Black Holes in Pseudo-Topological Gravity

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners

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Abstract

In the following, we build on previous work done on higher derivative gravity, in particular Lovelock gravity. The latter is a family of theories in higher space-time dimensions in which interactions involving higher powers of curvature are introduced, but the equations of motion remain second order in derivatives. We develop a new theory involving cubic terms in the curvature. We then show that the equations of motion for graviton fluctuations remain second order. The curvature cubed term is shown not to be a topological object, contrary to the belief that dimensionally extended Euler densities provided the only stable dimensionally continued theories of gravity (Lovelock gravity). Black hole solutions are studied in this new gravitational framework.

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For their love and support,

this thesis is dedicated to my family

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Chapter 1

Introduction

At the heart of the modern scientific revolution which occurred over the first half of the 20th century was the construction of the two main pillars of physics that we have since consumed ourselves with the task of reconciling: General Relativity and Quantum Mechanics/Quantum Field Theory. Quantum field theory has given us a picture of physical interactions on subatomic scales where the electromagnetic, strong and weak nuclear forces are the central interactions. The description that arises from quantum field theories of these forces interacting with matter has proved to be extremely accurate [1]. In spite of all of the successes that quantum field theory describing nature through the Standard Model has enjoyed, there is one force that has as yet proven reluctant to fit nicely into this framework, gravity.

This is where the other part of the modern physics revolution comes in to play. General relativity has thus far provided the best description of gravity that we have been able to formulate. This theory provides a remarkably accurate description of physics on astronomical and cosmological scales. However as a quantum field theory, general relativity has been shown to be non-renormalizable, and so it is beyond our current paradigm for quantum physics. However, there are tantalizing suggestions that there will be a full quantum theory of gravity. Some of those clues come in the area of the general relativity that has captured the attention of researchers and lay people alike, the regions of the universe in which gravity is most extreme: black holes. It was only realized nearly forty years after the inception of general relativity that the theory gives rise to regions where gravity is so strong that not even light can escape. Since this realization, a tremendous amount of work has been done on characterizing black holes as they represent the most exotic and confounding objects in the theory. A remarkable discovery by Hawking was that black holes appear to emit thermal radiation from quantum effects [2, 3]. Finding that quantum effects play an important role near black holes provides some suggestion that we may use these objects to explore possible theories of quantum gravity. More over the resulting connection between gravity, quantum physics, and thermodynamics [4] seems to hint that just as statistical physics provides a quantum description of thermodynamic processes a quantum theory of gravity will hold a statistical description of black holes and spacetime.

It is a lofty goal to write down a consistent quantum theory of gravity, and it is one that is unlikely to be achieved in the near future. That is not to say there has not been due attention paid to the problem. There have been scores of ideas to try to join the two wildly successful theories, but there have been precious few that have gained and sustained some traction. Among the ideas for finding a consistent renormalizable quantum theory of gravity was that the Einstein-Hilbert action represents an effective theory of gravity and must be supplemented by interactions involving of higher powers of curvature. Though the idea of adding terms of higher order in curvature to the action is one that dates back to the time around the origins of general relativity [5], serious work on the problem only began in the latter half of the 20^{th} century [6]. The original attempts to formulate a proper quantum theory of gravity this way have been shown to be problematic because while the higher curvature theories may be renormalizable, they suffer a loss of S-Matrix unitarity (see Stelle in [6]). Considering higher curvature theories outside of the context of fixing the ultraviolet divergences of gravity still provides interesting corrections in gravitational theories in higher spacetime dimensions .

The connection of higher order theories of gravity to string theory has generated a considerable amount of interest. The past decade in string theory has been dominated by work concerned with an idea put forth in the late 1990's, the AdS/CFT correspondence. At the heart of this revolutionary idea is that the physics of a string theory including gravity on a bulk ten dimensional space whose non-compact directions are AdS_5 can be replaced by a description in terms of a strongly coupled conformal field theory which lives on the conformal boundary the AdS_5 space[7, 8]. This correspondence was conjectured based on examining the large N behavior of closely stacked D3-branes in the near horizon limit. In this regime, low energy excitations are seen to come in two varieties that are depending on an order of limits. One order yields the strongly coupled field theory states on the D3-branes, while the other gives the long wavelength gravitational modes in the bulk of the D3-brane throat. This holographic description of gravity has enormous power in its usefulness in probing regimes of gauge theories where we previously had a dearth of computational methods. Furthermore, studying black holes in the bulk AdS gravitational theory allows us to study conformal field theories at non-zero temperatures [9, 10].

There has been recent work in exploring the implications of considering a bulk spacetime with higher curvature theory gravity on the dual conformal boundary theory. A particular result coming out of that work has motivated the main thesis project [11, 12]. It had been shown by Kotvun, Starinets, and Son that in the dual thermal field theory for a black hole solution in Einstein gravity using the AdS/CFT correspondence that the ratio of shear viscosity, η, to entropy density, s, satisfies $\frac{\eta}{s} = \frac{1}{4\pi}$ [13]. Further, they conjectured that this should be a lower bound for η/s of any physical system. The KSS bound was shown to hold universally for Einstein gravity coupled to a variety of matter fields. However, it was shown in [11] that for a bulk gravitational theory with curvature squared interactions this bound may be violated. In a string theory context, these calculations are done perturbatively in the coupling of the new interaction [11, 14, 12]. However, this analysis can be extended to finite coupling with Gauss-Bonnet gravity, in which the curvature squared term is given by the Euler density of a four dimensional manifold. In this case, the KSS bound is (non-perturbatively) corrected by the interaction parameter λ as $\frac{\eta}{s} = \frac{1}{4\pi}(1-4\lambda)$. The bound in Gauss-Bonnet gravity is no longer strictly greater than $\frac{1}{4\pi}$ but it is always positive as $\lambda < \frac{1}{4}$. In fact, the authors of [11] argue based on the grounds of causality that $\lambda < \frac{9}{100}$. The fact that a consistent higher curvature theory provides a finite modifications to η/s in the dual thermal field theory has motivated us to ask whether we can further alter the KSS bound by adding a linear combination of scalar monomials of three curvature tensors. It is not clear whether such a consistent theory can be constructed with these new interactions can be treated non-perturbatively. Hence being able to write down a consistent third order theory in five dimensions is the first step in exploring this problem.

In short, the goal of the thesis is to write down the most general consistent bulk theory of gravity that includes terms of curvature cubed. What we would like most to see is that we can do this in five dimensions, specifically AdS_5 since these are related to four-dimensional gauge theories by the AdS/CFT correspondence. Being able to do so we create a theory of gravity with a term that seems to mimic the behavior of a topological invariant in six dimensions. However, we ultimately prove that this is not the case, and hence, we call our new cubic order in curvature theory pseudo-topological gravity. In the five dimensional pseudo-topological theory, we would hope to find a new class of black hole solutions that exist beyond those found for Gauss-Bonnet gravity [15, 16, 17]. After finding these black hole solutions, we will characterize the behavior of the black hole solutions based on their behavior with respect to the new interaction parameter for the higher order terms. More importantly we would like to determine the thermodynamic properties of the solutions, and we will explore the consequences of having solutions with horizons with different topologies [9, 18, 19].

In Chapter 2, we will lay out the foundation for the considerations made in the thesis. We will begin by laying out the basic tools of general relativity that we will need to understand before moving on to more general cases. We consider the extensive work that has been done in studying the nature of black holes. After introducing black holes, we will present work that had been done that exposed their connection to thermodynamics [20, 3, 22, 21].

Later in Chapter 2, we will present previous work that has been done in considering Lovelock gravity. We briefly review results coming from the renewed interest in considering higher curvature gravity that began in the 1980's in connection with string theory. Since its development the work done in Lovelock gravity has mainly focused on the second order theory, Gauss-Bonnet gravity [23, 17, 15]. We will look at some of the major developments like interesting black hole solutions that have come from Gauss-Bonnet gravity with a focus on those in AdS [16, 24].

In Chapter 3, we begin to write down a candidate third order action by assembling all irreducible curvature cubed terms [25, 26]. We discuss a method to determine the coefficients that would give us a term that we consistently could add to the Einstein-Gauss-Bonnet action. We first work in five dimensions and then generalize to an arbitrary dimensional theory. In appendix A, we compare the integrated value of the cubic order term with the integrated six dimensional Euler density on different example six dimensional manifolds to determine if what we have found is a topological invariant. After writing down the D dimensional action, we compute the field equations for the pseudo-topological theory. Finding the field equations allows us then to write down the linearized theory by taking the first variation of these equations. The goal of writing down the linearized theory is to determine if the equations of motion for the graviton contain more than two derivatives. If there are non-vanishing four derivative terms, then the theory would be pathological [23, 17].

In Chapter 4, we go about the process of finding and characterizing possible black hole solutions in pseudo-topological gravity. The calculation involves determining the constraint equation for the metric function for a given ansatz. Solving cubic polynomial constraint shows that there is a rich space of solutions for the metric function. The form of the constraint equation allows us to determine the type of vacua for the theory [17]. Moreover, the constraint equation shows which of the vacua are stable and allow black hole solutions. After finding the black hole admitting vacua, we calculate the explicit forms for the metric function by solving the constraint equation. We then generalize our calculations from planar horizons *i.e.,* 'black brane' solutions, to include black holes with curved horizons.

In Chapter 5, we determine the thermodynamic properties of the pseudo-topological black holes found in Chapter 4. We concern ourselves first with calculating the temperature of the black holes. The next major thermodynamic quantity we examine is the entropy. There are several different approaches to calculating black hole entropy [20, 21, 18], but any method used should yield the same result [27]. Lastly, we determine the free energy of the cubic black holes. We note in an appendix that there are some difficulties with different approaches to calculating free energy, or energy density, of black holes in higher derivative gravity due to some asymptotic ambiguities. In the final chapter we discuss and summarize the work that has been presented in the main text, and we then provide an outlook for future work to be done.

Chapter 2

An Introduction to General Relativity

One of the most important and celebrated developments in physics over the past century was the formulation of the most successful theory of gravity that we have yet been able write down, General Relativity. A key feature of general relativity was that it predicted new phenomena in the universe like cosmic expansion on top of solving contemporary problems such as the anomalous perihelion shift of Mercury. Experimental confirmation in a wide range of tests and observations has firmly entrenched general relativity as the benchmark for accuracy in theories of gravity. The vast importance of general relativity is best seen in its philosophical shift from the previously held Newtonian view of gravity being an abstract force that acts on all massive bodies to one that views gravity as being intrinsically linked to the distortion of geometry spacetime by energy and matter. This view of the nature of gravity has given us a new way to think about the how the universe originated, evolved, and subsequently became populated with different large scale structures. In this section, we review the fundamental concepts of general relativity starting with defining elementary objects used in studying manifolds and ending with a discussion of the basic concepts in black hole thermodynamics

2.1 Mathematical Preliminaries

To begin our review of general relativity, we need to define some of the mathematical machinery behind Einstein's beautiful geometric equations. Here we will provide a very brief review of mathematical topics and the notation that will be used throughout the text. Since general relativity is a theory based on differential geometry, topology and tensor calculus, we will start with a brief discussion of manifolds. A more detail discussion can be found in [22], whose notation and conventions we adopt. In a broad definition, manifolds are topological spaces that in a neighborhood around any point locally look like $\mathbb{R}^n$. A manifold being a topological space allows it to be endowed with a metric and so a metric tensor. The metric tensor g_{ab} is a symmetric tensor that allows us to define inner products between vectors living in the tangent space of the manifold [28]. We can also use the metric tensor to raise and lower indices on other tensors *e.g.,* $A^a = g^{ac}A_c$. The infinitesimal line element for a manifold with coordinates x^a is given by

$$ds^2 = g_{ab}dx^a dx^b. \tag{2.1}$$

We will use the 'mostly positive' $(- + + \ldots +)$ signature for the metric [29]. We will denote the determinant of the metric tensor by g. With the meaning of distance between points on a manifold being defined by the metric, we would like to have a notion of how to move vectors, and in general tensors, around on the manifold *e.g.,* how to compare vectors at different points. We then need to define a connection on the manifold and the idea of covariant differentiation. The connection on a manifold ensures that infinitesimal displacements of a vector along a curve lying in the manifold transform the vector in way the maintains its tensorial nature. This means that after displacing the tensor it behaves the same way under coordinate transformations as it did prior to the transport [22]. Let us then define the operation of covariant differentiation of a tensor by the following

$$A^b{}_{c;a} = \nabla_a A^b{}_c = \partial_a A^b{}_c + \Gamma^a{}_{db} A^d{}_c - \Gamma^d{}_{ca} A^b{}_d, \tag{2.2}$$

where the terms $\Gamma^a{}_{lb}$ are the 'Christoffel symbols' which are defined as a symmetric, metric compatible ($g_{ab;c} = 0$) connection given by

$$\Gamma^a{}_{cb} = \frac{1}{2} g^{ad} \left(\partial_c g_{db} + \partial_b g_{dc} - \partial_d g_{bc} \right). \tag{2.3}$$

Here we must note that the connection does not transform as a tensor under a change of coordinates. We will see that we can build tensorial objects out of Γ's and its derivatives despite its non-tensorial nature. With the Christoffel symbols telling us how tensors transform moving along a curve in a manifold, we can then ask which curves provide the 'straightest' possible paths. To determine this, we employ the the geodesic equation in affine parameterization [29]

$$A^a{}_{;b} A^b = 0, \tag{2.4}$$

or in coordinate form with affine parameter α and the tangent vector to the geodesic $A^a = \frac{dx^a}{d\alpha}$

$$\frac{d^2 x^a}{d\alpha^2} + \Gamma^a{}_{bc} \frac{dx^b}{d\alpha} \frac{dx^c}{d\alpha} = 0. \tag{2.5}$$

These equations will prove useful in studying the paths of light rays in black hole spacetimes [22]. Moving on, using the definition of the covariant derivative and Christoffel symbols we can define another fundamental object that plays a crucial role in describing a manifold. The Riemann curvature tensor is defined by evaluating the commutator of covariant derivatives acting on a tensor

$$\left[\nabla_a, \nabla_b \right] A^c{}_d = R^c{}_{eab} A^e{}_d + R^e{}_{cba} A^c{}_e. \tag{2.6}$$

Eqs. (2.2) and (2.6) allow us to express $R^a{}_{bcd}$ in terms of the Christoffel symbols

$$R^a{}_{bcd} = \partial_c \Gamma^a{}_{bd} - \partial_d \Gamma^a{}_{bc} + \Gamma^a{}_{ec} \Gamma^e{}_{bd} + \Gamma^a{}_{ed} \Gamma^e{}_{bc}. \tag{2.7}$$

Now using eq. (2.3) we can see that the Riemann tensor contains two derivatives of the metric tensor. R_{abcd} also enjoys a number of useful symmetries and identities

$$R_{abcd} = R_{[ab][cd]} = R_{cdab}, \tag{2.8}$$

$$R_{abcd} + R_{adbc} + R_{acdb} = 0, \tag{2.9}$$

$$R_{abcd;e} + R_{abec;d} + R_{abde;c} = 0. \tag{2.10}$$

Furthermore, we can define other curvature tensors by contracting over the indices of the Riemann tensor, $R^a{}_{bcd}$

$$R_{ab} = g^{cd} R_{cadb}, \qquad R = g^{ab} R_{ab}, \tag{2.11}$$

which are known as the Ricci curvature tensor and Ricci scalar respectively.

2.2 The Einstein-Hilbert Action

Defining the bare essentials allows us to move on to examining the core concepts of general relativity. The action for general relativity was formulated separately and nearly simultaneously by Albert Einstein and David Hilbert [22] who were able to write down an amazingly simple and elegant action principle for gravity. The Einstein-Hilbert action is given by

$$I = \frac{1}{16\pi G} \int_M \mathrm{d}^4x \sqrt{-g} R + I_{bdry}. \tag{2.12}$$

Eq. (2.12) did not originally have the boundary term when written in 1915. As explained below, it was shown that the so-called Gibbons-Hawking boundary term is required to provide for well defined variational principle [20]. The region of spacetime over which the integral is taken is an arbitrary, connected, finite volume with boundary ∂M possessing a metric h_{ab} induced by the embedding in the spacetime. As in classical mechanics, we vary the action eq. (2.12) with respect to the dynamical field of which it is a functional to obtain the equations of motion for the theory. For general relativity, here we vary the action with respect to the inverse metric tensor g^{ab} for simplicity. Were we to vary by the metric g_{ab} directly, we would only incur an extra overall negative sign as $\delta g_{ab} = -g_{ac}g_{bd}\delta g^{cd}$. The necessity for the boundary action comes from the fact that the Lagrangian density contains two derivatives of the metric. In varying $\sqrt{-g}R$, we would end up with boundary terms containing $h^{ab}\delta g_{ab,c}n^c$ that do not vanish by the boundary conditions, $\delta g_{ab}|_{\partial M} = 0$ and $h^{ab}\delta g_{ca,b}|_{\partial M} = 0$ [22]. Here we have denoted the unit normal vector to the boundary n^a. The explicit form of I_{bdry} is given by

$$I_{bdry} = \frac{1}{8\pi G}\oint_{\partial M} d^3y\,\varepsilon\sqrt{-\varepsilon h}\tilde{K}, \tag{2.13}$$

where, $\varepsilon = \pm 1$ depending on whether ∂M is timelike $(+)$ or spacelike $(-)$, h is the determinant of the induced metric on the boundary ∂M, and $\tilde{K}$ is the regularized extrinsic curvature, $K = n^c{}_{;c}$, of ∂M. While the variational principle does not demand a regularization of K, this is typically introduced in the context of Euclidean quantum gravity where one wants to evaluate the action as a finite quantity [30]. This regularization of K is necessary in order to remove the divergence of K as we let $\partial M \to \infty$. If we do not fix the divergence of the boundary term, then the gravitation action eq. (2.12) is infinite [22]. A common choice of regularization, known as background subtraction, is obtained by subtracting off the value for extrinsic curvature, K_0, found upon embedding ∂M in a flat spacetime. That is, $\tilde{K} = K - K_0$.

Care must be taken in the variation because we are not only varying R. We must also vary the measure term in the integral $\sqrt{-g}$. To compute the variation of $\sqrt{-g}$ we employ the formula [29]

$$\delta \ln|g| = \frac{1}{g}\delta|g| = -g_{ab}\delta g^{ab}. \tag{2.14}$$

After computing the variation of eq. (2.12) we arrive at the famed vacuum Einstein

Field Equations

$$G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab} = 0. \tag{2.15}$$

The Einstein tensor G_{ab} has the properties that it is symmetric in its indices $G_{ab} = G_{ba}$, divergence free $G^{ab}{}_{;b} = 0$, and contains only up to two derivatives of the metric. The action given in eq. (2.12) can be generalized by including matter fields and the cosmological constant term, Λ

$$I = \frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{-g}\left(R - 2\Lambda + \mathcal{L}_m\right), \tag{2.16}$$

where Ł$_m$ is a function of g_{ab} and any finite number of matter fields. Varying eq. (2.16) with respect to the metric gives the field equations

$$G_{ab} + \Lambda g_{ab} = R_{ab} + \left(\Lambda - \frac{1}{2}R\right)g_{ab} = 8\pi G T_{ab}, \tag{2.17}$$

where the stress energy tensor is defined as $T_{ab} = 2\frac{\partial\mathcal{L}_m}{\partial g^{ab}} - \mathcal{L}_m g_{ab}$. T_{ab} is also be symmetric by definition and divergence free, which expresses the local conservation of energy and momentum.

2.3 Vacuum Solutions

After writing down the Einstein-Hilbert action and finding the vacuum field equations by varying with respect to the metric tensor, the problem becomes finding the metrics that solve $G_{ab} = 0$. Here we are beginning with the simplest case where we have set the cosmological constant $\Lambda = 0$. Later we will examine solutions with non-vanishing Λ. However, we will focus our attention to the case where $\Lambda < 0$ for reasons that we will discuss. By including the cosmological constant, we will see that we obtain metrics that solve eq. (2.17) that have interesting properties. Beyond the simple vacuum solutions, we will explore the metrics solving the field equations that are black hole solutions.

Starting with the simplest case of a metric that solves $G_{ab} = 0$, we find the trace eq. (2.15)

$$R^a{}_a - \frac{1}{2} g^a{}_a R = 0 \quad \Rightarrow \quad R = 0. \tag{2.18}$$

Substituting this back into the field equations, this of course implies that $R_{ab} = 0$. Manifolds that satisfy this relationship are called Ricci flat. An example of a Ricci flat solution to the Einstein field equations is a metric familiar from special relativity representing Minkowski spacetime,

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = -dt^2 + dx^2 + dy^2 + dz^2. \tag{2.19}$$

While important in its own right, eq. (2.19) is a somewhat trivial example of a solution to the Einstein field equations. A far more interesting case was found by Schwarzschild shortly after the publication of the Einstein-Hilbert action [31, 29]. The metric for Schwarzschild's solution is

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{1 - \frac{2M}{r}} dr^2 + r^2 \left(d\theta^2 + \sin\theta^2 d\phi^2\right). \tag{2.20}$$

It is a simple exercise to calculate the Ricci tensor and show that all of its components are zero *i.e.,* the metric is Ricci flat. Eq. (2.20) does not look flat unless one goes to the limit of large r (asymptotic flatness) where the metric looks like eq. (2.19) with spatial directions in spherical polar coordinates. Note we have adopted conventions where $G = 1 = c$. Then examining the geodesic equations of particles moving in this asymptotically flat region, one can explicitly verify that M is the mass of the solution (as suggested by the notation).

More interesting is the fact that there are values of r for which the metric function $f(r) = 1 - \frac{2M}{r} = g_{tt} = g_{rr}^{-1}$ becomes problematic: $r = 2M$ and $r = 0$. Because of the pathology of the metric at these values for r, not much thought was given to the full geometry described by these of solutions until 40 years after they were first written down. A major advance came when it was shown by Kruskal and Szekeres that the singularity at $r = 2M$ is the result of a poor choice of coordinates. The pathology of the metric at $r = 2M$ can be cured by a simple coordinate transformation [22, 29]. To remove this 'coordinate singularity', we start by defining the 'Tortoise coordinate', r^* with

$$r^* = \int g_{rr} dr = \int \frac{1}{\left(1 - \frac{2M}{r}\right)} dr = r + 2M \ln \left| \frac{2M}{r} - 1 \right|. \tag{2.21}$$

which allows us to define coordinates useful in describing incoming (outgoing) light rays by

$$u = t - r^* \quad , \quad v = t + r^*. \tag{2.22}$$

In this way, we have mapped the coordinate singularity from $r = 2M$ to $u - v = \infty$ and transformed eq. (2.20) to

$$ds^2 = -f(r)\, du dv + r^2 \left(d\theta^2 + \sin\theta^2 d\phi^2\right). \tag{2.23}$$

This process has not yet entirely cured the coordinate pathology, and in order to do so, we need to define the Kruskal coordinates U, V

$$U = -e^{-\frac{u}{4M}} \quad , \quad V = e^{\frac{v}{4M}}, \tag{2.24}$$

which then gives the maximally extended Schwarzschild spacetime as

$$ds^2 = \left(-\frac{32M^3}{r} e^{-\frac{r}{2M}}\right) dU dV + r^2 \left(d\theta^2 + \sin\theta^2 d\phi^2\right). \tag{2.25}$$

This choice of coordinates gives the Kruskal extension of the Schwarzschild spacetime which covers the entire manifold.

However, the Kruskal extension does not remove all of the pathologies of the Schwarzschild solution. The singularity at $r = 0$ cannot be eliminated by any coordinate transformation and is interpreted as a singularity in the spacetime manifold. We call $r = 0$ a curvature singularity because at that point the non-vanishing components of the Riemann tensor diverge, whereas at $r = 2M$ they remain finite. More importantly, scalars constructed from the Riemann tensor will diverge at $r = 0$ *e.g.,*

$R_{abcd}R^{abcd} = \frac{48M^2}{r^6}$. As these scalars are invariant under coordinate transformations, we can conclude that no choice of coordinates will remove the pathology at $r = 0$.

Passing into the region $r < 2M$ the metric function, $f(r)$, becomes negative. In this way the t and r coordinates seem to exchange roles within the bounding surface $r = 2M$. The surface $r = 2M$ is indeed special, and we can see this by performing a different coordinate transformation on the Schwarzschild metric. By making the substitution u (v) from eq. (2.22) into eq. (2.20), we arrive at the null outgoing (incoming) Eddington-Finkelstein coordinates respectively [22]. Using the incoming Eddington-Finkelstein coordinates transforms the metric to

$$ds^2 = -f(r)dv^2 + 2dvdr + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{2.26}$$

Calculating the path of radial null geodesics, meaning $ds^2 = 0 = d\theta^2 = d\phi^2$, we find

$$-f(r)dv^2 + 2dvdr = 0 \Rightarrow -f(r)\left(\frac{dv}{dr}\right)^2 + 2\frac{dv}{dr} = 0, \tag{2.27}$$

where we have parameterized the curves by r. The above equation gives the radial null geodesics in the incoming Eddington-Finkelstein coordinates as curves satisfying

$$dv = 0, \tag{2.28}$$

$$\frac{dv}{dr} = \frac{2}{f(r)}. \tag{2.29}$$

We note that we could have done the above calculation in the 'outgoing' coordinates replacing v with u. In this case, the difference between (2.26) and the outgoing metric has $-2dudr$ instead of a $+$ sign. The disadvantage of using the outgoing coordinates is that the description of null geodesics incoming to $r = 0$ has the same trouble with $r = 2M$ as the original Schwarzschild metric [32]. Eq. (2.26) suffers the same pathology for outgoing null geodesics exiting $r = 0$ through $r = 2M$, but as we will explain below, this singular behavior for outgoing null geodesics at $r = 2M$ has a deep meaning.

The outgoing light rays originating outside of $r = 2M$ in the (v, r) half-plane have positive slope and extend out to infinity. Since $f(r) < 0$ for $r < 2M$, the 'outgoing' light rays follow geodesics with negative slope. This means that the path of the light ray for $r < 2M$ is bent towards and and will hit $r = 0$ in some finite amount of advanced time,v. The slope of the null geodesics for $r = 2M$ is infinite, and this indicates that light rays originating on the surface $r = 2M$ are trapped on there for infinite advanced time. The fact that light rays on and inside at $r = 2M$ never escape beyond that point lends itself the moniker of 'event horizon', or just horizon. We then define what we mean by 'black hole' as an object that is surrounded by a horizon from which light cannot escape to infinity.

There are many more solutions to eq. (2.15) with $\Lambda = 0$ that we could explore. However, the main focus of the thesis will be on a solution where $\Lambda < 0$. More specifically, we will consider the case where $\Lambda = -\frac{(D-1)(D-2)}{2L^2}$. The solution to the vacuum field equations with this value for the cosmological constant gives us a D dimensional Anti de-Sitter (AdS) spacetime whose metric can be expressed indirectly using a $D+1$ dimensional embedding space with

$$ds^2 = -dx_0^2 - dx_1^2 + \sum_i^{D-1} dx_i^2. \tag{2.30}$$

Now we consider the D-dimensional hyperboloid described by

$$L^2 - x_0^2 - x_1^2 + \sum_i^{D-1} x_i^2 = 0. \tag{2.31}$$

This geometry describes a D-dimensional maximally symmetric space with constant negative curvature. A notable difference from the Minkowski or Schwarzschild metrics is the signature of eq. (2.30) is $(- - + + \ldots +)$. The metric is given by the metric on the surface (2.31) induced by the embedding geometry (2.30). We can illustrate this more clearly by changing coordinates $x_0 = L \cosh r \cos t$, $x_1 = L \cosh r \sin t$, $x_i =$

$L \sinh r\, \Omega_i$ where $\sum_i \Omega_i^2 = 1$ which transforms the metric to

$$ds^2 = L^2 \left(-\cosh^2 r\, dt^2 + dr^2 + \sinh^2 r\, d\Omega_i^2 \right). \tag{2.32}$$

Note that we have that the signature is $(- + + \ldots +)$ as desired. A property that we will exploit later is that AdS space is 'maximally symmetric'. We can thus express the curvature tensors of a D-dimensional AdS spacetime as [7]

$$\begin{aligned} R_{abcd} &= -\frac{1}{L^2} \left(g_{ac} g_{bd} - g_{ad} g_{bc} \right), & (2.33) \\ R_{ab} &= -\frac{(D-1)}{L^2} g_{ab}, & (2.34) \\ R &= -\frac{D\,(D-1)}{L^2}. & (2.35) \end{aligned}$$

There is a slight problem with eq. (2.32) in that upon making the coordinate transformation, we have $0 \leq t \leq 2\pi$ which would produce closed timelike curves. However, we can instead 'unwrap' the manifold by simply letting t run from $-\infty$ to ∞. In that case, eq. (2.32) fives a many-sheeted cover of the hyperboloid and does not have the problems with causality. The metric describing an AdS spacetime that we will be interested in most arises in the context of studies of the AdS/CFT correspondence. The near horizon metric for the D3-brane solution to ten dimensional supergravity describes a product of a five dimensional AdS space (AdS_5) and a compact five dimensional sphere (S^5) which is written: [7]

$$ds^2 = \frac{r^2}{L^2} \left(-dt^2 + dx^2 + dy^2 + dz^2 \right) + \frac{L^2}{r^2} dr^2 + R^2 d\Omega_5^2. \tag{2.36}$$

where $d\Omega_5^2$ is the line element for S^5. See [10] for a detailed description of the relationship beteween the AdS portion of eq. (2.36) and the metric eq. (2.32). In general, we will only be interested in the non-compact portion of the metric. So, we will will neglect the S^5 part of the metric and focus on the AdS_5 portion. Motivated by eq. (2.36) being a vacuum solution to general relativity we would like to find if it admits black hole solutions respecting the same translational symmetries in x, y, and

z. We begin by postulating an ansatz for such black brane or planar horizon black hole. [16, 24]

$$ds^2 = \frac{r^2}{L^2}\left(-f(r)N(r)^2 dt^2 + dx^2 + dy^2 + dz^2\right) + \frac{L^2}{r^2 f(r)} dr^2, \tag{2.37}$$

where we have added the metric function $f(r)$ to eq. (2.36) as well as the function $N(r)$ to scale the time coordinate. We can solve the equations of motion for (2.37), but instead we will use a variational approach that will be employed again in the main thesis for the higher curvature theories. That is, we begin by computing the value for the action and finding its dependence on r, $f(r)$, and $N(r)$. We then integrate by parts so that we are left with, up to total derivatives, the following form for the action

$$I = \frac{1}{16\pi G_5}\int d^5x \frac{3N(r)}{L^5}\left[r^4\left(1 - f(r)\right)\right]'. \tag{2.38}$$

where $'$ denotes a derivative with respect to r. Varying with respect to N will give a constraint equation for $f(r)$

$$\left[r^4\left(1 - f(r)\right)\right]' = 0, \tag{2.39}$$

which upon solving gives

$$f(r) = 1 - \frac{\omega^4}{r^4}, \tag{2.40}$$

where ω^4 is an arbitrary integration constant. The horizon is given by $r = \omega$ which causes $f(r)$ to vanishing. Varying by δf

$$N(r)' r^4 = 0. \tag{2.41}$$

We then determine that $N(r) = N_\sharp$ is some arbitary constant. We could in principle set $N_\sharp$ to one by simply rescaling the t coordinate If instead of restricting our attention to five dimensions we had considered an AdS_D black brane metric ansatz

$$ds^2 = \frac{r^2}{L^2}\left(-N(r)^2 f(r) dt^2 + (dx^{\bar{a}})^2\right) + \frac{L^2}{r^2 f(r)} dr^2, \tag{2.42}$$

where $x^{\bar{a}}$ are the planar coordinates. The action and solutions to the constraint equations are then given by

$$I = \frac{1}{16\pi G_D}\int d^D x \frac{(D-2)N(r)}{L^D}\left[r^{D-1}\left(1-f(r)\right)\right]', \tag{2.43}$$

$$f(r) = 1-\frac{\omega^{D-1}}{r^{D-1}}, \tag{2.44}$$

where we also arrive at the conclusion that $N(r) = N_\sharp$ constant. Generalizing eq (2.37) by considering the case where the horizon in eq. (2.37) is curved

$$ds^2 = -\left(k+\frac{r^2}{L^2}f(r)\right)N(r)^2dt^2 + r^2 d\Sigma_k^2 + \frac{1}{k+\frac{r^2}{L^2}f(r)}dr^2. \tag{2.45}$$

One sets $k=\pm 1$ corresponding to spherical (hyperbolic) horizons respectively, while $k=0$ recovers the planar black holes. The spatial section $d\Sigma_k$ for the different values of k is given by

$$k=1: \qquad d\Sigma_1^2 = d\Omega_{D-2}^2, \tag{2.46}$$

$$k=0: \qquad d\Sigma_0^2 = \frac{1}{L^2}(dx^{\bar{a}})^2, \tag{2.47}$$

$$k=-1: \qquad d\Sigma_{-1}^2 = d\mathcal{H}_{D-2}^2, \tag{2.48}$$

where $d\Omega_{D-2}^2$ ($d\mathcal{H}_{D-2}^2$) is the metric for a $D-2$ dimensional unit sphere (hyperboloid). One finds the solution for the metric function $f(r)$ is precisely the same as before. In this case, the horizon appears at $g_{tt}=0 \Rightarrow f(r) = -\frac{kL^2}{r^2}$ [19].

2.4 Black Hole Thermodynamics

In the 1970's there were several major discoveries relating to the fundamental properties of black holes as they relate to thermodynamics. These results built on the following where the four Laws of black hole mechanics , which summarized some of the general properties of black holes in general relativity [4]

- 0^{th} *Law*: Surface Gravity κ is constant on the horizon

- 1^{st} *Law*: $\delta M = \frac{\kappa}{8\pi}\delta A + \Omega_H \delta J$ where Ω_H is the angular velocity of the horizon of a rotating black hole and $J$is the associated angular momentum.
- 2^{nd} *Law*: $\delta A \geq 0$
- 3^{rd} *Law*: In no finite time in the future can a physical process make κ vanish

These four Laws above were realized to have striking similarities to the Laws governing thermodynamics. These mere similarities then crystalized into a precise equivalence with the realization by Hawking that through quantum effects an observer at infinity would see a black hole 'emitting' particles with a blackbody spectrum with temperature T related to the surface gravity by $\kappa = 2\pi T$ [3, 2]. Hence, it follows from the 1^{st} law that the black hole horizon carries an intrinsic entropy given by $\frac{A}{4G}$. The latter had already been alluded to in the work of Jacob Beckenstein [3].

We will begin this section deriving a general expression for black hole temperature, which we will be able to apply in pseudo-topological gravity. Afterwards, we will calculate the free energy and entropy of a black hole. To start our considerations, we first consider the metric eq. (2.37) with the time coordinate analytically continued to $\tau = -\imath t$ [33, 34]

$$ds^2 = \frac{r^2}{L^2}\left(f(r)N(r)^2 d\tau^2 + dx^2 + dy^2 + dz^2\right) + \frac{L^2}{r^2 f(r)}dr^2. \tag{2.49}$$

This Euclidean metric remains a solution of $R_{ab} = 0$ Drawing from one's experience with thermal field theory, we anticipate that the Euclidean time is periodically identified $\tau = \tau + \beta$ with $\beta = \frac{1}{T}$. Note that we began with a black hole with horizon where $f(r = r_h) = 0$. The Euclidean metric cannot have a horizon, and so we must pay special attention to interpretting the geometry at $r = r_h$. By Taylor expanding the metric function $f(r)$ around the horizon radius r_h

$$f(r) = f(r_h) + f'|_{r_h}(r - r_h) + \ldots = f'|_{r_h}(r - r_h) + \ldots, \tag{2.50}$$

but the first term vanishes since $f(r_h) = 0$. The analytically continued AdS black brane metric near the horizon then becomes to first order

$$\begin{aligned} ds^2 &\approx \frac{r_h^2}{L^2}\left(f'|_{r_h}(r-r_h)\,N(r)^2 d\tau^2 + dx^2 + dy^2 + dz^2\right) + \frac{L^2}{r_h^2 f'|_{r_h}(r-r_h)}dr^2, \\ &= \frac{r_h^2}{L^2}\left(f'|_{r_h}N(r)^2\rho^2 d\tau^2 + dx^2 + dy^2 + dz^2\right) + \frac{4L^2}{r_h^2 f'|_{r_h}}d\rho^2, \end{aligned} \tag{2.51}$$

where $\rho^2 = r - r_h$. Focusing on the terms of the metric with factors of f' as

$$ds^2 \approx \frac{4L^2}{r_h^2 f'|_{r_h}}\left(\rho^2\left(\frac{r_h^2 f'|_{r_h}N(r)}{2L^2}\right)^2 d\tau^2 + d\rho^2\right) + \frac{r_h^2}{L^2}d\vec{x}^2. \tag{2.52}$$

Here we can see that the geometry at $r = r_h$ will be smooth with the appropriate interpretation. The ρ and τ coordinates describe a two-plane in polar coordinates with radial direction ρ and angular direction $\theta = \frac{r_h^2 f'|_{r_h}N(r)}{2L^2}\tau$. The origin is smooth if the latter is periodically identified with $\theta = \theta + 2\pi$. Recalling that τ has period given by $\beta = \frac{1}{T}$ we find

$$\begin{aligned} 2\pi &= \frac{r_h^2 f'|_{r_h}N(r)}{2L^2}\beta, \\ \Rightarrow T &= \frac{1}{4\pi}\frac{r_h^2 f'|_{r_h}N(r)}{L^2}. \end{aligned} \tag{2.53}$$

Going back to the constraint equation for $f(r)$, eq. (2.40), we can see that differentiating and evaluating at the horizon gives

$$f'|_{r_h} = \frac{4\omega^4}{r_h^5} = \frac{4}{\omega}, \tag{2.54}$$

and so the temperature of the planar black hole is

$$T = \frac{\omega N_\sharp}{\pi L^2}. \tag{2.55}$$

The natural choice here is to set $N_\sharp = 1$ so that $\frac{g_{tt}}{g_{xx}} \to -1$ asymptotically. With this choice, $T = \frac{\omega}{\pi L^2}$.

Now that we have derived the temperature of a black hole described by eq. (2.37), we will move on to deriving other thermodynamic quantities. First, we will calculate the free energy of the black hole, and then derive from that an expression for black hole entropy using familiar thermodynamic arguments [20, 35]. To calculate the free energy, we work in the Euclidean framework, and we regard the black hole as a thermal system. Then using path integral techniques as described in [33], we calculate the partition function as

$$Z = e^{-\frac{F}{T}} = \int \mathcal{D}g e^{I_E(g)} \simeq e^{I_E(g_{cl})}, \tag{2.56}$$

where in the last step we have approximated the path integral over metrics by the value of the integrand at the saddle point. That is, we evaluate I_E for the Euclidean black hole solution with temperature T. Thus, we can identify the free energy of the thermal ensemble, F, with the Euclidean black hole action by

$$I_E\left[T\right] = \frac{F}{T}. \tag{2.57}$$

The I_E for the black brane metric is given by

$$I_E = -\frac{1}{16\pi G}\int_{r_h}^{\infty} dr \int_0^{\frac{1}{T}} dt_E \int d^3x \sqrt{g_E}\left(R - 2\Lambda\right). \tag{2.58}$$

Note that in the present case with asymptotically AdS boundary conditions, the Gibbons-Hawking term, eq (2.13) that normally is the relevant contribution in flat solutions ($\Lambda = 0$) will not contribute here, and so we have dropped it. However, I_E is still divergent for $r \to \infty$ because the bulk contribution no longer vanishes with $\Lambda \neq 0$. To fix this divergence we instead integrate over the radial coordinate up to a boundary radius r_+. We then regularize the Euclidean action by subtracting off the value given by computing the corresponding action of empty AdS space, $I_E^0[T'(T)]$. That is, we obtain the background by setting $\omega = 0$ in the metric function $f(r)$. We must be careful to choose the function, $T'(T)$ *i.e.,* the periodicty of τ in the AdS background, so that the asymptotic geometries of the background and black hole spacetimes match [11]. This procedure allows us to compute the free energy of a black hole as

$$F[T] = T\left(I_E[T] - I_E^0[T'(T)]\right) = -\frac{V_3\,\omega^4 N_\sharp}{16\pi G L^5}. \tag{2.59}$$

Expressing the free energy in terms of the temperature

$$F[T] = -\frac{V_3}{4G}\frac{(\pi L T)^3}{N_\sharp^3}\frac{T}{4}. \tag{2.60}$$

Other than the discovery that black holes were thermal objects, the calculation of black hole entropy was one of the most interesting results concerning black holes coming out of the 1970's. For a stationary, axisymmetric black hole the entropy is given by $S = \frac{k_B A}{4 l_p^2}$ where k_B is Boltzmann's constant and Planck length $l_p = \sqrt{\frac{G\hbar}{c^3}}$. We usually work in units where $k_B = c = \hbar = 1$, and so the area formula for black hole entropy is given by $S = \frac{A}{4G}$. Thus, we have that a black hole has entropy roughly expressed as the area of the horizon in units of l_p^2 [3].

From the standard expression coming from thermodynamics, we have the entropy of the black hole is given by

$$S[T] = -\frac{d}{dT}F[T] = \frac{V_3}{4G}\frac{(\pi L T)^3}{N_\sharp^3} = \frac{V_3}{4G}\frac{\omega^3}{L^3}. \tag{2.61}$$

Hence, we have recovered the expected result noted above, namely $S = \frac{A}{4G}$ with $A = \frac{V_3 \omega^3}{L^3}$. Note this result is proportional to $\int d^3x$ which in principle is infinite. In the language of the AdS/CFT, this corresponds to the volume in which the dual CFT lives. So, it is convenient to consider the entropy and (free energy) density

$$s = \frac{S}{\int d^3x} = \frac{\omega^3}{4GL^3}, \qquad f[T] = -\frac{1}{4G}\frac{(\pi L T)^3}{N_\sharp^3}\frac{T}{4}. \tag{2.62}$$

The thermodynamics describing black hole solutions in the bulk AdS space are directly related by the AdS/CFT correspondence to the same properties of the dual CFT plasma. Using the above expressions, we can then calculate other useful thermodynamic quantities in the dual CFT. We start with the simple relation for the pressure of a system absent a chemical potential in terms the free energy density

$$p = -f[T] = \frac{(\pi L T)^3}{4GN_\sharp^3}\frac{T}{4}. \tag{2.63}$$

Further, we can calculate the energy density ρ in the CFT, which matches the total energy density of the black hole spacetime

$$\rho = -T^2 \frac{\partial}{\partial T}\left(\frac{f[T]}{T}\right) = \frac{3\pi^3 L^3 T^4}{16 G N_\sharp^3}. \tag{2.64}$$

We can use the known relationships involving the above thermodynamic results to check the consistency of our approach. Using the traceless property of the stress tensor, we arrive at an equation relating the energy density and pressure

$$T^a{}_a = \rho - 3p = 0. \tag{2.65}$$

The above result is for a four-dimensional CFT and can be generalized to D dimensions by replacing $-3p$ by $-(D-1)p$. Using eqs. (2.63) and (2.64), we see that this consistency check is satisfied for the Euclidean action approach to the thermodynamics of the black branes. Another check on the value of the energy density is given by substituting the relations $\rho = 3p$ and $p = -f[T]$ into the expression for energy density

$$\rho = f + sT \quad \Rightarrow \quad \rho = \frac{3}{4} sT. \tag{2.66}$$

Again inserting the values of the quantities above that we have calculated for the black branes, we find that eq (2.66) is satisfied. These checks will prove to be useful when dealing with higher curvature theories as we will see below. In particular, eq. (2.66) provides a relationship that will be used to underscore some of the difficulties in the computation of energy density of pseudo-topological black holes using the quasilocal formulation of [36].

2.5 Higher Curvature Gravity

After the theory was written down, it was thought that Einstein gravity may not be the entire picture. In particular, the pervasive appearance of singularities inside black holes suggests a more fundamental theory is required for a complete description of these systems. One simple idea is to add higher order powers of curvature to the action, and this may provide the necessary non-trivial corrections to the theory, as initially proposed in 1919 [5]. Higher curvature gravity sat of the shelf until it started to garner some consideration again in the middle of the 20^{th} century in the context of a possible solution to the non-renomalizability of quantum gravity [6]. Further work was done by Lovelock in [37] by showing that adding dimensionally continued Euler densities for manifolds of dimension $2n$ to the Lagrangian density for Einstein gravity yielded a non-trivial extension of the Einstein tensor for $D = 2n + 1$. Note that the two dimensional Euler density is the Ricci scalar, which is the Lagrangian density for the Einstein-Hilbert action for general relativity. We will focus our attention on second order Lovelock, or Gauss-Bonnet, gravity obtained by adding four-dimensional Euler density to the Einstein-Hilbert term. Of course, this can only effect the gravitational field equations in theories where the spacetime dimensiona is greater than four. Explorations into the presence of these higher curvature terms received a certain amount of attention in string theory and led to a growth in interest in the 1980's [17, 23, 38, 39]. In this section we will review results for Gauss-Bonnet gravity including black hole solutions, their thermodynamic properties, and briefly mention work done in exploring the implications of higher curvature gravity using the AdS/CFT correspondence.

2.5.1 Gauss-Bonnet Gravity

We will begin our consideration of higher curvature theories by examining Gauss-Bonnet gravity. This theory is built out of the Einstein-Hilbert Lagrangian density with the addition of the Euler density of a four dimensional manifold $\mathcal{X}_4$. However before we get to Gauss-Bonnet gravity, we must note that $\mathcal{X}_4$ is a very special case for a specific linear combination of curvature squared terms. We will begin by writing down the most general theory that contains an arbitrary linear combination of scalar monomials quadratic in the curvature tensors and then show why we choose only to discuss $\mathcal{X}_4$ among the possible choices [38]. The general second order action is given by

$$I = \frac{1}{16\pi G}\int \mathrm{d}^D x\sqrt{-g}\left(R - 2\Lambda + \alpha_1 R^2 + \alpha_2 R_{ab}R^{ab} + \alpha_3 R_{abcd}R^{abcd}\right). \tag{2.67}$$

The linearized theory of eq. (2.67) were studied in [23]. Varying the field equations for (2.67) by $g_{ab} = g^0_{ab} + h_{ab}$ where the background g^0_{ab} is a solution to the field equations, the authors found that quadratic terms with arbitrary coefficients give rise to kinetic terms of the form $h_{ab}\Box^2 h^{ab}$ in the linearized theory. These terms are certainly problematic for the initial value problem in general relativity. A quantum version of theses problems is that the inclusion of these four derivative terms leads to ghost modes in the graviton propagator with $m^2 = \frac{1}{l_p^2}$. The quantum theory is found then to be non-unitary [6]. The presence of ghost modes for the graviton in the theory indicates a sickness in the full non-perturbative theory or alternatively that the theory described by eq. (2.67) must be incomplete.

In this regard, the Gauss-Bonnet theory proves special amongst the curvature squared theories. By choosing the coefficients of the quadratic terms to be $\alpha_1 = \alpha_3$, $\alpha_2 = -4\alpha_1$, the $h_{ab}\Box^2 h^{ab}$ terms in the second variation vanish up to total derivatives. Thus we find that the unique curvature-squared theory that is stable in the non-linear regime is given by the addition of $\sqrt{-g}\mathcal{X}_4$ to the Einstein-Hilbert term. However, since $\mathcal{X}_4$ is topological in $D = 4$ and vanishes for $D \leq 3$, this theory only provides a non-trivial extension of Einstein gravity in dimension greater than or equal to five. The full Gauss-Bonnet action is then

$$I = \frac{1}{16\pi G}\int \mathrm{d}^D x\sqrt{-g}\left(R - 2\Lambda + \tilde{\lambda}\left(R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd}\right)\right). \tag{2.68}$$

Varying eq. (2.68) by δg^{ab}, we find the field equations for Gauss-Bonnet gravity are given by:

$$G_{ab} + \tilde{\lambda}G^{(2)}_{ab} = 0, \tag{2.69}$$

where G_{ab} is the usual Einstein tensor and $G^{(2)}_{ab}$ is given by [40]

$$\begin{aligned} G^{(2)}_{ab} &= 2\left(RR_{ab} - 2R^{dc}R_{dacb} - 2R_{ad}R^{d}{}_{b} + R_{a}{}^{dcm}R_{bdcm}\right) - \frac{1}{2}g_{ab}\left(R^2\right. \\ &\qquad \left. -4R_{dc}R^{dc} + R_{dcmn}R^{dcmn}\right). \end{aligned} \tag{2.70}$$

Note that there are no terms involving derivatives of curvature that survive here *e.g.,* $\Box R_{ab}$. Given our interest in exploring higher curvature gravity in the framework of the AdS/CFT correspondence we should look for black hole solutions using the metric ansatz in eq. (2.37). Working in five dimensions yields a value for the $\Lambda = -\frac{6}{L^2}$. Using the variational approach described above with Einstein gravity, we calculate the action (2.68) and, integrating by parts, we find up to total derivatives

$$I = \frac{1}{16\pi G}\int d^5x \frac{3N(r)}{L^5}\left[r^4\left(1 - f(r) + \lambda f(r)^2\right)\right]', \tag{2.71}$$

where we have made the substitution $\tilde{\lambda} \to \frac{\lambda L^2}{2}$. First finding the equations of motion for N by varying with respect to f, we see again that $N(r) = N_\sharp = \mathit{constant}$. Varying by $\delta N(r)$, we obtain an equation for $f(r)$ which upon integrating once gives

$$\lambda f(r)^2 - f(r) + 1 - \frac{\omega^4}{r^4} = 0. \tag{2.72}$$

We can easily solve this quadratic equation with

$$f(r) = \frac{1}{2\lambda}\left[1 \pm \left(1 - 4\lambda\left(1 - \frac{\omega^4}{r^4}\right)\right)^{\frac{1}{2}}\right]. \tag{2.73}$$

Observe that with the $-$ branch of the solution above, upon taking the limit $\lambda \to 0$ we find

$$\lim_{\lambda\to 0} f(r) \approx \frac{1}{2\lambda}\left(1 - \left(1 - 2\lambda\left(1 - \frac{\omega^4}{r^4}\right) - \ldots\right)\right) = 1 - \frac{\omega^4}{r^4} + \ldots, \tag{2.74}$$

where the terms in $\ldots$ are $\mathcal{O}(\lambda)$ and vanish in the limit. We have recovered the result for the metric function that we had in Einstein gravity in the limit that the Gauss-Bonnet interaction vanishes.

However, the $\pm$ sign in eq. (2.73) emphasises a key distinction from Einstein gravity. Consider the case where $\omega = 0$. We then have a quadratic equation for the metric function which will now simply be a constant

$$h(f) = \lambda f^2 - f + 1 = 0. \tag{2.75}$$

We thus have two distinct vacua corresponding to the solutions

$$f_\pm = \frac{1}{2\lambda}\left(1 \pm \sqrt{1-4\lambda}\right). \tag{2.76}$$

We find that only the $-$ branch has a sensible limit as $\lambda \to 0$. The $+$ has leading $\mathcal{O}(\frac{1}{\lambda})$ terms and thus diverges in the limit. The problem becomes how to interpret these vacua. In [17], the authors found that in solving the field equations for Gauss-Bonnet gravity the two branches of the solution can be classified by the slope of the polynomial $h(f) = \lambda f^2 - f + 1$ as it passes through its roots. This sign determined the sign of the kinetic term of the graviton in the second variation of the action. That is for the backgrounds with negative (positive) slope the sign of the kinetic term is positive (negative). According to present conventions,a negative sign kinetic term corresponds to a ghost. Hence, the negative slope solutions are stable vacua whereas solutions with positive slope have ghost graviton. As an example, in fig. (2.1) when we plot out $h(f) = \lambda f^2 - f + 1$ for $\lambda = 0.162$. We note that no vacua exist $\lambda > \frac{1}{4}$ as $f_\pm$ becomes complex. At the critical point $\lambda = \frac{1}{4}$ the roots coalesce, $f_\pm = 2$. Examining fig. (2.1) we find that both roots $f_\pm$ are positive, corresponding to AdS vacua. The negative branch of in eq. (2.76) is the smaller root with negative slope and is thus stable. The positive branch of $f_\pm$ is the larger root with positive slope and has a ghost graviton.

We now look to find which vacua admit black hole solutions by letting $\omega \neq 0$ with $h(f)$

$$h(f) = \lambda f^2 - f + 1 - \frac{\omega^4}{r^4} = 0. \tag{2.77}$$

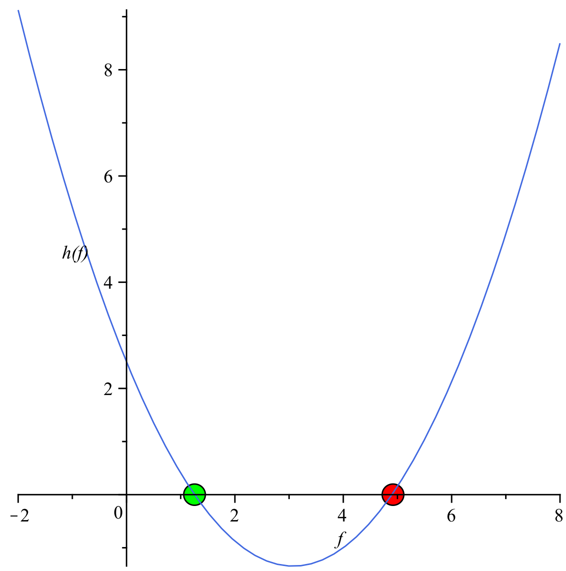

Figure 2.1: Plot of h(f) with stable vacuum highlighted with green circle, ghosty vacuum with red circle

Here we see that as r decreases from ∞, the parabola is dragged downward as in fig. (2.2). At $r = \omega$ we see that the root of the stable branch occurs at $f = 0$. Recall,however, that $f = 0$ corresponds to the event horizon, and so we have determined that $r_h = \omega$. This result is readily seen by examining (2.73) for the $-$ sign

$$f_{-}(r) = \frac{1}{2\lambda}\left[1 - \left(1 - 4\lambda\left(1 - \frac{\omega^4}{r^4}\right)\right)^{\frac{1}{2}}\right]. \tag{2.78}$$

Here we can easily see that $f(r_h = \omega) = 0$. Hence, we can interpret theses metrics as black hole solutions for Gauss-Bonnet gravity in AdS_5.

If we consider the + of (2.73) instead, we can examine fig. (2.2) and find that no matter the value of r the root of the positive branch never hits $f = 0$. Thus as we have discussed immediately above, the metrics with f_+ do not have black hole solutions. Moreover for the positive branch, we find that solutions with $\omega^4 > 0$ have negative energy.

Now we examine the thermodynamic properties of the above higher curvature

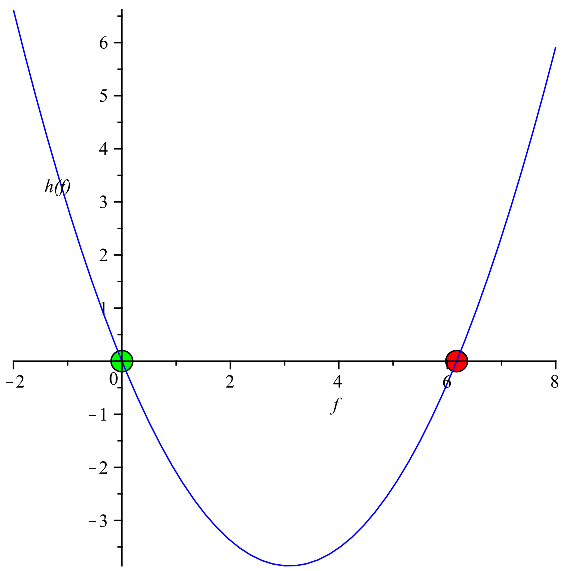

Figure 2.2: Plot of h(f) for $r = \omega \neq 0$ with stable vacuum with black holes highlighted with green circle, ghosty vacuum with red circle

black holes [15]. The calculation for temperature that was done for Einstein gravity, eq. (2.53), remains the same, but the result is modified by the higher curvature terms by making a standard choice for $N_\sharp^2 = \lim_{r\to\infty} f(r)$ so that asymptotically $\frac{g_{tt}}{g_{xx}} \to -1$ as we saw before:

$$T = \frac{\omega N_\sharp}{\pi L^2} = \frac{\omega}{\pi L^2}\left[\frac{1}{2\lambda}\sqrt{1-4\lambda}\right]. \tag{2.79}$$

It was shown in [18] and[19] that by integrating the 1^{st} Law of Black Hole Mechanics the entropy for Gauss-Bonnet black holes for planar 'black branes' does not change. Black holes with curved horizons coming from eq. (2.45) do see correction to their thermodynamic properties through the higher curvature terms. These qualities extend beyond Gauss-Bonnet gravity and apply to Lovelock gravity in general [18].

Motivated by considering the 1^{st} Law for higher curvature black holes, Wald in [21] determined a method to compute the entropy of a black hole in a gravitational theory with arbitrary higher order curvature terms. The calculation for entropy can be stated briefly as follows. Let $\mathcal{L} = \tilde{\mathcal{L}}\bar{\varepsilon}$ be the Lagrangian form of a gravitational theory. For simplicity, we will consider the case where we have no matter fields. The entropy of a black hole can then be calculated as

$$S = -2\pi \oint Y^{abcd} \hat{\varepsilon}_{ab} \hat{\varepsilon}_{cd} \varepsilon, \tag{2.80}$$

where,

$$Y^{abcd} = \frac{\partial \tilde{\mathcal{L}}}{\partial R_{abcd}}, \tag{2.81}$$

and $\hat{\varepsilon}_{ab}$ is the binormal to the horizon and ε is the volume form evaluated on the horizon. If $Y = Y^{abcd} \hat{\varepsilon}_{ab} \hat{\varepsilon}_{cd}$ is constant on the horizon, the entropy is given simply as

$$S = -2\pi Y A, \tag{2.82}$$

where $A = \oint \bar{\varepsilon}$. As an example which will be relevant later, let us take our metric to be eq. (2.37). Then since we are dealing with Einstein gravity our Lagrangian is $\tilde{\mathcal{L}} = \frac{1}{16\pi G}(R - 2\Lambda)$ which when plugged into the steps above yields

$$Y^{abcd} = \frac{1}{16\pi G} \frac{\partial R}{\partial R_{abcd}} = \frac{1}{32\pi G} \left(g^{ac} g^{bd} - g^{ad} g^{bc} \right). \tag{2.83}$$

In an orthonormal frame, we have $Y = 4Y^{trtr} = -4Y^{tr}{}_{tr}$ leading to the standard result of black hole entropy for Einstein gravity:

$$S = -2\pi A \left(-\frac{1}{8\pi G} \left(g^{t}{}_{t} g^{r}{}_{r} - g^{t}{}_{r} g^{r}{}_{t} \right) \right) = \frac{A}{4G}. \tag{2.84}$$

This agrees with the previous result for entropy obtained in the Euclidean framework as we had expected [27]. Applying this formalism to Gauss-Bonnet gravity, we find that in addition to the Einstein contribution:

$$Y_2 = Y_2^{abcd} \hat{\varepsilon}_{ab} \hat{\varepsilon}_{cd} = -\frac{1}{4\pi G} \left(R - 2 \left(R^{t}{}_{t} + R^{r}{}_{r} \right) + 2 R^{tr}{}_{tr} \right), \tag{2.85}$$

where $Y_2^{abcd} = \frac{\partial \tilde{\mathcal{L}}}{\partial R_{abcd}}$. Integrating this over the horizon gives the value for the entropy as

$$S = \frac{A}{4G} \left(1 + \lambda L^2 \left(R - 2 \left(R^{t}{}_{t} + R^{r}{}_{r} \right) + 2 R^{tr}{}_{tr} \right) \right). \tag{2.86}$$

Calculating eq. (2.86) for the metric eq. (2.37) gives

$$S = \frac{A}{4G}\left(1 - 6\lambda f(r)\right). \tag{2.87}$$

However, evaluating this on the horizon $r_h = \omega$ we retrieve the result that $S = \frac{A}{4G}$. Considering the case of black holes with curved horizons, we note that at the horizon radius r_h the metric function $f(r)$ does not vanish. Instead by eq. (2.45), $f(r_h) = -\frac{kL^2}{r_h^2}$, and so the curved horizon entropy formula

$$S_k = \frac{A}{4G}\left(1 + 6\lambda k \frac{L^2}{r_h^2}\right). \tag{2.88}$$

Continuing to explore the thermodynamics of the Gauss-Bonnet black brane, we can calculate the free energy using the Euclidean action approach presented in the previous section. Including, the Gauss-Bonnet interaction in eq. (2.58) where we have again dropped the boundary terms which do not contribute [11]

$$I_E = -\frac{1}{16\pi G}\int_{r_h}^{\infty} dr \int_0^{\frac{1}{T}} dt_E \int d^3x \sqrt{g_E}\left(R + \frac{12}{L^2} + \frac{\lambda L^2}{2}\left(R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd}\right)\right). \tag{2.89}$$

Calculating eq. (2.89) for the metric ansatz eq. (2.37), we find that

$$I_E[T] = \frac{1}{16\pi G}\frac{V_3\,\omega^4 N_\sharp}{TL^5\lambda}\left(\frac{r_+^4}{\omega^4}\left(12\lambda - 5 + 5\sqrt{1-4\lambda}\right) - 4\lambda + \frac{2\lambda}{\sqrt{1-4\lambda}}\right), \tag{2.90}$$

where V_3 is the volume obtained by integrating over the planar directions. However as we have discussed previously, the divergence of I_E as we take $r_+ \to \infty$ necessitates regularization. Considering the solution for the pure Gauss-Bonnet AdS vacuum by letting $\omega = 0$ in the metric function $f_-(r)$, and then calculating the form of $T'[T]$ [11]

$$T' = T\left(\frac{1 - \sqrt{1-4\lambda}}{1 - \sqrt{1 - 4\lambda\left(1 - \frac{\omega^4}{r_+^4}\right)}}\right)^{\frac{1}{2}}. \tag{2.91}$$

Now calculating the empty Gauss-Bonnet AdS space Euclidean action

$$I_E^0[T'(T)] = \frac{1}{16\pi G}\frac{V_3\,\omega^4 N_\sharp}{T'L^5\lambda}\left(\frac{r_+^4}{\omega^4}\left(12\lambda - 5 + 5\sqrt{1-4\lambda}\right)\right). \tag{2.92}$$

Subtracting $I_E - I_E^0$ and taking the limit $r_+ \to \infty$, we find that F is given by

$$F[T] = -\frac{1}{16\pi G}\frac{1}{L^5}V_3\,\omega^4 N_\sharp = -\frac{V_3}{4G}\frac{(\pi L T)^3}{N_\sharp^3}\frac{T}{4}. \tag{2.93}$$

Then calculating the entropy density, s, of the Gauss-Bonnet black brane

$$s[T] = -\frac{1}{V_3}\frac{d}{dT}F[T] = \frac{(\pi L T)^3}{4GN_\sharp^3} = \frac{1}{4G}\frac{\omega^3}{L^3}. \tag{2.94}$$

We see that the Euclidean action approach directly above matches the result using Wald's method [27]. As we had done in the case of the black branes in Einstein gravity, we check that the above thermodynamic results satisfy the relationships eqs. (2.65) and (2.66). We first calculate the pressure and energy density as in eqs. (2.63) and (2.64) and find

$$p = \frac{1}{4G}\frac{(\pi L T)^3}{N_\sharp^3}\frac{T}{4}, \qquad \rho = \frac{3(\pi L)^3 T^4}{16GN_\sharp^3}. \tag{2.95}$$

We can see from the above expressions that eq. (2.65) is satisfied for the Gauss-Bonnet black brane solutions. The values above for temperature and entropy density show that eq. (2.66) is satisfied as well. We note that the contribution of the higher curvature terms comes from the factors of $N_\sharp$.

Chapter 3

Pseudo-Topological Gravity

Given the rich area of research that Gauss-Bonnet gravity has become both as a theory of gravity on its own and also in the context of AdS/CFT, it would follow that we should ask if there is any further correction to the gravitational action, eqs. (2.16), (2.68), that would be at least third order and remain ghost free. That is not to say theories of gravity with terms of cubic order in curvature tensors have been neglected. However, the work done has been mostly in the context of Lovelock gravity, which are non-trivial at third order only for spacetimes with seven or more dimensions [39, 41]. What we would like to do is explore whether or not we can write down a nice, non-trivial third order theory in five dimensions and find black hole solutions therein. If this is at all possible, it will obviously not be part of the Lovelock group of theories

In this chapter, our goal is to study gravitational theories with curvature cubed interactions that will allow us to explore possible black hole solutions. We begin by writing down the most general irreducible combination of contractions of three curvature tensors with arbitrary coefficients. We then apply the variational method, as we had in Gauss-Bonnet gravity, to determine the values of the coefficients. We find that there does exist a linear combination of the cubic terms that have a non-trivial contribution to the equations of motion in five dimensions. After writing down the action in five dimensions, we generalize the theory to an arbitrary dimensional spacetime. The surprising result that in six dimensions the cubic order action yields only boundary terms suggests that it may be a new topological object. However as we show in Appendix A with explicit examples, this is not the case. We then derive the field equations for this 'pseudo-topological' theory. From the field equations, we write down the linearized pseudo-topological theory, and we find that it has two-derivative equations of motion by choosing a certain amount of symmetry of the space time and using a standard gauge choice. Thus, we see that pseudo-topological gravity is free from the sicknesses that usually plague higher curvature theories.

3.1 Finding a Cubic Order Action

Being inspired by the simplicity of the black hole solutions for Gauss-Bonnet gravity, found in section 2.5, we would like to see if we can reproduce a similarly nice results in a curvature cubed theory in five dimensions. Finding that it is indeed possible to write down a curvature cubed theory in five dimensions, we will go back and check the linearized equations of motion to verify that these are in fact 2^{nd} order.

Let us begin by listing a basis of the possible six-derivative interactions, which then appear at the same order as the curvature-cubed terms. Considering the work don in [42, 25, 26] we see that the basis for our theory is given by the following scalar combinations:

1. $R_{a\ b}^{\ c\ d}\, R_{c\ d}^{\ e\ f}\, R_{e\ f}^{\ a\ b}$
2. $R_{ab}{}^{cd}\, R_{cd}{}^{ef}\, R_{ef}{}^{ab}$
3. $R_{abcd}\, R^{abc}{}_{e}\, R^{de}$
4. $R_{abcd}\, R^{abcd}\, R$
5. $R_{abcd}\, R^{ac} R^{bd}$
6. $R_a{}^b R_b{}^c R_c{}^a$
7. $R_a{}^b R_b{}^a R$
8. R^3
9. $\nabla_a R_{bcde} \nabla^a R^{bcde}$
10. $\nabla^a \nabla^c R_{abcd}\, R^{ab}$
11. $\nabla_a R_{bc}\, \nabla^a R^{bc}$
12. $\nabla^a R_{ab}\, \nabla^b R$
13. $\nabla_a R\, \nabla^a R$

In assembling this list, we have discarded any total derivatives, *e.g.,* $\nabla^a \nabla_a \nabla^b \nabla^c R_{bc}$ and we have simplified various expressions using the index symmetries of the Ricci and Riemann tensors from eqs. (2.9) and (2.10). In particular, these symmetries allow us to reduce any other index contraction of three Riemann tensors to some combination of terms 1 and 2. Further, term 12 can be reduced to term 13, using $\nabla^a R_{ab} = \frac{1}{2}\nabla_b R$. Similarly, using the Bianchi identities, terms 9 and 10 can be shown to be reducible to other terms and total derivatives as well. Hence, we are left with a list of 10 independent cubic curvature terms to consider. Combining all of these terms together in a single expression gives:

$$\begin{aligned}\sqrt{-g}\mathcal{Z}_5 \;\; = \;\; & \sqrt{-g}\Big(c_1\, R_{a\ b}^{\ c\ d} R_{c\ d}^{\ e\ f} R_{e\ f}^{\ a\ b} + c_2\, R_{ab}^{\ \ cd} R_{cd}^{\ \ ef} R_{ef}^{\ \ ab} + c_3\, R_{abcd} R^{abc}_{\ \ \ e} R^{de} \\ & + c_4\, R_{abcd} R^{abcd} R + c_5\, R_{abcd} R^{ac} R^{bd} + c_6\, R_a^{\ b} R_b^{\ c} R_c^{\ a} + c_7\, R_a^{\ b} R_b^{\ a} R \\ & + c_8\, R^3 + c_{11}\, \nabla_a R_{bc} \nabla^a R^{bc} + c_{13}\, \nabla_a R \nabla^a R\Big)\,. \qquad (3.1)\end{aligned}$$

Using the discussed method, we must isolate the highest derivative of N and tune the coefficients present to allow us to integrate by parts. The only part of eq. (3.1) that will give rise to terms with three derivatives acting on N will be $c_{11}\, \nabla_a R_{bc} \nabla^a R^{bc} + c_{13}\, \nabla_a R \nabla^a R$ which when calculated give

$$\begin{aligned}& \frac{(N''')^2 r^9 f^3}{N L^9}(2c_{11}+4c_{13}) + \Bigg(\Bigg(\frac{-4r^9 f^3 N'}{N^2 L^9}(2c_{13}+c_{11}) + \frac{r^7 f^2}{L^9 N}\left(4c_{13}\left(5r^2 f' + 16 r f^2\right)\right. \\ & \left.\left.\left. + \frac{c_{11}}{2}\left(20 r^2 f' + 52 r f^2\right)\right)\right) N'' + \frac{(N')^2 r^7 f^2}{L^9 N^2}\Bigg(4c_{13}\left(-12rf - 3r^2 f'\right) + \frac{c_{11}}{2}(-36rf \\ & -12r^2 f')\Bigg) + \frac{r^7 f^2 N'}{L^9 N}\left(4c_{13}\left(3r^2 f'' + 12 f + 18 r f'\right) + \frac{c_{11}}{2}\left(12 r^2 f'' + 60 r f' + 36 f\right)\right) \\ & + \frac{r^7 f^2}{L^9}\left(c_{11}\left(2r^2 f''' + 18 r f'' + 30 f'\right) + 4c_{13}\left(12 r f'' + r^2 f''' + 30 f'\right)\right) N''' + \Bigg(\frac{r^7 N' f^2}{L^9 N^2}\times \\ & \times\left(-4c_{13}\left(5r^2 f' + 16 r f\right) + \frac{c_{11}}{2}\left(-52 r f - 20 r^2 f'\right)\right) + \frac{r^9 f^3 (N')^2}{L^9 N^3}(2c_{11}+4c_{13}) \\ & + \frac{r^5 f}{L^9 N}\left(c_{13}\left(5r^2 f + 16 r f\right)^2 + \frac{c_{11}}{2}\left(196 r^2 f^2 + 130 r^3 f f' + 25 r^4 (f')^2\right)\right)\Bigg)(N'')^2 \\ & + \Bigg(\frac{r^5 f N'}{N L^9}\left(c_{13} f\left(-4r^2\left(12 r f'' + r^2 f''' + 30 f'\right) + 2\left(5r^2 f' + 16 r f\right)\left(3r^2 f'' + 12 f\right.\right.\right. \\ & \left.\left.\left. + 18 r f'\right)\right) + \frac{c_{11}}{2}\left(486 r^2 f f' + 150 r^3 (f')^2 + 30 r^4 f'' f' + 312 r f^2 - 4 r^4 f f''' + 42 r^3 f f''\right)\right)\end{aligned}$$

$$
\begin{aligned}
&+\frac{r^7 f^2 (N')^3}{L^9 N^3}\left(\frac{c_{11}}{2}\left(12r^2 f' + 36rf\right) - 4c_{13}\left(-12rf - 3r^2 f'\right)\right) + \frac{2f(N')^2}{L^9 N^2}\left(-2r^7 f c_{13}\times\right.\\
&\times\left(3r^2 f'' + 12f + 18fr'\right) + r^5 c_{13}\left(-12rf - 3r^2 f'\right)\left(5r^2 f' + 16rf\right) + \frac{c_{11} r^5}{2}\left(-12r^4 f f''\right.\\
&\left.\left. 30r^4 (f')^2 - 300r^2 f^2 - 228r^3 f f'\right)\right) + \frac{c_{11} f r^5}{2L^9}\left(150 (f')^2 f^2 + 90r^3 f' f'' + 10r^4 f' f'''\right.\\
&\left.+510rff' + 258r^2 f'' + 26r^3 f f'''\right) + \frac{2r^5 c_{13} f}{L^9}\left(5r^2 f' + 16rf\right)\left(12rf'' + r^2 f''' + 30f'\right)\Bigg)\times\\
&\times N'' + \frac{fr^5}{L^9}\left(\frac{c_{11}}{2}\left(r^4 (f''')^2 + 90r^2 (f'')^2 + 18r^3 f'' f''' + 450 (f')^2 + 30r^2 f' f''' + 360r\times\right.\right.\\
&\left.\left.\times f' f''\right) + c_{13}\left(12rf'' r^2 f''' + 30f'\right)^2\right) N + \frac{(N')^4 r^5 f}{L^9 N^3}\left(\frac{c_{11}}{2}\left(132r^2 f^2 + 9r^4 (f')^2 + 54r^3\times\right.\right.\\
&\left.\left.\times f f'\right) + c_{13}\left(-12rf - 3r^2 f'\right)^2\right) + \frac{(N')^3 r^5 f}{L^9}\left(2c_{13}\left(-12rf - 3r^2 f'\right)\left(3r^2 f'' + 12f\right.\right.\\
&\left.\left.+18rf'\right) + \frac{c_{11}}{2}\left(-318r^2 f f' - 120rf^2 - 90r^3 (f')^2 - 54r^3 f f'' - 18r^4 f'' f'\right)\right) + \frac{N' r^5 f}{L^9}\times\\
&\times\left(2c_{13}\left(3r^2 f'' 12f + 18f' r\right)\left(12rf'' + r^2 f''' + 30f'\right) + \frac{c_{11}}{2}\left(18fr^2 f''' + 54r^3 (f'')^2\right.\right.\\
&\left.\left.+390r^2 f' f'' + 6r^4 f'' f''' + 450 f f' + 600 (f')^2 + 30r^3 f' f''' + 198frf''\right)\right) + \frac{r^5 f (N')^2}{NL^9}\times\\
&\times\left(c_{13}\left(2\left(-12rf - 3r^2 f'\right)\left(12rf'' + r^2 f''' + 30f'\right) + \left(3r^2 f'' + 12f + rf'\right)^2\right)\right.\\
&+\frac{c_{11}}{2}\left(9r^4 (f'')^2 - 18r^3 f f''' + 78rff' - 120r^2 f f'' + 186 (f')^2 r^2 - 6r^4 f' f''' + 36r^3 f' f''\right.\\
&\left.\left.+180f^2\right)\right). \qquad (3.2)
\end{aligned}
$$

As is obvious from eq. (3.2) tuning c_{11} and c_{13} to allow for partial integration with respect to r would require both to be set to zero. Iterating the process to successively lower numbers of derivatives on N and integrating by parts so that we arrive at the following expression with ambiguous coefficients, up to total derivatives, involving only $N(r)$ and powers of r and derivatives of f :

$$
\begin{aligned}
\sqrt{-g}\mathcal{Z} \;=\; &\frac{-N}{4L^9}\left(\left(8r^9 (c_{11} + 2c_{13}) f^2 f^{(6)} + \left(r^8 (236c_{11} + 496c_{13}) f^2 + r^9 (56c_{13} + 28c_{13})\;\times\right.\right.\right.\\
&\times\; ff')) f^{(5)} + \left(r^9 (24c_3 + 48c_2 + 20c_{11} + 48c_8 + 12c_6 + 24c_7 + 12c_5 + 48c_4\right.\\
&+c_{13}) ff'' + 8r^9 (c_{11} + 2c_{13}) (f')^2 + r^7 (5056c_{13} + 384c_4 + 96c_6 + 2264c_{11}\\
&+96c_2 + 96c_3 + 288c_7 + 960c_8 + 72c_5) f^2 + r^8 (72c_5 + 288c_4 + 120c_3
\end{aligned}
$$

$$
\begin{aligned}
&+1312c_{13} + 192c_2 + 480c_8 + 192c_7 + 608c_{11} + 84c_6)\, f f')\, f^{(4)} + \big(r^8\,(48c_4 \\
&+12c_5 + 48c_8 + 24c_7 + 16c_{13} + 12c_6 + 8c_{11} + 24c_3 + 48c_2)\, f\,(f''')^2 + \big(r^8 \times \\
&\times (240c_8 + 96c_7 + 60c_3 + 100c_{11} + 96c_2 + 224c_{13} + 36c_5 + 144c_4 + 42c_6)\,(f')^2 \\
&+r^7\,(1494c_5 + 6069c_4 + 3054c_2 + 4380c_7 + 4012c_{11} + 9776c_{13} + 12000c_8 \\
&+1794c_6 + 2340c_3 + 18c_1)\, f f' + \big(r^9\,(6c_5 + 24c_2 + 24c_4 + 12c_7 + 24c_8 + 12c_3 \\
&+6c_6)\, f' + r^8\,(1296c_2 + 920c_{13} + 384c_5 + 708c_3 + 388c_{11} + 2016c_8 + 1536c_4 \\
&+414c_6 + 888c_7)\, f)\, f'' + r^6\,(1584c_6 + 8140c_{11} + 36c_1 + 1440c_2 + 1212c_5 \\
&+4944c_7 + 1488c_3 + 5952c_4 + 17280c_8 + 19712c_{13})\, f^2\big)\, f''' + r^9\,(-2c_5 - 8c_8 \\
&-2c_6 - 4c_7 - 8c_4 - 4c_3 - 8c_2)\,(f'')^3 + \big(r^8\,(42c_3 + 84c_8 + 21c_6 + 84c_2 + 42c_7 \\
&+21c_5 + 84c_4)\, f' + r^7\,(10896c_8 + 4260c_7 + 18c_1 + 6384c_4 + 1308c_{11} + 3696c_{13} \\
&+1608c_5 + 2700c_3 + 4488c_2 + 1842c_6)\, f)\,(f'')^2 + \big(r^6\,(26544c_{13} + 306c_1 \\
&+28092c_7 + 8790c_5 + 84000c_8 + 10740c_6 + 8916c_{11} + 17472c_2 + 12600c_3 \\
&+35088c_4)\, f f' + r^5\,(324c_1 + 5436c_5 + 564c_2 + 6960c_6 + 82560c_8 + 8676c_{11} \\
&+22608c_7 + 6069c_3 + 24384c_4 + 24192c_{13})\, f^2 + r^7\,(264c_{11} + 252c_5 + 1056c_4 \\
&+720c_7 + 300c_6 + 624c_2 + 1920c_8 + 672c_{13} + 408c_3)\,(f')^2\big)\, f'' + r^3\,(-128c_3 \\
&-640c_4 - 6400c_8 - 1280c_7 - 256c_6 - 48c_1 - 256c_5 - 64c_2)\, f^3 + r^6\,(434c_3 \\
&+1240c_4 + 301c_5 + 950c_7 + 592c_2 + 2800c_8 + 9c_1 + 361c_6)\,(f')^3 + r^5\,(10122c_6 \\
&+10236c_3 + 558c_1 + 10080c_{13} + 2160c_{11} + 29040c_7 + 12480c_2 + 8202c_5 \\
&+31296c_4 + 96000)\,(f')^2 + r^4\,(-900c_{11} + 5904c_3 + 24240c_7 + 504c_1 \\
&+23520c_4 + 7248c_6 + 5748c_5 + 91200c_8 + 5232c_2))\, f^2 f'. \qquad (3.3)
\end{aligned}
$$

By the method we used to eliminate the c_{11} and c_{13} coefficients *i.e.,* requiring that we can integrate by parts on $N(r)$, we choose the c_i's to take the following values

1. $c_3 = -\frac{9}{7}\,c_1 - \frac{60}{7}\,c_2$
2. $c_4 = \frac{3}{8}c_1 + \frac{3}{2}c_2$
3. $c_5 = \frac{15}{7}\,c_1 + \frac{72}{7}\,c_2$
4. $c_6 = \frac{64}{7}\,c_2 + \frac{18}{7}\,c_1$
5. $c_7 = -\frac{33}{14}\,c_1 - \frac{54}{7}\,c_2$
6. $c_8 = \frac{15}{56}c_1 + \frac{11}{14}c_2$
7. $c_{11} = 0$
8. $c_{13} = 0$

Inserting the values for the coefficients in the above table into eq. (3.3), we find that that the third order Lagrangian simplifies to

$$\sqrt{-g}\mathcal{Z} = \frac{12}{7}\frac{N(r)}{L^9}(c_1 + 2c_2)(r^4 f^3)'. \tag{3.4}$$

While it is disheartening to find that we do not have a complete specification of the coefficients in terms a single parameter, we are free to choose the values of the two remaining c_i's. Explicitly then, if we choose $c_1 = 1,\ c_2 = 0$, the new curvature-cubed interaction takes the form

$$\begin{aligned}\mathcal{Z}_5 \;&=\; R_{a\ b}^{\ c\ d}R_{c\ d}^{\ e\ f}R_{e\ f}^{\ a\ b} + \frac{1}{56}\big(21\,R_{abcd}R^{abcd}R - 72\,R_{abcd}R^{abc}{}_{e}R^{de} \\ &\quad +120\,R_{abcd}R^{ac}R^{bd} + 144\,R_a{}^bR_b{}^cR_c{}^a - 132\,R_a{}^bR_b{}^aR + 15\,R^3\big)\,, \end{aligned}\tag{3.5}$$

or with $c_1 = 0,\ c_2 = 1$,

$$\begin{aligned}\mathcal{Z}'_5 \;&=\; R_{ab}{}^{cd}R_{cd}{}^{ef}R_{ef}{}^{ab} + \frac{1}{14}\big(21\,R_{abcd}R^{abcd}R - 120\,R_{abcd}R^{abc}{}_{e}R^{de} \\ &\quad +144\,R_{abcd}R^{ac}R^{bd} + 128\,R_a{}^bR_b{}^cR_c{}^a - 108\,R_a{}^bR_b{}^aR + 11\,R^3\big)\,. \end{aligned}\tag{3.6}$$

Note that the six-dimensional Euler density can be inferred by setting $c_1 = -2c_2$ in which case eq. (3.4) vanishes, as it mustif evaluated for the six-dimensional Euler density. A standard normalization for the six-dimensional Euler density is [37, 26]:

$$\begin{aligned}\mathcal{X}_6 \;&=\; \frac{1}{8}\,\varepsilon_{abcdef}\,\varepsilon^{ghijkl}\,R_{ab}{}^{gh}\,R_{cd}{}^{ij}\,R_{ef}{}^{kl} \\ &=\; 4\,R_{ab}{}^{cd}R_{cd}{}^{ef}R_{ef}{}^{ab} - 8\,R_{a\ b}^{\ c\ d}R_{c\ d}^{\ e\ f}R_{e\ f}^{\ a\ b} - 24\,R_{abcd}R^{abc}{}_{e}R^{de} + 3\,R_{abcd}R^{abcd}R \\ &\quad +24\,R_{abcd}R^{ac}R^{bd} + 16\,R_a{}^bR_b{}^cR_c{}^a - 12\,R_a{}^bR_b{}^aR + R^3, \end{aligned}\tag{3.7}$$

where in the first line, ε is the completely antisymmetric tensor in six dimensions and hence the second expression only applies for $D = 6$. However, the first line also makes clear that this expression should vanish when evaluated in five (or lower) dimensions. This normalization corresponds to the choice $c_2 = 4$ and $c_1 = -8$. We also note that $\mathcal{X}_6 = 4\mathcal{Z}_5' - 8\mathcal{Z}_5$.

3.2 Generalizing to Higher Dimensions

At this point we can move our attention to what this process yields in dimensions greater than five, and then use that to obtain an expression for what the values of these coefficients should be for an arbitrary, D-dimensional spacetime. What we find is that after fixing the c_i's to allow judicious integration by parts as above we are consistently left with a form for $\sqrt{-g}\mathcal{Z}$ looking like

$$\sqrt{-g}\mathcal{Z} \sim \frac{N(r)}{L^D} \big(a_1 c_1 + a_2 c_2)(r^{D-1} f(r)^3\big)',$$

where a_1, a_2 are certain numerical coefficients which depend on the dimension of the spacetime. Further, this result requires the values of the other c_i's to be specified in terms of c_1 and c_2 with a form that can schematically be written:

$$\frac{aD+b}{cD^2+eD+f}\, c_1 + \frac{a'D+b'}{c'D^2+e'D+f'}\, c_2. \tag{3.8}$$

For simplicity, we take $c_1 = 1$ and $c_2 = 0$, since we are free to choose them to be anything we want, and we shall do so from here on out. At this point we are left with a simple exercise of determining the coefficients in eq. (3.8). Taking for example c_5, we have 5 unknowns in the general formula, and so we calculate the values for $c_5\,(c_1 = 1, c_2 = 0)$ for $D = 6 \ldots 10$. In doing so, we then have a system of equations completely determining the constants a, b, c, e, f in (3.8) for c_5. Repeating for each of the c_i's gives us that for an arbitrary, D, dimensional space time our new R^3 theory has :

1. $c_3(D) = -\frac{3(D-2)}{(2D-3)(D-4)}$
2. $c_4(D) = \frac{3(3D-8)}{8(2D-3)(D-4)}$
3. $c_5(D) = \frac{3D}{(2D-3)(D-4)}$
4. $c_6(D) = \frac{6(D-2)}{(2D-3)(D-4)}$
5. $c_7(D) = -\frac{3(3D-4)}{2(2D-3)(D-4)}$
6. $c_8(D) = \frac{3D}{8(2D-3)(D-4)}$
7. $c_{11} = 0$
8. $c_{13} = 0$

These expressions for the coefficients allow us to write down a general form of $\mathcal{Z}_D$ with the choice $c_1 = 1,\ c_2 = 0$:

$$\begin{aligned}\mathcal{Z}_D \;=\; & R_{a\ b}^{\ c\ d} R_{c\ d}^{\ e\ f} R_{e\ f}^{\ a\ b} + \frac{1}{(2D-3)(D-4)}\left(\frac{3(3D-8)}{8} R_{abcd}R^{abcd}R - 3(3D-2)\times\right. \\ & \times R_{abcd}R^{abc}{}_{e}R^{de} + 3D R_{abcd}R^{ac}R^{bd} + 6(D-2)R_a{}^{b}R_b{}^{c}R_c{}^{a} - \frac{3(3D-4)}{2}\times \\ & \left.\times R_a{}^{b}R_b{}^{a}R + \frac{3D}{8}R^3\right). \end{aligned} \tag{3.9}$$

It is straight forward to verify that this result reduces to eq. (3.5) for $D = 5$. In principle, one can generalize this expression for $D > 6$ by adding another component proportional to the six-dimensional Euler character (3.7). This would be equivalent to leaving c_2 arbitrary in the above analysis.

3.3 Field Equations

In this section we take the next logical step after writing down a gravitational action. We calculate the field equations for pseudo-topological gravity as we had for Einstein and Gauss-Bonnet gravity. That is we want to vary the action including Einstein, Gauss-Bonnet, and pseudo-topological terms with respect to the inverse metric tensor g^{ab}. From the field equation, we can determine if the theory contains ghost gravitons by computing the linearized variation of these equations around a suitable background and looking at the highest order of derivatives acting on the metric perturbation h_{ab}. We will only see up to four derivatives acting on h_{ab} since the terms that would generate six derivatives have had their coefficients tuned to zero, $c_{11} = c_{13} = 0$. To start we write down the action for the pseudo-topological theory with arbitrary coefficients which we will then fix to our values arbitrary dimensions

$$
\begin{aligned}
I \;\; = \;\; & \int \mathrm{d}^D x \sqrt{-g} \Big(R - 2\Lambda + \tilde{\lambda}_D \; \left(R_{abdc} R^{abdc} - 4 R_{ab} R^{ab} + R^2 \right) + \tilde{\mu}_D \; \big(c_8 R^3 \\
& + c_2 R_{ab}{}^{dc} R_{dc}{}^{mn} R_{mn}{}^{ab} + c_1 R_{a}{}^{d}{}_{b}{}^{c} R_{c}{}^{m}{}_{d}{}^{n} R_{m}{}^{a}{}_{n}{}^{b} + c_3 R_{abdc} R^{abd}{}_{m} R^{cm} \qquad (3.10) \\
& + c_5 R_{abdc} R^{ad} R^{bc} + c_4 R_{abdc} R^{abdc} R + c_6 R_a{}^b R_b{}^d R_d{}^a + c_7 s R_a{}^b R_b{}^a R \big) \Big) \,,
\end{aligned}
$$

where

$$
\tilde{\lambda}_D = \frac{\lambda L^2}{(D-3)(D-4)}. \tag{3.11}
$$

and

$$
\tilde{\mu}_D = \frac{24\,(2D-3)\,\mu L^4}{(D-3)(D-6)(D-2)^2(3D^2-15D+16)}, \tag{3.12}
$$

are the values of the interaction parameters for arbitrary dimension. We do not have to be concerned about the divergences of $\tilde{\mu}_D$ and $\tilde{\lambda}_D$ because they occur when the interaction terms are topological or total derivatives. In order to obtain the field equations for eq. (3.11), we calculate the functional derivative with respect to the metric of this action, where we have used results obtained in [26]:

$$
\begin{aligned}
\frac{1}{\sqrt{-g}} \frac{\delta I}{\delta g^{ab}} \;\; = \;\; & \left[\Lambda\, g_{ab} + R_{ab} - \frac{1}{2} R\, g_{ab} \right] + \tilde{\lambda}_D \Big[2 \big(R R_{ab} - 2 R^{dc} R_{dacb} - 2 R_{ad} R^{d}{}_{b} + R_{a}{}^{dcm} R_{bdcm} \\
& - \frac{1}{2} g_{ab} \left(R^2 - 4 R_{dc} R^{dc} + R_{dcmn} R^{dcmn} \right) \Big] + \tilde{\mu}_D \; \big[\left(6c_8 + c_7 + 2c_4 \right) R\, R_{;ab} \\
& + \left(6c_8 + c_7 + \frac{1}{4} c_5 \right) R_{;a} R_{;b} - 3c_8 R^2 R_{ab} + \left(-c_7 - \frac{1}{2} c_5 \right) \left(\Box R \right) R_{ab} \\
& + \frac{1}{2} \left(2c_7 + \frac{3}{2} c_6 + c_3 + c_5 \right) \left(R_{;da} R^{d}_{\ b} + R_{;db} R^{d}_{\ a} \right) + \left(-c_7 - 4c_4 \right) R \Box R_{ab} \\
& + \frac{1}{2} \left(2c_7 + c_5 \right) R^{dc} \left(R_{dc;ab} + R_{dc;ba} \right) + \frac{1}{2} \left(2c_7 + \frac{3}{2} c_6 + 8c_4 + c_3 \right) R_{;d} \big(R^{d}{}_{a;b} \\
& + R^{d}{}_{b;a} \big) - s R^{dc} R_{dc} R_{ab} + \left(-2c_7 - c_5 - 8c_4 - c_3 \right) R_{;d} R_{ab}{}^{;d} + \big(2c_7 + 2c_5 \\
& + 3c_1 \big) R^{dc}{}_{;a} R_{dc;b} + \left(c_5 + 2c_3 \right) R^{dc} R_{da} R_{cb} + \left(-2c_7 - 4c_4 \right) R R^{dc} R_{dacb} \\
& + \frac{1}{2} \left(-3c_6 - 2c_3 \right) \left(R_{da} \Box R^{d}{}_{b} + R_{db} \Box R^{d}{}_{a} \right) + \frac{1}{2} \left(3c_6 + 2c_3 \right) R^{dc} \big(R_{da;bc} \\
& + R_{db;ac} \big) + \frac{1}{2} \left(3c_6 - 2c_5 + 2c_3 - 6c_1 \right) \left(R^{dc}{}_{;;a} R_{bd;c} + R^{dc}{}_{;b} R_{ad;c} \right) + \big(-3c_6
\end{aligned}
$$

$$
\begin{aligned}
&-4c_3 - 12c_2)\, R^d{}_{a;c} R_{db}{}^{;c} + \frac{1}{2}\left(3c_6 + 2c_5 + 2c_3\right) R^{dc}\left(R^m{}_a R_{mcdb} + R^m{}_b R_{mcda}\right) \\
&+ \left(-2c_5 - 2c_3\right) R^{dc} R_{ab;dc} + \left(-c_5 - 3c_1\right) \Box R^{dc} R_{dacb} + \frac{1}{2}\left(-2c_5 - 6c_1\right) \times \\
&\times \left(R^{dc;m}{}_{;a} R_{mcdb} + R^{dc;m}{}_{;b} R_{mcda}\right) + \frac{1}{2}\left(-2c_5 - 4c_3 + 6c_1\right) R^{dc;m}\left(R_{mcda;b}\right. \\
&\left.+ R_{mcdb;a}\right) + \left(2c_5 + \frac{3}{2} - 6c_2\right) R^{dc} R^{mn}{}_{da} R_{mncb} + \left(-2c_5 - 2c_3 + 3c_1\right) \times \\
&\times R^{dm} R^c{}_m R_{dacb} + \left(-2c_5 - 3c_1\right) R_{dc} R^{dcm}{}_a R^c{}_{mnb} + \left(-4c_4 - c_3 + \frac{3}{2}c_1\right) \times \\
&\times R^{;dc} R_{dacb} + \frac{1}{2}\left(2c_4 + \frac{1}{2}c_3 - \frac{3}{4}c_1\right) R^{dcmn}\left(R_{dcmn;ab} + R_{dcmn;ba}\right) + \left(2c_4\right. \\
&\left.+ \frac{1}{2}c_3 + \frac{3}{4}c_1 + 3c_2\right) R^{dcmn}{}_{;a} R_{dcmn;b} + 4c_4 R R_{da} R^d{}_b - 2c_4 R R^{dcm}{}_a R_{dcmb} \\
&- k R_{ab} R^{dcmn} R_{dcmn} + \frac{1}{2}\left(4c_3 + 24c_2\right)\left(R^d{}_a{}^{;cm} R_{dcmb} + R^d{}_b{}^{;cm} R_{dcma}\right) + \left(-c_3\right. \\
&\left.-6c_2\right) R^{dcm}{}_{a;n} R_{dcmb}{}^{;n} + \left(2c_3 + c_5 + 3c_1 + 12c_2\right) R^d{}_{a;c} R^c{}_{b;d} + \left(2c_3 - 3c_1\right) \times \\
&\times R_{dc} R^{dcmn} R_{manb} + \left(c_3 - \frac{3}{2}c_1\right) R^{dcmn} R_{dcqa} R_{mn}{}^q{}_b + \left(4c_3 - 9c_1 + 12c_2\right) \times \\
&\times R^{dmcn} R^q{}_{dca} R_{qmnb} + \left(-c_3 + \frac{3}{2}\right) R^{dcm}{}_n R_{dcmq} R^n{}_a{}^q{}_b + \left(-2c_5 - 2c_3\right) \times \\
&\times R^{dc;m} R_{dacb;m} + \left(-2c_3 + 3c_1\right) R^d{}_{(a} R^{cmn}{}_{|d|} R_{|cmn|b)} + g_{ab}\left[\left(-6c_8 - \frac{1}{2}c_7\right) \times\right. \\
&\times R \Box R + \left(-6c_8 - c_7 - \frac{3}{8}\right) R_{;d} R^{;d} + \frac{1}{2} c R^3 + \left(-c_7 - \frac{3}{2}c_6 + \frac{1}{2}c_5\right) R_{;dc} R^{dc} \\
&+ \left(-2c_7 - c_5\right) R_{dc} \Box R^{dc} + \left(-2c_7 - 2c_5 - c_3\right) R_{dc;m} R^{dc;m} + \frac{1}{2} s R R_{dc} R^{dc} \\
&+ \left(-\frac{3}{2}c_6 + 2c_5 + c_3\right) R_{dc;m} R^{dm;c} + \left(c_5 - c_6\right) R_{dc} R^d_m R^{cm} + \left(\frac{3}{2}c_6 - \frac{1}{2}c_5\right) \times \\
&\times R_{mc} R_{mn} R^{dmcn} + \left(-c_5 - 8c_4 - c_3\right) R_{dc;mn} R^{dmcn} + \left(-2c_4 - \frac{1}{4}c_3\right) \times \\
&\times R_{dcmn;q} R^{dcmn;q} + \frac{1}{2} c_4 R R_{dcmn} R^{dcmn} - 4k R_{dc} R^d{}_{mnq} R^{cmnq} + \left(2c_4 + \frac{1}{4}c_3\right. \\
&\left.\left. + \frac{1}{2}c_2\right) R_{dcmn} R^{dcqs} R^{mn}{}_{qs} + \left(8c_4 + c_3 + \frac{1}{2}c_1\right) R_{dmcn} R^d{}_q{}^c{}_s R^{mqns}\right]\Bigg] . \qquad (3.13)
\end{aligned}
$$

While the above form is nice to write down, it is more illuminating to consider the values of the coefficients for an arbitrary dimensional spacetime that was found in section (3.2). Using the D dimensional values for the c_i's will allow us to find the field equations by substituting in whatever the dimension of the spacetime we are considering is. Swapping out the c_i's (with $c_1 = 1$, $c_2 = 0$) and simplifying the coefficients gives the field equations expressed as:

$$
\begin{aligned}
\frac{1}{\sqrt{-g}}\frac{\delta I}{\delta g^{ab}} = & \left[\Lambda\, g_{ab} + R_{ab} - \frac{1}{2} R\, g_{ab}\right] + \tilde{\lambda}_D \Big[2\left(R R_{ab} - 2R^{dc} R_{dacb} - 2R_{ad} R^{d}{}_{b} + R_{a}{}^{dcm} R_{bdcm}\right. \\
& \left.\left. -\frac{1}{2} g_{ab}\left(R^2 - 4R_{dc}R^{dc} + R_{dcmn}R^{dcmn}\right)\right] + \frac{\tilde{\mu}_D}{(2D-3)(D-4)} \times \right. \\
& \times \left[\left(\frac{-3(D-4)}{2}\right) R_{;a}R_{;b} - \left(\frac{9}{8}D\right) R^2 R_{ab} + (3(D-2))(\Box R) R_{ab}\right. \\
& + 6R\Box R_{ab} - 3(D-2) R^{dc}\left(R_{dc;ab} + R_{dc;ba}\right) + (3(D-4)) R_{;d}\left(R^{d}{}_{a;b}\right. \\
& \left. + R^{d}{}_{b;a}\right) + 6R_{;d}R_{ab}{}^{;d} + 6\left(D^2 - 6D + 8\right) R^{dc}{}_{;a}R_{dc;b} - 3(D-4) R^{dc}R_{da}R_{cb} \\
& + \frac{3(3D-4)}{2} R^{dc}R_{dc}R_{ab} + \left(\frac{9D}{2}\right) R R^{dc} R_{dacb} - 6(D-2)\left(R_{da}\Box R^{d}_{b}\right. \\
& \left. + R_{db}\Box R^{d}_{a}\right) + 6(D-2) R^{dc}\left(R_{da;bc} + R_{db;ac}\right) - 6\left(D^2 - 6D + 8\right) \times \\
& \times \left(R^{dc}{}_{;a}R_{bd;c} + R^{dc}{}_{;b}R_{ad;c}\right) - 6(D-2) R^{d}{}_{a;c}R_{db}{}^{;c} + 3(3D-4) R^{dc}\left(R^{m}_{a}R_{mcdb}\right. \\
& \left. + R^{m}_{b}R_{mcda}\right) - 12R^{dc}R_{ab;dc} - 6\left(D^2 - 5D + 6\right)\Box R^{dc}R_{dacb} - 6\left(D^2\right. \\
& \left. -5D + 6\right)\left(R^{dc;m}{}_{a}R_{mcdb} + R^{dc;m}{}_{b}R_{mcda}\right) + 6\left(D^2 - 5D + 4\right) R^{dc;m}\left(R_{mcda;b}\right. \\
& \left. + R_{mcdb;a}\right) + \frac{3\left(2D^2 - 7D + 12\right)}{2} R^{dc}R^{mn}{}_{da}R_{mncb} + 3\left(2D^2 - 11D + 8\right) \times \\
& \times R^{dm}R^{c}{}_{m}R_{dacb} - 3\left(2D^2 - 11D + 16\right) R_{dc}R^{dmn}{}_{a}R^{c}{}_{mnb} - 3\left(D^2 - 6D\right. \\
& \left. + 8\right) R^{;dc}R_{dacb} - \frac{3}{4}\left(D^2 - 6D + 8\right) R^{dcmn}\left(R_{dcmn;ab} + R_{dcmn;ba}\right) + \frac{3}{2}\left(D^2\right. \\
& \left. -5D + 4\right) R^{dcmn}{}_{;a}R_{dcmn;b} + \left(\frac{3(3D-8)}{2}\right) R R_{da}R^{d}_{b} - \left(\frac{3(3D-8)}{4}\right) \times \\
& \times R R^{dcm}{}_{a}R_{dcmb} - \frac{3(3D-8)}{8} R_{ab}R^{dcmn}R_{dcmn} - 6(D-2)\left(R^{d}{}_{a}{}^{;cm}R_{dcmb}\right. \\
& \left. + R^{d}{}_{b}{}^{;cm}R_{dcma}\right) + 3(D-2) R^{dcm}{}_{a;n}R_{dcmb}{}^{;n} + 6\left(D^2 - 6D + 8\right) R^{d}{}_{a;c}R^{c}{}_{b;d} \\
& - 3\left(2D^2 - 9D + 8\right) R_{dc}R^{dcmn}R_{manb} - \left(\frac{3}{2}\left(2D^2 - 9D + 8\right)\right) R^{dcmn}R_{dcqa}R_{mn}{}^{q}{}_{b} \\
& - \left(3\left(6D^2 - 29D + 28\right)\right) R^{dmcn}R^{q}{}_{dca}R_{qmnb} + \left(\frac{3}{2}\left(2D^2 - 9D + 8\right)\right) \times
\end{aligned}
$$

$$\times R^{dcm}{}_{n} R_{dcmq} R^{n\ q}_{\ a\ b} + g_{ab}\left[-3R\Box R - \frac{3}{2} R_{;d} R^{;d} + \frac{3D}{16} R^3 - 3\,(D-4)\,\times\right.$$
$$\times R_{;dc} R^{dc} + 6\,(D-2)\,R_{dc}\Box R^{dc} + 6\,(D-3)\,R_{dc;m} R^{dc;m} - \frac{3\,(3D-4)}{4}\times$$
$$\times R R_{dc} R^{dc} - 6\,(D-4)\,R_{dc;m} R^{dm;c} - 3\,(D-4)\,R_{dc} R^{d}{}_{m} R^{cm}$$
$$-6\,(D-2)\,R_{dc;mn} R^{dmcn} - \frac{3\,(D-3)}{2} R_{dcmn;q} R^{dcmn;q} + \frac{3\,(3D-8)}{16}\times$$
$$\times R R_{dcmn} R^{dcmn} + \left(\frac{3\,(5D-12)}{2}\right) R_{dc} R_{mn} R^{dmcn} - \frac{3\,(3D-8)}{2}\times$$
$$\times R_{dc} R^{d}{}_{mnq} R^{cmnq} + \left(\frac{3\,(D-3)}{2}\right) R_{dcmn} R^{dcqs} R^{mn}{}_{qs} + \left(\frac{2D^2+D-24}{2}\right)\times$$
$$\left.\times R_{dmcn} R^{d\ c}_{\ q\ s} R^{mqns}\right]\Big]\,. \tag{3.14}$$

A useful comparison to make is between eq. (3.14) and the field equations for 3^{rd} order Lovelock gravity where $\mathcal{Z}_D$ is replaced by $\mathcal{X}_6$ by choosing $c_1 = -2c_2$, $c_2 = 4$ in eq. (3.11). The field equations for the variation of $\mathcal{X}_6$ with respect to the metric are, as expected, much simpler than those for $\mathcal{Z}_D$:

$$\begin{aligned}
\frac{1}{\sqrt{-g}}\frac{\delta I}{\delta g^{ab}} \;=\;& \left[\Lambda\, g_{ab} + R_{ab} - \frac{1}{2} R\, g_{ab}\right] + \tilde{\lambda}_D \Big[2\left(R R_{ab} - 2R^{dc} R_{dacb} - 2R_{ad} R^{d}{}_{b} + R_{a}{}^{dcm} R_{bdcm}\right.\\
& \left.\left. -\frac{1}{2} g_{ab}\left(R^2 - 4R_{dc} R^{dc} + R_{dcmn} R^{dcmn}\right)\right]\right] + \tilde{\mu}_D\left[-\frac{3}{4} R^2 R_{ab} - 6R^{dc} R_{da} R_{cb}\right.\\
& +3R^{dc} R_{dc} R_{ab} + 3R R^{dc} R_{dacb} + 6R^{dc}\left(R^{m}_{a} R_{mcdb} + R^{m}_{b} R_{mcda}\right)\\
& +3R^{dc} R^{mn}{}_{da} R_{mncb} - 6R_{dc} R^{dmn}{}_{a} R^{c}{}_{mnb} + 3R R_{da} R^{d}_{b} - \frac{3}{2} R R^{dcm}{}_{a} R_{dcmb}\\
& -6R^{dm} R^{c}{}_{m} R_{dacm} - \frac{3}{4} R_{ab} R^{dcmn} R_{dcmn} - 6R_{dc} R^{dcmn} R_{manb}\\
& -3R^{dcmn} R_{dcqa} R_{mn\ b}^{\ \ \ q} + 3\left(R^{d}{}_{a} R^{cmn}{}_{d} R_{cmnb} + R^{d}{}_{b} R^{cmn}{}_{d} R_{cmna}\right)\\
& +6R^{dmcn} R^{q}{}_{dca} R_{qmnb} + 3R^{dcm}{}_{n} R_{dcmq} R^{n\ q}_{\ a\ b} + g_{ab}\left(\frac{1}{8} R^3 - \frac{3}{2} R R_{dc} R^{dc}\right.\\
& +2R_{dc} R^{d}_{m} R^{cm} + 3R_{dc} R_{mn} R^{dmcn} + \frac{3}{8} R R_{dcmn} R^{dcmn} - 3R_{dc} R^{d}{}_{mnq} R^{cmnq}\\
& \left.\left. +\frac{1}{2} R_{dcmn} R^{dcqs} R^{mn}{}_{qs} - R_{dmcn} R^{d\ c}_{\ q\ s} R^{mqns}\right)\right]\,.
\end{aligned} \tag{3.15}$$

The simpler form for eq. (3.15) is expected because of the special nature of the theory based on topological invariants. Eq. (3.15) matches the calculations of the field equations for the third order Lovelock gravity by [26, 41]. Here we note that the main difference between the pseudo-topological and Lovelock theories are that eq. (3.15) does not contain any derivatives of curvature tensors whereas eq. (3.14) does. The appearance of ∇R terms in eq. (3.14) indicates that there may be higher derivative terms that will display a pathology similar to other non-topological theories of gravity. However in the next section, we will explicitly show that there are situations where this is not the case.

3.3.1 The Linearized Theory

After having found field equations for the pseudo-topological action by taking the functional derivative with respect to the metric, we would like to determine if the second variation of the generic D dimensional cubic theory, obtained by substituting by $g_{ab} = g^0_{ab} + h_{ab}$ into eq. (3.14) where g^0_{ab} is a solution to eq. (3.14), contains non-vanishing terms of $\mathcal{O}(\Box^2 h)$ due to the presence $R\Box R$ terms . In contrast, we know that using the coefficients that give us $\mathcal{X}_6$ has at most 2 derivatives of the metric perturbation, $\mathcal{O}(\Box h)$. However we shall see that in the general case if we make a few motivated choice of the symmetries of the spacetime and fix the gauge, then there is a very nice cancelation of the troublesome terms giving us second order field equations for h_{ab}.

The Four Derivative Terms

The easiest way to approach the problem at hand is to focus our attention first on the terms with the highest number of derivatives acting on h_{ab}. Our goal is to elicit cancelation while trying to keep our consideration as general as possible. That is, we would like to make as few specifications of the terms and coefficients as possible. First, we gather all of the four derivative terms manifest in the second variation based on the ways the six indices of $h_{ab;cdef}$ can be contracted:

1. $h_{ab;cdef}$: (3.16)

$$\frac{3}{4}\frac{(D-4)(D-2)}{(2D-3)(D-4)}\left(2h_{dc;[nm](ab)}+h_{dn;cm(ab)}-h_{dm;cn(ab)}-h_{cn;dm(ab)}\right.$$
$$\left.+h_{cm;dn(ab)}\right)R^{dcmn}$$

2. $\Box(h_{ab;cd})$: $None$

3. $(\Box h_{ab})_{;cd}$: (3.17)

$$\frac{3(D-2)}{(2D-3)(D-4)}\left(R^{dc}(\Box h_{dc})_{;(ab)}-2R^{dc}\left(\Box h_{d(a}\right)_{;b)c}+\frac{2}{D-2}R^{dc}(\Box h_{ab})_{;dc}\right.$$
$$\left.+2\left(\Box h^{d}{}_{(a}\right)^{;cm}R_{|dcm|b)}+g_{ab}(\Box h_{dc})_{;mn}R^{dmcn}\right)$$

4. $h_{;abcd}$: (3.18)

$$\frac{3(D-2)}{(2D-3)(D-4)}\left(R^{dc}h_{;dc(ab)}-2R^{dc}h_{;d(ab)c}+\frac{2}{(D-2)}R^{dc}h_{;abdc}+2h^{;d}{}_{(a}{}^{cm}R_{|dcm|b)}\right.$$
$$\left.+g_{ab}h_{;dcmn}R^{dmcn}\right)$$

5. $h^{a}{}_{b;cade}$: (3.19)

$$\frac{3(D-2)}{(2D-3)(D-4)}\left(2R^{dc}\left(h^{m}{}_{d;(a|m|b)c}+h^{m}{}_{(a;|dm|b)c}-h^{m}{}_{d;cm(ab)}-h^{m}{}_{c;dm(ab)}\right)\right.$$
$$-\frac{4}{(D-2)}R^{dc}h^{m}{}_{(a;b)mdc}-2\left(h^{pd}{}_{;(a|p|}{}^{cm}R_{|dcm|b)}+h^{p}{}_{(a}{}^{;d}{}_{|p|}{}^{cm}R_{|dcm|b)}\right.$$
$$\left.\left.+g_{ab}h^{p}{}_{(d;c)pmn}R^{dcmn}\right)\right)$$

6. $\Box^2 h_{ab}$: (3.20)

$$\frac{3(D-2)}{(2D-3)(D-4)}\left(R_{d(a}\Box^2 h^{d}{}_{b)}+(D-3)\Box^2 h^{dc}R_{dacb}-\frac{1}{2}g_{ab}R_{dc}\Box^2 h^{dc}\right.$$
$$\left.-\frac{R}{(D-2)}\Box^2 h_{ab}\right)$$

7. $(\Box h)_{;cd}$: (3.21)

$$\frac{3(D-4)}{(2D-3)(D-4)}\left((D-2)(\Box h)^{;dc}R_{dacb}+g_{ab}(\Box h)_{;dc}R^{dc}\right)$$

8. $\Box(h_{;bc})$: (3.22)

$$\frac{3(D-2)}{(2D-3)(D-4)}\left(\frac{-R}{(D-2)}\Box(h_{;ab})+2R_{d(a}\Box\left(h^{;d}{}_{b)}\right)+(D-3)R_{dacb}\Box\left(h^{;dc}\right)\right.$$
$$\left.-g_{ab}R_{dc}\Box\left(h^{;dc}\right)\right)$$

9. $\quad \Box\left(h^{a}{}_{b;ca}\right):$ (3.23)

$$\frac{3\left(D-2\right)}{\left(2D-3\right)\left(D-4\right)}\left(\frac{2R}{\left(D-2\right)}\Box\left(h^{d}{}_{(a;b)d}\right)+R_{d(a}\Box\left(h^{md}{}_{;b)m}\right)+R_{d(a}\Box\left(h^{m}{}_{b)}{}^{;d}{}_{m}\right)\right.$$
$$\left.+2g_{ab}R_{dc}\Box\left(h^{n(d;c)}{}_{n}\right)-2\left(D-3\right)\Box\left(h^{n(d;c)}{}_{n}\right)R_{dacb}\right)$$

10. $\quad h^{ab}{}_{;abcd}:$ (3.24)

$$\frac{-3\left(D-4\right)}{\left(2D-3\right)\left(D-4\right)}\left(\left(D-2\right)h^{mn}{}_{;mn}{}^{dc}R_{dacb}+g_{ab}R^{dc}h^{mn}{}_{;mndc}\right)$$

11. $\quad \Box\left(h^{ab}{}_{;ab}\right):$ (3.25)

$$\frac{3\left(D-2\right)}{\left(2D-3\right)\left(D-4\right)}\left(R_{ab}-\frac{1}{\left(D-2\right)}g_{ab}R\right)\Box\left(h^{dc}{}_{;dc}\right)$$

12. $\quad \Box^{2}h:$ (3.26)

$$\frac{-3\left(D-2\right)}{\left(2D-3\right)\left(D-4\right)}\left(R_{ab}-\frac{1}{\left(D-2\right)}g_{ab}R\right)\Box^{2}h.$$

Whereupon, we find that using the fact that h_{ab} is symmetric and by the various symmetries of R_{abcd}, eq. (2.8), we have the outright cancelation of terms like $R^{dcmn}h_{dc;[nm](ab)}$ and $g_{ab}(\Box h_{dc})_{;mn}R^{dcmn}$, while the terms $R_{dcmb}h^{p}{}_{a}{}^{;d}{}_{p}{}^{cm}$ can eliminated from consideration among the 4-derivative terms by recasting it as 2-derivatives acting on a sum of products of R_{abdc} and h_{ab} by the definition of the Riemann Tensor $[\nabla_a,\nabla_b]u^c=R^{c}{}_{dba}u^d$. After the dust settles there is not as much cancelation as we would have liked, and we will have to make some further choices to try and deal with the problematic terms.

At this point, we can fix the gauge by making a familiar choice that h_{ab} have vanishing divergence ($h^{ab}{}_{;b}=0$) and be trace free ($h=0$) [29]. As is obvious from the expressions above, there will be a large number of terms killed off by making this choice. The terms with indices given by the contractions $h_{;abcd}$, $(\Box h)_{;cd}$, $\Box(h_{;cd})$, $h^{ab}{}_{;abcd}$, $\Box(h^{ab}{}_{;ab})$, $\Box^2 h$ vanish immediately. Furthermore, the contractions $h^{a}{}_{b;cade}$ and $\Box(h^{a}{}_{b;ca})$ vanish up to terms like $([\nabla_a,\nabla_c]h^{a}{}_{b})_{;de}$ and $\Box([\nabla_a,\nabla_c]h^{a}{}_{b})$. So, in the transverse traceless gauge we find that we are left with the following 4-derivative terms to reckon with

1. $$\frac{3\left(D-4\right)\left(D-2\right)}{4\left(2D-3\right)\left(D-4\right)}R^{dcmn}\left(h_{dn;cm(ab)}-h_{dm;cn(ab)}-h_{cn;dm(ab)}+h_{cm;dn(ab)}\right)\qquad(3.27)$$

2. $$\frac{3(D-2)}{(2D-3)(D-4)}\left(R^{dc}\left(\Box h_{dc}\right)_{;(ab)}-2R^{dc}\left(\Box h_{d(a}\right)_{;b)c}+\frac{2}{D-2}R^{dc}\left(\Box h_{ab}\right)_{;dc}\right. \tag{3.28}$$
$$\left.+2\left(\Box h^{d}_{(a}\right)^{;cm}R_{|dcm|b)}+g_{ab}\left(\Box h_{dc}\right)_{;mn}R^{dmcn}\right)$$

3. $$\frac{3(D-2)}{(2D-3)(D-4)}\left(2R_{d(a}\Box^2 h^{d}_{b)}+(D-3)\Box^2 h^{dc}R_{dacb}-\frac{1}{2}g_{ab}R_{dc}\Box^2 h^{dc}\right. \tag{3.29}$$
$$\left.-\frac{R}{(D-2)}\Box^2 h_{ab}\right).$$

In spite of the number of terms killed off by the choice of gauge, we still have four derivative terms remaining. A possible choice to make in order to eliminate the remaining four derivative terms is hinted at by that our work on the previous calculations has been performed in AdS which is a maximally symmetric spacetime. Furthermore, we can motivate this by the fact that in the section where we are deciding whether $\mathcal{Z}$ is a topological invariant or not we found that on maximally symmetric manifolds it does indeed look topological while in general deformations of the space will cause changes in the value of integrating $\sqrt{-g}\mathcal{Z}$ over the manifold. For a maximally symmetric spacetime we have a very simple form for the curvature tensors [7]

$$R_{abcd} = -\frac{1}{\tilde{L}^2}(g_{ac}g_{bd}-g_{ad}g_{bc}), \tag{3.30}$$
$$R_{ab} = -\frac{D-1}{\tilde{L}^2}g_{ab}, \tag{3.31}$$
$$R = -\frac{D(D-1)}{\tilde{L}^2}. \tag{3.32}$$

Here we note that we are using $\tilde{L}$ to denote the AdS length scale. The distinction in notation is that while we would ordinarily use L to denote the AdS length, in our metric ansatz eq. (2.37) and eq. (2.45) L is not the AdS length scale. Rather as one takes the limit $r \to \infty$ we would see that $\frac{L}{\sqrt{f_\infty}}$ is the true AdS length. It is a subtle distinction, but it is one that would cause confusions and errors in any further calculations if we were not careful. Using the expressions for the curvature tensors, eqs. (3.30)- (3.32), in the remaining four derivative terms while keeping mind the gauge constraints already imposed

$$
\begin{aligned}
1. \qquad & -\frac{3\left(D-2\right)}{4\left(2D-3\right)}\left(\frac{1}{L^2}\left(g^{dm}g^{cn}-g^{dn}g^{cm}\right)\right)\left(h_{dn;cm}-h_{dm;cn}+h_{cm;dn}-h_{cn;dm}\right)_{;(ab)} \\
& = -\frac{3\left(D-2\right)}{\left(2D-3\right)\tilde{L}^2}\left(h^{mc}{}_{;cm}-\Box h\right)_{;(ab)} \\
& = 0 \qquad (3.33) \\
2. \qquad & -\frac{3\left(D-2\right)}{\left(2D-3\right)\left(D-4\right)\tilde{L}^2}\Bigg((D-1)\left(\Box h\right)_{;(ab)}-2\left(D-1\right)\left(\Box h^c_{(a}\right)_{;b)c}+\frac{2\left(D-1\right)}{\left(D-2\right)}\Box^2 h_{ab} \\
& \qquad +2\left(\Box h^d{}_a\right)^{;cm}\left(g_{dm}g_{cb}-g_{db}g_{cm}\right)+\left(\Box h^d_b\right)^{;cm}\left(g_{dm}g_{ca}-g_{da}g_{cm}\right)-g_{ab}\left(\Box h^{dc}\right)_{;dc}\Bigg) \\
& = -\frac{3\left(D-2\right)}{\left(2D-3\right)\left(D-4\right)\tilde{L}^2}\left(\frac{2\left(D-1\right)}{\left(D-2\right)}\Box^2 h_{ab}-2\Box^2 h_{ab}\right) \\
& = -\frac{6}{\left(2D-4\right)\left(D-4\right)\tilde{L}^2}\Box^2 h_{ab} \qquad (3.34) \\
3. \qquad & -\frac{3\left(D-2\right)}{\left(2D-3\right)\left(D-4\right)\tilde{L}^2}\Bigg(2\left(D-1\right)\Box^2 h_{ab}-\left(D-3\right)\Box^2 h_{ab}-\frac{\left(D-1\right)}{2}g_{ab}\Box^2 h \\
& \qquad -\frac{D\left(D-1\right)}{\left(D-2\right)}\Box^2 h_{ab}\Bigg) \\
& = -\frac{3\left(D-2\right)}{\left(2D-3\right)\left(D-4\right)\tilde{L}^2}\left(\left(D+1\right)-\frac{D\left(D-1\right)}{\left(D-2\right)}\right)\Box^2 h_{ab} \\
& = \frac{6}{\left(2D-3\right)\left(D-4\right)\tilde{L}^2}\Box^2 h_{ab}, \qquad (3.35)
\end{aligned}
$$

where the second term in 2. generates lower derivative by commuting the derivatives terms in order to get it into the form of a divergence, which vanishes by gauge constraints. Thus, we have found that by imposing the transverse traceless gauge conditions in a maximally symmetric spacetime, in our case AdS_D, there are no terms with four derivatives acting on h_{ab}. We do not find any three or one derivative terms because we have made the choice of being in a maximally symmetric spacetime, which implies that R_{abcd} is proportional to the metric. For a metric compatible connection, the covariant derivative of the metric vanishes $g_{ab;c}=0$. By explicit calculation, one can show that all of the odd number of derivatives acting on h_{ab} occur with factors of a derivative acting on a curvature tensor. This shows that the theory is more stable than expected because the graviton equation of motion is only second order, and thus pseudo-topological gravity may have a well-posed initial value problem. However, we have only computed this for perturbing around the vacuum spacetime, and work must be done to show that the black hole spacetimes are similarly to metric perturbations.

Lower Derivative Terms

Now that we have shown the linearized equations of motion do not contain any three or four derivative terms, we can now move on to collecting and simplifying the two and zero derivative terms. Of course, we have to keep track of the two derivative terms generated by commuting the covariant derivatives in the four derivative terms above. Similarly, the two derivative terms of the form $h^c{}_{(a;b)c}$ can be turned into divergences, vanishing by the gauge constraints, and will generate h_{ab} terms. After collecting and simplifying everything, we are left with the pseudo-topological contribution to the linearized equations of motion being

$$\frac{\alpha}{\tilde{L}^4}\Box h_{ab} - \frac{\beta}{\tilde{L}^6} h_{ab}, \tag{3.36}$$

where it remains to be seen what the form of α and β are.

Chapter 4

Black Holes in Pseudo-Topological Gravity

We have thus far constructed a new theory of gravity that includes interactions up to cubic order in the curvature and have written an action for the theory in arbitrary dimensions. Furthermore, we have seen that the theory is stable, in that it does not contain a ghost graviton, at least for maximally symmetric spacetimes in the transverse traceless gauge. Our original inspiration for this construction was that the black hole metric ansatz eq. (2.37) yield a particular form when evaluated in the action. In this chapter, we complete the study of the black holes in this new theory. We begin by finding black hole solutions with the original ansatz eq. (2.37) for black holes with horizon topology $\mathbb{R}^3$ [7, 16]. We see that by solving a cubic polynomial equation for the metric function of the ansatz we find a rich phase space of the theory's interaction parameters populated by different vacua, some of which admit black holes and others that are not even stable. We finish this chapter by considering the possibility of finding black hole solutions with curved horizons, focussing on spherical or hyperbolic geometries, and where the vacua that admit them are located in the phase space. In considering the curved horizon black holes, we have to determine where the horizon is and if the solutions allow for negative mass black holes.

4.1 Planar Black Holes

In order to find black hole solutions, we recall that we have evaluated $\int \sqrt{-g}\mathcal{Z}_5$ with the metric ansatz eq. (2.37). Hence, we need only to add it to the contribution already calculated for the five dimensional AdS-Gauss-Bonnet action, eq. (2.68):

$$\begin{aligned} I &= \frac{1}{16\pi G_5}\int \mathrm{d}^5x\,\sqrt{-g}\left[\frac{12}{L^2}+R+\frac{\lambda L^2}{2}\mathcal{X}_4+\frac{7\mu L^4}{4}\mathcal{Z}_5\right], \\ &= \frac{1}{16\pi G_5}\int \mathrm{d}^5x\,\frac{3N(r)}{L^5}\left[r^4(1-f+\lambda f^2+\mu f^3)\right]', \end{aligned} \tag{4.1}$$

where, G_5 is the five dimensional gravitational constant, $\mathcal{X}_4$ is the Gauss-Bonnet term, R is the Ricci Scalar, the prime denotes a derivative with respect to r, and μ is the interaction parameter for the pseudo-topological term. Following what was done for the same situation in Gauss-Bonnet gravity, the next step is to find and solve the constraint equations obtained by varying with respect to $f(r)$ and $N(r)$. Solving these equations will help us explore the problem of finding possible bounds on the μ and λ that give physically relevant non-perturbative solutions for pseudo-topological gravity. Beginning with the equation of motion due to variation with respect to f we find:

$$\delta f: \qquad N'\Big(3\mu f^2+2\lambda f-1\Big)=0, \tag{4.2}$$

$$\Rightarrow \qquad N'=0 \quad \mathit{or} \quad 3\mu f^2+2\lambda f-1=0. \tag{4.3}$$

From this we arrive at $N'=0 \Rightarrow N=\mathit{constant}$. By choosing the value of N appropriately, we can set the speed of light to unity at the boundary *i.e.*, $N=\frac{1}{\sqrt{f_\infty}}$ as we had seen for solutions of this type in Gauss-Bonnet gravity. Moving on to the constraint equation for $f(r)$ from varying eq. (4.1) with respect to N, we find that similar to the constraint equation for Gauss-Bonnet gravity we have to solve a polynomial equation for f

$$\begin{aligned}\delta N: \qquad\qquad [r^4(1-f+\lambda f^2+\mu f^3)]' &= 0, \\ \Rightarrow r^4(1-f+\lambda f^2+\mu f^3) &= \omega^4, \\ \Rightarrow f^3+\frac{\lambda}{\mu}f^2-\frac{1}{\mu}f+\frac{1}{\mu}(1-\frac{\omega^4}{r^4}) &= 0. \qquad (4.4)\end{aligned}$$

Hence we are now left with a cubic equation to solve for $f(r)$. In order for us to do so, we first make the substitution $f = x - \frac{\lambda}{3\mu}$ which enables us to write eq. (4.4) as:

$$x^3+3(\frac{-3\mu-\lambda^2}{9\mu^2})x+2(\frac{2\lambda^3+9\lambda\mu+27\mu^2(1-\frac{\omega^4}{r^4})}{54\mu^3}) = 0. \qquad (4.5)$$

Simplifying this expression by defining

$$p = \frac{-3\mu-\lambda^2}{9\mu^2}, \qquad\qquad q = \frac{2\lambda^3+9\mu\lambda+27\mu^2(1-\frac{\omega^4}{r^4})}{54\mu^3}. \qquad (4.6)$$

We then arrive at a depressed form for the equation:

$$x^3+3px+2q = 0. \qquad (4.7)$$

The most prudent method will be to compartmentalize our approach to three cases based on the sign of the discriminant of the depressed cubic, $\mathcal{D} = q^2 + p^3$, with the following results:

1. $q^2 + p^3 > 0 \Rightarrow$ 1 real root, and 2 complex roots conjugate to one another
2. $q^2 + p^3 < 0 \Rightarrow$ 3 unequal real roots
3. $q^2 + p^3 = 0 \Rightarrow$ 3 real roots, at least 2 of which must be equal

In this way, we can characterize the space of solutions to eq. (4.7) by the behavior of its discriminant in the different regions of the $\mu - \lambda$ plane shown in fig. (4.1).

Proceeding with the requirement that $p \neq 0$ (we will explore the case later on), we define

$$\begin{aligned}\alpha &= (-q + \sqrt{q^2 + p^3})^{\frac{1}{3}}, \\ \beta &= (-q - \sqrt{q^2 + p^3})^{\frac{1}{3}},\end{aligned}$$

which then, by Cardano's Formula, allows us to write the roots of eq. (4.7) in the simple forms

$$x_1 = \alpha + \beta, \tag{4.8}$$

$$x_2 = -\frac{1}{2}(\alpha + \beta) + \imath\frac{\sqrt{3}}{2}(\alpha - \beta), \tag{4.9}$$

$$x_3 = -\frac{1}{2}(\alpha + \beta) - \imath\frac{\sqrt{3}}{2}(\alpha - \beta). \tag{4.10}$$

First, let us take the simple case of $q^2 + p^3 = 0$, which then implies:

$$\mu = \frac{2}{27} - \frac{\lambda}{3} \pm \frac{2}{27}\sqrt{1 - 9\,\lambda + 27\,\lambda^2 - 27\,\lambda^3}. \tag{4.11}$$

Eq. (4.11) generates the two uppermost curves in the μ-λ plane shown in fig. (4.1). The zero value for the discriminant gives us that $\alpha = \beta = (-q)^{\frac{1}{3}}$, and the roots take the form:

$$f_1 = -\frac{(8\lambda^3+36\lambda\mu+108\mu^2(1-\frac{\omega^4}{r^4}))^{\frac{1}{3}}+\lambda}{3\mu}, \tag{4.12}$$

$$f_2 = \frac{(8\lambda^3+36\lambda\mu+108\mu^2(1-\frac{\omega^4}{r^4}))^{\frac{1}{3}}-2\lambda}{6\mu}, \tag{4.13}$$

where f_2 is the degenerate root.

Moving on, we next consider the case $q^2+p^3 < 0$ which yields 3 unequal real roots and puts in the region of the phase diagram 'sandwiched' between the two $\mathcal{D} = 0$ lines generated by the positive and negative branches. Since we no longer have a vanishing discriminant, we cannot write a simple expression for α and β. However, if we make the substitution that $B = -(q^2 + p^3) > 0$ then we can recast the expression for the roots into a more workable form.

$$
\begin{aligned}
f_1 &= (q + \imath\sqrt{B})^{\frac{1}{3}} + (q - \imath\sqrt{B})^{\frac{1}{3}} - \frac{\lambda}{3\mu}, \\
f_2 &= -\frac{1}{2}((q + \imath\sqrt{B})^{\frac{1}{3}} + (q - \imath\sqrt{B})^{\frac{1}{3}}) - \frac{\lambda}{3\mu} + \frac{\imath\sqrt{3}}{2}((q + \imath\sqrt{B})^{\frac{1}{3}} - (q - \imath\sqrt{B})^{\frac{1}{3}}), \\
f_3 &= -\frac{1}{2}((q + \imath\sqrt{B})^{\frac{1}{3}} + (q - \imath\sqrt{B})^{\frac{1}{3}}) - \frac{\lambda}{3\mu} - \frac{\imath\sqrt{3}}{2}((q + \imath\sqrt{B})^{\frac{1}{3}} - (q - \imath\sqrt{B})^{\frac{1}{3}}).
\end{aligned}
$$

In general, we know that finding the roots of a cubic equation will require taking the cube root of a complex number, and now we must. We can recast the complex root as $(-q \pm i\sqrt{B})^{\frac{1}{3}}$ as $|-q \pm i\sqrt{B}|^{\frac{1}{3}} e^{\frac{\theta + 2k\pi}{3}}$ where $|-q \pm i\sqrt{B}| = (-p)^{\frac{3}{2}}$ and $\tan\theta = \frac{\mp\sqrt{B}}{q}$. The solutions to the cubic equation are then:

$$
\begin{aligned}
f_1 &= 2\sqrt{-p}\cos\frac{\theta}{3} - \frac{\lambda}{3\mu}, \\
f_2 &= -\sqrt{-p}\cos\frac{\theta}{3} - \frac{\lambda}{3\mu} - \sqrt{3}\sqrt{-p}\sin\frac{\theta}{3}, \\
f_3 &= -\sqrt{-p}\cos\frac{\theta}{3} - \frac{\lambda}{3\mu} + \sqrt{3}\sqrt{-p}\sin\frac{\theta}{3}.
\end{aligned}
$$

Where, $\cos\theta = -q(-p)^{\frac{-3}{2}}$ and $\sin\theta = \sqrt{B}(-p)^{\frac{-3}{2}}$. Since we have only an algebraic expression for $\cos\theta$, $\sin\theta$, and $\tan\theta$, it is difficult to find an explicit expression for the above solutions containing a trigonometric function of $\frac{\theta}{3}$.

Now that we have characterized the solutions based on the behavior of the discriminant, we note that there is special case of eq. (4.5) we wish to consider. First, we recognize that eq. (4.5) can be turned into a perfect cubic equation by setting $p = 0$, which requires $\mu = -\frac{\lambda^2}{3}$. This generates the lowermost curve in fig. (4.1). With this constraint on μ, we can eliminate μ from the expression for q and find a solution depending only on the parameter λ:

$$\begin{aligned} q &= \frac{2\lambda^3 + 9\lambda(-\frac{\lambda^2}{3}) + 27(\frac{-\lambda^2}{3})^2(1-\frac{\omega^4}{r^4})}{54(-\frac{\lambda^2}{3})^3}, \\ &= \frac{-\lambda^3 + 3\lambda^4(1-\frac{\omega^4}{r^4})}{-2\lambda^6} = \frac{1}{2\lambda^3}[1 - 3\lambda(1-\frac{\omega^4}{r^4})]. \end{aligned}$$

Furthermore if we go back and insert this relationship into eq. (4.5), we have:

$$\begin{aligned} 0 &= x^3 + 2q, \\ \Rightarrow x &= -\frac{1}{\lambda}[1 - 3\lambda(1-\frac{\omega^4}{r^4})]^{\frac{1}{3}}. \end{aligned}$$

Recalling that $x = f + \frac{\lambda}{3\mu}$ and that $\mu = -\frac{\lambda^2}{3}$, the above equation becomes:

$$f(r) = \frac{1}{\lambda}\Big(1 - [1 - 3\lambda(1-\frac{\omega^4}{r^4})]^{\frac{1}{3}}\Big). \tag{4.14}$$

Whereupon taking the limit of the interaction parameter $\lambda \to 0$ we see the following behavior of f:

$$\begin{aligned} f(r)_{\lambda\to 0} &\approx \frac{1}{\lambda}\Big(1 - (1 - \frac{1}{3}(3\lambda(1-\frac{\omega^4}{r^4})) + \ldots)\Big), \\ \Rightarrow f(r)_{\lambda\to 0} &\approx 1 - \frac{\omega^4}{r^4} - \ldots, \end{aligned} \tag{4.15}$$

where the terms contained in the ellipses are of order λ. Thus, upon turning off λ which controls the higher derivative terms in the action (seeing as μ is now expressed in terms of a power of λ), we arrive at the original solution of Einstein's equations of motion for $f(r)$ in the case of the planar black hole solution for AdS_5.

Examining fig. (4.1) more closely, the curves generated by eq. (4.11) in the $\mu - \lambda$ plane give us a clear picture of where we are in the phase space of the parameters of the theory when we talk about the sign of the discriminant. The region that is outside the region bounded by the positive and negative branch lines is the part of the diagram where $\mathcal{D} > 0$. In between the two branches is a region of $\mathcal{D} < 0$. Note that the three curves meet at $(\lambda, \mu) = (1/3, -1/27)$, as shown in fig. (4.2). It is at

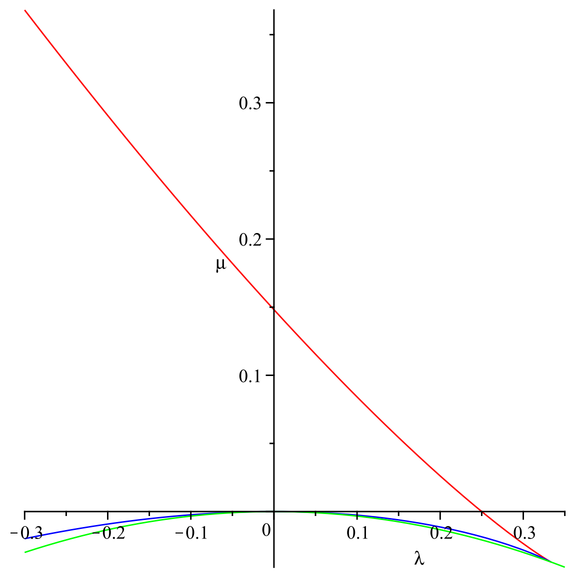

Figure 4.1: Diagram of the behavior of $\mathcal{D}$ in $\mu - \lambda$ plane. Where $\mathcal{D} < 0$ is contained in the region between the red and blue curves generated by the positive and negative branches of eq. (4.11) respectively. $\mathcal{D}$ lies outside these red and blue curves. The green curve is generated by the case where $p = 0$.

this point that we see the two $\mathcal{D} = 0$ curves become complex while the lower curve continues on to $-\infty$. Following the positive branch of the $\mathcal{D} = 0$ curves down to the point of coincidence we will see that it provides a boundary between regions of the phase space in which the theory has vacua containing black holes (to the left of the curve) and AdS vacua that do not support black holes (right of the curve). Beyond the point of coincidence the boundary is given by the lower parabola whose significance will be discussed momentarily.

As an aside we make the following interesting note that the positive branch of eq. (4.11) crosses the λ-axis at $\lambda = \frac{1}{4}$ and extends to $\lambda = \frac{1}{3}$. The λ–axis ($\mu = 0$) corresponds to the Gauss-Bonnet theory. Recall that in this case, $\lambda = \frac{1}{4}$ was seen to be a pathological point. For $\lambda > \frac{1}{4}$, the graviton was a ghost and there were no black holes [15, 11]

At this point we should pause to discuss how we are to decide if any of the

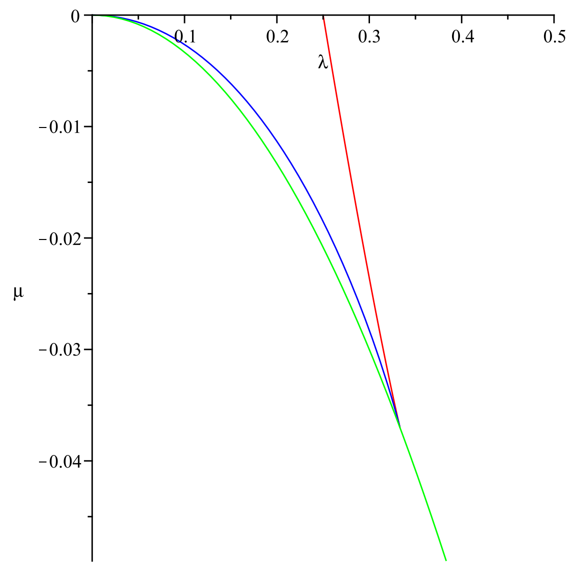

Figure 4.2: A closer examination of the two nearly coinciding roots (negative and p=0 branches). The two places of intersection for the lines are at $\lambda = 0$, $\mu = 0$ and $\lambda = \frac{1}{3}$, $\mu = -\frac{1}{27}$. However, aside from those two points, the negative branch is always above the $p = 0$ branch. After the $\lambda = \frac{1}{3}$ point of coincidence, the negative branch ceases to be real, while the $p = 0$ branch continues to $-\infty$.

solutions to the equations of motion, eq. (4.4), indeed describe black holes. One way to go about doing so would be to follow the arguments made by Boulware and Desser [17] in their exploration of the stability of curvature squared theories as we had seen in section 2.2. We have indications that we may be able to extend their work from Gauss-Bonnet to Pseudo-Topological gravity, but this relationship that the slope of the polynomial equation of motion, eq. (4.4), determines the sign of the kinetic term for the graviton equation of motion is not concretely established at present. However, we will proceed with the reasoning with the noted caveat in mind. That is, we will say that for a positive slope the graviton is regarded as a ghost, and if the slope is negative, the branch of the theory is stable. To start, let us first look at the polynomial of the vacuum solutions where $\omega = 0$

$$h(f) \;\; = \;\; \mu f^3 + \lambda f^2 - f + 1, \tag{4.16}$$

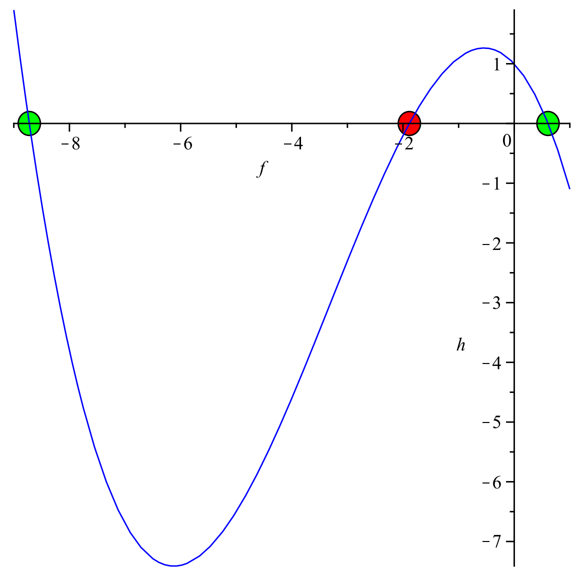

Figure 4.3: Graph of $h(f)$ for $\mu = -0.1$, $\lambda = -1$, and $\omega = 0$. The stable (negative slope) vacua are indicated by a green circle and ghosty, unstable (positive slope) vacua by a red circle

$$h'(f) \;\;=\;\; 3\mu f^2 + 2\lambda f - 1. \tag{4.17}$$

By looking at the value of eq. (4.17) for the values of parameters in the different regions of fig. (4.1), we find the following:

[region] μ, λ $(+/-)$	$\mathcal{D}\,(+/-)$	(#, *Type*) Stable Vacua	(#, *Type*) Ghosty Vacua
$[a]$ $(+,+)$	$+$	None	(1,dS)
$[b]$ $(+,-)$	$+$	None	(1,dS)
$[c]$ $(+,-)$	$-$	(1,AdS)	(1,dS), (1,AdS)
$[d]$ $(-,+)$	$+$	(1,AdS)	None
$[e]$ $(-,+)$	$-$	(2,AdS)	(1,AdS)
$[f]$ $(-,-)$	$+$	(1,AdS)	None
$[g]$ $(-,-)$	$-$	(1,AdS)	(2,dS)

Now that we have characterized the vacua of the theory, in order to decide which vacua do or do not contain black holes we consider the polynomial eq. (4.16) with the change that $\omega \neq 0$

$$h(f) = \mu f^3 + \lambda f^2 - f + 1 - \frac{\omega^4}{r^4}. \tag{4.18}$$

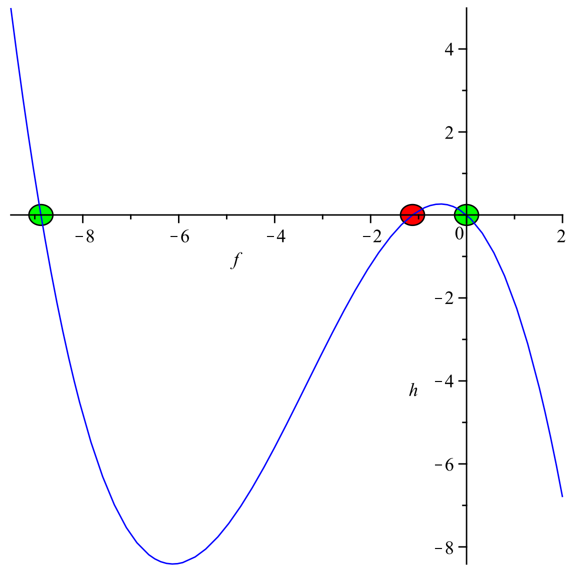

Figure 4.4: Graph of $h(f)$ for $\mu = -0.1$, $\lambda = -1$, and $r = \omega = 1$. The right most zero is interpreted as a vacuum containing black holes as it intersects the f-axis at $f = 0$ when $r = \omega$.

The vacua containing black holes will be determined as one decreases r from infinity. In doing, the $\frac{\omega^4}{r^4}$ contribution makes the constant term smaller and smaller dragging the graph of eq. (4.18) downward. From the idea that there is a black hole with a horizon when the metric function vanishes $f = 0$, we only need to find which root of eq. (4.18) hits $f = 0$ as r decreases. (For example see fig. (4.3) and fig. (4.4))

μ, λ $(+/-)$	$\mathcal{D}\,(+/-)$	(#, *Type*) Black Hole	(#, *Type*) Non-Black Hole [stable/ghost]
[a] $(+,+)$	$+$	None	(1,dS)[ghost]
[b] $(+,-)$	$+$	None	(1,dS)[ghost]
[c] $(+,-)$	$-$	(1,AdS)	(1,dS)[ghost],(1,AdS)[stable]
[d] $(-,+)$	$+$	(1,AdS)	None
[e] $(-,+)$	$-$	(1,AdS)	(1,AdS)[ghost], (1, AdS)[stable]
[f] $(-,-)$	$+$	(1,AdS)	None
[g] $(-,-)$	$-$	(1,AdS)	(1,dS)[stable], (1,dS)[ghost]

Graphing the roots we see that in the different regions we can see which solutions to eq. (4.4) correspond to which vacua in the tables above. In the region $\mathcal{D} < 0$, $\mu < 0$, fig. (4.6), we find that that both f_2 and f_3 are negative and f_1 is the lone positive root. In the region of $\mathcal{D} < 0$, $\mu > 0$, fig. (4.5), f_2 is negative and f_1 and f_3 are positive with f_3 being the smallest of the positive roots. Lastly, we have the situation where $\mathcal{D} > 0$ which leaves us in the region outside of the area swept out by the $\mathcal{D} = 0$ curves generated by the $p \neq 0$ solutions. In this case we do not have to take the cube root of a complex number and are left with the general solutions in terms of α and β which can easily be expanded out to show their dependence on λ, μ, and r. Plotting the solutions in the region $\mu < 0$, we find that the only real solution is given by $f_1 = \alpha + \beta - \frac{\lambda}{3\mu}$. In the region where $\lambda < 0$, f_1 does not give a black hole solution, but it does give a non-ghosty AdS vacuum. On the other hand when $\lambda < 0$, $\mathcal{D} > 0$, and $\mu > -\frac{\lambda^2}{3}$, f_1 gives an AdS vacuum with black holes, fig. (4.7). Continuing through to the region inside the lowermost curve, we see that f_1 is the only real root corresponding to an AdS vacuum admitting black holes, while the other two roots are again residing in the complex plane, fig. (4.8).

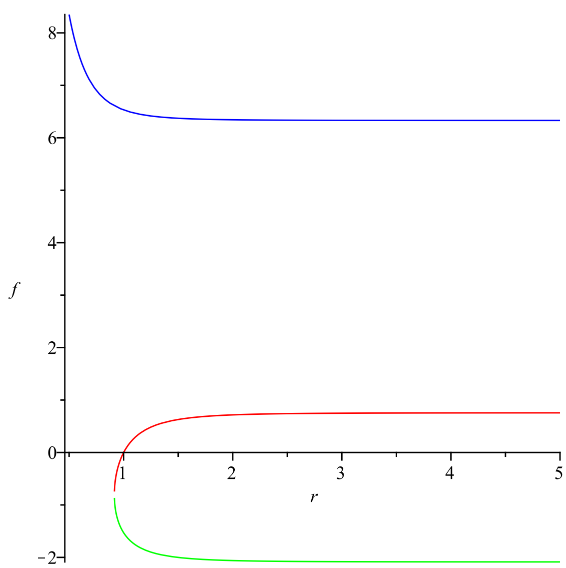

Figure 4.5: The graph of the solutions f_1 (blue line,ghosty AdS vacuum), f_2 (green line, ghosty dS vacuum), and f_3 (red line, AdS vacuum with black holes). $\mathcal{D} < 0$, $\mu > 0$, specifically $\lambda = -0.5$, $\mu = 0.1$, and setting the $\omega = 1$ for convenience.

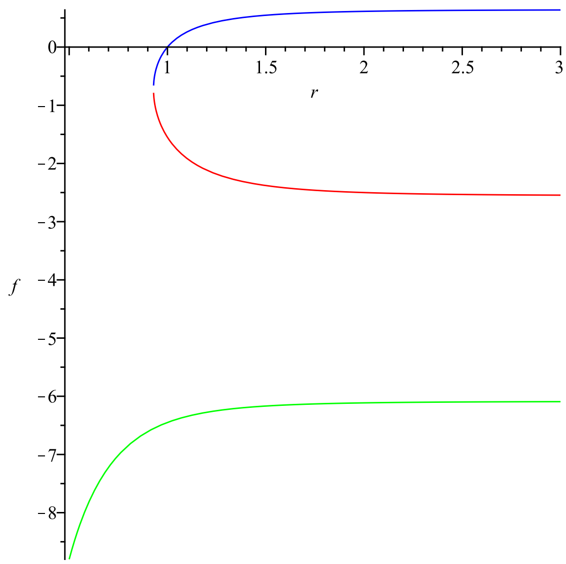

Figure 4.6: The graph of f_1 (blue) (AdS vacuum with black holes), f_2 (green) (dS vacuum without black holes), and f_3 (red) (ghosty dS vacuum) for $\lambda = -0.8$, $\mu = -0.1$, and again $\omega = 1$.

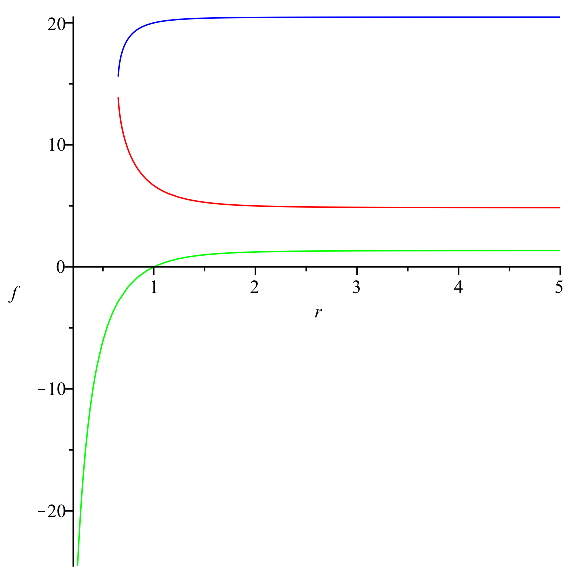

Figure 4.7: The graph of f_1 (blue) (AdS vacuum without black holes), f_2 (green) (AdS vacuum with black holes), and f_3 (red) (ghosty AdS vacuum) for $\lambda = 0.2$, $\mu = -0.0075$, and $\omega = 1$

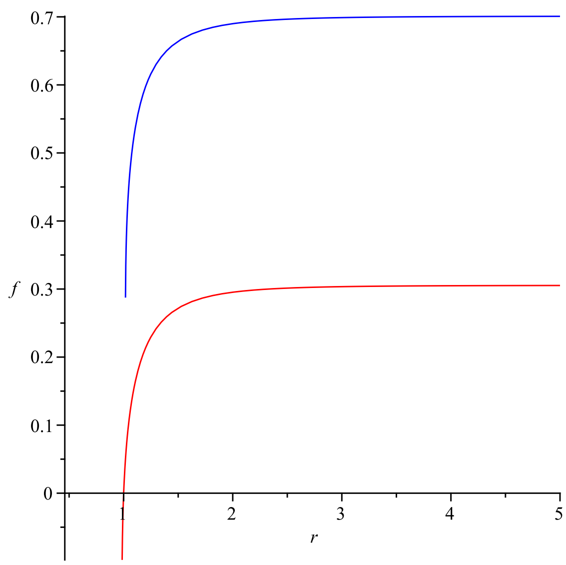

Figure 4.8: Graph of f_1 in the regions where $\mathcal{D} > 0$ and $p > 0$. The case where $\lambda < 0$ (red) we see an AdS vacuum with black holes, $\lambda > 0$ we find an AdS vacuum without black holes. The upper graph corresponds to the region where $D > 0$ and $p < 0$ where no black hole solutions exist. The other two roots in both cases are again complex.

4.2 Curved Horizons

Generalizing our discussion beyond considering only the planar horizons, we consider the metric ansatz to allow for the possibility of adding curvature to the horizon as in eq. (2.45). The analysis from the previous section follows through to our current, more general, case in that the equations of motion now yields

$$\mu f^3 + \lambda f^2 - f + 1 - \frac{\omega^4}{r^4} = 0.$$

So, we see that finding the roots follows exactly the same process as with the planar black hole, and one would arrive at the same solutions for f as previously found. The difference between planar and curved horizons is that the usual horizon equation $g_{tt} = 0$ now becomes $f = -k\frac{L^2}{r^2}$, which gives the horizon radius as the solution to the cubic equation in r^2:

$$r^6 + k^2L^4\lambda + kL^2r^4 - \omega^4r^2 - \mu L^6k = 0, \tag{4.19}$$

which upon solving we find that the only real root is

$$\begin{aligned} r_k^2 \;=\; & \frac{1}{6}\left(36kL^6\lambda - 36kL^2\omega^4 + 108\mu L^6k - 8kL^6 + 12\left(12\lambda^3L^{12}k^2 - 36\lambda^2L^8k^2\omega^4\right.\right. \\ & 3\lambda^2L^{12}k^2 + 36\lambda L^4k^2\omega^8 + 6\lambda L^8k^2\omega^4 - 12\omega^{12} - 3\omega^8k^2L^4 + 54k^4L^{12}\omega\mu \\ & \left.\left.-54k^2L^8\omega^4\mu + 81\mu^2L^{12}k^2 - 12\mu L^{12}k^4\right)^{\frac{1}{2}}\right)^{\frac{1}{3}} + 2\left(-\lambda L^4k^2 + \omega^4 + \frac{1}{3}k^2L^4\right) \times \\ & \times\left(36kL^6\lambda - 36kL^2\omega^4 + 108\mu L^6k - 8kL^6 + 12\left(\lambda^3L^{12}k^2 - 36\lambda^2L^8k^2\omega^4\right.\right. \\ & -3\lambda^2L^{12}k^2 + 26\lambda L^4k^2\omega^8 + 6\lambda L^8k^2\omega^4 - 12\omega^{12} - 3\omega^8k^2L^4 + 54k^4L^{12}\lambda\mu \\ & \left.\left.-54k^2L^8\omega^4\mu + 81\mu^2L^{12}k^2 - 12\mu L^{12}k^4\right)^{\frac{1}{2}}\right)^{-\frac{1}{3}} - \frac{kL^2}{3}. \end{aligned} \tag{4.20}$$

Note that eq. (4.20) is the square of the horizon radius for $k = \pm 1$. However, we note that substituting $k = 0$ in eq. (4.20) yields zero. Although, the horizon radius for $k = 0$ is obvious from eq. (4.19), $r = \omega$. Exploring eq. (4.20) in the different regions of fig. (4.1), we find that for $k = 1$ the horizon radius is positive and real in the regions of $\mathcal{D} < 0$ from the tables in the previous section, and for $\mathcal{D} > 0\, \mu > 0$ for all λ, and for $k = -1$ the horizon radius is real and positive in the regions $\mathcal{D} < 0$ excluding $[e]$ and for $\mathcal{D} > 0$, $\mu < 0$.

Alternative to the above calculation, we could have also noticed that at the horizon $\frac{1}{r^4} = \frac{f^2}{L^4}$ for $k = \pm 1$. That is, we are ignoring the $k = 0$ case for the moment. Inserting the relationship back into the constraint yields a slightly modified cubic equation

$$\mu f^3 + \left(\lambda - \frac{\omega^4}{L^4}\right) f^2 - f + 1 = 0. \tag{4.21}$$

Solving this gives several roots, but testing each in the various regions of fig. (4.1) shows that the only real root is

$$\begin{aligned} f_0 \;=\; & \frac{1}{6\mu L^4}\big(-36\mu L^{12}\lambda + 36\mu L^8\omega^4 - 108\mu^2 L^{12} - 8\lambda^3 L^{12} + 24\lambda^2 L^8\omega^4 \\ & -24\lambda L^4\omega^8 + 8\omega^{12} + 12\sqrt{3}\left(-L^{12}\lambda^2 + 18\mu L^{12}\lambda + 2\lambda L^8\omega^4 - 4\mu L^{12}\right. \\ & \left.+12\lambda L^4\omega^8\right)^{\frac{1}{2}}\mu L^6\Big)^{\frac{1}{3}} + \frac{2}{3\mu L^4}\left(3\mu L^8 + \lambda^2 L^8 - 2\lambda L^4\omega^4 + \omega^8\right) \times \\ & \times \big(-36\mu L^{12}\lambda + 36\mu L^8\omega^4 - 108\mu^2 L^{12} - 8\lambda^3 L^{12} + 24\lambda^2 L^8\omega^4 \\ & -24\lambda L^4\omega^8 + 8\omega^{12} + 12\sqrt{3}\left(-L^{12}\lambda^2 + 18\mu L^{12}\lambda + 2\lambda L^8\omega^4 - 4\mu L^{12}\right. \\ & \left.+12\lambda L^4\omega^8\right)^{\frac{1}{2}}\mu L^6\Big)^{-\frac{1}{3}} - \frac{\lambda L^4 - \omega^4}{3\mu L^4}. \end{aligned} \tag{4.22}$$

To make sure that f_0 has the 'correct' sign, we substitute it back into the original equation for the horizon *i.e.*, , $r_k^2 = -\frac{kL^2}{f_0}$. Noting that we should expect a real value for the horizon radius, $r^2 > 0$, we investigate the $\lambda - \mu$ plane to determine where $-\frac{k}{f_0} > 0$ for $k = \pm 1$. We find that $-\frac{1}{f_0} > 0$ (spherical horizon) is satisfied for $\mathcal{D} < 0$, $\mu < 0$, $\lambda < 0$. For hyperbolic horizon geometry, $\frac{1}{f_0} > 0$ is satisfied for $\mathcal{D} < 0$, $\mu > 0$.

Since we now do not have a simple relation relating the horizon radius to the mass function of the black hole, $\omega^4 = r_h^4 > 0$, as we had in the planar case, we might ask if the black hole solutions found allow for a negative mass. To do so, we go back to eq. (4.4), evaluate at $f = -k\frac{L^2}{r^2}$ and rearrange:

$$-\mu\frac{kL^6}{r^2} + kL^2r^2 + r^4 + \lambda k^2 L^4 = \omega^4. \tag{4.23}$$

Finding the minimum of the left hand side will provide a lower bound for the mass parameter ω. Differentiating and setting to zero, we find a cubic equation in $\rho = r^2$:

$$\rho^3 + \frac{kL^2}{2}\rho^2 + \mu\frac{kL^6}{2} = 0. \tag{4.24}$$

Solving for ρ (for $k = \pm 1$ as the solution for $k = 0$ is obvious) and testing the type of extremum each solution, we find the minimum is given by

$$\begin{aligned}\rho_0 \;=\; & \frac{1}{6}\left(\left(-54kL^6\mu - kL^6 + 6\sqrt{3k^2L^{12}\mu\,(27\mu+1)}\right)^{\frac{2}{3}} + k^2L^4 - kL^2\left(-54kL^6\mu - kL^6\right.\right. \\ & \left.\left.+6\sqrt{3k^2L^{12}\mu\,(27\mu+1)}\right)^{\frac{1}{3}}\right)\left(-54kL^6\mu - kL^6 + 6\sqrt{3k^2L^{12}\mu\,(27\mu+1)}\right)^{-\frac{1}{3}}\end{aligned} \tag{4.25}$$

which when inserted back into eq. (4.23) gives the lower bound on mass for the black holes with curved horizons in terms of μ, λ and k as

$$\begin{aligned}\omega^4 \;\geq\; & -\frac{1}{72k^2L^{16}\mu}\left(\left(-54kL^6\mu - kL^6 + 6\left(3k^2L^{12}\mu\,(27\mu+1)\right)^{\frac{1}{2}}\right)^{\frac{1}{3}} \left(\left(-54kL^6\mu - kL^6\right.\right.\right. \\ & \left.\left.+6\left(3k^2L^{12}\mu\,(27\mu+1)\right)^{\frac{1}{2}}\right)^{\frac{1}{3}} + kL^2\right)\left(27kL^6\mu\,(1+36\mu) + (1+108\mu)\left(3k^2L^{12}\mu\times\right.\right. \\ & \left.\left.\times\,(27\mu+1)\right)^{\frac{1}{2}}\right)\left(1944kL^{10}\mu^2 - 18kL^{10}\mu - 216k^2L^4\mu\sqrt{3k^2L^{12}\mu\,(27\mu+1)}\right. \\ & -24\lambda k^2L^4\left(3k^2L^{12}\mu\,(27\mu+1)\right)^{\frac{1}{2}} - 18kL^6\mu\left(-54kL^6\mu - kL^6 + 6\left(3k^2L^{12}\mu\times\right.\right. \\ & \left.\left.\times\,(27\mu+1)\right)^{\frac{1}{2}}\right)^{2/3} + 6k^2L^4\left(3k^2L^{12}\mu\,(27\mu+1)\right)^{\frac{1}{2}} - kL^6\left(-54kL^6\mu - kL^6\right. \\ & \left.+6\left(3k^2L^{12}\mu\,(27\mu+1)\right)^{\frac{1}{2}}\right)^{2/3} + 216\lambda kL^{10}\mu + 4\lambda kL^{10} + 2\left(3k^2L^{12}\mu\,(27\mu+1)\right)^{\frac{1}{2}}\times \\ & \times\left(-54\,kL^6\mu - kL^6 + 6\left(3k^2L^{12}\mu\,(27\mu+1)\right)^{\frac{1}{2}}\right)^{2/3} - kL^{10} + 36\mu k^2L^8\times \\ & \times\left(-54kL^6\mu - kL^6 + 6\left(3k^2L^{12}\mu\,(27\mu+1)\right)^{\frac{1}{2}}\right)^{\frac{1}{3}} + k^2L^8\left(-54kL^6\mu - kL^6\right. \\ & \left.+6\left(3k^2L^{12}\mu\,(27\mu+1)\right)^{\frac{1}{2}}\right)^{\frac{1}{3}} - 216\mu\lambda k^2L^8\left(-54kL^6\mu - kL^6 + 6\left(3k^2L^{12}\mu\times\right.\right. \\ & \left.\left.\times\,(27\mu+1)\right)^{\frac{1}{2}}\right)^{\frac{1}{3}} - 4kL^2\left(3k^2L^{12}\mu\,(27\mu+1)\right)^{\frac{1}{2}}\left(-54kL^6\mu - kL^6 + 6\left(3k^2L^{12}\mu\times\right.\right.\end{aligned}$$

$$
\begin{aligned}
&\times(27\mu+1))^{\frac{1}{2}}\Big)^{\frac{1}{3}} - 4\lambda k^2 L^8\left(-54kL^6\mu - kL^6 + 6\left(3k^2L^{12}\mu(27\mu+1)\right)^{\frac{1}{2}}\right)^{\frac{1}{3}} \\
&+4\lambda kL^6\left(-54kL^6\mu - kL^6. + 6\left(3k^2L^{12}\mu(27\mu+1)\right)^{\frac{1}{2}}\right)^{2/3} + 24\lambda kL^2\Big(3k^2L^{12}\mu\times \\
&\times(27\mu+1))^{\frac{1}{2}}\left(-54kL^6\mu - kL^6 + 6\left(3k^2L^{12}\mu(27\mu+1)\right)^{\frac{1}{2}}\right)^{\frac{1}{3}}\Big)\Big).
\end{aligned}
\tag{4.26}
$$

Plotting the right hand side of eq. (4.26) in the different regions indicates for which black hole solutions negative mass is not forbidden and which strictly have positive mass. Interestingly the only areas in fig. (4.9) that allow $\omega^4 < 0$ are where $\mathcal{D} < 0$ while $\mathcal{D} > 0$ restricts $\omega > 0$ with the boundary between the two areas given by the same curves as in fig. (4.1). However, we emphasize that the regions restricting $\omega > 0$ do not coincide with the regions in which we found real, positive radii for spherical and hyperbolic horizons. That is easily seen if we recall that $r_k^2 = -\frac{kL^2}{f_0} > 0$ only in the region $\mathcal{D} > 0$, and this also restricted the types of horizon geometry with $\mu < 0$ giving spherical horizons and $\mu > 0$ yielding a hyperbolic black hole.

As a caution, we note that the analysis here is preliminary, and while the results may indicate that a horizon is possible, we have by no means guaranteed it to exist. We can illustrate this by a simple example. Suppose we look ate fig. (4.9) in the region where $\lambda > 0$ and $-\frac{\lambda^2}{3} < \mu < 0$. The analysis of the bound on ω^4 seems to indicate that there are black hole solutions with $\omega^4 > 0$. We consider the cubic polynomial eq. (4.18) first in the limit of $r \to \infty$ and find that f_0 is the non-ghost AdS vacuum in the region (eq. (4.10)). If we then decrease r, we begin dragging the plot down and eventually we could hit the extrema located at f_{crit}. However if we reach f_{crit}, then the space becomes singular. Hence if a horizon (f_h) forms, it must form for $0 < f_{crit} < f_h < f_0$ in order not to have a naked singularity. The equation determining the spherical horizon demands that $r^2 = -\frac{L^2}{f_h} > 0$ and so $f_h < 0$. So, we may find solutions of eq. (4.21) for large $\omega^4 > 0$ where $f_h < 0$, but they will no correspond to black hole solutions. Thus, the present analysis does not entirely characterize the black holes with curved horizons in pseudo-topological gravity.

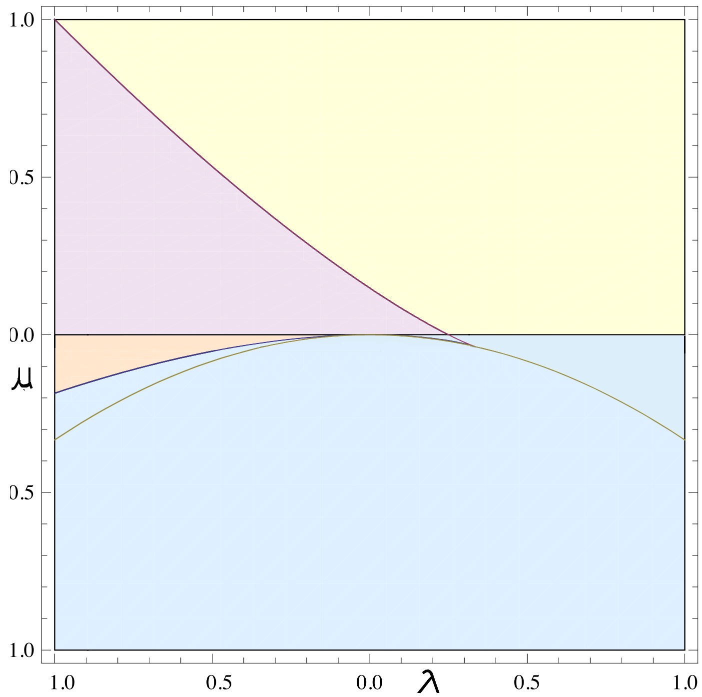

Figure 4.9: Region plot of curved horizon black hole solutions indicating either strictly positive or allowing negative masses. The regions colored blue and orange correspond to $k = 1$ black holes solutions admitting strictly positive or possibly negative masses, respectively. The yellow and purple correspond to $k = 1$ horizon black hole solutions forbidding and allowing negative mass, respectively.

While we have specifically done the calculations in five dimensions, we could just as easily generalize the results above to an arbitrary, D-dimensional case as we had with the results in Gauss-Bonnet gravity. The calculation follows just as the above with the action

$$\begin{aligned} I \;=\; & \frac{1}{16\pi G_D}\int \mathrm{d}^D x\,\sqrt{-g}\,\Bigg[\frac{(D-1)(D-2)}{L^2} + R + \frac{\lambda L^2}{(D-3)(D-4)}\mathcal{X}_4 \\ & -\frac{8(2D-3)}{(D-6)(D-3)(3D^2-15D+16)}\mu L^4 \Bigg(R_{a\;b}^{\;c\;d} R_{c\;d}^{\;e\;f} R_{e\;f}^{\;a\;b} \\ & +\frac{1}{(2D-3)(D-4)}\Bigg(\frac{3(3D-8)}{8}R_{abcd}R^{abcd}R - 3(3D-2)R_{abcd}R^{abc}{}_{e}R^{de} \\ & +3D R_{abcd}R^{ac}R^{bd} + 6(D-2)R_a{}^b R_b{}^c R_c{}^a - \frac{3(3D-4)}{2}R_a{}^b R_b{}^a R + \frac{3D}{8}R^3\Bigg)\Bigg)\Bigg] \end{aligned} \tag{4.27}$$

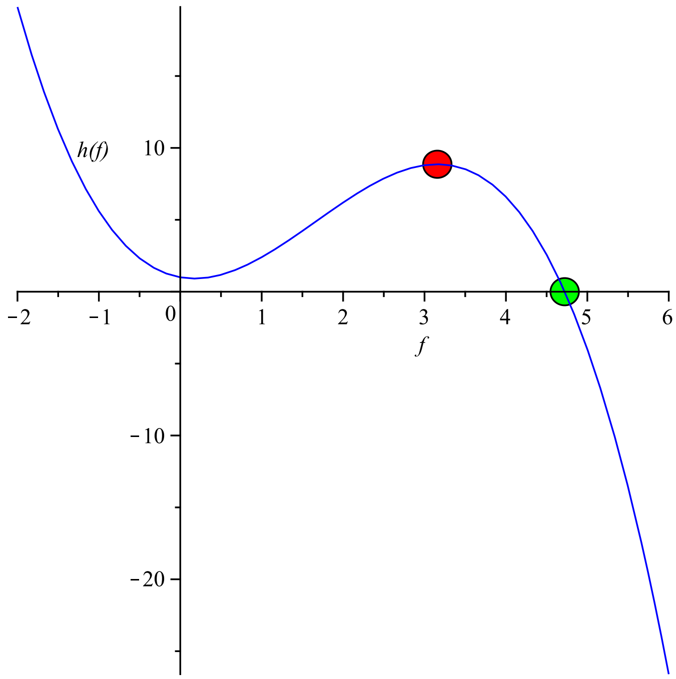

Figure 4.10: $h(f)$ plotted for $\mathcal{D} > 0$, $-\frac{\lambda^2}{3} < \mu < 0$ where f_0 given in eq. (4.22) is the stable AdS vacuum. The red disk indicates the location of f_{crit}, which is the point where the space becomes singular.

$$= \frac{1}{16\pi G_D} \int \mathrm{d}^D x \, \frac{(D-2)\, N(r)}{L^D} \left[r^{D-1} \left(1 - f + \lambda f^2 + \mu f^3 \right) \right]' .$$

The difference from the black hole solutions that were found in the five-dimensional case is that $\frac{\omega^4}{r^4} \rightarrow \frac{\omega^{D-1}}{r^{D-1}}$ as we had expected from the Gauss-Bonnet case. As noted above, this applies for $D > 6$, but in those dimensions, one already has a cubic order theory in Lovelock gravity. Hence black holes in our higher dimensional theory, eq (4.28), would essentially be the same as those already found [41, 43].

Chapter 5

Thermodynamics of Pseudo-Topological Black Holes

In the previous chapter, we found black hole solutions to the pseudo-topological theory that we have constructed. After having characterized the black holes based on the values of the interaction parameters, we now explore their thermodynamics. The calculations in this chapter will be done primarily for the five-dimensional black brane solutions using the metric ansatz eq. (2.37), where we note any possible generalizations. Following the calculations done Chapter 2, we use the formula derived for black hole temperature, eq. (2.55), and apply it to the pseudo-topological black branes. We then use the Euclidean action approach to calculate the free energy and then entropy of the black branes. Using the thermodynamic relations eqs. (2.63) and (2.64) we then are able to calculate pressure and energy density. In considering Gauss-Bonnet gravity, we had also used Wald's approach to calculating entropy and shown that the result matched the Euclidean action formulation, and we will do the same for the pseudo-topological theory.

5.1 Temperature

Following the logic discussed in Section 2.4, we can use the result obtained by analytically continuing the metric eq. (2.37) to Euclidean signature $\tau = -\imath t$ and periodically identifying τ. Interpreting the period of τ as inverse temperature, we find that the temperature of the black brane can be expressed as (noting to keep track of the $N(r)^2 = 1/f_\infty$ term):

$$T_H = \frac{1}{4\pi}\frac{\omega^2 f'_{r=\omega}}{L^2\sqrt{f_\infty}}. \tag{5.1}$$

However, we can evaluate this simply by referring back to the constraint equation that determined f and recalling at $r = \omega$, f vanishes. Then a simple calculation yields $f'|_\omega = \frac{4}{\omega}$, giving:

$$T_H = \frac{\omega}{\pi L^2\sqrt{f_\infty}}. \tag{5.2}$$

Now, calculating the temperature for the various black holes yields, with [x] indicating the region in which the solution describes a black hole as listed in the table on page 57:

$$[g] \quad T_{H_{f_1}} = \frac{\sqrt{3}\omega}{L^2\pi}\left(-\frac{\lambda}{\mu} + 2\sqrt{\frac{\lambda^2+3\mu}{\mu^2}}\cos\left(\frac{1}{3}\arccos\left(-\frac{2\lambda^3+9\mu\lambda+27\mu^2}{2\mu^3\left(\frac{\lambda^2+3\mu}{\mu^2}\right)^{3/2}}\right)\right)\right)^{-\frac{1}{2}}$$

$$[e] \quad T_{H_{f_2}} = \frac{\sqrt{3}\omega}{L^2\pi}\left(-\frac{\lambda}{\mu} - 2\sqrt{\frac{\lambda^2+3\mu}{\mu^2}}\cos\left(\frac{1}{3}\arccos\left(\frac{2\lambda^3+9\mu\lambda+27\mu^2}{2\mu^3\left(\frac{\lambda^2+3\mu}{\mu^2}\right)^{3/2}}\right)\right)\right)^{-\frac{1}{2}}$$

$$\begin{aligned}[c] \quad T_{H_{f_3}} = \frac{\sqrt{3}\omega}{L^2\pi}&\left(-\frac{\lambda}{\mu} - \sqrt{\frac{\lambda^2+3\mu}{\mu^2}}\cos\left(\frac{1}{3}\arccos\left(-\frac{2\lambda^3+9\mu\lambda+27\mu^2}{2\mu^3\left(\frac{\lambda^2+3\mu}{\mu^2}\right)^{3/2}}\right)\right)\right. \\ &\left. +\sqrt{3}\sqrt{\frac{\lambda^2+3\mu}{\mu^2}}\sin\left(\frac{1}{3}\arccos\left(-\frac{2\lambda^3+9\mu\lambda+27\mu^2}{2\mu^3\left(\frac{\lambda^2+3\mu}{\mu^2}\right)^{3/2}}\right)\right)\right)^{-\frac{1}{2}}\end{aligned}$$

$$[f] \quad T_{H_{D>0}} = \frac{\sqrt{6}\omega}{L^2\pi}\left(2^{2/3}\left(\left(\left(\frac{-2\lambda^3 - 9\mu\lambda + 3\mu^2\left(\sqrt{3}\mu\sqrt{\frac{4\lambda^3-\lambda^2+18\mu\lambda+\mu(27\mu-4)}{\mu^4}} - 9\right)}{\mu^3}\right)^{\frac{1}{3}}\right.\right.\right.$$

$$\left.\left.\left. - \left(\frac{2\lambda^3 + 9\mu\lambda + 3\mu^2\left(\sqrt{3}\sqrt{\frac{4\lambda^3-\lambda^2+18\mu\lambda+\mu(27\mu-4)}{\mu^4}}\mu + 9\right)}{\mu^3}\right)^{\frac{1}{3}}\right) - \frac{2\lambda}{\mu}\right)^{-\frac{1}{2}}$$

$$T_{H_{p=0}} = \frac{\omega}{L^2\pi}\left(\frac{1 - \sqrt[3]{1-3\lambda}}{\lambda}\right)^{-\frac{1}{2}},$$

which is much messier than one would hope. However, this aesthetic difficulty is just resulting from the $\frac{1}{\sqrt{f_\infty}}$ normalizing the temperature to the proper units. The plots of the temperature versus μ, λ (with ω and L set to 1), shows that the temperature of each black hole is positive in its respective valid region.

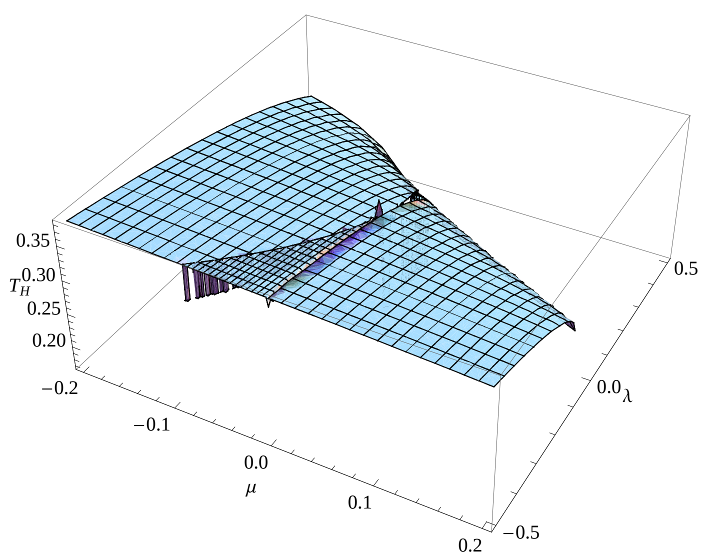

Figure 5.1: Temperature plots of all black hole solutions together. Looking along the edges of the plots, the temperature decreases to a small positive value but does not extend to zero. The discontinuities along certain lines are due to the fact that we have plotted all of the temperatures together, while each solution is only valid in a certain region of the $\mu - \lambda$ plane. So, we see sharp edges where a solution approaches its boundary of validity.

5.2 Energy Density

In this section, we calculate the entropy and energy densities of the pseudo-topological black holes by finding the free energy with the Euclidean action approach and following standard thermodynamic arguments. That is, we follow the arguments in section 2.4 to arrive at the correspondence:

$$e^{-\frac{1}{T}F[T]} \simeq e^{-I_E}, \tag{5.3}$$

where I_E is the Euclidean action for the black hole solution. Regarding the black hole as a thermal system at temperature T, we express I_E as

$$I_E[T] = -\frac{1}{16\pi G}\int_\omega^R \mathrm{d}r \int_0^{1/T} \mathrm{d}t_E \int \mathrm{d}^3x\sqrt{g_E}\left(R - 2\Lambda + \frac{\lambda L^2}{2}\chi_4 + \frac{7\mu}{4}\mathcal{Z}_5\right). \tag{5.4}$$

Calculating the integrand and using that N is a constant (which we choose to be $\frac{1}{\sqrt{f_\infty}}$), we find that eq. (5.4) reduces to:

$$\begin{aligned} I_E[T] \quad = \quad & -\frac{V_3}{16\pi G}\frac{1}{TL^5\sqrt{f_\infty}}\left[r^4\left(3-5f(r)+15\left(\lambda f(r)^2-\mu f(r)^3\right)\right)+r^5\left(-1+6\lambda f(r)\right.\right. \\ & \left.\left.-9\mu f(r)^2\right)f'(r)\right]_\omega^R . \end{aligned} \tag{5.5}$$

Evaluating this can be simplified by using the constraint and and the asymptotic expansion of $f(r)$:

$$f \sim f_\infty - \frac{\omega^4}{r^4}\frac{1}{(1-2\lambda f_\infty - 3\mu f_\infty^2)} + \dots . \tag{5.6}$$

Keeping only the divergent and finite terms in the limit $R \to \infty$, eq. (5.5) reduces to:

$$I_E[T] = -\frac{V_3}{16\pi G}\frac{\omega^4}{TL^5\sqrt{f_\infty}}\left[\frac{R^4}{\omega^4}\left(10f_\infty - 30\mu f_\infty^3 - 12\right) - \frac{(5f_\infty - 15\mu f_\infty^3 - 6)}{f_\infty(1-2\lambda f_\infty - 3\mu f_\infty^2)} + 1\right]. \tag{5.7}$$

Regulating this by subtracting off the value of eq. (5.4) for pure AdS (obtained by taking $\omega = 0$ in eq. (2.37)):

$$I_E^0[T'] = -\frac{V_3}{16\pi G}\frac{1}{T'L^5\sqrt{f_\infty}}\left[R^4\left(10f_\infty - 30\mu f_\infty^3 - 12\right)\right], \tag{5.8}$$

where T' is chosen so that the asymptotic geometries of the pure and black hole AdS spaces match:

$$\frac{1}{T'} = \frac{1}{T}\frac{\sqrt{f(R)}}{\sqrt{f_\infty}} \sim \frac{1}{T}\left(1 - \frac{\omega^4}{2R^4 f_\infty \left(1 - 2\lambda f_\infty - 3\mu f_\infty^2\right)}\right). \tag{5.9}$$

The free energy is then expressed as:

$$F[T] = T\left(I_E[T] - I_E^0[T]\right) = -\frac{V_3\omega^4}{16\pi G L^5\sqrt{f_\infty}} = -\frac{V_3 T}{16G}\left(\pi L T\sqrt{f_\infty}\right)^3. \tag{5.10}$$

The entropy given by the familiar relationship, and then dividing by the volume gives the entropy density as:

$$s[T] = -\frac{1}{V_3}\frac{d}{dT}F[T] = \frac{1}{4G}\frac{\omega^3}{L^3}. \tag{5.11}$$

Recalling eq. (5.2), we can use the thermodynamic relation for a system at temperature T in absence of chemical potential, $\rho = \frac{3}{4}Ts[T]$ to express the energy density of the pseudo-topological black holes as:

$$\rho = \frac{3\omega^4}{16\pi G L^5\sqrt{f_\infty}}. \tag{5.12}$$

5.3 Noether Charge Approach to Entropy Density

In this section we will follow the prescription developed by Wald and Iyer [21] to compute the entropy density of our new black holes in cubic gravity. We should note that this computation should match that done by the Euclidean action approach computed in the previous section. [27]. Using the method discussed in Section 2.4

and applying it to pseudo-topological gravity:

$$
\begin{aligned}
Y_3^{abcd} &= \frac{1}{16\pi G}\Bigg[\frac{3}{2}c_8 R^2\left(g^{ac}g^{bd}-g^{ad}g^{bc}\right)+c_4\left(2R^{abcd}R+\frac{1}{2}R_{mnpq}R^{mnpq}\left(g^{ac}g^{bd}-g^{ad}g^{bc}\right)\right)\\
&\quad+\frac{c_7}{2}\left(R_m^{\ n}R_n^{\ m}\left(g^{ac}g^{bd}-g^{ad}g^{bc}\right)+R\left(g^{ac}R^{bd}+g^{bd}R^{ac}-g^{ad}R^{bc}-g^{bc}R^{ad}\right)\right)\\
&\quad+\frac{3}{4}c_6\left(g^{ac}R^{dm}R_m^{\ b}+g^{bd}R^{cm}R_m^{\ a}-g^{bc}R^{dm}R_m^{\ a}-g^{ad}R^{cm}R_m^{\ b}\right)+\frac{c_5}{2}\left(R^{ac}R^{bd}\right.\\
&\quad\left.-R^{ad}R^{bc}+\left(g^{ac}R^{b\ d}_{\ m\ n}R^{mn}+g^{bd}R^{a\ c}_{\ m\ n}R^{mn}-g^{ad}R^{b\ c}_{\ m\ n}R^{mn}-g^{bc}R^{a\ d}_{\ m\ n}R^{mn}\right)\right)\\
&\quad+c_3\left(R^{abc}{}_{m}R^{md}-R^{abd}{}_{m}R^{mc}+\frac{1}{4}\left(g^{ac}R_{mnp}{}^{b}R^{mnpd}+g^{bd}R_{mnp}{}^{a}R^{mnpc}\right.\right.\\
&\quad\left.\left.-g^{ad}R_{mnp}{}^{b}R^{mnpc}-g^{bc}R_{mnp}{}^{a}R^{mnpd}\right)\right)+3c_2R^{abmn}R_{mn}^{\ \ \ cd}+\frac{3c_1}{2}R^{a\ c}_{\ m\ n}R^{bmdn}\Bigg],
\end{aligned}
$$

which leads to,

$$
\begin{aligned}
Y_3 &= -\frac{1}{4\pi G}\Bigg[\frac{3}{2}c_8R^2+c_4\left(2R^{tr}{}_{tr}+\frac{1}{2}R_{mnpq}R^{mnpq}\right)+\frac{3}{4}c_6\left(R^{rm}R_{mr}+R^{tm}R_{tm}\right)+\frac{c_7}{2}\times\\
&\quad\times\left(R_m^{\ n}R_n^{\ m}+R\left(R^r{}_r+R^t{}_t\right)\right)+\frac{c_5}{2}\left(R^t{}_tR^r{}_r-R^t{}_rR^r{}_t+\left(R^r{}_{mrn}R^{mn}\right.\right.\\
&\quad\left.\left.+R^t{}_{mtn}R^{mn}\right)\right)+c_3\left(R^{tr}{}_{tm}R_r^{\ m}-R^{tr}{}_{rm}R_t^{\ m}+\frac{1}{4}\left(R_{mnp}{}^{r}R^{mnp}{}_{r}+R_{mnp}{}^{t}R^{mnp}{}_{t}\right)\right)\\
&\quad+3c_2R^{trmn}R_{mntr}+\frac{3c_1}{2}R^t{}_{mtn}R^{rm\ n}{}_{r}\Bigg],
\end{aligned}
$$

and thus the entropy of a black hole in pseudo-topological gravity is given by the following:

$$
\begin{aligned}
S &= \frac{A}{4G}\Bigg(1+2\lambda L^2\left(R-2\left(R^t{}_t+R^r{}_r\right)+2R^{tr}{}_{tr}\right)+2\mu L^4\Bigg[\frac{3}{2}c_8R^2+c_4\left(2R^{tr}{}_{tr}\right. \qquad (5.13)\\
&\quad\left.+\frac{1}{2}R_{mnpq}R^{mnpq}\right)+\frac{3}{4}c_6\left(R^{rm}R_{mr}+R^{tm}R_{tm}\right)+\frac{c_7}{2}\left(R_m^{\ n}R_n^{\ m}+R\left(R^r{}_r\right.\right.\\
&\quad\left.\left.+R^t{}_t\right)\right)+\frac{c_5}{2}\left(R^t{}_tR^r{}_r-R^t{}_rR^r{}_t+\left(R^r{}_{mrn}R^{mn}+R^t{}_{mtn}R^{mn}\right)\right)+c_3\times\\
&\quad\times\left(R^{tr}{}_{tm}R_r^{\ m}-R^{tr}{}_{rm}R_t^{\ m}+\frac{1}{4}\left(R_{mnp}{}^{r}R^{mnp}{}_{r}+R_{mnp}{}^{t}R^{mnp}{}_{t}\right)\right)\\
&\quad+3c_2R^{trmn}R_{mntr}+\frac{3c_1}{2}\left(R^t{}_{mtn}R^{rm\ n}{}_{r}-R^t{}_{mrn}R^{rm\ n}{}_{t}\right)\Bigg]\Bigg).
\end{aligned}
$$

Evaluating this for the metric ansatz that we have set out with recalling N is constant

$$
S=\frac{A}{4G}\left(1-6\lambda f(r)+9\mu f(r)^2\right). \qquad (5.14)
$$

Evaluating at the horizon yields the familiar result, $S = \frac{A}{4G}$, which holds regardless of the curvature of the horizon. The entropy density is then given by $s = \frac{\omega^3}{4GL^3}$. For the case of pseudo-topological black holes with curved horizons, we find that the entropy is given by

$$S_k = \frac{A}{4G}\left(1 + 6\lambda k \frac{L^2}{r_h^2} + 9\mu k^2 \frac{L^4}{r_h^4}\right). \tag{5.15}$$

Chapter 6

Discussion and Outlook

In the previous sections, we have seen a variety of unexpected results. We began by considering the most general form for a gravitational interaction built out of curvature cubed terms. From that point, we have been able to apply a variational method to constrain the coefficients to produce a theory that has some of the nice properties of higher curvature gravity theories built out of topological invariants. However, it is shown in Appendix A that $\mathscr{Z}_6$ is not a true topological quantity. Moreover having found that there are no problematic, three or four derivative, terms in the linearized theory was an important step in showing that pseudo-topological gravity did not suffer from some of the pathologies that plague other non-Lovelock higher curvature theories [23, 17]. We thus found that the equations of motion are second order in derivatives of the metric perturbation given that we restrict the calculation to highly symmetric spaces. As known from classical field theory, having equations of motion contain two time derivatives of the dynamical field indicates that given sufficient initial data there is a unique solution depending continuously on the initial values. So, we have indications that the initial value problem for pseudo-topological gravity is well posed, but a complete proof of this has not been formulated [29]. Solving the equations of motion obtained by the variational approach, we determined the behavior of the solutions by their location in the phase space of the interaction parameters. What we found was a rich landscape of vacua with some of the stable spacetimes admitting interesting black hole solutions. In these cases, we determined the thermodynamics of the pseudo-topological black holes, and we found that the same formulae for temperature, entropy and energy densities of the black brane solutions as in Gauss-Bonnet and Einstein gravity. The pseudo-topological interaction effected the values of the energy density and temperature of the black branes by the presence of the factor of $\frac{1}{\sqrt{f_\infty}}$. The same could not be said for the black holes with curved horizons, but that was to be expected as Gauss-Bonnet interaction was already shown to non-trivially modify their thermodynamic properties [18].

With the pseudo-topological gravitational theory written down, the next phase of research will focus on its dual description in the boundary CFT. As we had been motivated by the results in [11, 12], we would like to find out if this new interaction provides some further modification to the KSS bound. As the authors in [11] had found, the possibility of causality violation the dual CFT could provide bounds on the values of the parameters λ and μ giving physically relevant theories. We could also return to the gravity side and explore what are the consequences of adding electric charge or rotation to the pseudo-topological black holes. In such a case, we could determine if such solutions have multiple horizons. More interestingly, we could look for and explore the behavior of extremal solutions. Pushing further, we could ask if we can construct pseudo-topological black holes other properties like NUT charge. I feel that beyond the theory we have presented in the main thesis, we should question if it is possible to write down an arbitrary order pseudo-topological theory of gravity in five dimensions. That is, can we determine the behavior of the coefficients for an arbitrary order interaction that will give two-derivative field equations? We have seen some very interesting results so far, but as we can see, there are still many more questions to answer on the subject of pseudo-topological gravity.

A. New Topological Object in Six Dimensions?

As we have been motivated by Gauss-Bonnet, and in general Lovelock, gravity, we should make a point about the case of $\mathcal{Z}_6$. As $\mathcal{Z}_6$ does not contribute to the equations of motion in six dimensions eq. (3.14), one might ask if this expression is itself related to new a topological invariant in six dimensions. However, we will show that with an explicit example, this is not the case. Taking the new curvature cubed out of the AdS-black brane setting in which we were exploring in order to examine its generic properties in $D = 6$ to determine whether or not it is a topological term.

As a simple test, we evaluate $\int \mathcal{Z}_6$ on some simple six-dimensional geometries. First so we can take a six-sphere with a particular deformation, and ask if the $\int \mathcal{Z}_6$ depends on this deformation parameter. Let us consider the following deformed six-sphere metric:

$$\begin{aligned} ds^2 \;\; &= \;\; R^2 \, \left(d\theta^2 + sin^2(\theta) \, \left(1 + a\, sin^2(\theta)\right)^n \, \left(d\phi_1^2 + sin^2(\phi_1) \, \left(d\phi_2^2 + sin^2(\phi_2) \times \right.\right.\right. \\ & \quad \left.\left.\left. \times \left(d\phi_3^2 + sin^2(\phi_3) \, \left(d\phi_4^2 + sin^2(\phi_4)\, d\psi^2\right)\right)\right)\right)\right) , \end{aligned} \tag{1}$$

where R is the radius of the six-sphere, $n = 1$ or 2, and $\theta,\ \phi_i,\, i = 1 \ldots 4$ range from 0 to π and ψ takes values over 0 to 2π. As a check, we can evaluate the six dimensional Euler character, eq. (3.7), of over this deformed sphere to show that the known topological invariant does not depend on the smooth deformation of the manifold. Following suit, $\int \mathcal{Z}_6$ can also be evaluated with the following results

$$\int_{\mathcal{S}^6} \sqrt{g}\; \mathcal{X}_6 = \;\; 768\pi^3, \tag{2}$$

$$\int_{\mathcal{S}^6} \sqrt{g}\; \mathcal{Z}_6 = \;\; \tfrac{544}{3}\pi^3, \tag{3}$$

where we have normalized the Euler density as in eq. (3.7). Hence we find that $\int \mathcal{Z}_6$ is also independent of the deformation parameter. This is by no means a conclusive result, but it is suggestive that $\mathcal{Z}_6$ may be topological.

However, we should make sure that the lack of dependence on the deformation parameter for $\mathcal{Z}_6$ integrated over S^6 is not just a property of the combination of curvature cubed terms in a very symmetric space. Let us then consider the same calculation on a manifold with much less symmetry by taking the the following deformed $S^2 \times S^4$ metric:

$$\begin{aligned} ds^2 &= L^2 \left(d\phi_2^2 + sin^2(\phi_2) \left(1 + b\, sin^2(\phi_2)\right)^2 \left(d\phi_3^2 + sin^2(\phi_3) \left(d\phi_4^2 + sin^2(\phi_4)\, d\psi^2 \right)\right)\right) \\ &\quad + R^2 \left(d\theta^2 + sin^2(\theta) \left(1 + a\, sin^2(\theta)\right)^2 d\phi_1^2 \right), \end{aligned} \tag{4}$$

where R (L) is the radius of the two(four)-sphere, and $\theta,\ \phi_i,\ i = 2 \ldots 4$ range from 0 to π and $\phi_1,\ \psi$ take values over 0 to 2π. We again evaluate the six dimensional Euler density (3.7) as a check of our calculations, and then calculate the value of $\mathcal{Z}_6$ integrated over this manifold.

$$\int_{S^2 \times S^4} \sqrt{g}\, \mathcal{X}_6 = 1536\pi^3, \tag{5}$$

$$\begin{aligned} \int_{S^2 \times S^4} \sqrt{g}\, \mathcal{Z}_6 = -\frac{8}{15}\pi^3 \Big(& -89280\sqrt{b}\sqrt{b+1}R^2 - 2392\, b^{9/2}\sqrt{b+1}L^2 - 3464\, L^2 b^{7/2}\sqrt{b+1} \\ & +655380\, R^2 \tanh^{-1}\left(\frac{\sqrt{b}}{\sqrt{b+1}}\right) b^2 - 447156\, b^{5/2}\sqrt{b+1}R^2 - 1920 \times \\ & \times\sqrt{b}\sqrt{b+1}L^2 - 6120\, b^{3/2}\sqrt{b+1}L^2 - 576\, L^2 b^{13/2}\sqrt{b+1} - 1512\, b^{11/2} \times \\ & \times\sqrt{b+1}R^2 - 73638\, b^{9/2}\sqrt{b+1}R^2 - 1968\, b^{11/2}\sqrt{b+1}L^2 + 555255\, R^2 \times \\ & \times b^3 \tanh^{-1}\left(\frac{\sqrt{b}}{\sqrt{b+1}}\right) + 38880\, R^2 b^5 \tanh^{-1}\left(\frac{\sqrt{b}}{\sqrt{b+1}}\right) - 324240 \times \\ & \times b^{3/2}\sqrt{b+1}R^2 + 5040 \tanh^{-1}\left(\frac{\sqrt{b}}{\sqrt{b+1}}\right) L^2 b^3 + 89280\, R^2 \times \\ & \times \tanh^{-1}\left(\frac{\sqrt{b}}{\sqrt{b+1}}\right) + 1920 \tanh^{-1}\left(\frac{\sqrt{b}}{\sqrt{b+1}}\right) L^2 + 233280\, R^2 \times \\ & \times b^4 \tanh^{-1}\left(\frac{\sqrt{b}}{\sqrt{b+1}}\right) - 283071\, b^{7/2}\sqrt{b+1}R^2 + 383760\, R^2 \times \end{aligned}$$

$$
\begin{aligned}
&\times \tanh^{-1}\left(\frac{\sqrt{b}}{\sqrt{b+1}}\right) b + 1080 \tanh^{-1}\left(\frac{\sqrt{b}}{\sqrt{b+1}}\right) L^2 b^4 + 1296\, b^{13/2} \times \\
&\times \sqrt{b+1} R^2 + 8760 \tanh^{-1}\left(\frac{\sqrt{b}}{\sqrt{b+1}}\right) L^2 b^2 + 6720 \tanh^{-1}\left(\frac{\sqrt{b}}{\sqrt{b+1}}\right) \times \\
&\times\, L^2 b - 6664\, b^{5/2} \sqrt{b+1} L^2 \Big)\, b^{-3/2} \left(b+1\right)^{-5/2} L^{-2}, \qquad (6)
\end{aligned}
$$

where for simplicity we have evaluated the integral of $\mathcal{Z}_6$ with the S^2 deformation parameter $a = 0$. Even with $R = L$, there is still a dependence on the S^4 deformation, b. We thus conclude the new action $\mathcal{Z}_6$ does not produce a topological term in six dimensions. Its mimicry of a topological term, especially on very symmetric manifolds, leads to our choice of name for the theory of gravity that we construct using $\sqrt{-g}\mathcal{Z}$, Pseudo-Topological Gravity.

B. Quasilocal Formulation of Black Hole Thermodynamics

Here we attempt to calculate the energy density of the pseudo-topological AdS black holes by finding the boundary stress energy tensor via the quasilocal formulation of Brown and York [36] and presented in the setting of the AdS/CFT correspondence in [44]. In the quasilocal approach, we consider a region M in the spacetime manifold with boundary ∂M foliated by spacelike hypersurfaces Σ whose boundaries B foliate ∂M. For our purposes, the boundary ∂M is timelike hypersurface at constant radius R. We denote spacetime coordinates by $u, v \ldots$, Σ coordinates by $i, j, \ldots$, and ∂M coordinates by $a, b, \ldots$. Considering the gravitational (and matter) action in M

$$I = \frac{1}{16\pi G}\int d^D x\sqrt{-g}\,(R-2\Lambda)+\frac{1}{8\pi G}\int d^{D-1}y\sqrt{h}K-\frac{1}{8\pi G}\int d^{D-1}z\sqrt{-\gamma}\Theta+I_{matter}, \tag{7}$$

where h_{ij} (γ_{ab}) is the induced metric on M (∂M), y^i (z^a) are the coordinates, and K_{ij} (Θ_{ab}) is the extrinsic curvature tensor. The quasilocal stress tensor τ^{ab} is defined as the functional derivative of the action evaluated for the classical solution $I_{cl} = I(g_{cl})$ with respect to the γ_{ab}

$$\tau^{ab} = \frac{2}{\sqrt{-\gamma}}\frac{\delta I_{cl}}{\delta \gamma_{ab}}, \tag{8}$$

where $\gamma = \det(\gamma_{ab})$. Denoting the normal to ∂M by n^a, we observe the relationship $\mathcal{D}_a\tau^{ab} = -T^{nb} = -T^{uv}n_a\gamma^a{}_u$ where T^{uv} is the familiar matter stress energy tensor. Calculating τ^{ab} for eq. (7) [44]

$$\tau_{ab} = \frac{1}{8\pi G}(\Theta_{ab} - \gamma_{ab}\Theta). \tag{9}$$

As discussed in section 2.2, the boundary terms in eq. (7) must be regulated, and the method used to do so is by subtracting the value for the boundary actions obtained by embedding M in some background spacetime. The boundary stress tensor, τ^{ab}, will see contributions, τ_0^{ab}, from the background term. We then define the regulated boundary stress tensor to be:

$$\hat{\tau}_{ab} = \tau_{ab} - \tau^0_{ab}. \tag{10}$$

In the context of the AdS/CFT correspondence, the total energy of the bulk AdS spacetime should match the total energy measured in the dual CFT. To find this we note that, in the limit of taking ∂M to spatial infinity, $R \to \infty$, the geometry of ∂M is equivalent to the background geometry of the CFT up to a conformal transformation. Taking this into account, the expectation value of the dual CFT is related to the regulated boundary stress tensor $\hat{\tau}^{ab}$ by [44]

$$\sqrt{-h}\, h^{ab} \langle T_{bc} \rangle = \lim_{R\to\infty} \sqrt{-\gamma}\, \gamma^{ab}\, \hat{\tau}_{bc}. \tag{11}$$

The energy density of the AdS space is then found by calculating $\langle T_{tt} \rangle$.

Now, we apply this method to the five dimensional pseudo-topological black branes in section 4.1 with metric eq. (2.37). The hypersurface at R, ∂M, then has metric components given by

$$\gamma_{tt} = -\frac{R^2}{L^2 f_\infty} f(R), \qquad \gamma_{\bar{a}\bar{b}} = \frac{R^2}{L^2}\delta_{\bar{a}\bar{b}}, \tag{12}$$

where $x^{\bar{a}}$ are the planar coordinates and δ_{ab} is the metric for $\mathbb{R}^p$ where $p = D - 2$. We have also taken $N_\sharp^2 = \frac{1}{f_\infty}$. The normal vector to ∂M is given by

$$n^a = \frac{R}{L}\sqrt{f(R)}\delta^a{}_r. \tag{13}$$

We then find that the extrinsic curvature is given by

$$\Theta_{tt} = \frac{R\sqrt{f(R)}}{2L^3 f_\infty}\left(2Rf(R) + R^2 f'(R)\right), \qquad \Theta_{\bar{a}\bar{b}} = -\frac{R^2\sqrt{f(R)}}{L^3}\delta_{\bar{a}\bar{b}}.$$

Computing the tt component of the boundary stress tensor yields

$$\tau_{tt} = -\frac{pR^2}{8\pi G L^3}\frac{f(R)^{\frac{3}{2}}}{f_\infty}. \tag{14}$$

To regulate τ_{ab}, we use as our background empty AdS space with metric given by

$$ds_0^2 = \frac{r^2}{L^2}\left(-\frac{f(R)}{f_\infty}dt^2 + (dx^{\bar{a}})^2\right) + \frac{L^2}{r^2 f_\infty}dr^2. \tag{15}$$

We calculate the regulated boundary stress tensor, and extracting the tt component gives:

$$(\hat{\tau}_{tt}) = \frac{3f(R)R^2}{8\pi G_5 L^3\sqrt{f_\infty}}\left(\sqrt{f_\infty} - \sqrt{f(R)}\right), \tag{16}$$

where in the regions where the f's corresponded to valid black hole solutions, the corresponding $\hat{\tau}_{tt}$'s are positive. In order to now calculate $\langle T_{tt}\rangle$, we define h_{ab} by

$$h_{ab} = \lim_{R\to\infty}\frac{L^2}{R^2}\gamma_{ab}. \tag{17}$$

We then find after routine calculation that the $\langle T_{tt}\rangle$ is given by:

$$\langle T_{tt}\rangle = \frac{3\omega^4}{16\pi G_5 L^5\sqrt{f_\infty}}\frac{1}{(1-2\lambda f_\infty - 3\mu f_\infty^2)}. \tag{18}$$

where we have used the asymptotic expansions of f, eq. (5.6), to simplify the calculation. However, we not here that the result is different from eq. (5.12).

As with the Euclidean approach to black hole thermodynamics in section 5, we should have the satisfaction of the thermodynamic relationship given by eq. (5.12). In this case, we have $\rho = \langle T_{tt}\rangle$ and s being the entropy density as seen in the dual CFT given by

$$s = \frac{\omega^3}{4GL^3}. \tag{19}$$

Evaluating for the values of $\langle T_{tt}\rangle$, s, and T already calculated, we find that the right hand side of eq. (5.12) and eq. (18) only agree if

$$\sqrt{f_\infty} = 1 - 2\lambda f_\infty - 3\mu f_\infty^2.$$

We should issue a warning about the discrepancy between the Euclidean and quasilocal approaches. We are confident in the results given in section 5 because of the satisfaction of the thermodynamic consistency check, eq. (5.12). The problem in the quasilocal formulation appears to be due to the simple prescription eq. (11) in the presence of higher curvature terms. That is, the problem is not due to the pseudo-topological interaction. We can see this by letting $\mu \to 0$ in the above analysis. The consistency relation would only be satisfied when $\sqrt{f_\infty} = 1 - 2\lambda f_\infty$. It is not clear how to resolve this issue at this time, and it may be revisited in later work.

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