

NOTE TO USERS

**Page(s) missing in number only; text follows.
Microfilmed as received.**

12, 56, 162, 244, 250

This reproduction is the best copy available.

UMI

The Distribution of the Pension Plan Surplus

by

Claire Bilodeau

A thesis

presented to the University of Waterloo

in fulfilment of the

thesis requirement for the degree of

Doctor of Philosophy

in

Statistics

Waterloo, Ontario, Canada, 1999

©Claire Bilodeau 1999



National Library
of Canada

Acquisitions and
Bibliographic Services

395 Wellington Street
Ottawa ON K1A 0N4
Canada

Bibliothèque nationale
du Canada

Acquisitions et
services bibliographiques

395, rue Wellington
Ottawa ON K1A 0N4
Canada

Your file *Votre référence*

Our file *Notre référence*

The author has granted a non-exclusive licence allowing the National Library of Canada to reproduce, loan, distribute or sell copies of this thesis in microform, paper or electronic formats.

The author retains ownership of the copyright in this thesis. Neither the thesis nor substantial extracts from it may be printed or otherwise reproduced without the author's permission.

L'auteur a accordé une licence non exclusive permettant à la Bibliothèque nationale du Canada de reproduire, prêter, distribuer ou vendre des copies de cette thèse sous la forme de microfiche/film, de reproduction sur papier ou sur format électronique.

L'auteur conserve la propriété du droit d'auteur qui protège cette thèse. Ni la thèse ni des extraits substantiels de celle-ci ne doivent être imprimés ou autrement reproduits sans son autorisation.

0-612-44752-9

Canada

The University of Waterloo requires the signatures of all persons using or photocopying this thesis. Please sign below, and give address and date.

Abstract

Private defined-benefit pension plans must, by law, pre-fund the benefits. The actual experience of the pension fund rarely matches the assumptions underlying the contributions and liabilities. If the fund is worth more than the liabilities, the plan has a surplus; otherwise, it has a deficit, called an unfunded liability.

At times, the surplus can reach such a high level that there is pressure to distribute some of it. This raises two questions. First, how much can we part with without jeopardizing the future of the pension plan? Second, to whom and in what proportion should the amount to be parted with be given? This thesis represents an attempt at answering these two questions.

We begin with the definition of a (simplified) model pension plan. A review of pension plan mathematics follows, based on the projected unit credit method. We first consider all the elements which require assumptions to value the liabilities. These assumptions, which form the valuation basis, are kept constant through time for the sake of simplicity. We then give the formulae determining the contributions and liabilities.

We continue with the definition of a criterion to determine the amount of surplus to distribute, both for the whole plan and for any subgroup. Under that criterion, the amount must be such that, with a given level of probability, there will not be an unfunded liability at the next actuarial valuation. Applying the criterion to subgroups requires an initial asset allocation. Whereas decrements are assumed to be deterministic, economic variables are simulated using the Wilkie model. These

variables impact both the assets and the liabilities.

Finally, we propose a methodology that could be used to distribute the disposable surplus. This approach is based on cooperative game theory. We review this theory and discuss its applicability in the present context. We also explore bargaining theory as an alternative approach. To illustrate the approach, we apply the method to our model pension plan and discuss the properties of four sharing rules that have been proposed in the economics literature.

Acknowledgements

To the Natural Sciences and Engineering Research Council, the Society of Actuaries, and the University of Waterloo:

Thanks for your financial support through the NSERC 67 Scholarship, the SoA Ph.D. Grant, and various scholarships, respectively.

To my supervisor, Dr. Phelim P. Boyle:

Thanks for the lessons of perseverance and determination. Your willingness to help me see it through, despite the delays, was immensely appreciated.

To the other committee members, Dr. Keith P. Sharp and Dr. Lutz-Alexander Busch:

Thanks to you for your precious guidance and your invaluable insights, particularly in the later stages of my thesis.

To Dr. Michel Jacques, chair of the School of Actuarial Science at Laval University:

Thanks for your confidence in me and for giving me the means to complete my degree; they helped bridge the gap between the dream and the reality.

To all my friends, especially my few but much cherished best friends:

Thanks for your friendship, which has helped me see it through the rough times. I now wish to share with you the joy of better times.

To my Fall 98 students at Université Laval:

Sincere thanks for your inspiration. Given how rewarding teaching you turned out to be and because teaching again was conditional on getting my Ph.D., you provided me with every reason in the world to work toward that degree.

Last but most importantly, to my mother Lénora, my father Charles-Henri, and my three brothers, Camil, Jacques and Jean:

From the bottom of my forever grateful heart. thanks for your loving and unfailing support. Your love for me and your eagerness to help me in any way you can make me feel like the luckiest girl in the world. Knowing how proud you are of me, and how proud I am of you, I could not disappoint you. This thesis, though mostly Greek to you, and its dedication to you are my simple way of thanking you and telling you, once more, how much I love you and everything you mean to me.

Dedication

To my beloved family, who has taught me to look skyward while keeping my feet on the ground.

À ma famille bien-aimée, qui m'a appris à regarder vers le ciel tout en gardant les deux pieds sur terre.

Contents

1	Background Information	1
1.1	Introduction	1
1.2	Types of Pension Plans	2
1.2.1	Defined-Benefit versus Defined-Contribution Plans	2
1.2.2	Ways of Funding the Plan	3
1.2.3	Funding Vehicles for Pre-Funded Plans	4
1.3	Importance of Trusteed Pension Plans	5
1.4	Emergence of Surpluses in the 1980s	6
1.5	Outline of Thesis	9
2	Traditional Valuation	13
2.1	Introduction	13
2.2	Legislation	14
2.3	Model Pension Plan	15
2.3.1	Retirement Benefits	16
2.3.2	Termination Benefits	19
2.3.3	Other Benefits	21

2.4	Assumptions Required for the Valuation	23
2.4.1	Economic Assumptions	24
2.4.2	Demographic Assumptions	29
2.4.3	Other Assumptions	34
2.5	Mathematics of the Pension Plan	34
2.5.1	Timing	35
2.5.2	Pension Plan Members	36
2.5.3	Initial Computations	37
2.5.4	Accrued Liabilities	42
2.5.5	Assets	49
2.5.6	Contributions	49
2.6	Conclusion	54
3	Choice of the Amount to Be Distributed	57
3.1	Introduction	57
3.2	Valuation of Assets	58
3.2.1	Valuation Methods Used in Practice	59
3.2.2	Alternative Valuation Methods	62
3.2.3	Valuation Method to Be Used	63
3.3	Criterion for the Amount to Distribute	64
3.3.1	Key Concerns of a Pension Plan	64
3.3.2	Suggested Criterion	65
3.4	Simulated Projections of the Pension Plan	66
3.4.1	Fixed and Random Elements	66

3.4.2	Model Used for the Economic Variables	71
3.4.3	Evolution of Assets and Liabilities	76
3.5	Valuation Basis versus Simulation Results	84
3.5.1	Algebraic Expression of Suggested Criterion	84
3.5.2	Simulation Results Identical to Valuation Basis	85
3.5.3	Simulation Results Differing from Valuation Basis	87
3.6	Adapting the Criterion for Subgroups	93
3.6.1	Choice and Definition of Five Subgroups	93
3.6.2	Allocating Initial Assets to Each of the Subgroups	98
3.6.3	Evolution of Assets and Liabilities	103
3.7	Conclusion	108
4	Distribution of the Surplus	111
4.1	Introduction	111
4.2	Entitlement to a Share of the Surplus	112
4.2.1	Entitlement of the Members	113
4.2.2	Entitlement of the Sponsor	116
4.3	Cooperative Game Theory	121
4.3.1	Definition of a Cooperative Game	122
4.3.2	Transferable Utility versus Non-Transferable Utility	124
4.4	Allocations, Sharing Rules and Their Properties	125
4.4.1	Definition of an Allocation	126
4.4.2	Definition of a Sharing Rule	126
4.4.3	Properties of Sharing Rules and Their Desirability	127

4.4.4	Relationships between Properties	143
4.5	Possible Sharing Rules	144
4.5.1	Types of Games	145
4.5.2	Shapley Value	149
4.5.3	Nucleolus	151
4.5.4	Proportional Sharing Rule	154
4.5.5	Serial Sharing Rule	155
4.5.6	Summary of Properties Enjoyed by Sharing Rules	157
4.6	Ease of Computation of Sharing Rules	159
4.7	Conclusion	160
5	Application	163
5.1	Introduction	163
5.2	Stationary Plan Membership	166
5.3	Evolution of Assets for One Member	168
5.4	Past Evolution of the Pension Plan	170
5.5	Shares, 3-Year Horizon, Liabilities plus Surplus	172
5.5.1	Disposable Surplus for Each Combination of Subgroups	172
5.5.2	Surplus Shares if Sponsor Is Treated Separately	178
5.5.3	Surplus Shares if Sponsor Is Treated as Essential	182
5.5.4	Surplus Shares if Sponsor Is Excluded	185
5.6	Shares, 3-Year Horizon, Liabilities Only	187
5.6.1	Surplus Shares if Sponsor Is Treated Separately	189

5.6.2	Surplus Shares if Sponsor Is Treated as Essential	191
5.6.3	Surplus Shares if Sponsor Is Excluded	191
5.7	Shares, 3-Year Horizon, Accumulated Fund	193
5.7.1	Surplus Shares if Sponsor Is Treated Separately	195
5.7.2	Surplus Shares if Sponsor Is Treated as Essential	196
5.7.3	Surplus Shares if Sponsor Is Excluded	196
5.8	Choice of Initial Asset Allocation	198
5.8.1	Impact of Asset Allocation if Sponsor Is Treated Separately	198
5.8.2	Impact of Asset Allocation if Sponsor Is Treated as Essential	201
5.8.3	Impact of Asset Allocation if Sponsor Is Excluded	202
5.8.4	Use of Game Theory to Determine Asset Allocation	203
5.8.5	Our Choice of Asset Allocation	205
5.9	Shares, 10-Year Horizon, 5% Probability	207
5.10	Shares, 10-Year Horizon, 15% Probability	208
5.10.1	Surplus Shares if Sponsor Is Treated Separately	209
5.10.2	Surplus Shares if Sponsor Is Treated as Essential	210
5.10.3	Surplus Shares if Sponsor Is Excluded	211
5.11	Shares, 3-Year Horizon, Real Starting Values	213
5.11.1	Surplus Shares if Sponsor Is Treated Separately	214
5.11.2	Surplus Shares if Sponsor Is Treated as Essential	215
5.11.3	Surplus Shares if Sponsor Is Excluded	216
5.12	Shares, 10-Year Horizon, Real Starting Values	217
5.12.1	Surplus Shares if Sponsor Is Treated Separately	218

5.12.2	Surplus Shares if Sponsor Is Treated as Essential	219
5.12.3	Surplus Shares if Sponsor Is Excluded	221
5.13	Choice of Treatment of the Sponsor	222
5.14	Choice of Sharing Rule	225
5.15	Impact of Nature of Distribution	226
5.16	Conclusion	229
6	Conclusion	233
6.1	Criterion for the Amount to Distribute	233
6.2	Distribution of that Amount	234
6.3	Further Work	236
A	Tables of Decrements	239
B	Salary Scale	245
C	Economic Data from 1924 to 1997	247
D	Parameters for Economic Models	251
E	Bargaining Theory	253
F	Tables of Results	257
	Bibliography	279

List of Tables

3.1	Impact of the Choice of Valuation Basis on the Average and Actual Disposable Surpluses; Initial Valuation Surplus Equal to \$10,000,000	89
4.1	List of Properties of Sharing Rules	143
4.2	Implications between Properties	144
4.3	Summary of Properties of Sharing Rules for General Games	157
4.4	Summary of Properties of Sharing Rules for the Subset of Superadditive Games	158
4.5	Summary of Properties of Sharing Rules for the Subset of Convex Games	158
5.1	Examples Included in Chapter 5	165
5.2	Initial Subgroup Sizes	166
5.3	Accumulation of Contributions for Someone Aged 25 in 1965	169
5.4	Evolution of the Plan from 1965 to 1986	171
5.5	Assets Allocated by Subgroup, 3-Year Horizon, Initial Assets Proportional to Actuarial Liability	174

5.6	Disposable Surpluses, 3-Year Horizon, Initial Assets Proportional to Actuarial Liability	175
5.7	Assets Allocated by Subgroup, 3-Year Horizon, Initial Assets Equal to Actuarial Liability	188
5.8	Assets Allocated by Subgroup, 3-Year Horizon, Initial Assets Equal to Accumulated Fund	194
5.9	Asset Allocation Determined by Game Theory	204
5.10	Number of Scenarios with Negative Disposable Surplus	227
A.1	Retirement Rates	240
A.2	Termination Rates	241
A.3	Disability Rates	242
A.4	Non-Disabled Mortality Rates	243
B.1	Salary Scale	246
C.1	Weights Given to Each Category of Assets	248
C.2	Economic Data from 1924 to 1997	249
D.1	Parameters for Economic Models	252
F.1	Disposable Surpluses, 3-Year Horizon, Initial Assets Proportional to Actuarial Liability	258
F.2	Shares, 3-Year Horizon, Initial Assets Proportional to Actuarial Liability, Sponsor Separate	259

F.3	Shares, 3-Year Horizon, Initial Assets Proportional to Actuarial Liability, Sponsor Essential	259
F.4	Shares, 3-Year Horizon, Initial Assets Proportional to Actuarial Liability, Sponsor Excluded	260
F.5	Disposable Surpluses, 3-Year Horizon, Initial Assets Equal to Actuarial Liability	261
F.6	Shares, 3-Year Horizon, Initial Assets Equal to Actuarial Liability, Sponsor Separate	262
F.7	Shares, 3-Year Horizon, Initial Assets Equal to Actuarial Liability, Sponsor Essential	262
F.8	Shares, 3-Year Horizon, Initial Assets Equal to Actuarial Liability, Sponsor Excluded	263
F.9	Disposable Surpluses, 3-Year Horizon, Initial Assets Equal to Accumulated Fund	264
F.10	Shares, 3-Year Horizon, Initial Assets Equal to Accumulated Fund, Sponsor Separate	265
F.11	Shares, 3-Year Horizon, Initial Assets Equal to Accumulated Fund, Sponsor Essential	265
F.12	Shares, 3-Year Horizon, Initial Assets Equal to Accumulated Fund, Sponsor Excluded	266
F.13	Shares under Different Asset Allocations, Sponsor Separate	266
F.14	Shares under Different Asset Allocations, Sponsor Essential	267
F.15	Shares under Different Asset Allocations, Sponsor Excluded	268

F.16 Disposable Surpluses, 10-Year Horizon, 5% Probability	269
F.17 Disposable Surpluses, 10-Year Horizon, 15% Probability	270
F.18 Shares, 10-Year Horizon, 15% Probability, Sponsor Separate	271
F.19 Shares, 10-Year Horizon, 15% Probability, Sponsor Essential	271
F.20 Shares, 10-Year Horizon, 15% Probability, Sponsor Excluded	272
F.21 Disposable Surpluses, 3-Year Horizon, Real Starting Values	273
F.22 Shares, 3-Year Horizon, Real Starting Values, Sponsor Separate	274
F.23 Shares, 3-Year Horizon, Real Starting Values, Sponsor Essential	274
F.24 Shares, 3-Year Horizon, Real Starting Values, Sponsor Excluded	275
F.25 Disposable Surpluses, 10-Year Horizon, Real Starting Values	276
F.26 Shares, 10-Year Horizon, Real Starting Values, Sponsor Separate	277
F.27 Shares, 10-Year Horizon, Real Starting Values, Sponsor Essential	277
F.28 Shares, 10-Year Horizon, Real Starting Values, Sponsor Excluded	278

List of Figures

5.1	Shares, 3-Year Horizon, Initial Assets Proportional to Actuarial Liability, Sponsor Separate	180
5.2	Shares, 3-Year Horizon, Initial Assets Proportional to Actuarial Liability, Sponsor Essential	184
5.3	Shares, 3-Year Horizon, Initial Assets Proportional to Actuarial Liability, Sponsor Excluded	186
5.4	Shares, 3-Year Horizon, Initial Assets Equal to Actuarial Liability, Sponsor Separate	190
5.5	Shares, 3-Year Horizon, Initial Assets Equal to Actuarial Liability, Sponsor Excluded	192
5.6	Shares, 3-Year Horizon, Initial Assets Equal to Accumulated Fund, Sponsor Separate	195
5.7	Shares, 3-Year Horizon, Initial Assets Equal to Accumulated Fund, Sponsor Excluded	197
5.8	Shares under Different Asset Allocations, Sponsor Separate	199
5.9	Shares under Different Asset Allocations, Sponsor Excluded	202
5.10	Shares, 10-Year Horizon, 15% Probability, Sponsor Separate	210

5.11	Shares, 10-Year Horizon, 15% Probability, Sponsor Essential	211
5.12	Shares, 10-Year Horizon, 15% Probability, Sponsor Excluded	212
5.13	Shares, 3-Year Horizon, Real Starting Values, Sponsor Separate . .	215
5.14	Shares, 3-Year Horizon, Real Starting Values, Sponsor Essential . .	216
5.15	Shares, 3-Year Horizon, Real Starting Values, Sponsor Excluded . .	217
5.16	Shares, 10-Year Horizon, Real Starting Values, Sponsor Separate . .	219
5.17	Shares, 10-Year Horizon, Real Starting Values, Sponsor Essential . .	220
5.18	Shares, 10-Year Horizon, Real Starting Values, Sponsor Excluded .	221

Chapter 1

Background Information

1.1 Introduction

This introductory chapter will serve two purposes. We will introduce the subject matter of this thesis. More importantly though, we will motivate it by showing that the problem addressed here is both current and important.

We begin with a brief overview of pension plans, their nature and their importance. We also discuss, in a Canadian context, the impact that the high surpluses of the 1980s, and in particular the court cases regarding their ownership, had in the pensions arena. We then quickly introduce, in turn, each of the main topics to be covered in the following chapters.

1.2 Types of Pension Plans

Many employers set up some kind of pension plan for their employees. As is the case with any retirement savings plan, the goal is to save money during the working years and to use the accumulated money to provide income after retirement.

In this section, we will discuss the two main types of pension plans and how they can be funded.

1.2.1 Defined-Benefit versus Defined-Contribution Plans

There are two main types of pension plans. Some are defined-benefit while others are defined-contribution. In Canada¹, as of January 1, 1996, 53.7% of the registered pension plans were of the latter type, 44.6% were of the former type, and 1.7% were composite (see *Pension Plans in Canada* [65]). If we focus on membership instead, we find that only 10.5% of the members belonged to a defined-contribution plan, in contrast with 88.1% of the members having a defined-benefit pension plan (and all others in a composite one).

From 1986 to 1996, in Canada, the number of both defined-benefit and defined-contribution plans has decreased substantially (from 8,215 to 6,884, and from 12,637 to 8,276, respectively). However, over the same period, the membership has increased for both types of plans, though more for the defined-contribution type (325,320 in 1986 compared to 540,369 in 1996) than for the defined-benefit type

¹Similar statistics concerning pension plans in the United States can be found in *The Future of Pensions in the United States* [59].

(4,295,691 in 1986 versus 4,535,388 in 1996).

The defined-contribution plans are very much like individual registered retirement savings plans. The employer (and employee) contributes a certain amount to the pension fund on a regular basis. That amount may be expressed as a percentage of salary or as a fixed amount per unit of work (hour or week). At retirement, the monies accumulated on behalf of the employee are given to him so that he can then buy an annuity or set up a retirement income fund.

As the name of these plans implies, the contributions are defined by the plan document, but the retirement income generated by these contributions is neither defined nor guaranteed. As a result, such plans cannot have a surplus/deficit since what the plan owes its participants is always equal to the funds it holds on their behalf.

In contrast, the defined-benefit plans guarantee the income during retirement according to the rules set in the plan document, while the contributions that need to be made may change from time to time. As for the defined-contribution plan, the employer (and employee) contributes a certain amount to the pension fund on a regular basis. Again, that amount may be expressed as a percentage of salary or as a fixed amount per unit of work. Such pension plans may also provide limited disability and death benefits.

1.2.2 Ways of Funding the Plan

The level of contributions to be made in a defined-benefit plan depends, for one thing, on the type of funding to be used. In particular, we must differentiate

between pre-funded plans and pay-as-you-go plans.

In the former case, the total of the contributions and investment income accumulated on behalf of an employee should be enough to fund his annuity at retirement. In other words, plan participants pay today for what *they* will get *later*.

In the latter case, the total of the contributions made by the employer and employees in a given year, perhaps along with some investment income, should provide the benefits promised to the current retirees. So, plan participants pay today for what *others* receive *today*.

The focus here will be on pre-funded plans. In Canada, private defined-benefit plans must, by law, be pre-funded. In contrast, government-sponsored plans often are of the pay-as-you-go type, or a close variation thereof with relatively little emphasis on the accumulation of funds. For pay-as-you-go plans, contributions vary as the ratio of payors to payees changes. In their pure form, such plans cannot experience a surplus or deficit. When funds are allowed to accumulate or are borrowed, this is usually a temporary measure meant to lessen the variability in contributions.

1.2.3 Funding Vehicles for Pre-Funded Plans

In the case of pre-funded plans, there are two main vehicles for funding². On one hand, the plan may be insured. In that case, contributions are used to pay the premiums charged by the insurance company in order to provide the benefits promised in the contract. On the other hand, the plan may be trusteeed. In that

²There also is a third funding vehicle for plans in the public sector: the government consolidated revenue arrangement.

case, contributions are deposited into a fund (separate from all other company operations), and benefits are paid directly out of that fund. The money in the fund is invested in stocks, bonds, and other types of investment securities, according to the investment policy adopted by the pension committee.

In both cases, contributions may vary, from time to time, because of changes in several different factors. Likewise, for trustee funds, the funding level – the value of the accumulated fund relative to the value of the promises made to the participants – may fluctuate for different reasons. The variety of factors influencing pension plan valuation will be discussed later.

From now on, the focus will be on trustee, defined-benefit pension plans. By their very nature, these plans are the only ones which can experience a surplus or a deficit. All other ones, in their pure form, are either not funded at all or are always fully funded. Henceforth, the phrase “pension plan” shall be taken to mean “trustee, defined-benefit, pre-funded pension plan.”

1.3 Importance of Trustee Pension Plans

In Canada, trustee pension plans are of considerable significance in terms of the volume of assets they represent. Out of the trustee plans, the defined-benefit ones accounted for 94.9% of the assets at the end of 1996 [67].

At the end of the first quarter of 1998, the funds of Canadian trustee pension plans held close to two thirds of the registered pension plan (RPP) monies. Also, these funds had close to 90% of the RPP assets which are directly invested in financial markets. These estimates were based on 179 funds with assets of at least

\$275 million. Although these funds accounted for only 5% of the total number of trustee pension funds, they controlled 86% of total assets (see the *Quarterly Estimate of Trustee Pension Funds* [66]).

Out of all the assets held in retirement savings vehicles³ in Canada on December 31, 1996, 43.7% belonged to a trustee pension fund (see *Trustee Pension Funds* [67]). Besides, as reported by Statistics Canada [66], with assets at a book value of \$393 billion at the end of 1997, trustee pension funds were second in size, with the chartered banks being first with \$820 billion. Hence, not only are trustee pension plans important when we consider only pension monies, they also come out as a key player in the financial markets when compared with all the other players.

1.4 Emergence of Surpluses in the 1980s

In recent years, surpluses have arisen in a number of Canadian pension plans. This section examines the recent history of the emergence of surpluses in Canadian defined-benefit pension plans. We will outline the institutional background and mention some of the issues that have been raised.

In the 1980s, the economic situation was such that surpluses naturally formed in pension funds. For some pension plans, this was the occasion to enhance the retirement benefits being paid, or to better the benefit formula found in the plan document. For others, the sponsor may have, also or instead, taken a contribution holiday.

³These include the public plans, registered pension plans and registered retirement savings plans.

In some cases, though, the surplus was so sizable that a mere contribution holiday would have depleted it only marginally. Of course, large benefit enhancements remain a possibility in this case. However, there comes a point when the sponsor, having had to make up for the past deficits, starts thinking that it should reap the reward of a surplus.

When money comes into play, and in large amounts, we do not expect everything to get settled without some discussion. In fact, claims to the ownership of the surplus made it to the courts throughout Canada. This led to a heated debate between unions and employers, whose diametrically opposed views did not offer the promise of a reasonable compromise.

Hoping to provide for a more peaceful setting in which to negotiate, many provinces, including Ontario and Quebec, imposed a moratorium on the use of surpluses. Ontario first targeted its moratorium at the ongoing plans in 1986, and then extended it to wound-up plans in 1988. Quebec imposed its moratorium on all plans in 1988.

Moratoria also gave jurisdictions much needed time to come up with guidelines or legislation to deal with surpluses. Three key ideas surfaced.

First, the sponsor is not entitled to the surplus unless either the plan document states so or the sponsor gets a court order to that effect. This is the standard approach, adopted by all jurisdictions except Newfoundland (see the August 1992 issue of *Commentaires Mercer* [21]). There is the additional requirement, in Ontario, Quebec and British Columbia, that participants approve (or not disapprove) of the surplus withdrawal.

Second, members should share in the surplus in proportion to the value of their accrued benefits. According to the discussion document produced by the Régie des rentes du Québec [48], this is the general rule on wind-up. Still, Ontario and Quebec (and possibly other jurisdictions) allow for other surplus allocations such as in proportion to accumulated contributions with interest (see the *PCO Bulletin* v. 5 no. 1 [51]) or relatively more advantageous to the inactive lives (article 230.2 of the *Loi sur les régimes complémentaires de retraite* [35]).

Third, the sharing is subject to an agreement among all the interested parties. That is clearly the case in Ontario, Quebec and British Columbia where the members' approval must be sought. In the other jurisdictions, insofar as any decision regarding the surplus depends on the plan document, this is also the case. Besides, some articles of pension plan legislation even indicate that they are to apply *unless the plan document says otherwise*. This last element reinforces the idea that the pension plan exists, first and foremost, because of a contract, the pension plan document, entered by two willing parties, the sponsor and the members.

The large surpluses of the 1980s also led to pension legislation reforms in many provinces. These reforms greatly added to the members' rights in a pension plan and thus made the plan much more valuable from their vantage point.

All the reforms or original pension legislations adopted over the last twenty years featured these six main improvements [48]:

- minimum interest rate on employee contributions;
- earlier vesting of accrued benefits;
- minimum employer contribution towards 50% of the commuted value;

- addition or enhancement of pre-retirement death benefits;
- joint pension as the default annuity;
- portability of deferred pensions at termination of employment.

In improving members' rights, it was felt that large surpluses were less likely to form. For instance, increased layoffs at the beginning of the 1980s contributed to the emergence of large surpluses since, at the time, vesting requirements were quite stringent. Today's vesting requirements are such that, though surplus still would emerge in a similar situation, the impact would be much less important.

Actually, as long as contributions paid into the pension fund are larger than the expected pension cost, a surplus is likely to emerge over time. While Quebec does not allow for a surplus distribution when the plan is ongoing other than through contribution holidays, other provinces provide that the excess over a certain percentage of liabilities or multiple of annual sponsor contributions may be withdrawn from the fund. In these provinces, the withdrawal of surplus amounts for an ongoing plan is subject to requirements similar to those for a terminating one.

1.5 Outline of Thesis

While pension plan legislation prescribes what amount can be withdrawn from the fund and how to allocate that amount among the interested parties, we have found no theoretical foundation for the rules it puts forward.

In this thesis, we will begin by defining a simple model pension plan. We will study its evolution from 1965 to 1986, at which point, given the size of the

surplus, we will have to ask ourselves what to do with it. This plan will be funded according to the projected unit method and we will review the relevant pension plan mathematics in Chapter 2.

Faced with a surplus, we must ask ourselves what portion of it, if any, we can safely part with. Unlike the *ad hoc* criterion given in the legislation, we will define one which has the property of letting us decide to what level of risk we wish to expose ourselves in the future. We will focus on the risk of running into an unfunded liability at the next valuation. Defining this criterion is the object of Chapter 3.

Since the application of the criterion requires simulations, that chapter will define the models to be used for the random elements. It will also feature all the formulae that describe the evolution of the assets and liabilities. These will be provided for the plan as a whole as well as for any subgroup, since we will need subgroup results in the next step.

Once we have agreed on the amount to give away, we must decide how we will allocate it among all the interested parties. (We will discuss the entitlement of the different parties to a share of the surplus.) Doing so in proportion to the value of accrued benefits is simple and appears intuitively defensible. In Chapter 4, we will resort to cooperative game theory and extract from it four sharing rules which can be shown to have certain properties. When necessary, we will show how to adapt these rules to the sharing of a pension plan surplus.

In Chapter 5, we will apply our criterion and sharing rules to our model pension plan. We will compare the shares we obtain under different asset allocations among

the subgroups. We will also consider different assumptions for the sponsor's role. In addition, we will look at the impact of changing the parameters (length of simulation period and target probability) of our criterion. Of course, we will also compare the shares obtained with the different sharing rules. This will allow us to make specific comments regarding the use of our suggested criterion as well as concerning the choice of a sharing rule. Finally, we will compute the probability of an unfunded deficit if, rather than being distributed in cash, the disposable surplus is distributed as benefit increases.

We will briefly conclude, in Chapter 6, with a summary of our findings. At the same time, we will indicate directly related areas where there is potential for further research.

NOTE TO USERS

**Page(s) missing in number only; text follows.
Microfilmed as received.**

12

This reproduction is the best copy available.

UMI

Chapter 2

Traditional Valuation

2.1 Introduction

In this chapter, we will examine how pension plan contributions and liabilities are traditionally determined. We will start with a brief overview of some items found in the legislation relating to pension plans. Others will be introduced as we define the model pension plan we will work with. We will then enumerate all the variables that enter into the valuation. We will also provide a short survey of pension plan mathematics, based on the projected unit credit method.

We will end this chapter with a brief discussion of other approaches that could be taken to calculate the contributions and liabilities. In the application discussed in the thesis, we will use the traditional method of valuing assets and liabilities, but we will use a percentile-based approach to the determination of disposable surplus.

2.2 Legislation

As required by law, the liabilities of a pension plan must be evaluated at least every three years in Canada. In the United States, the Employee Retirement Income Security Act (ERISA) says that such valuation must be done annually. On the one hand, triennial valuation may be frequent enough for a plan in good financial health. On the other hand, annual valuation will be required to monitor a plan in difficulty. In fact, annual valuation is valuable in that it allows one to keep better track of the evolution of the financial situation of the pension plan, whether it be positive or negative.

Thus, in order to carry out the analysis without a great loss of generality, we will assume that valuation is conducted every year. Indeed, in Canada, even if not required to, many pension plans are valued annually. However, since reporting is normally done only every three years, contribution levels (and surplus amounts to be distributed) will be determined at that frequency.

Typically, the participant's contribution will be a set percentage of his wages, and that percentage will not change unless the total contribution changes considerably. As for the sponsor, he will pay the difference between the total contribution to be made and the sum of the participants' contributions.

Theoretically, the participant's contribution could be anywhere between nil and the entire cost. However, there is a so-called 50% rule which prescribes, in general terms, that the participant should not contribute towards more than 50% of the cost of the benefits assigned to him. All Canadian jurisdictions adopted such a rule with respect to post-reform contributions. In order to comply with the 50% rule,

the sponsor may choose the participant's flat contribution rate to be equal to half the smallest total contribution rate over all ages. For the sake of simplicity, we will assume that either there is no 50% rule or the contributions are split in such a way that it is not effective.

We assume that the total contribution only covers increases in accrued benefits; that is, it does not account for an existing surplus or deficit. When a deficit occurs, an amortization payment shall be made by the sponsor. Depending on the source of the deficit, in Quebec, the maximum repayment period can be 5 or 15 years (see *Loi sur les régimes complémentaires de retraite* [35], articles 129 and 130). In any year, we will take the additional payment to be the deficit divided by the value of a 15-year annuity-certain, valued at the valuation interest rate¹. As to the case when a surplus occurs, we will address it in the later sections, as it is the main concern of this thesis.

2.3 Model Pension Plan

At its name indicates, a pension plan exists to provide pensions. It also offers a variety of complementary benefits. We will look at each type of benefit in turn and specify, for each of them, a particular form. The selected combination of benefits will define our model pension plan, to be used throughout the examples that appear in this thesis.

¹By law, only those deficits which stem from modifications to the pension plan have a maximum amortization period of 5 years. Since we keep the pension plan characteristics unchanged, we are justified in using a 15-year amortization period.

2.3.1 Retirement Benefits

With respect to retirement benefits, the pension plan document must indicate how to calculate the annual pension. It also must specify the conditions of eligibility and the default form of annuity to which this annual pension applies. Finally, it must prescribe how to calculate the benefits for other ages at retirement and under other forms of annuity.

The three main types of defined-benefit plans are final earnings, career average earnings, and flat benefit. According to Statistics Canada [65], as of January 1, 1996, they accounted for 52.7%, 30.0% and 17.3% of all such plans, respectively. If, instead, we consider the number of members for each type of plan, we find that 71.4% of the members had a final earnings plan, 8.8% had a career average earnings plan, and the rest, 19.8%, had a flat benefit plan.

Under the flat benefit arrangement, which is found mostly in a unionized setting, the annual amount to be paid during retirement is the product of the fixed amount per year or hour of service, times the number of years or hours of service, depending over which unit the fixed amount is defined. As was mentioned earlier, *ad hoc* increases in benefits are periodically negotiated and result in all accumulated benefits (for participants and annuitants as well) being upgraded.

With a career average earnings plan, the annuitant annually receives an amount based on his average earnings over his whole career. In periods of high wage inflation, the value of the initial earnings is eroded. As a remedy to that, some career average earnings plans will adjust the earnings in one way or another.

According to the final earnings formula, the annual amount to be paid during

retirement is the product of the number of years of service times the average of the last n years of earnings² times a certain so-called generosity factor. The number n usually is between 1 and 5. As to the generosity factor, it is a fixed percentage, often set to 2%³.

Due to the existence of the Canada and Quebec Pension Plans (C/QPP), to which employers have to contribute, many pension plans allow for an integration of contributions and benefits with those of these public plans. Most that do do so using the excess method; the offset method is much less common.

Under the excess method (also known as the step rate method), a certain percentage applies to the salary up to the Yearly Maximum Pensionable Earnings (YMPE). A larger percentage then applies to the portion of the salary in excess of the YMPE, if any.

Under the offset method, a certain percentage is applied to the whole salary. Contributions (or benefits) are then reduced by all or part of the actual contributions to (benefits from) the Canada or Quebec Pension Plan.

Both methods of integration are equivalent. However, the offset method is sensitive to changes in the C/QPP contribution rates and schedule of benefits. The particular choice of method usually depends on which one is deemed most easily understandable by sponsor and participants.

It would be ideal to incorporate all possible variables in the definition of the pension plan. That would be more realistic and would show the potential user

²Under some plans, the average is actually taken over the years with the *best* earnings.

³According to Statistics Canada [64], as of January 1, 1994, for 80% of the members whose pension was calculated as a percentage of earnings, the generosity factor was exactly 2%.

how to apply the techniques introduced in this thesis to a full-fledged pension plan. However, for the sake of simplicity and tractability, we define a simplified pension plan which captures the main elements but leaves out the precise details an actual plan would feature.

With these considerations in mind, and so as to incorporate a reasonably large number of variables into the benefit formula, we will work with a final-earnings formula. However, we will not use an integrated formula, even though most plans are integrated, because of the added level of complexity it would entail.

Henceforth, we shall assume we have a pension plan offering, upon retirement at age 65, an annuity of 2% of the earnings in the final year, multiplied by the number of years of service. Age 65 is called the normal retirement age. So, if we denote by Sal_x^t the wages received by a participant aged x in year t , a person retiring in year z after n years of service shall receive $2\% \times Sal_{64}^{z-1} \times n$ per year until death. For example, if someone starts working at age 25 and retires at age 65, after earning \$80,000 in the last year, he will receive a pension of $2\% \times \$80,000 \times 40 = \$64,000$ per year for the rest of his life.

There are other ways of determining entitlement to full retirement benefits. For example, the plan document may stipulate that a certain number of years of service or a combination of service and age may serve as alternatives. In that case, the participant is eligible for full benefits as soon as he meets one of the criteria. We will not make use of alternative criteria here since this would only add to complexity; it would not add to the number of variables needed to be defined since age and service are already required.

We will take the default type of annuity to be a single annuity without any guaranteed period. We will ignore other types of annuity (offered as optional forms), assuming that their value would be determined by actuarial equivalence. This choice could be debated on the grounds that, by law, if there is a spouse, the default annuity must be a joint one. However, this would require assumptions regarding the spouse's age and mortality, and we do not feel that the gain in generality justifies the greatly added complexity.

We will not, however, ignore early and postponed retirements, as retirees will not be treated like participants when it comes to surplus distribution. When someone retires at an age other than the normal retirement age, we first must calculate the accrued benefit based on *actual* service and earnings as though he was then 65. We then must adjust that accrued benefit by a factor accounting for the fact that payments begin earlier (or later) and hence are payable over a longer (shorter) period of time. We allow retirement starting at age 55 and make it mandatory at age 70.

2.3.2 Termination Benefits

Many change jobs in the course of a lifetime. Leaving a job usually means withdrawing from the pension plan. Exceptions include, among others, changing employers with both in the same multi-employer pension plan or with both involved with the same union in the case of a union-sponsored pension plan.

In the general case, what happens when quitting depends on whether or not the benefits are vested. We say that benefits are vested if, when quitting, the

participant has acquired the right to a pension under the pension plan.

If quitting without vested benefits, the participant is not entitled to a deferred pension and usually will only get back his own contributions, accumulated with interest. Hence, he loses what the sponsor has contributed on his behalf.

When quitting with vested benefits, the participant is entitled to a deferred pension. Typically, the benefit will be determined by using *actual* service and final salary in the benefit formula, and it will become payable at normal retirement age. Some plans may allow the payment of the value of the deferred pension provided it is transferred to some retirement savings vehicle. Some may also offer the possibility of an early retirement or offer some benefit enhancement in the deferral period.

Allowing for transfers could raise some very complex issues, which this thesis does not aim to tackle. In particular, we may have to account for adverse selection, particularly at the higher ages. Moreover, legislation may prescribe the rates to use when calculating the transfer value. More importantly though, this would dramatically alter the composition of our membership and likely would have a great impact on the resulting surplus distribution.

Rules in the plan document specify when benefits become vested. Vesting may be full or graded. Under the former, after a fixed number of years, benefits are vested. Under the latter, an increasing portion of the benefits becomes vested as time goes by. Permissible vesting rules vary by jurisdiction. Currently, most Canadian jurisdictions require vesting after two years, though some prescribe it only after five years.

Rules regarding vesting changed quite a bit when pension law was revised in

Canada⁴ and the United States. In practice, this gives rise to situations where contributions and accrued benefits have to be split into pre-revision and post-revision portions so that appropriate laws be applied to them. Prior to revision, vesting often required a certain age (usually 45) and number of years of service (usually 10) which were quite high.

For plans with employee contributions, some plans do not allow participation until after a certain age or a certain number of years of service. Some other plans do not make participation mandatory until such time.

For our model pension plan, we assume participation starts at the time of employment. Full vesting occurs after two years of service, without regards to pre- and post-reform years. Prior to vesting, we return contributions with interest. After vesting, we provide a pension deferred to the normal retirement age, without any indexation in the deferral period.

2.3.3 Other Benefits

Pension plans provide other benefits in addition to the retirement benefits, namely death and disability benefits. By law, these benefits are often limited so that they remain incidental to the plan and be directly related to the retirement benefits.

⁴For more details concerning the changes in Canadian pension plan legislation, see Appendix 1 of *Pension Plans in Canada* [65] or visit Standard Life's web site at <http://www.standardlife.ca/eng/montre/legis/index.html>.

Death Benefits

For death benefits, we must distinguish between death before and after retirement. Before retirement, returning the participant's contributions with interest is a bare minimum imposed by law. If benefits are vested and there is a surviving spouse, the law may require that the spouse be offered a deferred annuity. We will assume that, upon death, the participant's accumulated contributions, and the sponsor's as well if vested, are payable as a lump sum. We will accumulate contributions with the rate of return realized on the pension fund, even though, in practice, the plan document may provide for a different rate of accumulation to be used.

In the case of members who have withdrawn from the plan with vested benefits, they shall be entitled to a death benefit equal to the actuarial value of the vested benefits as of the time immediately preceding their death. In other words, the death benefit is the value of the vested benefits to a life of the same age.

After retirement, some plans pay a modest amount upon death. However, most plans do not provide anything in addition to what is payable under the kind of annuity elected. Since we only consider single life annuities without guarantees under our model pension plan, we will not provide any post-retirement death benefits.

Disability Benefits

If disability is permanent, the disabled participant may be eligible for early retirement without reduction according to the terms of the pension plan. If the onset of permanent disability happens before early retirement is possible, the plan must at least provide a deferred annuity as in the case of a withdrawal when vested.

However, if, in addition to this pension plan, the employer provides a program of disability insurance which indemnifies the disabled employee until the normal retirement age, all that the pension plan then has to provide is an annuity deferred to age 65.

Questions remain as to the salary on which the pension will be based and as to the service that will be credited. Ideally, the last salary of the disabled person should be adjusted to account for increases in wages that have taken place between the onset of disability and the normal retirement age. Moreover, since the person would have worked had it not been for the disability, crediting service while disabled seems desirable.

For our model pension plan, we will provide annuities deferred to age 60 to anyone who becomes permanently disabled before age 60, and early retirement benefits, without reduction, to anyone who does so at age 60 or later. Moreover, in the period of deferment, service will continue to accrue. However, the salary on which the retirement benefit is based will not be adjusted but remain at the pre-disability level.

We will assume there is no short-term disability in our application. In real life, however, the plan document should contain provisions so that the accrued benefits be protected during this break in service.

2.4 Assumptions Required for the Valuation

The purpose of the pension plan valuation is twofold: First, we wish to calculate the actuarial present value of the different promises we have made so far under the

pension plan, via all the benefits we provide. Second, we want to find how much money we should set aside in the following years, until the next valuation, so as to fund the increases in accrued benefits that will take place over that period of time. In other words, we wish to value the liabilities and the contributions.

To value the liabilities of a pension plan, we need to project the future cash flows, arising from past or current benefit accruals, and discount them. We must be careful to account for all types of benefits and all participants.

To estimate these cash flows and their present value, we need to make assumptions, which we usually split into two groups: demographic and economic. As the choice of assumptions is an essential step of the actuarial valuation process, we will review the key assumptions in this section. For a more thorough discussion of the different factors affecting the valuation, the reader is referred to Chapter 22 of *Fundamentals of Private Pensions* [37].

2.4.1 Economic Assumptions

There are three important economic assumptions required for the pension plan valuation. They relate to the interest rate, the inflation rate, and the wage increases. We define and discuss each in turn.

interest rate (i) The rate at which cash flows are discounted.

It is also often called the valuation rate. The valuation results are very sensitive to this assumption. Hence, it needs to be set carefully. Normally, it is related to the rate of return that is expected to be earned on the pension fund in the long run. It may be treated as a single variable or as the composition of

several variables such as the rates of return on bills, on bonds and on stocks, for example. It is also possible to work with the real rate of return instead of with this nominal rate. The choice between the two is discretionary since they are related through the inflation rate.

inflation rate (inf) The increase in price of a certain typical basket of goods defined by some national statistical agency, meant to track increases in the cost of living.

It is necessary to make assumptions about inflation if we wish to work with real rates or if the plan offers some kind of indexation. It is also needed if the plan is integrated and the values with respect to which the integration is made are indexed in some fashion or other. Neither applies to the plan we consider and hence we will not have to make any related assumption.

wage increases (w and s_x) The rate of increase in wages varies with the year and with the age x (or the experience).

The component of the wage increase that has to do with the year is called wage inflation and is denoted by w . The other component, accounting for age or experience, is referred to as the salary scale and is given by s_x .

In general terms, the wage inflation compensates for price inflation and increases in productivity while the salary scale constitutes an appraisal of merit. The benefit formula and the valuation method will determine whether or not we need to make assumptions for these two components. In general, we need to when either we have a final average earnings plan or the funding method calls for the use of projected benefits.

Since we have a final salary plan and will be using the projected unit credit method, we will have to include these elements in our valuation basis.

As mentioned earlier, in setting an assumption for the interest rate, we must look at the long-term expected return on the assets. Since the future is not known, the interest rate really is a random variable over whose components the pension plan has virtually no control⁵. By components we mean the rates of return that are realized on the different asset categories. In fact, those values are mostly determined by the economy as a whole.

However, through its investment policy, the pension plan can decide to what extent it wishes to be exposed to each sector and, in each sector, how diversified it wants its position to be. The choice of asset mix may depend, among other things, on the size of the plan, on the age distribution of the members, and on the expertise of the committee responsible for the investment policy. Moreover, regulations usually place bounds on the investment policy through maximum percentages in any given sector or particular asset. However, the effectiveness of such restrictions is seriously undermined by the availability of derivative securities⁶ which basically allow an investor to build a portfolio whose current asset holdings differ appreciably from its actual exposure to the markets.

⁵The claim that the pension plan has no impact on the financial markets becomes questionable when we consider plans for which their sheer size, in terms of assets, is such that they control a non-negligible share of certain markets. Two examples of such plans are the Ontario Teachers Pension Plan (OTPP) whose administrator is the OTPP Board, and the Québec Pension Plan whose assets are managed by the Caisse de dépôt et placement du Québec.

⁶For more information of the nature and types of derivative securities, one good reference is *Options, Futures, and Other Derivatives* [28].

As a result, in coming up with the valuation rate, one will, first of all, acquaint himself with the investment policy. Second, he will either look at the average past returns in different asset classes, or study the past returns of the fund, or come up with a model to estimate the expected future returns. Canadian economic statistics are published annually by the Canadian Institute of Actuaries⁷, and the finance literature provides different models for each asset class. In selecting the rate, while no one method stands out as being the one to use, the emphasis must remain on the long term. In that respect, the valuation rate should not change very much from one valuation to the next.

Even though we do not need to set an inflation rate assumption to value our model pension plan, we still wish to provide some indication as to how we would set it if we had to. The inflation rate is another random variable on whose value the pension plan has no influence. However, in designing the pension plan, a decision must be made as to what extent the plan provisions will be affected by inflation. More precisely, the pension plan may offer different types of indexation: before and/or after retirement, full or partial. Before retirement, indexation may apply only to certain categories of pension plan members (*e.g.*, those who quit with vested benefits). As for partial indexation, it can take on many forms: percentage, excess of, capped, or any combination thereof.

In setting this assumption, the actuary can always look at the past and into the future. For the inflation rate, however, he also has to account for the actual impact it has on the different cash flows. This may call for different inflation rates

⁷American economic statistics are published annually by the Society of Actuaries.

to apply to different benefits.

Of all three main economic variables, it is over the increase in wages that the pension plan, through the sponsor and employer, has the most control. It does not have full control though since competition from companies in the same industry as well as from the economy as a whole comes into play. Since wages form only part of the whole remuneration package, an employer retains latitude with respect to wage increases insofar as he can offer equivalent enhancements through, say, revamped fringe benefits.

In setting assumptions for the total wage increase, we will consider the wage inflation and the salary scale separately. The first component is typically much more variable than the second. Besides, in any given year, whereas the first one is defined by a single value, the second one requires a vector in order to be fully specified.

This is not to mean that the salary scale is static. It too can vary each year. In fact, it generally has a given age for which the value of s_x is set at 1.00, while all other values are subject to change. Thus, the wage inflation captures the change in wages which is not accounted for by the restructuring of the salary scale.

In choosing a rate of wage inflation to be used in the valuation, the actuary should be concerned not only with what happens in the economy in general, but more particularly with what happens in other closely related companies. As always, the past may provide a good indication for the future.

As for the salary scale, it requires the study of the ratio between the wages earned at different ages in the past. Since the scale depends not only on the

industry but perhaps even more so on the type of work, more than one scale may be needed if the participants belong to different working classes (*e.g.*, secretaries and actuaries).

2.4.2 Demographic Assumptions

In pension plan valuation, there are also seven distinct sets of decrements for which assumptions must be made. Each decrement determines the type of benefit to which the participant is entitled. While active, a participant may terminate, become disabled, retire or die. In any other status, the only decrement a participant is subject to is death.

For each set of decrements, the actuarial organizations perform studies and publish tables which the actuary can use with or without modifications. Some tables even come with different projections so as to update or adapt them easily to a different year or a different situation.

termination rates ($q_x^{(t)}$) The probability of leaving the pension plan (and company) before retiring, dependent on the age x .

They are sometimes called withdrawal rates. These rates may also depend on the number of years of service, through incentives embedded in the pension plan and other fringe benefits.

disability rates ($q_x^{(d)}$) The probability of becoming permanently disabled before retiring, dependent on the age x .

For general pension plans, for which short-term disability cannot be ignored, we would need two sets of disability rates (short-term and long-term). In

addition, in the case of short-term disability, we would need probabilities of recovery, dependent on the age and on the time since the onset of disability. Long-term disability does not preclude recovery either, but we make the assumption that someone who is disabled for the long term will not work again.

retirement rates ($q_x^{(r)}$) The probability of retiring, dependent on the age x .

In our model pension plan, entitlement to retirement benefits depends only on age. If that were not the case, we would have to express these rates as a function of the number of years of service as well.

Even in our plan, the decision to retire may depend indirectly on the number of years of service through its impact on the value of the annual benefit to be received.⁸

active mortality rates ($q_x^{(m)}$) The probability of dying while active (contributing), dependent on the age x .

These rates depict the mortality experienced by healthy lives. Typically, they would be lower than for the population at large.

terminated mortality rates ($q_x^{(T)}$) The probability of dying after withdrawing with vested benefits, dependent on the age x .

We perhaps should use different mortality rates after someone has left the company instead of active mortality rates. On the one hand, there is no

⁸For a more in-depth look at the determinants of the retirement decision, we suggest *Pension Incentives and Job Mobility* [23], *Pensions, Economics and Public Policy* [29], or *Pensions, Labor, and Individual Choice* [72].

way to know if someone becomes disabled. On the other hand, as far as the pension plan is concerned, death becomes the only competing decrement. Nevertheless, the active mortality rates will also be applied to the members who have quit with vested benefits but not retired yet. Still, we retain the distinction in notation so as to keep track of the status of any member.

disabled mortality rates ($q_x^{(D)}$) The probability of dying while disabled, dependent on the age x .

These rates are significantly greater than those for active lives. They apply only to those lives which have become disabled while contributing to the pension plan. In other words, they apply only to those members for which the disabled status is made known automatically to the pension plan.

retired mortality rates ($q_x^{(R)}$) The probability of dying after retirement for a previously active participant, dependent on the age x .

It is important to distinguish between the mortality of someone who retired after having led an active life and that of another who retired following disability. For that reason, disabled mortality rates are provided for all ages, up to the one for which $q_x^{(D)} = 1.00$.

As for these rates, they are defined from the age at which early retirement becomes possible (here, age 55) to the age for which $q_x^{(R)} = 1.00$.

At any point in time and at any age, for an active participant, the difference between 1 and the sum of the first four rates gives the probability that the participant remains active under the pension plan. It is denoted by

$$p_x^{(\tau)} = 1 - q_x^{(d)} - q_x^{(t)} - q_x^{(r)} - q_x^{(m)}.$$

Likewise, at any point in time and at any age, for a terminated, disabled or retired participant, the difference between 1 and the applicable mortality rate gives the probability that that participant is still alive and, if in his retirement years, continues collecting retirement benefits. We denote these probabilities by

$$p_x^{(T)} = 1 - q_x^{(T)},$$

$$p_x^{(D)} = 1 - q_x^{(D)},$$

and

$$p_x^{(R)} = 1 - q_x^{(R)},$$

respectively.

The first four sets of rates are very important when valuing a pension plan, unless all benefits are determined by actuarial equivalence. Otherwise, depending on the decrement causing the person to cease being an active participant, the actuarial value of the benefits may differ from the liability accrued in that person's name. In that latter case, it becomes important to have four separate sets of rates deemed representative for the pension plan. Each of these four sets of rates is influenced to some extent, among others, by the type of work, the level of safety in the workplace, the working conditions found in competing companies, and the design of the pension plan itself. Some of these elements are under the direct control of the sponsor and employer.

The last three sets of rates are also important, but not very likely to be impacted by the pension plan or the sponsor and employer. As mentioned earlier, these three sets of rates are likely to be quite different and it may not be appropriate to blend them into one set.

Typically, for each set of rates, we would need two series of numbers, one for males and one for females. Instead of working with two series, it is possible to blend them into one so as to reflect the male/female ratio. We will greatly simplify our analysis by assuming that all members are male.

For closed group valuations, those which only account for the current pension plan members, we need not make assumptions about any other demographic variables. However, because open group valuations incorporate new entrants, we must make additional assumptions. We then need the following set of increments:

distribution of new entrants (ne_x) The number of new entrants at age x .

This distribution may vary over time. Besides, a split should be made with respect to gender. Moreover, to characterize the new entrants fully, we also need to know their entrance salary. To simplify matters, we assume that the pension plan membership is stationary⁹; hence ne_x is constant over time. Moreover, we will assume that entry occurs at only one age and all entrants at that age start with the same salary¹⁰.

Sometimes, an alternative increment assumption is made as to the total number of active participants. At first glance, it may seem to be redundant since we specify the number of new entrants and decrements for active members. On the contrary, because there actually is variability in the decrements, a fixed number of new entrants will not provide for a number of workers that can be calculated ahead

⁹The definition of a stationary population and related concepts can be found in *Introduction to the Mathematics of Demography* [12].

¹⁰All salaries, including the entrance salary, will be found using Equation 2.3 on page 39.

of time. Instead, if the company's work force is determined, the number of new entrants is left to fluctuate to compensate for losses of workers to termination, disability, retirement and death.

2.4.3 Other Assumptions

Finally, a third set of assumptions, related to expenses, could be required. Typically, investment expenses are considered as an offset to rates of return realized on the assets. There are other expenses as well. To mention a few, pension plan-related expenses include the cost of keeping the plan records, and the actuaries' fees for valuing the plan. Often, these other expenses are paid directly by the sponsor. Even when they are paid out of the plan assets, they rarely make the object of specific assumptions in the valuation. Rather, they are ignored based on the consideration that they usually represent a relatively small percentage of the total cost of the plan. We, too, will forgo making expense assumptions. In fact, we will actually ignore them altogether.

2.5 Mathematics of the Pension Plan

After assumptions have been made concerning each and every of the variables entering the valuation, we are almost ready to compute the contributions and liabilities. The assumptions are sometimes termed "conservative estimates"¹¹ and include a

¹¹The term "conservative estimates" is not universally accepted. The term "best estimates" is also often used. Nevertheless, based on a study published by the Society of Actuaries in 1996 [56], whereas economists held the view that long-term inflation would be 4% or less, only about 10% of the plans included in the study used an assumed inflation rate in that range. This seems

safety margin. They actually are best estimates which have been adjusted upward or downward so as to lower the probability of an unfunded liability arising in the near future.

Different values could be used over different years. However, it is more customary to use a single value assumed to hold over the entire number of years over which the valuation is carried out. For example, rather than assume that the inflation will be at a certain level for the next five years and then at another for the rest of the time horizon, we shall pick a single value for the whole. The notation introduced for the different variables will be used to denote the single value associated with each of them, when no superscript is used to indicate the year.

2.5.1 Timing

Before proceeding with the valuation, however, we will make some assumptions as to the timing of the payments and events so as to simplify the calculations. Here is what happens according to the time of year:

- All new employees, then aged 25, start working at the *beginning* of the year. Moreover, January 1 is the birthday of every plan member as well as the day on which wages are increased.
- Contributions and retirement benefits are paid at the *beginning* of the year.
- All deaths, withdrawals, disabilities and retirements take place at the *end* of the year. It is also at that time that death benefits are paid and that

to indicate that, indeed, most estimates used in pension plan valuation *are* conservative to some extent.

contributions are returned when applicable.

As for the valuation itself, it will be performed at the beginning of the year, right after all decrements have taken place and just before new entrants start working. Basically, we can imagine the computations being made in the wee hours of the morning, before business hours. At that time, the members have just turned one year older.

Of course, these assumptions represent important simplifications. It would be more realistic to assume a uniform distribution of increments and decrements over each year, except for retirements, particularly around ages 60, 65 and 70. A similar assumption would be more sensible with respect to the ages of the members. A distribution for the ages at entry would also be more sensible. As for contributions, the employee's portion is levied directly on the pay, which may be weekly, biweekly, bimonthly, or even monthly, but definitely not yearly. It could be though that the sponsor's portion is put into the pension fund on a much less regular basis.

2.5.2 Pension Plan Members

We will also distinguish between four groups of members of the pension plan. That will be helpful when we start calculating the liabilities and the costs. It will also provide us with broad categories of members which we may or may not want to consider as owners of the surplus. They are as follows:

active (A) Members of the pension plan who are still working for the sponsor and paying contributions.

terminated (T) Members of the pension plan who have left the employer with vested benefits.

disabled (D) Members of the pension plan who have stopped working because of long-term disability.

retired (R) Members of the pension plan who are receiving retirement benefits and were active prior to retirement.

2.5.3 Initial Computations

We are now ready to deal with the actual computations. Two of the books dealing with the mathematics of the pension plan are those by Anderson [5] and Berin [9]. The books by Aitken [3] and McGill [37] are the references used in SoA Course 210. *Actuarial Mathematics* [11] may also provide an introduction to the subject. The main reference used for the following is *Pension Mathematics with Numerical Illustrations* [71].

We will define terms and introduce notation as we go along. We will attempt to use symbols which are meaningful. In general, the right-hand subscript will represent the age while the right-hand superscript will give the year. Notable exceptions are the status indicators – (τ) , (d) , (t) , *etc.* – which will take the place of the year. If symbols with such an indicator require a year, it shall be separated from the status by a comma. For example, if survival probabilities varied over time, we would then denote by $p_x^{(\tau),t}$ the probability that an active life aged x survives in year t .

Every time we perform a valuation of the pension plan, we need to know how

many people there are. We will denote by n_x the total number of members aged x . Superscripts will be used to split this number into the four categories:

$$n_x = n_x^{(A)} + n_x^{(T)} + n_x^{(D)} + n_x^{(R)}.$$

We also need to know the benefit B_x^t that has accrued to each participant as of January 1, year t . Without projection, the accrued benefit would be equal to 2% times the wages earned in the previous year times the number of years of service completed. In symbols, this gives the following equation:

$$B_x^t = 2\% \times \text{Sal}_{x-1}^{t-1} \times (x - 25), \quad 26 \leq x \leq 70. \quad (2.1)$$

For people who are already receiving retirement benefits, we simply look at what is being paid annually to know B_x^t , for $x \geq 55$. Hence, if a person began collecting a pension at age θ in year y , then

$$B_x^{y+x-\theta} = B_\theta^y, \quad \forall x \geq \theta. \quad (2.2)$$

Likewise, we can define b_x^t , the unit benefit accrued to an active person aged x during year t . It is given by 2% of Sal_x^t . We wish to emphasize that b_x^t is *not* the difference between B_{x+1}^{t+1} and B_x^t ; this would be the case only if some kind of projection were used. (What we mean by that will be made clear in the following paragraphs.)

Since we will be using the *projected unit credit method*, we will project the accrued benefit to account for future wage increases. This choice of method is justified by the kind of pension plan we are working with. Because the retirement

benefit depends on the age at retirement as well as on the disability status, we cannot come up with a single figure for the projected accrued benefit. Rather, we need to do it independently for each combination of events. Of course, for benefits that are already in payment or deferred due to withdrawal or disability, no projection is needed.

To project the benefit, we essentially project the salary used to determine the benefit. In the case of early or postponed retirement, we also need to know the actuarial factors used to adjust the benefit.

The salaries in future years are linked to the current one by the following formula:

$$\text{Sal}_{x+k}^{t+k} = \text{Sal}_x^t \left(\prod_{i=1}^k (1 + w^{t+i}) \right) \frac{s_{x+k}}{s_x}, \quad 25 \leq x \leq 69, \quad 0 \leq k \leq 69 - x. \quad (2.3)$$

In words, the current salary (at time t) is multiplied by the wage inflation over the next k years and adjusted for projected increases due to merit in order to get the salary to be earned k years from now. In the actuarial valuation, the unknown wage inflation factors are replaced by their assumed value of w .

Actuarial factors are determined using the valuation interest rate i and the mortality rates provided by the $q_x^{(R)}$'s. First, we need to find the value of life annuities of one per year¹², denoted by $\ddot{a}_x^{(R)}$, for ages 55 to 70. Denoting the

¹²In fact, for the calculations to follow, we need to find the value of life annuities not only for people who went directly from active to retired, but also for people who terminated ($\ddot{a}_x^{(T)}$) or became disabled ($\ddot{a}_x^{(D)}$) prior to retiring. Equation 2.4 still holds when changing status superscripts on both sides, and does so for any age.

reciprocal of $(1 + i)$ by v , we find the following:

$$\ddot{a}_x^{(R)} = \sum_{k=0}^{\omega-x-1} v^k {}_k p_x^{(R)}, \quad x \geq 55, \quad (2.4)$$

where

$${}_k p_x^{(R)} = \begin{cases} 1 & , \quad k = 0, \\ \prod_{j=0}^{k-1} p_{x+j}^{(R)} & , \quad 1 \leq k \leq \omega - x, \end{cases} \quad (2.5)$$

for $55 \leq x \leq \omega - 1$, and ω is the ultimate age for which $q_{\omega-1}^{(R)} = 1.00$ ¹³. We also need to calculate survivorship factors, denoted by ${}_{65-x}E_x^{(R)}$ and ${}_{x-65}E_{65}^{(R)}$, from those ages lower than 65 to age 65 and from age 65 to those ages greater than 65, respectively. These factors also account for interest and are given by the general formula:

$${}_{y-x}E_x^{(R)} = v^{(y-x)} \times {}_{y-x}p_x^{(R)}, \quad 55 \leq x \leq \omega - 1, \quad x \leq y \leq \omega. \quad (2.6)$$

Since we use actuarial equivalence to calculate early and postponed retirement benefits, the actuarial factors are given by

$$\begin{cases} \frac{{}_{65-\theta}E_{\theta}^{(R)} \ddot{a}_{65}^{(R)}}{\ddot{a}_{\theta}^{(R)}} & , \quad \theta < 65; \\ 1 & , \quad \theta = 65; \\ \frac{\ddot{a}_{65}^{(R)}}{{}_{\theta-65}E_{65}^{(R)} \ddot{a}_{\theta}^{(R)}} & , \quad \theta > 65, \end{cases} \quad (2.7)$$

where θ is the age at which retirement benefits become payable.

¹³Actually, we will assume that the ultimate age ω is the same for all lives, regardless of their status.

We can now give some examples. For someone now aged x (at time t) that would quit at age $x + k$, the projected accrued benefit is

$$2\% \times \text{Sal}_{x+k}^{t+k} \times (x - 25), \quad 25 \leq x \leq 69, \quad 1 \leq k \leq 69 - x.$$

Disability at age $x + k$ would lead to the same basic value, which would have to be adjusted for the continued service accrual. For example, for someone now aged x , $25 \leq x \leq 58$, that would become disabled at age $x + l$, $0 \leq l \leq 58 - x$, and die at age $x + k$, $l < k \leq 59 - x$, the projected accrued benefit is

$$2\% \times \text{Sal}_{x+l}^{t+l} \times (x - 25) \frac{(x + k + 1 - 25)}{(x + l + 1 - 25)}.$$

In the retirement case, the projected accrued benefit depends on the age at retirement. Hence, for someone now aged x that would retire at age $\theta - 1$, it is given by

$$\begin{cases} 2\% \times \text{Sal}_{\theta-1}^{t+\theta-1-x} \times (x - 25) \times \frac{{}_{65-\theta}E_{\theta}^{(R)} \bar{a}_{65}^{(R)}}{\bar{a}_{\theta}^{(R)}} & , \quad \theta < 65; \\ 2\% \times \text{Sal}_{\theta-1}^{t+\theta-1-x} \times (x - 25) & , \quad \theta = 65; \\ 2\% \times \text{Sal}_{\theta-1}^{t+\theta-1-x} \times (x - 25) \times \frac{\bar{a}_{65}^{(R)}}{{}_{\theta-65}E_{65}^{(R)} \bar{a}_{\theta}^{(R)}} & , \quad \theta > 65. \end{cases}$$

Not only do we need to know the accrued benefit, we also need to know the value of the accumulated contributions, both total and, at least for the first year of employment, the employee's portion. We will denote by AC_x^t the total of the contributions made on behalf of or by an employee aged x at time t , together with investment income. We will denote by EAC_x^t the employee's portion of that total. These values are needed to determine the amount to be paid to employees that

quit before vesting, and will find further use once we get to determining the actual distribution of the surplus.

2.5.4 Accrued Liabilities

We are now ready to calculate the accrued liabilities, denoted by AL , for each member of the pension plan, according to the group to which he belongs. In each of the equations to be given, we will multiply the discount factor by the probability of the event, and then by the value of the benefit paid at of the time of the event.

Retired Lives

We will start with the retired lives. For the member j , $1 \leq j \leq n_x^{(R),t}$, we will denote by $\rho(j)$ the age on the January 1 following retirement. The associated actuarial liability is given by

$$AL_x^{(R),t}(j) = B_{\rho(j)}^{t-x+\rho(j)} \bar{a}_x^{(R)}, \quad 55 \leq x \leq \omega - 1, \quad (2.8)$$

which is the actuarial present value of the retirement benefits to be paid to j ¹⁴.

The actuarial liability for all of the $n_x^{(R),t}$ retired males of age x is

$$AL_x^{(R),t} = \sum_{j=1}^{n_x^{(R),t}} AL_x^{(R),t}(j). \quad (2.9)$$

¹⁴The equation would be much more complicated if other pension forms had been made available to the retiree. In addition, if joint forms could have been elected, we would have had an actuarial liability for the survivors.

Finally, summing over all retired males gives the total liability for the retired members:

$$AL^{(R),t} = \sum_{x=55}^{\omega-1} AL_x^{(R),t}. \quad (2.10)$$

Disabled Lives

Next, we consider the group of the disabled lives. As we calculate the accrued liability, we must remember that, under the special terms of our model pension plan, a disabled life is entitled to a full pension (that is, without actuarial reduction) starting at age 60. We will denote by $\varrho(j)$ the age which individual j , now aged x , turned the day following disablement. The accrued liability associated with that individual is

$$AL_x^{(D),t}(j) = \left\{ \begin{array}{l} \sum_{k=0}^{59-x} v^{(k+1)} {}_kP_x^{(D)} q_{x+k}^{(D)} \\ \quad B_{\varrho(j)}^{t-x+\varrho(j)} \left(\frac{(x+k+1)-25}{\varrho(j)-25} \right) \\ \quad v^{(60-x-k-1)} {}_{60-x-k-1}P_{x+k+1}^{(D)} \ddot{a}_{60}^{(D)} \\ \\ + v^{(60-x)} {}_{60-x}P_x^{(D)} B_{\varrho(j)}^{t-x+\varrho(j)} \left(\frac{60-25}{\varrho(j)-25} \right) \ddot{a}_{60}^{(D)} \quad , \quad x < 60, \\ \\ B_{\varrho(j)}^{t-x+\varrho(j)} \left(\frac{60-25}{\varrho(j)-25} \right) \ddot{a}_x^{(D)} \quad , \quad x \geq 60 \text{ and } \varrho(j) < 60, \\ \\ B_{\varrho(j)}^{t-x+\varrho(j)} \ddot{a}_x^{(D)} \quad , \quad x \geq 60 \text{ and } \varrho(j) \geq 60, \end{array} \right. \quad (2.11)$$

where $1 \leq j \leq n_x^{(D),t}$. When $x < 60$, the next event may be either death or retirement. The first term captures the death benefit while the second takes care of

the retirement benefit. The former is the commuted value of the deferred annuity, discounted and multiplied by the probability of the event. The latter is simply the value of a deferred annuity with the appropriate benefit. When $x \geq 60$, the disabled person has already retired and the liability is limited to the actuarial value of the pension being paid. As before, the accrued liability of the $n_x^{(D),t}$ disabled individuals is given by

$$AL_x^{(D),t} = \sum_{j=1}^{n_x^{(D),t}} AL_x^{(D),t}(j) \quad (2.12)$$

while that for all disabled is given by

$$AL^{(D),t} = \sum_{x=26}^{\omega-1} AL_x^{(D),t}. \quad (2.13)$$

Terminated Lives

The liabilities associated with the group of terminated lives are very similar to those of the disabled lives. Indeed, both are entitled to a deferred annuity and, in case of death before retirement, to the payment of the commuted value. The key differences are the age to which the annuity is deferred and the service being credited during the deferral period. As a result, we simply have to modify Equations 2.11 to 2.13 accordingly. Hence, denoting by $\tau(j)$ the age on the day after which individual j withdrew with vested benefits, we obtain, for the individual accrued liability:

$$AL_x^{(T),t}(j) = \begin{cases} \sum_{k=0}^{64-x} v^{(k+1)} {}_kP_x^{(T)} q_{x+k}^{(T)} \\ \quad B_{\tau(j)}^{t-x+\tau(j)} v^{(65-x-k-1)} {}_{65-x-k-1}P_{x+k+1}^{(T)} \bar{a}_{65}^{(T)} \\ + v^{(65-x)} {}_{65-x}P_x^{(T)} B_{\tau(j)}^{t-x+\tau(j)} \bar{a}_{65}^{(T)} & , \quad x < 65, \\ B_{\tau(j)}^{t-x+\tau(j)} \bar{a}_x^{(T)} & , \quad x \geq 65, \end{cases} \quad (2.14)$$

where $1 \leq j \leq n_x^{(T),t}$. The accrued liability for all terminated lives now aged x is

$$AL_x^{(T),t} = \sum_{j=1}^{n_x^{(T),t}} AL_x^{(T),t}(j), \quad (2.15)$$

and that for all terminated lives is

$$AL^{(T),t} = \sum_{x=27}^{\omega-1} AL_x^{(T),t}. \quad (2.16)$$

Active Lives

For the model plan described here, the accrued liabilities of the active lives are, by far, the hardest to compute. Of the four groups identified, that one is the only whose members are subject to decrements other than death. In spite of this major addition in complexity, these accrued liabilities have the particularity of being equal for any two individuals of the same age – under the restrictive assumptions of our model. Hence, multiplying by $n_x^{(A),t}$ the quantity $AL_x^{(A),t}(j)$, where j can be any of $1, \dots, n_x^{(A),t}$, yields the accrued liability for all of the $n_x^{(A),t}$ active members who are aged x .

The following equation is unquestionably a complex one. We broke it down so as to identify its parts easily, according to the first status (dead, disabled, terminated or retired) to which the active life moves to. In symbols, the accrued liability for any active life j aged x , $26 \leq x \leq 69$, is given by

$$AL_x^{(A),t}(j) = {}^M AL_x^{(A),t}(j) + {}^T AL_x^{(A),t}(j) + {}^D AL_x^{(A),t}(j) + {}^R AL_x^{(A),t}(j), \quad (2.17)$$

where the left-hand superscript indicates the first decrement to happen (M for mortality, T for termination, D for disability, and R for retirement).

The portion of the accrued liability that is due to mortality alone is

$${}^M AL_x^{(A),t}(j) = \sum_{k=0}^{69-x} v^{(k+1)} {}_k P_x^{(\tau)} q_{x+k}^{(m)} AC_x^t (1+i)^{(k+1)}, \quad 26 \leq x \leq 69, \quad (2.18)$$

in which case accumulated contributions are returned.

The portion due to termination, followed by death or retirement, is

$${}^T AL_x^{(A),t}(j) = \begin{cases} \left. \begin{aligned} & \sum_{k=1}^{64-x} v^{(k+1)} \sum_{l=0}^{k-1} {}_l P_x^{(\tau)} q_{x+l}^{(t)} {}_{k-l-1} P_{x+l+1}^{(T)} q_{x+k}^{(T)} \\ & B_x^t (1+w)^{l+1} \frac{s_{x+l}}{s_{x-1}} \\ & v^{(65-x-k-1)} {}_{65-x-k-1} P_{x+k+1}^{(T)} \ddot{a}_{65}^{(T)} \end{aligned} \right\} + \sum_{k=0}^{63-x} v^{(k+1)} {}_k P_x^{(\tau)} q_{x+k}^{(t)} \\ \left. \begin{aligned} & v^{(65-x-k-1)} {}_{65-x-k-1} P_{x+k+1}^{(T)} \\ & B_x^t (1+w)^{k+1} \frac{s_{x+k}}{s_{x-1}} \ddot{a}_{65}^{(T)} \end{aligned} \right\} , \quad x \leq 63, \\ 0 \quad , \quad x \geq 64, \end{cases} \quad (2.19)$$

where we did not account for any termination past age 63 since, at the end of the year throughout which the member is 64, he is entitled to full retirement benefits.

The portion due to disability, for which death or retirement can be the second decrement, is

$${}^D AL_x^{(A),t}(j) = \left\{ \begin{array}{l} \sum_{k=1}^{59-x} v^{(k+1)} \sum_{l=0}^{k-1} {}_l P_x^{(\tau)} q_{x+l}^{(d)} {}_{k-l-1} P_{x+l+1}^{(D)} q_{x+k}^{(D)} \\ \quad B_x^t (1+w)^{l+1} \frac{s_{x+l}}{s_{x-1}} \frac{(x+k+1-25)}{(x+l+1-25)} \\ \quad v^{(60-x-k-1)} {}_{60-x-k-1} P_{x+k+1}^{(D)} \ddot{a}_{60}^{(D)} \\ \\ + \sum_{k=0}^{58-x} v^{(k+1)} {}_k P_x^{(\tau)} q_{x+k}^{(d)} \\ \quad v^{(60-x-k-1)} {}_{60-x-k-1} P_{x+k+1}^{(D)} \\ \quad B_x^t (1+w)^{k+1} \frac{s_{x+k}}{s_{x-1}} \frac{(60-25)}{(x+k+1-25)} \ddot{a}_{60}^{(D)} \\ \\ + \sum_{k=59-x}^{69-x} v^{(k+1)} {}_k P_x^{(\tau)} q_{x+k}^{(d)} B_x^t (1+w)^{k+1} \frac{s_{x+k}}{s_{x-1}} \ddot{a}_{x+k+1}^{(D)} \quad , \quad x \leq 58 \\ \\ \sum_{k=0}^{69-x} v^{(k+1)} {}_k P_x^{(\tau)} q_{x+k}^{(d)} B_x^t (1+w)^{k+1} \frac{s_{x+k}}{s_{x-1}} \ddot{a}_{x+k+1}^{(D)} \quad , \quad x \geq 59 \end{array} \right. \quad (2.20)$$

where the pension is deferred only if the member becomes disabled before age 59. Otherwise, as defined for our model pension plan, the pension starts on the following January 1. Since we do not enhance the retirement benefits past age 65 in case of disability, it would be fair to assume that $q_x^{(d)} = 0$, $x \geq 65$ and that people becoming disabled after that age retire instead.

The portion due to retirement, which can be early, normal or postponed, is

$${}^R AL_x^{(A),t}(j) = \left\{ \begin{array}{l} \sum_{k=\max(0,54-x)}^{63-x} v^{(k+1)} {}_k P_x^{(\tau)} q_{x+k}^{(\tau)} \\ \quad v^{(65-x-k-1)} {}_{65-x-k-1} P_{x+k+1}^{(R)} \\ \quad B_x^t (1+w)^{k+1} \frac{s_{x+k}}{s_{x-1}} \ddot{a}_{65}^{(R)} \\ \\ + v^{(65-x)} {}_{64-x} P_x^{(\tau)} q_{64}^{(\tau)} B_x^t (1+w)^{(65-x)} \frac{s_{64}}{s_{x-1}} \ddot{a}_{65}^{(R)} \\ \\ + \sum_{k=65-x}^{69-x} v^{(k+1)} {}_k P_x^{(\tau)} q_{x+k}^{(\tau)} \\ \quad v^{(65-x-k-1)} B_x^t (1+w)^{k+1} \frac{s_{x+k}}{s_{x-1}} \ddot{a}_{65}^{(R)} / {}_{x+k-64} P_{65}^{(R)} \quad , \quad x \leq 63, \\ \\ v^{(65-x)} {}_{64-x} P_x^{(\tau)} q_{64}^{(\tau)} B_x^t (1+w)^{(65-x)} \frac{s_{64}}{s_{x-1}} \ddot{a}_{65}^{(R)} \\ \\ + \sum_{k=65-x}^{69-x} v^{(k+1)} {}_k P_x^{(\tau)} q_{x+k}^{(\tau)} \\ \quad v^{(65-x-k-1)} B_x^t (1+w)^{k+1} \frac{s_{x+k}}{s_{x-1}} \ddot{a}_{65}^{(R)} / {}_{x+k-64} P_{65}^{(R)} \quad , \quad x = 64, \\ \\ \sum_{k=0}^{69-x} v^{(k+1)} {}_k P_x^{(\tau)} q_{x+k}^{(\tau)} \\ \quad v^{(65-x-k-1)} B_x^t (1+w)^{k+1} \frac{s_{x+k}}{s_{x-1}} \ddot{a}_{65}^{(R)} / {}_{x+k-64} P_{65}^{(R)} \quad , \quad x \geq 65, \end{array} \right. \quad (2.21)$$

where we need to trisect the age interval to account for the declining number of retirement options open to the individual.

As is always the case, the total liability for the active lives is given by the sum over all ages and individuals:

$$AL^{(A),t} = \sum_{x=26}^{69} n_x^{(A),t} AL_x^{(A),t}(j). \quad (2.22)$$

Likewise, the total liability for all members is found by summing the accrued liability of each group:

$$AL^t = AL^{(A),t} + AL^{(T),t} + AL^{(D),t} + AL^{(R),t}. \quad (2.23)$$

This completes the determination of the liabilities.

2.5.5 Assets

On the asset side, we assume that market value is used, for the time being. In practice, a smoothing of the rates of return realized in the past few years is done so as to avoid major ups or downs in the value of the assets. We will revisit this issue in the next chapter, when we look at the alternative asset valuation methods used in practice.

2.5.6 Contributions

We still have to determine the contributions that will have to be made over the next three years. Contributions could include extra amounts to pay for unfunded liabilities. Here, we only wish to determine the normal cost, this part of the contributions that would be incurred if all assumptions made in the past had been borne out by experience. We assume that the assumptions that we make for this year's valuation are also valid for the next two years¹⁵. Hence, the normal cost for any active member in the next two years can easily be found by adjusting the current

¹⁵In fact, since the valuation basis is kept constant, the normal cost never changes as a percentage of the salary for a given age.

normal cost associated with a member of that same age for changes in salary. For example, if, this year, the normal cost for a member aged 35 whose salary is \$35,000 is \$3,500, then, it would be \$3,700 for a member aged 35 in two years whose salary is then \$37,000. The normal cost may be split in different ways between sponsor and member. It is common to have a fixed contribution percentage of the salary for the member, with the sponsor paying the rest. We will denote by α the percentage of his salary contributed by the member.

We denote the normal cost for someone aged x in year t by NC_x^t . We do not need to use any status as a right-hand superscript since contributions are paid only by active lives. However, as we did for the actuarial liability of active lives, we will use the left-hand superscript to indicate the first decrement to take place. Overall, the formula for the normal cost is similar to that for $AL_x^{(A),t}(j)$, with AC_x^t and B_x^t replaced by NC_x^t (or α prior to vesting) and b_x^t , respectively.

The component of the cost that comes from mortality being the first decrement is given by

$${}^M NC_x^t = \begin{cases} vq_x^{(m)}\alpha(1+i) \\ + \sum_{k=26-x}^{69-x} v^{(k+1)} {}_k p_x^{(\tau)} q_{x+k}^{(m)} NC_x^t (1+i)^{(k+1)} & , \quad x = 25, \\ \sum_{k=0}^{69-x} v^{(k+1)} {}_k p_x^{(\tau)} q_{x+k}^{(m)} NC_x^t (1+i)^{(k+1)} & , \quad 26 \leq x \leq 69, \end{cases} \quad (2.24)$$

where we need to distinguish between ages before and after benefits are vested.

Termination as a first decrement accounts for the following portion of the normal cost:

$${}^TNC_x^t = \begin{cases} vq_x^{(t)}\alpha(1+i) \\ + \sum_{k=1}^{64-x} v^{(k+1)} \sum_{l=0}^{k-1} {}_lP_x^{(\tau)} q_{x+l}^{(t)} {}_{k-l-1}P_{x+l+1}^{(T)} q_{x+k}^{(T)} \\ \quad b_x^t (1+w)^{l \frac{s_{x+l}}{s_x}} v^{(65-x-k-1)} {}_{65-x-k-1}P_{x+k+1}^{(T)} \ddot{a}_{65}^{(T)} \\ + \sum_{k=26-x}^{63-x} v^{(k+1)} {}_kP_x^{(\tau)} q_{x+k}^{(t)} \\ \quad v^{(65-x-k-1)} {}_{65-x-k-1}P_{x+k+1}^{(T)} b_x^t (1+w)^{k \frac{s_{x+k}}{s_x}} \ddot{a}_{65}^{(T)} \quad , \quad x = 25, \\ \sum_{k=1}^{64-x} v^{(k+1)} \sum_{l=0}^{k-1} {}_lP_x^{(\tau)} q_{x+l}^{(t)} {}_{k-l-1}P_{x+l+1}^{(T)} q_{x+k}^{(T)} \\ \quad b_x^t (1+w)^{l \frac{s_{x+l}}{s_x}} v^{(65-x-k-1)} {}_{65-x-k-1}P_{x+k+1}^{(T)} \ddot{a}_{65}^{(T)} \\ + \sum_{k=0}^{63-x} v^{(k+1)} {}_kP_x^{(\tau)} q_{x+k}^{(t)} \\ \quad v^{(65-x-k-1)} {}_{65-x-k-1}P_{x+k+1}^{(T)} b_x^t (1+w)^{k \frac{s_{x+k}}{s_x}} \ddot{a}_{65}^{(T)} \quad , \quad 26 \leq x \leq 63. \\ 0 \quad , \quad 64 \leq x \leq 69, \end{cases} \quad (2.25)$$

where there is no cost associated with termination past age 63 since the member would presumably retire instead.

The portion of the normal cost that funds the benefits paid to the disabled lives is given by

$${}^D NC_x^t = \left\{ \begin{array}{l} \sum_{k=1}^{59-x} v^{(k+1)} \sum_{l=0}^{k-1} {}_l p_x^{(\tau)} q_{x+l}^{(d)} {}_{k-l-1} p_{x+l+1}^{(D)} q_{x+k}^{(D)} \\ \quad b_x^t (1+w)^t \frac{s_{x+l}}{s_x} \frac{(x+k+1-25)}{(x+l+1-25)} v^{(60-x-k-1)} {}_{60-x-k-1} p_{x+k+1}^{(D)} \ddot{a}_{60}^{(D)} \\ \\ + \sum_{k=0}^{58-x} v^{(k+1)} {}_k p_x^{(\tau)} q_{x+k}^{(d)} \\ \quad v^{60-x-k-1} {}_{60-x-k-1} p_{x+k+1}^{(D)} b_x^t (1+w)^k \frac{s_{x+k}}{s_x} \frac{(60-25)}{(x+k+1-25)} \ddot{a}_{60}^{(D)} \\ \\ + \sum_{k=59-x}^{69-x} v^{(k+1)} {}_k p_x^{(\tau)} q_{x+k}^{(d)} b_x^t (1+w)^k \frac{s_{x+k}}{s_x} \ddot{a}_{x+k+1}^{(D)} \quad , \quad x \leq 58. \\ \\ \sum_{k=0}^{69-x} v^{(k+1)} {}_k p_x^{(\tau)} q_{x+k}^{(d)} b_x^t (1+w)^k \frac{s_{x+k}}{s_x} \ddot{a}_{x+k+1}^{(D)} \quad , \quad x \geq 59, \end{array} \right. \quad (2.26)$$

where we need to distinguish between ages for which retirement is deferred and those for which it is immediate. Again, as we mentioned earlier, we will ignore disability past age 65 since, under our model pension plan, a disabled life would then be disadvantaged compared to a retired life.

The last portion of the normal cost, that for retirement, amounts to

$${}^R NC_x^t = \left\{ \begin{array}{l} \sum_{k=\max(0,54-x)}^{63-x} v^{(k+1)} {}_k P_x^{(\tau)} q_{x+k}^{(\tau)} \\ \quad v^{(65-x-k-1)} {}_{65-x-k-1} P_{x+k+1}^{(R)} b_x^t (1+w)^k \frac{s_{x+k}}{s_x} \ddot{a}_{65}^{(R)} \\ \\ + v^{(65-x)} {}_{64-x} P_x^{(\tau)} q_{64}^{(\tau)} b_x^t (1+w)^{(64-x)} \frac{s_{64}}{s_x} \ddot{a}_{65}^{(R)} \\ \\ + \sum_{k=65-x}^{69-x} v^{(k+1)} {}_k P_x^{(\tau)} q_{x+k}^{(\tau)} \\ \quad v^{65-x-k-1} b_x^t (1+w)^k \frac{s_{x+k}}{s_x} \ddot{a}_{65}^{(R)} / {}_{x+k-64} P_{65}^{(R)} \quad , \quad 25 \leq x \leq 63, \\ \\ v^{(65-x)} {}_{64-x} P_x^{(\tau)} q_{64}^{(\tau)} b_x^t (1+w)^{(64-x)} \frac{s_{64}}{s_x} \ddot{a}_{65}^{(R)} \\ \\ + \sum_{k=65-x}^{69-x} v^{(k+1)} {}_k P_x^{(\tau)} q_{x+k}^{(\tau)} \\ \quad v^{65-x-k-1} b_x^t (1+w)^k \frac{s_{x+k}}{s_x} \ddot{a}_{65}^{(R)} / {}_{x+k-64} P_{65}^{(R)} \quad , \quad x = 64, \\ \\ \sum_{k=0}^{69-x} v^{(k+1)} {}_k P_x^{(\tau)} q_{x+k}^{(\tau)} \\ \quad v^{65-x-k-1} b_x^t (1+w)^k \frac{s_{x+k}}{s_x} \ddot{a}_{65}^{(R)} / {}_{x+k-64} P_{65}^{(R)} \quad , \quad 65 \leq x \leq 69. \end{array} \right. \quad (2.27)$$

where, once again, we split the age interval into three sections so as to account for the declining number of retirement options.

Combining these four components, we obtain that the normal cost is

$$NC_x^t = {}^M NC_x^t + {}^T NC_x^t + {}^D NC_x^t + {}^R NC_x^t, \quad 25 \leq x \leq 69. \quad (2.28)$$

Though not apparent, α is implicitly dependent on NC_x^t since we definitely would

not want the former to be larger than the latter. This will require us to be careful in our choice of α and to double-check the relationship between the two percentages. This completes our triennial actuarial valuation, performed in the traditional way.

2.6 Conclusion

We have reviewed some elements of pension plan regulation. We have also defined a model pension plan, which will be the basis for the examples to be given in the following chapters. We have defined the variables that enter the actuarial valuation of that plan and derived the formulae needed to compute the liabilities and contributions. We have performed a traditional actuarial valuation, using the projected unit credit method. Other valuation methods could be devised.

We know that calculations are sensitive to the assumptions. We also know that variability exists among the valuation assumptions used by pension plans (see the *Study of Public Employees Retirement Systems* [56]). Of course, differences between pension plans (such as job sector and investment policy) explain some of this variability. However, they do not explain it all. For example, in that *Study*, for the 78 Retirement Systems included, the assumed inflation rate lay between 3.5% and 6.5%. Based on these considerations, one may suggest that distributional assumptions be used in lieu of single values. It naturally would add in complexity, but would have the advantage of yielding a distribution of potential values rather than a single one.

A further level of sophistication could be brought in the valuation process

through the use of the Bayesian approach¹⁶. Under that approach, not only would one make distributional assumptions for each of the variables, but distributions over the parameters of those distributions would also have to be picked. (In actuarial science, the Bayesian approach is already used in the theory of credibility [31, 32].) This approach too would yield a distribution of potential values rather than a single one.

Of course, we still would wish to report a single value for the liabilities and use one single set of contribution rates. We could do so by picking percentiles. Alternatively, we could use statistical decision theory¹⁷, which is the theory of making decisions, when faced with uncertainty but possessing some statistical knowledge about the problem.

There certainly exist other possible ways to perform actuarial valuations. We did not strive to be exhaustive, but simply to hint at other potentially valid approaches. These remain subjects for further research, as little has been ventured away from the traditional path.

¹⁶Robert provides an excellent coverage of the Bayesian theory in his book *The Bayesian Choice: A Decision-Theoretic Motivation* [54].

¹⁷The reader may wish to consult Clemen [15] for an introduction to the topic or Berger [8] for a more in-depth exposition.

NOTE TO USERS

**Page(s) missing in number only; text follows.
Microfilmed as received.**

56

This reproduction is the best copy available.

UMI

Chapter 3

Choice of the Amount to Be Distributed

3.1 Introduction

In the last chapter, we have determined how to calculate the value of the liabilities, using the traditional actuarial method. This alone does not suffice in establishing the surplus, let alone the actual amount to be distributed. We also need to know the value of the assets. In a wind-up situation, that would complete the picture since all of the surplus has to be distributed. The same approach could, in theory, be adopted for ongoing plans, although more care may be taken in coming up with a value for the assets. However, on an ongoing basis, we will come up with a more detailed criterion, which will require simulations of the evolution of the plan.

In this chapter, first, we will look at how assets are, or could be, valued to determine the surplus for both wind-up and ongoing situations. Then, we will

present our suggested criterion for determining the amount to distribute, when the plan is an ongoing concern. The application of this criterion will require us to develop some tools. In particular, we will present the model we will use to simulate economic scenarios. We will also derive the equations that determine the evolution of the assets and liabilities through time.

We will see what happens when assumptions and reality are one and the same. We will also see that there should be some relationship between the valuation basis and the expected simulation results; otherwise, the suggested criterion may produce amounts to be distributed that are of unreasonable value.

Moreover, we will adapt the criterion and its related tools so as to make them applicable to any subgroup of the pension plan. Most adaptations will be straightforward, save for the determination of the initial value of the assets to be allocated to the subgroup. *A priori*, we would not need to know all these hypothetical amounts to be distributed to subgroups. Yet, we will see in the next chapter how they come into play as we decide how to split the global amount.

To end this chapter, we will offer some ideas of other possible ways of determining the amount to distribute. Many of them will be dependent upon alternative valuation methods such as those mentioned at the end of the last chapter.

3.2 Valuation of Assets

As mentioned earlier, if the plan is winding up, the surplus, both on paper and available for distribution, is merely the excess of the assets over the liabilities. In that case, assets are naturally appraised at market value, since it represents the

total amount available.

However, if the plan is ongoing, giving away all of the surplus based on a volatile market value may appear too risky. Rather, we may want to value the assets in some other way. We will first look at how pension plan assets are valued in practice. We will then mention some valuation methods found in the finance literature. We will end this section by reiterating our choice for the present purpose of surplus sharing and try to give some justification for this choice.

3.2.1 Valuation Methods Used in Practice

There are basically three ways of valuing the assets found in actuarial practice. One possibility is *market valuation*. In that case, on valuation date, the assets are valued based on what their current price is on the stock, bond, or other corresponding market. For valuation purposes, it makes sense insofar as the valuation is meant to provide a snapshot of the financial status of the pension plan. Still, this snapshot may be distorted unless the liabilities are appraised at market value as well¹.

Out of the 183 plans that participated in the *Study of Public Employees Retirement Systems* [56], only 24 used market value for equities, while 14 used it for fixed income². Hence, the majority of the plans used another method.

Since market value is about the easiest value to obtain – especially when the

¹In the United States, there is a trend towards market value accounting on the liabilities side. Whereas finding the market value of the assets is fairly straightforward, determining the market value of the liabilities is much harder since there is no active market from which the prices can be extracted. As a result, this problem has become part of the actuarial literature.

²The valuation method for equities is analyzed separately from the valuation method for fixed income. As a result, we cannot determine if the 14 plans that used market value for fixed income also used it for equities.

funds are invested with an outside manager that generates regular statements containing this information, other considerations must explain the relative lack of popularity of this method in practice. Indeed, the inherent volatility of market value poses a particular problem to the sponsor: it also makes the amortization payment very variable. For the sponsor, since this additional payment is written off the company's balance sheet and represents money that could have been invested in the company's operations, there is a natural desire that such payments be rare and small.

Market-smoothing and *cost basis* are the other two alternatives. For equities, about two thirds of the plans included in the study used smoothing, with about one half smoothing over more than three years. For fixed income, more than one half used smoothing; out of those who used smoothing, we find that about three quarters smoothed over more than three years. Also, about one third of the plans used the cost basis for fixed income.

The main objective associated with market-smoothing is stability. On the one hand, when asset value is relatively high, there may be pressure for an enhancement in the benefits. On the other hand, when asset value is relatively low, additional contributions may be required. Neither is desirable if it is solely caused by variability in the rates of return and does not result from a material change in the expected rates of return over the long run.

For equities and fixed income, market-smoothing determines the value of the assets by calculating some weighted average of the actual rates of return over the past n years, where n is the number of years over which the smoothing is done.

In other words, instead of including the total market losses and gains for a year in that same year, they are spread over time. As net losses and gains alternate, this method generates a more stable value for the assets, thus helping to avoid the two pitfalls indicated earlier. As larger values of n instill more stability but, at the same time, delay the inclusion of actual experience, consideration of these two conflicting objectives will direct the choice of the smoothing period.

As for cost basis, the actual objective varies according to the assets that are valued using it. For fixed income, it is again stability. Valuation of fixed income based on cost³ is done by amortizing the discount or premium included in the original bond price so as to obtain a smooth progression of the value of the bond from its purchase to its maturity. Cost basis is a valid valuation method for fixed income, particularly when bonds are kept to maturity. In these circumstances, the pension plan knows exactly what cash flows it will receive from its fixed income investments. Hence, in keeping with its objective of stability, cost basis makes sense.

In the case of equities, the future cash flows are not known with certainty. Thus, using cost basis to value them is done not really for stability (although it does have a "modest smoothing effect" [56, p. 18]), but for conservatism. Using a modest estimate for the rate of return to be realized in the long run on the equities, cash flows can be calculated, from year to year, using this underestimate, until assets are actually sold.

³This valuation method may actually be simpler to use for fixed income, instead of market value, depending on the liquidity of the fixed income instruments held in the portfolio. Whereas cost and cash flows are always known for such assets, it is not necessarily the case that they are traded actively enough to have access to a current market value at the time of valuation.

In summary, market valuation is the method to use if we wish to know the current status of the pension plan, as if the plan were to be “sold.” However, the other two methods bring stability to the value, which is very desirable from the plan’s perspective.

Presumably, some kind of stability would also be desirable when we determine the amount of surplus to be given away. Moreover, smoothed values are typically below market values. Hence, using the former in the determination of the amount to distribute would automatically instill some prudence, though to an unknown degree.

3.2.2 Alternative Valuation Methods

Besides the three methods commonly used in actuarial practice, we may also want to derive some kind of function that would attempt to extract the fundamentals from the market information. In other words, we could try to come up with the “true” asset value underlying the market value.

Such an endeavor really amounts to research in finance. For stocks, there exist different discounted dividend and capitalized earnings models aimed at assessing the intrinsic value. The theory of bubbles [13, 57] may also be applied in this case. This theory starts with the premise that values observed in the markets are not necessarily equal to the “true” values and attempts to remove the so-called bubbles from the market prices.

For bonds, pricing involves selecting an appropriate rate (or term structure) at which to discount the cash flows. Furthermore, the probability of default has to

be accounted for. Other bond features, such as callability or convertibility, when present, add to the complexity of the valuation.

As was the case for the liabilities, we could also come up with distributional assumptions for the different elements determining the value of the different assets. This would generate a distribution for the value of the assets, out of which we would then have to pick a single value.

However the assets may be valued, distributing the difference between the assets and the liabilities is one possibility. While this approach has the advantage of being simple, it does not provide any indication of the extent to which it may jeopardize the future financial health of the pension plan⁴.

3.2.3 Valuation Method to Be Used

As briefly mentioned in the last chapter, we will be valuing the assets according to the market. It does, of course, greatly simplify matters. Still, we believe this is a sensible approach since we potentially will be distributing part of the assets currently held and, to distribute them, we will have to sell them at market value (unless the distribution is in a form other than cash).

As mentioned earlier, volatility is an issue when working with market values. Nonetheless, we consider that the criterion to be introduced will allow us to avoid the potential downside of volatility, namely, giving away more than we actually can

⁴In the context of life insurance, Pansera [45] has shown that the use of dynamic solvency testing to quantify the mismatch risk – the interest rate risk – does not lead to values corresponding to a fixed short-term solvency probability. This suggests that, unless we adopt a stochastic approach, we will not know, with any degree of accuracy, to what extent we actually are protecting our insurance portfolio against a given risk.

afford to.

As for the issue of comparability between the values for assets and liabilities, we do not pretend to derive a market value of liabilities. However, we will be using a realistic valuation basis, based on expected future economic conditions. This, in some sense, shall relate the value of the liabilities to the markets.

3.3 Criterion for the Amount to Distribute

In elaborating a criterion for the amount to distribute, we will have to determine what the key concerns of a pension plan are. Based on these concerns, we will then come up with a criterion that attempts to address them in a satisfactory way.

3.3.1 Key Concerns of a Pension Plan

In their works on pension plans, Dufresne [19, 20] and Haberman [24] have studied the variability of pension plan contributions and funding level when only the rate of return on the pension fund is modeled as a random variable. In doing so, they identified two key concerns of pension plans: contributions and funding level⁵.

On the one hand, we wish total contributions (normal cost plus any additional payment) to be stable through time. On the other hand, we want to keep the funding level (assets as a percentage of liabilities) within a reasonable interval, outside of which there is either a deficit to fund or a large surplus to be depleted in some way.

⁵At the same time, they also worked with the implicit assumption that the randomness of rates of return is the main source of variability for pension plans.

Greenwood and Keogh [22] were more explicit in their identification of a pension plan's key concerns: they suggested targeting "a sufficient margin above the current Minimum Funding Requirement⁶ to make the chances of unexpected failure at valuations 3 and 6 years hence acceptably low." However, they have not produced mathematical results that showed what that meant in practical terms.

3.3.2 Suggested Criterion

Combining Dufresne's, Haberman's, and Greenwood and Keogh's observations on a pension plan's key concerns, we propose the following criterion for the determination of the amount to be distributed:

The amount must be such that, with a given level of probability, there will not be an unfunded liability at the next actuarial valuation.

In other words, taking into account the monies that are to be deposited into or withdrawn from the fund, we want to be reasonably sure that we will not require an additional payment from the sponsor over the next three years the next time we value assets and liabilities. Of course, we could choose to look further than three years hence. In our application of this criterion to our model pension plan, we will compare the results obtained for two different lengths of time.

This suggested criterion requires simulations of possible scenarios between this and the next valuation date (or other time at which we want to check the funding level). Both the assets and the liabilities need to be projected in time. At the end of the projection period, the surplus is calculated and discounted back to time 0

⁶The Minimum Funding Requirement valuation is a required valuation in the United Kingdom.

using the rates of return that led to this surplus. All the discounted surpluses are then ordered and the quantile corresponding to the desired probability is identified as the amount to be shared. If this amount is negative, we simply do not distribute anything.

Nevertheless, if there is an unfunded liability (deficit), and hence an additional payment to be made by the sponsor over the next three years, we do not need any criterion. Indeed, it would be rather odd to prescribe any kind of distribution of the assets when they are deemed insufficient at that very time⁷.

3.4 Simulated Projections of the Pension Plan

In this section, we will see how the simulations are carried out. We will first identify what is assumed fixed and what is assumed random. Then, we will describe the model used for the random elements. Finally, we will show how assets and liabilities evolve through time with respect to these elements.

3.4.1 Fixed and Random Elements

We need to decide which elements, out of the many that are subject to assumptions⁸, we wish to consider random. In making that choice, we want to assess which ones contribute the most to the variability of contributions and funding level in a pension plan.

⁷As we will see, though, if the valuation basis is much more conservative than the expected outcomes would dictate, we could end up with such nonsense.

⁸All the items for which we must make assumptions for the pension plan valuation were listed and defined in Section 2.4.

Whereas demographic risks are diversifiable, economic ones are not⁹. Hence, we can predict with more precision the number of members who will retire in the next year for a large plan than for a small plan. However, we cannot make any better predictions for the return on a pension fund based merely on of its size.

The above statements about precision are both made on the basis of a known distribution for the element being projected. In both cases, there is also uncertainty about the actual distribution and parameters. In fact, from the actuary's point of view, this type of uncertainty may be more important and relevant to him as it relates directly to his choice of valuation assumptions¹⁰. Nevertheless, in view of the goal of surplus sharing that we have and because of the criterion we will develop to meet that goal, we are more preoccupied with variability through time than sensitivity to assumptions.

For life insurance, Parker [46, 47] has shown that, as the number of insureds increases, the economic risk eventually explains more of the variance in the total cost of insurance than the demographic risk does. The same still has not been shown for pension plans, but we believe the general result would hold in that case too¹¹. This would mean that, for pension plans that are large enough, more variability

⁹We hope we do not mislead the reader when we write that economic risks are not diversifiable. Of course, a diversified portfolio of assets is less risky than a single asset. However, the sheer size of a portfolio has no impact at all on its level of risk.

¹⁰Insofar as the mean (or median or mode) provides a starting point in setting valuation assumptions, any source of uncertainty affecting the mean will be important to the valuation actuary. Uncertainty in the distribution or in its parameters both lead to uncertainty in the mean.

¹¹Indeed, it appears reasonable to think that results obtained for life insurance would also apply to pension plans. There is an important difference though. The results derived were based on a single decrement. The multiplicity of decrements in pension plans may mean that the number of members needed for economic risks to take over in relative importance is so high that the result in itself has little applicability.

is induced in the results by economic risks than by demographic risks. Intuitively, this can be justified by the degree of diversifiability of each type of risk.

One may then wonder if we should not distinguish between small and large pension plans in this study. Because small plans are vulnerable to departures from assumptions, most of them are pre-funded through insurance. In fact, according to Statistics Canada [65], as of January 1, 1996, 67.4% of all plans were insured; yet, these plans only covered 12.9% of the members. For the sake of comparison, while only 32.4% of all plans were trustee, these plans covered 78.5% of all members.

We choose to focus on large plans mainly for two reasons. First of all, a large percentage of pension plan members belong to a large plan. Indeed, as of January 1, 1996, 55.8% of all such members were in a plan with 10,000 members or more. More importantly perhaps, at that same time, nearly 70% of the members in the public sector were in a plan with 30,000 members or more (see [65]). This is important if we believe the public sector to work as a benchmark for the private sector in terms of working conditions and benefits.

Second, the existence and nature of large plans are less sensitive to changes in legislation. In fact, the provincial reforms of pension plan legislation appear to have caused the disappearance of several small plans. In Canada, reforms have taken place in different years between 1984 and 1993 in the different provinces. From 1986 to 1996, the number of plans with 1 to 9 members went from 11,200 to 5,288, a substantial decrease of 52.8% (see [65]). The number of plans of other sizes remained relatively unaffected over that same period.

It may be possible to offer one more justification for focusing on large plans.

They are more likely to generate large absolute surpluses which catch the participants' as well as the sponsor's attention, and even in some cases the public's attention. Large plans are also more likely to go to court in order to seek approval for a particular distribution of the surplus. In such a case, it would be nice to have some notion of what a desirable distribution might be.

For plans of a smaller size, it seems reasonable to think that a surplus, although relatively large, will not generate the same interest. On the one hand, it may be that the sponsor and participants will decide together on the best course of action. On the other hand, it could be that the participants feel powerless and that the sponsor virtually has full control. Besides, the resources needed to go to court may very well then appear large in relation to the surplus at stake.

For large pension plans, since we suspect the economic risk to be of much greater importance than the demographic risk, we will assume the decrements to be fixed. Hence, we will treat all values of $q_x^{(*)}$ as deterministic for any superscript $*$. (Tables are given in Appendix A.) Likewise, we will assume increments to be deterministic; that is, we will assume a stationary pension plan membership. Out of the different economic variables, we will treat the salary scale, given by s_x , as static. (The scale is given in Appendix B.) However, we will consider the rate of return, the inflation rate and the wage inflation to be random variables.

Since the model we will adopt accounts for different components of the rate of return, we could use dynamic asset allocation. In practice, the investment policy, which the pension committee must elaborate, will contain guidelines as to how to split the assets among the different categories. While these guidelines would not

be so stringent as to specify exact proportions, they would not be so loose as to be ineffective.

Notwithstanding, since optimal asset allocation is not the focus of this thesis¹², we will work with a fixed investment policy. Every year, there shall be a reweighting of the portfolio so as to have 50% of the assets in stocks, 40% in bonds, and 10% in bills (short-term investments)¹³. Hence, the rate of return realized on the pension fund will be a simple weighting of the rates of return realized for each asset category.

This allocation may appear rather arbitrary. Actually, for the first quarter of 1998, the allocation of the total assets of the trustee pension funds was as follows (see [66]):

Asset category	Percentage of market value	Corresponding weights
Stocks	44.7%	52.2%
Bonds	36.5%	42.7%
Short-term	4.4%	5.1%
Mortgages	1.6%	
Real estate	3.5%	
Other	9.3%	

The last column presents the relative weights of the first three categories; this would be the resulting allocation if all the assets held in the other categories would be reinvested proportionally in the first three. Thus, we appear to err on the side of

¹²Wright [73] has studied the issue of optimal asset allocation for pension plans under Wilkie's model for the United Kingdom.

¹³Transaction costs are not accounted for, whether it be on the initial purchase or on the reallocation.

caution since we give a larger weight to short-term investments. In fact, over the past 14 years, stocks have gained in importance at the expense of bonds, mortgages and short-term investments (see [64])¹⁴.

3.4.2 Model Used for the Economic Variables

In order to project our pension plan through time, we need to generate scenarios for the inflation rate, rate of return and wage inflation. To do so, we will use Wilkie's model [70] for the first two, complemented by Sharp's wage inflation model [61].

The model developed by Wilkie has the advantage of modeling inflation and the different asset categories in an integrated fashion¹⁵. As a result, it accounts for the relationship among the different variables. In fact, it does so through a "cascade" structure, which we will follow in our presentation of the model¹⁶.

The first building block is the force of (price) inflation, denoted by $I(t)$, where t denotes the year¹⁷. Its model is

$$I(t) = QMU + QN(t), \quad (3.1)$$

$$QN(t) = QA \cdot QN(t-1) + QE(t), \quad (3.2)$$

$$QE(t) = QSD \cdot \epsilon_1(t), \quad (3.3)$$

¹⁴The trend is the same if we look at book value. However, in proportion, bonds then represent a larger segment, while stocks come out at a disadvantage.

¹⁵For a review of Wilkie's model for the United Kingdom, see the paper by Huber [27]. While we are using the model developed with Canadian data, we suspect similar criticisms would apply.

¹⁶We also use the same symbols for the parameters as did Wilkie [70].

¹⁷Typically, a model for the force of inflation would be in continuous time. What we really have here is a model for the rate of inflation in year t , which is $e^{I(t)} - 1$.

where the parameters are QMU , QA and QSD . The values for these parameters, and all others included in the economic models, are given in Appendix D. As for the term $\epsilon_1(t)$, it represents a standardized normal random variable. All other variables included in Wilkie's model will depend on $I(t)$.

The dividend yield in year t is given by $Y(t)$. It is the ratio of the dividends paid in year t to the value of the stock at the beginning of that year. The model for this second building block, on which all the ones to follow will depend as well, is

$$\ln Y(t) = YW \cdot I(t) + \ln YMU + YN(t), \quad (3.4)$$

$$YN(t) = YA \cdot YN(t-1) + YE(t), \quad (3.5)$$

$$YE(t) = YSD \cdot \epsilon_2(t), \quad (3.6)$$

where $\epsilon_2(t)$ is also a standardized normal random variable. In fact, every $\epsilon_i(t)$ will represent such a variate and all these variates are mutually independent. We again have a first-order autoregressive model, with an added element to account for the dependency of $\ln Y(t)$ on $I(t)$. The long-term mean of the dividend yield is $YMU \cdot e^{YW \cdot QMU}$.

Alone, the dividend yield does not provide us with a rate of return on stocks. We also need a model either for the stock price or for the dividends. Wilkie has modeled the latter, with $D(t)$ as the dividend amount in year t :

$$\begin{aligned} \ln D(t) - \ln D(t-1) = & DW \cdot DM(t) + (1 - DW) \cdot I(t) + DMU \\ & + DY \cdot YE(t-1) + DB \cdot DE(t-1) + DE(t), \end{aligned} \quad (3.7)$$

$$DM(t) = DD \cdot I(t) + (1 - DD) \cdot DM(t - 1), \quad (3.8)$$

$$DE(t) = DSD \cdot \epsilon_3(t). \quad (3.9)$$

Hence, what really is being modeled is the force of dividend increase in year t . It depends on the force of inflation through a weighted average of current and past rates. It also depends on the previous dividend yield through its deviation from the expected value.

If we let $SPI(t)$ be the value of stocks at the end of year t , then we get that the rate of return on stocks is given by, in simplistic terms,

$$r_s(t) = \frac{D(t) + SPI(t)}{SPI(t - 1)} - 1. \quad (3.10)$$

This is simplistic since it implicitly assumes that all the dividends are payable at the end of the year. It would be more realistic to assume the dividends to be paid uniformly over the year. We would then have to make an assumption as to the rate of return earned on the reinvestment of the dividends, which itself would be questionable. As for the assumption of uniformity, reality does not verify it. Rather, at least for the S&P 100 Index, the payment of dividends turns out to be highly seasonal (see Harvey and Whaley [26]).

We still are missing the rates of return on bonds and bills (our short-term investment). To find that on bonds, we start by looking at the model for the yield on bonds in year t , denoted by $C(t)$ ¹⁸. As is clear in the following model, $C(t)$

¹⁸The notation $C(t)$ for the yield on bonds comes from “consols”, the name given to perpetual bonds in the United Kingdom.

depends on the history of inflation rates, as well as on the current dividend yield:

$$C(t) = CW \cdot CM(t) + CMU \cdot e^{CN(t)}, \quad (3.11)$$

$$CM(t) = CD \cdot I(t) + (1 - CD) \cdot CM(t - 1), \quad (3.12)$$

$$CN(t) = CA \cdot CN(t - 1) + CY \cdot YE(t) + CE(t). \quad (3.13)$$

$$CE(t) = CSD \cdot \epsilon_4(t). \quad (3.14)$$

The long-term median of the bond yield is $CW \cdot QMU + CMU$.

The yield on a bond is the single interest rate which, if coupons and capital are discounted at that rate, reproduces the current price. It does not actually give us the rate of return on the bond over a one-year period. To obtain it, we make the following assumptions:

- At the beginning of each year, we invest 40% of the assets in 15-year bonds, the maturity for which we assume $C(t)$ to give us the yield.
- The nominal semi-annual coupon rate is the multiple of 0.01% that makes the price of the bond closest to par.
- At the end of each year, we sell the bonds held using the price then dictated by the market for 14-year bonds. We use linear interpolation between yields on 15-year bonds and 1-year bonds¹⁹ to find the required yield.

Based on these assumptions, we get that the rate of return on bonds, denoted by $r_b(t)$, is the internal rate of return realized on the transactions described above for

¹⁹The short-term yield is the next random variable whose model we will present.

year t .

In order to find the rate of return on the assets, we still are missing one piece, the bill yield. Treasury bills are three-month investments. We will assume that, when bills owned mature, all the money received goes toward the purchase of new bills. This process is repeated every three months, until the end of the year, at which time reallocation takes place. We also will assume that the bill yield will be equivalent to both the annual rate of return on bills and the yield on a 1-year bond.

Letting $B(t)$ denote the yield on bills in year t , Wilkie provides us with the following model:

$$\ln B(t) = \ln C(t) + BMU + BN(t), \quad (3.15)$$

$$BN(t) = BA \cdot BN(t-1) + BC \cdot CE(t) + BE(t), \quad (3.16)$$

$$BE(t) = BSD \cdot \epsilon_5(t). \quad (3.17)$$

Hence, the current yield on bills is directly related to the current yield on bonds, and thus indirectly related to the other “upstream” random variables. The long-term median of the yield on bills is given by e^{BMU} times the long-term median of the yield on bonds.

Combining all the preceding definitions with our fixed investment policy, we find that the rate of return $r(t)$ on the total assets in year t is the following weighted sum:

$$r(t) = 0.5r_s(t) + 0.4r_b(t) + 0.1B(t). \quad (3.18)$$

While this completes our presentation of Wilkie’s model, we still are missing a

model for the rate of wage inflation. Let $W(t)$ represent the wage indexation in year t . Sharp [61] modeled it as follows:

$$W(t) = WMU + WW \cdot (e^{I(t)} - 1) + WN(t) \quad (3.19)$$

$$WN(t) = WA \cdot WN(t-1) + WE(t) \quad (3.20)$$

$$WE(t) = WSD \cdot \epsilon_6(t). \quad (3.21)$$

According to this model, wage inflation depends on price inflation. Sharp modeled the latter as well and obtained results comparable to Wilkie's.

All of the preceding equations provide us with the models for the variables we have chosen to depict as random. We will use them to generate economic scenarios for the future, while we will use the statistics contained in the *Report on Canadian Economic Statistics* [17] (and reproduced in Appendix C) as the actual scenario for the past. When we need values before 1924 (first year included in the report), we will assume they are equal to those assumed in the valuation basis.

3.4.3 Evolution of Assets and Liabilities

If we know all the economic outcomes, whether they be actual or simulated results, we can determine exactly how assets and liabilities evolve over time.

We will assume that, at the time we start our study of the plan, we already have a stationary plan membership. This means that either we then have a mature plan²⁰ or we have a more recent one which, at inception, granted all past service

²⁰Mature plans are likely to be rare in Canada. Though the first pension plan came into effect in 1870, it is only in 1940 that pension plans really started to gain in popularity (see [64]).

benefits²¹.

At the onset of the study, we determine all the liabilities, whether they be total or individual, using the formulae found in Section 2.5. As for the total assets, we will assume them to be a certain percentage of the liabilities. This is equivalent to assuming a given initial funding level. Starting with these initial liabilities and assets, we must find how to project them in time.

Liabilities

We will begin with the liabilities, which, for the most part, increase with the wages, albeit with a delay, except for the portion of the actuarial liability which covers the return of the accumulated contributions.

We will proceed in the same order that we did when we calculated the liabilities for any given year. This time, we will show the relationships between values for year $t + 1$ and those for year t .

For the retired, disabled and terminated lives, the following equations hold for any x and t :

$$AL_x^{(R),t+1}(j) = AL_x^{(R),t}(j)(1 + w^{t-x+\rho(j)}); \quad (3.22)$$

$$AL_x^{(D),t+1}(j) = AL_x^{(D),t}(j)(1 + w^{t-x+e(j)}); \quad (3.23)$$

$$AL_x^{(T),t+1}(j) = AL_x^{(T),t}(j)(1 + w^{t-x+\tau(j)}). \quad (3.24)$$

Technically, we do not have the same individual j on both sides of the equations.

²¹It probably is unrealistic to think that a pension plan would be so generous, but plans may offer members the possibility of purchasing past service.

Rather, we mean to compare two people who both retired, got disabled or withdrew at the same age, but who were born in two consecutive years. Since one left the active status one year later than the other, the salary on which the accrued benefits is based must account for the additional wage increase.

For the active lives, projecting the liabilities is not as straightforward. Instead, we need to consider the portion with a return of contributions separate from the portion with benefits depending directly on the salary. It turns out that the former is given by the portion due to mortality alone, as defined in Equation 2.18, while the latter is given by the sum of all the other components.

The equation that links AC_x^t to AC_x^{t+1} is not particularly nice, since both the rate of return and the increase in wages come into play. In fact, it is simpler to find AC_x^{t+1} from AC_{x-1}^t , as follows:

$$AC_x^{t+1} = (AC_{x-1}^t + NC_{x-1}^t)(1 + r(t)), x \geq 26, \quad (3.25)$$

and use this result in Equation 2.18, which is quite simple once we notice how v^{k+1} and $(1 + i)^{k+1}$ cancel one another. (The starting value for this recursion is $AC_{25}^t = 0, \forall t$.)

For all of the other components of the liabilities accrued to an active life, we find a relationship similar to that we found for the other liabilities:

$${}^T AL_x^{(A),t+1} = {}^T AL_x^{(A),t}(1 + w^t); \quad (3.26)$$

$${}^D AL_x^{(A),t+1} = {}^D AL_x^{(A),t}(1 + w^t); \quad (3.27)$$

$${}^R AL_x^{(A),t+1} = {}^R AL_x^{(A),t}(1 + w^t). \quad (3.28)$$

Finally, the summing equations 2.9, 2.10, 2.12, 2.13, 2.15, 2.16, 2.17, 2.22 and 2.23 are still valid for any year t and are used to generate the total liabilities for any group or in total.

Assets

We still have to take care of the assets, whose evolution depends on several components. We will denote by A^t the total assets at the beginning of year t , before any contributions or pension benefits are paid. We will denote by C^t the contributions (based on the normal cost only) to be made at the beginning of year t , by RB^t the retirement benefits to be paid at the same time, and by DB^t the death benefits to be paid at the end of that year²².

In any year t , the contributions to be paid are given by

$$C^t = \sum_{x=25}^{69} n_x^{(A),t} NC_x^t. \quad (3.29)$$

Since $n_x^{(A),t}$ does not depend on t (the membership is stationary) and NC_x^t represents an age-dependent but time-independent fraction of the salary, we have the following recursion linking contributions in consecutive years:

$$C^{t+1} = C^t(1 + w^{t+1}). \quad (3.30)$$

Retirement benefits to be paid at the beginning of any year are more complicated to determine since we must keep track of the benefits that were accrued to the

²²The term DB^t actually also includes the termination benefits paid to those who quit before satisfying the vesting requirements, that is, at the end of their first year of participation.

member when he left active service.

$$\begin{aligned}
 RB^t &= \sum_{x=55}^{\omega-1} \sum_{j=1}^{n_x^{(R),t}} B_{\rho(j)}^{t-x+\rho(j)} \\
 &+ \sum_{x=60}^{\omega-1} \sum_{j=1}^{n_x^{(D),t}} B_{\varrho(j)}^{t-x+\varrho(j)} \max\left(\frac{60-25}{\varrho(j)-25}, 1\right) \\
 &+ \sum_{x=65}^{\omega-1} \sum_{j=1}^{n_x^{(T),t}} B_{\tau(j)}^{t-x+\tau(j)} \tag{3.31}
 \end{aligned}$$

Again, since the membership profile is constant, we could eliminate the time superscript for the number of retired, disabled and terminated lives. While we cannot come up with a neat relationship between RB^{t+1} and RB^t , we can establish simple equations relating individual retirement benefits, similar to the ones we derived for individual liabilities. If we let $\theta(j)$ be the age the member turned right after leaving active service, whatever be the reason (disability, retirement or termination),

$$B_{\theta(j)}^{t+1-x+\theta(j)} = B_{\theta(j)}^{t-x+\theta(j)} (1 + w^{t-x+\theta(j)}), \tag{3.32}$$

where, once again, we actually do not have the same individual j on both sides of the equation, but two persons with the same length of service and in the same status but born a year apart²³.

Unlike contributions and retirement benefits, death benefits are paid at the end of the year. While some are in the form of a return of accumulated contributions (member's and sponsor's, or member's only prior to vesting), others are in the form

²³Striving to find simple relationships between cash flows from one year to the next may appear to be a futile endeavor to the reader, but savings in computational time are precious when the program must repeat the same operations for every simulation.

of a commuted value. As we saw earlier in the case of liabilities, it is simpler to treat the two types separately.

Equation 3.25 provided us with a recursion to calculate the total accumulated contributions AC_x^t . We also need to find a similar recursion for the member's contributions only. We recall from Chapter 2 that they are denoted by EAC_x^t for someone aged x at the beginning of year t and that the member contributes a fixed portion α of his salary in every year.

$$EAC_x^{t+1} = \begin{cases} 0 & , x = 25 \\ (EAC_{x-1}^t + \alpha Sal_{x-1}^t)(1 + r(t)) & , x \geq 26 \end{cases} \quad (3.33)$$

We are now able to find the benefits payable to those who die while active or quit before benefits are vested. We will denote them by ${}^c DB^t$ to indicate that these benefits are related to contributions.

$$\begin{aligned} {}^c DB^t &= n_{25}^{(A),t} (q_{25}^{(m)} + q_{25}^{(t)}) EAC_{26}^{t+1} \\ &+ \sum_{x=26}^{69} n_x^{(A),t} q_x^{(m)} AC_{x+1}^{t+1} \end{aligned} \quad (3.34)$$

One may naturally wonder why we are using the accumulated contributions of a member apparently one year older and one year later than the one for whom we want to pay death benefits. Actually, we must recall that EAC_{x+1}^{t+1} and AC_{x+1}^{t+1} are accumulated contributions at the very beginning of the year, *before* payment of that very year's contributions. Hence, they also are the accumulated contributions for someone aged x at the *end* of year t .

All of the other death benefits are paid to disabled or terminated lives that die

in the deferral period. The amount paid to the beneficiary is the actuarial present value of the annuity that would have been payable to the member based on his currently accrued benefit. We will denote the sum of all these amounts payable at the end of year t by ${}^B DB^t$ to show their dependence on the accrued benefit.

$$\begin{aligned} {}^B DB^t &= \sum_{x=27}^{64} \sum_{j=1}^{n_x^{(T),t}} q_x^{(T)} B_{\tau(j)}^{t-x+\tau(j)} v^{(65-x-1)} {}_{65-x-1} p_{x+1}^{(T)} \ddot{a}_{65}^{(T)} \\ &+ \sum_{x=26}^{59} \sum_{j=1}^{n_x^{(D),t}} q_x^{(D)} B_{\varrho(j)}^{t-x+\varrho(j)} \left(\frac{x+1-25}{\varrho(j)-25} \right) v^{(60-x-1)} {}_{60-x-1} p_{x+1}^{(D)} \ddot{a}_{60}^{(D)} \end{aligned} \quad (3.35)$$

Again, we have $x+1$ where we might otherwise have been tempted to put x , but someone who dies at age x at the end of year t would have been $x+1$ the very next day.

Just as was the case for liabilities for non-active lives, we need to look at individual death benefits so as to be able to derive a recursive relationship from one year to the next. In fact, we can group all those lives who have been inactive for the same number of years, regardless of their attained age. We then obtain the death benefits to be paid to members of a particular group in year $t+1$ by multiplying the benefits that were paid to members of the equivalent group in year t by $(1+w^{t+1-k})$, where k is the number of years elapsed since becoming inactive up to the end of the year under consideration. By “equivalent group,” we mean all the members who had been inactive for the same number k of years at the end of year t .

Taking the sum of the two components we have derived, we find the total death

benefits payable at the end of year t :

$$DB^t = {}^C DB^t + {}^B DB^t. \quad (3.36)$$

The picture may appear to be complete, but we are still missing any additional contributions the sponsor may be called on to make if there is an unfunded liability in order to be able to link assets from one year to the next.

We will denote by ADJ^t the additional contribution paid at the beginning of year t , along with the normal contributions. We will calculate it at every valuation for the next three years. If we do a valuation at the beginning of year t ,

$$ADJ^t = \frac{\max(AL^t - A^t, 0)}{\ddot{a}_{15|i}}. \quad (3.37)$$

Otherwise, $ADJ^t = ADJ^{t-1}$. Hence, ADJ^t represents a certain fraction of the unfunded liability as of the last valuation. In fact, it would be the payment required to bring the funding level back to 100% at the end of fifteen years if all the valuation assumptions were borne out by reality.

Now that we have defined all the cash flows that take place in any given year, we are able to link A^t and A^{t+1} :

$$A^{t+1} = (A^t + C^t - RB^t + ADJ^t)(1 + r(t)) - DB^t. \quad (3.38)$$

This completes our exposition of the way we will project assets and liabilities, using economic scenarios based on the Wilkie model and treating all the other factors entering the valuation as deterministic.

3.5 Valuation Basis versus Simulation Results

When projecting assets and liabilities, not only must we generate economic scenarios, we must also pick a valuation basis (i and w). We will see how these elements interact in the calculation of the disposable surplus.

Beforehand, however, we will re-express our suggested criterion in algebraic terms. We will also see what would happen if all valuation assumptions were borne out by reality.

3.5.1 Algebraic Expression of Suggested Criterion

Using the expressions we defined above, we can give a better idea of the computations involved in the application of our suggested criterion for the amount to distribute.

Assume we want to calculate the disposable surplus at the beginning of year t . We first generate a given number of economic scenarios for the next three years; each scenario will give us the rate of return and the wage increase for the years t , $t + 1$ and $t + 2$. Using the equations given earlier in this chapter, we then project AL^t and A^t to obtain AL^{t+3} and A^{t+3} for each scenario. Dividing $A^{t+3} - AL^{t+3}$ by $\prod_{u=t}^{t+2}(1 + r(u))$, we obtain the divisible surplus for a particular scenario. After ordering the surpluses resulting from each scenario, we pick the one corresponding to the desired probability of a deficit in three years' time.

If that quantile turns out to be negative, there is nothing to be given away²⁴.

²⁴In fact, there is not much point in going through the whole exercise if $A^t - AL^t$ itself is negative. This is not to say that we would necessarily end up with nothing to give away according to the criterion – as we shall see, the relationship between the valuation basis and the simulation

Otherwise, if the quantile is positive, we do have some amount to give away and we will have to determine whom to give it to.

Later in this chapter, we will give an adaptation of our suggested criterion to subgroups of the pension plan. This will pave the way to Chapter 4, in which we will introduce cooperative game theory as the tool we will use to allocate the disposable surplus among the subgroups.

Before doing so though, we still have to look into the question of what valuation basis to use.

3.5.2 Simulation Results Identical to Valuation Basis

Of course, what we assume will happen according to our valuation basis is unlikely to materialize in the next year, let alone in all of the next three years. Nevertheless, it is relevant to ask what would happen if all simulation results conformed to the valuation basis. This will provide us with initial insights into the correspondence that should prevail between the valuation basis and the simulation results.

If all scenarios reproduce our assumptions, we will then be paying (normal) contributions that, together with interest, cover all the benefits to be paid as well as the increase in liabilities. This means that, if we started out with assets equal to liabilities, we still would have 100% funding at the next valuation.

If, instead, we started with an unfunded liability, the sponsor would have had to make additional contributions for the next three years. This would lead to an unfunded liability, after three years, equal to the present value of twelve more pay-

results is critical – but how could one justify the distribution of assets to the members when the sponsor is being called on to make an additional contribution over the next three years?

ments in the same amount²⁵. Alternatively, if we started with a surplus, since the contributions alone would suffice to pay benefits and fund the increase in liabilities, the surplus would grow with the rate of return. Discounting it would produce, as disposable surplus, exactly the value $A^t - AL^t$.

Hence, if we knew exactly what would happen in the future and if we calculated contributions and liabilities accordingly, the disposable surplus would simply be equal to the difference between the value of the assets and that of the liabilities at the time of valuation.

Based on these observations, we can then ask what the disposable surplus would be if the future would still be known with certainty but the valuation basis was either more conservative or lax than called for.

With conservative assumptions, contributions would be more than sufficient to cover benefits and increase in liabilities. As a result, even if we had no surplus on paper at the time of valuation (*i.e.*, $A^t = AL^t$), we still would come up with an amount to distribute according to our criterion. As a matter of fact, we even could have a small deficit as of time t and determine that a certain amount may be given away without affecting the security of the pension plan.

With slack assumptions, we naturally expect to reach the opposite conclusion. Indeed, in that case, contributions will not suffice to pay for the benefits and fund the increase in liabilities. Unless the existing surplus is large enough to cover the

²⁵We note that, according to the definition we gave for the additional contribution, the payments to be made in the fourth to sixth years would not be the same as those that were made in the first three years. Rather, they would be equal to the reduced unfunded liability divided by a 15-year annuity, not by a 12-year annuity. Hence, using our definition, the unfunded liability would never completely disappear.

shortfall, we would actually need additional funds to be injected in order to ensure the pension plan's survival. Hence, we could not part with any money unless the existing surplus was more than sufficient to compensate for the inadequacy of the contributions.

In summary, if the future were known with certainty, we could give away any amount not needed so that assets actually held, together with future contributions and interest, be sufficient to pay for all the promised benefits. If assumptions and the future are one and the same, $\max(A^t - AL^t, 0)$ will be that amount; if not, and assumptions are more conservative, the amount will be more; otherwise, the amount will be less.

3.5.3 Simulation Results Differing from Valuation Basis

As we indicated earlier, we do not think we can predict the future with certainty. In fact, we do not even pretend to be able to model the future with a high degree of accuracy. So, instead of comparing our valuation basis with a supposedly known future, we will only be able to compare it with an expected or average future.

We expect the same conclusions to be reached when comparing the average disposable surplus to the one given by the current snapshot ($A^t - AL^t$). We also expect the actual disposable surplus, as determined by the appropriate quantile, to increase with the component of the assumption which reflects the conservative padding.

To verify that this is indeed the case, we will check the results we obtain, based

on one set of 1000 simulations²⁶, when we change the valuation basis, through either i or w . For the particular set of economic scenarios that we used, based on the Wilkie model with neutral starting values²⁷, the average annual rate of return over the three-year period was 8.002%. Over the same period, the average annual increase in wages was 4.853%.

We will use an 8%/5% basis to generate the results against which we will compare the others. The numbers we will give are based on 100 entrants at age 25 in every year. Pre-1986 economic results come from the *Report on Canadian Economic Statistics* [17], while post-1985 results are simulated.

Using the 8%/5% basis, we find that the liabilities at the beginning of 1986 amount to \$50,390,312. We will assume initial assets of \$10,000,000 over the value of the liabilities, regardless of the valuation basis. In the following table, we find the average disposable surplus (top number) as well as the 5th percentile (bottom number) for each of the valuation bases used.

Before trying to interpret the results in the table, we first have to agree on which bases are more conservative than the others. For the valuation interest rate i , we are more prudent when we assume a smaller value since we effectively are counting less on interest revenue to generate the necessary cash flows. As for the rate of wage increase w , assuming a larger value is more conservative since we then are funding for relatively larger benefits²⁸. So, in the table, we get more conservative

²⁶All the results to be given in this section are based on the *same* set of 1000 simulations.

²⁷For the Wilkie model, we consider as neutral starting values those values which are the long-term values obtained when all the noise terms are nil.

²⁸This does not mean that the benefits to be paid actually will be larger. It only means that we fund for them as though we anticipated them to be larger. Again, what actually turns out

Table 3.1: Impact of the Choice of Valuation Basis on the Average and Actual Disposable Surpluses; Initial Valuation Surplus Equal to \$10,000,000

	$w = 4.5\%$	$w = 5.0\%$	$w = 5.5\%$
$i = 7.5\%$	9,603,898	9,955,505	10,339,413
	-4,118,556	-4,087,269	-4,047,280
$i = 8.0\%$	8,979,293	9,282,670	9,613,439
	-3,751,291	-3,731,071	-3,703,951
$i = 8.5\%$	8,448,539	8,711,157	8,997,068
	-3,395,829	-3,384,216	-3,367,150

valuation bases as we move up or to the right.

As we anticipated, the more conservative we are in our choice of valuation assumptions, the larger the average disposable surplus. The results for the percentiles puzzle us though. Based on the numbers given in the table, it appears that a more conservative valuation basis with respect to the wage increase leads to an increase in the percentile whereas a more conservative valuation interest rate leads to a decrease instead.

This impression is only partly true. What really is the case is the following. While a more conservative basis, with respect to either valuation assumption, leads to a larger disposable surplus for the majority of the scenarios, *it does not do so for all*. Rather, for the adverse scenarios that lead to the smallest surpluses, the opposite takes place: conservatism is penalized.

This is a very interesting phenomenon, for which we can offer the following insights. As we will see, it is very much dependent on the nature of the criterion

to be the case does not necessarily have something to do with what we anticipated, although we would hope that there be some relationship between the two.

we have suggested.

We can, without loss of generality, assume that we start off with no assets at all since any amount added to the assets will also automatically be added to the disposable surplus. For instance, in the table of surpluses given earlier, if we had started out with \$20,000,000 more in assets than in liabilities, all the results would have been \$10,000,000 higher.

Let us now consider Equation 3.38. Since the assets are nil, we simply have to consider the balance of contributions and benefits. Contributions are sensitive to both w and i . Retirement benefits do not depend on w but do change when i changes through the actuarial equivalents used to determine early and postponed retirement benefits. As for the death benefits, we must consider returns of contributions and payments of commuted values separately. Since contributions are returned with interest, this portion of the death benefits will grow insofar as the contributions themselves grow. As for the commuted values, they are insensitive to changes in w but vary with i since present values need to be calculated.

Based on the simulations we carried out, the more conservative the basis, the smaller the present value of the net monetary shortfalls²⁹, as we expected. This statement only relates to actual money coming in and out of the fund; it does not say anything about the liabilities. Thus, if we determined the disposable surplus based on all future cash flows, our intuition would not be countered and we would find that more conservative bases do indeed lead to larger disposable surpluses.

However, our criterion only takes us a few years into the future, at which point

²⁹We necessarily have shortfalls when we assume no initial assets since the investment income on the assets typically accounts for a large portion of a pension plan's monetary inflows.

we must compare assets and liabilities to find out what amount, if any, can be distributed. Without question, the value of the liabilities depends on both w and i , as well as on the past rates of return and wage increases. What we must find out is how the assets required to fund the increase in liabilities over the simulation horizon vary with a change in the valuation basis.

At time 0, the liabilities are larger with the more conservative basis. Hence, if we wish to compare equal initial surpluses, we cannot assume assets to be nil. Rather, we will assume assets to be equal to liabilities at that time, and we will analyze how assets and liabilities, ignoring all cash flows but investment income on the assets, evolve over the selected horizon.

Assets will grow with the rate of return and liabilities will, by and large, grow with the increase in wages. Typically, $r(u)$ will be larger than w^u in a given year u , or will at least be so on average over the simulated future. Whenever that is the case, comparing only assets to liabilities and ignoring all intervening cash flows, we end up with a larger surplus with the more conservative basis. Indeed, at time 3, the ratio of assets to liabilities will be about the same under both bases³⁰. Hence, the surplus will be larger for the basis yielding the larger liabilities, namely the more conservative one. Since that same basis led to the smaller present value of net monetary shortfalls, combining the two elements of our analysis leads to a larger disposable surplus with the more conservative basis, in the case considered.

There are cases, however, when the increase in assets will be less than the increase in liabilities. We then end up with the reverse situation: the deficit (un-

³⁰That ratio, the funding level, actually would be the same if liabilities increased only with respect to the most recent wage inflation.

funded liability) will be larger with the more conservative basis. In those cases, the disposable surplus may be larger for the more conservative basis or for the other one, depending on how monetary shortfalls and increases in liabilities add up in their effects.

In practice, if we are to err on one side, we prefer that it be on the side of caution. Hence, we would favor a conservative basis to a lax basis. For our criterion, however, choosing a conservative basis may lead to an apparent contradiction: while we may have, on paper, a larger surplus when using the conservative basis³¹, we could prescribe a smaller surplus disposal based on our criterion.

This leads us to recommend a realistic valuation basis, not unlike some actuaries working in the pensions area (see Thornton and Wilson [69]). This is not to say that we do not want to be prudent. Actually, we can be more prudent simply by targeting a smaller probability of an unfunded liability or by lengthening the forecasting period³².

³¹We are not saying here that the assets are the same. Indeed, those for the valuation with the more conservative basis would have to be larger if a larger surplus is to exist. Presumably, these assets would be larger because of the higher contribution rates warranted by a more conservative basis.

³²Targeting a given probability of an unfunded liability over a longer period is more conservative than targeting the same probability over a shorter period. Indeed, repeated application of the criterion over small periods leads to an effectively higher probability of deficit than applying the criterion once over the total period since $1 - (1 - p)^n > p$, $n \geq 2$, where p is the targeted probability of a deficit at the end of the horizon.

3.6 Adapting the Criterion for Subgroups

While the criterion we have introduced answers the question as to what to give away, it does not tell us anything as to whom we should give it to. As we will see in the next chapter, using cooperative game theory to determine how to allocate the divisible surplus among given entities requires us to find the surplus that such entities would have generated on their own, or in any combination thereof.

We will start this section by defining the five subgroups we are considering. We will then address the issue of what assets to earmark for them at the beginning of the simulations. Finally, we will describe how assets and liabilities evolve for any given subgroup and, in doing so, will show how to adapt our suggested criterion to find the corresponding disposable surplus.

3.6.1 Choice and Definition of Five Subgroups

Typically, in game theory, we would treat as separate entities all the individuals or independent units involved in the problem. For most problems, either the number of individuals is fairly small or the structure of the problem is such that determining the surplus generated by each combination of individuals is relatively easy.

Certainly, that is not the case here. By definition, large pension plans cannot have only a small number of members. Moreover, as we shall see, determining the disposable surplus for a subgroup will be almost as computationally intensive as doing so for the whole group. Besides, if we were to work with individuals, our assumption of fixed decrements no longer would be tenable.

So as to keep the number of computations manageable and to avoid having to

complicate our analysis by introducing random decrements, we will work with five subgroups, four of them being the four categories of members we have worked with so far, and the fifth one being the sponsor. We will define each subgroup in turn so that we have a clear picture of who belongs to each subgroup at time t (beginning of simulations) and in any subsequent year.

Active Lives

At time t , the subgroup of active lives is simply composed of all the members who are active as of that time. Their number is given by $\sum_{x=25}^{69} n_x^{(A),t}$.

We assume that this subgroup is closed; that is, there will be no entries (no new members) in that subgroup and the only exits will be through death. Hence, in any later year, this subgroup will include all four categories of members as some of the active lives become disabled, terminate or retire.

The number of members found in this subgroup in any future year $t+n$, $n \geq 1$, is given by:

$$\begin{aligned}
 \sum_{x=25+n}^{69} n_x^{(A),t+n} &+ \sum_{x=\max(25+n,26)}^{\omega-1} n_x^{(D),t+n} \sum_{j=1}^{n_x^{(D),t+n}} I[x - \varrho(j) \leq n-1] \\
 &+ \sum_{x=\max(25+n,27)}^{\omega-1} n_x^{(T),t+n} \sum_{j=1}^{n_x^{(T),t+n}} I[x - \tau(j) \leq n-1] \\
 &+ \sum_{x=\max(25+n,55)}^{\omega-1} n_x^{(R),t+n} \sum_{j=1}^{n_x^{(R),t+n}} I[x - \rho(j) \leq n-1], \quad (3.39)
 \end{aligned}$$

where $I[C]$ is an indicator function taking on the value 1 if condition C is met and 0 otherwise. Hence, at any point in time, the members who belong to this subgroup

are those who, regardless of their current status, were active at time t . In other words, the name given to this subgroup refers to the status of its members not at the time that we count them but at time t . This is a very important distinction as it emphasizes the fact that we are keeping track of a certain subgroup, not a certain type, of members.

Disabled Lives

Naturally, at time t , the subgroup of disabled lives is made up of all those members that are disabled, their total number being $\sum_{x=26}^{\omega-1} n_x^{(D),t}$.

We also assume this subgroup to be closed. Since the only decrement facing disabled lives is death, this subgroup will, at all times, include disabled lives only. This is not to say, though, that it will include *all* disabled lives.

The members in this subgroup in year $t + n$ are those who have been disabled for at least n years (*i.e.*, those who were disabled at time t) and their number is:

$$\sum_{x=26+n}^{\omega-1} n_x^{(D),t+n} \sum_{j=1}^n I[x - \varrho(j) \geq n]. \quad (3.40)$$

Retired Lives

(This subgroup and the following one are very similar to the subgroup of disabled lives in general terms.)

All (and only) the members that are retired at that time are considered to be part of the subgroup of retired lives at time t . The initial number of members in the subgroup is $\sum_{x=55}^{\omega-1} n_x^{(R),t}$.

For this subgroup as well, the only decrement is death. As a result, as it evolves through time, it will be made up only of retired lives, although not of all retired lives.

At time $t + n$, the composition of this subgroup is given by:

$$\sum_{x=55+n}^{\omega-1} n_x^{(R),t+n} \sum_{j=1} I[x - \rho(j) \geq n]. \quad (3.41)$$

Terminated Lives

When we start our simulations, the number of members that belong to the subgroup of terminated lives is $\sum_{x=27}^{\omega-1} n_x^{(T),t}$, that is, the sum of those who have quit with vested benefits and who are, at that time, either in the deferral period or receiving retirement benefits.

For that subgroup too, the only decrement is death and only (though not all) terminated lives will be found in it in later years. In year $t + n$, the members left in this subgroup add up to this number:

$$\sum_{x=27+n}^{\omega-1} n_x^{(T),t+n} \sum_{j=1} I[x - \tau(j) \geq n]. \quad (3.42)$$

Sponsor

At a first glance, one may think that, at any time, we have all pension plan members in one of the four subgroups. That is not quite the case: we still are missing those who join the plan after time t . Unlike the other subgroups, this one will be very small initially. We did not think however that we should ignore it since we aim to

apply our criterion on an ongoing basis.

One could argue that we could have included them in the subgroup of active lives. However, our idea for the four subgroups described so far is to find out the surplus they could dispose of among themselves if they were to form their own “mini-plan” with the assets and liabilities allocated to them.

So, we did not think it was natural to add these new entrants to any of the four subgroups. Rather, we thought it made more sense to make the sponsor responsible for them. That explains why we chose to make this last subgroup that of the sponsor.

This subgroup, unlike the other ones, starts out very small and increases in size. If the simulation period were long enough, it eventually would include all the pension plan members.

For this particular subgroup, the assumption of fixed decrements may appear more questionable than ever. Yet, with 100 new entrants every year, we quickly have an important number of members in this subgroup.

This particular subgroup will have no members at time t and $n_{25}^{(A),t+1}$ members at time $t + 1$. In year $t + n$, $n \geq 2$, the number of members in this subgroup will be given by

$$\sum_{x=25}^{25+n-1} n_x^{(A),t+n} + \sum_{x=26}^{25+n-1} n_x^{(D),t+n} + \sum_{x=27}^{25+n-1} n_x^{(T),t+n} + \sum_{x=55}^{25+n-1} n_x^{(R),t+n}, \quad (3.43)$$

where we understand that any sum whose upper bound is less than its lower bound is to be removed from the calculation. We could also have found the number of members in the group by subtracting the numbers in the four other subgroups from

the total number of plan members.

We observe that we need only look at the age of the member to determine whether or not he belongs to this subgroup. Unlike for the other subgroups, we need not worry about how long the member has been in his current status to make this assessment.

This completes our definition of the five subgroups we will work with. Theoretically, we could have used other divisions, finer or coarser, of the plan membership. While we cannot make it much coarser, we could refine it, for instance, by working with age groups instead. Whatever segmentation we use, we must be careful when we trace its evolution over the simulation period.

3.6.2 Allocating Initial Assets to Each of the Subgroups

As we mentioned earlier, for each subgroup, we wish to determine the surplus that it could dispose of based on the assets and liabilities that are allocated to it.

For the liabilities, the allocation is straightforward and we simply have to add the individual liabilities of each of the members of a subgroup to derive the total liabilities for that subgroup. Hence, at time t , we allocate $AL^{(A),t}$ to the active lives, $AL^{(D),t}$ to the disabled lives, $AL^{(R),t}$ to the retired lives, and $AL^{(T),t}$ to the terminated lives. Since the sum of these four components yields all of the pension plan liabilities at time t , this leaves none for the subgroup we identified as that of the sponsor.

For the assets, however, no single allocation comes out as being the only plausible approach. Rather, we come up with three possibilities, which we do not claim

to be exhaustive. In fact, we will revisit the issue in Chapter 5, when we consider the possibility of applying game theory not only to the distribution of the surplus, but also to the allocation of the assets.

The first possibility is to make the initial assets of a subgroup equal to its liabilities. This would almost appear to be a bare minimum. We would expect such an allocation to yield no disposable surplus according to our criterion unless an exaggeratedly conservative basis were used.

The second possibility is to allocate the assets in proportion to the liabilities. Hence, each subgroup would start out with a funding level equal to that of the whole group. This is probably the most natural approach and would be in line with Quebec's pension law [35] which recommends, on a pension plan wind-up, that the surplus be allocated in proportion to the value of the accrued benefits³³.

The third possibility is "giving" to each group the assets it would have accumulated if:

- all of its members' past contributions had been deposited into a separate fund;
- this fund had earned the same rate of return as the pension fund (or, equivalently, this fund had the same investment policy as the pension fund); and
- all of the retirement benefits payable to its members in the past were paid out of the fund.

³³For the active and disabled lives, the present value of the accrued benefits is less than the actuarial liability since the latter projects the benefits to account for future increases in salary or credited service, respectively. We still elect to work with the actuarial liability rather than with the present value of the accrued benefits since we are considering an ongoing plan, not a terminating one.

This is, in essence, assuming that the subgroup has been on its own all along. In order to generate this allocation, we will determine individual accumulated contributions and benefits, and then combine them as appropriate to constitute each subgroup's assets.

At an individual level, we make the implicit assumption that the plan existed at the time the member joined the company. Hence, the total accumulated contributions, AC_x^t , for an active member aged x at the beginning of year t (before the payment of this year's contributions) amount to

$$AC_x^t = \sum_{k=25}^{x-1} NC_k^{t-x+k} \prod_{u=t-x+k}^{t-1} (1+r(u)), \quad 26 \leq x \leq 70. \quad (3.44)$$

We can use Equation 3.25 to calculate these values recursively.

We also have accumulated contributions for the inactive lives. They are then a function of the attained age and of the age at which the inactive status began. If we let θ be the age the member (aged x at time t) turned right after ceasing to be an active life³⁴, then

$$IAC_x^t(\theta) = \begin{cases} AC_\theta^t & , \quad x = \theta, \\ AC_\theta^{t-x+\theta} \cdot \prod_{u=t-x+\theta}^{t-1} (1+r(u)) & , \quad \theta < x \leq \omega - 1, \end{cases} \quad (3.45)$$

represents his accumulated contributions.

As for the accumulated benefits, they are nil for active lives. For the inactive lives, we need to identify the decrement which took the member out of the ac-

³⁴We used different symbols in Chapter 2 according to whether the member ceased being active because of termination ($\tau(j)$), retirement ($\rho(j)$) or disability ($\varrho(j)$). The distinction is not necessary here because everybody of the same age pays the same contributions while active.

tive status. Upon termination or retirement, we know exactly the annual pension amount which will become payable upon actual retirement and which shall be used to calculate the death benefit if death precedes retirement. We will denote this pension amount, as we did in Chapter 2, by $B_{\tau(j)}^{t-x+\tau(j)}$ or $B_{\rho(j)}^{t-x+\rho(j)}$, respectively. However, in the case of disability, we have used $B_{e(j)}^{t-x+e(j)}$ to denote the accrued benefit as of the disabling event, but the actual pension amount on which death or retirement benefits are based changes every year until retirement.

We will denote by $ATB_x^t(j)$ the termination benefits individual j , now aged x , $j = 1, \dots, n_x^{(T),t}$, has received, accumulated with interest.

$$ATB_x^t(j) = B_{\tau(j)}^{t-x+\tau(j)} \sum_{k=65}^{x-1} \prod_{u=t-x+k}^{t-1} (1+r(u)), \quad 65 < x \leq \omega - 1 \quad (3.46)$$

A similar equation holds for $ARB_x^t(j)$, the accumulated retirement benefits individual j , now aged x , $j = 1, \dots, n_x^{(R),t}$, has received as of the very beginning of year t :

$$ARB_x^t(j) = B_{\rho(j)}^{t-x+\rho(j)} \sum_{k=\rho(j)}^{x-1} \prod_{u=t-x+k}^{t-1} (1+r(u)), \quad 55 < x \leq \omega - 1. \quad (3.47)$$

As for the accumulated disability benefits paid to individual j , now aged x , $j = 1, \dots, n_x^{(D),t}$, up to but not including the payment in year t , they are given by:

$$ADB_x^t(j) = \begin{cases} B_{e(j)}^{t-x+e(j)} \left(\frac{60-25}{e(j)-25} \right) \sum_{k=60}^{x-1} \prod_{u=t-x+k}^{t-1} (1+r(u)) & , \quad e(j) < 60, \\ B_{e(j)}^{t-x+e(j)} \sum_{k=e(j)}^{x-1} \prod_{u=t-x+k}^{t-1} (1+r(u)) & , \quad e(j) \geq 60, \end{cases} \quad (3.48)$$

where $60 < x \leq \omega - 1$.

For the ages for which we did not define accumulated benefits, be they termination or disability benefits, they are nil since these ages fall in the deferral period.

We now simply have to combine the individual amounts to generate the allocation of the funds. They are as follows, with the same status superscripts as before:

$$A^{(A),t} = \sum_{x=25}^{69} n_x^{(A),t} AC_x^t; \quad (3.49)$$

$$A^{(D),t} = \sum_{x=26}^{\omega-1} \sum_{j=1}^{n_x^{(D),t}} \left(IAC_x^t(\varrho(j)) - ADB_x^t(j) \right); \quad (3.50)$$

$$A^{(R),t} = \sum_{x=55}^{\omega-1} \sum_{j=1}^{n_x^{(R),t}} \left(IAC_x^t(\rho(j)) - ARB_x^t(j) \right); \quad (3.51)$$

$$A^{(T),t} = \sum_{x=27}^{\omega-1} \sum_{j=1}^{n_x^{(T),t}} \left(IAC_x^t(\tau(j)) - ATB_x^t(j) \right). \quad (3.52)$$

It may seem that we have forgotten to allocate assets to the sponsor's subgroup. Insofar as the initial assets are proportional to the initial liabilities, no assets are allocated to the sponsor initially. However, if we use our first or last approach, the following assets would become the sponsor's, which we will identify with the superscript (S):

$$A^{(S),t} = A^t - \left[A^{(A),t} + A^{(D),t} + A^{(R),t} + A^{(T),t} \right]. \quad (3.53)$$

As for combinations of subgroups, except for that combination which is the whole pension plan membership, we find the initial assets and liabilities by adding those of the subgroups that are in the combination.

3.6.3 Evolution of Assets and Liabilities

We will apply the same criterion to subgroups (and combinations thereof) as we applied to the whole group. Therefore, we will have to project a subgroup's assets and liabilities a given number of years into the future and discount back the surplus available at that time in order to determine its value at the time of the valuation. Repeating the process for each simulated scenario and ordering the subgroup's results will allow us to identify the desired percentile as the disposable surplus for that subgroup.

For the liabilities, we have already seen how to project them from one year to the next individually in the subsection starting on page 77. Hence, if we want to know the liabilities of any subgroup at any point in time, we just have to add up the corresponding individual ones. Also, if we want to know the liabilities of a combination of subgroups, we need simply add the liabilities of the subgroups included in the combination.

Here are the liabilities at the beginning of year $t+n$, $n \geq 1$, for each subgroup. In the left-hand side of the equations, the status superscript indicates the *initial* status of the members of the subgroup at time t . In the right-hand side, it corresponds to their *actual* status at time $t+n$.

$$AL^{(A),t+n} = \begin{cases} \sum_{x=25+n}^{69} AL_x^{(A),t+n} \\ + \sum_{x=\max(25+n,26)}^{\omega-1} \sum_{j=1}^{n_x^{(D),t+n}} AL_x^{(D),t+n}(j) I[x - \varrho(j) \leq n - 1] \\ + \sum_{x=\max(25+n,27)}^{\omega-1} \sum_{j=1}^{n_x^{(T),t+n}} AL_x^{(T),t+n}(j) I[x - \tau(j) \leq n - 1] \\ + \sum_{x=\max(25+n,55)}^{\omega-1} \sum_{j=1}^{n_x^{(R),t+n}} AL_x^{(R),t+n}(j) I[x - \rho(j) \leq n - 1] \end{cases} \quad (3.54)$$

$$AL^{(D),t+n} = \sum_{x=26+n}^{\omega-1} n_x^{(D),t+n} \sum_{j=1}^{n_x^{(D),t+n}} AL_x^{(D),t+n}(j) I[x - \varrho(j) \geq n] \quad (3.55)$$

$$AL^{(R),t+n} = \sum_{x=55+n}^{\omega-1} n_x^{(R),t+n} \sum_{j=1}^{n_x^{(R),t+n}} AL_x^{(R),t+n}(j) I[x - \rho(j) \geq n] \quad (3.56)$$

$$AL^{(S),t+n} = \begin{cases} 0 & , \quad n = 1 \\ \sum_{x=26}^{25+n-1} AL_x^{(A),t+n} + \sum_{x=26}^{25+n-1} AL_x^{(D),t+n} & , \quad n \geq 2 \\ + \sum_{x=27}^{25+n-1} AL_x^{(T),t+n} + \sum_{x=55}^{25+n-1} AL_x^{(R),t+n} & \end{cases} \quad (3.57)$$

$$AL^{(T),t+n} = \sum_{x=27+n}^{\omega-1} n_x^{(T),t+n} \sum_{j=1}^{n_x^{(T),t+n}} AL_x^{(T),t+n}(j) I[x - \tau(j) \geq n] \quad (3.58)$$

As was the case for the whole group, projecting assets is more complicated than projecting liabilities. Equation 3.38 still describes the evolution of the assets over one year, with the addition of a superscript on each of the terms to indicate which subgroup we are working with. This equation is applicable regardless of the actual asset allocation among the subgroups.

Depending on the particular subgroup, some of the terms may be nil. Also, when we project the assets of a combination of subgroups, only the additional contribution ADJ^t requires special consideration. For all the other cash flows (contributions and benefits), we simply have to add those of the subgroups in the combination.

We will now, in turn, show how to calculate each of the cash flows for each of the subgroups.

No contributions are paid by the disabled, retired, and terminated lives; so, this

term is nil for all of these subgroups. For the other two subgroups, in year $t + n$, the contributions paid are:

$$C^{(A),t+n} = \sum_{x=25+n}^{69} n_x^{(A),t+n} N C_x^{t+n}, \quad n \geq 0; \quad (3.59)$$

$$C^{(S),t+n} = \begin{cases} 0 & , \quad n = 0, \\ \sum_{x=25}^{25+n-1} n_x^{(A),t+n} N C_x^{t+n} & , \quad n \geq 1. \end{cases} \quad (3.60)$$

Retirement benefits are eventually nonzero for all subgroups. However, in the case of the sponsor's subgroup, it takes 31 years before any of its members receive such benefits³⁵. Since we do not intend to consider simulation periods that long, we will not define any retirement benefits for the sponsor's subgroup.

For the other subgroups, retirement benefits to be paid in year $t + n$ are given by:

$$RB^{(A),t+n} = \begin{cases} 0 & , \quad n = 0, \\ \sum_{x=\max(25+n,60)}^{\omega-1} \sum_{j=1}^{n_x^{(D),t+n}} B_{\varrho(j)}^{t+n-x+\varrho(j)} \\ \quad \max\left(\frac{60-25}{\varrho(j)-25}, 1\right) I[x - \varrho(j) \leq n - 1] \\ + \sum_{x=\max(25+n,65)}^{\omega-1} \sum_{j=1}^{n_x^{(T),t+n}} B_{\tau(j)}^{t+n-x+\tau(j)} \\ \quad I[x - \tau(j) \leq n - 1] \\ + \sum_{x=\max(25+n,55)}^{\omega-1} \sum_{j=1}^{n_x^{(R),t+n}} B_{\rho(j)}^{t+n-x+\rho(j)} \\ \quad I[x - \rho(j) \leq n - 1] & , \quad n \geq 1; \end{cases} \quad (3.61)$$

³⁵The first members aged 25 enter this subgroup in year $t + 1$ and they cannot receive retirement benefits until age 55, 30 years later.

$$RB^{(D),t+n} = \sum_{x=\max(26+n,60)}^{\omega-1} n_x^{(D),t+n} \sum_{j=1}^{n_x^{(D),t+n}} B_{\varrho(j)}^{t+n-x+\varrho(j)} \max\left(\frac{60-25}{\varrho(j)-25}, 1\right) I[x - \varrho(j) \geq n]; \quad (3.62)$$

$$RB^{(R),t+n} = \sum_{x=55+n}^{\omega-1} n_x^{(R),t+n} \sum_{j=1}^{n_x^{(R),t+n}} B_{\rho(j)}^{t+n-x+\rho(j)} I[x - \rho(j) \geq n]; \quad (3.63)$$

$$RB^{(T),t+n} = \sum_{x=\max(27+n,65)}^{\omega-1} n_x^{(T),t+n} \sum_{j=1}^{n_x^{(T),t+n}} B_{\tau(j)}^{t+n-x+\tau(j)} I[x - \tau(j) \geq n]. \quad (3.64)$$

The subgroup of retired lives does not have any death benefits to pay. The death benefits³⁶ paid in year $t+n$ for each of the other subgroups are given below:

$$DB^{(A),t+n} = \begin{cases} n_{25}^{(A),t} (q_{25}^{(m)} + q_{25}^{(t)}) EAC_{26}^{t+1} \\ + \sum_{x=26}^{69} n_x^{(A),t} q_x^{(m)} AC_{x+1}^{t+1} & , \quad n = 0, \\ \\ \sum_{x=25+n}^{69} n_x^{(A),t+n} q_x^{(m)} AC_{x+1}^{t+n+1} \\ + \sum_{x=\max(25+n,26)}^{59} \sum_{j=1}^{n_x^{(D),t+n}} q_x^{(D)} B_{\varrho(j)}^{t+n-x+\varrho(j)} \left(\frac{x+1-25}{\varrho(j)-25}\right) \\ \quad v^{(60-x-1)} {}_{60-x-1}p_{x+1}^{(D)} \ddot{a}_{60}^{(D)} I[x - \varrho(j) \leq n-1] \\ + \sum_{x=\max(25+n,27)}^{64} \sum_{j=1}^{n_x^{(T),t+n}} q_x^{(T)} B_{\tau(j)}^{t+n-x+\tau(j)} \\ \quad v^{(65-x-1)} {}_{65-x-1}p_{x+1}^{(T)} \ddot{a}_{65}^{(T)} I[x - \tau(j) \leq n-1] & , \quad n \geq 1; \end{cases} \quad (3.65)$$

$$DB^{(D),t+n} = \sum_{x=26+n}^{59} \sum_{j=1}^{n_x^{(D),t+n}} q_x^{(D)} B_{\varrho(j)}^{t+n-x+\varrho(j)} \left(\frac{x+1-25}{\varrho(j)-25}\right) \\ v^{(60-x-1)} {}_{60-x-1}p_{x+1}^{(D)} \ddot{a}_{60}^{(D)} I[x - \varrho(j) \geq n]; \quad (3.66)$$

³⁶Again, the death benefits actually include the termination benefits paid to those who leave the plan before completing two years of participation.

$$DB^{(S),t+n} = \begin{cases} 0 & , n = 0, \\ n_{25}^{(A),t+1} (q_{25}^{(m)} + q_{25}^{(t)}) EAC_{26}^{t+2} & , n = 1 \\ n_{25}^{(A),t+n} (q_{25}^{(m)} + q_{25}^{(t)}) EAC_{26}^{t+n+1} \\ + \sum_{x=26}^{25+n-1} n_x^{(A),t+n} q_x^{(m)} AC_{x+1}^{t+n+1} \\ + \sum_{x=26}^{\min(25+n-1,59)} \sum_{j=1}^{n_x^{(D),t+n}} q_x^{(D)} B_{\varrho(j)}^{t+n-x+\varrho(j)} \left(\frac{x+1-25}{\varrho(j)-25} \right) \\ \quad v^{(60-x-1)} {}_{60-x-1}p_{x+1}^{(D)} \bar{a}_{60}^{(D)} \\ + \sum_{x=27}^{\min(25+n-1,64)} \sum_{j=1}^{n_x^{(T),t+n}} q_x^{(T)} B_{\tau(j)}^{t+n-x+\tau(j)} \\ \quad v^{(65-x-1)} {}_{65-x-1}p_{x+1}^{(T)} \bar{a}_{65}^{(T)} & , n \geq 2 \end{cases} \quad (3.67)$$

$$DB^{(T),t+n} = \sum_{x=27+n}^{64} \sum_{j=1}^{n_x^{(T),t+n}} q_x^{(T)} B_{\tau(j)}^{t+n-x+\tau(j)} v^{(65-x-1)} {}_{65-x-1}p_{x+1}^{(T)} \bar{a}_{65}^{(T)} I[x - \tau(j) \geq n]. \quad (3.68)$$

As mentioned earlier, we simply have to add the contributions, retirement benefits, or death benefits of the subgroups included in a combination to find the corresponding cash flows for that particular combination. However, for the additional contribution, we will calculate one, using Equation 3.37, *only if* the sponsor is included; otherwise, we will assume it is zero.

The definition of the additional contribution is almost only of theoretical interest for our present purpose. Truly, when we simulate the future, we should not be considering giving anything away if we know that an additional contribution should be made to start with.

At this point, we have all the tools we need to generate the disposable surplus

according to our suggested criterion, not only for the whole plan, but also for any of the $2^n - 1$ non-empty combinations of subgroups. This provides us with the elements necessary to the application of game theory, which we introduce in the next chapter.

Before we end this section, we wish to point out that the applicability of our criterion to combinations of subgroups that do not include the sponsor may be called into question. When active lives are included, one could argue that the unfunded liability could be addressed through increased employee contributions as a substitute to the usual sponsor's amortization payment. However, in the absence of members in the active status, one is left to wonder how an unfunded liability would be handled if it arose, as it may with a probability equal to the targeted one. We will revisit this criticism as we consider an application in Chapter 5.

3.7 Conclusion

We have started this chapter with a discussion of the different methods used, or that could be used, to value the assets. Comparing assets with liabilities gives us a first, but very vague, indication of the surplus money there is in a pension fund.

The main objective of this chapter was to come up with a criterion for deciding on what amount we could distribute out of the pension fund without jeopardizing the plan's future. We were guided in our endeavor by a pension plan's two key concerns: solvency and stability.

The criterion we have come up with – small probability of a deficit at the next actuarial valuation – requires projection of the assets and liabilities between the

two valuations.

We have limited the randomness in our model to the components of the rate of return and the increase in wages. We introduced the models developed by Wilkie [70] and Sharp [61] as those we would use to simulate economic scenarios. As for the number of entrants, the salary scale, and all the decrements, we chose to make them deterministic.

Given an economic scenario, we showed how the assets and liabilities of the plan, or of one of five subgroups, evolve through time. By way of an example, we have also indicated how important the correspondence between the valuation basis and the expected economic outcomes was.

Once we have the results for each of the simulations, applying our suggested criterion is then a matter of ordering the results and selecting the desired percentile.

We believe our suggested criterion has the advantage of accounting explicitly for a plan's objectives of solvency and stability. Other methods for determining the amount to distribute could be devised. Among the possibilities are making the amount equal to a certain percentage of the valuation surplus³⁷, or to the excess of that surplus over a certain amount. However, though simple to apply and understand, these *ad hoc* rules lack a formal justification to make them truly appealing.

Statistical decision theory may prove to be an interesting alternative here. If we can come up with a probability distribution for the actual value of the liabilities as well as for the assets, then we can also derive a probability distribution for the

³⁷By "valuation surplus," we mean the difference between the assets and the liabilities as of the valuation date.

actual value of the surplus.

To use this theory, we would have to define a loss function of the surplus given away versus the actual surplus. We would then pick as disposable surplus the value that minimizes the expected loss. Undoubtedly, the choice of a loss function would be a sensitive issue since it should capture the relative consequences of giving away more or less than actually was available.

Another issue which we have not really discussed is that of the choice of economic models. Not all actuaries consider Wilkie's model as a valid choice. Hence, a possible area for further research would be the study of the sensitivity of the results to the particular economic model used.

Moreover, assuming decrements to be deterministic is perhaps a much stronger assumption than we believe it to be. A theoretical analysis of the relative importance of demographic and economic risks in a pension plan would be needed to shed light on this question.

Chapter 4

Distribution of the Surplus

4.1 Introduction

In the last chapter, we have developed a criterion to decide on the amount of surplus we could give away. While this provides the answer as to what to give, it does not tell us whom to give it to. In preparation for this chapter, we then also adapted the criterion to subgroups so that we could determine the disposable surplus for any combination of them.

We will begin by discussing the entitlement of each of the five subgroups to a share of the surplus. As we shall see, the entitlement of the sponsor versus that of the members is the most sensitive issue.

If we can settle the question of the claimants to the surplus, we can then use cooperative game theory to determine their shares. We will start our exposition of the theory by a definition of cooperative games, along with a distinction between transferable and non-transferable utility. We will also offer reflections on the

applicability of this theory to the sharing of a pension plan surplus.

We will continue with the definition of an allocation, which is precisely what we wish to do with the disposable surplus. We will also introduce the broader concept of a sharing rule. We will then present some of the different properties sharing rules may satisfy. As we define these properties, we will try to give an indication of their desirability in the context of a pension plan. On a more theoretical note, we will give the relationships that exist between certain of these properties.

We will complete this chapter with the definition of four sharing rules. For each of them, we will identify the properties satisfied. We will conclude with some comments about the computations involved in applying them. These sharing rules will constitute the tools to use as we go about solving the whole problem for our model pension plan in the next chapter.

4.2 Entitlement to a Share of the Surplus

Potential claimants to the disposable surplus are the five subgroups we defined: the sponsor, the active lives, the disabled lives, the past members who left with vested benefits, and the retired lives. In real life, we would also have to include the beneficiaries who currently receive a pension from the plan or are entitled to a deferred pension from that plan. However, in our model pension plan, we have assumed single life annuities without guarantees; hence, we *do not* have beneficiaries. If we had beneficiaries, the simplest way to account for them would be to include them with the subgroup the related pension plan member belonged to. For example, if a retired life had elected to receive a joint pension and died, then the beneficiary

would be included in the subgroup of retired lives.

In other words, the potential claimants are those people whose names are kept on record by the plan administrator, and the plan sponsor. For instance, if a retiree dies and has no further entitlement (through a beneficiary) to the pension plan, we consider his name to have been removed from the records and no one is eligible for any share of the surplus in his name.

These are *potential* claimants. As a result, it may be that only some subset of the potential claimants share in the surplus to be distributed. We will comment on the entitlement of each of them in turn.

4.2.1 Entitlement of the Members

We defer the discussion of the entitlement of the members as a whole to the next section, which deals with the entitlement of the sponsor. In that section, we will see that some of the arguments in favor of the sponsor automatically exclude all the members. In this section, we will focus on the entitlement of each subgroup of members, given that we have not excluded all the members already.

Active Lives

One argument to justify excluding the active lives from sharing in the surplus *a priori* might be that, until we know exactly what we will have to pay them (*i.e.*, until they go into an inactive status), they should not receive any portion of the surplus. The underlying justification would be that the surplus share that would accrue to them will potentially be eaten away by increases in benefits, brought

about by higher wages, larger than anticipated.

Still, excluding the active lives would mean that some pension plan members, namely those that die in active service, will never share in the surplus¹, regardless of the surplus amounts that are distributed while they belong to the plan. It appears drastic to us to annihilate the right to a share in the surplus for a particular segment of the members. As a result, we will include the active lives as claimants. In doing so, we will ensure that all pension plan members, at one time or another, have the right to a surplus that may arise.

Inactive Lives

Whichever subgroup composed of inactive lives we wish to consider, we could come up with similar reasons for excluding it.

For instance, if vested members that quit, or disabled members, are deemed to relinquish any claim to the pension plan other than that to their deferred benefits, they should not receive any part of the surplus.

Similarly, if it is considered that retirees are getting the benefits they were promised, according to the terms of the contract, and that, moreover, they could share in the surplus while active, we may be tempted to conclude that they, too, should be excluded from the surplus sharing.

However, all inactive lives have paid contributions at some point in time. Hence,

¹Of course, some members may never share in the surplus for the simple reason that the economic situation turns out to be such that there never is any surplus to give away while they belong to the plan. We would consider these particular members to have been excluded *a posteriori*. It would not necessarily follow that, in the future, members with the same status would not share in the disposable surplus at that time.

even though they no longer directly contribute to the generation of a surplus², they still do so indirectly insofar as some or all of their past contributions are still part of the assets.

Moreover, we may wish to consider that, under our model non-indexed final-earnings pension plan, the benefits they receive or will receive are subject to erosion due to inflation³. Still, as appealing as this argument may appear, it is a valid one only insofar as one believes surpluses to be driven by inflation. In other words, unless inflation is the main explanatory variable for the emerging surpluses, we will not deal effectively with the erosion of the benefits simply by including the inactive lives in the surplus sharing.

Rather, it might be wise to include provisions for wage and price indexation in the plan document to avoid these built-in penalties. We do not have statistics related to indexation in the deferral period. However, as of January 1, 1996, 28.3% of all defined-benefit pension plans (covering 49.7% of all members) provided for some automatic adjustment of pension benefits⁴ (see [65]). The situation in the public sector is quite different from that in the private sector though. In the public sector, 49.0% of the plans (covering 79.7% of the members in that sector) offered indexation, in contrast with 26.9% of the plans in the private sector (covering 17.1%

²In fact, they rather contribute to its depletion through the retirement benefits they collect.

³During the deferral period, benefits are eroded through wage inflation. Had the member remained active, and with the same sponsor, the benefit accrued for the service performed before terminating or becoming disabled would have increased with the wages. During retirement, benefits are eroded through price inflation. At retirement, the member has a certain purchasing power. If prices rise, though the retiree still receives the same amount in nominal terms, he will be adversely affected in real terms.

⁴The adjustment is not necessarily full CPI-linked indexation. Other possibilities are partial CPI-linked indexation, excess interest earnings, percentage increase and flat dollar increase.

of the members in that sector).

In any case, without strong reasons to exclude the inactive lives, we will make them eligible. Nonetheless, it will be clear in what follows how to exclude any or all of the subgroups of inactive lives if so desired.

4.2.2 Entitlement of the Sponsor

Of all potential claimants, the one for which it is hardest to determine entitlement is the sponsor. Two diametrically opposed views are found in the literature (see Adell [1], Ascah [6], or Bodie [10] for an exposition to both views). The Régie's discussion paper regarding the fair sharing of surpluses [48] repeats these arguments and forms the basis of the following two paragraphs.

At one end of the spectrum, it can be argued that pension plan contributions are deferred wages and that, as such, all of the pension plan assets belong to the participants. It can also be argued that the contributions are only one element of the total compensation package. Insofar as, regardless of the total contributions it pays, the sponsor ends up spending the same total amount on remuneration, we can argue that these contributions, not unlike the rest of the remuneration (which benefits the employees immediately), are the employees' property. Under this line of reasoning, defended by plan members, often via their union, the sponsor would not get any share⁵.

At the other end of the spectrum, it can be asserted that the promises made by

⁵Although this would eliminate one subgroup from consideration, it would leave the four subgroups composed of current members among which to share the surplus. The approach developed in this chapter also applies to this situation since it can accommodate any number of entities.

the sponsor are only with respect to the benefits (as opposed to the contributions) and that, moreover, the sponsor fully assumes the deficit risk. Indeed, the sponsor has to pay additional contributions when the plan has an unfunded liability. This indicates that, from a legal point of view, the members' entitlement is first and foremost to benefits, not to contributions. Since the sponsor assumes the deficit risk, it arguably should get the reward too. Consequently, as claimed by some sponsors, any arising surplus should revert to them⁶.

In practice, in Ontario, on wind-up, all of the surplus reverts to the members by default (see [52]). If the sponsor wishes to get a portion of the surplus, it must receive the consent of the Financial Services Commission of Ontario⁷ (FSCO). To obtain that consent, the sponsor must show that the plan document provides for the payment of surplus to the sponsor *and* obtain the written agreement of at least two thirds of the members or a court order concerning surplus entitlement and distribution.

To give us an idea of the actual shares received by sponsors, we can look at the recent decisions rendered by the FSCO regarding surpluses of terminating pension plans⁸ (visit www.ontarioinsurance.com/Pensions/PCO-Home.nsf, the pension sector of the FSCO's web site). For the decisions rendered between October 1, 1997 and November 25, 1998, we find that the sponsor's share ranged between 20% and

⁶If we took that viewpoint, the question of what to give away would be the only question to answer since there would then be a single claimant to the surplus.

⁷This is a new organization amalgamating the former Ontario Insurance Commission, the Pension Commission of Ontario, and the Deposit Institutions Division of the Ministry of Finance.

⁸No such decision is necessary if the sponsor does not request a share of the surplus. Hence, these decisions do not give us an idea of the percentage of plans having a surplus on wind-up which distribute it all to the members.

100%, 100% being the most common, followed by 50%.

In Quebec, the law governing pension plans requires an agreement to be reached between the sponsor and the participants before a surplus can be distributed on wind-up (see [35]). This agreement, proposed by the sponsor, must not be rejected by 30% or more of the participants⁹; otherwise, arbitration is necessary.

In all other provinces, except British Columbia and Newfoundland, unless the sponsor proves its right to the surplus according to the plan document or obtains a court ruling to that effect, all of the surplus reverts to the members. Thus, in the next chapter, we will also show how to use cooperative game theory to allocate the entirety of the disposable surplus to the members.

Before legislation regarding the use of surpluses came into effect, some claims to surplus ownership made it to the courts. In Quebec, the most famous case is definitely the Singer case. In that case, not only was the surplus determined to belong to the participants, but the court ordered Singer to pay back the value of all contribution holidays it had taken (see the March 14, 1995 *Communiqué Mercer* [30]). A similar judgement was given in the Hockin v. The Bank of British Columbia case, in British Columbia (see the April 17, 1995 *Communiqué Mercer* [2]). In Ontario, in the Dominion Stores case, the sponsor was denied withdrawal of the surplus (see Claridge's article in the *Globe and Mail* [14]). Also, in the CUPE v. Ontario Hydro case, the Ontario Court of Appeal declared that the sponsor could not take contribution holidays as these were, in effect, a surplus withdrawal (see

⁹It may appear that acceptance is equivalent to absence of rejection. However, whereas, in Ontario, participants must accept the agreement in writing for it to come into effect, in Quebec, participants must object to the agreement to prevent it from coming into effect.

Salter's article in *Maclean's* [55]).

However, in the *Schmidt v. Air Products Canada Ltd.* case, the Supreme Court of Canada stated that Air Products was entitled to the whole surplus that came from the Stearns plan (one of the two plans involved in the fusion) since that plan gave discretionary power to the sponsor regarding use of the surplus upon termination. It also confirmed the right of the sponsor to take contribution holidays. Similarly, in Ontario, in the *Otis Canada* case (in respect of the pension plan for Steel Workers Local 7062), the Weekly Court granted legal entitlement of the surplus to the sponsor (see the March 1991 issue of the *PCO Bulletin* [50]).

These few court cases lead us to two observations. On the one hand, the pension plan document is central to the determination of the ownership of the surplus. On the other hand, court decisions appear to have been largely of an "all or nothing" nature. The new legislation adopted in Ontario and Quebec as moratoria on the use of surplus were lifted favors intermediate positions.

Deciding whether or not to give a share to the sponsor remains a delicate issue. At the time of joining the plan, since all members change status over time, they all have some positive probability of sharing in the surplus provided some subgroup other than the sponsor's is included in the sharing exercise. However, because the sponsor does not change status – indeed it remains the sponsor, it never can hope to share in the surplus if it is excluded.

This thesis does not pretend to settle the question as to which of the two opposing views presented at the beginning of this subsection, or any compromise in between, is to be adopted. It is a delicate, yet very important, issue. Precisely

because it is so important and central to the whole exercise of surplus sharing, we wish to offer some suggestions, being guided in part by some statements made by the Régie des rentes du Québec.

In determining the amount to give away before deciding how to split it among the claimants, we make the implicit assumption that the second decision is independent of the first. Nevertheless, if the sponsor is, in actual fact, the one picking the amount, it naturally will want to consider what it stands to gain, through a share in the surplus, or to lose, through later unfunded liabilities to be financed. In that case, the first decision actually would be quite dependent on the second one. Hence, instead of dealing with two separate games (choosing the amount to give and allocating that amount), we would have to deal with one large game in which the incentives given to the sponsor (surplus share versus deficit share) would largely impact the amount identified as the disposable surplus and even the contribution rate.

In that respect, if the sponsor knew it never would get any share of the surplus, it would have a very strong disincentive to fund the pension plan adequately. Instead, it would contribute as little as possible and allowable, thereby minimizing the probability and size of future surpluses. Since adequate funding underpins the guarantee of the defined benefits, it appears rather undesirable to encourage the sponsor, even though indirectly, not to fund appropriately. This leads us to believe that the sponsor should be entitled to at least a portion of any disposable surplus.

As to the actual portion that should revert to the sponsor, we will present two approaches in the next chapter. One will define the sponsor's share to be the ratio

of its contributions to the total cost of the pension plan. This is the suggested approach. on the wind-up of a pension plan, in a discussion paper published by the Régie des rentes du Québec¹⁰ [48]. The other approach will consist in assuming that the sponsor is essential to the existence of a pension plan and, hence, that the disposable surplus is nil unless the sponsor is part of the combination of subgroups. Under the first approach, cooperative game theory is used to determine only the members' shares whereas, under the second one, it is used to determine all shares, including the sponsor's.

In the application of our suggested approach to our model pension plan, we focus on a specific set of claimants to the surplus. However, our exposition of game theory will not assume a particular number of claimants. We thus wish to convey the idea that our approach is easily adaptable to other splits of the plan membership into subgroups.

4.3 Cooperative Game Theory

As indicated earlier, we will use cooperative game theory, as developed in Moulin's book [39], to answer the question of how to share the disposable surplus. This theory provides us with a way to solve decision problems under the assumption that the involved parties can get together and write binding contracts on their

¹⁰This assumes that the plan is contributory. For non-contributory plans, the document indicates that the law would prescribe a maximum percentage for the sponsor's share. However, we were unable to locate the corresponding item in the law. It is interesting to note that this approach stems from the view that both members and sponsor have obligations to the pension plan and that these obligations are in proportion to their contributions.

actions¹¹.

In the broader field of actuarial science, Lemaire [33, 34] was one of the very first ones to use this theory in insurance applications. More recently, Alegre and Claramunt [4], and Suijs, de Waegenare and Borm [68] have used cooperative game theory to allocate the cost of group annuities and insurance portfolios, respectively.

Throughout our discussion of cooperative game theory, we will focus on surplus sharing. In the literature, the problem of cost sharing is also very often considered. The two problems are equivalent: a cost is a negative surplus; likewise, a surplus is a negative cost.

Therefore, if we wished to assume that unfunded liabilities are to be shared as well, we could apply the same principles, although the set of potential payors would probably then have to be restricted to the sponsor and the active participants. Even though, technically, it would be possible to reduce the payments being made to the other claimants, some may consider it to undermine the very nature of a *defined-benefit* pension plan. Nevertheless, in practice, in the case of a wound-up pension plan, if the funding level is below 100%, all members, be they active or inactive, see their benefits reduced in the same proportion.

4.3.1 Definition of a Cooperative Game

In a general cooperative game, there are n agents, indexed by $i \in N = \{1, 2, \dots, n\}$. Any non-empty subset of agents $S \subseteq N$ can decide to form a coalition and receive collectively the surplus generated by it, denoted by $v(S)$. All agents know the

¹¹This is in contrast to non-cooperative game theory where the parties' actions cannot be contracted on.

surplus $v(S)$ associated with each $S \in \mathcal{P}(N) \setminus \{\emptyset\}$, where $v(S)$ is called the worth of the coalition, $\mathcal{P}(N)$ is the power set of N , and \emptyset is the empty coalition, with no agents in it.

The function v is called the characteristic function, with domain $\mathcal{P}(N)$ and range \mathfrak{R} . It specifies what any given subset of agents can achieve, independent of what the other agents do. The coalition containing all n agents, namely the set N , is called the grand coalition.

We assume the grand coalition to form¹² and we thus have to decide how to divide $v(N)$ among the n agents.

As mentioned earlier, cooperative game theory assumes that the agents can write a contract prescribing how they will behave and what they will get in return for their cooperation. We note that the sponsor and members have already done so via the plan document with respect to contributions and benefits. In view of the “all or nothing” court decisions rendered in the past, we believe that the sponsor and members have an incentive to meet together and elaborate a binding contract with respect to the distribution of the surplus too. This view justifies our use of cooperative game theory.

Moreover, since our approach is based on axioms (called properties hereafter), it will allow us to see how the choice of desirable properties interacts with a subgroup’s characteristics to impact its share.

In the rest of this chapter, within the context of the pension plan, agents will refer to subgroups, and coalitions, to combinations of subgroups.

¹²In fact, the grand coalition is formed simply by virtue of the fact that participation in the pension plan usually is a *sine qua non* of employment.

4.3.2 Transferable Utility versus Non-Transferable Utility

At first glance, it may appear that cooperative game theory only applies to problems of sharing money. That is not so. As a matter of fact, cooperative games are of two types: transferable utility (TU) and non-transferable utility (NTU).

In the first case, we basically are dealing with money, or some other divisible good. As money is the medium of exchange *par excellence*, it is easy to transfer it among the participants. In such a setting, the loss of one dollar by an agent translates directly into a net gain of one dollar by the remaining agents.

In the second case, however, in which we typically deal with utility itself, the loss of one unit of utility by an agent does not necessarily translate into a net gain of one unit of utility by the rest of the agents.

One clarification must be made about the nature of TU games. Unless all agents give the same value to a dollar and unless the value given to a dollar by an agent does not depend on what he already has (money *and* other goods), the game considered is *not* a TU game: we actually are dealing with an NTU game with some money to divide.

For the two conditions introduced here to be satisfied, agents must have quasi-linear utilities. Such agents have utility functions that are linear in money and additively separable in money and other goods. It is only under these two conditions that we truly can speak of money as a means of transferring utility.

The sharing of the pension plan surplus could be described as a game of either type. We will treat it as a transferable utility game, which is simpler to work with for four reasons. First, it is generally difficult to determine an agent's utility

function. Second, more variables would potentially come into play as more goods would have to be taken into account. Third, working with results applicable to the TU game only requires information that should be kept in the records of the pension plan. Fourth, assuming agents to have quasi-linear utilities allows us to work with groups instead of individuals since their utilities are additive in the good being allocated. However, these reasons should not overshadow the fact that, underlying any TU game, are the strong assumptions mentioned in the previous paragraph.

We will henceforth consider cooperative games with transferable utility. The reader interested in NTU games and examples thereof is referred to books by Moulin [39], Owen [44] and Shubik [62]. These same three books also provide additional information regarding TU games and could be used to gain a different exposition to the topic.

4.4 Allocations, Sharing Rules and Their Properties

In a cooperative game, whatever coalition/s is/are formed, agents have to decide among themselves how to split the surplus generated by the coalition they belong to. This split is called an allocation.

In this section, we will give theoretical definitions of “allocation” and “sharing rule.” We will then look at some of the properties allocations and sharing rules can have. For each property, we will comment on its meaning and desirability, in general as well as from a pension plan’s perspective. We will conclude with a brief

summary of the properties that shows how they relate to one another.

4.4.1 Definition of an Allocation

An **allocation** is a vector of shares $(x_1, x_2, \dots, x_n) \in \mathbb{R}_+^n$, where x_i is the amount received by agent i . For the allocation to be feasible, we must have that $\sum_{i=1}^n x_i \leq v(N)$. Also, the allocation is efficient only if none of the surplus to be distributed is wasted. In all that follows, we will assume allocations to be feasible and efficient: that is, we will assume $\sum_{i=1}^n x_i = v(N)$.

We will focus on the grand coalition N since we ultimately want to determine how to split the surplus among all the potential claimants. As indicated earlier, potential claimants may not receive anything in a given allocation. Indeed, zero shares do satisfy the definition of an allocation.

4.4.2 Definition of a Sharing Rule

Whereas an allocation prescribes the agents' shares for *one* particular game, a sharing rule does so for *any* particular game.

Given the set of agents N , of size n , we denote by Γ^N the set of TU games having N as grand coalition. We then define a **sharing rule** to be a mapping

$$\begin{aligned} \phi: \Gamma^N &\rightarrow \mathbb{R}_+^n \\ v &\rightarrow \phi(v), \end{aligned}$$

that is, one that associates to any game v an allocation $\phi(v)$ of the total surplus

$v(N)$. Hence, $\phi(v)$ satisfies the equality $\sum_{i \in N} \phi_i(v) = v(N)$, where $\phi_i(v)$ denotes the share received by agent i . (Sharing rules are also called value operators.)

4.4.3 Properties of Sharing Rules and Their Desirability

Sharing rules may enjoy a number of properties, some of which we introduce here. At the same time, we will try to offer some insight as to what they mean, and how desirable they are, in the problem at hand. We will also indicate relationships among certain properties, or the absence thereof. While we will provide some intuition for these results, we refer the reader interested in formal proofs to Moulin's book [39].

The first two properties are given as properties of allocations. A sharing rule will be said to enjoy either property if, for any game, the allocation it yields satisfies that property.

Property 4.1 *An allocation is said to be **individually rational (IR)** if each agent i receives at least $v(\{i\})$, his opportunity surplus. That is, in symbols, $x_i \geq v(\{i\})$, $\forall i \in N$.*

This property requires that each agent get at least as much when belonging to N as he would get on his own. A much stronger version of this property follows.

Property 4.2 *An allocation is said to satisfy the **stand-alone principle (SAP)** if, for every coalition $S \subset N$, $\sum_{i \in S} x_i \geq v(S)$. In other words, each coalition must receive at least the value of the surplus it would generate on its own.*

This principle ensures that the allocation is such that no coalition has an incentive to deviate from the grand coalition. Indeed, if a coalition were to secede, this principle guarantees that it would not be better off.

While an allocation satisfying the stand-alone principle necessarily is individually rational, the converse does not hold.

Depending on the actual game, there may or may not exist allocations satisfying Property 4.2. If they exist, the set of all such allocations is defined to be the core. If not, the core is said to be empty.

In terms of the sharing of the pension plan surplus, the stand-alone principle requires that each combination of subgroups receive at least as much as it would have received had it formed a pension plan by itself, with the same characteristics as the one under study.

Often, in real-life situations, this principle is considered very important, for it is easy for coalitions other than the grand one to form. Here, however, forming smaller coalitions would be rather difficult, if not simply impossible.

Several impediments exist, from the law as well as from the sheer consideration of numbers. For a pension plan to be registered, and thus enjoy a preferred tax treatment, certain conditions have to be met. Among them, one states that the relationship between the sponsor and the participants must be of a certain type (*e.g.*, employer-employees, union-members). A plan brought about by employees without sponsorship from their employer or union hardly would qualify. Besides, the cost associated with maintaining a pension plan would be prohibitive for most employee coalitions.

Nevertheless, it may appear that the sponsor could restrict the plan to fewer participants, or split the plan into two or more smaller plans. Again, rules exist as to the conditions that may exist in a pension plan to restrict participation to it. Moreover, if the smaller plans were to be quasi-replicas of the original one, it is hard to conceive that the sponsor would gain anything but an increase in the cost of plan administration from such a move.

As a result, even though the stand-alone principle may appear to be a natural and desirable requirement, we will not seek to satisfy it at all costs. For the same reasons, we will not insist on individual rationality. We will, however, indicate whether the rules to be studied enjoy either property or not.

Property 4.3 *A sharing rule is said to be **anonymous (AN)** if it commutes with permutations of agents. This means that for all bijections τ of N onto itself and for all games in Γ^N , $\tau(\phi(v)) = \phi(\tau(v))$. In terms of notation, $\tau(v)(S) = v(\tau(S))$ for $S \subset N$ and $\tau(x)_i = x_{\tau(i)}$ for $x \in \mathfrak{R}_+^n$.*

In layman's terms, a rule is anonymous (or, equivalently, symmetric) if the names or labels given to agents do not have any impact on the allocation. Hence, each agent receives a certain share of the surplus which depends on the game and on the sharing rule, but which does not vary with the particular label i he may be given. To summarize this property, economists often say, "names do not matter."

Viewed from another angle, if all agent i 's characteristics that are relevant to the game are exchanged with those of agent j ($i \neq j$), their shares are exchanged as well under anonymity.

In real life, we typically want to use anonymous rules so as to avoid partiality.

As was indicated earlier, we might want to exclude a subgroup of inactive lives, or the sponsor. That is somewhat arbitrary; yet, it depends on the label given to the subgroup only insofar as the label itself refers to a characteristic that is quite relevant to the pension plan. So, to introduce any degree of arbitrariness, we will limit ourselves to excluding particular subgroups; we will not exclude particular individuals within a subgroup.

This is an obscure point but, when we indicate whether a sharing rule is anonymous or not, we do not imply anything about the way in which the inputs to the problem we consider were determined in the first place. Rather, we only are saying something about the way the rule uses the inputs (the characteristic function) to generate the shares. For instance, if the disposable surplus of coalitions including the sponsor was found with a larger target probability, we could not pretend that the inputs were found without reference to the “names.” This would favor the sponsor and it would be reflected in the shares of any anonymous sharing rule.

Property 4.4 *A sharing rule is said to satisfy the **dummy axiom (DA)** if, for any $i \in N$, and for all $v \in \Gamma^N$,*

$$\{v(S \cup \{i\}) = v(S), \forall S \subseteq N \setminus \{i\}\} \Rightarrow \phi_i(v) = 0.$$

This last axiom gets its name from the fact that an agent that does not contribute anything to any coalition is called a dummy¹³. According to the axiom, he should not receive any share of the surplus.

¹³Actually, though an agent might not do anything, his consent may be necessary for the distribution of the surplus. Hence, in a sense, he does do something after all, namely agree, and should share in the surplus too.

Property 4.4 is a quite desirable one. Indeed, why should someone get a portion of the surplus for simply being there? The dummy axiom ensures that agents who do not “do” anything do not get anything. However, it is important to realize that it does not rule out the possibility that agents that *do* “do” something get nothing.

For the problem at hand, we desire the dummy axiom to hold. Otherwise, it would be hard to justify that pseudo-participants who do not contribute anything to a pension plan (directly or indirectly through the sponsor) get something out of it.

By virtue of the fact that people joining the plan do pay contributions, we may be led to believe we cannot have dummy agents in the problem at hand. Actually, in the subgroup of active lives, we have one hundred new entrants aged 25 who have not contributed anything yet. Hence, under the dummy axiom, they should not get anything¹⁴. However, if that subgroup’s share is distributed via a reduced contribution rate¹⁵, and if *all* employees contribute at the same rate, we indeed would have given something to dummy agents. As a result, whether the dummy axiom truly is satisfied will depend much more on the mode of distribution of the shares than on the sharing rule itself since none of the subgroups as a whole is dummy.

Instead of focusing on the individual, we now focus on the aggregate and introduce a seemingly unrelated property. As a matter of fact, the last property and the following one are both weaker versions of Property 4.7, to be introduced later on.

¹⁴We note that the subgroup as a whole is not dummy.

¹⁵Just as the dividends in participating life insurance can be distributed in a number of ways, so too can the surplus shares.

Property 4.5 A sharing rule is said to be **monotonic in the aggregate (AM)** if, for any $v, w \in \Gamma^N$,

$$\{v(N) \geq w(N) \text{ and } v(S) = w(S), \forall S \subset N\}$$

implies that

$$\{\phi_i(v) \geq \phi_i(w), \forall i \in N\}.$$

This form of monotonicity guarantees that if the worth of the grand coalition increases while the worth of all other ones remains the same, then no agent is penalized. In such a situation, the better overall performance cannot be tracked down to a particular agent and it would indeed be awkward not to have everyone gain from it, let alone have someone lose because of it. While aggregate monotonicity prevents the latter, it does not prevent the former, as all of the gain could go to only a few of the n agents.

In real life, this property is of particular interest when the outcome of a venture is different from what was expected. Since the worth of all coalitions other than the grand coalition cannot be determined to have changed, we basically are concerned with what happens in the aggregate.

Based on the criterion we gave to establish the worth (as given by the disposable surplus) of any coalition, we hardly can imagine how only the worth of the grand coalition could increase. In that regard, it may be that this property does not relate to the problem we consider. However, it is a weaker version of the next property to be defined. Besides, the next property is very important and we will want it to hold. In the cases for which the next property does not hold, aggregate monotonicity will provide an indicator of the extent to which it is violated.

As one may suspect, Property 4.5 is a somewhat weak form of monotonicity, since it only prescribes how the allocation should be affected when only $v(N)$ changes. A stronger form of monotonicity is defined next, while a still much stronger version is introduced afterwards.

Property 4.6 *A sharing rule is said to be **coalitionally monotonic (CM)** if, for all $v, w \in \Gamma^N$, for any $i \in N$,*

$$\{v(S) \geq w(S), \forall S \text{ such that } i \in S\}$$

and

$$\{v(S) = w(S), \forall S \text{ such that } i \notin S\}$$

imply that $\phi_i(v) \geq \phi_i(w)$.

Property 4.6 says that if the worth of every coalition to which a particular agent i belongs increases or stays the same, then his share x_i should not decrease.

That aggregate monotonicity is a weaker form of coalitional monotonicity should be clear as the grand coalition is just one of the 2^{n-1} coalitions to which any particular agent i belongs.

In general, coalitional monotonicity is a very desirable property in that it provides the agents with an incentive to increase their worth. In the context of a pension plan, though agents cannot set their contributions themselves, we will want this property to hold. Indeed, if we were to look at the same pension plan at two different points in time (or at two different pension plans), we would hope that the subgroups who see their worth increase do not get smaller shares. Whether or not this is the case will depend on whether or not the sharing rule is coalitionally monotonic.

As mentioned before, coalitional monotonicity, itself a strengthening of aggregate monotonicity, exists in a yet stronger form, fittingly called strong monotonicity.

To help with the discussion that follows, we define the **marginal contribution** of i to the coalition S in game v , where $i \notin S$, to be $v(S \cup \{i\}) - v(S)$. By convention, the worth of the empty coalition \emptyset is 0.

Property 4.7 *A sharing rule is said to be **strongly monotonic (SM)** if, for all $v, w \in \Gamma^N$, for any $i \in N$,*

$$\{v(S \cup \{i\}) - v(S) \geq w(S \cup \{i\}) - w(S), \forall S \text{ such that } i \notin S\}$$

implies that $\phi_i(v) \geq \phi_i(w)$.

The last three properties (4.5 to 4.7) provide monotonicity conditions; that is, they indicate how the allocation should be affected by a change in the characteristic function. Properly speaking, the dummy axiom provides an independence condition: hence, it identifies cases for which part of the allocation is not affected by changes in the characteristic function. Strong monotonicity provides an independence condition as well. According to that property, if an agent's marginal contributions are the same in two games, his share should remain the same, independent of changes in other agents' marginal contributions.

Under strong monotonicity, an agent's share depends on his contributions *only*. Hence, since a dummy agent has zero marginal contributions, strong monotonicity prescribes a zero share for him, just as did the dummy axiom. Nevertheless, a sharing rule satisfying the dummy axiom is not necessarily strongly monotonic since that axiom does not deal with any set of marginal contributions other than the set of zeroes.

Strong monotonicity is stronger than coalitional monotonicity since it indicates how an agent's share should be affected not only if he increases his marginal contributions but also if he keeps his constant while all other agents change theirs.

Should we wish strong monotonicity to hold? In a general problem, if we want to avoid agents benefiting from the others' contributions and prefer to have agents reap the full rewards of their efforts, then we indeed should look for a sharing rule that is strongly monotonic. However, if we do not want to rule out redistribution among the agents, then we should not want to enforce strong monotonicity.

Once more, it can be said that pension plan participants do not have much say on their contributions. However, through time, we certainly could encounter two games which are such that we end up with a situation addressed in Property 4.7. In such a case, we believe that, although giving less to those whose marginal contributions to the surplus have increased would be extremely hard to justify¹⁶, precluding redistribution may not be that desirable. In other words, we will not specifically want this property to hold.

Property 4.8 *A sharing rule is said to be **zero independent (ZI)** if for all v , $w \in \Gamma^N$, and all $\beta \in \mathfrak{R}^n$,*

$$\{w(S) = v(S) + \sum_{i \in S} \beta_i, \forall S \subseteq N\} \Rightarrow \{\phi_i(w) = \phi_i(v) + \beta_i, \forall i \in N\}.$$

Under zero independence, if all of an agent's marginal contributions increase (or decrease) by the same constant, then so does his share. If we think in terms of utility, Property 4.8 says that if the zero of an agent's utility function shifts by

¹⁶Coalitional monotonicity deals with this eventuality.

some constant, his share of utility will also be shifted in the same direction and by the same constant.

We thus have one more independence property, which is implied by strong monotonicity. Indeed, the situation addressed by zero independence is a particular case of the conditions to which strong monotonicity applies, case in which all of an agent's marginal contributions increase by the same quantity.

However, neither does zero independence imply any of Properties 4.4 to 4.6, nor is it implied by any of them. This should be clear for the dummy axiom as well as for aggregate monotonicity, since the conditions for their application differ appreciably from those found in Property 4.8.

We still have to consider Property 4.6. We first note that the cases addressed by zero independence are a subset of those addressed by coalitional monotonicity. Hence, the former cannot imply the latter, but the latter still *may* imply the former. Whereas zero independence dictates by what quantity the shares change, coalitional monotonicity would allow the increase in agent i 's share to be anywhere between 0 and β_i . As a result, no implication holds between Property 4.8 and any other property introduced except for strong monotonicity.

Naturally, if we think that strong monotonicity is desirable, so too do we concerning zero independence. Still, without regards to strong monotonicity, in and of itself, Property 4.8 may or may not be desirable. The ultimate question is whether or not we want individuals to get the full reward for their extra effort in going from game v to game w . It is assumed here that any individual's extra effort translates into the same added marginal contribution for all coalitions he belongs to.

In these circumstances, since it is possible to link the improvement to the agents and to allocate it exactly among them, it seems natural to increase the shares accordingly, unless some redistribution is desired. This involves incentive considerations as well.

In general cases, we may prefer to enforce coalitional monotonicity and make sure that incentives to perform better (when possible) are present, rather than enforce zero independence which rules out redistribution, when we consider monetary quantities.

However, if concerned with utility values, then we will want zero independence to hold; otherwise, agents could change their zero of utility so as to advantage themselves. Besides, since utility functions are unique up to a positive linear transformation, any sharing rule which does not allocate identical shares for equivalent utility functions encourages misreporting.

For the pension plan surplus, we are dealing with money. It is not clear whether or not we should want to enforce zero independence. In other words, we have to determine whether or not we want redistribution in this particular case. In the end, this may depend on the particular type of pension plan and on the degree of redistribution already found in the formulation of the pension plan benefits.

Property 4.9 *A sharing rule is said to be **additive (ADD)** if, for all $v, w \in \Gamma^N$, $\phi(v + w) = \phi(v) + \phi(w)$.*

This property says that if we express a certain game as the sum of two games and if the rule is additive, any agent's share under the combined game is equal to the sum of his shares under the two separate games.

It may appear rather counterintuitive that the property should not hold in practice. The reasoning is that if two games add up to one game in terms of every single coalitional surplus, then so should they in terms of every surplus share. There do exist, however, sharing rules which are not additive.

Certainly, it is desirable that, whether two pension plans covering the same set of people be kept separate or combined into one, the surplus sharing rule applying to them be additive. Unfortunately, it is hardly ever the case that, when a pension plan is split or two are combined, the whole is exactly equal to the sum of the parts, and additivity then does not apply. On the one hand, when a plan is split, it is usually because the two newly created plans are to apply to two different groups, although transfers between groups may take place. On the other hand, when two plans are combined, most differences that existed between them are likely to be done away with as much as possible.

Before introducing the next property, we need to know what a reduced game is. Given a game $v \in \Gamma^N$, an allocation x , and a proper coalition $S \subset N$, the **reduced game on S at x** is denoted by (S, v_x) and is defined (in Peleg [49]) as follows:

$$v_x^S(S) = \sum_{i \in S} x_i,$$

$$v_x^S(T) = \max_{\emptyset \subsetneq R \subsetneq N \setminus S} \left\{ v(T \cup R) - \sum_{i \in R} x_i \right\}, \quad \forall T \subset S.$$

Therefore, in order to define a reduced game, we define the worth of the “grand” coalition to be its total share in the game v . Also, we define the worth of any sub-coalition to be the maximum worth it could achieve by joining with any of the

agents not included in the reduced game and giving them their original share (*i.e.*, their share in the game v).

Basically, the idea is, for the agents in the reduced game, to maximize their worths without changing the shares received by the agents outside the reduced game. This may be done with or without solicitation of the collaboration of agents in $N \setminus S$, collaboration which can be secured at the price indicated by x .

For all of the previous properties introduced, we only needed to define a sharing rule for a fixed size of N . However, we now need to redefine the concept of a sharing rule so that, not only does it apply to various TU games, but also to various sizes of N . That generalization is required so as to accommodate the change in size of the grand coalition in going from the original game to the reduced game.

We denote by Γ the set of all TU games (N, v) and, as usual, by n the size of N . We then define a **sharing rule** to be a mapping

$$\begin{aligned} \phi : \quad \Gamma &\rightarrow \cup_{n \in \mathcal{N}} \mathfrak{R}_+^n \\ (N, v) &\rightarrow \phi(N, v), \end{aligned}$$

where $\phi(N, v) \in \mathfrak{R}_+^n$; that is, one that associates to any game (N, v) an allocation $\phi(N, v)$ of the total surplus $v(N)$. As before, $\phi(N, v)$ satisfies the equality $\sum_{i \in N} \phi_i(N, v) = v(N)$.

From here on, we shall write $\phi(v)$ when N is clearly identified and fixed in size. Otherwise, we shall work with the more general notation $\phi(N, v)$.

Property 4.10 *A sharing rule ϕ , defined for TU games of any size, satisfies the reduced-game property (RGP1) if, for any game $v \in \Gamma^N$ and any coalition*

$S \subset N$.

$$\mathbf{x} = \phi(N, v) \Rightarrow \phi(S, v_{\mathbf{x}}^S) = \pi_S(\mathbf{x}),$$

where π_S is the projection of \mathfrak{R}^n onto \mathfrak{R}^s , s is the size of S , and $v_{\mathbf{x}}^S$ is the reduced game as defined by Peleg.

This property is very interesting, but also delicate to interpret. It says that, whether a subset of agents remains in the grand game or decides to play a reduced game in which it can buy the cooperation of the outside agents for the price of their share in the grand game, the share of each reduced game agent remains the same. A sharing rule satisfying that property is also said to be consistent.

Computationally, it may appear that this property allows us to focus on a subset of agents without really worrying about what happens to the other agents. However, it should be noted that in determining the worth of any coalition in the reduced game, we need to compute and compare 2^{n-s} values, where s denotes the number of agents in S . Moreover, we still need to know the shares assigned to the outside agents by \mathbf{x} . In the end, Property 4.10 creates more interest from the theoretical side than from the computational side.

It may also appear that the reduced-game property would allow us to concentrate first on subgroups, then on the members of a given subgroup. In other words, we may think that this property justifies our working with five subgroups rather than individual members. That is not quite the case. What the property actually means is that, if we computed all the *individual* shares separately, then gave a subgroup the sum of the shares we calculated for its members, and members of that subgroup then found their shares via the reduced game as defined by Peleg, these

members would end up with the same shares that were computed in the grand game originally.

Still, we should try to come up with some indication of its desirability. Say we are given a sharing rule that is to apply to the grand coalition as well as to any of its subsets in the context of a reduced game. If agents are allowed to negotiate not only as a whole but also as subsets, and if the cooperation of any agent outside a given subset can be secured for the price of his share in the grand coalition, we may want the sharing rule to be such that subsets have no incentive to form. Subsets have no incentive to form if the shares they determine for their members turn out to be the same as those determined in the grand coalition. This amounts to requiring that the reduced-game property holds.

Another interpretation of reduced games, which is somewhat simpler, is worth introducing here. We end up with a reduced game if some subset of agents make their contribution, take their share, and leave. In that case, the participation of any of the "outside" agents has already been secured at the price set by x . Since the overall game has been modified, we need to know whether or not we have to recalculate the shares of the people that remain. Recalculation is not necessary when the sharing rule satisfies the reduced-game property.

This new interpretation readily applies to the context of a pension plan. Should participants who contributed to the plan, received their share of the surplus as they were entitled to, and then quit the pension plan have an impact on the distribution of the surplus to the participants who remain in the plan? We believe that it should not be the case and that, in light of this, the reduced-game property is appropriate

and desirable.

Peleg's definition of a reduced game is not the only one given in the literature. We will have another reduced-game property if we adopt another definition of "reduced game," as given by Hart and Mas-Colell [25]. The alternative definition of the **reduced game on S , given ϕ** , is the following:

$$v^S(T) = \sum_{i \in T} \phi_i(T \cup (N \setminus S), v), \quad \forall T \subseteq S.$$

Unlike Peleg's definition which depends on the allocation x , this one depends on the sharing rule itself. As a result, cooperation of outside agents simply costs their total share according to the sharing rule as applied to the game involving the group of (inside and outside) agents that are to cooperate. We note that, under this definition, every subset of S seeks the cooperation of *all* agents outside S . Despite that simplification, the calculations for this type of reduced game still get very computationally intensive when the number of agents is large.

Using Peleg's definition, the total of the shares to be given to agents in S is determined by the allocation x . Using Hart and Mas-Colell's definition, this same total depends on the total share of $v(N)$ assigned to S according to the sharing rule. For both, the corresponding reduced-game property addresses the issue of redistribution of the total share within that subset. When the sharing rule used to calculate x is the same as the one used for the reduced game, both total shares for S are the same. Even then, the redistribution happening in the reduced game needs not be the same in both cases.

Property 4.11 *A sharing rule ϕ , defined for TU games of any size, satisfies the reduced-game property (RGP2) if, for any game $v \in \Gamma^N$ and any coalition*

$S \subset N$,

$$x = \phi(N, v) \Rightarrow \phi(S, v^S) = \pi_S(x),$$

where π_S is the projection of \mathbb{R}^n onto \mathbb{R}^s , s is the size of S , and v^S is the reduced game as defined by Hart and Mas-Colell.

4.4.4 Relationships between Properties

Before closing up this section on properties, we wish to list the properties introduced and highlight implications that exist between them. Table 4.1 gives the list of properties, along with their numbers and symbols, as they were introduced in the previous subsection. Out of these eleven properties, we have identified three (namely DA, AM and CM) as being particularly desirable in the context of a pension plan.

Table 4.1: List of Properties of Sharing Rules

Number	Name of property	Symbol
4.1	Individual rationality	IR
4.2	Stand-alone principle	SAP
4.3	Anonymity	AN
4.4	Dummy axiom	DA
4.5	Aggregate monotonicity	AM
4.6	Coalitional monotonicity	CM
4.7	Strong monotonicity	SM
4.8	Zero independence	ZI
4.9	Additivity	ADD
4.10	Reduced-game property (Peleg)	RGP1
4.11	Reduced-game property (Hart and Mas-Colell)	RGP2

Table 4.2 contains the properties that correspond to varying degrees of rationality, independence, or monotonicity. For each, we show which property implies which other one. Hence, in going from left to right, we go from more stringent to less stringent conditions to meet in order to satisfy the property.

Table 4.2: Implications between Properties

Property	Implications
Rationality	SAP \Rightarrow IR
Independence	SM \Rightarrow DA SM \Rightarrow ZI
Monotonicity	SM \Rightarrow CM \Rightarrow AM

These two lists shall come in handy as we, from now on, refer to these properties and determine whether or not they are satisfied. Other implications may exist between the properties, namely when two or more suffice to imply another one. However, these have not been included in the list. Instead, only those which are strictly weaker versions of another one have been singled out.

4.5 Possible Sharing Rules

Now that we have defined eleven properties that sharing rules may or may not satisfy, we are in a position to look at some of the sharing rules found in the literature. For each one, we will state which properties are satisfied.

Moreover, whenever possible, we will state which properties are necessary and sufficient to describe the sharing rule under consideration. These characterization results are particularly valuable in that they tell us that, if we want a certain set

of properties to hold, the only sharing rule available that meets these requirements is the one found in the result.

Some impossibility results are equally important. These tell us that certain properties are incompatible and, hence, that we must choose which subset of properties we wish to favor over the others. For example, it may be that a certain set of four properties is impossible to satisfy, whereas any subset of three can be satisfied by some sharing rule.

While some properties may not be satisfied by certain sharing rules for all TU games in Γ^N , they may be satisfied when restricting ourselves to a subset of games sharing a common characteristic. Consequently, we start this section by defining certain types of games. (Whenever no restrictions are made with respect to the type of game, we shall refer to such games as “general games” or, simply, “games.”)

4.5.1 Types of Games

Type of game 4.1 *A game is **superadditive** if and only if the worth of the union of any two disjoint coalitions is greater than or equal to the sum of their worths. In symbols, for any game $v \in \Gamma^N$, this translates to requiring that*

$$v(S \cup T) \geq v(S) + v(T), \forall S, T \in \mathcal{P}(N), S \cap T = \emptyset.$$

Games are not necessarily superadditive. In the event that a game is not, it means that, for some two disjoint coalitions, we have that their union is worth less than the two separately. For them, there is a natural deterrent to joining forces.

In the case of a pension plan, the larger the number of members, the smaller we anticipate the administration costs per member to be as the fixed costs get divided

between more people. Moreover, when the size of the fund is small, the investment policy is usually more conservative than when the fund is large. Hence, at least in expectation, the rate of return on the assets tends to increase with the number of participants. However, there may be periods when, in fact, conservative portfolios enjoy better returns than aggressive ones and, indeed, we then will have the opposite relationship between the rate of return and the number of plan members. Therefore, *on average*, we expect the net rate of return to increase with the size of the plan, but not for every single scenario.

As a result, we might expect that the game depicted by our model pension plan will be superadditive. However, we will see that it is not. For one thing, we use the same investment policy for all subgroups and combinations of subgroups, and hence we cannot count on gains expected from a more aggressive portfolio. We cannot count on economies of scale for the administration of the plan either since we do not take the associated costs into account. More importantly though, the criterion yielding the worth of each coalition is based on a percentile, and percentiles located in the tail of a distribution often behave quite differently from those closer to the center.

Type of game 4.2 *A game is **convex** if and only if, for any $i \in N$, and for all $S, T \subseteq N \setminus \{i\}$,*

$$\{S \subset T\} \Rightarrow \{v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)\}.$$

So to speak, convex games are games with increasing returns to cooperation. More precisely, the larger the coalition that an agent considers joining is, the larger his marginal contribution to that coalition would be (or at least the contribution

never gets smaller). Moreover, convex games always have a non-empty core. To convince ourselves of that, we simply have to come up with one allocation which, for any convex game, satisfies the stand-alone principle. We will consider the following:

$$x_i = v(\{1, \dots, i\}) - v(\{1, \dots, i-1\}), \quad i = 1, \dots, n, \quad (4.1)$$

where $v(\{1, \dots, i-1\}) = v(\emptyset)$, $i = 1$. These shares do satisfy the definition of an allocation. Moreover, they do satisfy the stand-alone principle whenever the game is convex. Since this allocation is in the core, the core is not empty, and this proves our claim.

Whereas it appeared reasonable to think that the game defined by a pension plan be superadditive, it does not seem as plausible that it be convex, at least not over the entire range of possible pension plan sizes. Of course, the game associated with our model pension plan and criterion cannot possibly be convex since it is not superadditive. However, for games defined by pension plans in general, returns to cooperation may be increasing as the plan gains in size up to a certain point, but then be decreasing for any further increase in size. This would come about since, on the efficient frontier¹⁷, the larger the risk already assumed, the smaller the increase in expected return for a fixed increase in the assumed risk (as given by the variance of the return) will be.

As mentioned earlier, certain properties may hold only in a particular subset of games. In particular, in the set of general TU games, it is impossible for a

¹⁷The efficient frontier, in the graph of the expected return versus its variance, identifies the minimum variance portfolio for each expected rate of return.

sharing rule to satisfy individual rationality since some games do not even satisfy the following inequality:

$$\sum_{i=1}^n v(\{i\}) \leq v(N),$$

which is a condition required for the *existence* of an individually rational allocation. For superadditive games, there always is at least one such an allocation. Then, in the set of such games, whether the rule is IR or not will depend on which allocation it picks.

Likewise, it is impossible for a rule to pick an allocation satisfying the stand-alone principle if none is feasible for a given game. We recall that such allocations form the core of a game. Hence, games with an empty core cannot possibly meet SAP. As a result, rules will not satisfy the stand-alone principle in the set of general or even superadditive games. Only the set of convex games will allow for the meaningful verification of that property.

At this point, we wish to emphasize that superadditivity is sufficient but not necessary for the existence of an IR allocation¹⁸. Hence, a sharing rule which is individually rational in the set of superadditive games *does not necessarily* pick an individually rational allocation whenever there is one available¹⁹. Likewise, the set of convex games does not contain all the games that have a core²⁰. Thus, a

¹⁸For example, take the three-person game for which all non-empty coalitions are worth 1 except $v(\{2, 3\}) = 2$ and $v(\{1, 2, 3\}) = 3$. This game is not superadditive but admits the IR allocation (1,1,1).

¹⁹Such a rule is the Shapley value (see Sharing rule 4.1), which would yield the allocation $(\frac{2}{3}, \frac{7}{6}, \frac{7}{6})$ in the above three-person example.

²⁰For example, take the three-person game for which $v(\{1, 2\}) = 27$, $v(\{1, 3\}) = 24$, $v(\{2, 3\}) = 21$, $v(\{1, 2, 3\}) = 37.5$, and all smaller coalitions have a worth of 0. The core contains (15,12,10.5), among others, but the game is not convex.

sharing rule which meets the stand-alone principle in the set of convex games *does not* necessarily select an allocation in the core whenever the core is not empty²¹. When the latter happens to hold, we say that the sharing rule is a core selection.

We will now define the four sharing rules we will apply in the next chapter. They are, in order of presentation, the Shapley value, the nucleolus, the proportional sharing rule, and the serial sharing rule.

4.5.2 Shapley Value

Sharing rule 4.1 (*Shapley [60]*) Given the set of agents N and a TU game $v \in \Gamma^N$, where n is the number of agents, the **Shapley value** ϕ^Y allocates $v(N)$ as follows:

$$\phi_i^Y = \sum_{S:i \notin S} \frac{(n-s-1)! s!}{n!} [v(S \cup \{i\}) - v(S)], \forall i \in N.$$

In the formula, s denotes the size of the coalition S and, by convention, $0! = 1$ and $v(\emptyset) = 0$.

Upon looking at the formula, we recognize the Shapley value as a weighted average of the individual's marginal contributions. The weight assigned to the marginal contribution of i to coalition S is the probability that, in a random ordering of the elements of N , the elements of S come first (in any order), then i , and then the elements of $N \setminus (S \cup \{i\})$ (again, in any order).

For general games, the Shapley value is not individually rational, let alone satisfies the stand-alone principle. Because of its very definition, however, the Shapley

²¹Again, we take the Shapley value. In the last example given, it would yield (14,12.5,11), which does not lie in the core.

value satisfies anonymity, strong monotonicity, zero independence and additivity. It does not satisfy the reduced-game property based on Peleg's definition, but it does satisfy the reduced-game property based on Hart and Mas-Colell's definition.

For superadditive games, the Shapley value is individually rational, but still fails to satisfy the stand-alone principle. For convex games, it does satisfy the stand-alone principle.

Since this sharing rule possesses, for general games, so many of the properties we have introduced, one may wonder if there exists one which possesses them all. As a matter of fact, it turns out that no sharing rule can satisfy both Properties 4.2 and 4.6, let alone all of the eleven introduced.

Impossibility 4.1 (*Young [74]*) *Given N , with $n \geq 5$, no core selection satisfies coalitional monotonicity.*

This impossibility result also holds for superadditive games. However, we no longer have that problem when we restrict ourselves to convex games. Indeed, as was remarked earlier, in the subset of convex games, the Shapley value picks allocations *in* the core.

If we wish to put more emphasis on the core property, we then need to look for other sharing rules, the most well-known of which is the nucleolus (see Sharing rule 4.2 on page 152). That would not be necessary if the game described by the pension plan surplus were convex, but we have already indicated that the games in our application would not be convex.

As was just noted, the Shapley value satisfies many of the properties introduced. Nonetheless, only a few of these properties are needed to characterize it.

Characterization 4.1 (Young [74]) *There is only one anonymous and strongly monotonic sharing rule. It is the Shapley value.*

This characterization result may appear surprisingly concise at first. Insofar as only two properties are involved, it is indeed. However, one ought to recall that strong monotonicity is a very strong property.

Characterization 4.2 (Shapley [60]) *There is only one sharing rule satisfying anonymity, additivity, and the dummy axiom. It is the Shapley value.*

This second characterization of the Shapley value uses the dummy axiom, which is a weak version of strong monotonicity. We will introduce a third and last one, although some others can be found in the literature.

Characterization 4.3 (Hart and Mas-Colell [25]) *There is only one sharing rule satisfying anonymity, zero independence, and the reduced-game property as defined by Hart and Mas-Colell [25]. It is the Shapley value.*

4.5.3 Nucleolus

In order to define the nucleolus, we need to define the **leximin ordering**. For any two vectors x and z , denote by x^* and z^* the corresponding ordered vectors in which the coordinates have been sorted according to increasing size. We then say that x and z are *leximin indifferent* if $x^* = z^*$. Also, we say that x is *leximin preferred* to z if there exists an integer $k < n$ such that $x_i^* = z_i^*, \forall i \leq k$, but $x_{k+1}^* > z_{k+1}^*$.

The leximin ordering firstly concerns itself with the elements that are worst off, then the next worst off, and so on. Underlying it is a search for equality. This

ordering is at the heart of the following definition.

Sharing rule 4.2 (*Schmeidler [58]*) *Given are the set of agents N and a TU game $v \in \Gamma^N$, where n is the number of agents. We denote by \mathcal{B} the set of allocations $x \in \mathfrak{R}_+^n$. We then define the excess vector $e(x) \in \mathfrak{R}^{2^n - 2}$ associated with the allocation $x \in \mathcal{B}$ as*

$$e(x; S) = \sum_{i \in S} x_i - v(S), \forall S \subset N, S \neq \emptyset.$$

*The **nucleolus** is the unique allocation $\phi^N \in \mathcal{B}$ such that, for every other $x \in \mathcal{B}$, $e(\phi^N)$ is lexicimin preferred to $e(x)$.*

As mentioned earlier, the nucleolus is a core selection²². As a matter of fact, it lies at the center of the core whenever it is not empty. (Here, we mean “center” in the physical sense. Hence, we consider the core as a hyperplane made of matter having uniform density and find its center of gravity.) This centrality comes about because of the egalitarian distribution of the excesses among all coalitions. The excesses measure the benefit enjoyed by each coalition S beyond its own opportunity surplus $v(S)$.

Equality of excesses should in no way be confused with equality of shares. It all depends on the location of the core. In fact, the Shapley value usually leads to shares which lie closer to the center of the simplex than those of the nucleolus.

For general games, the nucleolus is not individually rational, but it is anonymous. It satisfies the dummy axiom, but not aggregate monotonicity²³. It is also

²²We remind the reader that a sharing rule is defined to be a core selection if it picks an allocation in the core whenever the core is *not* empty.

²³A counterexample with $n = 9$ can be found in Megiddo’s article [38].

not additive. It does however enjoy zero independence and the reduced-game property based on Peleg's definition. However, it does not satisfy the reduced-game property based on Hart and Mas-Colell's definition since the Shapley value is the only anonymous and zero independent sharing rule that does.

Out of the eleven properties introduced, the nucleolus satisfies amazingly few of them in the general case. In fact, out of the four it has, three are required to characterize it.

Characterization 4.4 (Sobolev [63]) *There is only one sharing rule satisfying anonymity, zero independence, and the reduced-game property (as defined in Property 4.10). It is the nucleolus.*

This characterization result is arguably very restrictive, since the nucleolus boasts of only one more out of the other properties introduced. That the nucleolus is not individually rational while it is a core selection may come as a surprise. To address this apparent contradiction, we note that a core selection prescribes an allocation which is individually rational *whenever the core is not empty*, but not necessarily so in the case of an empty core.

Just as was the case for the Shapley value, the nucleolus also is individually rational if we restrict ourselves to superadditive games. Moreover, in the subset of convex games, the nucleolus meets the stand-alone principle, simply because convexity implies a non-empty core.

Even in the case of convex games, the nucleolus is not monotonic in the aggregate. That appears quite undesirable. There exists a modified nucleolus, called the **per capita nucleolus**, which, in addition to being a core selection, is monotonic

in the aggregate. Instead of considering the excesses, it looks at the per capita excesses $\bar{e}(x: S) = \frac{e(x; S)}{s}$, and performs the leximin ordering over the latter.

As for additivity, it still fails in the case of superadditive games as well as in that of convex games. This may come as a surprise because of the centrality of this allocation inside the core. However, in the case of games with a non-empty core, the non-additivity comes from adding games which do not have the same set of coalitions defining the limits of their core.

4.5.4 Proportional Sharing Rule

Both of the next two rules to be introduced have been defined for and applied to the problem of splitting the costs of a single homogeneous good. This problem has the particularity that quantities demanded by each agent are known. As a result, we have to modify the definitions found in the literature for our surplus sharing problem.

Sharing rule 4.3 (adapted from Moulin and Shenker [40]) Given the set of agents N and a TU game $v \in \Gamma^N$, where n is the number of agents, denote by c_i agent i 's contribution and by c_N the sum of the contributions. The **proportional sharing** rule ϕ^P allocates $v(N)$ as follows:

$$\phi_i^P(v) = \frac{c_i}{c_N} v(N).$$

By “contributions,” we mean the inputs to the activity generating the surplus. Typically, they are the amounts invested by each of the agents in the venture.

The proportional sharing rule (also called average sharing rule in the literature) is anonymous and satisfies the dummy axiom. It is also coalitionally monotonic

since the only way to increase the share of an agent, when c is fixed, is by increasing $v(N)$. Hence, in this case, unless $v(N)$ goes up, all shares remain unaffected by a change in the characteristic function. It is not strongly monotonic though, since the Shapley value is the only anonymous sharing rule that is.

Whether or not the proportional sharing rule is considered to be additive depends on what we assume with respect to the contributions when combining games. If we assume that the contributions are the same, which would require a redefinition of Property 4.9, then the rule is additive. Otherwise, whatever the set of games, the rule is not additive. It is not zero independent either. However, it always satisfies both reduced-game properties.

As has been the case so far, the proportional sharing rule is not individually rational for general games. In fact, neither superadditivity nor convexity suffices to ensure individual rationality. That is a major drawback to the proportional sharing rule if agents are free to withdraw their contributions.

There exist a few characterizations of the proportional sharing rule (see Moulin and Shenker [40, 41, 42]). Unfortunately, they are all formulated with respect to games corresponding to the production of a single homogeneous good. Besides, they call on properties which make sense only for that particular kind of game and are not merely refinements of existing general properties.

4.5.5 Serial Sharing Rule

Sharing rule 4.4 (adapted from Moulin and Shenker [40]) *Given the set of agents N and a TU game $v \in \Gamma^N$, where n is the number of agents, re-label the agents 1 to*

n in increasing order of marginal contributions to the whole surplus²⁴. The **serial sharing** rule ϕ^C allocates $v(N)$ as follows:

$$\begin{aligned}\phi_1^C(v) &= \frac{v(\{1, 1, \dots, 1\})}{n}; \\ \phi_i^C(v) &= \phi_{i-1}^C(v) + \frac{v(\{1, 2, \dots, i, \dots, i\}) - v(\{1, 2, \dots, i-1, \dots, i-1\})}{n+1-i}, \forall i > 1.\end{aligned}$$

There is a very nice interpretation of the serial sharing rule, particularly if we consider the case of a project in which agents invest different amounts. Under that rule, each agent i receives a share based on all agents' contributions up to what i himself contributes. In other words, in determining agent i 's share, we calculate the surplus based on the actual contributions for agents with index less than or equal to i , and on contributions equal to those of i for agents with index greater than i . That is what is meant by the notation $v(\{1, 2, \dots, i, \dots, i\})$.

For general games, the serial sharing rule is not individually rational. However, we do not know whether superadditivity would suffice to enforce individual rationality and, similarly, if convexity would be enough to satisfy the stand-alone principle.

This sharing rule is anonymous and also satisfies the dummy axiom and aggregate monotonicity. For these three properties, a careful look at the formula should be convincing. However, through counterexamples, one can show that it is not coalitionally monotonic.

²⁴This ordering requires that, for any $i = 1, \dots, n$, assuming we have identified the $i-1$ preceding agents, we find for which agent j (out of the $n-i+1$ not ordered yet) we get the smallest $v(\{1, 2, \dots, i-1, j, \dots, j\})$.

Just as was the case for the proportional sharing rule, the serial sharing rule is additive when we consider the contributions fixed; it is not additive in general. It is not zero independent. As for the reduced-game properties, we could not establish if it satisfied either, both or none.

Just as in the case of the proportional sharing rule, characterizations of the serial sharing rule found in the literature are based on properties not introduced here, some of which are particular to the sharing of production costs of one good (see Moulin and Shenker [40, 41, 42]).

4.5.6 Summary of Properties Enjoyed by Sharing Rules

To simplify the comparison between the four sharing rules introduced, we simply wish to summarize, in the following three tables, the properties that are satisfied by the sharing rules, for general, superadditive and convex games, respectively. A check mark indicates that the property is satisfied; a blank, that it is not; and a question mark, that we do not know.

Table 4.3: Summary of Properties of Sharing Rules for General Games

Sharing rule	IR	SAP	AN	DA	AM	CM	SM	ZI	ADD	RGPI	RGPI2
Shapley value			✓	✓	✓	✓	✓	✓	✓		✓
Nucleolus			✓	✓				✓		✓	
Proportional sharing			✓	✓	✓	✓				✓	✓
Serial sharing			✓	✓	✓					?	?

The choice of a particular rule, for a given surplus-sharing game, can be made

based on these three tables. Preference of some properties over others will be the key determinant. Of course, other properties than the ones looked at exist in the literature and may be considered. Besides, incompatibility of certain properties will also have to be taken into account.

Table 4.4: Summary of Properties of Sharing Rules for the Subset of Superadditive Games

Sharing rule	IR	SAP	AN	DA	AM	CM	SM	ZI	ADD	RGP1	RGP2
Shapley value	✓		✓	✓	✓	✓	✓	✓	✓		✓
Nucleolus	✓		✓	✓				✓		✓	
Proportional sharing			✓	✓	✓	✓				✓	✓
Serial sharing	?		✓	✓	✓					?	?

Table 4.5: Summary of Properties of Sharing Rules for the Subset of Convex Games

Sharing rule	IR	SAP	AN	DA	AM	CM	SM	ZI	ADD	RGP1	RGP2
Shapley value	✓	✓	✓	✓	✓	✓	✓	✓	✓		✓
Nucleolus	✓	✓	✓	✓				✓		✓	
Proportional sharing			✓	✓	✓	✓				✓	✓
Serial sharing	?	?	✓	✓	✓					?	?

Based on the preferences we have expressed for the dummy axiom and coalitional monotonicity, the Shapley value and proportional sharing rule come out as the potentially most interesting sharing rules to apply to our pension plan. Insofar as the former satisfies more properties and provided we do not consider any of them to be undesirable, the Shapley value actually appears to be the most promising. We

also note that, if we dealt with superadditive or convex games, the Shapley value would gain some rationality property whereas the proportional sharing rule would not; in general, this represents a major shortcoming for the latter.

4.6 Ease of Computation of Sharing Rules

While properties represent a theoretical concern, there are practical issues to consider as well when choosing a sharing rule. Hence, not only may the sharing rule itself be hard to apply, but it may require a high number of inputs (worths). We will give an idea of the computational complexity for each sharing rule.

Both the Shapley value and the nucleolus require that we know the worths of all $2^n - 1$ coalitions. Since this number grows exponentially, unless the problem has a simple structure, the calculations will quickly become prohibitively numerous.

Whereas the computation of the Shapley value is relatively straightforward, that of the nucleolus is quite difficult (see, for example, Owen [44, p. 331]). Formulae easy to apply are available only for special cases or for very small sizes of N .

One hardly can imagine a sharing rule which requires fewer computations than the proportional sharing rule. Indeed, to apply it, we only need to identify n contributions and the worth of the grand coalition. The calculation of the shares is also easy to perform.

As for the serial sharing rule, the number of inputs needed is only n . However we must realize that we actually have to compare $\frac{n(n+1)}{2} - 1$ worths in total in order to come up with the n worths we need. Calculating the shares using this rule is very simple though.

So, if we focus on computations rather than on properties, the proportional sharing rule comes out as the clear winner²⁵. Hence, for particularly large problems, its shortcomings in terms of the actual properties satisfied may be more than compensated by the particularly simple computations that it requires.

Since we consider only five subgroups in our application, computations are not a big concern, except when solving for the nucleolus. Actually, we picked that number of subgroups in part because it provided for a manageable problem.

4.7 Conclusion

We have started this chapter with a discussion of the entitlement of each of our five subgroups to a share in the disposable surplus. The right of the sponsor is, by far, the most subject to debate because of the opposing views that exist about the nature of its participation in the plan.

Then, we have introduced cooperative game theory as the tool we would use to determine the shares of each of them. Finding a way to allocate the surplus actually was the main objective of this chapter.

We have identified a number of properties sharing rules may enjoy. At the same time, we gave their meaning and desirability both in general and in the particular context of a pension plan.

Afterwards, we defined four sharing rules, indicating which properties they satisfied and how easy they were to compute. We will be using these four sharing rules in the next chapter as we apply the tools we developed to our model pension plan.

²⁵In fact, any simple rule, such as equal shares or weighted shares, would be as easy to compute.

Assuredly, these four sharing rules are not the only ones found in the cooperative game theory literature. Hence, others could have been explored as well. We simply chose to focus on the two most popular ones (Shapley value and nucleolus), a very simple one (proportional sharing) and a relatively new one (serial sharing).

Besides, other rules could have been found in another branch of economics, non-cooperative game theory. As the name of the theory indicates, instead of allowing and encouraging cooperation among the agents, it prohibits it. Whereas the pension plan is not an actual non-cooperative game, though the court cases may lead us to think otherwise, it still may be valuable to explore some of the possibilities offered by this other theory in the future.

Yet another branch of economics may provide a valuable alternative: bargaining theory, under the axiomatic approach. Under this theory, the agents discuss to reach an agreement; if they fail to, the default agreement (disagreement event) is enforced. We provide a quick introduction to this theory in Appendix E and we will come back to it in Chapter 5 when we discuss the initial asset allocation used to find the disposable surplus for subgroups.

NOTE TO USERS

**Page(s) missing in number only; text follows.
Microfilmed as received.**

162

This reproduction is the best copy available.

UMI

Chapter 5

Application

5.1 Introduction

In order to assess the potential interest and validity of our suggestions, we will apply them to our model pension plan. We gave the plan characteristics in Chapter 2 and must remind the reader that it is simple compared to a real, full-fledged pension plan. In that regard, the contents of this chapter should be considered as illustrative only. Moreover, we stress the fact that, even though we will apply our suggestions only once, they really are meant to be applied at every valuation.

We will begin our study of the plan in 1965, assuming it is then mature and with a funding level of 150%. We admit that such a funding level is unrealistic, since plans hardly were funded at the time. In fact, Ontario and Quebec were the first two jurisdictions to adopt an act regarding pension plans in 1965 and 1966, respectively. Before these laws were adopted, the only funding requirement consisted in the purchase of the promised annuity at retirement. We will study the

past evolution of this plan until 1986, thus going over 7 triennial valuations. We selected this year because of the surplus that then emerges¹.

We will start by establishing the initial size of each of the five subgroups we consider. We will then look at the evolution of the assets for one member joining at age 25 in 1965. We will also go over the evolution of the assets for the whole plan over the 21-year period.

Having surveyed the past of our pension plan, we will then simulate its evolution over the next three years. For each subgroup and combination thereof, we will use the criterion developed in Chapter 3 to determine its disposable surplus. Afterwards, we will apply the sharing rules defined in Chapter 4 to determine each subgroup's share. We will be calculating the shares with three different treatments of the sponsor: First, we will determine its share independently from the others by making it equal to the ratio of its accumulated contributions to the total contributions. Second, we will force shares of coalitions missing the sponsor to be nil. Third, we will exclude the sponsor from the sharing.

We will repeat this whole exercise for each of the three ways of allocating assets to subgroups (accumulated fund, liabilities, liabilities plus share of surplus). This will allow us to see the impact of the initial asset allocation on the eventual surplus allocation. Given the large influence it has, we will try to justify the use of one over the others. We will even examine the possibility of using game theory to come up

¹For our plan, since decrements are assumed to be deterministic, a surplus can only emerge because of favorable economic experience. However, surpluses that mushroomed in the 1980s did so not only because of the high returns from 1982 to 1985, but also because of the increased layoffs at the beginning of the decade due to the unfavorable economic conjuncture [48]. At that time, prior to legislation reforms, vesting requirements were harder to meet and, thus, most people that left contributed to increasing the surplus.

with the initial asset allocation.

We will also repeat the exercise if we instead consider a ten-year horizon. We will do so with both the original target probability of a deficit and with a probability accounting for the corresponding number of three-year periods. This will show us the impact of the chosen horizon on the size of the disposable surplus.

Unless indication to the contrary is given, we will use neutral starting values in our economic models for all simulations. However, we will also use actual 1985 values as inputs to the models so as to see how sensitive these, and the resulting shares, are to different starting values.

Table 5.1 shows the characteristics of all the different examples we will consider in this chapter along with the corresponding section numbers. In each of these sections, we will apply the three different treatments of the sponsor in three different subsections, and we will use as many of the four sharing rules introduced as possible in each of these subsections.

Table 5.1: Examples Included in Chapter 5

Section	Initial Asset Allocation	Target Probability of a Deficit	Horizon	Starting Values
5.5	Prop. liab.	5%	3	Neutral
5.6	Equal liab.	5%	3	Neutral
5.7	Acc. fund	5%	3	Neutral
5.9	Prop. liab.	5%	10	Neutral
5.10	Prop. liab.	15%	10	Neutral
5.11	Prop. liab.	5%	3	Real
5.12	Prop. liab.	15%	10	Real

Throughout the presentation of the various results obtained, we will comment on the various treatments of the sponsor as well on the different sharing rules. By the end of the chapter, based on these observations, we will make recommendations as to which treatment to adopt and which sharing rule to apply.

Finally, noting that our criterion for the determination of the disposable surplus makes the implicit assumption that all shares will be paid in cash, we will compute the actual probability of a deficit if the members' shares are paid as benefit enhancements instead. As we will see, the mode of distribution greatly impacts the potential for future shortfalls.

5.2 Stationary Plan Membership

We assume we have 100 entrants aged 25 every year. Using the fixed decrements provided in Appendix A, we find that, on January 1, 1986, the subgroups have the initial sizes given in Table 5.2. The table also gives the relative size of each subgroup of members.

Table 5.2: Initial Subgroup Sizes

Subgroup	Size	Percent
Sponsor	1	-
Active lives	1359	28.04%
Disabled, vested lives	88	1.82%
Disabled, retired lives	108	2.23%
Retired lives	275	5.67%
Terminated, vested lives	1995	41.16%
Terminated, retired lives	1022	21.08%

Through time, the number of people with a given status remains the same. As for the actual subgroups that we use, we indeed consider only five different ones. We have shown the split of disabled and terminated lives between those having a deferred pension and those currently receiving retirement benefits simply to give a rough idea of the inner composition of these two groups.

Besides, this split suggests that four subgroups could have been used instead of five: active, sponsor, vested inactive, and retired (*i.e.*, receiving pension benefits). Nonetheless, we will use our five subgroups because their members naturally fall into different categories which are treated differently according to the plan document.

Except for the subgroup that we labeled as the sponsor's, which all future new entrants join, all other subgroups decrease in size over the projection horizon. Nevertheless, each time we would repeat the surplus sharing exercise, at each actuarial valuation for example, we would start afresh with groups of that same initial size (though with different people in them).

At this point, we simply wish to remind the reader that the subgroup identified as the sponsor's really is the subgroup for those who will be hired between the two valuation dates. This subgroup is somewhat of a theoretical construct so that we do not ignore the people that will join the plan in the future and indeed work with the pension plan as an ongoing concern. We will actually account for the sponsor's role in the plan in more than one way and not only via this subgroup. It is worth noting however that this subgroup will allow us to find the sponsor's share along with the four others, rather than separately.

5.3 Evolution of Assets for One Member

To illustrate the evolution of assets for individual members, we will take someone aged 25 in 1965 and show how his accumulated contributions AC_x^t and accrued liability $AL_x^{(A),t}(1)$ evolve up to 1986. (Chapter 2 provides all the required formulae.) We assume that employees contribute 2.9% of their salary each year, and that the sponsor pays the difference between the normal cost and that percentage. As before, we have an 8%/5% valuation basis, which we keep constant throughout.

Salaries likely appear very low. We assumed $Sal_{65}^{1924} = 1000$ as the initial value on which we based all others. Though it may not be realistic, it is not material to the analysis since changing this assumption would inflate or deflate all the other results accordingly.

We recall that accumulated contributions and accrued liability are determined at the very beginning of the year. In Table 5.3, the normal cost is expressed as a percentage of salary, unlike in Equation 2.28 where it was the actual contribution.

We note that the normal cost quickly rises with age. In fact, it reaches 27.410% for someone aged 69. In comparison, the 2.9% employee contribution rate appears ridiculously low². We picked that percentage simply to avoid having the employee contribute more than the normal cost. There are funding methods which would avoid this sharp increase in normal cost: these methods allocate the cost over time so that, as a percentage of salaries, it remains constant.

We also note that no salary is given for the year 1986. At the time of the

²Actually, the overall contribution rate (total normal cost divided by salary roll) is 6.2%. In that regard, a 2.9% employee contribution rate does not appear so low.

Table 5.3: Accumulation of Contributions for Someone Aged 25 in 1965

Year	Age	Normal Cost in %	Salary	Accumulated Contributions	Accrued Liability
1965	25	2.911659	2,733.44	0.00	0.00
1966	26	2.900558	2,906.92	82.37	83.90
1967	27	3.117710	3,128.78	162.86	191.07
1968	28	3.355775	3,383.45	280.19	331.84
1969	29	3.602334	3,630.28	430.75	513.99
1970	30	3.850494	3,999.51	543.55	736.01
1971	31	4.101659	4,482.89	706.62	1,035.11
1972	32	4.358730	4,891.03	1,001.81	1,438.10
1973	33	4.619742	5,294.17	1,438.55	1,903.98
1974	34	4.884329	6,078.96	1,647.78	2,442.59
1975	35	5.154834	7,044.98	1,697.72	3,265.46
1976	36	5.432762	7,933.15	2,332.92	4,385.03
1977	37	5.717619	8,680.05	3,106.63	5,669.11
1978	38	6.009327	9,355.74	3,916.38	7,060.61
1979	39	6.309070	10,290.42	5,083.20	8,611.83
1980	40	6.616970	11,614.56	6,592.30	10,651.23
1981	41	6.931609	13,105.15	8,685.78	13,444.44
1982	42	7.252318	14,539.64	9,747.68	16,795.95
1983	43	7.578694	15,858.73	13,243.43	20,674.69
1984	44	7.910632	16,559.19	17,334.37	24,910.51
1985	45	8.247060	17,427.86	20,359.59	28,559.55
1986	46	8.586370		26,940.94	33,048.58

actuarial valuation, we assume that it is not known. Rather, it will depend on the wage increase obtained in the simulation of the future.

5.4 Past Evolution of the Pension Plan

Before we perform simulations of the future evolution of the pension plan, we need to know its current situation. As mentioned earlier, we assume 150% funding in 1965. (Again, here, Chapter 2 contains all the formulae needed to generate the numbers.)

Table 5.4 gives all the payments in and out of the fund in each year. (We notice that, were it not for the assets already in the fund that generate investment income, the monetary shortfalls quickly would lead to an unmanageable deficit.) This table allows us to track the evolution of the assets. Technically, the liabilities would be calculated only once every three years. Yet, we have calculated them at the beginning of every year. That does not change the fact that the amortization payment, if necessary, is determined only once every three years and remains constant over that period. Finally, the salary roll is given to provide an idea of the relative importance of the pension plan. Besides, if we multiply this figure by 2.9%, we obtain the employee contributions.

As we can see from the table, even though the initial surplus is quite large, it eventually is eaten away because of the economic outcomes that fall short of our expectations under the assumptions we made. The plan eventually runs a deficit from 1975 to 1982. Hence, though the initial funding level was questionably high, it actually only delayed the emergence of the unfunded liability. The situation then

Table 5.4: Evolution of the Plan from 1965 to 1986

Year	Normal Cost	Amortization Payment	Retirement Benefits	Death Benefits	Assets	Liabilities	Salary Roll
1965	264,400	0	588,123	32,721	17,190,208	11,460,139	4,262,333
1966	278,863	0	612,836	32,912	17,424,091	12,052,566	4,495,482
1967	297,769	0	639,066	34,820	16,664,132	12,627,169	4,800,276
1968	319,477	0	667,310	37,085	17,528,550	13,321,791	5,150,216
1969	339,668	0	697,896	37,295	18,758,620	14,099,386	5,475,710
1970	369,898	0	730,685	38,624	17,774,284	14,857,653	5,963,048
1971	409,367	0	766,795	42,068	17,601,248	15,845,599	6,599,306
1972	441,052	0	807,264	46,829	19,357,229	17,073,371	7,110,092
1973	471,396	0	851,272	47,417	22,438,535	18,220,134	7,599,266
1974	534,375	0	898,669	46,279	21,548,010	19,344,881	8,614,528
1975	611,431	0	952,992	51,478	18,447,104	21,120,295	9,856,743
1976	679,912	0	1,015,801	56,704	20,443,997	23,316,182	10,960,699
1977	734,577	315,555	1,086,208	61,445	22,544,809	25,461,878	11,841,939
1978	781,663	315,555	1,162,731	68,028	24,405,547	27,434,162	12,601,007
1979	848,886	315,555	1,244,535	75,729	27,557,911	29,353,350	13,684,694
1980	946,084	22,850	1,333,791	85,599	31,523,761	31,734,999	15,251,591
1981	1,054,032	22,850	1,433,769	89,210	36,681,907	34,808,065	16,991,798
1982	1,154,587	22,850	1,545,667	103,852	36,817,010	38,164,383	18,612,816
1983	1,243,374	0	1,668,712	118,531	44,582,352	41,637,102	20,044,141
1984	1,281,919	0	1,801,647	127,499	52,869,886	45,030,036	20,665,510
1985	1,332,298	0	1,939,020	147,943	57,038,874	47,519,223	21,477,664
1986					69,602,196	50,390,312	

goes back to a surplus and we begin 1986 with a 138% funding level and a valuation surplus of \$19,211,884. It will be interesting to see, out of that amount, how much we actually end up distributing.

5.5 Surplus Sharing with a 3-Year Horizon and Assets Allocated in Proportion to Liabilities

As we conduct our first surplus sharing exercise, we will try to show clearly how we go about it. In the subsequent sections, we will present the results with much less introduction except that differences with this one will be highlighted.

We first need to find the disposable surplus for each and every combination of subgroups as required for the application of the different sharing rules. We then use these as inputs for the sharing rules under different assumptions regarding the sponsor's share.

5.5.1 Disposable Surplus for Each Combination of Subgroups

In Chapter 3, we developed a criterion to determine the amount of disposable surplus. We provided all the related formulae, both for the plan as a whole and for subgroups of the plan. We will apply this criterion using a three-year horizon and targeting a 5% probability of a deficit. Hence, for each subgroup and combination of subgroups, we will find the amount we can give away, if any, so that the probability of an unfunded liability in three years' time is no more than 5%.

All the results in this chapter are based on 10,000 simulations of possible economic scenarios³. We generated *one* set of 10,000 scenarios, using the models given in Chapter 3, and used the same set for each subgroup and combination of subgroups. For each subgroup or combination thereof, we projected assets and liabilities using the formulae given in Chapter 3. We then ordered the resulting surpluses and picked the 5th percentile as the disposable one. If the 5th percentile turned out to be negative, we actually faced a probability greater than 5% that there be an unfunded liability at the end of the projected horizon and we then would consider the disposable surplus to be zero. Since the same scenarios were used in each case, differences between subgroups cannot be due to differences in scenarios; they can only be due to the subgroups' characteristics.

As noted in Chapter 3, when working with subgroups, we must allocate the assets in some fashion among them. In this first application, we endow them with assets in proportion to their liabilities; in other words, we give them an initial funding level equal to that for the whole plan. This assigns no assets to the sponsor's subgroup. Table 5.5 contains the resulting asset allocation.

The Shapley value and the nucleolus require the knowledge of the disposable surplus for each of the thirty-one non-empty combinations of subgroups. These values are given in Table 5.6⁴. Along with the disposable surplus we found for each combination of subgroups, we provide a confidence interval. Since the disposable surplus actually is the 5th percentile if we rearrange the simulation results in

³This number of simulations may appear quite small. Unfortunately, limited computer resources did not allow us to perform more at the time.

⁴They also are reproduced in Appendix F, which actually contains all the characteristic functions and tables of shares underlying the figures to be presented in this chapter.

Table 5.5: Assets Allocated by Subgroup, 3-Year Horizon, Initial Assets Proportional to Actuarial Liability

Subgroup	Assets
Active lives	33,871,795.43
Disabled lives	7,355,306.44
Sponsor	0.00
Retired lives	14,930,744.29
Terminated lives	13,444,350.41

increasing order, we can use the results found in David's book [18] in order to find a confidence interval for it.

Hence, if n is the number of simulations, a 95% confidence interval around the $100p$ -th percentile (the np -th ordered value) is given by the interval between the $(np - 1.96\sqrt{np(1-p)})$ -th and $(np + 1.96\sqrt{np(1-p)})$ -th ordered values. As a result, as the number of simulations increases, the confidence interval around the desired percentile tightens. Indeed, if we ran an infinite number of simulations, we would end up with the actual probability density function of the surpluses and we would know the value of every percentile exactly.

We must provide some indication as to how to interpret the binary numbers found in the first column of Table 5.6. Whereas a "0" indicates that the subgroup in that position is not included, a "1" indicates it is. As to the subgroups' positions, we have the active lives first, the disabled lives second, the sponsor third, the retired lives fourth, and the terminated lives last. Hence, the number "01010" refers to the combination of the disabled and retired subgroups, while the number "10100" refers to the active lives together with the sponsor.

Table 5.6: Disposable Surpluses, 3-Year Horizon, Initial Assets Proportional to Actuarial Liability

adsrt	Disposable Surplus	95% Confidence Interval	
		Lower Bound	Upper Bound
10000	1,511,303.02	1,266,853.96	1,746,289.06
01000	715,865.45	684,749.34	768,171.38
00100	-21,368.78	-22,001.29	-20,889.85
00010	1,700,242.26	1,623,802.37	1,794,048.15
00001	1,136,876.27	1,073,895.25	1,223,248.27
11000	2,333,133.25	2,048,393.50	2,630,603.79
10100	1,501,812.94	1,248,363.24	1,734,584.79
10010	3,396,443.94	3,121,538.34	3,697,320.60
10001	2,804,297.04	2,556,165.13	3,123,938.51
01100	704,718.50	672,110.02	756,316.37
01010	2,399,953.36	2,303,700.17	2,569,377.76
01001	1,861,692.31	1,754,348.11	2,001,247.88
00110	1,686,272.85	1,611,167.19	1,783,028.75
00101	1,132,651.40	1,063,849.36	1,210,004.97
00011	2,822,144.84	2,701,031.08	3,025,800.40
11100	2,320,151.51	2,030,613.29	2,618,227.57
11010	4,230,124.85	3,797,976.60	4,494,821.90
11001	3,642,842.65	3,293,700.64	3,933,229.18
10110	3,380,904.20	3,105,982.31	3,684,435.91
10101	2,791,872.28	2,542,087.73	3,113,433.42
10011	4,673,601.81	4,277,510.06	4,973,816.04
01110	2,397,570.14	2,296,584.34	2,556,244.45
01101	1,856,158.08	1,743,355.05	1,985,887.54
01011	3,536,815.22	3,385,569.59	3,792,267.55
00111	2,811,976.99	2,690,201.09	3,017,675.56
11110	4,213,575.21	3,795,606.02	4,483,268.41
11101	3,626,969.37	3,290,834.29	3,918,778.61
11011	5,392,042.47	4,974,599.25	5,825,589.44
10111	4,657,659.32	4,261,008.18	4,957,487.31
01111	3,526,234.98	3,376,340.69	3,785,846.94
11111	5,361,093.61	4,967,671.27	5,808,146.81

Not only is the game defined by these worths not convex, it is not even super-additive. This may come as a surprise since, for a given scenario, the subgroups' results are additive⁵. However, we must realize that percentiles are not. Indeed, different scenarios lead to the smallest values for the different subgroups and combinations. For instance, whereas subgroups that still contribute are vulnerable to high wage inflation combined with low rates of return, subgroups that are waiting for or receiving retirement benefits are only affected by poor rates of return.

If the scenarios led to the same ordering of surpluses for all subgroups, we would have a strictly additive characteristic function provided we did not change negative values to zeroes. In other words, if we had perfect correlation between the subgroups, strict additivity would result. We actually have coefficients of correlation greater than 0.9 among all subgroups except the sponsor's, for which it is around 0.6 with any of the other four. Hence, we may be surprised to see, for instance, that the combination of the retired and terminated lives leads to a smaller disposable surplus than the sum of their individual worths. What happens is that, though the bad scenarios tend to be the same for both, we get negative correlation over some subsets of scenarios. This "internal" negative correlation is at the root of the lack of superadditivity⁶.

⁵The reader may remember that we calculate an amortization payment only for those combinations including the sponsor. That is true but, since the plan starts with a surplus, no amortization is called for and hence, for a given scenario, results are strictly additive.

⁶Here is a simple example to illustrate the phenomenon. Suppose that the same five scenarios lead to the values (1,2,3,4,5) for the first group and to (2,1,3,4,5) for the second group. The coefficient of correlation is 0.9 and the two groups together generate (3,3,6,8,10). If we take the second smallest value as the worth of the group, the two groups are worth 2 and 2 separately, but only 3 jointly. Hence, though the two vectors were almost perfectly correlated, the negative correlation in the two lowest values led to the subadditivity.

For the proportional sharing rule, we need to come up with some proxy for the contributions of the subgroups to the surplus. We will use the assets allocated to the subgroups for that purpose (see Table 5.5). We recall that they are allocated in proportion to the subgroup's liabilities.

As for the serial sharing rule, as outlined in Chapter 4, it requires the ordering of the subgroups with respect to their "marginal contributions." In fact, it often suffices simply to order them with respect to their worths when taken singly. In this case, we have, in increasing order, the sponsor, the disabled lives, the terminated lives, the active lives, and the retired lives (labeled s , d , t , a and r , respectively). Based on this ordering, we find the following disposable surpluses for artificial combinations and replications of subgroups:

$$v(\{s, s, s, s, s\}) = -106,843.89$$

$$v(\{s, d, d, d, d\}) = 2,865,781.07$$

$$v(\{s, d, t, t, t\}) = 4,136,213.79$$

$$v(\{s, d, t, a, a\}) = 5,180,988.15$$

$$v(\{s, d, t, a, r\}) = 5,361,093.61$$

Again, because of the lack of additivity, we had to find the disposable surplus for each scenario and each set of subgroups so as to identify the required percentile.

We recall that our criterion calls for no distribution if the resulting amount is negative. Indeed, it would be odd to ask some subgroups to contribute additional amounts of money while others are withdrawing money from the pension fund. Accordingly, we will replace negative disposable surpluses with nil amounts throughout this chapter in the characteristic function and in the worths defined for serial sharing.

Based on our results, we will be distributing \$5,361,094. This represents 27.9% of the valuation surplus. It also is equal to the surplus in excess of 27.5% of the liabilities. We will, in the next subsections, establish the surplus shares, comparing the four sharing rules and accounting for the sponsor's role in three different ways.

5.5.2 Surplus Shares if Sponsor Is Treated Separately

As a first possibility, we will determine the sponsor's share separately from the others. First, as recommended by both Ontario and Quebec pension plan legislators, we will make the sponsor's share equal to the percentage of all its payments, accumulated with interest, divided by the accumulated value of all monies put into the fund, from plan inception until now. This corresponds to 56.419% of the disposable surplus, or \$3,024,654.

This way of determining the sponsor's share has the advantage of explicitly accounting for the risk assumed by the sponsor, at least retrospectively, by accumulating its amortization payments along with its regular contributions. However, it is sensitive to the split of the normal cost between the sponsor and the members. If we agree that the split is somewhat artificial and that it actually does not impact the overall remuneration, this sensitivity is not desirable. Nevertheless, if we wish to consider each party's contributions as their separate and actual engagements towards the pension plan, the split then comes out as natural and relevant.

In order to split the remaining \$2,336,440, we will apply the notion of reduced game seen in Chapter 4. For the Shapley value, we will use Hart and Mas-Colell's definition. Hence, every combination of subgroups will pick, as its disposable sur-

plus, the one it would achieve with the sponsor, minus the sponsor's share. As usual, negative surpluses are made nil. For example, we will define

$$v'(\{a, d, r\}) = v(\{a, d, s, r\}) - 3,024,654 = 1,188,921.21.$$

For the nucleolus, we use Peleg's notion of a reduced game. So, for each combination of subgroups, we decide whether or not it is profitable to secure the sponsor's cooperation at the price of its share. With the values we have here, we always end up excluding the sponsor. For instance,

$$v'(\{a, d, r\}) = \max(v(\{a, d, r\}), v(\{a, d, s, r\}) - 3,024,654) = 4,230,124.85.$$

Concerning the proportional sharing rule, we simply apply the weights, as given by the assets, to the amount left to distribute in order to get the shares. This rule, together with the separate treatment of the sponsor, produces the shares that the pension legislators in Ontario and Quebec recommend.

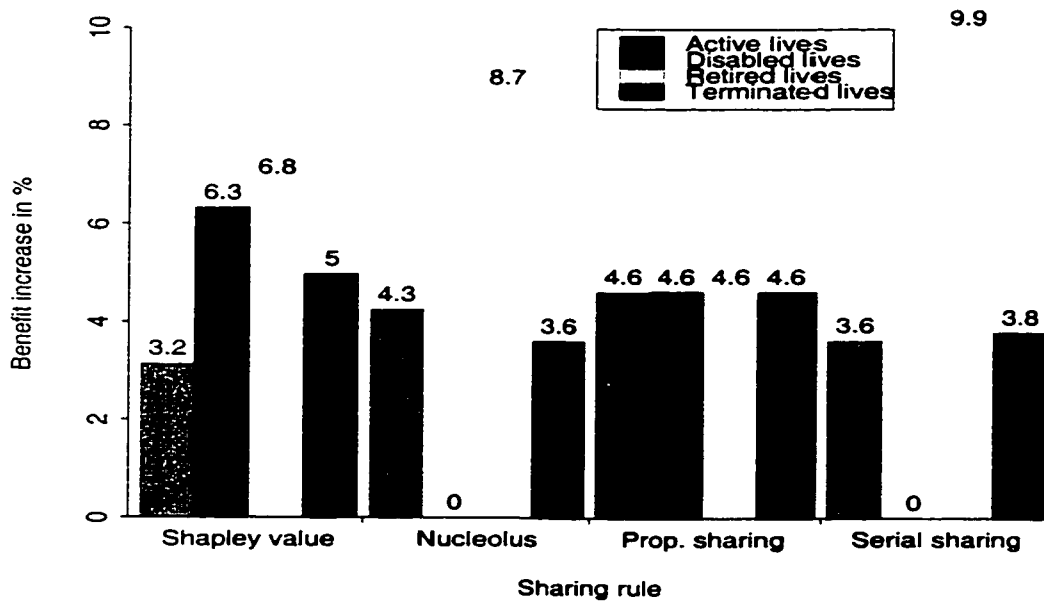
As for the serial sharing rule, we cannot really exclude the sponsor in computing a disposable surplus since this rule compares the worths of coalitions having the *same* number of agents. To include the sponsor, we will bring it in first, whether or not it generates the smallest surplus. We will then bring in the other subgroups in order of increasing "marginal contributions." In computing the shares of the four subgroups, we will only use the last four worths computed under this approach, making any negative one equal to zero as usual. We must remember to subtract the sponsor's share from each value obtained. In the game we have here, the sponsor

naturally was brought in first. To illustrate what we have tried to explain, here are the worths, as given by the characteristic function v' , the serial sharing rule will make use of:

$$\begin{aligned}
 v'(\{d, d, d, d\}) &= v(\{s, d, d, d, d\}) - 3,024,654 = 0.00 \\
 v'(\{d, t, t, t\}) &= v(\{s, d, t, t, t\}) - 3,024,654 = 1,111,559.79 \\
 v'(\{d, t, a, a\}) &= v(\{s, d, t, a, a\}) - 3,024,654 = 2,156,334.15 \\
 v'(\{d, t, a, r\}) &= v(\{s, d, t, a, r\}) - 3,024,654 = 2,336,439.61
 \end{aligned}$$

Now that we have explained how we modify the worths to account for the fixed sponsor's share, for each of the sharing rules, we can look at the resulting shares in Figure 5.1.

Figure 5.1: Shares, 3-Year Horizon, Initial Assets Proportional to Actuarial Liability, Sponsor Separate



While actual dollar figures may mean something, we believe that the enhance-

ments in retirement benefits that these amounts could fund⁷ are much more relevant for interpretation. These allowable enhancements are given in parentheses below each subgroup's share in Table F.2 and represent the information depicted in Figure 5.1. None is given for the sponsor since its share rather would represent a cash amount or some contribution holiday⁸.

We find it interesting to note that both the nucleolus and the serial sharing rule end up giving nothing to the disabled. In fact, as we shall see, though these two sharing rules do not give the same results, they yield somewhat similar shares. This comes somewhat as a surprise since these two rules do not enjoy the same properties (see Table 4.3). Rather, we suspect that the reduction of the game drives these zero shares⁹. Still, insofar as the nucleolus is much more difficult to compute than the serial shares, and since the outputs cannot be any more precise than the actual inputs are, we would favor using serial sharing out of the two.

Since the proportional sharing rule allocates the surplus in proportion to the assets, which happen to be proportional to liabilities, it should come as no surprise

⁷According to pension legislation in Quebec [35], surplus shares should be used to increase the benefits unless the maximum benefits allowable have already been attained, in which case the excess would be paid in cash. Moreover, unless we were to refine our definition of subgroups, allocating the subgroup's share among its members in proportion to their liabilities appears to be the simplest thing to do at this time. Ideally, however, given sufficient computer resources, one should look at the shares that would result from treating each member individually.

⁸These two forms of payment of the sponsor's share are equivalent from a theoretical point of view. Indeed, if the sponsor receives the amount in cash, it can set it aside and use it to pay for future contributions. However, these two forms of payment may not be equivalent if we take into account their fiscal treatment.

⁹One may think that the reduced-game properties (Properties 4.10 and 4.11) should address this suspicion. They actually do so only in the case that the sponsor's share would equal its share in the grand game. That is not what we have here and hence we do not know how shares in the reduced game are supposed to be affected.

that this sharing rule leads to the same percentage increase for all subgroups.

For the three other sharing rules, we have that the retired lives gain relatively more than the others. Out of the four subgroups with members at the onset, that is the one with the smallest coefficient of variation for its simulated surpluses. As a result, applying the criterion to this subgroup alone would identify as disposable a larger portion of its valuation surplus than it would for the other subgroups.

If we agree that the coefficients of variation provide a natural ordering for the shares, we would then expect that retired lives come out first, followed by the disabled lives, then the terminated lives, and ending with the active lives. Indeed, we think it would be prudent to release relatively more of the surplus to those subgroups which are least likely to run into a deficit in the short run. Only the Shapley value leads to this ordering in Figure 5.1. We had already pointed to this rule as a promising one in the last chapter because of its monotonicity properties.

For this particular game, the resulting shares vary quite a bit from one rule to the next and, until we see more results, we will not know if the differences come from the rules themselves or if, as we suspect, the use of a reduced game is at the root of the variability.

5.5.3 Surplus Shares if Sponsor Is Treated as Essential

Another possible way of accounting for the sponsor's role is to make the worths of the coalitions that do not include the sponsor equal to zero. It is equivalent to stating that, without the sponsor, we cannot have any kind of pension plan. In practical terms, in Table 5.6, we would change all the values for combinations with

a "0" in third position to zeroes.

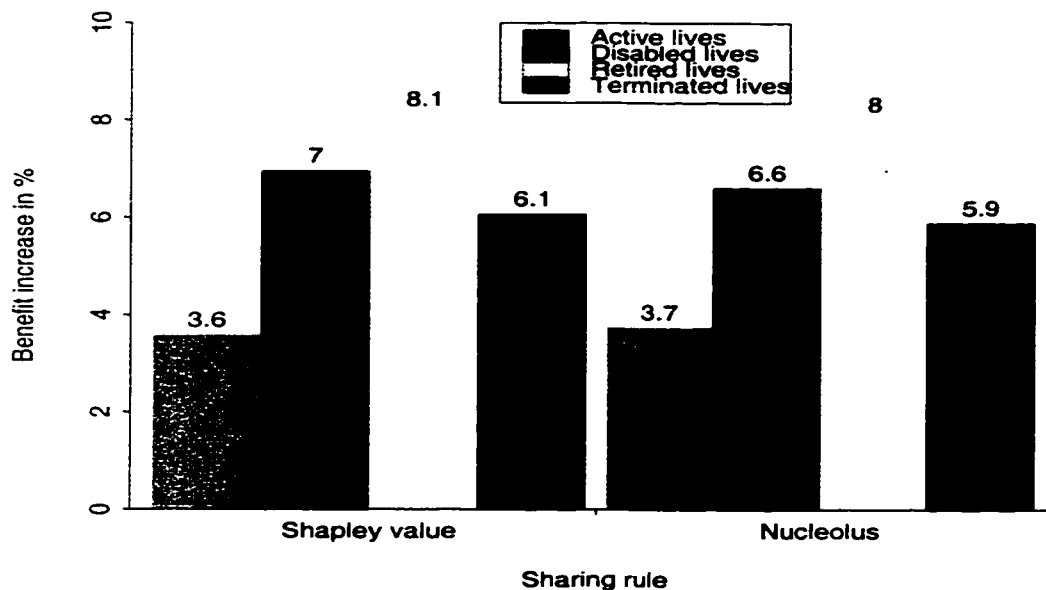
This approach to defining the characteristic function is interesting in many ways. In practical terms, it reduces from thirty-one to fifteen the number of worths that we have to compute. It also avoids applying the criterion to combinations of subgroups for which it does not seem sensible. For instance, if we apply the criterion to the subgroup of retired lives and give them the resulting disposable surplus, who will take care of the unfunded liability at the next valuation date in the 5% of cases that it arises?

For this particular treatment of the sponsor, one may wonder if there is a way to account explicitly for the deficit risk it assumes. As we will see, treating the sponsor as essential to the game actually already ensures it a good share of the overall disposable surplus. However, we actually can account for the risk assumed prospectively. Indeed, we could calculate the expected unfunded liability and allocate this amount as initial assets to the sponsor instead of zero. We would then reduce the funding level of the other subgroups by the same percentage. As mentioned in Chapter 3, adding a given amount to the assets adds the same amount to the disposable surplus. Hence, for sharing rules which exhibit zero independence (Property 4.8), namely the Shapley value and the nucleolus, we can adjust the sponsor's share directly via an adjustment in its allocated assets.

It turns out that we can use these modified shares only with the Shapley value and the nucleolus. It is not clear how we would bring in the sponsor as an essential agent in the proportional sharing rule. However, if we were to use this approach with the serial sharing rule, we simply would give everything away to the sponsor since

it alone would have the largest “marginal contribution” and all other subgroups would come out as contributing nothing.

Figure 5.2: Shares, 3-Year Horizon, Initial Assets Proportional to Actuarial Liability. Sponsor Essential



In looking at Figure 5.2, we find similar results for both sharing rules. As for the sponsor, as we can see in Table F.3, the Shapley value gives it \$2,643,335 (49.306% of the total) while the nucleolus gives it \$2,651,125 (49.451% of the total). In both cases, the sponsor’s share is less than under our previous treatment of the sponsor – it then was 56.419%. So, while making the sponsor essential to the game might have appeared extreme, it actually does not produce results that unduly favor it over the other subgroups. As mentioned earlier, if we wanted to, we could adjust the sponsor’s share through a modification of the asset allocation.

We notice that the retired lives are again the ones getting the best increase in

benefits, followed by the disabled and terminated lives, respectively. This actually is the ordering we thought was natural based on coefficients of variation.

We actually appreciate that the active lives get less than the others in the way of enhanced benefits for two more reasons, in addition to the higher coefficient of variation. First, as we mentioned earlier, future wage increases naturally will lead to increases in benefits. Second, active lives will eventually be eligible for potentially better improvements when they change status.

5.5.4 Surplus Shares if Sponsor Is Excluded

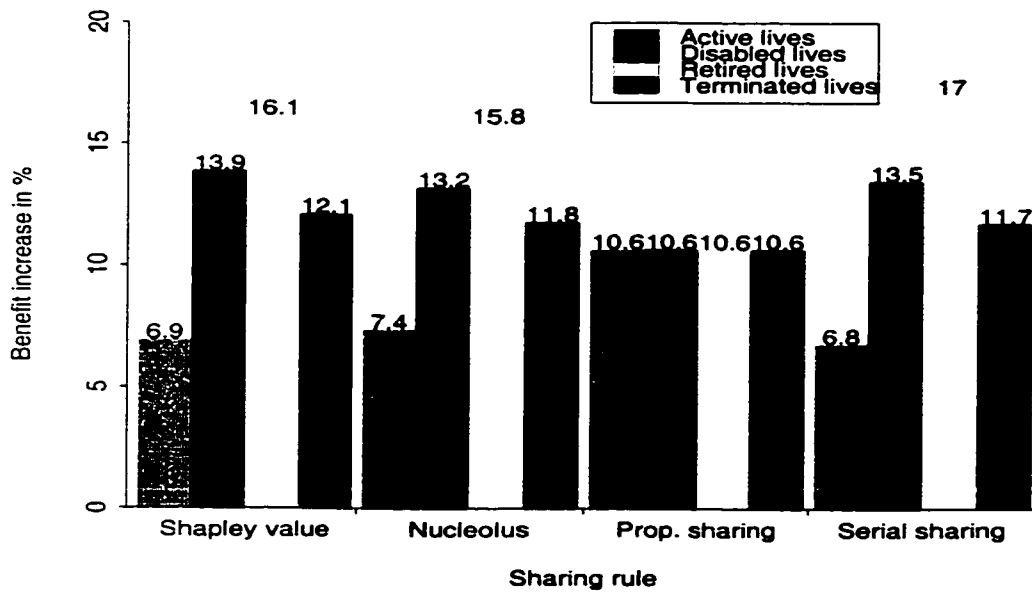
In the event that the sponsor is not entitled to any share of the surplus, we end up with a case similar to fixing the sponsor's share separately from the others with the only difference that its share is now set to zero. Hence, we apply the same methods that we used to generate Figure 5.1. We thus generate Figure 5.3.

For this particular game, we actually get that the sponsor's share is nil unless we force it, in either of the two ways presented in the previous subsections, to take on a positive value. Hence, we do not need to play a reduced game.

Since we do not give anything back to the sponsor, we naturally end up with more for each of the other subgroups, regardless of the sharing rule used. We note that, with respect to the percentage increases in benefits, the subgroups rank in the same order for all sharing rules. (We ignore the proportional sharing since, as noted before, it leads to the same percentage increase for everyone.)

Insofar as the ordering appears natural, and if we wish to avoid some of the shuffling we saw under our first treatment of the sponsor, we may adapt this ap-

Figure 5.3: Shares, 3-Year Horizon, Initial Assets Proportional to Actuarial Liability. Sponsor Excluded



proach so as to give the sponsor a share too. Indeed, if the sponsor is to be given a certain portion p of the whole surplus, we could then multiply the shares we just found by $1 - p$.

5.6 Surplus Sharing with a 3-Year Horizon and Assets Allocated Equal to Liabilities

In the last section, we found the surplus shares with different sharing rules and for different treatments of the sponsor, with all the assets allocated among the members in proportion to their liabilities. This left no assets in the sponsor's name.

While being allocated such assets is about the best a subgroup could hope for, being granted an amount equal to the liabilities is the very least it can expect to get¹⁰. So, in this section, we will actually give members assets equal to liabilities, and leave the rest to the sponsor, as indicated in Table 5.7. Of course, what is left for the sponsor is the valuation surplus.

Allocating assets in this fashion leaves all subgroups but the sponsor's with no valuation surplus. Hence, for combinations of subgroups which do not include the sponsor, since the average disposable surplus will be around zero, we should not anticipate that our criterion lead to a positive value. This is confirmed in Table F.5, which contains the characteristic function for this new game.

This game, like the first one, is neither convex nor superadditive. In fact, none

¹⁰We work in the context of a valuation surplus. Of course, if the plan was under-funded, we could not expect each subgroup to receive assets equal to liabilities. Rather, we would anticipate each subgroup to have its benefits decreased by the same percentage.

Table 5.7: Assets Allocated by Subgroup, 3-Year Horizon, Initial Assets Equal to Actuarial Liability

Subgroup	Assets
Active lives	24,522,363.24
Disabled lives	5,325,064.52
Sponsor	19,211,884.38
Retired lives	10,809,498.89
Terminated lives	9,733,385.55

of the games to be introduced will be. To the economist, this definitely would be counterintuitive since, according to Moulin [39, p. 122], “virtually all economic examples yield superadditivity.” (This is not to say, though, that the reduced or modified games, used to generate the shares, necessarily are neither convex nor superadditive.)

As usual, the Shapley value and nucleolus make use of the characteristic function, while the proportional sharing rule uses the allocated assets (in Table 5.7) as proxies for the contributions. As for the serial sharing rule, if we did not force the sponsor to be in first, as we have to when we fix its share, we would obtain the following worths as inputs:

$$\begin{aligned}
 v(\{a, a, a, a, a\}) &= -39,190,645.89 \\
 v(\{a, t, t, t, t\}) &= -17,657,311.97 \\
 v(\{a, t, r, r, r\}) &= -17,221,474.69 \\
 v(\{a, t, r, d, d\}) &= -15,119,367.49 \\
 v(\{a, t, r, d, s\}) &= 5,361,093.61
 \end{aligned}$$

However, since we need the sponsor to be in, we define the following worths instead

to be used with the serial sharing rule:

$$\begin{aligned}
 v(\{s, a, a, a, a\}) &= -12,150,122.41 \\
 v(\{s, a, t, t, t\}) &= 3,979,343.63 \\
 v(\{s, a, t, r, r\}) &= 4,293,293.03 \\
 v(\{s, a, t, r, d\}) &= 5,361,093.61
 \end{aligned}$$

As in the last section, we have an amount of \$5,361,094 to distribute. However, as we will see in the following subsections, the shares we will obtain are quite different.

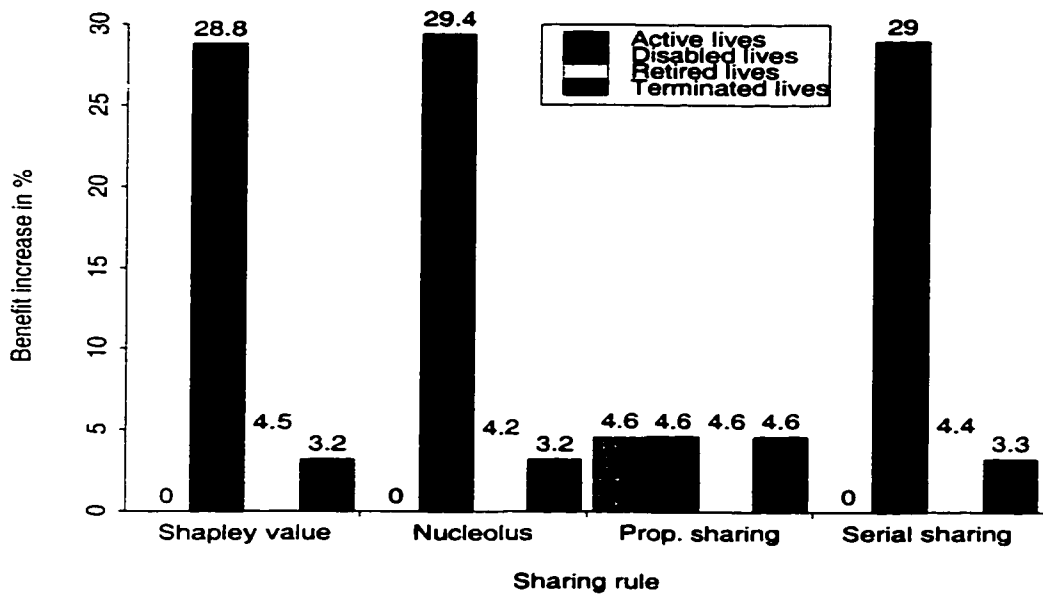
5.6.1 Surplus Shares if Sponsor Is Treated Separately

In the first set of shares we provide, we start by giving the sponsor a share proportional to its accumulated contributions. Again, this represents 56.419% of the total, or \$3,024,654.

We then find the worths in the corresponding reduced games. Applying the four sharing rules to these produces the shares illustrated in Figure 5.4. Except with proportional sharing, the share going to the disabled lives appears excessively high.

Whereas we thought that the ordering of the shares based on the coefficients of variation was defensible, we find that, for this game, the ordering is with respect to the variance of the simulated surpluses. This is not surprising since the average disposable surplus for all subgroups other than the sponsor is close to nil.

Figure 5.4: Shares, 3-Year Horizon, Initial Assets Equal to Actuarial Liability, Sponsor Separate



5.6.2 Surplus Shares if Sponsor Is Treated as Essential

As indicated earlier, to make the sponsor's role essential to the plan, we make equal to zero the worths of the coalitions that do not contain the sponsor. A careful glance at Table F.5 confirms that the game already has the sponsor in that role. Indeed, the worth of every combination with a "0" in third position is negative, and thus was forced to zero already. Hence, we need not change anything.

As we could have anticipated, giving the sponsor the valuation surplus as initial assets greatly advantages him. As a result, as confirmed in Table F.7, the sponsor receives all of the surplus if we do not fix its share separately from the others. While some may see these results as grounds for giving all the surplus to the sponsor, we rather consider them to point to the inappropriateness of our initial asset allocation. We will make further comments on the choice of asset allocation in Section 5.8.

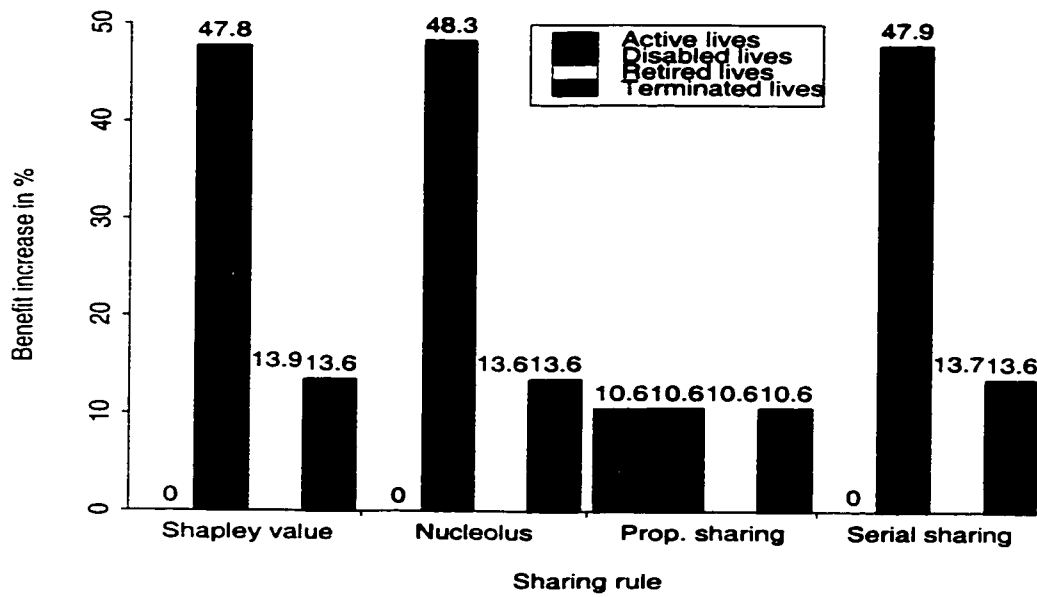
5.6.3 Surplus Shares if Sponsor Is Excluded

We still have to determine the members' shares if the sponsor is excluded. This requires the definition of some reduced games, to which we apply each of the four sharing rules. We plot the resulting benefit increases in Figure 5.5.

We are not surprised to see, once more, that the disabled lives get so much out of the deal, and that the active lives receive nothing (always with the exception of the proportional sharing rule).

Based on all the shares we calculated under an initial asset allocation equal to liabilities for the members, we are led to conclude that allocating assets this way is, if not incorrect in principle, undesirable in its repercussions. In actual fact, one

Figure 5.5: Shares, 3-Year Horizon, Initial Assets Equal to Actuarial Liability, Sponsor Excluded



could argue that the disabled lives should get more because they face the erosion of their benefits, both before and after retiring, due to circumstances out of their control (in contrast with termination). This argument, however sensible it may be, does not relate in any way to the variance of the surpluses. Hence, we cannot use it to justify the results we obtain under this asset allocation.

5.7 Surplus Sharing with a 3-Year Horizon and Assets Allocated Equal to Accumulated Fund

As a third alternative, instead of allocating assets with reference to liabilities, we suggest giving to each subgroup the money it would have if its members had paid their contributions in and withdrawn their retirement benefits from a separate fund enjoying the same rates of return as the pension fund. Since the active lives have not withdrawn anything yet, the size of their fund is appreciably large (see Table 5.8). For disabled and retired lives, their contributions, together with interest income, were not sufficient to fund their benefits. We would not expect them to be either, as these lives are getting the most out of the pension plan, unlike terminated lives which lose out on anticipated salary growth. As for the sponsor, which keeps the rest of the assets, it ends up with even more than the valuation surplus.

Starting out with these assets puts the disabled and retired lives at a clear disadvantage. It also hurts the active lives since they end up with a fund worth less than their liabilities. Accordingly, we anticipate no disposable surplus for any combination of these three subgroups. Table F.9, which exhibits the characteristic

Table 5.8: Assets Allocated by Subgroup, 3-Year Horizon, Initial Assets Equal to Accumulated Fund

Subgroup	Assets
Active lives	19,823,046.72
Disabled lives	-1,680,230.88
Sponsor	37,990,153.17
Retired lives	-5,281,220.97
Terminated lives	18,750,448.55

function for this game. confirms our suspicions.

As before. the allocated assets serve as proxies for contributions when using the proportional sharing rule, while the characteristic function is the input necessary for the Shapley value and nucleolus. We still are missing the ingredients needed for the serial sharing rule. If there was no particular role given to the sponsor, the required worths would be:

$$v(\{r, r, r, r, r\}) = -92,558,615.06$$

$$v(\{r, a, a, a, a\}) = -68,568,868.44$$

$$v(\{r, a, d, d, d\}) = -55,652,350.60$$

$$v(\{r, a, d, t, t\}) = -26,114,614.00$$

$$v(\{r, a, d, t, s\}) = 5,361,093.61$$

However. since we must account for the sponsor as belonging to each and every combination of subgroups, we rather will be using the following elements:

$$v(\{s, r, r, r, r\}) = -36,080,284.74$$

$$v(\{s, r, a, a, a\}) = -17,973,697.49$$

$$v(\{s, r, a, d, d\}) = -9,362,593.30$$

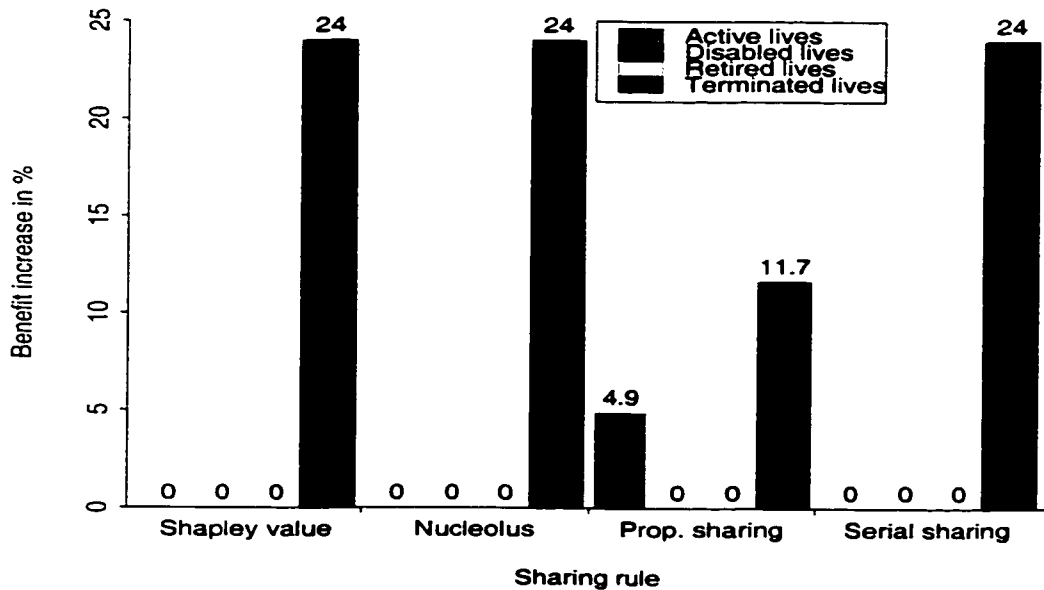
$$v(\{s, r, a, d, t\}) = 5,361,093.61$$

Since we still have the same overall plan and we apply our criterion over the same horizon and with the same probability, the amount to distribute remains \$5,361,094.

5.7.1 Surplus Shares if Sponsor Is Treated Separately

The sponsor's share, proportional to its accumulated contributions, is \$3,024,654. Properly defining the reduced games that result from fixing the sponsor's share, we find the relative shares for the other subgroups, as shown in Figure 5.6.

Figure 5.6: Shares, 3-Year Horizon, Initial Assets Equal to Accumulated Fund, Sponsor Separate



Given our earlier observations about each subgroup's assets, we are not surprised to see that all of the remaining amount, \$2,336,440, goes to the terminated lives,

except with proportional sharing. Because we no longer have liabilities as proxies for the contributions in determining the proportional shares, we no longer have equal benefit increases under that rule.

5.7.2 Surplus Shares if Sponsor Is Treated as Essential

Forcing the worth of coalitions not including the sponsor to be nil only affects one of them in the present game: the one with the terminated subgroup only.

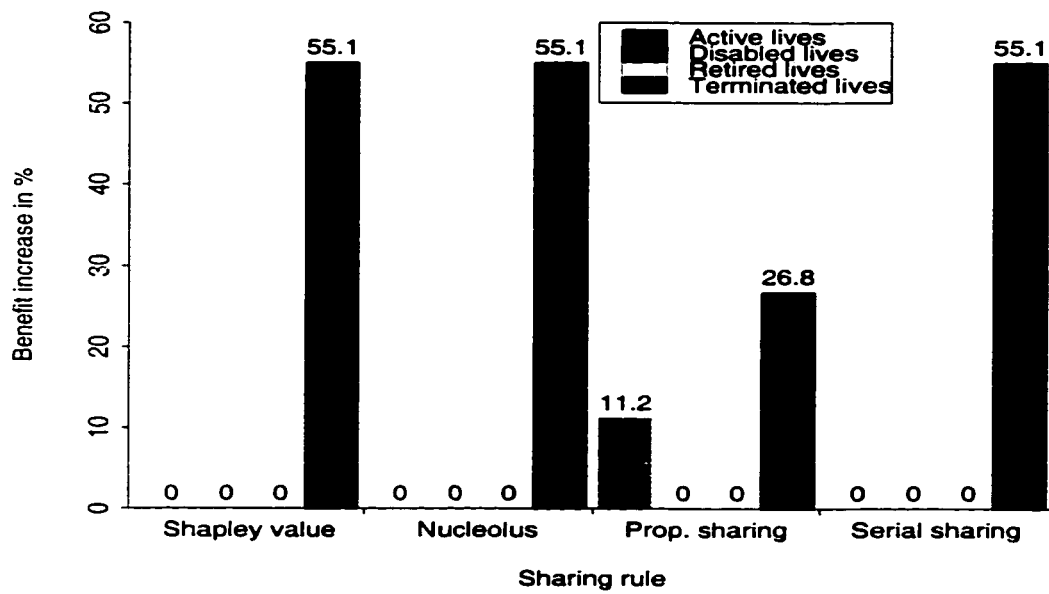
Unlike the usual similarity we found between the two sets of shares with this way of accounting for the sponsor in the last two sections, we obtain quite different shares this time (see Table F.11). While the Shapley value splits the total amount equally between the sponsor and the terminated lives, the nucleolus, once more, returns the total amount to the sponsor.

5.7.3 Surplus Shares if Sponsor Is Excluded

Even if we exclude the sponsor from the sharing, we still obtain shares that are very polarized. As we see in Figure 5.7, all the disposable surplus goes to the terminated lives, except with proportional sharing.

Unless we think we should favor the terminated lives because they get less out of the pension plan than the other subgroups, none of the shares calculated in this section supports the use of accumulated funds as initial subgroup assets. We did not think that assets merely equal to liabilities led to viable results either. Nevertheless, before concluding that we will want to allocate assets in proportion to the liabilities, we will deal with the issue of asset allocation at greater length in

Figure 5.7: Shares, 3-Year Horizon, Initial Assets Equal to Accumulated Fund, Sponsor Excluded



the next section.

5.8 Choice of Initial Asset Allocation

As we could observe, the distribution of the surplus is very sensitive to the way we divide the assets among the subgroups. Unless we are willing to favor one over the others, we will be left in an uncomfortable position: one legitimately could pick an asset allocation that would be some combination of the three introduced so as to meet a certain goal of surplus sharing.

In order to provide some indication of what a sensible asset allocation might be, we will compare the shares we obtained in the last three sections, for each of the three different roles conferred upon the sponsor. We will also explore the possibility of using game theory to derive the initial asset allocation. We will conclude this section by identifying our chosen asset allocation and mentioning some variations thereof which may be applicable.

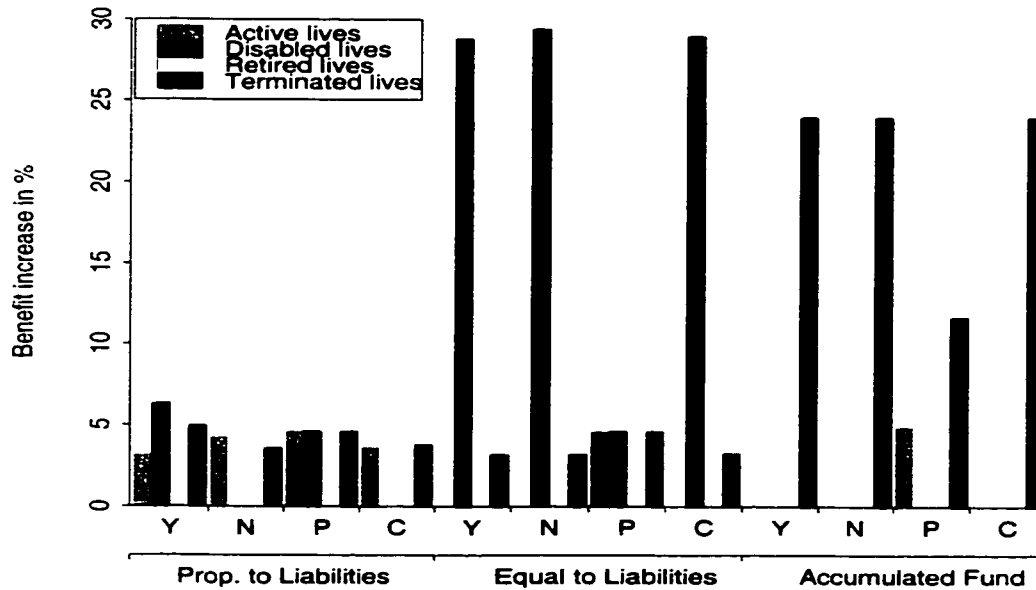
5.8.1 Impact of Asset Allocation if Sponsor Is Treated Separately

In our first treatment of the sponsor, we have assumed we give it the percentage of the disposable surplus that corresponds to the percentage represented by its accumulated contributions over the total. Hence, the particular asset allocation has no impact on the sponsor's share.

To view the impact the asset allocation has on the other subgroups' shares, we

have combined the results found in Figures 5.1, 5.4 and 5.6 to generate Figure 5.8.

Figure 5.8: Shares under Different Asset Allocations, Sponsor Separate



Each asset allocation leads to a particular winner out of the four subgroups. If assets are proportional to liabilities, the most favored subgroup is that of the retired lives. If assets are equal to liabilities, the disabled lives are greatly advantaged. If assets are equal to the accumulated fund, the terminated lives come out as the clear winner.

We now attempt to explain how each asset allocation leads to the reported phenomenon. With assets equal to the accumulated fund, the terminated lives are advantaged because they are the ones who get less than their money's worth out of the pension fund. Whereas they had contributed towards a pension that potentially would be based on their salary at retirement, they will receive a pension based on

their salary at termination, which is often quite smaller. For the other subgroups, as we pointed out earlier, their fund is worth less than their liabilities; in some sense, they are the ones gaining from those who terminate, as well as from those that die.

In general, the reserve is larger than the accumulated premiums of those who are still alive, simply because a reserve accrues not just from investment gains, but also from survivorship gains. Based on that observation, making assets equal to the accumulated fund seems to run counter to the essence of any insurance scheme. Naturally, one might argue that, when we consider sharing the surplus, we no longer consider the risk sharing nature of the plan, but some additional and extraneous feature of the plan.

When assets are equal to liabilities, the disabled lives are advantaged because their simulated surpluses exhibit the smallest variance. With an initial funding level of 100%, the mean of the distribution for the surpluses will be around zero. Hence, if the distribution is close to normal¹¹, the 5th percentile will be a multiple of the standard deviation. Since we take a negative multiple, the subgroups with less variance will have a greater worth associated with them. In other words, if we had two groups, one of which was twice the size of the other, but otherwise identical in all respects, the larger group would be penalized. We do not think that the sheer size of the liabilities should dictate which will receive more in relative value. Rather, we believe that these two groups should have been receiving the same percentage of benefit increase.

¹¹For each of the five subgroups, we generated a Q-Q plot and found that the distribution is quite close to normal, except for the smallest and largest 1% of simulated values.

With assets proportional to liabilities, the retired lives get relatively more than the other subgroups. As we mentioned earlier, this makes sense to us insofar as that subgroup obtains surpluses which are *relatively* less variable than those of other subgroups. We consider intuitively appealing and theoretically relevant to release more and more of the valuation surplus as we assert with more and more precision the value of our engagements. Although we did not compute the duration of the liabilities for the subgroups, we believe that it would lead to the same ordering of subgroups as the coefficients of variation.

So, for this particular treatment of the sponsor, we are led to favor an asset allocation proportional to the liabilities. We still should confirm our preference for the other two treatments of the sponsor that we have considered.

5.8.2 Impact of Asset Allocation if Sponsor Is Treated as Essential

In our second treatment of the sponsor, we have considered him to be essential to the existence of the pension plan. We recall that, in that case, we only could use the Shapley value and the nucleolus. As pointed out earlier, using serial sharing would have resulted in all of the disposable surplus reverting to the sponsor.

Table F.14 combines all the results we have derived so far for this particular treatment of the sponsor. We did not generate a graph for these results as we did not feel it would shed further light. Except for the asset allocation proportional to liabilities, we end up giving the disposable surplus almost exclusively to the sponsor. For proponents of the view that the sponsor should receive all of the surplus, these

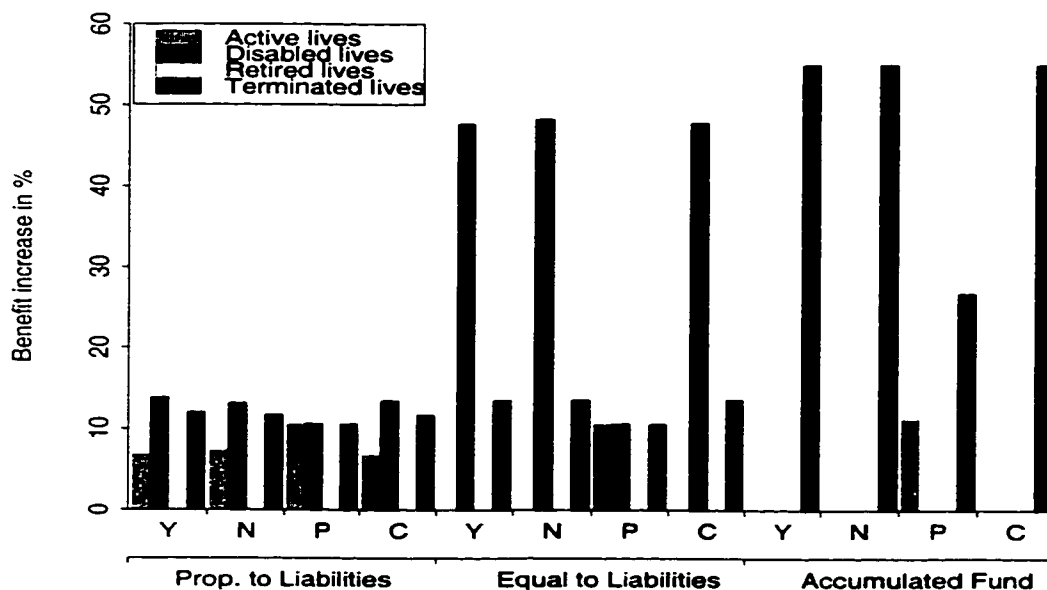
particular results are welcome. However, if we believe that at least the portion of the contributions which does not change from year to year (the normal cost as a percentage of salary) can be considered as part of the total remuneration, we think we should question giving all the surplus to the sponsor.

Thus, for this second treatment of the sponsor, we still favor allocating assets in proportion to the liabilities.

5.8.3 Impact of Asset Allocation if Sponsor Is Excluded

In our third and last treatment of the sponsor, we excluded it from the sharing exercise. To guide our comments and ease the comparison, we have combined Figures 5.3, 5.5 and 5.7 to generate Figure 5.9.

Figure 5.9: Shares under Different Asset Allocations, Sponsor Excluded



The overall appearance of this bar plot is strikingly similar to the one we obtained under our first treatment of the sponsor (see Figure 5.8). Hence, we are led to the same conclusions and, based on all our observations, would privilege the allocation of assets in proportion to the liabilities.

5.8.4 Use of Game Theory to Determine Asset Allocation

We noted that endowing subgroups of members with assets equal to their liabilities was a bare minimum, in our opinion. As for the sponsor, we have mentioned the idea of giving it the value of the expected unfunded liability¹² as initial assets. This would generate the initial allocation given in the first column of numbers in Table 5.9.

These initial amounts add up to less than the total assets available. Just as we used cooperative game theory to distribute the disposable surplus in the previous sections, we also can resort to this theory to allocate the excess assets¹³.

To generate the characteristic function for this game, we define the worths of the coalitions as follows: For each combination of subgroups that does not include all five of them, we make its worth equal to the sum of the initial assets given in the table for those subgroups included in the combination. For the full combination of all five subgroups, we make its worth equal to the total assets available.

¹²The expected unfunded liability is given by the absolute value of the sum of all negative disposable surpluses obtained in the scenarios, divided by the total number of scenarios.

¹³Using game theory to allocate the assets does not preclude its use to distribute the disposable surplus. Indeed, we actually have two distinct steps when determining the surplus shares. First, we must allocate the assets among the subgroups. Second, we must distribute the disposable surplus among the subgroups.

Table 5.9: Asset Allocation Determined by Game Theory

Subgroup	Initial Allocation	Shapley Value	Proportional Sharing
Active lives	24,522,363.24	28,319,603.24	33,720,769.35
Disabled lives	5,325,064.52	9,122,304.52	7,322,510.91
Sponsor	225,684.40	4,022,924.40	310,339.24
Retired lives	10,809,498.89	14,606,738.89	14,864,171.75
Terminated lives	9,733,385.55	13,530,625.55	13,384,405.33

The characteristic function thus leads to a surplus which can be allocated only for the grand coalition. This is a special kind of game which is normally called a bargaining game. The two most popular cooperative solution concepts for bargaining games are the Nash bargaining solution and the Kalai-Smorodinsky bargaining solution. They are introduced in Appendix E.

We use the characteristic function with the Shapley value and the nucleolus. We only use the worths of the singletons and that of the grand coalition with the Nash and Kalai-Smorodinsky bargaining solutions. All these four ways of allocating the excess assets lead to an equal split among the five subgroups. For the sharing rules, that is because only the grand coalition has additional value; for the bargaining solutions, that is because we have a game with transferable utility.

For the proportional sharing rule, we use the initial assets as the weights with respect to which to allocate the excess assets. This leads to an asset allocation similar to the very first one we used, as given in Table 5.5. Of course, the main difference is the non-zero value of assets allocated to the sponsor.

As for the serial sharing rule, we do not know how we should define the worths

of the different coalitions of five subgroups since they all are, in some sense, grand coalitions. We do not even know how to order the subgroups with respect to their marginal contributions. Hence, we do not use this sharing rule to allocate assets.

Table 5.9 contains all the resulting asset allocations. We had noticed, in Chapter 3, that any increase in initial assets directly translates into the same increase in the value of the disposable surplus. Since the asset allocation obtained with proportional sharing is similar to the one we have identified as our preference, we will compare the other with that one.

When all subgroups are given the same in absolute value, as they are with the Shapley value, the subgroups with the smallest liabilities are the ones getting the most in relative value and they end up with the largest funding levels. Since having more assets leads to getting more of the disposable surplus, the disabled lives and the sponsor have everything to gain while the active lives stand to lose. We do not see why groups with smaller liabilities should be earmarked relatively more assets than the others. Besides, if excess assets were allocated in this fashion in actual fact, this would create a natural incentive to form the smallest possible groups; eventually, if feasible, groups with equal liabilities would result and we effectively would be back to proportional sharing.

5.8.5 Our Choice of Asset Allocation

While game theory offers an alternative in determining the initial asset allocation, we do not find it offers a viable one. Rather, we still establish proportional allocation as the most appropriate way of allocating assets.

More precisely, we believe that giving the same funding level to each subgroup other than the sponsor is most appropriate. As for the sponsor, we need not necessarily endow it with no assets. While we did not think we should give it all of the valuation surplus, as was the case when making assets equal to liabilities for the other subgroups, we consider appropriate to endow it with an amount in relation to the expected unfunded liability.

Regardless of what we decide to endow the sponsor with, insofar as the worths of the coalitions that do not include the sponsor all are positive, we have flexibility in our way of accounting for the sponsor. On one hand, we can increase its share of the assets and reduce all the other ones proportionally. On the other hand, we can find each subgroup's share with the chosen asset allocation and adjust the resulting sponsor's share if, for some reason, we believe it should have been different. In that case, we would reduce the other subgroups' shares proportionally.

In conclusion, regarding the initial asset allocation, we will allocate assets in proportion to liabilities, as we did in the first of the three examples given so far, for the rest of this chapter. Hence, no further reference shall be made concerning the asset allocation used. Rather, for all that follows, we allocate assets according to Table 5.5.

5.9 Surplus Sharing with a 10-Year Horizon and a 5% Probability of Deficit

While using a three-year horizon makes sense intuitively since that is the time between valuations, we can use other horizons as well. In order to provide an idea of the impact of the length of the horizon on the surplus sharing, we will consider forecasts over 10 years.

In theory, we could choose the forecast period to be anything. However, because we consider that this period should also determine the number of years between surplus sharing exercises, we would advise against making it too long. Frequent surplus sharing does not lead to more money being distributed, but it does ensure that a greater percentage of the members potentially share in arising surpluses.

Initially, we will keep the target probability of a deficit at 5%. Nevertheless, since we simulate over a longer period, a wider range of outcomes becomes possible and we expect the disposable surplus to be smaller for every coalition.

Indeed, as Table F.16 reveals, every element of the characteristic function has decreased. Particularly affected are the worths of the coalitions containing the active lives. This should not come as a surprise since, as mentioned earlier, this subgroup's disposable surplus is sensitive to both the rate of return and the wage inflation.

Since the overall disposable surplus is negative (it is \$-117,803), we need not go any further since there is nothing to distribute. Hence, we should be careful in picking our target probability of a deficit.

5.10 Surplus Sharing with a 10-Year Horizon and a 15% Probability of Deficit

Since, over a ten-year horizon, targeting a 5% probability of a deficit turned out to be too restrictive, we will try to find a target probability over that period which would be equivalent, in some way, to the one we used over a three-year horizon.

As there are three and one-third three-year periods in ten years, we will use, as our target probability, $1 - (1 - 0.05)^{3\frac{1}{3}} \approx 0.15$. If this probability is indeed some sort of equivalent¹⁴, we would expect, in general, disposable surpluses to be larger with a ten-year period than with a three-year period since the frequency of the surplus sharing is then reduced.

Table F.17 contains the disposable surpluses using a ten-year period. Comparing this table with Table 5.6 confirms that all worths have increased except that of the sponsor when considered alone. Two elements explain this fact: First, this particular coalition is the only one to start with no surplus or deficit; second, it increases in size over time.

Hence, instead of having \$5,361,094 to share, we now have \$8,071,440. Again, we will compare the shares produced by different sharing rules and under different assumptions regarding the sponsor's role. We will use the characteristic function in computing the Shapley value and the nucleolus. Based on our different observations about the impact of the initial asset allocation, we will use Table 5.5 to provide the proxies for contributions when we use proportional sharing, for this case and

¹⁴It would be equivalent if distributions were independent over time, but we do not believe this to be the case.

all the ones to follow in this chapter. Finally, for serial sharing, we will be using the following worths:

$$v(\{s, s, s, s, s\}) = -1,022,705.49$$

$$v(\{s, d, d, d, d\}) = 4,512,271.44$$

$$v(\{s, d, t, t, t\}) = 6,263,926.08$$

$$v(\{s, d, t, a, a\}) = 7,276,965.66$$

$$v(\{s, d, t, a, r\}) = 8,071,439.60$$

Since this set of worths already has the sponsor coming in first, we need not generate another one.

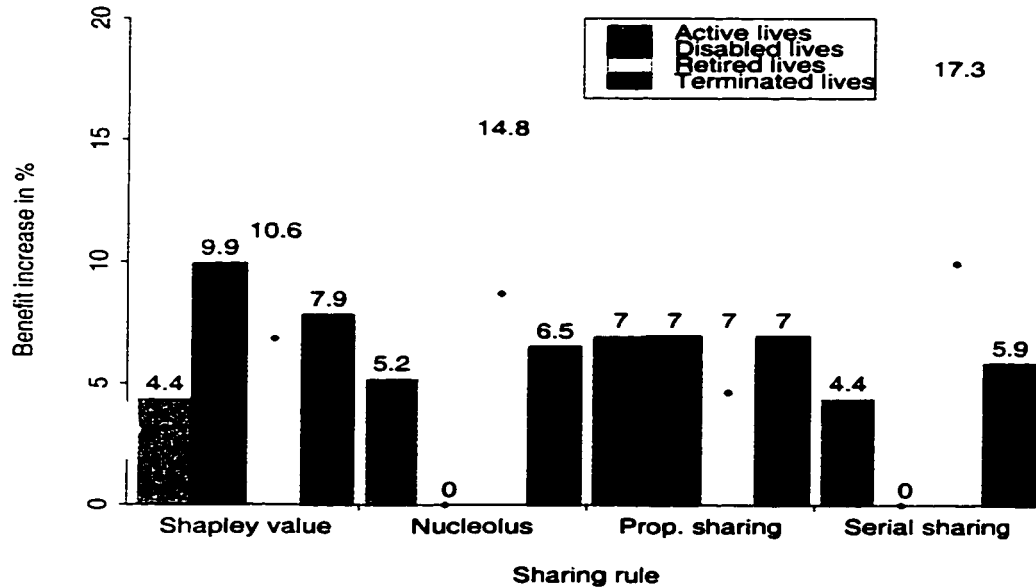
5.10.1 Surplus Shares if Sponsor Is Treated Separately

While the percentage of the total surplus that is paid to the sponsor remains the same (56.419%), since the total is larger, so too will be its share, now at \$4,553,794.

The shares of the other subgroups are shown in Figure 5.10. The filled diamonds correspond to the shares depicted in Figure 5.1; these shares were obtained under similar conditions except that the horizon was then only three years.

We again notice that the disabled lives receive nothing according to the nucleolus and serial sharing rule. Also, we can see that lengthening the horizon has benefited all subgroups, although relatively more those which are less sensitive to economic variations. Hence, all sharing rules but the proportional one lead to a greater benefit enhancement for the retired lives, and to a smaller one for active lives.

Figure 5.10: Shares, 10-Year Horizon, 15% Probability, Sponsor Separate



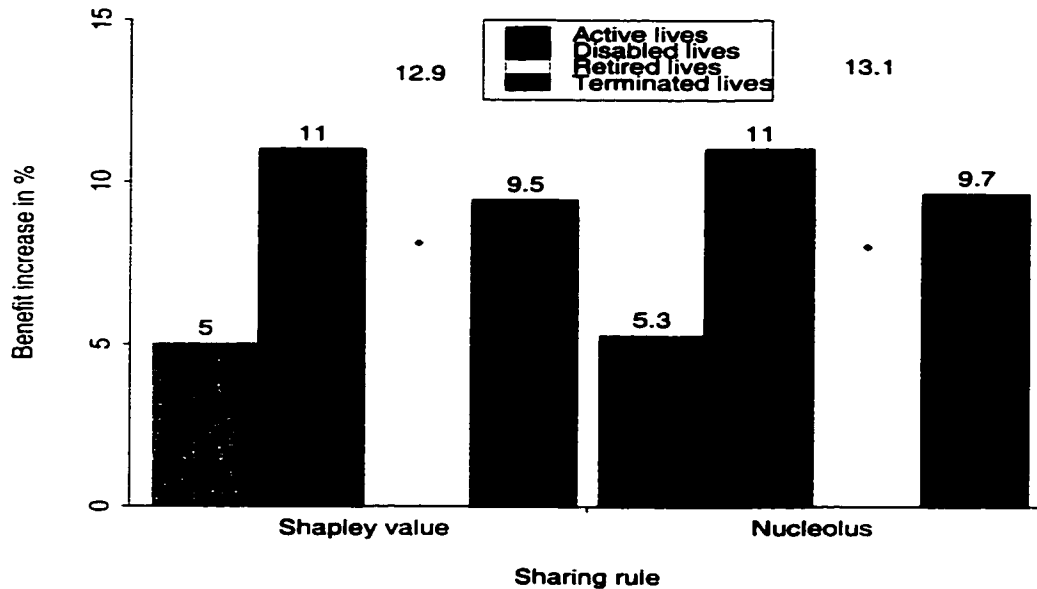
5.10.2 Surplus Shares if Sponsor Is Treated as Essential

If we consider that, without the sponsor, there cannot be any pension plan and hence any disposable surplus, we can modify the characteristic function accordingly and compute new shares using the Shapley value and the nucleolus.

These new shares, exhibited in Figure 5.11, are similar for the two sharing rules. Compared to treating the sponsor separately, we now give it a share that is smaller under either rule (48.746% of the total disposable surplus with the Shapley value, and 47.407% with the nucleolus, versus 56.419% otherwise).

Just as in Figure 5.2, whose information is repeated here via the filled diamonds, we have that the percentage increases are more profitable to the retired lives, then the disabled lives, followed by the terminated lives, and with the actives lives last.

Figure 5.11: Shares, 10-Year Horizon, 15% Probability, Sponsor Essential

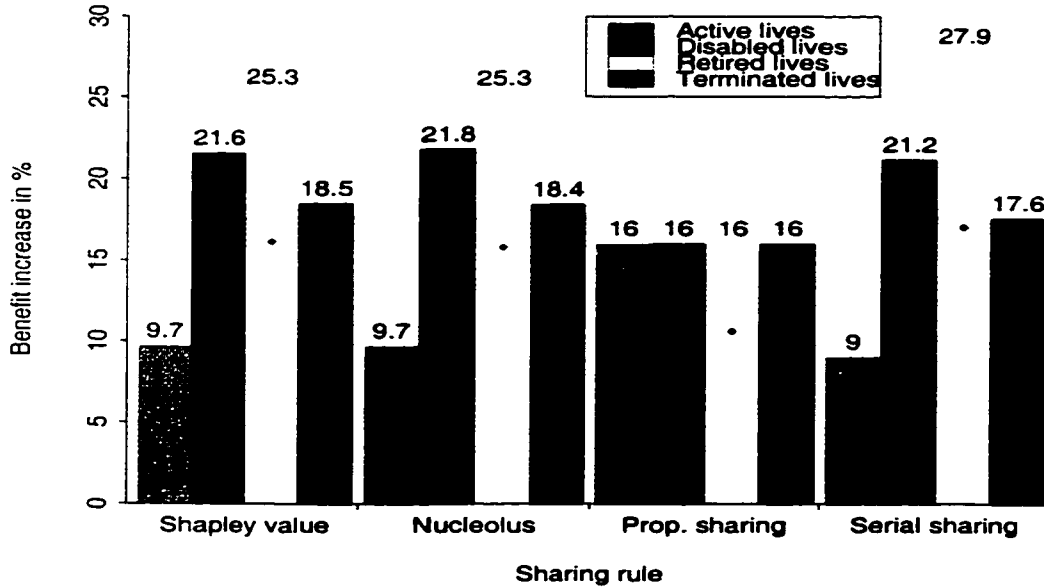


5.10.3 Surplus Shares if Sponsor Is Excluded

We still have to compute the shares in the case that the sponsor is to receive nothing at all. Once more, we obtain results, given in Figure 5.12, that are similar to those we obtained with a three-year horizon using the same initial asset allocation (shown originally in Figure 5.3 and represented here by the filled diamonds). Naturally, since the total amount available for distribution is larger here, all amounts and percentages increases are also larger. (Table F.20 provides the dollar figures.)

The ordering of the percentage increases for the different subgroups remains the same as when we treated the sponsor as essential to the game, in the last subsection. As always, the proportional sharing rule leads to the same percentage increase for all since we made assets proportional to liabilities.

Figure 5.12: Shares, 10-Year Horizon, 15% Probability, Sponsor Excluded



In conclusion, if we compare the results obtained in this section, with a ten-year horizon, to those obtained in Section 5.5, under the same conditions but with a three-year horizon, we find qualitatively similar results. Quantitatively, we get larger amounts since we distribute the shares that otherwise would have been split over four payments (the fourth one being smaller since it accounts for the last of the ten years).

5.11 Surplus Sharing with a 3-Year Horizon and Real Starting Values

Typically, the economic models we are using (Wilkie for the rate of return and Sharp for the increase in wages) are quite sensitive to the starting values we provide them with. In all of the previous cases, we have performed the 10,000 simulations starting with neutral values. These values led to an average annual rate of return of 8.27% and an average annual rate of wage inflation of around 4.92% over either simulation period¹⁵. If we use actual 1985 values as initial inputs into our models, we obtain an average annual rate of return of 8.00% and an average annual wage increase of 4.38%.

The lower returns should adversely affect the disposable surplus of those coalitions that contain inactive members only. As for the subgroups of the active lives and the sponsor, since the wage increases are much lower (and by a greater margin than the returns), we expect their disposable surpluses to be larger.

Our conjectures are validated in Table F.21 (compared to Table 5.6). We also see that the lower wage increase has the overall larger impact since we end up with more to distribute here (\$5,644,629) than we did with neutral starting values (\$5,361,094).

As proxies for contributions for use with the proportional sharing rule, we still use Table 5.5. As for the worths needed for the serial sharing rule, they are:

¹⁵If we carry more decimals the average annual rates are actually different for the two periods. For the rate of return, we have an average of 8.267587% over three years, but 8.267435% over ten years. As for the increase in wages, it averages 4.913039% over three years, but 4.916924% over ten years.

$$\begin{aligned}
v(\{s, s, s, s, s\}) &= -104,049.29 \\
v(\{s, d, d, d, d\}) &= 2,769,922.41 \\
v(\{s, d, t, t, t\}) &= 4,000,136.24 \\
v(\{s, d, t, r, r\}) &= 5,091,586.24 \\
v(\{s, d, t, r, a\}) &= 5,644,628.80
\end{aligned}$$

Finally, all the values we need for the Shapley value and the nucleolus are given by the characteristic function.

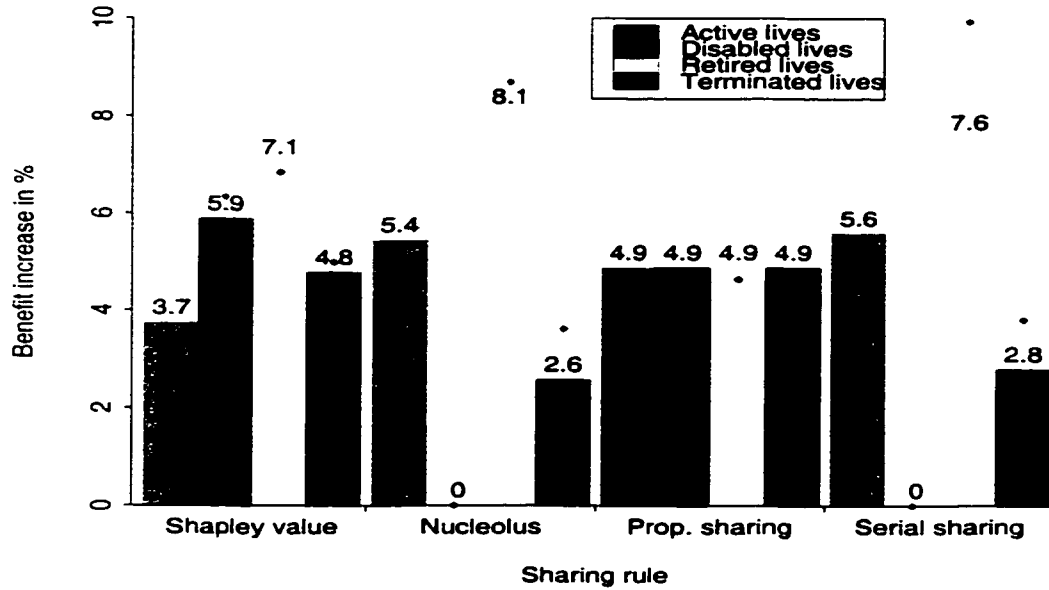
5.11.1 Surplus Shares if Sponsor Is Treated Separately

Like we did for the previous cases, we begin by treating the sponsor separately, giving it 56.419% of the total surplus. It thus receives \$3,184,621. We then use reduced games to find the shares of the other subgroups.

Not unlike shares derived previously with the same treatment of the sponsor, those given in Figure 5.13 provide a greater benefit enhancement for the retired lives. We also have zero shares for the disabled lives under both the nucleolus and the serial sharing rule. Moreover, these two rules place the active lives second for increases in benefits, unlike the Shapley value which places it last.

In addition, if we closely compare Figures 5.1 and 5.13, or equivalently the diamonds with the bars in the latter figure (or their corresponding Tables F.2 and F.22), we realize that the shares of the inactive lives generally are smaller while those for the active lives and the sponsor are larger. This of course comes about because of the opposite effects of the decreases in return and wage inflation.

Figure 5.13: Shares, 3-Year Horizon, Real Starting Values, Sponsor Separate

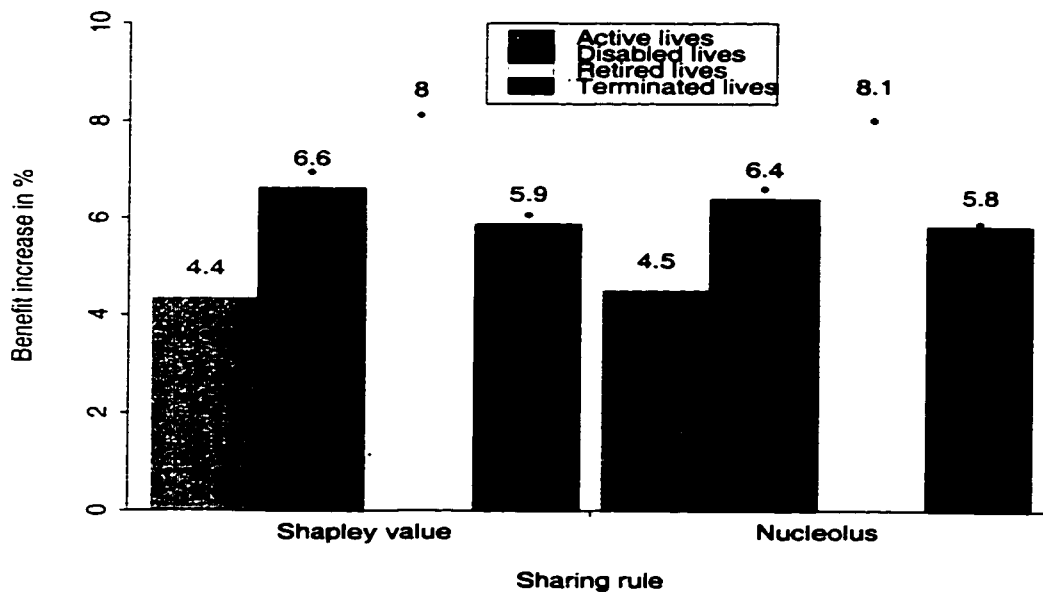


5.11.2 Surplus Shares if Sponsor Is Treated as Essential

If we treat the sponsor as essential to the pension plan, we obtain the shares depicted in Figure 5.14. The diamonds reproduce the information given in Figure 5.2.

This treatment of the sponsor leaves it, as usual, with a smaller share than if we treat it separately from the others. While it received 56.419% of the total disposable surplus when we treated it separately, the Shapley value now grants it 49.292% of the total while the nucleolus gives it 48.728%. We also recover the usual ordering of the subgroups with respect to the percentage increases in benefits.

Figure 5.14: Shares, 3-Year Horizon, Real Starting Values, Sponsor Essential



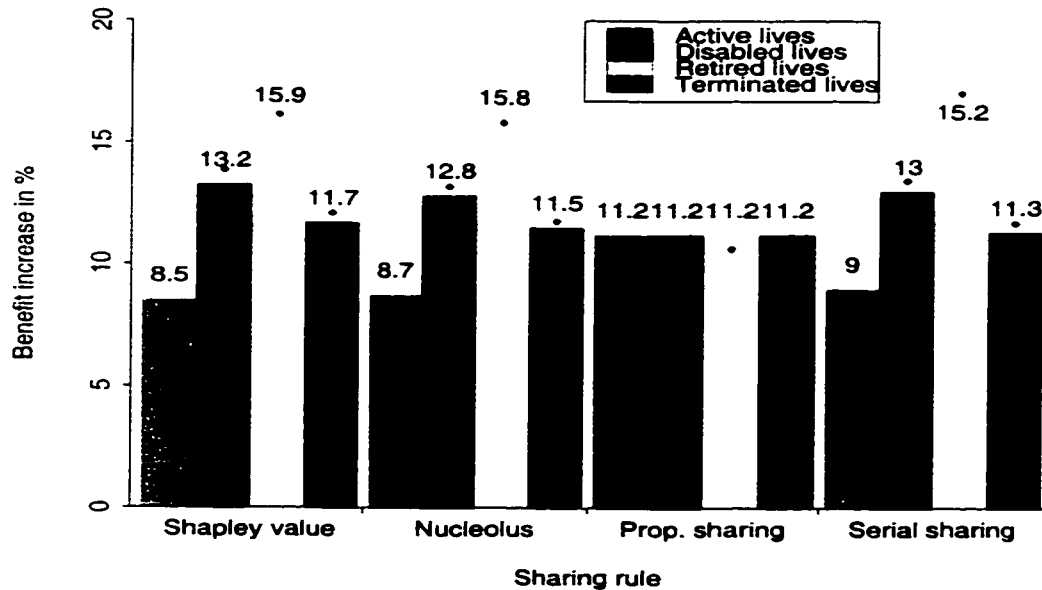
5.11.3 Surplus Shares if Sponsor Is Excluded

In the event that the sponsor is to receive none of the disposable surplus, we need to define reduced games appropriately. We then end up with the shares displayed in Figure 5.15. Had we used neutral starting values instead, as in Section 5.5, the shares indicated by the diamonds would have resulted.

Except for the proportional sharing rule, which grants all lives the same relative increase in benefits, all other rules favor the retired lives, followed by the disabled lives, then the terminated lives, with the active lives last.

Overall, if we compare the results in this section to those obtained in Section 5.5, for which the only difference lies in the inputs to the scenario-generating models, we find that subgroups for which the new scenarios are more favorable than

Figure 5.15: Shares, 3-Year Horizon, Real Starting Values, Sponsor Excluded



the original ones receive more, and vice versa. This is with the exception of the proportional sharing rule which treats everyone equally. This observation concerning the proportional rule actually provides grounds against its use if we wish some monotonicity to hold.

5.12 Surplus Sharing with a 10-Year Horizon and Real Starting Values

Just as we did for the three-year simulation period, we will also compare results obtained for a ten-year horizon with 1985 starting values to those obtained with neutral starting values. Over ten years, using real starting values, we obtain an

average annual rate of return of 8.89% and an average increase in wages of 4.67%.

As we could infer and as we find in Table F.25, these larger return and smaller wage inflation positively impact all disposable surpluses. These surpluses are our inputs to the Shapley value and nucleolus. The total disposable surplus itself turns out to be \$11,811,583, in contrast with \$8,071,440 under neutral starting values.

Table 5.5 still provides the values we use as proxies for contributions when computing proportional shares. As for the serial shares, they will depend on the following worths:

$$\begin{aligned}
 v(\{s, s, s, s, s\}) &= -711,087.36 \\
 v(\{s, d, d, d, d\}) &= 5,360,810.75 \\
 v(\{s, d, t, t, t\}) &= 8,004,943.74 \\
 v(\{s, d, t, r, r\}) &= 9,488,982.10 \\
 v(\{s, d, t, r, a\}) &= 11,811,583.43
 \end{aligned}$$

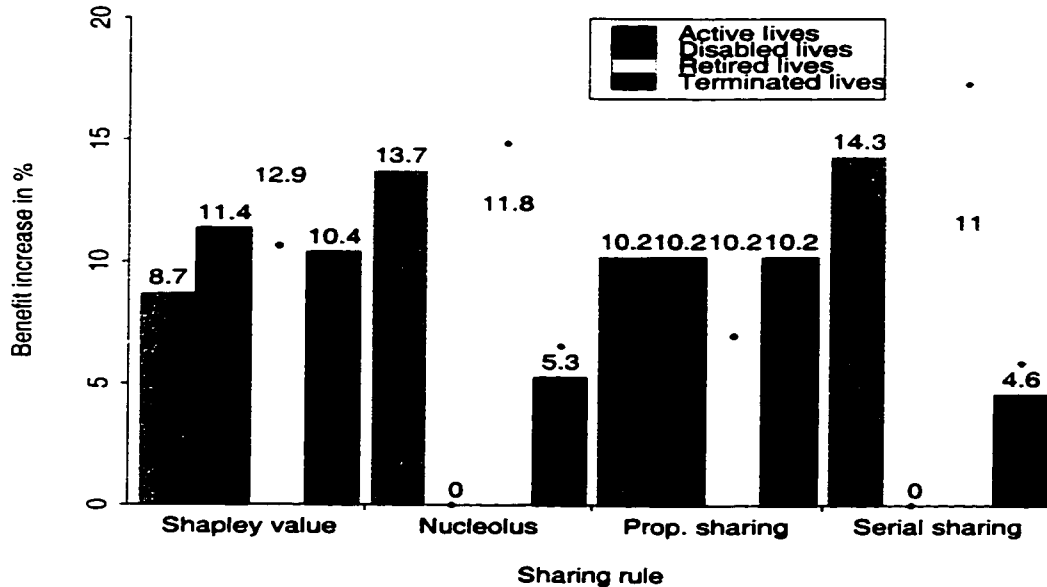
We will derive, one more time, the subgroups' shares using the four sharing rules introduced in Chapter 4 and the three different treatments of the sponsor's role.

5.12.1 Surplus Shares if Sponsor Is Treated Separately

We start by treating the sponsor separately, giving it \$6,663,930, which is 56.419% of the total surplus. We then compute the shares of the other subgroups under the resulting reduced game.

All the resulting benefit increases are given in Figure 5.16, along with diamonds for the ones we originally obtained with neutral starting values in Figure 5.10. Naturally, proportional shares lead to equal percentage increases in benefits for all

Figure 5.16: Shares, 10-Year Horizon, Real Starting Values, Sponsor Separate



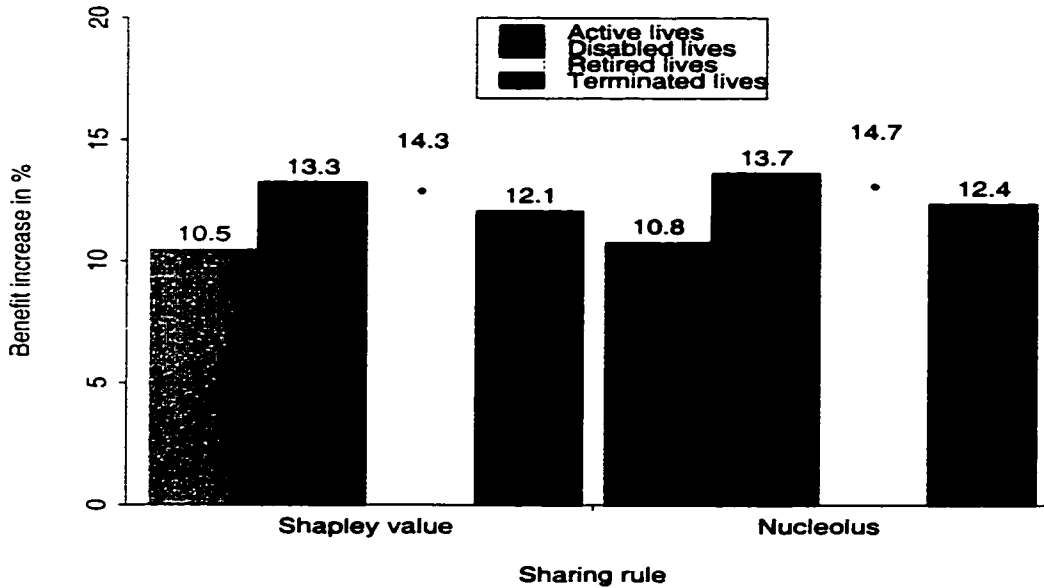
lives. The Shapley value leads to the usual ordering of subgroups with respect to benefit enhancements (retired, disabled, terminated, active). However, the nucleolus and serial sharing rule not only give nothing to the disabled lives, but they also provide relatively more to the active lives. That is in contrast with the shares these rules produced for all of the other games under this initial asset allocation. Otherwise, this mimics all the results we derived so far under this asset allocation and treatment of the sponsor.

5.12.2 Surplus Shares if Sponsor Is Treated as Essential

The other approach we used to give the sponsor a share of the surplus was to treat it as essential to the existence of any pension plan. Applying it once again produces

the shares in Figure 5.17.

Figure 5.17: Shares, 10-Year Horizon, Real Starting Values, Sponsor Essential

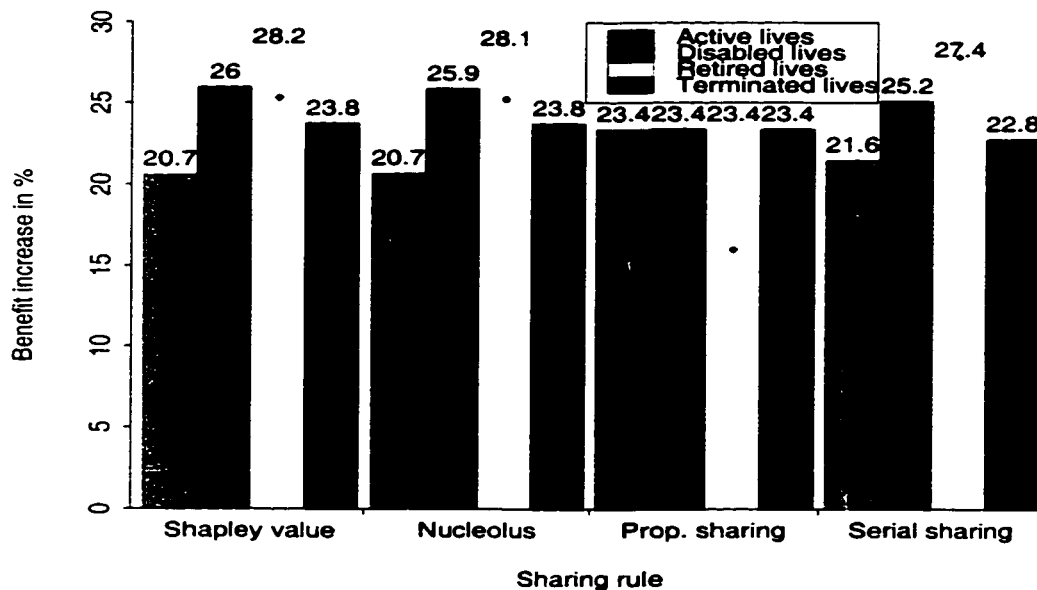


Once more, we end up giving less to the sponsor under this approach than under the previous one (49.175% with the Shapley value and 47.738% with the nucleolus, versus 56.419% under the previous approach). We also recover the same ordering of the subgroups in terms of benefit enhancements, although the differences in the percentages are not as high as they were when we used neutral starting values (compare with the diamonds, which recover the information provided by Figure 5.11).

5.12.3 Surplus Shares if Sponsor Is Excluded

The third treatment we make of the sponsor is its exclusion from the surplus sharing. It requires the definition of reduced games, which lead to the shares shown in Figure 5.18. We again provide, in the form of diamonds, the shares that we obtained under similar conditions except that we used neutral starting values (shares that we originally plotted in Figure 5.12).

Figure 5.18: Shares, 10-Year Horizon, Real Starting Values, Sponsor Excluded



As in the previous subsection, while the ordering of the subgroups according to their benefit increases is the same for the three sharing rules other than the proportional one, the actual percentages are actually closer to one another than they were when we used neutral starting values. Of course, the percentages are also higher in this case than in the other one since we have a larger surplus to

distribute.

5.13 Choice of Treatment of the Sponsor

Based on the few examples we gave, we now wish to make summary comments on the choice of a treatment for the sponsor. We have considered three possibilities, each of which we will review in turn.

First of all, we have treated the sponsor separately from the other subgroups. Under that treatment, we fix its share of the disposable surplus equal to its share of the accumulated contributions. This has the advantage of explicitly accounting for the deficit risk it has assumed in the past through the inclusion of the amortization payments in the accumulated contributions. Furthermore, insofar as we consider the engagements of the members and sponsor to be in proportion to their contributions, this initial split between members and sponsor appears quite natural.

Fixing the sponsor's share in this fashion requires us to define reduced games. Whereas the reduced-game properties give us a clear indication of what the shares in the reduced game are if the shares of the agents excluded from the reduced game are equal to their shares in the grand game, these properties give us no information if the shares of the excluded agents are different from those shares.

Actually, while the Shapley value and nucleolus typically lie close to one another in the simplex, we found them to be appreciably different when we played reduced games. One may think that the explanation resides in the different ways of reducing the game for these two sharing rules. Actually, we tested both types of reduced games on both sharing rules only to find that that did not affect the shares very

much. Rather, we suspect that reducing the game should be avoided since we do not know enough about its theoretical impact on the surplus shares. As we have seen, there are other ways of having the sponsor share in the surplus.

Second, we treated the sponsor as essential to the game. This means that we forced the shares of all coalitions missing the sponsor to be nil, with the understanding that “mini” pension plans could not exist without a sponsor. As we noted earlier, this treatment actually ensures that we define a worth only for coalitions for which it is sensible to do so, namely those that include the sponsor. Indeed, in the absence of a sponsor, our suggested criterion loses much of its meaning as there is then no one to make the amortization payments that will be required with a 5% probability.

With this treatment, we always ended up giving less to the sponsor than when we treated it separately. This needs not be a deterrent. Since increases in assets result in increases in worths, and because the two sharing rules which lend themselves to this treatment (namely, the Shapley value and the nucleolus) enjoy zero independence, we easily can modify the initial asset allocation so as to reproduce the sponsor’s share we obtained when we treated it separately. Similarly, if, instead of accounting for the deficit risk retrospectively, we prefer to account for it prospectively, we can do so as well by modifying the asset allocation as we indicated at the end of Section 5.8.

We consider the split of the normal cost between the participant and the sponsor to be somewhat arbitrary. This split drives the surplus sharing in the separate treatment of the sponsor. However, it has no impact if we treat the sponsor as

essential. We actually believe this to be a great advantage of this treatment of the sponsor since, presumably, in negotiating wages and fringe benefits, employer and employees take both employer and employee contributions into account to determine the value of the compensation package they offer or are being offered. Hence, given that the split of the normal cost is even artificial, we should not want to have it play a role in the distribution of the disposable surplus.

Third, we excluded the sponsor from the sharing. We remind the reader that we might end up in a situation where, by virtue of the pension plan document or of a court order, the sponsor is not entitled to a share in the surplus. Excluding the sponsor is then necessary. When we allocated assets in proportion to the liabilities, the sponsor naturally was given a share of zero. However, under the other asset allocations, we had to play a reduced game to make it zero. As we noted for the separate treatment of the sponsor, reducing games does not appear to be an approach to be privileged.

As a result, unless we opt for treating the sponsor as essential, we suggest that the grand game be played and the resulting shares, computed. If the share going to the sponsor is not that wished, we simply have to make it equal to the desired one and reallocate the difference among the other subgroups in proportion to what they already were allocated. (The difference may be either positive or negative.) This would be a simple way of giving the sponsor a target share without disturbing the relative positions of the other subgroups in the game.

In the end, we suggest that the grand game be solved, with or without making the worth of the coalitions not involving the sponsor equal to zero. If the resulting

shares do not produce the sought one for the sponsor, we then simply have to perform a reallocation which preserves the shares of the other subgroups in relative terms.

5.14 Choice of Sharing Rule

In all the examples we gave, whenever feasible, we applied four sharing rules. Based on our results, we now want to offer our comments and suggestions regarding the choice of one sharing rule in particular.

We recall that, considering the properties we deemed desirable in the context of a pension plan, we ended up singling the Shapley value as the most appropriate one. If we focused on ease of computation, the proportional sharing rule then had no real contender.

Bargaining theory¹⁶ might provide an alternative to the four sharing rules we used. Both the Nash solution and the Kalai-Smorodinsky solutions are remarkably easy to compute. In fact, they split the disposable surplus among the subgroups into five equal shares. (This is not the same as the proportional sharing rule which splits the disposable surplus in proportion to liabilities, and hence into five equal *relative* shares.) As we noted when we discussed the initial asset allocation, giving equal shares gives an advantage to the subgroups with the smallest liabilities. We did not find any justification to do so. Besides, since bargaining theory uses as inputs only the worths of singletons and the worth of the grand coalition, it cannot account for the interactions between the subgroups.

¹⁶We introduce bargaining theory quickly in Appendix E.

Unless we want to treat everyone equally, in which case we naturally should resort to the proportional sharing rule, we must consider using one of the other three rules we presented in Chapter 4. Out of these three rules, the Shapley value is our choice for several reasons. First, we can use it for all treatments of the sponsor. Second, reduced games did not seem to alter the shares produced by the Shapley value to any extent comparable to those obtained with the nucleolus or the serial sharing rule. Third, it enjoys zero independence and that allows us to adjust the sponsor's share directly via an asset reallocation if we want to. Fourth and most importantly, it is strongly monotonic; hence, those who contribute more to the emergence of a surplus receive more, while those who contribute less receive less.

The nucleolus is appealing in that it aims at pleasing, or displeasing, all subgroups and combinations of subgroups equally. However, its failure to satisfy aggregate monotonicity represents a major drawback. As for the serial sharing rule, while it enjoys aggregate monotonicity, it does not satisfy coalitional monotonicity. Moreover, our inability to verify certain properties results in our inability to assess fully the theoretical interest of this rule from an axiomatic point of view.

5.15 Impact of Nature of Distribution

Our criterion for the determination of the amount of surplus we can distribute implicitly assumes that the distribution will be made in cash. If that is the case, regardless of which particular subgroups receive the amount, the resulting probability of a deficit at the end of the horizon is exactly as targeted.

However, if, as would appear more sensible in the case of a pension plan, the members' shares would be distributed in the form of enhanced benefits, we no longer preserve our target probability of a deficit. To illustrate this, we have used the shares obtained when excluding the sponsor from the sharing¹⁷, enhanced benefits accordingly and checked how many simulations then resulted in a deficit at the end of the horizon. In all cases, we have allocated assets at the onset among the members only, in proportion to their liabilities. The results are given in Table 5.10. If we had retained our target probability, we would have expected 499 negative scenarios with the three-year period, and 1499 with the ten-year period.

Table 5.10: Number of Scenarios with Negative Disposable Surplus

Years in Horizon	Starting Values	Shapley Value	Nucleolus	Proportional Sharing	Serial Sharing
3	Neutral	1029	1047	1296	1018
3	Real	1135	1148	1358	1168
10	Neutral	2195	2195	2646	2160
10	Real	2714	2717	2915	2782

As we can see, we effectively increase the probability of a deficit when we use the shares to enhance the benefits. This is so because, when we use money to enhance benefits, we do so on an expected basis. We equate the amount we have to give with what we expect a certain benefit increase to be worth. However, the actual cost of a benefit increase is only known when the future unfolds and benefits become payable. Hence, for unfavorable scenarios, we end up in a situation in which the increases granted are actually worth more than we had thought and, if

¹⁷We used the shares given in Tables F.4, F.24, F.20 and F.28.

the difference is high enough, we end up with a deficit.

By comparing the shares underlying the results in a given row, we find that, the more we give to the active lives, the more we increase the likelihood of a deficit. As noted earlier, unlike the other subgroups composed of members, this one is sensitive to variability in both the rate of return and the wage increase. This greater sensitivity translates into a wider range for the realized cost of the benefit increase, and hence a larger number of scenarios in which this realized cost will wipe away the initial surplus.

Given that pension legislation will not allow cash distributions to members unless they have already reached certain maxima pertaining to accrued benefits, this puts back into question the way we apply our criterion. We must, at the very least, be aware that the fashion in which we will make the distribution will impact the resulting probability of a deficit.

Of course, since we gave nothing to the sponsor in the examples we have just reported on, we have come up with the worst possible cases. Whether the sponsor receives its share in cash or uses it up via contribution holidays, the effect on the probability of a deficit is nil. Hence, giving some share of the surplus to the sponsor can only decrease the probability of a deficit compared to the extreme cases given above.

Assessing the probability of a deficit *ex post* is one thing, but it would be better to account for the mode of distribution *ex ante*, when we decide on the particular amount we choose to part with. This shows, once more, that the decisions of what to give and whom to give it to are intimately linked and we cannot assume the first

to be independent from the second. Since the second clearly depends on the first (repeated for each coalition), we end up with somewhat of a circular problem that would justify further research in the future.

5.16 Conclusion

We have begun the chapter with an overview of the stationary membership and past evolution of the pension plan, from 1965 to 1986, at both the individual and aggregate levels. We ended up with a valuation surplus of \$19,211,884, of which we gave different proportions depending on the horizon and target probability we wished to make use of.

We tested three different ways of allocating assets among the subgroups at the start of the simulations. We recall that we need to perform such an allocation in order to be able to apply our criterion for the disposable surplus to subgroups as well as to combinations.

We found that making the members' assets merely equal to the liabilities leads to an ordering of the shares with respect to the variance of the simulated surpluses. It also greatly advantages the sponsor when we do not fix its share before playing the grand game.

Making members' assets equal to their accumulated fund greatly disadvantages all members but the terminated lives. With such an asset allocation, the subgroups that gain are those that, under the terms of the plan, end up "subsidizing" the other groups. Typically, in the examples we gave, the terminated lives shared the surplus with the sponsor.

Out of the three alternatives, this leaves us with, as the one to use, allocating all the assets to the members in proportion to their liabilities. While this leaves no assets for the sponsor, we can remedy to it by endowing it with an amount in relation to the expected unfunded liability. We can also provide for the sponsor through one the three treatments used in each example.

Forcing the sponsor's share to be proportional to its accumulated contributions and using reduced games led to the exclusion of the disabled lives for two of the four sharing rules. This would appear to be a case against either these two rules or this way of setting the sponsor's share.

Making the sponsor play an essential role in our surplus sharing games turned out to produce qualitatively similar results without excluding any subgroup. We noted that, even though we excluded the sponsor in calculating the shares under the third approach, we could decide to give the sponsor a certain share and adjust the other shares proportionally. Either approach would appear reasonable to us. The final choice would depend on the importance we give to the split of the normal cost as well as to the actual interpretation of our suggested criterion.

For all the different cases we studied, whenever possible, we computed the shares using the four sharing rules introduced in Chapter 4. We noticed that the nucleolus and serial sharing rule led to the same exclusions, when there were any. For their lack of coalitional monotonicity, we would reject them. We also would reject the proportional sharing rule, unless we really wish to enhance all the benefits equally.

This leaves us with the Shapley value. This rule (like the nucleolus) has the advantage of allowing us to treat the sponsor as essential to the plan. It also appears

to respond well to reduced games. It is zero independent and strongly monotonic. For all these reasons, we would single out the Shapley value as the best sharing rule to use out of the four we have used.

Though we worked with both a three-year and a ten-year forecasting period, we would advise against the use of the latter on the grounds that this effectively excludes more people from the sharing. If we want to be more certain that we will not be facing a deficit in the long run, it may be better to actually repeat the sharing exercise at three-year intervals over time and see what happens. Besides, it appears natural to us to make the sharing coincide with the valuation.

We also looked at the impact of using real starting values instead of neutral ones in our models. We found different shares and disposable surpluses depending on the starting values used. While the differences may not have appeared that considerable with a three-year horizon, this still highlights the fact that we should try to come up with a model for the future with which we are comfortable. Through the criterion used to calculate the disposable surplus, the simulated futures have a sizable impact on the resulting shares.

In summary, we recommend using a forecasting period corresponding to the time between valuations, allocating all assets to the members in proportion to their liabilities, and determining shares using the Shapley value. As for the actual treatment of the sponsor, since we picked the Shapley value, all the suggested approaches actually remain valid, even though we favor treating it as essential. Also, whether we should use neutral or real starting values depends on what we believe about the future.

We wish to make one final point to conclude this chapter. The criterion and distribution method we developed really are meant to be applied over time to an ongoing plan. (They naturally could and would be applied only once at the termination of a pension plan.) Thus, a full assessment of the potential interest of the tools we developed would require a study of results over time. This remains a subject for future research.

Chapter 6

Conclusion

We started out with two objectives in this thesis. First, we wanted to come up with a criterion to determine the amount of surplus we could dispose of at any point in time. Second, we wanted to devise a way to allocate that amount among the interested parties.

We will summarize our work on these two fronts and indicate further work one could undertake to complement what has been done here.

6.1 Criterion for the Amount to Distribute

In meeting our first goal, we wanted a criterion that would address two key concerns of a pension plan: solvency, and stability of contributions. This led us to formulating the following criterion for the amount to distribute:

The amount must be such that, with a given level of probability, there will not be an unfunded liability at the next actuarial valuation.

This is equivalent to wishing that the sponsor not have to make any payment in addition to the normal cost in the years following the next valuation.

Instead of focusing on the next actuarial valuation, one could work with a longer horizon, thus increasing the level of prudence. Likewise, one could choose to target a higher probability of remaining above a certain funding level. Though we have applied our criterion with horizons of three and ten years, we still think it is preferable to make the distribution of the surplus coincide with the valuations both for practical reasons and to allow as many as possible to share in surpluses arising over time.

Since our suggested criterion looks into the future, it requires us to model the evolution of the pension plan over time. For this thesis, we have treated the rate of return and the increase in wages as the only two random variables, assuming all the others to be deterministic. Different economic scenarios led to different disposable surpluses, the collection of which we could then apply our criterion to. The particular choice of models for the random variables, and even the choice of seed values, is very material to the ensuing shares. Hence, care should be taken in picking models that are deemed to generate plausible futures.

6.2 Distribution of that Amount

To meet the second goal, we surveyed the theory of cooperative games. While a pension plan does not qualify as a cooperative game in the purely theoretical sense of the term, it does rely heavily on the idea of an enforceable contract (the plan document) for its existence, the same idea that underpins cooperation.

In general terms, sharing rules require the knowledge of the surplus that any subset of players could have achieved. In the case we are studying, this calls for a repeated application of our criterion to different subgroups of plan members. It itself requires an initial asset allocation among the subgroups, to which the final shares are very sensitive.

We considered three different asset allocations, as well as using game theory to allocate the assets. We ended up favoring allocating all the assets to the current members proportionally to their liabilities. Out of all alternatives, it was the only one that did not favor one subgroup in particular over all the others, without a valid reason to support the bias.

Because this left the sponsor with no assets, we suggested allocating it the expected unfunded liability. We also indicated that we could increase the assets of the sponsor and deplete those of the other subgroups proportionally so as to achieve the desired sponsor's share.

We also observed that we could fix the sponsor's share by using one of three different treatments. Treating it separately presented some irregularities, due possibly to the need to define reduced games, which we thought we should avoid. Excluding the sponsor or, rather, playing the grand game without any modifications and then adjusting the sponsor's share and the other ones proportionally offered a valid alternative. Otherwise, we still favor treating the sponsor as essential. This does not depend on the split of the normal cost. More importantly, it does not require the application of our criterion to combinations of subgroups for which its application would be dubious.

Taking the many surpluses defining the characteristic function as inputs, sharing rules then prescribe how to share the total surplus among the different players. Different rules lead to potentially different shares and the choice of a specific rule can be based on the particular properties it satisfies.

Out of the four rules we applied to our model pension plan, the Shapley value stood out. It consistently provided the best benefit increases to the retired lives, followed by the disabled and terminated lives, with the active lives last. This ordering appears natural if we consider the coefficients of variation of the simulated surpluses. Moreover, it allowed us to find the sponsor's share *along with* the shares of the members by treating the sponsor as essential¹. In terms of important axioms it satisfies, it is zero independent and strongly monotonic. One of its main drawbacks, though, is that it requires inputs whose number increases exponentially with the number of players considered. It is however very easy to compute.

6.3 Further Work

We believe that the application we made of our suggested approach reveals it has potential. Still, further work is needed to show its applicability and relevance.

To start with, one should apply our approach to pension plans with characteristics that differ from those of our model pension plan to see if the results then obtained still appear to be reasonable. In particular, one should consider different pension plan memberships so as to study the impact of plan composition on the

¹We say that the sponsor is essential in the sense that a pension plan could not exist without it and hence there can be no disposable surplus if the sponsor is not one of the players.

resulting shares.

As mentioned in our thesis, the approach we suggest really is meant to be applied at every valuation (or at the end of each simulation period). Since we performed applications at a single point in time only, we cannot make observations on the validity of our approach through time. In repeating the application, we believe one should look not only at the shares for particular subgroups, but also for particular members as they potentially move from one subgroup to another. We also note that a repeated application would require special care in coming up with the initial asset allocation so as to account for past distributions.

One more hurdle that one should attempt to surmount is the necessity to account for the mode of distribution of the shares in computing the actual disposable surplus for each combination of subgroups. We noted that enhancing benefits instead of giving cash effectively increased the probability of an unfunded liability. Properly resolving this apparent circularity is essential to the distribution of a surplus amount that, in actual fact, does not jeopardize the financial health of the pension plan to an extent greater than that which we are willing to assume.

While we believe our approach to be of interest, we admit there may be other ways to accomplish the two goals we set forth. In determining the amount to distribute, one could explore statistical decision theory, which looks into making decisions when faced with uncertainty. Non-cooperative game theory may also offer an alternative insofar as we can identify competing interests among the many parties to the plan, including the taxation authority and the legislative body.

In choosing how to distribute the amount, one could apply other sharing rules

found in the theory of cooperative games. Again, non-cooperative game theory may provide an alternative.

In conclusion, we at least hope we have shown that the sharing of a pension plan surplus could lend itself to theoretical approaches rather than only to *ad hoc* treatments for which we can only provide intuitive justification, and that these approaches were interesting in their own right.

Appendix A

Tables of Decrements

Following are the different decrement tables used to generate the results found in the thesis. The first table contains the retirement rates, adapted from [71]. These rates really are $q_x^{(r)}$ in the theory of multiple decrements (see *Actuarial Mathematics* [11]). To obtain $q_x^{(r)}$, we used the following equation:

$$q_x^{(r)} = q_x^{(r)}(1 - q_x^{(d)} - q_x^{(m)} - q_x^{(t)}).$$

Second is the termination table, which is labeled TTW02 in *Pension Tables for Actuaries* [53]. Third is the disability table, taken from *Fundamentals of Private Pensions* [37].

We applied the same mortality rates to active, retired and terminated lives. These rates are given by the 1983 Group Annuity Mortality Table developed by the Society of Actuaries Committee on Annuities [16], and are reproduced in the fourth table.

As for the mortality of a disabled life, we assumed it was equal to that of a non-disabled life 5 years older. Hence, $q_x^{(D)} = q_{x+5}^{(m)}$. To keep the limiting age the same, however, we have linearly interpolated between $q_{104}^{(D)}$ and $q_{110}^{(D)} = 1$.

Table A.1: Retirement Rates

Age	$q_x^{(r)}$
54	0.0500
55	0.0500
56	0.0500
57	0.0500
58	0.0500
59	0.2000
60	0.3000
61	0.4000
62	0.3000
63	0.3000
64	0.7500
65	0.7500
66	0.7500
67	0.7500
68	0.7500
69	1.0000

Table A.2: Termination Rates

Age	$q_x^{(w)}$	Age	$q_x^{(w)}$
20	0.1121		
21	0.1061	46	0.0428
22	0.1018	47	0.0416
23	0.0975	48	0.0403
24	0.0933	49	0.0387
25	0.0891	50	0.0365
26	0.0851	51	0.0328
27	0.0810	52	0.0269
28	0.0770	53	0.0198
29	0.0731	54	0.0120
30	0.0692	55	0.0036
31	0.0654	56	0.0000
32	0.0618	57	0.0000
33	0.0588	58	0.0000
34	0.0565	59	0.0000
35	0.0546	60	0.0000
36	0.0531	61	0.0000
37	0.0519	62	0.0000
38	0.0508	63	0.0000
39	0.0498	64	0.0000
40	0.0488	65	0.0000
41	0.0478	66	0.0000
42	0.0468	67	0.0000
43	0.0459	68	0.0000
44	0.0449	69	0.0000
45	0.0439	70	0.0000

Table A.3: Disability Rates

Age	$q_x^{(d)}$	Age	$q_x^{(d)}$
20	0.0013		
21	0.0013	46	0.0070
22	0.0013	47	0.0075
23	0.0014	48	0.0085
24	0.0015	49	0.0096
25	0.0016	50	0.0106
26	0.0017	51	0.0116
27	0.0017	52	0.0126
28	0.0019	53	0.0144
29	0.0020	54	0.0163
30	0.0021	55	0.0181
31	0.0022	56	0.0199
32	0.0023	57	0.0218
33	0.0025	58	0.0230
34	0.0027	59	0.0242
35	0.0029	60	0.0255
36	0.0031	61	0.0267
37	0.0033	62	0.0280
38	0.0037	63	0.0280
39	0.0040	64	0.0280
40	0.0043	65	0.0000
41	0.0046	66	0.0000
42	0.0049	67	0.0000
43	0.0054	68	0.0000
44	0.0060	69	0.0000
45	0.0065	70	0.0000

Table A.4: Non-Disabled Mortality Rates

Age	$q_x^{(m)}$
20	0.000377
21	0.000392
22	0.000408
23	0.000424
24	0.000444
25	0.000464
26	0.000488
27	0.000513
28	0.000542
29	0.000572
30	0.000607
31	0.000645
32	0.000687
33	0.000734
34	0.000785
35	0.000860
36	0.000907
37	0.000966
38	0.001039
39	0.001128
40	0.001238
41	0.001370
42	0.001527
43	0.001715
44	0.001932
45	0.002183
46	0.002471
47	0.002790
48	0.003138
49	0.003513
50	0.003909

Age	$q_x^{(m)}$
51	0.004324
52	0.004755
53	0.005200
54	0.005660
55	0.006131
56	0.006618
57	0.007139
58	0.007719
59	0.008384
60	0.009158
61	0.010064
62	0.011133
63	0.012391
64	0.013868
65	0.015592
66	0.017579
67	0.019804
68	0.022229
69	0.024817
70	0.027530
71	0.030354
72	0.033370
73	0.036680
74	0.040388
75	0.044597
76	0.049388
77	0.054758
78	0.060678
79	0.067125
80	0.074070

Age	$q_x^{(m)}$
81	0.081484
82	0.089320
83	0.097525
84	0.106047
85	0.114836
86	0.124170
87	0.133870
88	0.144073
89	0.154859
90	0.166307
91	0.178214
92	0.190460
93	0.203007
94	0.217904
95	0.234086
96	0.248436
97	0.263954
98	0.280803
99	0.299154
100	0.319185
101	0.341086
102	0.365052
103	0.393102
104	0.427255
105	0.469531
106	0.521945
107	0.586518
108	0.665268
109	0.760215
110	1.000000

NOTE TO USERS

**Page(s) missing in number only; text follows.
Microfilmed as received.**

244

This reproduction is the best copy available.

UMI

Appendix B

Salary Scale

The following table gives the portion of the salary increases that accounts for merit only. It is labeled SA2 in *Pension Tables for Actuaries* [53].

Table B.1: Salary Scale

Age	s_x	Age	s_x
20	0.61875		
21	0.62635	46	0.83585
22	0.63355	47	0.84642
23	0.64012	48	0.85710
24	0.64642	49	0.86782
25	0.65235	50	0.87860
26	0.65777	51	0.88945
27	0.66302	52	0.90037
28	0.66827	53	0.91128
29	0.67440	54	0.92207
30	0.68227	55	0.93277
31	0.69100	56	0.94332
32	0.69975	57	0.95352
33	0.70867	58	0.96315
34	0.71782	59	0.97202
35	0.72705	60	0.97995
36	0.73625	61	0.98685
37	0.74562	62	0.99252
38	0.75525	63	0.99665
39	0.76492	64	0.99905
40	0.77465	65	1.00000
41	0.78455	66	1.00080
42	0.79462	67	1.00130
43	0.80482	68	1.00170
44	0.81510	69	1.00210
45	0.82542	70	1.00230

Appendix C

Economic Data from 1924 to 1997

The *Report on Canadian Economic Statistics* [17] provides a variety of economic data, including the pension median rate of return, rate of inflation, and wage increase, as reproduced in Table C.2. Several other economic variables are included in that *Report*, some of which are needed if we wish to use actual data as inputs to the economic models we use.

The pension median rate of return actually is given only for the years 1960 and later. For the previous years, we used the historical returns on stocks, bonds, bills and mortgages to generate these rates of return in the same way that Maynard [36] did. The actual weights given to each asset category vary over the period 1924-1960 since returns for the four categories are not all available throughout. These weights are given in the following table, while the table after that one contains the historical data.

Table C.1: Weights Given to Each Category of Assets

Period	Stocks	Bonds	Bills	Mortgages
1924 - 1945	37%	63%		
1946 - 1951	37%	58%	5%	
1952 - 1959	37%	38%	10%	15%

Table C.2: Economic Data from 1924 to 1997

Year	$r(\text{Year})$	inf^{Year}	w^{Year}	Year	$r(\text{Year})$	inf^{Year}	w^{Year}
1924	9.10	-2.13	0.11	1961	13.30	0.42	3.11
1925	13.89	2.90	-0.22	1962	2.00	1.67	1.53
1926	12.43	-1.41	1.41	1963	8.10	1.64	4.08
1927	23.03	-1.43	1.72	1964	11.10	2.02	4.77
1928	12.53	0.72	1.48	1965	3.50	3.16	6.48
1929	-2.82	2.88	1.15	1966	-2.30	3.45	5.47
1930	-5.60	-6.29	-1.35	1967	7.60	4.07	6.78
1931	-15.33	-9.70	-5.26	1968	9.40	3.91	7.29
1932	3.01	-8.26	-6.12	1969	-3.20	4.79	6.32
1933	23.75	-1.80	-2.08	1970	1.30	1.31	8.90
1934	19.88	0.92	1.90	1971	12.50	5.16	10.67
1935	11.86	2.73	2.35	1972	18.40	4.91	7.74
1936	16.39	0.88	4.64	1973	-2.10	9.36	6.88
1937	-6.22	4.39	5.21	1974	-12.70	12.30	13.36
1938	6.93	-2.52	1.68	1975	13.20	9.52	14.42
1939	-1.81	2.59	2.13	1976	12.40	5.87	11.20
1940	-1.60	5.04	6.63	1977	8.70	9.45	8.04
1941	3.11	6.40	7.12	1978	13.50	8.44	6.41
1942	7.12	3.01	7.49	1979	15.00	9.69	8.60
1943	9.72	1.46	5.49	1980	18.00	11.20	11.45
1944	6.97	-1.44	2.01	1981	1.60	12.20	11.41
1945	16.60	1.46	0.98	1982	22.60	9.23	9.54
1946	2.96	5.76	6.28	1983	20.00	4.51	7.69
1947	1.98	14.97	11.06	1984	9.20	3.77	3.10
1948	3.13	8.88	8.95	1985	23.60	4.38	3.93
1949	11.20	1.09	6.08	1986	13.40	4.19	2.73
1950	17.88	5.91	7.93	1987	3.80	4.12	4.60
1951	7.12	10.66	9.86	1988	10.40	3.96	4.19
1952	1.48	-1.38	7.22	1989	15.90	5.17	5.00
1953	2.66	0.00	4.17	1990	-0.80	5.00	4.68
1954	19.52	0.00	3.01	1991	17.60	3.78	4.23
1955	11.31	0.47	4.47	1992	7.00	2.14	3.27
1956	3.44	3.24	5.48	1993	21.40	1.70	0.82
1957	-4.31	1.79	4.51	1994	-0.70	0.23	2.05
1958	10.84	2.64	3.97	1995	17.40	1.75	1.17
1959	0.66	1.29	3.77	1996	18.80	2.17	3.10
1960	9.50	1.27	3.24	1997	14.80	0.73	0.85

NOTE TO USERS

**Page(s) missing in number only; text follows.
Microfilmed as received.**

250

This reproduction is the best copy available.

UMI

Appendix D

Parameters for Economic Models

The parameters we used for our economic simulations are the same Wilkie [70] and Sharp [61] gave in their papers. They are given in the table below, by model component, and are based on Canadian data.

Table D.1: Parameters for Economic Models

Model Component	Parameter	Value
Price inflation	<i>QMU</i>	0.034
	<i>QA</i>	0.640
	<i>QSD</i>	0.032
Dividend yield	<i>YW</i>	1.170
	<i>YA</i>	0.700
	<i>YMU</i>	0.038
	<i>YSD</i>	0.190
Dividends	<i>DW</i>	0.190
	<i>DD</i>	0.260
	<i>DMU</i>	0.001
	<i>DY</i>	-0.110
	<i>DB</i>	0.580
	<i>DSD</i>	0.070
Bond yield	<i>CW</i>	1.000
	<i>CD</i>	0.040
	<i>CA</i>	0.950
	<i>CMU</i>	0.037
	<i>CY</i>	0.100
	<i>CSD</i>	0.185
Bill yield	<i>BMU</i>	-0.260
	<i>BA</i>	0.380
	<i>BC</i>	0.730
	<i>BSD</i>	0.210
Wage inflation	<i>WW</i>	0.408
	<i>WMU</i>	0.035
	<i>WA</i>	0.703
	<i>WSD</i>	0.017

Appendix E

Bargaining Theory

There are different approaches to bargaining, but we will focus on the axiomatic approach. Under this approach, we formulate desirable properties and find solutions which satisfy these properties.

The following introduction to bargaining theory is based on Chapter 2 of *Bargaining and Markets* [43]. It assumes that the reader is familiar with cooperative game theory as introduced in Chapter 4.

Bargaining refers to a situation in which people, with competing interests, can write a mutually beneficial contract. However, no individual may be forced to give his approval.

To define a bargaining problem, we must know three elements:

- the set of bargainers (or agents) N ;
- the set of feasible utility vectors $S \in \mathfrak{R}^n$;
- a disagreement point $d \in S$.

Typically, bargaining theory applies to NTU games and the set S contains the utility vectors associated with every possible bargaining outcome. It is assumed that S is convex, closed and bounded. In the context of TU games, with a finite amount of good to divide, we then have that S is a simplex in the positive quadrant, and thus satisfies our assumptions.

We also assume that there is an element $s \in S$ such that $s_i > d_i$, $i = 1, \dots, n$. Hence, there is an incentive to reach an agreement since there exists an outcome which is strictly better than the disagreement outcome for each and every agent. The disagreement outcome will prevail in the case that the agents fail to reach an agreement.

When we go from a cooperative game to a bargaining game, we keep the same set of agents N and define S to be $\{x \in \mathbb{R}_+^n \mid \sum_{i=1}^n x_i \leq v(N)\}$. Also, we normally define the disagreement point to be the vector of the worths of the singletons.

In the context of a pension plan, at the disagreement point, not all of the disposable surplus is being distributed. This would leave more to be distributed at the next valuation. To properly account for the incentives bargainers have to reach an agreement now rather than wait for the next time they will play the game, we would need to consider sequential games. Nevertheless, for this thesis, as we indicated earlier, we focus on one-time applications and keep the extension to several periods as a subject for further work.

A bargaining solution is a mapping

$$\begin{aligned} f: \quad B &\rightarrow \mathbb{R}^n \\ (S, d) &\rightarrow f(S, d) \end{aligned}$$

which associates to every bargaining problem in \mathcal{B} (the set of all such problems) a particular outcome $f(S, d) \in S$.

Here are five properties that a bargaining solution may enjoy (axioms we may wish to hold):

INV We say that f satisfies *invariance to equivalent utility representations* if, for any $\alpha \in \mathbb{R}_{++}^n$ and $\beta \in \mathbb{R}^n$,

$$s'_i = \alpha_i s_i + \beta_i, \quad \forall i$$

and

$$d'_i = \alpha_i d_i + \beta_i, \quad \forall i$$

imply that

$$f_i(S', d') = \alpha_i f_i(S, d) + \beta_i, \quad \forall i.$$

SYM We say that f satisfies *symmetry* if, whenever (S, d) is symmetric,

$$f_i(S, d) = f_j(S, d), \quad \forall i, j = 1, \dots, n.$$

IIA We say that f satisfies *independence of irrelevant alternatives* if, for two bargaining problems (S, d) and (T, d) ,

$$S \subset T \text{ and } f(T, d) \in S \Rightarrow f(S, d) = f(T, d).$$

PAR We say that f satisfies *Pareto efficiency* if, for any given bargaining problem (S, d) ,

$$t_i > s_i \quad \forall i, \quad s, t \in S \Rightarrow f(S, d) \neq s.$$

MON We say that f satisfies *monotonicity in the ideal point*¹ if, for $S \subset T$,

¹ The ideal point a for a bargaining problem (S, d) is defined via its components:
 $a_i(S, d) = \max_s \{s_i \mid s \in S, s \geq d\}.$

$$a(S, d) = a(T, d) \Rightarrow f(T, d) \geq f(S, d).$$

There is only one bargaining solution satisfying the axioms INV, SYM, IIA and PAR. It is the *Nash solution*. This solution is given by

$$f^N(S, d) = \arg \max_{d \leq s \in S} \prod_{i=1}^n (s_i - d_i).$$

There is also only one bargaining solution satisfying the axioms INV, SYM, PAR and MON. It is the *Kalai-Smorodinsky solution*. This solution $f^{KS}(S, d)$ is given by the maximal member of S on the line joining d and $a(S, d)$.

In general, these two solutions do not yield the same results. However, in the case of TU games, since these games are symmetric, we always end up with an equal division of the amount in excess of that already allocated at the disagreement point.

Appendix F

Tables of Results

We provide here the characteristic functions and the tables of results that correspond to the figures given in Chapter 5. Whereas the figures focus on percentages of benefit increase, the tables also give the dollar share of each subgroup. Because subgroups vary in size, we felt that benefit increases were more meaningful and relevant.

Table F.1: Disposable Surpluses, 3-Year Horizon, Initial Assets Proportional to Actuarial Liability

adsrt	Disposable Surplus	95% Confidence Interval	
		Lower Bound	Upper Bound
10000	1,511,303.02	1,266,853.96	1,746,289.06
01000	715,865.45	684,749.34	768,171.38
00100	-21,368.78	-22,001.29	-20,889.85
00010	1,700,242.26	1,623,802.37	1,794,048.15
00001	1,136,876.27	1,073,895.25	1,223,248.27
11000	2,333,133.25	2,048,393.50	2,630,603.79
10100	1,501,812.94	1,248,363.24	1,734,584.79
10010	3,396,443.94	3,121,538.34	3,697,320.60
10001	2,804,297.04	2,556,165.13	3,123,938.51
01100	704,718.50	672,110.02	756,316.37
01010	2,399,953.36	2,303,700.17	2,569,377.76
01001	1,861,692.31	1,754,348.11	2,001,247.88
00110	1,686,272.85	1,611,167.19	1,783,028.75
00101	1,132,651.40	1,063,849.36	1,210,004.97
00011	2,822,144.84	2,701,031.08	3,025,800.40
11100	2,320,151.51	2,030,613.29	2,618,227.57
11010	4,230,124.85	3,797,976.60	4,494,821.90
11001	3,642,842.65	3,293,700.64	3,933,229.18
10110	3,380,904.20	3,105,982.31	3,684,435.91
10101	2,791,872.28	2,542,087.73	3,113,433.42
10011	4,673,601.81	4,277,510.06	4,973,816.04
01110	2,397,570.14	2,296,584.34	2,556,244.45
01101	1,856,158.08	1,743,355.05	1,985,887.54
01011	3,536,815.22	3,385,569.59	3,792,267.55
00111	2,811,976.99	2,690,201.09	3,017,675.56
11110	4,213,575.21	3,795,606.02	4,483,268.41
11101	3,626,969.37	3,290,834.29	3,918,778.61
11011	5,392,042.47	4,974,599.25	5,825,589.44
10111	4,657,659.32	4,261,008.18	4,957,487.31
01111	3,526,234.98	3,376,340.69	3,785,846.94
11111	5,361,093.61	4,967,671.27	5,808,146.81

Table F.2: Shares, 3-Year Horizon, Initial Assets Proportional to Actuarial Liability, Sponsor Separate

Subgroup	Shapley Value	Nucleolus	Proportional Sharing	Serial Sharing
Active lives	773,756 (3.155%)	1,045,259 (4.262%)	1,137,025 (4.637%)	892,907 (3.641%)
Disabled lives	337,239 (6.333%)	0 (0.000%)	246,907 (4.637%)	0 (0.000%)
Sponsor	3,024,654	3,024,654	3,024,654	3,024,654
Retired lives	740,178 (6.847%)	939,231 (8.689%)	501,202 (4.637%)	1,073,013 (9.927%)
Terminated lives	485,267 (4.986%)	351,950 (3.616%)	451,306 (4.637%)	370,520 (3.807%)

Table F.3: Shares, 3-Year Horizon, Initial Assets Proportional to Actuarial Liability, Sponsor Essential

Subgroup	Shapley Value	Nucleolus
Active lives	879,330 (3.586%)	917,430 (3.741%)
Disabled lives	370,129 (6.951%)	351,717 (6.605%)
Sponsor	2,643,335	2,651,125
Retired lives	877,645 (8.119%)	867,062 (8.021%)
Terminated lives	590,655 (6.068%)	573,760 (5.895%)

Table F.4: Shares, 3-Year Horizon, Initial Assets Proportional to Actuarial Liability, Sponsor Excluded

Subgroup	Shapley Value	Nucleolus	Proportional Sharing	Serial Sharing
Active lives	1,700,983 (6.936%)	1,804,566 (7.359%)	2,608,968 (10.639%)	1,662,310 (6.779%)
Disabled lives	738,302 (13.865%)	701,901 (13.181%)	566,541 (10.639%)	716,445 (13.454%)
Sponsor	0	0	0	0
Retired lives	1,745,146 (16.145%)	1,709,247 (15.812%)	1,150,037 (10.639%)	1,842,416 (17.044%)
Terminated lives	1,176,663 (12.089%)	1,145,380 (11.768%)	1,035,548 (10.639%)	1,139,923 (11.711%)

Table F.5: Disposable Surpluses, 3-Year Horizon, Initial Assets Equal to Actuarial Liability

adsrt	Disposable Surplus	95% Confidence Interval	
		Lower Bound	Upper Bound
10000	-7,838,129.18	-8,082,578.23	-7,603,143.13
01000	-1,314,376.46	-1,345,492.58	-1,262,070.53
00100	19,190,515.60	19,189,883.08	19,190,994.52
00010	-2,421,003.15	-2,497,443.04	-2,327,197.26
00001	-2,574,088.59	-2,637,069.61	-2,487,716.59
11000	-9,046,540.86	-9,331,280.61	-8,749,070.32
10100	11,364,265.12	11,110,815.42	11,597,036.97
10010	-10,074,233.67	-10,349,139.26	-9,773,357.00
10001	-10,256,100.02	-10,504,231.92	-9,936,458.55
01100	17,886,360.97	17,853,752.48	17,937,958.83
01010	-3,751,533.96	-3,847,787.15	-3,582,109.56
01001	-3,879,514.47	-3,986,858.66	-3,739,958.89
00110	16,776,911.82	16,701,806.16	16,873,667.72
00101	16,633,570.91	16,564,768.87	16,710,924.48
00011	-5,010,065.43	-5,131,179.19	-4,806,409.87
11100	10,152,361.77	9,862,823.56	10,450,437.84
11010	-11,270,794.67	-11,702,942.92	-11,006,097.62
11001	-11,447,796.32	-11,796,938.33	-11,157,409.79
10110	9,122,110.98	8,847,189.08	9,425,642.69
10101	8,943,359.61	8,693,575.06	9,264,920.74
10011	-12,508,040.65	-12,904,132.40	-12,207,826.42
01110	15,457,967.19	15,356,981.40	15,616,641.50
01101	15,326,835.69	15,214,032.65	15,456,565.14
01011	-6,325,636.96	-6,476,882.59	-6,070,184.63
00111	14,191,651.10	14,069,875.19	14,397,349.67
11110	7,924,540.07	7,506,570.88	8,194,233.27
11101	7,748,214.78	7,412,079.70	8,040,024.02
11011	-13,819,841.90	-14,237,285.13	-13,386,294.94
10111	6,687,901.23	6,291,250.10	6,987,729.23
01111	12,875,667.17	12,725,772.88	13,135,279.14
11111	5,361,093.61	4,967,671.27	5,808,146.81

Table F.6: Shares, 3-Year Horizon, Initial Assets Equal to Actuarial Liability. Sponsor Separate

Subgroup	Shapley Value	Nucleolus	Proportional Sharing	Serial Sharing
Active lives	0 (0.000%)	0 (0.000%)	1,137,025 (4.637%)	0 (0.000%)
Disabled lives	1,534,848 (28.823%)	1,566,227 (29.412%)	246,907 (4.637%)	1,543,050 (28.977%)
Sponsor	3,024,654	3,024,654	3,024,654	3,024,654
Retired lives	489,565 (4.529%)	456,777 (4.226%)	501,202 (4.637%)	475,160 (4.396%)
Terminated lives	312,027 (3.206%)	313,436 (3.220%)	451,306 (4.637%)	318,230 (3.269%)

Table F.7: Shares, 3-Year Horizon, Initial Assets Equal to Actuarial Liability. Sponsor Essential

Subgroup	Shapley Value	Nucleolus
Active lives	0	0
Disabled lives	0	0
Sponsor	5,361,094	5,361,094
Retired lives	0	0
Terminated lives	0	0

Table F.8: Shares, 3-Year Horizon, Initial Assets Equal to Actuarial Liability, Sponsor Excluded

Subgroup	Shapley Value	Nucleolus	Proportional Sharing	Serial Sharing
Active lives	0 (0.000%)	0 (0.000%)	2,608,968 (10.639%)	0 (0.000%)
Disabled lives	2,543,066 (47.757%)	2,574,445 (48.346%)	566,541 (10.639%)	2,551,268 (47.911%)
Sponsor	0	0	0	0
Retired lives	1,497,783 (13.856%)	1,464,995 (13.553%)	1,150,037 (10.639%)	1,483,378 (13.723%)
Terminated lives	1,320,245 (13.564%)	1,321,654 (13.579%)	1,035,548 (10.639%)	1,326,448 (13.628%)

Table F.9: Disposable Surpluses, 3-Year Horizon, Initial Assets Equal to Accumulated Fund

adsrt	Disposable Surplus	95% Confidence Interval	
		Lower Bound	Upper Bound
10000	-12,537,445.70	-12,781,894.75	-12,302,459.65
01000	-8,319,671.87	-8,350,787.98	-8,267,365.94
00100	37,968,784.39	37,968,151.87	37,969,263.31
00010	-18,511,723.01	-18,588,162.90	-18,417,917.12
00001	6,442,974.41	6,379,993.39	6,529,346.41
11000	-20,751,152.78	-21,035,892.53	-20,453,682.25
10100	25,443,217.39	25,189,767.69	25,675,989.24
10010	-30,864,270.04	-31,139,175.64	-30,563,393.38
10001	-5,938,353.54	-6,186,485.44	-5,618,712.07
01100	29,659,334.35	29,626,725.86	29,710,932.21
01010	-26,847,549.23	-26,943,802.42	-26,678,124.82
01001	-1,867,746.88	-1,975,091.07	-1,728,191.30
00110	19,464,460.75	19,389,355.09	19,561,216.65
00101	44,428,902.70	44,360,100.66	44,506,256.27
00011	-12,083,722.29	-12,204,836.05	-11,880,066.73
11100	17,226,018.64	16,936,480.42	17,524,094.70
11010	-39,066,126.45	-39,498,274.70	-38,801,429.40
11001	-14,135,345.25	-14,484,487.26	-13,844,958.72
10110	7,110,343.39	6,835,421.49	7,413,875.10
10101	32,039,374.87	31,789,590.32	32,360,936.01
10011	-24,281,014.03	-24,677,105.78	-23,980,799.80
01110	11,140,220.71	11,039,234.92	11,298,895.03
01101	36,116,872.06	36,004,069.03	36,246,601.52
01011	-20,404,589.23	-20,555,834.86	-20,149,136.90
00111	25,896,263.03	25,774,487.12	26,101,961.59
11110	-1,092,522.92	-1,510,492.12	-822,829.73
11101	23,838,934.63	23,502,799.56	24,130,743.88
11011	-32,598,110.69	-33,015,553.92	-32,164,563.73
10111	13,693,196.64	13,296,545.50	13,993,024.63
01111	17,574,983.69	17,425,089.40	17,834,595.66
11111	5,361,093.61	4,967,671.27	5,808,146.81

Table F.10: Shares, 3-Year Horizon, Initial Assets Equal to Accumulated Fund.
Sponsor Separate

Subgroup	Shapley Value	Nucleolus	Proportional Sharing	Serial Sharing
Active lives	0 (0.000%)	0 (0.000%)	1,200,704 (4.896%)	0 (0.000%)
Disabled lives	0 (0.000%)	0 (0.000%)	0 (0.000%)	0 (0.000%)
Sponsor	3,024,654	3,024,654	3,024,654	3,024,654
Retired lives	0 (0.000%)	0 (0.000%)	0 (0.000%)	0 (0.000%)
Terminated lives	2,336,440 (24.004%)	2,336,440 (24.004%)	1,135,736 (11.668%)	2,336,440 (24.004%)

Table F.11: Shares, 3-Year Horizon, Initial Assets Equal to Accumulated Fund.
Sponsor Essential

Subgroup	Shapley Value	Nucleolus
Active lives	0 (0.000%)	0 (0.000%)
Disabled lives	0 (0.000%)	0 (0.000%)
Sponsor	2,680,547	5,361,094
Retired lives	0 (0.000%)	0 (0.000%)
Terminated lives	2,680,547 (27.540%)	0 (0.000%)

Table F.12: Shares, 3-Year Horizon, Initial Assets Equal to Accumulated Fund, Sponsor Excluded

Subgroup	Shapley Value	Nucleolus	Proportional Sharing	Serial Sharing
Active lives	0 (0.000%)	0 (0.000%)	2,755,084 (11.235%)	0 (0.000%)
Disabled lives	0 (0.000%)	0 (0.000%)	0 (0.000%)	0 (0.000%)
Sponsor	0	0	0	0
Retired lives	0 (0.000%)	0 (0.000%)	0 (0.000%)	0 (0.000%)
Terminated lives	5,361,094 (55.079%)	5,361,094 (55.079%)	2,606,010 (26.774%)	5,361,094 (55.079%)

Table F.13: Shares under Different Asset Allocations, Sponsor Separate

Subgroup	Asset Allocation	Shapley Value	Nucleolus	Proportional Sharing	Serial Sharing
Active lives	Liab. Surp.	773,756	1,045,259	1,137,025	892,907
	Liab. Only	0	0	1,137,025	0
	Acc. Fund	0	0	1,200,704	0
Disabled lives	Liab. Surp.	337,239	0	246,907	0
	Liab. Only	1,534,848	1,566,227	246,907	1,543,050
	Acc. Fund	0	0	0	0
Sponsor	Liab. Surp.	3,024,654	3,024,654	3,024,654	3,024,654
	Liab. Only	3,024,654	3,024,654	3,024,654	3,024,654
	Acc. Fund	3,024,654	3,024,654	3,024,654	3,024,654
Retired lives	Liab. Surp.	740,178	939,231	501,202	1,073,013
	Liab. Only	489,565	456,777	501,202	475,160
	Acc. Fund	0	0	0	0
Terminated lives	Liab. Surp.	485,267	351,950	451,306	370,520
	Liab. Only	312,027	313,436	451,306	318,230
	Acc. Fund	2,336,440	2,336,440	1,135,736	2,336,440

Table F.14: Shares under Different Asset Allocations, Sponsor Essential

Subgroup Subgroup	Asset Allocation	Shapley Value	Nucleolus
Active lives	Liab. Surp.	879,330	917,430
	Liab. Only	0	0
	Acc. Fund	0	0
Disabled lives	Liab. Surp.	370,129	351,717
	Liab. Only	0	0
	Acc. Fund	0	0
Sponsor	Liab. Surp.	2,643,335	2,651,125
	Liab. Only	5,361,094	5,361,094
	Acc. Fund	2,680,547	5,361,094
Retired lives	Liab. Surp.	877,645	867,062
	Liab. Only	0	0
	Acc. Fund	0	0
Terminated lives	Liab. Surp.	590,655	573,760
	Liab. Only	0	0
	Acc. Fund	2,680,547	0

Table F.15: Shares under Different Asset Allocations, Sponsor Excluded

Subgroup Subgroup	Asset Allocation	Shapley Value	Nucleolus	Proportional Sharing	Serial Sharing
Active lives	Liab. Surp.	1,700,983	1,804,566	2,608,968	1,662,310
	Liab. Only	0	0	2,608,969	0
	Acc. Fund	0	0	2,755,084	0
Disabled lives	Liab. Surp.	738,302	701,901	566,541	716,445
	Liab. Only	2,543,066	2,574,445	566,541	2,551,268
	Acc. Fund	0	0	0	0
Sponsor	Liab. Surp.	0	0	0	0
	Liab. Only	0	0	0	0
	Acc. Fund	0	0	0	0
Retired lives	Liab. Surp.	1,745,146	1,709,247	1,150,307	1,842,416
	Liab. Only	1,497,783	1,464,995	1,150,037	1,483,378
	Acc. Fund	0	0	0	0
Terminated lives	Liab. Surp.	1,176,663	1,145,380	1,035,548	1,139,923
	Liab. Only	1,320,245	1,321,654	1,035,548	1,326,448
	Acc. Fund	5,361,094	5,361,094	2,606,010	5,361,094

Table F.16: Disposable Surpluses, 10-Year Horizon, 5% Probability

adsrt	Disposable Surplus	95% Confidence Interval	
		Lower Bound	Upper Bound
10000	-3,883,031.33	-4,325,506.21	-3,456,127.16
01000	541,169.45	500,463.47	590,386.70
00100	-376,150.47	-388,821.10	-365,097.39
00010	1,584,263.36	1,508,942.65	1,676,501.51
00001	376,781.33	236,442.20	489,547.38
11000	-3,012,837.09	-3,509,096.73	-2,551,271.65
10100	-3,610,572.37	-3,958,487.24	-3,157,823.43
10010	-1,538,961.94	-2,132,865.34	-1,164,019.95
10001	-3,035,987.28	-3,559,426.00	-2,570,098.01
01100	314,305.22	276,553.65	368,260.53
01010	2,132,774.16	1,996,799.34	2,269,494.15
01001	927,903.56	792,768.25	1,093,519.98
00110	1,471,190.48	1,391,036.59	1,546,749.67
00101	164,759.35	57,591.87	258,012.92
00011	2,104,345.25	1,897,462.51	2,252,053.40
11100	-2,868,090.91	-3,296,668.51	-2,394,530.70
11010	-775,445.66	-1,461,215.26	-400,577.06
11001	-2,349,046.93	-2,868,313.54	-1,778,360.46
10110	-1,445,126.62	-2,027,232.91	-1,043,352.46
10101	-2,758,415.05	-3,428,077.46	-2,257,086.48
10011	-997,313.44	-1,574,349.30	-451,810.11
01110	2,034,313.67	1,882,464.64	2,162,387.83
01101	760,572.36	583,856.83	900,750.00
01011	2,655,668.73	2,401,673.47	2,864,071.63
00111	1,901,137.12	1,746,957.46	2,103,737.41
11110	-717,051.85	-1,225,753.34	-318,760.08
11101	-2,086,743.74	-2,701,115.04	-1,582,120.78
11011	-372,955.42	-1,051,674.70	327,580.84
10111	-824,950.26	-1,367,064.92	-276,992.91
01111	2,479,339.26	2,275,784.51	2,695,841.28
11111	-117,803.02	-811,159.02	411,745.06

Table F.17: Disposable Surpluses, 10-Year Horizon, 15% Probability

adsrt	Disposable Surplus	95% Confidence Interval	
		Lower Bound	Upper Bound
10000	2,045,682.51	1,824,080.29	2,333,762.14
01000	1,149,629.55	1,116,315.17	1,178,638.13
00100	-204,541.10	-213,118.23	-197,284.33
00010	2,629,566.80	2,585,771.96	2,678,349.33
00001	1,736,542.68	1,670,015.64	1,800,388.20
11000	3,328,840.59	3,045,852.31	3,648,695.66
10100	2,048,526.29	1,820,958.71	2,288,189.25
10010	4,981,043.11	4,691,057.66	5,259,511.18
10001	4,008,106.88	3,672,524.24	4,303,423.81
01100	1,015,873.24	986,549.79	1,050,480.70
01010	3,800,675.30	3,711,178.68	3,873,090.33
01001	2,886,413.04	2,794,386.42	2,982,579.93
00110	2,545,141.04	2,495,225.37	2,598,989.46
00101	1,610,453.00	1,543,710.47	1,677,553.69
00011	4,431,046.37	4,309,478.24	4,543,361.17
11100	3,314,139.88	3,027,447.28	3,616,331.01
11010	6,218,981.57	5,920,452.98	6,518,442.63
11001	5,251,103.96	4,904,482.65	5,566,592.75
10110	4,961,736.12	4,607,034.96	5,240,234.37
10101	4,002,740.04	3,647,937.89	4,315,544.53
10011	6,890,825.58	6,510,639.90	7,262,108.08
01110	3,696,338.77	3,623,871.13	3,770,689.36
01101	2,790,813.58	2,685,502.52	2,890,884.72
01011	5,579,100.89	5,430,138.21	5,724,820.35
00111	4,315,875.07	4,197,042.03	4,426,451.21
11110	6,191,130.75	5,852,275.99	6,489,237.50
11101	5,239,960.74	4,834,424.05	5,547,415.37
11011	8,108,263.83	7,821,211.37	8,445,171.86
10111	6,897,538.57	6,466,615.40	7,203,857.92
01111	5,467,164.09	5,332,898.06	5,600,885.29
11111	8,071,439.60	7,730,973.17	8,482,470.51

Table F.18: Shares, 10-Year Horizon, 15% Probability, Sponsor Separate

Subgroup	Shapley Value	Nucleolus	Proportional Sharing	Serial Sharing
Active lives	1,074,001 (4.380%)	1,276,510 (5.205%)	1,711,857 (6.981%)	1,076,564 (4.390%)
Disabled lives	529,220 (9.938%)	0 (0.000%)	371,732 (6.981%)	0 (0.000%)
Sponsor	4,553,794	4,553,794	4,553,794	4,553,794
Retired lives	1,149,736 (10.636%)	1,604,507 (14.843%)	754,589 (6.981%)	1,871,038 (17.309%)
Terminated lives	764,689 (7.856%)	636,629 (6.541%)	679,468 (6.981%)	570,044 (5.857%)

Table F.19: Shares, 10-Year Horizon, 15% Probability, Sponsor Essential

Subgroup	Shapley Value		Nucleolus	
Active lives	1,236,467	(5.042%)	1,302,138	(5.310%)
Disabled lives	586,375	(11.012%)	586,951	(11.022%)
Sponsor	3,934,465		3,826,457	
Retired lives	1,392,596	(12.883%)	1,415,740	(13.097%)
Terminated lives	921,537	(9.468%)	940,154	(9.659%)

Table F.20: Shares, 10-Year Horizon, 15% Probability, Sponsor Excluded

Subgroup	Shapley Value	Nucleolus	Proportional Sharing	Serial Sharing
Active lives	2,382,597 (9.716%)	2,385,550 (9.728%)	3,927,953 (16.018%)	2,218,473 (9.047%)
Disabled lives	1,148,700 (21.572%)	1,161,766 (21.817%)	852,961 (16.018%)	1,128,068 (21.184%)
Sponsor	0	0	0	0
Retired lives	2,739,758 (25.346%)	2,729,623 (25.252%)	1,731,448 (16.018%)	3,012,946 (27.873%)
Terminated lives	1,800,385 (18.497%)	1,794,501 (18.437%)	1,559,078 (16.018%)	1,711,953 (17.588%)

Table F.21: Disposable Surpluses, 3-Year Horizon, Real Starting Values

adsrt	Disposable Surplus	95% Confidence Interval	
		Lower Bound	Upper Bound
10000	1,925,009.54	1,663,987.32	2,114,717.83
01000	696,651.89	657,336.35	751,064.98
00100	-20,809.86	-21,419.28	-20,365.52
00010	1,655,411.02	1,583,646.73	1,738,069.48
00001	1,101,271.91	1,025,991.41	1,183,509.04
11000	2,672,664.18	2,478,297.22	2,968,335.53
10100	1,913,639.13	1,658,680.04	2,120,326.92
10010	3,746,498.51	3,453,087.83	4,032,912.36
10001	3,166,562.29	2,879,637.01	3,449,579.58
01100	684,255.80	642,504.95	736,680.78
01010	2,359,552.87	2,236,399.57	2,488,965.51
01001	1,802,471.20	1,674,788.93	1,936,404.29
00110	1,643,694.00	1,569,130.17	1,729,198.79
00101	1,091,834.37	1,011,671.13	1,172,310.05
00011	2,745,883.96	2,600,596.51	2,947,058.91
11100	2,661,218.77	2,457,959.20	2,954,033.84
11010	4,527,975.65	4,198,019.63	4,821,067.55
11001	3,904,961.15	3,605,451.38	4,263,431.54
10110	3,740,504.51	3,441,482.06	4,018,920.51
10101	3,146,348.53	2,864,631.79	3,435,294.57
10011	4,979,092.88	4,590,178.23	5,305,233.74
01110	2,348,111.34	2,227,122.31	2,479,189.44
01101	1,792,344.79	1,664,111.09	1,925,560.72
01011	3,442,535.84	3,257,955.66	3,698,123.88
00111	2,729,198.82	2,576,513.49	2,936,035.49
11110	4,508,124.87	4,182,986.51	4,804,082.35
11101	3,893,161.05	3,581,042.99	4,240,746.53
11011	5,659,460.25	5,308,837.67	6,101,220.40
10111	4,963,094.28	4,576,069.08	5,295,301.46
01111	3,425,850.71	3,231,640.37	3,687,837.08
11111	5,644,628.80	5,301,089.21	6,088,512.42

Table F.22: Shares, 3-Year Horizon, Real Starting Values, Sponsor Separate

Subgroup	Shapley Value	Nucleolus	Proportional Sharing	Serial Sharing
Active lives	918,561 (3.746%)	1,335,958 (5.448%)	1,197,159 (4.882%)	1,370,606 (5.589%)
Disabled lives	313,500 (5.887%)	0 (0.000%)	259,965 (4.882%)	0 (0.000%)
Sponsor	3,184,621	3,184,621	3,184,621	3,184,621
Retired lives	762,791 (7.057%)	873,532 (8.081%)	527,709 (4.882%)	817,564 (7.563%)
Terminated lives	465,156 (4.779%)	250,518 (2.574%)	475,175 (4.882%)	271,838 (2.793%)

Table F.23: Shares, 3-Year Horizon, Real Starting Values, Sponsor Essential

Subgroup	Shapley Value	Nucleolus
Active lives	1,068,450 (4.357%)	1,109,389 (4.524%)
Disabled lives	352,824 (6.626%)	340,767 (6.399%)
Sponsor	2,782,366	2,750,487
Retired lives	868,273 (8.033%)	875,734 (8.102%)
Terminated lives	572,716 (5.884%)	568,252 (5.838%)

Table F.24: Shares, 3-Year Horizon, Real Starting Values, Sponsor Excluded

Subgroup	Shapley Value	Nucleolus	Proportional Sharing	Serial Sharing
Active lives	2,085,022 (8.503%)	2,135,457 (8.708%)	2,746,949 (11.202%)	2,201,319 (8.977%)
Disabled lives	705,080 (13.241%)	681,094 (12.790%)	596,504 (11.202%)	692,481 (13.004%)
Sponsor	0	0	0	0
Retired lives	1,717,561 (15.889%)	1,709,064 (15.811%)	1,210,860 (11.202%)	1,648,277 (15.248%)
Terminated lives	1,136,966 (11.681%)	1,119,014 (11.497%)	1,090,316 (11.202%)	1,102,552 (11.328%)

Table F.25: Disposable Surpluses, 10-Year Horizon, Real Starting Values

adsrt	Disposable Surplus	95% Confidence Interval	
		Lower Bound	Upper Bound
10000	4,796,524.35	4,555,283.90	5,025,141.49
01000	1,346,209.69	1,314,098.50	1,370,579.53
00100	-142,217.47	-150,078.39	-135,644.97
00010	2,903,670.40	2,853,444.72	2,946,066.11
00001	2,231,661.24	2,185,383.90	2,286,330.41
11000	6,244,397.82	6,013,158.47	6,503,849.11
10100	4,806,894.98	4,563,320.87	5,053,575.65
10010	7,975,377.65	7,714,434.03	8,238,709.16
10001	7,223,814.10	6,936,776.62	7,483,964.96
01100	1,281,321.07	1,249,623.97	1,313,831.20
01010	4,258,072.45	4,178,795.17	4,341,231.80
01001	3,576,264.52	3,493,701.57	3,679,922.87
00110	2,890,023.77	2,841,758.17	2,932,719.41
00101	2,185,157.71	2,109,353.31	2,244,353.16
00011	5,228,347.82	5,084,301.77	5,311,324.24
11100	6,239,785.46	5,987,404.65	6,484,797.48
11010	9,415,029.91	9,174,039.11	9,707,295.46
11001	8,656,459.28	8,400,669.55	8,927,686.36
10110	7,937,258.63	7,672,714.85	8,241,477.73
10101	7,194,135.27	6,909,939.29	7,444,189.76
10011	10,399,112.56	10,097,206.88	10,687,635.99
01110	4,243,235.18	4,168,895.77	4,301,892.52
01101	3,542,851.26	3,451,729.09	3,630,867.84
01011	6,566,282.54	6,394,415.39	6,675,812.89
00111	5,154,780.36	5,029,038.72	5,254,561.71
11110	9,401,346.45	9,102,497.52	9,669,991.79
11101	8,629,465.21	8,341,942.86	8,931,949.74
11011	11,840,747.80	11,509,529.23	12,157,197.69
10111	10,357,319.44	10,064,863.84	10,619,551.08
01111	6,514,276.29	6,372,522.20	6,637,856.43
11111	11,811,583.43	11,456,895.14	12,111,688.43

Table F.26: Shares, 10-Year Horizon, Real Starting Values, Sponsor Separate

Subgroup	Shapley Value	Nucleolus	Proportional Sharing	Serial Sharing
Active lives	2,136,903 (8.714%)	3,362,192 (13.711%)	2,505,097 (10.216%)	3,511,625 (14.320%)
Disabled lives	605,184 (11.365%)	0 (0.000%)	543,985 (10.216%)	0 (0.000%)
Sponsor	6,663,930	6,663,930	6,663,930	6,663,930
Retired lives	1,393,357 (12.890%)	1,272,016 (11.768%)	1,104,251 (10.216%)	1,189,024 (11.000%)
Terminated lives	1,012,209 (10.399%)	513,445 (5.275%)	994,320 (10.216%)	447,004 (4.592%)

Table F.27: Shares, 10-Year Horizon, Real Starting Values, Sponsor Essential

Subgroup	Shapley Value	Nucleolus
Active lives	2,572,658 (10.491%)	2,648,654 (10.801%)
Disabled lives	705,991 (13.258%)	727,132 (13.655%)
Sponsor	5,809,342	5,639,619
Retired lives	1,547,797 (14.319%)	1,591,059 (14.719%)
Terminated lives	1,175,795 (12.080%)	1,205,119 (12.381%)

Table F.28: Shares, 10-Year Horizon, Real Starting Values, Sponsor Excluded

Subgroup	Shapley Value	Nucleolus	Proportional Sharing	Serial Sharing
Active lives	5,064,545 (20.653%)	5,083,453 (20.730%)	5,748,088 (23.440%)	5,286,201 (21.557%)
Disabled lives	1,384,122 (25.993%)	1,379,340 (25.903%)	1,248,205 (23.440%)	1,340,203 (25.168%)
Sponsor	0	0	0	0
Retired lives	3,047,875 (28.196%)	3,034,683 (28.074%)	2,533,766 (23.440%)	2,963,599 (27.417%)
Terminated lives	2,315,041 (23.785%)	2,314,107 (23.775%)	2,281,524 (23.440%)	2,221,580 (22.824%)

Bibliography

- [1] Bernard L. Adell. Pension plan surpluses and the law: Finding a path for reform. Reprint series 75, Industrial Relations Centre, Queen's University, Kingston, Ont., 1988.

- [2] Affaire Hockin contre The Bank of British Columbia. *Communiqué Mercer*, April 17, 1995.

- [3] William H. Aitken. *A Problem-Solving Approach to Pension Funding and Valuation*. ACTEX Publications, Winsted, CT, 1996.

- [4] Antoni Alegre and M. Mercè Claramunt. Allocation of solvency cost in group annuities: Actuarial principles and cooperative game theory. *Insurance: Mathematics and Economics*, 17(1):19–34, 1995.

- [5] Arthur W. Anderson. *Pension Mathematics for Actuaries*. ACTEX Publications, Winsted, CT, second edition, 1992.

- [6] Louis Ascah. Pension fund surpluses: An economic perspective. Technical report, Canadian Centre for Policy Alternatives, Ottawa, Ont., 1991.

- [7] Robert J. Aumann and Sergiu Hart, editors. *Handbook of Game Theory with Economic Applications*, volume 1. Elsevier Science Publishers B. V., Amsterdam, Netherlands, 1992.
- [8] James O. Berger. *Statistical Decision Theory: Foundations, Concepts and Methods*. Springer-Verlag, New York, NY, 1980.
- [9] Barnet N. Berin. *The Fundamentals of Pension Mathematics*. Society of Actuaries, New York, NY, revised edition, 1989.
- [10] Zvi Bodie. Pension fund investment policy. Working paper 2752, National Bureau of Economic Research, Cambridge, MA, 1988.
- [11] Newton L. Bowers et al. *Actuarial Mathematics*. Society of Actuaries, Itasca, IL, second edition, 1997.
- [12] Robert L. Brown. *Introduction to the Mathematics of Demography*. ACTEX Publications, Winsted, CT, third edition, 1997.
- [13] Robert S. Chirinko and Huntley Schaller. Bubbles, fundamentals, and investment: A multiple equation testing strategy. *Journal of Monetary Economics*, 38(1):47–76, 1996.
- [14] Thomas Claridge. Pension surplus no reason for Hydro not to pay share, Ontario appeal court rules. *Globe and Mail*, pages A1–A2, May 4, 1989.
- [15] Robert T. Clemen. *Making Hard Decisions: An Introduction to Decision Analysis*. Duxbury Press, Belmont, CA, second edition, 1996.

- [16] Committee on Annuities. Development of the 1983 Group Annuity Mortality Table. *Transactions of the Society of Actuaries*, 35:859–899, 1983.
- [17] Committee on Economic Statistics. *Report on Canadian Economic Statistics, 1924 – 1997*. Canadian Institute of Actuaries, Ottawa, ON, 1998.
- [18] Herbert A. David. *Order Statistics*. Wiley Series in Probability and Mathematical Statistics. John Wiley & Sons, Inc., New York, NY, second edition, 1981.
- [19] Daniel Dufresne. Moments of pension plan contributions and fund level when rates of return are random. *Journal of the Institute of Actuaries*, 115(3):535–544, 1988.
- [20] Daniel Dufresne. Stability of pension systems when rates of return are random. *Insurance: Mathematics and Economics*, 8(1):71–76, 1989.
- [21] Excédent à la liquidation d'un régime de retraite: confusion ou compromis. *Commentaires Mercer*, 42(8), August 1992.
- [22] P. M. Greenwood and T. W. Keogh. Pension funding and expensing in the minimum funding requirement environment. *British Actuarial Journal*, 3:497–582, 1997.
- [23] Alan L. Gustman and Thomas L. Steinmeier. *Pension Incentives and Job Mobility*. W.E. Upjohn Institute for Employment Research, Kalamazoo, MI, 1995.

- [24] Steven Haberman. Autoregressive rates of return and the variability of pension contributions and fund levels for a defined benefit pension scheme. *Insurance: Mathematics and Economics*, 14(3):219–240, 1994.
- [25] Sergiu Hart and Andreu Mas-Colell. Potential, value, and consistency. *Econometrica*, 57(3):589–614, 1989.
- [26] Campbell R. Harvey and Robert E. Whaley. Dividends and S&P valuation. *Journal of Futures Markets*, 12(2):123–137, 1992.
- [27] P. P. Huber. A review of Wilkie's stochastic asset model. *British Actuarial Journal*, 3:181–210, 1997.
- [28] John Hull. *Options, Futures, and Other Derivatives*. Prentice Hall, Upper Saddle River, NJ, third edition, 1997.
- [29] Richard A. Ippolito. *Pensions, Economics and Public Policy*. Pension Research Council, Wharton School, University of Pennsylvania, Homewood, IL, 1986.
- [30] Le jugement Singer: nouvelle décision en matière de propriété des excédents et d'interruptions de cotisations. *Communiqué Mercer*, March 14, 1995.
- [31] Stuart A. Klugman. *Bayesian Statistics in Actuarial Science: With Emphasis on Credibility*. Huebner International Series on Risk, Insurance, and Economic Security. Kluwer Academic Publishers, Boston, MA, 1992.
- [32] Stuart A. Klugman, Harry H. Panjer, and Gordon E. Willmot. *Loss Models: From Data to Decisions*. Wiley Series in Probability and Statistics. John Wiley & Sons, Inc., New York, NY, 1998.

- [33] Jean Lemaire. An application of game theory: Cost allocation. *ASTIN Bulletin*, 14(1):61–81, 1984.
- [34] Jean Lemaire. Cooperative game theory and its insurance applications. *ASTIN Bulletin*, 21(1):17–40, 1991.
- [35] Loi sur les régimes complémentaires de retraite, 1997.
- [36] John C. Maynard. Pricing defined-benefit pension plans with indexed benefits. *Transactions of the Society of Actuaries*, 44:193–246, 1992.
- [37] Dan Mays McGill et al. *Fundamentals of Private Pensions*. University of Pennsylvania Press, Pennsylvania, PA, seventh edition, 1996.
- [38] Nimrod Megiddo. On the monotonicity of the bargaining set, the kernel, and the nucleolus of a game. *SIAM Journal on Applied Mathematics*, 27:355–358, 1974.
- [39] Hervé Moulin. *Axioms of Cooperative Decision Making*. Cambridge University Press, Cambridge, MA, 1988.
- [40] Hervé Moulin and Scott Shenker. Cost sharing of an homogeneous good: Average versus serial methods. Unpublished manuscript, Duke University, 1991.
- [41] Hervé Moulin and Scott Shenker. Average cost pricing versus serial cost sharing: An axiomatic comparison. Unpublished manuscript, Duke University, 1992.

- [42] Hervé Moulin and Scott Shenker. Serial cost sharing. *Econometrica*, 60(5):1009–1037, 1992.
- [43] Martin J. Osborne and Ariel Rubinstein. *Bargaining and Markets*. Academic Press, San Diego, CA, 1990.
- [44] Guillermo Owen. *Game Theory*. Academic Press, San Diego, CA, third edition, 1995.
- [45] Jérôme Pansera. Le coût de disparité d'une compagnie d'assurance-vie dans des modèles aléatoires de taux d'intérêt. Master's thesis, Université Laval, 1999.
- [46] Gary Parker. Distribution of the present value of future cash flows. In *Proceedings of the 3rd AFIR Colloquium*, pages 831–843, Rome, Italy, 1993.
- [47] Gary Parker. Stochastic analysis of the interaction between investment and insurance risks. *North American Actuarial Journal*, 1(2):55–71, 1997.
- [48] Le partage équitable des excédents d'actif. Document de consultation, Régie des rentes du Québec.
- [49] Bezalel Peleg. On the reduced game property and its converse. *International Journal of Game Theory*, 15(3):187–200, 1986.
- [50] Pension Commission of Ontario. Court decisions. *PCO Bulletin*, 2(1):6–7, 1991.

- [51] Pension Commission of Ontario. Allocation of surplus distributed to members and former member on wind up. *PCO Bulletin*, 5(1):28–30, 1994.
- [52] Pension Commission of Ontario. Surplus distribution to an employer. *PCO Bulletin*, 5(2):32–45, 1994.
- [53] *Pension Tables for Actuaries*. American Society of Pension Actuaries, Washington, D.C., 1977.
- [54] Christian P. Robert. *The Bayesian Choice: A Decision-Theoretic Motivation*. Springer-Verlag, New York, NY, 1994.
- [55] Michael Salter. Battle of the surplus. *Maclean's*, page 30, September 1, 1986.
- [56] Michael J. Samet, Timothy P. Peach, and W. Paul Zorn. *A Study of Public Employees Retirement Systems*. Society of Actuaries, Schaumburg, IL, 1996. SOA Monograph M-RS96-1.
- [57] Manuel S. Santos and Michael Woodford. Rational asset pricing bubbles. *Econometrica*, 65(1):19–57, 1997.
- [58] David Schmeidler. The nucleolus of a characteristic function game. *SIAM Journal on Applied Mathematics*, 17:1163–1170, 1969.
- [59] Ray Schmitt, editor. *The Future of Pensions in the United States*. Pension Research Council of the Wharton School of the University of Pennsylvania, Philadelphia, PA, 1993.

- [60] L. S. Shapley. A value for n-person games. In H. W. Kuhn and A. W. Tucker, editors, *Contributions to the Theory of Games*, number 28 in *Annals of Mathematics Studies*, pages 307–317. Princeton University Press, Princeton, NJ, 1953.
- [61] Keith P. Sharp. Modelling canadian price and wage inflation. *Proceedings of the Canadian Institute of Actuaries*, 24:565–587, 1994.
- [62] Martin Shubik. *Game Theory in the Social Sciences: Concepts and Solutions*. The MIT Press, Cambridge, MA, 1982.
- [63] Aleksandr Ivanovich Sobolev. Characterization of the principle of optimality for cooperative games through functional equations. In N. N. Vorby'ev, editor, *Mathematical Methods in the Social Sciences*, volume 6, pages 92–151. VILNIUS. USSR, 1975.
- [64] Statistics Canada. *Canada's Retirement Income Programs: A Statistical Overview*. Minister of Industry, Ottawa, Ont., 1996. Catalogue number 74-507-XPB.
- [65] Statistics Canada. *Pension Plans in Canada*. Minister of Industry, Ottawa, Ont., 1998. Catalogue number 74-401-XPB.
- [66] Statistics Canada. *Quarterly Estimates of Trusteed Pension Funds. First Quarter 1998*, volume 26.1. Minister of Industry, Ottawa, Ont., 1998. Catalogue number 74-001-XPB.

- [67] Statistics Canada. *Trusteed Pension Funds: Financial Statistics*. Minister of Industry, Ottawa, Ont., 1998. Catalogue number 74-201-XPB.
- [68] Jeroen Suijs, Anja de Waegenaere, and Peter Borm. Stochastic cooperative games in insurance. *Insurance: Mathematics and Economics*, 22(3):209–228, 1998.
- [69] P. N. Thornton and A. F. Wilson. A realistic approach to pension funding. *Journal of the Institute of Actuaries*, 119(2):229–312, 1992.
- [70] A. David Wilkie. More on a stochastic asset model for actuarial use. *British Actuarial Journal*, 1:777–964, 1995.
- [71] Howard E. Winklevoss. *Pension Mathematics with Numerical Illustrations*. Pension Research Council, Wharton School, University of Pennsylvania, Philadelphia, PA, second edition, 1977.
- [72] David A. Wise, editor. *Pensions, Labor, and Individual Choice*. University of Chicago Press, Chicago, IL, 1985.
- [73] Ian Douglas Wright. *A Stochastic Approach to Pension Scheme Funding & Assets Allocation*. PhD thesis, Heriot-Watt University, 1997.
- [74] H. P. Young. Monotonic solutions of cooperative games. *International Journal of Game Theory*, 14(2):65–72, 1985.