Operator Theoretic Methods in Nevanlinna-Pick Interpolation

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

This Master's thesis will develops a modern approach to complex interpolation problems studied by Carathéodory, Nevanlinna, Pick, and Schur in the early 20^{th} century. The fundamental problem to solve is as follows: given complex numbers $z_1, z_2, ..., z_N$ of modulus at most 1 and $w_1, w_2, ..., w_N$ additional complex numbers, what is a necessary and sufficiency condition for the existence of an analytic function $f : \mathbb{D} \to \mathbb{C}$ satisfying $f(z_i) = w_i$ for $1 \leq i \leq N$ and $|f(z)| \leq 1$ for each $z \in \mathbb{D}$? The key idea is to realize bounded, analytic functions (the algebra H^∞) as the *multiplier algebra* of the Hardy class of analytic functions, and apply dilation theory to this algebra.

This operator theoretic approach may then be applied to a wider class of interpolation problems, as well as their matrix-valued equivalents. This also yields a fundamental distance formula for H^∞, which provides motivation for the study of completely isometric representations of certain quotient algebras.

Our attention is then turned to a related interpolation problem. Here we require the interpolating function f to satisfy the additional property $f'(0) = 0$. When $z_i = 0$ for some i, we arrive at a special case of a problem class studied previously. However, when 0 is not in the interpolating set, a significant degree of complexity is inherited. The dilation theoretic approach employed previously is not effective in this case. A more function theoretic viewpoint is required, with the proof of the main interpolation theorem following from a factorization lemma for the Hardy class of analytic functions. We then apply the theory of completely isometric maps to show that matrix interpolation fails when one imposes this constraint.

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Chapter 1

Introduction

Given N points in the complex disk and N additional points in the complex plane, under what conditions does there exist a bounded, analytic function that interpolates the two data sets? This question was answered affirmatively by Pick in 1916 [**Pic16**] and independently by Nevalinna in 1919 [**Nev19**]. Let $\mathcal{M}_N$ denote the algebra of N by N complex-valued matrices.

Theorem 1.0.1 (**Pick**). *Suppose $z_1, ..., z_N \in \mathbb{D}$ and $W_1, ..., W_N$ are $n \times n$ matrices. There is a bounded, analytic function $F : \mathbb{D} \to \mathcal{M}_{n\times n}$ such that $\|F\|_\infty := \sup_{\lambda\in\mathbb{D}} \|F(\lambda)\|_{\mathcal{M}_n} \leq 1$ and $F(z_i) = W_i$, for each i, if and only if the Pick matrix*

$$\left[\frac{I_n - W_i W_j^*}{1 - z_i \overline{z_j}}\right]_{i,j=1}^N \tag{1.1}$$

has a real, non-negative spectrum.

Matrices and operators with real, non-negative spectra will be refered to as *positive* or *positive semidefinite* operators. If the Pick matrix is positive, parameterizing the solutions is referred to as the Nevanlinna problem. For this reason, this area of mathematics is called Nevanlinna-Pick interpolation. In Chapter 2, we will prove Theorem 1.0.1 using Foias and Sz. Nagy's *commutant lifting theorem* [**FSN68**]. Sarason proved Pick's theorem this way in his seminal paper on the subject [**Sar67**] where he proved the commutant lifting theorem in a specific context. We will develop the commutant lifting approach in its full generality, and then apply it to a related interpolation problem known as the Carathéodory problem.

In Chapter 3, the Pick problem will be re-posed with a derivative constraint and the corresponding interpolation results will be proven. Even the simplest of constraints increases the relative complexity of these problems significantly, which will be the central topic of Chapter 4. Here we will examine the completely isometric representation theory of certain quotient algebras associated to interpolation problems .

The dimension and complexity of these representations often correspond to the tractability of an interpolation problem. The study of these representations is often called the C^* approach to interpolation, as it makes use of an enormously powerful tool in operator space theory known as the C^*-envelope. We will compute C^*-envelopes associated with several interpolation problems and show how even the slightest variation of Pick's original problem can lead to large and surprising C^*-envelopes. Presently, we devote the remainder of this chapter to developing the relevant terminology and tools used in Nevanlinna-Pick interpolation.

1.1 Hilbert Function Spaces

Definition 1.1.1. A *Hilbert function space* is a Hilbert space $\mathcal{H}$ consisting of complex-valued functions on a set X such that point evaluations at each point of X is a non-zero, bounded linear functional.

As a convention, we say that a function $f : X \to \mathbb{C}$ is identically 0 if $f(\lambda) = 0$ for each $\lambda \in X$.

Example 1.1.2. The sequence space ℓ^2 is trivially a Hilbert function space, since for $\mathbf{x} = \{x_i\}_{i\geq 0} \in \ell_2$ we have $|x_i| \leq \|\mathbf{x}\|$ for each i.

Example 1.1.3. The Hardy space H^2: the set of all analytic functions on the open complex disk $\mathbb{D}$ such that $\sum_{k=0}^{\infty} |\hat{f}(n)|^2 < \infty$ where $\hat{f}(n)$ is the n^{th} Taylor coefficient of f at zero. The Hardy space is the most important Hilbert function space; the last two sections of this chapter will be devoted to its properties.

Example 1.1.4. We define a family of Hilbert function spaces of analytic functions on the disk whose members satisfy

$$\|f\|_s = \sum_{n=0}^{\infty} (n+1)^{-s} |\hat{f}(n)|^2 < \infty$$

where $\hat{f}(n)$ is the n^{th} Taylor coefficient of f at 0. This induces an inner product given by

$$\langle f, g\rangle = \sum_{n=0}^{\infty} (n+1)^{-s} \hat{f}(n)\overline{\hat{g}(n)}.$$

Powers of z form an orthogonal basis for these spaces. When $s = 0$ we get the Hardy space. When $s = 1$ we have the *Bergman space* and when $s = -1$ we have the *Dirichlet* space.

Example 1.1.5. $L^2[0, 1]$ is not a Hilbert function space. To see this, suppose evaluation at 1 was a bounded linear functional, say δ_1. Then $1 = \delta_1(x^n) \leq \|\delta_1\|\|x^n\| = \|\delta_1\|\frac{1}{2n+1}$

for each n, which is impossible. This, in part, tells us that it is essential to view Hilbert function spaces as spaces of concrete functions rather than equivalence classes of functions.

Let $\mathcal{H}$ be a Hilbert function space on X and let $\lambda \in X$. Since evaluation at λ is continuous, the Riesz representation theorem implies the existence of $k_\lambda \in \mathcal{H}$ such that $f(\lambda) = \langle f, k_\lambda \rangle$. Since k_λ is itself a function, we have $k_\lambda(\zeta) = \langle k_\lambda, k_\zeta \rangle$. Let $k(\zeta, \lambda) := \langle k_\lambda, k_\zeta \rangle$ and call it the *kernel function* for $\mathcal{H}$. Given an orthonormal basis $\{e_i\}_{i \in \mathcal{I}}$ of $\mathcal{H}$, Parseval's formula gives:

$$k(\zeta, \lambda) = \langle k_\lambda, k_\zeta \rangle = \sum_{i \in \mathcal{I}} \langle k_\lambda, e_i \rangle \langle e_i, k_\zeta \rangle = \sum_{i \in \mathcal{I}} \overline{e_i(\lambda)} e_i(\zeta) \tag{1.2}$$

Using this, we may explicitly compute kernel functions for given spaces. In the case of the Hardy space, its kernel function is given by

$$k_{H^2}(z, w) = \sum_{k=1}^{\infty} z^k \overline{w}^k = \frac{1}{1 - \overline{w} z}$$

We call k_{H^2} the *Szegö kernel*.

The kernel functions k_λ have a dense linear span in a Hilbert function space since $f = 0$ if and only if $\langle f, k_\lambda \rangle = f(\lambda) = 0$ for each $\lambda \in X$. The next theorem shows that given a kernel k on some set X, we may construct one and only one Hilbert function space on X.

Theorem 1.1.6. *Let X be a set. There is a bijective correspondence between Hilbert function spaces on X and kernels on X.*

Proof. Suppose k is a kernel on X and let $\mathcal{V}$ be the vector space of finite linear combinations of the functions $k_\lambda(x) := k(x, \lambda)$ for $\lambda, x \in X$. Define a sesqui-linear form on $\mathcal{V}$ by

$$\left\langle \sum_i a_i k_{\lambda_i}, \sum_j b_j k_{\delta_j} \right\rangle = \sum_{i,j} a_i \overline{b_j} k(\delta_j, \lambda_i)$$

Let $\mathcal{N} = \{v \in \mathcal{V} : \langle v, v \rangle = 0\}$. It is easily seen that $\mathcal{N}$ is a closed subspace of $\mathcal{V}$ and $\langle v, \eta \rangle = 0$ for $v \in \mathcal{V}$ and $\eta \in \mathcal{N}$. Let $\mathcal{H}$ be the completion of the pre-Hilbert space $\mathcal{V}/\mathcal{N}$. We claim that $\mathcal{H}$ is a Hilbert function space with kernel k. Let $f \in \mathcal{H}$ and set

$$f(\lambda) := \langle f, k_\lambda + \mathcal{N} \rangle_{\mathcal{H}}.$$

The above equation realizes every element of $\mathcal{H}$ as a function on X and the equivalence class of k_λ is the reproducing kernel for λ. Since

$$\langle k_\lambda + \mathcal{N}, k_\delta + \mathcal{N} \rangle_{\mathcal{V}/\mathcal{N}} = \langle k_\lambda, k_\delta \rangle_{\mathcal{V}} = k(\delta, \lambda),$$

we see that the kernel function on $\mathcal{H}$ is precisely k.

If $\mathcal{H}$ and $\mathcal{K}$ are both Hilbert function spaces on X with kernel k, then the map $U : \mathcal{H} \to \mathcal{K}$ which sends $\sum_i a_i k_{\lambda_i}$ to $\sum_i a_i k_{\lambda_i}$ is an isometry on dense sets, and therefore extends to a unitary that satisfies $(Uf)(\lambda) = f(\lambda)$ for any $f \in \mathcal{H}$. $\square$

Let $\mathcal{H}_k$ denote the Hilbert function space $\mathcal{H}$ with the (unique) kernel function k.

1.2 Multiplier Algebras and Restatement of the Pick Theorem

For a Hilbert function space $\mathcal{H}_k$, define $\mathrm{Mult}(\mathcal{H}_k)$, the *multiplier algebra* of $\mathcal{H}_k$, as

$$\{\phi : \phi f \in \mathcal{H}_k, f \in \mathcal{H}_k\}$$

where each ϕ is understood to be a $\mathbb{C}$-valued function on X. Let M_ϕ denote the linear transformation that multiplies functions on $\mathcal{H}_k$ by ϕ. We claim that M_ϕ is a bounded operator on $\mathcal{H}_k$. To see this, suppose $f_n \in \mathcal{H}_k$ such that $f_n \to 0$, and $M_\phi f \to g \in \mathcal{H}_k$. Then

$$g(\lambda) = \langle g, k_\lambda \rangle = \lim_{n\to\infty} \langle \phi f_n, k_\lambda \rangle = \phi(\lambda) \langle \lim_{n\to\infty} f_n, k_\lambda \rangle = 0.$$

for each $\lambda \in X$. Thus, M_ϕ is bounded by the closed graph theorem. The multiplier algebra is an *operator algebra*: a subalgebra of the bounded operators on $\mathcal{H}_k$. The operator algebraic structure of the multiplier algebra will be the central topic in Chapter 4. The most remarkable and useful property of multiplication operators is the effect of their adjoint on kernel functions. For $\lambda \in X$ compute

$$\langle f, M_\phi^* k_\lambda \rangle = \langle \phi f, k_\lambda \rangle = \phi(\lambda) f(\lambda) = \langle f, \overline{\phi}(\lambda) k_\lambda \rangle.$$

Hence $M_\phi^* k_\lambda = \overline{\phi(\lambda)} k_\lambda$ for each λ, and k_λ is an eigenvector for M_ϕ^*. This gives a lower bound on the norm of M_ϕ

$$\|M_\phi\| \geq \sup_{\lambda \in X} |\phi(\lambda)|. \tag{1.3}$$

In particular, this inequality implies that the quantity on the right is always finite for any multiplier. Equation (1.3) becomes an equality for any Hilbert function space of the form $L^2(X, \mu)$ where μ is a measure on X. Indeed, for any function g in such a space, we have

$$\|M_\phi g\|_2 = \|\phi g\|_2 \leq \sup_{\lambda \in X} |\phi(\lambda)| \|g\|_2$$

Any operator that has each kernel function as an eigenvector is, surprisingly, the adjoint of a multiplication operator. Indeed, if R is a bounded operator on $\mathcal{H}_k$ with each k_λ as an eigenvector, define $\phi : X \to \mathbb{C}$ by

$$Rk_\lambda = \overline{\phi(\lambda)} k_\lambda.$$

Then for any $f \in \mathcal{H}_k$ we have

$$\langle R^* f, k_\lambda \rangle = \langle f, \overline{\phi(\lambda)} k_\lambda \rangle = \langle \phi(\lambda) f, k_\lambda \rangle = \langle M_\phi f, k_\lambda \rangle.$$

for all λ. Hence $R = M_\phi^*$ and ϕ is a multiplier on $\mathcal{H}_k$.

Given a Hilbert function space $\mathcal{H}_k$ and an n dimensional Hilbert space $\mathcal{K}$ (allowing the possibility that n is infinite), consider the Hilbert space $\mathcal{H}_k \otimes \mathcal{K}$. That is, the completion of the algebraic tensor product of $\mathcal{H}_k$ and $\mathcal{K}$ with respect to the unique inner product given by $\langle h_1 \otimes k_1, h_2 \otimes k_2 \rangle = \langle h_1, h_2 \rangle \langle k_1, k_2 \rangle$. We may think of this space as n-tuples of functions on X. If $F \in \mathcal{H}_k \otimes \mathcal{K}$, we may write it as the column vector $F = [f_i]_{i=1}^n$, where each $f_i \in \mathcal{H}_k$, and $\|F\|^2_{\mathcal{H}_k \otimes \mathcal{K}} = \sum_{i=1}^n \|f_i\|^2_{\mathcal{H}_k}$. We refer to this space as a *vector-valued Hilbert function space.*

Suppose $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces such that $S \in B(\mathcal{H})$ and $T \in B(\mathcal{K})$. Define the operator $S \otimes T$ on the elementary tensors of $\mathcal{H} \otimes \mathcal{K}$ by $(S \otimes T)(h \otimes k) = Sh \otimes Tk$. Extend $S \otimes T$ linearly and continuously to all of $\mathcal{H} \otimes \mathcal{K}$ and notice that $\|S \otimes T\| = \|S\| \|T\|$ (the so called *spatial* tensor product of two C^*- algebras). We shall make use of these types of operators shortly.

Definition 1.2.1. Let $\mathcal{H}_k \otimes \mathcal{K}$ be a vector-valued Hilbert function space. A *multiplier* of $\mathcal{H}_k \otimes \mathcal{K}$ is a function $\Phi : X \to B(\mathcal{K})$ such that $\Phi F \in \mathcal{H}_k \otimes \mathcal{K}$ for each $F \in \mathcal{H}_k \otimes \mathcal{K}$. The multiplication ΦF is defined pointwise as $\Phi F(\lambda) = \Phi(\lambda) F(\lambda)$ for $\lambda \in X$.

Let $\mathrm{Mult}(\mathcal{H}_k \otimes \mathcal{K})$ denote the algebra of these multipliers, and let M_Φ be the bounded operator that multiplies vectors in $\mathcal{H}_k \otimes \mathcal{K}$ by Φ. Just as in the scalar case, the action of adjoints of multipliers on kernel functions uniquely determines their action. For a vector-valued Hilbert function space $\mathcal{H}_k \otimes \mathcal{K}$, any function of the form $k_\lambda \otimes u$ is the correct analogue of a reproducing kernel at λ. Indeed, a simple vector calculation shows

$$\langle F, k_\lambda \otimes u \rangle_{\mathcal{H}_k \otimes \mathcal{K}} = \langle F(\lambda), u \rangle_{\mathcal{K}}. \tag{1.4}$$

Since the span of kernel functions is dense in $\mathcal{H}_k$, simple tensors of the form $k_\lambda \otimes u$ are dense in $\mathcal{H}_k \otimes \mathcal{K}$. We are now in a position to prove several fundamental facts about multipliers on $\mathcal{H}_k \otimes \mathcal{K}$ analogous to the scalar case. First note the action of M_Φ on kernel functions:

$$\begin{aligned} \langle F, M_\Phi^*(k_\lambda \otimes u) \rangle_{\mathcal{H}_k \otimes \mathcal{K}} &= \langle M_\Phi F, k_\lambda \otimes u \rangle_{\mathcal{H}_k \otimes \mathcal{K}} \\ &= \langle \Phi(\lambda) F(\lambda), u \rangle_{\mathcal{K}} \\ &= \langle F(\lambda), \Phi(\lambda)^* u \rangle_{\mathcal{K}} \\ &= \langle F, k_\lambda \otimes \Phi(\lambda)^* u \rangle_{\mathcal{H}_k \otimes \mathcal{K}} \end{aligned}$$

Hence

$$M_\Phi^*(k_\lambda \otimes u) = k_\lambda \otimes \Phi(\lambda)^* u. \tag{1.5}$$

The elementary tensors $k_\lambda \otimes u$ are dense in $\mathcal{H}_k \otimes \mathcal{K}$, so any multiplier Φ is completely determined by (1.5). Just as in the scalar case, any bounded operator on $\mathcal{H}_k \otimes \mathcal{K}$ satisfying (1.5) is actually a multiplier.

Proposition 1.2.2. *Suppose $\mathcal{H}_k \otimes \mathcal{K}$ is a vector-valued Hilbert function space and R is a bounded operator on $\mathcal{H}_k \otimes \mathcal{K}$ satisfying*

$$R(k_\lambda \otimes u) = k_\lambda \otimes W_\lambda u \tag{1.6}$$

for operators $W_\lambda \in B(\mathcal{K})$, $\lambda \in X$. Then $R = M_\Phi^*$ for some $\Phi \in \mathrm{Mult}(\mathcal{H}_k \otimes \mathcal{K})$.

Proof. Define $\Phi : X \to B(\mathcal{K})$ by $\Phi(\lambda) = W_\lambda^*$. By density, it suffices to show that for each $\lambda \in X$ and $u \in \mathcal{K}$ we have $\Phi(k_\lambda \otimes u) \in \mathcal{H}_k \otimes \mathcal{K}$. Reversing the calculation preceding Equation (1.5) shows this to be the case. □

An operator $T \in B(\mathcal{H}_k \otimes \mathcal{K})$ has norm at most 1 if and only if $I - TT^* \geq 0$, i.e., $\langle (I - TT^*)F, F\rangle \geq 0$ for all $F \in \mathcal{H}_k \otimes \mathcal{K}$. It suffices to check this condition for vectors in $\vee\{k_{\lambda_i} \otimes u : \lambda_i \in X, u \in \mathcal{K}\}$. For convenience write $k_i := k_{\lambda_i}$.

Lemma 1.2.3. *Suppose $\Phi \in \mathrm{Mult}(\mathcal{H}_k \otimes \mathcal{K})$, $\alpha_i \in \mathbb{C}$, and $\lambda_i \in X$. Then $\|M_\Phi\| \leq 1$ if and only if*

$$\sum_{i,j=1}^{N} \langle (I_{\mathcal{K}} - \Phi(\lambda_i)\Phi(\lambda_j)^*)u_i, u_j\rangle_{\mathcal{K}} \langle k_i, k_j\rangle_{\mathcal{H}_k} \geq 0 \tag{1.7}$$

for each $u_i \in \mathcal{K}$.

Proof. The norm of M_ϕ is at most 1 if and only if for all vectors of the form $\sum_{i=1}^N \alpha_i(k_i \otimes u_i)$, where the α_i are scalars, we have

$$\begin{aligned} 0 &\leq \left\langle (I - M_\phi M_\phi^*) \sum_{i=1}^N \alpha_i k_i \otimes u_i, \sum_{j=1}^N \alpha_j k_j \otimes u_j \right\rangle_{\mathcal{H}_k \otimes \mathcal{K}} \\ &= \sum_{i,j=1}^N \alpha_i \overline{\alpha_j} \langle (I - M_\Phi M_\Phi^*) k_i \otimes u_i, k_j \otimes u_j\rangle_{\mathcal{H}_k \otimes \mathcal{K}} \\ &= \sum_{i,j=1}^N \alpha_i \overline{\alpha_j} \langle (I_{\mathcal{K}} - \Phi(\lambda_i)\Phi(\lambda_j)^* u_i, u_j\rangle_{\mathcal{K}} \langle k_i, k_j\rangle_{\mathcal{H}_k}. \end{aligned}$$

The last equality follows from Equation (1.6). Since the α_i and u_i are arbitrary, we have the result. □

We are now able to prove the necessity of the positivity of the Pick matrix for vector-valued Hilbert function spaces. The next chapter will develop the tools to prove sufficiency in the case of the Szegö kernel.

Theorem 1.2.4. *Let $\mathcal{H}_k \otimes \mathcal{K}$ be a vector-valued Hilbert function space on a set X. Suppose $\lambda_1, ..., \lambda_N \in X$ and $W_1, .., W_N \in B(\mathcal{K})$. If there is a function $\Phi \in \mathrm{Mult}(\mathcal{H}_k \otimes \mathcal{K})$ such that $\|M_\Phi\| \leq 1$ and $\Phi(\lambda_i) = W_i$ for each i then the Pick matrix*

$$\left[(I_{\mathcal{K}} - W_i W_j^*)\langle k_i, k_j\rangle\right]_{i,j=1}^{N} \tag{1.8}$$

is positive semidefinite.

Proof. Suppose such a $\Phi \in \mathrm{Mult}(\mathcal{H}_k \otimes \mathcal{K})$ exists. We have $\langle (I - M_\Phi M_\Phi^*)F, F\rangle_{\mathcal{H}_k \otimes \mathcal{K}} \geq 0$ for $F \in \vee_{i=1}^{n}\{k_i \otimes u_i\}$, where $u_i \in \mathcal{K}$ are arbitrary but fixed. Then

$$\sum_{i,j=1}^{N} \langle (I_{\mathcal{K}} - W_i W_j^*)u_i, u_j\rangle_{\mathcal{K}} \langle k_i, k_j\rangle_{\mathcal{H}_k} = \sum_{i,j=1}^{N} \langle (I_{\mathcal{K}} - \Phi(\lambda_i)\Phi(\lambda_j)^*)u_i, u_j\rangle_{\mathcal{K}} \langle k_i, k_j\rangle_{\mathcal{H}_k} \geq 0.$$

By Lemma 1.2.3, this is precisely the positivity of the Pick matrix. □

Let $\mathcal{H}_k$ be a Hilbert function space and suppose $\mathcal{K} = \mathbb{C}^n$. If the converse to Theorem 1.2.4 holds, we say that k has the $n \times n$ *Pick property.* If k has the $n \times n$ Pick property for all n, we say that k has the *complete Pick property.* The original Pick theorem is equivalent to the Szegö kernel having the 1×1, or *scalar*, Pick property. The Szegö kernel has the complete Pick property, which we will prove in the following chapter.

It is worth noting that all of these theorems and their proofs hold *mutatis mutandis* for non-square operators. In this case let $\mathcal{H}_k \otimes \mathcal{K}_1$ and $\mathcal{H}_k \otimes \mathcal{K}_2$ be vector-valued Hilbert function spaces. A multiplier from $\mathcal{H}_k \otimes \mathcal{K}_1$ to $\mathcal{H}_k \otimes \mathcal{K}_2$ is a function $\Phi : X \to B(\mathcal{K}_1, \mathcal{K}_2)$ such that $\Phi F \in \mathcal{H}_k \otimes \mathcal{K}_2$ for all $F \in \mathcal{H}_k \otimes \mathcal{K}_1$. Let $\mathrm{Mult}(\mathcal{H}_k \otimes \mathcal{K}_1, \mathcal{H}_k \otimes \mathcal{K}_2)$ denote the set of such multipiers. The most important observation to make is that, in general, the set of multipliers is a subspace of $B(\mathcal{K}_1, \mathcal{K}_2)$ but *not* an algebra. For this reason, we restrict ourselves to the case where $\mathcal{K}_1 = \mathcal{K}_2$, as $\mathrm{Mult}(\mathcal{H}_k \otimes \mathcal{K}_1)$ is a subalgebra of $B(\mathcal{K}_1)$.

Finally, it is often convenient to identify $\mathcal{H}_k \otimes \mathcal{K}$ with $\mathcal{H}_k^{(n)} := \bigoplus_{i=1}^{n} \mathcal{H}_k$ when $\mathcal{K}$ is $n-$dimensional. With this notation in mind, we may identify $\mathrm{Mult}(\mathcal{H}_k \otimes \mathcal{K})$ with $\mathcal{M}_n(\mathrm{Mult}(\mathcal{H}_k))$: the $n \times n$ matrices with $\mathrm{Mult}(\mathcal{H}_k)$-valued entries. In this context, a multiplier acts on $\mathcal{H}_k \otimes \mathcal{K}$ by matrix multiplication.

1.3 Hardy Spaces

The goal of this section is to examine the basic functional theoretic and operator theoretic properties of the Hardy spaces. For our purposes, we will only be developing this theory to the extent that we require it in interpolation and most theorems will be stated without proof. The reader is directed to **[Hof62]** and **[Nik85]** for much more detailed treatments of this deep and remarkable theory.

Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ denote the complex unit circle. Suppose $f : \mathbb{D} \to \mathbb{C}$ and set

$f_r(e^{i\theta}) := f(re^{i\theta})$. Define the *mean of order p at radius r of f* as

$$M_p(r, f) := \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\right)^{1/p},$$

where $1 \le p < \infty$ and $0 < r < 1$. For $p = \infty$ we define it as

$$M_\infty(r, f) := \sup_{0\le\theta\le 2\pi} |f(re^{i\theta})|.$$

The *Hardy space*, H^p, is the Banach space of *analytic* functions on the disk satisfying:

$$H^p = \{f : \sup_{0<r<1} M_p(r, f) < \infty\}. \tag{1.9}$$

It is easily verified that $M_2(r, f)^2 = \sum_{n=0}^{\infty} r^{2n}|\hat{f}(n)|^2$ (use the orthonormality of the trigonometric polynomials on $L^2(\mathbb{T})$), which shows that the above norm on H^2 agrees with the one given in Example 1.1.3. Even though these norms are formally equivalent, it remains to be seen that an analytic function on $\mathbb{D}$ with square summable Taylor series may be identified with a function in $L^2(\mathbb{T})$. Indeed, it is true that functions in H^2 have radial limits almost everywhere, and one can identify H^2 functions on $\mathbb{D}$ with L^2 functions on $\mathbb{T}$ whose negative Fourier coefficients are equal to 0. We merely state the required results, as their proofs make heavy use of function theoretic tools not developed here.

Define the *Poisson kernel* as

$$P_r(e^{i\theta}) := \frac{1 - r^2}{1 + r^2 - 2r\cos\theta}.$$

If u is a harmonic function on $\mathbb{D}$ and continuous on $\mathbb{T}$, then

$$u(re^{i\theta}) = \int_{\mathbb{T}} u(e^{it})P_r(e^{i(\theta-t)})d\sigma(e^{it}),$$

where σ is the normalized Lebesgue measure on $\mathbb{T}$. The following proposition shows that u need not be continuous on $\mathbb{T}$, but rather has bounded means of order p.

Proposition 1.3.1. *Suppose $1 < p \le \infty$, u is a harmonic function on $\mathbb{D}$, and the set $\{M_p(r, u) : 0 < r < 1\}$ is bounded. Then there is a function $F \in L^p(\sigma)$ such that*

$$u(re^{i\theta}) = \int_{\mathbb{T}} F(e^{it})P_r(e^{i(\theta-t)})d\sigma(e^{it})$$

We say that u is the *Poisson integral* of F. For $F \in L^p(\sigma)$, let $\hat{F}(n) := \int_{\mathbb{T}} F(e^{i\theta})e^{-in\theta}d\sigma$ denote the n^{th} Fourier coefficient of F. These are not be confused with $\hat{f}(n)$ (the n^{th} Taylor coefficients of f about the point 0) though it turns out that these coefficients are the same for H^p functions. The next result is primarily due to Fatou, and establishes the required identification of H^p functions on the torus.

Theorem 1.3.2 (**Fatou**). *Let $1 \le p \le \infty$, $F \in L^p(\sigma)$, and let f be the Poisson integral of F. If $\hat{F}(n) = 0$ for all $n < 0$, then*

- *$f \in H^p$ and $\|f\|_{H^p} = \|F\|_{L^p}$,*
- *$\hat{f}(n) = \hat{F}(n)$ for $n \ge 0$,*
- *for $1 \le p < \infty$, $\|f_r - F\|_{L^p} \to 0$ as $r \nearrow 1$,*
- *and for $p = \infty$, f_r tends to F in the weak* topology on $L^\infty(\sigma)$.*

A few minor consequences on the above theorem are immediately evident. For $1 \le p < \infty$, each H^p is a closed subspace of $L^p(\sigma)$ and H^∞ is a weak* closed subalgebra of $L^\infty(\sigma)$. Moreover, the norm on H^2 is actually an L^2 norm, which implies that $\|M_f\| = \|f\|_\infty$ for any $f \in \mathrm{Mult}(H^2)$. It remains to be shown that the algebra of multipliers on H^2 is, in fact, H^∞.

One of the most important properties of the Hardy spaces is the property that every H^p function may be uniquely factorized into very special types of functions.

Definition 1.3.3. A function $\omega : \mathbb{D} \to \mathbb{C}$ is *inner* if $|\omega(z)| \le 1$ for $z \in \mathbb{D}$ and

$$\lim_{r \nearrow 1} |\omega(re^{i\theta})| = 1$$

almost everywhere. A function $k : \mathbb{D} \to \mathbb{C}$ is *outer* if there is a unimodular constant c and a function $w \in L^2(\sigma)$ with $\log(w) \in L^1(\sigma)$ such that

$$k(z) = c \exp\left[- \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log(w(\theta)) d\theta \right].$$

Theorem 1.3.4 (**Inner-Outer Factorization**). *Suppose $1 \le p \le \infty$ and f is a function in H^p. There exists a unique (up to multiplication of scalars) decomposition $f = \omega k$, where ω is inner and k is outer. Moreover, $f \in H^p$ if and only if $k \in H^p$.*

Corollary 1.3.5. *Suppose $g \in H^1$. There exist functions $h, k \in H^2$ such that $g = hk$ where k is outer and*

$$\|g\|_1 = \|h\|_2^2 = \|k\|_2^2$$

Proof. Let $g = h_0 k_0$ be the inner-outer factorization of g where h_0 is inner and k_0 is outer. The function k is non-vanishing; hence, it has a well-defined analytic square root in H^2. Write $k_o^{1/2} = \omega k$, where ω is inner and k is outer (in fact, $\omega = 1$, but we do not require this here). Let $h = \omega^2 h_0 k$. Then we have $g = h_0 k_0 = h_0 \omega^2 k^2 = hk$ and

$$\|g\|_1 = \left(\int |g| \right) = \left(\int |k_0| \right) = \left(\int |k|^2 \right) = \|k|_2^2.$$

Clearly $\|h\|_2 = \|k\|_2$, which completes the proof. $\square$

We now introduce the most important bounded operator on H^2. Let U be the operator that multiplies a function in $L^2(\sigma)$ by the independent variable, $e^{i\theta}$. Let $S = U|_{H^2}$ denote the *shift operator* on H^2 (interchangeably called the unilateral shift or forward shift). Using the identification made in Fatou's theorem, S is the operator on H^2 that multiplies by z. Beurling classified **[Beu49]** the invariant subspaces of S as special classes H^2 subspaces.

Theorem 1.3.6 (Beurling). *A subspace $\mathcal{N}$ of H^2 is invariant under S if and only if $\mathcal{N} = \omega H^2$ for some inner function ω.*

Recall that a vector h in a Hilbert space $\mathcal{H}$ is *cyclic* if there is an operator T acting on $\mathcal{H}$ such that $\bigvee\{T^n h : n \geq 0\}$ is dense in $\mathcal{H}$. As a consequence of Beurling's theorem, the cyclic vectors in H^2 are precisely the outer functions.

Corollary 1.3.7. *Suppose $k \in H^2$. Then $\overline{H^\infty k} = H^2$ if and only if k is an outer function.*

Proof. Suppose k is outer. Since $\overline{H^\infty k}$ is invariant under the shift, we have $\overline{H^\infty k} = \psi H^2$ for some inner function ψ. In particular, we have $k = \psi g$ for some $g \in H^2$. Since inner-outer factorization is unique, it must be the case that $\psi = 1$.

Conversely, suppose $\overline{H^\infty k} = H^2$. Write $k = \psi f$ where ψ is inner and f is outer.Then $H^2 = \overline{H^\infty \psi f} = \psi H^2$ (f is outer), which implies $\psi = 1$. □

The above result indicates that if an outer function is bounded, its multiplication operator corresponds to an operator with dense range. On the other hand, multiplication by an inner function is an isometric map. Consequently, any multiplication operator on H^2 is the product of an isometric multiplier and a multiplier with dense range. We are now able to identify the multiplication algebra of the Hardy space as H^∞.

Theorem 1.3.8. *The multiplier algebra of H^2 is isometrically isomorphic to H^∞.*

Proof. Suppose ϕ is a multiplier of H^2. Since $1 \in H^2$, we have $\phi \in H^2$ and therefore it is analytic. As we have seen, $\sup_{\lambda \in \mathbb{D}} |\phi(\lambda)| \leq \|M_\phi\|$ which shows that $\phi \in H^\infty$. The reverse inclusion is obvious since for any $\phi \in H^\infty$ and $f \in H^2$ we have $\|\phi f\|_{H^2} \leq \|\phi\|_\infty \|f\|_{H^2}$. Equality of the operator norm and supremum norm of a multiplier follows from the fact that H^2 is a Hilbert function space, and so $\|M_f\| \geq \sup_{z \in \mathbb{D}} |f(z)|$. □

Suppose that $\mathcal{H}$ is a Hilbert space and $B(\mathcal{H})$ its bounded operators. In addition to being a C^*-algebra, $B(\mathcal{H})$ is also a dual space and hence inherits a weak-* topology from its predual. One may identify the predual of $B(\mathcal{H})$ with the ideal $\mathcal{T}$ of *trace class operators*: the set of compact operators T satisfying $\sum_{n=0}^\infty |\langle (T^*T)^{1/2} e_n, e_n\rangle| < \infty$ where $\{e_i\}$ is any orthonormal basis of $\mathcal{H}$. We define the *trace* of such an operator as

$$\mathrm{tr(T)} := \sum_{\mathrm{n}=0}^{\infty} \langle \mathrm{Te_n}, \mathrm{e_n} \rangle.$$

The trace of an operator is independent of the choice of orthonormal basis. The continuous linear functionals on $\mathcal{T}$ defined by $\tau_A(T) = \mathrm{tr}(\mathrm{TA})$ for $A \in B(\mathcal{H})$ identify the dual of $\mathcal{T}$ as $B(\mathcal{H})$. If $\{h_n\}_{n\geq 0}$ and $\{k_n\}_{n\geq 0}$ are sequences in $\mathcal{H}$ satisfying $\sum_n \|h_n\|^2 < \infty$ and $\sum_n \|k_n\|^2 < \infty$, a net of operators $\{T_\alpha\}$ on $\mathcal{H}$ converge to an operator T in the weak-* topology if and only if $\sum_n \langle T_\alpha h_n, k_n\rangle \to \sum_n \langle T h_n, k_n\rangle$.

If $\mathcal{H}_k$ is a Hilbert function space, then $\mathrm{Mult}(\mathcal{H}_k)$ is always a commutative subalgebra of the bounded operators on $\mathcal{H}_k$. Fatou's theorem shows that H^∞ may be identified with a weak* closed subalgebra of $L^\infty(\sigma)$. The next result shows that the multiplier algebra of H^2 is closed in the weak* topology on $B(H^2)$, and that the weak* topologies of H^∞ and $\mathrm{Mult}(H^2)$ coincide.

Theorem 1.3.9. *The multiplier algebra of H^2 is weak-* homeomorphic to H^∞.*

Proof. We first show that the isometric isomorphism of H^∞ onto $\mathrm{Mult}(H^2)$ given by $\phi \mapsto M_\phi$ is weak* continuous. Suppose $\{\phi_\alpha\} \in H^\infty$ is a net such that $\phi_\alpha \to 0$ in the weak* topology. Regarding ϕ_α as an element in $L^\infty(\sigma)$, this is equivalent to $\int_{\mathbb{T}} \phi_\alpha h d\sigma \to 0$ for each $h \in L^1(\sigma)$. We must show that $M_{\phi_\alpha} \to 0$ in the weak* topology on $B(H^2)$. Suppose f_n and g_n are sequences of functions in H^2 satisfying $\sum_n \|f_n\|^2 + \|g_n\|^2 < \infty$. Then

$$\begin{aligned}
\sum_{n=1}^{\infty} \langle M_{\phi_\alpha} f_n, g_n\rangle_{H^2} &= \sum_{n=1}^{\infty} \langle \phi_\alpha f_n, g_n\rangle_{L^2(\sigma)} \\
&= \sum_{n=1}^{\infty} \int_{\mathbb{T}} \phi_\alpha f_n \overline{g_n} d\sigma \\
&= \int_{\mathbb{T}} \phi_\alpha \sum_{n=1}^{\infty} f_n \overline{g_n} d\sigma.
\end{aligned}$$

The final equality follows from the easily verified fact that $\sum_{n=1}^{\infty} |f_n \overline{g_n}| \in L^1(\sigma)$. This also shows that the last integral converges to 0, which proves the isometry is weak* continuous.

For the moment, let weak$_1^*$ and weak$_2^*$ denote the respective topologies on H^∞ and $\mathrm{Mult}(H^2)$. We have shown that the isometry $\phi \mapsto M_\phi$ is (weak$_1^*$, weak$_2^*$) continuous. If it were not a homeomorphism, then there would be a linear functional γ on H^∞ that is weak$_1^*$ continuous but not weak$_2^*$ continuous. The kernel of γ is then weak$_1^*$ closed, but not weak$_2^*$ closed. By the Banach-Alaoglu theorem, the intersection of the kernel of γ with any ball of radius r is weak$_1^*$ compact. By continuity, these intersections are also weak$_2^*$ compact. Now apply the Krein-Smulian theorem to see that the kernel of γ must be weak$_2^*$ closed. □

The most useful application of the weak* topology in interpolation is the notion of building a sequence of functions that converge precisely to a desired interpolant. As an example, consider the case of a sequences of functions $\{h_n\} \subset H^\infty$ with $\|h_n\| \leq 1$ for all n, and a sequence of points $\{z_n\} \subset \mathbb{D}$ converging to a point z that satisfy $h_n(z_n) = w$ for some fixed $w \in \mathbb{D}$. The unit ball of H^∞ is compact in the weak* topology by the Banach-Alaoglu theorem, and hence is metrizable. It follows that there is some weak*-convergent subsequence of $\{h_n\}$, say h in the unit ball of H^∞, that satisfies $h(z) = w$ (weak* converging sequences of functions will preserve point evaluations, for if f_n convergences weakly to f, then by definition $\langle f_n, k_\lambda \rangle$ converges to $\langle f, k_\lambda \rangle$). This type of argument will be useful in Chapter 4.

1.4 A Distance Formula for H^∞

In Sarason's seminal paper on interpolation, the essential tools used to prove his prototypical commutant lifting theorem are a certain distance estimate for H^∞, and the duality between L^1 and L^∞. Both of these tools will be developed in this section. The scalar version of Pick's theorem is an easy consequence of the distance formula. However, we will still develop the full generality of the commutant lifting theorem in chapter 2 in order to solve a wider class of interpolation problems (including their matrix-valued analogues). Our immediate goal is to explore the properties of the Hardy spaces that unify the fields of interpolation and operator theory. Moreover, very close analogues of these results will be used in Chapter 3, where we examine a constrained interpolation problem.

Before we begin, a technical lemma about distances in a Banach space is required. Recall that given a Banach dual space X^*, the *pre-annihilator* of a set $S \in X^*$ is the subspace $S_\perp := \{f \in X : s(f) = 0 \text{ for each } s \in S\} \subset X$.

Lemma 1.4.1. *Suppose Y is a Banach space and $X \subset Y^*$ a weak* closed subspace. The distance from $y^* \in Y^*$ to X is given by*

$$d(y^*, X) = \sup_{\zeta \in X_\perp, \|\zeta\| \le 1} |y^*(\zeta)|.$$

Proof. Suppose $x \in X$. By the Hahn-Banach theorem we have

$$\begin{aligned} \|y^* + x\|_{Y^*} &= \sup_{\zeta \in Y, \|\zeta\| \le 1} |(y^* + x)(\zeta)| \\ &\ge \sup_{\zeta \in X_\perp, \|\zeta\| \le 1} |y^*(\zeta)|. \end{aligned}$$

On the other hand, we may realize the coset $y^* + X$ as a functional via the identification

$Y^*/X \cong (Y/X_\perp)^*$, since X is weak* closed. It follows that

$$\begin{aligned}
\|y^* + X\| &= \sup_{\delta \in Y/X_\perp, \|\delta\| \le 1} |(y^* + X)(\delta)| \\
&\le \sup\{|(y^* + X)(\zeta + X_\perp)| : \zeta \in Y, \|\zeta\| \le 1\} \\
&= \sup\{|y^*(\zeta)| : \zeta \in Y, \|\zeta\| \le 1, X(\zeta) = 0\} \\
&= \sup_{\zeta \in X_\perp, \|\zeta\| \le 1} |y^*(\zeta)|,
\end{aligned}$$

which proves the reverse inequality. □

Let $E = \{z_1, ..., z_N\}$ be a subset of $\mathbb{D}$ and $\mathcal{I}^E$ as the ideal of functions in H^∞ vanishing at each z_i. Form the quotient algebra $\mathcal{Q} = H^\infty/\mathcal{I}^E$, and let $\mathcal{M}$ be the subspace of H^2 spanned by the kernel functions k_i for $1 \le i \le N$. It is easy to determine a lower bound for the norm of an element in $\mathcal{Q}$. Indeed, for $f \in H^\infty$ and $g \in \mathcal{I}^E$ we have $M_g^* k_i = 0$, hence

$$\|f - g\|_\infty \ge \|M_{f-g}^*|_{\mathcal{M}}\| = \|M_f^*|_{\mathcal{M}}\|,$$

and

$$d(f, \mathcal{I}^E) = \|f + \mathcal{I}_E\|_Q = \inf_{g \in \mathcal{I}^E} \{\|f - g\|_\infty\} \ge \|M_f^*|_{\mathcal{M}}\|.$$

In fact, it is true that we actually have equality in the above equation. We shall use a fundamental factorization technique in H^∞ to prove this. Let H_0^1 denote the subspace of functions in H^1 with $f(0) = 0$. It is can be subsequently verified that $(H^\infty)_\perp = H_0^1$.

Let $\phi_\lambda(z) = \frac{z-\lambda}{1-\bar{\lambda}z}$ denote the elementary Möbius map that sends λ to 0, and let $B_E = \prod_{i=1}^N \phi_{z_i}$ be the finite Blaschke product with simple zeroes in E. Since $\mathcal{I}^E = B_E H^\infty$, we may explicitly compute its pre-annihilator: $(\mathcal{I}^E)_\perp = \overline{B_E} H_0^1$. We now state and prove a factorization lemma for this pre-annihilator. Write $\mathcal{N} := \mathcal{M}^\perp = \{f \in H^2 : f(z) = 0$ for $z \in E\}$.

Lemma 1.4.2. *Suppose $g \in (\mathcal{I}^E)_\perp$. There exist functions $k \in H^2$ and $h \in L^2(\mathbb{T})$ such that the following hold:*

- *h is orthogonal to $\mathcal{N}$ in $L^2(\mathbb{T})$,*
- *$g = \overline{h}k$,*
- *$\|g\|_1 = \|h\|_2^2 = \|k\|_2^2$.*

Conversely, if $g \in L^1(\mathbb{T})$ may be factored as $\overline{h}k$ where $k \in H^2$ and $h \in L^2(\mathbb{T})$ is orthogonal to $\mathcal{N}$, then $g \in (\mathcal{I}^E)_\perp$.

Proof. Let $g \in (\mathcal{I}^E)_\perp$ and write $g = \overline{B_E} g_0$ where $g_0 \in H_0^1$. Factor $g_0 = kh_0$, as in Corollary 1.3.5, where $k, h_0 \in H^2$, k is outer, and

$$\|k\|_2^2 = \|h_0\|_2^2 = \|g_0\|_1 = \|g\|_1.$$

Let $h = B_E\overline{h_0}$ so that $g = \overline{h}k$. The first statement will be proved if we can show that h is orthogonal to $\mathcal{N}$ in $L^2(\mathbb{T})$. Observe that $\mathcal{N} = \overline{\mathcal{I}^E H^2} = B_E H^2$ where the bar denotes closure in H^2 (in situations where the potential for ambiguity exists, it will be stated if the bar operation denotes closure or complex conjugation). One inclusion is clear, and equality is established since the codimesion of both is precisely N. By Corollary 1.3.7, $H^\infty k$ is dense in H^2 since k is outer. Combining these reults shows that functions in $H^\infty k\mathcal{I}^E = k\mathcal{I}^E$ ($\mathcal{I}^E$ is an ideal) are dense in H^2. Therefore, to check orthogonality of h to $\mathcal{N}$, it suffices to show $\langle fk, h\rangle = 0$ for $f \in \mathcal{I}^E$. We have

$$\langle fk, h\rangle = \langle fk, B_E\overline{h_0}\rangle = \int f\overline{B_E}kh_0 = \int fg = 0.$$

The second statement follows by reversing the above calculation and observing that any such factorization annihilates $\mathcal{I}^E$. $\square$

We may now prove the desired distance formula.

Theorem 1.4.3. *Let $E = \{z_1, ..., z_N\} \subset \mathbb{D}$ and $\mathcal{M} = \bigvee\{k_i\}_{i=1}^N$. For any $f \in H^\infty$*

$$d(f, \mathcal{I}^E) = \|M_f^*|_{\mathcal{M}}\|.$$

Proof. By the Lemma 1.4.1, we have

$$d(f, \mathcal{I}^E) = \sup\left\{\left|\int fg\right| : g \in (\mathcal{I}^E)_\perp, \|g\|_1 \le 1\right\}.$$

Now factor $g = \overline{h}k$ as in Lemma 1.4.2. Let P denote the orthogonal projection of $L^2(\mathbb{T})$ into H^2. It follows that

$$\begin{aligned}
\sup\left\{\left|\int fg\right| : g \in (\mathcal{I}^E)_\perp, \|g\|_1 \le 1\right\} &= \sup\left\{\left|\int fk\overline{h}\right|\right\} \\
&= \sup|\langle fk, h\rangle| \\
&= \sup|\langle Pfk, h\rangle| \\
&= \sup|\langle k, M_f^*Ph\rangle| \\
&\le \|M_f^*|_{\mathcal{M}}\|,
\end{aligned}$$

since $Ph \subset \mathcal{M}$, where the supremums on the right hand side are taken over the set:

$$\{k \in H^2, h \in L^2(\mathbb{T}) : \|k\|_2 \le 1, \|h\|_2 \le 1, h \perp \mathcal{N}\}.$$

The reverse inequality was proved in the discussion preceding Lemma 1.4.2. $\square$

We also obtain a simple proof of Nehari's theorem using the factorization technique developed here.

Theorem 1.4.4 (Nehari). *If $f \in L^\infty(\mathbb{T})$ and P is the orthogonal projection of $L^2(\mathbb{T})$ into H^2, then*

$$d(f, H^\infty) = \|(I-P)M_f P\|.$$

Proof. Applying Lemma 1.4.2 to the case where E is empty, we may factor $g \in (H^\infty)_\perp$ as $g = \overline{h}k$, where $k \in H^2$ for and h is orthogonal to $\mathcal{N}(:= H^2$, in this case). Moreover, $\|k\|_2^2 = \|h\|_2^2 = \|g\|_1$. Conversely, every product of this form lies in $(H^\infty)_\perp$. It follows that

$$\begin{aligned} d(f, H^\infty) &= \sup_{g \in (H^\infty)_\perp, \|g|_1 \le 1} \left| \int fg \right| \\ &= \sup\{|\langle fk, h\rangle| : h \perp H^2, \|h\|_2 \le 1; k \in H^2, \|k\|_2 \le 1\} \\ &= \|(I-P)M_f P\|, \end{aligned}$$

as desired. □

Observe that $\mathcal{I}^E$ is an intersection of kernels of evaluation functionals, each of which are weak* continuous on H^∞. This implies that the unit ball of $\mathcal{I}^E$ is closed and hence weak* compact, which in turn implies that the infimum $d(f, \mathcal{I}^E)$ is attained.

To see this, fix $f \in H^\infty$, suppose that $d(f, \mathcal{I}_E) = 1$, and find a sequence of functions $g_m \in \mathcal{I}_E$ satisfying $\|f - g_m\| \le \frac{1}{m} + 1$. The functions $f - g_m$ are contained in the ball of radius 2 about 0 in H^∞, hence there is some weak* converging subsequence (the unit ball of H^∞ is weak* metrizable) converging to an element h. By construction, $\|h\| \le 1$ and $h(z_i) = f(z_i)$. It follows that $f - h \in \mathcal{I}_E$ and $\|f - (f-h)\| \le 1 = d(f, \mathcal{I}_E)$.

The scalar version of Pick's theorem follows from this. Indeed, let p be any polynomial that interpolates the given data. The positivity of the Pick matrix implies that $\|M_p^*\| \le 1$, which is equal to $d(p, \mathcal{I}^E)$ by Theorem 1.2.5. Let $g \in \mathcal{I}_E$ be the function where the distance is attained. It follows that $p - g$ has the required norm and interpolates the given data. This illustrates the deep connection between distances to ideals and interpolation. On the other hand, a Hilbert function space whose kernel has the scalar Pick property always satisfies the analogous distance formula.

Proposition 1.4.5. *Suppose $\mathcal{H}_k$ is a Hilbert function space, $F = \{\lambda_1, ..., \lambda_N\} \subset X$, and $\mathcal{J}^F = \{\sigma \in \mathrm{Mult}(\mathcal{H}_k) : \sigma(\lambda_i) = 0, 1 \le i \le N\}$. If the kernel k has the scalar Pick property, then the distance between a function $\phi \in \mathrm{Mult}(\mathcal{H}_k)$ to $\mathcal{J}^F$ is given by*

$$d(\phi, \mathcal{J}^F) = \|M_\phi^*|_{\mathcal{M}}\|,$$

where $\mathcal{M}$ is the subspace of $\mathcal{H}_k$ generated by the kernel functions k_{λ_i}.

Proof. It is routine to verify that $d(\psi, \mathcal{J}^E) \ge \|M_\psi^*|_{\mathcal{M}}\|$. To establish the reverse inequality, suppose that, without loss of generality, $\|M_\phi^*|_{\mathcal{M}}\| = 1$. This implies that the Pick matrix $[(1 - \phi(\lambda_i)\overline{\phi(\lambda_j)})k(\lambda_j, \lambda_i)]$ is positive semidefinite. Since k has the scalar Pick property, there is some $\psi \in \mathrm{Mult}(\mathcal{H}_k)$ of norm at most 1 satisfying $\phi(\lambda_i) = \psi(\lambda_i)$ for $1 \leq i \leq N$. Let $\sigma = \phi - \psi \in \mathcal{J}^F$. Then $d(\phi, \mathcal{J}^F) \leq \|M_{\phi-\sigma}\| \leq 1$, which is what we wanted. $\square$

Chapter 2

Commutant Lifting and Interpolation

In 1967, Sarason's seminal paper *Generalized Interpolation in* H^∞ **[Sar67]** provided an operator-theoretic approach to interpolation problems. His work was later generalized by Foias and Sz.Nagy with their *commutant lifting theorem* **[FSN68]**. Consider the problem of trying to determine if the Szegö kernel has the scalar Pick property. Suppose

$$\left[\frac{1 - w_i\overline{w_j}}{1 - z_i\overline{z_j}}\right]_{i,j=1}^{N} \geq 0. \tag{2.1}$$

Recall our notational convention $k_i := k_{z_i}$, and consider $\mathcal{M} = \bigvee_{I=1}^{N}\{k_i\} \subset H^2$. As we have seen in the last chapter, (2.1) is equivalent to the property that the map $R : k_i \mapsto \overline{w_i}k_i$ is a contraction on $\mathcal{M}$.

Let $S = M_z$ denote the forward shift on H^2 and recall that $S^*k_{\lambda_i} = \overline{\lambda_i}k_{\lambda_i}$ for each i. Clearly S^* leaves $\mathcal{M}$ invariant and $S^*|_{\mathcal{M}}$ commutes with R. Sarason's result has two facets. First, is it possible that the operator R is always the restriction of an operator on H^2 commuting with S^*? Second, is this *lifting* possible without an increase in norm? The answer to the first question can be answered immediately, since the commutant of the shift is precisely the multiplication algebra of H^2.

Proposition 2.0.6. *Suppose T is an operator on H^2 that commutes with S. Then $T = M_f$ for some $f \in H^\infty$.*

Proof. Let $T\mathbf{1} = f$. Then

$$Tz^n = TS^n\mathbf{1} = S^nT\mathbf{1} = f(z)z^n.$$

So $Tp = fp$ for any polynomial p and, by boundedness of T and passing through to limits, we have $Th = fh$ for all $h \in H^2$. This implies $\phi f \in H^2$, hence $\phi \in H^\infty$. □

A similar result for the vector-valued case is worth recording.

Corollary 2.0.7. *Suppose T is an operator on $H^2 \otimes \mathbb{C}^n$ that commutes with $S \otimes I_n$. Then $T = M_F$ for some $F \in H^\infty \otimes \mathcal{M}_n$.*

This shows that the operators that commute with S^* are precisely the adjoints of multiplication operators. If the answers to both questions are affirmative, then there would be a function $f \in H^\infty$ such that $M_f|_{\mathcal{M}}^* = R$ and $\|R\| = \|f\|_\infty$. Moreover $M_f^* k_{\lambda_i} = Rk_{\lambda_i} = \overline{w_i} k_{\lambda_i}$ and so f interpolates the data (and $\|f\|_\infty \leq 1$). Sarason proved that such an extension is always possible in H^2.

Theorem 2.0.8 (Sarason). *Suppose R is an operator on $\mathcal{M}$ commuting with $S^*|_{\mathcal{M}}$. There is a function $f \in H^\infty$ such that $M_f^*|_{\mathcal{M}} = R$ and $\|f\|_\infty = \|R\|$.*

Before we prove the more general result, some preliminaries in dilation theory are required.

2.1 Proof of the Theorem

Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{K}$ be a Hilbert space containing $\mathcal{H}$. Suppose $T \in B(\mathcal{K})$ and write $R = P_{\mathcal{H}} T|_{\mathcal{H}}$. If $R^n = P_{\mathcal{H}} T^n|_{\mathcal{H}}$ for all $n \geq 0$ we say that T is a *dilation* of R. R is called a *compression* of T.

In his pioneering paper on dilations of operators, Sz. Nagy proved that any contraction may be dilated to a unitary **[SNa53]**. We prove that a unitary dilation always exists in two steps. It is not difficult to see that any isometry may be dilated to a unitary. Let V be an isometry on $\mathcal{H}$ and define U in $B(\mathcal{H} \oplus \mathcal{H})$ by

$$\begin{bmatrix} V & I_{\mathcal{H}} - VV^* \\ 0 & V^* \end{bmatrix}.$$

U is the required unitary dilation. Similarly, we are able to dilate any contraction to an isometry. Let T be a contraction on $\mathcal{H}$ and let $D_T = (I_{\mathcal{H}} - T^*T)^{1/2}$. For h in H we have

$$\|Th\|^2 + \|D_T h\|^2 = \langle T^*Th, h\rangle + \langle D_T^2 h, h\rangle = \|h\|^2.$$

Let $\mathcal{K} = \mathcal{H} \otimes \ell^2$, the Hilbert space of square-summable sequences in $\mathcal{H}$. Define $V \in B(\mathcal{K})$ by $V(h_1, h_2, h_3, ...) = (Th_1, D_T h_1, h_2, h_3, ...)$. V is an isometry since

$$\|V(h_1, h_2, h_3, ...)\|^2 = \|Th_1\|^2 + \|D_T h_1\|^2 + \|h_2\|^2 + ... = \|h_1\|^2 + \|h_2\|^2 + ... = \|(h_1, h_2, h_3, ...)\|^2.$$

Then, by identifying $\mathcal{H}$ with $\mathcal{H} \oplus 0 \oplus 0 \oplus ...$, we have $T^n = P_{\mathcal{H}} V^n|_H$ for any $n \geq 0$. By simply observing that a dilation of a dilation is a dilation, we may combine these two results to prove Sz. Nagy's result. For our purposes, it is often convenient to work with extensions rather than dilations.

Definition 2.1.1. Suppose $\mathcal{H}$ and $T \in B(\mathcal{H})$, that there is a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ as a subspace, and that $W \in B(\mathcal{K})$. If $\mathcal{H}$ is an invariant subspace of W and $T = P_{\mathcal{H}}W|_{\mathcal{H}}$ we say that W is an *extension* of T and that T is a *part* of W.

An extension is automatically a dilation since $P_{\mathcal{H}}W|_{\mathcal{H}} = W|_{\mathcal{H}}$. The unitary dilation of an isometry is actually an extension, but in general the isometric dilation of a contraction is not.

The standard approach to proving commutant lifting relies heavily upon Ando's theorem [**And63**], which states that any two commuting contractions may be simultaneously dilated to commuting unitaries. It is relatively easy to prove that two commuting isometries may be dilated to commuting unitaries; the difficulty is dilating commuting contractions to commuting isometries. Given the subspace $\mathcal{M}$ we are trying to extend the operator R (which commutes with $S^*|_{\mathcal{M}}$) to an operator (hence the adjoint of a multiplier) commuting with S^*. Since S^* is a co-isometry, it is convenient to extend R to a co-isometry rather than an isometry. A simple change in notation is all that is needed.

Proposition 2.1.2. *Suppose that $\mathcal{H}$ is a Hilbert space and $T \in B(\mathcal{H})$ is a contraction. Then T has a co-isometric extension.*

Proof. Let $\mathcal{K} = \mathcal{H} \otimes \ell^2$ and define $W \in B(\mathcal{K})$ by

$$W(h_1, h_2, h_3, h_4, ...) = (Th_1, (I_{\mathcal{H}} - TT^*)^{\frac{1}{2}}h_1, h_2, h_3, ...).$$

By identifying $\mathcal{H}$ with $\mathcal{H} \oplus 0 \oplus 0 \oplus ...$ it is clear that W leaves $\mathcal{H}$ invariant. We also have

$$\|T^*h\|^2 + \|(I_{\mathcal{H}} - TT^*)^{\frac{1}{2}}h\|^2 = \|h\|^2$$

for $h \in \mathcal{H}$, which implies that W^* is an isometry. □

It suffices, then, to prove the analogue of Ando's theorem for co-isometric extensions (we forego proving that commuting isometries may be dilated to commuting unitaries). For convenience of notation, we write the above co-isometric extension of T as $(W, \mathcal{K})$. The extension $(W, \mathcal{K})$ is said to be the *minimal co-isometric extension of* T if

$$\mathcal{K} = \bigvee\{(W^*)^n h : h \in \mathcal{H}, n \in \mathbb{Z}^+\}.$$

Any co-isometric extension of T will be a direct sum of the minimal extension and some other co-isometry. If $(W_1, \mathcal{K}_1)$ and $(W_2, \mathcal{K}_2)$ are minimal co-isometric extensions of T, they will be unitarily equivalent. To see this, define the map $U : \mathcal{K}_1 \to \mathcal{K}_2$ by

$$U((W_1^*)^n h) = (W_2{}^*)^n h$$

for $n \geq 0$. U is a well defined isometry since, for $n \geq m$ and $h, k \in \mathcal{H}$, we have

$$\begin{aligned}
\langle (W_1{}^*)^n h, (W_1{}^*)^m k \rangle &= \langle (W_1{}^*)^{n-m} h, k \rangle \\
&= \langle h, T^{n-m} k \rangle \\
&= \langle (W_2{}^*)^{n-m} h, k \rangle \\
&= \langle (W_2{}^*)^n h, (W_2{}^*)^m k \rangle.
\end{aligned}$$

U is then unitary since it maps a dense set to a dense set. Moreover, $U|_{\mathcal{H}} = I$ and $UW_1U^* = W_2$. Because of this equivalence, we shall now refer to $(W, \mathcal{K})$ as *the* minimal co-isometric extension of T. We now have the tools to prove the most fundamental result of this section.

Theorem 2.1.3. ***(Ando)*** *If T_1 and T_2 are commuting contractions on a Hilbert space $\mathcal{H}$, then they may be simultaneously extended to commuting co-isometries on a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$.*

Proof. Given two commuting contractions T_1 and T_2 on some Hilbert space $\mathcal{H}$, it is easy to extend them to co-isometries on the same space. Indeed if $(V_1, \mathcal{H} \oplus \mathcal{K}_1)$ and $(V_2, \mathcal{H} \oplus \mathcal{K}_2)$ are the minimal co-isometric extensions of T_1 and T_2, respectively, then the operators $W_1 = V_1 \oplus I_{\mathcal{K}_2}$ and $W_2 = V_2 \oplus I_{\mathcal{K}_1}$ on the space $\mathcal{K} = \mathcal{H} \oplus \mathcal{K}_1 \oplus \mathcal{K}_2$ are the required co-isometric extensions to operators on $\mathcal{K}$.

In general W_1 and W_2 may not commute. However, W_1W_2 and W_2W_1 are both co-isometric extensions of $T_1T_2 = T_2T_1$. Let W be the minimal co-isometric extension of T_1T_2. Then W_1W_2 and W_2W_1 are unitarily equivalent to $W \oplus W_{12}$ and $W \oplus W_{21}$, respectively, where W_{12} and W_{21} are co-isometries acting on Hilbert spaces $\mathcal{H}_{12}$ and $\mathcal{H}_{21}$, respectively. Let $\mathcal{L} = \mathcal{K} \oplus ((\mathcal{H}_{12} \oplus \mathcal{H}_{21}) \otimes \ell^2)$. Now define co-isometric extensions of T_1 and T_2 on $\mathcal{L}$ as follows:

$$\begin{aligned}
W_1' &= W_1 \oplus I_{(\mathcal{H}_{12} \oplus \mathcal{H}_{21}) \otimes \ell^2} \\
W_2' &= W_2 \oplus ((W_{12} \oplus W_{21}) \otimes I_{\ell^2}).
\end{aligned}$$

Then we have

$$\begin{aligned}
W_1'W_2' &= W_1W_2 \oplus ((W_{12} \oplus W_{21}) \otimes I_{\ell^2}) = W \oplus W_{12} \oplus ((W_{12} \oplus W_{21}) \otimes I_{\ell^2}) \\
W_2'W_1' &= W_2W_1 \oplus ((W_{12} \oplus W_{21}) \otimes I_{\ell^2}) = W \oplus W_{21} \oplus ((W_{12} \oplus W_{21}) \otimes I_{\ell^2})
\end{aligned}$$

which implies $W_1'W_2'$ are $W_2'W_1'$ are unitarily equivalent. By the discussion preceding this theorem, let U be a unitary on $\mathcal{L}$ satisfying $W_1'W_2' = UW_2'W_1'U^*$ and $U|_{\mathcal{K}} = I$. Now define $W_1'' = W_1'U^*$ and $W_2'' = UW_2'$. These operators clearly commute, and are co-isometric extensions of T_1 and T_2, respectively, since

$$\begin{aligned} W_1''|_{\mathcal{H}} &= W_1'|_{\mathcal{H}} = T_1 \\ W_2''|_{\mathcal{H}} &= UW_2'|_{\mathcal{H}} = UT_2 = T_2. \end{aligned}$$

The last line follows from the fact that U is the identity on $\mathcal{K}$, hence on its subspace $\mathcal{H}$. □

The desired theorem is an easy consequence of Ando's result.

Theorem 2.1.4. ***(Commutant lifting theorem)*** *Let T be a contraction on $\mathcal{H}$ and let $(W, \mathcal{K})$ be its minimal co-isometric extension. Suppose R is an operator on $\mathcal{H}$ commuting with T. Then there is an operator R' on $\mathcal{K}$ such that*

$$\begin{aligned} R'|_{\mathcal{H}} &= R \\ R'W &= WR' \\ \|R'\| &= \|R\|. \end{aligned}$$

Proof. Rescaling if necessary, suppose $\|R\| = 1$. Apply Ando's theorem to T and R to get commuting co-isometries W_T and W_R acting on some Hilbert space $\mathcal{K}_1$ containing $\mathcal{H}$ as a subspace. Let $(W, \mathcal{K})$ be the minimal co-isometric extension of T. Relative to the decomposition $\mathcal{K}_1 = \mathcal{K} \oplus \mathcal{K}^\perp$, we have

$$W_T = \begin{bmatrix} W & 0 \\ 0 & * \end{bmatrix}$$

$$W_R = \begin{bmatrix} R' & * \\ * & * \end{bmatrix}$$

for some R'. Since W_T and W_R commute, so do W and R'. Moreover, $R'|_{\mathcal{H}} = P_{\mathcal{K}} W_R P_{\mathcal{K}}|_{\mathcal{H}} = R$. It is clear that $\|R'\| = \|R\|$ since $1 = \|R'\| \leq \|W_R\| \leq 1$. □

Note that, in general, the operator R' is not a co-isometry (it is a compression of a co-isometric extension of R). We now have the required tool to prove Pick's theorem.

Corollary 2.1.5. *The Szegö kernel has the complete Pick property.*

Proof. Suppose $z_1, ..., z_N \in \mathbb{D}$, $W_1, ..., W_n \in \mathcal{M}_n(\mathbb{C})$, and that

$$\left[\frac{1 - W_i W_j^*}{1 - z_i \overline{z_j}}\right]_{i,j=1}^N \geq 0.$$

As before, let $\mathcal{M} = \bigvee\{k_i : 1 \leq i \leq N\}$. Let $\{e_j\}_{i=1}^n$ denote the canonical orthonormal basis of $\mathbb{C}^n$, consider the following subspace of $H^2 \otimes \mathbb{C}^n$:

$$\mathcal{M} \otimes \mathbb{C}^n = \vee\{k_i \otimes e_j : 1 \leq i \leq N, 1 \leq j \leq n\}.$$

It follows that $\mathcal{M} \otimes \mathbb{C}^n$ is invariant under the contraction $S^* \otimes I_n$, which is the miminal co-isometric extension of $S^* \otimes I_n|_{\mathcal{M}\otimes\mathbb{C}^n}$. Define an operator R on $\mathcal{M} \otimes \mathbb{C}^n$ as follows:

$$R(k_i \otimes e_j) = k_i \otimes W_i^* e_j.$$

R is easily seen to commute with $S^* \otimes I_n$ and the positivity of the Pick matrix precisely means that R is a contraction. Apply commutant lifting to R and $S^* \otimes I_n|_{\mathcal{M}\otimes\mathbb{C}^n}$. Since the lifting of R commutes with $S^* \otimes I_n$, it is the adjoint of a multiplier, say M_F. Moreover, it satisfies

$$M_F^*(k_i \otimes u_i) = k_i \otimes F(z_i)^* u_i = k_i \otimes W_i^* u_i$$

and so interpolates the given data. □

Note that this implies a vector-valued analogue of the H^∞ distance estimate in Chapter 1.

Corollary 2.1.6. *Let $E = \{z_1, ..., z_N\} \subset \mathbb{D}$ and let $\mathcal{I}^E$ denote the ideal of functions in H^∞ vanishing on E . For any $F \in \mathcal{M}_n(H^\infty)$ we have*

$$d(F, \mathcal{M}_n(\mathcal{I}^E)) = \|M_F^*|_{\mathcal{M}\otimes\mathbb{C}^n}\|.$$

Proof. If $G \in \mathcal{M}_n(\mathcal{I}^E)$, then

$$\|F - G\|_\infty \geq \left\|M_{F-G}^*|_{\mathcal{M}\otimes\mathbb{C}^n}\right\| = \|M_F^*|_{\mathcal{M}\otimes\mathbb{C}^n}\|,$$

which implies one inequality. Conversely, suppose $\|M_F^*|_{\mathcal{M}\otimes\mathbb{C}^n}\| = 1$. By the previous theorem, there is some $\Phi \in \mathcal{M}_n(H^\infty)$ such that $\|M_\Phi\| \leq 1$ and $\Phi(z_i) = F(z_i)$ for $1 \leq i \leq N$. It follows that $d(F, \mathcal{M}_n(\mathcal{I}^E)) \leq \|F - (F - \Phi)\|_\infty \leq 1$, as desired. □

On the other hand, the complete Pick property of the Szegö kernel follows readily from this distance formula. As we have previously seen, the infimum $d(F, \mathcal{M}_n(\mathcal{I}^E))$ is actually attained. If P is any interpolant (a matrix polynomial may always be chosen to interpolate), it follows that $d(P, \mathcal{M}_n(\mathcal{I}^E)) = \|P_F^*|_{\mathcal{M}\otimes\mathbb{C}^n}\| \leq 1$. Let G be any function satisfying $\|P - G\| = d(P, \mathcal{M}_n(\mathcal{I}^E))$. Then $P - G$ is the required interpolant.

As a final note, we remark that suitable analogues of Ando's theorem and commutant lifting hold for isometric dilations of contractions. This can be done by simply adjusting the proofs here, or by applying the theorems to adjoints of contractions. This will be useful in the following section, where it is more convenient to use the shift S as our model rather than its adjoint.

2.2 The Carathéodory Problem

The Pick problem is concerned with finding a bounded, analytic function that maps N values to N points, whereas the Carathéodory problem asks under what conditions a bounded, analytic function has its first $M+1$ Taylor coefficients specified. That is, given $a_0, a_1, ..., a_M \in \mathbb{C}$ when does there exist a $f \in H^\infty$ such that $\|f\| \leq 1$ and

$$f(z) = \sum_{k=1}^{M} a_k z^k + O(z^{M+1})?$$

A necessary and sufficient condition for such a function to exist is the contractivity of the following Toeplitz matrix:

$$C := \begin{bmatrix} a_0 & 0 & 0 & \cdots & 0 \\ a_1 & a_0 & 0 & \cdots & 0 \\ a_2 & a_1 & a_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_M & a_{M-1} & a_{M-2} & \cdots & a_0 \end{bmatrix},$$

which we will prove by commutant lifting. A vector-valued analogue for the Carathéodory problem is easily formulated. Indeed, any multiplier $F \in H^\infty \otimes \mathcal{M}_n$ has a Taylor series expansion

$$F(z) = \sum_{k=0}^{\infty} z^k \otimes A_k,$$

where $A_k \in \mathcal{M}_n$.

If $f(z) = \sum_{i=0}^{\infty} a_n z^n \in H^\infty$, we may represent the operator M_f by the infinite matrix

$$\begin{bmatrix} a_0 & 0 & 0 & \cdots \\ a_1 & a_0 & 0 & \cdots \\ a_2 & a_1 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

written with respect to the orthonormal basis $\{1, z, z^2, ...\}$ of H^2. The matrix C is simply the compression of M_f to the finite dimensional subspace $\mathcal{R} := \bigvee\{1, z, ..., z^M\}$. Note that for any multiplier M_f, the subspace $\mathcal{R}$ is co-invariant, i.e., $P_\mathcal{R} M_f|_\mathcal{R} = P_\mathcal{R} M_f$.

Theorem 2.2.1. *Let $A_0, A_1, ..., A_M \in \mathcal{M}_n$. There is a function $F \in \mathcal{M}_n(H^\infty)$ such that*

$\|F\| \leq 1$ *and* $F(z) = A_0 + z \otimes A_1 + ... + z^M \otimes A_M + \sum_{k=M+1}^{\infty} z^k \otimes B_k$ *if and only if*

$$C := \begin{bmatrix} A_0 & 0 & 0 & \cdots & 0 \\ A_1 & A_0 & 0 & \cdots & 0 \\ A_2 & A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_M & A_{M-1} & A_{M-2} & \cdots & A_0 \end{bmatrix}$$

is a contraction.

Proof. Suppose such an F exists. Then we need only verify that C represents the desirable compression of M_F, from which it follows that C is automatically a contraction. With $\{e_i\}_{i=1}^n$ the canonical orthonormal basis of $\mathbb{C}^n$ consider the subspace

$$\mathcal{R} \otimes \mathbb{C}^n = \bigvee \{z^k \otimes e_i : 0 \leq k \leq M, 1 \leq i \leq n\}$$

of $H^2 \otimes \mathbb{C}^n$. Let P denote the orthogonal projection of $H^2 \otimes \mathbb{C}^n$ onto $\mathcal{R} \otimes \mathbb{C}^n$ and note that co-invariance implies $PM_F P = PM_F$. Then we have

$$\begin{aligned} PM_F(z^m \otimes e_i) &= \sum_{k=0}^{M-m} (z^k \otimes A_k)(z^m \otimes e_i) \\ &= \sum_{k=0}^{M-m} (z^{k+m} \otimes A_k e_i) \end{aligned}$$

for $0 \leq m \leq M$ and $0 \leq i \leq n$. The $(i,j)^{th}$-entry of the $(r,m)^{th}$-block of $P_{\mathcal{R}\otimes\mathbb{C}^n} M_F$ is then given by

$$\begin{aligned} \langle R(z^m \otimes e_j), z^r \otimes e_i \rangle_{H^2 \otimes \mathbb{C}^n} &= \left\langle \sum_{k=0}^{M-m} z^{k+m} \otimes A_k e_j, z^r \otimes e_i \right\rangle_{H^2 \otimes \mathbb{C}^n} \\ &= \sum_{k=0}^{M-m} \langle z^{k+m}, z^r \rangle_{H^2} \langle A_k e_j, e_i \rangle_{\mathbb{C}^n} \\ &= \begin{cases} 0 & : r < m \\ (A_{r-m})_{ij} & : r \geq m \end{cases}, \end{aligned}$$

which is precisely the matrix C.

Conversely, suppose C is a contraction. It is easily verified that C commutes with $P(S \otimes I_n)$. Applying commutant lifting to C and $P(S \otimes I_n)$ yields a contractive multiplier M_F, and the compression to the subspace $\mathcal{R} \otimes \mathbb{C}^n$ is precisely the matrix C, as the above calculation shows. It follows from the same calculation that the first M Taylor coefficients of F are the A_i for $0 \leq i \leq M$. $\square$

As in the case of the Pick problem, the solution to the Carathéodory problem allows one to make a distance estimate in H^∞.

Corollary 2.2.2. *Fix $M > 0$ and let $\mathcal{J}$ be the weak* closed ideal of functions in H^∞ generated by the monomials $\{z^j\}_{j \geq M+1}$. Then for $\mathcal{R} = \bigvee\{z_1, ..., z^M\} \subset H^2$ we have*

$$d(f, \mathcal{J}) = \|P_{\mathcal{R}} M_f\|,$$

for any f in H^∞.

Proof. For $f \in H^\infty$ $g \in \mathcal{J}$, we have

$$\|M_f - M_g\| \geq \|P_{\mathcal{R}} M_{f-g}\| = \|P_{\mathcal{R}} M_f\|,$$

which follows from the fact that the range of M_g is contained in the orthogonal complement of $\mathcal{R}$.

On the other hand, the ideal $\mathcal{J}$ is weak* closed, and so the infimum $d(f, \mathcal{J})$ is attained. Without loss of generality, we may assume $\|P_{\mathcal{R}} M_f\| = 1$. By the previous theorem, there is a contractive multiplier h with the same initial $M+1$ Taylor coefficients as f. It follows that $f - h \in \mathcal{J}$, and so

$$d(f, \mathcal{J}) \leq \|f - (f - h)|_\infty = \|h\|_\infty \leq 1,$$

which establishes the reverse inequality. □

The matrix-valued analogue follows similarly.

Corollary 2.2.3. *Fix $M > 0$ and let $\mathcal{J}$ be the ideal of functions in H^∞ generated by the monomials $\{z^j\}_{j \geq M+1}$. Then for $\mathcal{R} = \vee\{z_1, ..., z^M\}$ we have*

$$d(F, \mathcal{M}_n(\mathcal{J})) = \|P_{\mathcal{R} \otimes \mathbb{C}^n} M_F\|,$$

for any F in $H^\infty \otimes \mathcal{M}_n$.

Suppose now we wish to find a function $F \in \mathcal{M}_n(H^\infty)$ that not only interpolates N data points, but also has its first $M+1$ Taylor coefficients specified. We shall refer to such a problem as a *mixed Carathéodory- Pick interpolation problem.* Commutant lifting handles this problem as easily as the previous two. Let C denote the Carathodory matrix

$$\begin{bmatrix} A_0 & 0 & 0 & \cdots & 0 \\ A_1 & A_0 & 0 & \cdots & 0 \\ A_2 & A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_M & A_{M-1} & A_{M-2} & \cdots & A_0 \end{bmatrix}$$

and let $\mathbb{P} = \left[\frac{1-W_i W_j^*}{1-z_i\overline{z_j}}\right]_{i,j=1}^{N}$ be the Pick matrix.

Corollary 2.2.4. *Suppose $W_1, ...W_N, A_0, A_1, ..., A_M \in \mathcal{M}_n$ and $z_1, ...z_N \in \mathbb{D}$, where each z_i is not 0. There is a function $F \in \mathcal{M}_n(H^\infty)$ such that $\|F\| \le 1$, $F(z_i) = W_i$ for each i and $F(z) = \sum_{k=0}^{M} z^k \otimes A_k + ...$ if and only if =*

$$\mathbb{M} = \begin{bmatrix} 1 - CC^* & T \\ T^* & \mathbb{P} \end{bmatrix}$$

is positive semidefinite, where the $(ij)^{th}$ block of T is given by $z_i^j \otimes I_n - \sum_{k=0}^{j}\left(z_i^{j-k} \otimes W_i A_k^*\right)$.

Proof. For clarity, we first prove the theorem for the scalar case. Recall our convention that $k_i := k_{z_i}$.

Suppose a multiplier f satisfies the hypotheses of the theorem and define the following subspaces of H^2:

$$\begin{aligned} \mathcal{M} &= \bigvee\{k_1, ..., k_N\} \\ \mathcal{R} &= \bigvee\{1, z, ..., z^M\}. \end{aligned}$$

Consider the operator PM_f^*P where P is the orthogonal projection of H^2 onto $\mathcal{M}\bigvee\mathcal{R}$. Let $g + h$ be an arbitrary element in $\mathcal{S}$ where $g = \sum_{m=0}^{M} \alpha_m z^m$ and $h = \sum_{i=1}^{N} \beta_i k_i$. Then PM_f^*P is a contraction if and only if

$$\begin{aligned} 0 &\le \left\langle (I - PM_fPM_f^*P)(g+h), g+h \right\rangle \\ &= \langle (I - PM_fPM_f^*P)g, g\rangle + 2Re\left\langle (I - PM_fPM_f^*P)g, h\right\rangle + \left\langle (I - PM_fPM_f^*P)h, h\right\rangle \\ &= \langle (I - M_fM_f^*)g, g\rangle + 2Re\left\langle (I - M_fM_f^*)g, h\right\rangle + \left\langle (I - M_fM_f^*)h, h\right\rangle, \end{aligned}$$

where the last line follows from the fact that M_f^* leaves $\mathcal{M}$ and $\mathcal{S}$ invariant. Our goal is to show that the contractivity of PM_fP is equivalent to the positivity of $\mathbb{M}$. Indeed, computing the middle term in the above sum yields:

$$\begin{aligned} \langle (I - M_fM_f^*)g, h\rangle &= \langle g, h\rangle - \langle M_f^*g, M_f^*h\rangle \\ &= \sum_m\sum_i \alpha_m\overline{\beta_i}\left(\langle z^m, k_i\rangle - \left\langle \sum_{k=0}^{m} \overline{a_k} z^{m-k}, \overline{w_i} k_i \right\rangle\right) \\ &= \sum_m\sum_i \alpha_m\overline{\beta_i}\left(z_i^m - w_i\sum_{k=0}^{m}\overline{a_k} z_i^{m-k}\right), \end{aligned}$$

which is evidently equivalent to the positivity of $\mathbb{M}$.

Conversely, suppose the given matrix is positive semidefinite. Define the operator R

on $\mathcal{M} \vee \mathcal{S}$ by the actions

$$\begin{aligned} Rk_i &= \overline{w_i}k_i,\ 1 \leq i \leq N, \\ Rz^j &= \sum_{k=0}^{j} \overline{a_k} z^{j-k},\ 0 \leq j \leq M, \end{aligned}$$

ad extend linearly. By hypothesis, R is a contraction, and so we may apply commutant lifting to R and $S^*|_{\mathcal{M}\vee\mathcal{S}}$, as they evidently commute. This yields a contractive multiplier f that satisfies the required conditions.

For the vector-valued case, define the operator R on $(\mathcal{M} \vee \mathcal{R}) \otimes \mathbb{C}^n$ by the actions

$$\begin{aligned} R(k_i \otimes e_j) &= k_i \otimes W^* e_j;\ 1 \leq i \leq N;\ 1 \leq j \leq n, \\ R(z^m \otimes e_k) &= \sum_{k=0}^{m} z^{m-k} \otimes A_k^* e_k,\ 0 \leq m \leq M;\ 1 \leq k \leq n. \end{aligned}$$

Now proceed as in the scalar case. □

Combining the two existing distance estimates for H^∞ and using the above result, we obtain a more general distance estimate.

Corollary 2.2.5. *Suppose N and M are positive integers and let $E = \{z_1, ..., z_N\}$ be a set of non zero points in $\mathbb{D}$. Let $\mathcal{M} = \bigvee\{k_i : 1 \leq i \leq N\}$ and $\mathcal{R} = \bigvee\{1, z, ..., z^M\}$. If $\mathcal{I}^E$ is the ideal of functions in H^∞ vanishing on E and J is the weak* closed ideal of H^∞ genereated by the monomials $\{z^j\}_{j\geq M+1}$, then for any $F \in \mathcal{M}_n(H^\infty)$ the following distance formula holds:*

$$d(F, \mathcal{M}_n(\mathcal{I}^E \cap \mathcal{J})) = \|PM_f^* P\|,$$

where P is the orthogonal projection of H^2 onto $\mathcal{M} \vee \mathcal{R}$.

Proof. The proof will follow the same heuristic as all of distance formulas proved previously. If $\|PM_f^* P\| = 1$, then the previous result implies the existence of a $G \in \mathcal{M}_n(H^\infty)$ of norm at most 1 such that $F - G \in \mathcal{M}_n(\mathcal{I}^E \cap \mathcal{J})$ for all $F \in \mathcal{M}_n(H^\infty)$. It follows that

$$d(F, \mathcal{M}_n(\mathcal{I}^E \cap \mathcal{J})) \leq \|F - (F - G)\| \leq 1.$$

The reverse inclusion is similar to previous formulae. □

This leads one to conjecture a more general distance formula for H^∞. Its proof follows from commutant lifting that is identical to that of the other fomulae.

Theorem 2.2.6. *Suppose $\mathcal{J}$ is a weak* closed ideal of H^∞ and let $\mathcal{N}$ be the closure of $J\mathbf{1}$ in H^2. For any $F \in \mathcal{M}_n(H^\infty)$ the following distance formula holds:*

$$d(F, \mathcal{M}_n(\mathcal{J})) = \|PM_F P\|,$$

where P is the orthogonal projection of H^2 onto $\mathcal{N}^\perp \otimes \mathbb{C}^n$.

Proof. Since $\mathcal{J}$ is an ideal, the subspace $\mathcal{N}$ is H^∞-invariant. It follows that $PM_FP = M_FP$ for any $F \in \mathcal{M}_n(H^\infty)$. If $G \in \mathcal{M}_n(\mathcal{J})$, then $M_GP = 0$, which implies the easy inequality.

On the other hand, suppose $\|M_FP\| = 1$. The operators $(S \otimes I_n)P$ and M_FP commute, so we may apply the commutant lifting theorem to this pair. A contractive multiplier M_H is obtained such that $F - G \in \mathcal{M}_n(\mathcal{J})$. The reverse inequality follows. $\square$

Chapter 3

A Constrained Problem

The goal of this chapter is to solve the Pick problem with the additional constraint that the interpolating function f must satisfy $f'(0) = 0$. This is the simplest of derivative constraints that one could impose on the standard Pick problem, and is perhaps the first step in obtaining a larger generalization of Nevanlinna-Pick interpolation results. This problem was first studied by Davidson, Paulsen, Raghupathi, and Singh [**DPRS07**].

Let $E = \{z_1, ..., z_N\}$ be the interpolating set. If it so happens that $0 \in E$, then this problem is equivalent to the mixed Carathéodory-Pick problem studied in Chapter 2 when $M = 2$. When $0 \notin E$ the picture is much different. While an appropriate analogue to Pick's theorem can be proved, it requires that the interpolating function belongs to a certain subalgebra of H^∞. This algebra turns out to be simultaneously the multiplier algebra of a large family of Hilbert function spaces. We will introduce this algebra and prove its various properties (analogous to the standard results for H^∞) in Section 1 and prove the appropriate Pick interpolation theorems in Section 2. In Chapter 4, we will use the C^* approach to interpolation to show why matrix-valued interpolation fails for this algebra.

3.1 The Algebra H_1^∞

Let H_1^∞ denote the algebra of functions f in H^∞ satisfying $f'(0) = 0$. Our constrained interpolation problem is equivalent to standard Nevanlinna-Pick interpolation on this algebra. H_1^∞ is the weak-* closed, unital subalgebra of H^∞ generated by the functions z^2 and z^3. Unless stated explicitly, we make no assumptions as to whether or not $0 \in E$. Our first goal is to define a family of Hilbert function spaces for which H_1^∞ is the multiplier algebra. Let α and β be complex numbers satisfying $|\alpha|^2 + |\beta|^2 = 1$. Define the subspace $H^2_{\alpha,\beta}$ of H^2 by

$$H^2_{\alpha,\beta} := \bigvee\{\alpha + \beta z, z^2 H^2\}.$$

A few observations are immediate. $H^2_{\alpha,\beta}$ is invariant for H^∞_1 and $H^2_{\alpha,\beta} = H^2_{\delta,\gamma}$ if and only if (α,β) is a scalar multiple of (γ,δ). If $\alpha \neq 0$, then $H^2_{\alpha,\beta}$ is *not* invariant under H^∞ (try multiplication by z) and when $\alpha = 0$ we have $H^2_{\alpha,\beta} = zH^2$, which is invariant under H^∞. Each $H^2_{\alpha,\beta}$ is a closed subspace of H^2 and therefore is a Hilbert function space. Let $k_{\alpha,\beta}(z,w)$ denote its reproducing kernel. By Equation 1.3 we may explicitly construct these kernels:

$$\begin{aligned} k_{\alpha,\beta}(z,w) &= (\alpha+\beta z)\overline{(\alpha+\beta w)} + \sum_{n=2}^{\infty} z^n \overline{w^n} \\ &= (\alpha+\beta z)\overline{(\alpha+\beta w)} + \frac{z^2\overline{w}^2}{1-z\overline{w}}. \end{aligned}$$

We now prove the analogue of Beurling's theorem for the invariant subspaces of H^∞_1.

Theorem 3.1.1. *Let $\mathcal{N}$ be a closed subspace of H^2 which is invariant for H^∞_1 but not for H^∞. Then there are constants $\alpha \neq 0$ and β and an inner function J such that $\mathcal{N} = JH^2_{\alpha,\beta}$.*

Proof. Let $\mathcal{N}' = \overline{H^\infty \cdot \mathcal{N}}$. The subspace $\mathcal{N}'$ is invariant under H^∞, and therefore by Beurling's theorem there is an inner function J satisfying $\mathcal{N}' = JH^2$. Compute

$$z^2\mathcal{N}' = \overline{z^2H^\infty \cdot \mathcal{N}} \subset H^\infty_1 \cdot \mathcal{N} = \mathcal{N} \subset \mathcal{N}'. \tag{3.1}$$

By Equation 3.1, we have $z^2JH^2 \subset \mathcal{N} \subset JH^2$. Both containments are strict ($\mathcal{N}$ is not invariant under H^∞) which implies that $\mathcal{N}$ is a codimension 1 subspace of H^2. Thus, there exist $\alpha,\beta \in \mathbb{C}$ such that $\mathcal{N} = \vee\{\alpha+\beta z, z^2H^2\}$. Moreover, we may choose α and β from the unit sphere of $\mathbb{C}^2$, proving the theorem. $\square$

We use this characterization of the invariant subspaces of H^∞_1 to find the multiplier algebra of $H^2_{\alpha,\beta}$.

Proposition 3.1.2. *The multiplier algebra of $H^2_{\alpha,\beta}$ for $\alpha \neq 0$ is isometrically isomorphic to H^∞_1.*

Proof. Suppose $f \in \mathrm{Mult}(H^2_{\alpha,\beta})$. Since $z^2 \in H^2_{\alpha,\beta}$ it follows that $z^2 f$ is analytic. Write $f(z) = \sum_{n=-2}^{\infty} a_n z^n$ and note that $(\alpha+\beta z)f(z) \in H^2_{\alpha,\beta}$. Examining the Taylor expansion about 0 implies that $a_{-2} = a_{-1} = 0$. Moreover, equating the zeroth and first terms of this Taylor expansion yields

$$a_0\alpha + (a_0\beta + a_1\alpha)z = \gamma(\alpha+\beta z)$$

for some γ. This implies $a_0 = \gamma$ and $a_1 = 0$, as desired. Since $H^2_{\alpha,\beta}$ is a Hilbert function space, we have $\|f\|_\infty = \|M_f\|$.

Conversely, the previous theorem shows that any $f \in H^\infty_1$ is a multiplier of $H^2_{\alpha,\beta}$. $\square$

3.2 Proof of the Interpolation Theorems

The goal of this section is to prove a Pick interpolation theorem for H_1^∞. Unfortunately, the positivity of a single matrix is not sufficient to ensure the existence of an interpolating function. We will show that the existence of an interpolating function of norm at most 1 is equivalent to the positivity of the Pick matrix

$$\mathbb{P}_{\alpha,\beta} := [(1 - w_i\overline{w_j})k_{\alpha,\beta}(z_i, z_j)] \geq 0$$

for each Hilbert function space $H^2_{\alpha,\beta}$. Our proof of this will be in the spirit of the original approach used by Sarason, which relies heavily upon the duality between L^1 and L^∞. The factorization methods developed in Section 4 of Chapter 1 will be applied to this problem with a few minor changes.

The natural question to ask is whether the generalized methods developed in Chapter 2 provide a direct method of proving this result. In the case of H^∞, a single operator (the shift) generated the whole algebra, and in order to apply commutant lifting the operator R needed to commute only with the shift restricted to the finite dimensional subspace $\mathcal{M}$. In our present context, we would need a similar operator to commute with analogous restrictions of M_{z^2} and M_{z^3}, and then lift all three operators simultaneously to operators on $H^2_{\alpha,\beta}$. As Ando's theorem fails for more than two commuting contractions, there is perhaps little to gain by applying the theory of commuting contractions.

Let $E = \{z_1, ..., z_N\}$ and let $\mathcal{I}_1^E$ denote the ideal of functions in H_1^∞ vanishing on E. If $0 \in E$, then, as a convention, we will always assume that $z_1 = 0$. Let $\phi_\lambda(z) = \frac{z-\lambda}{1-\overline{\lambda}z}$ denote the elementary Möbius map that sends λ to 0. Let $B_E = \Pi_{i=1}^N \phi_{z_i}$ be the finite Blaschke product with simple zeroes in E, and let c_0 and c_1 denote its zeroth and first Taylor coefficients about 0, respectively, normalized so that $|c_0|^2 + |c_1|^2 = 1$. For $1 \leq p \leq \infty$, define $H^p_{a,b}$ as the closure of $\vee\{a + bz, z^2H^p\}$ in the relevant topology on H^p.

Lemma 3.2.1. $\mathcal{I}_1^E = B_E H^\infty_{c_0,-c_1}$.

Proof. Suppose first that $0 \notin E$. Then we have

$$\mathcal{I}_1^E = H_1^\infty \cap B_E H^\infty = z^2 B_E H^\infty + \mathbb{C}B_E(c_0 - c_1 z),$$

which follows from the fact that $c_0 - c_1 z$ is the function that multiplies B_E into H_1^∞. On the other hand, if $0 \in E$, write $E = E' \cup \{0\}$. Then $B_E = zB_{E'}$ and

$$\mathcal{I}_1^E = H_1^\infty \cap B_E H^\infty = z^2 B_{E'} H^\infty = B_E H^\infty_{0,-1} = B_E H^\infty_{0,1},$$

as desired. □

This allows us to explicitly compute the pre-annhilator of $\mathcal{I}_1^E$ in $L^1(\mathbb{T})$.

Lemma 3.2.2. *The pre-annihilator of* $\mathcal{I}_1^E$ *in* $L^1(\mathbb{T})$ *is* $\overline{zB_E}H^1_{c_0,c_1}$.

Proof. Since $z^2 B_E H^\infty \subset \mathcal{I}_1^E$, we have

$$(\mathcal{I}_1^E)_\perp \subset (z^2 B_E H^\infty)_\perp = \overline{z^2 B_E} H_0^1 = \overline{z B_E} H^1.$$

With $g \in (\mathcal{I}_1^E)_\perp$ write $g = \overline{z B_E} g_0$ where $g_o \in H^1$. Any function in $(\mathcal{I}_1^E)_\perp$ must be orthogonal to $B_E(c_0 - c_1 z)$; hence, we have

$$\begin{aligned} 0 &= \int B_E(c_0 - c_1 z) g = \int B_E(c_0 - c_1 z)\overline{z B_E} g_0 \\ &= c_0 \int \overline{z} g_0 - c_1 \int g_0 = c_0 g_0'(0) - c_1 g_0(0) \end{aligned}$$

hence $g_0 \in H^1_{c_0,c_1}$. The reverse inclusion is clear by the previous lemma. □

We will now prove a factorization lemma that is pivotal to the main result. Let $\mathcal{N}_{\alpha,\beta}$ denote the set of functions in $H^2_{\alpha,\beta}$ vanishing on E and let $k_{\alpha,\beta}(z_i) := k_{\alpha,\beta}(z,, z_i)$ be the reproducing kernel at z_i. The orthogonal complement of $\mathcal{N}_{\alpha,\beta}$ in $H^2_{\alpha,\beta}$ is easily seen to be $\mathcal{M}_{\alpha,\beta} := \vee\{k_{\alpha,\beta}(z_i)\}$.

Lemma 3.2.3. *If g is in the pre-annhilator of $\mathcal{I}_1^E$, then there are scalars $\alpha \neq 0$ and β such that $g = \overline{h} k$ where $k \in H^2_{\alpha,\beta}$, $h \in L^2(\mathbb{T})$ and h is orthogonal to $\mathcal{N}_{\alpha,\beta}$. Moreover, $\|g\|_1 = \|h\|_2^2 = \|k\|_2^2$.*

Conversely, if $g \in L^1(\mathbb{T})$ satisfies $g = \overline{h} k$ where $k \in H^2_{\alpha,\beta}$ and $h \in L^2(\mathbb{T})$ is orthogonal to $\mathcal{N}_{\alpha,\beta}$, then g is in the pre-annihilator of $\mathcal{I}_1^E$.

Proof. Let $g \in (\mathcal{I}_1^E)_\perp$ where $\|g\|_1 = 1$ and write $g = \overline{z B_E} g_0$ where $g_0 \in H^1_{c_0,c_1}$. As in Corollary 1.3.5, factor $g_0 = k h_0$ where $k, h_0 \in H^2$ and k is outer, so that

$$\|k\|_2^2 = \|h_0\|_2^2 = \|g_0\|_1 = \|g\|_1.$$

Since k is outer, it must be the case that $\hat{k}(0) \neq 0$. Define

$$\begin{aligned} \alpha &= \frac{\hat{k}(0)}{\sqrt{|\hat{k}(0)|^2 + |\hat{k}(1)|^2}} \\ \beta &= \frac{\hat{k}(1)}{\sqrt{|\hat{k}(0)|^2 + |\hat{k}(1)|^2}} \end{aligned}$$

so that $k \in H^2_{\alpha,\beta}$. Write $h = z B_E \overline{h_0}$, from which it follows that $g = \overline{h} k$. Now we prove the orthogonality conditions. Since outer functions are cyclic for H^∞, we have

$$\overline{H_1^\infty k} = \mathbb{C} k + \overline{z^2 H^\infty k} = \mathbb{C} k + z^2 H^2 = H^2_{\alpha,\beta};$$

hence k is cyclic for H_1^∞ on $H^2_{\alpha,\beta}$. Also, $\mathcal{I}_1^E H^2_{\alpha,\beta}$ is dense $\mathcal{N}_{\alpha,\beta}$. Indeed, $\mathcal{I}_1^E H^2_{\alpha,\beta}$ vanishes on E and so is contained in $\mathcal{N}_{\alpha,\beta}$ and the closure of $\mathcal{I}_1^E H^2_{\alpha,\beta}$ has codimension N. This shows that the set $\mathcal{I}_1^E k$ is dense in $\mathcal{N}_{\alpha,\beta}$. It follows that h is orthogonal to $\mathcal{N}_{\alpha,\beta}$ by taking an arbitrary element $f \in \mathcal{I}_1^E$ and calculating:

$$\langle fk, h\rangle = \langle fk, zB_E\overline{h_0}\rangle = \int f\overline{zB_E}kh_0 = \int fg = 0.$$

To prove the converse, note that if a function $g = \overline{h}k$ has the above decomposition, then reversing the above calculation shows that g must annihilate $\mathcal{I}_1^E$. $\square$

Theorem 3.2.4. *Suppose $z_1, ..., z_N \in \mathbb{D}$ and $w_1, ..., w_N \in \mathbb{C}$. There is a function $f \in H_1^\infty$ such that $\|f\|_\infty \leq 1$ and $f(z_i) = w_i$ for each i if and only if*

$$[(1 - w_i\overline{w_j})k_{\alpha,\beta}(z_i, z_j)] \geq 0$$

for all $|\alpha|^2 + |\beta|^2 = 1$.

Proof. Necessity is clear by Theorem 1.2.5. For sufficiency, note that since the algebra of polynomials satisfying $p'(0) = 0$ separates points on the disk, we may choose a polynomial $p \in H_1^\infty$ that interpolates the given data. As we have seen previously, the positivity of the matrix stated in the theorem implies that $\|M_p^*|_{\mathcal{M}_{\alpha,\beta}^E}\| \leq 1$ for any α and β.

Let $P_{\alpha,\beta}$ be the orthogonal projection of $L^2(\mathbb{T})$ onto $H^2_{\alpha,\beta}$ for all α and β. Define a linear functional ζ on $(\mathcal{I}_E)_\perp$ by $\zeta(f) = \int pf$. We claim that ζ has norm at most 1. Indeed, take any $g \in (\mathcal{I}_E)_\perp$ and factor it as $g = \overline{h}k$ as in Lemma 3.2.3. Now we must *fix* α and β as precisely those obtained in the factorization. Since h is orthogonal to $\mathcal{N}_{\alpha,\beta}$, we have $P_{\alpha,\beta}(h) \in \mathcal{M}_{\alpha,\beta}$. Now compute

$$\begin{aligned}\zeta(g) &= \int pk\overline{h} = \langle pk, h\rangle = \langle P_{\alpha,\beta}pk, h\rangle = \langle pk, P_{\alpha,\beta}h\rangle \\ &= \langle k, M_p^* P_{\alpha,\beta}h\rangle = \langle k, M_p^*|_{\mathcal{M}_{\alpha,\beta}} P_{\alpha,\beta}h\rangle.\end{aligned}$$

Since $\|g\|_1 = \|h\|_2^2 = \|k\|_2^2$ we have

$$|\zeta(g)| \leq \|M_p^*|_{\mathcal{M}_{\alpha,\beta}^E}\| \leq 1$$

for $\|g\|_1 \leq 1$. By the Hahn-Banach theorem, we may extend ζ to a linear functional on all of $L^1(\mathbb{T})$ of norm at most 1. As L^∞ is the dual of L^1, there is an $f \in L^\infty(\mathbb{T})$ such that $\|f\|_\infty \leq 1$ and $\zeta(g) = \int fg = \int pg$ for each $g \in (\mathcal{I}_1^E)_\perp$. Thus, $f - p \in ((\mathcal{I}_1^E)_\perp)^\perp = \mathcal{I}_1^E$, which implies that $f \in H_1^\infty$ and $f - p$ vanishes on E. This proves the theorem. $\square$

Note that in the above proof, the choice of α and β depends on the factorization of the function g in $(\mathcal{I}_1^E)_\perp$. The proof of this factorization is existential; it is unknown what choice of α and β is used in the lemma. The natural question to ask is if every choice of $|\alpha|^2 + |\beta|^2 = 1$ (or at least a dense subset) is required. Indeed, it can be shown that even in the case where $N = 2$, every $|\alpha| > 2^{-1/2}$ is required **[DPRS07]**. Confirming the positivity of an infinite number of matrices is not always computationally feasible, and therefore we provide the following complementary theorem.

Theorem 3.2.5. *Suppose $z_1, ..., z_N \in \mathbb{D}$ and $w_1, ..., w_N \in \mathbb{C}$. There is a function $f \in H_1^\infty$ such that $\|f\|_\infty \leq 1$ and $f(z_i) = w_i$ for each i if and only if there exists $\lambda \in \mathbb{D}$ such that*

$$\left[\frac{z_i^2\overline{z_j^2} - \phi_\lambda(w_i)\overline{\phi_\lambda(w_j)}}{1 - z_i\overline{z_j}}\right] \geq 0.$$

Proof. Suppose there is an $f \in H_1^\infty$ that interpolates the given data and $\|f\|_\infty \leq 1$. First assume that $0 \notin E$. Let $\lambda := f(0) \in \mathbb{D}$ and set $g(z) = \phi_\lambda(f(z))$. Then $g \in H^\infty$ since $\|g\|_\infty \leq 1$ and $g'(0) = \phi(f(0))f'(0) = 0$. Moreover, $g(0) = \phi_\lambda(\lambda) = 0$ so we may define $h(z) = z^{-2}g(z) \in H^\infty$ (note that $\|h\|_\infty \leq 1$). Since the Szegö kernel has the scalar Pick property, the matrix

$$\left[\frac{1 - h(z_i)\overline{h(z_j)}}{1 - z_i\overline{z_j}}\right]$$

is positive semidefinite. It follows that

$$\left[\frac{z_i^2\overline{z_j^2} - \phi_\lambda(w_i)\overline{\phi_\lambda(w_j)}}{1 - z_i\overline{z_j}}\right] = \left[\frac{z_i^2\overline{z_j^2} - g(z_i)\overline{g(z_j)}}{1 - z_i\overline{z_j}}\right]$$

which is equal to

$$\begin{bmatrix} z_1^2 & 0 & 0 & 0 \\ 0 & z_2^2 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & z_N^2 \end{bmatrix} \left[\frac{1 - h(z_i)\overline{h(z_j)}}{1 - z_i\overline{z_j}}\right] \begin{bmatrix} \overline{z_1}^2 & 0 & 0 & 0 \\ 0 & \overline{z_2}^2 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \overline{z_N}^2 \end{bmatrix},$$

is positive semidefinite as well. Note that this decomposition is still valid if $0 \in E$ since the first column and first row of this matrix will be zero.

Conversely, if the matrix stated in the theorem is positive semidefinite, then the previous calculation shows that

$$\left[\frac{1 - z_i^{-2}\phi_\lambda(w_i)\overline{z_j^{-2}\phi_\lambda(w_j)}}{1 - z_i\overline{z_j}}\right] \geq 0. \tag{3.2}$$

for $0 \notin E$. By the Pick theorem, this implies there is a function $h \in H^\infty$ such that $\|h\|_\infty \leq 1$ and $h(z_i) = z_i^{-2}\phi_\lambda(w_i)$. It follows that the function $f(z) = \phi_{-\lambda}(z^2h(z))$ is the required interpolant in H_1^∞. If $0 \in E$, note that the matrix stated in the theorem is positive if and only if $\lambda = w_1$. The same argument may then be used (with the first row and column of the matrix in (3.2) being equal to 0). □

3.3 Distance Formulae

The goal of this section is to prove the H_1^∞ analogues of the distance formulae established in Section 4 of Chapter 1. For $f \in H_1^\infty$, it is easily verified that $\|M_f^*|_{\mathcal{M}_{\alpha,\beta}}\|$ is a lower bound for $d(f, \mathcal{I}_1^E)$ for any $|\alpha|^2 + |\beta|^2 = 1$. Also note that this infimum is attained since $\mathcal{I}_1^E$ is weak* closed. Just as in Theorem 1.4.3, it turns out that this is the best we can do.

Theorem 3.3.1. *Let $E = \{z_1, ..., z_N\} \in \mathbb{D}$. For any $f \in H_1^\infty$*

$$d(f, \mathcal{I}_1^E) = \sup_{|\alpha|^2+|\beta|^2=1} \|M_f^*|_{\mathcal{M}_{\alpha,\beta}}\|. \tag{3.3}$$

Proof. By Lemma 1.4.1, we have

$$d(f, \mathcal{I}_1^E) = \sup\left\{\left|\int fg\right| : g \in (\mathcal{I}_1^E)_\perp, \|g\|_1 \leq 1\right\}.$$

Now factor $g = \overline{h}k$ as in Lemma 3.2.3. It follows that

$$\begin{aligned}
\sup\left\{\left|\int fg\right| : g \in (\mathcal{I}_1^E)_\perp, \|g\|_1 \leq 1\right\} &= \sup\left\{\left|\int fk\overline{h}\right|\right\} \\
&= \sup|\langle fk, h\rangle| \\
&= \sup|\langle P_{\alpha,\beta} fk, h\rangle| \\
&= \sup|\langle k, M_f^* P_{\alpha,\beta} h\rangle| \\
&\leq \sup_{|\alpha|^2+|\beta|^2=1} \|M_f^*|_{\mathcal{M}_{\alpha,\beta}}\|,
\end{aligned}$$

since $P_{\alpha,\beta}h \in \mathcal{M}_{\alpha,\beta}$, where the supremums on the right hand side are taken over the set:

$$\{\alpha, \beta \in \mathbb{C}, k \in H^2_{\alpha,\beta}, h \in L^2(\mathbb{T}) : |\alpha|^2 + |\beta|^2 = 1, \|k\|_2 \leq 1, \|h\|_2 \leq 1, h \perp \mathcal{N}_{\alpha,\beta}\}.$$

The reverse inequality is identical to the H^∞ case. □

Since $\sup_{|\alpha|^2+|\beta|^2=1} \|M_f^*|_{\mathcal{M}_{\alpha,\beta}}\| \leq 1$ is equivalent to the positivity of the H_1^∞ Pick matrices, Theorem 3.3.1 implies that there exists an interpolating function if and only if $d(f, \mathcal{I}_1^E) \leq 1$ for all $f \in H_1^\infty$. The natural question to ask is whether or not the vector-valued distance formula also holds:

$$d\left(F, \mathcal{M}_n(\mathcal{I}_1^E)\right) \stackrel{?}{=} \sup_{|\alpha|^2+|\beta|^2=1} \|M_F^*|_{\mathcal{M}_{\alpha,\beta}\otimes\mathbb{C}^n}\|. \tag{3.4}$$

As we have seen, this is equivalent to matrix interpolation in the algebra H_1^∞. In Chapter 4, we will show indirectly that there are functions for which this distance formula fails.

To conclude this chapter, we compute the H_1^∞ analogue of Nehari's theorem.

Corollary 3.3.2. *If $f \in L^\infty(\mathbb{T})$, then*

$$d(f, H_1^\infty) = \sup_{|\alpha|^2+|\beta|^2=1} \|(I - P_{\alpha,\beta})M_f P_{\alpha,\beta}\|.$$

Proof. Applying Lemma 3.2.3 to the case where E is empty, we may factor $g \in (H_1^\infty)_\perp$ as $g = \overline{h}k$, where $k \in H^2_{\alpha,\beta}$ for some α, β and h is orthogonal to $\mathcal{N}_{\alpha,\beta}(= H^2_{\alpha,\beta})$. Moreover, $\|k\|_2 = \|h\|_2 = \|g\|_1$. Conversely, every product of this form lies in $(H_1^\infty)_\perp$. It follows that

$$\begin{aligned}
d(f, H_1^\infty) &= \sup_{g \in (H_1^\infty)_\perp \cdot \|g\|_1 \le 1} \left| \int fg \right| \\
&= \sup_{|\alpha|^2+|\beta|^2=1} \{|\langle fk, h\rangle| : h \perp H^2_{\alpha,\beta}, \|h\|_2 \le 1; k \in H^2_{\alpha,\beta}, \|k\|_2 \le 1\} \\
&= \sup_{|\alpha|^2+|\beta|^2=1} \|(I - P_{\alpha,\beta})M_f P_{\alpha,\beta}\|.
\end{aligned}$$

□

Chapter 4

The C^* Approach

Using our notation from previous chapters, let $\mathcal{Q}$ denote the quotient algebra

$$H^\infty/\mathcal{I}_E$$

where E is a set of N distinct points in the disk and $\mathcal{I}_E$ is the ideal of functions in H^∞ vanishing on E. Our first goal is to represent $\mathcal{Q}$ on $\mathbb{C}^n$. This is relatively simple since the finite dimensional operator R completely determines the optimal multiplier.

Theorem 4.0.3. *Let $Q = [k(z_i, z_j)]_{i,j=1}^N$ and $D_f = \text{diag}\{f(z_1), ..., f(z_N)\}$ be matrices in $\mathcal{M}_N$. The homomorphism $\rho : \mathcal{Q} \to \mathcal{M}_N$ given by*

$$\rho(f + \mathcal{I}^E) = Q^{-1/2} D Q^{1/2}$$

is an isometry.

Proof. First note that the matrix Q is positive since it is the Pick matrix for the zero function. Hence the matrix $Q^{1/2}$ exists. As before, let $\mathcal{M} := \vee\{k_1, .., k_N\}$ and define an invertible operator $S : \mathcal{M} \to \mathbb{C}^n$ by $S = [k_1, ..., k_N]^*$ where the action of S is given by $Sf = (\langle f, k_1\rangle, ..., \langle f, k_N\rangle) = (f(z_1), ..., f(z_N))$ for $f \in \mathcal{M}$. Note that $SS^* = Q$. Consider the representation $\Phi : H^\infty \to \mathcal{M}_N$ given by $\Phi(f) = P_\mathcal{M} M_f|_\mathcal{M}$. Since $\Phi(f) = P_\mathcal{M} M_f^*|_\mathcal{M}$ is the operator sending k_i to $\overline{f(z_i)}k_i$, it follows that $\Phi(f) = P_\mathcal{M} M_f^*|_\mathcal{M} = S^{-1}D_f^* S$. Hence

$$\Phi(f) = S^* D_f (S^{-1})^*.$$

Let $S^* = UQ^{1/2}$ be the polar decomposition of S^* where U is a unitary onto $\mathcal{M}$. We have

$$\Phi(f) = S^* D(S^{-1})^* = UQ^{1/2} D(UQ^{1/2})^{-1} = UQ^{1/2} DQ^{-1/2} U^*.$$

Finally, Φ induces a natural representation $\tilde{\Phi}$ of the quotient $\mathcal{Q}$ on $\mathbb{C}^n$, which is an isometry by the H^∞ distance formula proved in Theorem 1.4.3 since $\|f + \mathcal{Q}\| = \|\Phi(f)\|$. As ρ is unitarily equivalent to $\tilde{\Phi}$, we are done. $\square$

The analogous result holds for the vector-valued case as well. First, we must briefly discuss the notion of a *completely* isometric map. If $\mathcal{A}$ is a C^*-algebra, we may form the tensor product $\mathcal{M}_n \otimes \mathcal{A}$, which can be naturally identified with $\mathcal{M}_n(\mathcal{A})$: the algebra of n by n matrices with entries in $\mathcal{A}$. Let $[A_{ij}]$ denote an element in $\mathcal{M}_n(\mathcal{A})$, where each $A_{ij} \in \mathcal{A}$. If $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras and $\nu : \mathcal{A} \to \mathcal{B}$ is a bounded linear map, the map $\nu^{(n)} : \mathcal{M}_n(\mathcal{A}) \to \mathcal{M}_n(\mathcal{B})$ is defined by

$$\nu^{(n)}([A_{ij}]) = [\nu(A_{ij})].$$

Note that each $\nu^{(n)}$ is bounded with $\|\nu^{(m)}\| \leq \|\nu^{(n)}\|$ for $m \leq n$. If each $\nu^{(n)}$ is a contraction (resp. isometry), we say that ν is a *complete contraction* (resp. *complete isometry*). Similarly, we say ν is *positive* if $\nu(A^*A)$ is a positive element of $\mathcal{B}$ for each $A \in \mathcal{A}$, and ν is *completely positive* if each $\nu^{(n)}$ is positive.

Suppose $\mathcal{H}_k \otimes \mathbb{C}^n$ is a vector-valued Hilbert function space and recall that we may naturally identify $\mathrm{Mult}(\mathcal{H}_k \otimes \mathbb{C}^n)$ with $\mathcal{M}_n(\mathrm{Mult}(\mathcal{H}_k))$. With $\mathcal{Q} = H^\infty/\mathcal{I}_E$ we may endow the algebra $\mathcal{M}_n(\mathcal{Q})$ with a norm via the identification

$$\mathcal{M}_n(\mathcal{Q}) \cong \mathcal{M}_n(H^\infty)/\mathcal{M}_n(\mathcal{I}_E).$$

Suppose $F \in \mathcal{M}_n(H^\infty)$ and let $D = \mathrm{diag}\{F(z_1), ..., F(z_N)\} \in \mathcal{M}_N(\mathcal{M}_n)$. Let $Q \in \mathcal{M}_N(\mathcal{M}_n)$ where the $(m,l)^{th}$-entry of the $(j,i)^{th}$-block of Q is given by $\langle k_i \otimes e_m, k_j \otimes e_l \rangle$. Using the same argument as in Theorem 4.0.3, we have that the homomorphism

$$\pi : F + \mathcal{M}_n(\mathcal{I}_E) \mapsto Q^{-1/2} D Q^{1/2} \tag{4.1}$$

is an isometry. Let ρ be the homomorphism from $\mathcal{Q}$ into $\mathbb{C}^n$ given in Theorem 4.0.3. By applying the canonical shuffle, which identifies the unitary equivalence between $\mathcal{M}_N(\mathcal{M}_n)$ and $\mathcal{M}_n(\mathcal{M}_N)$, we see that π is unitarily equivalent to $\rho^{(n)}$. We summarize this result in the following theorem.

Theorem 4.0.4. *Let $Q = [k(z_i, z_j)]_{i,j=1}^N$ and $D = \mathrm{diag}\{f(z_1), ..., f(z_N)\}$ be matrices in $\mathcal{M}_N$. The homomorphism $\rho : \mathcal{Q} \to \mathcal{M}_N$ given by*

$$\rho(f + \mathcal{I}^E) = Q^{-1/2} D Q^{1/2}$$

is a complete isometry.

It turns out that the homomorphism ρ is, in some sense, the *minimal* completely isometric representation of $\mathcal{Q}$ on a Hilbert space. Roughly speaking, any C^*-algebra containing a completely isometric image of $\mathcal{Q}$ has $\mathcal{M}_N$ as a quotient. We will formalize this notion in the following section, where we develop the *C^*-envelope* of an operator algebra. The existence of such an object was first demonstrated by Hamana **[Ham79]**, and it was first applied to interpolation problems by McCullough and Paulsen [**McP02**]. Determining the C^*-envelope of an interpolation problem reveals many underlying complexities, as we shall observe in the last section of this chapter.

In general, computing C^*-envelopes is difficult, but allows one to manipulate the underlying algebra in a more tractable way. In the case of interpolation problems, finding completely isometric representations is motivated by the more general problem of representing quotient algebras. In Section 2 of this chapter, we will compute the C^*-envelopes for the constrained interpolation problem in Chapter 3 for the case where $0 \in E$. We will also discuss the case where $0 \notin E$ and why the computation of the associated envelope is so difficult. This allows us to indirectly prove that matrix interpolation fails in this case.

4.1 Operator Algebras and Completely Isometric Representations

The goal of this section is to introduce the C^*-envelope of an operator algebra and apply it immediately to interpolation problems. This rich and beautiful theory is developed fully in [**Pau02**], and we will primarily restrict ourselves to only the statements of theorems.

An *operator algebra* will be regarded simply as a subalgebra of a C^*-algebra. In general, operator algebras are not self-adjoint. For instance, the only normal elements in H^∞ are the scalars. It is often useful to somehow embed an operator algebra $\mathcal{A}$ into a C^*-algebra in order to gain a better understanding of $\mathcal{A}$. A principle example of this would be the algebra $\mathcal{Q} = H^\infty/\mathcal{I}^E$. Without the interpolation theory developed in the previous chapters, the problem of computing norms in $\mathcal{M}_n(\mathcal{Q})$ would be extremely complex.

Definition 4.1.1. A C^*-cover of an operator algebra $\mathcal{A}$ is a pair $(\mathcal{C}, \phi)$ such that $\mathcal{C}$ is a C^*-algebra and $\phi : \mathcal{A} \to \mathcal{C}$ is a completely isometric homomorphism.

Example 4.1.2. Consider the disk algebra $A(\mathbb{D}) = \{f : \mathbb{D} \to \mathbb{C} : f$ is analytic on $\mathbb{D}$ and $f|_{\mathbb{T}}$ is continuous $\}$. We may regard $A(\mathbb{D})$ as a uniformly closed subalgebra of either $C(\mathbb{D}^-)$ or $C(\mathbb{T})$. $A(\mathbb{D})$ separates the points of both spaces, so the C^*-algebra it generates can be either. That is, the C^*-algebra generated by an operator algebra depends on its representation. There is still an obvious connection between these two C^*-algebras; the restriction $C(\mathbb{D}^-)|_{\mathbb{T}}$ defines a $*$-homomorphism from $C(\mathbb{D}^-)$ onto $C(\mathbb{D})$ that is the identity on $A(\mathbb{D})$. Thus, $C(\mathbb{T})$ is a *smaller* representation of $A(\mathbb{D})$ in the sense that it is a quotient of $C(\mathbb{D}^-)$.

In the above example, it is not entirely clear if $C(\mathbb{T})$ is the best we can do. Our goal is to find the smallest C^*-cover for any operator algebra $\mathcal{A}$. Stated precisely, we wish to find a C^*-cover $(C_e^*(\mathcal{A}), \gamma)$ of $\mathcal{A}$ that satisfies the following universal property: if $\rho : \mathcal{A} \to B(\mathcal{K})$ is any completely isometric homomorphism for some Hilbert space $\mathcal{K}$, then there exists a surjective $*$-homomorphism $\pi : C^*(\rho(\mathcal{A})) \to C^*(\gamma(\mathcal{A}))$ such that

$\pi(\rho(a)) = \gamma(a)$ for all $a \in \mathcal{A}$. Hamana **[Ham79]** proved that $C_e^*(\mathcal{A})$ always exists, and we call it the *C^*-envelope of $\mathcal{A}$.*

Theorem 4.1.3 (**Hamana**). *Let $\mathcal{A}$ be a unital operator algebra and let $\rho : \mathcal{A} \to B(\mathcal{H})$ be a completely isometric homomorphism. There exists a surjective $*$-homomorphism $\pi : C^*(\rho(\mathcal{A})) \to C_e^*(\mathcal{A})$ with $\pi(\rho(a)) = a$ for all a in $\mathcal{A}$.*

The C^*-envelope of an operator algebra is necessarily unique, up to $*$-isomorhpism. To see this, suppose γ_1 and γ_2 are two completely isometric homomorphisms acting on $\mathcal{A}$ satisfying the above universal property. Then there are surjective $*$-homomorphisms $\pi_1 : C^*(\gamma_2(\mathcal{A})) \to C^*(\gamma_1(\mathcal{A}))$ and $\pi_2 : C^*(\gamma_1(\mathcal{A})) \to C^*(\gamma_2(\mathcal{A}))$ satisfying $\pi_1(\gamma_1(a)) = \gamma_2(a)$ and $\pi_2(\gamma_2(a)) = \gamma_1(a)$ for $a \in \mathcal{A}$. It follows that $\pi_2\pi_1 = I_{C^*(\gamma_1(\mathcal{A}))}$ and $\pi_1\pi_2 = I_{C^*(\gamma_2(\mathcal{A}))}$, and so π_1 and π_2 are $*$-isomorphisms. Also note that if $C^*(\mathcal{A})$ is simple, then $C_e^*(\mathcal{A}) = C^*(\mathcal{A})$. This follows from the fact that there is a surjective homomorphism $\pi : C^*(\mathcal{A}) \to C_e^*(\mathcal{A})$, and simplicity implies that π is also an injection.

We may now calculate the C^*-envelope of the algebra $\mathcal{Q}$.

Corollary 4.1.4. *Suppose $z_1, ..., z_N$ are distinct points in $\mathbb{D}$ and let $\mathcal{I}_E$ denote the ideal of functions in H^∞ vanishing at each z_i. The C^*-envelope of the quotient algebra $\mathcal{Q} = H^\infty/\mathcal{I}^E$ is $\mathcal{M}_N$.*

Proof. By Theorem 4.0.4, there is a completely isometric homomorphism $\rho : \mathcal{Q} \to \mathcal{M}_N$. Since $\mathcal{M}_N$ is simple, it suffices to show that $\mathcal{B} := C^*(\rho(\mathcal{Q})) = \mathcal{M}_N$. Consider the Lagrange polynomials $p_i = \prod_{j\neq i}(z - z_j)(z_i - z_j)^{-1}$ for $1 \leq i \leq N$. The commuting idempotents $p_i + \mathcal{I}^E$ generate all of $\mathcal{Q}$. Let E_{ij} denote the matrix units of $\mathcal{M}_N$. We have $\rho(p_i) = Q^{-1/2}E_{ii}Q^{1/2} \in \mathcal{B}$ which implies

$$\rho(p_i)\rho(p_i)^* = Q^{-1/2}E_{ii}QE_{ii}Q^{-1/2} = q_{ii}Q^{-1/2}E_{ii}Q^{-1/2} \in \mathcal{B},$$

where $q_{ii} = (1 - |z_i|^2)^{-1}$ is the $(i,i)^{th}$-entry of Q. Thus $Q = \sum_i q_{ii}^{-1}\rho(p_i)\rho(p_i)^* \in \mathcal{B}$, and so Q^{-1} and $Q^{1/2}$ are in $\mathcal{B}$ as well. Thus $E_{ii} \in \mathcal{B}$ for $1 \leq i \leq N$, and since $q_{ij}E_{ij} = E_{ii}QE_{jj} \in \mathcal{B}$ and the q_{ij} are never 0, we also have $E_{ij} \in \mathcal{B}$ for $1 \leq i, j \leq N$. This proves the result. ∎

4.2 The C^* Approach to Interpolation in H_1^∞

Recall that the kernel function for $H^2_{\alpha,\beta}$ is given by

$$k_{\alpha,\beta}(z,w) = (\alpha + \beta z)\overline{(\alpha + \beta w)} + \frac{z^2\overline{w}^2}{1 - z\overline{w}}.$$

It will be important to use each choice of α and β exactly once. Since, for any $0 \leq \theta < 2\pi$, the kernels $k_{\alpha,\beta}$ and $k_{e^{i\theta}\alpha, e^{i\theta}\beta}$ coincide, we may parameterize these kernels by the set of complex lines in $\mathbb{C}^2$. This set may be identified topologically with the complex projective 2-sphere, which we denote PS^2.

We now turn to matrix interpolation. Given $z_1, ...z_N \in \mathbb{D}$ and $W_1, ..., W_N \in \mathcal{M}_n$, the existence of an interpolating function in $\mathcal{M}_n(H_1^\infty)$ of norm at most 1 implies that the Pick matrices

$$\left[(I - W_i W_j^*) k_{\alpha,\beta}(z_i, z_j)\right]$$

are positive semidefinite for all $|\alpha|^2 + |\beta|^2 = 1$. Our goal is to use the C^*-approach to show that the converse of this fails.

Let $\mathcal{Q}_1$ denote the quotient algebra $H_1^\infty / \mathcal{I}_1^E$ and let (α, β) be a point in PS^2. For $f \in H_1^\infty$, consider the compression $P_{\mathcal{M}_{\alpha,\beta}} M_f|_{\mathcal{M}_{\alpha,\beta}}$, where $\mathcal{M}_{\alpha,\beta} = \vee\{k_{\alpha,\beta}(z, z_i)\}_{i=0}^N$ as before. We may regard this operator as a continuous function from PS^2 into $\mathcal{M}_N$ since the map $(\alpha, \beta) \mapsto P_{\mathcal{M}_{\alpha,\beta}}$ is continuous. We may then define a map $\Phi_E : H_1^\infty \to C(PS^2, \mathcal{M}_N)$ by

$$\Phi_E(f)(\alpha, \beta) = P_{\mathcal{M}_{\alpha,\beta}} M_f|_{\mathcal{M}_{\alpha,\beta}}.$$

Since the $\ker(\Phi_E) = \mathcal{I}_1^E$, we induce the natural quotient map $\tilde{\Phi}_E : \mathcal{Q}_1 \to C(PS^2, \mathcal{M}_N)$, the continuous $\mathcal{M}_N$-valued functions acting on PS^2. Theorem 3.3.1 says that $\tilde{\Phi}_E$ is isometric.

Consider now the case where $0 \in E$. In this instance, $\mathcal{I}_1^E = zB_E H^\infty$, which is an ideal in H^∞ as well. Consequently, we may compute $d(f, \mathcal{I}_1^E)$ directly for $f \in H_1^\infty$. Letting P be the orthogonal projection of $L^\infty(\mathbb{T})$ onto H^∞ and using Nehari's theorem (Theorem 1.4.4), compute:

$$\begin{aligned}
d(f, \mathcal{I}_1^E) &= d(\overline{zB_E} f, H^\infty) \\
&= \left\| (I - P) M_{\overline{zB_E} f} P \right\| \\
&= \left\| (I - P) M_{zB_E}^* M_f P \right\| \\
&= \left\| (I - P) M_{zB_E}^* P M_f P \right\| \\
&= \left\| P M_f^* P M_{zB_E} (I - P) \right\| \\
&= \left\| P M_f^* |_{H^2 \ominus zB_E H^2} \right\|.
\end{aligned}$$

Write $\mathcal{K} = H^2 \ominus zB_E H^2$. That there is another map Ψ_E from H_1^∞ into $\mathcal{M}_{N+1}$ given by

$$\Psi_E(f) = P_{\mathcal{K}} M_f|_{\mathcal{K}}.$$

It is easily verified that $\mathcal{K} = \vee\{1, z, k_2, ..., k_N\}$, where k denotes the Szegö kernel of H^2. Since $\ker(\Psi_E) = \mathcal{I}_1^E$, there is an induced quotient map $\tilde{\Psi}_E : \mathcal{Q}_1 \to \mathcal{M}_{N+1}$, which is an isometry by the previous computation. The map Ψ_E extends in a natural way to all of H^∞, and we keep the same name for this extension. The new representation Ψ_E also has $\mathcal{I}_1^E = zB_E H^\infty$ as its kernel, which is an ideal of H^∞ as well. Consequently, we may factor through by this quotient to obtain the isometry $\tilde{\Psi}_E$. It turns out that this representation is actually completely isometric.

Lemma 4.2.1. *The restriction of the representation $\tilde{\Psi}_E$ to the quotient $\mathcal{Q}_1$ is a complete isometry.*

Proof. Since $0 \in E$, this interpolation problem is of the mixed Carathéodory-Pick type, and so the vector-valued analogue of the distance estimate preceding this lemma holds by Corollary 2.2.5. Consequently, $\tilde{\Psi}_E : H^\infty/\mathcal{I}_1^E \to \mathcal{M}_{N+1}$ is a *complete* isometry. The inclusion map of $\mathcal{Q}_1$ into $H^\infty/\mathcal{I}_1^E$ is evidently a complete isometry, and so it follows that the map $\tilde{\Psi}_E$ restricted to $\mathcal{Q}_1$ is also a complete isometry. $\square$

We have now obtained a completely isometric representation of the quotient algebra $\mathcal{Q}_1$ into a simple C^*-algebra. Except for the case when $N = 2$, we now show that this representation generates all of $\mathcal{M}_{N+1}$.

Theorem 4.2.2. *Let $E = \{0, z_2, ...z_N\} \subset \mathbb{D}$ where $N \geq 3$. Then $C^*_e(\mathcal{Q}_1) = \mathcal{M}_{N+1}$. If $N = 2$, then $C^*_e(\mathcal{Q}_1) = \mathcal{M}_2$.*

Proof. For $f \in H_1^\infty$ we wish to write down the operator $\tilde{\Psi}_E(f)$ with respect to a more suitable basis of $\mathbb{C}^{N+1}$ (as in Theorem 4.0.3). With respect to the basis $\{1, z, k_2, ..., k_N\}$, $\tilde{\Psi}_E(f)$ is the diagonal operator satisfying:

$$\begin{aligned} \tilde{\Psi}_E(f)^* 1 &= \overline{w_1} \\ \tilde{\Psi}_E(f)^* z &= \overline{w_1} z \\ \tilde{\Psi}_E(f)^* k_i &= \overline{w_i} k_i \end{aligned}$$

for $2 \leq i \leq N$. Write $D_f = \text{diag}[f(z_1), f(z_1), f(z_2), ..., f(z_N)] \in \mathcal{M}_{N+1}$. For convenience, write the standard orthonormal basis of $\mathbb{C}_{N+1}$ as $\{e_0, ..., e_N\}$. Define the invertible operator $S := [1, z, k_2, ..., k_N]^* : \mathcal{K} \to \mathbb{C}^{N+1}$. Then $\tilde{\Psi}_E(f)^* = S^{-1} D_f^* S$, and so we may apply the polar decomposition technique used in Theorem 4.0.3. With $Q = SS^*$, we have that $\tilde{\Psi}_E(f)$ is unitarily equivalent to the map $\pi(f) = Q^{1/2} D_f Q^{-1/2}$.

For $1 \leq i \leq N$, the quotient algebra $\mathcal{Q}_1$ is generated by the idempotents $f_i + \mathcal{I}_1^E$ where $f_i(z_j) = \delta_{ij}$ (the f_i may be chosen to be polynomials). Let $\{E_{ij}\}_{i,j=0}^N$ denote the standard matrix units in $\mathcal{M}_{N+1}$. Then the following hold:

$$\begin{aligned} \pi(f_1 + \mathcal{I}_1^E) &= Q^{1/2}(E_{00} + E_{11}) Q^{-1/2} \\ \pi(f_j + \mathcal{I}_1^E) &= Q^{1/2}(E_{jj}) Q^{-1/2}, \end{aligned}$$

for $2 \leq j \leq N$. For any $1 \leq j \leq N$, we have $\pi(f_j)^* \pi(f_j) \in \mathcal{A}$. It follows that

$$\begin{aligned} \pi(f_1)^* \pi(f_1) &= Q^{-1/2}(E_{00} + E_{11}) Q (E_{00} + E_{11}) Q^{-1/2} \\ &= Q^{-1/2}(E_{00} + E_{11}) Q^{-1/2}, \end{aligned}$$

and for $2 \leq j \leq N$,

$$\pi(f_j)^*\pi(f_j) = Q^{-1/2}(E_{jj})QE_{jj})Q^{-1/2} = Q_{jj}Q^{-1/2}E_{jj}Q^{-1/2},$$

where $Q_{jj} = (1-|z_j|^2)^{-1}$ is the $(j,j)^{th}$-entry of Q. It follows that $\mathcal{A}$ contains the operator

$$\sum_{j=1}^{N} Q_{jj}^{-1}\pi(f_j)^*\pi(f_j) = Q^{-1/2}(E_{00}+E_{11})Q^{-1/2} + \sum_{j=2}^{N} Q^{-1/2}E_{jj}Q^{-1/2} = Q^{-1}.$$

Thus Q, $Q^{1/2}$ and $Q^{-1/2}$ are in $\mathcal{A}$ as well. Therefore $E_{00}+E_{11} \in \mathcal{A}$ and $E_{jj} \in \mathcal{A}$ for $2 \leq j \leq N$. We also have $E_{ii}QE_{jj} = Q_{ij}E_{ij}$ for $2 \leq i,j \leq N$, and so $E_{ij} \in \mathcal{A}$ since $Q_{ij} \neq 0$. Similarly, for $2 \leq j \leq N$, we have

$$(E_{00}+E_{11})QE_{jj} = E_{0j} + z_jE_{1j} \tag{4.2}$$

belonging to $\mathcal{A}$. If $N \geq 3$, we may multiply Equation 4.2 by $E_{j,j-1}$, for $j \geq 3$, which yields $(z_{j-1}-z_j)E_{1,j-1} \in \mathcal{A}$. Since the z_j are chosen to be distinct, this implies $E_{1,j-1} \in \mathcal{A}$ for $3 \leq j \leq N$. That the rest of the matrix units are contained in $\mathcal{A}$ follows.

Now suppose $N = 2$. In this case, $\mathcal{A}$ is generated by Q, $E_{00}+E_{11}$, and E_{22}. Define

$$C := (E_{00}+E_{11})QE_{22} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & z_2 \\ 0 & 0 & 0 \end{bmatrix}.$$

It follows that the algebra generated by $\{CC^*, C, D_2\}$ acting on the subspace $\mathcal{R} = \vee\{(1,z_2,0),(0,0,1)\}$ is isomorphic to $\mathcal{M}_2$. On the other hand, the matrix

$$E_0 := D_1 - (1-|z_2|^2)^{-1}CC^*$$

spans a copy of $\mathbb{C}$ on $\mathcal{R}^\perp = \vee\{(\overline{z_2},-1,0)\}$. Since $Q = D_1 + D_2 + C + C^*$, these two subalgebras must generate all of $\mathcal{A}$. Hence $\mathcal{A} \cong \mathcal{M}_2 \oplus \mathbb{C}$. To show that $C_e^*(Q_1) = \mathcal{M}_2$ in this case, it suffices to show that the quotient map onto the $\mathcal{M}_2$ direct summand induces a completely isometric representation of $\mathcal{Q}_1$.

The algebra $\mathcal{M}_n(\mathcal{Q}_1)$ is generated by functions of the form $F = f_1 \otimes A_1 + f_2 \otimes A_2$ where $A_1, A_2 \in \mathcal{M}_n$. The matrices E_0, $E_1 := D_1 - E_0$, and $E_2 := D_2$ act as the diagonal matrix units with respect to this decomposition of $\mathcal{A}$. Moreover, we may write $Q = E_0 \oplus Q_1$ where $Q_1 = (E_2+E_3)Q(E_2+E_3)|_{\mathcal{R}}$; hence $Q^{1/2} = E_0 \oplus Q_1^{1/2}$. With $\pi^{(n)}$, the ampliation of π on $\mathcal{M}_n(\mathcal{Q}_1)$, calculate

$$\begin{aligned}
\pi^{(n)}(F) &= (Q^{-1/2} \otimes I_k)((E_0 + E_1) \otimes A_1 + E_2 \otimes A_2)(Q^{1/2} \otimes I_k) \\
&= (Q^{-1/2} \otimes I_k)((E_0 \otimes A_1 + E_1 \otimes A_1 + E_2 \otimes A_2)(Q^{1/2} \otimes I_k) \\
&= (Q^{-1/2} E_0 Q^{1/2} \otimes A_1) \oplus \left[(Q_1^{-1/2} \otimes I_k)(E_1 \otimes A_1 + E_2 \otimes A_2)(Q^{1/2} \otimes I_k)\right] \\
&= (E_0 \otimes A_1) \oplus \left[(Q_1^{-1/2} \otimes I_k)(E_1 \otimes A_1 + E_2 \otimes A_2)(Q^{1/2} \otimes I_k)\right] \\
&= (E_0 \otimes A_1) \oplus (P_1 \otimes A_1 + P_2 \otimes A_2),
\end{aligned}$$

where $P_1 = Q_1^{-1/2} E_1 Q^{1/2}$ is idempotent and $P_2 = I - P_1$. Therefore, to show that the quotient map $\tilde{\pi}$ onto the $\mathcal{M}_2$ direct summand is a complete isometry on $\mathcal{Q}_1$, it suffices to show

$$\|\pi^{(n)}(X)\| = \|(P_1 \otimes A_1 + P_2 \otimes A_2)\|.$$

To see this, choose a unit vector e in the range of P_1 and a unit vector x on which A_1 attains its norm. It follows that

$$\|(P_1 \otimes A_1 + P_2 \otimes A_2)\| \geq \|((P_1 \otimes A_1 + P_2 \otimes A_2)(e \otimes x)\| = \|e \otimes A_1 x\| = \|A_1\|.$$

This proves the theorem. $\square$

The universal property of the C^*-envelope provides an indirect way of proving that the homomorphism $\tilde{\Phi}_E$ is not completely isometric. Before proving this, it is necessary to show that any irreducible representation of $C(PS^2, \mathcal{M}_N)$ acts on a space of dimension at most N. We require the following result of Arveson **[Arv69]** which deals with extensions of completely positive maps. The proof may be found in **[Pau02]**.

Theorem 4.2.3 (**Arveson's Extension Theorem for CP Maps**)**.** *Let $\mathcal{A}$ be a C^*-algebra and suppose $\mathcal{S}$ is a self-adjoint subspace of $\mathcal{A}$ containing the identity. If $\phi : \mathcal{S} \to B(\mathcal{H})$ is a completely positive map, then there exists a completely positive map $\psi : \mathcal{A} \to B(\mathcal{H})$ so that $\psi|_{\mathcal{S}} = \phi$.*

The next theorem is due to Stinespring **[Sti55]** which characterizes every completely positive map into $B(\mathcal{H})$) as 'part' of a representation. The proof follows the GNS construction, and we briefly sketch the details here.

Theorem 4.2.4 (**Stinespring's Dilation Theorem**)**.** *Let $\mathcal{A}$ be a unital C^*-algebra and let $\phi : \mathcal{A} \to B(\mathcal{H})$ be a completely positive map. Then there is a Hilbert space $\mathcal{K}$, a unital *-homomorphism $\pi : \mathcal{A} \to B(\mathcal{K})$ and a bounded operator $V : \mathcal{H} \to \mathcal{K}$ with $\|\phi(I)\| = \|V\|^2$ such that*

$$\phi(a) = V^* \pi(a) V$$

for each $a \in \mathcal{A}$.

Proof. Consider the vector space tensor product $\mathcal{A} \otimes \mathcal{H}$. Define a symmetric, bilinear function $\langle , \rangle$ on this space by setting $\langle a \otimes x, b \otimes y\rangle := \langle \phi(b^*a)x, y\rangle_{\mathcal{H}}$ and extend linearly. The complete positivity of ϕ implies that this function is positive semidefinite.

Positive semidefinite bilinear forms satisfy the Cauchy-Schwarz inequality, so by letting

$$\mathcal{N} = \{u \in A \otimes H : \langle u, u\rangle = 0\} = \{u \in A \otimes H : \langle u, v\rangle = 0,\ v \in A \otimes H\},$$

we may form the quotient space $(\mathcal{A} \otimes \mathcal{H})/\mathcal{N}$ with an inner product defined by

$$\langle u + \mathcal{N}, v + \mathcal{N}\rangle := \langle u, v\rangle.$$

Let $\mathcal{K}$ denote the completion of this quotient space with respect to the given inner product.

For $a \in \mathcal{A}$ define a linear map $\pi(a) : \mathcal{A} \otimes \mathcal{H} \to \mathcal{A} \otimes \mathcal{H}$ by

$$\pi(a)(\sum a_i \otimes x_i) = \sum (aa_i) \otimes x_i.$$

It is routine to show that $\pi(a)$ leaves $\mathcal{N}$ invariant, and so induces a quotient linear map on $(\mathcal{A}\otimes\mathcal{H})/\mathcal{N}$, which we still denote $\pi(a)$. We also have $\|\pi(a)\| \leq \|a\|$, so we may extend $\pi(a)$ to all of $\mathcal{K}$. An elementary computation shows that the map $\pi : A \to B(\mathcal{K})$ is a unital *-homomorphism.

Now define $V : \mathcal{H} \to \mathcal{K}$ by $Vx = 1 \otimes x + \mathcal{N}$. The operator V is bounded since $\|Vx\|^2 = \langle 1\otimes x, 1\otimes x\rangle = \langle \phi(I)x, x\rangle_H \leq \|\phi(I)\|\,\|x\|^2$, which also shows that $\|V\|^2 = \|\phi(I)\|$. For $x, y \in \mathcal{H}$ we have $\langle V^*\pi(a)Vx, y\rangle_{\mathcal{H}} = \langle \pi(a)(I \otimes x), (I \otimes y)\rangle_{\mathcal{K}} = \langle \phi(a)x, y\rangle_{\mathcal{H}}$. Hence $V^*\pi(a)V = \phi(a)$ for each $a \in A$. This completes the proof. □

Note that if the completely positive map ϕ in the above theorem is unital, then the operator V is an isometry. By identifying the Hilbert space $\mathcal{H}$ with $V\mathcal{H}$, V^* becomes the projection of $\mathcal{K}$ onto $\mathcal{H}$ which yields

$$\phi(a) = P_{\mathcal{H}}\pi(a)|_{\mathcal{H}}, \tag{4.3}$$

for $a \in \mathcal{A}$. Since any compression of a $*$-homomorphism is certainly completely positive, Stinespring's theorem classifies every *unital* completely positive map of a C^*-algebra into $B(\mathcal{H})$ as a compression of a representation.

Lemma 4.2.5. *Suppose X is a locally compact Hausdorff space, $\mathcal{A}$ is a unital subalgebra of $C(X, \mathcal{M}_n)$, and that $\pi : \mathcal{A} \to B(\mathcal{H})$ is an irreducible, unital representation. Then $dim(\mathcal{H}) \leq n$.*

Proof. First suppose that $\mathcal{A} = C(X, \mathcal{M}_n)$. Let $Z = C(X) \otimes I_n$ denote the center of $\mathcal{A}$. Since π is irreducible, its range is weak operator dense in $B(\mathcal{H})$, and hence $\pi(Z) = \mathbb{C}I_{\mathcal{H}}$ by an application of von Neumann's double commutant theorem. Let $\alpha(f)$ be the complex number satisfying $\pi(f \otimes I_n) = \alpha(f)I_{\mathcal{H}}$. It is easily verified that α is a bounded, multiplicative linear functional on $C(X)$, and hence there is some $x_0 \in X$ so that $f(x_0) = \alpha(f)$. Define a unital representation $\pi_m : \mathcal{M}_n \to B(\mathcal{H})$ by setting $\pi_m(A) := \pi(\mathbf{1} \otimes A)$. It follows that $\pi(f \otimes A) = f(x_0)\pi_m(A)$. In particular, π_m is irreducible. It is elementary to verify that the only irreducible self-representation of $\mathcal{M}_n$ is the identity representation (up to unitary equivalence). It follows that $\pi_m(A) \in \mathcal{M}_n$, and so $\dim(\mathcal{H}) \leq n$.

If $\mathcal{A}$ is a proper subalgebra, then we may apply Arveson's extension theorem and obtain a completely positive map $\psi : C(X, \mathcal{M}_n) \to B(\mathcal{H})$ which extends π. By Stinespring's theorem, there is a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a unital $*$-homomorphism $\rho : C(X, \mathcal{M}_N) \to B(\mathcal{K})$ such that $\psi(F) = P_{\mathcal{H}}\rho(F)|_{\mathcal{H}}$ for $F \in C(X, \mathcal{M}_n)$. By the previous case, $\dim(\mathcal{K}) \leq n$, which implies $\dim(\mathcal{H}) \leq n$. □

We are now able to prove that the homomorphism $\tilde{\Phi}_E$ is not a complete isometry.

Theorem 4.2.6. *Let E be a set of $N \geq 3$ distinct points in $\mathbb{D}$ containing 0. The isometric homomorphism $\tilde{\Phi}_E : \mathcal{Q}_1 \to C(PS^2, \mathcal{M}_N)$ is not completely isometric.*

Proof. If $\tilde{\Phi}_E$ were a complete isometry, then there would be a surjective $*$-homomorphism $\rho : C^*(\tilde{\Phi}_E(\mathcal{Q}_1)) \to C^*_e(Q_1) = \mathcal{M}_{N+1}$. By the previous lemma, this is impossible. □

Fix $\alpha, \beta \in \mathbb{C}$ such that $|\alpha|^2 + |\beta|^2 = 1$, and let $k_i^{\alpha,\beta} := k_{\alpha,\beta}(z, z_i)$ for convenience. Our first goal is to express the homomorphism $\tilde{\Phi}_E$ in a more useful form, as we did with the homomorphism $\tilde{\Psi}_E$. Consider the operator $V_{\alpha,\beta} := [k_1^{\alpha,\beta}, ..., k_N^{\alpha,\beta}]^* : \mathcal{M}_{\alpha,\beta} \to \mathbb{C}^N$. It follows that $\Phi_E(f)(\alpha, \beta) = V_{\alpha,\beta}^{-1} D_f V_{\alpha,\beta}$, and so we may define

$$Q_{\alpha,\beta} := V_{\alpha,\beta}V_{\alpha,\beta}^* = \left[\langle k_j^{\alpha,\beta}, k_i^{\alpha,\beta}\rangle\right] = \left[k_{\alpha,\beta}(z_i, z_j)\right],$$

from which the unitary equivalent of Φ_E and the representation π given by

$$\pi(f)(\alpha, \beta) = Q_{\alpha,\beta}^{1/2} D_f Q_{\alpha,\beta}^{-1/2}$$

follows. The representation π induces a representation on the quotient $\mathcal{Q}_1$, which we will also call π.

We are now in position to prove that matrix interpolation fails for sets that do not contain the point 0. The principal tool used here will be the fact that $\tilde{\Phi}_E$ is not a complete isometry on certain sets that do contain 0, and then invoke a continuity argument to show that the same result holds when $0 \notin E$.

Theorem 4.2.7. *Let $\{z_2, z_3, ...z_N\}$ be a set of $N - 1$ distinct points in $\mathbb{D}$ **not** containing 0. There exists an $r > 0$ so that if $|z_1| \leq r$ and $E = \{z_1, ..., z_N\}$, then the homomorphism $\tilde{\Phi}_E : Q_1 \to C(PS^2, \mathcal{M}_N)$ is not completely isometric.*

Proof. Suppose to the contrary. Then there exists a sequence $\{z_1^{(m)}\} \subset \mathbb{D}$ converging to 0 such that the corresponding set $E_m := \{z_1^{(m)}, z_2, ..., z_N\}$ induces the completely isometric representation $\tilde{\Phi}_{E_m} : H^\infty/\mathcal{I}_1^{E_m} \to C(PS^2, \mathcal{M}_N)$.

As per the discussion preceding this theorem, let $Q_{\alpha,\beta}$ denote the matrix corresponding to the set E, and $Q_{\alpha,\beta}^{(m)}$ are the matrices corresponding to each E_m, respectively. Regarding $Q_{\alpha,\beta}$ as a function from PS^2 into $\mathcal{M}_N$, one can deduce that the continuity of the kernel functions $k_{\alpha,\beta}(z_i, z_j) : PS^2 \to \mathbb{C}$ implies that the $Q_{\alpha,\beta}^{(m)}$ converge uniformly to $Q_{\alpha,\beta}$. Since $Q_{\alpha,\beta}$ is nonzero and PS^2 is compact, there is some $\delta > 0$ such that $Q_{\alpha,\beta}^{(m)} \geq \delta I$ for all $(\alpha, \beta) \in PS^2$ and $m \geq 1$.

Suppose $W_1, ..., W_N \in \mathcal{M}_n$ satisfy

$$\Omega(\alpha, \beta) := \left[(I_n - W_i W_j^*) k_{\alpha,\beta}(z_i, z_j)\right] \geq 0,$$

for all $(\alpha, \beta) \in PS^2$, and let $\Omega_m(\alpha, \beta) := \left[(I_n - W_i W_j^*) k_{\alpha,\beta}(z_i^{(m)}, z_j^{(m)})\right]$. The sequence $\Omega_m(\alpha, \beta) - \Omega(\alpha, \beta)$ converges uniformly to 0, so in particular there is a sequence of positive scalars ϵ_m tending to 0 such that

$$\Omega_m(\alpha, \beta) - \Omega(\alpha, \beta) \geq -\epsilon_m \delta I_n \geq -\epsilon_m Q_{\alpha,\beta}^{(m)} = -\epsilon_m \left[k_{\alpha,\beta}(z_i^{(m)}, z_j^{(m)})\right].$$

Therefore

$$\left[((I_n + \epsilon_m) - W_i W_j^*) k_{\alpha,\beta}(z_i, z_j)\right] = \Omega_m(\alpha, \beta) + \epsilon_m \left[k_{\alpha,\beta}(z_i^{(m)}, z_j^{(m)})\right] \geq \Omega(\alpha, \beta) \geq 0.$$

But this implies that there are functions $F_m \in \mathcal{M}_n(H_1^\infty)$ with $\|F\|_\infty \leq 1 + \epsilon_m$ for $m \geq 1$, $F_m(z_1^{(m)}) = W_1$, and $F_m(z_i) = W_i$ for $2 \leq i \leq N$. The sequence F_m has a weak* convergent subsequence with limit F. It follows that $\|F\|_\infty \leq 1$ and $F(z_i) = W_i$ for $1 \leq i \leq N$. This proves that π *is* a complete isometry, contradicting the previous corollary. □

This proves that matrix interpolation fails in H_1^∞.

Corollary 4.2.8. *There exists a set $E = \{z_1, z_2, z_3\}$ of distinct points in $\mathbb{D}$, an integer n, and matrices $W_1, W_2, W_3 \in \mathcal{M}_n$ such that the Pick matrices $[(I - W_i W_j^*)) k_{\alpha,\beta}(z_i, z_j)]$ are positive semidefinite for all pairs (α, β), but there does not exist a function $F \in \mathcal{M}_n(H_1^\infty)$ such that F interpolates the given data and $\|F\|_\infty \leq 1$.*

Proof. If no such function existed, then the positivity of the Pick matrices

$$\left[(I - W_i W_j^*)) k_{\alpha,\beta}(z_i, z_j)\right]$$

would always imply the existence of an interpolating function of norm at most 1, which implies that π is a complete isometry. This contradicts the previous theorem. □

As a final result, the matrix-valued analogue of the H_1^∞ distance estimate must also fail.

Corollary 4.2.9. *There exists a set $E = \{z_1, z_2, z_3\}$ of distinct points in $\mathbb{D}$, an integer $n \geq 2$, and a function $F \in \mathcal{M}_n(H_1^\infty)$ such that*

$$d\left(F, \mathcal{M}_n(\mathcal{I}_1^E)\right) \neq \sup_{|\alpha|^2+|\beta|^2=1} \|M_F^*|_{\mathcal{M}_{\alpha,\beta}\otimes\mathbb{C}^n}\|.$$

Theorem 4.2.7 demonstrates that there are certain sets E where matrix interpolation fails. However, it certainly does not rule out the possibility that there are other sets where matrix interpolation is possible. Unfortunately, no set E has been found such that the homomorphism π is completely isometric (or even 2-isometric). In addition, no completely isometric representation of $\mathcal{Q}_1$ has ever been demonstrated for sets not containing 0.

Chapter 5

Discussion and Conclusion

Sarason's remarkable paper on H^∞ interpolation intertwined the fields of operator theory and complex interpolation in a dramatic way. The commutant lifting method provides an elegant method of handling interpolation problems on algebras that are singly generated. In this case, one needs only to verify that the positivity of the Pick matrix yields a commuting pair of contractions on a finite dimensional space.

However, we saw in Chapter 3 that this method is not always applicable. The algebra H_1^∞ is not singly generated (more precisely, it is generated by the square and cube of the unilateral shift), and so the commutant lifting does not immediately apply. Moreover, an example of Varopoulos [**Var74**] shows that Ando's theorem does not generalize to three or more contractions. One would then not expect that such an approach is possible in doubly generated algebras.

The algebra H_1^∞ also has the property that it is simultaneously the multiplier algebra of a large family of Hilbert function spaces. We proved that a necessary and sufficient condition for interpolation is the positivity of the associated Pick matrix for each space. Certainly, no single kernel $k_{\alpha,\beta}$ has the scalar Pick property, but we could label the set of all such kernels as having a 'scalar Pick-like' property. This idea, it turns out, is prominent in modern interpolation theory.

Abrahamse made use of families of kernels in [**Abr79**], where he proved a Nevanlinna-Pick interpolation theorem for functions with finitely connected domains. He too observed that the positivity of a collection of Pick matrices is required for interpolation. In fact, this family of kernels has a 'complete Pick-like' property, in the sense that the simultaneous positivity of all the Pick matrices implies the existence of a matrix-valued interpolant as well. Federov and Vinnikov showed that if one *fixes* the interpolating data (in the scalar case), only a *finite* number of Pick matrices are required to be positive definite [**FV98**]. On the other hand, McCullough showed that all kernels are required for matrix interpolation [**McC01**]. This leads one to ask a similar question for H_1^∞:

Problem 5.0.10. *Suppose $z_1, ..., z_N \in \mathbb{D}$ and $w_1, ..., w_N \in \mathbb{C}$ are arbitrary but fixed. Does there exist a finite set of pairs $\{\alpha_k, \beta_k\}_{k=1}^m$ so that there is a function $f \in H_1^\infty$ satisfying*

$\|f\|_\infty \leq 1$ *and* $f(z_i) = w_i$ *for each* i *if and only if*

$$[(1 - w_i\overline{w_j})k_{\alpha_k,\beta_k}(z_i, z_j)] \geq 0$$

for each $k = 1, ..., m$*?*

In Chapter 4, we showed that if one chooses an interpolating set with a point sufficiently close to 0, then matrix interpolation fails in H_1^∞. This behavior is somewhat mysterious, and indicates an inherent instability in the quotient algebra $\mathcal{Q}_1$ for such sets. When $0 \in E$, we found the C^*-envelope of $\mathcal{Q}_1$ by showing that the representation $\tilde{\Psi}_E\ \mathcal{Q}_1 \to \mathcal{M}_{N+1}$ is a complete isometry. However, we saw that the representation $\tilde{\Phi}_E : \mathcal{Q}_1 \to C(PS^2, \mathcal{M}_N)$ *cannot* be completely isometric. This is a surprising duality, since we regard the case where $0 \in E$ as being well-behaved. This leads to the natural questions:

Problem 5.0.11. *Given a subset* $E = \{z_1, z_2, ...z_N\}$*, when is the isometric representation* $\tilde{\Phi}_E$ *a complete isometry? When is it a* 2*-isometry?*

Problem 5.0.12. *What is the* C^**-envelope of the quotient algebra* $\mathcal{Q}_1$ *when* $0 \notin E$*?*

Perhaps the strongest point to be made here is the extreme delicacy needed when imposing constraints on the classic Nevanlinna-Pick interpolation problem. The C^* approach is not in itself a solution to any particular interpolation problem, but instead provides a way of quantifying the complexity of a constraint. If one were successful in finding the C^*-envelope of $\mathcal{Q}_1$ for $0 \notin E$, perhaps it would elucidate why matrix interpolation fails.

McCullough and Paulsen **[McP02]** apply the C^* approach to Nevanlinna-Pick interpolation of the annulus (a subcase of Abrahamse's result). Here, the C^*-envelope of the appropriate quotient algebra is $C(\mathbb{T}, \mathcal{M}_N)$, where $N \geq 3$. This motivates the expectation that $C_e^*(\mathcal{Q}_1)$ is an infinite dimensional algebra for $0 \notin E$. This would not be surprising, since a finite dimensional C^*-envelope would presumably indicate a single matrix condition sufficient for interpolation.

We also direct the reader to a recent work of Raghupathi **[Rag08]** that generalizes the constrained problem studied in Chapter 3 and Chapter 4. Here, the author studies subalgebras of H^∞ of the form $\mathbb{C}\mathbf{1} + BH^\infty$, where B is an inner function. When $B = z^2$, we obtain the algebra H_1^∞. Interpolation in these algebras provides additional constraints. Consider the case where B is a finite Blaschke product with two distinct zeroes a and b. Then any function in the algebra $\mathbb{C}\mathbf{1} + BH^\infty$ must be equal at these two points. Most of the results concerning the algebra H_1^∞ may be generalized to $\mathbb{C}\mathbf{1} + BH^\infty$. Most importantly, one obtains a Buerling-type theorem for invariant subspaces, and an infinite family of Hilbert function spaces which have $\mathbb{C}\mathbf{1} + BH^\infty$ as their multiplier algebra.

The analogous Nevanlinna-Pick interpolation theorem holds in this context. That is, the positivity of every Pick matrix associated to each underlying Hilbert function space is sufficient for interpolation. Similar distance estimates are formulated, and the C^*-envelopes associated to the relevant quotient algebras are calculated for the case where B is a finite Blaschke product and at least one interpolation node is a zero of B.

At time of press, an important result by Ball, Bolotnikov, and Ter Horst [**BBH09**] has been released. Here, the authors were successfully able to find a sufficient condition for the existence of a matrix-valued interpolating function in H_1^∞ when $0 \notin E$. Their idea was to enrich the collection of kernel functions for which the associated Pick matrix was positive semidefinite. We summarize the basic construction here.

Fix a positive integer n and let ℓ and ℓ' denote a pair of positive integers satisfying $1 \leq \ell \leq \ell' \leq n$, and let $\mathbb{G}(\ell' \times \ell)$ denote the set of pairs (α, β) of $\mathcal{M}_{\ell \times \ell'}$ satisfying $\alpha\alpha^* + \beta\beta^* = I_{\ell'}$. We may then define an *operator-valued* kernel function as follows:

$$K^{\alpha,\beta}(z,w) = (\alpha^* + \overline{w}\beta^*)(\alpha + z\beta) + \frac{\overline{w^2}z^2}{1 - \overline{w}z} I_\ell, \tag{5.1}$$

where α is injective. When $n = 1$, these kernels clearly correspond to the kernel functions for the Hilbert spaces $H^2_{\alpha,\beta}$. It turns out that this enriched collection of kernel functions contain enough information in order to determine the existence of an interpolating matrix-valued function.

Theorem 5.0.13 (**Ball, Bolotnikov, Ter Horst**). *Given a subset $\{z_1, ..., z_N\}$ of $\mathbb{D}$ and $\{W_1, ...W_N\} \subset \mathcal{M}_n$, there is a function F in the unit ball of $\mathcal{M}_n(H_1^\infty)$ interpolating the given data if and only if*

$$\sum_{i,j=1}^{n} \text{Trace}\left[X_j K^{\alpha,\beta}(z_i, z_j) X_i^* - W_j^* X_j K^{\alpha,\beta}(z_i, z_j) X_i^* W_i\right] \geq 0, \tag{5.2}$$

for all $(\alpha, \beta) \in \mathbb{G}(\ell' \times \ell)$ and n-tuples $\{X_1, ...X_n\}$ of $n \times \ell$ matrices.

It is straightforward to show that this theorem generalizes the interpolation theorem for H_1^∞ in the scalar case. A distance formula follows from this theorem, as one would expect. However, it is still unknown what the C^*-envelope of $\mathcal{I}_1^E$ is.

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