $t, \rho\phi z, \psi$ for cylindrical coordinates), while Greek super/subscripts run over the four dimensional spacetime *subspace*, $\alpha, \beta, ... = 0, 123 = t, r\theta\phi$ ($t, \rho\phi z$ for cylindrical coordinates). Since there are two metrics in each of the main chapters, 3, 4 and 5, the Roman numeral subscripts I and II on calculated quantities (such as $T_{\mu}{}^{\nu}$) denote the respective metric number for that calculated quantity in that chapter.

Five dimensional quantities are denoted by circumflexes (*e.g.*, $d\hat{s}^2$ represents the five dimensional metric, while ds^2 represents the four dimensional metric). The fifth coordinate is denoted by: $x^5 \equiv \psi$. The signature of the metric is taken to be $(+, -, -, -, \epsilon)$, where ϵ is the signature of the fifth dimension and is either "$+$" or "$-$" depending on whether the fifth dimension is taken to be timelike or spacelike (the latter usually being the case in most work in Induced Matter theory, ref. [8]; however see ref. [10] for work involving a *timelike* fifth dimension). Units are chosen such that $c = 8\pi G = \frac{1}{4\pi\epsilon_0} = 1$.

Chapter 2

5D GR Theory

2.1 Inertia: Mach's Principle

In developing a relativistic form of gravity, Einstein was motivated to the General Theory of Relativity by two main principles: the Equivalence Principle and Mach's Principle (refs. [14]).

By the Equivalence Principle, Einstein meant the inability of any local, non-gravitational experiment to distinguish between an external gravitational field and acceleration. The most notable prediction of the Equivalence Principle was that rays of light should bend in a gravitational field; since a horizontally-projected beam of light would appear to trace out a curved path in a vertically-accelerated frame, then, by the Equivalence Principle, the same thing should happen in a gravitational field. Thus, light should bend in a gravitational field (ref. [15]).

Since the images by which we see the universe are made up of light rays, then light rays can trace out space(time). If light rays, indeed, bend in a gravitational field, then gravity can be understood as a *bending* of space(time).

Mach's Principle represents the (positivist) view held by Mach (and others) that position and motion (*i.e.*, space and time) should only be regarded in a 'relative' context; that is, any reference to 'absolute space' (or 'absolute time'), as Newton would have preferred, was meaningless since 'absolute space' (or time) could not be detected nor measured against. Instead, Mach insisted that the motion of any object be regarded only relative to the rest of the universe. Mach further stipulated that the laws of physics should be formulated in such a way as to make irrelevant whether it was the object that was moving with respect to the universe, or whether it was the universe moving about the object (ref. [16]).

A typical example of this difference in perspective is given by consideration of a rotating bucket of water. If a bucket of water is rotating about a vertical axis through its center, the water will assume a parabolic shape about that axis. While the experiment is simple, its interpretation is not. How does the water 'know' that it is in a rotational (*i.e.*, *non*inertial) frame so that it might assume a parabolic shape?

According to the traditional Newtonian perspective, this is caused by the interaction of the water with *absolute space*; that is, 'absolute space' provides the reference frame with respect to which the water can 'sense' that it is *not* in an inertial reference frame, and react to it (*i.e.*, assume a parabolic shape) (ref [15]).

According to Mach's point-of-view, however, 'absolute space' is a meaningless mathematical concept since it can't be measured. Instead, Mach argued that the rotating water experiences an 'interaction' with respect to the rest of the matter distribution in the universe (so the water's noninertia (*rotation*) is measured relative to the totality of all the matter in the universe). It is this interaction (with respect to the rest of the universe), therefore, that causes the water to assume a parabolic shape (ref [15]).

The greatest predicted difference from these two perspectives (Newtonian and Machian) on the rotating bucket of water then comes from considering what would happen if the bucket were *not* rotating, but the *rest of the universe* were rotating about it. According to the Newtonian view, there would be no effect on the water. Since the *water* is not rotating with respect to absolute space, then the water's surface will remain flat.

According to Mach's view, however, all motion should be regarded as relative, and the situation of having the bucket remain 'at rest' while the universe is rotating about it is identical to the situation where the universe is 'at rest' and the bucket is rotating with respect to *it*. Therefore, Mach's Principle says that the universe rotating about a bucket of water 'at rest' should *cause* the surface of the water to assume a parabolic shape. It is interesting to note that, before Einstein, Mach and his followers considered *themselves* 'relativists' (ref. [15]).

Intimately connected with Mach's ideas on the relativity of motion, are his ideas on the relativity of *inertia* (that is, the *resistance* of an object to changes in motion; *mass*). In the same way that Mach believed that space (and time) should be regarded in a functional (*positivist*) sense, so he also believed that mass (inertia) should be regarded from a functional point-of-view. Mass, as the *resistance* of an object to (changes in) motion, can then be understood, according to Mach, by comparing the relative motion/acceleration imparted to an object for a given impulse of momentum/force. If, for a given force, an object attains twice the acceleration than another object, then the first object would have half the mass (inertia) of the second (ref. [17]).

Since motion/acceleration are, themselves, only to be understood from a relative point-of-view (from Mach's perspective), then inertia must, therefore, also be understood only in a relative fashion (relative to the rest of the universe). Particularly, since the standards of nonacceleration and nonrotation are defined by the totality of the matter in the universe, then the mass (inertia) of a given object must be related in some (unspecified) relative way to the distribution of matter in the universe (ref [15]).

For example, an object in an otherwise empty universe would possess no 'self'-inertia. Since, by Mach's view, there is nothing for the object to move with respect to, it can*not* possess motion, thus, inertia for it is meaningless. Only if there is an appreciable distribution of matter within this universe can the standards of nonacceleration and nonrotation shift appreciably such that the object can obtain inertia (ref. [15]). Although Mach doesn't specify *how* this is to be accomplished, some (unknown?) interaction between matter must therefore give rise to inertia. This is seen in the statement that "matter there governs inertia here" (ref. [16]).

There are, of course, numerable objections relating to Mach's Principle that have been raised (refs. [17], [15]). For example, Mach's contention was that space was not a "thing" in its own right, but, rather, dependent on the material objects within it (*i.e.*, he saw space as a 'creation' of the totality of all the distance-relations within that space; ref. [15]). So, for a single object in an otherwise empty universe, Mach would have regarded the extension of space as meaningless. In other words, the dimensionality of such a space would be *zero*. Similarly, for only *two* objects within an otherwise empty universe, only distances along their direction of motion would have any meaning; thus, the dimensionality of such a space is *one*. Likewise, the dimensionality of a universe with *three* objects would be *two*, and the dimensionality of a universe with *four* objects would be *three*.

But if one extends the number of objects to more than four, then one might expect that the dimensionality to extend to more than three. But this is clearly not the case; the dimensionality of space is observed to be finite at *three*[1]. Therefore, one may conclude that space possesses some 'reality' *independent* of the matter within it[2]. Nevertheless, certain (other) aspects of Mach's Principle were seen to be aesthetically pleasing to Einstein (and others), who then tried to incorporate Mach's Principle with the Equivalence Principle into a relativistic formulation of gravity (refs. [14], [15]).

Another way of stating Mach's objection to Newtonian physics (which is particularly relevant to Einstein's development of General Relativity) was the observation that, in Newton's view, a test particle was completely *separate* from the space (manifold) in which it traveled, and yet could experience *inertial effects* as if there *was* some connection to its (absolute) motion through space. This despite the fact that there was no *a priori* connection (in Newtonian physics, at least) between the *extrinsic* (the particle's motion in space) and the *intrinsic* (the inertial state of the particle) (ref. [13]). This was considered to be the most serious Machian critique of Newtonian physics; that (absolute) space could act on matter, but that matter could not act on space, and it was this point that was most readily resolved by Einstein in his General Relativity (ref. [15]).

Einstein tried to satisfy this aspect of Mach's Principle by positing interaction between space(time) and matter. As noted above, the Equivalence Principle

[1]Of course, there is the possibility that space may possess *hidden* dimensions which are somehow collapsed. Indeed, this is the very *study* of this thesis; *five*-dimensional gravity. Nevertheless, even in the most extended (supersymmetry, superstring) theories, the number of spatial dimensions is taken to be *finite*, and *limited*, in contrast to the nearly *unlimited* numbers of particles observed in our universe.

[2]One could argue that, if Mach viewed matter as being intrinsically *dependent* on space (for both its position/motion and inertia), then, from Mach's perspective, matter and space *might be incorporated as the the same entity*, another idea which is explored in this thesis on 5D GR.

implies that gravitation could be understood in terms of curved space(time). Einstein, therefore, tried to couple the geometry of *curved space(time)* to the matter distribution within that space(time).

Einstein already had the four-dimensional view of space-time as developed by Minkowski to explain the effects of Special Relativity, which described the motions of objects in absence of a gravitational field. To then develop a relativistic form of gravity, Einstein applied the laws of Riemannian geometry (the geometry of curved spaces) to the four-dimensional space-time manifold developed by Minkowski to derive the formalism of General Relativity. Written in terms of his postulated Einstein tensor, $G_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R$ (where $g_{\alpha\beta}$ is the usual metric tensor, $R_{\alpha\beta}$ is the usual Ricci tensor, and R is the usual Ricci scalar) [representing the geometry of curved spacetime], coupled to an energy-momentum tensor, $T_{\alpha\beta}$ [representing the local distribution of energy, momentum and matter in space-time and acting as the *source* of the space-time curvature], Einstein wrote his law of gravitation as:

$$G_{\alpha\beta} = T_{\alpha\beta} \tag{2.1}$$

Although Einstein's (four-dimensional) General Relativity did satisfy Mach's critique of 'space acting on matter, but matter unable to act on space', it, nevertheless, fell short of satisfying Mach's Principle completely. For example, in absence of any matter/energy, Mach's view was that there should be *no* space (the aforementioned dependency of space on the material within it). However, the Minkowski solution, which describes an *empty* universe, is a valid solution to the vacuum field equations, $G_{\alpha\beta} = (R_{\alpha\beta} =)0$. Additionally, the Kerr solution, which describes an *isolated* rotating object in an otherwise empty universe is similarly in conflict with Mach's Principle (if the universe is otherwise empty, then the object in question *defines* the state of nonacceleration, *nonrotation*; so, by *definition*, it can*not* rotate) (ref. [15]). (Also see ref. [18] for work analyzing Mach's Principle from the perspective of finding exact solutions to the field equations.)

Objections to this aspect of Einstein's General Relativity may be resolved by imposing Mach's Principle as a sort of 'boundary condition' to select against solutions which are not 'Machian' (ref. [15]). Additionally, while there are anti-Machian solutions to General Relativity, there are also Machian solutions as well. For example, as noted above, the standards of nonrotation will be *defined* by the total mass distribution in the universe. So, near a massive object, one might expect that the (local) standards of nonrotation will be influenced by that object. Especially if such an object is rotating, it should 'drag' the local standards of 'rest' along with it. This effect is predicted in General Relativity, and is known as the Lense-Thirring effect (refs. [14]). Physically, it could be measured, for example, by the precession of a gyroscope in the presence of a strong gravitational field of a rotating object [19]).

However, there are other problems with Mach's Principle which are still not satisfied. Most notably: although there *is* an interaction between the matter and the spatial manifold, the matter (inertia) of the 'source' in 4D GR (represented by the $T_{\alpha\beta}$) is still *separate* from the manifold. There is no *intrinsic connection* between matter and space(time), despite their obvious interaction (ref. [13]). This problem was noted by Einstein, himself, when he referred to the geometrical portion of the field equations (the left-hand side of eq. 2.1; the $G_{\alpha\beta}$) as the "marble" of his theory, but derided to the material portion (the right-hand side of the equation, the $T_{\alpha\beta}$) as its "base-wood" (ref. [20]). Einstein (and others) were, thus, led to try to find a way of *incorporating* (possibly *geometrizing*) the matter (the right-hand side) into the *geometry* (the left-hand side) of the theory.

2.2 Kaluza-Klein Theory

Shortly after the publication of Einstein's General Relativity, Theodor Kaluza, in 1921, expanded Einstein's four-dimensional theory of General Relativity to *five* dimensions, in an attempt to extend Einstein's idea of geometrizing *gravity* to include *electromagnetism*. Kaluza's approach was to incorporate the 4-vector potentials of Maxwell's electromagnetism, the A_μ, along the fifth (off-diagonal) column and row of an extended 5D metric tensor, $\hat{g}_{ab}$, in an expanded 5D general relativistic manifold (refs. [3], [4], [9]);

$$\hat{g}_{ab} = \begin{bmatrix} (g_{\alpha\beta} - A_\alpha A_\beta) & -A_\alpha \\ -A_\beta & -1 \end{bmatrix} \tag{2.2}$$

where the circumflex over the $\hat{g}_{ab}$ distinguishes it as a 5D metric, and where, again, Latin indices such as a and b range over the full five dimensions, $0, 123, 5$, while Greek indices such as α and β range over the four-dimensional subset $0, 123$.

With this metric and a Riemannian geometry expanded by an extra dimension, Kaluza was able to show that the 4D field equations of General Relativity (in the presence of electromagnetic field) emerged as a 4×4 subset of the complete 5×5 set of equations in vacuum. More specifically, for a metric of the type 2.2, which had a constant $\hat{g}_{55}$ and was independent of the fifth coordinate (the 'cylinder condition'), he found that the vacuum equations $\hat{G}_{\alpha\beta} = (\hat{R}_{\alpha\beta} =)0$ (where $\hat{G}_{\alpha\beta}$ are the 4×4 *subset of the* 5D Einstein tensors) could be manipulated into the form (refs. [3], [4], [2], [21]):

$$\hat{G}_{\alpha\beta} = 0 \ \rightarrow \ G_{\alpha\beta} = T_{EM\ \alpha\beta} \tag{2.3}$$

where the $G_{\alpha\beta}$ is the (standard) 4D Einstein tensor and the $T_{EM\ \alpha\beta}$ is the *effective* or *induced* energy-momentum tensor usual for electromagnetism;

$$T_{EM\ \alpha\beta} \equiv -\frac{1}{2}\left[F_{\alpha\lambda}F^{\beta\lambda} - \frac{1}{4}g_{\alpha\beta}F_{\mu\nu}F^{\mu\nu}\right] \tag{2.4}$$

The Einstein summation convention over repeated upper and lower indices is implicitly employed ($F_{\alpha\lambda}F^{\beta\lambda} \equiv \sum_\lambda F_{\alpha\lambda}F^{\beta\lambda}$), and the $F_{\mu\nu}$ is the usual Faraday-Maxwell tensor;

$$F_{\mu\nu} \equiv A_{\nu,\mu} - A_{\mu,\nu} \tag{2.5}$$

In standard comma notation, commas denote ordinary partial derivatives ($A_{\nu,\mu} \equiv \partial_\mu A_\nu \equiv \partial A_\nu/\partial x^\mu$), and where indices on 4D quantities (such as $F_{\mu\nu}$) are raised and lowered by the 4D metric ($g^{\alpha\beta}$ and $g_{\alpha\beta}$).

Kaluza then went on to show that the source-free Maxwell's equations emerged from the '5th component' "off-diagonal" set of the vacuum field equations, $\hat{G}_{5\mu} = (\hat{R}_{5\mu} =)0$ (refs. [3], [4], [2], [21]);

$$\hat{G}_{5\mu} = 0 \ \rightarrow \ F^{\nu}_{\mu;\nu} = 0 \tag{2.6}$$

In standard semicolon notation, the semicolon denotes the *four dimensional* covariant derivative ($F^{\nu}_{\mu;\nu} \equiv \partial_\nu F^{\nu}_{\mu} + F^{\alpha}_{\mu}\Gamma^{\nu}_{\alpha\nu} - F^{\nu}_{\beta}\Gamma^{\beta}_{\mu\nu}$, where $\Gamma^{\lambda}_{\mu\nu}$ is the *four dimensional* Christoffel term; $\Gamma^{\lambda}_{\mu\nu} \equiv \frac{1}{2}g^{\lambda\sigma}[g_{\sigma\mu,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma}]$).

As was discussed in the Introduction, Kaluza and Klein attempted to explain the obvious observed lack of any fifth dimension by 'compactifying' the fifth dimension to the dimensions of Planck length. This 'cylinder condition' then forced the metric to be independent of the fifth coordinate, and also yielded a constant (*minus unity*) for the 5-5 component of the metric if only a single electric or magnetic field component was present (because of the vanishing of $F_{\mu\nu}F^{\mu\nu}$; see eq. 2.7 below). However, as was also discussed in the Introduction, this constraint forced the 'natural' mass scale for particles to then be on the order of Planck mass, an obvious contradiction with reality. Additionally, this approach yields a cosmological constant which is also much larger than is observed (refs. [3], [4]).

Perhaps the most damaging thing for Kaluza-Klein theory was that the final equation worked out from the field equations, $\hat{G}_{55} = (\hat{R}_{55} =)0$, yields (ref. [3], [4], [2]):

$$\hat{G}_{55} = 0 \;\rightarrow\; F_{\mu\nu}F^{\mu\nu} = 0 \tag{2.7}$$

which constituted an obviously unacceptable restriction: a sole electric or magnetic field *could not exist* under this constraint; this would cause A_μ, and, thus, $T_{EM\ \alpha\beta}$, to vanish, creating a *4D* vacuum (ref. [2]). These and other problems caused the original version of Kaluza-Klein theory to fall out of favor with theoretical physicists (ref. [9]).

2.3 Induced Matter Theory

In recent times, contemporary physicists have reexamined Kaluza-Klein theories, even extending them to ten or eleven dimensions to try to unify all the interactions and fundamental particles of nature. In order to address the problems posed by the compactifying of the extra dimensions in Kaluza-Klein theory, some of these contemporary theories have avoided the compactifying problem by avoiding *compactification* altogether. The cylinder condition is, therefore, relaxed, and our reality exists on an *approximate* 4D hypersurface of constant $x^5 \equiv \psi$ in an otherwise 5D manifold (refs. [3], [4], [5]).

Since the theory developed by Kaluza and Klein specifically involved this compactification, then, strictly speaking, it is not appropriate to label these 'non-compactified' multidimensional theories as "Kaluza-Klein". Instead, such theories are better referred to as "extended gravity theories" as they are essentially dimensionally-extended versions of General Relativity.

The specific version of five dimensional extended gravity theory examined here is usually referred to as "Induced Matter" theory, and is due to P. Wesson, J. Ponce de Leon, B. Mashhoon, H. Liu, *etc.* In this version, unlike in the original Kaluza-Klein theory, the 5-5 component of the 5D metric is *not* set to *minus unity*, as it was in eq. 2.2. Instead, the 5-5 component of the metric is allowed to vary, and is usually represented by the square of a scalar field, Φ (ref. [5]);

$$\hat{g}_{ab} = \begin{bmatrix} (g_{\alpha\beta} + \epsilon\Phi^2 A_\alpha A_\beta) & \epsilon\Phi^2 A_\alpha \\ \epsilon\Phi^2 A_\beta & \epsilon\Phi^2 \end{bmatrix} \tag{2.8}$$

where, again, ϵ is the signature of the fifth dimension, and is either "+" or "−" depending on whether the fifth dimension is timelike or spacelike (the latter usually being chosen, ref. [8]).

The corresponding inverse metric is given as:

$$\hat{g}^{ab} = \begin{bmatrix} g^{\alpha\beta} & -A^\alpha \\ -A^\beta & (\epsilon\Phi^{-2} + A_\lambda A^\lambda) \end{bmatrix} \tag{2.9}$$

Additionally, the Induced Matter metric is allowed to depend on the fifth coordinate, ψ. [This potential dependency of the metric on the fifth coordinate allows one to describe states of matter other than radiation. As was discussed in the Introduction, and is shown later in the next section, when the metric is independent of ψ, the natural equation of state for the matter induced is that of radiation[3]. By allowing the metric to depend on the fifth coordinate can one induce states of 'matter' that aren't pure radiation (ref. [8]).]

In the Induced Matter theory, the universe is again assumed to be a vacuum in five dimensions, $\hat{G}_{ab} = (\hat{R}_{ab} =)0$, and these 15 equations are again split into the 10 equations of 4D GR, 4 Maxwellian equations, and a scalar field equation. This time, though, because $\hat{g}_{55} = \epsilon\Phi^2$ and because the metric $\hat{g}_{ab}$ depends on ψ, the results are *not* so restrictive as to force a 4D vacuum. For the 4×4 spacetime subset of the vacuum field equations, $\hat{G}_{\alpha\beta} = (\hat{R}_{\alpha\beta} =)0$, we get:

$$G_{\alpha\beta} = T_{\alpha\beta} \tag{2.10}$$

where, in *this* case, $T_{\alpha\beta}$ is given by:

$$T_{\alpha\beta} = -\epsilon\Phi^2 T_{EM\ \alpha\beta} + \Phi^{-1}\left[\Phi_{,\alpha;\beta} - g_{\alpha\beta}\Phi_{;\mu}^{;\mu}\right] + (\psi - dependent\ terms) \tag{2.11}$$

where $T_{EM\ \alpha\beta}$ is as defined in eq. 2.4 (ref. [2]).

The equations found from $\hat{G}_{5\mu} = (\hat{R}_{5\mu} =)0$ can then be rewritten as:

$$F^{\nu}_{\mu;\nu} = -3\Phi^{-1}\Phi^{,\nu}F_{\nu\mu} + (\psi - dependent\ terms) \tag{2.12}$$

[3]The theorem proving a radiative equation of state for metrics independent of the fifth coordinate applies *only* to *diagonal* metrics. As will be discussed in the next section, and in the next chapter, the existence of *off*-diagonal metric elements can *also* allow for nonradiative equations of state, even if the metric does *not* depend on the fifth coordinate.

and the final equation, derived from $\hat{G}_{55} = (\hat{R}_{55} =)0$, can be rewritten as a scalar wave equation;

$$\Phi_{;\lambda}^{;\lambda} = -\frac{1}{4}\Phi^3 F_{\mu\nu}F^{\mu\nu} + (\psi - dependent\ terms) \tag{2.13}$$

Clearly, if the metric is independent of ψ and $\epsilon\Phi^2 = -1$, then the eqs. 2.10, 2.12 and 2.13 (and 2.11) reduce to their Kaluza-Klein counterparts, eqs. 2.3, 2.6 and 2.7 (and 2.4). But here, one does not have the unacceptable restriction of having $F_{\mu\nu}F^{\mu\nu}$ being zero.

2.3.1 Induced Matter

As was mentioned in the previous section, the allowance of the metric to depend on the fifth coordinate, ψ, permits one to describe states of matter other than radiation. In ref. [8], this theorem was demonstrated by assuming a 5D spherically-symmetric, diagonal metric defined as:

$$d\hat{s}^2 = e^{\nu}dt^2 - e^{\lambda}dr^2 - R^2 d\Omega^2 + \epsilon e^{\mu}d\psi^2 \tag{2.14}$$

where $d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2$ and where ν, λ, R and μ are all general functions of r, t and ψ.

The energy-momentum tensor can then be calculated for this metric. Since the metric is diagonal, the energy-momentum tensor can be written as (ref. [8]):

$$\begin{aligned} T_{\alpha\beta} = \tfrac{\Phi_{,\alpha;\beta}}{\Phi} - \tfrac{\epsilon}{2\Phi^2}\Big[\tfrac{\overset{*}{\Phi}\overset{*}{g}_{\alpha\beta}}{\Phi} - \overset{**}{g}_{\alpha\beta} + g^{\mu\nu}\, \overset{*}{g}_{\alpha\mu}\overset{*}{g}_{\beta\nu} \\ -\tfrac{1}{2}g^{\mu\nu}\, \overset{*}{g}_{\mu\nu}\overset{*}{g}_{\alpha\beta} + \tfrac{1}{4}g_{\alpha\beta}\, \overset{*}{g}{}^{\mu\nu}\overset{*}{g}_{\mu\nu} + \tfrac{1}{4}g_{\alpha\beta}(g^{\mu\nu}\, \overset{*}{g}_{\mu\nu})^2\Big] \end{aligned} \tag{2.15}$$

where overstars are partial derivatives with respect to ψ, and the trace can be found as (ref. [8]):

$$T^{\mu}_{\mu} = -2\epsilon e^{-\mu}\left[\frac{\overset{*}{\nu}\overset{*}{\lambda}}{4} + \frac{\overset{**}{R}\overset{*}{\nu}}{R} + \frac{\overset{**}{R}\overset{*}{\lambda}}{R} + \frac{\overset{*}{R}^2}{R^2}\right] \tag{2.16}$$

In order to physically interpret the energy-momentum tensor, it is customary to model it to some physically known (and realistic) model. If one models the energy-momentum tensor with that of a fluid (whether perfect or imperfect, charged or neutral), the trace of the energy-momentum tensor can then be identified as: $T^{\mu}_{\mu} = \wp - 3P$, where $\wp$ is the density and P is the pressure of the fluid (the density $\wp$ is written in *script* in order to distinguish it from the cylindrical coordinate ρ used in chapter 4 to represent axial-symmetric solutions). Comparing this with eq. 2.16, one can write:

$$\wp - 3P = -2\epsilon e^{-\mu}\left[\frac{\overset{*}{\nu}\overset{*}{\lambda}}{4} + \frac{\overset{**}{R}\overset{*}{\nu}}{R} + \frac{\overset{**}{R}\overset{*}{\lambda}}{R} + \frac{\overset{*}{R}^2}{R^2}\right] \tag{2.17}$$

Since the equation of state of radiation is: $P = \wp/3$, it is clear that if the 4D metric coefficients (ν, λ and R) are independent of ψ, the metric will simply represent the state of radiation. Conversely, if these metric coefficients depend on ψ, it will be possible to describe states of matter other than radiation, such as 'dust' ($P = 0$, $\wp \neq 0$), vacuum ($P = -\wp$), 'absolute vacuum' ($P = \wp = 0$) and 'stiff matter' ($P = \wp$) (ref. [8]).

[Note: Since pressure is kinetic energy density, negative pressure is usually considered to be unphysical. Negative pressure can arise, however, when the kinetic energy density of a given medium is *less* than that of the background space. (In GR, it is possible for the space to possess an energy density.)]

[Additionally, the equation of state: $P = -\wp/3$ has also appeared elsewhere in the analysis of 5D GR (see appropriate references in [8]). This state of matter is of interest since the (4D GR) gravitational mass is proportional to $3P + \wp$, and its vanishing indicates the presence of matter which has no (4D) gravitational effects. It is noted here because it appears as a limiting case to the solutions found in chapter 3. Note: Though this solution represents matter which has no (4D) gravitational mass, it is still *matter*, and not *radiation*, which is represented by the equation: $P = \wp/3$.]

It should be noted, of course, that it is possible for the metric to depend on ψ, and yet *still* have the equation of state of radiation. This is possible if the bracketed expression on the right-hand side of eq. 2.17, $\frac{\overset{*}{\nu}\overset{*}{\lambda}}{4} + \frac{\overset{*}{R}\overset{*}{\nu}}{R} + \frac{\overset{*}{R}\overset{*}{\lambda}}{R} + \frac{\overset{*}{R}{}^2}{R^2}$, happens to vanish. It is also possible for the metric to be independent of ψ, and yet *not* possess a radiative equation of state. This is possible if the metric happens to be *off*-diagonal, since the theorem as it's outlined here (and as it was derived it ref. [8]) is only relevant to *diagonal* metrics. The fact that off-diagonal metrics can yield *non*radiative equations of state is an important discovery for this thesis, and will be noted again in the next chapter.

2.3.2 Machian (Inertial) Aspects of Induced Matter Theory

This approach of the Induced Matter Theory not only has the advantage of unifying gravitation and electromagnetism (as the original Kaluza-Klein theory attempted) *without* the unacceptable restrictions of the original Kaluza-Klein theory, but also has the potential of addressing Mach's concern of ensuring that the 'local' properties of matter (as defined by the energy-momentum tensor, T^{ν}_{μ}) depend on the *global* distribution of matter.

By defining the 'local' properties of matter (the $T_{\mu\nu}$) in terms of (curvature in) the fifth dimension (see eq. 2.11), then the local (inertial) properties of matter can be related to the $\hat{g}_{ab}$ (the *global* solutions found to solve the 5D field equations). Since these global solutions (the $\hat{g}_{ab}$) represent the *total* 5D manifold curvature, this creates an intimate relation between the the inertial properties of matter at any one point and the *total* distribution of matter throughout the manifold (represented as the curvature throughout that manifold) (ref. [6]).

This differs from the analogous 4D approach in which the $T_{\mu\nu}$ can be defined solely in terms of the 4D $G_{\mu\nu}$. In that approach, the matter/energy (the source) is separate from the manifold. As such, the difference $G_{\mu\nu} - T_{\mu\nu}$ (equal to $\hat{G}_{\mu\nu}$ in 5D) is *not* constrained as it is in 5D (in 5D, $G_{\mu\nu} - T_{\mu\nu} = \hat{G}_{\mu\nu} = 0$; in 4D, no such constraint exists). Because of this constraint in 5D, it is typical to find an equation of state (*e.g.*, a relation between P and $\wp$) imposed on the source, which, in 4D, has to be imported from outside.

In fact, the Induced Matter Theory goes even further, answering Mach's concern of a lack of fundamental *connection* between the *intrinsic* (inertial) state of matter and its *extrinsic* position/motion in space(time). Whereas in both Newtonian *and* Einsteinian gravity, the sources of gravitational interaction (*i.e.*, the *matter*) are introduced in the theory in a somewhat "ad hoc" 'external' manner into their spatial (and temporal) background, for the Induced Matter Theory, there *are no* sources; the universe is assumed to be *vacuum*, with matter on a 4D level *appearing* to result from the curvature in the fifth dimension.

As was discussed in section 2.1, Mach's assumption that space(time) could not exist without matter, *and* his assumption that matter was dependent on *space*(time) for its position/motion *and* inertia could lead one to imagine that space(time) *and* matter are *effectively* the *same thing*. This is *precisely* the tack taken by the

Induced Matter Theory as it assumes that matter *results* from the curvature of an (otherwise) empty 5D manifold. This approach also has the aesthetically-pleasing effect of finally addressing Einstein's concern of *geometrizing* the 'basewood' (right-hand side) of his equations.

It must be noted that not *all* 5(or higher)D gravity theories possess this Machian aspect. The Induced Matter Theory is Machian in the sense outlined here because it assumes a vacuum state in 5D which then *appears* to contain matter in 4D. Not all 5D theories assume a vacuum state. For example, the 5D gravity theory worked out by E. Leibowitz and N. Rosen assumes the existence of a 5D energy-momentum tensor, $\hat{T}_{\mu\nu}$, which couples to the 5D Einstein tensor as (ref. [22]; see also refs. in [23]):

$$\hat{G}_{\mu\nu} = \hat{T}_{\mu\nu} \tag{2.18}$$

Though this approach allows for many more, less restrictive solutions than those required for the *vacuum* field equations of the Induced Matter Theory, it, nevertheless, has the (d)effect of having to introduce a new 5D source term into the equations, $\hat{T}_{\mu\nu}$. Because of this, this version of 5D gravity theory is *not* Machian in the sense that the Induced Matter Theory is. It is precisely *because* it is vacuum that the Induced Matter Theory is Machian as outlined in this section.

In this thesis, special concentration shall be given to metrics which describe *matter* as deduced from the Induced Matter Formalism. In describing such distributions of matter, it will be seen how their *local* inertial properties (such as density, pressure, *etc.*) relate to the *global* solutions derived as solutions to the 5D vacuum field equations, which are the basis of the Induced Matter formalism.

In considering Mach's Principle as a motivation for research, it is noted that

Mach, himself, concentrated his concerns specifically on trying to relate the inertial properties of *matter* to the global distribution of the universe. The idea of relating *radiation* to the global distribution of the universe, or even in trying to *explain* radiation in any such terms seems to have been overlooked by Mach in much, if not all of his writing (refs. [15], [17], [14]). Not that it isn't possible to apply Mach's Principle to radiation in *some* sense; by Special Relativity, radiation can be endowed with an *effective* mass resulting from its energy content, which could then, in principle, be related to the global distribution of the universe. But Mach applied the term 'mass' in its *inertial* context as "resistance to motion" (ref. [17]), which is *inapplicable* to radiation. Since radiation (*e.g.*, photons) travel with*out* inertial resistance (*i.e.*, at the maximal speed, c), they are presummed to be without inertial mass of the kind in ordinary matter.

It is, therefore, traditional to speak of Mach's Principle as applying to the inertial properties of *matter*, and not radiation. For that reason, the metrics that are studied in this thesis represent solutions that are *not* those of radiation. Additionally, metrics recognized as describing radiation (*e.g.*, the Gross-Perry/Davidson-Owen solutions) have been largely analyzed in depth (ref. [10]; see also refs. in [8]), while nonradiative solutions have not (see refs. [24] and [10] for few examples). For that reason, also, nonradiative solutions, and their inertial properties shall be the study of this thesis.

As well, most metrics studied (in the Induced Matter formalism) have also been *neutral*, without any electromagnetic fields (see refs. [25] and [2] for few examples). As it turns out, the electromagnetic component of mass is an important factor in associating inertia (mass) to solutions which are *independent* of ψ; such solutions are examined in both chapters 3 and 4. It is because such electromagnetic metrics have not been not been studied (or not studied *well*) that charged/magnetized metrics will also figure prominently in this thesis.

2.3.3 Energy-Momentum Tensor

In order to fully describe the physical characterististics of the matter represented locally by the (induced) energy-momentum tensor, $T_{\mu\nu}$, it is necessary to model the induced energy-momentum tensor with a physically realistic form. As noted above, this has traditionally been the model of a perfect fluid;

$$T^{\nu}_{fl\ \mu} = \wp u_{\mu} u^{\nu} + P h^{\nu}_{\mu} \tag{2.19}$$

where $h^{\nu}_{\mu} \equiv u_{\mu} u^{\nu} - \delta^{\nu}_{\mu}$ (δ^{ν}_{μ} being the Kronecker delta, $= 1$ if $\mu = \nu$, $= 0$ otherwise) and the u^{μ} ($\equiv \frac{dx^{\mu}}{ds}$) are the *four* velocities of the fluid (the relation between the four velocities, $u^{\alpha} \equiv \frac{dx^{\alpha}}{ds}$, and the *five* velocities, $\hat{u}^{a} \equiv \frac{dx^{a}}{d\hat{s}}$, is discussed in the next section).

If one then chooses a co-moving reference frame such that $u^0 \neq 0$, while $u^1 = u^2 = u^3 = 0$, *i.e.*, no spatial motion[4], one can then show (assuming $u^{\alpha} u_{\alpha} = 1$; see next section) that:

$$\wp = T^{0}_{fl\ 0} \tag{2.20}$$

and:

$$P = -T^{1}_{fl\ 1} = -T^{2}_{fl\ 2} = -T^{3}_{fl\ 3} \tag{2.21}$$

[4]Even if this is not strictly the case, it can still be seen that the spatial velocities, u^1, u^2 and u^3 will typically be so small compared to u^0 that $u^1 \simeq u^2 \simeq u^3 \simeq 0$ will be a reasonable approximation.

However, for metrics which are not isotropic (such as the ones in chapters 3 and 4), it can be seen that, in *general*, $T^1_{fl\ 1} \neq T^2_{fl\ 2} \neq T^3_{fl\ 3}$, so that the definition of eq. 2.21 becomes problematic.

Three possible solutions are used to resolve this. First, one can define an *effective* pressure, P_{eff}, which is equal to (see refs. in [11]):

$$P_{eff} = -\frac{1}{3}(T^1_{fl\ 1} + T^2_{fl\ 2} + T^3_{fl\ 3}) \tag{2.22}$$

Alternatively, one can define orthogonal 'components' to the pressure, so that one has a pressure 'parallel' to the radial line of sight, $P_{\|}$, and another 'perpendicular' to it, $P_{\perp}$ (ref. [4]). With this, one can obtain results analogous to those obtained by assuming an effective pressure (described above). For example, for a diagonal (nonisotropic) metric independent of ψ, one obtains the equation of state: $\wp = P_{\|} + 2P_{\perp}$, which, for $P_{eff} = \frac{1}{3}(P_{\|} + 2P_{\perp})$, yields the usual equation of state of radiation, $\wp = 3P_{eff}$ (ref. [4]).

However, a more complete and more satisfying resolution to this problem is to consider an *imperfect* fluid model, with the introduction of *anisotropic stresses*. In ref. [11] (and also in ref. [12]), such an imperfect fluid model is used for the first time in connection with the Induced Matter formalism. There, the energy-momentum tensor is written out as:

$$T_{fl\ \mu\nu} = \wp u_\mu u_\nu + P h_{\mu\nu} + 2q_{(\mu}u_{\nu)} + \tau_{\mu\nu} \tag{2.23}$$

where q_μ is the heat flux and $\tau_{\mu\nu}$ is the anisotropic stress tensor.

Heat flux, in the context of a(n imperfect) fluid, is well-understood (see refs. [16], [26] and [27]), and vanishes in the co-moving cases studied here (see below).

The stress tensor, as it pertains to fluid mechanics, traditionally includes both *normal* stresses (pressure) and *shear* stresses (strains) (refs. [26], [27]). The *off*-diagonal components of the stress tensor correspond to shear strain, while the *diagonal* components correspond to pressure.

However, if the three 'components' of the pressure are not all equal (as in the cases to be studied here), it is assumed that there are *anisotropic normal stresses* in the fluid. These *anisotropic normal stresses* are then included in the diagonal components of the stress tensor. The diagonal components of the stress tensor then correspond to a *mean* pressure plus individual components of *normal stress* which may cause the individual 'components' of pressure to differ from the mean pressure (ref. [26]).

In this sense, the approach here is similar to the one outlined previously, in which one assumes the existence of (parallel and perpendicular) 'components' of pressure whose average corresponds to a 'mean' pressure. Where this approach is superior is in its methodology; because it sets up a stress tensor it allows for the possible introduction of *shear* stresses, as opposed to just the normal stresses and pressure encountered here. The 'parallel/perpendicular' splitting of pressure (though it answers the problem of nonidentical pressure components) has no recourse to the concept of shear stress. In future work, this might be of some use were one to examine imperfect fluid properties for which shear stresses were present.

So, in the approach outlined here, the 'mean' pressure, P, is (mathematically) separated out from the rest of the stress tensor; the diagonal components of the stress tensor are left representing the components of *anisotropic normal stress.* (The off-diagonal components of the stress tensor continue to represent the shear strain.) The pressure, P, is explicitly written out separately from the anisotropic stress tensor, $\tau_{\mu\nu}$, as shown in eq. 2.23 (ref. [11]).

Physically, the (remaining) components of the stress tensor of a fluid are assumed to be related to interactions between the fluid particles. These interactions define the nature of the fluid, destroying the otherwise 'perfect' nature of the fluid and, macroscopically, giving rise to viscosity effects (refs. [26], [27]). In standard Newtonian fluids, the stress tensor is *assumed* to be proportional to the shear tensor (defined as the rotational rate of deformation of a fluid element); the proportionality constant being (twice) the (Newtonian) viscosity. In relativistic notation, the shear tensor can be written out as (refs. [11], [16]):

$$\sigma_{\alpha\beta} = u_{(\alpha;\beta)} - \overset{\odot}{u}_{(\alpha}\, u_{\beta)} - \frac{1}{3} u^{;\mu}_{\mu} h_{\alpha\beta} \tag{2.24}$$

where the over*circle-dot* denotes a derivative with respect to the *four* dimensional interval, s (*elsewhere* in this thesis, such derivatives are with respect to the *five* dimensional interval, $\hat{s}$, and are represented by over*dots* only). (Semi-colons again represent covariant 4D derivatives, and, again, the Einstein summation is implicitly employed over repeated upper and lower indices.) Here, it can be worked out that: $\overset{\odot}{u}_{\alpha} = u_{\alpha;\gamma} u^{\gamma}$.

However, the assumption that the stress tensor is proportional to the shear tensor is an *assumption* made by studying classical fluids (ref. [26]). It is known that *non*Newtonian fluids exist which do *not* satisfy this relation, and, therefore, we should be wary about implicitly assuming that the fluids to be studied here are *Newtonian* (ref. [27]). (The assumption that the stress tensor is proportional to the shear tensor is the fluid mechanical equivalent of the assumption in mechanics that friction is proportional to the normal force.)

In fact, it turns out that one can*not* write $\tau_{\alpha\beta} \propto \sigma_{\alpha\beta}$ for any of the solutions found in this thesis. Indeed, for the cases examined here (in chapters 3 and 4), it is found that $\sigma_{\alpha\beta}$ *vanishes.* Consider: assume one has a co-moving frame, such that $u^1 = u^2 = u^3 = 0$ and $u^0(\neq 0) = g_{00}^{-1/2}$ (deducible from a co-moving 4D metric), where g_{00} is dependent on the radial coordinate (and not on time). Then one can show, for the terms in eq. 2.24, that: (i) $u_{(\alpha;\beta)}$ vanishes for all components, except: $u_{(0;1)} = u_{(1;0)} = \frac{1}{2}u_{0,1} = \frac{1}{4}u_0 g^{00} g_{00,1}$, (ii) $\overset{\odot}{u}_{(\alpha}\, u_{\beta)}$ vanishes for all components except: $\overset{\odot}{u}_{(0}\, u_{1)} = \overset{\odot}{u}_{(1}\, u_{0)} = \frac{1}{4}u_0 g^{00} g_{00,1}$, and (iii) $u^{;\mu}_{\mu} = 0$.

Therefore, $u_{(\alpha;\beta)}$ and $\overset{\odot}{u}_{(\alpha}\, u_{\beta)}$ substituted into eq. 2.24, for $\sigma_{10} = \sigma_{01}$, will cancel out, thereby causing $\sigma_{\alpha\beta}$ to vanish completely ($u^{;\mu}_{\mu}$, by (iii), vanishes also, so it does not contribute).

The reason that none of these solutions can be written in standard Newtonian form is because the velocity cannot be written as a gradient of a scalar (velocity) potential, Υ, as: $u^\alpha = g^{\alpha\beta}\Upsilon_{,\beta}$, which is standard for Newtonian fluids (refs. [27], [26]). This does not indicate a deficiency of this approach, but shows that the viscosity effects (presummed to exist because of the presence of anisotropic stress) can*not* be modeled in the simple, Newtonian fashion. Since the solutions of both chapters 3 and 4 involve 'nonstandard' electric and magnetic fields, respectively, we can assume that the *intra-fluid interactions* are more complicated than in simple (viscous) Newtonian fluids.

However, going so far as to *model* such interactions would require a detailed set of assumptions about the structure and nature of the fluid *particles*, and would bring one down to molecular and atomic levels which are beyond the scope of this treatment (this being a 5D *classical* treatment, not a *quantum* treatment). Therefore, the stresses (which *represent* these intra-fluid interactions) for all cases will be calculated and mentioned, though no modeling of them will be made.

In ref. [11], it is shown, by taking projections along and orthogonal to the velocity field and taking traces, that:

$$\wp = T_{fl\ \mu\nu} u^\mu u^\nu \tag{2.25}$$

$$q_\alpha = -T_{fl\ \mu\nu} u^\mu h^\nu_\alpha \tag{2.26}$$

$$P = -\frac{1}{3}\Pi^\alpha_\alpha \tag{2.27}$$

$$\tau_{\alpha\beta} = \Pi_{\alpha\beta} - P h_{\alpha\beta} \tag{2.28}$$

where $\Pi_{\alpha\beta}$ is:

$$\Pi_{\alpha\beta} \equiv P h_{\alpha\beta} + \tau_{\alpha\beta} = T_{fl\ \mu\nu} h^\mu_\alpha h^\nu_\beta \tag{2.29}$$

[In deriving these relations (eqs. 2.25 to 2.28), it was assumed (among other things) that the trace of the stress tensor vanishes ($\tau^\mu_\mu = 0$) (ref. [11]). As noted above, the diagonal components of the stress tensor are *normal* stresses which cause the individual 'components' of pressure to differ from the mean pressure, P. However, the average of these three 'components', $\tau^1_1 + P$, $\tau^2_2 + P$, $\tau^3_3 + P$, should yield the mean pressure, P. This is only possible if $\tau^1_1 + \tau^2_2 + \tau^3_3 = 0$.]

Applying these relations to the co-moving reference frame just outlined (and assuming the $T^\nu_{fl\ \mu}$ are diagonal), one can see that the pressure *can* then be shown to be: $P = -\frac{1}{3}(T^1_{fl\ 1} + T^2_{fl\ 2} + T^3_{fl\ 3})$ (where the differences in $T^1_{fl\ 1} \neq T^2_{fl\ 2} \neq T^3_{fl\ 3}$ can now be attributed to the presence of anisotropic stresses, and, hence, viscosity). In such a co-moving frame (with diagonal $T^\nu_{fl\ \mu}$'s), the stresses will be: $\tau^1_1 = \frac{2}{3}T^1_{fl\ 1} - \frac{1}{3}\left[T^2_{fl\ 2} + T^3_{fl\ 3}\right]$, $\tau^2_2 = \frac{2}{3}T^2_{fl\ 2} - \frac{1}{3}\left[T^1_{fl\ 1} + T^3_{fl\ 3}\right]$, $\tau^3_3 = \frac{2}{3}T^3_{fl\ 3} - \frac{1}{3}\left[T^1_{fl\ 1} + T^2_{fl\ 2}\right]$, with all other τ^ν_μ's equal to zero.

It can also be seen, incidentally, that for a co-moving reference frame, $\wp$ will still be $T^0_{fl\ 0}$, while q_α will be zero.

For the metrics examined in chapters 3 and 4, electromagnetic fields are present, so it will be necessary to modify the expression of the energy-momentum tensor to account for this. In ref. [28], the energy-momentum tensor of a charged perfect fluid is shown to be a linear combination of the electromagnetic energy-momentum tensor (as given in eq. 2.4) and the perfect fluid energy-momentum tensor. The presence of (viscous) stresses is the only difference here between perfect and imperfect fluids. Since (viscous) stresses should not affect the linear superposition of the electromagnetic field and the fluid in the energy-momentum tensor, we will assume an energy-momentum tensor of the form:

$$T^{\nu}_{\mu} = T^{\nu}_{EM\ \mu} + T^{\nu}_{fl\ \mu} \tag{2.30}$$

where $T^{\nu}_{fl\ \mu}$ is the imperfect fluid energy-momentum tensor defined by eq. 2.23 and where $T^{\nu}_{EM\ \mu}$ is the electromagnetic energy-momentum tensor defined by:

$$T^{\nu}_{EM\ \mu} \equiv \frac{1}{2}\epsilon\Phi^2 \left[F_{\alpha\lambda}F^{\beta\lambda} - \frac{1}{4}g_{\alpha\beta}F_{\mu\nu}F^{\mu\nu}\right] \tag{2.31}$$

Incidentally, this definition of $T^{\nu}_{EM\ \mu}$ differs from the electromagnetic energy-momentum tensor given by eq. 2.4 by a factor of $-\epsilon\Phi^2$. This is because the electromagnetic component of the Induced Matter energy-momentum tensor, as defined by eq. 2.11, can be seen to be just this quantity (eq. 2.31). In making identifications of certain quantities in five dimensions (such as $T_{EM\ \mu\nu}$), we note that they may have different definitions than those same quantities defined in four dimensions. This is considered reasonable provided they correspond in the appropriate 5D $\rightarrow$ 4D limit (ref. [5]).

This point cannot be overstressed. All the solutions and all work that are derived here are done in *five* dimensions. The metrics satisfy *five* dimensional vacuum field equations, and they are presumed to describe *five* dimensional geodesic motion. It is the 5D quantities that are *exact*. As such, when these solutions are then perceived from a *four* dimensional perspective, quantities that are usually defined in 4D (such as the electromagnetic energy-momentum tensor) cannot be expected to possess *precisely* the same form as the corresponding quantities in 5D. In a 5D manifold, quantities defined in 4D are regarded as the *approximations*. Provided, however, that such definitions coincide in the appropriate limit, the 4D reality we *appear* to live in is then believed to be a reasonable approximation of the underlying 5D manifold.

From the assumed form of the energy-momentum tensor given in eq. 2.31, the expressions for $T^{\nu}_{fl\ \mu}$ used in the definitions of $\wp$, P, τ and q_α can then be replaced by: $T^{\nu}_{fl\ \mu} = T^{\nu}_{\mu} - T^{\nu}_{EM\ \mu}$ where T^{ν}_{μ} is calculated from the metric, using either eq. 2.10 or eq. 2.11, and $T^{\nu}_{EM\ \mu}$ is calculated from eq. 2.31. The resulting expressions for density, pressure and stress in a co-moving reference frame then become:

$$\wp = T^{0}_{fl\ 0} = T^{0}_{0} - T^{0}_{EM\ 0} \tag{2.32}$$

$$P = -\frac{1}{3}T^{i}_{fl\ i} = -\frac{1}{3}T^{i}_{i} + \frac{1}{3}T^{i}_{EM\ i} \tag{2.33}$$

$$\tau^{1}_{1} = \frac{2}{3}T^{1}_{1} - \frac{2}{3}T^{1}_{EM\ 1} - \frac{1}{3}\left[T^{2}_{2} + T^{3}_{3}\right] + \frac{1}{3}\left[T^{2}_{EM\ 2} + T^{3}_{EM\ 3}\right] \tag{2.34}$$

$$\tau^{2}_{2} = \frac{2}{3}T^{2}_{2} - \frac{2}{3}T^{2}_{EM\ 2} - \frac{1}{3}\left[T^{1}_{1} + T^{3}_{3}\right] + \frac{1}{3}\left[T^{1}_{EM\ 1} + T^{3}_{EM\ 3}\right] \tag{2.35}$$

$$\tau^{3}_{3} = \frac{2}{3}T^{3}_{3} - \frac{2}{3}T^{3}_{EM\ 3} - \frac{1}{3}\left[T^{1}_{1} + T^{2}_{2}\right] + \frac{1}{3}\left[T^{1}_{EM\ 1} + T^{2}_{EM\ 2}\right] \tag{2.36}$$

For a radially-dependent, spherically-symmetric system, it can be deduced that (ref. [16]): $T^{0}_{EM\ 0} = T^{1}_{EM\ 1} = -T^{2}_{EM\ 2} = -T^{3}_{EM\ 3}$ (where, again, sub/superscripts $0, 123$ represent $t, r\theta\phi$ for a spherically-symmetric system), and $T^{3}_{3} = T^{2}_{2}$. For such a system (which will be examined in the next chapter), the eqs. 2.32 to 2.36 become:

$$\wp = T_0^0 - T_{EM} \tag{2.37}$$

$$P = -\frac{1}{3}T_i^i - \frac{1}{3}T_{EM} \tag{2.38}$$

$$\tau_1^1 = \frac{2}{3}(T_1^1 - T_2^2) - \frac{4}{3}T_{EM} \tag{2.39}$$

$$\tau_2^2 = \tau_3^3 = \frac{1}{3}(T_2^2 - T_1^1) + \frac{2}{3}T_{EM} = -\frac{1}{2}\tau_1^1 \tag{2.40}$$

where: $T_{EM} \equiv T^0_{EM\ 0}$.

For a radially-dependent, cylindrically-symmetric system, it can similarly be deduced that (ref. [16]): $T^0_{EM\ 0} = -T^1_{EM\ 1} = T^2_{EM\ 2} = -T^3_{EM\ 3}$ (where, for a cylindrically-symmetric system, sub/superscripts $0, 123$ represent $t, \rho\phi z$). For such a system (which will be examined in the chapter 4), the eqs. 2.32 to 2.36 become:

$$\wp = T_0^0 + T_{EM} \tag{2.41}$$

$$P = -\frac{1}{3}T_i^i + \frac{1}{3}T_{EM} \tag{2.42}$$

$$\tau_1^1 = \frac{2}{3}T_1^1 - \frac{1}{3}(T_2^2 + T_3^3) - \frac{2}{3}T_{EM} \tag{2.43}$$

$$\tau_2^2 = \frac{2}{3}T_2^2 - \frac{1}{3}(T_1^1 + T_3^3) + \frac{4}{3}T_{EM} \tag{2.44}$$

$$\tau_3^3 = \frac{2}{3}T_3^3 - \frac{1}{3}(T_1^1 + T_2^2) - \frac{2}{3}T_{EM} \tag{2.45}$$

where here: $T_{EM} \equiv -T^0_{EM\ 0}$.

From these expressions for density, pressure and stress, the physical (and inertial) nature of the fluid can be understood.

2.3.4 Geodesic Motion

Finally, we consider the geodesic motion of a test object traveling through the manifold. First we note from the metric definition given in eq. 2.8, one can write the 5D interval as:

$$d\hat{s}^2 \equiv \hat{g}_{ab} dx^a dx^b = ds^2 + \epsilon\Phi^2(d\psi + A_\mu dx^\mu)^2 \tag{2.46}$$

where the 4D interval, ds^2, is given by:

$$ds^2 \equiv g_{\mu\nu} dx^\mu dx^\nu \tag{2.47}$$

Then, we note that the 5D geodesic equation for a test particle traveling through the manifold can be written as:

$$\frac{d^2x^a}{d\hat{s}^2} + \hat{\Gamma}^a_{bc}\frac{dx^b}{d\hat{s}}\frac{dx^c}{d\hat{s}} = 0 \tag{2.48}$$

where the $\hat{\Gamma}^a_{bc}$ are the five-dimensional Christoffel terms defined by: $\hat{\Gamma}^a_{bc} \equiv \frac{1}{2}\hat{g}^{ad}[\hat{g}_{db,c} + \hat{g}_{dc,b} - \hat{g}_{bc,d}]$.

The fifth component of the 5D geodesic equation can be written in the form:

$$\frac{dB}{d\hat{s}} = \frac{1}{2}\frac{\partial \hat{g}_{ab}}{\partial\psi}\frac{dx^a}{d\hat{s}}\frac{dx^b}{d\hat{s}} \tag{2.49}$$

where B is a scalar function defined by (ref. [5]):

$$B \equiv \epsilon\Phi^2\left(\frac{d\psi}{d\hat{s}} + A_\mu\frac{dx^\mu}{d\hat{s}}\right) \tag{2.50}$$

and which is a constant if the metric is independent of the fifth coordinate, ψ.

In terms of this scalar function B, the four-components of the 5D geodesic equation, eq. 2.48, can then be written out as (ref. [5]):

$$\begin{aligned}\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = &\frac{B}{(1-\epsilon B^2/\Phi^2)^{1/2}}\left[F^\mu_\nu\frac{dx^\nu}{ds} - \frac{A^\mu}{B}\frac{dB}{ds} - g^{\mu\lambda}\frac{\partial A_\lambda}{\partial\psi}\frac{d\psi}{ds}\right] \\ &+\frac{\epsilon B^2}{(1-\epsilon B^2/\Phi^2)\Phi^3}\left[\Phi^{;\mu} + \left(\frac{\Phi}{B}\frac{dB}{ds} - \frac{d\Phi}{ds}\right)\frac{dx^\mu}{ds}\right] - g^{\mu\lambda}\frac{\partial g_{\lambda\nu}}{\partial\psi}\frac{dx^\nu}{ds}\frac{d\psi}{ds}\end{aligned} \tag{2.51}$$

The left-hand side of this equation is the 'standard' 4D geodesic equation representing the acceleration of a test particle due to 'standard' Einsteinian gravity (all quantities on the left-hand side are 4D quantities). The terms on the right-hand side of the equation, therefore, represent *modifications* to the classical 4D geodesic motion due to the fact that we are actually dealing with five dimensional motion. (Again, it is noted that we are rewriting quantities *properly* defined in 5D into forms that *resemble* those defined in 4D.) By inspection, the first term on the right-hand side of the equation is clearly the form of the Lorentz force, assuming one identifies the factor out in front as the charge-to-mass ratio of the test particle;

$$\frac{q}{m} \Rightarrow \frac{B}{(1-\epsilon B^2/\Phi^2)^{1/2}} \tag{2.52}$$

This shows how one might relate the charge-to-mass ratio of a test particle within a given manifold to the effects of curvature in the fifth dimension. The modification of an object's motion through the 5D manifold can be observed and then described in familiar 4D terms (such as the Lorentz force) via eq. 2.51.

The identification 2.52 is permissible in the cases where the 4D part of the metric is independent of the fifth coordinate (so that B is a constant), and the fifth component of the metric is 'flat' (so $\Phi \equiv \sqrt{\epsilon\hat{g}_{55}} = const$). However, in more general cases, the identification of eq. 2.52 forces certain relations between the velocity components of the particle and the potentials of the source which are known to be violated in certain cosmological solutions. Nevertheless, the identification of eq. 2.52 *should* be acceptable in appropriate limiting cases (ref. [5], see also discussion in ref. [29]).

Identifying the 4D *gravitational* mass and charge/current of the *source* then comes from matching the radial 4D gravitational and Lorentz portions of eq. 2.51 with their Newtonian and Coulomb/Ampéric approximations, respectively.

It is again stressed that quantities defined in 4D may not be constant under transformations in 5D (if, for example, the mass or charge/current, depends on the fifth coordinate). (ref. [5]). However, as long as they agree in the limit $5D \rightarrow 4D$, such identifications may be taken as reasonable.

The radial portion of the (4D) gravitational field from eq. 2.51 can then be given as:

$$\Gamma^1_{\alpha\beta} u^\alpha u^\beta \simeq \Gamma^1_{00} (u^0)^2 \tag{2.53}$$

where the second step was made by assuming low spatial velocities ($u^i \ll u^0$), and where $u^\mu \equiv \frac{dx^\mu}{ds}$ are, again, the four-velocities, taken with respect to the 4D interval, ds. Typically, however, equations of motion would be given in terms of derivatives with respect to the 5D interval, $d\hat{s}$. The relation between ds and $d\hat{s}$ must then be taken into account (from eqs. 2.46 and 2.47) as:

$$ds = d\hat{s} \left[1 - \hat{g}_{55} \left(\dot{\psi} + A_\mu \dot{x^\mu} \right)^2 \right]^{\frac{1}{2}} \tag{2.54}$$

where overdots denote derivatives with respect to $d\hat{s}$.

As noted at the beginning of this section, the approach taken in this thesis assumes that our universe exists on an *approximate* hypersurface of constant ψ. This approximation ensures that $\dot{\psi} \equiv \frac{d\psi}{d\hat{s}}$ will be $\ll$ unity[5]. Additionally, any A_μ, if they exist, will also likely be small (neutral metrics *can* exist for which $A_\mu = 0$ for all μ). Therefore, in most situations, $ds \simeq d\hat{s}$, and $u^0 \simeq \hat{u}^0$, where $\hat{u}^0 \equiv \frac{dx^0}{d\hat{s}}$ is the zero-component of the five-velocity, $\hat{u}^a$ (in most cases, in any event, $u^0 \simeq \hat{u}^0 \simeq 1$).

The expression for radial 4D gravity (eq. 2.53) can then be equated (*approximately*) to the Newtonian gravitational potential;

$$\Gamma^1_{00}(u^0)^2 \simeq \frac{M}{r^2} \tag{2.55}$$

to solve for the Newtonian gravitational mass, M.

Similarly, we match the radial Lorentz portion, F^1_μ, from the right-hand side of eq. 2.51 with the Coulomb field for an electrostatic charge;

$$F^1_0 = \frac{Q}{r^2} \tag{2.56}$$

or with Ampéric current expression for a magnetic field;

$$F^1_j = \epsilon^1{}_j{}^k B_k \tag{2.57}$$

where $\epsilon^1{}_j{}^k$ *here* is the Levi-Civita "antisymmetric permutation" tensor.

These allow one to then calculate the charge, Q, or current, I (implicit in B_k) for the given source. Again, care must be taken to emphasize that such identifications

[5] As well, all fields dependent on ψ, *e.g.* $\Upsilon(x^\mu, \psi)$, existing on the manifold can be *approximately* given by their ψ_0-hypersurface values. That is, in Taylor expanding Υ as: $\Upsilon(\psi) = \Upsilon(\psi_0) + (\psi - \psi_0)\frac{\partial \Upsilon}{\partial \psi} + ...$; all additional terms above the zero-order $\Upsilon(\psi_0)$ term will be *small*.

(for mass and charge/current) are *approximations*, relating familiar (but inexact) 4D quantities to the (exact) 5D components found here.

Chapter 3

Charged Particle Solutions

As discussed in the previous chapter, it was desired to find solutions which describe matter, preferably charged matter, with an eye to examing its inertial properties. Building on work previously done on 5D charged solutions, therefore, a study has been made of various charged solutions with analysis given to such properties.

In ref. [30], H. Liu and P. Wesson derived the general r-dependent 5D charged metric, corresponding to a spherically symmetric static charge distribution (*i.e.*, a 'charged particle'). The Liu-Wesson solution can be written out as:

$$\begin{aligned} d\hat{s}^2 = \left[\tfrac{(1-k)B^{\mathbf{a}}}{(1-kB^{\mathbf{a}-\mathbf{b}})} - \tfrac{(B^{\mathbf{b}}-kB^{\mathbf{a}})A^2}{(1-k)}\right] dt^2 - 2\tfrac{(B^{\mathbf{b}}-kB^{\mathbf{a}})}{(1-k)} A dt d\psi \\ -B^{-\mathbf{a}-\mathbf{b}} dr^2 - r^2 B^{1-\mathbf{a}-\mathbf{b}} d\Omega^2 - \tfrac{(B^{\mathbf{b}}-kB^{\mathbf{a}})}{(1-k)} d\psi^2 \end{aligned} \tag{3.1}$$

where $d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2$ and A and B are defined via:

$$A \equiv \frac{-\sqrt{k}(1 - B^{\mathbf{a}-\mathbf{b}})}{(1 - kB^{\mathbf{a}-\mathbf{b}})} \tag{3.2}$$

$$B \equiv 1 - \frac{2M(1-k)}{r} \tag{3.3}$$

where A is also the electrostatic vector potential, A_0, of this metric and where $\mathbf{a}$ and $\mathbf{b}$ are parameters constrained by:

$$\mathbf{a}^2 + \mathbf{ab} + \mathbf{b}^2 = 1 \tag{3.4}$$

The Liu-Wesson solution is a generalization of the Gross-Perry solution (ref. [30]), which, itself, is a (*5D*) generalization of the Schwarzchild solution (ref. [7]). For $k \to 0$, the Liu-Wesson solution reduces to the Gross-Perry solution, for which $\mathbf{a} \to 1$ and $\mathbf{b} \to 0$ reduces to the Schwarzchild solution (with an extra flat fifth dimension).

Unlike in 4D, however, the conditions of Birkhoff's Theorem do not apply in this case (or in any of the other metrics examined in this thesis). Birkhoff's Theorem applies only when there is general spherical symmetry amongst *all* the space-like coordinates. Here, there is spherical symmetry amongst r, θ and ϕ, but not ψ[1]. Therefore, insofar that Birkhoff's Theorem prohibits the radially-dependent, spherically-symmetric (*Schwarzchild*) solution from depending on any coordinates *other* than r, this is expected not to apply here. One may, therefore, find radially-dependent, spherically-symmetric (*3D* spherically-symmetric) solutions which depend on *other* coordinates. The Liu-Wesson solution is, therefore, *not* expected to

[1]In order for an $(n+1)$-dimensional spacetime to possess *general* spherical symmetry, the n spacelike dimensions must be written out as:

$x_1 = r\cos\varphi_1$

$x_2 = r\sin\varphi_1\cos\varphi_2$

$x_3 = r\sin\varphi_1\sin\varphi_2\cos\varphi_3$

...

$x_{n-1} = r\sin\varphi_1\sin\varphi_2\sin\varphi_3 ... \cos\varphi_{n-1}$

$x_n = r\sin\varphi_1\sin\varphi_2\sin\varphi_3 ... \sin\varphi_{n-1}$

where $r = x_i x^i$, $0 \le \varphi_{1..n-2} \le \pi$ and $0 \le \varphi_{n-1} \le 2\pi$.

represent all possible (3D) spherically-symmetric charge states in 5D[2]. It is such alternative solutions that are derived in appendix A, two of which are examined here. (The 'nonapplicability' of Birkhoff's Theorem in 5(or higher)D has been noted in the 5D literature; ref. [31].)

As an aside, it should be noted that, until this work here was explicitly done, it was *assumed* that the Liu-Wesson solution, like the Gross-Perry solution before it, possessed the equation of state of *radiation* (ref. [2]). As charged *radiation*, the nature of the Liu-Wesson solution was thought to be, fundamentally, *unphysical*, representing, at best, an *approximation* to perceived reality (perhaps as a distribution of *ultra-relativistic* charged particles). This was, originally, one of the strongest motivations for examining the *r- and* ψ-dependent charged solutions studied in this chapter.

However, the assumption that the Liu-Wesson solution represented *radiation* was based on the fact that the Liu-Wesson metric is independent of ψ, and hence, by the theorem of ref. [8] (outlined in section 2.3.1 of the previous chapter), should represent radiation. However, as was explicitly noted in the previous chapter, that theorem was derived *solely* for *diagonal* metrics. As such, it *does not apply* to the Liu-Wesson metric, which is *off*-diagonal. The exact physical nature of the Liu-Wesson metric (which was *not* examined in ref. [2] since it was assumed to be that of charged radiation) is elucidated here for the first time, at the end of this chapter.

[2]Birkhoff's Theorem fails to apply in another sense; it requires the *Schwarzchild* solution to be unique (ref. [30]). But in 5D, the Liu-Wesson and Gross-Perry solutions, which are the 5D *extensions* to the Schwarzchild solution, represents an infinite *class* of solutions, parametrized by either **a** or **b**.

3.1 The Metrics

To further the Liu-Wesson line of research, therefore, and to try to find metrics representing charged matter *not* solely dependent on the radial coordinate, r, we therefore seek a 5D spherically-symmetric metric, off-diagonal in $dtd\psi$ (*i.e.*, electrically charged), and dependent on r and ψ. [In this sense, we are imposing *3D* spherical symmetry by requiring $\hat{g}_{ij}$ (where $i, j = 0, 1, 2, 3, 5$) be invariant under the action of SO(4) acting on S^3. (The range of i, j has to include 0 since it is desired that $\hat{g}_{11}$ depend only on R.)] Two such solutions were found for this (see appendix A for derivation), and they are:

$$d\hat{s}_I^2 = \mathcal{F}dt^2 - \frac{dr^2}{\mathcal{F}} - R^2 d\Omega^2 + 2\epsilon\frac{b}{a}dtd\psi - (1 - \frac{b^2}{a^2\mathcal{F}})d\psi^2 \tag{3.5}$$

$$d\hat{s}_{II}^2 = \mathcal{F}dt^2 - dr^2 - R^2 d\Omega^2 + 2(\epsilon + \mathcal{F})dtd\psi + (2\epsilon + \mathcal{F})d\psi^2 \tag{3.6}$$

where, as mentioned in the Introduction, the Roman numeral subscripts I and II denote which metric we are taking about, and where $\mathcal{F}$ and R are defined by:

$$\mathcal{F} \equiv 1 + \frac{K}{R} \tag{3.7}$$

$$R \equiv ar + b\psi \tag{3.8}$$

where K is a constant and a and b are parameters (not to be confused with the a and b parameters from the Liu-Wesson metric!) constrained by:

$$a^2 + b^2 = 1 \tag{3.9}$$

As will be seen below, K is related to the 4D gravitational mass.

The first metric can be seen to be an extension of the Schwarzchild metric. When $b \to 0$, the 4D part of the metric becomes completely Schwarzchild assuming one identifies K as $-2M$. Since the vector potential, A_0, is given as the ratio of $\hat{g}_{50}/\hat{g}_{55}$ (see eq. 2.8), then the vanishing of b would cause A_0 to disappear as well. Thus, b may be identified as being related to the charge.

The second metric not only does *not* possess a Schwarzchild form, but $b \to 0$ does *not* cause the vanishing of its vector potential, A_0. However, if one sets K to zero (and ϵ to -1, for a spacelike fifth dimension), the resulting metric then becomes Minkowskian, suggesting that the K in the second metric might be related to *both* mass *and* charge.

Both metrics are functions of the two main variables, r and ψ. An important point to consider is that both metrics can be transformed into functions of just one variable, R say, if one substitutes for r:

$$\begin{aligned} r &= \tfrac{1}{a}R - \tfrac{b}{a}\psi \\ dr &= \tfrac{1}{a}dR - \tfrac{b}{a}d\psi \\ dr^2 &= \tfrac{1}{a^2}dR^2 - 2\tfrac{b}{a^2}dRd\psi + \tfrac{b^2}{a^2}d\psi^2 \end{aligned} \tag{3.10}$$

These transformations alter the metrics 3.5 and 3.6 to the forms:

$$d\hat{s}_I^2 = \mathcal{F}dt^2 - \frac{1}{a^2\mathcal{F}}dR^2 - R^2d\Omega^2 + 2\epsilon\frac{b}{a}dtd\psi + 2\frac{b}{a^2\mathcal{F}}dRd\psi - d\psi^2 \tag{3.11}$$

$$d\hat{s}_{II}^2 = \mathcal{F}dt^2 - \frac{1}{a^2}dR^2 - R^2d\Omega^2 + 2(\epsilon+\mathcal{F})dtd\psi + 2\frac{b}{a^2}dRd\psi + (2\epsilon+\mathcal{F}-\frac{b^2}{a^2})d\psi^2 \tag{3.12}$$

The original forms of both metrics were functions of r and ψ and possessed one off-diagonal term, $dtd\psi$. These transformed versions of the metrics, however, suggest that the original metrics (3.5 and 3.6) are, actually, more *naturally* thought of as belonging to a class of metrics which are functions of a single coordinate, R, and possessing *two* off-diagonal terms, $dtd\psi$ and $dRd\psi$. The derivation of the *general class* of such a solution type (analogous to the Liu-Wesson general solution), however, would be difficult, possessing, as it does, *two* off-diagonal elements (a *general* solution with *one* off-diagonal is usually difficult enough; see appendix B for such an example).

However, since we endeavor to use the original forms of the metrics, the possibility of the transformed metrics being more 'natural' should at least be considered. For a constant-ψ manifold, $d\psi = 0$, and both the transformed forms of the metrics, as well as the original forms of the metrics, reduce to (effective 4D) metrics of the form:

$$d\hat{s}^2_{\psi=const} \Rightarrow \hat{g}_{tt}dt^2 + \hat{g}_{rr}dr^2 + \hat{g}_{\Omega\Omega}d\Omega^2 \tag{3.13}$$

which possesses both spherical symmetry and a limiting Minkowski (SR) form. As such, there is no necessary reason to prefer the *original* forms of the metrics over the *transformed* forms, in the case $\psi = const$. The choice of the original forms of the metrics over the transformed forms is then an *arbitrary* choice between two otherwise acceptable forms in this limit.

Of course, as noted at the beginning of the previous section, the aforementioned cylinder condition is *relaxed* so that our reality exists on an *approximate* hypersurface of constant-ψ. $d\psi$, then, is merely *small*, not *zero*. It is still *mathematically* possible that one could use the transformed forms of the metrics, though it is not clear what *physical* meaning one could take from such solutions from within the Induced Matter formalism.

Physically, the extra off-diagonal term, $dRd\psi$, would correspond, within the

Induced Matter formalism, to a radial vector potential, A_R. However, since this vector potential would be a function of R *only* ($A_R(R)$), then the only non-zero derivative of A_R would be $A_{R,R}$. But $A_{R,R}$ would only show up in F_{RR} which, by the definition of $F_{\mu\nu}$, would be *zero*. Therefore, the existence of such an off-diagonal R-dependent $dRd\psi$ term would seem extraneous from a physical point-of-view.

Nevertheless, one could still use the transformed forms of the metrics if one wished, and ignore the apparent superfluity provided by the $dRd\psi$ term. In this thesis, however, we prefer to avoid this superfluity and simply use the original forms of the metrics, which are the more physically reasonable. (If, one wished, however, one could examine the transformed versions of the metrics, instead.)

3.1.1 Comparison to Liu-Wesson Solution

In order to demonstrate the independence of these two metrics from their Liu-Wesson counterparts (that is, to show that these metrics are not coordinate-transformations of (special cases of) the Liu-Wesson class of metrics), it is desirable to compare the 5D Kretschmann scalars for both these metrics with the Kretschmann scalar for the Liu-Wesson metric. As noted at the beginning of this section, both metrics 3.5 and 3.6 can be transformed into functions of one variable, R, like the Liu-Wesson metric (metric forms 3.11 and 3.12). This fact suggests a comparison with the Liu-Wesson class is in order, if only to ensure that they are, in fact, distinct.

In 4D, the Kretschmann scalar is the only scalar invariant of the Riemann tensor in an empty spherically-symmetric spacetime (in the sense that the nonzero invariant, which is cubic in the Riemann tensor, is functionally related to the Kretschmann scalar), and is important in determining the nature of singularities. In 5D, there may be other such scalars, but, for our purposes, we shall focus only on the Kretschmann. (As well, GRTensor II was set up to calculate *only* the Kretschmann!) The 5D Kretschmann scalars, $\hat{R}_{abcd}\hat{R}^{abcd}$, were then calculated by GR Tensor II (ref. [1]) on Maple for both metrics to be:

$$\left(\hat{R}_{abcd}\hat{R}^{abcd}\right)_I = \frac{12K^2a^4}{R^6} \tag{3.14}$$

$$\left(\hat{R}_{abcd}\hat{R}^{abcd}\right)_{II} = \frac{12K^2b^4}{R^6} \tag{3.15}$$

[As an aside, we note that the Kretschmann scalars blow up at $R = 0$, whereas they are well-behaved at $R = -K$. In the original forms of the metrics, 3.5 and 3.6, $R = 0$ and $R = -K$ (for the first metric, at least) yielded singularities. The forms of the Kretschmanns here (eqs. 3.14 and 3.15) suggest that $R = 0$ is the only *real* singularity, with $R = -K$ (for the first metric, at least) being some kind of 'coordinate singularity' (or *horizon*); *i.e.*, an *apparent* singularity resulting from the choice of 'bad' coordinates, exactly analogous to the kind 'coordinate singularity' (*horizon*) present in the Schwarzchild solution for $r = 2M$ (ref. [15]). Indeed, since the first metric reduces to the Schwarzchild metric in the limit $b \to 0$, then, by continuity, we would expect such an analogous correspondence. (To properly analyze the nature of these singularities(/horizons), however, would require an extensive Kruskal-type analysis which is beyond the scope of this work.)]

In comparison with the Liu-Wesson metric class, if the metrics 3.11 and 3.12 were coordinate transformations of Liu-Wesson cases, it would require some kind of matching between the two radial coordinates, R from these metrics, and r from the Liu-Wesson metric. This would then, naturally, indicate some kind of match between the $\mathcal{F}$ of these metrics and the B of the Liu-Wesson metric. Assuming, then, a match between R and r and also between $\mathcal{F}$ and B yields:

$$1 + \frac{K}{R} = 1 - \frac{2M(1-k)}{r} \rightarrow K = -2M(1-k) \tag{3.16}$$

Additionally, it appears reasonable to assume that any matching (if any were possible) would have to entail transformations on R, t and ψ *only*; that is, *no* transformations on θ or ϕ. This means that the coefficients of $d\Omega^2$ from both sets of metrics should match exactly.

Since the $\hat{g}_{\Omega\Omega}$ in the metrics 3.11 and 3.12 are equal to $-R^2$, while in the Liu-Wesson metric they are equal to $-B^{(1-\mathbf{a}-\mathbf{b})}r^2$ (and assuming matches between R and r, and between $\mathcal{F}$ and B), this requires $1-\mathbf{a}-\mathbf{b}=0$. Along with $\mathbf{a}^2+\mathbf{ab}+\mathbf{b}^2 = 1$, this means either $\mathbf{a}=1$ and $\mathbf{b}=0$, or $\mathbf{a}=0$ and $\mathbf{b}=1$. In both cases, the Kretschmann scalar for the Liu-Wesson metric becomes:

$$\hat{R}_{abcd}\hat{R}^{abcd} = \frac{48M^2(1-k)^2}{r^6} \tag{3.17}$$

(In general, the Kretschmann scalar for the Liu-Wesson metric is a *very* complicated function of r, ref. [30].) Equating this (eq. 3.17) with eq. 3.14, and $R=r$ and $K=-2M(1-k)$, forces: $a^4=1$.

In other words, only if $a=\pm 1$ (or $\pm i$) can the two Kretschmann's, and hence the two metrics, be equal. In general, a will *not* be ± 1 (or $\pm i$), and the first metric will then be distinct from the Liu-Wesson class.

Similarly, for the second metric; equating eq. 3.17 with eq. 3.15, forces: $b^4=1$, or $b=\pm 1$ (or $\pm i$), which, again, is not, in *general* satisfied.

Thus, only in special cases can the metrics 3.11 and 3.12 (and, hence, metrics 3.5 and 3.6) be regarded as transformations of the Liu-Wesson class. In general, the metrics presented here are independent of the Liu-Wesson class.

3.2 Mass and Charge

To study these metrics more fully, we next examine appropriate limiting cases in order to determine the 4D (*limiting*) identifications for the mass, M, and charge, Q, of the *source* as represented by each metric.

As discussed in section 2.3.4 of the previous chapter, we can identify, in an *approximate* sense, the 4D ('Newtonian') gravitational mass, M, by equating $\Gamma^1_{00}(u^0)^2$ with M/r^2. First, writing out the *effective 4D metrics* for the two metrics 3.5 and 3.6 gives us:

$$ds_I^2 = \frac{\mathcal{F}}{(1-\frac{b^2}{a^2\mathcal{F}})}dt^2 - \frac{dr^2}{\mathcal{F}} - R^2 d\Omega^2 \tag{3.18}$$

$$ds_{II}^2 = \frac{-dt^2}{(2\epsilon + \mathcal{F})} - dr^2 - R^2 d\Omega^2 \tag{3.19}$$

where, again, the Roman numeral subscripts I and II denote the metric number.

From these 4D metrics, the expressions for $\Gamma^1_{00}(u^0)^2$ can then be calculated to give:

$$\left[\Gamma^1_{00}(u^0)^2\right]_I \simeq -\frac{1}{2}(u^0)^2\mathcal{F}\frac{\left[1-2\frac{b^2}{a^2\mathcal{F}}\right]}{\left[1-\frac{b^2}{a^2\mathcal{F}}\right]^2}\frac{Ka}{R^2} \tag{3.20}$$

$$\left[\Gamma^1_{00}(u^0)^2\right]_{II} \simeq -\frac{(u^0)^2}{2(2\epsilon+\mathcal{F})^2}\frac{Ka}{R^2} \tag{3.21}$$

Taking the extreme radial limit for each of these, $r \to \infty$ (so that $ar \gg b\psi$, $R \simeq ar$ and $\mathcal{F} \simeq 1 + K/ar \approx 1$), and equating them with M/r^2 then allows K to be solved in terms of M. Doing this for both metrics then yields:

$$K_I \simeq -2M\frac{(a^2-b^2)^2}{a(a^2-2b^2)(u^0)^2} \tag{3.22}$$

$$K_{II} \simeq -2Ma\frac{(2\epsilon+1)^2}{(u^0)^2} \tag{3.23}$$

The zero-components of 4-velocity, u^0, represent first order corrections over the base-level terms, which are found by setting u^0 to unity;

$$K_I \simeq -2M\frac{(a^2-b^2)^2}{a(a^2-2b^2)} \tag{3.24}$$

$$K_{II} \simeq -2Ma(2\epsilon+1)^2 \tag{3.25}$$

Setting u^0 to unity is reasonable, and necessary in this instance given that the equations of motion have not been solved. If one were to solve the equations of motion (for $\hat{u}^a$), one would be able to approximate u^0, but this is beyond the scope of this work. Instead, we will simply approximate u^0 as unity.

From eq. 3.25, the expression for K_I is perfectly consonant with the observation that $b \to 0$ should yield the Schwarzchild limit; as $b \to 0$, then $a \to 1$, and $K_I \to -2M$, the appropriate Schwarzchild mass term.

For the second metric, the value of K_{II} also reduces to $-2M$ for $b \to 0$ ($a \to 1$) and $\epsilon = -1$. However, it must be recalled from section 3.1 that b need not be associated with the charge of the second metric, and that $b \to 0$ need not yield the only reasonable 'limiting approximation'. Nevertheless, as far as the Newtonian gravitational mass is concerned, $b \to 0$ (and $\epsilon = -1$) does, in fact, afford a reasonable limit.

For the identification of charge, Q, we need to identify F_0^1 (the electric fields) with Q/r^2 in the same extreme radial limit ($r \to \infty$). First, we note the expressions for the vector potentials, $A_0 = \hat{g}_{50}/\hat{g}_{55}$, explicitly for each metric (3.5 and 3.6) as:

$$A_{I\,0} = \frac{-\epsilon b/a}{(1-\frac{b^2}{a^2\mathcal{F}})} \tag{3.26}$$

$$A_{II\ 0} = \frac{(\epsilon + \mathcal{F})}{(2\epsilon + \mathcal{F})} = 1 - \frac{\epsilon}{(2\epsilon + \mathcal{F})} \tag{3.27}$$

The calculation of $F_0^1 = g^{11}\partial_1 A_0$ for each of the two metrics gives:

$$F_{I\ 0}^1 = \frac{\epsilon b^3/a^3}{\left[1 - \frac{b^2}{a^2\mathcal{F}}\right]^2} \frac{Ka}{\mathcal{F}R^2} \tag{3.28}$$

$$F_{II\ 0}^1 = \frac{\epsilon}{(2\epsilon + \mathcal{F})^2} \frac{Ka}{R^2} \tag{3.29}$$

and the corresponding calculations of Q (in the extreme radial limit, $r \to \infty$) then gives:

$$Q_I \simeq \frac{\epsilon b^3}{(a^2 - b^2)^2} K \tag{3.30}$$

$$Q_{II} \simeq \frac{\epsilon}{(2\epsilon + 1)^2} \frac{K}{a} \tag{3.31}$$

From the eqs. 3.24 and 3.25 and eqs. 3.30 and 3.31, it is then possible to calculate a charge-to-mass ratio, Q/M, of the source for each of the given metrics. For the first metric, this ratio works out to be:

$$(\frac{Q}{M})_I \simeq \frac{-2\epsilon b^3}{a(a^2 - 2b^2)} \tag{3.32}$$

Based on the observed accuracy of the Schwarzchild metric in describing our world, and given that, the Schwarzchild metric is solely dependent on r, it is reasonable to presume that b, which represents the contribution of ψ to the metric, should be vanishingly small in most physically real situations. This view is strengthened by the observation that b^3, appearing in the numerator of eq. 3.32, is coupled to the charge-to-mass ratio, Q/M, which is *very* small on macroscopic scales (for the Earth, which has an electric field of about 100 V/m (ref. [32]), $Q/M \simeq 9 \times 10^{-10}$

here [3]; for *microscopic* matter the situation is reversed: $Q/M \simeq 2 \times 10^{21}$ for the electron, but such regions, where *quantum mechanics* would be involved, are outside the domain of a classical theory such as 5D GR). Therefore, we make the further assumption that $b \sim 0$ ($a \sim 1$) and find, for the first metric:

$$(\frac{Q}{M})_I \approx -2\epsilon(\frac{b}{a})^3 \;\rightarrow\; b_I \approx a\left(\frac{-Q}{2\epsilon M}\right)^{\frac{1}{3}} \tag{3.33}$$

This approximation is true mainly for the limit $b \approx 0$, which, physically-speaking, occurs primarily in the macroscopic regime (especially for astronomical-type objects, such as planets), where the charge density is reasonably small. In the *extreme* limit $b \Rightarrow 0$ ($a \Rightarrow 1$) gives:

$$b_I \sim \left(\frac{-Q}{2\epsilon M}\right)^{\frac{1}{3}} \tag{3.34}$$

Since the right-hand side of eq. 3.34 is *very* close to zero in most macroscopic situations, then this is consistent with the assumption that $b \Rightarrow 0$, and, further, that b is directly related to the charge-to-mass ratio of the source for the first metric.

For the second metric, the charge-to-mass ratio works out to be:

$$(\frac{Q}{M})_{II} \simeq -2\epsilon \tag{3.35}$$

which is 2 for $\epsilon = -1$ (spacelike fifth dimension) or -2 for $\epsilon = +1$ (timelike fifth dimension).

The fact that the charge-to-mass ratio for the second metric is on the order of unity (twice unity, in fact) is truly unexpected. As noted above, the charge-to-mass

[3] $\frac{Q}{M} = \frac{q}{m}\sqrt{\frac{k_C}{G}}$ in restored units, where k_C is the Coulomb electric constant and G is Newton's gravitational constant, and $\sqrt{\frac{k_C}{G}} \simeq 1.2 \times 10^{10} kg/C$.

ratio for a *macroscopic* system is vanishingly small ($\ll 1$), while for *microscopic* systems, it is huge ($\gg 1$). In this case, the ratio falls in the *middle* of these two disparate ranges. Since we are concerned with classical systems, this indicates that the second metric represents matter with an extreme or saturated level of charge distribution.

3.3 Maxwell's Equations

To further examine the expressions of charge (in the form of charge density) of these two metrics, we examine Maxwell's Equations, given by: $F^{\alpha\beta}{}_{;\beta} = J^\alpha$ where J^α is the Maxwellian current density (although here we will be most interested in the zeroth component of J^α, the *charge* density J^0).

For both metrics, the charge density, J^0, can be given by: $J^0 = F^{0\beta}{}_{;\beta}$, which, for both metrics, can be written out as:

$$J^0 = F^{01}{}_{,1} + \frac{1}{2} F^{01} g^{\alpha\alpha} g_{\alpha\alpha,1} \tag{3.36}$$

with all the other J^α's being zero for both metrics.

For the two metrics, the expression for J^0 can be calculated from this (or, alternatively, using the computer algorithm listed in appendix D) as:

$$J^0_I = \frac{-\epsilon(2 - \frac{b^2}{2a^2\mathcal{F}})b^3/a^3}{(1 - \frac{b^2}{a^2\mathcal{F}})^2} \frac{K^2a^2}{\mathcal{F}^3R^4} \tag{3.37}$$

$$J^0_{II} = \frac{3\epsilon/2}{(2\epsilon + \mathcal{F})^2} \frac{K^2a^2}{R^4} \tag{3.38}$$

again Roman numeral subscripts denoting the metric number.

If one again assumes a large r, then $R \to ar$ and $\mathcal{F} \to 1$. Additionally, the approximation $b \ll 1$ ($a \approx 1$), which is realistic for most situations, is made, so that the two charge densities become:

$$J_I^0 \simeq \frac{-2\epsilon b^3 K^2}{R^4} \approx \frac{-2\epsilon b^3 K^2}{r^4} \tag{3.39}$$

$$J_{II}^0 \simeq \frac{3\epsilon K^2}{2(2\epsilon+1)^2 R^4} \approx \frac{3\epsilon K^2}{2(2\epsilon+1)^2 r^4} \tag{3.40}$$

These expressions show the general dependency of the charge densities of each of the two solutions, as being inversely proportional to the fourth power of r, at least at large values of r, which matches that of other solutions (see Liu-Wesson analysis below in section 3.5.1). Of course, at *small* values of r, the difference between R and r becomes significant, and this dependency may not hold.

In order to compare these expressions of charge density with the charge-to-mass ratios of the previous section (eqs. 3.33 and 3.35) requires next examining the mass densities, among other things, of these metrics.

3.4 Density, Pressure and Stress

In this section, we examine the induced energy-momentum tensor for each of the two metrics and their resulting mass densities, pressures and stresses. This will give us the inertial properties of the matter (*fluid*) described by the two metrics, as well as allowing us to reexamine the expressions for the charge-to-mass ratios for both metrics.

Although we could have calculated the induced energy-momentum tensor, T_μ^ν, from eq. 2.11, we instead used the 4D $g_{\mu\nu}$ from eqs. 3.18 and 3.19 to calculate the

4D G^{ν}_{μ} (using GRTensor II on Maple, ref. [1]) and the fact that $T^{\nu}_{\mu} = G^{\nu}_{\mu}$ to deduce T^{ν}_{μ}. For the first metric, the nonzero energy-momentum tensor components are:

$$T^{0}_{I\ 0} = \frac{b^2}{R^2} \tag{3.41}$$

$$T^{1}_{I\ 1} = \frac{b^2(a^2 - b^2)}{(a^2\mathcal{F} - b^2)R^2} \tag{3.42}$$

$$T^{2}_{I\ 2} = T^{3}_{I\ 3} = \frac{Ka^2b^2(a^2 - b^2)}{2(a^2\mathcal{F} - b^2)^2\mathcal{F}R^3} + \frac{K^2a^2b^2(3a^2 - 4b^2)}{4(a^2\mathcal{F} - b^2)^2\mathcal{F}R^4} + \frac{K^3a^4b^2}{4(a^2\mathcal{F} - b^2)^2\mathcal{F}R^5} \tag{3.43}$$

while, for the second metric, the nonzero components of the energy-momentum tensor are:

$$T^{0}_{II\ 0} = \frac{b^2}{R^2} \tag{3.44}$$

$$T^{1}_{II\ 1} = \frac{b^2}{R^2} - \frac{Ka^2}{(2\epsilon + \mathcal{F})R^3} \tag{3.45}$$

$$T^{2}_{II\ 2} = T^{3}_{II\ 3} = \frac{Ka^2}{2(2\epsilon + \mathcal{F})R^3} - \frac{3K^2a^2}{4(2\epsilon + \mathcal{F})^2R^4} \tag{3.46}$$

Also, the electromagnetic component of the energy-momentum tensor, $T_{EM} \equiv T^{0}_{EM\ 0} = T^{1}_{EM\ 1} = -T^{2}_{EM\ 2} = -T^{3}_{EM\ 3}$, is, for the first metric:

$$T_{I\ EM} = \frac{\frac{1}{4}b^6/a^6}{[1 - \frac{b^2}{a^2\mathcal{F}}]^2}\frac{K^2a^2}{\mathcal{F}^4R^4} = \frac{b^6K^2}{4(a^2\mathcal{F} - b^2)\mathcal{F}^2R^4} \tag{3.47}$$

while, for the second metric, it is:

$$T_{II\ EM} = \frac{1/4}{(2\epsilon + \mathcal{F})^2}\frac{K^2a^2}{R^4} \tag{3.48}$$

From section 2.3.3 of the previous chapter, we next use $\wp = T^0_0 - T_{EM}$ to define the density, $P = -\frac{1}{3}(T^1_1 + T^2_2 + T^3_3 + T_{EM})$ to define the pressure, and

$\tau_1^1 = \frac{2}{3}(T_1^1 - T_2^2) - \frac{4}{3}T_{EM}$ (and $\tau_2^2 = \tau_3^3 = -\frac{1}{2}\tau_{I\ 1}^{1}$) to define the anisotropic stress. For the first metric, the density, pressure and stress are:

$$\wp_I = \frac{b^2}{4}[4(a^2-b^2)^2R^4 + 8(2a^2-b^2)(a^2-b^2)KR^3 + 3(8a^4-8a^2b^2+b^4)K^2R^2 \\ +8a^2(2a^2-b^2)K^3R + 4a^4K^4]/[(a^2-b^2)R^3 + (2a^2-b^2)KR^2 + a^2K^2R]^2 \tag{3.49}$$

$$P_I = \frac{-b^2}{12}[4(a^2-b^2)^2R^4 + 8(2a^2-b^2)(a^2-b^2)KR^3 + (22a^4-28a^2b^2+5b^4)K^2R^2 \\ +12a^2(a^2-b^2)K^3R + 2a^4K^4]/[(a^2-b^2)R^3 + (2a^2-b^2)KR^2 + a^2K^2R]^2 \tag{3.50}$$

$$\tau_{I\ 1}^{1} = \frac{b^2}{6}[4(a^2-b^2)^2R^4 + 2(5a^2-4b^2)(a^2-b^2)KR^3 + (7a^4-10a^2b^2+2b^4)K^2R^2 \\ -a^4K^4]/[(a^2-b^2)R^3 + (2a^2-b^2)KR^2 + a^2K^2R]^2 \tag{3.51}$$

the relation $\tau_{I\ 2}^{2} = \tau_{I\ 3}^{3} = -\frac{1}{2}\tau_{I\ 1}^{1}$ following *directly* from the spherical symmetry of the solution (see eq. 2.40).

The density, pressure and stress of the second metric are then:

$$\wp_{II} = \frac{1}{4R^2}[4(41+40\epsilon)b^2R^4 + 16(13+14\epsilon)b^2KR^3 + (24b^2-a^2)(5+4\epsilon)K^2R^2 \\ +2(8b^2-a^2)(2\epsilon+1)K^3R + (4b^2-a^2)K^4]/[(2\epsilon+1)R+K]^4 \tag{3.52}$$

$$P_{II} = \frac{-1}{12R^2}[4(41+40\epsilon)b^2R^4 + 16(13+14\epsilon)b^2KR^3 + (24b^2-5a^2)(5+4\epsilon)K^2R^2 \\ +2(8b^2-5a^2)(2\epsilon+1)K^3R + (4b^2-5a^2)K^4]/[(2\epsilon+1)R+K]^4 \tag{3.53}$$

$$\tau_{II\ 1}^{1} = \frac{1}{6R^2}[4(41+40\epsilon)b^2R^4 + \{16(13+14\epsilon)b^2 - 2(39+42\epsilon)a^2\}KR^3 \\ +(24b^2-17a^2)(5+4\epsilon)K^2R^2 + 16(b^2-a^2)(2\epsilon+1)K^3R \\ +(4b^2-5a^2)K^4]/[(2\epsilon+1)R+K]^4 \tag{3.54}$$

with, again, $\tau_{II\ 2}^{2} = \tau_{II\ 3}^{3} = -\frac{1}{2}\tau_{II\ 1}^{1}$.

3.4.1 Simple Equations of State

In special consideration of equations of state of the form $P = n\wp$, where n is a dimensionless number (independent of r or ψ), we consider the expressions for both metrics given by dividing each P by its corresponding $\wp$. For the first metric, this is:

$$\begin{aligned}(P/\wp)_I = -\tfrac{1}{3}[&4(a^2-b^2)^2R^4 + 8(2a^2-b^2)(a^2-b^2)KR^3 + (22a^4-28a^2b^2+5b^4)K^2R^2 \\ &+12a^2(a^2-b^2)K^3R + 2a^4K^4]/[4(a^2-b^2)^2R^4 + 8(2a^2-b^2)(a^2-b^2)KR^3 \\ &+3(8a^4-8a^2b^2+b^4)K^2R^2 + 8a^2(2a^2-b^2)K^3R + 4a^4K^4]\end{aligned} \tag{3.55}$$

while, for the second metric, it is:

$$\begin{aligned}(P/\wp)_{II} = -\tfrac{1}{3}[&4(41+40\epsilon)b^2R^4 + 16(13+14\epsilon)b^2KR^3 + (24b^2-5a^2)(5+4\epsilon)K^2R^2 \\ &+2(8b^2-5a^2)(2\epsilon+1)K^3R + (4b^2-5a^2)K^4]/[4(41+40\epsilon)b^2R^4 + 16(13+14\epsilon)b^2KR^3 \\ &+(24b^2-a^2)(5+4\epsilon)K^2R^2 + 2(8b^2-a^2)(2\epsilon+1)K^3R + (4b^2-a^2)K^4]\end{aligned} \tag{3.56}$$

From examinations of both expressions on the right-hand sides of eq. 3.55 and 3.56 of $P/\wp$, it is clear that there is no *general simple* relation between P and $\wp$; that is, there is no general expression the form of $P = n\wp$ for which n is a constant (independent of R). It is, however, possible to examine the relations between P and $\wp$ for special cases.

For the first metric, all quantities (P, $\wp$, and τ_μ^ν) are proportional to b which, for the first metric, is related to the charge. Thus, $b = 0$ yields $P = \wp = \tau_\mu^\nu = 0$ (for all μ, ν). This indicates that all the matter (given by $\wp$), all the kinetic motion (given by P) and all the internal interactions (*i.e.*, viscosity, given by τ_μ^ν) are of *electromagnetic* origin, and its vanishing (*i.e.*, $b = 0$) causes these quantities to vanish too.

For the first metric, an interesting equation of state results from setting $K = 0$; $P = -\wp/3$. As noted in the previous chapter, the right-hand side of this equation possesses the *opposite* sign of the standard radiation equation of state, $P = \wp/3$. Nevertheless, such equations of state have been studied in other contexts by several authors (refs. [8], [33], [34], [35], [36], [8]). Of particular note is that, because the 4D GR gravitational mass is proportional to $3P + \wp$, this indicates the existence of matter which exerts no gravitational effects. Indeed, since K has been identified as the *mass term* for these metrics, it, therefore, stands to reason that $K = 0$ would yield such a state.

For the second metric, setting $K = 0$ again yields the equation of state $P = -\wp/3$. Interestingly, setting $a = 0$ ($b = 1$) *also* yields the equation of state $P = -\wp/3$. In the first metric, $\hat{g}_{50} = \epsilon\frac{b}{a}$, so setting $a = 0$ was not possible. However, for the second metric, it is possible (mathematically, at least) to set $a = 0$ (though this does have the effect of rendering the metric a sole function of ψ, and not r).

At first glance, it may seem unusual that setting $a = 0$ will yield $P = -\wp/3$ ("vanishing gravitational mass") since, for the second metric, a was not proportional to the mass, but *inversely* proportional to it (from $K_{II} \simeq -2Ma(2\epsilon+1)^2$, eq. 3.25). However, given that a represents the r-dependence of the metric, its vanishing causes the metric to become independent of r, which then causes $\Gamma^1_{00}(u^0)^2$ (which gives the gravitational field) to become zero automatically. This then forces M to be zero, which may then be taken to be consistent with the "matterless state" indicated by $P = -\wp/3$.

Finally, for $b = 0$ for the second metric, we obtain $P = -\frac{5}{3}\wp$, which is interesting because, at first glance, it *suggests* the existence of matter which is gravitationally *repulsive* ($3P + \wp = -4\wp$). However, upon further examination, this turns out *not* to be the case. For the second metric, for $b = 0$, the density works out to be:

$$\wp \rightarrow \frac{-K^2}{4R^2[(2\epsilon + 1)R + K]^2} \tag{3.57}$$

In this case, the density is *negative definite*, so that $3P + \wp = -4\wp$, and, thus, the gravitational mass, becomes *positive definite*.

Generally, we note, however, that for all metrics, the dominating terms in the density (and pressure and stress) are the $1/R^2 \simeq 1/r^2$ terms, which are typical of isothermal states of fluid (see refs. in [10]), and generally indicate that we are dealing with fluids with their densities, (*etc.*), concentrated at their origins.

3.4.2 Charge-to-Mass Ratio (Reconsidered)

In order to make use of the expressions for charge density from the previous section (eqs. 3.39 and 3.40) and to then make a comparison with the charge-to-mass ratios calculated in section 3.2, we calculate the (gravitational) mass density from these expressions for density and pressure (eqs. 3.49 and 3.50, and eqs. 3.52 and 3.53). As just noted, the 4D gravitational mass is proportional to $3P + \wp$. This is, in fact, the 'relativistic' 4D gravitational mass density, with the $3P$ acting as the relativistic 'correction' term to the base-level density $\wp$. If we calculate this for each of the two metrics from the existing expressions for density and pressure, we get:

$$\begin{aligned}(3P + \wp)_I = \tfrac{b^2}{4}[(2a^4 + 4a^2b^2 - 2b^4)K^2R^2 + 4a^2(a^2 + b^2)K^3R + 2a^4K^4]/ \\ [(a^2 - b^2)R^3 + (2a^2 - b^2)KR^2 + a^2K^2R]^2\end{aligned} \tag{3.58}$$

$$(3P + \wp)_{II} = \frac{K^2 a^2}{R^2[(2\epsilon + 1)R + K]^2} \tag{3.59}$$

where, for the second case, use was made of the fact that: $(2\epsilon + 1)^2 = (5 + 4\epsilon)$.

For the extreme radial limit, $r \to \infty$, these can then be approximated as:

$$(3P + \wp)_I \simeq \frac{K^2 b^2}{2R^4} \approx \frac{K^2 b^2}{2r^4} \tag{3.60}$$

$$(3P + \wp)_{II} \simeq \frac{K^2 a^2}{(2\epsilon + 1)^2 R^4} \approx \frac{K^2}{(2\epsilon + 1)^2 r^4} \tag{3.61}$$

When we consider the *charge* densities for both metrics as given by eqs. 3.39 and 3.40, we can divide them by these expressions for the gravitational *mass* densities to get alternative expressions for charge-to-mass ratios (alternative to eqs. 3.33 and 3.35). Doing this yields:

$$(Q/M)_I \simeq -4\epsilon b \tag{3.62}$$

$$(Q/M)_{II} \simeq \frac{3\epsilon}{2a^2} \approx \frac{3}{2}\epsilon \tag{3.63}$$

Although these expressions yield constant values for Q/M, as do eqs. 3.33 and 3.35, they are not *identical* to those expressions. This is due, in part, to the differing approximations which were used in arriving at these expressions. In section 3.3, the charge density was found as a fit to what we observe as a 4D *Maxwellian* electric field. In section 3.2, however, the charge was found by using a *Coulombic* fit. Differences of such approaches may naturally lead to differing results.

Additionally, as noted earlier, we are dealing with quantities which are normally defined in 4D, and trying to match them to effects from the 5D manifold. In both cases, the (Maxwellian and Coulombic) electric fields are 4D approximations to the underlying 5D curvature, which gives rise to the fields in question. We see this 5D curvature in 4D and try to fit fields to it familiar to us from 4D. Ultimately, this can lead to results which differ for differing approaches.

Nevertheless, we note that, for the first metric, at least, the two expressions, 3.33 and 3.62, *do* agree in the limit $b \to 0$, which, as noted before, is the reasonable limiting case for this metric.

3.5 Epilogue: Analysis of the Liu-Wesson Metric

As discussed at the beginning of this chapter, one of the prime motivations for examining 5D charged metrics dependent on (r and) ψ was to extend the investigations of charge distributions, initiated by Liu and Wesson, to that of charged *matter*. It was assumed by both, and published in ref. [2], that their solution, the Liu-Wesson metric (eq. 3.1), possessed the equation of state of *radiation*. This assumption was based upon the theorem, published in ref. [8] and outlined in section 2.3.1 of chapter 2, which stated that metrics which are independent of ψ must possess the equation of state of radiation.

As such, the Liu-Wesson metric would have been regarded as unphysical, since it would have ended up representing charged *radiation*. Only in the limiting case of ultrarelativistic charge would the Liu-Wesson metric have been applicable. (Therefore, extending their line of research to a *material* charge distribution seemed highly reasonable, especially when coupled with the independent desire to examine solutions representative of *matter*.)

However, as was also mentioned in section 2.3.1 of chapter 2, this theorem has been derived *only* for *diagonal* metrics. As the Liu-Wesson solution is an *off*-diagonal metric, the theorem can*not* be expected to apply to it. In the course of doing this thesis, I (the author) realized that the Liu-Wesson metric, *contrary* to what was published in ref. [2], does *not* possess the equation of state of radiation. It does, in fact, represent, a charged *fluid*, just as the two metrics of this chapter do (though with different inertial properties). Therefore, it seems appropriate that the equation of state, *etc.*, of the Liu-Wesson metric be analyzed here and compared with the two solutions of this chapter.

3.5.1 Physical Aspects of the Liu-Wesson Solution

As is suggested from the written form of the Liu-Wesson metric, the quantity M appearing in eq. 3.3 can be identified with the (4D) Schwarzchild mass in the 4D limit. Additionally, k, the 'charge' of the metric, has been identified (in the '5D-Schwarzchild' case, at least[4]) as the *square of the charge-to-mass ratio*; $k = \frac{Q^2}{4M^2}$. This case is discussed at length in ref. [2], and, as such, will not be examined any further here.

Maxwell's equations for the general form of the Liu-Wesson metric can also be given in this context as:

$$J^0 = \frac{6(1-k)^2\sqrt{k}(\mathbf{a}-\mathbf{b})(\mathbf{b}-\mathbf{a}kB^{\mathbf{a}-\mathbf{b}})}{(1-kB^{\mathbf{a}-\mathbf{b}})}\frac{M^2}{r^4} \approx 6\sqrt{k}(\mathbf{a}-\mathbf{b})\mathbf{b}\frac{M^2}{r^4} \qquad (3.64)$$

where the second step was made for $k \approx 0$ and $r \gg M$.

As can be seen, the Liu-Wesson solution possesses the same r-dependency at large r as the solutions found here (but will differ as $r \to 0$).

To finally calculate the *correct* nature of the Liu-Wesson solution, we calculate energy-momentum tensor for the Liu-Wesson metric here from the ψ-independent

[4]That is, for the case $\mathbf{a} = 1$ and $\mathbf{b} = 0$.

expression for T_μ^ν of the previous chapter. The effective 4D metric from the Liu-Wesson metric is:

$$ds^2 = \left[\frac{(1-k)B^{\mathrm{a}}}{(1-kB^{\mathrm{a-b}})}\right] dt^2 - B^{-\mathrm{a-b}}dr^2 - r^2B^{1-\mathrm{a-b}}d\Omega^2 \tag{3.65}$$

from which the components for T_μ^ν can be calculated as:

$$T_0^0 = -\mathrm{ab}(1-k)^2B^{(\mathrm{a+b-2})}\frac{M^2}{r^4} \tag{3.66}$$

$$\begin{aligned} T_1^1 = &\tfrac{(1-k)^2B^{(\mathrm{a+b-2})}}{(1-kB^{(\mathrm{a-b})})^2}\left[-(2+\mathrm{a}+2\mathrm{b})\mathrm{b}+2(1+\mathrm{a}+\mathrm{b})kB^{(\mathrm{a-b})}-(2+2\mathrm{a}+\mathrm{b})\mathrm{a}k^2B^{2(\mathrm{a-b})}\right]\tfrac{M^2}{r^4} \\ &+\tfrac{2(1-k)B^{(\mathrm{a+b-2})}}{(1-kB^{(\mathrm{a-b})})}(\mathrm{b}-\mathrm{a}kB^{(\mathrm{a-b})})\tfrac{M}{r^3} \end{aligned} \tag{3.67}$$

$$\begin{aligned} T_2^2 = &\tfrac{(1-k)^2B^{(\mathrm{a+b-2})}}{(1-kB^{(\mathrm{a-b})})^2}\left[(1+\mathrm{a}+\mathrm{b})\mathrm{b}-(4+\mathrm{a}-8\mathrm{ab}+\mathrm{b})kB^{(\mathrm{a-b})}-(1+\mathrm{a}+\mathrm{b})\mathrm{a}k^2B^{2(\mathrm{a-b})}\right]\tfrac{M^2}{r^4} \\ &-\tfrac{(1-k)B^{(\mathrm{a+b-2})}}{(1-kB^{(\mathrm{a-b})})}(\mathrm{b}-\mathrm{a}kB^{(\mathrm{a-b})})\tfrac{M}{r^3} \end{aligned} \tag{3.68}$$

with $T_3^3 = T_2^2$.

For the Liu-Wesson metric, the electromagnetic component of the energy-momentum tensor, $T_{EM} \equiv T_{EM\ 0}^0 = T_{EM\ 1}^1 = -T_{EM\ 2}^2 = -T_{EM\ 3}^3$, is then:

$$T_{EM} = \frac{(1-k)^2B^{(\mathrm{a+b-2})}}{(1-kB^{(\mathrm{a-b})})^2}(\mathrm{a}-\mathrm{b})^2kB^{(\mathrm{a-b})}\frac{M^2}{r^4} \tag{3.69}$$

Again, from section 2.3.3 of the previous chapter, we use $\wp = T_0^0 - T_{EM}$ to define the density, $P = -\frac{1}{3}(T_1^1 + T_2^2 + T_3^3 + T_{EM})$ to define the pressure, and $\tau_1^1 = \frac{2}{3}(T_1^1 - T_2^2) - \frac{4}{3}T_{EM}$ (and $\tau_2^2 = \tau_3^3 = -\frac{1}{2}\tau_1^1$) to define the anisotropic stress. For the Liu-Wesson metric, therefore, the density, pressure and stress are:

$$\wp = \frac{(1-k)^2B^{(\mathrm{a+b-2})}}{(1-kB^{(\mathrm{a-b})})^2}\left[-\mathrm{ab}+(-1+5\mathrm{ab})kB^{(\mathrm{a-b})}-\mathrm{ab}k^2B^{2(\mathrm{a-b})}\right]\frac{M^2}{r^4} \tag{3.70}$$

$$P = \frac{(1-k)^2 B^{(\mathbf{a}+\mathbf{b}-2)}}{3(1-kB^{(\mathbf{a}-\mathbf{b})})^2}\left[-\mathbf{ab} + (5 - 13\mathbf{ab})kB^{(\mathbf{a}-\mathbf{b})} - \mathbf{ab}k^2 B^{2(\mathbf{a}-\mathbf{b})}\right]\frac{M^2}{r^4} \tag{3.71}$$

$$\tau_1^1 = \tfrac{(1-k)^2 B^{(\mathbf{a}+\mathbf{b}-2)}}{3(1-kB^{(\mathbf{a}-\mathbf{b})})^2}[-2(3+2\mathbf{a}+3\mathbf{b})\mathbf{b} + 2(4+3\mathbf{a}-2\mathbf{ab}+3\mathbf{b})kB^{(\mathbf{a}-\mathbf{b})} -2(3+3\mathbf{a}+2\mathbf{b})\mathbf{a}k^2 B^{2(\mathbf{a}-\mathbf{b})}]\tfrac{M^2}{r^4} + \tfrac{2(1-k)B^{(\mathbf{a}+\mathbf{b}-2)}}{(1-kB^{(\mathbf{a}-\mathbf{b})})}(\mathbf{b} - \mathbf{a}kB^{(\mathbf{a}-\mathbf{b})})\tfrac{M}{r^3} \tag{3.72}$$

Surprisingly, therefore, the pressure and densities as given here follow $1/r^4$ expressions, as opposed to the $1/r^2$ expressions given by the metrics studied in this chapter. This is a notable result as it affords a very significant difference between the two sets of solutions.

Again, special consideration is given to equations of state of the form $P = n\wp$, where n is a dimensionless number (independent of r). We consider the expression given by dividing each P by $\wp$ which is:

$$P/\wp = \frac{\left[-\mathbf{ab} + (5 - 13\mathbf{ab})kB^{(\mathbf{a}-\mathbf{b})} - \mathbf{ab}k^2 B^{2(\mathbf{a}-\mathbf{b})}\right]}{\left[-3\mathbf{ab} + (-3 + 15\mathbf{ab})kB^{(\mathbf{a}-\mathbf{b})} - 3\mathbf{ab}k^2 B^{2(\mathbf{a}-\mathbf{b})}\right]} \tag{3.73}$$

Upon examining of the right-hand side of eq. 3.73, it is clear that, as with the two metrics studied here, there is no *general simple* relation between P and $\wp$; that is, there is no general expression the form of $P = n\wp$ for which n is a constant (independent of R). The only special cases are those for which the metric is neutral (either $k = 0$, or $\mathbf{a} = \mathbf{b}$, which causes A to vanish), or the unusual state $n = -\frac{5}{3}$, which has been previously examined.

Finally, we calculate the 4D gravitational mass density, $3P + \wp$, which is to be:

$$\begin{aligned}(3P + \wp) &= \tfrac{(1-k)^2 B^{(\mathbf{a}+\mathbf{b}-2)}}{(1-kB^{(\mathbf{a}-\mathbf{b})})^2}\left[-2\mathbf{ab} + 4(1-2\mathbf{ab})kB^{(\mathbf{a}-2\mathbf{b})} - \mathbf{ab}k^2 B^{2(\mathbf{a}-\mathbf{b})}\right]\tfrac{M^2}{r^4} \\ &\approx -2\mathbf{ab}\tfrac{M^2}{r^4}\end{aligned} \tag{3.74}$$

where the second step was made for $k \approx 0$ and $r \gg M$. Taking the ratio, then, of the charge density found from Maxwell's equations above with this expression for the 4D gravitational mass density then yields:

$$\frac{Q}{M} \approx -3\sqrt{k}\frac{(\mathbf{a}-\mathbf{b})}{\mathbf{a}} \tag{3.75}$$

which, for $\mathbf{a} = 1$ and $\mathbf{b} = 0$ (the '5D Schwarzchild' case), gives:

$$\frac{Q}{M} \approx -3\sqrt{k} \;\rightarrow\; k = \frac{Q^2}{9M^2} \tag{3.76}$$

which, again, is similar (though not identical) to the $k = \frac{Q^2}{4M^2}$ value found for k by approximating a Coulombic field to $\mathcal{F}_0^1$ for the '5D Schwarzchild' case examined in ref. [2].

Chapter 4

Magnetized Wire Solutions

While 5D metrics describing unified gravitational and electric fields have been examined here (chapter 3) and elsewhere (ref. [30]), few have been examined (from within the Induced Matter framework, at least) which describe unified gravitational and magnetic fields (see ref. [25] as one example).

In ordinary (4D) physics, the magnetic field about an axial-symmetric line source (a 'wire') is described by assuming that charges of one sign (*e.g.*, electrons) possess a velocity relative to charges of another sign (*e.g.*, positive copper ions). In 5D Kaluza-Klein/Induced Matter physics, however, an axially-symmetric field is assumed to result from curvature in the fifth dimension, coupled to the axial-symmetric (z) axis. This is represented in the metric by the presence of a $dzd\psi$ term, which represents the electromagnetic vector potential, A_3.

In this chapter, therefore, we shall examine two axially-symmetric metrics possessing $dzd\psi$ terms (corresponding to A_3 vector potentials). Such metrics will, therefore, be able to describe axially-symmetric, or 'wire', distributions of matter possessing an axially-symmetric magnetic field, B_2. We note that we are using the sub/superscripts of $0, 123, 5$ to represent the *cylindrical* coordinates of $t, \rho\phi z, \psi$ in this chapter (so that $A_3 \equiv A_z$ and $B_2 \equiv B_\phi$).

4.1 The Metrics

In addition to finding metrics with off-diagonal $dzd\psi$ terms, thereby allowing the description of axially-symmetric magnetic fields (B_2), we desire to find metrics which represent static (and therefore t-independent), axially-symmetric (and therefore ϕ-independent) distributions of matter that are infinite[1] in the z-direction (and therefore z-independent). This means that the metric coefficients may be functions of the radial coordinate, ρ, and/or the fifth coordinate, ψ. Since we desire the 4D part of the metric to at least have *some* connection with its Minkowski counterpart, $ds^2 = dt^2 - d\rho^2 - \rho^2 d\phi^2 - dz^2$, which possesses a ρ-dependency at least in $g_{22}(= -\rho^2)$, we seek metrics in which the coefficients are functions of ρ only, or of ρ and ψ (but not of ψ only). We have found one of each, and their derivation is described in appendix B.

The first metric is solely ρ-dependent and is:

$$\begin{aligned} d\hat{s}_I^2 = \rho^\lambda dt^2 - (1+\gamma/2)^2 \rho^\gamma d\rho^2 - \rho^{(2+\delta)} d\phi^2 + \left[-\tfrac{\rho^{(\eta-\mu)/2}}{(1-J/\rho^\mu)} + \tfrac{\rho^{(\eta+\mu)/2}(k-J/\rho^\mu)^2}{(1-J/\rho^\mu)J(1-k)^2}\right] dz^2 \\ \pm 2\sqrt{\epsilon}\tfrac{\rho^{(\eta+\mu)/2}(k-J/\rho^\mu)}{\sqrt{J}(1-k)} dz d\psi + \epsilon \rho^{(\eta+\mu)/2}(1 - J/\rho^\mu) d\psi^2 \end{aligned} \tag{4.1}$$

with k and J arbitrary constants and ϵ again the signature of the fifth coordinate ($= +1$ for a timelike fifth dimension and -1 for a spacelike fifth dimension), and where I denotes the metric number. μ is a parameter defined by:

[1]'Infinite in the z-direction' being taken partly for simplicity and partly for correspondence with other (4D diagonal) axial-symmetric metrics (ref.[37]).

$$\mu \equiv \pm\sqrt{\eta^2 + 8(\lambda + \eta) + 4(\lambda\eta + \lambda\delta + \eta\delta)} \tag{4.2}$$

and δ, λ, η and γ are parameters obeying the relation:

$$\delta + \lambda + \eta - \gamma = 0 \tag{4.3}$$

and are otherwise arbitrary.

This metric, eq. 4.1, as its derivation in appendix B clearly demonstrates, is the most general ρ-dependent metric possible for describing an axially-symmetric magnetized 'wire' solution, and bears a certain resemblance to its 4D counterpart (see ref. [37]). For $J = k = \delta = \lambda = \eta = \gamma = \mu = 0$, the two relations 4.2 and 4.3 are satisfied and the metric (eq. 4.1) reduces to:

$$d\hat{s}_I^2 \Rightarrow -d\rho^2 - \rho^2 d\phi^2 - dz^2 - dt^2 + \epsilon d\psi^2 \tag{4.4}$$

which is essentially the 5D axially-symmetric Minkowski metric. If, on the other hand, one lets only $J = k = 0$ and sets $-\mu = \eta$ (so that $\hat{g}_{55} = \pm 1$[2]), one obtains:

$$d\hat{s}_I^2 \Rightarrow \rho^\lambda dt^2 - (1 + \gamma/2)^2 \rho^\gamma d\rho^2 - \rho^{(2+\delta)} d\phi^2 - \rho^\eta dz^2 + \epsilon d\psi^2 \tag{4.5}$$

which must satisfy eq. 4.3 and the *reduced* relation:

$$2(\lambda + \eta) + \lambda\eta + \lambda\delta + \eta\delta = 0 \tag{4.6}$$

[2] Ensuring that $\hat{g}_{55} = \pm 1$ with $\hat{g}_{5\mu} = 0$ (ensured by $J = k = 0$) and $\hat{g}_{\mu\nu}$ independent of $x^5 = \psi$ forces the 4D portion of the metric to become the standard 4D solution (ref. [15]).

Eq. 4.5 satisfying relations 4.3 and 4.6 essentially constitute the 4D (diagonal) axially-symmetric 'wire' solution (with an extra flat fifth dimensional component and with an arbitrary factor in front of the $d\rho^2$ term, which can be absorbed by a simple coordinate transformation on ρ) (ref. [37]).

Although this metric (eq. 4.1) was found as the most general radial (ρ)-dependent, axial-symmetric solution with a single off-diagonal ($dzd\psi$) term (as shown in appendix B), this metric possesses the unexpected attribute that it can be transformed into a *diagonalized* form by a 'simple' coordinate transformation which does not affect the metric's purely ρ-dependency.

Of course, it is well known that any metric subspace of the form: $ds^2 = g_{11}(dx^1)^2 + 2g_{12}dx^1dx^2 + g_{22}(dx^2)^2$, where g_{11}, g_{12} and g_{22} all possess the same sign and are functions of x^1 and x^2, can be put into the diagonal form: $ds^2 = g[(dx^1)^2 + (dx^2)^2]$ (ref. [38]; see also axial-symmetric approach in ref. [39]). This can then be generalized to three dimensions, so that the metric subspace: $ds^2 = g_{11}(dx^1)^2 + 2g_{12}dx^1dx^2 + g_{22}(dx^2)^2 + 2g_{23}dx^2dx^3 + g_{33}(dx^3)^2 + 2g_{31}dx^3dx^1$ can be put into the form: $ds^2 = g[(dx^1)^2 + (dx^2)^2 + (dx^3)^2]$. But this requires that the new (diagonal) g's to be, in *general*, functions of *all* the coordinates, x^1, x^2 and x^3, irrespective of the forms of the original g_{11}, g_{12}, g_{22}, g_{23}, g_{33} and g_{31}.

If one identifies here $x^1 = \rho$, $x^2 = z$ and $x^3 = \psi$, one can see that, while this *can* diagonalize the metric 4.1, this diagonalization technique has the *potential* to introduce factors of z (and ψ) into the metric coefficients. This would violate the metric's basic premise as being representative of a wire infinite in the z direction (as discussed above, being infinite in the z-direction would require that the metric coefficients be independent of z). Surprisingly, however, in the case of metric 4.1, the metric *can* be diagonalized *without* introducing extra coordinates into the metric coefficients.

If one performs the coordinate transformation on the original metric 4.1:

$$\psi \to \psi \mp \frac{z}{\sqrt{\epsilon J}(1-k)} \tag{4.7}$$

followed by the transformation:

$$z \to \sqrt{J}z \pm \sqrt{\epsilon J}\psi \tag{4.8}$$

the transformed metric will become:

$$d\hat{s}_I^2 = \rho^\lambda dt^2 - (1+\gamma/2)^2\rho^\gamma d\rho^2 - \rho^{(2+\delta)}d\phi^2 + \rho^{(\eta+\mu)/2}dz^2 - \epsilon J\rho^{(\eta-\mu)/2}d\psi^2 \tag{4.9}$$

which is clearly a diagonal metric *still* of purely radial(ρ)-dependence, and, so, *still* represents a static, axial-symmetric matter distribution infinite in the z-direction.

This also means that one could have *started* from the diagonalized metric 4.9 and then *off*-diagonalized it by applying the *reverse* of the transformations 4.8 and 4.7. Normally, such an approach would not, in *principle*, yield the most general solution possible. But, as is shown in appendix B, this metric (eq. 4.1) *does* represent the most general form of a static ρ-dependent axially-symmetric metric possessing a single off-diagonal ($dzd\psi$) term.

The fact that the metric 4.1 *can* be diagonalized in this manner merely indicates that we are dealing with a 'wire' metric in which the magnetic field is *implicit* in the (5D) gravitational field. That is, the $dzd\psi$ crossterm (containing the A_3 vector potential) can be generated from a (simple) ψ-transformation on a diagonal metric (as above). [This is in contrast to *explicit* electric/magnetic fields (such as those of the previous chapter) in which the crossterms in $d\psi$ can*not* be generated from (simple) ψ-transformations on a diagonal metric.] The physical interpretation, however, within the Induced Matter formalism, is still the same, and, hence, the metric 4.1 is *physically* a respectable solution.

We decide to use a form of the metric given by making the transformation 4.7 on the original metric 4.1, but *not* transformation 4.8. The reason is that, while the vector potential, A_3, taken from metric 4.1, depends on both J and k, the corresponding magnetic field, $B_2 = F_{13}$, turns out to be *independent* of k. k is, therefore, an extraneous constant from the point of view of describing observable fields. The transformation 4.7 removes k from the metric, thereby reducing the metric to a 'least extraneous' form while still allowing it to possess the off-diagonal $dzd\psi$ term.

The specific transformations made on the original metric 4.1 begin with the transform 4.7. (We also replace $\mu \to -\mu$, which is allowed by the ambiguity of the sign of μ as defined by eq. 4.2. This change of sign simplifies terms like J/ρ^μ to $J\rho^\mu$.) We then perform the transformations:

$$t \to (1+\gamma/2)^{\lambda/(2+\gamma)} t \tag{4.10}$$

$$\rho \to (1+\gamma/2)^{-2/(2+\gamma)} \rho \tag{4.11}$$

$$\phi \to (1+\gamma/2)^{(2+\delta)/(2+\gamma)} \phi \tag{4.12}$$

$$z \to \sqrt{-J}(1+\gamma/2)^{(\eta-\mu)/[2(2+\gamma)]} z \tag{4.13}$$

$$\psi \to \frac{1}{\sqrt{-J}}(1+\gamma/2)^{(\eta+\mu)/[2(2+\gamma)]} \psi \tag{4.14}$$

and introduce:

$$j \equiv \frac{1}{\sqrt{-J}}(1+\gamma/2)^{\mu/(2+\gamma)} \tag{4.15}$$

as a new constant into the metric. The resulting metric we will then use is:

$$d\hat{s}_I^2 = \rho^\lambda dt^2 - \rho^\gamma d\rho^2 - \rho^{(2+\delta)} d\phi^2 - \rho^{(\eta-\mu)/2} dz^2 \mp 2\sqrt{-\epsilon} j \rho^{(\eta-\mu)/2} dz d\psi + \epsilon \rho^{(\eta-\mu)/2}(j^2 + \rho^\mu) d\psi^2 \tag{4.16}$$

where the new constant j can now be seen to represent the magnitude of A_3 (and hence B_2), and may be likened to the current magnitude. (Note, also, that the coefficient in front of the $d\rho^2$ term, the factor of $(1+\gamma/2)^2$, has vanished. This is due to the specific rescaling of the constants in the manner prescribed by eqs. 4.10 to 4.14.)

In analyzing wire metrics, we also seek a 'wire' metric which is *not* so easily diagonalized (*i.e.*, in which the magnetic field is *not* 'implicit' in the 5D gravitational field). Since we have the possibility of axially-symmetric metrics that are functions of ρ *and* ψ, we, therefore, seek an axial-symmetric, off-diagonal ($dzd\psi$) metric which is a function of ρ and ψ. The metric of this type found was derived as (see appendix 4):

$$d\hat{s}_{II}^2 = \rho^{\mathsf{NL}}\psi^{\mathsf{n}} dt^2 - \mathsf{K}\mathsf{L}^2\psi^2\rho^{2(\mathsf{L}-1)} d\rho^2 - \rho^{\mathsf{AL}}\psi^{\mathsf{a}} d\phi^2 - \rho^{\mathsf{BL}}\psi^{\mathsf{b}}(1+\mathsf{C}) dz^2 \pm 2\sqrt{-\epsilon \mathsf{C}}\rho^{(1+\mathsf{B}/2)\mathsf{L}}\psi^{\mathsf{b}/2} dz d\psi + \epsilon\rho^{2\mathsf{L}} d\psi^2 \tag{4.17}$$

where II denotes this as the second metric for this chapter, and where the new constants K, C, B, b, A and N are defined by:

$$\mathsf{K} \equiv \frac{\epsilon \mathsf{N}(2-\mathsf{B})}{(1-\epsilon \mathsf{C})\mathsf{n}(2-\mathsf{b})} \tag{4.18}$$

$$\mathsf{C} \equiv \frac{2\epsilon(\mathsf{N}\mathsf{b} - \mathsf{n}\mathsf{B})}{\mathsf{n}(2-\mathsf{B})} \tag{4.19}$$

$$B \equiv 1 - \frac{1}{2}(A + N) \tag{4.20}$$

$$b \equiv 1 - \frac{1}{2}(a + n) \tag{4.21}$$

$$A \equiv a f(a, n) \tag{4.22}$$

$$N \equiv n f(a, n) \tag{4.23}$$

where $f(a, n)$ is a function of a and n given by:

$$f(a, n) \equiv \frac{(2 + a + n)}{[-(a + n) + \frac{1}{6}(5a^2 + 2an + 5n^2)]} \tag{4.24}$$

and where a and n (and L) are free parameters. Again, ϵ is ± 1 depending on whether the fifth dimension is time-like or space-like. Note that the off-diagonal ($dzd\psi$) term has a further sign ambiguity (resulting from the square root of C) coupled to its ϵ-term. This corresponds to differences in direction of the current.

From these definitions, it can be seen that this metric (eq. 4.17) depends on the three parameters, a, n and L. a and n are 'free parameters', resulting from the derivation of the metric, while L is a constant which results solely from a coordinate transformation on ρ of the form: $\rho \to \rho^L$. This transformation is done to include a ρ-dependence in $\hat{g}_{11}$ (this is done to make this metric 'comparable' with other (4D) metrics which possess ρ-dependencies in their g_{11} terms; see discussion in section B.2 of appendix B). Setting $L = 1$ effectively removes the result of this transformation.

Substituting definition 4.24 into the definitions 4.18, 4.19 and 4.20 one obtains, as functions of a and n;

$$K = \frac{\epsilon f(a, n)[2 + (a + n) f(a, n)]^2}{(2 + a + n)[6 + (a + n - 4) f(a, n)]} \tag{4.25}$$

$$\mathsf{C} = \frac{-2\epsilon[1 - f(\mathsf{a},\mathsf{n})]}{[1 + \frac{1}{2}(\mathsf{a}+\mathsf{n})f(\mathsf{a},\mathsf{n})]} \tag{4.26}$$

$$\mathsf{B} = 1 - \frac{1}{2}(\mathsf{a}+\mathsf{n})f(\mathsf{a},\mathsf{n}) \tag{4.27}$$

It can be seen that the magnetic field for this metric is coupled to C, as its vanishing will cause A_3 (given by $\hat{g}_{35}/\hat{g}_{55}$) to vanish. Additionally, setting either $\mathsf{L} = 0$ or $\mathsf{B} = 2$ will cause the magnetic field to vanish as such settings will cause A_3 to become independent of ρ (so that $B_2 = F_{13} = A_{3,1} = 0$). However, setting $\mathsf{L} = 0$ or $\mathsf{B} = 2$ will yield singularities in the metric. Setting $\mathsf{L} = 0$ will cause $\hat{g}_{11} = 0$, so that the inverse metric component $\hat{g}^{11}$ will become infinite.

Setting $\mathsf{B} = 2$ causes the denominator in C to vanish, leading to infinities in the metric components $\hat{g}_{33}$ and $\hat{g}_{35}$. [To ensure that C really vanishes for $\mathsf{B} = 2$, one has to ensure that the *numerator* of C does not *also* vanish. Setting $\mathsf{B} = 2$ in view of the definitions 4.20 to 4.24, yields the constraint $\mathsf{a}^2 + \mathsf{an} + \mathsf{n}^2 = 0$. Setting the numerator of C to 0 with $\mathsf{B} = 2$ then requires $\mathsf{a} + \mathsf{n} = -2$. These two conditions require $\mathsf{a} = -1 \pm 3i$, and $\mathsf{n} = -1 \mp 3i$. If we insist that all exponents be real, then this is not possible, C's numerator could not vanish, and $\mathsf{B} = 2$ must necessarily cause C to become infinite.]

Thus, $\mathsf{L} = 0$ and $\mathsf{B} = 2$ may be taken as unphysical cases, and that $\mathsf{C} = 0$ is the only physically realistical way to make the magnetic field here vanish.

4.2 Linear Mass and Current Densities

To get our first understanding of these two metrics, we examine their limiting cases for (4D) gravitational and magnetic fields. As in the previous chapter, Roman numeral subscripts, I and II, represent the metric in question; I representing the first, ρ-dependent metric (eq. 4.16), with II representing the second, ρ- and ψ-dependent metric (eq. 4.17).

As discussed in chapter 2, we can identify the Newtonian gravitational limiting case with $\Gamma^1_{00}(u^0)^2$, where u^0 can be given by (see eq. 2.46): $u^0 = \dot{t}[1 - \hat{g}_{55}(\dot{\psi} + A_\mu \dot{x^\mu})^2]^{-1/2}$. In the previous chapter, we set u^0 to unity as a zero-order approximation. In this case, however, it is essential that we make *some* kind of approximation on u^0 before proceeding.

The reason is that we expect the metrics here (as their counterparts in the previous chapter) to be close to their Minkowski limits for most physically realistic situations. Normally, metrics of the type examined in the previous chapter (Schwarzchild-like) will have metric coefficients which are very close to their Minkowski limits in most physically reasonable situations. For Schwarzchild space (as a specific example), this is not a concern, as the non-Minkowski parts of the metric coefficients depend on factors of $(1 - 2M/r)$ which is ≈ 1 for any physically realistic (macroscopic) values of r. In other words, the 1 in the $(1 - 2M/r)$ *dominates* in most reasonable situations.

In these metrics, however, the metric coefficients are proportional to pure powers of ρ, the radial coordinate, and do not possess terms like the "1" which can dominate over the ρ portions in limiting cases. This indicates that, in order that the metrics (especially metric 4.16) reduce to Minkowski-like forms, one would expect the powers of ρ in both metrics to be vanishingly small. The smallness of these powers indicates that, even though u^0 differs from unity by only a small factor, we might expect that this factor to be *on the order of the smallness of the powers of* ρ, and therefore have the ability to be, itself, significant. For these metrics, the expression for u^0 will become:

$$u^0 = \dot{t}[1 - \hat{g}_{55}(\dot{\psi} + A_3\dot{z})^2]^{-1/2} \tag{4.28}$$

If we then assume $\dot{t} \approx 1 \gg \dot{x}^i$, for $i = 1,2,3,5$, then the square-root term in eq. 4.28 reduces to unity, and we obtain: $u^0 \simeq \dot{t}$. This is then easy to solve by using the Lagrangian approach; both metrics are independent of t, and contain no off-diagonal terms in dt. As a result, the Lagrangians may be written as:

$$\mathcal{L} = \hat{g}_{00}\dot{t}^2 + \hat{g}_{11}\dot{\rho}^2 + \hat{g}_{22}\dot{\phi}^2 + \hat{g}_{33}\dot{z}^2 + 2\hat{g}_{35}\dot{z}\dot{\psi} + \hat{g}_{55}\dot{\psi}^2 \tag{4.29}$$

and where the equation of motion for $\dot{t}$ may be found as:

$$\frac{\partial \mathcal{L}}{\partial \dot{t}} = 2\hat{g}_{00}\dot{t} = const \equiv 2E \tag{4.30}$$

so that: $\dot{t}_I = E_I\rho^{-\lambda}$ for the first metric, and $\dot{t}_{II} = E_{II}\rho^{\mathsf{NL}}\psi^{\mathsf{n}}$ for the second metric, where the E's are the constants of motion for the respective $\dot{t}$'s, and are close to unity for limiting cases.

From these values of $u^0 \simeq \dot{t}$ and by calculating Γ^1_{00} from the effective 4D metrics for each case (see eqs. 4.50 and 4.51 in section 4.4 for the effective 4D metrics), the weak gravitational fields can then be calculated. For the first metric, this is:

$$(\Gamma^1_{00}(u^0)^2)_I \simeq \frac{1}{2}E_I^2\lambda\rho^{-(\lambda+\gamma+1)} \tag{4.31}$$

while, for the second metric, it is:

$$(\Gamma^1_{00}(u^0)^2)_{II} \simeq \frac{\mathsf{N}E_{II}^2}{2\mathsf{KL}}\rho^{-\mathsf{NL}-\mathsf{L}+1}\psi^{-\mathsf{n}} \tag{4.32}$$

In both cases, we might expect the gravitational fields to fall off as $1/\rho$, as that is the form of a field surrounding a line source. Of course, we may imagine, in a curved space, a field which does *not* fall off *exactly* as $1/\rho$, perhaps as $1/\rho^{(1+\Delta)}$, where Δ is vanishingly small. But assuming we do impose the restriction $1/\rho$ form on the fields, we can then say:

$$\gamma = -\lambda \tag{4.33}$$

for the first metric, and

$$\mathsf{L} = \frac{2}{(1+\mathsf{N})} \tag{4.34}$$

for the second metric.

Additionally, since we expect the fields in question to be proportional to the linear mass density, we can then identify the factors in eqs. 4.31 and 4.32 with the linear mass densities. Taking $E_I \simeq E_{II} \simeq 1$, this identifies λ in the first metric with the linear mass density ($4\times$ the linear mass density, in fact, assuming a form of $2(\textit{linear mass density})/\rho$ for the (Newtonian) gravitational limit about a line source), while N/KL is ($4\times$) the linear mass density for the second metric.

In further consideration of the weak-field approximation of these metrics, we next examine the radial components of their magnetic fields, which is also known to have a flat-space limit of $1/\rho$. The magnetic field components considered are those from the first term on the right-hand side of the expanded geodesic equation, eq. 2.51, which, for the radial component of the magnetic field, are: F^1_3.

For the first metric, the vector potential A_3 is:

$$A_{I\ 3} = \frac{\pm j}{\sqrt{-\epsilon}(j^2 + \rho^\mu)} \tag{4.35}$$

while, for the second metric, A_3 is:

$$A_{II\ 3} = \pm\sqrt{\mathsf{C}/\epsilon}\,\rho^{(-1+\mathsf{B}/2)\mathsf{L}}\psi^{\mathsf{b}/2} \tag{4.36}$$

and where all other A_μ's are zero for both metrics. Calculating $F^1_3 = g^{11}F_{13} = g^{11}A_{3,1}$ gives, for the first metric;

$$F^1_{I\ 3} = \frac{\pm j\mu\rho^{(\mu-\gamma-1)}}{\sqrt{-\epsilon}(j^2 + \rho^\mu)^2} \tag{4.37}$$

and, for the second metric;

$$F^1_{II\ 3} = \frac{\pm\sqrt{-\mathsf{C}/\epsilon}}{\mathsf{KL}}(1 - \mathsf{B}/2)\rho^{(1-3\mathsf{L}+\mathsf{BL}/2)}\psi^{(-2+\mathsf{b}/2)} \tag{4.38}$$

For the first metric, we note that $j = 0$ causes the magnetic field to vanish, which is consistent with the identification of j with the magnitude of the current. For the limit $j \to 0$, we note that the first metric's magnetic field becomes:

$$F^1_{I\ 3} \simeq \frac{\pm 1}{\sqrt{-\epsilon}} j\mu\rho^{-(\mu+\gamma+1)} \tag{4.39}$$

For the second metric, we note again that the field is, indeed, coupled to C (the square root of C), as well as $(1 - \mathsf{B}/2)$. However, as noted above, B can never be 2, as this is an unphysical case, so the second metric's magnetic field can only vanish when C vanishes. As a result, we might identify $\sqrt{\mathsf{C}}$ with the magnitude of the current in its case.

As in the case of the weak gravitational field (just examined), it may be possible to imagine that, in a weak curved space, the magnetic field does *not* fall off as $1/\rho$ *exactly*, but perhaps as $1/\rho^{(1+\Delta)}$, where Δ is again very close to zero. However, if, as in the weak gravitational field case, we insist that the magnetic field falls off *exactly* as $1/\rho$ (which, of course, should be its limiting value), we find, for the first metric;

$$\gamma = -\mu \tag{4.40}$$

and:

$$\mathsf{L} = \frac{2}{(3 - \mathsf{B}/2)} \tag{4.41}$$

for the second metric. If one then combines these results with those obtained from fitting the weak gravitational field to a $1/\rho$ form (eqs. 4.33 and 4.34), one can then reduce both metrics to functions of just one variable.

For the first metric, eqs. 4.33 and 4.40 force:

$$\gamma = -\lambda = -\mu \rightarrow \mu = \lambda \tag{4.42}$$

When coupled with the constraints 4.3 and 4.2, this forces:

$$\eta = -2\lambda - \delta \tag{4.43}$$

with δ being given by:

$$\delta = \frac{1}{6}\left[-8 - 7\lambda \pm \sqrt{64 + 16\lambda - 11\lambda^2}\right] \tag{4.44}$$

For $\lambda = 0$, δ is consistent with zero (assuming one takes the "+" sign from the square root), thereby causing η, μ and γ to vanish, resulting in a Minkowskian expression for the 4D portion of the first metric. Since we have already identified λ with ($4\times$) the linear mass density, then its vanishing, forcing the 4D portion of the metric to become Minkowskian, is thereby consistent with this identification.

For the second metric, the combination of constraints 4.34 and 4.41 force:

$$\mathsf{L} = \frac{2}{(1+\mathsf{N})} = \frac{2}{(3-\mathsf{B}/2)} \rightarrow \mathsf{N} = 2 - \mathsf{B}/2 \tag{4.45}$$

which constrains L and introduces the constraint $2\mathsf{a} + 6\mathsf{n} - 3\mathsf{a}^2 - \mathsf{n}^2 = 0$ (if these relations are accepted).

4.3 Maxwell's Equations

To further examine the expressions of current (and current density) of these two metrics, we examine Maxwell's Equations, given by: $F^{\alpha\beta}{}_{;\beta} = J^\alpha$ where J^α is the desired current (density) which we seek. Again Roman numeral subscripts denote the metric number.

From Maxwell's Equations, J^3 can be calculated from $J^3 = F^{3\beta}{}_{;\beta}$, which, for both metrics, can be written out as:

$$J^3 = F^{31}_{,1} + \frac{1}{2} F^{31} g^{\alpha\alpha} g_{\alpha\alpha,1} \tag{4.46}$$

For the first metric, the expression for J^3 can be calculated (see computer algorithm listed in appendix D) as:

$$J^3_I = \frac{\mp\frac{3}{4}j\mu\rho^{[-2-\gamma+(\mu-\eta)/2]}}{\sqrt{-\epsilon}(j^2+\rho^\mu)}\left[\mu-\eta-\frac{2\mu\rho^\mu}{(j^2+\rho^\mu)}\right] \tag{4.47}$$

while, for the second metric, J^3 can be found as:

$$J^3_{II} = \frac{\pm 2\sqrt{-\mathsf{C}/\epsilon}}{\mathsf{K}}(\mathsf{B}-2)(\mathsf{B}+1)\rho^{(-3-\mathsf{B}/2)\mathsf{L}}\psi^{(-2-\mathsf{b}/2)} \tag{4.48}$$

with all the other J^α's being zero for both metrics.

For the first metric, it has already been established that j should be related to the magnitude of the current. For a weak current then, $j \to 0$, and J^3 becomes:

$$J^3_I \simeq \pm\frac{3}{4\sqrt{-\epsilon}}j\mu(\mu+\eta)\rho^{[-2-\gamma-(\mu+\eta)/2]} \tag{4.49}$$

As can be seen here, the current density of the first solution couples to j, while the current density of the second solution (see eq. 4.48) couples to $\sqrt{\mathsf{C}}$, thereby confirming the interpretation of j and $\sqrt{\mathsf{C}}$ as the magnitudes of the currents of the first and second metrics, respectively.

For a magnetic field falling off roughly as $1/\rho$, one would expect J^3 to fall off roughly as $1/\rho^2$. This is *approximately* the case for the first metric, and, *possibly* the case for the second metric, depending on the choice of parameters.

Indeed, the choice of parameters available in both wire solutions makes solving for mass and current density *difficult*. Additionally, though we expect the magnetic and gravitational fields to fall off *roughly* as $1/\rho$, we can*not* expect that these will be *exactly* the cases, especially in curved manifolds. As a result, we do not calculate such quantities as *current*-to-mass ratios for these wires as they would be highly dependent on the choice of parameters, as well as undoubtedly being sensative to the smallness of the powers of ρ (*i.e.*, sensative to the fact that ρ will not differ much from unity; see discussion in relation to u^0 above).

4.4 Density, Pressure and Stress

Next, we examine the induced energy-momentum tensor for each of the two metrics and their resulting equations of state. Again, we use the 4D $g_{\mu\nu}$ to calculate the 4D G_μ^ν (using GRTensor II on Maple, ref. [1]) and using the fact that $T_\mu^\nu = G_\mu^\nu$ to deduce T_μ^ν. With this in mind, we explicitly note the 4D portion of both metrics; for the first metric, this is:

$$ds_I^2 = \rho^\lambda dt^2 - \rho^\gamma d\rho^2 - \rho^{(2+\delta)} d\phi^2 - \frac{\rho^{(\eta+\mu)/2}}{(j^2+\rho^\mu)} dz^2 \tag{4.50}$$

while, for the second metric, it is:

$$ds_{II}^2 = -\mathsf{K}\mathsf{L}^2\rho^{2(\mathsf{L}-1)}\psi^2 d\rho^2 - \rho^{\mathsf{AL}}\psi^{\mathsf{a}} d\phi^2 - \rho^{\mathsf{BL}}\psi^{\mathsf{b}} dz^2 + \rho^{\mathsf{NL}}\psi^{\mathsf{n}} dt^2 \tag{4.51}$$

From these 4D metrics, the effective energy-momentum tensors can then be found as outlined above. For the first metric, the nonzero energy-momentum tensor components are:

$$\begin{aligned} T_{I\;0}^{0} &= \frac{(\eta-\mu)[2+\gamma+(\mu-\eta)/2]j^2+(\eta+\mu)[2+\gamma-(\mu+\eta)/2]\rho^\mu}{8\rho^{(2+\gamma)}(j^2+\rho^\mu)} \\ &= \frac{(\mu-\eta)\lambda j^4+(6\mu^2-2\lambda\eta)j^2\rho^\mu-(\eta+\mu)\lambda\rho^{2\mu}}{8\rho^{(2+\gamma)}(j^2+\rho^\mu)^2} \end{aligned} \tag{4.52}$$

$$\begin{aligned} T_{I\;1}^{1} = \{&(\eta-\mu)[2+\gamma-(\eta-\mu)/2]j^4+[2(2+\gamma)\eta-(\eta^2+\mu^2)]j^2\rho^\mu \\ &+(\eta+\mu)[2+\gamma-(\eta+\mu)/2]\rho^{2\mu}]\}/[8\rho^{(2+\gamma)}(j^2+\rho^\mu)^2] \end{aligned} \tag{4.53}$$

$$T^2_{I\ 2} = \{(\mu-\eta)[2+\gamma-\lambda-\eta]j^4 + [6\mu^2 - 2(2+\gamma-\lambda-\eta)\eta]j^2\rho^\mu \\ -(\eta+\mu)[2+\gamma-\lambda-\eta]\rho^{2\mu}]\}/[8\rho^{(2+\gamma)}(j^2+\rho^\mu)^2] \tag{4.54}$$

$$T^3_{I\ 3} = \frac{\frac{1}{2}(\mu^2-\eta^2)}{8\rho^{(2+\gamma)}} = \frac{\frac{1}{2}(\mu^2-\eta^2)j^4 + (\mu^2-\eta^2)j^2\rho^\mu + \frac{1}{2}(\mu^2-\eta^2)\rho^{2\mu}}{8\rho^{(2+\gamma)}(j^2+\rho^\mu)^2} \tag{4.55}$$

while, for the second metric, the nonzero components of the energy-momentum tensor are:

$$T^0_{II\ 0} = \frac{-1}{4\mathsf{K}\rho^{2\mathsf{L}}\psi^2}[\mathsf{AB}+\mathsf{AN}+\mathsf{BN}] \tag{4.56}$$

$$T^1_{II\ 1} = \frac{-1}{4\mathsf{K}\rho^{2\mathsf{L}}\psi^2}\left[-2\mathsf{B}+\mathsf{B}^2-2\mathsf{N}+\mathsf{N}^2+\mathsf{BN}\right] \tag{4.57}$$

$$T^2_{II\ 2} = \frac{-1}{4\mathsf{K}\rho^{2\mathsf{L}}\psi^2}\left[-2\mathsf{A}+\mathsf{A}^2-2\mathsf{N}+\mathsf{N}^2+\mathsf{AN}\right] \tag{4.58}$$

$$T^3_{II\ 3} = \frac{-1}{4\mathsf{K}\rho^{2\mathsf{L}}\psi^2}\left[-2\mathsf{A}+\mathsf{A}^2-2\mathsf{B}+\mathsf{B}^2+\mathsf{AB}\right] \tag{4.59}$$

Again, the electromagnetic component of the energy-momentum tensor can be calculated (see discussion at end of section 2.3.3 for the *cylindrically*-symmetric case,) as: $T_{EM} \equiv -T^0_{EM\ 0} = T^1_{EM\ 1} = -T^2_{EM\ 2} = T^3_{EM\ 3}$. This is, for the first metric;

$$T_{I\ EM} = \frac{\mu^2 j^2 \rho^\mu}{\rho^{(2+\gamma)}(j^2+\rho^\mu)^2} \tag{4.60}$$

while, for the second metric, it is:

$$T_{II\ EM} = \frac{-1}{4\mathsf{K}\rho^{2\mathsf{L}}\psi^2}\left[\mathsf{C}(\mathsf{B}-2)^2\right] \tag{4.61}$$

From section 2.3.3 of the previous chapter, we next use $\wp = T^0_0 + T_{EM}$ to define the density (and where we remind the reader not to confuse the *density*

$\wp$ with the *coordinate* ρ), $P = -\frac{1}{3}(T_1^1 + T_2^2 + T_3^3 - T_{EM})$ to define the pressure, and $\tau_1^1 = \frac{2}{3}T_1^1 - \frac{1}{3}(T_2^2 + T_3^3) - \frac{2}{3}T_{EM}$, $\tau_2^2 = \frac{2}{3}T_2^2 - \frac{1}{3}(T_1^1 + T_3^3) + \frac{4}{3}T_{EM}$ and $\tau_3^3 = \frac{2}{3}T_3^3 - \frac{1}{3}(T_1^1 + T_2^2) - \frac{2}{3}T_{EM}$ to define the anisotropic stress (all for a *cylindrically-symmetric system*). For the first metric, the density, pressure and stress are then:

$$\wp_I = \frac{(\mu-\eta)\lambda j^4 + (14\mu^2 - 2\lambda\eta)j^2\rho^\mu - (\eta+\mu)\lambda\rho^{2\mu}}{8\rho^{(2+\gamma)}(j^2+\rho^\mu)^2} \tag{4.62}$$

$$P_I = \frac{(\mu-\eta)\lambda j^4 + 2(\mu^2 - \lambda\eta)j^2\rho^\mu - (\eta+\mu)\lambda\rho^{2\mu}}{24\rho^{(2+\gamma)}(j^2+\rho^\mu)^2} \tag{4.63}$$

$$\begin{aligned}\tau_{I\ 1}^1 = &\left[(\eta-\mu)(6+3\gamma-\lambda) - \tfrac{3}{2}(\eta-\mu)^2\right]\tfrac{j^4}{24\rho^{(2+\gamma)}(j^2+\rho^\mu)^2}\\ &+\left[2\eta(6+3\gamma-\lambda) - 3\eta^2 - 25\mu^2\right]\tfrac{j^2\rho^\mu}{24\rho^{(2+\gamma)}(j^2+\rho^\mu)^2}\\ &+\left[(\eta+\mu)(6+3\gamma-\lambda) - \tfrac{3}{2}(\eta+\mu)^2\right]\tfrac{\rho^{2\mu}}{24\rho^{(2+\gamma)}(j^2+\rho^\mu)^2}\end{aligned} \tag{4.64}$$

$$\begin{aligned}\tau_{I\ 2}^2 = &\left[(\eta-\mu)(-6-3\gamma+2\lambda+3\eta)\right]\tfrac{j^4}{24\rho^{(2+\gamma)}(j^2+\rho^\mu)^2}\\ &+\left[2\eta(-6-3\gamma+2\lambda+3\eta) + 44\mu^2\right]\tfrac{j^2\rho^\mu}{24\rho^{(2+\gamma)}(j^2+\rho^\mu)^2}\\ &+\left[(\eta+\mu)(-6-3\gamma+2\lambda+3\eta)\right]\tfrac{\rho^{2\mu}}{24\rho^{(2+\gamma)}(j^2+\rho^\mu)^2}\end{aligned} \tag{4.65}$$

$$\begin{aligned}\tau_{I\ 3}^3 = &\left[(\mu-\eta)\lambda + \tfrac{3}{2}(\mu^2-\eta^2)\right]\tfrac{j^4}{24\rho^{(2+\gamma)}(j^2+\rho^\mu)^2}\\ &+\left[-3\eta^2 - 2\eta\lambda - 19\mu^2\right]\tfrac{j^2\rho^\mu}{24\rho^{(2+\gamma)}(j^2+\rho^\mu)^2}\\ &+\left[-(\mu+\eta)\lambda + \tfrac{3}{2}(\mu^2-\eta^2)\right]\tfrac{\rho^{2\mu}}{24\rho^{(2+\gamma)}(j^2+\rho^\mu)^2}\end{aligned} \tag{4.66}$$

while the density, pressure and stress of the second metric are:

$$\wp_{II} = \frac{-1}{4\mathsf{K}\rho^{2\mathsf{L}}\psi^2}\left[-2\mathsf{A} + \mathsf{A}^2 - 2\mathsf{B} + \mathsf{B}^2 + \mathsf{AB} + 4\mathsf{C} - 4\mathsf{CB} + \mathsf{CB}^2\right] \tag{4.67}$$

$$P_{II} = \frac{1}{12\mathsf{K}\rho^{2\mathsf{L}}\psi^2}\left[\mathsf{AB} + 2\mathsf{AN} + 2\mathsf{BN} - 2\mathsf{B} + \mathsf{B}^2 - 4\mathsf{N} + 2\mathsf{N}^2 - 2\mathsf{A} + \mathsf{A}^2 - 4\mathsf{C} + 4\mathsf{CB} - \mathsf{CB}^2\right] \tag{4.68}$$

$$\tau_{II\ 1}^1 = \frac{1}{12\mathsf{K}\rho^{2\mathsf{L}}\psi^2}\left[-2\mathsf{AB} - \mathsf{AN} - \mathsf{BN} - 2\mathsf{B} + \mathsf{B}^2 - 4\mathsf{N} + 2\mathsf{N}^2 - 2\mathsf{A} + \mathsf{A}^2 + 8\mathsf{C} - 8\mathsf{CB} + 2\mathsf{CB}^2\right] \tag{4.69}$$

$$\tau^{2}_{II\ 2} = \frac{-1}{12\mathsf{K}\rho^{2\mathsf{L}}\psi^{2}}\left[-4\mathsf{B}+2\mathsf{B}^{2}-2\mathsf{N}+\mathsf{N}^{2}+\mathsf{BN}-\mathsf{AB}-2\mathsf{AN}+2\mathsf{A}-\mathsf{A}^{2}+16\mathsf{C}-16\mathsf{CB}+4\mathsf{CB}^{2}\right] \tag{4.70}$$

$$\tau^{3}_{II\ 3} = \frac{-1}{12\mathsf{K}\rho^{2\mathsf{L}}\psi^{2}}\left[-4\mathsf{A}+2\mathsf{A}^{2}-2\mathsf{N}+\mathsf{N}^{2}+\mathsf{AN}-\mathsf{AB}-2\mathsf{BN}+2\mathsf{B}-\mathsf{B}^{2}-8\mathsf{C}+8\mathsf{CB}-2\mathsf{CB}^{2}\right] \tag{4.71}$$

In general, these forms will depend on the values of the various parameters of each metric. For both metrics, however, it appears as though there are singularities, at least, for $\rho \to 0$ (though, *un*like for the first metric in the previous chapter, there do not appear to be any potential *horizons* from the forms of the metrics here). Instead, we note that the equations given here seem to indicate fluids whose densities, *etc.*, increase towards $\rho \to 0$, so that, instead of pure *wire* forms, we are dealing with axially-symmetric *columns of fluid* concentrated along their axes.

4.4.1 Simple Equations of State

In consideration of equations of state of the form $P = n\wp$, where n is a number (independent of ρ or ψ), we consider the expressions for both metrics given by dividing the P by its corresponding $\wp$. For the first metric, this is:

$$(P/\wp)_I = \left[\frac{(\mu-\eta)\lambda j^4 + 2(\mu^2-\eta\lambda)j^2\rho^{\mu} - (\mu+\eta)\lambda\rho^{2\mu}}{3(\mu-\eta)\lambda j^4 + 6(7\mu^2-\eta\lambda)j^2\rho^{\mu} - 3(\mu+\eta)\lambda\rho^{2\mu}}\right] \tag{4.72}$$

while, for the second metric, it is:

$$(P/\wp)_{II} = \left[\frac{\mathsf{AB}+2\mathsf{AN}+2\mathsf{BN}-2\mathsf{B}+\mathsf{B}^{2}-4\mathsf{N}+2\mathsf{N}^{2}-2\mathsf{A}+\mathsf{A}^{2}-4\mathsf{C}+4\mathsf{CB}-\mathsf{CB}^{2}}{6\mathsf{A}-3\mathsf{A}^{2}+6\mathsf{B}-3\mathsf{B}^{2}-3\mathsf{AB}-12\mathsf{C}+12\mathsf{CB}-3\mathsf{CB}^{2})}\right] \tag{4.73}$$

In order that the metrics describe an equation of state of the form $P = n\wp$ with n a number, it is necessary for the expressions on the right-hand sides of eq. 4.72

and 4.73 be independent of ρ or ψ. This is automatically true for the second metric (eq. 4.73), but is not automatically true for the first metric (eq. 4.72).

In fact, it is not possible for the first metric to represent an equation of state of the form $P = n\wp$ that isn't nonmagnetic, with either j or μ being zero (the magnetic field being proportional to both j and μ), or 'gravitational-less', with λ being zero (since the 'Newtonian limit' of linear mass density is proportional to λ). In general, therefore, the equation of state for the first first metric will be complicated.

For the second metric, however, an equation of state of the form $P = n\wp$ is possible if one assigns:

$$\left[\frac{\mathrm{AB} + 2\mathrm{AN} + 2\mathrm{BN} - 2\mathrm{B} + \mathrm{B}^2 - 4\mathrm{N} + 2\mathrm{N}^2 - 2\mathrm{A} + \mathrm{A}^2 - 4\mathrm{C} + 4\mathrm{CB} - \mathrm{CB}^2}{6\mathrm{A} - 3\mathrm{A}^2 + 6\mathrm{B} - 3\mathrm{B}^2 - 3\mathrm{AB} - 12\mathrm{C} + 12\mathrm{CB} - 3\mathrm{CB}^2)}\right] = n \tag{4.74}$$

where n is $\frac{1}{3}$ for radiation, -1 for vacuum, $+1$ for stiff matter and 0 for dust. Therefore, the second metric, at least, has the potential to model various *simple* equations of state of the form $P = n\wp$.

Chapter 5

Cosmological Solutions

As discussed in chapter 2, if spacetime is truly five-dimensional, then our universe (which is perceived as four-dimensional) must essentially occupy a(n approximate) four-dimensional hypersurface on the five-dimensional manifold. Indeed, as written out in ref. [40], the standard 4D cosmological models (FRW solutions) can be written as conformally flat solutions (in 4D).

It is also known that the standard (FRW) 4D cosmological solutions can be embedded in a flat 5D manifold (see refs. in [41] and [42]). In refs. [12], [23], [41] and [42], 5D solutions are examined which possess 4D cosmological interpretations. Many of these solutions are *Riemann* flat, as well as Ricci flat (*vacuum*), but they are not *conformally* flat (*i.e.*, the solutions are not written out as *flat metrics possessing conformal factors*).

To further this line of research, therefore, we examine two 5D metrics which possess conformally flat sections. In the first, the 4D $(t, r\theta\phi)$ spacetime portion is conformally flat, while the 5-5 portion possesses an additional factor. In the second, the *entire* metric is (5D) conformally flat. For these reasons, the first metric will be referred to as 4D conformally flat (in a 5D manifold), while the second will be referred to as 5D conformally flat.

5.1 The Metrics

The first metric examined will be the 4D conformally flat metric. It was found (see appendix C) as the most general flat[1] solution to the metric form:

$$d\hat{s}_I^2 = \frac{[dt^2 - dr^2 - r^2 d\Omega^2]}{\mathbf{A}(t,r,\psi)^2} + \epsilon \frac{d\psi^2}{\mathbf{B}(t,r,\psi)^2} \tag{5.1}$$

where I denotes that this is the 4D conformally flat solution, and where $\mathbf{A}(t,r,\psi)$ and $\mathbf{B}(t,r,\psi)$ are general functions of t, r and ψ. To satisfy the vacuum field equations, one finds (from appendix C):

$$\mathbf{B}(t,r,\psi) = \frac{\mathbf{A}(t,r,\psi)}{\overset{*}{\mathbf{A}}(t,r,\psi)} \tag{5.2}$$

so that the metric can be written out as:

$$d\hat{s}_I^2 = \frac{[dt^2 - dr^2 - r^2 d\Omega^2]}{\mathbf{A}(t,r,\psi)^2} + \epsilon \frac{\overset{*}{\mathbf{A}}(t,r,\psi)^2 d\psi^2}{\mathbf{A}(t,r,\psi)^2} \tag{5.3}$$

where, again, an overstar denotes a partial derivative with respect to the fifth coordinate, ψ. To satisfy the vacuum field equations,$\mathbf{A}(t,r,\psi)$ must be given by:

$$\mathbf{A}(t,r,\psi) = \alpha(\psi)t^2 + \beta(\psi)t + \gamma(\psi) - \alpha(\psi)r^2 \tag{5.4}$$

with $\alpha(\psi)$, $\beta(\psi)$ and $\gamma(\psi)$ general functions of ψ which are constrained by:

[1]See discussion on this point in section C.2 of appendix C.

$$\beta(\psi)^2 - 4\alpha(\psi)\gamma(\psi) = -1/\epsilon \tag{5.5}$$

which equals +1 for $\epsilon = -1$ (spacelike fifth dimension).

The second metric to be examined is a 5D conformally flat metric corresponding to a solution of the general form:

$$d\hat{s}^2_{II} = e^{\tilde{\Phi}(t,r,\psi)}[dt^2 - dr^2 - r^2 d\Omega^2 + \epsilon d\psi^2] \tag{5.6}$$

where II denotes that this is the 5D conformally flat solution, and where $e^{\tilde{\Phi}(t,r,\psi)}$ is the conformal factor in question, and where $\tilde{\Phi}(t, r, \psi)$ is a general function of t, r and ψ (and where it is written with a *tilde* to distinguish it from the Φ in the general form of the 5-5 component of the Induced Matter metric, $\hat{g}_{55} = \epsilon\Phi^2$).

Upon solving the field equations for the r-dependencies for this metric (see appendix C), we find:

$$e^{\tilde{\Phi}(t,r,\psi)} = \frac{(kv_{t\psi})^4}{[u_{t\psi} - \frac{1}{2}(v_{t\psi}r)^2]^2} \tag{5.7}$$

where k is a constant, and where $u(t, \psi)$ and $v(t, \psi)$ are functions of t and ψ (but not of r), and which must satisfy the *reduced* field eqs. C.32, C.33, C.34 and C.36 from appendix C.

The simplist (nontrivial) solution is to set $v(t, \psi)$ to a constant, such as unity (and set k to $1/\sqrt{2}$ to properly normalize the conformal factor) and solve $u(t, \psi)$ to be $\frac{1}{2}(t^2 + \epsilon\psi^2)$ from the reduced field equations. This then yields a metric of the form:

$$d\hat{s}^2_{II} = \frac{[dt^2 - dr^2 - r^2 d\Omega^2 + \epsilon d\psi^2]}{[t^2 - r^2 + \epsilon\psi^2]^2} \tag{5.8}$$

which constitutes the *main* 5D conformally flat solution used in the bulk of this chapter.

There are, however, many other 5D conformally flat solutions one could find from the aforementioned reduced field equations. For example, one could also set $v_{t\psi} = 0$ and let $k \to 1/0$, so that $kv_{t\psi} = 1$, and solve $u(t, \psi)$ to be $(t + \epsilon\psi)$. This would give a metric of the form:

$$d\hat{s}^2_{overlap} = \frac{1}{(t+\epsilon\psi)^2}\left[dt^2 - dr^2 - r^2 d\Omega^2 + \epsilon d\psi^2\right] \tag{5.9}$$

However, this is essentially the 4D conformally flat metric (5.3) with $\alpha(\psi) = 0$, $\beta(\psi) = 1$ and $\gamma(\psi) = \epsilon\psi$. Metric 5.9, therefore, represents the *overlap* of the 4D and 5D conformally flat metrics, and is mentioned here mainly for completeness.

It is noted that both metrics 5.3 and 5.8 share a similar form to each other, and to other cosmological metrics, particularly in their r-dependency. The standard 4D cosmological (FRW) models can be written in isotropic form with $(1+kr^2)^{-2}$ as the metric coefficients for the 3-spatial portion of the metric (the $d\sigma^2 = dr^2 + r^2 d\Omega^2$) (ref. [16]). As well, in some of the 5D cosmological metrics studied in refs. [12], [23], [41] and [42], a similar factor appears in the 3-spatial portions of those metrics. This matches with the r-dependence of both metrics 5.3 and 5.8 in this chapter, and lends credence to the idea that these metrics (here) represent cosmologies.

It should also be noted, however, that these metrics (here) still *differ* from those studied in refs. [12], [23], [41] and [42]. Most of those metrics are, at most, written in the aforementioned isotropic form. They are not written in conformally flat form (either 4D or 5D), and, as such, these metrics (here) differ from those previously studied. (As well, these metrics also possess significant *mixed* t and ψ-dependencies in their conformal factors).

At this point, it should also be noted that, despite first appearances, the *main* 5D solution (eq. 5.8) is *not* a special case of the 4D solution (eq. 5.3). One might be tempted to assign $\mathbf{A} = (t^2 + \epsilon\psi^2 - r^2)$ to match the denominators in both metrics. However, this designation assigns $\alpha_\psi = 1$, $\beta = 0$ and $\gamma = \epsilon\psi^2$, which do *not* satisfy the constraint 5.5.

Additionally, such a designation causes $\overset{*}{\mathbf{A}}^2 = (2\epsilon\psi)^2$, which is *not* unity (or a constant), so that this solution would still not be 5D conformally flat. One *might* be tempted to transform: $\psi \to \sqrt{\psi/\epsilon}$ so that $\overset{*}{\mathbf{A}}\ d\psi \to d\psi$, however, this transformation would then transform the denominator $\mathbf{A} \to (t^2 + \psi - r^2)$, which is *not* the 5D metric's denominator.

The only solution which 'overlaps' the 4D and 5D conformal cases is given by the 'overlap' metric 5.9. The 5D solution is *not*, in general, a special case of the 4D solution due to the fact that there are *more* (differing) metric coefficients in the 4D solution, and, thus, greater differences amongst its Ricci tensors. This more greatly constrains the 4D's metric coefficients, and, thus, makes its solution more restrictive. The fact that there is an entire *class* of metrics for the 4D case, while the 5D case is not, is simply because, *due* to this increased restrictiveness, it was possible to constrain the 4D metric coefficients to allow for a complete derivation.

In the case of the 5D solution, as note above, it turns out that the equations were so *un*restrictive, that all that could be done was to reduce the field equations by removing their r-dependence (see appendix C). The 4D solution *was* the most general 4D conformally flat solution to be found, while the 5D solution found was *not* the most general 5D conformally flat solution. As noted in the derivation of the 'overlap' metric, there are *many* 5D solutions, of which eq. 5.8 is but one restrictive example. Thus, the two solutions here are unique, and allow for separate, if similar, analyses.

5.2 Equations of State

To examine the (4D) physical nature of these metrics, we again calculate the energy-momentum tensor, which, again, requires the effective 4D metrics of these 5D metrics, 5.3 and 5.8. For the 4D conformally flat metric, this is:

$$ds_I^2 = \frac{[dt^2 - dr^2 - r^2 d\Omega^2]}{\mathbf{A}(t, r, \psi)^2} \tag{5.10}$$

while, the corresponding 4D portion of the 5D conformally flat metric is:

$$ds_{II}^2 = \frac{dt^2 - dr^2 - r^2 d\Omega^2}{(t^2 - r^2 + \epsilon\psi^2)^2} \tag{5.11}$$

Again, using $T_\mu^\nu = G_\mu^\nu$, where G_μ^ν is the effective 4D Einstein tensor constructed from the effective 4D metrics (eqs. 5.10 and 5.11), one can calculate the energy-momentum tensor for the 4D conformally flat metric as:

$$T_{I\ \mu}^{\nu} = (G_{I\ \mu}^{\nu} =) 3\delta_\mu^\nu \tag{5.12}$$

and for the 5D conformally flat metric as:

$$T_{II\ \mu}^{\nu} = (G_{II\ \mu}^{\nu} =) - 12\epsilon\psi^2\delta_\mu^\nu \tag{5.13}$$

where, as in previous chapters, the δ_μ^ν is the Kronecker delta ($\delta_\mu^\nu = 1$ for $\mu = \nu$; 0 otherwise).

Because both metrics 5.3 and 5.8 are neutral (possessing no off-diagonal terms in $d\psi$), it is then possible to *directly* assign a density (see section 2.3.3 from chapter 2) as: $\wp = T_0^0$, and a pressure as: $P = -\frac{1}{3}[T_1^1 + T_2^2 + T_3^3]$. Since both metrics are *isotropic*, this definition of pressure becomes *exact*, and the anisotropic stresses, τ_μ^ν are zero for *all* μ, ν.

However, for the first (4D conformally flat) metric, this assigns $\wp = 3$ and $P = -3$, which seems a curious designation. To interpret this result somewhat more physically *realistically*, we consider the usual (4D) de Sitter cosmological solution:

$$ds^2 = dt^2 - e^{-2t/t_0}[dr^2 + r^2 d\Omega^2] \tag{5.14}$$

where e^{-2t/t_0} is the usual de Sitter cosmological expansion factor, with t_0 as a constant (curvature) factor related to the de Sitter cosmological constant, Λ, in the usual way (ref. [39]);

$$1/t_0 \equiv \sqrt{\frac{\Lambda}{3}} \tag{5.15}$$

The de Sitter solution can then be put into a conformally flat form by the transformation:

$$t \rightarrow t_0 \; \ln(t/t_0) \tag{5.16}$$

which yields a conformally flat metric of the form (ref. [39]):

$$ds^2 = \left(\frac{t_0}{t}\right)^2 [dt^2 - dr^2 - r^2 d\Omega^2] \tag{5.17}$$

In this form (eq. 5.17), the de Sitter metric bears a similar resemblance to the 'overlap' metric listed in eq. 5.9. If one imagines our universe existing on a hypersurface of (nearly) constant ψ, the $d\psi^2$ term can be ignored (dropped), and the ψ term in the conformal factor can be made to disappear by making a transformation on t as:

$$t \rightarrow t - \epsilon\psi \tag{5.18}$$

so that the 4D portion of metric 5.9 will appear as:

$$ds^2_{overlap} \Rightarrow \frac{1}{t^2}[dt^2 - dr^2 - r^2 d\Omega^2] \tag{5.19}$$

which is the de Sitter metric (5.17) with the curvature factor, t_0, normalized to unity.

When we examine the field equations of the usual (4D) de Sitter solution (either metric 5.14 or 5.17), we find (ref. [39]):

$$G^{\nu}_{\mu} = \Lambda\delta^{\nu}_{\mu} \tag{5.20}$$

where the cosmological constant, Λ, is constrained by:

$$\Lambda = 3t_0^{-2} \tag{5.21}$$

which, for t_0 set to unity, is precisely the solution of eq. 5.12.

Therefore, the metric 5.3 represents a 5D generalization of the kind of universe described by the 4D de Sitter solution, with a (normalized) cosmological constant given by: $\Lambda = 3$. It should be noted, however, that metric 5.3 represents a 5D generalization to the de Sitter solution, and not *just* the special case of metric 5.9. Metric 5.9 can be put into a de Sitter form by transformation 5.18, but in general, there will be no simple (obvious) transformation of this kind for metric 5.3. This indicates that the *general* 4D conformally flat solution represents a de Sitter-type solution.

For the second metric, the density and pressure can be rendered as:

$$\wp = -12\epsilon\psi^2 \tag{5.22}$$

$$P = 12\epsilon\psi^2 \tag{5.23}$$

which then have an equation of state as:

$$P = -\wp \tag{5.24}$$

which is the equation of state of a vacuum (ref. [8]). However, since neither the density nor the pressure are *actually zero* (unless the coordinate ψ is zero, which is arbitrary), then this metric, like the 4D conformally flat one before it, represents a de Sitter vacuum solution with a cosmological constant given by: $\Lambda = 3/t_0^2$, with t_0 given in this case by: $t_0 = -4\epsilon\psi^2$.

Again, the metric 5.8 represents a 5D generalization to the de Sitter solution even though the metric 5.11 can*not* be put into a pseudo-de Sitter form like eq. 5.19 by any kind of transformation like eq. 5.18.

We can also look specifically at the transformed form of the *overlap* metric, 5.9. If we transform:

$$\begin{gathered} t \to e^t - \epsilon\psi \\ (t + \epsilon\psi)^{-2} \to e^{-2t} \\ dt \to e^t dt - \epsilon d\psi \\ dt^2 \to e^{2t} dt^2 - 2\epsilon e^t dt d\psi + d\psi^2 \end{gathered} \tag{5.25}$$

then the overlap metric becomes:

$$d\hat{s}^2_{overlap} \Rightarrow dt^2 - e^{-2t}\left[dr^2 + r^2 d\Omega^2 - (1+\epsilon)d\psi^2\right] - 2\epsilon dt d\psi \qquad (5.26)$$

The 4D-portion of this metric is, again, that of de Sitter spacetime, with the curvature factor, t_0, again normalized to unity. (The presence of the off-diagonal $dtd\psi$-term indicates the presence of an electromagnetic vector potential, A_0. However, since $\hat{g}_{50}$, $\hat{g}_{55}$ and, thus, $A_0 \equiv \hat{g}_{50}/\hat{g}_{55}$ are constants, then no electromagnetic field exists here.)

Because both solutions shown here represent vacuum cosmological states (with cosmological constants), it may seem that they are not truly representative of the actual universe we inhabit. However, it must be remembered that our universe, which is, in this epoch, *sparsely* populated with galaxies, can be *approximated* as a vacuum state. Therefore, both metrics can be used as *approximations* of the current universe (though they would almost certainly *not* apply to the early universe).

[In connection with Mach's Principle, the cosmological constant can be regarded as the average combined field of the rest of the universe to the field of a given object at a certain point (ref. [13]). In this context, describing the Cosmological constant in terms of an underlying 5D manifold, as is done here, unifies it with the rest of space(time), and amounts to a Machian description of the Cosmological constant.]

Finally, because the cosmological constant in the 5D conformally flat solution is ψ-dependent, it is, thereby, possible to make the cosmological constant either disappear, by letting $\psi = 0$, or, at least, be very small (since ψ is arbitrary). This would be necessary in order to model our (current era) universe, where the observed cosmological constant is very small (ref. [13]). Thus, in this respect, the 5D conformally flat solution is easier to model to our universe by allowing its cosmological constant become very small.

Chapter 6

Conclusions

In conclusion, we wanted to study solutions of matter from within the Induced Matter Formalism which allowed analysis of inertial and electromagnetic properties. To that end, we have examined six five-dimensional metrics satisfying the 5D vacuum field equations, and analyzed their physical properties (both inertial and electromagnetic) from within the Induced Matter formalism. The solutions found were all complicated, possessing either off-diagonal terms (chapter 4), mixed (nonseparable) metric coefficients (chapter 5), or *both* (chapter 3). The analysis used here, for the first time ever, a *charged imperfect* fluid form to model the solutions (except for the cosmological solutions, which were modeled with a de Sitter spacetime).

In chapter 3, the first r- and ψ-dependent (mixed) solutions were found to represent charged particle solutions. These two metrics were found to represent spherically-symmetric, static charge distributions, whose charge and matter densities approached infinity towards the origin.

The first metric had a 4D limiting case of the Schwarzchild solution, which unfortunately would make differences between it and the Schwarzchild solution difficult to examine. The second solution, conversely, did *not* possess a near-Schwarzchild form. As such the second metric is more easily *testable*, while the first metric is more physically *realizable*.

It was also discovered that the previously analyzed Liu-Wesson solution had been *mis*read as radiation. In fact, it was fluid, like the two charge metrics studied here; the theorem 'proving' it radiation being inapplicable since that theorem was only applicable to *diagonal* solutions which the Liu-Wesson solution was not. The electromagnetic and inertial properties of the Liu-Wesson solution were then calculated.

In chapter 4, two solutions were found to represent axially-symmetric 'wire' distributions of matter possessing magnetic fields. The first constituted the most general ρ-dependent solution possible, but possessed the unusual ability that it could be diagonalized by a 'simple' coordinate transformation.

The second metric was less general, but, being dependent on ρ and ψ, it could not be diagonalized by a 'simple' coordinate transformation.

Physically, the solutions represented axially-symmetric columns of matter which were concentrated along their axes. Unfortunately, due to the arbitrariness in their parameters, it was difficult to uniquely analyze these solutions.

Finally, in chapter 5 conformally flat solutions, representing 4D cosmologies were found. Their metric coefficients were mixed combinations of all three main variables, t, r and ψ. The first metric was 4D conformally flat in a 5D manifold, and, as such, was the most general (flat) such metric possible.

Two 5D conformally flat solutions were also found, one of which 'overlapped' with the 4D conformally flat solution as a 'special case'. All solutions were found to represent the vacuum de Sitter space, which might *approximate* our universe in this epoch.

Appendix A

Derivation of Charged Particle Metrics

We wish to derive solutions which represent static, spherically-symmetric and radially-dependent charge solutions which have a *possible* limiting case in the Schwarzchild solution. In order to do this within the Induced Matter Formalism, we require a 5D metric which possesses an off-diagonal ($dtd\psi$) term, representing an electromagnetic vector component A_0, and in which the metric coefficients depend on r, and *possibly* on ψ. As discussed in chapter 3, we wish to examine such charged metrics that are both r- and ψ-dependent. [As also discused, we impose 3D spherical symmetry by requiring $\hat{g}_{ij}$ (where $i, j = 0, 1, 2, 3, 5$) be invariant under the action of SO(4) acting on S^3.] We, therefore, write out the general form of such a metric as:

$$d\hat{s}^2 = \hat{g}_{00}dt^2 + \hat{g}_{11}dr^2 + \hat{g}_{22}d\theta^2 + \hat{g}_{33}d\phi^2 + 2\hat{g}_{50}dtd\psi + \hat{g}_{55}d\psi^2 \qquad (A.1)$$

where the $\hat{g}_{ab}$ are the 5D metric coefficients of the Induced Matter Formalism, and are functions of both r and ψ.

A.1 Nonseparable Charged Particle Metrics

For the metrics used in chapter 3, the assumption of *nonseparability* of the metric coefficients is made. That is to say, we assume that the metric coefficients (the $\hat{g}_{ab}$) can *not* be written as products of factors solely dependent on r and solely dependent on ψ. Separability is often employed for the sake of simplicity, but it also has the effect of greatly restricting the resulting solutions. See section A.2 for such solutions.

From eq. 2.8, in section 2.3 of chapter 2, it can be seen that one can equate:

$$\hat{g}_{11} = g_{11} \tag{A.2}$$

$$\hat{g}_{22} = g_{22} \tag{A.3}$$

$$\hat{g}_{33} = g_{33} \tag{A.4}$$

where the $g_{\alpha\beta}$ are the corresponding 4D metrics. Additionally, from eq. 2.8, it can be seen that:

$$\hat{g}_{00} = g_{00} + \hat{g}_{55}A_0^2 = g_{00} + \epsilon\Phi^2 A_0^2 \tag{A.5}$$

along with:

$$\hat{g}_{50} = \hat{g}_{55}A_0 = \epsilon\Phi^2 A_0 \tag{A.6}$$

$$\hat{g}_{55} = \epsilon\Phi^2 \tag{A.7}$$

These identifications (particularly eq. A.5) suggest a 'natural' set-up for the charged metric (eq. A.1) in which the $\hat{g}_{00}$ metric coefficient is *split* into components for g_{00} and $\epsilon\Phi^2 A_0^2$. For example, using the identifications A.2 to A.7, we get for the metric:

$$d\hat{s}^2 = (g_{00} + \hat{g}_{55}A_0^2)dt^2 + g_{11}dr^2 + g_{22}d\theta^2 + g_{33}d\phi^2 + 2\hat{g}_{55}A_0 dt d\psi + \hat{g}_{55}d\psi^2 \quad \text{(A.8)}$$

(Indeed, this splitting is used as the basis for finding the general solution to the axial-symmetric radial-dependent ('wire') metric in appendix B.) One may then proceed by associating exponential factors for each of the metric coefficients (as is customary), and then write out the vacuum field equations.

However, despite this 'natural' splitting of $\hat{g}_{00}$, it is actually, here, *preferred* to split the $\hat{g}_{55}$ term, while keeping the $\hat{g}_{00}$ term intact. This is written out (along with the other metric coefficient definitions) as:

$$\hat{g}_{00} = g_{00} + \hat{g}_{55}A_0^2 \equiv e^{\nu(r,\psi)} \quad \text{(A.9)}$$

$$\hat{g}_{11} = g_{11} \equiv -e^{\lambda(r,\psi)} \quad \text{(A.10)}$$

$$\hat{g}_{22} = g_{22} \equiv -e^{\alpha(r,\psi)} \quad \text{(A.11)}$$

$$\hat{g}_{33} = g_{33} \equiv g22\sin^2\theta = -e^{\alpha(r,\psi)}\sin^2\theta \quad \text{(A.12)}$$

$$\hat{g}_{50} = \hat{g}_{55}A_0 \equiv e^{\kappa(r,\psi)} \quad \text{(A.13)}$$

$$\hat{g}_{55} \equiv -e^{\mu(r,\psi)} + e^{2\kappa(r,\psi)-\nu(r,\psi)} \quad \text{(A.14)}$$

where the ν, λ, α, κ and μ are all general functions of r and ψ.

The reason for this '*un*natural' splitting of the $\hat{g}_{55}$ is that it turns out to make the field equations (when written out) *symmetric* in derivatives of r and ψ, and in terms of λ and μ (the coefficients of dr^2 and $d\psi^2$), which will be crucial to finding the solutions listed in chapter 3. This is not apparent from just *looking* at the metric, but can be deduced from a great deal of trial and error (and is actually suggested by a similar form used in ref. [43]).

At first glance, it appears that one may have *over*specified the number of constants defined in eqs. A.9 to A.14. After all, $\hat{g}_{55}$ is *one* term, while it is defined in terms of *two* terms, $-e^{\mu}$ and $e^{2\kappa-\nu}$. However, the fact that $\hat{g}_{55}$ is defined in terms of *two* terms is compensated for by the fact that $\hat{g}_{00}$ (which *is* two terms) is defined as *one* term. Examining the original form of the metric (eq. A.1), one can see that there are five independent variables, $\hat{g}_{00}$, $\hat{g}_{11}$, $\hat{g}_{22}$, $\hat{g}_{50}$ and $\hat{g}_{55}$ (with $\hat{g}_{33} \equiv \hat{g}_{22}\sin^2\theta$). In the new system of terms, as defined by eqs. A.9 to A.14, there are *also* five variables introduced, ν, λ, α, κ and μ. Thus, the new system possesses the same (maximal) number of variables as the original.

The metric, then, as written out in this form is:

$$d\hat{s}^2 = e^{\nu(r,\psi)}dt^2 - e^{\lambda(r,\psi)}dr^2 - e^{\alpha(r,\psi)}d\Omega^2 + 2e^{\kappa(r,\psi)}dtd\psi + (-e^{\mu(r,\psi)} + e^{2\kappa(r,\psi)-\nu(r,\psi)})d\psi^2 \tag{A.15}$$

where $d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2$.

The five-dimensional (mixed) Ricci tensors, $\hat{R}^n_m$, for this metric can then be written out (with overprimes denoting partial derivatives w.r.t. r, and overstars denoting partial derivative w.r.t. ψ) as:

$$\begin{aligned}\hat{R}^0_0 = {} & \tfrac{1}{4}e^{-\lambda}(2\overset{\prime\prime}{\nu} + 2\overset{\prime}{\nu}\overset{\prime}{\alpha} + \overset{\prime}{\nu}\overset{\prime}{\mu} + \overset{\prime}{\nu}{}^{2} - \overset{\prime}{\nu}\overset{\prime}{\lambda}) \\ & + \tfrac{1}{4}e^{-\mu}(2\overset{**}{\nu} + 2\overset{*}{\nu}\overset{*}{\alpha} + \overset{*}{\nu}\overset{*}{\lambda} + \overset{*}{\nu}{}^{2} - \overset{*}{\nu}\overset{*}{\mu}) \\ & + \tfrac{1}{4}e^{2\kappa-\lambda-\mu-\nu}(2\overset{\prime\prime}{\kappa} - 2\overset{\prime\prime}{\nu} + 2\overset{\prime}{\kappa}\overset{\prime}{\alpha} - 2\overset{\prime}{\nu}\overset{\prime}{\alpha} - \overset{\prime}{\kappa}\overset{\prime}{\lambda} + \overset{\prime}{\nu}\overset{\prime}{\lambda} - \overset{\prime}{\kappa}\overset{\prime}{\mu} + \overset{\prime}{\nu}\overset{\prime}{\mu} + \overset{\prime}{\nu}{}^{2} - 5\overset{\prime}{\nu}\overset{\prime}{\kappa} + 4\overset{\prime}{\kappa}{}^{2})\end{aligned} \tag{A.16}$$

$$\begin{aligned}\hat{R}^1_1 = {} & \tfrac{1}{4}e^{-\lambda}(2\overset{\prime\prime}{\nu} + \overset{\prime}{\nu}{}^{2} + 4\overset{\prime\prime}{\alpha} + 2\overset{\prime}{\alpha}{}^{2} + 2\overset{\prime\prime}{\mu} + \overset{\prime}{\mu}{}^{2} - 2\overset{\prime}{\lambda}\overset{\prime}{\alpha} - \overset{\prime}{\lambda}\overset{\prime}{\nu} - \overset{\prime}{\lambda}\overset{\prime}{\mu}) \\ & + \tfrac{1}{4}e^{-\mu}(2\overset{**}{\lambda} + 2\overset{*}{\lambda}\overset{*}{\alpha} + \overset{*}{\lambda}{}^{2} + \overset{*}{\lambda}\overset{*}{\nu} - \overset{*}{\lambda}\overset{*}{\mu}) \\ & - \tfrac{1}{2}e^{2\kappa-\lambda-\mu-\nu}(\overset{\prime}{\nu}{}^{2} - 2\overset{\prime}{\kappa}\overset{\prime}{\nu} + \overset{\prime}{\kappa}{}^{2})\end{aligned} \tag{A.17}$$

$$\begin{aligned}\hat{R}^2_2 = {} & \tfrac{1}{4}e^{-\lambda}(2\overset{\prime\prime}{\alpha} + 2\overset{\prime}{\alpha}{}^{2} + \overset{\prime}{\alpha}\overset{\prime}{\mu} + \overset{\prime}{\alpha}\overset{\prime}{\nu} - \overset{\prime}{\alpha}\overset{\prime}{\lambda}) \\ & + \tfrac{1}{4}e^{-\mu}(2\overset{**}{\alpha} + 2\overset{*}{\alpha}{}^{2} + \overset{*}{\alpha}\overset{*}{\lambda} + \overset{*}{\alpha}\overset{*}{\nu} - \overset{*}{\alpha}\overset{*}{\mu}) \\ & - e^{-\alpha}\end{aligned} \tag{A.18}$$

$$\begin{aligned}\hat{R}^5_5 = {} & \tfrac{1}{4}e^{-\lambda}(2\overset{\prime\prime}{\mu} + 2\overset{\prime}{\mu}\overset{\prime}{\alpha} + \overset{\prime}{\mu}\overset{\prime}{\nu} + \overset{\prime}{\mu}{}^{2} - \overset{\prime}{\mu}\overset{\prime}{\lambda}) \\ & + \tfrac{1}{4}e^{-\mu}(2\overset{**}{\nu} + \overset{*}{\nu}{}^{2} + 4\overset{**}{\alpha} + 2\overset{*}{\alpha}{}^{2} + 2\overset{**}{\lambda} + \overset{*}{\lambda}{}^{2} - 2\overset{*}{\mu}\overset{*}{\alpha} - \overset{*}{\mu}\overset{*}{\nu} - \overset{*}{\mu}\overset{*}{\lambda}) \\ & - \tfrac{1}{4}e^{2\kappa-\lambda-\mu-\nu}(2\overset{\prime\prime}{\kappa} - 2\overset{\prime\prime}{\nu} + 2\overset{\prime}{\kappa}\overset{\prime}{\alpha} - 2\overset{\prime}{\nu}\overset{\prime}{\alpha} - \overset{\prime}{\kappa}\overset{\prime}{\lambda} + \overset{\prime}{\nu}\overset{\prime}{\lambda} - \overset{\prime}{\kappa}\overset{\prime}{\mu} + \overset{\prime}{\nu}\overset{\prime}{\mu} + \overset{\prime}{\nu}{}^{2} - 5\overset{\prime}{\nu}\overset{\prime}{\kappa} + 4\overset{\prime}{\kappa}{}^{2})\end{aligned} \tag{A.19}$$

$$\hat{R}^1_0 = \frac{1}{4}e^{\kappa-\lambda-\mu}\left[2(\overset{\prime *}{\kappa} - \overset{\prime *}{\nu}) + (\overset{\prime}{\kappa} - \overset{\prime}{\nu})(2\overset{*}{\alpha} - \overset{*}{\lambda} + \overset{*}{\nu} - \overset{*}{\mu} + 2\overset{*}{\kappa})\right] \tag{A.20}$$

$$\hat{R}^5_1 = \frac{1}{4}e^{-\mu}\left[2\overset{\prime *}{\nu} + \overset{\prime}{\nu}\overset{*}{\nu} + 4\overset{\prime *}{\alpha} + 2\overset{\prime}{\alpha}\overset{*}{\alpha} - \overset{*}{\lambda}(2\overset{\prime}{\alpha} + \overset{\prime}{\nu}) - \overset{\prime}{\mu}(2\overset{*}{\alpha} + \overset{*}{\nu})\right] \tag{A.21}$$

$$\hat{R}^5_0 = -\frac{1}{4}e^{\kappa-\lambda-\mu}\left[2(\overset{\prime\prime}{\kappa} - \overset{\prime\prime}{\nu}) + (\overset{\prime}{\kappa} - \overset{\prime}{\nu})(2\overset{\prime}{\alpha} - \overset{\prime}{\lambda} + \overset{\prime}{\nu} - \overset{\prime}{\mu} + 2\overset{\prime}{\kappa})\right] \tag{A.22}$$

with:

$$\hat{R}^3_3 = \hat{R}^2_2 \tag{A.23}$$

$$\hat{R}^1_5 = e^{\kappa}\hat{R}^1_0 + e^{\mu-\lambda}\hat{R}^5_1 \tag{A.24}$$

$$\hat{R}^0_1 = -e^{\lambda-\nu}\hat{R}^1_0 - e^{\kappa-\nu}\hat{R}^5_1 \tag{A.25}$$

$$\hat{R}^0_5 = e^{\kappa-\nu}(\hat{R}^0_0 - \hat{R}^5_5) + (e^{2\kappa-2\nu} - e^{\mu-\nu})\hat{R}^1_0 \tag{A.26}$$

and with all other (mixed) Ricci tensors zero.

There are, thus, seven independent nonzero Ricci tensors, $\hat{R}^0_0$ (eq. A.16), $\hat{R}^1_1$ (eq. A.17), $\hat{R}^2_2$ (eq. A.18), $\hat{R}^5_5$ (eq. A.19), $\hat{R}^1_0$ (eq. A.20), $\hat{R}^5_1$ (eq. A.21) and $\hat{R}^5_0$ (eq. A.22), and solving the vacuum field equations means solving these seven Ricci tensors equated to zero.

First, we note that we can integrate both eqs. A.20 ($\hat{R}^1_0$) and A.22 ($\hat{R}^5_0$) to yield the same result. Consider rewriting eqs. A.20 and A.22 as:

$$\hat{R}^1_0 = 0 \to 2\frac{(\overset{\prime *}{\kappa} - \overset{\prime *}{\nu})}{(\overset{\prime}{\kappa} - \overset{\prime}{\nu})} + (2\overset{*}{\alpha} - \overset{*}{\lambda} + \overset{*}{\nu} - \overset{*}{\mu} + 2\overset{*}{\kappa}) = 0 \tag{A.27}$$

$$\hat{R}^1_0 = 0 \to 2\frac{(\overset{\prime\prime}{\kappa} - \overset{\prime\prime}{\nu})}{(\overset{\prime}{\kappa} - \overset{\prime}{\nu})} + (2\overset{\prime}{\alpha} - \overset{\prime}{\lambda} + \overset{\prime}{\nu} - \overset{\prime}{\mu} + 2\overset{\prime}{\kappa}) = 0 \tag{A.28}$$

Both equations can be immediately integrated to give similar forms. Eq. A.27 can be integrated to give:

$$(\overset{\prime}{\kappa} - \overset{\prime}{\nu})^2 = e^{f_r-2\kappa-2\alpha+\mu+\lambda-\nu} \tag{A.29}$$

where f_r is a possible function of r, but independent of ψ. Eq. A.28 can then be integrated to give:

$$(\overset{\prime}{\kappa} - \overset{\prime}{\nu})^2 = e^{f_\psi-2\kappa-2\alpha+\mu+\lambda-\nu} \tag{A.30}$$

where f_ψ is a possible function of ψ, but independent of r. Clearly the only way these two functions can be true simultaneously is for:

$$f_r = f_\psi = const \equiv f_c \tag{A.31}$$

where f_c is now a *constant*. The simultaneous solution to making Ricci tensors A.20 and A.22 vanish, therefore, is:

$$(\overset{\prime}{\kappa} - \overset{\prime}{\nu})^2 = e^{f_c - 2\kappa - 2\alpha + \mu + \lambda - \nu} \tag{A.32}$$

The fact that this kind of redundancy occurs in the solution of the field equations is guaranteed by the Bianchi identities, $G^b_{a\ ;b} = 0$, which ensures that at least one of field equations is redundant with the others (ref. [31]).

The Ricci tensors overall possess a notable symmetry between derivatives in r (overprimes) and derivatives in ψ (overstars), as well as between λ and μ. This symmetry can be more fully appreciated by combining certain Ricci tensors in appropriate combinations. Consider the 'modified' $\hat{R}^0_0$:

$$\begin{aligned}\hat{R}^0_0 + e^{\kappa-\nu}\hat{R}^5_0 = {} & \tfrac{1}{4}e^{-\lambda}(2\overset{\prime\prime}{\nu} + 2\overset{\prime}{\nu}\overset{\prime}{\alpha} + \overset{\prime}{\nu}\overset{\prime}{\mu} + \overset{\prime}{\nu}{}^2 - \overset{\prime}{\nu}\overset{\prime}{\lambda}) \\ & + \tfrac{1}{4}e^{-\mu}(2\overset{**}{\nu} + 2\overset{*}{\nu}\overset{*}{\alpha} + \overset{*}{\nu}\overset{*}{\lambda} + \overset{*}{\nu}{}^2 - \overset{*}{\nu}\overset{*}{\mu}) \\ & + \tfrac{1}{2}e^{2\kappa-\lambda-\mu-\nu}(\overset{\prime}{\kappa} - \overset{\prime}{\nu})^2\end{aligned} \tag{A.33}$$

and next consider the 'modified' $\hat{R}^5_5$:

$$\begin{aligned}\hat{R}^5_5 - e^{\kappa-\nu}\hat{R}^5_0 = {} & \tfrac{1}{4}e^{-\lambda}(2\overset{\prime\prime}{\mu} + 2\overset{\prime}{\mu}\overset{\prime}{\alpha} + \overset{\prime}{\mu}\overset{\prime}{\nu} + \overset{\prime}{\mu}{}^2 - \overset{\prime}{\mu}\overset{\prime}{\lambda}) \\ & + \tfrac{1}{4}e^{-\mu}(2\overset{**}{\nu} + \overset{*}{\nu}{}^2 + 4\overset{**}{\alpha} + 2\overset{*}{\alpha}{}^2 + 2\overset{**}{\lambda} + \overset{*}{\lambda}{}^2 - 2\overset{*}{\mu}\overset{*}{\alpha} - \overset{*}{\mu}\overset{*}{\nu}) - \overset{*}{\mu}\overset{*}{\lambda}) \\ & - \tfrac{1}{2}e^{2\kappa-\lambda-\mu-\nu}(\overset{\prime}{\kappa} - \overset{\prime}{\nu})^2\end{aligned}$$

(A.34)

These expressions, eqs. A.33 and A.34, along with Ricci tensor A.17, all possess $(\kappa' - \nu')^2$ terms, which can be substituted for by eq. A.32. The results are:

$$\begin{aligned}\hat{R}^0_0 + e^{\kappa-\nu}\hat{R}^5_0 = {} & \tfrac{1}{4}e^{-\lambda}(2\,\nu'' + 2\,\nu'\alpha' + \nu'\mu' + \nu'^2 - \nu'\lambda') \\ & + \tfrac{1}{4}e^{-\mu}(2\,\ddot{\nu} + 2\,\dot{\nu}\dot{\alpha} + \dot{\nu}\dot{\lambda} + \dot{\nu}^2 - \dot{\nu}\dot{\mu}) \\ & + \tfrac{1}{2}e^{f_c-2\nu-2\alpha}\end{aligned} \tag{A.35}$$

$$\begin{aligned}\hat{R}^5_5 - e^{\kappa-\nu}\hat{R}^5_0 = {} & \tfrac{1}{4}e^{-\lambda}(2\,\mu'' + 2\,\mu'\alpha' + \mu'\nu' + \mu'^2 - \mu'\lambda') \\ & + \tfrac{1}{4}e^{-\mu}(2\,\ddot{\nu} + \dot{\nu}^2 + 4\,\ddot{\alpha} + 2\,\dot{\alpha}^2 + 2\,\ddot{\lambda} + \dot{\lambda}^2 - 2\,\dot{\mu}\dot{\alpha} - \dot{\mu}\dot{\nu}) - \dot{\mu}\dot{\lambda}) \\ & - \tfrac{1}{2}e^{f_c-2\nu-2\alpha}\end{aligned} \tag{A.36}$$

$$\begin{aligned}\hat{R}^1_1 = {} & \tfrac{1}{4}e^{-\lambda}(2\,\nu'' + \nu'^2 + 4\,\alpha'' + 2\,\alpha'^2 + 2\,\mu'' + \mu'^2 - 2\,\lambda'\alpha' - \lambda'\nu' - \lambda'\mu') \\ & + \tfrac{1}{4}e^{-\mu}(2\,\ddot{\lambda} + 2\,\dot{\lambda}\dot{\alpha} + \dot{\lambda}^2 + \dot{\lambda}\dot{\nu} - \dot{\lambda}\dot{\mu}) \\ & - \tfrac{1}{2}e^{f_c-2\nu-2\alpha}\end{aligned} \tag{A.37}$$

As noted previously, these expressions (along with eqs. A.18 and A.21) exhibit a high degree of symmetry between derivatives in r and ψ, and between λ and μ. In fact, eqs. A.36 and A.37 can be combined to yield the highly symmetric form:

$$\begin{aligned}\hat{R}^1_1 - (\hat{R}^5_5 - e^{\kappa-\nu}\hat{R}^5_0) = {} & \tfrac{1}{4}e^{-\lambda}\left[2\,\nu'' + \nu'^2 + 4\,\alpha'' + 2\,\alpha'^2 - \lambda'\,(2\,\alpha' + \nu') - \mu'\,(2\,\alpha' + \nu')\right] \\ & - \tfrac{1}{4}e^{-\mu}\left[2\,\ddot{\nu} + \dot{\nu}^2 + 4\,\ddot{\alpha} + 2\,\dot{\alpha}^2 - \dot{\lambda}\,(2\,\dot{\alpha} + \dot{\nu}) - \dot{\mu}\,(2\,\dot{\alpha} + \dot{\nu})\right]\end{aligned} \tag{A.38}$$

The two main terms in this expression, eq. A.38, $\left[2\,\nu'' + \nu'^2 + 4\,\alpha'' + 2\,\alpha'^2 - \lambda'\,(2\,\alpha' + \nu') - \mu'\,(2\,\alpha' + \nu')\right]$ and $\left[2\,\ddot{\nu} + \dot{\nu}^2 + 4\,\ddot{\alpha} + 2\,\dot{\alpha}^2 - \dot{\lambda}\,(2\,\dot{\alpha} + \dot{\nu}) - \dot{\mu}\,(2\,\dot{\alpha} + \dot{\nu})\right]$, bear a striking resemblance to the main term from eq. A.21,

$\left[2\overset{\prime *}{\nu} + \overset{\prime}{\nu}\overset{*}{\nu} + 4\overset{\prime *}{\alpha} + 2\overset{\prime}{\alpha}\overset{*}{\alpha} - \overset{*}{\lambda}(2\overset{\prime}{\alpha} + \overset{\prime}{\nu}) - \overset{\prime}{\mu}(2\overset{*}{\alpha} + \overset{*}{\nu})\right]$. In fact, these three terms may be made *identical* (to within a set of arbitrary proportionality factors) by assuming that the metric coefficients are functions of *linear combinations of* r *and* ψ. If we define R as:

$$R \equiv ar + b\psi \tag{A.39}$$

where a and b are arbitrary parameters, and then assume that the metric coefficients, ν, λ, α, μ (and κ) are all functions of R, then the three terms become proportional to each other;

$$\begin{aligned}&\left[2\overset{\prime\prime}{\nu} + \overset{\prime}{\nu}^{2} + 4\overset{\prime\prime}{\alpha} + 2\overset{\prime}{\alpha}^{2} - \overset{\prime}{\lambda}(2\overset{\prime}{\alpha} + \overset{\prime}{\nu}) - \overset{\prime}{\mu}(2\overset{\prime}{\alpha} + \overset{\prime}{\nu})\right] = \\ &a^2\left[2\nu_{,RR} + \nu_{,R}^2 + 4\alpha_{,RR} + 2\alpha_{,R}^2 - \lambda_{,R}(2\alpha_{,R} + \nu_{,R}) - \mu_{,R}(2\alpha_{,R} + \nu_{,R})\right]\end{aligned} \tag{A.40}$$

$$\begin{aligned}&\left[2\overset{**}{\nu} + \overset{*}{\nu}^{2} + 4\overset{**}{\alpha} + 2\overset{*}{\alpha}^{2} - \overset{*}{\lambda}(2\overset{*}{\alpha} + \overset{*}{\nu}) - \overset{*}{\mu}(2\overset{*}{\alpha} + \overset{*}{\nu})\right] = \\ &b^2\left[2\nu_{,RR} + \nu_{,R}^2 + 4\alpha_{,RR} + 2\alpha_{,R}^2 - \lambda_{,R}(2\alpha_{,R} + \nu_{,R}) - \mu_{,R}(2\alpha_{,R} + \nu_{,R})\right]\end{aligned} \tag{A.41}$$

$$\begin{aligned}&\left[2\overset{\prime *}{\nu} + \overset{\prime}{\nu}\overset{*}{\nu} + 4\overset{\prime *}{\alpha} + 2\overset{\prime}{\alpha}\overset{*}{\alpha} - \overset{*}{\lambda}(2\overset{\prime}{\alpha} + \overset{\prime}{\nu}) - \overset{\prime}{\mu}(2\overset{*}{\alpha} + \overset{*}{\nu})\right] = \\ &ab\left[2\nu_{,RR} + \nu_{,R}^2 + 4\alpha_{,RR} + 2\alpha_{,R}^2 - \lambda_{,R}(2\alpha_{,R} + \nu_{,R}) - \mu_{,R}(2\alpha_{,R} + \nu_{,R})\right]\end{aligned} \tag{A.42}$$

where $_{,R}$ indicates derivatives with respect to R. The third term here (eq. A.42) must vanish, by the vanishing of Ricci tensor A.21. Since all three terms are proportional to each other, then the first two terms (eqs. A.40 and A.41), and, thus, eq. A.38 must also vanish *if* we assume that the metric coefficients are functions of linear combinations of r and ψ.

Although it is not *necessary* to assume the metric coefficients to be functions of linear combinations of r and ψ, the obvious simplicity presented by assuming this in trying to solve the vacuum field equations is too great to ignore. Therefore, it will be assumed that the metric coefficients are functions of R, where R is defined as in eq. A.39. Further, since it is desired to find spherically-symmetric solutions representing charged *extensions* of the Schwarzchild case, we will assume that the *specific* forms of the metrics to be functions of $\mathcal{F}$, where $\mathcal{F}$ is defined as:

$$\mathcal{F} \equiv 1 + K/R \tag{A.43}$$

where K is an arbitrary constant, which can be identified, in the weak, neutral-field limit, as the Schwarzchild mass. We, therefore, *assume* that the metric coefficients are given by:

$$e^{\nu} = K_n \mathcal{F}^n \tag{A.44}$$

$$e^{\lambda} = K_\ell \mathcal{F}^\ell \tag{A.45}$$

$$e^{\alpha} = K_{\tilde{a}} \mathcal{F}^{\tilde{a}} R^2 \tag{A.46}$$

$$e^{\kappa} = K_k \mathcal{F}^k \tag{A.47}$$

$$e^{\mu} = K_m \mathcal{F}^m \tag{A.48}$$

with $\mathcal{F}$ and R given as previously defined, and with all of the Ks arbitrary factor constants (*not* to be confused with the K in the definition of $\mathcal{F}$). The tilde over the $\tilde{a}$ distinguishes *this* $\tilde{a}$ from the a used in the definition of R ($R \equiv ar + b\psi$).

From these assumptions, the metric can then be written out as:

$$d\hat{s}^2 = K_n \mathcal{F}^n dt^2 - K_\ell \mathcal{F}^\ell dr^2 - K_{\tilde{a}} \mathcal{F}^{\tilde{a}} R^2 d\Omega^2 + 2K_k \mathcal{F}^k dt d\psi + (-K_m \mathcal{F}^m + \frac{K_k^2}{K_n} \mathcal{F}^{2k-n}) d\psi^2 \tag{A.49}$$

The assumed forms of these metric coefficients (eqs. A.44 to A.48) were based on creating the most general metric coefficients possible, each with arbitrary constant factors (the K's). It is, however, possible to simplify things such that we remove a maximum of these constants through coordinate transformations, which do not further reduce the generality of the metric. If we transform:

$$t \to \frac{t}{\sqrt{K_n}} \tag{A.50}$$

$$r \to \frac{r}{\sqrt{K_\ell}} \tag{A.51}$$

$$\psi \to \frac{\psi}{\sqrt{K_m}} \tag{A.52}$$

$$a \to \sqrt{\frac{K_\ell}{K_{\bar{a}}}}a \tag{A.53}$$

$$b \to \sqrt{\frac{K_m}{K_{\bar{a}}}}b \tag{A.54}$$

$$K_k \to \sqrt{K_n K_m} \tag{A.55}$$

$$K \to \frac{K}{\sqrt{K_{\bar{a}}}} \tag{A.56}$$

then we remove all the arbitrary constants except for K_k, and the metric becomes:

$$d\hat{s}^2 = \mathcal{F}^n dt^2 - \mathcal{F}^\ell dr^2 - \mathcal{F}^{\bar{a}} R^2 d\Omega^2 + 2K_k \mathcal{F}^k dt d\psi + (-\mathcal{F}^m + K_k^2 \mathcal{F}^{2k-n}) d\psi^2 \tag{A.57}$$

Substituting these metric definitions into Ricci tensor A.21, with the tensor equated to zero, the results are:

$$(-2n + n^2 - 4\bar{a} + 2\bar{a}^2 - 2\bar{a}\ell - n\ell - 2\bar{a}m - nm)\frac{ab}{4\mathcal{F}^{2+m}R^4} + (n + \ell + m)\frac{ab}{\mathcal{F}^{1+m}R^3} = 0 \tag{A.58}$$

In order for this equation to vanish for arbitrary values (powers) in R, then both terms must separately be zero;

$$-2n + n^2 - 4\tilde{a} + 2\tilde{a}^2 - 2\tilde{a}\ell - n\ell - 2\tilde{a}m - nm = 0 \tag{A.59}$$

$$n + \ell + m = 0 \tag{A.60}$$

As noted previously, if this Ricci tensor (eq. A.21) is satisfied (*i.e.*, equal to zero), then so will the two terms (eqs. A.40 and A.41) of eq. A.38 (and, hence, the whole equation, itself) based on the assumed form of the metric. Since eq. A.38 is based on a combination of Ricci tensors A.17 and A.19, then it is only necessary to satisfy *either* eq. A.17 or eq. A.19 (since solving one will automatically solve the other if eq. A.38 is also solved). In fact, the equality of the two terms of eq. A.38 can then be used with either eq. A.17 or eq. A.19 to construct another (simpler) form. Consider:

$$\begin{aligned} eq.\ A.17 - eq.\ A.40 = eq.\ A.34 + eq.\ A.41 = {} & \tfrac{1}{4}e^{-\lambda}\left[2\overset{\prime\prime}{\mu} + 2\overset{\prime}{\mu}\overset{\prime}{\alpha} + \overset{\prime}{\mu}\overset{\prime}{\nu} + \overset{\prime}{\mu}{}^2 - \overset{\prime}{\mu}\overset{\prime}{\lambda}\right] \\ & + \tfrac{1}{4}e^{-\mu}\left[2\overset{**}{\lambda} + 2\overset{*}{\lambda}\overset{*}{\alpha} + \overset{*}{\lambda}{}^2 + \overset{*}{\lambda}\overset{*}{\nu} - \overset{*}{\lambda}\overset{*}{\mu}\right] \\ & - \tfrac{1}{2}e^{f_c - 2\nu - 2\alpha} \end{aligned} \tag{A.61}$$

Substituting the expressions for the metric coefficients (from eq. A.57) into this expression (eq. A.61), and setting it to zero, yields the result:

$$(-2+2\tilde{a}+n+m-\ell)\frac{ma^2}{4\mathcal{F}^{2+\ell}R^4} + (-2+\ell-m+n+2\tilde{a})\frac{\ell b^2}{4\mathcal{F}^{2+m}R^4} - \frac{e^{f_c}}{2\mathcal{F}^{-2n-2\tilde{a}}R^4} = 0 \tag{A.62}$$

Similarly, substituting the expressions for the metric coefficients into eq. A.35, one obtains:

$$(-2+2\tilde{a}+m+n-\ell)\frac{na^2}{4\mathcal{F}^{2+\ell}R^4}+(-2+2\tilde{a}+\ell+n-m)\frac{nb^2}{4\mathcal{F}^{2+m}R^4}+\frac{e^{f_c}}{2\mathcal{F}^{-2n-2\tilde{a}}R^4}=0 \quad \text{(A.63)}$$

Adding these two expressions, eqs. A.62 and A.63, then gives:

$$[-2+2\tilde{a}+m+n-\ell][n+m]\frac{a^2}{4\mathcal{F}^{2+\ell}R^4}+[-2+2\tilde{a}+\ell+n-m][n+\ell]\frac{b^2}{4\mathcal{F}^{2+m}R^4}=0 \quad \text{(A.64)}$$

In solving this expression, we note that the powers of $\mathcal{F}$ of the two terms may either be the same or different. If they are the same (*i.e.*, $\ell = m$), then either (a) $-2+2\tilde{a}+n=0$, (b) $n+m=n+\ell=0$, or (c) $a^2+b^2=0$. But one can see from eqs. A.63 and A.62 that if cases (a) or (c) are true then, from eqs. A.63 and A.62, e^{f_c} must be zero. Since $(\kappa' - \nu')$ is proportional to (the square root of) this factor (see eq. A.32), then its vanishing would indicate that κ would be equal to $\nu + h_\psi$ where h_ψ is an integration term which must be independent of r, but which may depend on ψ). If this is true, then the the metric could be diagonalized by a coordinate transformation of the form: $t \to t - \int e^{h_\psi} d\psi$. As is discussed in chapter 4, such a potential diagonalization does not invalidate the solution, since it merely treats the charged field *implicitly*, within the (5D) gravitational field. However, as was also discussed in that chapter, for this thesis, it was desired to find solutions which do not possess the 'over'simplicity of being diagonizable. Therefore, solutions with $\kappa = \nu + h_\psi$ were not looked at.

If case (b) were true, then one could see, from eq. A.60, that n, m *and* ℓ would have to be zero. This causes the (Newtonian) gravitational field to vanish (*i.e.*,

$\hat{g}_{00}$), and yields trivial results (*i.e.*, resulting in either Minkowski space or spaces that are transferable into Minkowski space by simple coordinate transformations). Thus, assuming the powers of the two $\mathcal{F}$ terms to be equal (*i.e.*, $\ell = m$) yields trivial (or diagonizable) results, which are not desired.

If we assume the powers of $\mathcal{F}$ to be different (*i.e.*, $\ell \neq m$), then the sets of factors of the different $\mathcal{F}$ powers (*i.e.*, $[-2+2\tilde{a}+m+n-\ell][n+m]$ and $[-2+2\tilde{a}+\ell+n-m][n+\ell]$) must both be zero (since different powers of $\mathcal{F}$ yield different powers of R, which all must be zero simultaneously). This gives four cases: (i) $-2+2\tilde{a}+m+n-\ell=0$ and $-2+2\tilde{a}+\ell+n-m=0$, (ii) $-2+2\tilde{a}+m+n-\ell=0$ and $n+\ell=0$, (iii) $-2+2\tilde{a}+\ell+n-m=0$ and $n+m=0$, or (iv) $n+m=0$ and $n+\ell=0$. However, cases (i) and (iv) yield $\ell = m$ which was already looked at above and rejected. Therefore, the only (nontrivial) viable cases are (ii) and (iii), which will be labeled cases (1) and (2). That is, case (1) is:

$$\begin{aligned} -2+2\tilde{a}+m+n-\ell &= 0 \\ n+\ell &= 0 \end{aligned} \tag{A.65}$$

while case (2) is:

$$\begin{aligned} -2+2\tilde{a}+\ell+n-m &= 0 \\ n+m &= 0 \end{aligned} \tag{A.66}$$

Combining these results with eq. A.60 then yields, for case (1):

$$\begin{aligned} m &= 0 \\ \ell &= -n \\ -2+2\tilde{a}+2n &= 0 \end{aligned} \tag{A.67}$$

and, for case (2):

$$\begin{aligned} \ell &= 0 \\ m &= -n \\ -2 + 2\tilde{a} + 2n &= 0 \end{aligned} \tag{A.68}$$

From these, one can see that the equation $-2 + 2\tilde{a} + 2n = 0$ is common to both cases. When combined with eq. A.60, the result is:

$$n = \pm 1 \rightarrow \tilde{a} = 0 \; or \; 2 \tag{A.69}$$

The existence to two possible values for n (and, hence, $\tilde{a}$), yields two possible *sub*cases for the two main cases given. For simplicity, these cases will be designated as (1+) and (2+) for $n = +1$, and (1-) and (2-) for $n = -1$.

If we then input these results into eq. A.63 to round out these calculations, we get, for both cases (1+) and (1-):

$$e^{f_c} = b^2 \tag{A.70}$$

and, for cases (2+) and (2-):

$$e^{f_c} = a^2 \tag{A.71}$$

Up to this point, we have not examined eq. A.18, which is the last Ricci tensor to be examined from the original set. Substituting the expressions for the metric coefficients into this term then gives:

$$[(-2+2\tilde{a}+m+n-\ell)\tilde{a}\tfrac{a^2}{\mathcal{F}^2R^4} - 2(2\tilde{a}+m+n-\ell)\tfrac{a^2}{\mathcal{F}R^3} + \tfrac{4a^2}{R^2}]/[4\mathcal{F}^\ell]$$
$$+[(-2+2\tilde{a}+\ell+n-m)\tilde{a}\tfrac{b^2}{\mathcal{F}^2R^4} - 2(2\tilde{a}+\ell+n-m)\tfrac{b^2}{\mathcal{F}R^3} + \tfrac{4b^2}{R^2}]/[4\mathcal{F}^\ell] - \tfrac{1}{\mathcal{F}^{\tilde{a}}R^2} = 0 \tag{A.72}$$

For all cases (1+, 1-, 2+ and 2-), the result is:

$$a^2 + b^2 = 1 \tag{A.73}$$

Finally, returning to eq. A.32, we must now solve this for κ and k. Substituting the expressions for the metric coefficients into this equation (eq. A.32), we get:

$$(k-n)^2\frac{a^2}{R^4}\mathcal{F}^{-2} = \frac{e^{f_\psi}}{K_k^2R^4}\mathcal{F}^{-2k-2\tilde{a}+m+\ell-n} \tag{A.74}$$

Since it is not permissible to have $e^{f_\psi} = 0$ (as just previously discussed), then the powers of $\mathcal{F}$ must be the same. Therefore, we can say:

$$-2k - 2\tilde{a} + m + \ell - n = -2 \tag{A.75}$$

In all four cases (1+, 1-, 2+ and 2-), $k = 0$, and $n^2a^2 = e^{f_\psi}/K_k^2$. For cases (1+) and (1-), we get:

$$a^2K_k^2 = b^2 \rightarrow K_k = \pm\frac{b}{a} \tag{A.76}$$

while, for cases (2+) and (2-), we get:

$$K_k^2 = 1 \rightarrow K_k = \pm 1 \tag{A.77}$$

At this point, we note that we have not used the "variable sign" ϵ ($= \pm 1$) which normally indicates the sign of the fifth dimension. This is because of the definition of the metric coefficients given in eq. A.15. We *could* have included a factor of ϵ in those definitions, but such a factor is just a constant and can easily be transformed away via a simple coordinate transformation. Such factors of ϵ are merely included for the sake of convenience for those wishing to consider *both* possibilities of timelike and spacelike fifth dimensions. Since it has not been used yet, we introduce now the ϵ-notation into the expressions for K_k. Also, we will use $-\epsilon$ to represent ± 1 for cases (2+) and (2-) (the reason for *this* specific designation is to minimize the number of negative signs appearing in the metrics of those two cases). For cases (1+) and (1-), we get:

$$K_k = \epsilon \frac{b}{a} \tag{A.78}$$

and, for cases (2+) and (2-), we get:

$$K_k = -\epsilon \tag{A.79}$$

Substituting all these evaluations of the metric terms back into the metric (eq. A.57), we get, for case (1+):

$$d\hat{s}^2 = \mathcal{F}dt^2 - \frac{dr^2}{\mathcal{F}} - R^2 d\Omega^2 + 2\epsilon \frac{b}{a} dt d\psi - \left[1 - \frac{b^2}{a^2 \mathcal{F}}\right] d\psi^2 \tag{A.80}$$

while for case (1-) we get:

$$d\hat{s}^2 = \frac{dt^2}{\mathcal{F}} - \mathcal{F}dr^2 - \mathcal{F}^2 R^2 d\Omega^2 + 2\epsilon \frac{b}{a} dt d\psi - \left[1 - \frac{b^2}{a^2}\mathcal{F}\right] d\psi^2 \tag{A.81}$$

For case (2+), though, we get:

$$d\hat{s}^2 = \mathcal{F}dt^2 - dr^2 - R^2 d\Omega^2 + 2\epsilon dt d\psi \tag{A.82}$$

and for case (2-) we get:

$$d\hat{s}^2 = \frac{dt^2}{\mathcal{F}} - dr^2 - \mathcal{F}^2 R^2 d\Omega^2 + 2\epsilon dt d\psi \tag{A.83}$$

In both cases (2+) and (2-), there is no $d\psi^2$ term. Although there is nothing *mathematically* wrong with this, the *physical* interpretation, from within the Induced Matter formalism, is that of a spherically-symmetric charge with an *infinite* vector potential A_0 ($A_0 = \hat{g}_{50}/\hat{g}_{55} = \hat{g}_{50}/0 = \infty$). In order to create metrics which have *reasonable* physical interpretations, therefore, we will perform coordinate transformations on t which create the required $d\psi^2$ terms. Transform t as: $t \to t + \psi$, and one gets, for case (2+):

$$d\hat{s}^2 = \mathcal{F}dt^2 - dr^2 - R^2 d\Omega^2 + 2\left[\epsilon + \mathcal{F}\right] dt d\psi + \left[2\epsilon + \mathcal{F}\right] d\psi^2 \tag{A.84}$$

and for case (2-) one gets:

$$d\hat{s}^2 = \frac{dt^2}{\mathcal{F}} - dr^2 - \mathcal{F}^2 R^2 d\Omega^2 + 2\left[\epsilon + \frac{1}{\mathcal{F}}\right] dt d\psi + \left[2\epsilon + \frac{1}{\mathcal{F}}\right] d\psi^2 \tag{A.85}$$

Although it appears here that there are *four* different metrics, it can be seen that a transformation of the form $R \to -R - K$ can cause metric (1-) to transform into metric (1+) *and* metric (2-) into metric (2+). Therefore, there are only *two* independent metrics in this set.

Thus, for completeness, the two metrics that are then used in chapter 3, are:

$$d\hat{s}^2 = \mathcal{F}dt^2 - \frac{dr^2}{\mathcal{F}} - R^2 d\Omega^2 + 2\epsilon\frac{b}{a}dtd\psi - \left[1 - \frac{b^2}{a^2\mathcal{F}}\right]d\psi^2 \qquad (A.86)$$

$$d\hat{s}^2 = \mathcal{F}dt^2 - dr^2 - R^2 d\Omega^2 + 2\left[\epsilon + \mathcal{F}\right]dtd\psi + \left[2\epsilon + \mathcal{F}\right]d\psi^2 \qquad (A.87)$$

where:

$$\mathcal{F} \equiv 1 + K/R \qquad (A.88)$$

with K an arbitrary constant (related to the Schwarzchild mass) and:

$$R \equiv ar + b\psi \qquad (A.89)$$

with a and b parameters arbitrary except for a and b obeying the relation: $a^2 + b^2 = 1$. Both of these solutions have then been verified on GRTensor II (ref. [1]).

A.2 Separable Charged Particle Metrics

In researching charged (off-diagonal in $dtd\psi$) metrics dependent on r and ψ, a number of other solutions were derived prior to the ones of the previous section. These solutions (shown in this section) all possess the property of *separability* in their metric coefficients. That is, the metric coefficients were derived to be products of purely r- and purely ψ-dependent factors. Such approaches are sometimes in accord with physical principles (as in quantum mechanics), but usually they are done as a matter of simplicity. Unfortunately, in the present case, the condition of separability yields 'questionable' results. They are quoted here merely for the sake of completeness.

In addition to separability, the metrics were assumed to be *isotropic*; that is, the 3-spatial portions of the metric, dr^2, $d\theta^2$ and $d\phi^2$, all possessed the same metric coefficients. Thus, the metrics were assumed to be of the form:

$$d\hat{s}^2 = e^{\nu(r)+\nu(\psi)}dt^2 - e^{\lambda(r)+\lambda(\psi)}[dr^2 + r^2 d\Omega^2] + 2\epsilon e^{\kappa(r)+\kappa(\psi)}dtd\psi + \epsilon e^{\mu(r)}d\psi^2 \quad (A.90)$$

where such exponential terms as $\nu(r)$, $\nu(\psi)$, etc., are *different* functions of r and ψ, despite their possessing the *same* Greek-letter term (see note in section B.2 of the next appendix).

In solving metric A.90, the constraints of separability allow greater ease in finding solutions to the field equations. This is because, due to the constraints of separability, terms dependent on r in the field equations must be separate of terms dependent on ψ. In each field equation, such terms can be separated out and set independently to zero.

However, this also greatly increases the number of equations needed in order to satisfy for vacuum, and this greatly constrains the metric coefficients. In general, the process makes it difficult to find *nontrivial* solutions for the metric coefficients. The metrics shown in this section are the only three such nontrivial results found in the course of this research. These solutions do not have 'reasonable' physical interpretations, and, as such, they are mentioned here for the sake of completeness only.

The three metrics found in this context are then:

$$d\hat{s}_I^2 = \left(\frac{1}{\alpha\psi + \beta} + \epsilon\right) dt^2 - \frac{(\alpha\psi + \beta)}{(a^2 r^2 + b)^2}\left[dr^2 + r^2 d\Omega^2\right] + 2\epsilon \frac{dtd\psi}{(\alpha\psi + \beta)} + \frac{d\psi^2}{(\alpha\psi + \beta)} \quad (A.91)$$

$$d\hat{s}^2_{II} = -\psi^{2(1+c)}\left[dr^2 + r^2 d\Omega^2\right] + 2\epsilon\psi^c dt d\psi + \epsilon(a + b/r)d\psi^2 \tag{A.92}$$

$$d\hat{s}^2_{III} = \frac{dt^2}{\psi^2} - \psi^2\left[dr^2 + r^2 d\Omega^2\right] + 2\epsilon dt d\psi \tag{A.93}$$

where Roman numeral subscripts "I", "II" and "III" represent the metric number (here), and where caution must be exercised not to confuse the various constants amongst the various metrics. The constants of the first metric (eq. A.91) are required to satisfy:

$$16a^2 b + \alpha^2 = 0 \tag{A.94}$$

in order to satisfy the vacuum field equations. All three metrics have also been verified on GRTensor II, ref. [1].

The first metric is the most 'physical'; the other two metrics are *absent* portions of their metrics. The second metric is absent a $\hat{g}_{00}$ term, which *appears* to indicate a lack of local 'clock'. The third metric is absent a $\hat{g}_{55}$ term, which, in the Induced Matter theory (where the vector potential would be given by: $A_0 = \hat{g}_{50}/\hat{g}_{55}$), would indicate an *infinite* vector potential, A_0.

However, while the first metric is the most 'physical', its electromagnetic vector potential, A_0, is a *constant* ($= -1$). This yields a vanishing electric field ($E_1 \equiv \partial_r A_0 = 0$). This could be interpreted as an *infinite, uniform* charge distribution, which yields no *net* electric field, despite the presence of electromagnetic matter.

Alternatively, one could note that the first metric can be *diagonalized* by the transformation:

$$\psi \to \psi - \epsilon t \tag{A.95}$$

for which the metric becomes:

$$d\hat{s}_I^2 \Rightarrow \epsilon dt^2 - \frac{[\alpha(\psi - \epsilon t) + \beta]}{(a^2r^2 + b)^2}\left[dr^2 + r^2 d\Omega^2\right] + \frac{d\psi^2}{[\alpha(\psi - \epsilon t) + \beta]} \tag{A.96}$$

which represents either an expanding or contracting fluid, depending on the sign of α. This metric, too, is subject to the same constraint (eq. A.94) as the former version of the metric.

When one calculates the induced equation of state for either this metric, or the original (off-diagonal) metric, one obtains:

$$\wp = \frac{3}{4}\frac{16a^2b + \alpha^2}{\alpha(\psi - \epsilon t) + \beta} \tag{A.97}$$

$$P = -\frac{1}{4}\frac{16a^2b + \alpha^2}{\alpha(\psi - \epsilon t) + \beta} \tag{A.98}$$

for the density and pressure, and:

$$P = -\wp/3 \tag{A.99}$$

for the equation of state.

As discussed in chapters 2 and 3, such equations of state have been studied by other authors, such as Davidson and Owen investigating alternative aspects of Kaluza-Klein physics (ref. [33]), Gott and Rees (ref. [34]) and Kolb (ref. [35]) in relation to cosmic strings, Ponce de Leon (ref. [36]) investigating certain ("limiting configurations") of sources in the Reissner-Nordström field and Wesson in relation to quantum zero-point fields (see refs. in [8]). They represent "gravitationaless matter", because the 4D gravitational mass, which is proportional to $3P + \wp$, vanishes in this case (see also ref. [8]).

However, a closer examination of these expressions of density and pressure along with the 5D constraint common to both the diagonal and off-diagonal forms of the metric (eq. A.94), clearly shows that both the density and pressure in eqs. A.97 and A.98 are *zero*. The diagonal form of the metric, therefore, represents the limiting case in which the matter *vanishes*.

The second metric appears somewhat more reasonable insofar as it possesses a vector potential which depends on r, $A_0 = \psi^c/(a + b/r)$, thereby yielding a reasonable expression for the electric field: $E_1 = [b\psi^c/(a + b/r)^2]r^{-2}$.

Despite this, the metric A.92 appears to be unacceptable owing to the fact that there *is no* g_{00} *term*, which would seem to exclude the possible definition of a local 'clock'. For example, The redshift of a photon as it travels from a point of emission to a point of reception in a (4D) stationary and static spacetime is given as (ref. [39]):

$$\omega = \sqrt{\frac{g_{00_{emit}}}{g_{00_{recpt}}}} \tag{A.100}$$

where $g_{00_{emit}}$ is the g_{00} value at the point of emission of the photon, and $g_{00_{recpt}}$ is the value of g_{00} at the point of reception of the photon.

This derivation (eq. A.100) is based on the constancy of the positions of both the emitter and receptors in (ordinary) 3-space (so that $dr^2 = d\Omega^2 = 0$). The remaining term, the $g_{00}dt^2$, then becomes the sole contributing term to the expression for the 4D metric (the ds^2). Since the interval, ds, is taken to represent the passage of *proper* time at a given point in spacetime, then the ratio of ds_{emit} for the emitter and ds_{recpt} for the receptor yields the ratios of the frequencies of a given photon passing through those points. And if the spacetime is both stationary and static, then one can say that $dt_{emit} = dt_{recpt}$, thereby yielding the derivation of eq. A.100

(ref. [39]).

The lack of any g_{00} term in eq. A.92 (*i.e.*, the lack of any *clock*) would, therefore, appear to deny the possibility of defining a redshift in this manner (putting $g_{00} = 0$ into eq. A.100 yields $\sqrt{0/0}$, an *undefined* expression).

The problem arises because the derivation of eq. A.100 effectively assumes that *all* the terms of the metric are zero *except* the $g_{00}dt^2$ term. This is *not necessarily* the case for a 5D metric, for which the constancy of position in the *fifth* dimension is not assured (it is reasonable to still assume the constancy in the positions of the emitter and the receptor in ordinary 3-space). In order to consider position in the fifth dimension, one must examine the equations of motion for this metric.

The 5th component of the geodesic equations ($\ddot{x}^5 + \hat{\Gamma}^5_{ab}\dot{x}^a\dot{x}^b = 0$) for the metric A.92 can be found to be:

$$\ddot{\psi} + \frac{c}{\psi}\dot{\psi}^2 = 0 \tag{A.101}$$

where overdots, again, denote derivatives with respect to $d\hat{s}$. Equation A.101 can then be solved to yield:

$$\psi^c\dot{\psi} = N \tag{A.102}$$

where N is a constant of motion. Integrating eq. A.102 yields an exact solution for ψ of the form (with k_0 an integration constant):

$$\psi = [(1+c)N\hat{s} + (1+c)k_0]^{1/(1+c)} \tag{A.103}$$

for $c \neq -1$. For $c = -1$, the solution is:

$$\psi = e^{N\hat{s}+k_0} \tag{A.104}$$

From these, it is clear that one can*not* in general set $d\psi = 0$, since that would mean forcing $\hat{s} = const.$ Therefore, the original objections to the metric A.92, assuming the lack of local 'clock' based on the lack of $\hat{g}_{00}$, are not valid.

However, it is, nevertheless, *preferred* that metrics (describing physical recognizable spacetimes) have a g_{00} term in order that they be made "compatible" with existing metrics to compare predictions. As such, the metric A.92 is considered critically deficient in this regard.

The third metric, eq. A.93, is even more unsalvageable insofar as its physical interpretation (within the Induced Matter Formalism) is that of a spherically-symmetric fluid possessing an infinite vector potential, $A_0 = \infty$. However, it might be possible to salvage this and the previous metric, eq. A.92, by making transformations on t and ψ.

For example, a transformation of the form: $\psi \to \psi + t$ will yield a $\hat{g}_{00}$ term for metric A.92, while a transformation of the form: $t \to t + \psi$ will yield a $\hat{g}_{55}$ term for metric A.92. However, for metric A.93, both the resulting $\hat{g}_{50}$ and $\hat{g}_{55}$ would be independent of r, yielding yet another fluid of infinite but uniform charge.

For metric A.92, the given transformation will introduce factors of t into the metric coefficients, destroying the basic premise of these metrics that they be *static* spherically-symmetric charged solutions.

Of course, other, more complicated transformations, yielding more interesting results, are possible for all three metrics. But the examination of such transformations are beyond the scope of this thesis.

Appendix B

Derivation of Magnetized Wire Metrics

In order to derive solutions which describe static, axially-symmetric 'wire' solutions, possessing an axially-symmetric magnetic field, within the Induced Matter Formalism, we require a 5D metric which possesses an off-diagonal ($dzd\psi$) term, representing an electromagnetic vector component A_3. Partially for simplicity, and partially because we desire that the 4D portion of this metric match up with the known 4D axially-symmetric metrics (refs. [39], [37]), the 4D portion of the metric will be assumed to be diagonal (like its 4D counterparts). The form of the metric is, then;

$$d\hat{s}^2 = \hat{g}_{00}dt^2 + \hat{g}_{11}d\rho^2 + \hat{g}_{22}d\phi^2 + \hat{g}_{33}dz^2 + 2\hat{g}_{35}dzd\psi + \hat{g}_{55}d\psi^2 \qquad \text{(B.1)}$$

In order that the resulting solution be static and axially-symmetric, the metric coefficients must be independent of both t and ϕ (ref. [39]). If we also desire the resulting solution to represent a 'wire' *infinite* in the z-direction, it is necessary that the metric coefficients be independent of z. The metric coefficients, therefore, may depend on ρ, and/or ψ. Since, again, we desire a correspondence between this metric and the 4D case, which possesses an explicit ρ-dependence ($g_{22}d\phi^2 = -\rho^2 d\phi^2$), then we must have an explicit ρ-dependence here. Therefore, the metric coefficients must depend on ρ and *may* depend on ψ.

We, therefore, derive two solutions, one dependent solely on ρ, and the other dependent on ρ and ψ. The first solution is completely general, but suffers from the fact that it can be 'simply diagonalized' (see section B.1.1), while the second solution is less general, but cannot be 'simply diagonalized'.

B.1 ρ-Dependent Magnetized Wire Metric

For the first solution, we assume a metric of the form:

$$d\hat{s}^2 = e^{\nu(\rho)}dt^2 - e^{\lambda(\rho)}d\rho^2 - e^{\alpha(\rho)}d\phi^2 + [-e^{\beta(\rho)} + \epsilon e^{2\kappa(\rho)-\mu(\rho)}]dz^2 + 2\epsilon e^{\kappa(\rho)}dz d\psi + \epsilon e^{\mu(\rho)}d\psi^2 \tag{B.2}$$

Because all the metric coefficients depend solely upon ρ, the metric coefficient $e^{\lambda(\rho)}$ can, without loss of generality, be absorbed into $d\rho^2$ by a coordinate transformation;

$$\rho \rightarrow \int e^{-\lambda(\rho)/2} d\rho \tag{B.3}$$

This puts the metric B.2 in the form:

$$d\hat{s}^2 = e^{\nu(\rho)}dt^2 - d\rho^2 - e^{\alpha(\rho)}d\phi^2 + [-e^{\beta(\rho)} + \epsilon e^{2\kappa(\rho)-\mu(\rho)}]dz^2 + 2\epsilon e^{\kappa(\rho)}dz\,d\psi + \epsilon e^{\mu(\rho)}d\psi^2 \tag{B.4}$$

Typically, 4D axially-symmetric ("wire") metrics are written with coefficients in front of the $d\rho^2$ terms, but such coefficients are (for purely radially-dependent solutions) powers of ρ (ref. [37]). It, therefore, should be simple enough to (re)introduce such a factor into the metric by a coordinate transformation (of the form: $\rho \to \rho^a$) at some later point in the analysis of the metric B.4.

The 5D Ricci tensors for this metric can then be calculated as:

$$\hat{R}_{00} = \frac{1}{4}e^{\nu}\left[2\nu'' + \nu'\alpha' + \nu'\beta' + \nu'^2 + \nu'\mu'\right] \tag{B.5}$$

$$\hat{R}_{11} = -\frac{1}{4}\left[2\alpha'' + \alpha'^2 + 2\beta'' + \beta'^2 + 2\nu'' + \nu'^2 + 2\mu'' + \mu'^2\right] + \frac{1}{2}\epsilon e^{2\kappa-\mu-\beta}\left[\mu' - \kappa'\right]^2 \tag{B.6}$$

$$\hat{R}_{22} = -\frac{1}{4}e^{\alpha}\left[2\alpha'' + \alpha'^2 + \alpha'\beta' + \alpha'\nu' + \alpha'\mu'\right] \tag{B.7}$$

$$\begin{aligned}\hat{R}_{33} = &-\tfrac{1}{4}e^{\beta}\left[2\beta'' + \beta'\alpha' + \beta'^2 + \beta'\nu' + \beta'\mu'\right] + \tfrac{1}{2}\epsilon^2 e^{4\kappa-2\mu-\beta}\left[\mu' - \kappa'\right]^2 \\ &+\tfrac{1}{4}\epsilon e^{2\kappa-\mu}\left[4\kappa'' + 2\kappa'\alpha' + 2\kappa'\nu' - 6\kappa'\mu' + 6\kappa'^2 - 2\mu'' - \mu'\alpha' + 3\mu'\beta' - \mu'\nu' + \mu'^2 - 2\kappa'\beta'\right]\end{aligned} \tag{B.8}$$

$$\hat{R}_{35} = \frac{1}{4}\epsilon e^{\kappa}\left[2\kappa'' + \kappa'\alpha' - \kappa'\beta' + \kappa'\nu' - \kappa'\mu' + 2\kappa'^2 + 2\mu'\beta'\right] + \frac{1}{2}\epsilon^2 e^{3\kappa-\mu-\beta}\left[\mu' - \kappa'\right]^2 \tag{B.9}$$

$$\hat{R}_{55} = \frac{1}{4}\epsilon e^{\mu}\left[2\mu'' + \mu'\alpha' + \mu'\beta' + \mu'\nu' + \mu'^2\right] + \frac{1}{2}\epsilon^2 e^{2\kappa-\beta}\left[\mu' - \kappa'\right]^2 \tag{B.10}$$

where overprimes denote differentiation with respect to ρ, and all terms (α, β, μ, ν, and κ) are dependent on ρ. Since there are six field equations, then any collection of six (independent) combinations of the various $\hat{R}$'s should constitute an *equivalent* set of field equations.

Combining eqs. B.7 and B.5 then yields:

$$-2\left[\frac{e^{-\alpha}}{\alpha'}\hat{R}_{22} + \frac{e^{-\nu}}{\nu'}\hat{R}_{00}\right] = \frac{\alpha''}{\alpha'} - \frac{\nu''}{\nu'} \tag{B.11}$$

which is the first of our equivalent field equations (and where it is implicitly assumed that α' and ν' do not vanish).

Since both $\hat{R}_{22}$ and $\hat{R}_{00}$ must be zero to satisfy the vacuum field equations, then eq. B.11 yields:

$$\frac{\alpha''}{\alpha'} - \frac{\nu''}{\nu'} = 0 \tag{B.12}$$

which can then be integrated to give:

$$\alpha' = c_1 \nu' \tag{B.13}$$

where c_1 is an integration constant.

Similarly, eqs. B.5, B.8, B.9 and B.10 can be combined to yield:

$$-\frac{2}{\mu' + \beta'}\left[\left(\frac{-1}{\epsilon}e^{-\mu} + e^{(2\kappa-2\mu-\beta)}\right)\hat{R}_{55} + e^{-\beta}\hat{R}_{33} - 2e^{\kappa-\mu-\beta}\hat{R}_{35}\right] - \frac{2e^{-\nu}}{\nu'}\hat{R}_{00} = \frac{\mu'' + \beta''}{\mu' + \beta'} - \frac{\nu''}{\nu'} \tag{B.14}$$

which is the second of our equivalent set of field equations.

Again, since $\hat{R}_{00}$, $\hat{R}_{33}$, $\hat{R}_{35}$ and $\hat{R}_{55}$ are all zero, then eq. B.14 is also zero;

$$\frac{\mu'' + \beta''}{\mu' + \beta'} - \frac{\nu''}{\nu'} = 0 \tag{B.15}$$

which can also then be integrated to give:

$$\mu' + \beta' = c_2 \nu' \tag{B.16}$$

where c_2 is another integration constant.

Our third equivalent field equation can be written directly in terms of $\hat{R}_{00}$:

$$4e^{-\nu} \hat{R}_{00} = 2\nu'' + (\alpha' + \beta' + \nu' + \mu')\nu' \tag{B.17}$$

Substituting in the results from eqs. B.13 and B.16 into eq. B.17 then yields:

$$2\nu'' + (1 + c_1 + c_2)\nu'^2 = 0 \tag{B.18}$$

which, of course, is zero since $\hat{R}_{00}$ must be zero. Eq. B.18 may then be integrated to yield:

$$\nu' = [\frac{1}{2}(1 + c_1 + c_2)\rho - k_1]^{-1} \tag{B.19}$$

for ν', and:

$$\nu = \frac{\ln[\frac{1}{2}(1 + c_1 + c_2)\rho - k_1]}{\frac{1}{2}(1 + c_1 + c_2)} + k_2 \tag{B.20}$$

for ν, where k_1 and k_2 are integration constants.

However, for the moment, we shall leave ν unaltered in the remaining field equations for the sake of simplicity in working out the rest of the equations. We shall only note here the form of $\nu(\rho)$ as a function of ρ.

Combining $\hat{R}_{55}$ and $\hat{R}_{33}$ then yields our fourth equivalent field equation:

$$4\left[\frac{1}{\epsilon}e^{-\mu}\hat{R}_{55}-\hat{R}_{33}\right]=4\,\mu''+2\,\mu'^2+2\,\beta''+\beta'^2+2\,\alpha''+\alpha'^2+2\,\nu''+\nu'^2+\alpha'\mu'+\beta'\mu'+\nu'\mu' \tag{B.21}$$

which is also zero by virtue of $\hat{R}_{55}$ and $\hat{R}_{33}$ being zero.

Substituting in the results of eqs. B.13, B.16 and B.18 into eq. B.21 then yields:

$$-2\,\mu''-2\,\mu'^2+(c_2-c_1-1)\,\mu'\nu'+2(c_1+c_2+c_1c_2)\,\nu'^2=0 \tag{B.22}$$

$\mu(\rho)$, a function of ρ, can be rewritten as a function of ν, $\mu(\nu)$, with the aid of eq. B.18;

$$\mu''=\frac{d\,\mu'}{d\rho}=\frac{d\,\mu'}{d\,\nu'}\frac{d\,\nu'}{d\rho}=\frac{d\,\mu'}{d\,\nu'}\,\nu''=-\frac{1}{2}(1+c_1+c_2)\,\nu'^2\,\frac{d\,\mu'}{d\,\nu'} \tag{B.23}$$

Eq. B.22 can then be written, with the help of eq. B.23, as:

$$A\frac{d\,\mu'}{d\,\nu'}+B\left(\frac{\mu'}{\nu'}\right)^2+C\frac{\mu'}{\nu'}+D=0 \tag{B.24}$$

where:

$$A\equiv 1+c_1+c_2 \tag{B.25}$$

$$B \equiv -2 \tag{B.26}$$

$$C \equiv c_2 - c_1 - 1 \tag{B.27}$$

$$D \equiv 2(c_1 + c_2 + c_1 c_2) \tag{B.28}$$

Eq. B.24 can then be rewritten as:

$$A\,d\mu' + \left[B\left(\frac{\mu'}{\nu'}\right)^2 + C\frac{\mu'}{\nu'} + D\right] d\nu' = 0 \tag{B.29}$$

for which an integrating factor, I_1, can be deduced as:

$$I_1 = \frac{1}{B\,\mu'^2/\nu' + (A+C)\,\mu' + D\,\nu'} \tag{B.30}$$

thereby rendering eq. B.29 as:

$$\frac{A\,d\mu' + \left[B\left(\frac{\mu'}{\nu'}\right)^2 + C\frac{\mu'}{\nu'} + D\right] d\nu'}{B\,\mu'^2/\nu' + (A+C)\,\mu' + D\,\nu'} = 0 \tag{B.31}$$

Eq. B.31 can then be integrated to give:

$$\frac{A}{Q}\ln\left[\frac{2B\frac{\mu'}{\nu'} + A + C - Q}{2B\frac{\mu'}{\nu'} + A + C + Q}\right] + \ln(\nu') = const = \frac{A}{Q}\ln(J) \tag{B.32}$$

where J is another integration constant, and Q is a constant defined by:

$$Q \equiv \pm\sqrt{(A+C)^2 - 4BD} \tag{B.33}$$

Eq. B.32 can then be rearranged to give:

$$\overset{\prime}{\mu} = -\left(\frac{A+C}{2B}\right)\overset{\prime}{\nu} + \frac{Q}{2B}\left[\frac{1+J\,\overset{\prime}{\nu}^{E}}{1-J\,\overset{\prime}{\nu}^{E}}\right]\overset{\prime}{\nu} \tag{B.34}$$

where E is a constant defined by:

$$E \equiv -Q/A \tag{B.35}$$

Returning to the field equations, can can next combine $\hat{R}_{55}$ and $\hat{R}_{00}$ to yield:

$$\frac{2}{\epsilon}e^{\beta}\,\overset{\prime}{\nu}\,\hat{R}_{55} - 2e^{-\nu+\mu+\beta}\,\overset{\prime}{\mu}\,\hat{R}_{00} = \left[\overset{\prime\prime}{\mu}\overset{\prime}{\nu} - \overset{\prime\prime}{\nu}\overset{\prime}{\mu}\right]e^{\mu+\beta} + \epsilon\,\overset{\prime}{\nu}\,(\overset{\prime}{\kappa} - \overset{\prime}{\mu})^2 e^{2\kappa} \tag{B.36}$$

which, again, is zero by virtue of $\hat{R}_{55}$ and $\hat{R}_{00}$ being zero. Eq. B.36 is now the fifth equivalent field equation.

From eq. B.34, the expression $(\overset{\prime\prime}{\mu}\overset{\prime}{\nu} - \overset{\prime\prime}{\nu}\overset{\prime}{\mu})$ can be written out as:

$$\overset{\prime\prime}{\mu}\overset{\prime}{\nu} - \overset{\prime\prime}{\nu}\overset{\prime}{\mu} = -\frac{Q^2 J}{4}\frac{\overset{\prime}{\nu}^{E+3}}{(1-J\,\overset{\prime}{\nu}^{E})^2} \tag{B.37}$$

If we introduce a constant c_3 defined as:

$$c_3 \equiv \frac{1}{2}(1+c_1+c_2) = \frac{1}{2}A \tag{B.38}$$

then, considering eq. B.18 and its first integration (eq. B.19), it's clear that we can write:

$$\overset{\prime\prime}{\nu} + c_3\,\overset{\prime}{\nu}^{2} = 0 \rightarrow e^{\nu} = e^{k_2}\,\overset{\prime}{\nu}^{-1/c_3} \tag{B.39}$$

where k_2 is the integration constant from eq. B.20. With eq. B.16, this yields:

$$e^{c_2 \nu} = e^{\mu+\beta} = e^{c_2 k_2 + \ell} \, {\nu'}^{-c_2/c_3} \tag{B.40}$$

where ℓ is another integration constant (from integrating eq. B.16).

Substituting eqs. B.37 and B.40 into eq. B.36 then yields:

$$(\kappa' - \mu')e^{\kappa} = \pm\sqrt{\frac{1}{\epsilon} e^{c_2 k_2 + \ell} J}\, \frac{Q}{2} \frac{{\nu'}^{1+E/2 \; -c_2/2c_3}}{(1 - J {\nu'}^{E})} \tag{B.41}$$

This can then be rewritten as:

$$\kappa' e^{\kappa} + \left(\frac{A+C}{2B}\right) \nu' e^{\kappa} - \frac{Q}{2B}\left[\frac{1 + J {\nu'}^{E}}{1 - J {\nu'}^{E}}\right] \nu' e^{\kappa} \mp \sqrt{\frac{1}{\epsilon} e^{c_2 k_2 + \ell} J}\, \frac{{\nu'}^{1+E/2 \; -c_2/2c_3}}{(1 - J {\nu'}^{E})} = 0 \tag{B.42}$$

Using a similar procedure as in eq. B.23, κ' can be rewritten using eq. B.18 as:

$$\kappa' = \frac{d\kappa}{d\rho} = \frac{d\kappa}{d\nu'}\frac{d\nu'}{d\rho} = \frac{d\kappa}{d\nu'}\nu'' = -\frac{1}{2}(1 + c_1 + c_2)\, {\nu'}^{2} \frac{d\kappa}{d\nu'} = -c_3 {\nu'}^{2} \frac{d\kappa}{d\nu'} \tag{B.43}$$

Substituting this into eq. B.42 then yields:

$$e^{\kappa} d\kappa + \left[-\left(\frac{A+C}{2Bc_3}\right) {\nu'}^{-1} e^{\kappa} + \frac{Q}{2Bc_3}\left[\frac{1 + J {\nu'}^{E}}{1 - J {\nu'}^{E}}\right] {\nu'}^{-1} e^{\kappa} \pm \sqrt{\frac{1}{\epsilon} e^{c_2 k_2 + \ell} J}\, \frac{Q}{2c_3} \frac{{\nu'}^{E/2 \; -1-c_2/2c_3}}{c_3(1 - J {\nu'}^{E})}\right] d\nu' = 0 \tag{B.44}$$

for which an integrating factor, I_2 can be found as:

$$I_2 = \frac{\nu'^{-(A+C-Q)/2Bc_3}}{(1-J\,\nu'^{E})^{Q/Bc_3E}} = \frac{\nu'^{-(A+C-Q)/2Bc_3}}{(1-J\,\nu'^{E})} \tag{B.45}$$

where the last step was allowed because $Q/Bc_3E = 1$. Inserting the integrating factor B.45 into eq. B.44 then yields:

$$\left[-\left(\frac{A+C}{2Bc_3}\right)\frac{\nu'^{-1-(A+C-Q)/2Bc_3}}{(1-J\nu'^{E})}e^{\kappa} + \frac{Q}{2Bc_3}\frac{(1+J\nu'^{E})}{(1-J\nu'^{E})^2}\,\nu'^{-1-(A+C-Q)/2Bc_3}\,e^{\kappa} \pm \sqrt{\tfrac{1}{\epsilon}e^{c_2k_2+\ell}J}\,\frac{Q}{2c_3}\frac{\nu'^{E-1}}{(1-J\nu'^{E})^2}\right]d\,\nu' + \frac{\nu'^{-(A+C-Q)/2Bc_3}}{(1-J\nu'^{E})}e^{\kappa}d\kappa = 0 \tag{B.46}$$

where use was made of the fact that $\frac{c_2}{c_3} + \frac{(A+C-Q)}{Bc_3} = -E$.

Eq. B.46 can then be integrated to give:

$$\frac{\nu'^{-(A+C-Q)/2Bc_3}}{(1-J\,\nu'^{E})}e^{\kappa} + \frac{\tilde{K}}{(1-J\,\nu'^{E})} = \hat{K} \tag{B.47}$$

where $\hat{K}$ is an integration constant, and $\tilde{K}$ is a constant defined by:

$$\tilde{K} \equiv \mp\sqrt{\frac{e^{c_2k_2+\ell}}{\epsilon J}} \tag{B.48}$$

Eq. B.47 can then be rearranged to give:

$$e^{\kappa} = \hat{K}\,\nu'^{(A+C-Q)/2Bc_3}\left[k - J\,\nu'^{E}\right] \tag{B.49}$$

where k is a constant defined by:

$$k \equiv 1 - \frac{\tilde{K}}{\hat{K}} \tag{B.50}$$

so that: $\hat{K} = \tilde{K}/(1-k)$.

The sixth and final equivalent field equation can be found by combining $\hat{R}_{35}$, $\hat{R}_{55}$ and $\hat{R}_{00}$ into the form:

$$\frac{2}{\epsilon}e^{-\mu}\left[e^{-\kappa+\mu}\hat{R}_{35} - \hat{R}_{55}\right] + \frac{2(\mu' - \kappa')}{\nu'}e^{-\nu}\hat{R}_{00} = \kappa'\left(\frac{\kappa''}{\kappa'} + \kappa' - \mu' - \beta' + \frac{\nu''}{\nu'}\right) + \mu'\left(\beta' - \frac{\mu''}{\mu'} + \frac{\nu''}{\nu'}\right) \tag{B.51}$$

which is then also set equal to zero.

However, when the solutions for κ (from eq. B.49) and μ (from eq. B.34) are substituted into eq. B.51, the resulting equation is *identical* to that of eq. B.44. This redundancy, is the result of the Bianchi identities,

$$G_{ab}{}^{;b} = 0 \tag{B.52}$$

which typically ensures that there is one redundancy within the set of field equations (ref. [31]).

The set of equivalent field equations for this metric (B.4) are therefore represented by eqs. B.11, B.14, B.17, B.21 and B.36 (with a redundancy coming in between eq. B.51 and B.36), whose solutions are then given by eqs. B.13, B.16, B.20, B.34 and B.49, respectively. As noted in eq. B.20, the solution for $\nu(\rho)$ can be given by:

$$\nu = \frac{\ln[\frac{1}{2}(1 + c_1 + c_2)\rho - k_1]}{\frac{1}{2}(1 + c_1 + c_2)} + k_2 = \frac{1}{c_3}\ln[c_3\rho - k_1] + k_2 \tag{B.53}$$

where the last step was made by eq. B.38.

However, this can, without loss of generality be simplified by making a simple coordinate transformation on ρ such that the integration constant k_1 is removed. This is done through the coordinate transformation:

$$\rho \rightarrow \frac{1}{c_3}(\rho + k_1) \tag{B.54}$$

so that eq. B.53 becomes:

$$\nu = \frac{1}{c_3}\ln(\rho) + k_2 \tag{B.55}$$

With this simplification, the solutions for the various metric coefficients can then be written out as:

$$e^{\nu} = e^{k_2}\rho^{1/c_3} \tag{B.56}$$

$$e^{\alpha} = e^{c_1 k_2 + \tilde{\ell}}\rho^{c_1/c_3} \tag{B.57}$$

$$e^{\mu} = e^{c_2 k_2 + \ell + \hat{\ell}}\rho^{(E + c_2/c_3)/2}(1 - \frac{J}{\rho^E}) \tag{B.58}$$

$$e^{\beta} = e^{-\hat{\ell}}\rho^{(-E + c_2/c_3)/2}(1 - \frac{J}{\rho^E})^{-1} \tag{B.59}$$

$$e^{\kappa} = \mp\frac{\sqrt{e^{c_2 k_2 + \ell}}}{\sqrt{\epsilon J}(1-k)}\rho^{(E + c_2/c_3)/2}(k - \frac{J}{\rho^E}) \tag{B.60}$$

where $\tilde{\ell}$ is an integration constant found from integrating eq. B.13, eq. B.58 (and, subsequently, eq. B.59) comes from integrating eq. B.34, and $\hat{\ell}$ is its integration constant.

Now, in order to make this metric compatible with 4D metrics (which typically have a ρ-factor in front of the $d\rho^2$ term; see comment after eq. B.4), we must now make a further transformation on ρ of the form:

$$\rho \to \rho^{a} \tag{B.61}$$

Additionally, we desire to remove as many of the 'arbitrary' constants from eqs. B.56 to B.60 as possible. To do this, we make the further transformations:

$$t \to e^{-k_2/2} t \tag{B.62}$$

$$\phi \to e^{-(c_1 k_2 + \hat{\ell})/2} \phi \tag{B.63}$$

$$z \to e^{\hat{\ell}/2} z \tag{B.64}$$

$$\psi \to e^{-(c_2 k_2 + \ell + \hat{\ell})/2} \psi \tag{B.65}$$

The metric B.2 then takes on the form:

$$\begin{aligned} d\hat{s}^2 = \rho^{(a/c_3)} dt^2 - a^2 \rho^{2(a-1)} d\rho^2 - \rho^{(ac_1/c_3)} d\phi^2 + \left[-\frac{\rho^{\frac{a}{2}(-E+c_2/c_3)}}{(1-\frac{J}{\rho^a E})} + \frac{\rho^{\frac{a}{2}(E+c_2/c_3)} \left(k - \frac{J}{\rho^a E}\right)^2}{\left(1-\frac{J}{\rho^a E}\right) J(1-k)^2} \right] dz^2 \\ \pm 2 \frac{\sqrt{\epsilon} \rho^{\frac{a}{2}(E+c_2/c_3)} \left(k - \frac{J}{\rho^a E}\right)}{\sqrt{J}(1-k)} dz d\psi + \epsilon \rho^{\frac{a}{2}(E+c_2/c_3)} \left(1 - \frac{J}{\rho^a E}\right) d\psi^2 \end{aligned} \tag{B.66}$$

which is dependent on five variables, c_1, c_2, a, k and J, and the only constraints being provided on c_3 from eq. B.38, $c_3 \equiv \frac{1}{2}(1 + c_1 + c_2)$. and on E (from eqs. B.35, B.33, B.25, B.26, B.27, and B.28) as:

$$E = \pm \frac{1}{c_3} \sqrt{c_2^2 + 4(c_1 + c_2 + c_1 c_2)} \tag{B.67}$$

Eq. B.66 is the most general form of the metric B.2 and has been verified to satisfy the field equations $\hat{R}_{ab} = 0$ by computer software (GRTensor II on Maple, ref. [1]).

In order to simplify this metric to make it more readable, we introduce five new constants, γ, δ, μ, η and λ, defined in terms of the previous constants, c_1, c_2, a, E and c_3, by:

$$2a - 2 \equiv \gamma \tag{B.68}$$

$$ac_1/c_3 \equiv 2 + \delta \tag{B.69}$$

$$aE \equiv \mu \tag{B.70}$$

$$ac_2/c_3 \equiv \eta \tag{B.71}$$

$$a/c_3 \equiv \lambda \tag{B.72}$$

so that the metric B.66 then takes on the form:

$$d\hat{s}^2 = \rho^\lambda dt^2 - (1+\gamma/2)^2 \rho^\gamma d\rho^2 - \rho^{(2+\delta)} d\phi^2 + \left[-\frac{\rho^{\frac{1}{2}(\eta-\mu)}}{(1-\frac{J}{\rho^\mu})} + \frac{\rho^{\frac{1}{2}(\eta+\mu)}\left(k-\frac{J}{\rho^\mu}\right)^2}{\left(1-\frac{J}{\rho^\mu}\right)J(1-k)^2} \right] dz^2 \pm 2\frac{\sqrt{\epsilon}\rho^{\frac{1}{2}(\eta+\mu)}\left(k-\frac{J}{\rho^\mu}\right)}{\sqrt{J}(1-k)} dz d\psi + \epsilon \rho^{\frac{1}{2}(\eta+\mu)} \left(1 - \frac{J}{\rho^\mu}\right) d\psi^2 \tag{B.73}$$

and the two constraints, eqs. B.38 and B.67, become:

$$\lambda + \eta + \delta - \gamma = 0 \tag{B.74}$$

$$\mu = \pm\sqrt{\eta^2 + 8(\lambda+\eta) + 4(\lambda\eta + \lambda\delta + \eta\delta)} \tag{B.75}$$

The form of the metric B.73 has also been verified on GRTensor II (ref.[1]).

B.1.1 Diagonalization of ρ-Dependent Magnetized Wire Metric

As discussed in chapter 4 and in the previous section, the metric B.73 can be diagonalized by a 'simple' coordinate transformation.

Of course, as was also discussed in chapter 4, it should be possible for a metric of this type to be diagonalized by a *general* (*i.e.*, 'nonsimple') transformation. In ref. [38], it is shown that any metric subspace of the form: $ds^2 = g_{11}(dx^1)^2 + 2g_{12}dx^1dx^2 + g_{22}(dx^2)^2$, where g_{11}, g_{12} and g_{22} are functions *of the same sign* of x^1 and x^2, can be transformed into the form: $ds^2 = g\left[(dx^1)^2 + (dx^2)^2\right]$, where g is a *new* function of x^1 and x^2. As discussed in chapter 4, this can then be expanded to three dimensions, so that: $ds^2 = g_{11}(dx^1)^2 + 2g_{12}dx^1dx^2 + g_{22}(dx^2)^2 + 2g_{23}dx^2dx^3 + g_{33}(dx^3)^2 + 2g_{31}dx^3dx^1$ can then diagonalized into the form: $ds^2 = g\left[(dx^1)^2 + (dx^2)^2 + (dx^3)^2\right]$, with g now a general function of x^1, x^2 and x^3. If one then associates $x^1 = \rho$, $x^2 = z$ and $x^3 = \psi$, one can then see that this could correspond to the ρ-z-ψ subspace of metric B.73.

However, because the resulting g must then be a *general* function of ρ, ψ *and* z (irrespective of the original forms of g_{11}, g_{12}, g_{22}, g_{23}, g_{33} and g_{31}), then the resulting diagonalized metric could end up being dependent on z. As discussed in the previous section, this would *not* correspond to a 'wire' solution *infinite* in the z-direction. As a result, the most general transformation cannot expect to preserve the original premise of the metric.

However, as was also noted in chapter 4, it turns out that one *can* diagonalize the metric B.73 'simply', and still have it remain a sole function of ρ, independent of z (and of ψ).

By suitable manipulation of terms, it can be shown that the metric B.73 can be put into the form:

$$\begin{aligned} d\hat{s}^2 &= \rho^\lambda dt^2 - (1+\gamma/2)^2\rho^\gamma d\rho^2 - \rho^{(2+\delta)}d\phi^2 - \tfrac{\rho^{\frac{1}{2}(\eta-\mu)}}{(1-\frac{J}{\rho^\mu})}dz^2 \\ &+\epsilon\rho^{\frac{1}{2}(\eta+\mu)}(1-\tfrac{J}{\rho^\mu})\left[d\psi \pm \tfrac{dz}{\sqrt{\epsilon J}(1-k)} \mp \tfrac{dz}{\sqrt{\epsilon J}(1-\frac{J}{\rho^\mu})}\right]^2 \end{aligned} \tag{B.76}$$

Now, by making the transformation:

$$\psi \to \psi \mp \frac{z}{\sqrt{\epsilon J}(1-k)} \tag{B.77}$$

we can then put the metric B.76 into the form:

$$\begin{aligned} d\hat{s}^2 &= \rho^\lambda dt^2 - (1+\gamma/2)^2\rho^\gamma d\rho^2 - \rho^{(2+\delta)}d\phi^2 - \tfrac{\rho^{\frac{1}{2}(\eta-\mu)}}{(1-\frac{J}{\rho^\mu})}dz^2 \\ &+\epsilon\rho^{\frac{1}{2}(\eta+\mu)}(1-\tfrac{J}{\rho^\mu})\left[d\psi \mp \tfrac{dz}{\sqrt{\epsilon J}(1-\frac{J}{\rho^\mu})}\right]^2 \end{aligned} \tag{B.78}$$

Note that in eq. B.78, we have transformed *away* the constant k, indicating that it was an artifact of a gauge-type transformation (reversing eq. B.77).

We can then rewrite eq. B.78 into the form:

$$\begin{aligned} d\hat{s}^2 &= \rho^\lambda dt^2 - (1+\gamma/2)^2\rho^\gamma d\rho^2 - \rho^{(2+\delta)}d\phi^2 + \tfrac{\rho^{\frac{1}{2}(\eta+\mu)}}{J}dz^2 \\ &\mp 2\sqrt{\tfrac{\epsilon}{J}}\rho^{\frac{1}{2}(\eta+\mu)}dz\,d\psi + \epsilon\rho^{\frac{1}{2}(\eta+\mu)}(1-\tfrac{J}{\rho^\mu})d\psi^2 \end{aligned} \tag{B.79}$$

which can then be reorganized into the form:

$$d\hat{s}^2 = \rho^\lambda dt^2 - (1+\gamma/2)^2\rho^\gamma d\rho^2 - \rho^{(2+\delta)}d\phi^2 + \rho^{\frac{1}{2}(\eta+\mu)}\left[\tfrac{1}{\sqrt{J}}dz \mp \sqrt{\epsilon}d\psi\right]^2 - \epsilon J\rho^{\frac{1}{2}(\eta-\mu)}d\psi^2 \tag{B.80}$$

By making another transformation;

$$z \rightarrow \sqrt{J}z \pm \sqrt{\epsilon J}\psi \tag{B.81}$$

the metric B.80 can again be simplified, this time to:

$$d\hat{s}^2 = \rho^{\lambda}dt^2 - (1+\gamma/2)^2\rho^{\gamma}d\rho^2 - \rho^{(2+\delta)}d\phi^2 + \rho^{\frac{1}{2}(\eta+\mu)}dz^2 - \epsilon J\rho^{\frac{1}{2}(\eta-\mu)}d\psi^2 \tag{B.82}$$

which is clearly a *diagonal* metric dependent *solely* on ρ. In other words, the most general form of the radially-dependent axially-symmetric metric (*i.e.*, eq. B.73) can be turned into a diagonal metric (through transformations B.77 and B.81). This, then, completes the proof of the diagonalizability of the radially-dependent axially-symmetric metric B.2.

B.2 ρ-ψ-Dependent Magnetized Wire Metric

As was discussed in chapter 4 and the previous section, there are two possible metric forms for an Induced Matter solution representing an axially-symmetric solution possessing a magnetic field: one in which the metric depends *only* on ρ (previous section) and one in which the metric depends on ρ and ψ. It is this solution which will be examined here.

We start by assuming a metric of the form:

$$\begin{aligned} d\hat{s}^2 = {} & e^{\nu(\rho,\psi)}dt^2 - e^{\lambda(\rho,\psi)}d\rho^2 - e^{\alpha(\rho,\psi)}d\phi^2 \\ & + \left[-e^{\beta(\rho,\psi)} + \epsilon e^{2\kappa(\rho,\psi)-\mu(\rho,\psi)}\right]dz^2 + 2\epsilon e^{\kappa(\rho,\psi)}dz\,d\psi + \epsilon e^{\mu(\rho,\psi)}d\psi^2 \end{aligned} \tag{B.83}$$

where the functions α, β, κ, λ, μ and ν are all general functions of ρ and ψ. Unfortunately, unlike in the previous section, there is no general method for solving for this metric when the metric coefficients are *general* functions of ρ and ψ. Instead, we assume the condition of separability for the metric coefficients; *i.e.*, that the metric coefficients can be written as products of purely ρ-dependent terms and purely ψ-dependent terms. This is achieved by writing the metric *exponents* as *sums* of ρ-dependent and ψ-dependent terms. Written out, this is:

$$\begin{aligned} d\hat{s}^2 &= e^{\nu(\rho)+\nu(\psi)}dt^2 - e^{\lambda(\rho)+\lambda(\psi)}d\rho^2 - e^{\alpha(\rho)+\alpha(\psi)}d\phi^2 \\ &\quad + \left[-e^{\beta(\rho)+\beta(\psi)} + \epsilon e^{2(\kappa(\rho)+\kappa(\psi))-(\mu(\rho)+\mu(\psi))}\right] dz^2 + 2\epsilon e^{\kappa(\rho)+\kappa(\psi)}dzd\psi + \epsilon e^{\mu(\rho)+\mu(\psi)}d\psi^2 \end{aligned} \tag{B.84}$$

where it must be stressed that the ρ- and ψ-dependent functions of a given exponent-type are *different* functions. So, for example, $\alpha(\rho)$ and $\alpha(\psi)$ are *different* functions of ρ and ψ.

By making the transformations:

$$\rho \to \int e^{-\lambda(\rho)/2} d\rho \tag{B.85}$$

$$\psi \to \int e^{-\mu(\psi)/2} d\psi \tag{B.86}$$

$$\kappa(\psi) \to \kappa(\psi) + \mu(\psi)/2 \tag{B.87}$$

the metric can, with loss of generality, be simplified to:

$$\begin{aligned} d\hat{s}^2 &= e^{\nu(\rho)+\nu(\psi)}dt^2 - e^{\lambda(\psi)}d\rho^2 - e^{\alpha(\rho)+\alpha(\psi)}d\phi^2 \\ &\quad + \left[-e^{\beta(\rho)+\beta(\psi)} + \epsilon e^{2\kappa(\rho)+2\kappa(\psi)-(\mu(\rho)}\right] dz^2 + 2\epsilon e^{\kappa(\rho)+\kappa(\psi)}dzd\psi + \epsilon e^{\mu(\rho)}d\psi^2 \end{aligned} \tag{B.88}$$

The nonzero Ricci tensors for this metric then are:

$$
\begin{aligned}
\hat{R}^0_0 = {} & \tfrac{1}{4}e^{(2\kappa_\rho+2\kappa_\psi-\beta_\rho-\beta_\psi-2\mu_\rho)}\left[\overset{*}{\alpha}\overset{*}{\nu} + \overset{*}{\lambda}\overset{*}{\nu} + 4\,\overset{*}{\kappa}\overset{*}{\nu} + 2\,\overset{**}{\nu} + \overset{*}{\nu}{}^2 - \overset{*}{\beta}\overset{*}{\nu}\right] \\
& - \tfrac{1}{4\epsilon}e^{-\mu_\rho}\left[\overset{*}{\alpha}\overset{*}{\nu} + \overset{*}{\lambda}\overset{*}{\nu} + 2\,\overset{**}{\nu} + \overset{*}{\nu}{}^2 - \overset{*}{\beta}\overset{*}{\nu}\right] \\
& + \tfrac{1}{4}e^{-\lambda_\psi}\left[\alpha'\nu' + 2\,\nu'' + \nu'^2 + \beta'\nu' + \mu'\nu'\right]
\end{aligned}
\tag{B.89}
$$

$$
\begin{aligned}
\hat{R}^1_1 = {} & \tfrac{1}{4}e^{(2\kappa_\rho+2\kappa_\psi-\beta_\rho-\beta_\psi-2\mu_\rho)}\left[\overset{*}{\alpha}\overset{*}{\lambda} + \overset{*}{\nu}\overset{*}{\lambda} + 4\,\overset{*}{\kappa}\overset{*}{\lambda} + 2\,\overset{**}{\lambda} + \overset{*}{\lambda}{}^2 - \overset{*}{\beta}\overset{*}{\lambda}\right] \\
& - \tfrac{\epsilon}{4}e^{(2\kappa_\rho+2\kappa_\psi-\beta_\rho-\beta_\psi-\mu_\rho-\lambda_\psi)}\left[2\,\kappa'^2 - 4\,\kappa'\mu' + 2\,\mu'^2\right] \\
& - \tfrac{1}{4\epsilon}e^{-\mu_\rho}\left[\overset{*}{\alpha}\overset{*}{\lambda} + \overset{*}{\nu}\overset{*}{\lambda} + 2\,\overset{**}{\lambda} + \overset{*}{\lambda}{}^2 - \overset{*}{\beta}\overset{*}{\lambda}\right] \\
& + \tfrac{1}{4}e^{-\lambda_\psi}\left[2\,\mu'' + \mu'^2 + 2\,\alpha'' + \alpha'^2 + 2\,\beta'' + \beta'^2 + 2\,\nu'' + \nu'^2\right]
\end{aligned}
\tag{B.90}
$$

$$
\begin{aligned}
\hat{R}^2_2 = {} & \tfrac{1}{4}e^{(2\kappa_\rho+2\kappa_\psi-\beta_\rho-\beta_\psi-2\mu_\rho)}\left[\overset{*}{\nu}\overset{*}{\alpha} + \overset{*}{\lambda}\overset{*}{\alpha} + 4\,\overset{*}{\kappa}\overset{*}{\alpha} + 2\,\overset{**}{\alpha} + \overset{*}{\alpha}{}^2 - \overset{*}{\beta}\overset{*}{\alpha}\right] \\
& - \tfrac{1}{4\epsilon}e^{-\mu_\rho}\left[\overset{*}{\nu}\overset{*}{\alpha} + \overset{*}{\lambda}\overset{*}{\alpha} + 2\,\overset{**}{\alpha} + \overset{*}{\alpha}{}^2 - \overset{*}{\beta}\overset{*}{\alpha}\right] \\
& + \tfrac{1}{4}e^{-\lambda_\psi}\left[\nu'\alpha' + 2\,\alpha'' + \alpha'^2 + \beta'\alpha' + \mu'\alpha'\right]
\end{aligned}
\tag{B.91}
$$

$$
\begin{aligned}
\hat{R}^3_3 = {} & \tfrac{1}{4}e^{(2\kappa_\rho+2\kappa_\psi-\beta_\rho-\beta_\psi-2\mu_\rho)}\left[2\,\overset{*}{\alpha}\overset{*}{\kappa} + 2\,\overset{*}{\lambda}\overset{*}{\kappa} + 2\,\overset{*}{\nu}\overset{*}{\kappa} + 4\,\overset{**}{\kappa} + 8\,\overset{*}{\kappa}{}^2 - 2\,\overset{*}{\beta}\overset{*}{\kappa}\right] \\
& - \tfrac{1}{4\epsilon}e^{-\mu_\rho}\left[\overset{*}{\alpha}\overset{*}{\beta} + \overset{*}{\lambda}\overset{*}{\beta} + 2\,\overset{**}{\beta} + \overset{*}{\beta}{}^2 - \overset{*}{\nu}\overset{*}{\beta}\right] \\
& + \tfrac{1}{4}e^{-\lambda_\psi}\left[\alpha'\beta' + 2\,\beta'' + \beta'^2 + \nu'\beta' + \mu'\beta'\right] \\
& - \tfrac{\epsilon}{4}e^{(2\kappa_\rho+2\kappa_\psi-\beta_\rho-\beta_\psi-\mu_\rho-\lambda_\psi)}[4\,\kappa'^2 + \mu'^2 - 5\,\mu'\kappa' - 2\,\mu'' + 2\,\kappa'' - \alpha'\mu' + \beta'\mu' + \alpha'\kappa' \\
& \qquad - \beta'\kappa' - \nu'\mu' + \kappa'\nu']
\end{aligned}
\tag{B.92}
$$

$$\begin{aligned}
\hat{R}^5_5 = {} & \tfrac{1}{4}e^{(2\kappa_\rho+2\kappa_\psi-\beta_\rho-\beta_\psi-2\mu_\rho)}[2\,\overset{**}{\lambda}+\overset{*}{\lambda}{}^2+2\,\overset{**}{\alpha}+\overset{*}{\alpha}{}^2+2\,\overset{*}{\alpha}\overset{*}{\kappa}-\overset{*}{\lambda}\overset{*}{\beta}+2\,\overset{*}{\lambda}\overset{*}{\kappa}-\overset{*}{\alpha}\overset{*}{\beta} \\
& \qquad -2\,\overset{*}{\beta}\overset{*}{\kappa}+4\,\overset{**}{\kappa}+8\,\overset{*}{\kappa}{}^2+2\,\overset{*}{\nu}\overset{*}{\kappa}+2\,\overset{**}{\nu}+\overset{*}{\nu}{}^2-\overset{*}{\beta}\overset{*}{\nu}] \\
& -\tfrac{1}{4\epsilon}e^{-\mu_\rho}\left[2\,\overset{**}{\lambda}+\overset{*}{\lambda}{}^2+2\,\overset{**}{\alpha}+\overset{*}{\alpha}{}^2+2\,\overset{**}{\beta}+\overset{*}{\beta}{}^2+2\,\overset{**}{\nu}+\overset{*}{\nu}{}^2\right] \\
& +\tfrac{1}{4}e^{-\lambda_\psi}\left[\overset{\prime}{\alpha}\overset{\prime}{\mu}+2\,\overset{\prime\prime}{\mu}+\overset{\prime}{\mu}{}^2+\overset{\prime}{\nu}\overset{\prime}{\mu}+\overset{\prime}{\beta}\overset{\prime}{\mu}\right] \\
& +\tfrac{\epsilon}{4}e^{(2\kappa_\rho+2\kappa_\psi-\beta_\rho-\beta_\psi-\mu_\rho-\lambda_\psi)}[4\,\overset{\prime}{\kappa}{}^2+\overset{\prime}{\mu}{}^2-5\,\overset{\prime}{\mu}\overset{\prime}{\kappa}-2\,\overset{\prime\prime}{\mu}+2\,\overset{\prime\prime}{\kappa}-\overset{\prime}{\alpha}\overset{\prime}{\mu}+\overset{\prime}{\beta}\overset{\prime}{\mu}+\overset{\prime}{\alpha}\overset{\prime}{\kappa} \\
& \qquad -\overset{\prime}{\beta}\overset{\prime}{\kappa}-\overset{\prime}{\nu}\overset{\prime}{\mu}+\overset{\prime}{\kappa}\overset{\prime}{\nu}]
\end{aligned} \tag{B.93}$$

$$\begin{aligned}
\hat{R}^1_3 = {} & \tfrac{\epsilon}{4}e^{(3\kappa_\rho+3\kappa_\psi-\beta_\rho-\beta_\psi-2\mu_\rho-\lambda_\psi)}\left[-\overset{*}{\lambda}\overset{\prime}{\mu}+6\,\overset{*}{\kappa}\overset{\prime}{\mu}+\overset{*}{\alpha}\overset{\prime}{\mu}+\overset{*}{\beta}\overset{\prime}{\kappa}-\overset{*}{\beta}\overset{\prime}{\mu}+\overset{*}{\nu}\overset{\prime}{\mu}-6\,\overset{*}{\kappa}\overset{\prime}{\kappa}-\overset{*}{\alpha}\overset{\prime}{\kappa}+\overset{*}{\lambda}\overset{\prime}{\kappa}-\overset{*}{\nu}\overset{\prime}{\kappa}\right] \\
& -\tfrac{1}{4}e^{(\kappa_\rho+\kappa_\psi-\mu_\rho-\lambda_\psi)}\left[\overset{*}{\lambda}\overset{\prime}{\beta}-\overset{*}{\lambda}\overset{\prime}{\kappa}-\overset{*}{\alpha}\overset{\prime}{\beta}+\overset{*}{\alpha}\overset{\prime}{\kappa}+\overset{*}{\beta}\overset{\prime}{\kappa}-\overset{*}{\beta}\overset{\prime}{\beta}+2\,\overset{*}{\kappa}\overset{\prime}{\kappa}-2\,\overset{*}{\kappa}\overset{\prime}{\beta}+\overset{*}{\nu}\overset{\prime}{\kappa}-\overset{*}{\nu}\overset{\prime}{\beta}\right]
\end{aligned} \tag{B.94}$$

$$\begin{aligned}
\hat{R}^3_1 = {} & \tfrac{1}{4}e^{(\kappa_\rho+\kappa_\psi-\beta_\rho-\beta_\psi-\mu_\rho)}[2\,\overset{*}{\kappa}\overset{\prime}{\beta}+\overset{*}{\alpha}\overset{\prime}{\beta}+\overset{*}{\beta}\overset{\prime}{\mu}+\overset{*}{\lambda}\overset{\prime}{\nu}+\overset{*}{\alpha}\overset{\prime}{\mu}-\overset{*}{\alpha}\overset{\prime}{\alpha}+\overset{*}{\lambda}\overset{\prime}{\alpha}-\overset{*}{\nu}\overset{\prime}{\mu} \\
& \qquad -\overset{*}{\nu}\overset{\prime}{\nu}-\overset{*}{\beta}\overset{\prime}{\kappa}+\overset{*}{\lambda}\overset{\prime}{\kappa}-2\,\overset{*}{\kappa}\overset{\prime}{\kappa}-\overset{*}{\alpha}\overset{\prime}{\kappa}-\overset{*}{\nu}\overset{\prime}{\kappa}+\overset{*}{\nu}\overset{\prime}{\beta}]
\end{aligned} \tag{B.95}$$

$$\begin{aligned}
\hat{R}^5_3 = {} & \tfrac{\epsilon}{4}e^{(3\kappa_\rho+3\kappa_\psi-\beta_\rho-\beta_\psi-2\mu_\rho-\lambda_\psi)}\left[-\overset{\prime}{\alpha}\overset{\prime}{\mu}+6\,\overset{\prime}{\kappa}{}^2+\overset{\prime}{\beta}\overset{\prime}{\mu}-\overset{\prime}{\beta}\overset{\prime}{\kappa}+3\,\overset{\prime}{\mu}{}^2-\overset{\prime}{\nu}\overset{\prime}{\mu}-9\,\overset{\prime}{\kappa}\overset{\prime}{\mu}-2\,\overset{\prime\prime}{\mu}+\overset{\prime}{\nu}\overset{\prime}{\kappa}+2\,\overset{\prime\prime}{\kappa}+\right. \\
& +\tfrac{1}{4}e^{(\kappa_\rho+\kappa_\psi-\mu_\rho-\lambda_\psi)}\left[2\,\overset{\prime\prime}{\kappa}-2\,\overset{\prime\prime}{\beta}+\overset{\prime}{\beta}\overset{\prime}{\mu}-\overset{\prime}{\kappa}\overset{\prime}{\beta}-\overset{\prime}{\mu}\overset{\prime}{\kappa}+2\,\overset{\prime}{\kappa}{}^2+\overset{\prime}{\kappa}\overset{\prime}{\nu}+\overset{\prime}{\kappa}\overset{\prime}{\alpha}-\overset{\prime}{\nu}\overset{\prime}{\beta}-\overset{\prime}{\alpha}\overset{\prime}{\beta}-\overset{\prime}{\beta}{}^2\right]
\end{aligned} \tag{B.96}$$

with the only other nonzero Ricci tensors, $\hat{R}^1_5$, $\hat{R}^5_1$ and $\hat{R}^3_5$, being given in terms of combinations of those above. In these Ricci tensors, overprimes denote partial derivatives with respect to ρ, while overstars denote partial derivatives with respect to ψ. Because of the assumed *splitting* of the metric exponents due to the assumption of separability (*e.g.*, $\alpha(\rho,\psi) = \alpha_\rho + \alpha_\psi$), it is understood that the *overprimed* terms are pure functions of ρ, while *overstarred* terms are pure functions of ψ (and, hence, why the ρ and ψ subscripts on the bulk of these terms have been dropped).

Despite the simplifying nature of the assumed separability of the metric coefficients, it still appears difficult to separate out purely ρ- and ψ-dependent terms from these equations for solving the field equations. Instead, a *further* simplifying assumption is made whereby *we assume that the general forms of the metric coefficients to be powers of* ρ *and* ψ. So, for example, we say: $e^{\alpha(\rho,\psi)} = e^{\alpha_\rho + \alpha_\psi} = \rho^{\mathsf{A}}\psi^{\mathsf{a}}$. The rationale for this approach is found in the previous section in which *the most general form for the* ρ*-dependent magnetic wire metric could be simply transformed into a (diagonal) solution in which all metric coefficients were powers of* ρ. Additionally, 'wire' solutions from refs. [37] and [44] show the same power form (especially in ref. [44] which describes a *neutral* ρ- and ψ-dependent wire metric).

Therefore, we assume a metric of the more *specific* form:

$$\begin{aligned} d\hat{s}^2 = {} & K_t \rho^{\mathsf{N}} \psi^{\mathsf{n}} dt^2 - K_\rho \psi^{\ell} d\rho^2 - K_\phi \rho^{\mathsf{A}} \psi^{\mathsf{a}} d\phi^2 \\ & + \left[-K_z \rho^{\mathsf{B}} \psi^{\mathsf{b}} + \epsilon \frac{K_\kappa^2}{K_\psi} \rho^{(2K-\mathsf{M})} \psi^{2k} \right] dz^2 + 2\epsilon K_\kappa \rho^K \psi^k dz d\psi + \epsilon K_\psi \rho^{\mathsf{M}} d\psi^2 \end{aligned} \tag{B.97}$$

If we then make the transformations:

$$t \rightarrow K_t^{-1/2} K_\psi^{\mathsf{n}/4} t \tag{B.98}$$

$$\phi \rightarrow K_\phi^{-1/2} K_\psi^{\mathsf{a}/4} \phi \tag{B.99}$$

$$z \rightarrow K_z^{-1/2} K_\psi^{\mathsf{b}/4} z \tag{B.100}$$

$$\psi \rightarrow K_\psi^{-1/2} \psi \tag{B.101}$$

and introduce new constants:

$$\mathsf{K} \equiv \epsilon K_\rho K_\psi^{-\ell/2} \tag{B.102}$$

$$\mathsf{C} \equiv -\epsilon K_\kappa^2 K_\psi^{(-1-k+\mathsf{b}/2)} K_z^{-1} \tag{B.103}$$

the metric can then be written in the simplified form:

$$\begin{aligned} d\hat{s}^2 = {}& \rho^{\mathsf{N}}\psi^{\mathsf{n}}dt^2 - \mathsf{K}\psi^{\ell}d\rho^2 - \rho^{\mathsf{A}}\psi^{\mathsf{a}}d\phi^2 \\ & - \left[\rho^{\mathsf{B}}\psi^{\mathsf{b}} + \mathsf{C}\rho^{(2K-\mathsf{M})}\psi^{2k}\right]dz^2 \pm 2\sqrt{-\epsilon\mathsf{C}}\rho^{K}\psi^{k}dz\,d\psi + \epsilon\rho^{\mathsf{M}}d\psi^2 \end{aligned} \tag{B.104}$$

It would also be possible to transform ρ such that K then be absorbed into $d\rho^2$. However, like the transformation on ψ, this will introduce factors of K into the other metric coefficients as ρ appears throughout the metric. Inevitably, this would leave *relative* differences between three of either the coefficients of $d\rho^2$, $d\psi^2$, $dzd\psi$ or the secondary portion of dz^2 (the portion containing the C in eq. B.104), depending on how one transforms ρ and/or ψ. We will leave the metric in the form it has in eq. B.104.

In consideration of the vacuum field equations (given by setting the Ricci tensors, eqs. B.89 to B.96, to zero), one notes that, in most of the equations, there are common groupings of terms which possess differing exponential factors (indeed, this is how the Ricci tensors in eqs. B.89 to B.96 were laid out). In order to properly combine terms within each of the Ricci tensors, it is then necessary to ensure that these relative terms are all *proportional* to each other, possessing the same powers of ρ and ψ. If each of these terms did *not* possess the same powers of ρ and ψ, then each of these terms would have to vanish independently. This would require many more restrictions on the metric coefficients and severely reduce the generality of the resulting solution.

In consideration of these terms, it is clear that the factor $e^{2\kappa(\rho,\psi)-\beta(\rho,\psi)-\mu(\rho,\psi)}$ figures quite prominently. A number of the coefficients differ by this relative amount. From the form of the metric given in eq. B.104, this term is: $\mathsf{C}\rho^{(2K-\mathsf{M}-\mathsf{B})}\psi^{(2k-\mathsf{b})}$. In order, then, that these terms possess the same *relative* form, it is necessary that this term become a constant. Therefore, we assume that:

$$\begin{gathered} 2K - \mathsf{M} - \mathsf{B} = 0 \rightarrow K = (\mathsf{M} + \mathsf{B})/2 \\ 2k - \mathsf{b} = 0 \rightarrow k = \mathsf{b}/2 \end{gathered} \tag{B.105}$$

so that the metric can be written as:

$$\begin{aligned} d\hat{s}^2 = {} & \rho^{\mathsf{N}}\psi^{\mathsf{n}}dt^2 - \mathsf{K}\psi^{\ell}d\rho^2 - \rho^{\mathsf{A}}\psi^{\mathsf{a}}d\phi^2 \\ & - \left[\rho^{\mathsf{B}}\psi^{\mathsf{b}} + \mathsf{C}\rho^{\mathsf{B}}\psi^{\mathsf{b}}\right]dz^2 \pm 2\sqrt{-\epsilon\mathsf{C}}\rho^{(\mathsf{M}+\mathsf{B})/2}\psi^{\mathsf{b}/2}dz\,d\psi + \epsilon\rho^{\mathsf{M}}d\psi^2 \end{aligned} \tag{B.106}$$

and where the dz^2 term can be more condensely written as: $-\rho^{\mathsf{B}}\psi^{\mathsf{b}}[1+\mathsf{C}]dz^2$.

The only other terms in the coefficients of the Ricci tensors that must be examined are the factors $e^{\lambda(\psi)}$ and $e^{\mu(\rho)}$. The factor $e^{\lambda(\psi)}$ appears in front of terms which possess derivatives with respect to ψ ("overstarred" terms), while $e^{\mu(\rho)}$ appears in front of term which possess derivatives with respect to ρ ("overprimed" terms). Since it is desired that *all* these terms (overstarred and overprimed) be *combinable*, with the same powers of ρ and ψ, it is then necessary that these two factors, $e^{\lambda(\psi)}$ and $e^{\mu(\rho)}$, cancel out the *relative* differences in powers of ρ and ψ.

In view of the form of the metric already given (eq. B.106), the overprimed terms can be seen to all be proportional to ρ^{-2} (relative to the overstarred terms). Similarly, the overstarred terms will all be proportional to ψ^{-2} (relative to the overprimed terms). Therefore, if one wishes the overprimed and overstarred terms to be *combinable* (with *no* relative differences in their respective powers of ρ and ψ), then $e^{\mu(\rho)}$ must be given by ρ^2, while $e^{\lambda(\psi)}$ must be given by ψ^2. Thus, $\mathsf{M} = \ell = 2$, and the metric can now be written as:

$$\begin{aligned} d\hat{s}^2 &= \rho^{\mathsf{N}}\psi^{\mathsf{n}}dt^2 - \mathsf{K}\psi^2 d\rho^2 - \rho^{\mathsf{A}}\psi^{\mathsf{a}}d\phi^2 \\ &\quad -\rho^{\mathsf{B}}\psi^{\mathsf{b}}(1+\mathsf{C})dz^2 \pm 2\sqrt{-\epsilon\mathsf{C}}\rho^{(1+\mathsf{B}/2)}\psi^{\mathsf{b}/2}dz d\psi + \epsilon\rho^2 d\psi^2 \end{aligned} \tag{B.107}$$

Substituting this form of the metric into the expression of the Ricci tensors, eqs. B.89 to B.96, set equal to zero (for vacuum solution) one then obtains (after dropping all common factors of ρ and ψ:

$$\epsilon\mathsf{Kn}(1+\mathsf{C})[\mathsf{a}+\mathsf{b}+\mathsf{n}] - \mathsf{NC}[\mathsf{A}+\mathsf{B}+\mathsf{N}] = 0 \tag{B.108}$$

$$-2\epsilon\mathsf{K}(1+\mathsf{C})[\mathsf{a}+\mathsf{b}+\mathsf{n}] + \frac{1}{2}[\mathsf{B}-2]^2 + \mathsf{C}[\mathsf{A}(\mathsf{A}-2)+\mathsf{B}(\mathsf{B}-2)+\mathsf{N}(\mathsf{N}-2)] = 0 \tag{B.109}$$

$$\epsilon\mathsf{Ka}(1+\mathsf{C})[\mathsf{a}+\mathsf{b}+\mathsf{n}] - \mathsf{AC}[\mathsf{A}+\mathsf{B}+\mathsf{N}] = 0 \tag{B.110}$$

$$-\epsilon\mathsf{Kb}(1+\mathsf{C})[\mathsf{a}+\mathsf{b}+\mathsf{n}] + \frac{1}{2}[(\mathsf{A}+\mathsf{B}+\mathsf{N})(\mathsf{B}-2)] + \mathsf{BC}[\mathsf{A}+\mathsf{B}+\mathsf{N}] = 0 \tag{B.111}$$

$$-\epsilon\mathsf{K}(1+\mathsf{C})[\mathsf{a}(\mathsf{a}-2)+\mathsf{b}(\mathsf{b}-2)+\mathsf{n}(\mathsf{n}-2)] + 2\mathsf{C}[\mathsf{A}+\mathsf{B}+\mathsf{N}] - \frac{1}{2}[(\mathsf{A}+\mathsf{B}+\mathsf{N})(\mathsf{B}-2)] = 0 \tag{B.112}$$

$$\epsilon(1+\mathsf{C})(\mathsf{a}+\mathsf{n}-2+2\mathsf{b})(\mathsf{B}-2) = 0 \tag{B.113}$$

$$\frac{1}{2}(\mathsf{a}+\mathsf{n}+2)(\mathsf{B}+2) - (\mathsf{aA}+\mathsf{nN}) + (\mathsf{A}+\mathsf{N}) = 0 \tag{B.114}$$

$$\epsilon(1+\mathsf{C})(\mathsf{A}+\mathsf{N}-2+2\mathsf{B})(\mathsf{B}-2) = 0 \tag{B.115}$$

One possible solution from this set is to assume $(1+\mathsf{C}) = 0$, and solve the remaining terms. However, this would render $\hat{g}_{33} = 0$, so that the metric would lose its dz^2 term. It is not clear what this would mean physically (see discussion at the end of the previous appendix, section A.2, regarding similar solutions), therefore, we avoid this case. Instead, we solve these equations, simultaneously, for K, C, B, b, A, a, N and n, and the results are:

$$K = \frac{\epsilon N(2-B)}{(1-\epsilon C)n(2-b)} \tag{B.116}$$

$$C = \frac{2\epsilon(Nb - nB)}{n(2-B)} \tag{B.117}$$

$$B = 1 - \frac{1}{2}(A+N) \tag{B.118}$$

$$b = 1 - \frac{1}{2}(a+n) \tag{B.119}$$

$$A = af(a,n) \tag{B.120}$$

$$N = nf(a,n) \tag{B.121}$$

where $f(a,n)$ is a function of a and n given by:

$$f(a,n) = \frac{(2+a+n)}{[-(a+n)+\frac{1}{6}(5a^2+2an+5n^2)]} \tag{B.122}$$

From this, it is seen that the metric depends on two independent variables, a and n.

Finally, in consideration of the axially-symmetric metrics given in refs. [37] and [44], it is obvious that the $d\rho^2$ portions of the metric *also* possess nonvanishing powers of ρ in the metric coefficient. In order to make this solution 'comparable' with these others, it is, therefore, desirable to consider introducing a ρ-term into the metric. This can be done through the transformation:

$$\rho \to \rho^L \tag{B.123}$$

which renders the metric B.107 as:

$$
\begin{aligned}
d\hat{s}^2 = {} & \rho^{\mathsf{NL}}\psi^{\mathsf{n}}dt^2 - \mathsf{KL}^2\psi^2 d\rho^2 - \rho^{\mathsf{AL}}\psi^{\mathsf{a}}d\phi^2 \\
& -\rho^{\mathsf{BL}}\psi^{\mathsf{b}}(1+\mathsf{C})dz^2 \pm 2\sqrt{-\epsilon\mathsf{C}}\rho^{(L+\mathsf{BL}/2)}\psi^{\mathsf{b}/2}dz\,d\psi + \epsilon\rho^{2\mathsf{L}}d\psi^2
\end{aligned} \tag{B.124}
$$

where the various terms are still defined by eqs. B.116 to B.122, since this transformation does not alter these definitions. For $\mathsf{L} = 1$, this metric goes back to its previous form, eq. B.107. And the metric can now be seen to depend on *three* independent variables, a, n *and* L.

Both forms of the metric, eqs. B.107 and B.124, have been verified on GRTensor II for the constants as defined in eqs. B.116 to B.122 (ref. [1]).

Appendix C

Derivation of Conformally Flat Metrics

As discussed in chapter 5, 4D cosmological solutions can be embedded in 5D flat (vacuum) solutions (refs. [12], [23], [41], [42]). Also in ref. [40], especially, it has been shown that the standard 4D cosmological (FRW) solutions can be written in terms of *conformally* flat metrics (in 4D). As such, we desire to extend such 4D conformally flat metrics to 5D to describe potential cosmological solutions.

However, because our manifolds here are in *5D*, then there exists an *ambiguity* as to whether such extensions should be represented as *4D* or *5D* conformally flat metrics (where the *4D* conformally flat metric would be embedded in a *5D* manifold). For the purposes of this analysis, then, we investigate *both*; the first metric will be 5D conformally flat, while the second will be 4D conformally flat embedded in a 5D manifold.

C.1 5D Conformally Flat Metrics

In ref. [39], calculations are performed for general conformally related solutions. A given metric, $g_{\mu\nu}$, which is conformally related to another metric, $g'_{\mu\nu}$, satisfies, by definition;

$$\begin{aligned} g_{\mu\nu} &= e^{\tilde{\Phi}} g'_{\mu\nu} \\ g^{\mu\nu} &= e^{-\tilde{\Phi}} g'^{\,\mu\nu} \end{aligned} \tag{C.1}$$

where $e^{\tilde{\Phi}}$ is said to be the *conformal factor*. The second line is the relation for the corresponding inverse metric. [Note: The conformal factor, $\tilde{\Phi}$, is written with a *tilde* in order to distinguish it from the Φ in $\epsilon\Phi^2$, the 5-5 element of the Induced Matter metric.]

In ref. [39], the relation between the Ricci tensors for these two metrics is then calculated to be, in general notation;

$$R_{\mu\nu} = R'_{\mu\nu} + \mathcal{A}^{\alpha}_{\alpha\mu;\nu} - \mathcal{A}^{\alpha}_{\mu\nu;\alpha} + \mathcal{A}^{\alpha}_{\mu\beta}\mathcal{A}^{\beta}_{\alpha\nu} - \mathcal{A}^{\alpha}_{\mu\nu}\mathcal{A}^{\beta}_{\alpha\beta} \tag{C.2}$$

where $R_{\mu\nu}$ is the Ricci tensor of $g_{\mu\nu}$, $R'_{\mu\nu}$ is the Ricci tensor of $g'_{\mu\nu}$ and $\mathcal{A}^{\lambda}_{\mu\nu}$ is another tensor given by:

$$\mathcal{A}^{\lambda}_{\mu\nu} \equiv \frac{1}{2}\left(\delta^{\lambda}_{\mu}\tilde{\Phi}_{,\nu} + \delta^{\lambda}_{\nu}\tilde{\Phi}_{,\mu} - g'_{\mu\nu}g'^{\,\lambda\alpha}\tilde{\Phi}_{,\alpha}\right) \tag{C.3}$$

with δ^{β}_{α} being the Kronecker delta ($= 1$ for $\alpha = \beta$; 0 otherwise).

If the second metric, $g'_{\mu\nu}$ is *flat* (*i.e.*, so that all its Riemann and Ricci tensors vanish), such as a Minkowski metric, then $R'_{\mu\nu}$ in eq. C.2 must vanish. The corresponding Ricci tensors, Ricci scalars and Einstein tensors can then be calculated.

In ref. [39], these calculations are then done for a 4D manifold. In what follows *here*, the calculations are done for a *5D* manifold. In principle, there *are* differences in these calculations, resulting, notably, from δ^{μ}_{μ}, which is *4* for a 4D manifold, but *5* for a 5D manifold.

(Actually, it turns out, in *this* case, that the 4D and 5D calculations *are* the same, but only because we are concerned with *vacuum* solutions; if they were *not* vacuum, then they *wouldn't* be the same.)

We take $g'_{\mu\nu} = \hat{\eta}_{\mu\nu}$, where $\hat{\eta}_{\mu\nu}$ is the 5D Minkowski metric $(= dt^2 - dr^2 - r^2 d\Omega^2 + \epsilon d\psi^2)$, and $\delta^{a}_{a} = 5$. Calculation of the various components of eq. C.2 (for *5D*) then gives:

$$\mathcal{A}^{a}_{am;n} = \frac{5}{2}\tilde{\Phi}_{,m;n} \tag{C.4}$$

$$\mathcal{A}^{a}_{mn;a} = \tilde{\Phi}_{,m;n} - \frac{1}{2}\hat{\eta}_{mn}\hat{\eta}^{ab}\tilde{\Phi}_{,a;b} \tag{C.5}$$

$$\mathcal{A}^{a}_{mb}\mathcal{A}^{b}_{na} = \frac{7}{4}\tilde{\Phi}_{,m}\tilde{\Phi}_{,n} - \frac{1}{2}\hat{\eta}_{mn}\hat{\eta}^{ab}\tilde{\Phi}_{,a}\tilde{\Phi}_{,b} \tag{C.6}$$

$$\mathcal{A}^{a}_{mn}\mathcal{A}^{b}_{ab} = \frac{5}{2}\tilde{\Phi}_{,m}\tilde{\Phi}_{,n} - \frac{5}{4}\hat{\eta}_{mn}\hat{\eta}^{ab}\tilde{\Phi}_{,a}\tilde{\Phi}_{,b} \tag{C.7}$$

From these calculations, the expression for the conformally transformed Ricci tensor becomes:

$$\hat{R}_{mn} = \frac{3}{2}\tilde{\Phi}_{,m;n} - \frac{3}{4}\tilde{\Phi}_{,m}\tilde{\Phi}_{,n} + \frac{1}{2}\hat{\eta}\hat{\eta}^{ab}\tilde{\Phi}_{,a;b} + \frac{3}{4}\hat{\eta}_{mn}\hat{\eta}^{ab}\tilde{\Phi}_{,a}\tilde{\Phi}_{,b} \tag{C.8}$$

To then satisfy the vacuum field equations of the Induced Matter Theory, the Ricci tensor must vanish. To assist with the calculations, we calculate the Ricci scalar:

$$\hat{R} \equiv \hat{g}^{mn}\hat{R}_{mn} = e^{-\tilde{\Phi}}\hat{\eta}^{mn}\hat{R}_{mn} = e^{-\tilde{\Phi}}\left[4\hat{\eta}^{ab}\tilde{\Phi}_{,a;b} + 3\hat{\eta}^{ab}\tilde{\Phi}_{,a}\tilde{\Phi}_{,b}\right] \tag{C.9}$$

The imposition of vacuum also requires that $\hat{R}$ vanish. From eq. C.9 this can be seen to yield:

$$\hat{\eta}^{ab}\tilde{\Phi}_{,a;b} = -\frac{3}{4}\hat{\eta}^{ab}\tilde{\Phi}_{,a}\tilde{\Phi}_{,b} \tag{C.10}$$

Substituting this result into the expression for the Ricci tensor, eq. C.8, then yields:

$$\hat{R}_{mn} = \frac{3}{2}\tilde{\Phi}_{,m;n} - \frac{3}{4}\tilde{\Phi}_{,m}\tilde{\Phi}_{,n} + \frac{3}{8}\hat{\eta}_{mn}\hat{\eta}^{ab}\tilde{\Phi}_{,a}\tilde{\Phi}_{,b} \Rightarrow 0 \tag{C.11}$$

For $m \neq n$, this yields:

$$\tilde{\Phi}_{,m;n} \equiv \tilde{\Phi}_{,m,n} - \hat{\Gamma}^{a}_{mn}\tilde{\Phi}_{,a} = \frac{1}{2}\tilde{\Phi}_{,m}\tilde{\Phi}_{,n} \tag{C.12}$$

At this point, it is assumed that $\tilde{\Phi}$ is a function of the three main coordinates, t, r and ψ (but not on θ and ϕ). As a result, $\hat{\Gamma}^{a}_{mn} = 0$ for $m \neq n$, so that eq. C.12 becomes:

$$\tilde{\Phi}_{,m,n} = \frac{1}{2}\tilde{\Phi}_{,m}\tilde{\Phi}_{,n} \tag{C.13}$$

Since $\tilde{\Phi}$ depends only on t, r and ψ, eq. C.13 can be explicitly written out as:

$$\overset{\circ\prime}{\tilde{\Phi}} = \frac{1}{2}\overset{\circ}{\tilde{\Phi}}\overset{\prime}{\tilde{\Phi}} \tag{C.14}$$

$$\overset{\prime *}{\tilde{\Phi}} = \frac{1}{2}\overset{\prime}{\tilde{\Phi}}\overset{*}{\tilde{\Phi}} \tag{C.15}$$

$$\overset{*\circ}{\bar{\Phi}} = \frac{1}{2} \overset{*}{\bar{\Phi}} \overset{\circ}{\bar{\Phi}} \tag{C.16}$$

where overprimes are partial derivatives with respect to r, overstars are partial derivatives with respect to ψ, and over*circles* are partial derivatives with respect to t.

For $m = n$ in eq. C.11, one then obtains:

$$\bar{\Phi}_{,m;m} - \frac{1}{2}\bar{\Phi}^2_{,m} + \frac{1}{4}\hat{\eta}_{mm}\hat{\eta}^{ab}\bar{\Phi}_{,a}\bar{\Phi}_{,b} \equiv \bar{\Phi}_{,m,m} - \hat{\Gamma}^a_{mm}\bar{\Phi}_{,a} - \frac{1}{2}\bar{\Phi}^2_{,m} + \frac{1}{4}\hat{\eta}_{mm}\hat{\eta}^{ab}\bar{\Phi}_{,a}\bar{\Phi}_{,b} = 0 \tag{C.17}$$

The only nonzero Christoffel terms, $\hat{\Gamma}^a_{mn}$, for $m = n$ turn out to be:

$$\hat{\Gamma}^r_{\theta\theta} = -r\hat{\Gamma}^r_{\phi\phi} = -r\sin^2\theta \tag{C.18}$$

Thus, writing eq. C.17 out explicitly yields:

$$\overset{\circ\circ}{\bar{\Phi}} - \frac{1}{2}\overset{\circ}{\bar{\Phi}}^2 + \frac{1}{4}\hat{\eta}^{ab}\bar{\Phi}_{,a}\bar{\Phi}_{,b} = 0 \tag{C.19}$$

$$\bar{\Phi}'' - \frac{1}{2}\bar{\Phi}'^2 - \frac{1}{4}\hat{\eta}^{ab}\bar{\Phi}_{,a}\bar{\Phi}_{,b} = 0 \tag{C.20}$$

$$\overset{**}{\bar{\Phi}} - \frac{1}{2}\overset{*}{\bar{\Phi}}^2 + \frac{\epsilon}{4}\hat{\eta}^{ab}\bar{\Phi}_{,a}\bar{\Phi}_{,b} = 0 \tag{C.21}$$

$$\hat{\Gamma}^r_{\theta\theta}\bar{\Phi}' - \frac{1}{4}\hat{\eta}_{\theta\theta}\hat{\eta}^{ab}\bar{\Phi}_{,a}\bar{\Phi}_{,b} = 0 \tag{C.22}$$

$$\hat{\Gamma}^r_{\phi\phi}\bar{\Phi}' - \frac{1}{4}\hat{\eta}_{\phi\phi}\hat{\eta}^{ab}\bar{\Phi}_{,a}\bar{\Phi}_{,b} = 0 \tag{C.23}$$

The last equation is redundant with the one prior to it since eq. C.23=eq. C.22$\times \sin^2\theta$. Writing out eq. C.22 (or eq. C.23) then yields:

$$\frac{\tilde{\Phi}'}{r} = \tilde{\eta}^{ab}\tilde{\Phi}_{,a}\tilde{\Phi}_{,b} \tag{C.24}$$

Substituting this into eq. C.20 then yields:

$$\tilde{\Phi}'' - \frac{1}{2}\tilde{\Phi}'^{2} - \frac{\tilde{\Phi}'}{r} = 0 \tag{C.25}$$

This equation can then be integrated to explicitly reveal the r-dependence of $\tilde{\Phi}$;

$$\tilde{\Phi} = w(t,\psi) + \ln[u(t,\psi) - \frac{1}{2}(v(t,\psi)r)^2]^{-2} \tag{C.26}$$

where $w(t,\psi)$, $u(t,\psi)$ and $v(t,\psi)$ are general functions of t and ψ, but independent of r. Substituting solution C.26 into eq. C.14 then yields, for the form of $w(t,\psi)$;

$$w(t,\psi) = K(\psi) + 4\ln[k(\psi)v(t,\psi)] \tag{C.27}$$

where $K(\psi)$ and $k(\psi)$ are (presumed) arbitrary functions of ψ, but independent of t and r.

Similarly, one can substitute eq. C.26 into eq. C.15 to get:

$$w(t,\psi) = K(t) + 4\ln[k(t)v(t,\psi)] \tag{C.28}$$

where $K(t)$ and $k(t)$ are *new* arbitrary functions of t, but independent of ψ and r. But since this is also an expression of $w(t,\psi)$, then equating eqs. C.27 and C.28 requires K and k to be *constants*, independent of t *and* ψ (*and* r). Thus;

$$w(t,\psi) = K + 4\ln[kv(t,\psi)] \tag{C.29}$$

and:

$$\bar{\Phi} = K + 4\ln[kv(t,\psi)] + \ln[u(t,\psi) - \frac{1}{2}(v(t,\psi)r)^2]^{-2} \qquad \text{(C.30)}$$

As the other terms (other than the K) on the right-hand side of this equation are *logarithmic* functions, then the K can be absorbed into them (into k for example). Thus, we have:

$$\bar{\Phi} = \ln[kv_{t\psi}]^4 + \ln[u_{t\psi} - \frac{1}{2}(v_{t\psi}r)^2]^{-2} \qquad \text{(C.31)}$$

Substituting eqs. C.31 and C.24 into eqs. C.19, C.21 and C.16 then yields:

$$\overset{\circ\circ}{u}\, v^2 - 2uv\, \overset{\circ\circ}{v} + 6u\, \overset{\circ}{v}^2 - 4\, \overset{\circ}{u}\, v\, \overset{\circ}{v} - v^4 = 0 \qquad \text{(C.32)}$$

$$\overset{**}{u}\, v^2 - 2uv\, \overset{**}{v} + 6u\, \overset{*}{v}^2 - 4\, \overset{*}{u}\, v\, \overset{*}{v} - \epsilon v^4 = 0 \qquad \text{(C.33)}$$

$$-2\, \overset{*\circ}{u}\, v^2 + 4\, \overset{\circ}{u}\, v\, \overset{*}{v} + 4\, \overset{*}{u}\, v\, \overset{\circ}{v} - 12u\, \overset{\circ}{v}\overset{*}{v} = 0 \qquad \text{(C.34)}$$

In eq. C.24, we have, on the right-hand side, the collection of terms: $\hat{\eta}^{ab}\bar{\Phi}_{,a}\bar{\Phi}_{,b}$. This can be expanded as:

$$\hat{\eta}^{ab}\bar{\Phi}_{,a}\bar{\Phi}_{,b} = \hat{\eta}^{aa}\bar{\Phi}^2_{,a} = \hat{\eta}^{tt}\bar{\Phi}^2_{,t} + \hat{\eta}^{rr}\bar{\Phi}^2_{,r} + \hat{\eta}^{\psi\psi}\bar{\Phi}^2_{,\psi} = \overset{\circ}{\bar{\Phi}}^2 - \overset{\prime}{\bar{\Phi}}^2 + \epsilon\, \overset{*}{\bar{\Phi}}^2 \qquad \text{(C.35)}$$

Substituting this expression, and eq. C.31, into eq. C.24, then yields the constraint:

$$\overset{\circ}{u}^2 v^2 - 4u\, \overset{\circ}{u}\, v\, \overset{\circ}{v} + 4u^2\, \overset{\circ}{v}^2 + \epsilon\, \overset{*}{u}^2 v^2 - 4\epsilon u\, \overset{*}{u}\, v\, \overset{*}{v} + 4\epsilon u^2\, \overset{*}{v}^2 - 2uv^4 = 0 \qquad \text{(C.36)}$$

Eqs. C.32, C.33, C.34 and C.36, therefore, constitute the total set of constraints on $u_{t\psi}$ and $v_{t\psi}$. They are, in fact, a set of *reduced* field equations, with the r-dependencies removed.

As noted in chapter 5, the simplest solution is to set $v_{t\psi} = 1$, $k = 1\sqrt{2}$, and solve $u_{t\psi}$ to be $\frac{1}{2}(t^2 + \epsilon\psi^2)$. The resulting metric becomes:

$$d\hat{s}^2 = \frac{[dt^2 - dr^2 - r^2 d\Omega^2 + \epsilon d\psi^2]}{[t^2 - r^2 + \epsilon\psi^2]^2} \tag{C.37}$$

Alternatively, as also noted in chapter 5, one could set $v_{t\psi} = 0$ and let $k \to \frac{1}{0}$ so that $kv_{t\psi} = 1$, and solve $u_{t\psi}$ to be $(t + \epsilon\psi)$. This gives a metric of:

$$d\hat{s}^2 = \frac{1}{(t + \epsilon\psi)^2}\left[dt^2 - dr^2 - r^2 d\Omega^2 + \epsilon d\psi^2\right] \tag{C.38}$$

Both solutions C.37 and C.38 have been verified on GRTensor II (ref. [1]). These, then, are the two 5D conformally flat metrics used in chapter 5 representing cosmological solutions. The first solution is the main 5D solution of interest, while the second solution *overlaps* with the 4D conformally flat solutions.

C.1.1 Riemann Flatness of (5D) Conformally Flat Metrics

As also noted in chapter 5 (and in the previous section), previously studied cosmological solutions have been Ricci *and* Riemann flat (refs. [12], [23], [41], [42]). It can be verified (also on GRTensor II) that the solutions C.37 and C.38 are both Riemann flat, but it is not clear whether *all* such possible solutions to eqs. C.31, C.32, C.33, C.34 and C.36 are necessarily flat.

In fact, it turns out that they are. Indeed, this turns out to be a property of solutions which are both *vacuum and conformally flat* (irrespective the dimensionality of the manifold).

In order to show the *general* flatness of any conformally flat metric which satisfies the vacuum field equations, it is first necessary to assume a *general* metric form, which includes *all* spatial coordinates. So, instead of assuming that the conformal factor depends on just r (and not on θ and ϕ) in spherical polar coordinates, we instead assume that it depends on x, y and z in Cartesian coordinates. Additionally, the conformal factor will also depend on t. [Note: For simplicity we shall do this in *4D* (not 5D), though the results can be generalized to any number of dimensions by simply extending the dimensionality of the (Minkowskian) metric and the number of (spacelike) coordinates on which the conformal factor can depend.]

We begin, then, with a general (4D) conformally flat metric of the form:

$$ds^2 = C(x, y, z, t)\left[dt^2 - dx^2 - dy^2 - dz^2\right] \tag{C.39}$$

The nonzero Riemann tensors for this metric can then be calculated using, in this case, GRTensor II. [Note: For maximal simplicity, the Riemann and Ricci tensors are shown as to what they are *proportional* to (using "$\Rightarrow$"s), not what they are *exactly equal* to. As will be seen, this does *not* affect the generality of the results.] The Riemann tensors are:

$$R_{xyxy} \Rightarrow -2C_{,y,y}C - 2C_{,x,x}C + 2C_{,y}^2 + 2C_{,x}^2 - C_{,z}^2 + C_{,t}^2 \tag{C.40}$$

$$R_{xzxz} \Rightarrow -2C_{,z,z}C - 2C_{,x,x}C + 2C_{,x}^2 + 2C_{,z}^2 - C_{,y}^2 + C_{,t}^2 \tag{C.41}$$

$$R_{xtxt} \Rightarrow 2C_{,t,t}C - 2C_{,x,x}C + 2C_{,x}^2 - 2C_{,t}^2 - C_{,y}^2 + C_{,z}^2 \tag{C.42}$$

$$R_{yzyz} \Rightarrow 2C_{,z,z}C + 2C_{,y,y}C - 2C_{,z}^2 - 2C_{,y}^2 - C_{,x}^2 + C_{,t}^2 \tag{C.43}$$

$$R_{ytyt} \Rightarrow 2C_{,t,t}C - 2C_{,y,y}C - 2C_{,t}^2 + 2C_{,y}^2 - C_{,x}^2 + C_{,z}^2 \tag{C.44}$$

$$R_{ztzt} \Rightarrow 2C_{,t,t}C - 2C_{,z,z}C - 2C_{,t}^2 + 2C_{,z}^2 - C_{,x}^2 + C_{,y}^2 \tag{C.45}$$

$$R_{xyxz} \Rightarrow R_{ytzt} \Rightarrow 3C_{,y}C_{,z} - 2C_{,y,z}C \tag{C.46}$$

$$R_{xyxt} \Rightarrow R_{yzzt} \Rightarrow 3C_{,y}C_{,t} - 2C_{,y,t}C \tag{C.47}$$

$$R_{xyyz} \Rightarrow R_{xtzt} \Rightarrow 3C_{,z}C_{,x} - 2C_{,x,z}C \tag{C.48}$$

$$R_{xyyt} \Rightarrow R_{xzzt} \Rightarrow 3C_{,t}C_{,x} - 2C_{,x,t}C \tag{C.49}$$

$$R_{xzxt} \Rightarrow R_{yzyt} \Rightarrow 3C_{,z}C_{,t} - 2C_{,t,z}C \tag{C.50}$$

$$R_{xzyz} \Rightarrow R_{xtyt} \Rightarrow 3C_{,y}C_{,x} - 2C_{,y,x}C \tag{C.51}$$

where, in standard relativistic notation, commas denote partial derivatives ($C_{,x} \equiv \partial C/\partial x$, *etc.*).

The nonzero Ricci tensors for this metric can then also be written out as:

$$R_{xx} \Rightarrow 3C_{,x}^2 - 3C_{,x,x}C - C_{,y,y}C - C_{,z,z}C + C_{,t,t}C \tag{C.52}$$

$$R_{yy} \Rightarrow 3C_{,y}^2 - 3C_{,y,y}C - C_{,x,x}C - C_{,z,z}C + C_{,t,t}C \tag{C.53}$$

$$R_{zz} \Rightarrow 3C_{,z}^2 - 3C_{,z,z}C - C_{,x,x}C - C_{,y,y}C + C_{,t,t}C \tag{C.54}$$

$$R_{tt} \Rightarrow -3C_{,t}^2 + 3C_{,t,t}C - C_{,x,x}C - C_{,y,y}C + C_{,z,z}C \tag{C.55}$$

$$R_{xy} \Rightarrow 3C_{,x}C_{,y} - 2C_{,x,y}C \tag{C.56}$$

$$R_{xz} \Rightarrow 3C_{,x}C_{,z} - 2C_{,x,z}C \tag{C.57}$$

$$R_{xt} \Rightarrow 3C_{,x}C_{,t} - 2C_{,x,t}C \tag{C.58}$$

$$R_{yz} \Rightarrow 3C_{,z}C_{,y} - 2C_{,z,y}C \tag{C.59}$$

$$R_{yt} \Rightarrow 3C_{,t}C_{,y} - 2C_{,x,t}C \tag{C.60}$$

$$R_{zt} \Rightarrow 3C_{,z}C_{,t} - 2C_{,z,t}C \tag{C.61}$$

In comparison of the Ricci's and Riemann's for this system, one finds:

$$R_{xzxz} \Rightarrow \frac{2}{3}R_{xx} + \frac{2}{3}R_{zz} - \frac{1}{3}R_{yy} - \frac{1}{3}R_{tt} \tag{C.62}$$

(and similarly for R_{xyxy}, R_{xtxt}, R_{yzyz}, R_{ytyt} and R_{ztzt}), as well as:

$$R_{xzyz} \Rightarrow R_{xtyt} \Rightarrow R_{xy} \tag{C.63}$$

(and similarly for the rest of the Riemann's).

Thus, if $R_{\alpha\beta} = 0$ (as for a vacuum), then so do all the $R_{\alpha\beta\gamma\delta} = 0$. As noted previously, though this was derived for a 4D system, it can be generalized to *any* number of dimensions by simply extending the number of (spacelike) dimensions of the metric in the conformal factor, C. Thus, a conformally flat metric which satisfies the vacuum field equations will *always* yield a Riemann-flat solution.

C.2 4D Conformally Flat Metric (Embedded in a 5D Manifold)

As was discussed in chapter 5 and in the previous section, finding an extension for 4D conformally flat solutions allows two possibilities when going to five dimensions: one in which the metric is 5D conformally flat (as the solutions found in the previous section were), and another in which the metric is 4D conformally flat embedded in a 5D manifold. In this section, we examine 4D conformally flat solutions embedded in a 5D manifold.

We first conceive a 5D metric which of the form:

$$d\hat{s}^2 = e^{U(t,r,\psi)} d\eta^2 + \epsilon e^{V(t,r,\psi)} d\psi^2 \tag{C.64}$$

where $U(t,r,\psi)$ and $V(t,r,\psi)$ are general functions of t, r and ψ, and $d\eta^2$ is the 4D Minkowski metric ($= dt^2 - dr^2 - r^2 d\Omega^2$). As can be seen, the 4D part of this metric is conformally flat, while the 5-5 portion of the metric possesses a different factor.

From this metric, the nonzero 5D Ricci tensors can be calculated as:

$$\begin{aligned} \hat{R}_{00} = & \left[2\overset{\prime\prime}{U} + 2\overset{\prime}{U}^{2} + 4\frac{\overset{\prime}{U}}{r} + \overset{\prime}{U}\overset{\prime}{V}\right]/4 \\ & - \left[6\overset{\circ\circ}{U} + 2\overset{\circ\circ}{V} - \overset{\circ}{U}\overset{\circ}{V} + \overset{\circ}{V}^{2}\right]/4 \\ & - \left[4\overset{*}{U}^{2} - \overset{*}{U}\overset{*}{V} + 2\overset{**}{U}\right] e^{U(t,r,\psi)-V(t,r,\psi)}/(4\epsilon) \end{aligned} \tag{C.65}$$

$$\begin{aligned} \hat{R}_{11} = & -\left[6\overset{\prime\prime}{U} + 2\overset{\prime\prime}{V} + \overset{\prime}{V}^{2} + 4\frac{\overset{\prime}{U}}{r} - \overset{\prime}{U}\overset{\prime}{V}\right]/4 \\ & + \left[2\overset{\circ\circ}{U} + \overset{\circ}{U}\overset{\circ}{V} + 2\overset{\circ}{U}^{2}\right]/4 \\ & + \left[4\overset{*}{U}^{2} - \overset{*}{U}\overset{*}{V} + 2\overset{**}{U}\right] e^{U(t,r,\psi)-V(t,r,\psi)}/(4\epsilon) \end{aligned} \tag{C.66}$$

$$\begin{aligned} \hat{R}_{22} = & -r^2\left[2\overset{\prime\prime}{U} + 2\overset{\prime}{U}^{2} + 8\frac{\overset{\prime}{U}}{r} + 2\frac{\overset{\prime}{V}}{r} + \overset{\prime}{U}\overset{\prime}{V}\right]/4 \\ & + r^2\left[2\overset{\circ\circ}{U} + \overset{\circ}{U}\overset{\circ}{V} + 2\overset{\circ}{U}^{2}\right]/4 \\ & + r^2\left[4\overset{*}{U}^{2} - \overset{*}{U}\overset{*}{V} + 2\overset{**}{U}\right] e^{U(t,r,\psi)-V(t,r,\psi)}/(4\epsilon) \end{aligned} \tag{C.67}$$

$$
\begin{aligned}
\hat{R}_{55} = & \left[2\overset{\prime\prime}{V} + \overset{\prime}{V}^{2} + 4\frac{\overset{\prime}{V}}{r} + 2\overset{\prime}{U}\overset{\prime}{V}\right] e^{V(t,r,\psi)-U(t,r,\psi)}/4 \\
& - \left[2\overset{\circ\circ}{V} + 2\overset{\circ}{U}\overset{\circ}{V} + \overset{\circ}{V}^{2}\right] e^{V(t,r,\psi)-U(t,r,\psi)}/4 \\
& - \left[4\overset{*}{U}^{2} - 4\overset{*}{U}\overset{*}{V} + 8\overset{**}{U}\right]/4
\end{aligned}
\qquad \text{(C.68)}
$$

$$
\hat{R}_{01} = \frac{1}{4}\left[-4\overset{\circ\prime}{U} + \overset{\circ}{U}\overset{\prime}{U} - 2\overset{\circ\prime}{V} - \overset{\circ}{V}\overset{\prime}{V} + \overset{\circ}{U}\overset{\prime}{V} + \overset{\prime}{U}\overset{\circ}{V}\right] \qquad \text{(C.69)}
$$

$$
\hat{R}_{15} = -\frac{3}{2}\left[\overset{\prime *}{U} - \frac{1}{2}\overset{*}{U}\overset{\prime}{V}\right] \qquad \text{(C.70)}
$$

$$
\hat{R}_{50} = -\frac{3}{2}\left[\overset{*\circ}{U} - \frac{1}{2}\overset{*}{U}\overset{\circ}{V}\right] \qquad \text{(C.71)}
$$

where, again, overprimes denote partial derivatives with respect to r, overstars denote partial derivatives with respect to ψ, and over*circles* denote partial derivatives with respect to t. The only other nonzero Ricci tensor is $\hat{R}_{\phi\phi}$ which can be given in terms of $\hat{R}_{\theta\theta}$ as: $\hat{R}_{\phi\phi} = \hat{R}_{\theta\theta}\sin^2\theta$.

Equating these Ricci tensors to zero then allows eqs. C.70 and C.71 to be integrated immediately, which gives:

$$
\overset{*}{U} = e^{V/2+c_{\psi}} \qquad \text{(C.72)}
$$

where c_{ψ} is an integration "constant", independent of r or t, but possibly dependent on ψ.

Unfortunately, there does not appear to be much more one can do with the equations at this point; no more *obvious* solutions exist, and substituting this expression for U into the rest of the field equations does *not* yield readily solvable results (it merely replaces equations in U and V with equations in c and V).

Instead, we note that, as discussed in chapter 5 and in the preceding section of this appendix, *Riemann flatness* is a characteristic feature of 4D cosmological solutions embedded in 5D. As a result, we might wish to examine *Riemann* tensors with an eye to making them vanish. Specifically, we examine one Riemann tensor in particular;

$$\hat{R}_{1220} = -\frac{1}{2}\left[\overset{\circ\prime}{U} - \frac{1}{2}\,\overset{\circ}{U}\overset{\prime}{U}\right] e^{U(t,r,\psi)} r^2 \tag{C.73}$$

If *this* is presumed to vanish, then one can immediately integrate it like eqs. C.70 and C.71. However, it should be stressed that there is no *necessary* reason to assume that this, or any other Riemann tensor vanishes, save for the argument outlined above. It merely affords a mathematical *convenience* which allows further calculation of (*possibly*) cosmological-type solutions, and is reasonable in light that we seek cosmological solutions. [Note: *No other* Riemann tensors are assumed to vanish; only this one. Interestingly, however, the resulting solution *does* turn out to be Riemann flat, so *all* Riemann tensors *do* end up vanishing.]

The integration of eq. C.73 then yields either:

$$\overset{\circ}{U} = e^{U/2 + c_{t\psi}} \tag{C.74}$$

or:

$$\overset{\prime}{U} = e^{U/2 + c_{r\psi}} \tag{C.75}$$

where $c_{t\psi}$ and $c_{r\psi}$ are *new* integration "constants", independent on r and t, respectively (but possibly dependent on t and ψ, and r and ψ, respectively). Care must be taken, however, *not to confuse $c_{t\psi}$ and $c_{r\psi}$ with c_ψ, nor with each other as they are all different, as indicated by their subscripts.*

Integrating eqs. C.74 and C.75, one then obtains:

$$\begin{aligned} e^{-U/2}dU &= e^{c_{t\psi}}dt \rightarrow -2e^{-U/2} = \int e^{c_{t\psi}}dt \Rightarrow -2(a_{t\psi} + a_{r\psi}) \\ e^{-U/2}dU &= e^{c_{r\psi}}dr \rightarrow -2e^{-U/2} = \int e^{c_{r\psi}}dr \Rightarrow -2(b_{r\psi} + b_{t\psi}) \end{aligned} \tag{C.76}$$

where $a_{t\psi}$, $a_{r\psi}$, $b_{t\psi}$ and $b_{r\psi}$ are *new* functions, each *different* from each other, resulting from the integrations of $e^{c_{t\psi}}dt$ and $e^{c_{r\psi}}dt$, respectively. $a_{t\psi}$, $a_{r\psi}$, $b_{t\psi}$ and $b_{r\psi}$ depend on, respectively, t and ψ, r and ψ, t and ψ, and r and ψ.

In order that these two expressions match, we require: $a_{t\psi} = b_{t\psi}$ and $a_{r\psi} = b_{r\psi}$, which then yields:

$$e^{U} = (a_{t\psi} + a_{r\psi})^{-2} \tag{C.77}$$

Substituting this expression into eq. C.72 then yields:

$$e^{V/2} = \frac{\sqrt{a_{\psi}}\left(\overset{*}{a}_{t\psi} + \overset{*}{a}_{r\psi}\right)}{(a_{t\psi} + a_{r\psi})} \tag{C.78}$$

where a_{ψ} is a *new* function, dependent on ψ, equal to $-2e^{-c_{\psi}}$, and *not* to be confused with the other a's.

Thus, one can write:

$$e^{V} = \frac{a_{\psi}\left(\overset{*}{a}_{t\psi} + \overset{*}{a}_{r\psi}\right)^2}{(a_{t\psi} + a_{r\psi})^2} \tag{C.79}$$

However, the factor e^{V} appears as the metric coefficient to $d\psi^2$, and a_{ψ} depends *solely* on ψ. Therefore one can transform this factor in the metric as:

$$\sqrt{a_{\psi}}d\psi \rightarrow d\psi \tag{C.80}$$

so that the *effect* 5-5 component of the metric becomes:

$$e^V = \frac{\left(\overset{*}{a}_{t\psi} + \overset{*}{a}_{r\psi}\right)^2}{\left(a_{t\psi} + a_{r\psi}\right)^2} \tag{C.81}$$

Thus, we can more concisely write:

$$e^U = \frac{1}{\mathbf{A}^2_{tr\psi}} \tag{C.82}$$

$$e^V = \frac{\overset{*}{\mathbf{A}}{}^2_{tr\psi}}{\mathbf{A}^2_{tr\psi}} \tag{C.83}$$

where $\mathbf{A}_{tr\psi}$ is a *new* function defined by:

$$\mathbf{A}_{tr\psi} \equiv a_{t\psi} + a_{r\psi} \tag{C.84}$$

When one considers eq. C.69 in the light of the assumed vanishing of eq. C.73, one can then say:

$$-2\overset{\circ\prime}{V} - \overset{\circ}{V}\overset{\prime}{V} + \overset{\circ}{U}\overset{\prime}{V} + \overset{\prime}{U}\overset{\circ}{V} = 0 \tag{C.85}$$

Surprisingly, this (portion of eq. C.69) is *automatically satisfied* from the expressions for U and V as given by eqs. C.82 C.83. This kind of 'redundancy' in the field equations is very similar to the kind which the Bianchi identities impose (ref. [31]), and *suggests* that requiring $\hat{R}_{1220} = 0$ is quite reasonable in this context.

If one sets eq. C.66=eq. C.67(= 0), one can write:

$$6\overset{\prime\prime}{U} + 2\overset{\prime\prime}{V} + \overset{\prime}{V}{}^2 + 4\frac{\overset{\prime}{U}}{r} - \overset{\prime}{U}\overset{\prime}{V} = 2\overset{\prime\prime}{U} + 2\overset{\prime}{U}{}^2 + 8\frac{\overset{\prime}{U}}{r} + 2\frac{\overset{\prime}{V}}{r} + \overset{\prime}{U}\overset{\prime}{V} \tag{C.86}$$

or:

$$+4\frac{\overset{'}{U}}{r} - 4\overset{''}{U} + 2\overset{'}{U}^{2} + 2\overset{'}{U}\overset{'}{V} + 2\frac{\overset{'}{V}}{r} - 2\overset{''}{V} - \overset{'}{V}^{2} = 0 \tag{C.87}$$

Substituting the expressions for U and V (and $A_{tr\psi}$) into this expression then yields:

$$\frac{3(\overset{''}{\mathbf{A}} - \overset{'}{\mathbf{A}}/r)}{\mathbf{A}} - \frac{(\overset{''*}{\mathbf{A}} - \overset{'*}{\mathbf{A}}/r)}{\mathbf{A}} = 0 \tag{C.88}$$

If we introduce a new function $B_{r\psi} \equiv \overset{''}{\mathbf{A}} - \overset{'}{\mathbf{A}}/r$, which *must* be a function of r and ψ *only* (which can deduced from noting that the function $\mathbf{A}_{tr\psi}$ is a sum of two parts, $a_{t\psi} + a_{r\psi}$, and when taking the derivative of $\mathbf{A}_{tr\psi}$ w.r.t r, only $a_{r\psi}$ will remain), we can then rewrite this expression as:

$$\frac{3B}{\mathbf{A}} = \frac{\overset{*}{B}}{\overset{*}{\mathbf{A}}} \tag{C.89}$$

which can then integrated to give:

$$\mathbf{A}^3_{tr\psi} C^3_{tr} = B_{r\psi} \tag{C.90}$$

where C_{tr} is an arbitrary constant in ψ, but a possible function of t and r. Examining the (cube-root) of this expression;

$$(a_{t\psi} + a_{r\psi})C_{tr} = B^{1/3}_{r\psi} \tag{C.91}$$

we find the right-hand side (of eq. C.91) to be independent of t. It is necessary, therefore, that the left-hand side be likewise independent of t. This yields three possibilities: Either (i) the t-dependence of C_{tr} cancels out with the t-dependence of $\mathbf{A}_{tr\psi}$, (ii) there *is no* t-dependence in $\mathbf{A}_{tr\psi}$ and C_{tr}, or (iii) $\mathbf{A}_{tr\psi}$ and/or C_{tr} are zero, along with $B_{r\psi}$.

For case (i), since one can write: $C_{tr} = B_{r\psi}^{1/3}/\mathbf{A}_{tr\psi}$, and since C_{tr} is *independent* of ψ, while $\mathbf{A}_{tr\psi}$ and $B_{r\psi}$ have potential dependencies on ψ, then, *assuming that* $\mathbf{A}_{tr\psi}$ *and* $B_{r\psi}$ *have nonvanishing dependencies on* ψ, it is necessary that the ψ-dependencies of $\mathbf{A}_{tr\psi}$ and $B_{r\psi}$ be *factorable*, and that C_{tr} cancel out with the remaining (tr-dependent) portions of $\mathbf{A}_{tr\psi}$.

(If $\mathbf{A}_{tr\psi}$ has *no* dependency on ψ, then, by eqs. C.82 and C.83, neither would e^U or e^V; $\overset{*}{U}=\overset{*}{V}=0$. This, however, would cause $\hat{g}_{55} = 0$, yielding infinites in the inverse metric.)

Writing out the a terms ($a_{t\psi}$ and $a_{r\psi}$) of $\mathbf{A}_{tr\psi}$ with the ψ-portions factorable gives:

$$
\begin{aligned}
a_{t\psi} &= b_t b_\psi \\
a_{r\psi} &= b_r b_\psi
\end{aligned}
\tag{C.92}
$$

where the b's are new functions, dependent on, respectively, t, ψ, r, and ψ (and where care must, again, be taken *not* to confuse these b's with the b terms *previously* defined in this section).

This then gives, for $\mathbf{A}_{tr\psi}$;

$$\mathbf{A}_{tr\psi} = b_\psi(b_t + b_r) \tag{C.93}$$

with the expression for C_{tr} being given by:

$$C_{tr} = (b_t + b_r) \tag{C.94}$$

However, the expression for $\mathbf{A}_{tr\psi}$, eq. C.93, then yields, for e^V;

$$e^V = \frac{\overset{*}{b}{}_\psi^2 \,(b_t + b_r)^2}{b_\psi^2 (b_t + b_r)^2} = \frac{\overset{*}{b}{}_\psi^2}{b_\psi^2} \tag{C.95}$$

which is a *sole* function of ψ. But because e^V is the coefficient of $d\psi^2$, this ψ-dependent expression can be absorbed into $d\psi^2$ by a coordinate transformation on ψ. One would then have $e^V = \pm 1$, and $\overset{\circ}{V} = V' = \overset{*}{V} = 0$, which would severely restrict the generality of the desired solution.

For case (ii), if $\mathbf{A}_{tr\psi}$ has *no* dependency on t, then, by eqs. C.82 and C.83, neither would e^U or e^V; $\overset{\circ}{U} = \overset{\circ}{V} = 0$. This, however, would, again, not be the fully t, r and ψ dependent metric desired.

Finally, in case (iii), we would assume $B_{r\psi} = 0$, and one of $\mathbf{A}_{tr\psi}$ or C_{tr} also be zero. Since $\mathbf{A}_{tr\psi} = 0$ would cause the metric *totally* to vanish, we assume that $C_{tr} = 0$. Letting $B_{r\psi}$ vanish;

$$B_{r\psi} \equiv \mathbf{A}'' - \mathbf{A}' / r = 0 \tag{C.96}$$

then allows immediate solution for $\mathbf{A}_{tr\psi}$, which can then be written as:

$$\mathbf{A}_{tr\psi} = a_{t\psi} - \alpha_\psi r^2 \tag{C.97}$$

where $-\alpha_\psi$ is an arbitrary function of ψ (taken to be *negative* for reasons of convenience which will become clear later).

If we equate eq. C.65=eq. C.67(= 0), we then get:

$$-4\frac{U'}{r} - 2\frac{V'}{r} + 2\,\overset{\circ}{U}{}^2 - 4\,\overset{\circ\circ}{U} + 2\,\overset{\circ}{U}\overset{\circ}{V} - \overset{\circ}{V}{}^2 - 2\,\overset{\circ\circ}{V} = 0 \tag{C.98}$$

Substituting the expressions for U and V, eqs. C.82 and C.83 (and $\mathbf{A}_{tr\psi}$, eq. C.84), into this then yields:

$$12\frac{\overset{\prime}{\mathbf{A}}/r}{\mathbf{A}}+12\frac{\overset{\circ\circ}{\mathbf{A}}}{\mathbf{A}}-4\frac{\overset{\prime *}{\mathbf{A}}/r}{\overset{*}{\mathbf{A}}}-4\frac{\overset{\circ\circ *}{\mathbf{A}}}{\overset{*}{\mathbf{A}}}=0 \tag{C.99}$$

If one then equates eq. C.65=eq. C.68(= 0), we get:

$$-2\overset{\prime\prime}{U}-2\overset{\prime}{U}^{2}-4\frac{\overset{\prime}{U}}{r}+\overset{\prime}{U}\overset{\prime}{V}+2\overset{\prime\prime}{V}-2\overset{\prime}{V}^{2}+4\frac{\overset{\prime}{V}}{r}+6\overset{\circ\circ}{U}-\overset{\circ}{U}\overset{\circ}{V}=0 \tag{C.100}$$

where use was made of the fact that $\overset{*}{U}\overset{*}{V}=2\overset{**}{U}$ (which can be deduced from the form of U and V given in eqs. C.82 and C.83).

Upon again substituting the expressions for U and V, eqs. C.82 and C.83 (and $\mathbf{A}_{tr\psi}$, eq. C.84), into eq. C.100, we get:

$$-12\frac{\overset{\prime}{\mathbf{A}}\overset{\prime *}{\mathbf{A}}}{\mathbf{A}\overset{*}{\mathbf{A}}}-12\frac{\overset{\circ\circ}{\mathbf{A}}}{\mathbf{A}}+12\frac{\overset{\circ}{\mathbf{A}}\overset{\circ *}{\mathbf{A}}}{\mathbf{A}\overset{*}{\mathbf{A}}}+12\frac{\overset{\prime *}{\mathbf{A}}/r}{\overset{*}{\mathbf{A}}}=0 \tag{C.101}$$

where use was made of the fact that $B_{r\psi}\equiv\overset{\prime\prime}{\mathbf{A}}-\overset{\prime}{\mathbf{A}}/r=0$.

By similar arguments as those in the vanishing of $B_{r\psi}\equiv\overset{\prime\prime}{\mathbf{A}}-\overset{\prime}{\mathbf{A}}/r$ discussed above, it can be seen, from eq. C.99, that $\overset{\circ\circ}{\mathbf{A}}+\overset{\prime}{\mathbf{A}}/r$ should also vanish. Therefore;

$$\overset{\circ\circ}{\mathbf{A}}=-\overset{\prime}{\mathbf{A}}/r=-\overset{\prime\prime}{\mathbf{A}} \tag{C.102}$$

where the last step was made by $B_{r\psi}\equiv\overset{\prime\prime}{\mathbf{A}}-\overset{\prime}{\mathbf{A}}/r=0$.

Integrating $\mathbf{A}_{tr\psi}$ from eq. C.102 (paying particular attention to the form $\mathbf{A}_{tr\psi}$ has been derived so far to have), then yields:

$$\mathbf{A}_{tr\psi} = \alpha_\psi t^2 + \beta_\psi t + \gamma_\psi - \alpha_\psi r^2 \tag{C.103}$$

where α_ψ, β_ψ and γ_ψ are arbitrary functions of ψ (and where it can, now, be seen why the factor of r^2 was chosen to be *negative*; so that the factor of t^2 would be *positive*).

Using this expression for $\mathbf{A}_{tr\psi}$ in eq. C.101 then yields, after much cancellation;

$$\beta \overset{*}{\beta} - 2(\alpha \overset{*}{\gamma} + \overset{*}{\alpha} \gamma) = 0 \tag{C.104}$$

which can then be integrated to give:

$$-2\alpha\gamma + \frac{1}{2}\beta^2 = k_1 \tag{C.105}$$

where k_1 is an arbitrary integration constant.

Finally, it becomes necessary to consider the last Ricci tensor equation, eq. C.68;

$$-2 \overset{\prime\prime}{V} - \overset{\prime}{V}^{2} - 4\frac{\overset{\prime}{U}}{r} - 2 \overset{\prime}{U}\overset{\prime}{V} + 2 \overset{\circ\circ}{V} + 2 \overset{\circ}{U}\overset{\circ}{V} + \overset{\circ}{V}^{2} + 16e^{U(t,r,\psi)}/\epsilon = 0 \tag{C.106}$$

where use was made of the fact that $\overset{*}{U}\overset{*}{V} = 2 \overset{**}{U}$ (see previous note) so that the ψ-derivative terms in eq. C.68 can be rendered:

$$\left[4 \overset{*}{U}^{2} - 4 \overset{*}{U}\overset{*}{V} + 8 \overset{**}{U}\right] e^{U(t,r,\psi)-V(t,r,\psi)} = \left[4 \overset{*}{U}^{2}\right] e^{U(t,r,\psi)} \frac{4}{\overset{*}{U}^{2}} = 16e^{U(t,r,\psi)} \tag{C.107}$$

Upon substituting the most recent expression for $\mathbf{A}_{tr\psi}$ (eq. C.103) into eq. C.107, one finds, after much algebra;

$$\left[(-32\alpha\gamma + 16\beta^2 + 16/\epsilon)\stackrel{*}{\alpha} + (-16\alpha\beta)\stackrel{*}{\beta} + (32\alpha^2)\stackrel{*}{\gamma}\right](t^2 - r^2) + \left[(32\beta\gamma)\stackrel{*}{\alpha} + (-64\alpha\gamma + 16/\epsilon)\stackrel{*}{\beta} + (32\alpha\beta)\stackrel{*}{\gamma}\right](t) + \left[(32\gamma^2)\stackrel{*}{\alpha} + (-16\beta\gamma)\stackrel{*}{\beta} + (-32\alpha\gamma + 16\beta^2 + 16/\epsilon)\stackrel{*}{\gamma}\right] = 0 \quad \text{(C.108)}$$

Since each of these [square-bracketed] factors are independent of t or r, then they must each vanish independently;

$$\left[(-32\alpha\gamma + 16\beta^2 + 16/\epsilon)\stackrel{*}{\alpha} + (-16\alpha\beta)\stackrel{*}{\beta} + (32\alpha^2)\stackrel{*}{\gamma}\right] = 0 \quad \text{(C.109)}$$

$$\left[(32\beta\gamma)\stackrel{*}{\alpha} + (-64\alpha\gamma + 16/\epsilon)\stackrel{*}{\beta} + (32\alpha\beta)\stackrel{*}{\gamma}\right] = 0 \quad \text{(C.110)}$$

$$\left[(32\gamma^2)\stackrel{*}{\alpha} + (-16\beta\gamma)\stackrel{*}{\beta} + (-32\alpha\gamma + 16\beta^2 + 16/\epsilon)\stackrel{*}{\gamma}\right] = 0 \quad \text{(C.111)}$$

Each of these expressions can be integrated to then give:

$$-8\alpha\beta^2 + 32\alpha^2\gamma - 8\alpha/\epsilon + k_2\alpha^3 = 0 \quad \text{(C.112)}$$

$$-32\alpha\beta\gamma + 8\beta/\epsilon + k_3\beta^3 = 0 \quad \text{(C.113)}$$

$$8\beta^2\gamma - 32\alpha\gamma^2 + 8\gamma/\epsilon + k_4\gamma^3 = 0 \quad \text{(C.114)}$$

where k_2, k_3 and k_4 are arbitrary integration constants.

These three equations, eqs. C.112, C.113 and C.114, along with eq. C.105, constrain the quantities α_ψ, β_ψ and γ_ψ sufficiently that the metric C.64 will satisfy the vacuum field equations.

If one assumes that neither α_ψ nor γ_ψ are zero, then eqs. C.112 and C.114 can be rewritten as:

$$-8\beta^2 + 32\alpha\gamma - 8/\epsilon + k_2\alpha^2 = 0 \tag{C.115}$$

$$8\beta^2 - 32\alpha\gamma + 8/\epsilon + k_4\gamma^2 = 0 \tag{C.116}$$

This is reasonable since, if either α_ψ or γ_ψ were zero, then, by eq. C.105, β_ψ would have to be a constant, which would remove all the ψ-dependence of the metric (and make $\hat{g}_{55} = 0$).

Substituting eq. C.105 into these two relations, eqs. C.115 and C.116, one obtains:

$$-16k_1 - 8/\epsilon + k_2\alpha^2 = 0 \tag{C.117}$$

$$16k_1 + 8/\epsilon + k_4\gamma^2 = 0 \tag{C.118}$$

Clearly, one might infer from both of these equations that both α_ψ and γ_ψ were *constant*, since the other terms in each equation are constant. However, recalling that both k_2 and k_4 are completely arbitrary, they can both be set to zero. The remainder of eqs. C.117 and C.118 then yield:

$$k_1 = -\frac{1}{2\epsilon} \tag{C.119}$$

This renders eq. C.105 as:

$$\beta_\psi^2 - 4\alpha_\psi\gamma_\psi = -\frac{1}{\epsilon} \tag{C.120}$$

which equals 1 for $\epsilon = -1$.

One can then see, by multiplying the constraint C.120 by 8β and comparing it with eq. C.113 (the final equation) that, in order for it be satisfied, that k_3 must equal 8.

Therefore, the metric C.64 with **A** defined by eq. C.103 and α_ψ, β_ψ and γ_ψ constrained by eq. C.105 then constitutes the solution for the 4D conformally flat metric. It has been verified by GRTensor II (ref. [1]), which also verified that the solution (as general as it is) is Riemann flat, *independent* of the choices of α_ψ, β_ψ and γ_ψ. As such it is the most general 4D conformally flat solution (in a 5D manifold) which is also Riemann flat (actually, with only the Riemann tensor $\hat{R}_{1220}$ being explicitly assumed to be zero).

Appendix D

Computer Subroutine: Maxwell's Equations

Shown in this appendix is a computer algorithm which can be used to calculate Maxwell's Equations ($F_{\mu\nu}{}^{;\nu} = J_\mu$) for this thesis. In is used with GRTensor II in Maple:

```
>with(linalg):readlib(grii):grtensor():
>grload(metric,'metric address'):
>q[1]:=r;q[2]:=theta;q[3]:=phi;q[4]:=t;q[5]:=psi;
>for i1 from 1 to 4 do for j1 from 1 to 4 do
 g4[i1,j1]:=simplify(grcomponent(metric,[i1,j1])
                        -grcomponent(metric,[i1,5])*grcomponent(metric,[5,j1])
                        /grcomponent(metric,[5,5]))
 od od;
>g4D:=array([[g4[1,1],g4[1,2],g4[1,3],g4[1,4]],
```

```
              [g4[2,1],g4[2,2],g4[2,3],g4[2,4]],
              [g4[3,1],g4[3,2],g4[3,3],g4[3,4]],
              [g4[4,1],g4[4,2],g4[4,3],g4[4,4]]]);
 h4D:=inverse(g4D);
>for i2 from 1 to 4 do for j2 from 1 to 4 do for k2 from 1 to 4 do
 Ch[i2,j2,k2]:=(1/2)*sum('h4D[k2,m2]*(diff(g4D[m2,i2],q[j2])
                                     +diff(g4D[m2,j2],q[i2])
                                     -diff(g4D[i2,j2],q[m2]))',m2=1..4)
 od od od;
>for i3 from 1 to 4 do
 A[i3]:=simplify(grcomponent(metric,[5,i3])/grcomponent(metric,[5,5]))
 od;
 Phi:=(grcomponent(metric[5,5])/epsilon)^(1/2);
>for i4 from 1 to 4 do for j4 from 1 to 4 do
 F[i4,j4]:=simplify(diff(A[j4],q[i4])-diff(A[i4],q[j4]))
 od od;
>for i5 from 1 to 4 do for j5 fromj 1 to 4 do for k5 from 1 to 4 do
 term[k5,j5,i5]:=h4D[i5,j5]*(diff(F[k5,j5],q[i5])
                             -sum('Ch[k5,i5,m5]*F[m5,j5]',m5=1..4)
                             -sum('Ch[j5,i5,n5]*F[k5,n5]',n5=1..4))
 od od od;
>for i6 from 1 to 4 do
 J[i6]:=simplify(sum('term[i6,s6,s6]',s6=1..4))
 od;
```

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