INFORMATION TO USERS

Multiple Time Scales in Survival Analysis

by

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Abstract

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In many survival analysis applications, there may not be a unique plausible scale in which to measure time to failure or assess performance. This is especially the case when several measures of usage or exposure are available on each unit or individual. For example, the age, the total number of flight hours, and the number of landings are usage measures that are often considered important in aircraft reliability. This thesis considers the definition of a "good" time scale, along with models that combine different usage or exposure measures into a single scale. New semiparametric inference methods for these models are developed and their efficiency in moderate size samples is investigated. Datasets are used to illustrate the implementation of these methods and of some model assessment procedures. Ideas for further applications and research are also given.

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Chapter 1

Introduction

In many survival analysis applications, there may not be a unique plausible scale in which to measure time until failure or assess performance. This is especially the case when several measures of usage or exposure (such as the total number of cycles, the cumulative exposure to some risk, etc.) are available on each unit. The most classic example of such a situation is an automobile warranty, where both cumulative mileage and time since purchase are virtually always considered[1]. Even though it is easier to think of problems arising from engineering, multiple time scales are also quite useful to model certain situations in biostatistics. For example, the cumulative exposure to pollution or the time since a first tumor may provide more information on the time-to-event distribution than calendar age itself.

Since most survival analysis methods assume that a time scale is already available, it is of obvious importance to come up with a "good" scale for each problem. In order to do so, a definition of "good" time scale and, in such cases, a statistical method for

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[1]At least this is the case in North America.

finding such a scale are required. But how should one define a "good" time scale? For one thing, a good time scale should preserve most of the "useful" information contained in the available usage measures. In the words of Farewell and Cox (1979), "we look for a time scale that accounts for as much of the variation as possible, in some sense". This principle is somewhat similar to the concept of *sufficiency* (c.f. Cox and Hinkley (1974)), in that we are searching for a method of reducing the dimension of the data without losing any relevant information about the failure time distribution. In their 1979 note on multiple time scales, Farewell and Cox do suggest a method to verify if a given scale represents a valid dimension reduction, but they do not give a precise definition of good time scale, nor do they propose any general method of finding a good time scale

Moreover, a good scale should be such that units with the same age in that scale have a similar survival probability, even if the values of their various usage measures are different. Kordonsky and Gertsbakh (1995a) use this very property in order to define what they term an *ideal monitoring scale*, or *load invariant scale*. Interestingly, they do not seem to use this definition when they propose a method to find a *best monitoring scale* from observed data. On the other hand, Oakes (1995) uses this desired property of good time scales albeit in a parametric setting. He also introduces the notion of *collapsibility*, which essentially implies that the probability of surviving beyond a given time is solely determined by the value of the usage measures at that time.

The objectives of this thesis are to study what constitutes a "good" time scale in further detail, to develop applications of alternative time scales, and to examine methods of selecting a time scale. The development is closely tied to the study of time-varying covariates in survival settings.

In this chapter, we will define what we call *ideal time scales* and investigate their

properties. We will also propose uses and applications for models based on those time scales.

1.1 Time scales

Before we give a formal definition of an *ideal time scale*, we first need to introduce a few concepts and describe the situation. We assume that we study a population of similar devices (brake pads, aircraft, animals or human beings, for example) that are used under different conditions. Along with the age in real time, we also observe several other measures of cumulative usage on each unit. We want to use this available information in order to estimate the distribution of time to failure for the devices under study. Let us now introduce some notation.

Let X be the random variable (rv) of real time to failure. Unless otherwise specified, we assume that X is a non-negative random variable. In what follows, $x \geq 0$ will represent real time, and sometimes also a realization of X.

Definition 1.1 *We say that $y(x)$ is a* **usage measure** *(or usage marker) if the mapping $x \to y(x)$ is left-continuous and if, for any two real times x_1, $x_2 \geq 0$ such that $x_1 > x_2$, $y(x_1) \geq y(x_2)$, i.e. $y(\cdot)$ is a left-continuous, non-decreasing function of real time.*

If p different usage measures are available at time x, we denote them by $y_1(x)$, $y_2(x)$, ..., $y_p(x)$. For convenience, we let $y_0(x) = x$ and define the rv's $Y_i = y_i(X)$, $i = 0, 1, \ldots, p$. In a similar fashion, we put $y_i = y_i(x)$, $i = 0, 1, \ldots, p$. Finally, we let $\mathbf{Y}$, $\mathbf{y}$ and $\mathbf{y}(x)$ denote $(Y_0, Y_1, \ldots, Y_p)^t$, $(y_0, y_1, \ldots, y_p)^t$ and $(y_0(x), y_1(x), \ldots, y_p(x))^t$, respectively. Sometimes, we also wish to introduce in our model (time-varying) covariates that are not necessarily usage measures. We denote the vector containing all the covariates, including

the usage measures **y**, by **z**. The elements of **z** that are not in **y** need not be non-decreasing in real time; all we require is that they are left-continuous.

We can now define what a *usage (covariate) path* (or *usage [covariate] history*) is.

Definition 1.2 *Let* $\mathcal{P}(x) = \{\mathbf{y}(u),\ 0 \leq u < x\}$, *where* **y** *is a vector of usage measures. We call* $\mathcal{P}(x)$ *the* **usage path up to time** x. *When* $\mathcal{P} = \lim_{x\to\infty} \mathcal{P}(x)$, *we say that* $\mathcal{P}$ *is the* **(whole) usage path.** *Finally, we denote the space of all possible usage paths,* $\mathcal{P}$, *by* **P**. *In a similar fashion, we define* $\mathcal{Z}(x) = \{\mathbf{z}(u),\ 0 \leq u < x\}$ *to be the* **covariate path up to time** x, *we use* $\mathcal{Z} = \lim_{x\to\infty} \mathcal{Z}(x)$, *and we denote the space of all possible covariate histories by* **Z**. *For convenience, we assume that* $\mathcal{P}(x) \subset \mathcal{Z}(x)$, *i.e. the usage history is just the part of the covariate history that contains only non-decreasing covariates.*

In much of our theory and applications, we will consider that $\{\mathbf{y}(x),\ x \geq 0\}$ and $\{\mathbf{z}(x),\ x \geq 0\}$ are given, i.e. we consider the covariate and usage history to be fixed for each unit under study. In situations in which the usage paths and covariate histories are random, this means that we will have conditioned on their realizations.

Before going any further, let us look at a few examples of usage measures and paths that can occur in practice.

Example 1.1.1 *Lawless, Hu and Cao (1995) study automobile reliability. The usage measure they consider is the cumulative mileage, and they assume that mileage is accumulated at a constant rate. Mathematically, let x represent real time measured from the date at which the car enters service, and $y(x)$ be the cumulative mileage at time x. Then they assume that $y(x) = \alpha x$, where $\alpha > 0$ is a constant usage rate that may vary from car to car. This type of usage path is quite common and conforms to Definitions 1.1 and 1.2. (See path 1 in Figure 1.1.)*

Example 1.1.2 *Suppose we want to assess the reliability of a system that has periods of up and down time. We could define x to be real time and $y(x)$ to represent either cumulative up time or cumulative down time at real time x. This would describe the up/down periods of the system completely and would still comply with Definitions 1.1 and 1.2. (See path 2 in Figure 1.1.)*

Example 1.1.3 *Kordonsky and Gertsbakh (1993) consider a model for aircraft reliability. One of the numerous usage measures considered is the cumulative number of landings. If we let x denote real time and $y(x)$ denote the total number of landings of an airplane at time x, we see that this type of usage path satisfies our definitions. (See path 3 in Figure 1.1.)*

Example 1.1.4 *Sometimes, we can consider a model where a given usage measure is just a location shift in real time. An example of this type of situation is given in Farewell and Cox (1979) where, for each woman under study, x is her calendar age and $y(x)$ is the elapsed time since the birth of her first child. In this case, $y(x) = x - a$, where a is the calendar age of the mother at the birth of her first child. (See path 4 in Figure 1.1.)*

Example 1.1.5 *The cumulative amount of pollutant an individual has been exposed to is often considered to be an important factor in the study of certain diseases. If we let x represent real time and $y(x)$ be the cumulative exposure to a substance up to time x, then this can be viewed as a usage path. Oakes (1995) uses this sort of approach to estimate the survivor function of miners exposed to asbestos dust. (See path 5 in Figure 1.1.)*
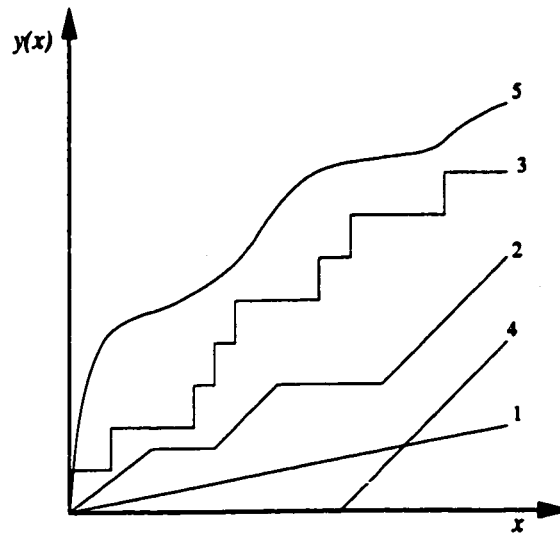
Figure 1.1: Examples of different types of usage paths seen in practice.

In most problems involving multiple time scales, the number of usage measures and covariates available on each unit is greater than one. This is why we adopt the vector notation described above. Here is an example.

Example 1.1.6 *A company interested in the reliability of its motorcycle engines keeps track of many usage measures on each engine: cumulative usage time (minutes), total number of low stress cycles, total number of high stress cycles, and cumulative mileage. In this case, we could let $y_0(x) = x$, $y_1(x)$, $y_2(x)$, $y_3(x)$ and $y_4(x)$ represent real time (x), cumulative usage time at real time x, cumulative number of low stress cycles at real time x, total number of high stress cycles up to real time x, and cumulative mileage at real time x, respectively. Let $\mathbf{y}(x) = (y_0(x), y_1(x), y_2(x), y_3(x), y_4(x))^t$. The usage path, $\mathcal{P}(x)$, is then denoted by $\mathcal{P}(x) = \{\mathbf{y}(u); \ 0 \le u < x\}$.*

As we have stated earlier, we wish to compute the distribution of real time to failure

under different conditions. In other words, we would like to know which proportion of its lifetime distribution a given unit has survived given its covariate history. Using the notation of Definition 1.2, this translates into finding

$$\Pr\left[X \geq x | \mathcal{Z}(x)\right], \quad \forall \mathcal{Z}(x) \in \mathbf{Z}(x), \quad \forall x \geq 0. \tag{1.1}$$

We now make this key assumption: all the covariates and usage measures in the covariate path (the z_i's and the y_i's) are *external* covariates, i.e. the covariate paths are determined independently of the time to failure X. Such covariates include environmental factors such as pollution level in the atmosphere, and usage measures whose value does not indicate if failure as occurred or not, like cumulative mileage on a car for example. We assume in this case that (1.1) is equal to

$$S_0[x | \mathcal{Z}(x)] \equiv \Pr\left[X \geq x | \mathcal{Z}\right], \quad \forall \mathcal{Z} \in \mathbf{Z}, \quad \forall x \geq 0. \tag{1.2}$$

Our concept of usage or exposure measures is that they are always external covariates. However, in some applications, internal covariates related to the "condition" of a unit or individual are of interest. For example, in reliability the amount of degradation or wear in a unit may be measurable, and related to the probability of failure. If we wish to include internal time-varying covariates in our model, conditional probabilities of the form (1.2) with internal covariates in $\mathcal{Z}$ are inadequate, because we would then be conditioning on future failure information. In that situation, a model would have to be based on the joint conditional distribution of the failure time and the internal covariates given the whole history of the external covariates. In more precise terms, let $\mathcal{W}(x)$ be the history of the internal time-varying covariates at time x, and $\mathcal{Z}(x)$ be the history of the external

covariates at x. Then the model would be based on probabilities of the form

$$\Pr\left[X \geq x,\ \mathcal{W}(x) = \mathbf{w}|\mathcal{Z}\right],\ \forall \mathcal{Z} \in \mathbf{Z},\ \forall \mathbf{w} \in \mathbf{W}(x),\ \forall x \geq 0. \qquad (1.3)$$

For most of the discussion which follows we ignore internal covariates.

Now that we have established the basic situation and defined the basic concepts, we can start working on a definition of *ideal time scale*. The first logical step towards our goal is to define what a *time scale* is. We first give a very general definition.

Definition 1.3 *We say that Φ is a* **time scale** *(TS) if Φ is a positive, real-valued functional $\Phi : I\!\!R^+ \otimes \mathbf{Z} \to I\!\!R^+$ such that for every $x \geq 0$ and every $\mathcal{Z} \in \mathbf{Z}$, $\Phi[x, \mathcal{Z}] = \Phi[x, \mathcal{Z}(x)]$ is non-decreasing in x.*

In other words, a TS is simply a functional that maps a given time x and covariate path $\mathcal{Z}$ onto the non-negative real line in such a way that when the value of x increases, the value of the time in the TS cannot decrease. From here on, we use the following notation: we let $T = \Phi[X, \mathcal{Z}(X)]$ be the r.v. of the time (in the Φ-scale) until failure and we let $t_{\mathcal{Z}}(x) = \Phi[x, \mathcal{Z}(x)]$. Often, $t_{\mathcal{Z}}(x)$ is referred to as the *operational time* at x.

1.2 Ideal time scales

Ideally, a good TS should enable us to compute the probability given by equation (1.1) for a given set of external covariates $z_1(x), \ldots, z_p(x)$. As a matter of fact, we base our definition of an *ideal time scale* on this very objective.

Definition 1.4 *We say that $t_{\mathcal{Z}}(x)$ defines an ideal time scale (ITS) if it is a TS such that for all $\mathcal{Z} \in \mathbf{Z}$ and all $x \geq 0$,*

$$Pr\,[X \geq x | \mathcal{Z}] = G(t), \qquad (1.4)$$

where $t = t_{\mathcal{Z}}(x)$ and $G(\cdot)$ is a strictly decreasing survivor function not depending on $\mathcal{Z}$.

In other words, an ITS is any TS that is a one-to-one function of the conditional survivor function $\Pr\,[X \geq x | \mathcal{Z}]$.

Simply put, an ITS is a TS in which we can directly compare the lifetimes of all the devices under study, no matter what their usage paths are. An ITS is therefore an extremely powerful tool since it allows us to analyze the lifetime of all the units with a single, univariate lifetime distribution. Of course, this is true only when the covariate path does not contain any internal covariates, $w(x)$, say. In that latter case, we would probably need scales of the form $T = T[x, \mathcal{W}(x), \mathcal{Z}(x)]$ in order to model probability (1.3). This last special case requires further research and is not considered in this thesis.

Thus, an ITS is "ideal" in the sense that the age in the ITS is the only information necessary in order to compute probability (1.1). We can hence think of an ITS as being "sufficient" in order to compute the age of the units. We must be careful in using the word "sufficient" here because as we will see in Chapter 3, some additional information is required in order to compute the likelihood when performing parametric estimation.

It is worth noting that we use the probability (1.4), i.e. the conditional probability of X given the covariate path, to define an ITS instead of the joint probability of X and $\mathcal{Z}$ like some authors do (e.g., Kordonsky and Gertsbakh (1993)). The advantage of a model based on a distribution of X conditional on the covariate path (conditional model) over a

model based on a joint distribution of X and Z (joint model) is that a specific covariate path distribution is intrinsic to the joint model, whereas the conditional model does not imply any distribution for the covariate paths. This means that a conditional model can generate a large set of distinct joint models just by being coupled with different covariate path distributions. Hence, conditional models are more comprehensive, more informative and more flexible than joint models. For instance, the conditional framework allows for time scale models to be used in situations where the paths are non-random or fixed by study design, as in accelerated life testing.

Different definitions of "good" TS are considered in the literature. For instance, Kordonsky and Gertsbakh (1993) define the *best* TS to be a TS that minimizes the "distance" from what they call the *load invariant scale*. Their idea of load invariant scale is the same as our idea of ITS: the load invariant scale is a TS in which same values of the time represent the same quantile of the lifetime distribution, independently of the covariate history (which they term *load*). However, their notion of a best TS is only defined when they consider semiparametric or parametric estimation. They base their inference on a conditional model of the form given by Definition 1.4 when dealing with censored data, in which case they use parametric methods and make strong assumptions on the survivor function G. When the data are complete, they use semiparametric estimation, but their method is based on a joint model rather than a conditional model. We will further discuss their method in subsequent chapters. Farewell and Cox (1979) use a slightly different approach in their treatment of problems involving multiple time scales. They do not give a precise definition of ITS, but rather derive a method with which they can verify that a given TS contains enough information in order to explain the failure pattern.

1.3 Properties of ideal time scales

Let us now assume that we want to model probability (1.1) when no internal covariates are present. Note that this can include cases in which X given Z may have a non-zero probability mass at certain x values; for example, if equipment is turned on and off then there may be a non-zero probability of failure at the instant a unit is turned on (e.g. Follmann (1990)). It is therefore convenient for general discussion to allow $\Pr[X \geq x|Z]$ to have jump discontinuities. The survivor function can then be written as (e.g. Kalbfleisch and Prentice (1980, Section 1.2.3))

$$\Pr[X \geq x|Z] = \exp\left\{-\int_0^x h_C(u|Z)\,du\right\} \prod_{u_j \leq x}[1 - h_D(u_j|Z)], \qquad (1.5)$$

where $h_C(u|Z)$ is an integrable hazard function corresponding to the continuous part of $\Pr[X \geq x|Z]$ and $u_1, u_2, \ldots$ are the jump points for $\Pr[X \geq x|Z]$. The values of $h_D(u_j|Z)$ are the discrete hazard function components,

$$h_D(u_j|Z) = \Pr[X = u_j|X \geq u_j, Z].$$

We can write (1.5) so as to provide an explicit ITS. We have

$$\Pr[X \geq x|Z] = \exp\left\{-\int_0^x h_C(u|Z)\,du + \int_0^x \log[1 - h_D(u|Z)]\,dN(u)\right\}, \qquad (1.6)$$

where $dN(u)$ equals 1 if a jump in $\Pr[X \geq x|Z]$ occurs at u, and 0 otherwise. Thus

$$t_Z(x) = \Phi[x, Z(x)] = \int_0^x h_C(u|Z)\,du - \int_0^x \log[1 - h_D(u|Z)]\,dN(u) \qquad (1.7)$$

is an ITS, with $G(t) = \exp(-t)$ in the format (1.4).

If $\Pr [X \geq x|Z]$ is continuous at all x then we will write $h(u|Z)$ for $h_C(u|Z)$ in (1.5), giving

$$\Pr [X \geq x|Z] = \exp \left\{ - \int_0^x h(u|Z) \, du \right\}. \tag{1.8}$$

We note that if $t_Z(x) = \Phi[x, Z(x)]$ is any ITS relative to Z then from (1.4) we have

$$\Pr [X \geq x|Z] = G[t_Z(x)]. \tag{1.9}$$

In the case where (1.8) holds and $\Pr [X \geq x|Z]$ is differentiable at x, we have

$$h(x|Z) = h_G[t_Z(x)]t'_Z(x), \tag{1.10}$$

where $h_G(t) = -G'(t)/G(t)$ is the hazard function corresponding to the survivor function $G(t)$.

Now let us take a closer look at the properties of ideal time scales. The first questions of interest concern the existence and uniqueness of ITSs.

Proposition 1.3.1 *There always exists an ITS for $Pr [X \geq x|Z]$.*

Proof: As mentioned by Oakes (1995) and shown by Kordonsky and Gertsbakh (1995a), we can always re-write

$$\Pr [X \geq x|Z] = \exp\{-H_0(x|Z)\}, \quad \forall Z \in \mathbf{Z}, \quad \forall x \geq 0,$$

where $H_0(\cdot|Z)$ is the conditional cumulative hazard function for X given Z. If we take $\Phi[x, Z(x)] = H_0(x|Z)$, then we have that $\Pr [X \geq x|Z] = G(t)$, where $t = t_Z(x)$ and $G(\cdot)$

is the survivor function of a standard exponential rv. Since $H_0(\cdot|\mathcal{Z})$ is non-decreasing, Φ is non-decreasing as well and hence fulfills the requirements of an ITS. Note that this argument is also valid for problems with jump discontinuities; all we have to do in this case is use $t_{\mathcal{Z}}(x)$ given by (1.7) instead of the cumulative hazard function H_0 mentioned above. ∎

Regarding the uniqueness issue, we know, by definition, that ITSs are unique up to one-to-one transformations.

The following theorem also follows immediately from the definition of an ITS. Its conclusion will be especially useful for semiparametric inference and model assessment.

Theorem 1.3.2 *Let $\Phi[x.\mathcal{Z}(x)]$ be an ITS for $Pr\ [X \geq x|\mathcal{Z}]$, $T = \Phi[X, \mathcal{Z}(X)]$ and $t = \Phi[x.\mathcal{Z}(x)]$. Then for all $\mathcal{Z} \in \mathbf{Z}$ and all $t = \Phi[x.\mathcal{Z}(x)]$ for some $x \geq 0^2$,*

$$Pr\ [T \geq t|\mathcal{Z}] = Pr\ [T \geq t].   \tag{1.11}$$

Proof: If $t = \Phi[x.\mathcal{Z}(x)]$ for a single value of x. then

$$
\begin{aligned}
\mathrm{Pr}\ [T \geq t|\mathcal{Z}] &= \mathrm{Pr}\ [X \geq x|\mathcal{Z}] \\
&= \mathrm{Pr}\ [T \geq t],
\end{aligned}
$$

the last equality following from the definition of an ITS. If $t = \Phi[x, Z(x)]$ for all x in some set A. then by definition of an ITS. $\mathrm{Pr}\ [X \geq x|\mathcal{Z}]$ remains constant for all $x \in A$ and, hence. $\mathrm{Pr}\ [T \geq t|\mathcal{Z}] = \mathrm{Pr}\ [X \geq x|\mathcal{Z}]$. But by definition of an ITS, $\mathrm{Pr}\ [X \geq x|\mathcal{Z}] = \mathrm{Pr}\ [T \geq t]$. which completes the proof. ∎

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[2] *Values of $t \neq \Phi[x, \mathcal{Z}(x)]$ for any $x \geq 0$ for a given $\mathcal{Z}$ are not a concern here as these values occur with probability zero. $G(\cdot)$ being a strictly decreasing survivor function in Definition 1.4.*

The theorem above indicates that, with probability equal to one, the failure times in the ITS, $T = \Phi[X, \mathcal{Z}(X)]$, will be statistically independent of the covariate histories, $\mathcal{Z}$, when these histories are random variables.

1.4 Uses and applications of ideal time scales

One may ask the question "Why not use one measure as the main scale and let the other usage measures enter the model as time-varying covariates?". As mentioned in Farewell and Cox (1979), this is the logical thing to do if one of the measures is of primary importance. However, when numerous usage measures are available, it may not be easy to determine which of the measures, or if a combination of those measures, is more relevant, and because models with covariates treat the time variable and the covariates quite asymmetrically, it is not recommended to choose an arbitrary scale as the main time scale and the other scales as covariates. Moreover, even if a particular scale appears to be more important than the others, the number of tractable models that can treat time-varying covariates is limited. As mentioned earlier, time scale based models may also possibly be used to include the information given by internal time-varying covariates. (Unfortunately, this last issue has not been investigated thoroughly yet and remains a potential research topic.)

One may also argue that because real world processes and their consequences occur in "real" or chronological time, it obviously plays an important role. Nevertheless, there are reasons for adopting an alternative time scale $t_{\mathcal{Z}}(x)$ in certain contexts.

For instance, in some cases the failure process may depend primarily on $t_{\mathcal{Z}}(x)$, so that it is scientifically "sufficient". For example, the probability that an item survives

past a certain chronological time x may only depend on its $t_Z(x)$ value.

Another important application of ITSs is what Kordonsky and Gertsbakh (1995a) call *indirect monitoring*. This means the monitoring of the internal state of a system by calculating its age. Since age in real time does not usually signify much for systems used under different conditions, Kordonsky and Gertsbakh combine various usage measures into a single TS in which they determine the age of the systems under study. Consequently, one sees that an ITS can be very useful in designing maintenance programs, especially if the failure distribution in this scale is concentrated over a short interval. Indeed, by the very definition of an ITS, one knows that the age in the ITS implies the same quantile of the lifetime distribution, no matter what the usage path is. This means that only one maintenance program is necessary, as long as the times of the different maintenance operations are scheduled according to the ITS. This turns out to be quite convenient in situations where direct monitoring of the internal state of the systems cannot be done or can only be done at a high cost (eg: nuclear power plants or workers exposed to asbestos dust, see Kordonsky and Gertsbakh (1995a) and Oakes (1995)). In addition, because of the generality in their definition, ITSs can be used to perform indirect monitoring and design maintenance programs in a wide range of applications. For instance, it is relatively simple to think of TS models that can be used in situations where usage paths are of the types discussed in Examples 1.1.1-1.1.5 and shown in Figure 1.1. We will consider such models in detail later on.

A time scale $t_Z(x)$ also is advantageous for experimental design and decision making. For example, if the failure pattern for some system component depends primarily on the accumulated voltage-time, then reliability studies in which such components are tested at very high voltage over short periods of time may be extrapolated to normal customer

usage conditions.

A further argument for using an alternative scale is that the effects of treatments or covariates may be most simply or directly expressed on some scale other than real time.

Another interesting feature of ITSs is the way they can handle more complicated censoring schemes. Such situations often occur in the analysis of warranty data (see Lawless, Hu and Cao (1995)), where we know the exact value of the time at failure only if the values of the usage measures are all within the warranty region at that time. Therefore, in those cases, censoring does not provide the lower (and/or upper) bound of an interval where failure has occurred, but rather the bounds of a multi-dimensional region. The nice property of an ITS in that kind of setup is that it brings back the problem to one dimension. Models based on ITSs are therefore quite useful for modeling the distribution of time to failure when the censoring scheme is multi-dimensional.

Censoring does not only cause problems when it is multivariate; heavy censoring[3] can make statistical inference extremely inefficient, and sometimes even almost impossible. However, if information on the covariate path is available, then looking at the distribution of X given Z or at the value of a given scale $T = \Phi[X, Z(X)]$ may turn out to be more informative than considering the marginal distribution of X, and hence improve on the precision and the efficiency of the inference.

Unfortunately ITSs are not always sufficient, for example if one wants to estimate warranty costs and calculate warranty reserves. This is due to the fact that knowledge about the distribution of the paths has to be taken under consideration. Actually, there are other cases in which a model for the distribution of the covariate path, Z, is relevant.

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[3]This is often the case in the study of aircraft or nuclear power plants, or any other device which, we hope, only rarely fails.

For instance, the case where the path includes internal time-varying covariates requires a probability model for those covariates which are internal. In fact, most applications of ITS models (e.g. conception of maintenance schedules, calculation of warranty costs, see Chapter 5) require knowledge of the usage path distribution if they involve prediction.

In the remainder of this thesis, we outline approaches to modeling using ideal time scales. We classify the various ITS models in Chapter 2, and we introduce a class of models considered by Oakes (1995) that has not been thoroughly investigated in the literature, namely, the *collapsible models*. In Chapter 3, we describe parametric inference based on maximum likelihood for collapsible models. We review the maximum likelihood method proposed by Oakes (1995) and we discuss issues concerning parameterization that have not been addressed in the literature in this type of framework previously. Semiparametric inference methods for those models are covered in Chapter 4; we survey the minimum coefficient of variation method of Kordonsky and Gertsbakh (1993), for which we give a proof of consistency that was lacking in the literature, and we derive quasi-likelihood and rank-based methods that yield new estimators that are often more efficient than the minimum coefficient of variation estimator and that can also handle censored samples. New ideas for nonparametric estimation are considered in Chapter 5, along with the problem of prediction and some applications. We apply the methods described in Chapters 3 and 4 to various datasets and propose model assessment procedures in Chapter 6. Finally, concluding remarks are given in Chapter 7. We remark that many of the most important applications of time scale methods are in reliability; for a recent exposition of this area see Meeker and Escobar (1998).

Chapter 2

Families of models

Various models that incorporate usage measures and time-varying covariates in general are proposed in the literature. In this chapter we review the main classes of such models.

Before getting into further detail, we need to first introduce parametric models generally. From here on, we will assume that probability (1.1) can be written as

$$\Pr\left[X \geq x | \mathcal{Z}(x)\right] = S_0[x | \mathcal{Z}(x); \beta], \tag{2.1}$$

where β is a vector of unknown parameters, possibly of infinite dimension. We also define $h_0(x|\mathcal{Z}(x);\beta)$ and $H_0(x|\mathcal{Z}(x);\beta)$, the hazard and cumulative hazard functions, respectively, as follows:

$$h_0(x|\mathcal{Z}(x);\beta) = \lim_{\Delta x \to 0} \frac{\Pr\left[X \in [x, x + \Delta x)|X \geq x, \mathcal{Z}(x);\beta\right]}{\Delta x};$$
$$\text{and } H_0(x|\mathcal{Z}(x);\beta) = \int_0^x h_0(u|\mathcal{Z}(u);\beta)\, du.$$

Of course,

$$S_0(x|\mathcal{Z}(x);\beta) = \exp\left\{-H_0(x|\mathcal{Z}(x);\beta)\right\}.$$

Let us now introduce the ITS model.

Definition 2.1 *An* **ideal time scale (ITS) model** *is a model where*

$$S_0[x|\mathcal{Z}(x);\beta] = G[t_{\mathcal{Z}}(x;\eta);\phi], \quad \forall \mathcal{Z} \in \mathbf{Z}, \ x \geq 0, \qquad (2.2)$$

where G is some absolutely continuous survivor function, $t_{\mathcal{Z}}(x;\eta)$ is an ITS as defined in Definition 1.4, and η and ϕ are two vectors of parameters that partition β.

In practical situations, G or $t_{\mathcal{Z}}(x)$ or both can be specified up to a vector of parameters or left arbitrary (i.e. η or ϕ of infinite dimension).

From Definition 1.4, we know that this family of models is extremely broad. Our interest in this chapter is to outline a few sub-classes of ITS models.

2.1   Transfer functional models

Bagdonavičius and Nikulin (1997) have proposed a series of regression models for problems involving time-varying covariates (*stress* in their terminology). Starting from differential equations that model the usage rate of what they term the *resource*, they obtain various probability models for the conditional survivor function $S_0[x|\mathcal{Z}(x);\beta]$, where $\mathcal{Z}$ is the history of a set of external covariates. Let us look at their Model 2 (see Bagdonavičius and Nikulin (1997, pp. 366-367)):

Definition 2.2 *Bagdonavičius and Nikulin's Model 2 specifies the following form for*
$t_Z(x)$:

$$t_Z(x) = \int_0^x \psi[u, Z(u)]\, dG^{-1}S(u),\qquad (2.3)$$

where $\psi[\cdot]$ is a positive, real-valued functional and S is a survivor function.

Bagdonavičius and Nikulin (1997) derive basic properties and also consider semiparametric estimation for such a model under different conditions.

In the remainder of this chapter, we will take a closer look at two special cases of Bagdonavičius and Nikulin's Model 2 that are quite useful in practice, namely the hazard based and the time transformation models.

2.2 Hazard based models

As the name suggests, hazard based models are models in which the covariate history, Z, modifies a so-called *baseline* hazard function.

Definition 2.3 *A hazard based model is a model where*

$$h_0(x|Z(x);\beta) = \psi[h(x), Z(x);\beta],\quad \forall Z \in \mathbf{Z},\ x \geq 0,\ \beta \in \Sigma,\qquad (2.4)$$

where Σ is the parameter space, and $h(\cdot)$ is a baseline hazard function that can be fully specified, specified up to a vector of parameters, or left arbitrary.

This definition is virtually useless in practice because equation (2.4) defines a family of models that is too general. Let us now consider a few special cases that are widely used.

2.2.1 Multiplicative hazards model

The class of multiplicative hazards models is one of the most popular in practice. Models of this form have been proposed by Cox (1972) and are discussed in great detail in the literature (see Lawless (1982), Kalbfleisch and Prentice (1980) or Cox and Oakes (1984), for example). The key feature of such models is that they are very tractable mathematically and hence allow for easy inference on β , even if $h(\cdot)$ is left arbitrary. Multiplicative hazards models are also quite versatile and can be used to model a wide variety of problems.

Definition 2.4 *A multiplicative hazards model is a model where*

$$h_0(x|Z(x);\beta) = h(x)\psi[Z(x);\beta], \quad \forall Z \in \mathbf{Z}, \ x \geq 0, \ \beta \in \Sigma, \qquad (2.5)$$

where $\psi[\cdot]$ is a positive, real-valued function.

If we take Model 2 and let $\psi[u, Z(u)] = \psi[Z(u); \beta]$ and $G(u) = \exp(-u)$, and if we let $S(\cdot)$ be an arbitrary survivor function with corresponding hazard function $h(\cdot)$, we obtain from (2.3) that

$$S_0[x|Z(x)] = \exp\left\{-\int_0^x h(u)\psi[Z(u);\beta] \, du\right\},$$

which is exactly the multiplicative hazards model.

Many choices for the function $\psi[\cdot]$ in equation (2.5) are possible. But there is one particular form of $\psi[\cdot]$ that makes inference on β in parametric and semiparametric cases particularly easier, namely, the *log-linear* form.

Definition 2.5 *Suppose that the problem is such that we can find mappings $z^*(x)$: $Z(x) \to I\!\!R^r$ such that*

$$h_0(x|Z(x); \beta) = h_0(x|z^*(x); \beta), \quad \forall Z \in \mathbf{Z}, \ x \geq 0, \ \beta \in \Sigma.$$

Then the model is said to be **log-linear** *if*

$$h_0(x|z^*(x); \beta) = h(x) \exp\{\beta^t z^*(x)\}, \quad \forall Z \in \mathbf{Z}, \ x \geq 0, \ \beta \in \Sigma. \tag{2.6}$$

2.2.2 Additive hazards model

Definition 2.6 *An* **additive hazards model** *is a model where*

$$h_0(x|Z(x); \beta) = h(x) + \psi[Z(x); \beta], \quad \forall Z \in \mathbf{Z}, \ x \geq 0, \ \beta \in \Sigma. \tag{2.7}$$

As is the case with multiplicative hazards models, some forms of the function $\psi[\cdot]$ are more convenient. One that is frequently used is the *linear form*.

Definition 2.7 *Suppose that the problem is such that we can find a mapping $z^*(x)$: $Z(x) \to I\!\!R^r$ such that*

$$h_0(x|Z(x); \beta) = h_0(x|z^*(x); \beta), \quad \forall Z \in \mathbf{Z}, \ x \geq 0, \ \beta \in \Sigma.$$

Then the model is said to be a **linear additive hazards model** *when*

$$h_0(x|z^*(x); \beta) = h(x) + \beta^t z^*(x) \quad \forall Z \in \mathbf{Z}, \ x \geq 0, \ \beta \in \Sigma. \tag{2.8}$$

Jewell and Kalbfleisch (1996) use a linear additive hazards model in order to incorporate the information provided by marker processes into a survival distribution. They assume that $z^*(x)$ is a (possibly multivariate) stochastic process that contains information that may influence the lifetime distribution, and they use a hazard function of the form given by equation (2.8) to include that information into the model. Unfortunately, the mathematics involved in inference procedures are quite challenging and solutions have been obtained only in a limited number of special cases. (See Singpurwalla (1995), Singpurwalla and Wilson (1995) or Jewell and Kalbfleisch (1996) for further discussion on this issue.)

In the case where we do not need to integrate over $\mathcal{Z}$, i.e. when we condition on a given value of the path $\mathcal{Z}$, we can handle very general linear additive hazards model. Assume that

$$h_0(x|\mathcal{Z}(x); \theta) = h_0(x|z^*(x); \theta) = \beta_0(x; \theta) + \beta(x; \theta)^t z^*(x), \quad \forall \mathcal{Z} \in \mathbf{Z}, \ x \geq 0.$$

Then it is possible to estimate the β functions either parametrically or nonparametrically (see Andersen et al. (1993, Sections VII.1 and VII.4)).

2.3 Time transformation models

Despite the flexibility of hazard based methods, in some situations it may be more natural to model the effect of the covariate history on the survival distribution in a different way. In this section, we consider an alternate approach in which the effect of the covariate path modifies the value of the time variable instead of the value of the hazard function.

Definition 2.8 *A* **time transformation model** *(also termed accelerated failure time (AFT) model) is an ITS model in which*

$$t_{\mathcal{Z}}(x) = \int_0^x \psi[u, \mathbf{z}(u); \beta]\, du, \quad \forall \mathcal{Z} \in \mathbf{Z}, \; x \geq 0, \tag{2.9}$$

where $\psi[\cdot]$ is a positive, real-valued function.

It is more convenient to define time transformation models in terms of the survivor or cumulative hazards functions because, since they involve a change of variable of the form $U = f[X, \mathcal{Z}(X); \beta]$, there is a differential element that needs to be considered in the hazard function. In fact one can observe that equation (2.9) is equivalent to

$$h_0(x|\mathcal{Z}(x); \beta) = h_G(t_{\mathcal{Z}}(x))\psi[x, \mathbf{z}(x); \beta], \quad \forall \mathcal{Z} \in \mathbf{Z}, \; x \geq 0, \; \beta \in \Sigma, \tag{2.10}$$

where $h_G(\cdot)$ is the hazard function corresponding to the survivor function $G(\cdot)$. One can easily see that the AFT model is a special case of the Model 2; this is trivially observed when one lets $G = S$ in equation (2.3).

A common choice for $\psi[x, \mathbf{z}(x); \beta]$ in Definition 2.8 is

$$\psi[x, \mathbf{z}(x); \beta] = \exp\{\mathbf{z}^t(x)\beta\}.$$

In this log-linear case, we can see that the effect of the covariate path is to multiply the value of the time variable by a certain quantity, in other words to "accelerate" time. This category of models is very important in engineering and other domains where *accelerated life experiments* are performed. One can find examples and a good treatment of these experiments and models and their properties in Lawless (1982, Chapter 6), Kalbfleisch

and Prentice (1980, Chapter 6) and Meeker and Escobar (1998) in the case of fixed covariates, and in Nelson (1990, Chapter 10) and Cox and Oakes (1984, Sections 5.2 and 6.3) in the case of time-varying covariates. These authors prefer to restrict their attention to the case in which $G(\cdot)$ is specified up to a vector of parameters, but Robins and Tsiatis (1992) and Lin and Ying (1995) consider a method of estimation for β in Definition 2.8 where $G(\cdot)$ can be an arbitrary survivor function.

A good example of an application of AFT models is given by Lawless, Hu and Cao (1995), where they study the reliability of automobile brake pads from warranty data. They use real time and cumulative mileage as their usage path. They assume that the usage path for a given unit is of the form $\mathcal{P}(x) = \{(u, \alpha u); 0 \leq u < x\}$, where αx is the cumulative mileage at time x, i.e. α is the mileage accumulation rate that they assume constant over time. Their model is an AFT model of the form (2.9) with, for the ith item,

$$S_0(x|\alpha_i; \beta) = G(x\alpha_i^\beta; \phi), \forall \mathcal{P} \in \mathbf{P}, \ x \geq 0, \ \beta \in [0,1], \ \phi \in \Sigma, \qquad (2.11)$$

where G is specified up to a vector of parameters, ϕ. An interesting property of model (2.11) is that if $\beta = 1$, the lifetime distribution depends on the cumulative mileage only, and if $\beta = 0$, it only depends on real time. The inference procedures for such models are slightly complicated, in this case, by the special censoring scheme common to warranty claim data and we refer the reader to Lawless, Hu and Cao (1995) for further information.

Another example of an application of AFT models is the problem in which a system is either "up" or "down" (see Example 1.1.2), and failure can only occur during an "up" period, sometimes with positive probability at the first moment of an up-period. If we define $u(x)$ to be the function indicating that the system is up at time x, and if we let $N_{\mathcal{Z}}(x)$ be the cumulative number of up-time periods up to and including time x, we can

put

$$t_{\mathcal{Z}}(x) = \int_0^x u(s)f(s,z(s);\beta_1)\,ds + \int_0^x k(s,z(s);\beta_2)\,dN_{\mathcal{Z}}(s), \qquad (2.12)$$

where $f(\cdot)$ and $k(\cdot)$ are two positive, real-valued functions. With $t_{\mathcal{Z}}(x)$ defined as in (2.12), we can see that the AFT model can be such that failures cannot occur during down-time and they can occur with probability masses at the beginning of up-time periods, and this even when the survivor function $G(\cdot)$ is absolutely continuous; the second integral on the right-hand side of (2.12) generates jump points in $t_{\mathcal{Z}}(x)$ when x is the first moment of an up-time period, and both integrals on the right-hand side of (2.12) imply that $t_{\mathcal{Z}}(x)$ will stay constant when the system is down at x.

2.4 Collapsible models

In some situations, it may not be necessary to use the whole path $\mathcal{Z}$ in order to compute probability (1.1). Oakes (1995) introduced the notion of collapsibility.

Definition 2.9 *Let $y_1(x),\ldots,y_p(x)$ be a specified set of usage factors. Then the distribution of $X|\mathcal{P}$ is collapsible in $y_1(x),\ldots,y_p(x)$ if*

$$Pr\,[X > x|\mathcal{P}] = S[\mathbf{y}(x)]. \qquad (2.13)$$

In other words, in a problem collapsible in $\mathbf{y}$, probability (1.1) depends on the path $\mathcal{P}(x)$ only through the endpoint $\mathbf{y}(x)$. In this case, ITS's are of the form

$$t_{\mathcal{P}}(x) = \Phi[x,y_1(x),\ldots,y_p(x)] \qquad (2.14)$$
$$= \Phi[\mathbf{y}(x)].$$

Geometrically, we can think of the ITS Φ as a function that traces *age surfaces* in $I\!R^{(p+1)+}$, i.e. surfaces on which the points represent the same quantile of the failure time distribution (see Figure 2.1).
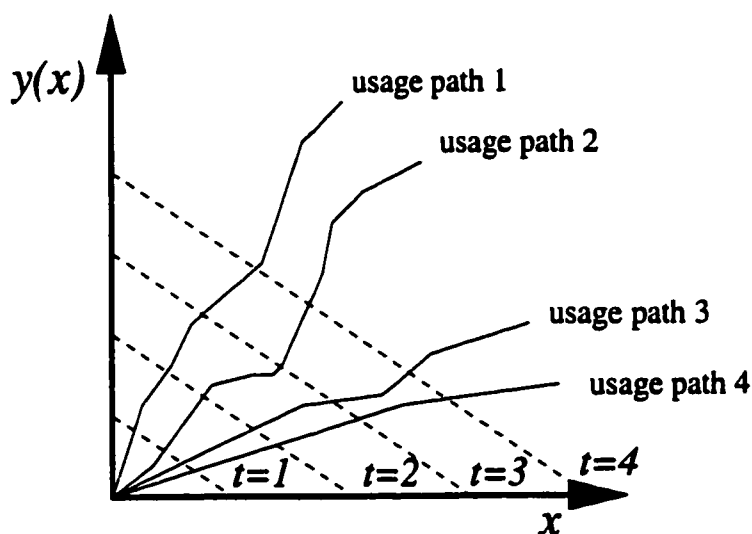
Figure 2.1: Example of age curves generated by the ITS $\Phi[x, y(x)] = x + \eta y(x)$, where the problem is collapsible in $y(x) = (x, y(x))^t$. All the points on a line parallel to the dotted lines (of slope $-1/\eta$) correspond to a same quantile of the failure time distribution.

It may seem that collapsibility is a very restrictive assumption. However, the possibility of defining measures $y_j(x)$ based on stress, usage or environmental factors allows considerable flexibility.

Note that collapsibility is defined in terms of usage measures, rather than general external covariates. This is due to the fact that it is not possible to define a function of $x, z_1(x), \ldots, z_q(x)$ that would be non-decreasing in x for all $Z \in Z$, unless the functions z_j's themselves are non-decreasing.

In some settings distinct usage paths never cross, e.g. when $y_j(x)$'s are linear in x. In that case all models for X given $\mathcal{P}$ are collapsible in the trivial sense that each $\mathcal{P}$ is identified by its $y(x)$ value at any given time x. However, the ITS (2.14) is not in general of simple functional form. The emphasis on modeling using collapsible models is to consider fairly simple parametric specifications $\Phi[y(x); \eta]$ in (2.14), thus yielding easily interpreted operational times. Oakes (1995) and Kordonsky and Gertsbakh (1993, 1995a, 1995a, 1997) consider models in which

$$t_{\mathcal{P}}(x) = \eta^t \mathbf{y}(x)$$

is assumed linear in x and $y_1(x), \ldots, y_p(x)$, and G in (2.2) has a specified parametric form. Semiparametric approaches can also be adopted. For example, Kordonsky and Gertsbakh consider a method of estimating η without a parametric model for G; this is examined in Chapter 4.

From Definition 2.9, we observe that a collapsible model is a special case of the class of models defined by equation (2.9). Indeed, let us set

$$\frac{d}{dx} t_{\mathcal{Z}}(x) = \psi[\mathbf{y}(x); \beta] \tag{2.15}$$

in equation (2.2) and we get a model that is collapsible, i.e. a model for which $S_0[x|\mathcal{P}(x)]$ only depends on the value of $\mathbf{y}$ at x.

Collapsible models are not AFT models in general; the derivative on the right of (2.15) implies that the ψ function in the integral on the right-hand side of (2.9) would have to be a function of both the usage measures $\mathbf{y}(x)$ and their derivatives $\mathbf{y}'(x)$.

A particular collapsible model of interest is the *linear scale model*.

Definition 2.10 *A linear scale model is a collapsible model such that*

$$S(\mathbf{y}^*(x); \eta) = G(\eta^t \mathbf{y}^*(x)), \quad \forall \mathcal{P} \in \mathbf{P}, \ x \geq 0, \ \eta \in \Sigma, \qquad (2.16)$$

where $\mathbf{y}^*(x)$ *is a mapping* $\mathbf{y}^*(x) : \mathbf{P}(x) \to I\!\!R^{+q}$, *as in Definition 2.7, and* $\eta \in I\!\!R^{+q}$ *is a vector of parameters.*

A very nice property of linear scale models, along with their collapsibility, is their interpretability.

Example 2.4.1 *Suppose that x is the age in calendar years and that y(x) is the number of packs of cigarettes smoked by an individual up to age x, and let's assume that*

$$Pr\left[X \geq x | \mathcal{P}(x)\right] = G(x + \frac{1}{365} y(x)), \quad \forall \mathcal{P} \in \mathbf{P}, \ x \geq 0.$$

Then we can say that for each pack of cigarettes smoked, an individual ages by one day. For example, in this model, and individual aged 50 that has smoked 3650 packs of cigarettes in his life would be "as old" as an individual aged 60 that has never smoked.

It is worth noting that the linear scale model and the additive hazard model do overlap in some situations. Indeed, if the covariates $\mathbf{z}^*(x)$ from Definition 2.7 do not change sign as x varies, we can let $z_j^{**}(x) = \int_0^x z_j^*(u) \, du$, $j = 1, \ldots, q$ and let $z_0^{**}(x) = \int_0^x h(u) \, du$. This can then be re-written as a linear scale model:

$$S_0(x|\mathcal{Z}; \beta) = \exp\left\{-\beta^t \mathbf{z}^{**}(x)\right\}.$$

Linear scale models are basically the only collapsible models used in practice so far. Kordonsky and Gertsbakh (1993), (1995a), (1995b) and (1997b) use a linear scale model

in order to study the reliability of aircraft; Oakes (1995) employs such models to obtain the lifetime distribution of miners exposed to asbestos dust; Farewell and Cox (1979) investigate the time to onset of breast cancer by combining real time and time since birth of the first child into a linear scale.

Another form of collapsible models that has been used in the literature is the multiplicative scale model.

Definition 2.11 *A* **multiplicative scale model** *is a collapsible model such that*

$$S(\mathbf{y}^*(x); \boldsymbol{\eta}) = G(y_0^*(x)^{\eta_0} y_1^*(x)^{\eta_1} \cdots y_q^*(x)^{\eta_q}), \quad \forall \mathcal{P} \in \mathbf{P}, \ x \geq 0, \ \boldsymbol{\eta} \in \Sigma, \qquad (2.17)$$

where $\mathbf{y}^*(x)$ *is a mapping* $\mathbf{y}^*(x) : \mathbf{P}(x) \to I\!\!R^{+q}$, *as in Definition 2.7, and* $\eta_0, \eta_1, \ldots, \eta_q$ *are model parameters.*

Unfortunately, there is virtually no mention of other forms of collapsible models in the current literature. Moreover, techniques of inference for the parameter η in equation (2.16) in the rather simple case of linear scales need to be refined, especially in the semiparametric setup (i.e. when $G(\cdot)$ is left arbitrary in (2.16)). We will try to fill this gap in Chapters 3 and 4 where we consider inference methods for collapsible models in a relatively general setting. We will also briefly explore the potential of collapsible models for prediction problems in Chapter 6.

Chapter 3

Parametric maximum likelihood estimation for collapsible models

Parametric estimation for the collapsible models of Section 2.4 is easy to implement. In this chapter we review maximum likelihood (ML) inference for the time scale parameters when a fully parametric model is specified. More precisely, we describe how to estimate η and ϕ in the collapsible model

$$\Pr\left[X > x|\mathcal{P}\right] = G[t_{\mathcal{P}}(x;\eta);\phi], \quad x > 0, \forall \mathcal{P} \in \mathbf{P}, \tag{3.1}$$

where η is the vector of time scale parameters and ϕ is the vector of parameters of the survivor function G. Of course, model (3.1) assumes that $t_{\mathcal{P}}(x;\eta)$ is an ITS for some η.

A first treatment of ML estimation for collapsible models was given by Oakes (1995). He showed that for a set of n observations $(x_1, \mathcal{P}_1, \delta_1), \ldots, (x_n, \mathcal{P}_n, \delta_n)$, where δ_i is 1 if x_i is a failure time and 0 if x_i is right-censored, the likelihood function under the model

(3.1) is

$$
\begin{aligned}
L(\phi, \eta) &= \prod_{i=1}^{n} \left\{ f_G[t_{\mathcal{P}_i}(x_i; \eta)] t'_{\mathcal{P}_i}(x_i) \right\}^{\delta_i} G[t_{\mathcal{P}_i}(x_i; \eta)]^{1-\delta_i} \\
&= \prod_{i=1}^{n} \left\{ -G'[t_{\mathcal{P}_i}(x_i; \eta)] t'_{\mathcal{P}_i}(x_i) \right\}^{\delta_i} G[t_{\mathcal{P}_i}(x_i; \eta)]]^{1-\delta_i},
\end{aligned}
\tag{3.2}
$$

where $f_G(t) = -G'(t)$ is the density corresponding to $G(t)$ and $t'_{\mathcal{P}}(x) = dt_{\mathcal{P}_i}(x)/dx$. Note that (3.1) involves $t'_{\mathcal{P}_i}(x_i)$ which itself usually involves $y'_i(x_i)$, even though the model is collapsible in $y_i(x_i)$.

3.1 Parametric usage paths

We sometimes make the assumption that the usage paths can be completely specified by a finite-dimensional vector of parameters, say θ. This assumption somewhat simplifies the calculations and, as we will see in the following chapters, makes semiparametric inference and model assessment easier to handle. In more precise terms, we assume that

$$
\mathcal{P} = \{ \mathbf{y}(u); u \geq 0 \} = \mathcal{P}(\theta),
$$

where θ is a vector of observed parameters that entirely describes $\mathcal{P}$. In this case, we observe $(x_1, \theta_1, \delta_1), \ldots, (x_n, \theta_n, \delta_n)$ and wish to estimate η and ϕ in

$$
\Pr[X > x | \mathcal{P}] = \Pr[X > x | \theta] = G[t_\theta(x; \eta); \phi].
\tag{3.3}
$$

In many cases, the parametric path representation is very reasonable. For example, Lawless, Hu and Cao (1995) consider usage paths of the form $\mathcal{P} = \{(x, y(x)); x \geq 0\}$,

where x is the age of an automobile in days since purchase and $y(x)$ is the accumulated mileage on that automobile at time x. They make the assumption that $y(x) = \theta x$, $\theta > 0$. Their underlying assumption is, thus, that the rate of mileage accumulation on an automobile is constant over time, which is usually close to reality during the warranty period.

Of course, we may not be able to observe the θ_i values directly in practice. Nevertheless, we can usually compute or estimate the value of the θ_i's from the data. For instance, Lawless, Hu and Cao (1995) observed the values x_i and y_i at failure and used $\theta_i = y_i/x_i$. If the usage path is of the form $y(x) = f(x, \theta_1, \theta_2)$, then, obviously, we need more than a single observation per usage path in order to get a unique value of (θ_1, θ_2).

3.2 Multiplicative scale model

A collapsible model encountered in the literature is the multiplicative scale model. In certain setups, this model is equivalent to the accelerated failure time model (e.g., Lawless, Hu and Cao (1995)). The multiplicative scale model is the one in which

$$\Pr[X > x|\mathcal{P}] = G[x^{\eta_0} y_1(x)^{\eta_1} \cdots y_p(x)^{\eta_p}; \phi], \quad x > 0, \forall \mathcal{P} \in \mathbf{P}. \tag{3.4}$$

If we restrict ourselves to the case where there is only one usage measure of the form $y(x) = \theta x$, then model (3.4) becomes

$$\Pr[X > x|\mathcal{P}] = \Pr[X > x|\theta] = G[x\theta^\eta; \phi], \tag{3.5}$$

which is simply the accelerated failure time model.

Direct application of equation (3.2) gives a simple formula of the likelihood for model (3.5):

$$L(\eta, \phi) = \prod_{i=1}^{n} \left\{ -G'(x_i \theta_i^{\eta}; \phi) \theta_i^{\eta} \right\}^{\delta_i} G(x_i \theta_i^{\eta}; \phi)^{1-\delta_i}. \tag{3.6}$$

The multiplicative scale model seems to yield more stable estimators of η than the linear scale model of the next section. The asymptotic variances obtained in Section 3.5 will confirm this last statement.

3.3 Linear scale model

Let us now introduce a particular collapsible model that has been considered a few times (e.g. Oakes (1995), Kordonsky and Gertsbakh (1997b)) in the literature, namely, the linear scale model:

$$\Pr\left[X > x | \mathcal{P}\right] = G[\eta^t \mathbf{y}(x); \phi], \quad x > 0, \ \forall \mathcal{P} \in \mathbf{P}. \tag{3.7}$$

The likelihood function for η and ϕ is obtained directly from (3.2):

$$L(\phi, \eta) = \prod_{i=1}^{n} \left\{ -G'[\eta^t \mathbf{y}_i(x_i); \phi] \eta^t \mathbf{y}_i'(x_i) \right\}^{\delta_i} G[\eta^t \mathbf{y}_i(x_i); \phi]^{1-\delta_i}, \tag{3.8}$$

where $\mathbf{y}_i'(x) = d\mathbf{y}_i(x)/dx$. A further simplification is possible when the accumulation rate is constant for each usage measure, i.e. $\mathbf{y}_i(x) = \boldsymbol{\theta}_i x$:

$$L(\phi, \eta) = \prod_{i=1}^{n} \left\{ -G'[\eta^t \boldsymbol{\theta}_i x_i; \phi] \eta^t \boldsymbol{\theta}_i \right\}^{\delta_i} G[\eta^t \boldsymbol{\theta}_i x_i; \phi]^{1-\delta_i}. \tag{3.9}$$

3.3.1 Parameterization

An important issue in parametric estimation is the parameterization. Indeed, ambiguities between the parameters in η and ϕ in (3.1) can arise. Furthermore, even when two parameters are not confounded it may be nearly impossible to estimate them separately in some cases. We will explain this through an example.

Suppose that we observe n failure times $x_1, \ldots, x_n$ along with n paths of the form $y_i(x) = \theta_i x$. We wish to fit the collapsible model

$$\Pr\left[X_i > x_i | \theta_i\right] = \exp\left\{-\left[\frac{x_i(1 + \eta\theta_i)}{\delta_1}\right]^{\delta_2}\right\}, \tag{3.10}$$

i.e. model (3.1) with $t_{p_i}(x_i)$ the linear scale and G the Weibull survivor function. Note that we do not write the time scale in (3.10) using two parameters, i.e. $x_i(\eta_1 + \eta_2\theta_i)$. This would actually define an irregular statistical model if the scale parameter, δ_1, in G is not set equal to one, as an infinite number of values of η_1, η_2 and δ_1 would lead to the exact same distribution. However, such confounding does not take place if we use the parameterization of (3.10). In fact, there is ambiguity in the parameterization of an ITS since 1-1 transformations of ITS's are also ITS's and since G can be defined in various ways. For example, (3.10) can be obtained by any of the following parameterizations:

1. $t = x_i(1 + \eta\theta_i)$ and $G(t) = \exp\left(-(t/\delta_1)^{\delta_2}\right)$, as in (3.10);

2. $t = x_i(1/\delta_1 + \eta\theta_i/\delta_1) = x_i(\eta_1 + \eta_2\theta_i)$ and $G(t) = \exp\left(-t^{\delta_2}\right)$;

3. $[x_i(1 + \eta\theta_i)]^{\delta_2}$ and $G(t) = \exp(-t/\lambda)$;

4. $[x_i(1 + \eta\theta_i)/\delta_1]^{\delta_2}$ and $G(t) = \exp(-t)$.

We usually try to maximize the number of parameters in G and minimize the number of parameters in $t_i(\eta)$; the latter will usually be necessary in order to get unique values of η such that $T(\eta)$ is an ITS, and hence, a unique estimator of η when G is left completely arbitrary (infinite number of parameters).

Despite the regularity of model (3.10), we may still encounter situations in which distinguishing between η and δ_1 is difficult. When $\eta\theta_i$ is large compared to one, the model for Pr $[X > x|\mathcal{P}]$ becomes approximately $\exp\left\{-[x(\eta/\delta_1)\theta]^{\delta_2}\right\}$. In this case we can only accurately estimate $\psi = \eta/\delta_1$ and δ_2. If there is not sufficient variability in the values of the slopes (θ_i's) in the sample or if the variation in X given θ is too large, inferences about the time scale parameter η may be very imprecise. Note also that in (3.10), η is not invariant to changes in the units of x or $y(x) = \theta x$.

Empirical observations, backed up by the asymptotic results from Section 3.5, show that two conditions must be satisfied in order to separate the part of the variability in the failure times represented by the age in the ITS and the part represented by the distribution G:

1. the observed variability in the usage paths must be large;

2. the failure times in the ITS must have a somewhat compact distribution, i.e. a distribution with small variability.

One should understand that these are conditions for precise inferences about the parameters in the model, not for precise predictions of future failure times. Note that since an ITS is a 1-1 function of the conditional survivor function of $X|\mathcal{P}$ (or $X|\Theta = \theta$ in this case), we could replace item 2. above by "the failure times, in real time, must have small variation, given the usage path, $\mathcal{P}$".

To illustrate, we simulated samples of size 100 from model (3.10), with

$$\Pr\left[X > x|\theta\right] = \exp\left\{-[x(1 + \eta\theta)/\delta_1]^{\delta_2}\right\}, \qquad (3.11)$$

where $\eta = 840$, $\delta_1 = 63800$ and $\delta_2 = 1$, so that the values of $\Pr\left[X > x|\theta\right]$ obtained were close to the ones from the multiplicative scale model used by Lawless, Hu and Cao (1995) to analyze automobile system reliability, i.e.

$$\Pr\left[X > x|\theta\right] = \exp\left\{-\left[\frac{x\theta^{\eta}}{\delta_1}\right]^{\delta_2}\right\}, \qquad (3.12)$$

with $\eta = 0.9$, $\delta_1 = 60$ and $\delta_2 = 1$. For both models, the usage paths θ's were generated from the same log normal distribution as in Lawless, Hu and Cao (1995), namely $\ln(\Theta) \sim N(2.37, 0.58^2)$.

As expected, it was nearly impossible to get precise estimates of η and δ_1 for samples generated from model (3.11) and the previous distribution of θ, and the MLE varied greatly from sample to sample. Adopting an alternate parameterization of the form $t(\eta) = (1 - \eta)x + \eta\theta x$ would not have solved the problem, as we would have obtained $\hat{\eta} = 1$ and a confidence interval that covers nearly the whole $[0, 1]$ interval. Figure 3.1 shows log-likelihood profiles for the time scale parameter η evaluated from a typical sample from each of model (3.12) and model (3.11). The shape of the profile log-likelihood curve obtained with model (3.11) implies only a precise lower bound for the value of η. The MLE's obtained for (3.11) were $\hat{\eta} = 3.04 \times 10^7$, $\hat{\delta}_1 = 2.29 \times 10^9$ and $\hat{\delta}_2 = 1.07$, for a maximized log-likelihood of -288.786.

With model (3.12), we have fairly precise estimation of η. The MLE's for the sample
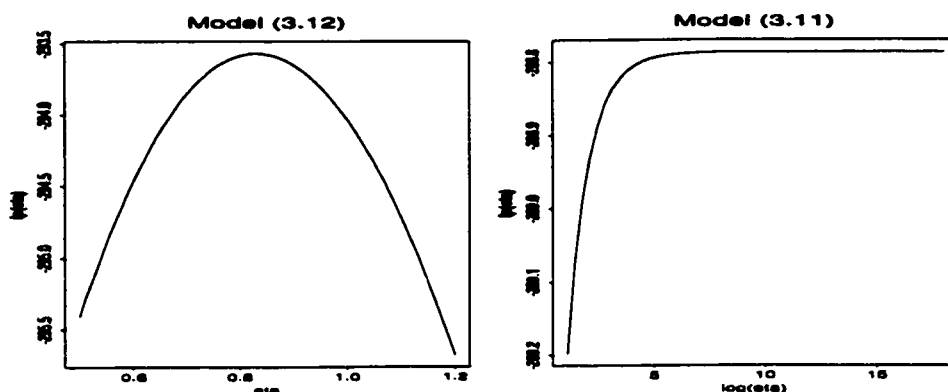
Figure 3.1: Profile log-likelihoods of η for typical samples from models (3.12) and (3.11).

used were $\hat{\eta} = 0.830$, $\hat{\delta}_1 = 50.4$ and $\hat{\delta}_2 = 0.969$.

The fact that we can only precisely estimate the quantity η/δ_1 in model (3.11) is clarified if we use the alternate parameterization

$$\Pr\left[X > x|\theta\right] = \exp\left\{-[x(\lambda + \psi\theta)]^{\delta_2}\right\},\qquad(3.13)$$

where $\psi = \eta/\delta_1$ and $\lambda = 1/\delta_1$. In this case we can estimate ψ accurately. However, inferences about λ remain imprecise. Figure 3.2 shows the log-likelihood contours in ψ and λ as well as the log-likelihood profile in ψ for the same sample as the one used above. The MLE's of the parameters are $\hat{\psi} = 0.0133$, $\hat{\lambda} = 4.37 \times 10^{-10}$ and $\hat{\delta}_2 = 1.07$. Those values are equivalent to the MLE's obtained under the parameterization (3.11).

Even though the parameterization in the linear scale model often does not allow for precise estimation of all the model parameters, the values of $\Pr\left[X > x|\theta\right]$ in a fully parametric model are usually well estimated. For example, if we evaluate (3.13) at different values of x and θ using the true values of ψ, λ and δ_2 and then using the MLE's for the sample associated with Figure 3.2, we obtain comparable values, as shown
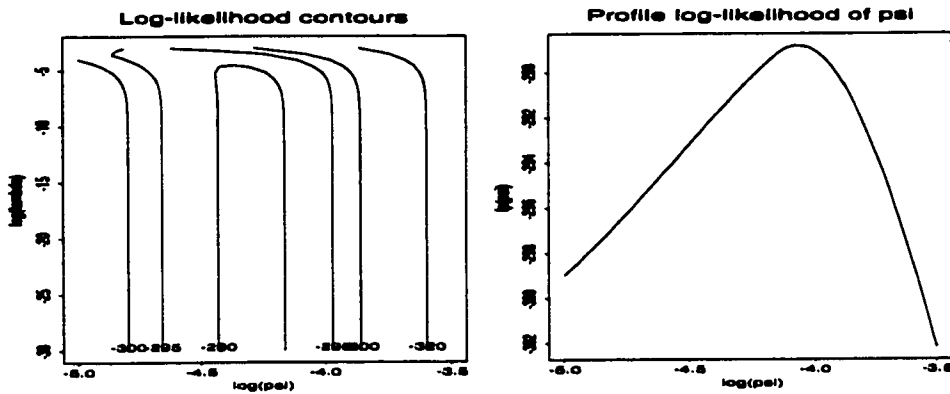
Figure 3.2: Contours of the log-likelihood evaluated at $\delta_2 = 1$ and profile log-likelihood of ψ for a typical sample from model (3.13).

in Table 3.1.

	$x = 1$ $\theta = 9$	$x = 1$ $\theta = 20$	$x = 5$ $\theta = 9$	$x = 5$ $\theta = 20$	$x = 30$ $\theta = 9$	$x = 30$ $\theta = 20$
true	0.89	0.77	0.55	0.27	0.029	0.00037
MLE	0.87	0.78	0.56	0.26	0.020	0.000098

Table 3.1: True and estimated values of Pr $[X > x|\theta]$ with model (3.13).

3.4   Maximization of the likelihood

Finding the values of η and ϕ that maximize the likelihood may not be as easy a task as it looks. As Oakes (1995) pointed out, the optimization can be simplified if performed through the profile log-likelihood of η. Proceeding in this way serves two purposes: maximizing the likelihood, of course, and building confidence regions for η. The following maximization algorithm was proposed by Oakes:

1. Let $l(\eta, \phi) = \ln L(\eta, \phi)$;

2. Fix η and let $l_{pro}(\eta) = \max_{\phi} l(\eta, \phi)$;

3. Find $\hat{\eta} = \arg\max_{\eta} l_{pro}(\eta)$;

4. Let $\hat{\phi} = \arg\max_{\phi} l(\hat{\eta}, \phi)$.

To build approximate confidence regions for η, one uses the likelihood ratio statistic based on $l_{pro}(\eta)$ described above. An approximate $(1-\alpha)100\%$ confidence region for η is given by

$$\left\{ \eta : \ 2[l_{pro}(\hat{\eta}) - l_{pro}(\eta)] \leq \chi^2_{k;1-\alpha} \right\}, \tag{3.14}$$

where $\hat{\eta}$ is the ML estimate of η, $k = \dim(\eta)$ and $\chi^2_{k;1-\alpha}$ is the $(1-\alpha)$th quantile of a chi-squared variate on k degrees of freedom. In a similar fashion, one can use the profile log-likelihood of η and the likelihood ratio statistic to test hypotheses about η.

3.5 Asymptotic properties

When the survivor function $G(\cdot; \phi)$ in (3.1) comes from a regular statistical model (see Cox and Hinkley (1974, Chapter 9)), then the parametric collapsible model is also a regular statistical model for most choices of $t_\theta(x; \eta)$. The ML estimators derived in this chapter are therefore consistent and asymptotically normal. The asymptotic distribution is given by

$$\sqrt{n}\left[\begin{pmatrix} \hat{\eta} \\ \hat{\phi} \end{pmatrix} - \begin{pmatrix} \eta_0 \\ \phi_0 \end{pmatrix} \right] \xrightarrow{d} N\left[0, I^{-1}\right], \tag{3.15}$$

where 0 represents a vector of zeros and I^{-1} is the inverse of the conditional Fisher information matrix, given the observed values of the paths.

Let us now derive the asymptotic variance matrix for the linear scale and multiplicative scale models. For convenience, let us assume that we have a sample of n uncensored observations from a collapsible model with $G(t;\mu) = \exp(-t/\mu)$, and that the paths are of the form $y_i(x) = \theta_i x$. For the linear scale model $t_\theta(x;\eta) = x(1-\eta+\eta\theta)$, the conditional Fisher information is

$$I[\eta,\mu|\theta_1,\ldots,\theta_n] = \begin{pmatrix} \sum_{i=1}^{n}\left(\frac{\theta_i-1}{1-\eta+\eta\theta_i}\right)^2 & -\frac{1}{\mu}\sum_{i=1}^{n}\frac{\theta_i-1}{1-\eta+\eta\theta_i} \\ -\frac{1}{\mu}\sum_{i=1}^{n}\frac{\theta_i-1}{1-\eta+\eta\theta_i} & \frac{n}{\mu^2} \end{pmatrix}. \tag{3.16}$$

Inversion of this matrix leads to the following asymptotic variances and covariance:

$$\mathrm{asvar}(\hat{\eta}) = \frac{1}{n\widehat{\mathrm{Var}}\left[\frac{\Theta-1}{1-\eta+\eta\Theta}\right]}, \tag{3.17}$$

$$\mathrm{asvar}(\hat{\mu}) = \frac{\mu^2\widehat{\mathrm{E}}\left[\left(\frac{\Theta-1}{1-\eta+\eta\Theta}\right)^2\right]}{n\widehat{\mathrm{Var}}\left[\frac{\Theta-1}{1-\eta+\eta\Theta}\right]}, \tag{3.18}$$

$$\mathrm{ascov}(\hat{\eta},\hat{\mu}) = \frac{\mu\widehat{\mathrm{E}}\left[\frac{\Theta-1}{1-\eta+\eta\Theta}\right]}{n\widehat{\mathrm{Var}}\left[\frac{\Theta-1}{1-\eta+\eta\Theta}\right]}, \tag{3.19}$$

where $\widehat{\mathrm{E}}$ and $\widehat{\mathrm{Var}}$[1] denote the sample mean and variance, respectively.

Similar calculations yield the following asymptotic variances and covariance for the multiplicative scale model:

$$\mathrm{asvar}(\hat{\eta}) = \frac{1}{n\widehat{\mathrm{Var}}\,[\ln\Theta]}, \tag{3.20}$$

$$\mathrm{asvar}(\hat{\mu}) = \frac{\mu^2\widehat{\mathrm{E}}\left[(\ln\Theta)^2\right]}{n\widehat{\mathrm{Var}}\,[\ln\Theta]}, \tag{3.21}$$

---

[1]Actually, $\widehat{\mathrm{Var}}$ is $(n-1)/n$ times the sample variance.

$$\mathrm{ascov}(\hat{\eta}, \hat{\mu}) = \frac{\mu \widehat{\mathrm{E}}\,[\ln \Theta]}{n \widehat{\mathrm{Var}}\,[\ln \Theta]}. \tag{3.22}$$

It is now obvious, from equations (3.17) and (3.20), that a fair amount of variability in the usage paths must be observed for precise estimation of η. For most distributions of Θ, the random variable $(\Theta - 1)/(1 - \eta + \eta\Theta)$ will tend to be more concentrated around its mean than the variable $\ln \Theta$, especially for large values of η. Indeed, for a random variable Θ having support on $I\!\!R^+$, $(\Theta - 1)/(1 - \eta + \eta\Theta)$ will take on values between zero and one, whereas $\ln \Theta$ will take on values over the whole real line. This implies that inference in the linear scale model will tend to be more sensitive to a lack of variability in the observed usage paths than inference in the multiplicative scale model. However, the value of η has a different meaning in both models and therefore one should not discriminate in favor of the multiplicative model solely based on this result; we cannot compare the values of η (and their respective variances) from the two models directly, and model assessment methods, such as the ones outlined in Chapter 6, should be used in choosing the model best suited to the data.

3.6 Missing information

In order to fit a collapsible model via ML, we need the following information:

Observed failures: real time at failure, x_i, usage measures at failure, $y_i(x_i)$ and the derivative of the usage measures at failure, $y_i'(x_i)$;

Censored observations: real time at point of censoring, x_i, and usage measures at point of censoring, $y_i(x_i)$.

In this section we briefly consider cases in which some of this required information is missing.

The most likely situation in which information is missing is when only failure times and the values of the usage measures at failure are available. This means that $y_i'(x_i)$ is not available and that we cannot compute $t'_{p_i}(x_i)$ and, hence, the likelihood. This situation may occur if $y_i'(x_i)$ is not measured, or if $y_i(x)$ is a step function.

When $y_i'(x_i)$ is missing because it has not been measured, we can think of two ways to remedy the situation. The first option is to parameterize the usage paths, then evaluate the derivative of the parameterized path at x_i. For example, we could assume that $y_i(x_i) = \theta_i x_i$ and then let $y_i'(x_i) = \theta_i$. The other option is to estimate $y_i'(x_i)$ through any other available means. For instance, we may be able to use some numerical differentiation method if we have more than one observation per path.

If $y_i(x_i)$ is a step function, the situation is somewhat different. In this case $t_{p_i}(x_i; \eta)$ may have jumps and, hence, so will $G[t_{p_i}(x_i; \eta)]$. We therefore need the following information in order to compute the likelihood contribution of a failure observed at x_i: $y_i(x_i)$ and $y_i'(x_i)$ if $y_i(\cdot)$ is continuous at x_i, and $y_i(x_i)$ and $\Delta y_i(x_i) = y_i(x_i + 0) - y_i(x_i - 0)$ if $y_i(\cdot)$ takes a jump at x_i.

Even though the value of $\Delta y_i(x_i)$ and $y_i'(x_i)$ are not of major importance once an ideal time scale has been obtained, they are still needed in order to estimate the parameters of the model. Parametric inference when those increments are missing is an important problem of interest (and a good avenue) for future research.

Chapter 4

Semiparametric inference for collapsible models

We have seen in Chapter 3 that statistical inference for collapsible models is relatively straightforward when a fully parametric model is assumed. Unfortunately, we must make strong assumptions about the survivor function G in (3.1). This is a problem in our setup as we do not directly observe the failure times in the ITS, and therefore cannot look at probability plots in order to find a suitable parametric form for G. Moreover, looking at probability plots of $G[t_{p_i}(x_i; \hat{\eta}); \hat{\phi}]$ may fail to identify misspecification of G as the $t_{p_i}(x_i; \hat{\eta})$ will tend to compensate for the wrongly specified G. Nevertheless, we can still build informative probability plots if we can estimate the failure times in the ITS *before* specifying a form for G. In order to do so, we need a good semiparametric estimate of η in (3.1), say $\tilde{\eta}$. Evaluating $t_p(x_i; \tilde{\eta})$ then yields a sample of failure times from which we can find the best form for G.

In this chapter, we will study various means of estimating η semiparametrically, i.e.

estimating η by assuming a parametric form for $t_P(x; \eta)$ and leaving G arbitrary in

$$\Pr\left[X > x | \mathcal{P}\right] = G[t_P(x; \eta)]. \tag{4.1}$$

The idea behind most of these methods is to exploit the orthogonality between the usage paths and the failure times in the ITS, as noted in Definition 1.4. Indeed, if $t_P(x; \eta)$ is an ITS for some value of η, then the failure times measured in this ITS are independent of the paths. Therefore, we can think of many ad hoc methods of estimating η in (4.1) simply by finding the value of η that minimizes the distance between the nonparametric distributions of $t_P(X; \eta)$ along each path, or that minimizes a measure of association between $t_P(X; \eta)$ and some path feature.

In the following sections, a particular class of collapsible models will be considered (but the methods outlined may be applied to more general models). Even though it is quite restrictive, this class of models is nevertheless broad enough to include the multiplicative and linear scale models when the paths are of the form $y_i(x) = \theta_i x$:

Definition 4.1 *A* separable scale model *is a collapsible model with an ITS of the form*

$$t_\theta(x; \eta) = u(x)v(\theta; \eta), \tag{4.2}$$

where $u(\cdot)$ and $v(\cdot; \eta)$ are strictly increasing functions of x and the elements of θ, respectively.

In the remainder of this chapter, we will study various semiparametric inference methods. For each method, we will derive results for the separable scale model, as well as for other forms of collapsible models when possible. The next section will examine the

minimum coefficient of variation method and Section 4.2 will cover a quasi-likelihood approach. We will look at a rank based method in Section 4.3. We conclude by investigating the efficiency of all these semiparametric estimators for samples of moderate size through a simulation study in Section 4.4.

4.1 Minimum coefficient of variation

The concept of small variation is widely considered by practitioners in areas such as reliability. The main approach to time scale selection using this concept is due to Kordonsky and Gertsbakh (1993, 1995ab, 1997); they suggest that if a parametric family of time scales $t_p(x; \eta) = \Phi[y_p(x); \eta]$ is being considered, then η be chosen so as to minimize the squared coefficient of variation of $T = t_p(X)$, that is, $\mathrm{Var}(T)/\mathrm{E}^2(T)$. This is used in practice by considering complete (uncensored) samples in which chronological times x_i and corresponding usage measures $y_i(x_i) = (y_{i0}(x_i) = x_i,\ y_{i1}(x_i)\ ,\ ...\ ,\ y_{ip}(x_i))$ are observed for units $i = 1, ..., n$. Then the operational times $t_i(\eta) = \Phi[y_i(x_i); \eta]$ are considered and η is estimated by minimizing

$$\widehat{CV}^2[t_i(\eta)] = \frac{\widehat{\mathrm{Var}}[t_i(\eta)]}{\widehat{\mathrm{E}}^2[t_i(\eta)]}, \tag{4.3}$$

where $\widehat{\mathrm{Var}}$ and $\widehat{\mathrm{E}}$ denote sample variance and sample mean, respectively.

Kordonsky and Gertsbakh apply this method to linear time scale families $t_p(x) = \eta^t y(x)$. In this case one easily obtains that the unrestricted minimum of (4.3) is obtained at

$$\tilde{\eta} \propto \widehat{\Sigma}^{-1} \hat{\mu}, \tag{4.4}$$

where $Y_{ij} = y_{ij}(X_i)$, $\widehat{\Sigma}$ is the sample covariance matrix for $(X_i, Y_{i1}, \ldots, Y_{ip})^t$ and $\widehat{\mu}$ is the sample mean vector for $(X_i, Y_{i1}, \ldots, Y_{ip})^t$. The coefficient of variation is invariant under scale changes to T, and so any η proportional to the right side of (4.4) may be used. Kordonsky and Gertsbakh assume that $\eta \geq 0$ and it may happen that the right side of (4.4) has negative elements. In the two-dimensional case Kordonsky and Gertsbakh then select either $\eta^t = (1, 0)$ or $\eta^t = (0, 1)$ (corresponding to $T = X$ or $T = Y$, respectively), according to whichever gives the smaller $\widehat{CV}^2$ for $T(\eta)$.

It is still not clear why this approach should give good estimators of the time scale parameter in certain situations; there is no obvious connection between the concept of minimum coefficient of variation and that of ideal time scale. Indeed, any monotone increasing function of T is an ITS if T is, whereas measures of variation or relative variation are not invariant under general transformations. Note also that the minimum CV method may be used without any information on the path $\mathcal{P}(x)$ except for the endpoint $(X_i, Y_{i1}, \ldots, Y_{ip})$ and that it seemingly applies to situations where the paths are randomly determined, rather than fixed.

4.1.1 Separable scale model

The following lemma implies that the min CV method yields consistent estimators of the time scale parameters in separable scale models. In this section, we make the assumption that the usage path parameters, Θ, are not fixed but rather independent and identically distributed non-negative random variables.

Lemma 4.1.1 *Consider the separable scale model, as defined in Definition 4.1. Also denote $t(\eta) = u(x)v(\eta, \theta)$ and $T(\eta) = u(X)v(\eta, \Theta)$, so that $t(\eta_0)$ is an ITS. Then for*

any η we have

$$CV^2[T(\eta)] \geq CV^2[T(\eta_0)],$$

and if $CV^2[T(\eta)] = CV^2[T(\eta_0)]$ then $t(\eta)$ is also an ITS.

Proof: Because $T(\eta_0)$ is ideal, we have that $E[T(\eta_0)] = E[T(\eta_0)|\Theta = \theta]$ and $\text{Var}[T(\eta_0)] = \text{Var}[T(\eta_0)|\Theta = \theta]$. Let $\mu_T = E[T(\eta_0)]$ and $\sigma_T^2 = \text{Var}[T(\eta_0)]$. Then

$$
\begin{aligned}
CV^2[T(\eta)] &= \text{Var}[T(\eta)]/E^2[T(\eta)] \\
&= \frac{\text{Var}_{\Theta}[E[T(\eta)|\Theta]] + E_{\Theta}[\text{Var}[T(\eta)|\Theta]]}{E_{\Theta}^2[E[T(\eta)|\Theta]]} \\
&= \left\{ \text{Var}_{\Theta}\left[\frac{v(\eta,\Theta)}{v(\eta_0,\Theta)} E[u(X)v(\eta_0,\Theta)|\Theta] \right] \right. \\
&\quad \left. + E_{\Theta}\left[\left(\frac{v(\eta,\Theta)}{v(\eta_0,\Theta)} \right)^2 \text{Var}[u(X)v(\eta_0,\Theta)|\Theta] \right] \right\} \\
&\quad \times E_{\Theta}^{-2}\left[\frac{v(\eta,\Theta)}{v(\eta_0,\Theta)} E[u(X)v(\eta_0,\Theta)|\Theta] \right] \\
&= \frac{\mu_T^2 \text{Var}_{\Theta}\left[\frac{v(\eta,\Theta)}{v(\eta_0,\Theta)} \right] + \sigma_T^2 E_{\Theta}\left[\left(\frac{v(\eta,\Theta)}{v(\eta_0,\Theta)} \right)^2 \right]}{\mu_T^2 E_{\Theta}^2\left[\frac{v(\eta,\Theta)}{v(\eta_0,\Theta)} \right]} \\
\Rightarrow CV^2[T(\eta)] &\geq \sigma_T^2/\mu_T^2 = CV^2[T(\eta_0)].
\end{aligned}
$$

In addition, equality can hold only if $v(\eta,\theta)/v(\eta_0,\theta)$ is constant with respect to θ. (We suppose the distribution of Θ is such that $\text{Var}[v(\eta,\theta)/v(\eta_0,\theta)] > 0$ implies this.) But in this case $t(\eta) = ct(\eta_0)$, where c does not depend on θ, which is also an ITS. ∎

Lemma 4.1.1 thus shows that we can identify ITS's within the family of separable scale models by finding η's that minimize $CV^2[T(\eta)]$. This covers several important situations. Suppose for instance that there is a single usage factor $y(x)$ and that usage

paths $\mathcal{P}$ are of the form

$$y(x) = \theta x, \quad x \geq 0, \tag{4.5}$$

where θ has some distribution in the population of interest. Linear time scales as in model (2.16) can in this case be expressed as

$$t_{\mathcal{P}}(x) = \eta_1 x + \eta_2 y(x) = x(\eta_1 + \eta_2 \theta),$$

which is of the form (4.2). Consider two ways of proceeding via Lemma 4.1.1. First, specify $T(\eta) = \eta_1 X + \eta_2 Y$, with $\eta = (\eta_1, \eta_2)^t$; alternatively, specify $T(\eta) = X + \eta Y$. With the second choice, minimization of $CV^2[T(\eta)]$ yields the unique value

$$\eta = \frac{\mu_Y \sigma_X^2 - \mu_X \sigma_{XY}}{\mu_X \sigma_Y^2 - \mu_Y \sigma_{XY}}. \tag{4.6}$$

With the first approach the minimum $CV^2[T(\eta)]$ is achieved for all $(\eta_1, \eta_2) = c(1, \eta)$, where $c \neq 0$ is an arbitrary constant. This is the same form as (4.4).

To estimate the parameter η in a separable scale model we can minimize the sample coefficient of variation (4.3) for a complete (uncensored) sample $t_1(\eta), \ldots, t_n(\eta)$. When η_0 is unique, assuming finiteness and smoothness of $E[T(\eta)]$ and $\text{Var}[T(\eta)]$ in the neighborhood of η_0 will ensure that $\bar{\eta}$ minimizing (4.3) is a consistent estimator for η_0.

There are some subtleties concerning minimum CV estimation. For example, if we assume $\eta_0 \geq 0$, which is plausible in most applications, then under the parameterization $T(\eta) = X + \eta Y$ the right side of (4.6) is non-negative. However, the estimate of η based on minimization of (4.3) is the value $\bar{\eta}$ obtained by replacing σ_X^2, σ_Y^2, σ_{XY}, μ_X and μ_Y in (4.6) with sample-based estimates, and it is possible for $\bar{\eta}$ to be negative. In particular,

it may be shown that $\bar{\eta} < 0$ if either of the following two conditions holds:

$$
\text{A:} \quad \frac{\hat{\sigma}_X^2}{\hat{\mu}_X^2} \;<\; \frac{\hat{\sigma}_{XY}}{\hat{\mu}_X\hat{\mu}_Y} \;<\; \frac{\hat{\sigma}_Y^2}{\hat{\mu}_Y^2}
$$

$$
\text{B:} \quad \frac{\hat{\sigma}_Y^2}{\hat{\mu}_Y^2} \;<\; \frac{\hat{\sigma}_{XY}}{\hat{\mu}_X\hat{\mu}_Y} \;<\; \frac{\hat{\sigma}_X^2}{\hat{\mu}_X^2}.
$$

If case A holds then $\widehat{CV}^2[t_i(\eta)]$ is minimized subject to $\eta \geq 0$ at $\eta = 0$ and if B holds then it is minimized at $\eta = \infty$, thus leading to the choice of X or Y, respectively, as the ideal time scale. As Kordonsky and Gertsbakh (1993) note, cases A and B are automatically ruled out if $\hat{\sigma}_{XY} < 0$.

The behavior of the minimum CV estimator is somewhat easier to understand when we take a closer look at the sample squared CV as a function of the time scale parameter η. One can easily show that

$$
\widehat{CV}^2[T(\eta)] \;=\; +\infty \quad \text{when} \quad \eta = -\frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} y_i}
$$

$$
\text{and} \quad \lim_{\eta \to \pm\infty} \widehat{CV}^2[T(\eta)] \;=\; \frac{n}{n-1}\left\{ \frac{n\sum_{i=1}^{n} y_i^2}{\left(\sum_{i=1}^{n} y_i\right)^2} - 1 \right\}.
$$

The minimum will thus be reached between $\eta = -\infty$ and $\eta = -\sum_{i=1}^{n} x_i / \sum_{i=1}^{n} y_i$ or between $\eta = -\sum_{i=1}^{n} x_i / \sum_{i=1}^{n} y_i$ and $\eta = +\infty$.

The squared sample CV function can take three forms, as shown in **Figure 4.1.**

The first form occurs when the minimum of $\widehat{CV}^2[T(\eta)]$ is at $\eta < -\sum_{i=1}^{n} x_i / \sum_{i=1}^{n} y_i$. This happens when we are in case B mentioned above. As we can see on the left most plot of Figure 4.1, the sample CV is minimized subject to $\eta \geq 0$ at $\eta = +\infty$. The second form is when the minimum occurs at $-\sum_{i=1}^{n} x_i / \sum_{i=1}^{n} y_i < \eta < 0$. This is when we are
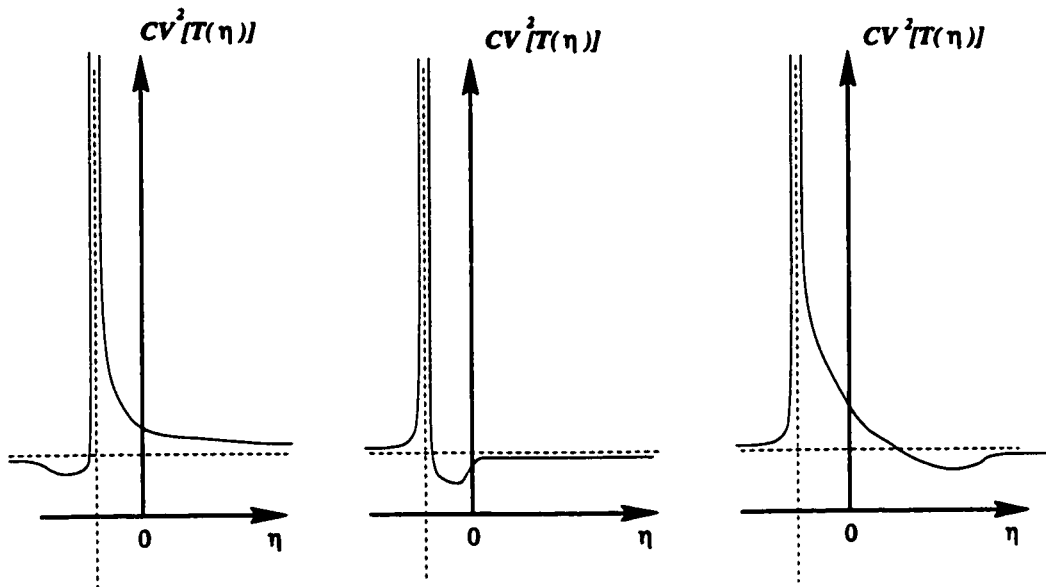
Figure 4.1: Possible forms of the squared sample CV function, linear scale model.

under case A. In this situation, we can see from the middle plot of Figure 4.1 that the sample CV is minimized subject to $\eta \geq 0$ at $\eta = 0$. Finally, when we are in neither case A nor case B, the form of the squared sample CV function is as shown in the right most plot of Figure 4.1. In this case, the sample CV is globally minimized at $\eta \geq 0$.

According to Lemma 4.1.1, the minimum CV estimator is also a consistent estimator of η in the multiplicative scale model $T(\eta) = x^{1-\eta}y(x)^{\eta}$, with $y(x) = \theta x$. However, there is no closed form expression for the minimum CV estimator in this case. Nevertheless, because the squared sample CV is smooth in η, it is still very easy to find the value of the minimum CV estimator using any of the popular minimization algorithms.

4.1.2 General collapsible model

The min CV method is flexible enough to be applicable to more general collapsible models. Indeed, once η is fixed, we can always compute the coefficient of variation in the $t[\eta]$ scale.

However, it is a lot more difficult to derive the properties of the min CV estimator when the model at hand is not a member of the separable scale family. Nonetheless, we know that the sample mean and variance of the $t_i[\eta]$'s are consistent estimators of the true mean and variance of $T[\eta]$ for every fixed value of η[1]. This implies that the min CV estimator will be consistent in every case where the true CV is uniquely minimized at the true value of η.

4.1.3 Censored observations

The min CV method is not as tractable when some of the observations in the sample are censored. Kordonsky and Gertsbakh (1997b) suggest a somewhat odd way of estimating η when censored observations are present. Their method is simply a hybrid of ML and min CV; they assume a parametric form for the survivor function, $G[\cdot; \phi]$, and they use the following algorithm:

1. fix $\eta = \eta_1$;

2. find $\phi_1 = \arg\max_\phi L(\eta_1, \phi)$;

3. compute the CV of $G[\cdot; \phi_1]$;

4. repeat steps 1, 2 and 3 until the CV in step 3 is minimized

Of course, proceeding this way defeats the purpose of semiparametric estimation, i.e. estimating η before making any strong assumption about the form of $G[\cdot]$. Moreover, why should one use ML to get estimates of some parameters of the model and then use another method to get estimates of the other parameters? Kordonsky and Gertsbakh claim that

---

[1] Under suitable regularity conditions, such as the smoothness of $E[T[\eta]]$ and $Var[T[\eta]]$ in η.

in the case where $G[\cdot]$ is the log-normal distribution, their method yields estimates of η in the linear scale model that are very close to the ML estimates. As it turns out, it can easily be shown that in this case, for a complete sample, Kordonsky and Gertsbakh's hybrid method is equivalent to maximum likelihood!

So how can we estimate η semiparametrically when censoring is present using the min CV method? Obviously, we cannot compute the usual sample mean and variance if we do not have complete data. However, if we see the squared coefficient of variation as a real functional, then it is possible to define a censored version of it. Let us first define $\mathcal{D}[0,1]$ to be the set of all mappings $F : \mathbb{R} \to [0,1]$ that are cdf's. Then we can define the squared CV as a mapping $CV^2 : \mathcal{D}[0,1] \to \mathbb{R}^+$ this way:

$$CV^2(F) = \frac{\int (u - \int u \, dF(u))^2 \, dF(u)}{[\int u \, dF(u)]^2} = \frac{\int u^2 \, dF(u)}{[\int u \, dF(u)]^2} - 1. \tag{4.7}$$

Using this approach, we can define the minimum CV estimator in an alternative, more general manner:

Definition 4.2 *Let $t_i[\eta] = \Phi[\mathcal{P}_i(x_i); \eta]$, $i = 1, 2, \ldots, n$ be the values of the observed sample translated in the new scale, which we denote $T[\eta]$ for some fixed η. Let $\hat{F}_\eta$ be the estimated cdf obtained from $t_1[\eta], \ldots, t_n[\eta]$. We define the min CV estimator to be the value of η that minimizes $CV^2(\hat{F}_\eta)$ as defined in equation (4.7).*

If there is no censoring, we can use the usual empirical cdf for $\hat{F}$ and this yields the usual min CV estimator, as defined by Kordonsky and Gertsbakh. However, when censoring is present, we have to be careful. If the largest time in the sample $t_i[\eta]$, $i = 1, 2, \ldots, n$ is a failure time, we can define $\hat{F}_\eta$ to be the usual product-limit estimator and then compute the min CV estimator according to Definition 4.2. When the largest value of the sample

is a censoring time, we need to define the estimator of F_η slightly differently. Let $t_{max}[\eta]$ be the largest observed failure time in the sample. We then define an estimator of F_η in this manner:

$$\hat{F}_\eta(t) = \begin{cases} \hat{F}_\eta^{PL}(t), & 0 \leq t \leq t_{max}[\eta]; \\ 1 - \exp\{-\hat{\lambda}t\}, & t > t_{max}[\eta]; \end{cases} \qquad (4.8)$$

where

$$\hat{\lambda} = \frac{-\ln[1 - \hat{F}_\eta^{PL}(t_{max}[\eta])]}{t_{max}[\eta]},$$

and $\hat{F}_\eta^{PL}$ is the usual product-limit estimator of F_η. One can see that we are just assuming that the tail of the distribution beyond $t_{max}[\eta]$ follows an exponential distribution. This will usually give a good estimate of the first two moments of F_η, hence we can use estimator (4.8) in Definition 4.2. Unfortunately, there is no closed form solution for $\hat{\eta}$ in this case and the numerical optimization is actually quite challenging, unless η is unidimensional, in which case the usual numerical methods, such as golden section search (see Press et al. (1992, Section 10.1)), seem to work relatively well.

The behavior of the estimators thereby obtained still needs to be studied. Indeed, the "empirical" cdf (4.8) in this case depends on the parameter to be estimated. Moreover, when the largest observation in the $t[\eta]$ scale is censored, the tail of the "empirical" cdf is approximated by the tail of an exponential distribution. Needless to say, the derivation of the asymptotic properties of this generalized minimum CV estimator represents quite a challenging problem.

4.1.4 Variance estimate

Despite its simplicity, it is very difficult to compute a good variance estimate for the min CV estimator. Even in the linear scale case, where a closed form expression for this

estimator is available, the variance that must be calculated is the variance of a ratio of products of correlated sums of random variables. Moreover, the min CV estimator cannot be expressed as an M-estimator, or a U-statistic or any type of functional for which the asymptotics have already been worked out.

Since the minimum CV method is based on the joint distribution of the real times and the usage paths, resampling based on this empirical joint distribution should at least give an idea of the precision of the estimate. More precisely, suppose we observe $(x_1, y_1), \ldots, (x_n, y_n)$ and that we estimate η in $T(\eta)$ via minimum CV. Then the variance estimate obtained via the following bootstrap algorithm is a good approximation of the true variance of the estimator:

1. For $b = 1$ to B:

   (a) sample n pairs with replacement from the original pairs $(x_1, y_1), \ldots, (x_n, y_n)$;

   (b) compute the minimum CV estimator of η based on this new sample and call this estimate $\hat{\eta}_b$;

2. The variance estimate of $\hat{\eta}$ is then given by

$$\sum_{b=1}^{B} \frac{\left(\hat{\eta}_b - \bar{\hat{\eta}}\right)^2}{B - 1},$$

where $\bar{\hat{\eta}} = \sum_{b=1}^{B} \hat{\eta}_b / B$.

According to Efron and Tibshirani (1993), a value of B around 200 is usually sufficient for variance estimates.

The algorithm above also gives a good approximation of the distribution of the estimator. We used the percentile bootstrap algorithm (see Efron and Tibshirani (1993) for a

description of this method) and the minimum CV to derive approximate 95% confidence intervals for the time scale parameter for the linear scale model. We used a value of $B = 1000$, as recommended by Efron and Tibshirani (1993). The results are given in Table 4.30 in Section 4.4.4.

4.2 Quasi-likelihood

When no censoring is present, we can use unbiased linear estimating equations (EE's) to estimate η in model (4.1). These EE's show good robustness properties and are relatively simple to implement. Moreover, some EE's are fully efficient for certain forms of G.

Suppose that we have a complete sample of real times at failure, x_i, along with the corresponding usage paths, $\mathcal{P}_i$, for $i = 1, \ldots, n$. In this section, we consider EE's of the form

$$\sum_{i}^{n} w_{i,j}\Big(h(x_i) - \mathrm{E}[h(X_i)|\mathcal{P}_i; \eta, \mu]\Big) = 0, \qquad j = 1, 2, \ldots, p+1, \tag{4.9}$$

where $p = \dim(\eta)$, $h(\cdot)$ is a strictly monotonic function and w_{ij} is some weight. In this chapter, we use a parameterization with the minimal number of parameters, p, in the ITS (see comment on parameterization following equation (3.10) in Section 3.3.1). This means that η does not contain a scale parameter; this scale parameter is accounted for in the value of the parameter μ. There is an extensive literature on such EE's (see Godambe and Thompson (1989)). For any fixed function $h(\cdot)$, it is possible to derive optimal weights w_{ij}'s. Unfortunately, the optimal $h(\cdot)$ function will depend on the form of G in (4.1). We will therefore limit our investigation to a few fixed choices of $h(\cdot)$.

4.2.1 Separable scale model

Let us now assume that $t_{P_i}(x_i; \eta) = u(x_i)v(\theta_i; \eta)$ and let $z_i = u(x_i)$. We wish to estimate η using EE's of the form

$$\sum_{i=1}^{n} w_{i,j}\Big(h(z_i) - E[h(Z_i)|\theta_i; \eta, \mu]\Big) = 0, \qquad j = 1, 2 \ldots, p+1. \qquad (4.10)$$

Note that even though $\dim(\eta) = p$, we need $p + 1$ EE's. This is due to the fact that the conditional expectation involved in (4.10) will be a function of an extra unknown parameter, namely the moment $\mu = -\int h(u) \, dG(u)$. We also must point out that care is needed in the parameterization here; EE's (4.10) are scale invariant, and therefore one must make sure that the parameters η will not be confounded with a scale parameter.

We first derive the best EE's for $h(u) = u$. First, we see that

$$\mu_i(\eta, \mu_T) = E[Z_i|\theta_i; \eta, \mu] = \mu_T/v(\theta_i; \eta),$$

where $\mu_T = -\int u \, dG(u)$, i.e. the first moment of distribution G. We also observe that

$$\sigma_i^2(\eta, \mu_T, \sigma_T^2) = \mathrm{Var}[Z_i|\theta_i; \eta, \mu] = \sigma_T^2/v(\theta_i; \eta)^2,$$

where $\sigma_T^2 = -\int(u - \mu_T)^2 \, dG(u)$, i.e. the variance of distribution G. We then notice that $\sigma_i^2(\eta, \mu_T, \sigma_T^2) = \phi\mu_i(\eta, \mu_T)^2$, where $\phi = \sigma_T^2/\mu_T^2$. Applying well known quasi-likelihood results (see McCullagh and Nelder (1989), for example), we obtain the following optimal EE's:

$$\sum_{i=1}^{n} \frac{\partial v(\theta_i; \eta)}{\partial \eta_j}\left(z_i - \frac{\mu_T}{v(\theta_i; \eta)}\right) = 0, \qquad j = 1, 2, \ldots, p \qquad (4.11)$$

$$\sum_{i=1}^{n} v(\theta_i; \eta)\left(z_i - \frac{\mu_T}{v(\theta_i; \eta)}\right) = 0. \tag{4.12}$$

Solving (4.12) for any fixed η yields $\hat{\mu}_T(\eta) = \overline{t[\eta]}$, the sample mean of the $z_i v(\theta_i; \eta)$ values. The system of EE's in η left to solve is thus

$$\sum_{i=1}^{n} \frac{\partial v(\theta_i; \eta)}{\partial \eta_j}\left(z_i - \frac{\overline{t[\eta]}}{v(\theta_i; \eta)}\right) = 0, \quad j = 1, 2, \ldots, p \tag{4.13}$$

or, equivalently, $\quad \sum_{i=1}^{n} \frac{\partial \ln v(\theta_i; \eta)}{\partial \eta_j}\left(t_i[\eta] - \overline{t[\eta]}\right) = 0, \quad j = 1, 2, \ldots, p, \tag{4.14}$

where $t_i[\eta] = z_i v(\theta_i; \eta)$. As it turns out, EE (4.14) is the ML equation for η when $G[\cdot; \mu]$ is the exponential survivor function.

Repeating the same procedure for $h(z_i) = \ln z_i$, we obtain the following optimal EE's:

$$\sum_{i=1}^{n} \frac{\partial \ln v(\theta_i; \eta)}{\partial \eta_j}\left(\ln z_i - \mu_{\ln T} + \ln v(\theta_i; \eta)\right) = 0, \quad j = 1, 2, \ldots, p \tag{4.15}$$

$$\sum_{i=1}^{n}\left(\ln z_i - \mu_{\ln T} + \ln v(\theta_i; \eta)\right) = 0, \tag{4.16}$$

where $\mu_{\ln T} = -\int \ln u \, dG(u)$. Then solving (4.12) for $\mu_{\ln T}$ and substituting in (4.15) yields

$$\sum_{i=1}^{n} \frac{\partial \ln v(\theta_i; \eta)}{\partial \eta_j}\left(\ln t_i[\eta] - \overline{\ln t[\eta]}\right) = 0, \quad j = 1, 2, \ldots, p, \tag{4.17}$$

where $\ln t_i[\eta] = \ln z_i + \ln v(\theta_i; \eta)$ and $\overline{\ln t[\eta]}$ is the sample mean of the $\ln t_i[\eta]$'s. This time, EE (4.17) is the ML equation for η when G is the lognormal survivor function.

Finally, another form for $h(\cdot)$ is $h(z_i) = z_i^\lambda$, for some fixed $\lambda > 0$. Popular choices of

λ in regression are $\lambda = 1/2$ or $\lambda = 2$. Going through the same calculations again yields

$$\sum_{i=1}^{n} \frac{\partial v(\theta_i; \eta)}{\partial \eta_j} v(\theta_i; \eta)^{\lambda-1} \left(z_i^\lambda - \frac{\mu_{T^\lambda}}{v(\theta_i; \eta)^\lambda} \right) = 0, \quad j = 1, 2, \ldots, p \quad (4.18)$$

$$\sum_{i=1}^{n} v(\theta_i; \eta)^\lambda \left(z_i^\lambda - \frac{\mu_{T^\lambda}}{v(\theta_i; \eta)^\lambda} \right) = 0, \quad (4.19)$$

where $\mu_{T^\lambda} = -\int u^\lambda \, dG(u)$. The next step gives

$$\sum_{i=1}^{n} \frac{\partial \ln v(\theta_i; \eta)}{\partial \eta_j} \left(t_i[\eta]^\lambda - \overline{t[\eta]^\lambda} \right) = 0, \quad j = 1, 2 \ldots, p, \quad (4.20)$$

where $t_i[\eta]^\lambda = z_i^\lambda v(\theta_i; \eta)^\lambda$ and $\overline{t[\eta]^\lambda}$ is the sample mean of the $t_i[\eta]^\lambda$'s. We can show that EE (4.20) is the ML equation for η when G is the survivor function of a Weibull distribution with shape parameter known and equal to λ. This should not come as a surprise; when T is Weibull with shape λ, then T^λ is exponential, and we have already seen that EE (4.14) is the ML equation for η when G is the exponential survivor function.

It is worth pointing out that for all our choices of the function $h(\cdot)$, the EE's for η are of the form

$$\sum_{i=1}^{n} \frac{\partial \ln v(\theta_i; \eta)}{\partial \eta} \left(h(t_i[\eta]) - \overline{h(t[\eta])} \right) = 0. \quad (4.21)$$

This is not necessarily true for all strictly monotone functions $h(\cdot)$, though.

One very interesting interpretation of (4.21) is that we are looking for a value of η such that the sample covariance between $\partial \ln v(\Theta; \eta)/\partial \eta$ and $h(T[\eta])$ is zero. This means that the quasi-likelihood EE's are consistent with the ITS definition in that they are looking for a value of η such that the failure times in the ITS are orthogonal to some path feature.

We can easily derive the asymptotic distribution of $\hat{\eta}$ obtained by solving the EE's

described by (4.21). We let

$$A(\eta, \mu_{h(T)})_{r,s} = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{g(\mu_i)} \left(\frac{\partial \mu_i}{\partial \eta_r}\right) \left(\frac{\partial \mu_i}{\partial \eta_s}\right), \quad r,s = 1,2,\ldots,p+1,$$

where $\mu_i = E[h(Z_i)|\theta_i; \eta, \mu_{h(T)}]$ and $g(\cdot)$ is such that $\text{Var}[h(Z_i)|\theta_i; \eta, \mu_{h(T)}] = \phi g(\mu_i)$ for some real number ϕ. (We can show that $\phi = -\int (h(u) - \mu_{h(T)})^2 \, dG(u)/\mu_{h(T)}^2$ when $h(u) = u^\lambda$ and $\lambda > 0$, and $\phi = -\int (h(u) - \mu_{h(T)})^2 \, dG(u)$ when $h(u) = \ln u$.)

The asymptotic distribution of $\begin{pmatrix} \hat{\eta} \\ \hat{\mu}_{h(T)} \end{pmatrix}$ is then given by

$$\sqrt{n} \left[\begin{pmatrix} \hat{\eta} \\ \hat{\mu}_{h(T)} \end{pmatrix} - \begin{pmatrix} \eta \\ \mu_{h(T)} \end{pmatrix} \right] \xrightarrow{d} N \left[0, \phi A^{-1}(\eta, \mu_{h(T)}) \right]. \qquad (4.22)$$

Let us now apply these results to the linear and multiplicative scale models in the case with a single usage measure of the form $y_i(x) = \theta_i x$. With the linear scale model, we have that $t_p(x; \eta) = (1 - \eta)x + \eta y(x) = x(1 - \eta + \eta\theta)$. We thus get $z_i = x_i$ and $v(\theta_i; \eta) = (1 - \eta + \eta\theta_i)$. Substituting these quantities in (4.21) yields

$$\sum_{i=1}^{n} \frac{\theta_i - 1}{1 - \eta + \eta\theta_i} \left(h(t_i[\eta]) - \overline{h(t[\eta])} \right) = 0, \qquad (4.23)$$

where $h(t_i[\eta]) = h(x_i(1 - \eta + \eta\theta_i))$ and $\overline{h(t[\eta])} = \sum_i h(t_i[\eta])/n$.

We can also use (4.22) to derive the asymptotic variance of $\hat{\eta}$, the root of (4.23).

Case $h(u) = u^\lambda$: In this case, we get that

$$\text{asvar}[\hat{\eta}] = \frac{\phi}{n\widehat{\text{Var}}\left[\frac{\Theta - 1}{1 - \eta + \eta\Theta}\right]}, \qquad (4.24)$$

where $\phi = -\int (u^\lambda - \mu_{T^\lambda})^2 \, dG(u)/\mu_{T^\lambda}^2$ is the squared CV of T^λ and $\widehat{\text{Var}}$ denotes the

sample variance.

Case $h(u) = \ln u$: For this choice of $h(\cdot)$, the asymptotic variance is

$$\text{asvar}[\hat{\eta}] = \frac{\sigma^2_{\ln T}}{n\widehat{\text{Var}}\left[\frac{\Theta-1}{1-\eta+\eta\Theta}\right]},\qquad (4.25)$$

where $\sigma^2_{\ln T} = -\int (\ln u - \mu_{\ln T})^2\, dG(u)$ is the variance of $\ln T$.

The results do not change much when we replace the linear scale by a multiplicative scale $t_{\mathcal{P}_i}(x_i; \eta) = x_i\theta_i^\eta$. Indeed, all that needs to be done is to replace $(\theta - 1)/(1 - \eta + \eta\theta)$ by $\ln\theta$ in (4.23), (4.24) and (4.25).

A very interesting conclusion that can be reached by looking at the asymptotic variances given in (4.24) and (4.25) is that in order to get a precise estimate of η, we need a distribution G with low coefficient of variation and large variability in the observed sample paths. We immediately understand the simulation results of Section 3.3.1 when we compute the asymptotic variance of the time scale parameter estimator for the multiplicative and linear time scale models used in the simulations. Using the equivalent reparameterization $t(\eta) = \frac{1}{1+\exp(\eta)}x + \frac{\exp(\eta)}{1+\exp(\eta)}\theta x$ for the linear scale, and $t(\eta) = x^{\frac{1}{1+\exp(\eta)}}(\theta x)^{\frac{\exp(\eta)}{1+\exp(\eta)}}$ for the multiplicative scale, we get, from (4.24), $\text{asvar}[\hat{\eta}] \approx 0.25$, whereas with the multiplicative scale model, we get $\text{asvar}[\hat{\eta}] \approx 0.0019$. We see that the standard deviation of the estimator in the linear scale model is ≈ 0.5, and the standard deviation in the multiplicative scale case is ≈ 0.0433. If we use $\eta^* = \frac{\exp(\eta)}{1+\exp(\eta)}$ (to map the parameter space back to $[0, 1]$) and an estimate of $\hat{\eta}^* = 0.5$, we obtain approximate 95% confidence intervals of $(0.27, 0.73)$ for the linear scale model and $(0.48, 0.52)$ for the multiplicative scale model. Notice how the confidence interval for the linear scale model covers a large proportion of the parameter space.

4.2.2 General collapsible model

We can still derive EE's for η when the ITS is not of the separable form. However, in this more general case, we do not necessarily have $\mathrm{Var}[h(X_i)|\mathcal{P}_i] = \phi g(\mathrm{E}[h(X_i)|\mathcal{P}_i])$, so we cannot use the quasi-likelihood EE's. We can nevertheless use linear EE's of the form (4.9) and derive weights w_{ij} that are optimal in some sense. However, the moments of X, the real time at failure, given a usage path $\mathcal{P}$ may be difficult to compute in practice. For this reason, semiparametric methods that do not require the calculation of conditional moments, such as min CV or rank based methods, seem to be more appropriate for more complicated collapsible models.

4.2.3 Censored observations

As was the case with the min CV method, quasi-likelihood and other methods based on comparing observations with sample moments are difficult to generalize so that they can handle censoring. Using an EM-type algorithm and replacing the censoring times, say x_i^*, with values of the form $\mathrm{E}[X_i|X_i > x_i^*, \theta_i; \eta, \mu]$ would not work here as we do not want to assume a parametric form for the survivor function G and, hence, cannot compute this last conditional expectation. Once again, we recommend the use of rank based methods when censoring is present.

4.3 Rank based methods

Semiparametric inference methods based on ranks are quite popular in survival analysis. These methods can easily be generalized to handle censoring, they are usually close to efficient and they are quite robust. In this section, we follow the methodology of Lin and

Ying (1995) in order to derive a rank based method for estimating semiparametrically the time scale parameter in a collapsible model.

4.3.1 General collapsible model

We first consider a rank based method for a general collapsible model with an ITS of the form $t_{\mathcal{P}_i}(x; \eta)$. We now write the log likelihood under some fully specified (but unknown) distribution G using the counting process notation of Andersen et al. (1993):

$$
l(\eta) = \sum_{i=1}^{n} \int_0^\infty \left\{ \left(\ln \lambda[t_{\mathcal{P}_i}(x; \eta)] + \ln t'_{\mathcal{P}_i}(x; \eta) \right) dN_i(x) \right. \tag{4.26}
$$
$$
\left. -Y_i(x)\lambda[t_{\mathcal{P}_i}(x; \eta)]t'_{\mathcal{P}_i}(x; \eta)dx \right\},
$$

where $\lambda[u]$ is the hazard function corresponding to $G(u)$, $N_i(u)$ is a counting process taking value 0 if $u < X_i$ and 1 if $u \geq X_i$, $Y_i(u)$ is 1 if individual i is at risk of failing at real time u (i.e., individual i is still alive and uncensored at the instant before u), 0 otherwise, and $t'_{\mathcal{P}_i}(x; \eta) = dt_{\mathcal{P}_i}(x; \eta)/dx$. Assuming that the usual regularity conditions hold, we can take the derivative with respect to η under the integral sign and obtain the following score function:

$$
U(\eta) = \sum_{i=1}^{n} \int_0^\infty \left\{ \left(\frac{\lambda'[t_{\mathcal{P}_i}(x; \eta)]}{\lambda[t_{\mathcal{P}_i}(x; \eta)]} \frac{\partial t_{\mathcal{P}_i}(x; \eta)}{\partial \eta} + \frac{\partial t'_{\mathcal{P}_i}(x; \eta)/\partial \eta}{t'_{\mathcal{P}_i}(x; \eta)} \right) dN_i(x) \right. \tag{4.27}
$$
$$
\left. -Y_i(x) \left(\lambda'(t_{\mathcal{P}_i}(x; \eta)] \frac{\partial t_{\mathcal{P}_i}(x; \eta)}{\partial \eta} t'_{\mathcal{P}_i}(x; \eta) + \lambda[t_{\mathcal{P}_i}(x; \eta)] \frac{\partial t'_{\mathcal{P}_i}(x; \eta)}{\partial \eta} \right) dx \right\}.
$$

Following Robins and Tsiatis (1992) and Lin and Ying (1995), we now make the assumption that $\lambda'[u] = 0$. This will yield maximum efficiency when G is the exponential

distribution. The score (4.27) then becomes

$$
\begin{aligned}
U(\eta) &= \sum_{i=1}^{n} \int_0^\infty \left\{ \frac{\partial t'_{p_i}(x;\eta)/\partial \eta}{t'_{p_i}(x;\eta)} dN_i(x) - Y_i(x)\lambda[t_{p_i}(x;\eta)] \frac{\partial t'_{p_i}(x;\eta)}{\partial \eta} dx \right\} \\
&= \sum_{i=1}^{n} \int_0^\infty \frac{\partial \ln t'_{p_i}(x;\eta)}{\partial \eta} \left(dN_i(x) - Y_i(x)\lambda[t_{p_i}(x;\eta)] t'_{p_i}(x;\eta) dx \right) \\
&= \sum_{i=1}^{n} \int_0^\infty \frac{\partial \ln t'_{p_i}(x;\eta)}{\partial \eta} \left(dN_i(x) - Y_i(x)\, d\Lambda[t_{p_i}(x;\eta)] \right),
\end{aligned}
\tag{4.28}
$$

where $\Lambda[u] = \int_0^u \lambda[t]dt = -\ln G[u]$ is the cumulative hazard function corresponding to G.

What Lin and Ying (1995) do at this stage is replace Λ by a "good" nonparametric estimator of it. This is straightforward once the value of η has been fixed; all we need to do is calculate the failure and censoring times in scale $t_{\mathcal{P}}(\cdot;\eta)$ and then estimate Λ using the Nelson-Aalen estimator. In more precise terms, let

$$
\widetilde{N}_i(t;\eta) = \begin{cases} 0, & \text{if } t < t_{p_i}(X_i;\eta) \\ 1, & \text{if } t \geq t_{p_i}(X_i;\eta) \end{cases}
$$

$$
\widetilde{Y}_i(t;\eta) = \begin{cases} 1, & \text{if individual } i \text{ is at risk of failing at time } t \text{ in scale } t_{\mathcal{P}}(\cdot;\eta) \\ 0, & \text{otherwise} \end{cases}
$$

$\widetilde{\Lambda}[t;\eta]$ = value of Nelson-Aalen estimator in scale $t_{\mathcal{P}}(\cdot;\eta)$ at time t.

Substituting these values in (4.28) yields

$$
\widetilde{U}(\eta) = \sum_{i=1}^{n} \int_0^\infty \left. \frac{\partial \ln t'_{p_i}(x;\eta)}{\partial \eta} \right|_{x=x^{-1}[t,\eta,\mathcal{P}_i]} \left(d\widetilde{N}_i(t;\eta) - \widetilde{Y}_i(t;\eta)\, d\widetilde{\Lambda}[t;\eta] \right),
\tag{4.29}
$$

where $x^{-1}[t,\eta,\mathcal{P}_i]$ is the value of x such that $t_{p_i}(x;\eta) = t$. This score functionis an unbiased linear rank estimating equation. Indeed, we can re-write (4.29) in the following

form:

$$\tilde{U}(\eta) = \sum_{i=1}^{n} \delta_i (Q_i - \overline{Q}),$$

where $Q_i = \partial \ln t'_{p_i}(x_i; \eta)/\partial \eta$ and $\overline{Q}$ is the average of the Q's still at risk when individual i fails. Note that if we set

$$t_p(x; \eta) = \int_0^x \exp \left\{ \eta^t y(u) \right\} \, du,$$

we obtain the exact same score as Lin and Ying (1995).

To obtain an estimator from score (4.29), we do as proposed by Lin and Ying (1995) and choose $\hat{\eta} = \arg\min_\eta \tilde{U}^t(\eta)\tilde{U}(\eta)$. The reason why we cannot simply solve $\tilde{U}(\eta) = 0$ for η is that this score is not a continuous function of η. It actually varies smoothly in η then takes a jump where η reaches a value that induces a change in the order of the times $t_{p_i}(\cdot; \eta)$'s. Figure 4.2 illustrates a typical behavior of $\tilde{U}(\eta)$ as a function of η.

Lin and Ying (1995) suggest the use of the *simulated annealing* algorithm (c.f. Press et al. (1992, Section 10.9)) in order to minimize the score function. When $\dim(\eta) = 1$, simpler algorithms, such as golden section search (Press et al. (1992, Section 10.1)), give good results.

4.3.2 Asymptotic properties

Through a fair amount of work, Lin and Ying (1995) show that the value of η that minimizes $\tilde{U}^t(\eta)\tilde{U}(\eta)$ is a consistent estimator of the true time scale parameter η_0 under regularity conditions satisfied when
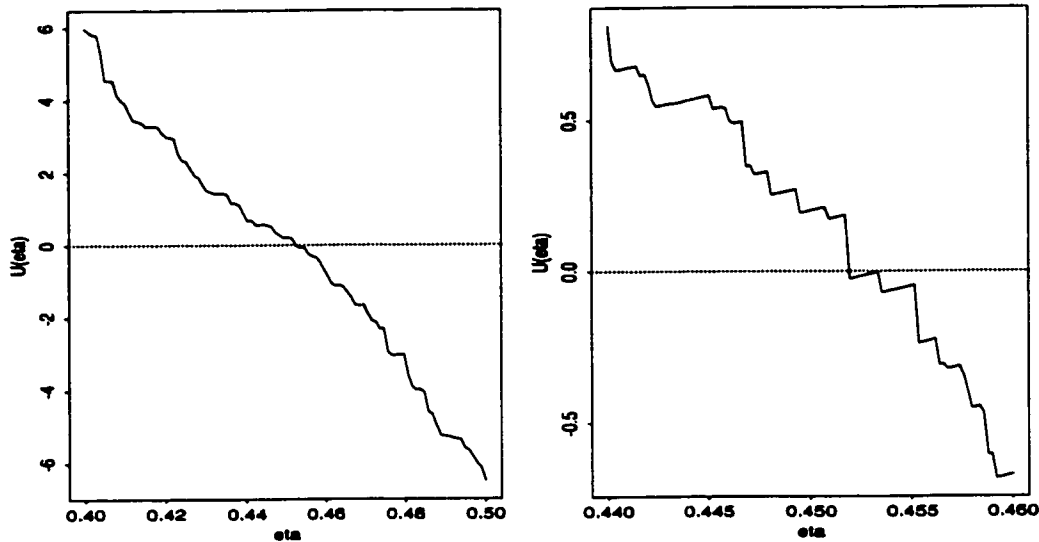
Figure 4.2: A typical form of the rank-based score function (4.29). (The plot on the right hand side is a magnification of the plot on the left hand side.)

1. the "covariates" $\partial \ln t'_{p_i}(x; \eta)/\partial \eta$ have uniformly bounded derivatives (w/r to x);

2. $-\int_0^\infty \left(d\ln f(u)/du \right)^2 dG(u) < \infty$, where $f(u)$ is the density corresponding to the survivor function G;

3. $-\int_0^\infty u^\varepsilon \, dG(u) < \infty$ for some $\varepsilon > 0$.

Conditions 2 and 3 depend only on the survivor function G. When G is a survivor function from one of the popular families in survival analysis (e.g., Weibull, lognormal, loglogistic), these conditions are satisfied. Condition 1 means that

$$\left| \frac{d}{dx} \frac{\partial \ln t'p(x; \eta)}{\partial \eta} \right| < K, \quad K < \infty, \; \forall x \geq 0.$$

Note that this condition is trivially satisfied in the case of the separable scale model, as $\partial \ln t'_{p_i}(x; \eta)/\partial \eta$ does not depend on x.

Under conditions 1-3, Lin and Ying (1995) show that the minimizer of $\tilde{U}^t(\eta)\tilde{U}(\eta)$ is strongly consistent and that

$$\psi(\eta) = \tilde{U}^t(\eta)V^{-1}(\hat{\eta})\tilde{U}(\eta) \overset{\text{asymp.}}{\sim} \chi_p^2, \tag{4.30}$$

where χ_p^2 represents a chi-squared distribution on $p = \dim(\eta)$ degrees of freedom and $n^{-1}V(\eta)$ is an estimate of the variance of $\tilde{U}(\eta)$ with

$$V(\eta) = \sum_{i=1}^n \int_0^\infty \left[\frac{\sum_{j=1}^n \tilde{Y}_j(t;\eta)Z_j(t;\eta)^{\otimes 2}}{\sum_{j=1}^n \tilde{Y}_j(t;\eta)} - \left(\frac{\sum_{j=1}^n \tilde{Y}_j(t;\eta)Z_j(t;\eta)}{\sum_{j=1}^n \tilde{Y}_j(t;\eta)} \right)^{\otimes 2} \right] d\tilde{N}_i(t;\eta), \tag{4.31}$$

where

$$Z_j(t;\eta) = \left. \frac{\partial \ln t_{p_j}'(x;\eta)}{\partial \eta} \right|_{x=x^{-1}(t,\eta,\mathcal{P}_j)}$$

and, for some column-vector z, $z^{\otimes 2}$ denotes the outer product zz^t.

We can thus use the quadratic form (4.30) to derive a (1-α)-level confidence region for η: $\{\eta : \psi(\eta) < \chi_{p;1-\alpha}^2\}$, where $\chi_{p;1-\alpha}^2$ is the (1-α)th quantile of a chi-squared distribution on p degrees of freedom. Asymptotically equivalent confidence regions can be built using $\{\eta : \tilde{\psi}(\eta) < \chi_{p;1-\alpha}^2\}$, where

$$\tilde{\psi}(\eta) = \tilde{U}^t(\eta)V^{-1}(\eta)\tilde{U}(\eta). \tag{4.32}$$

For large samples, quadratic forms (4.30) and (4.32) are equivalent. However, they may yield different confidence regions for moderate size samples in some situations. In the simulations study of Section 4.4, we consider both types of confidence regions and give guidelines as to which type to use for moderate size samples.

4.3.3 Separable scale model

We now apply the results derived above to the separable scale model. In this case, we have that

$$\frac{\partial \ln t'_{p_i}(x; \eta)}{\partial \eta} = \frac{\partial \ln v(\theta_i; \eta)}{\partial \eta}.$$

We then substitute this quantity into (4.28) to obtain

$$U(\eta) = \sum_{i=1}^{n} \int_0^\infty \frac{\partial \ln v(\theta_i; \eta)}{\partial \eta} \left(dN_i(x) - Y_i(x) \, d\Lambda[u(x)(\theta_i; \eta)] \right). \tag{4.33}$$

Replacing Λ in (4.33) by the Nelson-Aalen estimator computed in scale $t_p(\cdot; \eta)$ then gives the score

$$\tilde{U}(\eta) = \sum_{i=1}^{n} \int_0^\infty \frac{\partial \ln v(\theta_i; \eta)}{\partial \eta} \left(d\tilde{N}_i(t; \eta) - \tilde{Y}_i(t; \eta) \, d\tilde{\Lambda}[t; \eta] \right). \tag{4.34}$$

As we can see, this score is somewhat close to the score (4.21), obtained through quasi-likelihood. In the special case where there is no censoring, score (4.34) simplifies to

$$\tilde{U}(\eta) = \sum_{i=1}^{n} \frac{\partial \ln v(\theta_{(i)}; \eta)}{\partial \eta} \left(1 - \left\{ \frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{n-i+1} \right\} \right), \tag{4.35}$$

where $\theta_{(i)}$ is the path parameter corresponding to the ith smallest failure time under scale $t_p(\cdot; \eta)$. Note that we would have obtained the exact same score function had we wanted to test $\beta = 0$ in $\ln T_i[\eta] = \mu + \beta^t \partial \ln v(\theta_i; \eta)/\partial \eta + \varepsilon_i$, where ε_i's are i.i.d. and η is fixed, using rank regression and exponential scores (see Cox and Hinkley (1974) or Kalbfleisch and Prentice (1980)). This method is thus consistent with our definition of ITS, i.e. we are looking for a value of η such that the failure times in the scale $t_p(\cdot; \eta)$ are "as unrelated as possible" to some function of the usage path parameter θ.

To illustrate this method, let us see what the score and variance estimate look like in the case of the linear scale model. Substituting $v(\theta; \eta) = (1 - \eta + \eta\theta)$ in (4.33) gives

$$\tilde{U}(\eta) = \sum_{i=1}^{n} \int_{0}^{\infty} \frac{\theta_i - 1}{1 - \eta + \eta\theta_i} \left(d\tilde{N}_i(t; \eta) - \tilde{Y}_i(t; \eta) \, d\tilde{\Lambda}[t; \eta] \right). \tag{4.36}$$

We can then use the fact that $\tilde{U}^2(\eta)/\tilde{V}(\eta)$ is approximately chi-squared on one degree of freedom to derive an approximate $(1 - \alpha)100\%$ confidence interval for η:

$$\left\{ \eta : \tilde{U}^2(\eta)/V(\hat{\eta}) \leq \chi^2_{1;1-\alpha} \right\}, \tag{4.37}$$

where $\chi^2_{1;1-\alpha}$ is the $(1 - \alpha)$th quantile of a chi-squared distribution on one degree of freedom and $V(\hat{\eta})$ is obtained by substituting $Z_j(t; \eta) = (\theta_j - 1)/(1 - \eta + \eta\theta_j)$ in (4.31).

4.3.4 Application of the method

This rank based method can be used in most frameworks as long as we have enough data to compute the score function. When $\partial \ln t'_p(x; \eta)/\partial \eta$ does not depend on x, such as in the separable scale model, we only need to observe the real times of failure/censoring along with the path parameter values, θ. Unfortunately, when $\partial \ln t'_p(x; \eta)/\partial \eta$ does depend on x, we need more information. In that case, for each unit we need to know the values of the usage measures and their derivative with respect to x at all the different real times at failure present in the sample. In addition, for units that get censored, we need to know the value of their usage measures at censoring. This problem is not new though; it is actually the same problem as with partial likelihood inference for the proportional hazards model when time-varying covariates are present.

All the semiparametric inference methods mentioned in this chapter yield consistent

estimators. However, the variance of these estimators (and thus, probably, their efficiency) tends to vary with the choice of the survivor function G and the distribution of the observed usage paths. It would therefore be of interest to know which methods are more efficient for a given time scale model, and this for samples of moderate size. In the next section, we compare the efficiency of the semiparametric methods outlined in this chapter through simulation. We also use these simulations to assess the accuracy of the various variance estimates and/or confidence intervals.

4.4 Simulation study

Because of the complexity of the models at hand, we compare the efficiency of the various estimators proposed in this chapter through a simulation study. We also use the simulation results to assess the accuracy of the variance estimators and of the confidence intervals.

In Section 4.4.1, we generate data from the linear scale model. We use the multiplicative scale model in Section 4.4.2 and a collapsible model that is not of the separable form and for which the covariates vary in time in Section 4.4.3. For each sample simulated, we compute the ML, rank (minimizer of the square of (4.34), denoted "Rank"), quasi-likelihood (root of (4.14), denoted "Quasi-like.") and minimum CV (Min CV) estimators. For the samples obtained from the multiplicative scale model, we also compute the popular ordinary least squares (OLS) estimator of η based on the logarithmic time scale $\log T[\eta] = \log x + \eta \log \theta$ (see Lawless (1982), for example).

4.4.1 Linear scale model

We generated data from the linear scale model

$$\Pr\left[X > x|\theta\right] = G[(1 - \eta)x + \eta\theta x]. \tag{4.38}$$

We used a value of $\eta = 0.5$, four different distributions for G and two distributions for the usage paths, Θ. We simulated 2000 samples of size 102 in each batch.

The results of the simulations were in agreement with the theoretical results of the previous sections; with the linear scale model, large variability in the observed values of Θ and a more compact distribution G lead to much more precise inference. We also obtain that all the studied estimators have small bias, but their relative efficiencies vary with changes in G.

Batch 1

We simulated Batch 1 from the linear scale model $t_\theta(x) = (1 - 0.5)x + 0.5\theta x$ with

1. $G(u) = \exp\left\{-\left(\frac{u}{1000}\right)^3\right\}$;

2. $\text{atan}(\Theta) \sim \text{Uniform}(0, \pi/2)$.

The fifth, fiftieth and ninety-fifth percentiles of the distribution G are $t_{0.05} = 371.6$, $t_{0.50} = 885.0$ and $t_{0.95} = 1441.6$. The same percentiles for the distribution of the slope parameter are $\theta_{0.05} = 0.0787$, $\theta_{0.50} = 1.0$ and $\theta_{0.95} = 12.71$. To get more insight on whether or not we can estimate η accurately and on the plausibility of this scenario, we can look at the percentiles of real time at failure, x, conditional on the path parameter, θ, for a few values of η. These values are summarized in Table 4.1 and on Figure 4.3

	$\eta = 0.25$			$\eta = 0.50$			$\eta = 0.75$		
	$\theta_{0.05}$ (0.0787)	$\theta_{0.50}$ (1.0)	$\theta_{0.95}$ (12.71)	$\theta_{0.05}$ (0.0787)	$\theta_{0.50}$ (1.0)	$\theta_{0.95}$ (12.71)	$\theta_{0.05}$ (0.0787)	$\theta_{0.50}$ (1.0)	$\theta_{0.95}$ (12.71)
$x_{0.05}$	688.9	371.6	54.22	482.7	371.6	94.63	1203	371.6	37.99
$x_{0.50}$	1150	885.0	225.4	1641	885.0	129.1	2864	885.0	90.49
$x_{0.95}$	1873	1441	367.1	2673	1441	210.4	4665	1441	147.4

Table 4.1: Percentiles of real time at failure given usage path, linear scale model, Batch 1.

As we can see, different values of η yield fairly different conditional distributions for real time at failure given the usage path. We should therefore be able to estimate the value of η accurately from this model. Table 4.2 confirms this last statement. If we look at the standard deviations obtained from the simulations (Table 4.2, 2nd column), we see that roughly 95% of the times, the estimates will be in the range (0.43, 0.57), which is fairly close to the true value, 0.5. We also notice the extremely high efficiency of the rank and minimum CV estimators relative to ML.

Estimator	Average	Standard Deviation	Relative Efficiency
ML	0.5003	0.03199	100.0%
Rank	0.5003	0.03238	97.6%
Quasi-like.	0.5004	0.03544	81.5%
Min CV	0.5003	0.03202	99.8%

Table 4.2: Simulation results, linear scale model, Batch 1.

Batch 2

We simulated Batch 2 from the linear scale model $t_\theta(x) = (1 - 0.5)x + 0.5\theta x$ with

1. $G(u) = \exp\left\{-\frac{u}{1000}\right\}$;
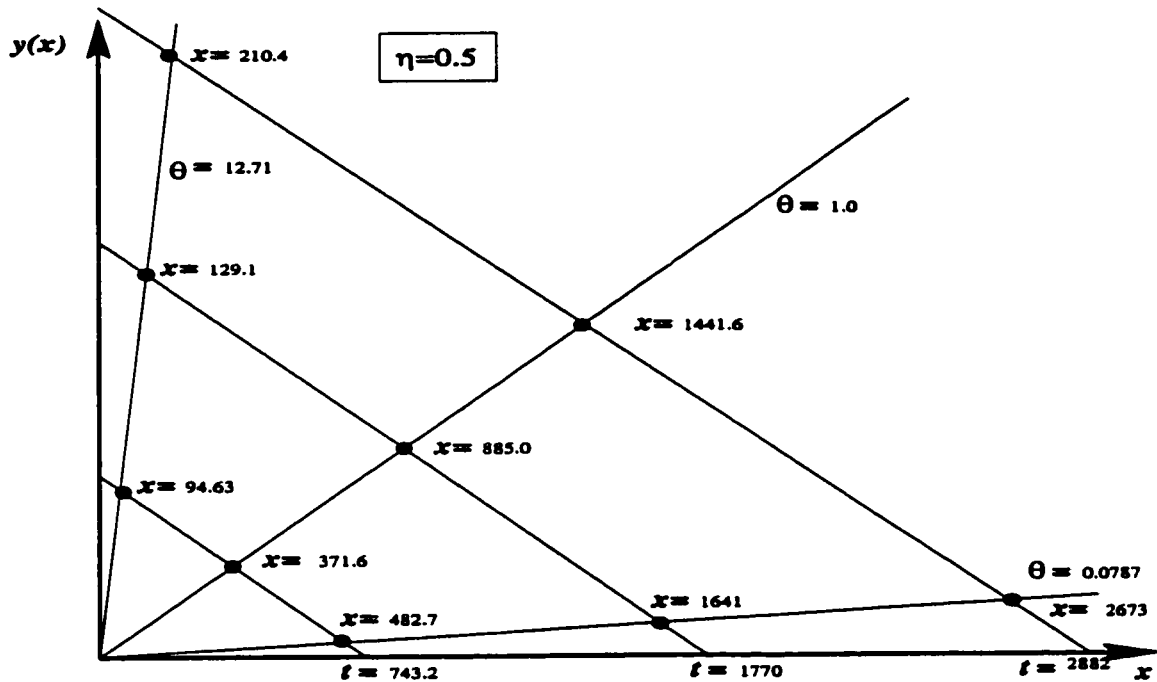
Figure 4.3: Percentiles of real time at failure given usage path, linear scale model, Batch 1.

2. $\operatorname{atan}(\Theta) \sim \operatorname{Uniform}(0, \pi/2)$.

The distribution for the usage paths is the same as in Batch 1. However, the distribution G has a shape parameter of value less than in Batch 1. This means that the CV of G is smaller and we therefore should observe less precision in the inference procedures. In fact, if one carries out the calculations, one finds that the CV of G in Batch 2 is about 2.75 times the CV of G in Batch 1.

To get a better idea of how this model is different from the model used in Batch 1, we compute the same conditional quantiles of real time at failure as in Table 4.1.

Table 4.4 summarizes the results. As expected, the validity of all the estimators was still good, but the variability increased. The simulation standard deviations were close to 2.75 times as large as they were for Batch 1, which is what we expected.

	$\eta = 0.25$			$\eta = 0.50$			$\eta = 0.75$		
	$\theta_{0.05}$ (0.0787)	$\theta_{0.50}$ (1.0)	$\theta_{0.95}$ (12.71)	$\theta_{0.05}$ (0.0787)	$\theta_{0.50}$ (1.0)	$\theta_{0.95}$ (12.71)	$\theta_{0.05}$ (0.0787)	$\theta_{0.50}$ (1.0)	$\theta_{0.95}$ (12.71)
$x_{0.05}$	66.64	51.29	13.06	95.10	51.29	7.485	166.0	51.29	5.245
$x_{0.50}$	900.6	693.1	173.5	1285	693.1	101.1	2243	693.1	70.88
$x_{0.95}$	3892	2996	762.9	5554	2996	437.1	9694	2996	306.3

Table 4.3: Percentiles of real time at failure given usage path, linear scale model, Batch 2.

Estimator	Average	Standard Deviation	Relative Efficiency
Quasi-like. (ML)	0.5007	0.0931160	100%
Rank	0.5007	0.09431	97.5%
Min CV	0.4993	0.1171	63.3%

Table 4.4: Simulation results, linear scale model, Batch 2.

Batch 3

We simulated Batch 3 from the linear scale model $t_\theta(x) = (1 - 0.5)x + 0.5\theta x$ with

1. $G(u) = \exp\left\{-\left(\frac{u}{1000}\right)^3\right\}$;

2. $\ln\Theta \sim N(2.37, 0.57^2)$.

The distribution G is the same as in Batch 1, but this time the usage path distribution is clustered around $\Theta = 11$ rather than uniform over the positive quadrant. We therefore expect our inferences with the linear scale model to be highly unprecise.

We notice that the three conditional distributions of x given θ in Table 4.5 do not vary as much in η. We can thus expect more variability in the estimates of η. This is confirmed in Table 4.6. Except for the ML estimator, all the other estimators had a

	$\eta = 0.25$			$\eta = 0.50$			$\eta = 0.75$		
	$\theta_{0.05}$ (4.121)	$\theta_{0.50}$ (10.70)	$\theta_{0.95}$ (27.77)	$\theta_{0.05}$ (4.121)	$\theta_{0.50}$ (10.70)	$\theta_{0.95}$ (27.77)	$\theta_{0.05}$ (4.121)	$\theta_{0.50}$ (10.70)	$\theta_{0.95}$ (27.77)
$x_{0.05}$	208.7	108.5	48.30	145.1	63.53	25.83	111.2	44.91	17.63
$x_{0.50}$	497.2	258.4	115.0	345.7	151.3	61.52	264.9	107.0	41.99
$x_{0.95}$	809.8	421.0	187.4	563.1	246.5	100.2	431.6	174.2	68.39

Table 4.5: Percentiles of real time at failure given usage path, linear scale model, Batch 3.

probability mass at $\hat{\eta} = 1.0$, the boundary of the parameter space. This makes efficiency comparisons by taking the ratio of the variances somewhat futile.

Estimator	Average	Standard Deviation	Mean Squared Error
ML	0.5270	0.1827	0.03410
Rank	0.8071	0.2083	0.1377
Quasi-like.	0.5375	0.1971	0.04026
Min CV	0.5278	0.1839	0.03461

Table 4.6: Simulation results, linear scale model, Batch 3.

As we can see in Table 4.6, the rank estimator seems to have substantial bias. This is due to the fact that the estimator yielded a value of $\hat{\eta} = 1$ for a large proportion of the samples, i.e. it identified Y as the ITS. Figure 4.4 illustrates the behavior of the rank based estimator in both Batch 1 and Batch 3. As was the case with Batch 1, the minimum CV estimator was again very close to the ML estimator in this case. This does not mean that the minimum CV estimator yields good results, however. A quick look at the standard deviations of Table 4.6 clearly shows that the estimators were all over the parameter space and did not zero in on 0.5 with any good precision. The same observations are also true for Batches 4, 7 and 8.
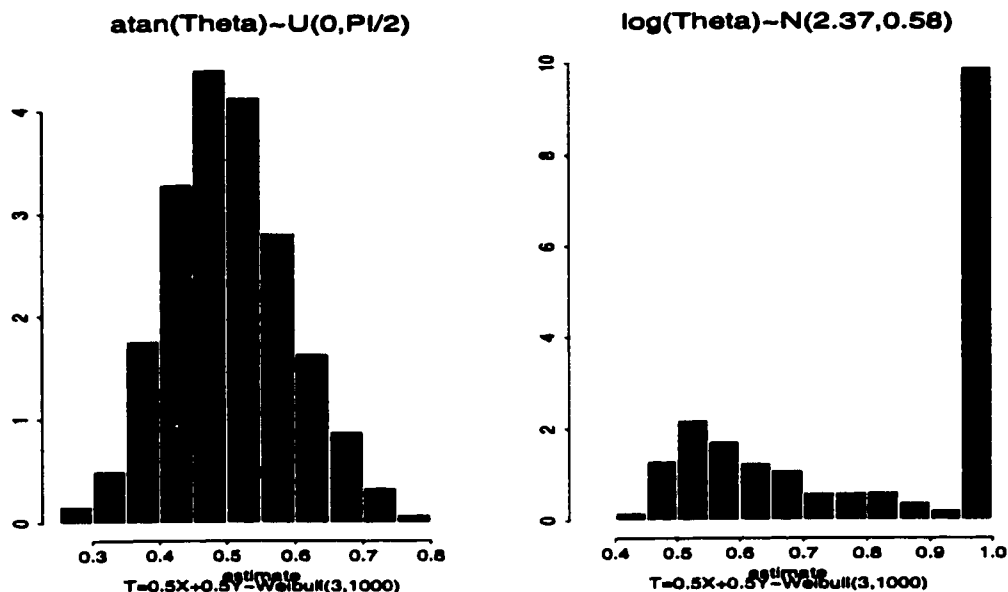
Figure 4.4: Histograms of the rank based estimates from Batch 1 (LHS) and Batch 3 (RHS).

Batch 4

We simulated Batch 4 from the linear scale model $t_\theta(x) = (1 - 0.5)x + 0.5\theta x$ with

1. $G(u) = \exp\left\{-\frac{u}{1000}\right\}$;

2. $\ln \Theta \sim N(2.37, 0.57^2)$.

We now have a distribution G with large coefficient of variation and clustered usage paths at the same time. The inferences should be very unprecise.

Once again, the rank estimator gave a value of $\hat{\eta} = 1$ for a large proportion of the samples. The minimum CV estimator was again very close to the ML estimator, but the standard deviations were around 0.32, which is not good at all when we wish to estimate $\eta \in [0, 1]$.

	$\eta = 0.25$			$\eta = 0.50$			$\eta = 0.75$		
	$\theta_{0.05}$ (4.121)	$\theta_{0.50}$ (10.70)	$\theta_{0.95}$ (27.77)	$\theta_{0.05}$ (4.121)	$\theta_{0.50}$ (10.70)	$\theta_{0.95}$ (27.77)	$\theta_{0.05}$ (4.121)	$\theta_{0.50}$ (10.70)	$\theta_{0.95}$ (27.77)
$x_{0.05}$	28.81	14.98	6.668	20.03	8.770	3.566	15.36	6.200	2.433
$x_{0.50}$	389.4	202.4	90.10	270.7	118.5	48.18	207.5	83.78	32.88
$x_{0.95}$	1683	874.8	389.4	1170	512.2	208.2	896.8	362.1	142.1

Table 4.7: Percentiles of real time at failure given usage path, linear scale model, Batch 4.

Estimator	Average	Standard Deviation	Mean Squared Error
Quasi-like. (ML)	0.5551	0.3276	0.1104
Rank	0.9087	0.1697	0.1958
Min CV	0.5350	0.3535	0.1262

Table 4.8: Simulation results, linear scale model, Batch 4.

Just as was the case with the ML estimator in Chapter 3, we can see that all the semiparametric methods for the linear scale model are sensitive to the choice of the distribution G and the distribution of the observed usage paths. Along with the asymptotic variances derived in Chapter 3 and Section 4.2.1, geometric arguments can explain this sensitivity. Figures 4.5 and 4.6 show how the variability in the path distribution and in G contribute to the precision of the time scale parameter estimate.

We are now entitled to wonder what kind of effect the form of the distribution G has on the properties of the estimator. Batches 5 to 8 have been simulated from the same models as Batches 1 to 4, but the Weibull distribution has been replaced by a lognormal
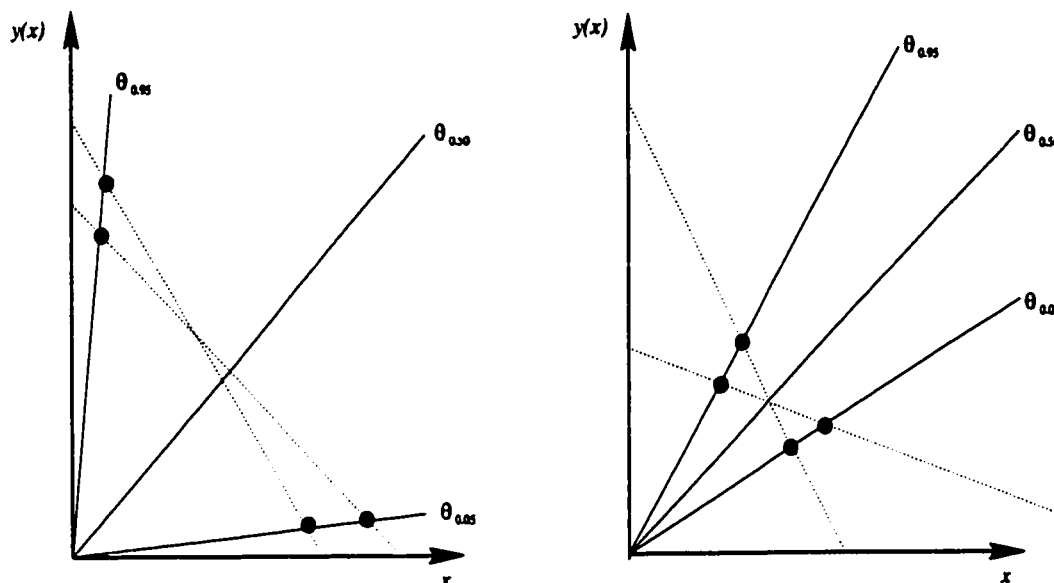
Figure 4.5: Impact of the observed path distribution on the precision of the estimator. Estimating η is similar to estimating the slope of the doted lines (linear ITS). If the usage paths are further apart (left hand side plot), we get more precision in the slope estimates than if they are tightly grouped (right hand side plot).

distribution with the same first two moments. As we will see, the results are very similar to the ones from Batches 1-4, except for the efficiencies, which differ by a good margin in some cases.

Batch 5

We simulated Batch 5 from the linear scale model $t_\theta(x) = (1 - 0.5)x + 0.5\theta x$ with

1. $G \sim \log \text{normal}(6.7, 0.35^2)$;

2. $\text{atan}(\Theta) \sim \text{Uniform}(0, \pi/2)$.

As in Batch 1, G had a large coefficient of variation and the usage paths are well spread over the positive quadrant, so we expect the inferences to be precise.
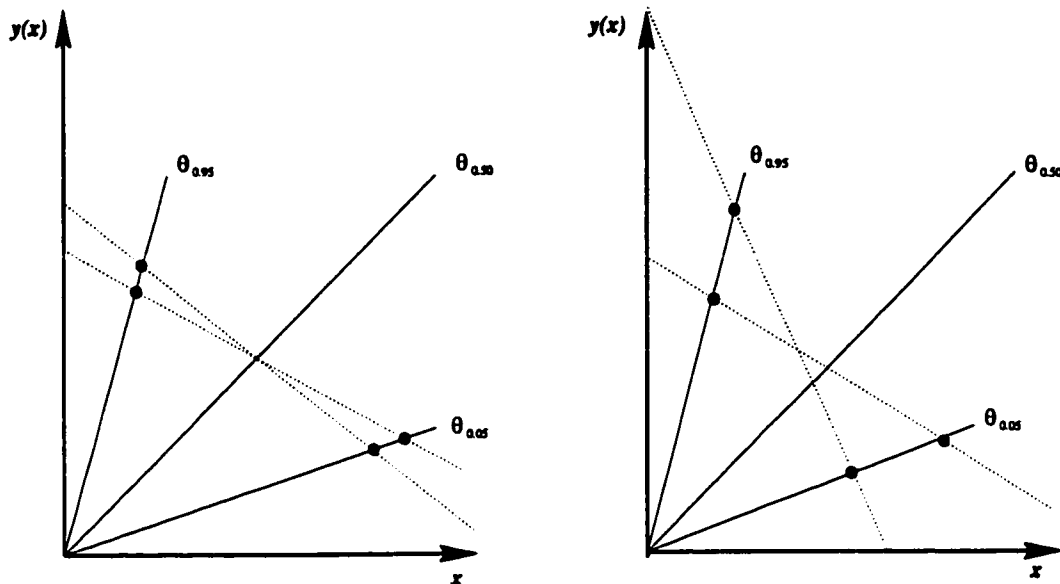
Figure 4.6: If G has a small variance (left hand side plot), then for a given usage path, θ, the observed failures along that path will be close and thus the precision in the estimates of the slope of the doted lines (the ITS) will be greater than if there is a lot of variability in G (right hand side plot).

The first conclusion that can be reached from the results presented in Table 4.9 is that the minimum CV loses an important part of its efficiency when the true G is the log normal distribution. The rank estimator is affected too, but to a much lesser extent. We also see that the standard deviations in Table 4.9 are close to those in Table 4.2.

Estimator	Average	Standard Deviation	Relative Efficiency
ML	0.4996	0.03402	100%
Rank	0.4997	0.0379	80.4%
Quasi-like.	0.4996	0.03529	92.9%
Min CV	0.4996	0.04369	60.6%

Table 4.9: Simulation results, linear scale model, Batch 5.

Batch 6

We simulated Batch 6 from the linear scale model $t_\theta(x) = (1 - 0.5)x + 0.5\theta x$ with

1. $G \sim$ log normal$(6.7, 0.83^2)$;

2. atan$(\Theta) \sim$ Uniform$(0, \pi/2)$.

From Table 4.10, it is clear that the minimum CV estimator is badly affected by the choice of a log normal distribution with high coefficient of variation, even if there is large variability observed usage paths.

Estimator	Average	Standard Deviation	Relative Efficiency
ML	0.4990	0.08137	100%
Rank	0.4993	0.08818	85.2%
Quasi-like.	0.4991	0.09323	76.2%
Min CV	0.4994	0.1696	23.0%

Table 4.10: Simulation results, linear scale model, Batch 6.

Batch 7

We simulated Batch 7 from the linear scale model $t_\theta(x) = (1 - 0.5)x + 0.5\theta x$ with

1. $G \sim$ log normal$(6.7, 0.35^2)$;

2. ln $\Theta \sim N(2.37, 0.57^2)$.

Once again, we observe the same probability mass at $\hat\eta = 1$ for the rank estimator and the same extreme volatility with the other estimators in Table 4.11 as we had in Table 4.6.

Estimator	Average	Standard Deviation	Mean Squared Error
ML	0.5449	0.1957	0.04029
Rank	0.8116	0.2081	0.1404
Quasi-like.	0.5418	0.1972	0.04064
Min CV	0.5423	0.2241	0.05199

Table 4.11: Simulation results, linear scale model, Batch 7.

Batch 8

We simulated Batch 8 from the linear scale model $t_\theta(x) = (1 - 0.5)x + 0.5\theta x$ with

1. $G \sim \log \text{normal}(6.7, 0.85^2)$;

2. $\ln \Theta \sim N(2.37, 0.57^2)$.

Estimator	Average	Standard Deviation	Mean Squared Error
ML	0.5784	0.3141	0.1048
Rank	0.9009	0.1749	0.1913
Quasi-like.	0.5610	0.3209	0.1067
Min CV	0.5209	0.3760	0.1418

Table 4.12: Simulation results, linear scale model, Batch 8.

If we compare the results from Batches 1 to 4 to the results from Batches 5 to 8, we can see that the averages and standard errors of the estimators are roughly the same for the ML and quasi-likelihood estimator. The fact that the standard deviations did not change much for the quasi-likelihood is not surprising; the asymptotic variance formulas of Section 4.2.1 only involve the first two moments of G and the observed variability of the usage paths. The rank based and minimum CV estimators seem to have lost some of their efficiency, though. In the case of the former, this was not surprising as it was

derived by substituting 0 for the derivative of the hazard function in the likelihood score. Nevertheless, the efficiency still remained good. The minimum CV estimator was badly affected by the change of form in G, and it could not keep the great efficiency shown under the Weibull parameterization for G.

4.4.2 Multiplicative scale model

We repeated the same analysis, but this time with the multiplicative scale model. We generated data from

$$\Pr[X > x|\theta] = G[x\theta^\eta], \tag{4.39}$$

where we used a value of $\eta = 0.5$ and a few different distributions for G and the distribution of the usage paths, Θ. We simulated 2000 samples of size 102 in each batch. The results were along the same line as with the linear scale model, except that the impact of the distribution of the usage paths was not as dramatic as in the linear scale model.

Batch 1

We simulated Batch 1 from the multiplicative scale model with

1. $G(u) = \exp\left\{-\left(\frac{u}{1000}\right)^3\right\}$;

2. $\mathrm{atan}(\Theta) \sim \mathrm{Uniform}(0, \pi/2)$.

Table 4.14 summarizes the simulation results for Batch 1. We can see that with G being Weibull with small coefficient of variation, the minimum CV is extremely efficient. The rank method also showed great efficiency.

	$\eta = 0.25$			$\eta = 0.50$			$\eta = 0.75$		
	$\theta_{0.05}$ (0.0787)	$\theta_{0.50}$ (1.0)	$\theta_{0.95}$ (12.71)	$\theta_{0.05}$ (0.0787)	$\theta_{0.50}$ (1.0)	$\theta_{0.95}$ (12.71)	$\theta_{0.05}$ (0.0787)	$\theta_{0.50}$ (1.0)	$\theta_{0.95}$ (12.71)
$x_{0.05}$	701.5	371.6	196.8	1324.4	371.6	104.2	2501	371.6	55.21
$x_{0.50}$	1671	885.0	468.7	3155	885.0	248.3	5956	885.0	131.5
$x_{0.95}$	2722	1442	763.5	5139	1442	404.4	9702	1442	214.2

Table 4.13: Percentiles of real time at failure given usage path, multiplicative scale model, Batch 1.

Estimator	Average	Standard Deviation	Relative Efficiency
ML	0.5005	0.02198	100.0%
Rank	0.5005	0.02210	98.9%
Quasi-like.	0.5004	0.02418	82.6%
Min CV	0.5005	0.02202	99.7%
OLS	0.5003	0.02849	59.5%

Table 4.14: Simulation results, multiplicative scale model, Batch 1.

Batch 2

We simulated Batch 2 from the multiplicative scale model with

1. $G(u) = \exp\left\{-\frac{u}{1000}\right\}$;

2. $\text{atan}(\Theta) \sim \text{Uniform}(0, \pi/2)$,

following the same pattern as with the linear scale model. Again, we expect the standard deviations to be about 2.75 times the ones from Table 4.14.

The minimum CV was seriously affected by the increase in the coefficient of variation of G in this batch. Except for the minimum CV estimator, the standard deviations from Table 4.16 are close to 2.75 times the ones in Table 4.14, which is what we expected.

	$\eta = 0.25$			$\eta = 0.50$			$\eta = 0.75$		
	$\theta_{0.05}$ (0.0787)	$\theta_{0.50}$ (1.0)	$\theta_{0.95}$ (12.71)	$\theta_{0.05}$ (0.0787)	$\theta_{0.50}$ (1.0)	$\theta_{0.95}$ (12.71)	$\theta_{0.05}$ (0.0787)	$\theta_{0.50}$ (1.0)	$\theta_{0.95}$ (12.71)
$x_{0.05}$	96.84	51.29	27.17	182.8	51.29	14.39	345.2	51.29	7.622
$x_{0.50}$	1309	693.1	367.1	2471	693.1	194.5	4665	693.1	103.0
$x_{0.95}$	5656	2996	1587	10680	2996	840.4	20160	2996	445.1

Table 4.15: Percentiles of real time at failure given usage path, multiplicative scale model, Batch 2.

Estimator	Average	Standard Deviation	Relative Efficiency
Quasi-like. (ML)	0.5014	0.06593	100%
Rank	0.5009	0.06628	98.9%
Min CV	0.5000	0.08919	54.7%
OLS	0.5009	0.08546	59.5%

Table 4.16: Simulation results, multiplicative scale model, Batch 2.

Batch 3

We simulated Batch 3 from the multiplicative scale model with

1. $G(u) = \exp\left\{-\left(\frac{u}{1000}\right)^3\right\}$;

2. $\ln\Theta \sim N(2.37, 0.57^2)$.

The results in Table 4.18 are somewhat surprising. Despite the clustering of the usage paths, the inferences still remained relatively precise, at least compared to Batches 3 and 4 from the linear scale model.

Batch 4

We simulated Batch 4 from the multiplicative scale model with

	$\eta = 0.25$			$\eta = 0.50$			$\eta = 0.75$		
	$\theta_{0.05}$ (4.121)	$\theta_{0.50}$ (10.70)	$\theta_{0.95}$ (27.77)	$\theta_{0.05}$ (4.121)	$\theta_{0.50}$ (10.70)	$\theta_{0.95}$ (27.77)	$\theta_{0.05}$ (4.121)	$\theta_{0.50}$ (10.70)	$\theta_{0.95}$ (27.77)
$x_{0.05}$	260.8	205.4	161.9	183.0	113.6	70.50	128.5	62.81	30.71
$x_{0.50}$	621.2	489.4	385.5	436.0	270.6	167.9	306.0	149.6	73.15
$x_{0.95}$	1012	797.1	628.0	710.2	440.8	273.5	498.4	243.7	119.2

Table 4.17: Percentiles of real time at failure given usage path, multiplicative scale model, Batch 3.

Estimator	Average	Standard Deviation	Relative Efficiency
ML	0.4987	0.06069	100%
Rank	0.4987	0.06136	97.8%
Quasi-like.	0.4988	0.06519	86.6%
OLS	0.4985	0.07591	63.9%
Min CV	0.4988	0.06131	98.0%

Table 4.18: Simulation results, multiplicative scale model, Batch 3.

1. $G(u) = \exp\left\{-\frac{u}{1000}\right\}$;

2. $\ln\Theta \sim N(2.37, 0.57^2)$.

Batch 5

We simulated Batch 5 from the multiplicative scale model with

1. $G \sim$ log normal$(6.7, 0.35^2)$;

2. $\operatorname{atan}(\Theta) \sim \operatorname{Uniform}(0, \pi/2)$.

	$\eta = 0.25$			$\eta = 0.50$			$\eta = 0.75$		
	$\theta_{0.05}$ (4.121)	$\theta_{0.50}$ (10.70)	$\theta_{0.95}$ (27.77)	$\theta_{0.05}$ (4.121)	$\theta_{0.50}$ (10.70)	$\theta_{0.95}$ (27.77)	$\theta_{0.05}$ (4.121)	$\theta_{0.50}$ (10.70)	$\theta_{0.95}$ (27.77)
$x_{0.05}$	36.00	28.36	22.27	25.27	15.68	9.733	17.74	8.672	4.240
$x_{0.50}$	486.5	383.3	301.9	341.5	211.9	131.5	239.7	117.2	57.30
$x_{0.95}$	2103	1656	1305	1476	915.9	568.5	1036	506.5	247.6

Table 4.19: Percentiles of real time at failure given usage path, multiplicative scale model, Batch 4.

Estimator	Average	Standard Error	Relative Efficiency
Quasi-like. (ML)	0.4990	0.1804	100%
Rank	0.4961	0.1825	97.7%
OLS	0.4954	0.2277	62.8%
Min CV	0.5213	0.2342	59.4%

Table 4.20: Simulation results, multiplicative scale model, Batch 4.

Batch 6

We simulated Batch 6 from the multiplicative scale model with

1. $G \sim \log \text{normal}(6.7, 0.85^2)$;

2. $\text{atan}(\Theta) \sim \text{Uniform}(0, \pi/2)$.

Estimator	Average	Standard Error	Relative Efficiency
OLS (ML)	0.4996	0.02301	100%
Rank	0.4996	0.02566	80.4%
Quasi-like.	0.4996	0.02386	93.0%
Min CV	0.4996	0.02932	61.5%

Table 4.21: Simulation results, multiplicative scale model, Batch 5.

Estimator	Average	Standard Error	Relative Efficiency
OLS (ML)	0.4991	0.05456	100%
Rank	0.4990	0.06084	80.4%
Quasi-like.	0.4991	0.06428	72.1%
Min CV	0.4998	0.1291	17.9%

Table 4.22: Simulation results, multiplicative scale model, Batch 6.

Batch 7

We simulated Batch 7 from the multiplicative scale model with

1. $G \sim$ log normal$(6.7, 0.35^2)$;

2. $\ln \Theta \sim N(2.37, 0.57^2)$.

Estimator	Average	Standard Error	Relative Efficiency
OLS (ML)	0.5012	0.06344	100%
Rank	0.5007	0.07024	81.6%
Quasi-like.	0.5013	0.06531	94.3%
Min CV	0.5012	0.08018	62.6%

Table 4.23: Simulation results, multiplicative scale model, Batch 7.

Batch 8

We simulated Batch 8 from the multiplicative scale model with

1. $G \sim$ log normal$(6.7, 0.85^2)$;

2. $\ln \Theta \sim N(2.37, 0.57^2)$.

Estimator	Average	Standard Error	Relative Efficiency
OLS (ML)	0.5027	0.1504	100%
Rank	0.5017	0.1661	82.0%
Quasi-like.	0.5055	0.1747	74.1%
Min CV	0.5683	0.2883	27.2%

Table 4.24: Simulation results, multiplicative scale model, Batch 8.

If we compare the simulation results from the multiplicative scale model to the ones from the linear scale model, some conclusions are apparent:

1. The estimators in the multiplicative scale model look less sensitive to the distribution of the usage paths θ;

2. The minimum CV estimator has small bias for both the linear and multiplicative scale models, but loses a lot of its efficiency when G is lognormal or has a high coefficient of variation. It is very efficient when G is Weibull with large shape parameter.

3. The efficiency of the rank based estimator does not seem to vary with a change in the values of the parameters of G.

4.4.3 Non separable scale model

We simulated data from a model that is not comprised in the separable scale family. The time scale was the linear scale $t_\mathcal{P}(x) = (1 - 0.5)x + 0.5y(x)$ and the paths were of the form $y(x) = x^\theta$, where θ took the values 1/2, 1 and 2, each with probability 1/3. Figure 4.7 shows the shape of the three possible usage paths.
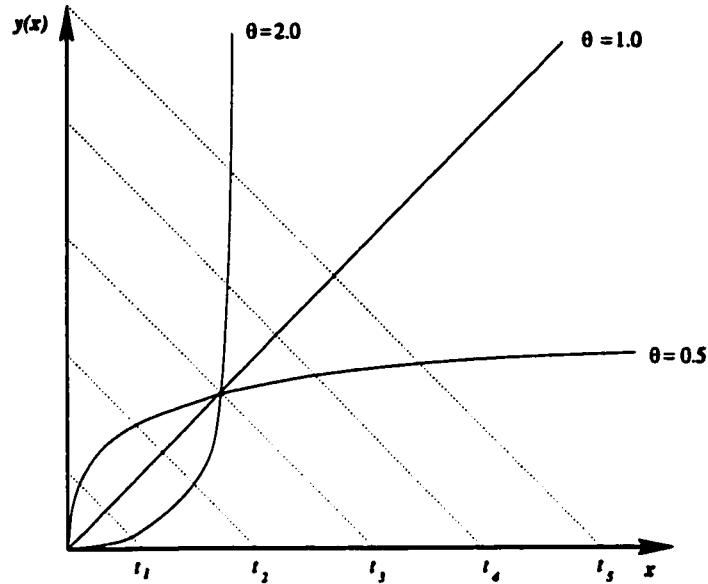
Figure 4.7: Shape of the usage paths for the model used in Section 4.4.3. Lines parallel to the doted lines represent various ages in the ideal time scale.

In Section 4.1, Lemma 4.1.1 gave sufficient conditions for the consistency of the minimum CV estimator. We cannot use the conclusion of Lemma 4.1.1 in this case because the model is not of the separable scale form. Therefore, we do not know if the squared CV of $T(\eta)$ is uniquely minimized at $\eta = 0.5$, and hence if the minimum CV estimator is consistent. However, if we compute $CV^2[T(\eta)]$ numerically for some fixed distribution G for this model, we can see that the value of η that minimizes it is $\eta = 0.5$. Figure 4.8 shows a plot of $CV^2[T(\eta)]$ for this non separable scale model with $G \sim \text{Weibull}(3, 1000)$.

From Figure 4.8, we expect the minimum CV estimator to be consistent. Also, we have that the regularity conditions for the consistency of the ML and rank based estimators are respected in this case, so we expect these two estimators to be consistent as well.
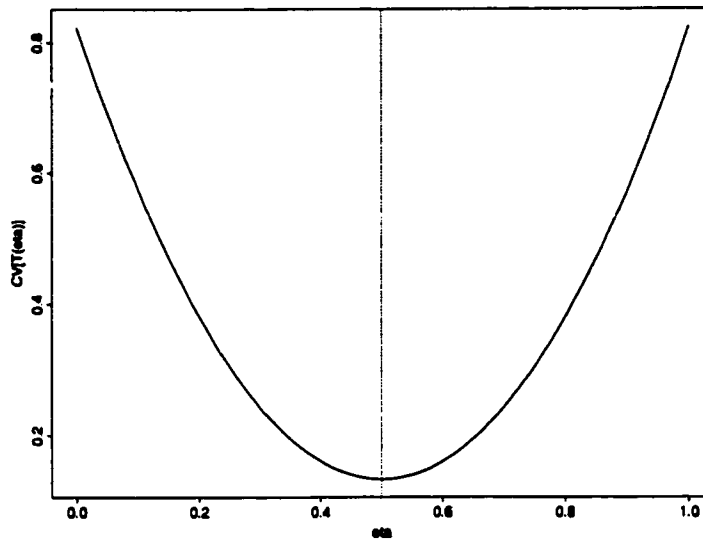
Figure 4.8: $CV^2[T(\eta)]$ as a function of η for the non separable scale model with G as in Batch 1. As we can see, the function seems to be uniquely minimized at $\eta = 0.5$. We thus expect the minimum CV estimator to be consistent.

Batch 1

We first simulated Batch 1 using $G(u) = \exp\left\{-(u/1000)^3\right\}$.

Estimator	Average	Standard Error	Relative Efficiency
ML	0.4994	0.02131	100%
Rank	0.4994	0.02175	96.0%
Min CV	0.4995	0.02147	98.5%

Table 4.25: Simulation results, nonseparable scale model, Batch 1.

Batch 2

We then simulated Batch 2 using $G(u) = \exp\left\{-u/1000\right\}$.

Estimator	Average	Standard Error	Relative Efficiency
ML	0.4983	0.06236	100%
Rank	0.4980	0.06361	96.1%
Min CV	0.4982	0.07903	62.3%

Table 4.26: Simulation results, nonseparable scale model, Batch 2.

Batch 3

We used the lognormal distribution to generate the failure times in Batches 3 and 4. In Batch 3, the value of the location parameter was 6.7 and the value of the scale parameter was 0.35.

Estimator	Average	Standard Error	Relative Efficiency
ML	0.5045	0.02194	100%
Rank	0.5020	0.02471	78.9%
Min CV	0.5022	0.02852	59.2%

Table 4.27: Simulation results, nonseparable scale model, Batch 3.

Batch 4

The value of the parameters of the lognormal distribution used to generate the failure times in Batch 4 were 6.7 for the location parameter and 0.85 for the scale parameter.

Estimator	Average	Standard Error	Relative Efficiency
ML	0.5071	0.05306	100%
Rank	0.5046	0.05783	84.2%
Min CV	0.5062	0.1094	23.5%

Table 4.28: Simulation results, nonseparable scale model, Batch 4.

As expected, all three estimators had small bias, with the minimum CV and rank estimators losing some efficiency when G was the log normal survivor function.

4.4.4 Variance estimates and confidence intervals

Another goal of this simulation study was to check the accuracy of the variance estimates and the coverage of the confidence intervals proposed in this chapter. Through simulations, we looked at the coverage proportions of approximate 95% confidence intervals derived from the quasi-likelihood, minimum CV and rank based methods for the separable scale models of Sections 4.4.1 and 4.4.2, and the rank-based method for the non separable scale model of Section 4.4.3.

Table 4.29 gives the coverage proportions of approximate 95% confidence intervals given by $\hat{\eta}_{QL} \pm 1.96\sqrt{\widehat{\text{Var}}[\hat{\eta}_{QL}]}$, where $\hat{\eta}_{QL}$ is the quasi-likelihood estimator and $\widehat{\text{Var}}[\hat{\eta}_{QL}]$ is the sample estimate of the variance given by (4.24). We used the same samples as the ones used to study the the validity and efficiency of the estimators, i.e. the same 2000 samples of size 102 used in the batches of Sections 4.4.1 and 4.4.2.

In addition to the data in Table 4.29, we observed that the intervals were not biased, i.e. they missed the true value to the left just as often as they missed it to the right, except for the "problem" Batches 3, 4, 7 and 8 for the linear scale model, where the true parameter value was larger than the upper bound of the interval whenever the interval did not include the true value. This tells us that the variance estimate tends to underestimate the true variance in most batches, and that the distribution of the estimator in the "problem" batches is not symmetric, as was shown in Figure 4.4 for the rank-based estimator.

To derive approximate confidence intervals using the minimum CV method, we used

	Linear scale model	Multiplicative scale model
Batch 1	93.75%	94.35%
Batch 2	92.70%	93.85%
Batch 3	91.70%	94.75%
Batch 4	84.35%	93.45%
Batch 5	93.10%	93.40%
Batch 6	91.65%	92.95%
Batch 7	91.40%	93.80%
Batch 8	84.35%	93.45%

Table 4.29: Coverage proportions of approximate 95% confidence intervals, quasi-likelihood estimator, Batches 1-8 of Sections 4.4.1 (linear scale model) and 4.4.2. (multiplicative scale model).

the percentile bootstrap method described by Efron and Tibshirani (1993). This method being rather computationally intensive (for each sample, we had to generate 1000 bootstrap samples), we limited ourselves to the first 1000 samples of each batch, rather than the full 2000 samples, and we only carried out the simulations for the eight batches of Section 4.4.1, the minimum CV estimator having a closed form solution under linear time scale models. The results are summarized in Table 4.30.

	Linear scale model
Batch 1	94.0%
Batch 2	91.1%
Batch 3	94.4%
Batch 4	92.6%
Batch 5	92.3%
Batch 6	91.8%
Batch 7	92.9%
Batch 8	90.6%

Table 4.30: Coverage proportions of approximate 95% confidence intervals, minimum CV estimator, Batches 1-8 of Sections 4.4.1 (linear scale model).

The results in Table 4.30 show that the coverage of the percentile bootstrap intervals is lower than 95%. However, these percentile intervals are rather crude and can easily be improved upon if one is willing to pay the computational price. Efron and Tibshirani (1993, Chapter 14) describe the BC_a method, which is an improvement of the percentile bootstrap method. It is extremely computationally intensive, and therefore its coverage is rather difficult to assess through a simulation study.

We also looked at coverage proportions of 95% confidence intervals obtained using the rank-based score. These intervals were derived using two versions of formula 4.31: one in which the interval was $\{\eta : U^2(\eta)/V(\eta) \leq \chi^2_{1;0.95}\}$ and one in which the interval was $\{\eta : U^2(\eta)/V(\hat{\eta}) \leq \chi^2_{1;0.95}\}$. The results are summarized in Table 4.31.

	Linear scale model		Multiplicative scale model	
	using $V(\eta)$	using $V(\hat{\eta})$	using $V(\eta)$	using $V(\hat{\eta})$
Batch 1	94.25%	94.75%	94.65%	94.90%
Batch 2	94.25%	96.55%	94.65%	94.90%
Batch 3	94.35%	55.25%	94.95%	95.20%
Batch 4	94.35%	47.70%	94.95%	95.20%
Batch 5	94.15%	95.20%	93.75%	94.55%
Batch 6	94.15%	96.50%	93.75%	94.55%
Batch 7	94.65%	55.40%	94.50%	95.00%
Batch 8	94.65%	47.70%	94.50%	95.00%

Table 4.31: Coverage proportion of 95% confidence intervals, rank-based estimator, Batches 1-8 of Sections 4.4.1 and 4.4.2.

Both types of rank-based confidence intervals seem to perform relatively well. From this small simulation study, we have no evidence that the intervals based on $V(\hat{\eta})$ are more appropriate than the intervals based on $V(\eta)$ in any case, except in the "problem cases" 3, 4, 7 and 8, where the coverage of the intervals based on $V(\eta)$ looks more stable and accurate. The bad coverage of the intervals based on $V(\hat{\eta})$ for those "problem batches"

is due to the fact that in a high proportion of the simulated samples (see histogram in Figure 4.4), the estimated value of η was 1.0, which is far from the true value of 0.5, hence the bad coverage of the intervals.

The rank-based intervals also have good coverage under the non separable scale model of Section 4.4.3. We give simulation results for this model in the next section on censored data simulations.

4.4.5 Censored data simulations

According to Lin and Ying (1995), the rank based method should also work for censored samples, as long as the real times at failure and the censoring real times are independent, given the usage paths. We simulated samples from a few of the models studied above, but we introduced censoring.

More precisely, we generated censoring real times independently of the failure times. We simulated the censoring times in the real time scale as censoring is more likely to happen in real time in most applications. The censoring times were normally distributed, with the mean and variance chosen so as to achieve specified censoring proportions. We simulated the same samples as the ones used in Batch 1 for both the linear and multiplicative scale models (Sections 4.4.1 and 4.4.2), but we censored those samples using normally distributed censoring real times. We did the same for the samples used in Batch 1 of the non-separable scale model of Section 4.4.3. The simulation results are summarized in Table 4.32.

It is also of interest to know if the coverage of the confidence intervals based on

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[1]These bad results are due to an important probability mass at $\hat{\eta} = 1$, i.e. the rank estimator identifies y as the ITS in a large proportion of samples.

	Linear scale		Mult. scale		Non-separable scale	
Censoring %	$\tilde{\eta}$	std. dev. $\tilde{\eta}$	$\tilde{\eta}$	std. dev. $\tilde{\eta}$	$\tilde{\eta}$	std. dev. $\tilde{\eta}$
0%	0.5004	0.03193	0.5005	0.02210	0.4991	0.02178
20%	0.5027	0.04271	0.5022	0.03179	0.5003	0.02933
60%	0.5154	0.1015	0.5046	0.05422	0.5871^1	0.1867^1

Table 4.32: Simulations results, censored samples.

the rank score function is good for both censored and uncensored samples. Table 4.33 summarizes the results of big batches of 10,000 simulations of samples of size 102 from the models used in Batch 1 of Sections 4.4.1 (linear scale model), 4.4.2 (multiplicative scale model) and 4.4.3 (non separable scale model).

	Linear scale		Mult. scale		Non-separable scale	
censored	using $V(\eta)$	using $V(\tilde{\eta})$	using $V(\eta)$	using $V(\tilde{\eta})$	using $V(\eta)$	using $V(\tilde{\eta})$
0%	94.58%	95.19%	94.90%	94.92%	94.88%	95.58%
20%	94.78%	94.74%	94.85%	95.17%	95.01%	95.04%
60%	94.95%	92.74%	94.21%	94.97%	95.63%	89.87%

Table 4.33: Coverage of rank score based 95% confidence intervals, censored samples.

From this small study, we see that the rank based method still gives good results when we have censored data. The coverage of the 95% confidence intervals was always very close to 95%, except when we used $V(\tilde{\eta})$ in highly censored samples, in which case the confidence intervals derived using $V(\eta)$ were still very accurate. If we consider the coverage proportions for all the simulations done with the rank-based method, it is probably safer to use the intervals computed using $V(\eta)$ rather than $V(\tilde{\eta})$; the former may give a coverage slightly lower than 95% in most situations, but their coverage was never worse than 93.75%, compared to some coverages that were as low as 47.70% or

55.25% for the latter.

4.4.6 Simulation summary

From these simulations, we can reach the following general conclusions:

1. The precision in the time scale parameter estimators increases with the variability in the observed usage paths;

2. The precision in the time scale parameter estimators increases when the coefficient of variability of the distribution G decreases;

3. Under the linear scale model, the properties of every estimator are seriously affected by changes in G or in the path distribution;

4. Under the multiplicative scale model, the precision of the estimators is still affected by changes in G or in the path distribution, but to a lesser extent;

5. The rank based method seems to be the semiparametric method of choice. The coverage of the confidence intervals was always good and the relative efficiency of the estimators was always high under all choices of G. Moreover, it can readily handle censoring, which is not the case with the other semiparametric methods studied.

From these conclusions, the first recommendation that stands out is to get samples with as much variability in the usage paths as possible. As we have seen, a sample size as large as 100 was nowhere near enough for precise inference with the linear scale when there was not much variability in the sample paths. However, when the variability in the sample paths is large, samples of size as small as 30 can lead to precise inference, as

will be seen in the data analyses of Chapter 6. If one wishes to design an experiment in order to identify an ITS, one must make sure that units will be run under as many different usage patterns as possible; trying to derive an ITS from a sample of items used under very similar conditions is useless. This implies that the use of collapsible models for inference is more appropriate when the data are from populations with heterogeneous usage histories, rather than populations where the units are all used in an homogeneous way. We give examples of those two types of populations in Chapter 6.

As far as the choice of inference method is concerned, the rank based estimator seems to be the most robust semiparametric estimator considered in this chapter, and this at a very low cost in efficiency. It did not lose much efficiency under changes in the value of the parameters of G, and when the form of G changed, it seemed to be the least affected of the estimators. In the few cases in which this estimator broke down, the other methods (including parametric ML) were also doing quite poorly, as precise inference was not possible. In addition, the rank based estimator readily handled censored samples, in which case it still had small bias. Moreover, the confidence intervals derived from the rank based estimator score function showed accurate coverage.

Chapter 5

Some other methods and applications

In this chapter, we briefly address a couple of issues that have been left out of our treatment of alternative time scales up to this point. More precisely, we consider nonparametric estimation of the time scale function, $t_p(x)$, in Section 5.1 and we look at potential applications of collapsible models in Section 5.2. Hopefully, further research will lead to a more thorough development of these methods and applications.

5.1   Ideas for nonparametric inference

A discussion of nonparametric estimation methods for the time scale function $t_p(x)$ is virtually absent from the literature. It would however be very useful as we do not know before hand which form of ITS is more appropriate for a given problem. In this section, we give some insight on how contours of the function $t_p(x) = \Phi[y_p(x)]$ could be traced from a sample sufficiently large. The idea remains the same as for semiparametric estimation:

exploit the fact that the distribution of the failure time in the ITS is the same along each usage path.

5.1.1 Method based on grouped usage paths

If we are lucky enough to have many observations from only a few sample paths, what we can do is link the corresponding quantiles of $X|\mathcal{P}$ from path to path and draw non-parametric age curves this way. We can also use this idea if we have several usage paths; all we need then is a method to group paths that are similar together and treat all the usage paths within a group as a single path. Here is the algorithm that describes this idea in more detail:

Algorithm 5.1

1. *Split the data into K groups so that observations with similar usage paths are in the same group.*

2. *For each group, identify a median or a mean path and then treat all the observations within the group as if they came from that median/mean usage path.*

3. *Choose R quantiles $\pi_1, \ldots, \pi_R$ and estimate these R quantiles of X on each of the K median/mean paths.*

4. *For each of the R quantiles, link the corresponding quantile of X from all K median/mean paths using segments, planes or hyper-planes (depending on the dimension of the problem). These segments, planes or hyper-planes trace "quantile (age) surfaces" that are similar to the elevation lines that we find on topographical maps.*

5. *These quantile (age) curves should have roughly the same form as the ITS.*

6. *The estimate of Pr $[X \geq x|\mathcal{P}]$ is then found by first finding to which of the K*

groups of paths $\mathcal{P}$ belongs, then by looking between which age surfaces x falls on the

median/mean path of that group, and finally by interpolating between the values of

the probabilities corresponding to those quantile surfaces.

Obviously, the properties of the estimators of Pr $[X \geq x|\mathcal{P}]$ obtained by applying Algorithm 5.1 are not trivial to derive. Nevertheless, we can still apply this algorithm to get a rough estimate of those probabilities, but more importantly, to visualize the form of the ITS. Let us take a look at a simple example.
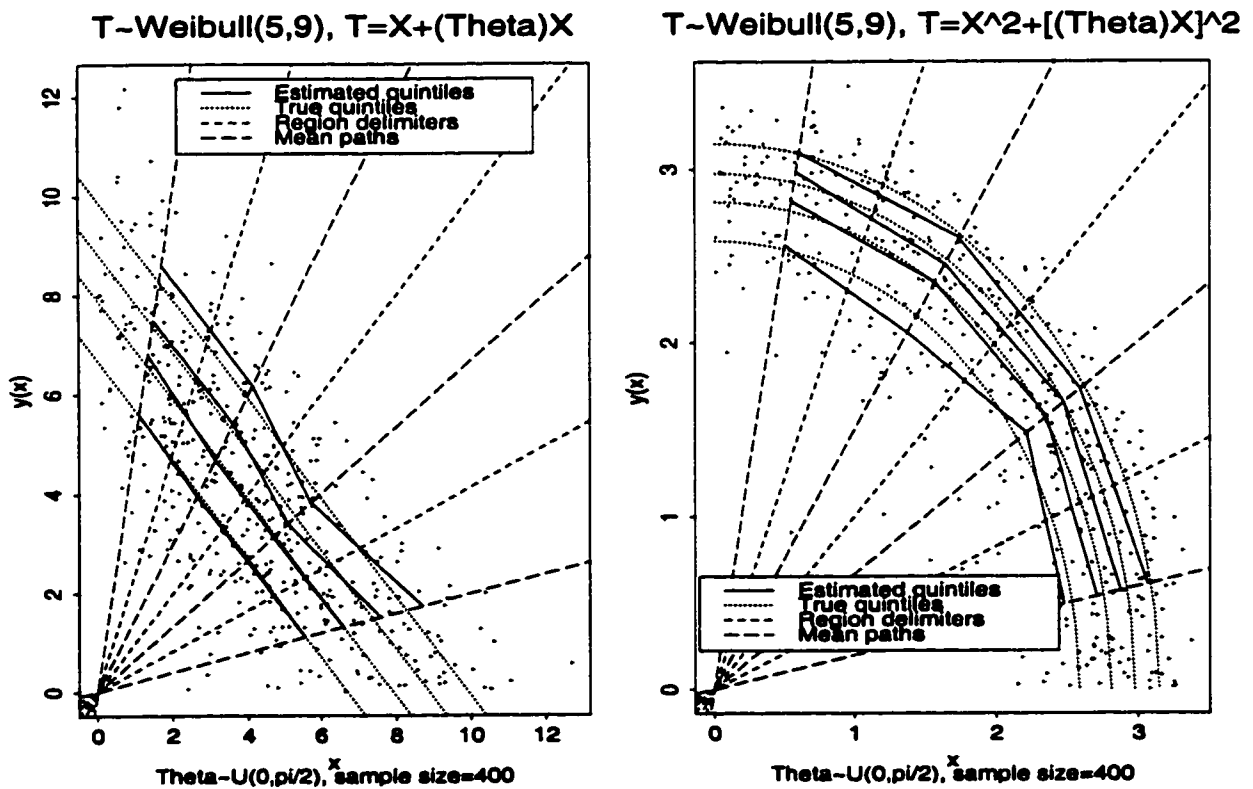
Figure 5.1: Age surfaces drawn using Algorithm 5.1.

Example 5.1.1 *Suppose we observe failure times coupled with one usage measure $y(x) = \theta x$ for each individual. We would like to know if a TS of the form $\Phi[x, y(x)] = x + \eta y$ is ideal. Using Algorithm 5.1, we can split the positive quadrant into, say, 4 "slices" (see Figure 5.1), treat all the data as if they were coming from the usage path at the center of their respective slice, and link the quintiles from slice to slice with segments. The first plot in Figure 5.1 shows this method applied to data simulated from $G(x + y)$, where $T = X + y(X)$ is generated from G, a Weibull distribution, and where $\Theta = \arctan[y(X)/X]$ comes from a uniform distribution on $(0, \pi/2)$. The second plot shows the same method applied to data coming from the same model except that now $T = X^2 + y(X)^2$. As we can see, the age curves on both plots roughly agree with the form of their corresponding ITS.*

The method described in Algorithm 5.1 needs refinement. Perhaps smoothing the age curves using polynomials or other means could help the efficiency. However, no matter how good the method is, it will always require substantial variability between the usage paths and a large number of observations in order to lead to sensible age curves.

5.1.2 Nonparametric quantile regression

Suppose that we have a pair of real random variables (X, Y), with

$$\Pr[Y \leq y | X = x] = F_x(y). \tag{5.1}$$

Nonparametric quantile regression consists in estimating the quantiles of F_x for all x without making any parametric assumptions about $F_x(\cdot)$. This problem has been considered by various authors in different frameworks. Good references on the topic include Lejeune and Sarda (1988), Dabrowska (1987), Bhattacharya and Gangopadhyay (1990)

and Cole and Green (1992).

In our collapsible model setup, we are interested in estimating the function $\Phi[x, y(x)]$ in

$$\Pr[X > x|\mathcal{P}] = G(\Phi[x, y(x)]).$$

If $y(x)$ is a parametric path of the form $y(x) = \theta x$, then our problem simplifies to nonparametrically estimating the quantiles of $\Pr[X > x|\theta]$. This is equivalent to the nonparametric quantile regression problem.

Unfortunately, the methods described in the literature exhibit some serious edge effects that deny us the possibility of identifying the form of the ITS. However, by the very principles of the quantile regression methods, there is reason to believe that proper fine tuning of these methods should lead to good nonparametric estimates of the ITS function, at least in the parametric usage path context.

Already, some methods have been proposed in the literature to remove edge effects in nonparametric regression. For example, Hall and Wehrly (1991) propose a geometric method for removing edge effects from kernel-based nonparametric regression estimators. Such methods do not quite apply to our problem, but they are a step in the right direction. Perhaps an hybrid of one of those nonparametric quantile regression methods with Algorithm 5.1 could lead to good estimates of the ITS function.

5.2 Applications of the collapsible model

Up to this point, the focus of this thesis has been on inference methods for collapsible models. In this section, we want to outline a few potential uses and applications of these models. We will assume that a collapsible model has been completely specified and we

will see how it can be used to solve practical problems such as the design of a maintenance program or the evaluation of warranty costs.

Unlike what was the case with inference methods, prediction using a collapsible model requires the specification of a probability model for the usage path process. Many such usage path models have been proposed in the literature, and we refer the reader interested in usage path modelling to Singpurwalla and Wilson (1993), Murthy, Iskandar and Wilson (1995), Singpurwalla (1995) or Jewell and Kalbfleisch (1996).

5.2.1 Maintenance programs

The great advantage of collapsible models is that they give a relatively simple way of comparing units used under quite different conditions (see Kordonsky and Gertsbakh (1993), Kordonsky and Gertsbakh (1994), Reinertsen (1996) or Gertsbakh and Kordonsky (1997)). At any given clock time, one can compare two machines that have been used differently simply by computing their age in the ITS. The advantage of collapsible models in this case is that they allow for an easy calculation of the age in the ITS.

Suppose we wish to schedule an inspection at the first quartile of the lifetime for a population of devices used under different conditions. Ideally, we would like a single inspection program that would be valid under all possible usage scenarios. Assume that the linear scale $x+5y(x)$ is an ITS and that the 25th percentile of the lifetimes in this scale is 100 units. Then for all the devices, the inspection should be held when $x+5y(x) = 100$. Kordonsky and Gertsbakh (1994, 1997a) show how a linear combination of alternative time scales can be used to derive maintenance programs.

Of course, it is possible to design such a maintenance program without a collapsible

model. As long as a model for $\Pr[X \leq x|\mathcal{P}]$ is available, all one has to do is monitor the value of $\Pr[X \leq x|\mathcal{P}]$ for each device and then proceed with the inspection when this probability reaches 0.25. However, device owners may have a harder time to calculate this probability than the age of the device in a simple time scale. Moreover, calculation of $\Pr[X \leq x|\mathcal{P}]$ may require all the usage information $\mathcal{P}(x)$, whereas calculation of the age in a collapsible models only requires the values of the usage measures at time x.

5.2.2 Prediction

Many applications of collapsible models involve the prediction of time to failure. When the usage path $\mathcal{P}$ is entirely known, this problem is straightforward. Suppose that $\Pr[X > x|\mathcal{P}] = G[t_\mathcal{P}(x)]$ for some (not necessarily collapsible) ITS $t_\mathcal{P}(x)$. Then

$$\begin{aligned} \Pr[X > x + s | X > s,\ \mathcal{P}] &= \frac{\Pr[X > x + s|\mathcal{P}]}{\Pr[X > x|\mathcal{P}]}, \quad \forall x,\ s > 0 \\ &= \frac{G[t_\mathcal{P}(x + s)]}{G[t_\mathcal{P}(x)]}. \end{aligned}$$

When $\mathcal{P}$ is unknown, a probability model for the usage process, $\{Y(x);\ x \geq 0\}$, is necessary. Indeed,

$$\begin{aligned} \Pr[X > x] &= E_{\mathcal{P}(x)}\{\Pr[X > x|\mathcal{P}(x)]\}, \quad \forall x > 0 \qquad (5.2) \\ &= E_{\mathcal{P}(x)}\{G[t_{\mathcal{P}(x)}(x)]\}, \end{aligned}$$

where the expectation is taken over all possible usage histories up to time x.

If $t_\mathcal{P}(x)$ is collapsible in $y(x)$, i.e. $t_\mathcal{P}(x) = \Phi[y(x)]$, the problem is greatly simplified as the expectation in (5.2) is now taken over all the possible values of $Y(x)$ rather than

over all the possible usage histories up to time x:

$$\Pr[X > x] = \mathbb{E}_{\mathcal{P}(x)}\{\Pr[X > x|\mathcal{P}(x)]\}, \quad \forall x > 0$$

$$= \mathbb{E}_{\mathcal{P}(x)}\{G[t_{\mathcal{P}(x)}(x)]\}$$

$$= \mathbb{E}_{\mathcal{P}(x)}\{G(\Phi[Y(x)])\}$$

$$= \mathbb{E}_{Y(x)}\{G(\Phi[Y(x)])\}.$$

In the case in which we observe a unit and its usage history up to time x and we wish to know the probability that this unit will survive up to time $x + s$, the probability is given by

$$\Pr[X > x + s|X > x, \mathcal{P}(x)] = \frac{\mathbb{E}_{Y(x+s)}\{G(\Phi[Y(x + s)])|\mathcal{P}(x)\}}{G(\Phi[Y(x)])}, \qquad (5.3)$$

where the expectation is taken over all possible values of $Y(x+s)$ given $\mathcal{P}(x) = \{y(u); 0 \leq u \leq x\}$. If $\{Y(u); u \geq 0\}$ is a Markovian process then (5.3) further simplifies to

$$\Pr[X > x + s|X > x, \mathcal{P}(x)] = \frac{\mathbb{E}_{Y(x+s)}\{G(\Phi[Y(x + s)])|Y(x) = y(x)\}}{G(\Phi[y(x)])}, \qquad (5.4)$$

where the expectation is now over all possible values of $Y(x + s)$ given $Y(x) = y(x)$.

5.2.3 An example of application: the warranty cost

A lot of attention has been devoted to methods for calculating warranty costs and warranty reserves in the literature. Singpurwalla and Wilson (1993) survey the basic principles underlying warranty cost calculations. In principle, calculating a warranty cost is fairly straightforward; all that is needed is a warranty region and a probability model to

compute the probability that a given item fails in that warranty region. In this section, we see how a collapsible model can be used to derive such a probability model.

Suppose that we only observe one usage measure, say $\mathcal{P} = \{y(x) : x \geq 0\}$, and that the warranty region is a compact set in the positive quadrant of $I\!R^2$, say W. Further, assume that $\Phi[x, y(x)]$ is an ITS, i.e.

$$\Pr[X \leq x | \mathcal{P}] = G(\Phi[x, y(x)])$$

for some distribution function G. Then the probability of failure in the warranty region is given by

$$\Pr[(X, Y(X)) \in W] = \int\!\!\!\int_W f(x, y)\, dx dy, \tag{5.5}$$

where

$$f(x, y) = \left[\frac{d}{dx} G(\Phi[x, y])\right] f_{Y(x)}(y),$$

$f_{Y(x)}(y)$ being the pdf of the value of the usage measure at time x.

If the boundary of W happens to be parallel to $\Phi[x, y(x)]$, i.e. $W = \{x, y : \Phi[x, y] \leq k\}$ for some positive constant k, then probability (5.5) simply becomes

$$\Pr[(X, Y(X)) \in W] = \Pr[\Phi[X, Y(X)] \leq k] = G(k). \tag{5.6}$$

For example, if we have an ITS of the form $\Phi[x, y(x)] = x + 10y(x)$ with distribution function $G(u) = 1 - \exp(-u \ln \lambda)$, and a warranty region of the form $x + 10y(x) \leq k$, then the probability that an item fails while under warranty is given by $1 - (1/\lambda)^k$.

A collapsible model will therefore not necessarily lead to easier warranty cost calculation, unless the warranty region is parallel to the ITS. However, if a collapsible ITS is

available, it could be used to design simple warranties that would lead to easy computation of warranty cost and reserve. Moreover, such warranties would be fair, in the sense that no matter how the devices under warranty are used, the warranty extends to the same proportion of the lifetime of the device for every user.

5.2.4 Accelerated testing

Another potential application of ITSs is in accelerated life testing. Nelson (1990, Chapter 10) and Meeker and Escobar (1998, Chapters 18-20) describe such procedures. Basically, accelerated life tests (ALT) consist in testing devices under usage conditions much harsher than "normal" usage in order to observe several failures during a relatively short test period. The attraction of ITS's in such a setup is that results obtained under the ALT can be extrapolated to normal usage conditions because of the independence of the failure times in an ITS and the usage histories.

For example, suppose that for some device, $t = x(1 + 5\theta)$ is an ITS, where x is real time in weeks, say, and θ is some usage rate. Then if we wish to design an ALT that will be equivalent to the first $2/3$ of the life of the devices under normal use, all that is needed is the 66.7th percentile of the distribution of T, the lifetime in the ITS. Suppose this percentile is $t_{0.667} = 51$ and that under normal conditions, a typical usage rate is around $\theta = 0.5$. Then an ALT at accelerated usage rate of $\theta = 10$ would only have to last one week, compared to 14.6 weeks under normal usage conditions. This ALT would actually be equivalent to $2/3$ of the lifetime distribution under any usage rate θ, as long as T is really an ITS.

Such tests are used in several fields. For example in automobile system reliability, test cars are driven under very harsh conditions, such as bumpy and twisty roads, highly

variable temperatures, hard acceleration and braking, etc. for 40,000 miles, say, which is supposed to be equivalent to, say, 150,000 miles under normal driving conditions. It would be interesting to see if simple collapsible models based on the usage conditions described in this example could be derived, and thus obtain equivalent mileages under different road/driving conditions.

Chapter 6

Data analysis and model assessment

In this chapter we analyze datasets available in the literature using some of the methods introduced in Chapters 3 and 4. We also use these datasets to illustrate how various hypotheses underlying collapsible models can be assessed.

Ideally, we would like to start every analysis by drawing nonparametric estimates of the ITS, as in Example 5.1.1. Unfortunately, the few datasets available in the literature are too small to allow for such nonparametric estimation. We can, nonetheless, fit models with different fixed forms for the ITS and assess their fit to the data. We then choose the model that seems to be most appropriate.

6.1 Fatigue life of steel specimens

Kordonsky and Gertsbakh (1995a) considered the lifetime of steel specimens subjected to periodic loading. The data are given in Appendix A.1. The 30 specimens were divided into 6 groups of 5 units each, and the loading program (i.e. usage path) for each group was a periodic sequence of high-level and low level stress cycles. Kordonsky and Gertsbakh

used the number of low stress cycles as their "real time", x, and the number of high stress cycles as their usage measure, $y(x)$. Because of the large number of cycles, the relationship between x and $y(x)$ was well approximated by a straight line through the origin. The slopes of the observed paths varied between 0.05 and 18. This large variability made the estimates of the time scale parameter relatively precise in spite of the small sample size.

The model proposed by Kordonsky and Gertsbakh is the linear scale model $T = X + \eta y(X)$. Using the minimum CV method, they obtain an estimate of η of 6.77. One could deduce from this result that one high stress cycle causes as much damage as 6.77 low stress cycles.

We can estimate η using other methods. Table 6.1 shows the values of estimates of η obtained through other procedures. The maximum likelihood estimators, along with

Method	Estimate	95% confidence interval
MLE, Weibull	6.61	$(5.33, 8.89)$[1]
MLE, lognormal	6.97	$(5.20, 9.67)$[1]
Quasi-likelihood	6.91	$(5.33, 9.75)$
Rank	6.59	$(5.40, 10.11)$
Minimum CV	6.77	$(4.99, 8.77)$[2]

Table 6.1: Linear scale parameter estimates, Kordonsky and Gertsbakh (1995a) data.

their corresponding likelihood ratio intervals, were computed using the method based on the profile log-likelihood of η outlined in Section 3.4. The quasi-likelihood confidence interval was simply $\hat{\eta}_{QL} \pm 1.96\sqrt{\widehat{\mathrm{Var}}[\hat{\eta}_{QL}]}$, where $\hat{\eta}_{QL}$ is the root of the score (4.23) with $h(u) = u$, and $\widehat{\mathrm{Var}}[\hat{\eta}_{QL}]$ is the asymptotic variance (4.24) evaluated at $\eta = \hat{\eta}_{QL}$. The rank

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[1]Likelihood ratio interval.

[2]Bootstrap percentile interval. See Efron and Tibshirani (1993, Chapter 13).

estimator was the minimizer of the square of the score (4.36), and the corresponding confidence interval was the one derived from the score equation, as described in (4.37). Finally, the minimum CV estimator was simply the value of η that minimized the squared sample CV of the $t_i(\eta)$'s, and the corresponding confidence interval was obtained by using the percentile bootstrap with 200 bootstrap samples, as described in Section 4.1.4.

We may argue that a multiplicative scale model, with $T = X^{1-\eta}y(X)^{\eta}$, would be more appropriate here. We can again let the "real time", x, be the number of low stress cycles and the usage measure, $y(x)$, be the number of high stress cycles at time x. For this multiplicative scale model, we obtain the estimates of η with their corresponding confidence intervals given in Table 6.2. These estimates and confidence intervals were computed with the same methods as in Table 6.1, except that the linear scale was replaced by a multiplicative scale in the formulas.

Method	Estimate	95% confidence interval
MLE, Weibull	0.800	$(0.715, 0.885)^3$
OLS (MLE, lognormal)	0.789	$(0.724, 0.854)$
Quasi-likelihood	0.793	$(0.736, 0.851)$
Rank	0.800	$(0.662, 0.930)$
Minimum CV	0.804	$(0.686, 0.896)^4$

Table 6.2: Multiplicative scale parameter estimates, Kordonsky and Gertsbakh (1995a) data.

We can also redo the analysis with the total number of cycles as our "real time" measure. (Note that we did not try this with the linear scale model as it would not have changed anything.) In this case we obtain the results summarized in Table 6.3.

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[3] Likelihood ratio interval.

[4] Bootstrap percentile interval.

Method	Estimate	95% confidence interval
MLE, Weibull	0.539	(0.460, 0.632)[5]
OLS (MLE, lognormal)	0.555	(0.466, 0.643)
Quasi-likelihood	0.553	(0.474, 0.631)
Rank	0.538	(0.450, 0.693)
Minimum CV	0.547	(0.469, 0.643)[6]

Table 6.3: Multiplicative scale parameter estimates, Kordonsky and Gertsbakh (1995a) data.

Now which of these three time scales is more appropriate? If we recall our definition of an ITS, we are looking for a scale such that the failure times in that scale will be independent of the usage paths. We should therefore be able to identify the best of these scales by plotting the failure times in each scale versus some features of the usage paths.

First, let us look at plots of the failure times in the scale $T = X + 6.7y(X)$, with X being the number of low stress cycles. Figure 6.1 shows a plot of the failure times in this scale versus (# of high stress cycles at failure)÷(# of low stress cycles at failure) on the left hand side and a plot of the same failure times against (# of high stress cycles at failure)÷(total # of cycles at failure) on the right hand side. As we can see, the failure times in the linear scale seem to have roughly the same distribution for all values of the path features. We would therefore not dismiss the linear scale from these two plots.

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[5]Likelihood ratio interval.

[6]Bootstrap percentile interval.
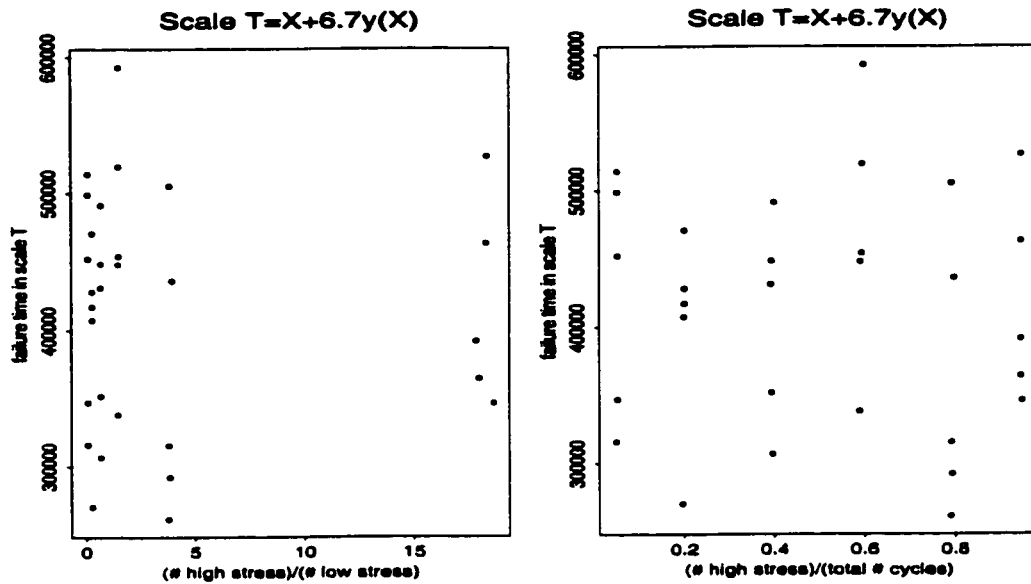
Figure 6.1: Failure times versus path features, scale $T = X + 6.7y(X)$.

From the data given by Kordonsky and Gertsbakh (1995a) we can compute other path features, like for instance the cumulative cycles×amplitude at failure, or the area under the cycles×amplitude curve. If we plot the failure times in the scale $T = X + 6.7Y$ versus these last two path features, as in Figure 6.2, we again cannot dismiss the linear time scale, as there does not seem to be any trend in the plots.
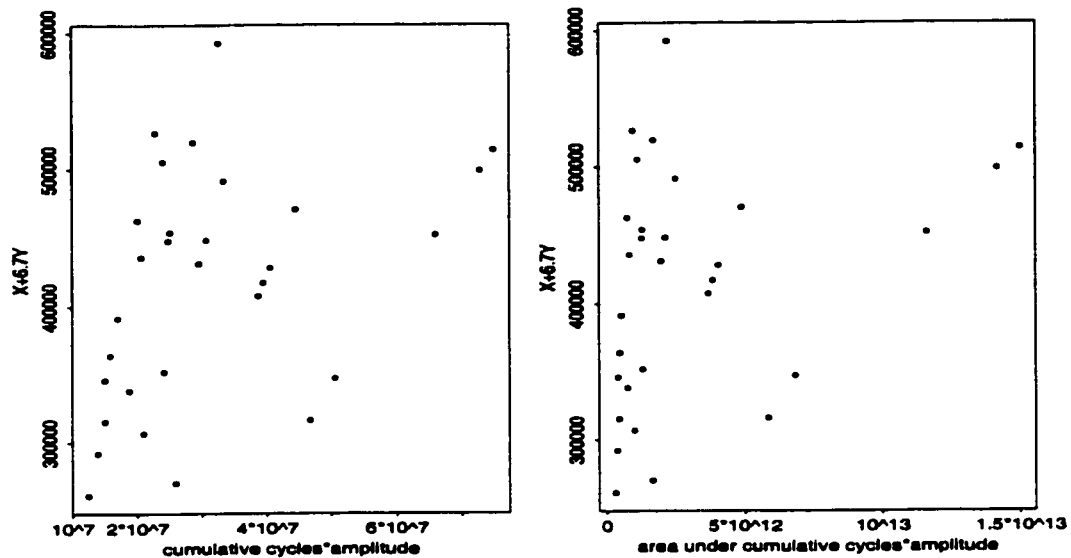
Figure 6.2: Failure times versus path features, scale $T = X + 6.7y(X)$.

Let us now repeat the same procedure with the multiplicative scale $T = X^{0.205}y(X)^{0.795}$, where X is, once again, the number of low stress cycles. The plot on the right hand side of Figure 6.3 shows a definite trend; the failure times for average usage paths look larger than the failure times for extreme usage paths. This means that the multiplicative scale with the number of low stress cycles as measure of real time is not ideal.
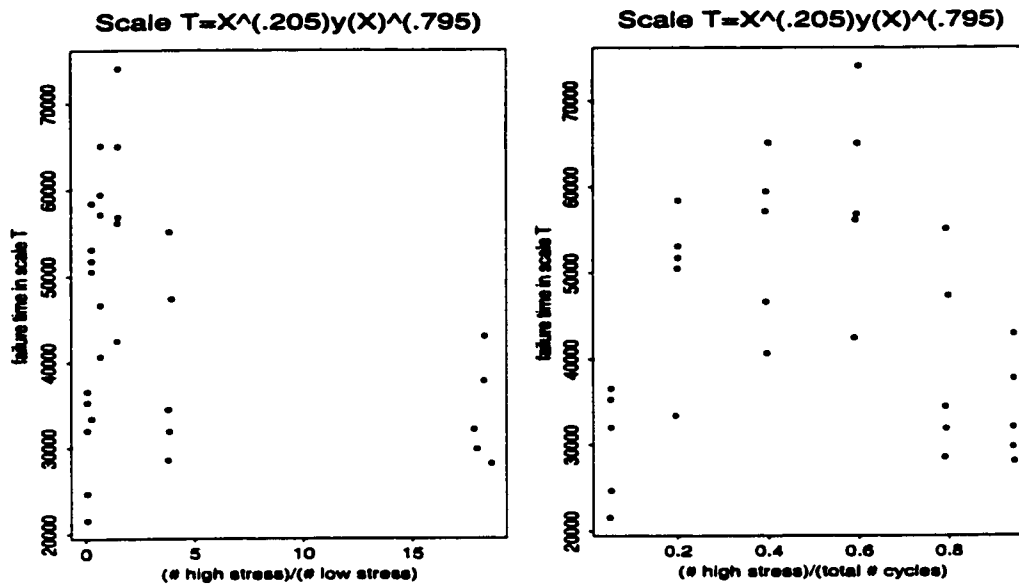
Figure 6.3: Failure times versus path features, scale $T = X^{0.205}y(X)^{0.795}$.

We can draw the same plots but this time with the failure times measured in the scale $T = X^{0.455}y(X)^{0.545}$, where X is the total number of cycles. Figure 6.4 shows that this multiplicative scale seems to be much closer to ideal than the multiplicative scale used in Figure 6.3. Nevertheless, the plot on the right hand side of Figure 6.4 still shows a small parabolic trend similar to the one in Figure 6.3. This means that of the three scales proposed, the linear scale seems to be the one that best fits the data.
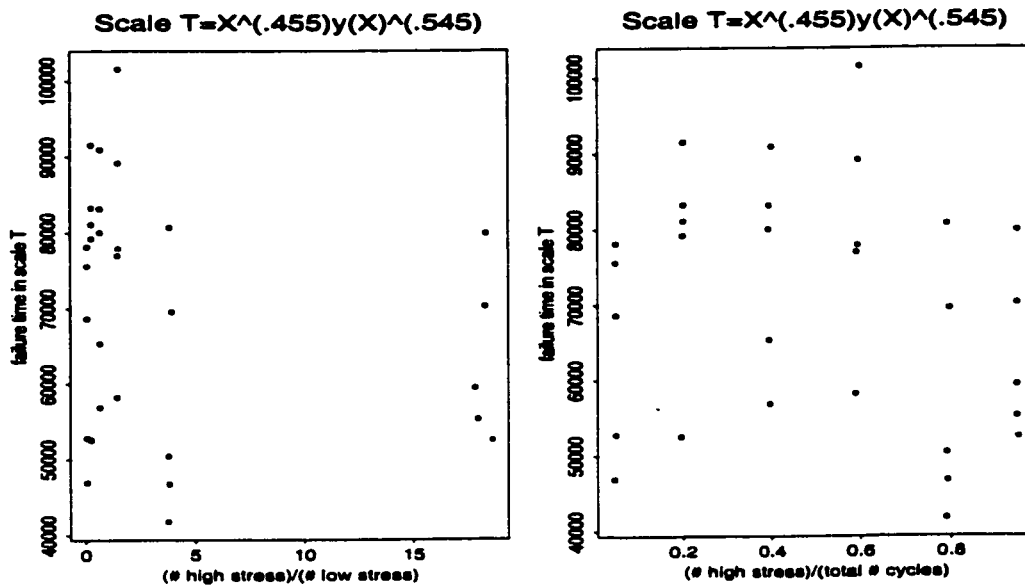
Figure 6.4: Failure times versus path features, scale $T = X^{0.455}y(X)^{0.545}$.

It is worth mentioning that in the case where $y(x) = \theta x$, a plot of $x_i\theta_i^{\eta_1}$ versus θ_i will have the shape of a parabola facing downwards if the true ITS is of the form $x_i(1 + \eta_2\theta_i)$. Similarly, a plot of $x_i(1 + \eta_2\theta_i)$ versus θ_i will have the shape of a parabola facing upwards when the true ITS is $x_i\theta_i^{\eta_1}$. The fact that the plot on the right hand side of Figure 6.3 has the shape of a parabola facing downwards is thus another argument favoring the choice of a linear time scale.

As we mentioned at the beginning of Chapter 4, we can look at probability plots of the failure times in the estimated ITS in order to find a suitable parametric form for G. Let us look at Weibull and lognormal probability plots of the failure times in the scale $T = X + 6.7y(X)$. As we can see on Figure 6.5, there doesn't seem to be any evidence against either distribution.
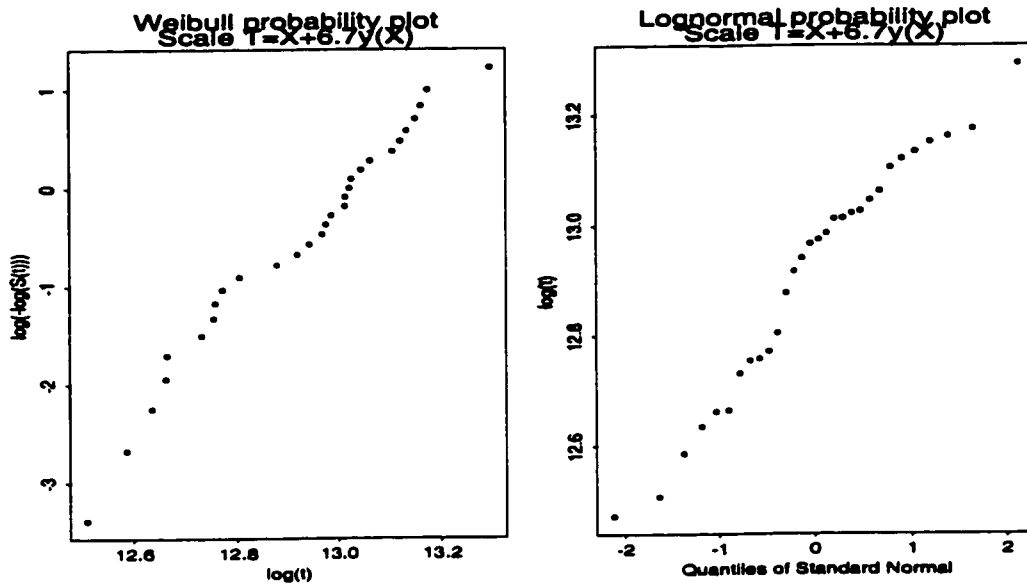
Figure 6.5: Weibull and lognormal probability plots, scale $T = X + 6.7y(X)$.

Such probability plots cannot, however, be used to verify if a time scale is ideal. Figure 6.6 shows Weibull and lognormal probability plots for the times in the multiplicative scale $T = X^{0.205}y(X)^{0.795}$, which, according to Figure 6.3, is not an ITS. As we can see, both plots look "just as straight" as the plots from Figure 6.5, even though this multiplicative scale is not ideal.
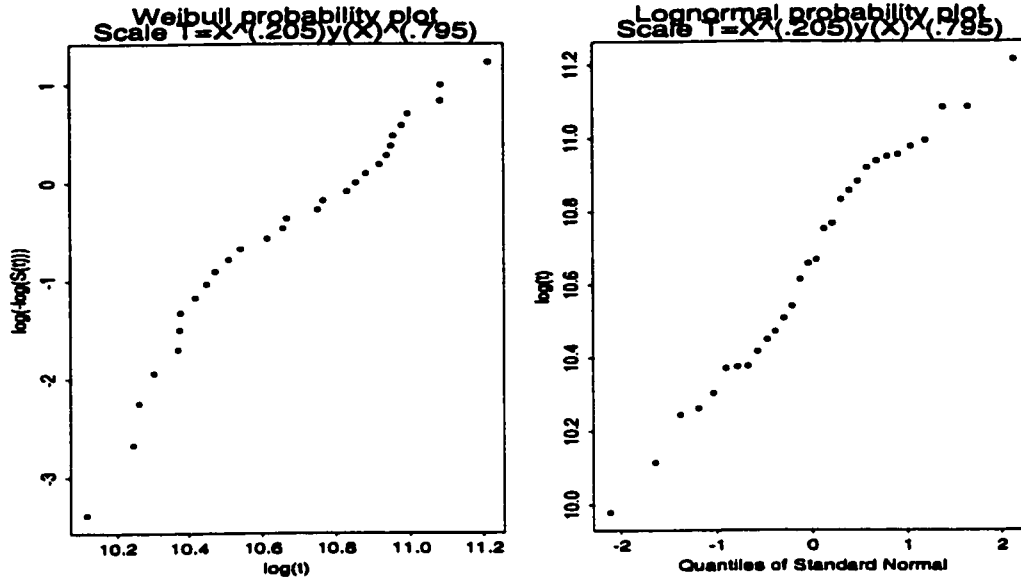
Figure 6.6: Weibull and lognormal probability plots, scale $T = X^{0.205}y(X)^{0.795}$.

To conclude, we can perform a test to verify the fit of the overall model. Suppose we choose the linear time scale $T = X + \eta y(X)$ and the Weibull distribution for G, i.e.

$$\Pr\left[X > x|\theta\right] = \exp\left\{-\left[\frac{x(1+\eta\theta)}{\phi_1}\right]^{\phi_2}\right\}.$$

We get $\hat{\eta} = 6.61$, $\hat{\phi}_1 = 443,190$ and $\hat{\phi}_2 = 5.61$ as our time scale, distribution scale and distribution shape parameter estimates, respectively. We can then divide the ideal time scale in 5 intervals, say $[0, 339,000)$, $[339,000, 393,000)$, $[393,000, 435,000)$, $[435,000, 481,000)$ and $[481,000, \infty)$. We then compute the Pearson chi-squared statistic using the times measured in the scale $X + \hat{\eta}y(X)$ as our observations. We obtain the results summarized in Table 6.4.

The value of the test statistic is 0.574. Because we do not actually observe the

Interval	Expected (E_i)	Observed (O_i)	$(O_i - E_i)^2/E_i$
$[0,\ 339,000)$	5.98	7	0.174
$[339,000,\ 393,000)$	6.00	5	0.167
$[393,000,\ 435,000)$	5.83	5	0.118
$[435,000,\ 481,000)$	6.03	6	0.000149
$[481,000,\ \infty)$	6.16	7	0.115
Total	30.0	30	0.574

Table 6.4: Chi-squared goodness-of-fit test, Kordonsky and Gertsbakh (1995a) data.

value of the times in the ITS, the usual chi-squared approximation cannot be used in order to derive the significance level of this test statistic. We can nonetheless estimate this significance level when needed using the model prescribed by the null hypothesis ($\Pr[X > x|\mathcal{P}] = G[x + \eta y(x)]$ with G Weibull) and the parametric bootstrap. In order to simulate the usage paths, we can use resampling. Note that such computations were not needed here, for the p-value is clearly high with a test statistic value as low as 0.574.

6.2   Fatigue life of aluminum specimen

Kordonsky and Gertsbakh (1997b) analyze data from another fatigue life experiment, but this time on aluminum specimen. The experiment is described by Schijve and Jakobs (1960) and the data are given in Appendix A.2. Kordonsky and Gertsbakh still use a linear combination of two time scales, but this time they propose two different ways of measuring x and y. In the first approach, they define x to be the cumulative number of cycles at amplitudes 1.6, 2.0 or 2.75, and y to be the cumulative number of cycles at amplitudes greater than 2.75 (see Appendix A.2, Table A.2). Using this approach and a linear time scale model of the form $T = X + \eta Y$, we obtain an estimate of η of about

11.5 (Weibull and log normal MLEs and minimum CV). But if we plot the failure times in this scale against the path number, we can see that the failure times seem to have different distributions for different paths. This can be verified by a Kruskal-Wallis test, which leads to a p-value of 0.013. A multiplicative scale does not look more ideal than this linear scale. If we use the model $T = X^{1-\eta}Y^{\eta}$, we get that the estimate of η is 0.53, and again the failure times in this scale look like they have different distributions for different usage paths. The p-value of the Kruskal-Wallis test in this case is 0.009. Figure 6.7 shows plots of the failure times in the two scales mentioned above against the path number.
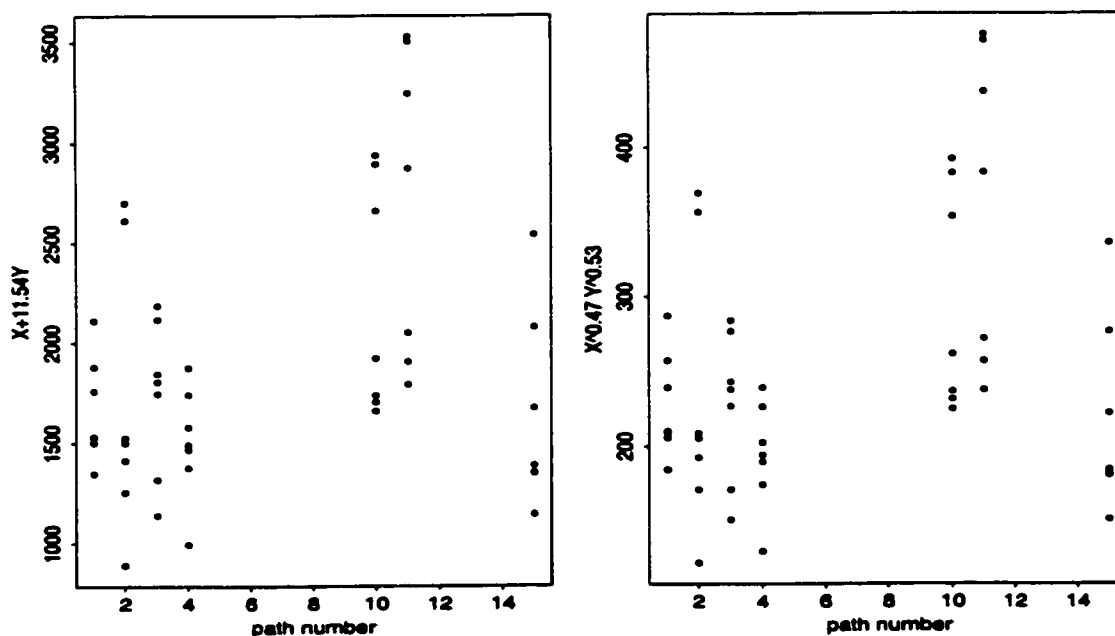
Figure 6.7: Failure times in the estimated scale versus usage path index, Kordonsky and Gertsbakh (1997b) data, first approach.

The second approach proposed by Kordonsky and Gertsbakh is to define x as the number of cycles at amplitude less than or equal to 3, and y as the number of cycles at amplitudes greater than 3. Once again, the results are similar. For the linear scale model $T = X + \eta Y$, we obtain an estimate of η equal to 24.2, and for the multiplicative scale model $T = X^{1-\eta}Y^{\eta}$, the estimate of η is 0.62. The same Kruskal-Wallis tests as above both yield p-values of about 0.003. The plots of the failure times in the two scales versus the path index shown in Figure 6.8 exhibit the same problem as with the first approach.
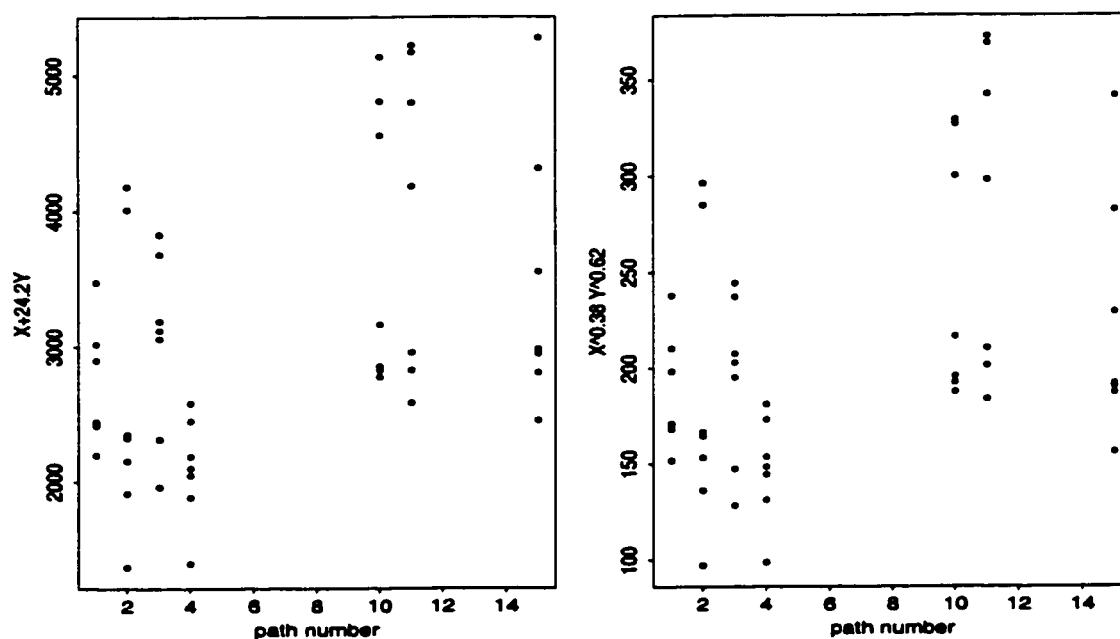
Figure 6.8: Failure times in the estimated scale versus usage path index, Kordonsky and Gertsbakh (1997b) data, second approach.

For both approaches used by Kordonsky and Gertsbakh, we fitted various types of time scales to the data, such as second degree polynomials in x and y, and linear and

multiplicative time scales of transformed x and y. The results were more or less the same. Perhaps x and y as defined by Kordonsky and Gertsbakh do not capture all the information contained in the usage paths that is relevant in order to describe failure.

Another approach that seems to be more natural is to define x as the cumulative number of cycles and y as the cumulative number of cycles×amplitude. Unfortunately, we cannot carry precise inference if we use these new measures. All the slopes y/x in this case are very close despite the different loading programs, as shown in Figure 6.9. This lack of variability means that we cannot precisely estimate the time scale parameters.
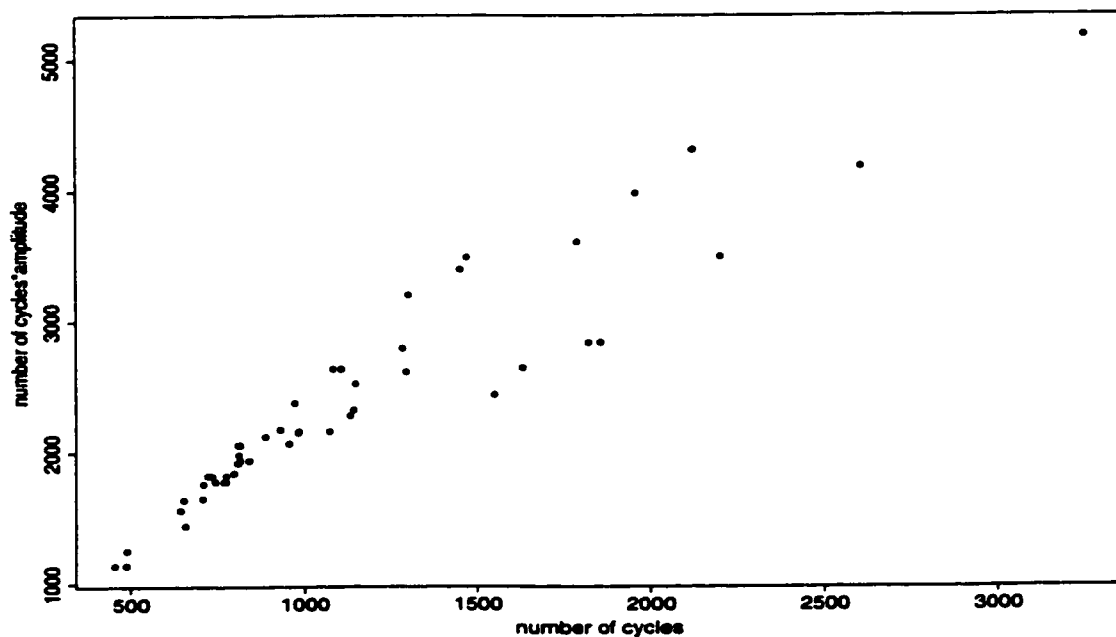
Figure 6.9: Number of cycles×amplitude versus number of cycles at failure.

This example illustrates that even when precise estimation of the time scale parameters is possible, as was the case in the two approaches used by Kordonsky and Gerts-

bakh (1997b), the estimated time scale obtained may not be ideal.

6.3   Cracks in the joint unit of the wing and fuselage

In aircraft reliability, engineers were interested in knowing what caused cracks in the joint unit of the wing and fuselage. Two hypotheses were discussed:

1. the cracks were caused by the high frequency resonance produced by the engine sound wave;

2. the cracks were due to the high stress sustained during landing.

The solution to the crack appearance problem under hypothesis 1. being completely different to that under hypothesis 2., it is obvious why the engineers were interested in finding which hypothesis was true.

Kordonsky and Gertsbakh (1993) consider this problem through a time scale approach. The reasoning of Kordonsky and Gertsbakh is the following: if the cracks are caused by high frequency resonance, then the total flight hours should be the ITS, whereas if cracks are caused by the high stress of landing, then the cumulative number of landings should be the ITS. The data given in Appendix A.3 are used by Kordonsky and Gertsbakh (1993). Because they only observe interval censored data, Kordonsky and Gertsbakh cannot use their usual semiparametric estimator of the time scale parameter. Their analysis thus consists in fitting a parametric distribution to the data under both scales (flight hours and number of landings) separately via maximum likelihood, then choosing the scale that yields the estimated distribution with the smaller coefficient of variation. The number of flight hours gives an estimated distribution with smaller coefficient of variation than

cumulative number of landings for all three parametric families of distributions fitted to the data. Kordonsky and Gertsbakh thus conclude that high frequency is the cause of the cracks in the joint units.

Kordonsky and Gertsbakh (1993) did not consider scales that are a function of both the number of flight hours and the number of landings. Let us assume a linear and multiplicative form for the ITS, i.e. scales of the form $(1 - \eta)(\#$ of flight hours$)+\eta(\#$ of landings$)$ and $(\#$ of flight hours$)^{1-\eta}(\#$ of landings$)^{\eta}$, respectively. If we fit a Weibull and a lognormal distribution using both these scales, we obtain the profile log likelihoods shown in Figures 6.10 and 6.11.
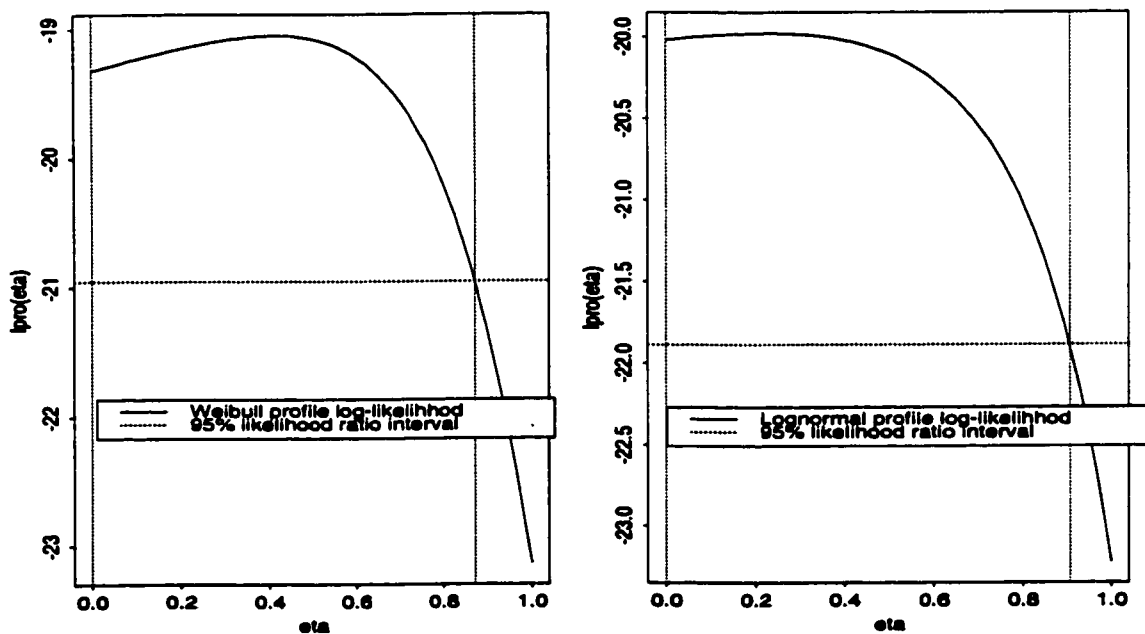
Figure 6.10: Profile log likelihood of the time scale parameter, linear scale model. The plot on the left hand side is assuming a Weibull distribution, the plot on the right hand side, a log normal.
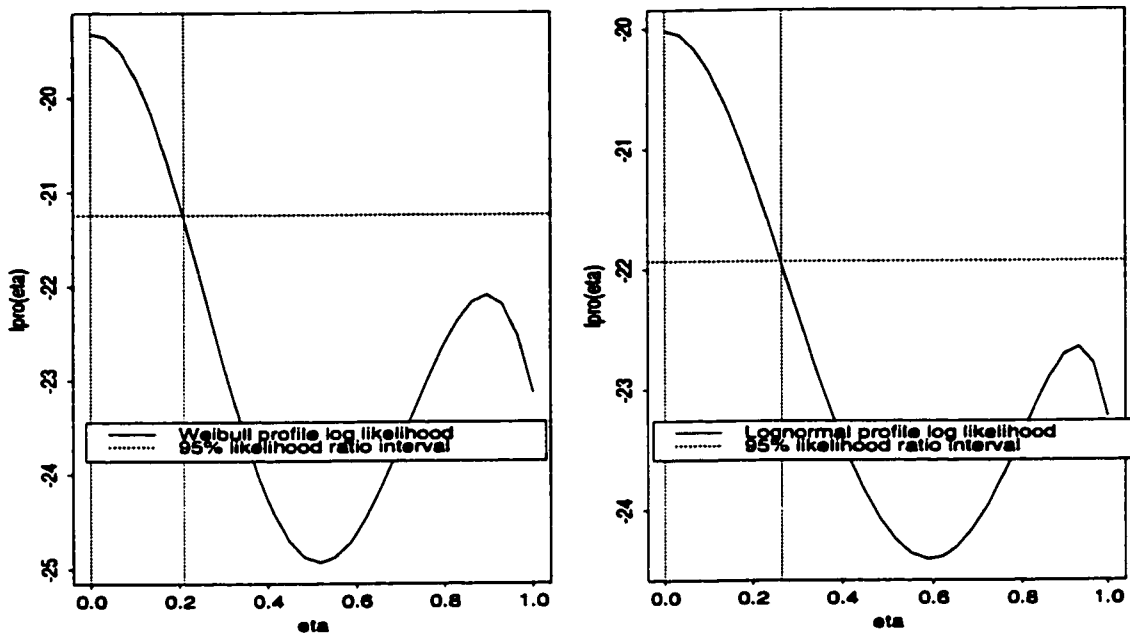
Figure 6.11: Profile log likelihood of the time scale parameter, multiplicative scale model. The plot on the left hand side is assuming a Weibull distribution, the plot on the right hand side, a log normal.

In all cases, it looks like $\eta = 0$ (i.e., the ITS is the number of flight hours) is included in the confidence interval. Moreover, in the multiplicative scale case (the case in which η is scale invariant), the likelihood over $[0, 1]$ is maximized at $\eta = 0$ and the confidence interval is very narrow. This points to the same conclusion as that reached by Kordonsky and Gertsbakh (1993).

6.4 Failure times and mileages for tractor motors

For predictive purposes, Singpurwalla and Wilson (1995) considered a joint model for the time and cumulative mileage at failure for locomotive traction engines. The data are given in Appendix A.4. We use this dataset to illustrate how data on time and cumulative usage at failure do not necessarily allow for precise estimation of an ITS.

If we look at Table A.5, we notice that there is not much variability in the slopes θ. This is even more obvious when we look at the scatter plot of mileage versus time at failure presented in Figure 6.12. Actually, if one fits a simple linear regression model through the origin to these data, one gets an R^2 statistic of 97.6%. This means that mileage and time at failure are almost constant multiples of each other and, therefore, it will be nearly impossible to estimate the ITS parameters no matter what the model is.

For instance, if we look at the score function for the rank-based method with the linear scale model $t_\theta = (1 - \eta)x + \eta\theta x$ sketched in Figure 6.13, we see that the score increases slowly as η increases, rather than increasing or decreasing sharply to cross 0 somewhere between $\eta = 0$ and $\eta = 1$. Choosing a multiplicative scale model does not solve the problem either, as shown on Figure 6.14. In this case the score slowly decreases as η increases, without crossing 0 between 0 and 1.
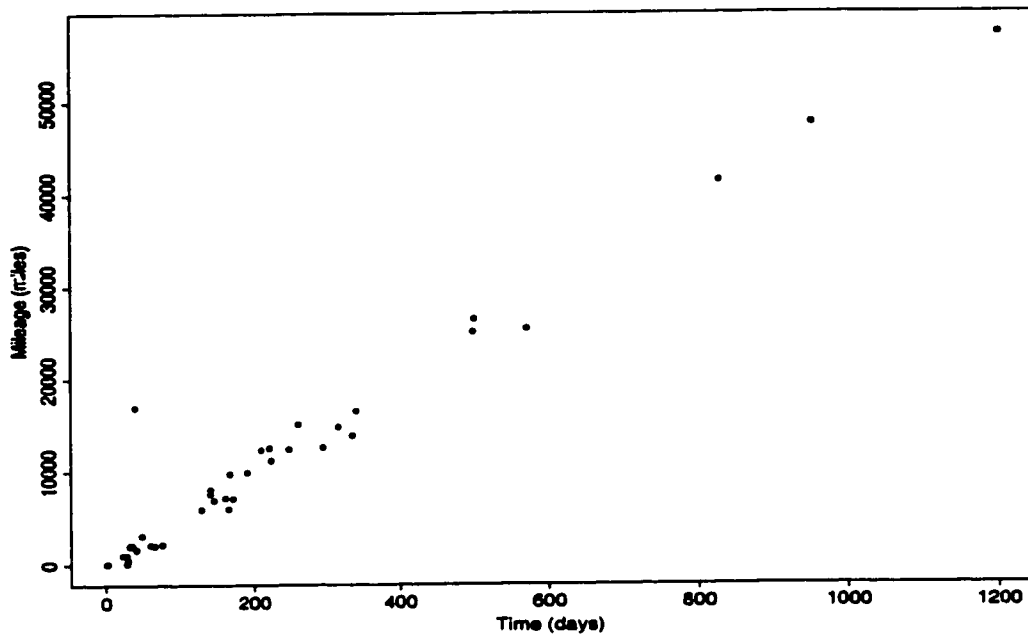
Figure 6.12: Mileage (miles) versus time (days) at failure for the locomotive traction motor data of Singpurwalla and Wilson (1995).
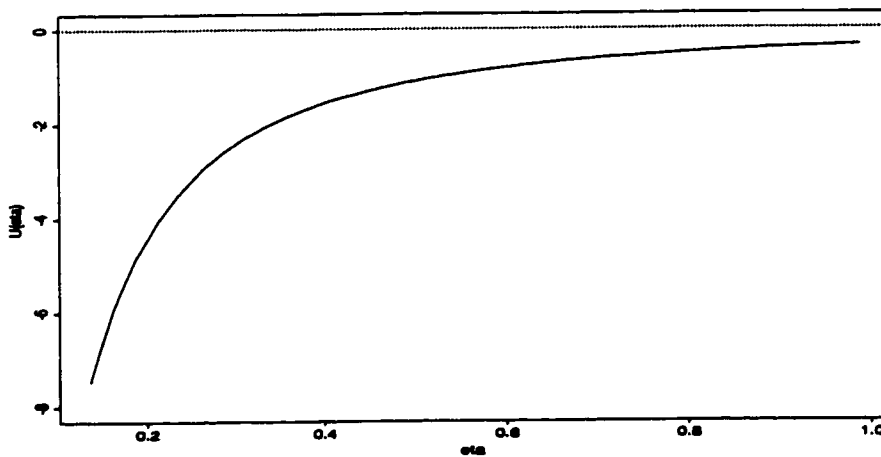
Figure 6.13: Rank score for the linear scale with the locomotive traction motor data.
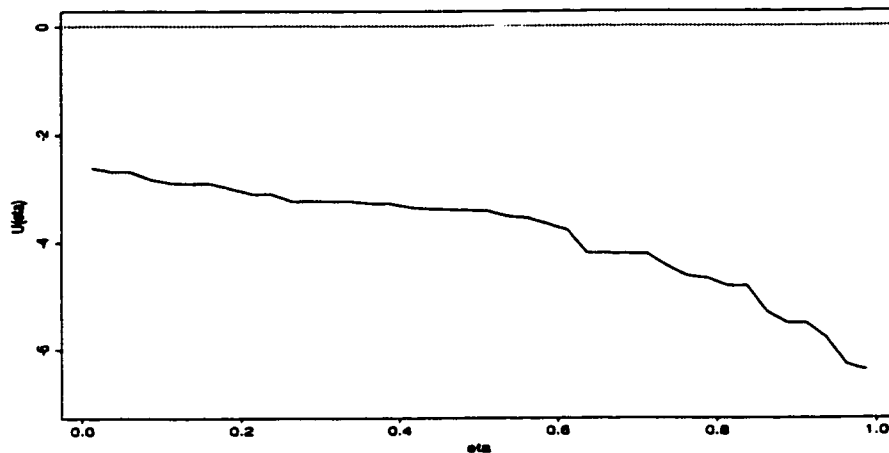
Figure 6.14: Rank score for the multiplicative scale with the locomotive traction motor data.

We get, as our estimate of η, $\hat{\eta} = 1$ with the linear scale model and $\hat{\eta} = 0$ with the multiplicative scale model. In both cases, and for all the different estimation methods, the 95% confidence intervals virtually covered the whole $[0, 1]$ interval. This is not surprising as knowing the value of time at failure is nearly equivalent to knowing the value of cumulative mileage at failure, which means that if one of them is an ideal time scale, almost any function of them also is.

In contrast, Lawless, Hu and Cao (1995) manage to estimate the parameter of their multiplicative time scale quite precisely from a sample of observed time and cumulative mileage at failure. But in their case, the items under study, namely automobile parts, are used at very different rates in the population, whereas the locomotive traction engines studied by Singpurwalla and Wilson (1995) tended to be used at roughly the same rate throughout the population of users.

This example thus confirms that if one wants to derive an ITS from a sample of

failure times and usage measures, one must observe a fair amount of variability in the usage paths.

Chapter 7

Conclusion

As stated in the proposal of this thesis, our main objectives in this research have been to define what constitutes a "good" time scale, sort out the different modeling approaches that are available for dealing with situations involving alternative time scales, and add to the few inference and model assessment methods that have been proposed in the literature.

We have managed to make good progress in all these areas, the main contribution being to semiparametric inference for the time scale parameters. The only method that had been proposed in the framework of alternative time scales prior to this thesis was the minimum coefficient of variation method of Kordonsky and Gertsbakh (1993). We have proposed various ways of carrying semiparametric inference that perform as well if not better than the minimum CV method on the basis of efficiency. Moreover, the semiparametric methods that we have derived are based on the ITS formulation of the model, and asymptotic variances and confidence intervals, which were lacking with the minimum CV method, are now available.

When there is no censoring in the sample, and under the separable scale model, we have proposed the use of a simple quasi-likelihood method that shows good efficiency. Another nice feature of this method is that it is easy to derive a formula for the asymptotic variance of the estimator. The very form of this variance formula sheds some light on the conditions for precise inference in an ITS setup: we need a distribution G of the failure times in the ITS with low coefficient of variation and, most importantly, we need substantial variability in the observed usage paths.

We have also derived what we term a rank-based method, inspired by the work on semiparametric inference for the accelerated failure time model of Lin and Ying (1995). The term "rank-based" originates from the fact that when we are dealing with the separable scale model, the score function for the time scale parameter reduces to the exponential score for rank regression, as described by Kalbfleisch and Prentice (1980) or Cox and Hinkley (1974). Among the three semiparametric methods studied in this thesis, this rank-based method seems to be the method of choice. It is the most flexible method as it needs not be modified when censoring is present, and it can easily handle models from outside the separable scale class. Moreover, the efficiencies observed during the simulation study indicate that this rank-based method performs very well relative to the other semiparametric methods considered.

In addition to deriving methods that were more closely related to the definition of ITS, we have developed new results for the minimum CV estimator. We have shown its consistency in the separable scale framework, and we have proposed a generalized version of the squared sample coefficient of variation that enables us to perform semiparametric minimum CV estimation in the presence of censored observations. We have looked at how accurate simple bootstrap standard errors are in estimating the standard deviation of the

minimum CV estimator. We have also revisited the fully parametric inference framework. We have issued some warnings about the potential for parameter confounding when the conditions for precise inference were not respected.

As data analysis is not complete without model checking, we have proposed model assessment methods for collapsible models. Oakes (1995) proposed a procedure based on the expansion of a fully parametric model to test the form of the ITS. Even though this is a quite sensible approach, it has good power only against a restricted number of alternatives. We have proposed a quite simple visual test that consists in plotting the failure times in the estimated ITS against as many path features as possible. If the estimated scale is indeed ideal, then these plots should show no trend at all. In our data analyses of Chapter 6, we have shown how such plots can be used to discriminate between various potential forms for the ITS. We have also described a way to test the overall fit of the model to the data using a modified Pearson chi squared statistic, with a parametric bootstrap method to estimate the significance level.

As maintenance and warranties are natural applications of this work on alternative time scales, we have outlined some advantages of collapsible models for prediction and warranty cost calculation. The methods outlined in this thesis can also be used to derive models with which to provide maintenance, decisions about warranties and accelerated life tests, as outlined in Section 5.2.4.

There are still many interesting research problems left to be solved in the study of alternative time scales. For instance, we have given ideas on how to carry out completely nonparametric inference in Chapter 5. However, there is still plenty of work left to be done in this area. For instance, one could investigate how an hybrid of the age curve method and a quantile regression method could be built in order to draw a clear picture of the

ITS nonparametrically. There is also the problem of testing the collapsibility assumption without assuming any parametric form for either of the ITS and the distribution of the failure times that has yet to be solved.

Another issue of interest is including more general time-varying covariates that are not necessarily usage measures. For instance, it would be of interest to practitioners working in reliability to include information given by measures of wear into the time scale. The same can be said of practitioners working in biostatistics, who would probably wish to include measures of health status in the time scale. For example, measures of the status of the immune system provide useful information about the progression of the HIV disease, and should therefore be included in a time scale used to predict time to onset of AIDS. Unfortunately, our current methods, that condition over the covariate history, cannot handle such measures because they are internal covariates, i.e. their value can indicate if failure has already occurred or not.

Finally, some methods are still needed in order to systematically identify which covariates should be part of the ITS and which should be left out of the scale and remain treated as covariates. At this stage, selection of the measures that should be part of the time scale is more of a judgment call. This is not too critical if there is already a good understanding of the scientific mechanism which causes failure, but in many cases, the search for an ITS may be due to the lack of such an understanding.

Appendix A

Datasets

A.1 Fatigue life of steel specimen data

Kordonsky and Gertsbakh (1995a) considered the choice of the best time scale to model the fatigue life of 30 steel specimens subjected to periodic loading. The specimens were divided into 6 groups of 5 units each. The usage path for each group was a periodic sequence of $5000(1-\theta)$ high-level cycles of amplitude $q_H = 282$ MPA and 5000θ low-level cycles of amplitude $q_l = 182$ MPA. Table A.1 gives the number of low-level and high-level cycles at failure along with the value of the path parameter, θ, for each specimen.

		Number of cycles				Number of cycles	
Specimen	Path (θ)	Low-level	High-level	Specimen	Path (θ)	Low-level	High-level
1	0.95	256800	13500	16	0.40	32000	45700
2	0.95	238500	11600	17	0.40	48000	70400
3	0.95	370150	19250	18	0.40	42000	61500
4	0.95	335100	17500	19	0.40	42000	60600
5	0.95	380300	20000	20	0.40	54000	80400
6	0.80	153000	38000	21	0.20	10000	37500
7	0.80	176000	44000	22	0.20	16000	62700
8	0.80	160300	40000	23	0.20	12000	45300
9	0.80	156000	39000	24	0.20	19000	72600
10	0.80	103000	25000	25	0.20	11000	42000
11	0.60	84000	54400	26	0.05	3000	53900
12	0.60	81000	52300	27	0.05	3750	68550
13	0.60	90000	59900	28	0.05	4250	77950
14	0.60	57000	37300	29	0.05	3250	57950
15	0.60	66000	42700	30	0.05	2750	51250

Table A.1: Fatigue life of 30 steel specimens.

A.2 Fatigue life of aluminum specimen

Kordonsky and Gertsbakh (1997b) study the data on the lifetime of aluminum specimen given in Schijve and Jakobs (1960). These data consisted in the lifetime of 49 aluminum specimen distributed among 7 different usage paths. The usage paths consisted in an alternance of blocks of cycles at different amplitudes and are described in Table A.2. For each specimen, Table A.3 gives the usage path number, the cumulative number of cycles at failure and the block number at which failure occurred.

	Amplitude						
Path	1.2	1.6	2.0	2.75	3.76	4.67	6.3
1	0	0	96	48	13	5	0
2	0	0	49	26	6.5	0	0
3	0	0	45.6	22.8	6.5	6.5	0
4	0	0	134.5	21.5	6	2	0
10	0	0	47.5	24.3	6.5	2.7	0.6
11	0	81	47.5	24.3	6.5	2.7	0.5
15	163	81.5	47.5	24.3	6.5	2.7	0.5

Table A.2: Description of the 7 different usage paths. Number of cycles per block at each amplitude.

Specimen	Path	Block number	Cycles	Specimen	Path	Block number	Cycles
1	1	4th	643	26	4	6th	955
2	1	5th	768	27	4	7th	1148
3	1	5th	744	28	4	8th	1283
4	1	6th	930	29	10	9th	734
5	1	5th	773	30	10	10th	806
6	1	6th	972	31	10	11th	887
7	1	5th	810	32	10	10th	774
8	2	9th	706	33	10	13th	1107
9	2	6th	488	34	10	13th	1083
10	2	10th	796	35	10	16th	1300
11	2	10th	815	36	11	7th	1142
12	2	11th	840	37	11	7th	1132
13	2	18th	1448	38	11	8th	1294
14	2	18th	1467	39	11	11th	1784
15	3	6th	489	40	11	12th	1956
16	3	6th	453	41	11	13th	2118
17	3	9th	722	42	11	13th	2120
18	3	9th	708	43	15	5th	1549
19	3	8th	651	44	15	5th	1632
20	3	10th	808	45	15	6th	1819
21	3	10th	814	46	15	7th	2201
22	4	4th	656	47	15	6th	1853
23	4	6th	982	48	15	8th	2604
24	4	7th	1073	49	15	10th	3250
25	4	6th	984				

Table A.3: Block number and cumulative number of cycles at failure.

A.3 Cracks in the joint unit of the wing and the fuse-lage

Kordonsky and Gertsbakh (1993) wish to find an appropriate scale in order to model the time of appearance of cracks in joint units between the wings and the fuselage of aircraft. They want to choose the better of the following two scales: cumulative horizontal flight time and cumulative number of landings. In order to do so, they analyze data arising from inspections of 16 aircraft. Each aircraft has two joints and is inspected twice. Table A.4 shows the number of cumulative flight hours at inspection, the cumulative number of landings at inspection and the number of cracked joints at inspection.

	Inspection 1			Inspection 2		
Plane	Flight time	Landings	Cracked joints	Flight time	Landings	Cracked joints
1	8401	2666	0	14691	4102	2
2	8210	2469	0	14496	3680	2
3	6605	1919	0	12898	3448	1
4	N/A	N/A	0	8993	1597	0
5	N/A	N/A	0	7258	1598	0
6	N/A	N/A	0	12809	2971	0
7	N/A	N/A	0	8998	2583	0
8	N/A	N/A	0	8823	1924	0
9	N/A	N/A	0	8370	1850	0
10	7699	1583	1	N/A	N/A	1
11	N/A	N/A	0	8398	2385	1
12	8013	2001	0	14296	3460	1
13	10525	5174	0	14458	5174	2
14	8377	2166	0	14393	4010	1
15	7394	2050	0	13611	3889	2
16	8390	2694	2	N/A	N/A	2

Table A.4: Number of cracks in wing joints and number of landings at inspection.

For example, if we look at aircraft number 1, it had 8401 flight hours, 2666 landings and no crack in either joint at inspection 1, and it had 14691 flight hours, 4102 landings and a crack in both joints at inspection 2.

A.4 Failure times and mileages for tractor motors data

Singpurwalla and Wilson (1995) considered a joint model for the time and mileage accumulated at failure of locomotive traction motors. They used the data provided in Table A.5 in order to estimate the parameters of their model.

| Motor | Path (θ) | At failure | | Motor | Path (θ) | At failure | |
		Days	Miles			Days	Miles
1	58.831325	166.00	9766	21	58.314286	35.00	2041
2	49.767068	249.00	12392	22	52.047368	190.00	9889
3	36.074074	27.00	974	23	38.878049	41.00	1594
4	36.067797	59.00	2128	24	28.773333	75.00	2158
5	50.165919	223.00	11187	25	50.063025	952.00	47660
6	41.274627	335.00	13827	26	36.536585	164.00	5992
7	47.758621	145.00	6925	27	41.635294	170.00	7078
8	53.950000	140.00	7553	28	50.228916	498.00	25014
9	44.448336	571.00	25380	29	52.971944	499.00	26433
10	48.511765	340.00	16494	30	44.762500	160.00	7162
11	46.265625	128.00	5922	31	63.677419	31.00	1974
12	31.230769	65.00	2030	32	56.705882	221.00	12532
13	46.822785	316.00	14796	33	44.500000	22.00	979
14	57.708812	261.00	15062	34	64.437500	32.00	2062
15	432.249808	39.07	16888	35	64.562500	48.00	3099
16	28.000000	1.00	28	36	3.518519	27.00	95
17	42.711864	295.00	12600	37	57.621429	140.00	8067
18	50.090689	827.00	41425	38	52.500000	2.00	105
19	58.861244	209.00	12302	39	15.413793	29.00	447
20	58.831325	166.00	9766	40	47.753333	1200.00	57304

Table A.5: Failure times and mileages of 40 traction motors.

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