Optimal Reinsurance Designs: from an Insurer's Perspective

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

The research on optimal reinsurance design dated back to the 1960's. For nearly half a century, the quest for optimal reinsurance designs has remained a fascinating subject, drawing significant interests from both academicians and practitioners. Its fascination lies in its potential as an effective risk management tool for the insurers. There are many ways of formulating the optimal design of reinsurance, depending on the chosen objective and constraints. In this thesis, we address the problem of optimal reinsurance designs from an insurer's perspective. For an insurer, an appropriate use of the reinsurance helps to reduce the adverse risk exposure and improve the overall viability of the underlying business. On the other hand, reinsurance incurs additional cost to the insurer in the form of reinsurance premium. This implies a classical risk and reward tradeoff faced by the insurer.

The primary objective of the thesis is to develop theoretically sound and yet practical solution in the quest for optimal reinsurance designs. In order to achieve such an objective, this thesis is divided into two parts. In the first part, a number of reinsurance models are developed and their optimal reinsurance treaties are derived explicitly. This part focuses on the risk measure minimization reinsurance models and discusses the optimal reinsurance treaties by exploiting two of the most common risk measures known as the Value-at-Risk (VaR) and the Conditional Tail Expectation (CTE). Some additional important economic factors such as the reinsurance premium budget, the insurer's profitability are also considered. The second part proposes an innovative method in formulating the reinsurance models, which we refer as the empirical approach since it exploits explicitly the insurer's empirical loss data. The empirical approach has the advantage that it is practical and intuitively appealing. This approach is motivated by the difficulty that the reinsurance models are often infinite dimensional optimization problems and hence the explicit solutions are achievable only in some special cases. The empirical approach effectively reformulates the optimal reinsurance problem into a finite dimensional optimization problem. Furthermore, we demonstrate that the second-order conic programming can be used to obtain the optimal solutions for a wide range of reinsurance models formulated by the empirical approach.

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Contents

List of Tables

List of Figures

Chapter 1

Introduction

1.1 Background

The research on optimal reinsurance design dated back to the 1960's (see Borch (1960), Kahn (1961), and Ohlin (1969)). For nearly half a century, the quest for optimal reinsurance designs has remained a fascinating subject, drawing significant interests from both academicians and practitioners. Its fascination lies in its potential as an effective risk management tool for insurer. The theme of this thesis is to study the optimal reinsurance design. In particular, we will consider various reinsurance models with the objective of deriving their solutions.

To introduce the concept of optimal reinsurance design, let us first recall the concept of it reinsurance. Generally speaking, reinsurance is an insurance on insurance or an insurance for the insurers. It is a contractual agreement between an insurer (cedent) and a reinsurer whereby, depending on the nature of the reinsurance arrangement, the reinsurer indemnifies part of the losses incurred on the insurer.

There are many reasons for the existence of the reinsurance. First, reinsurance can be employed by the insurance company to mitigate its risk exposure and hence stabilize the underwriting (or earnings) volatilities. Second, the reinsurance might be utilized by the insurer to avoid a large single loss, for example, claims resulting from a catastrophic risk, which might lead to the insurer's bankruptcy. Third, a newly established insurance company can obtain the business expertise from some reinsurance companies by relating to them through reinsurance contracts. Fourth, reinsurance also provides a mechanism allowing an insurance company to increase its capacity to accept risks.

In order to clarify the concept of *optimal reinsurance design*, let us further analyze the general effect of a reinsurance treaty on the insurer. Obviously, by spreading some of the risks to a reinsurer, the insurer incurs additional cost in the form of reinsurance premium which is payable to the reinsurer. Naturally, the higher the expected risk transfers to a reinsurer, the more costly the reinsurance premium is. Similarly, a cedent can lower the cost of reinsuring by exposing to higher expected retained risk. This demonstrates the trade-off between risk spreading and risk retaining. Such a trade-off right leads to the topic of optimal reinsurance design. It is a process of determining the optimal reinsurance contract according to some optimality criteria along with some constraints if necessary. In the nutshell, it deals with the optimal partitioning of a risk between insurer and reinsurer.

The optimal reinsurance design therefore entails specifying certain optimization problems and solving them for the optimal reinsurance treaties. We will term these optimization problems as *reinsurance models* and refer the risk on which the reinsurance is applied as the *underlying risk*. The studies of the reinsurance models therefore could provide important insights to the nature of the underlying risks to which the insurer is exposed and could also help to develop sound and prudent risk management tools for the insurance companies.

Let us now recall various types of reinsurance models that have been proposed in the literature. It is convenient to first divide the models into two major classes depending on the time periods. These are known as the *static models* and the *dynamic models*. In the former models, we are only concerned with reinsuring risk over a single time period and thus they are also called the *single period models*. The latter models address reinsurance in a multi-period setting which typically involves specifying a surplus process such as the classical compound Poisson model. We also refer the latter model as the *multi-period models*.

Among these models, one can further classify into either the *global models* or the *local models* depending on how reinsurance is structured. When reinsurance is only applied to the risk in aggregate, we call such models as global, otherwise local. Hence in the former case we only need to know the aggregate loss distribution while in the latter case, we need to know the joint distribution of the risks and how the reinsurance contract affects the resulting risk. Note that a substantial amount of the existing literature discusses the global models, although the local models and the combination of these two types models are common in practice. This thesis will focus on the global models.

Further classification of the reinsurance models is possible depending on how optimality is defined. For example in the *insurer-reinsurer-oriented models*, the optimal reinsurance is determined in such a way that it reflects jointly the interests of both insurer and reinsurer. In this case, the optimization is often formulated as a game-theoretic problem between both players and then determine the Pareto optimal reinsurance if it exists; see, for example, Borch (1960). On the other hand there are models, which are referred as the *insurer-oriented models*, that focus exclusively on the insurer in deriving the optimal reinsurance. The optimal reinsurance is determined solely from the point of view of the insurer. Hence, the insurer is the active player while the reinsurer is the passive counterpart. While the assumption of the reinsurer being passive is debatable, one can argue that the reinsurance market is competitive and the insurer can be demanding. Another advantage of focusing on the insurer-oriented model is that from the point of view of the insurer, the optimal reinsurance can become a benchmark or a guideline for the insurer, even if such optimal reinsurance contract may not be available from the market. Much of the research in recent years is devoted to the insurer-oriented models, which is also the focus of the present thesis.

We now list the following commonly-used reinsurance models, depending on the nature of the goal function:

(1) *variance minimization models*: if the insurer were to minimize the variance of its retained risk (or total risk);

(2) *expected utility maximization models*: if the insurer were to maximize its expected utility;

(3) *(convex) risk measure minimization models*: if the insurer were to minimize a (convex) risk measure of its retained risk (or total risk);

(4) *ruin probability minimization models*: if the insurer were to minimize ruin probability for its surplus process.

It should be noted that the first three models are nested in the sense that the variance minimization models can be considered as subset of the expected utility maximization models, which in turn is a special case of the risk measure minimization models. Note also that in studying the above optimization models, constraints such as the maximum premium budget or the minimum expected profits guarantee are often imposed. In these cases, one is dealing with constrained optimization models, as opposed to the unconstrained optimization models.

1.2 Literature Review

In this section, we provide a brief literature review and summarize some of the major results on optimal reinsurance that are relevant to the thesis. In particular, we will emphasize on the static global models with occasional reference to other models and techniques.

By examining the existing literature, there is a proliferate of research being conducted on the insurer-oriented models, particularly the static global insurer-oriented models. This type of model usually involves modelling the underlying risk as a non-negative random variable, say X. Suppose $f(X)$, with the conventional assumption $0 \leq f(x) \leq x$ for all $x \geq 0$, is the part of the underlying risk that is covered by the reinsurer, and Π denotes the premium principle adopted for determining the reinsurance premium for a given reinsurance arrangement $f \equiv f(X)$. Then the insurer retains the risk of $I_f \equiv I_f(X) := X - f(X)$ and pays $\Pi(f) \equiv \Pi(f(X))$ to the reinsurer in the form of reinsurance premium; hence its total cost or total risk, denoted by $T_f \equiv T_f(X)$, is the sum of $\Pi(f)$ and I_f, i.e., $T_f(X) = I_f(X) + \Pi(f(X))$. Note that the function $f(x)$ implies a partition of the initial risk X between insurer and reinsurer. This function is known as the *compensation function*, *indemnification function*, or *ceded loss function*, while $I_f(x)$ is referred as the *retained loss function*. It is reasonable to assume that the insurer has a preset reinsurance premium budget, say π. This implies that π is the maximum premium an insurer is willing to pay for reinsuring its risk. This is equivalent to imposing the constraint $\Pi(f(X)) \leq \pi$ in the reinsurance model.

Most of the static reinsurance models investigated to-date take either of the following formulations:

$$\begin{cases} \min & \mathsf{E}\left[w\big(I_f(X)\big)\right] = \mathsf{E}\left[w\big(X - f(X)\big)\right] \\ \text{s.t.} & 0 \leq f(x) \leq x, \ \text{ for all } \ x \geq 0, \ \text{ and } \ \Pi(f(X)) = \pi, \end{cases} \tag{1.2.1}$$

and

$$\begin{cases} \min & \mathsf{E}\left[w\big(\overline{I_f(X)}\big)\right] = \mathsf{E}\left[w\big(\overline{X - f(X)}\big)\right] \\ \text{s.t.} & 0 \leq f(x) \leq x, \ \text{ for all } \ x \geq 0, \ \text{ and } \ \Pi(f(X)) = \pi, \end{cases} \tag{1.2.2}$$

where w is a convex function and $\overline{Y}$ denotes $Y - \mathsf{E}Y$ for a random variable Y. The optimization models (1.2.1) and (1.2.2) are the general forms of the various reinsurance models including, for example, the following expected utility maximization model:

$$\begin{cases} \max & \mathsf{E}\left[u(W_0 - X + f(X) - \pi)\right] \\ \text{s.t.} & 0 \le I(x) \le x, \ \text{ for all } \ x \ge 0, \ \text{ and } \ \Pi(f(X)) = \pi, \end{cases} \tag{1.2.3}$$

where W_0 denotes the insurer's initial wealth so that $W_0 - X + f(X) - \pi$ represents the insurer's wealth after reinsurance arrangement. As the insurer is seeking a risk transfer, it is reasonable to assume that it is risk averse with a concave utility function. Suppose $u(t)$ is the corresponding concave utility function, then $u(-t + W_0 - \pi)$ is obviously convex as a function of t. Furthermore, by setting $w(t) = u(-t + W_0 - \pi)$, one recovers (1.2.3) from (1.2.1). For an excellent review of the utility function with respect to insurance applications, see Gerber and Pafumi (1998). Another important class of reinsurance model is obtained by letting $w(x) = x^2$ in (1.2.2). This leads to the classical variance minimization model:

$$\begin{cases} \min & \mathsf{Var}\big(I_f(X)\big) = \mathsf{Var}\big(X - f(X)\big) \\ \text{s.t.} & 0 \le f(x) \le x, \ \text{ for all } \ x \ge 0, \ \text{ and } \ \Pi(f(X)) = \pi. \end{cases} \tag{1.2.4}$$

The most classical and the most fundamental result on optimal reinsurance is that the stop-loss reinsurance treaty is the optimal solution that solves both expected utility maximization model (1.2.3) and variance minimization model (1.2.4). This key result assumes that the reinsurance premium is determined by the expectation premium principle. The result relevant to the utility is due to Arrow (1974); see also Bowers et al. (1997), Gerber and Pafumi (1998, section 6). The Arrow's result can be regarded as a generalization of the result established in earlier literature including Borch (1960), Kahn (1961) and Ohlin (1969). For detailed discussion on the variance minimization model (1.2.4), see, for example, Bowers, et al. (1997), Kaas, et al. (2001) and Gerber (1979).

In the 1980's one also observes numerous generalizations of Arrow's result. For example, Deprez and Gerber (1985) generalized this result in the sense that they established one sufficient and necessary condition for the optimal contract f^* under convex and Gâteaux differentiable[1] premium principle Π for reinsurance models without the premium budget constraint π. That is, they established the sufficient and necessary conditions for the solution to model (1.2.3) for convex and Gâteaux differentiable premium principles excluding the constraint $\Pi(f(X)) = \pi$.

Heerwaarden et al. (1989) subsequently generalized Arrow's result to the so-called tail-averse decision criteria, which is a class of criteria including, for example, maximizing the expected utility using a concave increasing utility function, minimizing variance, the zero-utility premium, or the mean-value premium for the retained risk, maximizing the adjustment coefficient or the ruin probability in a compound Poisson risk process, and so on. Young (1999) extended the work of Deprez and Gerber (1985) to the case with Wang's premium principle, which is convex but not Gâteaux differentiable.

In recent years, there appears to have been a surge of interests in optimal reinsurance, and many creative optimal reinsurance models have surfaced as a result. In conjunction with this, elegant mathematical tools and innovative optimization theories have also been used in deriving the optimal solutions to the proposed reinsurance models. The main developments on the recently proposed static models are as follows.

Gajeck and Zagrodny (2000) considered the variance minimization model (1.2.4) by changing the binding budget condition $\Pi(f(X)) = \pi$ to the unbinding constraint $\Pi(f(X)) \leq \pi$ and the expected premium principle to the standard deviation premium principle. Although these modifications introduced additional complexity to the optimization problems, they derived explicitly the optimal reinsurance contracts

[1]For a brief introduction to the concept of Gâteaux differentiability, see Subsection 4.4.1 in Chapter 4.

by relying on techniques that are based on the Lagrange multipliers method and the Gâteaux derivatives. In response to the criticism of using variance as a risk measure criterion, the same authors in their subsequent work (Gajeck and Zagrodny (2004)) developed a method for analyzing the optimal reinsurance contracts to model (1.2.2) with w defined as one of the so-called pseudoconvex functions. The pseudoconvex functions include a large class of asymmetric functions such as $h(t) = \max(0, t)$ and $h(t) = [\max(0, t)]^2$. In their paper, explicit forms of optimal contracts were derived in the case of absolute deviation and truncated variance risk measures. See also Zagrodny (2003) for related works.

A series of papers published by Kaluszka (2001, 2004a, 2004b, 2005) made undeniable important contributions to the optimal reinsurance design. In 2001, Kaluszka developed a technique for deriving explicit forms of the optimal reinsurance contract with the variance minimization model (1.2.4) for the mean-variance principles, i.e., the principles under which the reinsurance premium only relies on the expectation and variance of the ceded loss. Subsequently, based on his previous paper, Kaluszka (2004a) developed a method for the solutions to the more general model (1.2.2) under the same class of premium principles. The solutions under several specific functions for u, such as $u(x) = x_+^2$ and $u(x) = x_+$, were explored. For a specific function u, his method might turn out to be still very complicated, and the optimal solutions are more likely to be expressed as the solutions to a system of equations and hence needs to rely on numerical method to obtain the optimal solutions.

In his more recent work, Kaluszka (2005) considered more general models (i.e. the convex risk measure models) along with a wider class of premium principles (mainly the convex principles). By a convex principle, we mean that the premium amount p over a random loss Z can be determined through the equation $g(p) = H(Z)$, where g is an increasing function and H is a convex function. The author first established several highly general theorems and then in turn identified, case by case, the solutions for models with a specific risk measure and a specific premium principle. Although the results he obtained for each specific model are sufficiently explicit to be of practical use, his method could still turn out to be very complicated to identify the solutions for other models even if they are also based on a convex risk measure and a convex premium principle.

Another important paper in optimal reinsurance design is attributed to Promislow and Young (2005). In this paper, the authors discussed the optimal insurance purchase under a unifying framework with the criterion of minimizing a general risk measure. Their model can be shifted to the reinsurance design setting. While their results are applicable for a general Gâteaux differentiable risk measure minimization model, their conclusion is restricted to only determining whether a ceded loss function (or the corresponding retained loss) should have a deductible or not[2].

More recently, Cai and Tan (2007)[3] introduced two new reinsurance models. They determined the optimal retention of stop-loss contracts by, respectively, minimizing the risk measures VaR (Value-at-Risk) and CTE (Conditional Tail Expectation) of $T_f(X) \equiv I_f(X) + \Pi(f(X))$, the total risk exposure of an insurer. Later on, Cai et al. (2008), which I coauthored, generalized the results of Cai and Tan (2007) by considering the optimal reinsurance among all the increasing convex treaties. Note that the ceded loss functions in the stop-loss reinsurance, quota-share reinsurance, and their combination are all some special increasing convex functions. While the results obtained in these two papers are explicit and elegant, the criticism on their models relies on two aspects. First is that they only consider the expectation principle for the reinsurance premium. Second is that their model is only concerned with risk exposure minimization for the insurer, without taking into

[2]In some very specific cases Promislow and Young (2005) also identified the shape of the optimal ceded loss functions

[3]The paper by Cai and Tan (2007) was awarded one of the best-papers submitted to the 2006 Stochastic Modeling Symposium, April 3-4, Toronto.

account other important factors, such as the reinsurance premium budget or the insurer's profitability.

The models we have reviewed so far are all single-period global models. There are results pertaining to local models, which I now briefly mention. For example, Borch (1960), Deprez and Gerber (1985), and Aase (2002) discussed the conditions for achieving Pareto optimality during a risk sharing among a group of financial individuals. Another example is by Kaluszka (2004b) who discussed the optimal reinsurance contracts when the mean-variance premium principle is applied to the sum of the individual ceded losses with the criteria of minimizing the variance of the insurer's global retained loss while imposing the insurer's expected gain. There are also papers which are devoted to discussing the optimal contracts within several common types of reinsurance, such as the quota-share, surplus, stop-loss, and their combinations. See for example, Centeno (1985, 1986) or Verlaak and Beirlant (2003)

The dynamic optimal reinsurance design is also an area of active research in recent years. Some recent works are due to Schmidli (2001), Hipp and Vogt (2003), Hald and Schmidli (2004), Dickson and Waters (2006), and Kaishev and Dimitrova (2006). Most of these results define the optimal reinsurance design with the criterion of minimizing the ruin probabilities of the insurer's surplus process. Kaishev and Dimitrova (2006), on the other hand, derived the optimality by maximizing the joint survival probability of the surplus processes of both the insurer and reinsurer. In the dynamic setting, the problems are usually so complicated that one has to compromise to consider some specific type of reinsurance so that the problem boils down to determining several optimal parameters in the reinsurance models. For example, Hipp and Vogt (2003) employed stochastic control methods to determine the optimal excess-of-loss reinsurance under the assumption that the insurer's surplus follows a compound Poison process.

Finally, it is worth noting that the principle Π adopted for the reinsurance premium assumes a critical role in the optimal design of reinsurance. The shape of optimal ceded loss function can be dramatically different for different types of reinsurance premium principles. The complexity of solving the resulting reinsurance models can also differ substantially for different reinsurance premium principles.

1.3 Mathematical Background

1.3.1 Insurance Company Risks and Risk Measures

"Risk is Opportunity." – This has been a recent slogan of the Society of Actuaries in reminding actuaries that risk is the core of our business; the management of risk has been our expertise. In this thesis, we are concerned with effectively using reinsurance as a risk management tool for an insurer. In particular, we assume that the (aggregate) risk exposure of an insurer is denoted by the random variable X. Associated with the risk random variable X, we can define appropriate measures of measuring and quantifying X. This leads to the development of risk measure. Usually, it is simply defined as a mapping ρ from $\mathcal{X}$, a set of random variables representing certain risks, to the real numbers $\mathbb{R}$.

Premium principles used by insurance companies can be perceived as some kinds of risk measures. Subsection 1.3.2 of this chapter lists some commonly adopted premium principles. More broadly speaking, risk measures are used for setting provisions and capital requirements of a financial institution to ensure solvency. Value-at-Risk (VaR) and Conditional Tail Expectation (CTE) are two of the most popular risk measures for this purpose.

Definition 1.1 *The VaR of a loss random variable Z at a confidence level $1-\alpha$, $0<\alpha<1$, is formally defined as*

$$VaR_\alpha(Z) = \inf\{z \in \mathbb{R} : \Pr(Z \le z) \ge 1-\alpha\}. \tag{1.3.5}$$

In probabilistic terms, VaR is merely a quantile of the loss distribution of Z. In practice, α is usually chosen such a small value as 5% or even 1%. Consequently, $\mathrm{VaR}_\alpha(Z)$ can be interpreted as a level such that the loss Z is bounded by this level from above with a large probability $1-\alpha$, or equivalently VaR is the level such that the loss happens beyond this level with a small probability α. Moreover, $\mathrm{VaR}_\alpha(Z)$ is nonincreasing and right continuous[4] as a function of α on the interval $(0,1)$.

Note that the minimum in (1.3.5) is attained because $\Pr(Z \le z)$ is nondecreasing and right-continuous in z as the cumulative distribution function of the random variable Z. When $\Pr\{Z \le z\}$ is continuous and strictly increasing, $z = \mathrm{VaR}_\alpha(Z)$ is the unique solution to the equation $\Pr(Z \le z) = 1-\alpha$. Moreover, it is also obvious that $\mathrm{VaR}_\alpha(Z)$ is right continuous as a function of α.

The risk measure VaR possesses the following two properties:

Lemma 1.1 *Let Z be a real-valued random variable, and $0 < \alpha < 1$.*

(i) It holds that

$$VaR_\alpha(g(Z)) = g\left(VaR_\alpha(Z)\right)$$

for any nondecreasing and left continuous function g such that $VaR_\alpha(g(Z))$ is well defined.

(ii) If additionally Z has finite expectation, then

$$\mathsf{E}[Z] = \int_0^1 VaR_u(Z)du.$$

[4]Note that in some literature, VaR is defined by $\mathrm{VaR}_\alpha(Z) = \inf\{z \in \mathbb{R} : \Pr(Z \le z) \ge \alpha\}$ with a large value for α, say 95% or 99%. In this case, VaR is nondecreasing and left continuous as a function of α.

Proof. See Dhaene et al. (2002) for proof of (i). Next we prove (ii). Denote $F_Z^{-1}(u) = \inf\{\xi : \Pr(Z \leq \xi) \geq u\}$ for $0 < u < 1$. Then $F_Z^{-1}(1-U) = \mathrm{VaR}_U(Z)$ and $F_Z^{-1}(1-U)$ has the same distribution as Z for a uniformly distributed random variable U on the unit interval $(0,1)$. Thus,

$$\mathsf{E}[Z] = \mathsf{E}\left[F_Z^{-1}(1-U)\right] = \int_0^1 F_Z^{-1}(1-u)du = \int_0^1 \mathrm{VaR}_u(Z)du,$$

and we complete the proof. □

In view of the fact that VaR corresponds to a quantile of a loss distribution, it does not adequately reflect the potential catastrophic losses of the tail of the distribution. This is one of the commonly criticized shortcomings for VaR despite its prevalence as a risk measure among financial institutions. To overcome this drawback, other risk measure such as the CTE is proposed. CTE is defined as the expected loss given that the loss falls in the worst α part of the loss distribution.

Definition 1.2 *The CTE of a random variable Z at a confidence level $1-\alpha$, $0 < \alpha < 1$, is formally defined as the mean of its α-upper-tail distribution $\Psi_\alpha(\xi)$, which is constructed based on the α-tail of the loss distribution of Z and given by*

$$\Psi_\alpha(\xi) = \begin{cases} 0, & \text{for } \xi < VaR_\alpha(Z), \\ \dfrac{\Pr(Z \leq \xi) - (1-\alpha)}{\alpha}, & \text{for } \xi \geq VaR_\alpha(Z). \end{cases} \tag{1.3.6}$$

At this point, we caution the readers that the literature itself on risk measures can be quite confusing. One of the reasons is that different authors have adopted different terminologies even though many of these risk measures are essentially measuring the same quantity. For example, the term "Conditional Tail Expectation" is coined by Wirch and Hardy (1999) while others have used names such as the Tail Conditional Expectation (see Artzner et al. (1999)), Conditional Value-at-Risk (CVaR) (see Rockafellar and Uryasev (2002)), Tail Value-at-Risk (TVaR) (see

Dhaene et al. (2006) and Expected Shortfall (ES) (see Tasche (2002), McNeil et al. (2005)). The formal definition of CTE is also another area that has led to some confusions. For instance, many authors (see, for example, Dhaene et al. (2006)) have taken at face value that (1.3.10) defined below is the definition for CTE. This is, however, not quite correct. Wirch and Hardy (1999) explicitly make it clear that (1.3.10) is the definition for CTE only under the additional assumption that the loss random variable is continuous. Because of this confusion, CTE has been unfairly criticized as a relevant measure of risk.

To avoid any further confusion, we formally collect some of the properties associated with the risk measure CTE in the Proposition 1.1 below. A detailed proof of proposition is also provided.

Proposition 1.1 *Let Z be a nonnegative loss random variable and $0 < \alpha < 1$.*

(i) CTE and VaR of Z are related as

$$CTE_\alpha(Z) = VaR_\alpha(Z) + \frac{1}{\alpha}\int_{VaR_\alpha(Z)}^{\infty} S_Z(x)dx, \tag{1.3.7}$$

where S_Z denotes the survival function of Z, i.e., $S_Z(x) = \Pr\{Z > x\}$ for any $x \in \mathbb{R}$.

(ii) CTE can be equivalently defined as the average of VaR on the α-tail, i.e.,

$$CTE_\alpha(Z) = \frac{1}{\alpha}\int_0^{\alpha} VaR_q(Z)dq. \tag{1.3.8}$$

(iii) Let $\beta = \inf\{u : VaR_u(Z) = VaR_\alpha(Z)\}$, or equivalently $\beta = \Pr\{Z > VaR_\alpha(Z)\}$, then

$$CTE_\alpha(Z) = \frac{1}{\alpha}\Big((\alpha - \beta)\, VaR_\alpha(Z) + \beta \mathsf{E}\big[Z|Z > VaR_\alpha(Z)\big]\Big), \tag{1.3.9}$$

provided that $\{Z > VaR_\alpha(Z)\}$ has nonzero probability.

(iv) *If Z is a continuous random variable, then CTE has the following simple representation:*

$$CTE_\alpha(Z) = \mathsf{E}[Z|Z > VaR_\alpha(Z)]. \tag{1.3.10}$$

Proof. Let Y be a random variable with distribution function Ψ_α defined in (1.3.6), then

$$\begin{aligned}
\mathrm{CTE}_\alpha(Z) &= \mathsf{E}[Y] \\
&= -\int_{-\infty}^{0} \Psi_\alpha(\xi)d\xi + \int_0^\infty [1 - \Psi_\alpha(\xi)]\, d\xi \\
&= 0 + \int_0^{\mathrm{VaR}_\alpha(Z)} 1 \cdot d\xi + \int_{\mathrm{VaR}_\alpha(Z)}^\infty \left[1 - \frac{\Pr(Z \le \xi) - (1-\alpha)}{\alpha}\right] d\xi \\
&= \mathrm{VaR}_\alpha(Z) + \frac{1}{\alpha}\int_{\mathrm{VaR}_\alpha(Z)}^\infty S_Z(\xi)d\xi,
\end{aligned}$$

which proves (i).

In order to prove (ii), first note that $g(t) = (t - \mathrm{VaR}_\alpha(Z))_+$ is nondecreasing and continuous as a function of t, and thus it follows from (i) of Lemma 1.1 that

$$\mathrm{VaR}_u\left([Z - \mathrm{VaR}_\alpha(Z)]_+\right) = [\mathrm{VaR}_u(Z) - \mathrm{VaR}_\alpha(Z)]_+$$

for any $0 < u < 1$. Moreover, by (ii) of Lemma 1.1 we have

$$\mathsf{E}\left[(Z - \mathrm{VaR}_\alpha(Z))_+\right] = \int_0^1 \mathrm{VaR}_u\left([Z - \mathrm{VaR}_\alpha(Z)]_+\right) du.$$

Thus,

$$\begin{aligned}
\int_{\mathrm{VaR}_\alpha(Z)}^\infty S_Z(\xi)\cdot d\xi &= \mathsf{E}\left[(Z - \mathrm{VaR}_\alpha(Z))_+\right] \\
&= \int_0^1 \mathrm{VaR}_u\left([Z - \mathrm{VaR}_\alpha(Z)]_+\right) du \\
&= \int_0^1 [\mathrm{VaR}_u(Z) - \mathrm{VaR}_\alpha(Z)]_+\, du \\
&= \int_0^\alpha [\mathrm{VaR}_u(Z) - \mathrm{VaR}_\alpha(Z)]\, du \\
&= \int_0^\alpha \mathrm{VaR}_u(Z)du - \alpha \cdot \mathrm{VaR}_\alpha(Z),
\end{aligned}$$

which, together with (1.3.7), implies (1.3.8).

As for (iii), we will first show the identity $\inf\{u : \mathrm{VaR}_u(Z) = \mathrm{VaR}_\alpha(Z)\} = \Pr\{Z > \mathrm{VaR}_\alpha(Z)\}$. Since $\Pr(Z \le \xi)$ is nondecreasing and right continuous as a function of ξ, $\mathrm{VaR}_u(Z)$ is nonincreasing and right continuous as a function of u on the interval $(0,1)$, which in turn implies that $\inf\{u : \mathrm{VaR}_u(Z) = \mathrm{VaR}_\alpha(Z)\}$ is attainable. Thus, it is sufficient for us to show $\mathrm{VaR}_\gamma(Z) = \mathrm{VaR}_\alpha(Z)$ and $\mathrm{VaR}_{\gamma-\varepsilon}(Z) > \mathrm{VaR}_\alpha(Z)$ for any $\varepsilon \in (0,\gamma)$, where $\gamma = \Pr\{Z > \mathrm{VaR}_\alpha(Z)\}$. Indeed, we have

$$\begin{aligned}\mathrm{VaR}_\gamma(Z) &= \inf\{\xi : \Pr(Z \le \xi) \ge 1-\gamma\} \\ &= \inf\{\xi : \Pr(Z \le \xi) \ge \Pr(Z \le \mathrm{VaR}_\alpha(Z))\} \\ &= \mathrm{VaR}_\alpha(Z).\end{aligned}$$

Moreover, if there exists an $\varepsilon \in (0,\gamma)$ such that $\mathrm{VaR}_{\gamma-\varepsilon}(Z) = \mathrm{VaR}_\alpha(Z)$, then by the definition of VaR, we obtain

$$\Pr(Z \le \mathrm{VaR}_\alpha(Z)) \ge 1-(\gamma-\varepsilon) = \Pr(Z \le \mathrm{VaR}_\alpha(Z)) + \varepsilon,$$

which is an obvious contradiction, and thus $\mathrm{VaR}_{\gamma-\varepsilon}(Z) > \mathrm{VaR}_\alpha(Z)$ for any $\varepsilon \in (0,\gamma)$.

Now, we are ready to prove (1.3.9). Since $\beta \le \alpha$ and $\mathrm{VaR}_u(Z) = \mathrm{VaR}_\alpha(Z)$ for $u \in [\beta,\alpha]$, it follows from (1.3.8) that

$$\mathrm{CTE}_\alpha(Z) = \frac{1}{\alpha}\left[(\alpha-\beta)\mathrm{VaR}_\alpha(Z) + \int_0^\beta \mathrm{VaR}_u(Z)du\right].$$

After comparing the above equation with (1.3.9), we only need to show

$$\mathsf{E}[Z|Z > \mathrm{VaR}_\alpha(Z)] = \frac{1}{\beta}\int_0^\beta \mathrm{VaR}_u(Z)du.$$

To prove this fact, we first note that

$$\begin{aligned}\Pr[(Z|Z > \mathrm{VaR}_\alpha(Z)) \le \xi] &= \frac{\Pr(Z \le \xi, Z > \mathrm{VaR}_\alpha(Z))}{\Pr(Z > \mathrm{VaR}_\alpha(Z))} \\ &= \begin{cases} \dfrac{1}{\beta}\left[\Pr(Z \le \xi) - (1-\beta)\right], & \text{if } \xi > \mathrm{VaR}_\alpha(Z); \\ 0, & \text{if } \xi \le \mathrm{VaR}_\alpha(Z). \end{cases}\end{aligned}$$

Thus, for $u \in (0, 1)$

$$\begin{aligned}\mathrm{VaR}_u(S) &= \inf\left\{\xi > \mathrm{VaR}_\alpha(Z) : \frac{1}{\beta}\left[\Pr(Z \le \xi) - (1-\beta)\right]\right\} \\ &= \inf\{\xi > \mathrm{VaR}_\alpha(Z) : \Pr(Z \le \xi) \ge 1 - \beta u\} \\ &= \mathrm{VaR}_{\beta u}(Z),\end{aligned}$$

and applying (ii) of Lemma 1.1, we obtain

$$\mathsf{E}[Z|Z > \mathrm{VaR}_\alpha(Z)] = \int_0^1 \mathrm{VaR}_u(S)du = \int_0^1 \mathrm{VaR}_{\beta u}(Z)du = \frac{1}{\beta}\int_0^\beta \mathrm{VaR}_u du,$$

by which we prove (iii).

(iv) is trivial result by (iii), and thus the proof is complete. □

Finally, we note that both VaR and CTE satisfy the property of Translation Invariance. This property will be frequently used in the subsequent chapters, and it is formally stated for a risk measure ρ as follows.

[**A1**] Translation Invariance: $\rho(Z + m) = \rho(Z) + m$ for any scalar $m \in \mathbb{R}$.

The discussion on VaR and CTE cannot be concluded without mentioning the notion of "coherent risk measure". A risk measure ρ is said to be coherent if it satisfies property **A1** defined above and the following three additional axioms for any $Y, Z \in \mathcal{X}$:

[**A2**] Subadditivity: $\rho(Y + Z) \le \rho(Y) + \rho(Z)$;

[**A3**] Positive Homogeneity: $\rho(\lambda Z) = \lambda\rho(Z)$ for any scalar $\lambda \geq 0$;

[**A4**] Monotonicity: $\rho(Y) \geq \rho(Z)$ if $Y(\omega) \geq Z(\omega)$ for all $\omega \in \Omega$;

The concept of "coherent risk measure" was first introduced by Artzner et al. (1999). For comprehensive review on risk measures, we refer the readers to Schied (2006), Föllmer and Schied (2002), Dhaene et al. (2006), and many others. While CTE is a coherent measure of risk in that it satisfies all of of the above four axioms **A1**-**A4**, VaR is not coherent as the subadditivity property **A2** is violated. For further discussion on VaR and CTE, see also Rockafellar and Uryasev (2002) and Section 2.2 of the monograph by McNeil et al. (2005).

1.3.2 Insurance Premium Principles

As mentioned in the previous section, insurance premium principles can be viewed as some kinds of risk measures. These principles are used for determining the premium of insurance contracts. There is a lot of discussion on the axioms that a risk measure must satisfy to be an appropriate insurance premium principle; see, for example, Wang et al. (1997). The following gives a list of the common insurance premium principles.

P1 (Expectation principle): $\Pi(Z) = (1+\theta)\mathsf{E}[Z]$ with $\theta > 0$;

P2 (Standard deviation principle): $\Pi(Z) = \mathsf{E}[Z] + \beta\sqrt{\mathrm{D}[Z]}$, where $\beta > 0$ and $D[Z]$ denotes the variance of Z;

P3 (Mixed principle): $\Pi(Z) = \mathsf{E}[Z] + \beta\mathrm{D}[Z]/\mathsf{E}[Z]$, where $\beta > 0$;

P4 (Modified variation principle): $\Pi(Z) = \mathsf{E}[Z] + \beta\sqrt{\mathrm{D}[Z]} + \gamma\mathrm{D}[Z]/\mathsf{E}[Z]$, where $\gamma, \beta > 0$;

P5 (Mean value principle): $\Pi(Z) = \sqrt{\mathsf{E}[Z^2]} = \sqrt{(\mathsf{E}[Z])^2 + \mathrm{D}[Z]}$;

P6 (p-mean value principle): $\Pi(Z) = (\mathsf{E}[Z^p])^{1/p}$, where $p > 1$;

P7 (Semi-deviation principle): $\Pi(Z) = \mathsf{E}[Z] + \beta \left\{\mathsf{E}(Z - \mathsf{E}[Z])_+^2\right\}^{1/2}$ with $0 < \beta < 1$;

P8 (Dutch principle): $\Pi(Z) = \mathsf{E}[Z] + \beta\mathsf{E}(Z - \mathsf{E}[Z])_+$ with $0 < \beta \leq 1$;

P9 (Wang's principle): $\Pi(Z) = \int_0^\infty [\Pr(Z \geq t)]^p dt$ with $0 < p < 1$;

P10 (Gini principle): $\Pi(Z) = \mathsf{E}[Z] + \beta\mathsf{E}|Z - Z^{'}|$, where $\beta > 0$ and Z' is an independent copy of Z;

P11 (Generalized percentile principle): $\Pi(Z) = \mathsf{E}[Z] + \beta\{F_Z^{-1}(1-p) - \mathsf{E}[Z]\}$ with $0 < \beta, p < 1$;

P12 (CTE principle): $\Pi(Z) = \frac{1}{p}\int_{1-p}^1 F_Z^{-1}(x)dx$, where $0 < p < 1$;

P13 (Variance principle): $\Pi(Z) = \mathsf{E}[Z] + \beta\mathrm{D}[Z]$ with $\beta > 0$;

P14 (Semi-variance principle): $\Pi(Z) = \mathsf{E}[Z] + \beta\mathsf{E}(Z - \mathsf{E}[Z])_+^2$ with $\beta > 0$;

P15 (Quadratic utility principle): $\Pi(Z) = \mathsf{E}[Z] + \gamma - \sqrt{\gamma^2 - \mathrm{D}[Z]}$ with $\gamma > 0$ and $\gamma^2 \geq \mathrm{D}[Z]$.

P16 (Covariance principle): $\Pi(Z) = \mathsf{E}[Z] + 2\beta\mathrm{D}[Z] - \beta\mathsf{Cov}(Z, Y)$ where $\beta > 0$ and Y is a random variable;

P17 (Exponential principle): $\Pi(Z) = \frac{1}{\beta}\log\mathsf{E}[\exp(\beta Z)]$ with $\beta > 0$.

1.3.3 Notation

- The following notation will be used throughout the whole thesis:
  - X: the underlying risk to which the reinsurance is applied.
  - $a \wedge b = \min\{a, b\}$
  - $[a]_+ = \max[a, 0]$
  - $F_Z(\cdot)$: the distribution function of a random variable Z.
  - $S_Z(\cdot)$: the survival function of a random variable Z.
  - $\mathrm{VaR}_\alpha(Z)$: the Value-at-Risk at confident level $1 - \alpha$ of the loss random variable Z.

- $\text{CTE}_\alpha(Z)$: the Conditional Tail Expectation at confident level $1-\alpha$ of the loss random variable Z.
- θ denotes the loading factor in the expectation principle.
- $\theta^* = 1/(1+\theta)$.
- $\delta_{\theta^*} = S_X^{-1}(\theta^*) = S_X^{-1}\left(\frac{1}{1+\theta}\right)$.
- $\delta_\alpha = S_X^{-1}(\alpha)$.

• Chapter 2:

- $X_{R_{qs}}$ ($X_{R_{sl}}$): the ceded loss under quota-share (stop-loss) reinsurance.
- $X_{I_{qs}}$ ($X_{I_{sl}}$): the retained loss under quota-share (stop-loss) reinsurance.
- $X_{T_{qs}}$ ($X_{T_{sl}}$): the total loss of the insurer under quota-share (stop-loss) reinsurance.
- $u_\alpha = S_X^{-1}(\alpha) + \frac{1}{\alpha}\int_{S_X^{-1}(\alpha)}^{\infty} S_X(x)dx$
- $\phi_\alpha(d) = d + \frac{1}{\alpha}\int_d^\infty S_X(x)dx, \;\; d \in \mathbb{R}$.
- $G(d) = S_X^{-1}(\alpha) + \frac{1}{\alpha}\int_{S_X^{-1}(\alpha)}^{d} S_X(x)dx + \Pi([X-d]_+), \;\; d \in \mathbb{R}$, where Π denotes the reinsurance premium principle.

• Chapter 3:

- $\phi(t) = [\text{VaR}_\alpha(X) - t]_+$.
- $\psi(t) = \mathsf{E}[(X-t)_+]$.
- $\beta(d) = \frac{B}{\int_d^\infty S_X(x)dx}, \;\; d \in \mathbb{R}$.
- $\kappa(d) = d + (1+\theta)\int_d^\infty S_X(x)dx - \delta_\alpha, \;\; d \in \mathbb{R}$.
- $\lambda(d) = \int_d^\infty S_X(x)dx + S_X(d)[d-\delta_\alpha], \;\; d \in \mathbb{R}$.

- Chapter 4:
  - Ω dentes the set $[0, \infty)$ and $\mathcal{F}$ represents the Borel sigma field on Ω.
  - Pr is one probability measure on Ω such that the underlying risk X has a distribution function $F_X(t) = \Pr[0, t)$.
  - $\mathcal{L}^2 \equiv \mathcal{L}^2(\Omega, \mathcal{F}, \Pr)$: the space of all the Pr-a.s. equivalence classes of random variables with finite second moment.
  - $\mathcal{Q}_f = \left\{ f \in \mathcal{L}^2 : 0 \leq f(x) \leq x \text{ for } x \geq 0 \right\}$.
  - $\mathcal{Q}_\pi = \left\{ f \in \mathcal{L}^2 : 0 \leq (1+\theta)\mathsf{E}[f] \leq \pi \right\}$.
  - $\mathcal{Q} = \mathcal{Q}_f \bigcap \mathcal{Q}_\pi$.

1.4 The Objective and Outline

1.4.1 Objective of the Thesis

The main objective of this thesis is to develop theoretically sound and yet practical solution in the quest for optimal reinsurance designs. In order to achieve such an objective, this thesis broadly consists of two main parts. In the first part, a series of reinsurance models are developed and their optimal reinsurance treaties are derived explicitly. In the second part, we propose an innovative reinsurance model, which we refer as the empirical model since it exploits explicitly the insurer's loss empirical data. This model has the advantage of its practicality and being intuitively appealing.

With respect to the research conducted in the first part, we focus on the risk measure minimization reinsurance models and discuss the optimal reinsurance treaties by exploiting two of the most common risk measures known as Value-at-Risk (VaR) and Conditional Tail Expectation (CTE). Some additional important economic factors such as the reinsurance reinsurance premium budget, the insurer's profitability will be incorporated to analyze the optimal design of reinsurance.

There are several reasons addressing the optimal reinsurance designs involving risk measures such as VaR and CTE. One is inspired by their prominent uses in risk management among banks and insurance companies for risk assessment and risk capital allocation, as well as their wide uses by the regulatory authorities in regulating solvency requirement for banks and insurance companies. The other reason is motivated by the optimal reinsurance models of Cai and Tan (2007) which exploit explicitly VaR and CTE risk measures. It is worth noting that while Cai and Tan (2007) adopts $\mathrm{CTE}_\alpha(Z) = \mathsf{E}[Z|Z \geq \mathrm{VaR}_\alpha(Z)]$ for the definition of CTE, this thesis will use the formal Definition 1.2 for the risk measure.

The empirical approach is motivated by the fact that the reinsurance models are often infinite dimensional optimization problems and hence the explicit solutions are achievable only in some special cases. This approach is proposed for deriving practical solutions to the reinsurance models, of which the theoretical solutions are difficult to obtain. The reinsurance models formulated using the empirical approach are finite dimensional optimization problems and hence are much more tractable. We will discuss the empirical approach in greater details in Chapters 5 and 6, where we will also demonstrate many other advantages of the empirical approach to optimal design of reinsurance.

1.4.2 Executive Summary of the Thesis Chapters

This subsection provides an executive summary to each of the subsequent chapters.

Chapter 2: By formulating the reinsurance model using the criterion of minimizing either VaR or CTE of the insurer's total (or retained) risk, this chapter separately investigates the optimality of reinsurance designs under as many as seventeen different reinsurance premium principles and by confining to two popular reinsurance treaties: quota-share and stop-loss reinsurance. Our results illustrate that the complexity of analysis highly depends on the adopted reinsurance premium principle and hence highlight the critical role of the reinsurance premium principle in the determination of the optimal design. In this chapter, sufficient and necessary conditions (or just sufficient conditions for some cases) are established for the existence of the nontrivial optimal reinsurance in each case.

Chapter 3: While the results obtained in Chapter 2 are explicit and elegant, there are two critical restrictions. The first is that the optimal reinsurance is explored only by restricting to some specific type of reinsurance. In practice, however, there are many other important reinsurance treaties that have a ceded loss function not as simple as those considered in Chapter 2. The second restriction is that the reinsurance models considered in the last chapter focus entirely on minimizing the risk exposure of the insurer. In practice, insurer is concerned with not only risk minimization but also profitability maximization. In other words, a more desirable reinsurance model should take into consideration the level of risk exposure in the presence of reinsurance while also guarantee a minimum level of expected profit. In light of these two aspects of restriction, this chapter incorporates a constraint into the model to reflect an expected profit guarantee for the insurer and formulates the reinsurance model as a VaR minimization problem with the ceded loss function over the set of all the increasing convex functions. By reformulating the model into an optimization problem over a space of positive measures, this chapter obtained explicit solutions.

Chapter 4: By considering the CTE minimization reinsurance model, this chapter devotes to deriving explicit optimal reinsurance among all the general reinsurance treaties, instead of restricting to any specific class. By regarding each reinsurance contract as an element in a Hilbert space and using the Lagrangian method based on the directional derivative, this chapter obtained explicitly the optimal solutions.

The result shows that the stop-loss treaties are optimal for models under CTE criterion. This is a parallel result to the classic variance minimization with a different risk measure.

Chapter 5: A key assumption with respect to the reinsurance models investigated in Chapters 3 and 4 is that the reinsurance premium is calculated by the expectation principle. While it is tempting to generalize models in Chapters 3 and 4 to other reinsurance premium principles, the resulting optimization problems become not tractable due to the nonlinearity and infinite dimension. In view of these challenges, we propose a new method, which we refer as the "empirical approach". The proposed empirical approach allows us to address the optimal reinsurance designs under nonlinear reinsurance premium principles and various optimality objectives. By experimenting with many important reinsurance models, this approach turns to be very effective in addressing the optimal solutions.

Chapter 6: This chapter addresses the stability and the consistency of the solutions obtained from the empirical-based models proposed in Chapter 5. By stability, we mean that the empirical solutions always generate the same functional form of the optimal ceded loss function for independent random samples from the same loss distribution and over the same set of parameter values. By consistency, we mean that the empirical optimal ceded loss function converges to the true optimal ceded loss function as we increase the sample size. While it is challenging to provide a formal analysis on the stability and consistency of a general empirical-based reinsurance model we proposed, we address these issues by resorting to some numerical examples in this chapter. The numerical studies also allow us to gain important insights on the behavior of our proposed empirical solutions for small sample size.

Chapter 7: This chapter concludes the thesis by listing some possible areas of future research.

Chapter 2

VaR and CTE Minimization Models: Quota-Share and Stop-Loss Reinsurance

2.1 Introduction

This chapter aims to exploit the optimality of two kinds of common reinsurance—the quota-share and the stop-loss—by the criteria of minimizing, respectively, the VaR and CTE of the insurer's total risk for all of these seventeen premium principles P1-P17 listed in Subsection 1.3.2.

Let X be the aggregate nonnegative loss random variable (in the absence of reinsurance) on which the reinsurance is applied. Then under the quota-share reinsurance with quota-share coefficient $c \in [0, 1]$, the transformed losses to both cedent and reinsurer can be expressed respectively as:

$$X_{I_{qs}} = (1 - c)X, \qquad \text{and} \qquad X_{R_{qs}} = cX, \tag{2.1.1}$$

where $X_{I_{qs}}$ is the loss retained by the cedent and $X_{R_{qs}}$ is the loss absorbed by the reinsurer. In other words, the cedent transfers risk by retaining $1-c$ proportion of the aggregate loss while the reinsurer is liable for the remaining c proportion. Note that $c=0$ denotes the special case where the insurer retains all losses while $c=1$ represents the insurer transferring all losses to a reinsurer. Consequently, the former case implies no reinsurance while the latter case leads to full reinsurance protection. Under the stop-loss reinsurance, the corresponding losses to the cedent and the reinsurer, denoted respectively by $X_{I_{sl}}$ and $X_{R_{sl}}$, are represented as

$$X_{I_{sl}} = \begin{cases} X, & X \leq d \\ d, & X > d \end{cases} = X \wedge d, \tag{2.1.2}$$

and

$$X_{R_{sl}} = \begin{cases} 0, & X \leq d \\ X-d, & X > d \end{cases} = [X-d]_+, \tag{2.1.3}$$

where the parameter $d \geq 0$ is known as the retention, $a \wedge b = \min[a,b]$, and $[a]_+ = \max[a,0]$. With this treaty, the risk exposure of the cedent is capped at the retention while the reinsurer is liable for any losses in excess of the retention if any. Note again that when $d=0$ and $d \to \infty$, these two special cases represent, respectively, full reinsurance and no reinsurance.

Recall that the main objective of the chapter is to determine the optimal quota-share reinsurance and optimal stop-loss reinsurance under various types of premium principles. This implies that it boils down to determining the optimal quota-share coefficient $c^* \in [0,1]$ in the quota-share reinsurance and the optimal retention $d^* \in [0,\infty)$ in the stop-loss reinsurance. In terms of the solution to the optimal reinsurance model studied in this paper, we classify the optimal reinsurance as either trivial or nontrivial. By trivial optimal reinsurance we mean that it is optimal to have either zero reinsurance or full reinsurance. In other words trivial optimal reinsurance implies c^* is ether 0 or 1 in the quota-share treaty while either $d^* = 0$

Premium Principle	Quota Share VaR	Quota Share CTE	Stop Loss VaR	Stop Loss CTE
(P1) Expectation principle	T	T	NT	NT*
(P2) Standard deviation principle	T	T	-	-
(P3) Mixed principle	T	T	-	-
(P4) Modified variation principle	T	T	-	-
(P5) Mean value principle	T	T	T	T
(P6) *p*-mean value principle	T	T	T	T
(P7) Semi-deviation principle	T	T	T	-
(P8) Dutch principle	T	T	NT	NT*
(P9) Wang's principle	T	T	T	T
(P10) Gini principle	T	T	-	-
(P11) Generalized percentile principle	T	T	T	T
(P12) CTE principle	T	T	T	T
(P13) Variance principle	NT	NT	NT	NT*
(P14) Semi-variance principle	NT	NT	NT	NT*
(P15) Quadratic utility principle	NT	NT	NT	NT*
(P16) Covariance principle	NT	NT	-	-
(P17) Exponential principle	NT	NT	T	-

Table 2.1: Nontriviality of optimal reinsurance under VaR/CTE criterion.

or $d^* \to \infty$ in the stop-loss treaty. On the other hand, the optimal quota-share coefficient in the quota-share reinsurance is nontrivial if it lies on the open interval $(0, 1)$ and the optimal retention in the stop-loss reinsurance is nontrivial if it is a real number in the open interval $(0, \infty)$.

The main results of the chapter lie in establishing Theorems 2.1, 2.2, 2.3 and 2.4 for the nontriviality (and triviality) of the optimal quota-share and the optimal stop-loss reinsurance under general premium principle. Then by confining to specific premium principle, these theorems enable us to effectively analyze in greater details the conditions for the optimal quota-share coefficient c^* and the optimal retention d^*. Table 3.1 provides a sneak preview of our findings. In the table, the premium principle and the criterion identified with a "T" implies that the optimal solution is trivial. Similarly those with a "NT" means that sufficient and necessary conditions for the existence of nontrivial optimal reinsurance are established. On the other hand, "NT*" indicates that only the sufficient conditions are identified for the existence of the nontrivial optimal reinsurance. Note also that because of the complexity of the optimization problem for the stop-loss reinsurance, there are a few premium principles for which we are unable to determine analytically whether the optimal reinsurance exists or not for a general loss distribution. These cases are indicated with the notation "-". For these cases, additional numerical methods need to be used to further investigate their optimality. Our findings also highlight the importance of the reinsurance premium principle assumption. Depending on the adopted reinsurance premium principles, there are cases for which optimal reinsurance is nontrivial and there are other cases for which optimal reinsurance is trivial.

The rest of the chapter is organized as follows. Section 2.2 introduces the notation and provides some preliminary results. Sections 2.3 and 2.4 discuss, respectively, the optimality for the quota-share reinsurance treaty and the optimal stop-loss reinsurance treaty. Section 2.5 presents some numerical examples to illustrate the results obtained in the preceding sections. Section 2.6 collects the proofs of some results.

2.2 Preliminaries

Throughout this chapter, we use X exclusively to denote the random loss to which the reinsurance treaty is applied. We further assume that X has a continuous one-to-one distribution on $(0, \infty)$ but with a possible jump at 0 with finite moment(s). We use X_I and X_R to denote, respectively, the retained loss and the ceded loss random variables under a generic reinsurance arrangement. Note that $X = X_I + X_R$ so that X_I and X_R form a partition of X. When we need to distinguish between a quota-share reinsurance and a stop-loss reinsurance, we simply subscript the notation with "sl" and "qs" as we do in (2.1.1), (2.1.2) and (2.1.3) bellow.

Recall that, in presence of either of the quota-share and stop-loss reinsurance arrangements, the total risk of the insurer is the sum of its retained loss X_I and the reinsurance premium amount $\Pi(X_R)$, i.e.,

$$X_T = X_I + \Pi(X_R). \tag{2.2.1}$$

Thus, it follows from the property of invariance translation for both VaR and CTE (see Section 1.3.1) that

$$\begin{aligned} \mathrm{VaR}_\alpha(X_T) &= \mathrm{VaR}_\alpha(X_I) + \Pi(X_R) && (2.2.2) \\ \mathrm{CTE}_\alpha(X_T) &= \mathrm{CTE}_\alpha(X_I) + \Pi(X_R). && (2.2.3) \end{aligned}$$

By the relation of VaR and CTE in (1.3.7), the CTE of the insurer's retained loss can further be decomposed as

$$\mathrm{CTE}_\alpha(X_I) = \mathrm{VaR}_\alpha(X_I) + \frac{1}{\alpha}\int_{\mathrm{VaR}_\alpha(X_I)}^{\infty} S_{X_I}(x)dx, \tag{2.2.4}$$

which combining (2.2.3) leads to

$$\mathrm{CTE}_\alpha(X_T) = \mathrm{VaR}_\alpha(X_I) + \frac{1}{\alpha}\int_{\mathrm{VaR}_\alpha(X_I)}^{\infty} S_{X_I}(x)dx + \Pi(X_R). \tag{2.2.5}$$

So far we have established some general relations for the risk measures associated with the retained loss random variable and the total cost random variable of insuring risks in the presence of reinsurance. We now consider these relations in greater details by examining these two specific reinsurance contracts: the quota-share reinsurance and stop-loss reinsurance.

For the quota-share reinsurance, the survival function of the retained loss $X_{I_{qs}}$ is given by

$$S_{X_{I_{qs}}}(x) = \Pr((1-c)X > x) = \begin{cases} S_X\left(\frac{x}{1-c}\right), & 0 \le c < 1, \\ 0, & c = 1 \end{cases} \tag{2.2.6}$$

for $x \ge 0$, and its VaR at confidence level $1-\alpha$, denoted by $\mathrm{VaR}_\alpha(X_{I_{qs}}; c)$, is given by

$$\mathrm{VaR}_\alpha(X_{I_{qs}}; c) = (1-c)S_X^{-1}(\alpha). \tag{2.2.7}$$

The above equation and together with (2.2.2) give us an expression for $\mathrm{VaR}_\alpha(X_{T_{qs}}; c)$ which represents the VaR of the total cost under the quota-share arrangement. We state this formally in the following proposition:

Proposition 2.1 *For* $0 \le c \le 1$ *and* $0 < \alpha < S_X(0)$,

$$VaR_\alpha(X_{T_{qs}}; c) = (1-c)S_X^{-1}(\alpha) + \Pi(cX). \tag{2.2.8}$$

Similarly, it follows from (2.2.5) that the corresponding CTE of the total cost, $\mathrm{CTE}_\alpha(X_{T_{qs}}; c)$, under the quota-share arrangement can be represented in the following proposition:

Proposition 2.2 *For $0 \le c \le 1$ and $0 < \alpha < S_X(0)$,*

$$CTE_\alpha(X_{T_{qs}}; c) = (1-c)S_X^{-1}(\alpha) + \frac{1-c}{\alpha}\int_{S_X^{-1}(\alpha)}^{\infty} S_X(x)dx + \Pi(cX). \qquad (2.2.9)$$

Note that in the above notation associated with risk measures, the quota-share coefficient c is one of the arguments to emphasize the fact that under the quota-share reinsurance, these risk measures depend explicitly on c.

Now let us consider the stop-loss reinsurance. In this case, the survival function of the retained loss $X_{I_{sl}}$ is given by

$$S_{X_{I_{sl}}}(x) = \begin{cases} S_X(x), & 0 \le x < d, \\ 0, & x \ge d, \end{cases} \qquad (2.2.10)$$

so that its VaR can be represented as

$$\mathrm{VaR}_\alpha(X_{I_{sl}}) \equiv \mathrm{VaR}_\alpha(X_{I_{sl}}; d) = \begin{cases} d, & 0 \le d \le S_X^{-1}(\alpha), \\ S_X^{-1}(\alpha), & d > S_X^{-1}(\alpha). \end{cases} \qquad (2.2.11)$$

Then together with (2.2.2), we obtain an expression for $\mathrm{VaR}_\alpha(X_{T_{sl}}) \equiv \mathrm{VaR}_\alpha(X_{T_{sl}}; d)$ as shown in the following proposition:

Proposition 2.3 *For each $d \ge 0$ and $0 < \alpha < S_X(0)$,*

$$VaR_\alpha(X_{T_{sl}}; d) = \begin{cases} d + \Pi([X-d]_+), & 0 \le d \le S_X^{-1}(\alpha), \\ S_X^{-1}(\alpha) + \Pi([X-d]_+), & d > S_X^{-1}(\alpha). \end{cases} \qquad (2.2.12)$$

Moreover, by (2.2.10), (2.2.11) and the fact $0 < \mathrm{VaR}_\alpha(X_{I_{sl}}; d) \le d$, we have

$$\begin{aligned} \int_{\mathrm{VaR}_\alpha(X_{I_{sl}};d)}^{\infty} S_{X_{I_{sl}}}(x)dx &= \int_{\mathrm{VaR}_\alpha(X_{I_{sl}};d)}^{d} S_X(x)dx \\ &= \begin{cases} 0, & 0 \le d \le S_X^{-1}(\alpha), \\ \int_{S_X^{-1}(\alpha)}^{d} S_X(x)dx, & d > S_X^{-1}(\alpha). \end{cases} \qquad (2.2.13) \end{aligned}$$

Thus, by defining

$$G(d) = S_X^{-1}(\alpha) + \frac{1}{\alpha}\int_{S_X^{-1}(\alpha)}^{d} S_X(x)dx + \Pi([X-d]_+) \qquad (2.2.14)$$

and together with (2.2.5), (2.2.11) and (2.2.13), we obtain the following expression for $\mathrm{CTE}_\alpha(X_{T_{sl}}) \equiv \mathrm{CTE}_\alpha(X_{T_{sl}}; d)$:

Proposition 2.4 *For each $d \geq 0$ and $0 < \alpha < S_X(0)$,*

$$CTE_\alpha(X_{T_{sl}}; d) = \begin{cases} d + \Pi([X-d]_+), & 0 \leq d \leq S_X^{-1}(\alpha), \\ G(d), & d > S_X^{-1}(\alpha). \end{cases} \qquad (2.2.15)$$

Note again that these risk measures depend explicitly on the retention d under the stop-loss reinsurance. Note also that Propositions 2.3 and 2.4 reduce, respectively, to Propositions 2.1 and 3.1 of Cai and Tan (2007) under the special case that $\Pi(\cdot)$ is the expectation premium principle.

We now revisit the decomposition (2.2.1) which highlights the dilemma faced by the insurer. Note that, roughly speaking, the premium principle $\Pi(X_R)$ is expected to be an increasing function in X_R. This implies that the smaller the risk is ceded to a reinsurer, the less costly the reinsurance premium is. On the other hand, a small retained risk exposure can be achieved at the expense of higher reinsurance premium. Consequently, there is a trade-off between how much risk to retain and how much risk to cede. The problem of optimal reinsurance essentially addresses the optimal partitions X_I and X_R. When the reinsurance treaty is confined to either quota-share type or stop-loss type, the problem then boils down to the determination of the optimal quota-share coefficient c^* in the former case or the optimal retention d^* in the latter case. The explicit dependence of c and d (depending on the type of reinsurance treaty) on the risk measures in Propositions 2.1-2.4 suggests that one formulation of optimal reinsurance model is to seek optimal parameters c^* and d^* that minimize the respective risk measure. More specifically, the optimal quota-share reinsurance models can be formulated as seeking the optimal quota-share coefficients c^* that are the solutions to the following optimization problems, depending on the adopted risk measure:

$$\text{VaR-optimization:} \quad \mathrm{VaR}_\alpha(X_{T_{qs}}; c^*) = \min_{c\in[0,1]} \Big\{ \mathrm{VaR}_\alpha(X_{T_{qs}}; c) \Big\}, \tag{2.2.16}$$

$$\text{CTE-optimization:} \quad \mathrm{CTE}_\alpha(X_{T_{qs}}; c^*) = \min_{c\in[0,1]} \Big\{ \mathrm{CTE}_\alpha(X_{T_{qs}}; c) \Big\}. \tag{2.2.17}$$

Analogously, under the optimal stop-loss reinsurance models, the optimal retentions d^* are the solutions to the following optimization problems:

$$\text{VaR-optimization:} \quad \mathrm{VaR}_\alpha(X_{T_{sl}}; d^*) = \min_{d\in[0,\infty)} \Big\{ \mathrm{VaR}_\alpha(X_{T_{sl}}; d) \Big\}, \tag{2.2.18}$$

$$\text{CTE-optimization:} \quad \mathrm{CTE}_\alpha(X_{T_{sl}}; d^*) = \min_{d\in[0,\infty)} \Big\{ \mathrm{CTE}_\alpha(X_{T_{sl}}; d) \Big\}. \tag{2.2.19}$$

We now make the following three remarks with respect to the above optimal reinsurance models. First, the models are relatively simple and intuitively appealing. They exploit the basic thrust of a risk management practice that the insurer is interested in risk minimization. Under the above optimization models, the optimal reinsurance design ensures that the risk exposure of the insurer, as measured by its risk measure of the total cost, is optimally minimized. Second, by confining to stop-loss reinsurance and under the additional assumption of expectation premium principle, optimization problems (2.2.18) and (2.2.19) reduce to the optimization reinsurance models as analyzed in Cai and Tan (2007). Third, as pointed out in the previous section that when optimal solutions to the above reinsurance models are nontrivial, this implies that the optimal quota-share coefficient c^* is strictly on the interval $(0, 1)$ and the optimal retention d^* is finite and strictly greater than 0.

We conclude this section by introducing the following function ϕ_α and notation u_α:

$$\phi_\alpha(d) = d + \frac{1}{\alpha}\int_d^\infty S_X(x)dx, \quad u_\alpha = S_X^{-1}(\alpha) + \frac{1}{\alpha}\int_{S_X^{-1}(\alpha)}^\infty S_X(x)dx.$$

As we will soon discover, these two functions play critical roles in deriving the solutions to our optimal reinsurance models. Furthermore, it is also useful to point out the following two relations: $u_\alpha = \phi_\alpha\left(S_X^{-1}(\alpha)\right)$ and $u_\alpha = \lim_{d\to\infty} G(d)$ provided that $\lim_{d\to\infty}\Pi([X-d]_+) = 0$, which are immediate consequence of their definitions. Note that function $G(d)$ is defined in (2.2.14).

2.3 Quota-share Reinsurance Optimization

In this section, we discuss the optimal quota-share reinsurance with respect to the premium principles P1-P17 listed in Section 1.3.2. The key result of this section is stated in Theorems 2.1 and 2.2 which provide the optimality of the quota-share reinsurance under the general premium principle. The proof of the Theorem 2.1 is collected in Section 2.6, while we omit the proof of Theorem 2.2 as it is similar to the proof of Theorem 2.1.

Theorem 2.1 *Consider the VaR-optimization (2.2.16).*

(a) Assume the reinsurance premium $\Pi(\cdot)$ satisfies $\Pi(0) = 0$, and the property of positive homogeneity, i.e., $\Pi(cX) = c\Pi(X)$ for constant $c > 0$. Then the optimal quota-share reinsurance is trivial, and moreover, the optimal quota-share coefficient depends on the relative magnitude between $\Pi(X)$ and $S_X^{-1}(\alpha)$ as indicated below:

$$c^* = \begin{cases} 0, & \Pi(X) > S_X^{-1}(\alpha), \\ \text{any number in } [0,1], & \Pi(X) = S_X^{-1}(\alpha), \\ 1, & \Pi(X) < S_X^{-1}(\alpha). \end{cases} \tag{2.3.1}$$

(b) If $\Pi(cX)$ is strictly convex in c for $0 \le c \le 1$, then the nontrivial optimal quota-share reinsurance exists if and only if there exists a constant $c^* \in (0,1)$

such that

$$\Pi_c'(c^*X) - S_X^{-1}(\alpha) = 0, \tag{2.3.2}$$

where $\Pi_c'(\cdot)$ *denotes the derivative of* $\Pi(cX)$ *with respect to* c*. Furthermore,* c^* *that satisfies (2.3.2) is the optimal quota-share coefficient.*

Theorem 2.2 *Consider the CTE-optimization (2.2.17).*

(a) Assume that the reinsurance premium $\Pi(\cdot)$ *satisfies* $\Pi(0) = 0$*, and positive homogeneity, i.e.,* $\Pi(cX) = c\Pi(X)$ *for constant* $c > 0$*. Then the optimal quota-share reinsurance is trivial, and moreover, the optimal quota-share coefficient is determined depending on the quantities* $\Pi(X)$ *and* u_α *in the following way:*

$$c^* = \begin{cases} 0, & \Pi(X) > u_\alpha, \\ \textit{any number in } [0,1], & \Pi(X) = u_\alpha, \\ 1, & \Pi(X) < u_\alpha. \end{cases} \tag{2.3.3}$$

(b) If $\Pi(cX)$ *is strictly convex in* c *for* $0 \le c \le 1$*, then the optimal quota-share reinsurance exists if and only if there exists a constant* $c^* \in (0,1)$ *such that*

$$\Pi_c'(c^*X) - u_\alpha = 0, \tag{2.3.4}$$

and in this case, c^* *determined by (2.3.4) is the optimal quota-share coefficient.*

The above two theorems provide the optimality condition for the existence (or non-existence) of the nontrivial optimal quota-share reinsurance under general premium principle. We now refine these results by explicitly considering the seventeen premium principles. These results are shown the following sequences of three propositions. Proposition 2.5 states that the optimal quota-share reinsurance is trivial for premium principles P1-P12 while Propositions 2.6 and 2.7 study remaining premium principles for the VaR-optimization and CTE-optimization respectively. The proof of the first proposition is trivial and it follows from part (a) of the above two theorems (Theorems 2.1 and 2.2) and the fact that all the premium principles P1-P12 satisfy the property $\Pi(0) = 0$ and positive homogeneity. The proof of Proposition 2.6 is relegated to Section 2.6 while we omit the proof of Proposition 2.7 as it is similar to the proof of Proposition 2.6.

Proposition 2.5 *For both VaR-optimization (2.2.16) and CTE-optimization (2.2.17), the optimal quota-share reinsurance is trivial for premium principles P1-P12, and the optimal quota-share coefficient is determined according to (2.3.1) for VaR criterion and (2.3.3) for CTE criterion.*

Proposition 2.6 *Consider the VaR-optimization (2.2.16).*

(a) P13(variance principle): the optimal quota-share reinsurance is nontrivial if and only if

$$\mathsf{E}[X] < S_X^{-1}(\alpha) < \mathsf{E}[X] + 2\beta D[X], \tag{2.3.5}$$

in which case, the optimal quota-share coefficient is given by

$$c^* = \frac{S_X^{-1}(\alpha) - \mathsf{E}[X]}{2\beta D[X]}. \tag{2.3.6}$$

(b) P14 (semi-variance principle): the optimal quota-share reinsurance is non-trivial if and only if

$$\mathsf{E}[X] < S_X^{-1}(\alpha) < \mathsf{E}[X] + 2\beta \mathsf{E}[X - \mathsf{E}X]_+^2, \tag{2.3.7}$$

in which case, the optimal quota-share coefficient is given by

$$c^* = \frac{S_X^{-1}(\alpha) - \mathsf{E}[X]}{2\beta \mathsf{E}[X - \mathsf{E}X]_+^2}. \tag{2.3.8}$$

(c) *P15 (quadratic utility principle): the optimal quota-share reinsurance is nontrivial if and only if*

$$S_X^{-1}(\alpha) > \mathsf{E}[X], \quad \text{and} \quad \frac{(S_X^{-1}(\alpha) - \mathsf{E}[X])\gamma}{\sqrt{D[X]\{D[X] + (S_X^{-1}(\alpha) - \mathsf{E}[X])^2\}}} < 1, \quad (2.3.9)$$

in which case, the optimal quota-share coefficient is given by

$$c^* = \frac{(S_X^{-1}(\alpha) - \mathsf{E}[X])\gamma}{\sqrt{D[X]\{D[X] + (S_X^{-1}(\alpha) - \mathsf{E}[X])^2\}}}. \quad (2.3.10)$$

(d) *P16 (covariance principle): Y being a random variable, the optimal quota-share reinsurance exists if and only if*

$$\mathsf{E}[X] > \beta \mathsf{Cov}(X, Y), \quad (2.3.11)$$

and

$$\mathsf{E}[X] - \beta \mathsf{Cov}(X, Y) < S_X^{-1}(\alpha) < 4\beta D[X] + \mathsf{E}[X] - \beta \mathsf{Cov}(X, Y), \quad (2.3.12)$$

in which case, the optimal quota-share coefficient is given by

$$c^* = \frac{S_X^{-1}(\alpha) - \mathsf{E}[X] + \beta \mathsf{Cov}(X, Y)}{4\beta D[x]}. \quad (2.3.13)$$

(e) *P17 (exponential principle): the optimal quota-share reinsurance is nontrivial if and only if there exists a constant $c^* \in (0, 1)$ such that*

$$\mathsf{E}[X \exp(c^* \beta X)] = S_X^{-1}(\alpha) \mathsf{E}[\exp(c^* \beta X)], \quad (2.3.14)$$

in which case, the optimal quota-share coefficient c^* is determined by (2.3.14).

Proposition 2.7 *Consider the CTE-optimization (2.2.17).*

(a) *P13 (variance principle): the optimal quota-share reinsurance is nontrivial if and only if*

$$\mathsf{E}[X] < u(\alpha) < \mathsf{E}[X] + 2\beta D[X], \tag{2.3.15}$$

in which case, the optimal quota-share coefficient is given by

$$c^* = \frac{u(\alpha) - \mathsf{E}[X]}{2\beta D[X]}. \tag{2.3.16}$$

(b) *P14 (semi-variance principle): the optimal quota-share reinsurance is nontrivial if and only if*

$$\mathsf{E}[X] < u(\alpha) < \mathsf{E}[X] + 2\beta \mathsf{E}[X - \mathsf{E}X]_+^2, \tag{2.3.17}$$

in which case, the optimal quota-share coefficient is given by

$$c^* = \frac{u(\alpha) - \mathsf{E}[X]}{2\beta \mathsf{E}[X - \mathsf{E}X]_+^2}. \tag{2.3.18}$$

(c) *P15 (quadratic utility principle): the optimal quota-share reinsurance is nontrivial if and only if*

$$u(\alpha) > \mathsf{E}[X], \quad \text{and} \quad \frac{(u(\alpha) - \mathsf{E}[X])\gamma}{\sqrt{D[X]\{D[X] + (u(\alpha) - \mathsf{E}[X])^2\}}} < 1, \tag{2.3.19}$$

in which case, the optimal quota-share coefficient is given by

$$c^* = \frac{(u(\alpha) - \mathsf{E}[X])\gamma}{\sqrt{D[X]\{D[X] + (u(\alpha) - \mathsf{E}[X])^2\}}}. \tag{2.3.20}$$

(d) *P16 (covariance principle): Y being a random variable, the optimal quota-share reinsurance is nontrivial if and only if*

$$\mathsf{E}[X] > \beta \mathsf{Cov}(X, Y), \tag{2.3.21}$$

and

$$\mathsf{E}[X] - \beta \mathsf{Cov}(X, Y) < u(\alpha) < 4\beta D[X] + \mathsf{E}[X] - \beta \mathsf{Cov}(X, Y), \tag{2.3.22}$$

in which case, the optimal quota-share coefficient is given by

$$c^* = \frac{u(\alpha) - \mathsf{E}[X] + \beta \mathsf{Cov}(X,Y)}{4\beta D[X]}. \tag{2.3.23}$$

(e) *P17 (exponential principle): the optimal quota-share reinsurance is nontrivial if and only if there exists a constant $c^* \in (0,1)$ such that*

$$\mathsf{E}[X \exp(c^*\beta X)] = u(\alpha)\mathsf{E}[\exp(c^*\beta X)] \tag{2.3.24}$$

in which case, the optimal quota-share coefficient c^* is determined by (2.3.24).

2.4 Stop-loss Reinsurance Optimization

We now discuss the optimization problems (2.2.18) and (2.2.19) for the stop-loss reinsurance contract. As we will see shortly, if the reinsurance is a stop-loss, it is mathematically more challenging to analyze its optimality, particularly for CTE-optimization with premium principles P2-P4, P10 and P16. Subsection 2.4.1 devotes to the VaR-optimization (2.2.18) while Subsection 2.4.2 focuses on the CTE-optimization (2.2.19).

2.4.1 VaR-optimization for Stop-loss Reinsurance

We first present the following theorem, with its proof given in Section 2.6, regarding the general reinsurance premium principle for the optimal stop-loss reinsurance and VaR criterion.

Theorem 2.3 *Consider the VaR-optimization (2.2.18). Suppose $\Pi(\cdot)$ is a premium principle such that $\Pi([X-d]_+)$ is decreasing in d.*

(a) *The optimal stop-loss reinsurance is trivial if either of the following conditions is satisfied:*

(i) $d + \Pi([X-d]_+)$ *is an increasing function in* d *on* $[0, S_X^{-1}(\alpha)]$*, or*

(ii) there exists a constant $d_0 \in (0, S_X^{-1}(\alpha))$ *such that* $d + \Pi([X-d]_+)$ *is increasing in* d *on* $[0, d_0]$ *while decreasing on* $[d_0, S_X^{-1}(\alpha)]$*.*

Moreover, in either of the above (i) and (ii), the trivial optimal retention depends on the relative magnitude between $\Pi(X)$ *and* $S_X^{-1}(\alpha)$ *as indicated below:*

$$d^* = \begin{cases} 0, & \text{if } \Pi(X) < S_X^{-1}(\alpha); \\ 0, \ \ or \ \ +\infty, & \text{if } \Pi(X) = S_X^{-1}(\alpha); \\ +\infty, & \text{if } \Pi(X) > S_X^{-1}(\alpha). \end{cases} \tag{2.4.25}$$

(b) If the premium principle $\Pi(\cdot)$ *satisfies* $\lim_{d\to\infty} \Pi([X-d]_+) = 0$*, and there exists a positive constant* d_0 *such that* $d + \Pi([X-d]_+)$ *is decreasing in* d *on* $[0, d_0]$ *while increasing on* $[d_0, \infty]$*, then the optimal stop-loss reinsurance is nontrivial if and only if the following condition is satisfied:*

$$S_X^{-1}(\alpha) > d_0 + \Pi([X - d_0]_+). \tag{2.4.26}$$

Moreover, when the optimal stop-loss reinsurance is nontrivial, d_0 *is the optimal retention with the corresponding minimum value of* $VaR_\alpha(X_{T_{sl}}; d)$

$$\min_{d\geq 0} VaR_\alpha(X_{T_{sl}}; d) = d_0 + \Pi([X - d_0]_+). \tag{2.4.27}$$

Remark 2.1 *If the premium* $\Pi(\cdot)$ *satisfies the conditions stated in (b) of Theorem 2.3 and* d_0 *is the unique constant on interval* $[0, S_X^{-1}(\alpha)]$ *such that* $d + \Pi([X-d]_+)$ *is decreasing in* d *on* $[0, d_0]$ *while increasing on* $[d_0, S_X^{-1}(\alpha)]$*, then* d_0 *is the unique solution to VaR-optimization (2.2.18)*

Relying on Theorem 2.3, we now demonstrate that optimal retention is trivial for some of the premium principles, as shown in the following proposition:

Proposition 2.8 *Consider the VaR-optimization (2.2.18). The optimal stop-loss reinsurance is trivial and the trivial optimal retention* d^* *is determined as in (2.4.25) for the following premium principles:*

(a) *P5 (mean value principle)* $\Pi(X) = \sqrt{E[X^2]} = \sqrt{E^2[X] + D[X]}$*;*

(b) *P6 (p-mean value principle)* $\Pi(X) = (E[X^p])^{1/p}$*, where* $p > 1$*;*

(c) *P7 (semi-deviation principle)* $\Pi(X) = E[X] + \beta \left[E[X - EX]_+^2\right]^{1/2}$*, where* $0 < \beta < 1$*;*

(d) *P9 (Wang's principle)* $\Pi(X) = \int_0^\infty [\Pr(X \geq t)]^p dt$*, where* $0 < p < 1$*;*

(e) *P11 (generalized percentile principle)* $\Pi(X) = E[X] + \beta(F_X^{-1}(1-p) - E[X])$*, where* $0 \leq \beta \leq 1$*;*

(f) *P12 (CTE principle)* $\Pi(X) = (1/p)\int_{1-p}^1 F_X^{-1}(x)dx$*, where* $0 < p < 1$*;*

(g) *P17 (exponential principle)* $\Pi(X) = \frac{1}{\beta}\log E(\beta X)$ *with* $\beta > 0$.

See Section 2.6 for the proof of the above proposition. While the above proposition demonstrates the premium principles for which the optimal stop-loss reinsurance is trivial, the following proposition indicates that for some other premium principles, the VaR-based optimal stop-loss reinsurance is nontrivial under some mild conditions. We again relegate its proof to Section 2.6.

Proposition 2.9 *Consider the VaR-optimization (2.2.18).*

(a) *P1 (expectation premium principle): the optimal stop-loss reinsurance is nontrivial if and only if*

$$S_X^{-1}(\alpha) \geq d_0 + (1+\theta)\int_{d_0}^\infty S_X(x)dx, \tag{2.4.28}$$

where $d_0 = S_X^{-1}\left(\frac{1}{1+\theta}\right)$*; moreover, in this case* d_0 *is the unique optimal retention.*

(b) P8 (Dutch principle): if there exists a positive constant d_0 satisfying the equation $\beta S_X(d_0 + \mathsf{E}[X - d_0]_+) = 1$, then the optimal stop-loss reinsurance is nontrivial if and only if

$$S_X^{-1}(\alpha) \geq d_0 + \mathsf{E}[X - d_0]_+ + \beta \mathsf{E}\big[[X - d_0]_+ - \mathsf{E}[X - d_0]_+\big]_+, \quad (2.4.29)$$

where moreover d_0 is the unique optimal retention.

(c) P13 (variance principle): if there exists a positive constant d_0 satisfying the equation $2\beta \mathsf{E}[X - d_0]_+ = 1$, then the optimal stop-loss reinsurance is nontrivial if and only if

$$S_X^{-1}(\alpha) \geq d_0 + \mathsf{E}[X - d_0]_+ + \beta D[X - d_0]_+, \quad (2.4.30)$$

where moreover d_0 is the unique optimal retention.

(d) P14 (semi-variance principle): if there exists a positive constant d_0 satisfying the equation $2\beta \mathsf{E}\big[X - d_0 - \mathsf{E}[X - d_0]_+\big]_+ = 1$, then the optimal stop-loss reinsurance is nontrivial if and only if

$$S_X^{-1}(\alpha) \geq d_0 + \mathsf{E}[X - d_0]_+ + \beta \mathsf{E}\big[[X - d_0]_+ - \mathsf{E}[X - d_0]_+\big]_+^2, \quad (2.4.31)$$

where moreover d_0 is the unique optimal retention.

(e) P15 (quadratic utility principle): if there exists a positive constant d_0 satisfying the equation $\dfrac{E[X - d_0]_+}{\sqrt{\gamma^2 - D[X - d_0]_+}} = 1$, then the optimal stop-loss reinsurance is nontrivial if and only if

$$S_X^{-1}(\alpha) \geq d_0 + \mathsf{E}[X - d_0]_+ + \gamma - \sqrt{\gamma^2 - D[X - d_0]_+}, \quad (2.4.32)$$

where moreover d_0 is the unique optimal retention.

Remark 2.2 *(i) Part (a) of the above proposition is equivalent to Theorem 2.1 of Cai and Tan (2007).*

(ii) For these principles P2-P4, P10 and P16, other than those discussed in Propositions 2.8 and 2.9, the goal function in the optimization problem (2.2.19) is so complicated that general result about the optimality of the stop-loss reinsurance have not been derived. For these, a numerical approach might have to be employed.

2.4.2 CTE-optimization for Stop-loss Reinsurance

Unlike the optimal stop-loss reinsurance under the VaR criterion, the analysis for the corresponding CTE optimality is complicated by the fact that the optimal retention can occur for $d \in \big(S_X^{-1}(\alpha), \infty\big)$. For further discussion, we first present the following theorem. See Section 2.6 for its proof.

Theorem 2.4 *Consider the CTE-optimization (2.2.19).*

(a) If $d + \Pi([X-d]_+)$ is increasing in d on $[0, S_X^{-1}(\alpha)]$ and either of the following conditions holds, then the optimal stop-loss reinsurance is trivial.

(i) $G(d)$ is concave for $d \geq S_X^{-1}(\alpha)$, or

(ii) there exists a constant $d_0 > S_X^{-1}(\alpha)$ such that $G(d)$ is increasing for $d \in [S_X^{-1}(\alpha), d_0]$ while decreasing for $d \geq d_0$.

Moreover, in either of the above (i) and (ii), the trivial optimal retention depends on the relative magnitude between $\Pi(X)$ and u_α as indicated below:

$$d^* = \begin{cases} 0, & \text{if } \Pi(X) < u_\alpha; \\ 0, \ \text{or } \infty, & \text{if } \Pi(X) = u_\alpha; \\ +\infty, & \text{if } \Pi(X) > u_\alpha. \end{cases} \tag{2.4.33}$$

(b) If both of the following conditions hold, then the optimal stop-loss reinsurance is nontrivial.

(i) there exists a constant $d_0 \in (0, S_X^{-1}(\alpha))$ *such that* $d + \Pi([X-d]_+)$ *is decreasing in* d *on* $[0, d_0]$ *while increasing in* d *on* $[d_0, S_X^{-1}(\alpha)]$, *and*

(ii) $S_X^{-1}(\alpha) \geq d_0 + \Pi([X-d_0]_+)$.

Furthermore, when (i) and (ii) hold the optimal retention $d^* = d_0$ *with the corresponding minimum value of* $CTE_{T_{sl}}(d, \alpha)$

$$\min_{d \geq 0} CTE_\alpha(X_{T_{sl}}; d) = d_0 + \Pi([X-d_0]_+). \tag{2.4.34}$$

Based on Theorem 2.4, we now demonstrate that the optimal stop-loss reinsurance is trivial under some premium principles as shown in the following proposition with its proof collected in Section 2.6.

Proposition 2.10 *Consider the CTE-optimization (2.2.19). The optimal stop-loss reinsurance is trivial and the trivial optimal retention is determined as in (2.4.33) for the following premium principles:*

(a) P9 (Wang's principle) $\Pi(X) = \int_0^\infty [\Pr(X \geq t)]^p dt$, *where* $0 < p < 1$;

(b) P11 (Generalized percentile principle) $\Pi(X) = \mathsf{E}[X] + \beta(F_X^{-1}(1-p) - \mathsf{E}[X])$, *where* $0 \leq \beta, p \leq 1$.

(c) P12 (CTE principle) $\Pi(X) = (1/p)\int_{1-p}^1 F_X^{-1}(x)dx$, *where* $0 < p < 1$;

Based on (b) of Theorem 2.4, we find that the optimal contract with respect to CTE-optimization (2.2.19) does exist for some premium principles under certain conditions. The following Proposition 2.11 presents these principles along with the corresponding sufficient conditions. Actually, we can find that the sufficient conditions and the optimal retention for each principle with the CTE criterion are the same as that with VaR criterion. Nevertheless, the corresponding conditions are not only sufficient but also necessary for VaR criterion while just sufficient for the CTE criterion. Among these principles, however, the P1 (expectation principle) is an exception, the conditions given out in Proposition 2.11 for the existence of optimal stop-loss reinsurance is also necessary; see Cai and Tan (2007) for detail interpretation. We omit the proof of the following Proposition 2.11, since it is trivial by combining (b) of Theorem 2.4 and the proof of Proposition 2.9.

Proposition 2.11 *Consider the CTE-optimization (2.2.19).*

(a) *Under P1 (expectation premium principle)* $\Pi(x) = (1+\theta)E[X]$ *with* $\theta > 0$*, if both*

$$d_0 := S_X^{-1}\left(\frac{1}{1+\theta}\right) \in (0, S_X^{-1}(\alpha)) \tag{2.4.35}$$

and

$$S_X^{-1}(\alpha) \geq d_0 + (1+\theta)\int_{d_0}^{\infty} S_X(x)dx \tag{2.4.36}$$

hold, then the optimal stop-loss reinsurance is nontrivial with the optimal retention $d^* = d_0$.

(b) *Under P8 (Dutch principle)* $\Pi(X) = E[X] + \beta E[X - EX]_+$ *with* $\beta > 0$*, if there exists a constant* d_0 *satisfying the equation* $\beta S_X(d_0 + E[X - d_0]_+) = 1$ *such that both* $d_0 \in (0, S_X^{-1}(\alpha))$ *and*

$$S_X^{-1}(\alpha) \geq d_0 + E[X - d_0]_+) + \beta E\big\{[X - d_0]_+ - E[(X - d_0)_+]\big\}_+ \tag{2.4.37}$$

hold, then the optimal stop-loss reinsurance is nontrivial with the optimal retention $d^* = d_0$

(c) *Under P13 (variance principle)* $\Pi(X) = E[X] + \beta D[X]$ *with* $\beta > 0$*, if there exists a constant* d_0 *satisfying the equation* $2\beta E[X - d_0]_+ = 1$ *such that both*

$d_0 \in (0, S_X^{-1}(\alpha))$ and

$$S_X^{-1}(\alpha) \geq d_0 + \mathsf{E}[X - d_0]_+ + \beta D[X - d_0]_+ \tag{2.4.38}$$

hold, then the optimal stop-loss reinsurance is nontrivial with the optimal retention $d^* = d_0$.

(d) Under P14 (semi-variance principle) $\Pi(X) = \mathsf{E}[X] + \beta \mathsf{E}[X - EX]_+^2$ with $\beta > 0$, if there exists a constant d_0 satisfying the equation $2\beta \mathsf{E}\big\{X - d_0 - \mathsf{E}[(X - d_0)_+]\big\}_+ = 1$ such that both $d_0 \in (0, S_X^{-1}(\alpha))$ and

$$S_X^{-1}(\alpha) \geq d_0 + \mathsf{E}[X - d_0]_+ + \beta \mathsf{E}[X - d_0]_+^2 \tag{2.4.39}$$

hold, then the optimal stop-loss reinsurance is nontrivial with the optimal retention $d^* = d_0$.

(e) Under P15 (quadratic utility principle) $\Pi(X) = \mathsf{E}[X] + \gamma - \sqrt{\gamma^2 - D[X]}$ with $\gamma > 0$, if there exists a constant d_0 satisfying the equation

$$\frac{E[X - d_0]_+}{\sqrt{\gamma^2 - D[X - d_0]_+}} = 1$$

such that both $d_0 \in (0, S_X^{-1}(\alpha))$ and

$$S_X^{-1}(\alpha) \geq d_0 + \mathsf{E}[X - d_0]_+ + \gamma - \sqrt{\gamma^2 - D[X - d_0]_+} \tag{2.4.40}$$

hold, then the optimal stop-loss reinsurance is nontrivial with the optimal retention $d^* = d_0$.

2.5 Examples

In this section, we assume that the loss random variable X has a distribution similar to exponential one with a jump at 0 and then discuss the specific conditions for the existence of optimal contract for both VaR-optimization and CTE-optimization. Specifically, we suppose that the loss random variable X is distributed with survival function

$$S_X(x) = \delta \mathrm{e}^{-\lambda x}, \quad x \geq 0. \tag{2.5.1}$$

Hence, $S_X^{-1}(y) = -\frac{1}{\lambda} \ln\left(\frac{y}{\delta}\right)$, $y \in [0,1]$, and $\Pr\{X=0\} = 1 - S_X(0) = 1 - \delta$.

Below we present three numerical examples based on the distributions specified above corresponding to Propositions 2.6, 2.7, 2.9 and 2.11. Specifically, Example 2.1 corresponds to Proposition 2.6 for VaR-optimization (2.2.16), Example 2.2 exploits Proposition 2.7 for CTE-optimization (2.2.17), while Example 2.3 relates to Proposition 2.9 and Proposition 2.11, respectively, for VaR-optimization (2.2.18) and CTE-optimization (2.2.19).

Example 2.1 *Consider VaR-optimization (2.2.16). The following conditions are sufficient and necessary for the existence of the nontrivial optimal quota-share reinsurance for each reinsurance premium principle. The optimal quota-share coefficient c^* is also given for each case below.*

(1) P13 (variance principle) $\Pi(X) = \mathsf{E}[X] + \beta D[X]$ *with* $\beta > 0$.

(a) Conditions: $\delta e^{-\delta - 2\beta\delta(2-\delta)/\lambda} < \alpha < \delta e^{-\delta}$.

Optimal quota-share coefficient: $c^* = -\dfrac{\left[\ln\left(\frac{\alpha}{\delta}\right) + \delta\right]\lambda}{2\beta\delta(2-\delta)}$.

(b) By setting $\lambda = 0.001$, $\delta = 3/4$ *and* $\beta = 0.1$, *then the conditions for the existence of the nontrivial optimal insurance is* $1.3156 \times 10^{-82} < \alpha < 0.3543$. *Furthermore,* $\alpha = 0.05$ *implies optimal quota-share coefficient* $c^* = -\left[\ln(\frac{\alpha}{0.75}) + 0.75\right]/187.5 = 0.0104$.

(2) P14 (semi-variance principle) $\Pi(X) = \mathsf{E}[X] + \beta \mathsf{E}(X - \mathsf{E}[X])_+^2$ *with* $\beta > 0$.

(a) *Conditions:* $\delta \exp\{-\delta - \frac{4\beta\delta}{\lambda} e^{-\delta}\} < \alpha < \delta \exp\{-\delta\}$.

Optimal quota-share coefficient: $c^* = -\frac{\left[\ln\left(\frac{\alpha}{\delta}\right) + \delta\right]\lambda}{4\beta\delta e^{-\delta}}$.

(b) *By setting* $\lambda = 0.001$, $\delta = 3/4$ *and* $\beta = 0.1$, *then the conditions for the existence of the nontrivial optimal insurance is* $1.0127 \times 10^{-62} < \alpha < 0.3543$. *Furthermore,* $\alpha = 0.05$ *implies optimal quota-share coefficient* $-\left(\ln\frac{4\alpha}{3} + 0.75\right) e^{0.75}/300 = 0.0138$.

(3) *P15 (quadratic utility principle)* $\Pi(X) = E[X] + \gamma - \sqrt{\gamma^2 - D[X]}$ *with* $\gamma > 0$.

(a) *Conditions:* $\begin{cases} \alpha < \delta e^{-\delta} \\ \lambda^2\gamma^2 - \delta(2-\delta) \le 0, \end{cases}$

or

$$\begin{cases} \delta \exp\{-\delta - \delta(2-\delta)[\lambda^2\gamma^2 - \delta(2-\delta)]^{-1/2}\} < \alpha < \delta e^{-\delta} \\ \lambda^2\gamma^2 - \delta(2-\delta) > 0, \end{cases}$$

Optimal quota-share coefficient:

$$c^* = -\frac{(\ln\frac{\alpha}{\delta} + \delta)\lambda\gamma}{\sqrt{\delta(2-\delta)\left\{\delta(2-\delta) + [\ln(\alpha/\delta) + \delta]^2\right\}}}.$$

(b) *By setting* $\lambda = 0.001$, $\delta = 3/4$ *and* $\gamma = 1000$, *then* $\lambda^2\gamma^2 - \delta(2-\delta) = \frac{1}{16} > 0$. *This implies that the second set of conditions applies so that the conditions for the existence of the nontrivial optimal insurance reduces to* $0.0083 < \alpha < 0.3543$. *Furthermore,* $\alpha = 0.05$ *implies optimal quota-share coefficient*

$$c^* = -\frac{2}{\sqrt{30}} \cdot \frac{4\ln(4\alpha/3) + 3}{\sqrt{2\ln^2(4\alpha/3) + 3\ln(4\alpha/3) + 3}} = 0.9258.$$

(4) *P17 (exponential principle)* $\Pi(X) = \frac{1}{\beta}\log E[\exp\{\beta X\}]$ *with* $\beta > 0$.

(a) *Conditions:* $\alpha < \delta$.

Optimal quota-share coefficient: $c^* = \frac{\lambda}{\beta} - \frac{\lambda}{\beta} \cdot \frac{2\delta}{M}$, *with*

$$M = -\delta \ln(\alpha/\delta) + \sqrt{\delta^2 \ln^2(\alpha/\delta) - 4\delta(1-\delta)\ln(\alpha/\delta)}.$$

(b) *By setting* $\lambda = 0.001$, $\delta = 3/4$ *and* $\beta = 0.001$, *then the conditions for the existence of the nontrivial optimal insurance is* $\alpha < 0.75$. *Furthermore,* $\alpha = 0.05$ *implies optimal quota-share coefficient*

$$c^* = 1 - \frac{6}{-3\ln(4\alpha/3) + \sqrt{9\ln^2(4\alpha/3) - 12\ln(4\alpha/3)}} = 0.6676.$$

Example 2.2 *Consider CTE-optimization (2.2.17). The following conditions are sufficient and necessary for the existence of the nontrivial optimal quota-share reinsurance for each reinsurance premium principle. The optimal quota-share coefficient* c^* *is also given for each case below.*

(1) *P13 (variance principle)* $\Pi(X) = \mathsf{E}[X] + \beta D[X]$ *with* $\beta > 0$.

(a) *Conditions:* $\delta e^{1-\delta-2\beta\delta(2-\delta)/\lambda} < \alpha < \delta e^{1-\delta}$.

Optimal quota-share coefficient: $c^* = -\frac{\left[\ln\left(\frac{\alpha}{\delta}\right) + \delta - 1\right]\lambda}{2\beta\delta(2-\delta)}$.

(b) *By setting* $\lambda = 0.001$, $\delta = 3/4$ *and* $\beta = 0.1$, *then the conditions for the existence of the nontrivial optimal insurance is* $3.5762 \times 10^{-82} < \alpha < 0.9630$. *Furthermore,* $\alpha = 0.05$ *implies quota-share coefficient*

$$c^* = -2(\ln \frac{4\alpha}{3} - 0.25)/375 = 0.0158.$$

(2) *P14 (semi-variance principle)* $\Pi(X) = \mathsf{E}[X] + \beta \mathsf{E}[X - \mathsf{E}[X]]_+^2$ *with* $\beta > 0$.

(a) *Conditions:* $\delta \exp\left\{1 - \delta - \frac{4\beta\delta}{\lambda} e^{-\delta}\right\} < \alpha < \delta \exp\{1-\delta\}$.

Optimal quota-share coefficient: $c^* = -\frac{\left[\ln\left(\frac{\alpha}{\delta}\right) + \delta - 1\right]\lambda}{4\beta\delta e^{-\delta}}$.

(b) *By setting* $\lambda = 0.001$, $\delta = 3/4$ *and* $\beta = 0.1$, *then the conditions for the existence of the nontrivial optimal insurance is* $2.7528 \times 10^{-62} < \alpha < 0.9630$. *Furthermore,* $\alpha = 0.05$ *implies the optimal quota-share coefficient*

$$c^* = -(\ln \frac{4\alpha}{3} - 0.25)e^{0.75}/300 = 0.0209.$$

(3) *P15 (quadratic utility principle)* $\Pi(X) = E[X] + \gamma - \sqrt{\gamma^2 - D[X]}$ *with* $\gamma > 0$.

(a) *Conditions:* $\begin{cases} \alpha < \delta e^{1-\delta} \\ \lambda^2\gamma^2 - \delta(2-\delta) \le 0, \end{cases}$

or

$$\begin{cases} \delta \exp\{1 - \delta - \delta(2-\delta)[\lambda^2\gamma^2 - \delta(2-\delta)]^{-1/2}\} < \alpha < \delta e^{1-\delta} \\ \lambda^2\gamma^2 - \delta(2-\delta) > 0, \end{cases}$$

Optimal quota-share coefficient:

$$c^* = -\frac{\left[\ln\left(\frac{\alpha}{\delta}\right) + \delta - 1\right]\lambda\gamma}{\sqrt{\delta(2-\delta)\left\{\delta(2-\delta) + [\ln(\alpha/\delta) + \delta - 1]^2\right\}}}.$$

(b) *By setting* $\lambda = 0.001$, $\delta = 3/4$ *and* $\gamma = 1000$, *then* $\lambda^2\gamma^2 - \delta(2-\delta) = \frac{1}{16} > 0$. *This implies that the second set of conditions applies so that the conditions for the existence of the nontrivial optimal insurance reduces to* $0.0226 < \alpha < 0.9630$. *Furthermore,* $\alpha = 0.05$ *implies the optimal quota-share reinsurance coefficient*

$$c^* = -\frac{2}{\sqrt{30}} \cdot \frac{4\ln(4\alpha/3) - 1}{\sqrt{2\ln^2(4\alpha/3) - \ln(4\alpha/3) + 2}} = 0.9816.$$

(4) *P17 (exponential principle)* $\Pi(X) = \frac{1}{\beta} \log E[\exp\{\beta X\}]$ *with* $\beta > 0$.

(a) *Conditions:* $\alpha < \delta$.

Optimal quota-share coefficient: $c^* = \frac{\lambda}{\beta} - \frac{\lambda}{\beta} \cdot \frac{2\delta}{M}$, *with*

$$M = -\delta[\ln(\alpha/\delta) - 1] + \sqrt{\delta^2[\ln(\alpha/\delta) - 1]^2 - 4\delta(1-\delta)[\ln(\alpha/\delta) - 1]}.$$

(b) *By setting* $\lambda = 0.001$, $\delta = 3/4$, $\beta = 0.2$, *then the conditions for the existence of the nontrivial optimal insurance is* $\alpha < 0.75$. *Furthermore,* $\alpha = 0.05$ *implies the optimal quota-share coefficient*

$$c^* = 1 - \frac{6}{-3\big[\ln(\frac{4\alpha}{3}) - 1\big] + \sqrt{9\big[\ln(\frac{4\alpha}{3}) - 1\big]^2 - 12\big[\ln(\frac{4\alpha}{3}) - 1\big]}} = 0.7510.$$

Example 2.3 *Consider VaR-optimization (2.2.18) and CTE-optimization (2.2.19). The following conditions are sufficient and necessary for VaR-optimization (2.2.18), while they are only sufficient for CTE-optimization (2.2.19). The optimal stop-loss retention* d^* *is also given for each reinsurance premium principle.*

(1) *P1 (expectation principle)* $\Pi(X) = (1+\theta)E[X]$ *with* $\theta > 0$.

(a) *Conditions:* $\alpha \le \frac{1}{(1+\beta)e}$.

Optimal retention: $d^* = -\frac{1}{\lambda} \ln \frac{1}{\delta(1+\beta)}$.

Note that if the probability that the loss random variable X *takes the value of 0 is large, the loading safety* γ *must be large enough to ensure the existence of the nontrivial optimal stop-loss reinsurance. For example if* $\Pr\{X = 0\} = 1 - \delta = 0.2$, *then the loading safety* β *must be larger than* 1.25.

(b) *By setting* $\lambda = 0.001$, $\delta = 4/5$ *and* $\beta = 0.3$, *then the condition for the existence of the nontrivial optimal insurance is* $\alpha < 0.2830$ *with the optimal retention* $d^* = 39.2207$.

(2) *P13 (variance principle)* $\Pi(X) = E[X] + \beta D[X]$ *with* $\beta > 0$.

(a) Conditions: $\alpha \leq \frac{\lambda}{2\beta} \exp\{-(1+\lambda/4\beta)\}$

Optimal retention: $d^* = -\frac{1}{\lambda} \ln \frac{\lambda}{2\beta\delta}$.

(b) By setting $\lambda = \beta = 0.001$ *and* $\delta = 3/4$, *then the condition for the existence of the nontrivial optimal insurance is* $\alpha < 0.1839$ *with the optimal retention* $d^* = 405.4651$.

Remark 2.3 *As shown in the above three examples, for some premium principles, only when the tolerance probability is large enough can the existence of nontrivial optimal reinsurance be guaranteed.*

2.6 Appendix: Proofs

Proof of Theorem 2.1:

(a) If $\Pi(cX) = c\Pi(X)$ for $c > 0$, it follows from Propositions 2.1 that

$$\begin{aligned} \mathrm{VaR}_\alpha(X_{T_{qs}}; c) &= (1-c)S_X^{-1}(\alpha) + c\Pi(X) \\ &= S_X^{-1}(\alpha) + c\left[\Pi(X) - S_X^{-1}(\alpha)\right], \end{aligned} \tag{2.6.2}$$

which is linear in c. Therefore, if $\Pi(X) < S_X^{-1}(\alpha)$, $\mathrm{VaR}_\alpha(X_{T_{qs}}; c)$ attains its minimum value at $c = 1$. If $\Pi(X) > S_X^{-1}(\alpha)$, with c going down to 0, $\mathrm{VaR}_\alpha(X_{T_{qs}}; c)$ keeps decreasing to $S_X^{-1}(\alpha)$, which is exactly $\mathrm{VaR}_\alpha(X_{T_{qs}}; c)$ evaluated at $c = 0$; thus the optimal quota-share coefficient $c^* = 0$ in this case. When $\Pi(X) = S_X^{-1}(\alpha)$, $\mathrm{VaR}_\alpha(X_{T_{qs}}; c)$ remains constant at $S_X^{-1}(\alpha)$ for all $c \in [0, 1]$. Combining the above, we can then conclude that the optimal quota-share reinsurance is trivial and the optimal quota-share coefficient c^* is determined as in (2.3.1).

(b) If $\Pi(cX)$ is strictly convex in c, it follows from Proposition 2.1 that $\mathrm{VaR}_\alpha(X_{T_{qs}};c)$ is also strictly convex in c. Hence $\mathrm{VaR}_\alpha(X_{T_{qs}};c)$ attains its global minimum value at c^* that is the solution to

$$\frac{\partial}{\partial c}\mathrm{VaR}_\alpha(X_{T_{qs}};c)\Big|_{c=c^*} = \Pi'_c(c^*X) - S_X^{-1}(\alpha) = 0,$$

which yields (2.3.2). □

Proof of Proposition 2.6:

(a) Observe that under variance premium principle, the reinsurance premium $\Pi(cX) = c\mathsf{E}[X] + c^2\beta\mathrm{D}[X]$ is strictly convex in c. Hence it follows from Theorem 2.1(b) that the optimal quota-share reinsurance is nontrivial if and only if there exists a constant $c^* \in (0,1)$ such that

$$\Pi'_c(c^*X) - S_X^{-1}(\alpha) = \mathsf{E}[X] + 2c^*\beta\mathrm{D}[X] - S_X^{-1}(\alpha) = 0. \tag{2.6.3}$$

Consequently, the nontriviality of the optimal quota-share reinsurance is equivalent to

$$0 < c^* = \frac{S_X^{-1}(\alpha) - \mathsf{E}[X]}{2\beta\mathrm{D}[X]} < 1, \tag{2.6.4}$$

which implies (2.3.5). Thus the proof follows.

(b)-(d) We omit the proofs of (b)-(d) for premium principles P14-P16 since they are similar to (a).

(e) Under the exponential principle P17, we have $\Pi(cX) = \frac{1}{\beta}\log\mathsf{E}[\exp(c\beta X)]$. It is easy to verify that

$$\Pi'_c(cX) = \frac{\mathsf{E}[X\exp(c\beta X)]}{\mathsf{E}[\exp(c\beta X)]},$$

and

$$\Pi''_c(cX) = \frac{\beta\Big\{\mathsf{E}[X^2\exp(c\beta X)]\mathsf{E}[\exp(c\beta X)] - \big[\mathsf{E}[X\exp(c\beta X)]\big]^2\Big\}}{\big\{\mathsf{E}[\exp(c\beta X)]\big\}^2}.$$

Let

$$f(c) = \Big\{ \mathsf{E}[X^2 \exp(c\beta X)] \mathsf{E}[\exp(c\beta X)] - \big[\mathsf{E}[X \exp(c\beta X)]\big]^2 \Big\},$$

then $f(0) = \mathsf{Var}[X] > 0$, and

$$f'_c(c) = \beta \mathsf{E}[X^3 \exp(c\beta X)] \mathsf{E}[\exp(c\beta X)] - \beta \mathsf{E}[X^2 \exp(c\beta X)] \mathsf{E}[X \exp(c\beta X)].$$

Moreover, it follows from the Hölder's inequality that

$$\mathsf{E}[X^2 \exp(c\beta X)] \leq \{\mathsf{E}[X^3 \exp(c\beta X)]\}^{2/3} \{\mathsf{E}[\exp(c\beta X)]\}^{1/3}$$

$$\mathsf{E}[X \exp(c\beta X)] \leq \{\mathsf{E}[X^3 \exp(c\beta X)]\}^{1/3} \{\mathsf{E}[\exp(c\beta X)]\}^{2/3}$$

Hence, $f'_c(c)$ is nondecreasing in c, and therefore $f(c) > 0$, $\Pi''_c(cX) > 0$, for $c \in [0, 1]$. This implies that $\Pi(cX)$ is strictly convex in c and using Theorem 2.1(b), we conclude the proof. □

Proof of Theorem 2.3: The proof of the theorem is trivial by first recognizing that from Proposition 2.3, $\mathrm{VaR}_\alpha(X_{T_{sl}}; d)$ is decreasing in d for $d \in (S_X^{-1}(\alpha), \infty)$ and it tends to the limiting minimum $S_X^{-1}(\alpha)$ as $d \to \infty$ if $\lim_{d\to\infty} \Pi([X - d]_+) = 0$. This implies that to show the nontriviality (or the triviality) of the optimal stop-loss reinsurance, we only need to focus on interval $0 < d < S_X^{-1}(\alpha)$ for which $\mathrm{VaR}_\alpha(X_{T_{sl}}; d) = d + \Pi([X - d]_+)$. Hence if either condition (i) or (ii) of Part (a) is satisfied, then $\mathrm{VaR}_\alpha(X_{T_{sl}}; d)$ attains its minimum value at either $d = 0$ or $d = \infty$, which implies that the optimal stop-loss reinsurance is trivial. Indeed, (2.4.25) follows immediately by comparing the values of $\mathrm{VaR}_\alpha(X_{T_{sl}}; d)$ corresponding to $d = 0$ and $d = \infty$. Moreover, (2.4.26) implies that $d_0 \in (0, S_X^{-1}(\alpha))$ since $\Pi([X - d_0]_+) \geq 0$. Hence, if $d + \Pi([X - d]_+)$ is decreasing for $d \in [0, d_0]$ while increasing on $[d_0, \infty]$, (2.4.26) ensures that $\mathrm{VaR}_\alpha(X_{T_{sl}}; d)$ attains its global minimum at $d = d_0$, which means the optimal stop-loss reinsurance is nontrivial; conversely, if optimal stop-loss reinsurance is nontrivial, d_0 must be the global minimizer for $\mathrm{VaR}_\alpha(X_{T_{sl}}; d)$, and hence (2.4.26) and (2.4.27) hold. □

Before proving Proposition 2.8, let us first state the following relations which will be used extensively in the proof:

$$\frac{\partial}{\partial d}\mathsf{E}[X-d]_+ = \frac{\partial}{\partial d}\int_d^\infty S_X(x)dx = -S_X(d), \tag{2.6.5}$$

for $m > 1$

$$\begin{aligned}
\frac{\partial}{\partial d}\mathsf{E}[X-d]_+^m &= \frac{\partial}{\partial d}\Big\{m\int_d^\infty (x-d)^{m-1}S_X(x)dx\Big\} \\
&= -m(m-1)\int_d^\infty (x-d)^{m-2}S_X(x)dx \\
&= -m\mathsf{E}[X-d]_+^{m-1},
\end{aligned} \tag{2.6.6}$$

and

$$\begin{aligned}
\frac{\partial}{\partial d}\mathsf{Var}[X-d]_+ &= \frac{\partial}{\partial d}\Big\{\mathsf{E}[X-d]_+^2 - (\mathsf{E}[X-d]_+)^2\Big\} \\
&= -2[1-S_X(d)]\mathsf{E}[X-d]_+.
\end{aligned} \tag{2.6.7}$$

Proof of Proposition 2.8: Note that $\Pi([X-d]_+)$ is decreasing in d for all the premium principles in the proposition. Hence it follows from Theorem 2.3(a) that we only need to verify if $d+\Pi_d([X-d]_+)$ is either increasing or first increasing than decreasing for $d \in [0, S_X^{-1}(\alpha)]$. For the premium principles listed in the proposition, $d+\Pi_d([X-d]_+)$ is actually an increasing function in d (or equivalently $1+\Pi'_d([X-d]_+) > 0$) as demonstrated below:

(a) This is a special case of (b) with $p = 2$.

(b) For $p > 1$ and $d \geq 0$,

$$\begin{aligned}
1+\Pi'_d([X-d]_+) &= 1+\frac{\partial}{\partial d}\big\{\mathsf{E}[X-d]_+^p\big\}^{1/p} \\
&= 1+\frac{1}{p}\Big\{\mathsf{E}[X-d]_+^p\Big\}^{\frac{1-p}{p}}\frac{\partial}{\partial d}\mathsf{E}[X-d]_+^p \\
&= 1-\Big\{\mathsf{E}[X-d]_+^p\Big\}^{\frac{1-p}{p}}\mathsf{E}[X-d]_+^{p-1} \\
&> 0,
\end{aligned}$$

where the inequality follows from the Hölder's inequality.

(c) First note that $\mathsf{E}\big[[X-d]_+ - \mathsf{E}[X-d]_+\big]_+^2 = \mathsf{E}\big[X-d-\mathsf{E}[X-d]_+\big]_+^2$. Then under the semi-deviation premium principle, we have

$$\begin{aligned}
1+\Pi_d'([X-d]_+) &= 1+\frac{\partial}{\partial d}\left\{\mathsf{E}[X-d]_+ + \beta\sqrt{\mathsf{E}\big[X-d-\mathsf{E}[X-d]_+\big]_+^2}\right\}\\
&= 1-S_X(d)-\beta\frac{2[1-S_X(d)]\int_{d+\mathsf{E}[X-d]_+}^{\infty} S_X(x)dx}{2\sqrt{\mathsf{E}\big[X-d-\mathsf{E}[X-d]_+\big]_+^2}}\\
&= (1-S_X(d))\left[1-\beta\frac{\mathsf{E}[X-d-\mathsf{E}[X-d]_+]_+}{\sqrt{\mathsf{E}\big[X-d-\mathsf{E}[X-d]_+\big]_+^2}}\right]\\
&> [1-S_X(d)](1-\beta) > 0,
\end{aligned}$$

where the first inequality follows from the simple relation that $\mathsf{E}[Y^2] > (\mathsf{E}[Y])^2$ and the second inequality is due to the constraint $0<\beta<1$.

(d) For the Wang's premium princple with $0<p<1$ and $d\ge 0$,

$$\begin{aligned}
1+\Pi_d'([X-d]_+) &= 1+\frac{\partial}{\partial d}\left\{\int_0^{\infty}[\Pr([X-d]_+\ge t)]^p dt\right\}\\
&= 1+\frac{\partial}{\partial d}\left\{\int_d^{\infty}[\Pr(X\ge t)]^p dt\right\}\\
&= 1-[\Pr(X\ge d)]^p > 0.
\end{aligned}$$

(e) Let us first note that

$$F_{[X-d]_+}^{-1}(1-p) = \begin{cases} 0, & p > S_X(d),\\ F_X^{-1}(1-p)-d, & \text{otherwise.}\end{cases} \tag{2.6.8}$$

Then for $0 < \beta, p < 1$ and $d \geq 0$, we have

$$\begin{aligned}
& 1 + \Pi'_d([X-d]_+) \\
= \; & 1 + \frac{\partial}{\partial d}\Big\{\mathsf{E}[X-d]_+ + \beta\big[F^{-1}_{[X-d]_+}(1-p) - \mathsf{E}[X-d]_+\big]\Big\} \\
= \; & \begin{cases} 1 + \frac{\partial}{\partial d}\Big\{(1-\beta)\mathsf{E}[X-d]_+\Big\}, & p > S_X(d), \\ \\ 1 + \frac{\partial}{\partial d}\Big\{(1-\beta)\mathsf{E}[X-d]_+ + \beta\big[F_X^{-1}(1-p) - d\big]\Big\}, & \text{otherwise.} \end{cases} \\
= \; & \begin{cases} 1 - (1-\beta)S_X(d), & p > S_X(d), \\ \\ (1-\beta)(1 - S_X(d)), & \text{otherwise.} \end{cases}
\end{aligned}$$

Both the above expressions are positive and this concludes the proof.

(f) For $0 < p < 1$ and $d \geq 0$, we have

$$1 + \Pi'_d([X-d]_+) \;=\; 1 + \frac{\partial}{\partial d}\left\{\frac{1}{p}\int_{1-p}^{1} F^{-1}_{[X-d]_+}(x)dx\right\}.$$

It follows from (2.6.8) that the above expression is positive and hence concludes the proof.

(g) First note that

$$\begin{aligned}
\mathsf{E}\big[\exp\big(\beta[X-d]_+\big)\big] &= \int_0^\infty e^{\beta[X-d]_+} dF_X(x) \\
&= \int_0^d dF_X(x) + \int_d^\infty e^{\beta(x-d)} dF_X(x) \\
&= \int_d^\infty e^{\beta(x-d)} dF_X(x) + F_X(d),
\end{aligned}$$

and

$$\frac{\partial}{\partial d}\mathsf{E}\big[\exp\big(\beta[X-d]_+\big)\big] \;=\; -\beta\int_d^\infty e^{\beta(x-d)} dF_X(x)$$

Then for the exponential premium principle with $0 < \beta < 1$ and $d \geq 0$, we have

$$\begin{aligned} & 1 + \Pi'_d([X-d]_+) \\ = & \; 1 + \Big\{\beta \mathsf{E}\big[\exp\big(\beta[X-d]_+\big)\big]\Big\}^{-1} \cdot \frac{\partial}{\partial d}\Big\{\mathsf{E}\big[\exp\big(\beta[X-d]_+\big)\big]\Big\} \\ = & \; 1 - \frac{\int_d^\infty e^{\beta(x-d)} dF_X(x)}{F_X(d) + \int_d^\infty e^{\beta(x-d)} dF_X(x)} \; > \; 0. \end{aligned}$$

□

Proof of Proposition 2.9: The results in the proposition can easily be verified by resorting to Theorem 2.3(b) and also noticing that $\Pi([X-d]_+)$ is decreasing in d along with $\lim_{d\to\infty} \Pi([X-d]_+) = 0$ for the considered premium principles. Let us illustrate by just considering the proof to (c) for the variance premium principle. From Theorem 2.3, it suffices to verify d_0 that solves $2\beta\mathsf{E}[X-d_0]_+ = 1$ is the unique solution such that $d + \Pi([X-d]_+)$ is decreasing for $d \in [0, d_0]$ while increasing for $d \in [d_0, \infty]$. For this purpose, we investigate its derivative first:

$$\begin{aligned} 1 + \Pi'_d([X-d])_+ &= 1 + \frac{\partial}{\partial d}\Big\{\mathsf{E}[X-d]_+ + \beta \mathrm{D}[X-d]_+\Big\} \\ &= F_X(d)\Big(1 - 2\beta\mathsf{E}[X-d]_+\Big), \end{aligned}$$

which is positive if $\mathsf{E}[X-d]_+ < \frac{1}{2\beta}$ while negative if $\mathsf{E}[X-d]_+ > \frac{1}{2\beta}$. Now that X is assumed to have a continuous one-to-one distribution function on $[0,\infty)$, $\mathsf{E}[X-d]_+$ is strictly decreasing in d and the equation $\mathsf{E}[X-d]_+ = \frac{1}{2\beta}$ has a unique solution $d_0 > 0$. Hence, the proof is complete. □

Proof of Theorem 2.4:

(a) When (i) or (ii) holds, it follows from Proposition 2.4 that $\mathrm{CTE}_\alpha(X_{T_{sl}}; d)$ attains its minimum either at $d = S_X^{-1}(\alpha)$ or as $d \to \infty$ on interval $[S_X^{-1}(\alpha), \infty]$. Therefore, when $d + \Pi([X-d]_+)$ is increasing for $d \in [0, S_X^{-1}(\alpha)]$, $\mathrm{CTE}_\alpha(X_{T_{sl}}; d)$

attains its global minimum either at $d = 0$ or as $d \to \infty$, which means the optimal stop-loss reinsurance is trivial. In this case, (2.4.33) follows immediately only by noticing that $\text{CTE}_\alpha(X_{T_{sl}}; 0) = \Pi(X)$ and $\lim_{d\to\infty} \text{CTE}_\alpha(X_{T_{sl}}; d) = \lim_{d\to\infty} G(d) = u_\alpha$.

(b) With condition (i), we can conclude that d_0 is the minimizer of $\text{CTE}_\alpha(X_{T_{sl}}; d)$ for d on $[0, S_X^{-1}(\alpha)]$ and the corresponding minimum of $\text{CTE}_\alpha(X_{T_{sl}}; d)$ is $d_0 + \Pi([X - d_0]_+)$. Moreover, it follows from Proposition 2.4 that $\text{CTE}_\alpha(X_{T_{sl}}; d) > S_X^{-1}(\alpha)$ for $d > S_X^{-1}(\alpha)$. Therefore, when (ii) holds, d_0 is the global minimizer of $\text{CTE}_\alpha(X_{T_{sl}}; d)$, and hence we conclude the proof. □

Proof of Proposition 2.10: Since Proposition 2.8 has already established the increasing property of $d + \Pi([X - d]_+)$ for d on interval $[0, S_X^{-1}(\alpha)]$, it suffices to verify either (i) or (ii) of (a) in Theorem 2.4 for all of these premium principles. Now we turn to verify one principle by another.

(a) For $0 < p < 1$ and $d \geq 0$,

$$\begin{aligned} G'(d) &= \frac{S_X(d)}{\alpha} + \Pi'_d((X-d)_+) \\ &= \frac{S_X(d)}{\alpha} + \left\{ \int_0^\infty [\Pr((X-d)_+ \geq t)]^p dt \right\}'_d \\ &= \frac{S_X(d)}{\alpha} - [\Pr(X \geq d)]^p \\ &= \frac{S_X(d)}{\alpha} \left\{ 1 - \alpha [S_X(d)]^{p-1} \right\} \end{aligned} \tag{2.6.9}$$

Noticing that $1 - \alpha[S_X(d)]^{p-1}$ is continuous and decreasing in d and that $1 - \alpha[S_X(d)]^{p-1} > 0$ when $d = S_X^{-1}(\alpha)$, there must exist a constant $d_0 > S_X^{-1}(\alpha)$ such that (ii) of (a) in Theorem 2.4 holds.

(b) For $0 < \beta, p < 1$ and $d \geq 0$,

$$\begin{aligned}
& G''(d) \\
= & -\frac{f_x(d)}{\alpha} + \Pi''_d((X-d)_+) \\
= & -\frac{f_X(d)}{\alpha} + \frac{\partial^2}{\partial d^2}\Big\{\mathsf{E}[X-d]_+ + \beta\Big(F^{-1}_{(X-d)_+}(1-p) - \mathsf{E}[X-d]_+\Big)\Big\} \\
= & \begin{cases} -\frac{f_X(d)}{\alpha} + \frac{\partial^2}{\partial d^2}\Big\{\mathsf{E}[X-d]_+ - \beta\mathsf{E}[X-d]_+\Big\}, & p > S_X(d) \\ \\ -\frac{f_X(d)}{\alpha} + \frac{\partial^2}{\partial d^2}\Big\{\mathsf{E}[X-d]_+ + \beta\Big(F^{-1}_X(1-p) - d - \mathsf{E}[X-d]_+\Big)\Big\}, & \text{otherwise} \end{cases} \\
= & \begin{cases} -\frac{f_X(d)}{\alpha} - \frac{\partial}{\partial d}\Big\{(1-\beta)S_X(d)\Big\}, & p > S_X(d) \\ \\ -\frac{f_X(d)}{\alpha} - \frac{\partial}{\partial d}\Big\{(1-\beta)S_X(d) + \beta\Big\}, & \text{otherwise} \end{cases} \\
= & f_X(d)\left[(1-\beta) - \frac{1}{\alpha}\right] < 0,
\end{aligned}$$

which means (i) of (a) in Theorem 2.4 holds.

(c) For $0 < p < 1$ and $d \geq 0$,

$$\begin{aligned}
G''(d) &= -\frac{f_X(d)}{\alpha} + \Pi''_d((X-d)_+) \\
&= -\frac{f_X(d)}{\alpha} + \frac{\partial^2}{\partial d^2}\Big\{\frac{1}{p}\int_{1-p}^{1} F^{-1}_{(X-d)_+}(x)dx\Big\} \\
&= \begin{cases} -\frac{f_X(d)}{\alpha}, & p > S_X(d) \\ \\ -\frac{f_X(d)}{\alpha} + \frac{\partial^2}{\partial d^2}\Big\{\frac{1}{p}\int_{1-p}^{1}[F^{-1}_X(x) - d]dx\Big\}, & \text{otherwise} \end{cases} \\
&= -\frac{f_x(d)}{\alpha} \; < 0,
\end{aligned}$$

which means (i) of (a) in Theorem 2.4 holds. □

Chapter 3

VaR Minimization Model: Increasing Convex Reinsurance Treaties

3.1 Introduction and Reinsurance Model

The previous chapter analyzed two specific types of reinsurance: the quota-share and the stop-loss. This chapter generalizes to analyze the optimal reinsurance among all the increasing convex treaties. In doing so, we adopt the VaR minimization model and consider a constrained reinsurance model with a reinsurance premium budget constraint. We assume the expectation reinsurance premium principle and in this case the reinsurance premium principle constraint can also be interpreted as an expected profit guarantee for the insurer. By equivalently reformulating the model into an optimization problem over a space of σ-finite positive measures on certain measurable space, we derive the explicit optimal solutions.

To specify our model, let X denote the nonnegative random variable which represents the (aggregate) loss initially assumed by an insurer. To simplify our discussions, we assume that X has finite mean and that X has a continuous strictly increasing distribution function on $(0, \infty)$ with a possible jump at 0. Suppose now the insurer were to manage its risk exposure via a reinsurance treaty. Under this arrangement, the insurer cedes part of its loss, say $f(X)$ satisfying $0 \leq f(X) \leq X$, to a reinsurer. The insurer thus retains loss $I_f(X) = X - f(X)$, where the function $f(x)$ is known as the ceded loss function and $I_f(x) = x - f(x)$ is referred as the retained loss function. By transferring part of its losses to a reinsurer, the insurer incurs a cost in the form of reinsurance premium, denoted by $\Pi(f(X))$, that is payable to a reinsurer. This implies that the sum $I_f(X) + \Pi(f(X))$ can be interpreted as the total risk (or the total cost) of the insurer in the presence of reinsurance. Using $T_f(X)$ to denote the total cost, we have

$$T_f(X) = I_f(X) + \Pi(f(X)). \tag{3.1.1}$$

Therefore, if we exploit VaR as the risk measure, the optimal reinsurance model can be formulated based on the insurer's total cost T_f as follows.

$$\begin{cases} \min\limits_{f \in \mathcal{IC}} & \text{VaR}_\alpha(T_f(X)) \\ \text{s.t.} & \Pi[f] \leq \pi, \end{cases} \tag{3.1.2}$$

where $\mathcal{IC}$ denotes the class of all the increasing convex functions on $[0, \infty)$ such that $0 \leq f(x) \leq x$ for all $x \geq 0$. Also note that π in the above model is a preset positive constant standing for the reinsurance premium budget so that $\Pi[f] \leq \pi$ implies the assumption that the insurer is willing to pay the reinsurance premium no more than π.

Under the additional assumption that the reinsurance premium $\Pi(f(X))$ is determined by the expectation premium principle; i.e.,

$$\Pi(f(X)) = (1 + \theta)\mathsf{E}[f(X)], \tag{3.1.3}$$

where $\theta > 0$ is the safety loading factor, the reinsurance premium budget constraint is equivalent to $\mathsf{E}[f] \leq \mathsf{E}[X]/(1+\theta)$ and thus we end up with a reinsurance model in a more explicit form as follows.

$$\begin{cases} \min\limits_{f \in \mathcal{IC}} & \mathrm{VaR}_\alpha(T_f(X)) \\ \text{s.t.} & \mathsf{E}[f(X)] \leq B \\ & 0 \leq f(x) \leq x \ \text{ for all } \ x \geq 0, \end{cases} \tag{3.1.4}$$

where B denotes the constant $\pi/(1+\theta)$.

The constraint $\mathsf{E}[f] \leq B$ can also be interpreted as an expected profit guarantee for the insurer. This can be argued by taking into account the insurance premium collected by the insurer from the policyholders and explained as follows. Let p_0 denote the aggregate insurance premium charged. Reducing the total risk exposure by the amount of the insurance premium received, we obtain the *net cost* or the *net risk* of insuring risk X which will be denoted by $NC_f(X)$; i.e.

$$NC_f(X) = T_f(X) - p_0 = I_f(X) + \Pi(f(X)) - p_0. \tag{3.1.5}$$

Then, the quantity $\mathsf{E}[-NC_f(X)]$ can be interpreted as the insurer's expected profit and the constraint condition $\mathsf{E}[-NC_f(X)] \geq P$ ensures that the expected profit of the insurer under the ceded loss function f is at least P. Clearly, $\mathsf{E}[-NC_f(X)] \geq P$ is equivalent to the condition $\mathsf{E}[f] \leq B$ with $B = (p_0 - P - \mathsf{E}[X])/\theta$.

3.2 Model Reformulation

As pointed out in Gaivoronski and Pflug (Winter 2004-005), the optimization problem associated with VaR, in general, is a non-trivial exercise even in the finite dimension case. To derive the solutions, we reformulate (3.1.4) as a linear programming with respect to σ-finite positive measures on the Borel measurable space $([0,\infty),\mathcal{B})$, where $\mathcal{B}:=\mathcal{B}([0,\infty))$ denotes the Borel sigma over the positive half real line $[0,\infty)$. Then we obtain the solutions by an approximating procedure which is a commonly-used technique for solving a linear programming regarding positive measures on certain measurable space, and the critical point is to establish a sequence of programming which are solvable and with solutions converging to the original linear programming.

Before the reformulation, we need the following two lemmas; see Cardin and Pacelli (2007) for the proof of Lemma 3.1, and Section 3.5 for proof of Lemma 3.2.

Lemma 3.1 *An increasing convex function f defined on $[0,\infty)$ can be represented as the following form:*

$$f(x)=f(0)+\int[x-t]_+d\mu,\ \ \textit{for any fixed}\ \ x\geq 0, \tag{3.2.6}$$

for some positive σ-additive measure μ on $\mathcal{B}$.

Lemma 3.2 *For any $f(x)\in\mathcal{IC}$, $I_f(x)=x-f(x)$ is increasing and concave in x.*

Now we have the following important facts:

(i) Note that for any ceded loss function $f\in\mathcal{IC}$, $f(0)=0$ and hence by Lemma 3.1 the ceded loss function f has the following representation:

$$f(x)=\int(x-t)_+d\mu,\ \ \text{for any fixed}\ \ x\geq 0. \tag{3.2.7}$$

with a positive σ-finite measure μ on $\mathcal{B}$. Furthermore, by Fubini theory,

$$\mathsf{E}[f(X)]=\int\mathsf{E}[(X-t)_+]d\mu. \tag{3.2.8}$$

(ii) By Lemma 3.2, for any $f\in\mathcal{IC}$, the function $I_f(x)=x-f(x)$ is increasing and concave, and hence also continuous. Consequently, it follows from (i) of Lemma 1.1 in Chapter 1 that

$$\begin{aligned}\mathrm{VaR}_\alpha(T_f(X)) &= \mathrm{VaR}_\alpha\Big(X-f(X)+(1+\theta)\mathsf{E}[f(X)]\Big)\\ &= \mathrm{VaR}_\alpha(X)-f\big(\mathrm{VaR}_\alpha(X)\big)+(1+\theta)\mathsf{E}[f(X)]. \end{aligned}\tag{3.2.9}$$

Note that when $\alpha \geq S_X(0)$, then $\mathrm{VaR}_\alpha(X) = 0$, the goal function $\mathrm{VaR}_\alpha(T_f(X))$ in model (3.1.4) only depends on the value of $\mathsf{E}[f(X)]$, and hence it will be optimal for the insurer not to cede any risk. To avoid this trivial case, we assume $0 < \alpha < S_X(0)$ hereafter.

Together with (3.2.7), (3.2.8) and setting $\phi(t) = \Big(\mathrm{VaR}_\alpha(X) - t\Big)_+$ and $\psi(t) = \mathsf{E}[(X-t)_+]$ for $t \geq 0$, (3.2.9) can be rewritten as

$$\mathrm{VaR}_\alpha(T_f(X)) = \mathrm{VaR}_\alpha(X) - \int \Big[\phi(t) - (1+\theta)\psi(t)\Big] d\mu.$$

Consequently, our proposed optimal reinsurance model (3.1.4) is equivalent to the following linear programming with respect to the positive measure μ:

$$\begin{cases} \min\limits_{\mu \in \mathcal{M}^+} & \mathrm{VaR}_\alpha(X) - \displaystyle\int \big[\phi(t) - (1+\theta)\psi(t)\big] d\mu \\ \text{s.t.} & \int \psi(t) d\mu \leq B, \end{cases} \tag{3.2.10}$$

where $\mathcal{M}^+$ denotes the set of all σ-finite positive measure on the measurable space $([0,\infty), \mathcal{B})$ such that $0 \leq \displaystyle\int (x-t)_+ d\mu \leq x$ for all $x \geq 0$.

3.3 Optimal Solutions

Since models (3.1.4) and (3.2.10) are equivalent regarding their solutions, it suffices for us to focus on model (3.2.10) for deriving the optimal ceded loss functions. To identify the solutions to (3.2.10) is the objective of the present section. We shall use the approximating approach, which is a routine approach regarding the optimization over a measure space. To proceed, let us first introduce some notation and define some functions as follows first:

$$\theta^* = \frac{1}{1+\theta}, \quad \delta_{\theta^*} = S_X^{-1}(\theta^*), \quad \delta_\alpha = S_X^{-1}(\alpha). \tag{3.3.11}$$

$$\beta(d) = \frac{B}{\int_d^\infty S_X(x)dx}, \quad (3.3.12)$$

$$\kappa(d) = d + (1+\theta)\int_d^\infty S_X(x)dx - \delta_\alpha, \quad (3.3.13)$$

$$\lambda(d) = \int_d^\infty S_X(x)dx + S_X(d)[d - \delta_\alpha]. \quad (3.3.14)$$

The following Lemma 3.3 collects some properties of functions $\kappa(\cdot)$ and $\lambda(\cdot)$ defined above. These properties will be used frequently in the subsequent discussion; see Section 3.5 for the its proof.

Lemma 3.3 *(a) The continuous function $\kappa(d)$ defined in (3.3.13) is convex for $d \geq 0$. Moreover, if $\theta^* < S_X(0)$, then $\kappa(d)$ is decreasing on $[0, \delta_{\theta^*}]$ while increasing on $[\delta_{\theta^*}, \infty)$ and satisfies*

$$\min_{0\leq d\leq a}\{\kappa(d)\} = \kappa(a) \;\; \textit{for} \;\; 0 \leq a \leq \delta_{\theta^*}, \quad (3.3.15)$$

$$\min_{0\leq d\leq a}\{\kappa(d)\} = \kappa(\delta_{\theta^*}) \;\; \textit{for} \;\; \delta_{\theta^*} \leq a; \quad (3.3.16)$$

if $\theta^* \geq S_X(0)$, then $\kappa(d)$ is increasing on $[0, \infty)$ and satisfies

$$\min_{0\leq d\leq a}\{\kappa(d)\} = \kappa(0) \;\; \textit{for} \;\; a \geq 0. \quad (3.3.17)$$

(b) The continuous function $\lambda(d)$ defined in (3.3.14) is strictly increasing on $[0, \delta_\alpha]$. Moreover, when $\alpha < \theta^*$, and $\lambda(\delta_{\theta^*}) < 0$, there exists a unique root $d = d_o$ to the equation $\lambda(d) = 0$ on $(\delta_{\theta^*}, \delta_\alpha)$.

The outline of our procedure for solving (3.2.10) is as follows. First of all, in Subsection 3.3.1 we construct a series of linear programming (3.3.19) which are optimization problems over a set of discrete measures with a particular structure, and then reformulate these programming into some equivalent models (3.3.23) which are optimization problems over Euclidean space. In Subsection 3.3.2, we solve models (3.3.23) with explicit solutions identified. The results show that the identified solutions are common for all of these models (3.3.23). The solutions of (3.3.19) are then

Case	Conditions	μ^*	$f^*(x)$	VaR^*_α
(1)	$\alpha \geq \theta^*$	0	0	δ_α
(2)	$\alpha < \theta^*$, $\kappa(\delta_{\theta^*}) > 0$	0	0	δ_α
(3)	$\alpha < \theta^*$, $\kappa(\delta_{\theta^*}) = 0$	$c^*\mathcal{X}(\delta_{\theta^*}, \cdot)$ where $0 \leq c^* \leq \min\{\beta(\delta_{\theta^*}), 1\}$	$c^*(x - \delta_{\theta^*})_+$, for $0 \leq c^* \leq \min\{\beta(\delta_{\theta^*}), 1\}$	δ_α
(4)	$\alpha < \theta^*$, $\kappa(\delta_{\theta^*}) < 0$, $\beta(\delta_{\theta^*}) > 1$	$\mathcal{X}(\delta_{\theta^*}, \cdot)$	$(x - \delta_{\theta^*})_+$	$\delta_\alpha + \kappa(\delta_{\theta^*})$
(5)	$\alpha < \theta^*$, $\kappa(\delta_{\theta^*}) < 0$, $\beta(\delta_{\theta^*}) \leq 1$, $\lambda(\delta_{\theta^*}) \geq 0$	$\beta(\delta_{\theta^*})\mathcal{X}(\delta_{\theta^*}, \cdot)$	$\beta(\delta_{\theta^*})(x - \delta_{\theta^*})_+$	$\delta_\alpha + \beta(\delta_{\theta^*}) \cdot \kappa(\delta_{\theta^*})$
(6)	$\alpha < \theta^*$, $\kappa(\delta_{\theta^*}) < 0$, $\beta(\delta_{\theta^*}) \leq 1$, $\lambda(\delta_{\theta^*}) < 0$, $\beta(d_o) \leq 1$	$\beta(d_o)\mathcal{X}(d_o, \cdot)$	$\beta(d_o)(x - d_o)_+$	$\delta_\alpha - B\big(\frac{1}{S_X(d_o)} - \frac{1}{\theta^*}\big)$
(7)	$\alpha < \theta^*$, $\kappa(\delta_{\theta^*}) < 0$, $\beta(\delta_{\theta^*}) \leq 1$, $\lambda(\delta_{\theta^*}) < 0$, $\beta(d_o) > 1$	$\mathcal{X}(d_B, \cdot)$ where $B = \int_{d_B}^{\infty} S_X(x)dx$	$(x - d_B)_+$, where $B = \int_{d_B}^{\infty} S_X(x)dx$	$\delta_\alpha + u(d_B)$

Table 3.1: Optimal ceded loss functions and minimal VaR.

derived by its equivalence to models (3.3.23). Finally, in Subsection 3.3.3 we show that these solutions also solve model (3.2.10), which are reported in Table 3.1[1].

3.3.1 Approximation Models

For integer $n \geq 1$, let $\mathcal{M}_n^+$ denote the set of all measures on $([0,\infty),\mathcal{B})$ with the following structure:

$$\mu_n(\cdot) = \sum_{j=1}^{n} c_{n,j}\mathcal{X}(d_{n,j},\cdot), \tag{3.3.18}$$

where the coefficients $c_{n,j} \geq 0$ and $d_{n,j} \geq 0$, $j = 1,\cdots,n$, $\sum_{j=1}^{n} c_{n,j} \leq 1$, and $\mathcal{X}(d_{n,j},\cdot)$ denote the Dirac measure concentrated on the point $d_{n,j}$. Without any loss of generality, we assume $0 \leq d_{n,1} \leq d_{n,2} \leq \cdots \leq d_{n,n}$ for all $n = 1,2,\cdots$. Note that $\mathcal{M}_n^+ \subset \mathcal{M}^+$ for all $n = 1,2,\cdots$. Then, we consider the following problems:

$$\begin{cases} \min\limits_{\mu_n \in \mathcal{M}_n^+} & \mathrm{VaR}_\alpha(X) - \displaystyle\int \Big[\phi(t) - (1+\theta)\psi(t)\Big] d\mu_n \\ \text{s.t.} & \int \psi(t) d\mu_n \leq B. \end{cases} \tag{3.3.19}$$

By defining coefficient vectors $\boldsymbol{c} := (c_{n,1},\ \cdots,\ c_{n,n})$ and $\boldsymbol{d} := (d_{n,1},\cdots,d_{n,n})$,

[1]This table reports the optimal solutions for all of the possible seven cases. Column 2 lists the conditions used to define each case. Column 3 gives the optimal measure to model (3.2.10) while Column 4 presents the corresponding optimal ceded loss function to model (3.1.4) for each case. Column 4 can be recovered from Column 3 by formula (3.2.7). Finally, Column 5 tabulates the minimal value of $\mathrm{VaR}_\alpha(T_f(X))$ obtained under the optimal solutions.

and using (3.3.18), the goal function in (3.3.19) can be expressed as follows.

$$
\begin{aligned}
\mathrm{VaR}_\alpha(\boldsymbol{c},\boldsymbol{d}) &:= \mathrm{VaR}_\alpha(X) - \int \Big[\phi(t) - (1+\theta)\psi(t)\Big] d\mu_n \\
&= \mathrm{VaR}_\alpha(X) - \sum_{j=1}^{n} c_{n,j}\Big[\phi(d_{n,j}) - (1+\theta)\psi(d_{n,j})\Big] \\
&= \begin{cases} \delta_\alpha + \Pi_{\mu_n}(X), & \delta_\alpha \le d_{n,1}, \\ A_{n,i}\,\delta_\alpha + B_{n,i} + \Pi_{\mu_n}(X), & d_{n,i} \le \delta_\alpha \le d_{n,i+1}, \\ & i = 1,\cdots,n-1, \\ A_{n,n}\,\delta_\alpha + B_{n,n} + \Pi_{\mu_n}(X), & d_{n,n} \le \delta_\alpha, \end{cases}
\end{aligned}
\tag{3.3.20}
$$

where

$$
A_{n,i} = 1 - \sum_{j=1}^{i} c_{n,j}, \qquad B_{n,i} = \sum_{j=1}^{i} c_{n,j}\, d_{n,j},\ \ i = 1,\cdots,n, \tag{3.3.21}
$$

and

$$
\Pi_{\mu_n}(X) = (1+\theta)\left\{\sum_{j=1}^{n} c_{n,j}\int_{d_{n,j}}^{\infty} S_X(x)dx\right\}. \tag{3.3.22}
$$

Similarly, the constraint in the optimization problem (3.3.19) becomes

$$
\sum_{j=1}^{n} c_{n,j}\int_{d_{n,j}}^{\infty} S_X(x)dx \le B.
$$

Note that the objective function and the constraint depend explicitly on the variables $\boldsymbol{c}$ and $\boldsymbol{d}$. This explains the notation $\mathrm{VaR}_\alpha(\boldsymbol{c},\boldsymbol{d})$ with the arguments $\boldsymbol{c}$ and $\boldsymbol{d}$.

Let us now introduce the following sets:

$$
\begin{aligned}
C_n &= \left\{(c_{n,1},...,c_{n,n}) \in \mathbb{R}^n : c_{n,j} \ge 0, j = 1,2,\cdots,n,\ \text{ and } \sum_{j=1}^{n} c_{n,j} \le 1\right\}, \\
D_n &= \{(d_{n,1},...,d_{n,n}) \in \mathbb{R}^n : 0 \le d_{n,1} \le \cdots \le d_{n,n}\}, \\
S_n &= \left\{(\boldsymbol{c},\boldsymbol{d}) : \boldsymbol{c} \in C_n,\ \ \boldsymbol{d} \in D_n,\ \sum_{j=1}^{n} c_{n,j}\int_{d_{n,j}}^{\infty} S_X(x)dx \le B\right\}.
\end{aligned}
$$

Then the coefficient vectors $(\boldsymbol{c}, \boldsymbol{d})$ defined in (3.3.18) must satisfy $\boldsymbol{c} \in C_n$ and $\boldsymbol{d} \in D_n$. Furthermore, the set S_n comprises both the feasible values of $\boldsymbol{c}$ and $\boldsymbol{d}$ as well as their constrained condition of problem (3.3.19). Consequently, the optimization problem (3.3.19) can be expressed more compactly as

$$\min_{(\boldsymbol{c},\boldsymbol{d})\in S_n} \mathrm{VaR}_\alpha(\boldsymbol{c}, \boldsymbol{d}). \tag{3.3.23}$$

It should be emphasized that the above formulation is a constrained optimization problem as the constraint on $\boldsymbol{c}$ and $\boldsymbol{d}$ is embedded in the definition of S_n. We make some remarks on two special cases with respect to the optimization problem (3.3.23). First is when $\boldsymbol{c} = \mathbf{0}$, where $\mathbf{0}$ is a zero vector $(0, \dots, 0)$. In this case, the objective function and the constraint in (3.3.23) are constant, independent of $\boldsymbol{d}$. When $\boldsymbol{c} = \mathbf{0}$ is the optimal solution to (3.3.23), then it is never optimal to reinsure the insurer's risk. Second, when $d_{n,j} = d$ for $j = 1, 2, \cdots, n$ and a constant d, both the objective function and the constraint in (3.3.23) depend only on d and $c := \sum_{j=1}^{n} c_{n,j}$. This implies that the optimization problem (3.3.23) reduces to a two-dimensional problem in terms of c and d, down from $2n$ dimensions. In the sequel, we might simply denote $(c, d) \in S$ for this situation for a set structured in the same way as S_n.

3.3.2 Solutions to the Approximation Models

As pointed out earlier that the VaR-based optimization model, in general, is a non-trivial problem; see Gaivoronski and Pflug (Winter 2004-2005). It is, therefore, difficult to obtain the global minimizer of the constrained optimization problem (3.3.23) directly. On the other hand, the fact that (3.3.23) is now formulated as an optimization problem over the Euclidean space suggests that we can derive the optimal solution via the following approach. To explain this approach, let us first note that for $n = 1, 2, \cdots$, the sets D_n and S_n can be partitioned, respectively, as

$$\begin{aligned} D_n^0 &= \{(d_{n,1}, ..., d_{n,n}) \in \mathbb{R}^n : \delta_\alpha \le d_{n,1} \le \cdots \le d_{n,n}\}, \\ D_n^i &= \{(d_{n,1}, ..., d_{n,n}) \in \mathbb{R}^n : 0 \le d_{n,1} \le \cdots \le d_{n,i} \le \delta_\alpha \le d_{n,i+1} \le \cdots \le d_{n,n}\}, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i = 1, ..., n-1, \\ D_n^n &= \{(d_{n,1}, ..., d_{n,n}) \in \mathbb{R}^n : 0 \le d_{n,1} \le \cdots \le d_{n,n} \le \delta_\alpha\}, \end{aligned}$$

and

$$S_n^i = \left\{(\boldsymbol{c}, \boldsymbol{d}) : \boldsymbol{c} \in C_n, \ \ \boldsymbol{d} \in D_n^i, \ \sum_{j=1}^n c_{n,j} \int_{d_{n,j}}^\infty S_X(x)dx \le B\right\}, \ \ i = 0, 1, 2, \cdots, n.$$

In other words, we have $D_n = \bigcup_{i=0}^n D_n^i$ and $S_n = \bigcup_{i=0}^n S_n^i$. The partition of S_n into $S_n^i, i = 1, \ldots, n$ enables us to analyze, case by case, the solution to (3.3.23) with the feasible set replaced by $S_n^i, i = 0, \ldots, n$. The global solution to (3.3.23) over the feasible set S_n is then given by the partition that yields the lowest VaR of the insurer's total risk among all partitions $S_n^i, i = 0, \ldots, n$. More specifically, let $\text{VaR}_\alpha^*(S)$ denote the minimum value of $\text{VaR}_\alpha(\boldsymbol{c}, \boldsymbol{d})$ for $(\boldsymbol{c}, \boldsymbol{d})$ over the feasible set S and let $(\boldsymbol{c}^*, \boldsymbol{d}^*) \in S$ be the corresponding optimal vectors for which the minimum is attained. Adopting this notation, $\text{VaR}_\alpha^*(S_n)$ with optimal vector $(\boldsymbol{c}^*, \boldsymbol{d}^*) \in S_n$ is the optimal solution to (3.3.23). The argument provided above implies that the minimum value $\text{VaR}_\alpha^*(S_n)$ can be obtained indirectly via

$$\text{VaR}_\alpha^*(S_n) = \min\{\text{VaR}_\alpha^*(S_n^0), \text{VaR}_\alpha^*(S_n^1), \cdots, \text{VaR}_\alpha^*(S_n^n)\}, \tag{3.3.24}$$

with the optimal vector $(\boldsymbol{c}^*, \boldsymbol{d}^*)$ corresponding to the partition that yields the lowest VaR.

The rest of this section is devoted to analyzing the minimum value of $\text{VaR}_\alpha(\boldsymbol{c}, \boldsymbol{d})$ for $(\boldsymbol{c}, \boldsymbol{d})$ over feasible set S_n^i, $i = 0, 1, 2, \cdots, n$. It turns out that the optimality associated with the first n cases (i.e. for feasible set $S_n^i, i = 0, \ldots, n-1$) is relatively straightforward to determine, as we demonstrate in Proposition 3.1. The optimal

$\mathrm{VaR}^*_\alpha(S^n_n)$, on the other hand, is more complicated and it requires us to consider several additional subcases, as we will later elaborate.

Before presenting some key results, it is useful to give the following explicit expressions for $\mathrm{VaR}_\alpha(\boldsymbol{c}, \boldsymbol{d})$ pertaining a given confidence level $1-\alpha$ with $0 < \alpha < S_X(0)$:

(i) When $\delta_\alpha \le d_{n,1}$, i.e., $(d_{n,1}, ..., d_{n,n}) \in D^0_n$,

$$\mathrm{VaR}_\alpha(\boldsymbol{c}, \boldsymbol{d}) = \delta_\alpha + (1+\theta)\Big\{\sum_{j=1}^n c_{n,j} \int_{d_{n,j}}^\infty S_X(x)dx\Big\}; \tag{3.3.25}$$

(ii) When $d_{n,i} \le \delta_\alpha \le d_{n,i+1}$, i.e., $(d_{n,1}, ..., d_{n,n}) \in D^i_n$, for $i = 1, \cdots, n-1$,

$$\mathrm{VaR}_\alpha(\boldsymbol{c}, \boldsymbol{d}) = \delta_\alpha + \sum_{j=1}^i c_{n,j}[d_{n,j} - \delta_\alpha] + (1+\theta)\Big\{\sum_{j=1}^n c_{n,j} \int_{d_{n,j}}^\infty S_X(x)dx\Big\}; \tag{3.3.26}$$

(iii) When $d_{n,n} \le \delta_\alpha$, i.e., $(d_{n,1}, ..., d_{n,n}) \in D^n_n$,

$$\mathrm{VaR}_\alpha(\boldsymbol{c}, \boldsymbol{d}) = \delta_\alpha + \sum_{j=1}^n c_{n,j}\kappa(d_{n,j}). \tag{3.3.27}$$

We now present the following proposition, which summarizes the optimality for the first n partitions of S_n:

Proposition 3.1 *(a) On S^0_n, $\boldsymbol{c}^* = \mathbf{0}$ is one optimal solution of $VaR^*_\alpha(S^0_n)$ with optimal minimum value $VaR^*_\alpha(S^0_n) = \delta_\alpha$.*

(b) On S^i_n, $i = 1, 2, \cdots, n-1$, the optimal solutions $(\boldsymbol{c}^*, \boldsymbol{d}^*)$ of $VaR^*_\alpha(S^i_n)$ must satisfy either $d^*_{n,j} \to \infty$ for $j = i+1, \cdots, n$ or equivalently $c^*_{n,j} = 0$ for $j = i+1, \cdots, n$.

Proof. The proof is trivial by the expressions of $\mathrm{VaR}_\alpha(\boldsymbol{c}, \boldsymbol{d})$ in (3.3.25) and (3.3.26). □

What remains is to consider the optimal solution on the final partition S^n_n. As alluded earlier that the optimality associated with S^n_n is more complicated to analyze. It entails us to sub-partitioning the feasible set S_n^n into a few more subcases depending on the relative magnitude of α and θ^*, the sign of $\kappa(\delta_{\theta^*})$, and whether $\beta(\delta_{\theta^*})$ is greater or smaller than one. More specifically, there are seven subcases in total to be considered and these are listed below. A flowchart of these subcases is depicted in Figure 3.1. For ease of referencing, we also include the respective proportions that deal with each of these subcases.

Case (i): $\alpha \geq \theta^*$, see Proposition 3.2(a).

Case (ii): $\alpha < \theta^*$, $\kappa(\delta_{\theta^*}) > 0$, see Proposition 3.2(b).

Case (iii): $\alpha < \theta^*$, $\kappa(\delta_{\theta^*}) = 0$, see Proposition 3.3(a).

Case (iv): $\alpha < \theta^*$, $\kappa(\delta_{\theta^*}) < 0$, and $\beta(\delta_{\theta^*}) > 1$, see Proposition 3.3(b).

Case (v): $\alpha < \theta^*$, $\kappa(\delta_{\theta^*}) < 0$, $\beta(\delta_{\theta^*}) \leq 1$, and $\lambda(\delta_{\theta^*}) \geq 0$, see Proposition 3.4(a).

Case (vi): $\alpha < \theta^*$, $\kappa(\delta_{\theta^*}) < 0$, $\beta(\delta_{\theta^*}) \leq 1$, $\lambda(\delta_{\theta^*}) < 0$, and $\beta(d_o) \leq 1$, see Proposition 3.4b(i).

Case (vii): $\alpha < \theta^*$, $\kappa(\delta_{\theta^*}) < 0$, $\beta(\delta_{\theta^*}) \leq 1$, $\lambda(\delta_{\theta^*}) < 0$, and $\beta(d_o) > 1$, see Proposition 3.4b(ii).

We first present the trivial cases where it is never optimal for the insurer to reinsure its risk. These correspond to cases (i) $\alpha \geq \theta^*$ and (ii) $\alpha < \theta^*$ with $\kappa(\delta_{\theta^*}) > 0$, as we show in the following Proposition 3.2.

Proposition 3.2 *Consider minimizing* $VaR_\alpha(\boldsymbol{c}, \boldsymbol{d})$ *with feasible set* S_n^n. *When (i)* $\alpha \geq \theta^*$ *or (ii)* $\alpha < \theta^*$ *and* $\kappa(\delta_{\theta^*}) > 0$, $\boldsymbol{c}^* = \boldsymbol{0}$ *is one solution with* $VaR_\alpha^*(S_n^n) = \delta_\alpha$.

Proof. The condition $\alpha \geq \theta^*$ implies $\delta_{\theta^*} \geq \delta_\alpha$ and using part (a) of Lemma 3.3, we have

$$\min_{d \in [0, \delta_\alpha]} \kappa(d) = u(\delta_\alpha) = (1+\theta) \int_{\delta_\alpha}^{\infty} S_X(x)dx > 0.$$

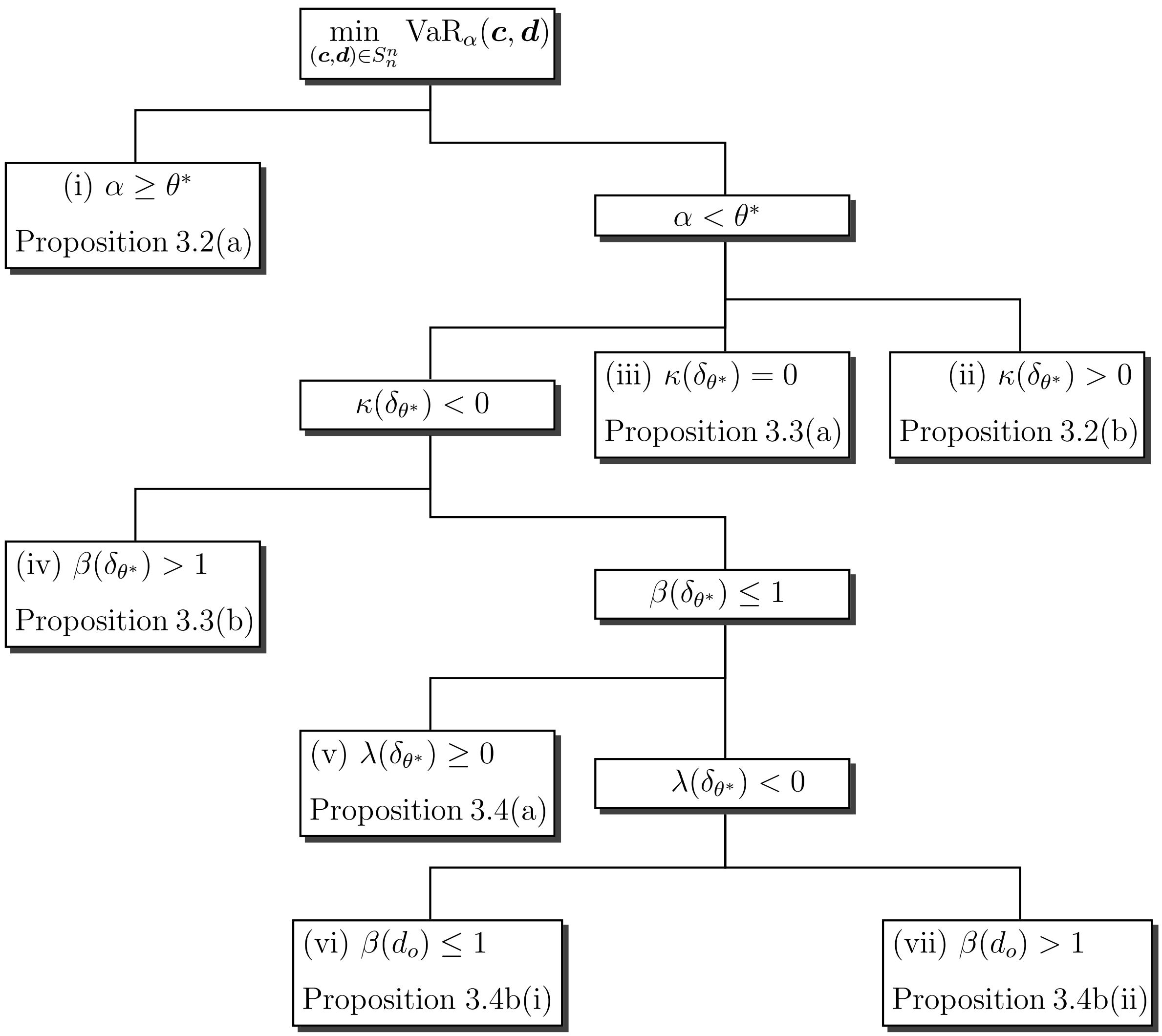


Figure 3.1: Subcases of S_n^n.

Thus, $\kappa(d) > 0$ for all $d \in [0, \delta_\alpha]$. Since on S_n^n we have $d_{n,j} \leq \delta_\alpha$ for $j = 1, 2, \cdots, n$, it follows from (3.3.27) that $\mathrm{VaR}_\alpha(\boldsymbol{c}, \boldsymbol{d})$ attains its minimum value when $\boldsymbol{c} = \boldsymbol{0}$. Hence $\mathrm{VaR}_\alpha^*(S_n^n) = \delta_\alpha$. The second case with $\alpha < \theta^*$ and $\kappa(\delta_{\theta^*}) > 0$ can be proved similarly since from Lemma 3.3(a) we have

$$\min_{d \in [0, \delta_\alpha]} \kappa(d) = \kappa(\delta_{\theta^*}) > 0.$$

□

To discuss the optimal solutions corresponding to Cases (iii) and (iv), it is essential to introduce the following sets:

$$\begin{aligned}
T_n^n &:= \{(\boldsymbol{c}, \boldsymbol{d}) : (\boldsymbol{c}, \boldsymbol{d}) \in S_n^n, \;\; d_{n,1} \geq \delta_{\theta^*}\}, \\
T_n^{\geq} &:= \Big\{(\boldsymbol{c}, \boldsymbol{d}) : (\boldsymbol{c}, \boldsymbol{d}) \in T_n^n, \;\; \sum_{j=1}^{n} c_{n,j} \geq \beta(\delta_{\theta^*})\Big\}, \\
T_n^{\leq} &:= \Big\{(\boldsymbol{c}, \boldsymbol{d}) : (\boldsymbol{c}, \boldsymbol{d}) \in T_n^n, \;\; \sum_{j=1}^{n} c_{n,j} \leq \beta(\delta_{\theta^*})\Big\}.
\end{aligned}$$

Recall that $\beta(\delta_{\theta^*})$ was defined through (3.3.11) and (3.3.12), and that $T_n^{\geq}$ is an empty set when $\beta(\delta_{\theta^*}) > 1$. Furthermore, $T_n^{\geq}$ and $T_n^{\leq}$ are the partitioned sets of T_n^n, i.e. $T_n^n = T_n^{\geq} \bigcup T_n^{\leq}$. Exploiting the partitioning, Lemma 3.4 in Section 3.5 establishes the following relation for $\alpha < \theta^*$:

$$\mathrm{VaR}_\alpha^*(S_n^n) = \mathrm{VaR}_\alpha^*(T_n^n) \tag{3.3.28}$$

The above result is useful in the sense that under the prescribed condition, for identifying solution on S_n^n it is sufficient to focus the optimality on T_n^n. This relation is used explicitly in deriving the optimal solutions for cases (iii) and (iv), as we show in the following proposition:

Proposition 3.3 *Consider minimizing $VaR_\alpha(\boldsymbol{c}, \boldsymbol{d})$ with feasible set S_n^n.*

(a) When $\alpha < \theta^*$ and $\kappa(\delta_{\theta^*}) = 0$, (c^*, δ_{θ^*}) such that $0 \leq c^* \leq \min\{\beta(\delta_{\theta^*}), 1\}$ is one solution with $VaR_\alpha^*(S_n^n) = \delta_\alpha$.

(b) *When* $\alpha < \theta^*$, $\kappa(\delta_{\theta^*}) < 0$ *and* $\beta(\delta_{\theta^*}) > 1$, $(1, \delta_{\theta^*})$ *is one solution with* $VaR^*_\alpha(S^n_n) = \delta_\alpha + \kappa(\delta_{\theta^*}) < \delta_\alpha$.

Proof. Lemma 3.4 shows that $\mathrm{VaR}^*_\alpha(S^n_n) = \mathrm{VaR}^*_\alpha(T^n_n)$ and for any vector $(\boldsymbol{c}, \boldsymbol{d}) \in T^n_n$,

$$\mathrm{VaR}_\alpha(\boldsymbol{c}, \boldsymbol{d}) = \delta_\alpha + \sum_{j=1}^{n} c_{n,j}\kappa(d_{n,j}) \geq \delta_\alpha + \kappa(\delta_{\theta^*}) \sum_{j=1}^{n} c_{n,j}. \tag{3.3.29}$$

(a) When $\alpha < \theta^*$ and $\kappa(\delta_{\theta^*}) = 0$, then (3.3.29) becomes $\mathrm{VaR}_\alpha(\boldsymbol{c}, \boldsymbol{d}) \geq \delta_\alpha$. The lower bound corresponds to $\mathrm{VaR}_\alpha(c^*, \delta_{\theta^*})$ with c^* as defined in the proposition, and the upper bound restriction on c^* ensures that $(c^*, \delta_{\theta^*}) \in T^n_n$; hence the results follow.

(b) To establish the optimality for $\alpha < \theta^*$, $\kappa(\delta_{\theta^*}) < 0$ and $\beta(\delta_{\theta^*}) > 1$, first recall that $T^n_n = T^{\geq}_n \bigcup T^{\leq}_n$ and $T^{\geq}_n$ is an empty set when $\beta(\delta_{\theta^*}) > 1$. Hence we only need to show that $(1, \delta_{\theta^*})$ is optimal over $T^{\leq}_n$. It follows from the condition $\kappa(\delta_{\theta^*}) < 0$ that the lower bound (3.3.29) can further be reduced to $\delta_\alpha + \kappa(\delta_{\theta^*})$, which is equal to $\mathrm{VaR}_\alpha(1, \delta_{\theta^*})$. Since $\beta(\delta_{\theta^*}) > 1$, $(1, \delta_{\theta^*})$ is obviously within $T^{\leq}_n$ and hence we obtain the required results. $\square$

To analyze the remaining three cases pertaining to conditions $\alpha < \theta^*$, $\kappa(\delta_{\theta^*}) < 0$, and $\beta(\delta_{\theta^*}) \leq 1$, we again employ the same approach as above except that in these cases, we consider the following set:

$$V := \left\{(c, d) \in \mathbb{R} : \ \delta_{\theta^*} \leq d \leq \delta_\alpha, \ \ c\int_d^\infty S_X(x)dx = B, \ \ 0 \leq c \leq 1\right\}. \tag{3.3.30}$$

Note that V is well defined since $\delta_{\theta^*} \leq \delta_\alpha$ due to the condition $\alpha < \theta^*$. Lemma 3.7 in Section 3.5 similarly shows that

$$\mathrm{VaR}^*_\alpha(S^n_n) = \mathrm{VaR}^*_\alpha(V) \tag{3.3.31}$$

and therefore it is sufficient to just focus on set V for the optimality on set S^n_n. This is demonstrated in the following proposition and hence completes the analysis for Cases (v), (vi) and (vii):

Proposition 3.4 *Suppose $\alpha < \theta^*$, $\kappa(\delta_{\theta^*}) < 0$, and $\beta(\delta_{\theta^*}) \leq 1$, and consider minimizing $VaR_\alpha(c,d)$ with feasible set S_n^n.*

(a) If $\lambda(\delta_{\theta^*}) \geq 0$, then $(\beta(\delta_{\theta^*}), \delta_{\theta^*})$ is one solution with $VaR_\alpha^*(S_n^n) = \delta_\alpha + \beta(\delta_{\theta^*}) \cdot \kappa(\delta_{\theta^*}) < \delta_\alpha$.

(b) Suppose $\lambda(\delta_{\theta^*}) < 0$.

(i) If $\beta(d_o) \leq 1$, then $(\beta(d_o), d_o)$ is one solution with $VaR_\alpha^*(S_n^n) = \delta_\alpha - B[\frac{1}{S_X(d_o)} - \frac{1}{\theta^*}] < \delta_\alpha$, where as defined in Lemma 3.3(b), $d_o \in (\delta_{\theta^*}, \delta_\alpha)$ satisfies $\lambda(d_o) = 0$.

(ii) If $\beta(d_o) > 1$, then $(1, d_B)$ is one solution with $VaR_\alpha^*(S_n^n) = \delta_\alpha + \kappa(d_B) < \delta_\alpha$, where d_B is determined through the equation

$$B = \int_{d_B}^{\infty} S_X(x)dx.$$

Proof. First note that V can be regarded as a subset of S_n^n, since any coefficient pair $(c,d) \in V$ will lead to the same VaR value as a vector $(\boldsymbol{c}, \boldsymbol{d}) \in S_n^n$ with $d_j = d$ and $c_j = c/n$ for $j = 1, 2, \cdots, n$. Thus, relation (3.3.31) implies that it is sufficient for us to focus on set V for the optimal solutions on S_n^n.

For any coefficient pair $(c,d) \in V$, it follows from (3.3.27) that

$$\begin{aligned} \mathrm{VaR}_\alpha(c,d) &= \delta_\alpha + c(d - \delta_\alpha) + (1+\theta)c \int_d^{\infty} S_X(x)dx \\ &= \delta_\alpha + (1+\theta)B + \frac{d - \delta_\alpha}{\int_d^{\infty} S_X(x)dx} B. \end{aligned} \tag{3.3.32}$$

Because of the condition $c\int_d^{\infty} S_X(x)dx = B$, the coefficient c uniquely determines d (and vice-versa) and hence $\mathrm{VaR}_\alpha(c,d)$ can be regarded as either a function of c or d. For the sake of our analysis, we will express $\mathrm{VaR}_\alpha(c,d)$ as a function d as we have shown in (3.3.32) and we will denote it by $\Gamma(d)$. The derivative of $\Gamma(d)$ with respect to d is

$$\Gamma'(d) = \frac{\lambda(d)}{[\int_d^{\infty} S_X(x)dx]^2} B. \tag{3.3.33}$$

(a) When $\lambda(\delta_{\theta^*}) \geq 0$, then $\Gamma'(d) > 0$ on $(\delta_{\theta^*}, \delta_\alpha]$ since $\lambda(d)$ is strictly increasing on $[0, \delta_\alpha]$ by Lemma 3.3(b). Hence $\Gamma(d)$ is strictly increasing on $[\delta_{\theta^*}, \delta_\alpha]$ and it attains its minimum value at $d = \delta_{\theta^*}$. This implies that $\mathrm{VaR}^*_\alpha(V) = \delta_\alpha + \beta(\delta_{\theta^*}) \cdot \kappa(\delta_{\theta^*}) < \delta_\alpha$. The last strict inequality is due to $\kappa(\delta_{\theta^*}) < 0$.

(b) Under the assumption $\lambda(\delta_{\theta^*}) < 0$, Lemma 3.3(b) assures that $\Gamma'(d) < 0$ for $d \in [\delta_{\theta^*}, d_o)$, and $\Gamma'(d) > 0$ for $d \in (d_o, \delta_\alpha]$. This in turn implies $\Gamma(d)$ is strictly decreasing on $[\delta_{\theta^*}, d_o]$ while strictly increasing on $[d_o, \delta_\alpha]$, and it attains the minimum value at $d = d_o$.

Recall that any coefficient pair $(c, d) \in V$ must satisfy the constraint

$$c \int_d^\infty S_X(x)dx = B$$

and $0 \leq c \leq 1$. Hence when $\beta(d_o) \leq 1$, $(\beta(d_o), d_o)$ is one optimal solution on V. Moreover because the minimum value of $\Gamma(d)$ is attained at $d = d_o$, we have

$$\mathrm{VaR}^*_\alpha(V) = \Gamma(d_o) < \Gamma(\delta_{\theta^*}) = \delta_\alpha + \beta(\delta_{\theta^*}) \cdot \kappa(\delta_{\theta^*}) < \delta_\alpha, \qquad (3.3.34)$$

and $\mathrm{VaR}^*_\alpha(V)$ is easily shown to be

$$\begin{aligned} \mathrm{VaR}^*_\alpha(V) &= \delta_\alpha + (1+\theta)B + B \cdot \frac{d_o - \delta_\alpha}{\int_{d_o}^\infty S_X(x)dx} \\ &= \delta_\alpha - B\left[\frac{1}{S_X(d_o)} - \frac{1}{\theta^*}\right], \qquad (3.3.35) \end{aligned}$$

as claimed in Part (i).

For Part (ii) with the condition $\beta(d_o) > 1$, first note that $(\beta(d_o), d_o)$ is no longer in V. Second, the property that $\Gamma(d)$ is strictly decreasing on $[\delta_{\theta^*}, d_o]$ while strictly increasing on $[d_o, \delta_\alpha]$ ensures that the minimum of $\mathrm{VaR}_\alpha(c, d)$ over V while subject to the constraint $0 < c \leq 1$ must occur at d_B. Finally, it is also obvious that $\mathrm{VaR}^*_\alpha(V) = \Gamma(d_B) \leq \Gamma(\delta_{\theta^*}) < \delta_\alpha$ and this completes the proof. □

Case	Conditions	$(\boldsymbol{c}^*, \boldsymbol{d}^*)$	VaR_α^*
(i)	$\alpha \geq \theta^*$	$(0, d)$, $d \geq 0$	δ_α
(ii)	$\alpha < \theta^*$,$\kappa(\delta_{\theta^*}) > 0$	$(0, d)$, $d \geq 0$	δ_α
(iii)	$\alpha < \theta^*$, $\kappa(\delta_{\theta^*}) = 0$	$(c^*, \delta_{\theta^*})_+$, for $0 \leq c^* \leq \min\{\beta(\delta_{\theta^*}), 1\}$	δ_α
(iv)	$\alpha < \theta^*$, $\kappa(\delta_{\theta^*}) < 0$, $\beta(\delta_{\theta^*}) > 1$	$(1, \delta_{\theta^*})$	$\delta_\alpha + \kappa(\delta_{\theta^*})$
(v)	$\alpha < \theta^*$, $\kappa(\delta_{\theta^*}) < 0$, $\beta(\delta_{\theta^*}) \leq 1$, $\lambda(\delta_{\theta^*}) \geq 0$	$(\beta(\delta_{\theta^*}), \delta_{\theta^*})$	$\delta_\alpha + \beta(\delta_{\theta^*}) \cdot \kappa(\delta_{\theta^*})$
(vi)	$\alpha < \theta^*$, $\kappa(\delta_{\theta^*}) < 0$, $\beta(\delta_{\theta^*}) \leq 1$, $\lambda(\delta_{\theta^*}) < 0$, $\beta(d_o) \leq 1$	$(\beta(d_o), d_o)$	$\delta_\alpha - B\left(\frac{1}{S_X(d_o)} - \frac{1}{\theta^*}\right)$
(vii)	$\alpha < \theta^*$, $\kappa(\delta_{\theta^*}) < 0$, $\beta(\delta_{\theta^*}) \leq 1$, $\lambda(\delta_{\theta^*}) < 0$, $\beta(d_o) > 1$	$(1, d_B)$, where $B = \int_{d_B}^{\infty} S_X(x)dx$	$\delta_\alpha + u(d_B)$

Table 3.2: Optimal ceded loss functions to the approximation models.

Now we are ready to present the solution to models (3.3.23) and (3.3.19). Recall that $S_n = \bigcup_{i=0}^{n} S_n^i$, S_n is the feasible set to (3.3.23), and Propositions 3.1-3.4 hold for any $n = 1, 2, \cdots$. It follows from Proposition 3.1(b) that $(\boldsymbol{c}_n^*, \boldsymbol{d}_n^*)$ is one solution to problem (3.3.23) if the following condition is satisfied:

$$\mathrm{VaR}_\alpha\{\boldsymbol{c}_n^*, \boldsymbol{d}_n^*\} = \min\left(\mathrm{VaR}_\alpha^*(S_n^0),\ \mathrm{VaR}_\alpha^*(S_n^n)\right) \qquad n = 1, 2, \cdots . \tag{3.3.36}$$

By Propositions 3.1(a), 3.4, and 3.3, we deduce that (c^*, δ_{θ^*}) such that $0 \le c^* \le \min\{\beta(\delta_{\theta^*}), 1\}$ are solutions to problem (3.3.23) when $\alpha < \theta^*$ and $\kappa(\delta_{\theta^*}) = 0$, and that $(1, \delta_{\theta^*})$ is one solution to (3.3.23) when $\alpha < \theta^*$, $\kappa(\delta_{\theta^*}) < 0$ and $\beta(\delta_{\theta^*}) > 1$. Moreover, after comparing $\mathrm{VaR}_\alpha^*(V)$ in Proposition 3.4 with $\mathrm{VaR}_\alpha^*(S_n^0)$ in Proposition 3.1, we conclude that the solutions over S_n^n solve problem (3.3.23) when $\alpha < \theta^*$, $\kappa(\delta_{\theta^*}) < 0$ and $\beta(\delta_{\theta^*}) \le 1$. For the optimal solutions in the remaining cases, we only need to compare Proposition 3.2 with Proposition 3.1. The results, together with the corresponding propositions, are summarized in Table 3.2, where $(\boldsymbol{c}^*, \boldsymbol{d}^*)$ is the solution to model (3.3.23). Note that the solution is independent of the dimension n. Then by formula (3.3.18) we derive the solution to (3.3.19), which is also independent of dimension n and is reported by the column entitled μ^* in Table 3.1.

3.3.3 Optimal Solutions to VaR Minimization Model

This subsection serves to show that the solution derived in the previous subsection for models (3.3.19) also solves model (3.2.10). This is proved as in the following proposition.

Proposition 3.5 *The solution μ^* summarized in Table 3.1 also solves model (3.2.10).*

Proof. Recall that μ^* in Table 3.1 solves models (3.3.19) for all $n = 1, 2, \cdots$. Let μ be any positive measure from the feasible set of problem (3.2.10), i.e., $\mu \in \mathcal{M}^+$

and $\int \gamma(t)d\mu \le B$. We need to show $\mathrm{VaR}_\alpha(\mu^*) \le \mathrm{VaR}_\alpha(\mu)$. Before we proceed, it might be helpful for us to recall the notation $\phi(t) = (\mathrm{VaR}_\alpha(X) - t)_+$ and $\psi(t) = \mathsf{E}[(X-t)_+]$.

By Lemma 3.8, there exists a sequence of measures $\{\mu_n, n = 1, 2, \cdots\}$ in $\mathcal{M}_n^+$ such that $\int (x-t)_+ d\mu_n$ converges pointwisely to $\int (x-t)_+ d\mu$ from below. This fact has two implications, which will end up the completeness of the proof. On the one hand, we have $\int (x-t)_+ d\mu_n \le B$ and hence $\int \psi(t) d\mu_n \le B$ by Fubini's Theorem for each $n = 1, 2, \cdots$. This implies that μ_n belongs to the feasible set of problem (3.3.19) for each $n = 1, 2, \cdots$, and consequently we have

$$\mathrm{VaR}_\alpha(\mu^*) \le \mathrm{VaR}_\alpha(\mu_n) \ \text{ for } \ n = 1, 2, \cdots . \tag{3.3.37}$$

On the other hand, by Fubini's Theorem and Lemma 3.8 we have

$$\begin{aligned} \int \phi(t) d\mu_n &= \int \left[\mathrm{VaR}_\alpha(X) - t\right]_+ d\mu_n \\ &\to \int \left[\mathrm{VaR}_\alpha(X) - t\right]_+ d\mu \\ &= \int \phi(t) d\mu_n \end{aligned}$$

and

$$\begin{aligned} \int \psi(t) d\mu_n &= \int \mathsf{E}\,(X-t)_+ \, d\mu_n \\ &= \mathsf{E}\left[\int (X-t)_+ \, d\mu_n\right] \\ &\to \mathsf{E}\left[\int (X-t)_+ \, d\mu\right] \\ &= \int \mathsf{E}\,(X-t)_+ \, d\mu \\ &= \int \psi(t) d\mu, \end{aligned}$$

where the first convergence result is due to Lemma 3.8 and the second convergence result is the combination of Lemma 3.8 and monotonic convergence theorem. These results in turn imply

$$\begin{aligned} \mathrm{VaR}_\alpha(\mu_n) &= \mathrm{VaR}_\alpha(X) - \int [\phi(t) - (1+\theta)\psi(t)]\, d\mu_n \\ &\to \mathrm{VaR}_\alpha(X) - \int [\phi(t) - (1+\theta)\psi(t)]\, d\mu \\ &= \mathrm{VaR}_\alpha(\mu). \end{aligned} \tag{3.3.38}$$

Finally, by combining (3.3.37) and (3.3.38), we immediately have $\mathrm{VaR}_\alpha(\mu^*) \leq \mathrm{VaR}_\alpha(\mu)$, by which the proof is complete. □

3.4 Some Remarks and Examples

As pointed out earlier that Cai et al. (2008) discusses the optimal reinsurance by unconstrained reinsurance models which are only concerned on minimizing certain risk measure of the insurer's total risk exposure. The approach described in this chapter, on the other hand, is a generalization of the VaR minimizing model in Cai et al. (2008) in the sense that we impose a new constraint in addition to the usual objective criterion for determining the optimal reinsurance. This new constraint can be interpreted as either a reinsurance premium budget or a profitability guarantee for the insurer. The present model is intuitively more appealing since it takes into account both risk and reward. We now make the following remarks to compare and contrast the results obtained in this paper with what derived in Cai et al. (2008) regarding the VaR minimization model.

Remark 3.1 *Except for the first two cases, the optimal ceded loss functions presented in Table 3.1 are all in the forms of stop-loss types. In fact, under certain conditions they will reduce to the quota-share treaties. For example, if $\theta^* > S(0)$, then $\delta_{\theta^*} \equiv S_X^{-1}(\theta^*) = 0$, which implies that the optimal ceded loss functions in Cases (3), (4), and (5) collapse to the quota-share type.*

Remark 3.2 *Recall that when $B > \mathsf{E}[X]$, the reinsurance premium budget constraint in the proposed constrained optimal reinsurance model (3.1.4) will have no impact on the solution. Consequently, this special case reduces to the unconstrained reinsurance model in Cai et al. (2008) for the VaR criterion. We can also exam the range of the insurer's expected profit P corresponding to this special case. Note that when $B > \mathsf{E}[X]$, the inequality $B > \int_{\delta_{\theta^*}}^{\infty} S_X(x)dx$ (or equivalently $\beta(\delta_{\theta^*}) > 1$) holds trivially. Furthermore, the condition $B > \int_{\delta_{\theta^*}}^{\infty} S_X(x)dx$ implies*

$$P < p_0 - \mathsf{E}[X] - \theta \int_{\delta_{\theta^*}}^{\infty} S_X(x)dx. \tag{3.4.39}$$

Hence if the expected profit P of the insurer is less than the quantity on the right-hand-side of the above inequality, the profitability constraint becomes redundant. In fact, in this situation, Case (4) of Table 3.1 recovers parts (a) and (c) of Theorem 3.1 in Cai et al. (2008), while Case (3) of Table 3.1 is equivalent to parts (b) and (d) of Theorem 3.1 in Cai et al. (2008).

Remark 3.3 *To understand the impact of imposing the profitability constraint on the optimal reinsurance model, let us, say, compare Case (4) to Case (5) of Table 3.1 and assuming $\lambda(\delta_{\theta^*}) \geq 0$. When an insurer becomes more aggressive so that it requires an expected profit greater than the quantity on the right-hand-side of the inequality (3.4.39) (i.e. $\beta(\delta_{\theta^*}) < 1$), the optimal ceded loss function is $f^*(x) = \beta(\delta_{\theta^*})(x - \delta_{\theta^*})_+$ with $VaR^*_\alpha = \delta_\alpha + \beta(\delta_{\theta^*}) \cdot \kappa(\delta_{\theta^*})$. A contrast of these results to the unconstrained model as in Case (4) of Table 3.1 imply that in the presence of the profitability constraint, the optimal reinsurance design is to retain greater losses while expose to a higher minimum attainable VaR^*_α. This is consistent with the classical risk and reward tradeoff.*

To conclude this section, we provide two examples to illustrate our results.

Example 3.1 *Assume X is exponentially distributed with mean $\mathsf{E}[X] = 1{,}000$. Then $S_X(x) = e^{-0.001x}$, $x \geq 0$ and $S_X(0) = 1$. Assume further that the loading* *factors for the reinsurer and insurer are 20% and 15%, respectively. This implies* $\theta = 0.2$ *and* $\delta_{\theta^*} = S_X^{-1}(1/(1+\theta)) = 182.32$*. Under the additional assumption of the expectation premium principle, we have* $p_0 = 1.15E[X] = 1,150$ *and* $\Pi(f(X)) = 1.2E[f(X)]$ *for a given ceded loss function* f*. In practice it is to be expected that the loading factor for the reinsurer is higher than the insurer's. Consequently, the achievable expected profits* P *are in the range* $[0, 150]$ *so that* $B \in [0, 750]$*. Table 3.3 reports the* VaR^*_α *and the corresponding optimal ceded loss function (as specified by c and d) for different combinations of* $P \in \{148, 145, 140, 100, 50, 0\}$ *and* $\alpha \in \{1\%, 5\%, 10\%\}$*. For these examples, conditions* $\alpha < \theta^*, \kappa(\delta_{\theta^*}) < 0$ *and* $\beta(\delta_{\theta^*}) \leq 1$ *are satisfied and hence Proposition 3.4 is used to determine the optimal solutions. For example, at* $\alpha = 10\%$ *the root of* $\lambda(d)$ *occurs at* $d_o = 1302.6$ *with* $\beta(d_o)$ *depending on the level of B; see equation (3.3.12). If we were to guarantee an expected profit of $145 (or equivalently* $B = 25$*), then the optimal ceded loss function in the class* $\mathcal{IC}$ *is a combination of quota-share and stop-loss reinsurance given by* $f(x) = c(x - d)_+$ *where* $c = \beta(d_o) = 0.09$*,* $d = d_o = 1,302.6$ *and with minimum attainable VaR $2,240.6.*

Note that when we increase the confidence level $1 - \alpha$*, the minimum VaR, the optimal values of c and d become larger as long as* $\beta(d_o) \leq 1$*. This implies that the higher level of confidence can be achieved at the expense of higher minimum VaR. Furthermore, the optimal reinsurance contract and the minimum attainable VaR are invariant to* α *as long as* $\beta(d_o) > 1$*.*

The impact of the expected profit P *(or equivalently* B*) on optimal reinsurance is also clearly demonstrated. First, if we were to decrease the minimum level of expected profits, the optimal retention d does not change as long as* $\beta(d_o) \leq 1$*. The optimal c, however, will increase accordingly as asserted by part (i) of Proposition 3.4(b) and also confirmed by our numerical results. Second, when the condition* $\beta(d_o) > 1$ *is satisfied as we further decrease* P*, the optimal reinsurance design becomes a pure stop-loss contract with the optimal retention d that also declines with*

B	P	return	$\alpha=1\%$	$\alpha=5\%$	$\alpha=10\%$
			VaR^*_α (c,d)	VaR^*_α (c,d)	VaR^*_α (c,d)
10	148	12.87%	4249.3 (0.37,3605.2)	2934.2 (0.07,1995.7)	2277.8 (0.04,1302.6)
25	145	12.61%	3715.5 (0.92,3605.2)	2841.8 (0.18,1995.7)	2240.6 (0.09,1302.6)
50	140	12.17%	3055.7 (1.00,2995.7)	2687.9 (0.37,1995.7)	2178.6 (0.18,1302.6)
250	100	8.70%	1686.3 (1.00,1386.3)	1686.3 (1.00,1386.3)	1682.9 (0.92,1302.6)
500	50	4.35%	1293.1 (1.00, 693.1)	1293.1 (1.00, 693.1)	1293.1 (1.00, 693.1)
750	0	0.00%	1187.7 (1.00, 287.7)	1187.7 (1.00, 287.7)	1187.7 (1.00, 287.7)
Unconstrained			1182.3 (1.00, 182.3)	1182.3 (1.00, 182.3)	1182.3 (1.00, 182.3)

Table 3.3: VaR^*_α and optimal ceded loss functions: Exponential risk.

P; see equation (3.3.12). Third, the minimum attainable VaR is an increasing function in P. This is the classical risk and reward tradeoff in the sense that higher expected profit can be achieved at the expense of higher minimum risk exposure (as measured by VaR); see Remark 3.3. Fourth, if we were to permit B to increase beyond 833.33 (and P is negative), then $\beta(\delta_{\theta^*}) > 1$ since $\int_{\delta_{\theta^*}}^{\infty} S_X(x)dx = 833.33$ and part (b) of Proposition 3.3 can be used to determine the optimal ceded loss function. In this case, the upper constraint B has no impact on the optimization problem and in fact it reduces to the unconstraint problem, as studied in Cai et al. (2007). In our example, VaR^*_α is \$1,182.3 with optimal retention $\delta_{\theta^*} = 182.3$. See also Remark 3.2. The unconstrained optimal reinsurance design also serves as a benchmark to our proposed constrained optimization problem. For instance at $\alpha = 1\%$, if the insurer were to seek an expected profit of \$145, the insurer needs to sustain more than three times the risk exposure relative to the unconstrained case (compare \$3,715.5 to 1,182.3).

Example 3.2 *In this example, we assume X has a Pareto distribution with $S_X(x) = \left(\frac{2{,}000}{x+2{,}000}\right)^3$, $x \geq 0$ so that its $E[X] = 1,000$ is the same as the previous example. We also assume $\theta = 0.2$ and $p_0 = 1150$. Table 3.4 produces the optimal reinsurance*

B	P	return	$\alpha = 1\%$ VaR$^*_\alpha$ (c, d)	$\alpha = 5\%$ VaR$^*_\alpha$ (c, d)	$\alpha = 10\%$ VaR$^*_\alpha$ (c, d)
10	148	12.87%	6998.9 (0.10,4188.8)	3381.6 (0.03,1619.2)	2291.2 (0.02, 872.6)
25	145	12.61%	6572.4 (0.24,4188.8)	3310.7 (0.08,1619.2)	2264.8 (0.05, 872.6)
50	140	12.17%	5861.7 (0.48,4188.8)	3192.5 (0.16,1619.2)	2220.7 (0.10, 872.6)
250	100	8.70%	2300.0 (1.00,2000.0)	2247.4 (0.82,1619.2)	1868.1 (0.52, 872.6)
500	50	4.35%	1428.4 (1.00, 828.4)	1428.4 (1.00, 828.4)	1428.4 (1.00, 828.4)
750	0	0.00%	1209.4 (1.00, 309.4)	1209.4 (1.00, 309.4)	1209.4 (1.00, 309.4)
Unconstrained			1188.0 (1.00, 125.3)	1188.0 (1.00, 125.3)	1188.0 (1.00, 125.3)

Table 3.4: VaR$^*_\alpha$ and optimal ceded loss functions: Pareto risk.

designs over the same set of parameter values as in the last example. Note that for a given P and α, the minimum attainable VaR is larger for the Pareto risk. This is to be expected since Pareto distribution is considered to be riskier than the corresponding exponential distribution in the sense that it has a heavier tail. Other than this, the discussions that we made earlier are equally applicable to the present example.

3.5 Appendix: Some Lemmas and Proof

Proof of Lemma 3.2. The concavity of $I_f(x)$ comes immediately from the fact that $f(x)$ is convex. Now suppose there exist two points x_1 and x_2 such that $0 \leq x_1 < x_2$ satisfying $I_f(x_1) - I_f(x_2) > 0$, i.e.,

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} > 1. \tag{3.5.40}$$

On the other hand, by the convexity of $f(x)$ we have

$$f(x_2) \leq \frac{x - x_2}{x - x_1} f(x_1) + \frac{x_2 - x_1}{x - x_1} f(x)$$

for $x \geq x_2$, or equivalently

$$f(x) \geq \frac{f(x_2) - f(x_1)}{x_2 - x_1} x + \frac{x_2 f(x_1) - x_1 f(x_2)}{x_2 - x_1}.$$

Hence, it follows from (3.5.40) that there exists a constant x_0 such that $f(x_0) > x_0$, which contradicts to the assumption that $f(x) \leq x$ for all $x \geq 0$. Therefore we conclude that $I_f(x)$ is increasing. □

Proof of Lemma 3.3

(a). First note that the derivatives of $\kappa(d)$, $\kappa'(d) = 1 - (1+\theta)S_X(d)$, and $\kappa''(d) = (1+\theta)f_X(d) \geq 0$ for $d \geq 0$. Hence $\kappa(d)$ is convex on $[0, \infty)$. Moreover, we can easily verify that if $\theta^* < S_X(0)$, then $\kappa'(d) < 0$ for $0 < d < \delta_{\theta^*}$ and $\kappa'(d) > 0$ for $d > \delta_{\theta^*}$, and that $\kappa'(d) > 0$ for $d > 0$ if $\theta^* \geq S_X(0)$. This justifies (3.3.15)-(3.3.17).

(b). It follows from (3.3.14) that $\lambda'(d) = [\delta_\alpha - d] f_X(d)$. By the assumption that X has strictly increasing distribution, we have $f_X(d) > 0$ for $d \geq 0$ so that $\lambda'(d) > 0$ for $d \in [0, \delta_\alpha)$. As a result, the function $\lambda(d)$ is strictly increasing on $[0, \delta_\alpha]$. Moreover, we have $\lambda(\delta_\alpha) = \int_{\delta_\alpha}^{\infty} S_X(x)dx > 0$, and $\delta_{\theta^*} < \delta_\alpha$ as $\alpha < \theta^*$. Hence, there must exist a unique root d_o to the equation $\lambda(d) = 0$ on $(\delta_{\theta^*}, \delta_\alpha)$ as we have $\lambda(\delta_{\theta^*}) < 0$. □

Lemma 3.4 *If $\alpha < \theta^*$, then $VaR_\alpha^*(S_n^n) = VaR_\alpha^*(T_n^n)$; i.e. (3.3.28) holds.*

Proof. If $\theta^* \geq S_X(0)$, then (3.3.28) holds trivially since $\delta_{\theta^*} = 0$ so that $S_n^n = T_n^n$. Suppose $\theta^* < S_X(0)$. Let $(\boldsymbol{c}, \boldsymbol{d})$ be any vector in S_n^n satisfying $d_{n,j} < \delta_{\theta^*}$, $j = 1, 2, \cdots, i$, for a fixed $i \in \{1, 2, \cdots, n\}$ and let $(\boldsymbol{c}, \boldsymbol{d'})$ be the corresponding vector constructed from $(\boldsymbol{c}, \boldsymbol{d})$ by merely replacing $d_{n,j}$ with δ_{θ^*} for all $j = 1, 2, \cdots, i$. Since

$$\sum_{j=1}^{i} c_{n,j} \int_{\delta_{\theta^*}}^{\infty} S_X(x)dx + \sum_{j=i+1}^{n} c_{n,j} \int_{d_{n,j}}^{\infty} S_X(x)dx \;\leq\; \sum_{j=1}^{n} c_{n,j} \int_{d_{n,j}}^{\infty} S_X(x)dx \leq B,$$

$(\boldsymbol{c}, \boldsymbol{d}')$ is in T_n^n. Moreover,

$$\begin{aligned}
\mathrm{VaR}_\alpha(\boldsymbol{c}, \boldsymbol{d}) &= \delta_\alpha + \sum_{j=1}^{n} c_{n,j}\kappa(d_{n,j}) \\
&\geq \delta_\alpha + \sum_{j=1}^{i} c_{n,j}\kappa(\delta_{\theta^*}) + \sum_{j=i+1}^{n} c_{n,j}\kappa(d_{n,j}) \\
&= \mathrm{VaR}_\alpha(\boldsymbol{c}, \boldsymbol{d}'), \qquad (3.5.41)
\end{aligned}$$

since the minimum value of the function $\kappa(x)$ is $\kappa(\delta_{\theta^*})$ by part (a) of Lemma 3.3 and this completes the proof. □

Lemma 3.5 *If $\alpha < \theta^*$, $\kappa(\delta_{\theta^*}) < 0$ and $\beta(\delta_{\theta^*}) \leq 1$, then the following relation holds*

$$VaR_\alpha^*(T_n^{\geq}) = VaR_\alpha^*(V). \qquad (3.5.42)$$

Proof. We prove (3.5.42) via the following two-step procedure. First we establish

$$\mathrm{VaR}_\alpha^*(T_n^{\geq}) = \mathrm{VaR}_\alpha^*(U), \qquad (3.5.43)$$

where U is a set defined as

$$U = \Big\{(\boldsymbol{c}, \boldsymbol{d}) : (\boldsymbol{c}, \boldsymbol{d}) \in T_n^{\geq},\ \sum_{j=1}^{n} c_{n,j} \int_{d_{n,j}}^{\infty} S_X(x)dx = B\Big\}. \qquad (3.5.44)$$

Then we show that

$$\mathrm{VaR}_\alpha^*(U) = \mathrm{VaR}_\alpha^*(V) \qquad (3.5.45)$$

to complete the proof.

Now for the first part of the proof of showing (3.5.43), we demonstrate the following two inequalities:

$$\mathrm{VaR}_\alpha^*(T_n^{\geq}) \leq \mathrm{VaR}_\alpha^*(U), \qquad (3.5.46)$$

$$\mathrm{VaR}_\alpha^*(T_n^{\geq}) \geq \mathrm{VaR}_\alpha^*(U). \qquad (3.5.47)$$

The first inequality is straightforward as it follows immediately from the definition that $U \subset T_n^{\geq}$. To justify the second inequality, first note that for any $(\boldsymbol{c}, \boldsymbol{d}) \in T_n^{\geq}$, we have $d_{n,j} \geq \delta_{\theta^*}$ for $j = 1, 2, \cdots, n$ and that both

$$\sum_{j=1}^{n} c_{n,j} \int_{\delta_{\theta^*}}^{\infty} S_X(x)dx \quad \geq \quad B \tag{3.5.48}$$

and

$$\sum_{j=1}^{n} c_{n,j} \int_{d_{n,j}}^{\infty} S_X(x)dx \quad \leq \quad B \tag{3.5.49}$$

hold simultaneously. The above two inequalities follow from the constraint on the sets $T_n^{\geq}$ and S_n^n, respectively. Moreover, $\sum_{j=1}^{n} c_{n,j} \int_{d_{n,j}}^{\infty} S_X(x)dx$ is continuous in $d_{n,j}$ for $j = 1, 2, \cdots, n$, since we assume X has a continuous distribution function on $[0, \infty)$. Thus, for any $(\boldsymbol{c}, \boldsymbol{d}) \in T_n^{\geq}$ satisfying (3.5.49), there exist constants $\delta_{\theta^*} \leq d'_{n,j} \leq d_{n,j}$, $j = 1, 2, \cdots, n$ such that $\sum_{j=1}^{n} c_{n,j} \int_{d'_{n,j}}^{\infty} S_X(x)dx = B$. If a vector $(\boldsymbol{c}, \boldsymbol{d}')$ is constructed by only replacing $d_{n,j}$ in $(\boldsymbol{c}, \boldsymbol{d})$ with $d'_{n,j}$, then $(\boldsymbol{c}, \boldsymbol{d}') \in U$. Furthermore, since $\kappa(x)$ is increasing on $[\delta_{\theta^*}, \infty)$ according to part (a) of Lemma 3.3, we have

$$\mathrm{VaR}_\alpha(\boldsymbol{c}, \boldsymbol{d}) \geq \mathrm{VaR}_\alpha(\boldsymbol{c}, \boldsymbol{d}'), \tag{3.5.50}$$

which in turn leads to (3.5.47) and together with (3.5.46), we prove (3.5.43).

For the second part of the proof of showing (3.5.45), we again use the same technique as above by demonstrating the following two inequalities:

$$\mathrm{VaR}_\alpha^*(U) \quad \leq \quad \mathrm{VaR}_\alpha^*(V) \tag{3.5.51}$$

$$\mathrm{VaR}_\alpha^*(U) \quad \geq \quad \mathrm{VaR}_\alpha^*(V). \tag{3.5.52}$$

To justify inequality (3.5.51), we first note that every coefficient pair $(c, d) \in V$ is a special case of the vector $(\boldsymbol{c}, \boldsymbol{d})$ in $T_n^{\geq}$. This can be seen by setting $c_{n,j} = \frac{c}{n}$ and $d_{n,j} = d$, $j = 1, 2, \cdots, n$. Note that the optimization problem in V becomes a two-dimensional problem, instead of $2n$ dimension as in the general case. Together with (3.5.43), we have inequality (3.5.51).

To verify inequality (3.5.52), we proceed as follows. For any vector $(\boldsymbol{c}, \boldsymbol{d}) \in U$, let us denote $c_n := \sum_{j=1}^{n} c_{n,j}$ and $d_n := \sum_{j=1}^{n} \frac{c_{n,j}}{c_n} d_{n,j}$. By treating $\int_d^\infty S_X(x)dx$ as a function of d, the convexity property ensures that

$$B = \sum_{j=1}^{n} c_{n,j} \int_{d_{n,j}}^{\infty} S_X(x)dx \;\; \geq \;\; c_n \int_{d_n}^{\infty} S_X(x)dx.$$

Consequently there exists a coefficient pair (c_n, d'_n) satisfying $\delta_{\theta^*} \leq d'_n \leq d_n$ and $c_n \int_{d'_n}^{\infty} S_X(x)dx = B$. As $U \subset T_n^{\geq} \subset T_n^n \subset D_n^n$, we see that $d_{n,j} \leq \delta_\alpha$ for $j = 1, 2, \cdots, n$, which together with the definition of d_n leads to $d_n \leq \delta_\alpha$. Hence $(c_n, d'_n) \in V$. Moreover,

$$\begin{aligned}
\mathrm{VaR}_\alpha(\boldsymbol{c}, \boldsymbol{d}) &= \delta_\alpha + \sum_{j=1}^{n} c_{n,j} \left[d_{n,j} + (1+\theta) \int_{d_{n,j}}^{\infty} S_X(x)dx - \delta_\alpha \right] \\
&= \delta_\alpha + (1+\theta)B + \sum_{j=1}^{n} c_{n,j} d_{n,j} - \sum_{j=1}^{n} c_{n,j} \delta_\alpha \\
&= \delta_\alpha + (1+\theta)c_n \int_{d'_n}^{\infty} S_X(x)dx + c_n \cdot d_n - c_n \cdot \delta_\alpha \\
&\geq \delta_\alpha + (1+\theta)c_n \int_{d'_n}^{\infty} S_X(x)dx + c_n \cdot d'_n - c_n \cdot \delta_\alpha \\
&= \delta_\alpha + c_n \kappa(d'_n) \\
&= \mathrm{VaR}_\alpha(c_n, d'_n), \qquad (3.5.53)
\end{aligned}$$

Since $(\boldsymbol{c}, \boldsymbol{d})$ is an arbitrary vector from U, inequality (3.5.52) follows immediately from (3.5.53) and this completes the proof. □

Lemma 3.6 *If $\alpha < \theta^*$, $\kappa(\delta_{\theta^*}) < 0$ and $\beta(\delta_{\theta^*}) \leq 1$, then the following inequality is true:*

$$VaR^*_\alpha(T_n^{\leq}) \;\; \geq \;\; VaR^*_\alpha(V). \qquad (3.5.54)$$

Proof. For any vector $(\boldsymbol{c}, \boldsymbol{d}) \in T_n^{\leq}$, it follows from (3.3.27) and part (a) of Lemma 3.3 that

$$\begin{aligned} \mathrm{VaR}_\alpha(\boldsymbol{c}, \boldsymbol{d}) &= \delta_\alpha + \sum_{j=1}^{n} c_{n,j}\kappa(d_{n,j}) \\ &\geq \delta_\alpha + \sum_{j=1}^{n} c_{n,j} \cdot \kappa(\delta_{\theta^*}) \\ &\geq \delta_\alpha + \beta(\delta_{\theta^*}) \cdot \kappa(\delta_{\theta^*}) \\ &= \mathrm{VaR}_\alpha(\beta(\delta_{\theta^*}), \delta_{\theta^*}). \end{aligned} \tag{3.5.55}$$

It is easy to see that $(\beta(\delta_{\theta^*}), \delta_{\theta^*}) \in V$, which, together with (3.5.55), implies inequality (3.5.54), as required. □

Lemma 3.7 *If $\alpha < \theta^*$, $\kappa(\delta_{\theta^*}) < 0$ and $\beta(\delta_{\theta^*}) \leq 1$, then we have $VaR_\alpha^*(S_n^n) = VaR_\alpha^*(V)$, i.e. equation (3.3.31) holds.*

Proof. The required relation follows immediately from Lemmas 3.4, 3.5, 3.6 and the partitioning that $T_n^n = T_n^{\geq} \cup T_n^{\leq}$. □

Lemma 3.8 *For any $\mu \in \mathcal{M}^+$, there exists a sequence of measures $\{\mu_n, n = 1, 2, \cdots\}$ in $\mathcal{M}_n^+$ such that $h_n(x)$ converges pointwisely to $f(x)$ from below, where $h_n(x) = \int (x-t)_+ d\mu_n$ and $f(x) = \int (x-t)_+ d\mu$ for $x \geq 0$.*

Proof. The proof is trivial when the measure μ is such that $f(x) \equiv 0$. Next, we suppose $f(x) \equiv 0$ does not hold. It is well known that for any nonnegative increasing convex function f defined on $[0, \infty)$, there exists a sequence of nonnegative functions $\{h_n, n = 1, 2, ...\}$ defined on $[0, \infty)$ such that $h_n(x) = \sum_{j=1}^n c_{n,j}(x - d_{n,j})_+$ for some constants $c_{n,j} \geq 0$ and $d_{n,j} \geq 0$ and $\lim_{n\to\infty} h_n(x) = f(x)$ from below for any $x \geq 0$. This implies

$$h_n(x) \leq f(x) \ \text{ for all } \ x \geq 0 \ \text{ and } \ n = 1, 2, \cdots. \tag{3.5.56}$$

See, for example, the proof to Case 1 of Theorem 1.5.7 of Müller and Stoyan (2002, p18).

By the definition of $\mathcal{M}^+$, for any $\mu \in \mathcal{M}^+$, we have $0 \le f(x) = \int (x-t)_+ d\mu \le x$, which together with (3.5.56), in return, implies that for any $x > 0$ and $n = 1, 2, ...$, we have

$$0 \le \frac{h_n(x)}{x} = \sum_{j=1}^{n} c_{n,j} \frac{(x - d_{n,j})_+}{x} \le 1. \tag{3.5.57}$$

Consequently, by letting $x \to \infty$ in (3.5.57) we have $0 \le \sum_{j=1}^{n} c_{n,j} \le 1$ for all $n = 1, 2, \cdots$. Thus, the sequence of the measures $\{\mu_n, n \ge n_0\}$ of the form (3.3.18) with these coefficients $c_{n,j}, d_{n,j}, j = 1, 2, \cdots, n$ satisfies the requirements of the lemma and hence the proof is complete. □

Chapter 4

CTE Minimization Model: General Reinsurance Contracts

4.1 Introduction and Reinsurance Models

In the previous chapters, the optimal ceded loss function is either assumed to have certain specific form or confined to some special class. For example, we considered the stop-loss or quota-share treaties in Chapter 2 and analyzed the class of increasing convex functions in Chapter 3. In this chapter we will extend our results by considering the optimality among all the possible ceded loss functions using the criterion of minimizing the CTE of the insurer's total risk. Because of the generality of the model, this significantly increases the mathematical complexity on identifying the optimal solutions. As we will soon present, our formulation of the optimal reinsurance model entails us to solve some convex optimization problem in a Hilbert space with a goal function which is directionally differentiable but not Gâteaux differentiable. Hence, the Lagrangian method based on the concept of directional derivative will be employed in searching for the optimal ceded loss functions.

Before specifying our reinsurance model, let us recall the general setup in a static reinsurance design and make some technical assumptions. Let X denote the (aggregate) loss initially assumed by an insurer. Suppose X is a nonnegative random variable, and identify it by a probability measure Pr on the measurable space $(\Omega, \mathcal{F})$ with $\Omega = [0, \infty)$ and $\mathcal{F}$ being the Borel σ-field on Ω, such that the distribution function of the underlying risk X is defined by $F_X(t) := \Pr\{[0, t]\}$ for $t \geq 0$. Denote by $f(X)$ the part of loss transferred from the insurer to a reinsurer in the presence of the reinsurance. f can be identified as such a function $f : [0, \infty) \mapsto [0, \infty)$, and it is called the ceded loss function, or the indemnification function. A conventional assumption for the ceded loss function f is that $0 \leq f(x) \leq x$ for all $x \geq 0$. With the ceded loss $f(X)$, the insurer will retain a loss of $I_f(X) := X - f(X)$. Similarly, I_f can also be recognized as a function $I_f : [0, \infty) \mapsto [0, \infty)$, called retained loss function. On the other hand, by transferring part of its loss to the reinsurer, the insurer is obligated to pay the reinsurance premium $\Pi(f(X))$ to the reinsurer according to a given premium principle Π. Consequently, the total cost or the total risk for the insurer in the presence of reinsurance, denoted by $T_f(X)$, is the sum of the retained loss and the reinsurance premium [1], i.e.,

$$T_f(X) = I_f(X) + \Pi(f(X)) = X - f(X) + \Pi(f(X)). \tag{4.1.1}$$

In what follows, we might omit "(X)" in notation like $f(X)$, $I_f(X)$ and $T_f(X)$, and simply use f, I_f and T_f to denote these random variables if it is clear in the corresponding context.

[1] Actually, the total loss random variable for the insurer in the business involving the risk X is the amount of $T_f - p_0$, where p_0 is the insurance premium payable to the insurer by the policyholders. However, p_0 is a constant; hence, by the property of translation invariance of the risk measure CTE, we can consider the optimal reinsurance design problem directly based on the random quantity T_f defined as in (4.1.1).

Suppose that the reinsurance premium uses the expectation principle with a safety loading $\theta > 0$, i.e., $\Pi(f) = (1+\theta)\mathsf{E}[f]$, and assume that the insurer is seeking the optimal reinsurance by minimizing the risk measure CTE based on its total risk T_f. Then the insurer's problem can be formulated as follows:

$$(\mathbf{P_0'}) \quad \begin{cases} \min_f & \mathrm{CTE}_\alpha(T_f) = \mathrm{CTE}_\alpha\Big(X - f(X) + (1+\theta)\mathsf{E}[f]\Big) \\ \text{s.t.} & 0 \le f(x) \le x \ \text{ for all } \ x \ge 0, \ \ \mathsf{E}[f(X)] \in [0, \pi/(1+\theta)]. \end{cases} \tag{4.1.2}$$

Note that the above optimal reinsurance model is very similar to those that we have analyzed before. The significant difference is that the minimization in model $(\mathbf{P_0'})$ is taken with respect to all possible ceded loss functions, instead of restricting to some special class.

Remark 4.1 *The constraint $\mathsf{E}[f(X)] \le \pi/(1+\theta)$ for the ceded loss function f has at least the following two economic interpretations:*

(i) The constraint can be interpreted as the reinsurance premium budget; i.e., the reinsurance premium that the insurer is willing to pay is no greater than π.

(ii) The constraint can also be understood as a minimum expected profitability guarantee. To see this, let p_0 denote the insurance premium received by the insurer for underwriting an insurance on the risk X and let B denote the insurer's net profit in the presence of the reinsurance. Then we have

$$B(f) = p_0 - T_f(X) = p_0 - X + f(X) - (1+\theta)\mathsf{E}[f],$$

so that the expected profit, $b(f) := \mathsf{E}[B]$, is given by

$$b(f) = \mathsf{E}[B(f)] = p_0 - \mathsf{E}[X] - \theta\mathsf{E}[f].$$

Consequently, a profitability constraint such that $b \ge l$ for a certain preset level b can be equivalently formulated as $\mathsf{E}[f] \le \dfrac{\pi}{1+\theta}$ where $\pi = \dfrac{1+\theta}{\theta}\big(p_0 - \mathsf{E}[X] - l\big)$.

We now continue to addressing the optimization problem ($\mathbf{P_0'}$). For mathematical convenience, we suppose that X has finite first two moments so that we can restrict to the space $\mathcal{L}^2 := \mathcal{L}^2(\Omega, \mathcal{F}, P)$ for the optimal ceded loss functions. Let $\mathcal{Q} = \mathcal{Q}_f \bigcap \mathcal{Q}_\pi$ where

$$\mathcal{Q}_f \quad := \quad \{f \in \mathcal{L}^2 : 0 \leq f(x) \leq x \ \text{ for } \ x \geq 0\}, \tag{4.1.3}$$

and

$$\mathcal{Q}_\pi \quad := \quad \{f \in \mathcal{L}^2 : 0 \leq (1+\theta)\mathsf{E}[f] \leq \pi\}, \tag{4.1.4}$$

respectively. Then the reinsurance model ($\mathbf{P_0'}$) can be equivalently reformulated as

$$(\mathbf{P_0}) \qquad \min_{f \in \mathcal{Q}} \ \ \mathrm{CTE}_\alpha(T_f) = \mathrm{CTE}_\alpha\Big(X - f(X) + (1+\theta)\mathsf{E}[f]\Big) \tag{4.1.5}$$

4.2 Optimal Reinsurance Treaties

The objective of this section is to discuss the optimal solutions to the reinsurance model ($\mathbf{P_0}$) defined in (4.1.5). The mathematical challenges of solving this problem directly lie on at least two aspects. First, the model is obviously an optimization problem of infinite dimension, involving searching for an optimal function instead of the optimal values of a finite number of parameters. Second, for a general feasible ceded loss function f there is no analytical expression for the goal function $\mathrm{CTE}_\alpha(T_f)$. Recognizing that solving ($\mathbf{P_0}$) directly can be very challenging, we resolve this by first introducing an auxiliary model ($\mathbf{P_T}$) (as defined in (4.2.10)). Then we will demonstrate shortly that model ($\mathbf{P_T}$) is more tractable. Furthermore, a key result in Rockafellar and Uryasev (2002) asserts that the solution to ($\mathbf{P_T}$) regarding the decision variable f is also the solution to ($\mathbf{P_0}$).

4.2.1 Auxiliary Model and the Optimality Conditions

To describe the auxiliary model ($\mathbf{P_T}$), it is convenient to introduce the mapping $G_\alpha(\xi, f) : \mathbb{R} \times \mathcal{L}^2 \mapsto \mathbb{R}$ such that

$$G_\alpha(\xi, f) := \xi + \frac{1}{\alpha}\mathsf{E}\left[\Big(X - f + (1+\theta)\mathsf{E}[f] - \xi\Big)_+\right] \tag{4.2.6}$$

with the same $\alpha > 0$ as the one associated with the risk measure CTE in model ($\mathbf{P_0}$). The significance of introducing $G_\alpha(\xi, f)$ can be deduced from the lemma below, which is a direction consequence of Rockafellar and Uryasev (2002, Theorem 14):

Lemma 4.1 *Minimizing $CTE_\alpha(T_f)$ with respect to $f \in \mathcal{Q}$ is equivalent to minimizing $G_\alpha(\xi, f)$ over all $(\xi, f) \in \mathbb{R} \times \mathcal{Q}$, in the sense that*

$$\min_{f\in\mathcal{Q}} CTE_\alpha(T_f) = \min_{(\xi,f)\in\mathbb{R}\times\mathcal{Q}} G_\alpha(\xi, f), \tag{4.2.7}$$

where moreover,

$$(\xi^*, f^*) \in arg\ min_{(\xi,f)\in\mathbb{R}\times\mathcal{Q}} G_\alpha(\xi, f) \tag{4.2.8}$$

if and only if

$$f^* \in arg\ min_{f\in\mathcal{Q}} CTE_\alpha(T_f), \xi^* \in arg\ min_{\xi\in\mathbb{R}} G_\alpha(\xi, f^*). \tag{4.2.9}$$

The above lemma formally states that minimizing $\text{CTE}_\alpha(T_f)$ over $\mathcal{Q}$ is equivalent to minimizing the function $G_\alpha(\xi, f)$ over the product space $\mathbb{R} \times \mathcal{Q}$. More importantly, this permits us to reformulate ($\mathbf{P_0}$) as follows.

$$(\mathbf{P_T})\begin{cases} \min\limits_{(\xi,f)\in\mathbb{R}\times\mathcal{Q}_f} & G_\alpha(\xi, f) \equiv \xi + \frac{1}{\alpha}\mathsf{E}\left[\Big(X - f + (1+\theta)\mathsf{E}[f] - \xi\Big)_+\right] \\ \text{s.t.} & \mathsf{E}[f] \in [0, \pi/(1+\theta)]. \end{cases} \tag{4.2.10}$$

By Lemma 4.1, if (ξ^*, f^*) is one solution to problem $\mathbf{P_T}$, then f^* solves the reinsurance model $\mathbf{P_0}$, i.e., f^* is one optimal ceded loss function.

While $(\mathbf{P_T})$ is equivalent to our original problem $(\mathbf{P_0})$, we have yet to solve $(\mathbf{P_T})$. Compared to problem $(\mathbf{P_0})$, one obvious advantage of model $(\mathbf{P_T})$ is that the goal function of the latter problem has analytical expression. However, it will be still quite mathematical involved in solving problem $(\mathbf{P_T})$. The challenge is that it is still an infinite dimensional optimization problem and the goal function is not Gâteaux differentiable, which implies that the widely-used Karush-Kuhn-Tucker Theorem is not helpful to tackle this problem.

As we will demonstrate in Section 4.4 the appendix of this chapter, $(\mathbf{P_T})$ is a convex problem. Moreover, its goal function $G_\alpha(\xi, f)$ is directionally differentiable with respect to (ξ, f) over its feasible set. This motivates us to use the Lagrangian-based directional derivatives method to solve problem $(\mathbf{P_T})$. In fact, by defining g^* and V as

$$g^* = X - f^* + (1+\theta)\mathsf{E}[f^*] - \xi^* \tag{4.2.11}$$

and

$$V = (1+\theta)\mathsf{E}[f] - \xi - f, \tag{4.2.12}$$

respectively, Section 4.4 formally establishes the optimality conditions for problem $(\mathbf{P_T})$ as shown in the following proposition:

Proposition 4.1 *An element $(\xi^*, f^*) \in \mathbb{R} \times \mathcal{Q}$ solves problem $(\mathbf{P_T})$ if and only if there exist a constant $r \in \mathbb{R}$ and a random variable $\lambda \in \mathcal{L}^2$ such that the following three conditions are satisfied:*

C1. $A(\xi, f) \equiv \alpha\Big[\xi + r(1+\theta)E[f] + E[\lambda f]\Big] + E[V\mathbf{1}_{\{g^*>0\}}] + E[V_+\mathbf{1}_{\{g^*=0\}}] \geq 0,$ $\forall\ (\xi, f) \in \mathbb{R} \times \mathcal{L}^2$;

C2. $E[\lambda(f - f^*)] \leq 0,\ f \in \mathcal{Q}_f$;

C3. $r\big(E[f] - E[f^*]\big) \leq 0$ *for every* $f \in \mathcal{Q}_\pi$.

Proof. See Section 4.4.3.

Armed with the above optimality conditions, we obtain an optimal solution to problem ($\mathbf{P_T}$) using the following strategy. First, we select some potential candidate. Second, we show that the candidate is indeed an optimal solution by verifying it with conditions **C1, C2** and **C3**. We emphasize that the above procedure of deriving an optimal solution is a non-trivial exercise, as confirmed by the theorems below.

4.2.2 Optimal Ceded Loss Functions

Throughout this subsection, we assume $\alpha(1+\theta) \leq 1$. We use the notation π_α to denote

$$\pi_\alpha = (1+\theta)\mathsf{E}\left[(X - d_\alpha)_+\right] \tag{4.2.13}$$

where

$$d_\alpha = \inf\left\{d : \Pr[X > d] \leq \alpha\right\}. \tag{4.2.14}$$

The notation π_θ and d_θ are defined analogously as

$$\pi_\theta = (1+\theta)\mathsf{E}\left[(X - d_\theta)_+\right] \tag{4.2.15}$$

where

$$d_\theta = \inf\left\{d : \Pr[X > d] \leq \frac{1}{1+\theta}\right\}. \tag{4.2.16}$$

We emphasize that the condition $\alpha(1+\theta) \leq 1$ is quite mild as in practice both α and θ are typically much smaller than one. The same condition also implies that $d_\alpha \geq d_\theta$ so that $\pi_\alpha \leq \pi_\theta$.

We now address the optimal solutions to problem ($\mathbf{P_T}$). We present the solutions depending on the level of the reinsurance premium budget. In particular, we consider the following three cases:

Case (i): $\pi \in (0, \pi_\alpha)$;

Case (ii): $\pi \in [\pi_\alpha, \pi_\theta]$; and

Case (iii): $\pi \in [\pi_\theta, \infty)$.

The solutions to these cases are formally stated in Theorems 4.1, 4.2, and 4.3, respectively.

Case (i): $\pi \in (0, \pi_\alpha)$

Theorem 4.1 *Suppose $\alpha(1+\theta) \leq 1$. Then all the ceded loss functions f^* of the following form are the optimal solutions to problem $(\mathbf{P_T})$:*

$$f^*(x) = \begin{cases} 0, & x < \hat{d}, \\ l(x), & x \geq \hat{d} \end{cases} \tag{4.2.17}$$

where the function $l(x)$ satisfies

$$0 \leq l(x) < x - d_\alpha, \qquad \text{for } x \geq \hat{d}, \tag{4.2.18}$$

and the retention $\hat{d} > 0$ is the solution to

$$E[f^*] = \frac{\pi}{(1+\theta)}. \tag{4.2.19}$$

Proof. To show that f^* defined in (4.2.17), with retention $\hat{d}$ and function l satisfying (4.2.18) and (4.2.19), is indeed an optimal solution to problem $(\mathbf{P_T})$, it is sufficient to verify conditions **C1, C2, C3** in Proposition 4.1 for appropriately chosen constants $\xi^*, r \in \mathbb{R}$, and random variable $\lambda \in \mathcal{L}^2$. Let us first focus on condition **C3**. By setting

$$r = \frac{1}{\alpha(1+\theta)} - 1, \tag{4.2.20}$$

we have $r \geq 0$ since $\alpha(1+\theta) \leq 1$. This implies that condition **C3** holds immediately.

To verify the remaining conditions, we notice that $d_\alpha < \hat{d}$ (see Remark 4.2 below) and choose $\xi^* = \pi + d_\alpha$. Then, (4.2.11) becomes

$$g^*(x) = \begin{cases} x - d_\alpha < 0, & x < d_\alpha, \\ x - d_\alpha \geq 0, & d_\alpha \leq x < \hat{d}, \\ x - l(x) - d_\alpha > 0, & x \geq \hat{d}. \end{cases} \tag{4.2.21}$$

Clearly, we have

$$\mathbf{1}_{\{g^*<0\}} = \mathbf{1}_{\{X<d_\alpha\}} \tag{4.2.22}$$

and

$$\mathbf{1}_{\{g^*=0\}} = \mathbf{1}_{\{X=d_\alpha\}}. \tag{4.2.23}$$

Let us now define

$$\beta_\alpha = \begin{cases} \dfrac{\alpha - \Pr\{X > d_\alpha\}}{\Pr\{X = d_\alpha\}}, & \text{if } \Pr\{X = d_\alpha\} \neq 0; \\ 0, & \text{if } \Pr\{X = d_\alpha\} = 0. \end{cases} \tag{4.2.24}$$

Note that $0 \leq \beta_\alpha \leq 1$ since from the definition of d_α in (4.2.14), we have

$$\Pr\{X > d_\alpha\} \leq \alpha, \text{ and } \Pr\{X \geq d_\alpha\} \geq \alpha.$$

Note also $\Pr\{X = d_\alpha\} = 0$ provided that $\Pr\{X > d_\alpha\} = \alpha$. Furthermore, by setting

$$\lambda = -\frac{1}{\alpha}\Big(\mathbf{1}_{\{X<d_\alpha\}} + (1-\beta_\alpha)\mathbf{1}_{\{X=d_\alpha\}}\Big), \tag{4.2.25}$$

so that together with (4.2.24), we obtain

$$\mathsf{E}[\lambda] = -\frac{1}{\alpha}\Big[\Pr\{X < d_\alpha\} + (1-\beta_\alpha)\Pr\{X = d_\alpha\}\Big] = -\frac{1-\alpha}{\alpha}.$$

The above result in turn leads to

$$\mathsf{E}[1+\alpha\lambda]=\alpha. \tag{4.2.26}$$

Accordingly, for any $f \in \mathcal{Q}_f$ we have

$$\begin{aligned}
\alpha\mathsf{E}[\lambda(f-f^*)] &= -\mathsf{E}\left[\left(\mathbf{1}_{\{X<d_\alpha\}}+(1-\beta_\alpha)\mathbf{1}_{\{X=d_\alpha\}}\right)(f-f^*)\right] \\
&= -\mathsf{E}\left[f\left(\mathbf{1}_{\{X<d_\alpha\}}+(1-\beta_\alpha)\mathbf{1}_{\{X=d_\alpha\}}\right)\right] \\
&\leq 0,
\end{aligned} \tag{4.2.27}$$

where the second equality follows from the definition that $f^*(x)=0$ for $x \leq d_\alpha$. Hence, condition **C2** is satisfied.

To demonstrate condition **C1**, we first establish the following relation:

$$\begin{aligned}
\mathsf{E}\Big[V\mathbf{1}_{\{g^*>0\}}+V_+\mathbf{1}_{\{g^*=0\}}\Big] &= \mathsf{E}\Big[V\mathbf{1}_{\{X>d_\alpha\}}+V_+\mathbf{1}_{\{X=d_\alpha\}}\Big] \\
&\geq \mathsf{E}\Big[V\mathbf{1}_{\{X>d_\alpha\}}+\beta_\alpha V_+\mathbf{1}_{\{X=d_\alpha\}}\Big] \\
&\geq \mathsf{E}\Big[V\Big(\mathbf{1}_{\{X>d_\alpha\}}+\beta_\alpha\mathbf{1}_{\{X=d_\alpha\}}\Big)\Big] \\
&= \mathsf{E}[V(1+\alpha\lambda)],
\end{aligned} \tag{4.2.28}$$

where the equality in the last step follows from (4.2.25). The above result, together with (4.2.20), (4.2.25), and (4.2.26) assert condition **C1** as shown below:

$$\begin{aligned}
A(\xi,f) &\geq \alpha\Big[\xi+r(1+\theta)\mathsf{E}[f]+\mathsf{E}(\lambda f)\Big]+\mathsf{E}[V(1+\alpha\lambda)] \\
&= \alpha\Big[\xi+r(1+\theta)\mathsf{E}[f]+\mathsf{E}(\lambda f)\Big]+\mathsf{E}\Big[\Big((1+\theta)\mathsf{E}[f]-f-\xi\Big)(1+\alpha\lambda)\Big] \\
&= \xi\Big(\alpha-\mathsf{E}[1+\alpha\lambda]\Big)+\mathsf{E}[f]\Big(\alpha(1+\theta)(1+r)-1\Big) \\
&= 0.
\end{aligned}$$

Since conditions **C1**, **C2** and **C3** hold with constants $\xi^*=\pi+d_\alpha$ and r as defined in (4.2.20), and the random variable $\lambda \in \mathcal{L}^2$ as defined in (4.2.25), f^*

defined in (4.2.17) is indeed the optimal ceded loss function and hence the proof is complete. □

We now make the following two remarks with respect to the above theorem.

Remark 4.2 *The constraint (4.2.18) states that for $x \geq \hat{d}$, the function $l(x) \geq 0$ is bounded from above by $x - d_\alpha$. Furthermore, when $x = \hat{d}$, we have $\hat{d} > l(\hat{d}) + d_\alpha \geq d_\alpha$. Consequently, the function (4.2.17) satisfying (4.2.18) and (4.2.19) defines a class of ceded loss functions which have a shape underneath the line $f(x) = x - d_\alpha$ with a retention larger than d_α and a resulting reinsurance premium exactly equal to the preset budget π. In Figure 4.1, the three lower curves (dashed lines) depict three samples of such cede ceded loss functions that are optimal.*

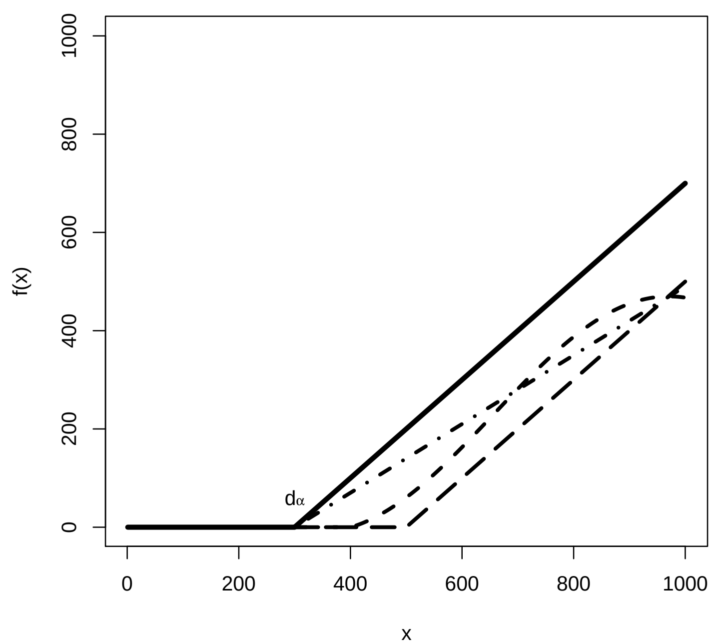


Figure 4.1: Three typical optimal ceded loss functions.

Remark 4.3 *We reiterate that Theorem 4.1 only provides solution, if it exists, for $\pi \in (0, \pi_\alpha)$. This is an immediate consequence of the conditions (4.2.18) and (4.2.19). More explicitly, suppose f^* is the optimal ceded loss function identified by Theorem 4.1, then we must have $\pi \in (0, \pi_\alpha)$ as can be justified as follows:*

$$\begin{aligned} \pi &= (1+\theta)\mathsf{E}[f^*(X)] \\ &= (1+\theta)\mathsf{E}[l(X)\cdot \mathbf{1}_{\{X\geq \hat{d}\}}] \\ &< (1+\theta)\mathsf{E}[(X-d_\alpha)\mathbf{1}_{\{X\geq d_\alpha\}}] \\ &= (1+\theta)\mathsf{E}[(X-d_\alpha)_+] \\ &= \pi_\alpha. \end{aligned}$$

Case (ii): $\pi_\alpha \leq \pi \leq \pi_\theta$

Theorem 4.2 *For a given underlying loss random variable X, if there exists a positive constant d^* such that*

$$(1+\theta)\mathsf{E}[(X-d^*)_+] = \pi, \tag{4.2.29}$$

$$\Pr\{X \geq d^*\} \leq \frac{1}{1+\theta}, \tag{4.2.30}$$

$$\Pr\{X \geq d^*\} \geq \alpha, \tag{4.2.31}$$

then $f^* = (X-d^*)_+$ is one optimal ceded loss function to the problem $(\mathbf{P_T})$.

Proof. Similar to the proof of Theorem 4.1, we need to show that the ceded loss function of the form $f^* = (X-d^*)_+$ satisfies the three sufficient conditions **C1**, **C2** and **C3** in Proposition 4.1 for appropriately chosen constants $\xi^*, r \in \mathbb{R}$, and random variable $\lambda \in \mathcal{L}^2$.

We begin by choosing $\xi^* = d^* + (1+\theta)\mathsf{E}[f^*]$ so that (4.2.11) simplifies to

$$g^*(x) = \begin{cases} x - d^*, & x < d^*; \\ 0, & x \geq d^*. \end{cases} \tag{4.2.32}$$

This implies that $\{g^* > 0\}$ is an empty set and that $\{g^* = 0\} = \{X \geq d^*\}$. For any constant δ such that $0 < \delta \leq 1$, we define $\lambda = -\frac{\delta}{\alpha}\mathbf{1}_{\{X<d^*\}}$. Then, with the selected ξ^* and λ, we obtain the following fact regarding condition **C1**:

$$\begin{aligned} A(\xi, f) &= \alpha\Big[\xi + r(1+\theta)\mathsf{E}[f] + \mathsf{E}[\lambda f]\Big] + \mathsf{E}[V_+ \cdot \mathbf{1}_{\{g^*=0\}}] \\ &\geq \alpha\Big[\xi + r(1+\theta)\mathsf{E}[f] + \mathsf{E}[\lambda f]\Big] + \delta\mathsf{E}[V_+ \cdot \mathbf{1}_{\{g^*=0\}}] \\ &\geq \alpha\Big[\xi + r(1+\theta)\mathsf{E}[f]\Big] - \delta\mathsf{E}[f \cdot \mathbf{1}_{\{X<d^*\}}] \\ &\quad +\delta\mathsf{E}\Big[\Big((1+\theta)\mathsf{E}[f] - \xi - f\Big) \cdot \mathbf{1}_{\{X\geq d^*\}}\Big] \\ &= \xi[\alpha - \delta \cdot \Pr(X \geq d^*)] + \mathsf{E}[f]\Big[\alpha r(1+\theta) - \delta + \delta(1+\theta) \cdot \Pr(X \geq d^*)\Big] \\ &= 0, \end{aligned} \tag{4.2.33}$$

provided that

$$\Pr\{X \geq d^*\} = \frac{\alpha}{\delta}, \tag{4.2.34}$$

and

$$r = \frac{\delta}{\alpha}\Big[\frac{1}{1+\theta} - \Pr\{X \geq d^*\}\Big]. \tag{4.2.35}$$

Moreover, for any $f \in \mathcal{Q}_f$ we have

$$\begin{aligned} \alpha\mathsf{E}[\lambda(f - f^*)] &= -\delta\mathsf{E}[\mathbf{1}_{\{X<d^*\}})(f - f^*)] \\ &= -\delta\mathsf{E}[f\mathbf{1}_{\{X<d^*\}}] \\ &\leq 0 \end{aligned} \tag{4.2.36}$$

so that condition **C2** is satisfied with the chosen ξ^* and λ. Because of (4.2.29), condition **C3** is trivially true for every $f \in \mathcal{Q}_\pi$ if $r \geq 0$.

In summary, the above analysis suggests that in order to fulfill all the optimality conditions **C1**, **C2** and **C3**, we need to verify that there exists a constant $\delta \in (0, 1]$ such that conditions (4.2.34), (4.2.35) and $r \geq 0$ are satisfied. From (4.2.35), condition $r \geq 0$ is equivalent to $\Pr\{X \geq d^*\} \leq 1/(1+\theta)$ which corresponds to the assumption (4.2.30) in the theorem. Moreover, the condition (4.2.31) guarantees the existence of δ satisfying (4.2.34) and $0 < \delta \leq 1$. Hence, the proof is complete. □

Remark 4.4 *The conditions (4.2.29), (4.2.30) and (4.2.31) in Theorem 4.2 imply that the following two conditions must hold to make the theorem applicable:*

(a) $\alpha(1+\theta) \leq 1$*;*

(b) $d_\alpha \geq d^* \geq d_\theta$*, where* d_α *and* d_θ *are defined in (4.2.14) and (4.2.16), respectively.*

The last condition, in turn, implies that the reinsurance premium budget π *must satisfy* $\pi_\alpha \leq \pi \leq \pi_\theta$*.*

Case (iii): $\pi \in [\pi_\theta, \infty)$

Theorem 4.3 *Suppose* $\alpha(1+\theta) \leq 1$ *and* $\pi \geq \pi_\theta$*. Then* $f^* = (X - d_\theta)_+$ *is an optimal ceded loss function to the problem* $(\mathbf{P_T})$*.*

Proof. The proof is very similar to that of Theorem 4.1. We will show that the stop-loss treaties f^* satisfies the conditions **C1**, **C2** and **C3** in Proposition 4.1 for appropriately chosen constants $\xi^*, r \in \mathbb{R}$, and random variable $\lambda \in \mathcal{L}^2$. We proceed by first introducing the variable β_θ, which is formally defined as follows:

$$\beta_\theta = \begin{cases} \dfrac{1/(1+\theta) - \Pr\{X > d_\theta\}}{\Pr\{X = d_\theta\}}, & \text{if } \Pr\{X = d_\theta\} \neq 0; \\ 0, & \text{if } \Pr\{X = d_\theta\} = 0. \end{cases} \tag{4.2.37}$$

Note that if follows from the definition of d_θ in (4.2.16) that

$$\Pr\{X > d_\theta\} \leq 1/(1+\theta), \text{ and } \Pr\{X \geq d_\theta\} \geq 1/(1+\theta).$$

Thus, $0 \leq \beta_\theta \leq 1$. Moreover, we also have $\Pr\{X = d_\theta\} = 0$ provided that $\Pr\{X > d_\theta\} = 1/(1+\theta)$.

By setting $\xi^* = d_\theta + (1+\theta)\mathsf{E}[f^*]$, then (4.2.11) becomes

$$g^*(x) = \begin{cases} x - d_\theta, & x < d_\theta; \\ 0, & x \geq d_\theta. \end{cases} \tag{4.2.38}$$

Hence, $\{g^*(X) > 0\}$ is an empty set and $\{g^*(X) = 0\} = \{X \geq d_\theta\}$.

Now set $r = 0$, then condition **C3** is trivially satisfied. Define the random variable λ as

$$\lambda = -(1+\theta)\Big\{\mathbf{1}_{\{X<d_\theta\}} + (1-\beta_\theta)\mathbf{1}_{\{X=d_\theta\}}\Big\}. \tag{4.2.39}$$

Then we have

$$\begin{aligned} \frac{1}{1+\theta}\mathsf{E}[\lambda(f-f^*)] &= -\mathsf{E}\left[(f-f^*)\left(\mathbf{1}_{\{X<d_\theta\}} + (1-\beta_\theta)\mathbf{1}_{\{X=d_\theta\}}\right)\right] \\ &= -\mathsf{E}\left[f\left(\mathbf{1}_{\{X<d_\theta\}} + (1-\beta_\theta)\mathbf{1}_{\{X=d_\theta\}}\right)\right] \\ &\leq 0 \ \text{ for any } \ f \in \mathcal{Q}_f, \end{aligned}$$

where the second equality is due to the fact that $f^*(x) = 0$ for $x \leq d_\alpha$. Thus, condition **C2** is satisfied with random variable λ as defined in (4.2.39).

To verify condition **C1**, first note that it follows from (4.2.37) and (4.2.39) that

$$\mathsf{E}[\lambda] = -\theta,$$

which in turn implies

$$\mathsf{E}\left[\left(1 + \frac{\lambda}{1+\theta}\right)\right] = \frac{1}{1+\theta}$$

and that

$$\begin{aligned}
\mathsf{E}[V_+\mathbf{1}_{\{g^*=0\}}] &\geq \alpha(1+\theta)\mathsf{E}[V_+\mathbf{1}_{\{g^*=0\}}] \\
&= \alpha(1+\theta)\mathsf{E}[V_+\mathbf{1}_{\{X\geq d_\theta\}}] \\
&= \alpha(1+\theta)\mathsf{E}[V_+(1-\mathbf{1}_{\{X<d_\theta\}})] \\
&\geq \alpha(1+\theta)\mathsf{E}[V(1-\mathbf{1}_{\{X<d_\theta\}}-(1-\beta_\theta)\mathbf{1}_{\{X=d_\theta\}})] \\
&= \alpha(1+\theta)\mathsf{E}\left[V\left(1+\frac{\lambda}{1+\theta}\right)\right] \\
&= \alpha(1+\theta)\mathsf{E}\left[\Big((1+\theta)\mathsf{E}[f]-f-\xi\Big)\left(1+\frac{\lambda}{1+\theta}\right)\right] \\
&= \alpha(1+\theta)\left[\mathsf{E}[f]-\mathsf{E}[f]-\mathsf{E}\left(\frac{f\lambda}{1+\theta}\right)-\frac{\xi}{1+\theta}\right] \\
&= -\alpha\Big(\mathsf{E}[f\lambda]+\xi\Big).
\end{aligned}$$

The above result implies that $A(\xi, f) \geq 0$, i.e. condition **C1** is satisfied with the chosen constants ξ^* and r, and random variable λ. Hence the proof is complete. □

Remark 4.5 *Suppose* $\pi \leq \pi_\alpha$ *and there exists a constant* d^* *such that* $(1+\theta)E[(X-d^*)_+] = \pi$*. Then Theorem 4.1 asserts that the stop loss treaty* $f^*(x) = (x-d^*)_+$ *is an optimal ceded loss function. Hence, combining this fact with Theorems 4.2 and 4.3, we see that a stop-loss treaty* $f^*(X) = (X-d^*)_+$ *is optimal for a general reinsurance premium budget* π*, where the retention* d^* *is determined by* $(1+\theta)E[(X-d^*)_+] = \min\{\pi, \pi_\theta\}$.

Remark 4.6 *Suppose an insurer is willing to spend up to* π *with* $\pi \geq \pi_\theta$*, to transfer part of its risk to a reinsurer. Theorem 4.3 asserts that the insurer should only be optimally spending a reinsurance premium budget of* π_θ*. It is not possible to reduce its risk (in terms of smaller CTE) by spending more than* π_θ*.*

4.2.3 Some Numerical Examples

In this section, we present some numerical examples to illustrate the results obtained in the previous section. More specifically, Example 4.1 draws results from Theorem 4.1, Example 4.2 is based on Theorem 4.2, and Example 4.3 uses results from both Theorems 4.2 and 4.3.

Example 4.1 *Assume the loss random variable X is exponentially distributed with mean μ = 1,000, so that it has a survival function S_X and a probability density function f_X as follows.*

$$S_X(x) = e^{-\frac{x}{\mu}}, \quad f_X(x) = \frac{1}{\mu}e^{-\frac{x}{\mu}}, \ \ \textit{for} \ \ x \geq 0.$$

Thus, it follows from the definition in (4.2.14) that

$$d_\alpha = S_X^{-1}(\alpha) = -\mu \ln \alpha, \ \ \textit{for} \ \ 0 < \alpha \leq 1.$$

Consequently, it is easy to verify that

$$\pi_\alpha = (1+\theta)E[(X - d_\alpha)_+] = (1+\theta)\mu\alpha,$$

which represents the maximum level of the reinsurance premium budget for Theorem 4.1 to be applicable. We further assume that $\theta = 0.2$ and $\alpha \in \{0.01, 0.05, 0.10\}$. With these values, the corresponding π_α and d_α are:

$$\begin{cases} \pi_\alpha = 12 \ \ \textit{and} \ \ d_\alpha = 4,605.170, & \textit{for} \ \ \alpha = 1\% \\ \pi_\alpha = 60 \ \ \textit{and} \ \ d_\alpha = 2,995.732, & \textit{for} \ \ \alpha = 5\% \\ \pi_\alpha = 120 \ \ \textit{and} \ \ d_\alpha = 2,302.585, & \textit{for} \ \ \alpha = 10\%. \end{cases}$$

Since Theorem 4.1 applies to the case $\pi \leq \pi_\alpha$, we therefore set $\pi = 10$ so that the condition $\pi \leq \pi_\alpha$ is satisfied for all these three levels α. To illustrate Theorem 4.1, seven reinsurance treaties are considered. The first three treaties are

$$\begin{aligned} &\textit{Tr1}: \ \ f(x) = c^*(x - d_\alpha)_+, \\ &\textit{Tr2}: \ \ f(x) = (x - \hat{d})_+, \\ &\textit{Tr3}: \ \ f(x) = (x - d_\alpha)_+ \wedge l_\alpha, \end{aligned}$$

	$\alpha = 1\%$	$\alpha = 5\%$	$\alpha = 10\%$
d_α	4,605.170	2,995.732	2,302.585
π_α	12	60	120
Minimal CTE	4,781.837	3,839.066	3,229.252
Tr1: $f(x) = c^*(x - d_\alpha)_+$	$c^* = \frac{10}{12}$	$c^* = \frac{2}{12}$	$c^* = \frac{1}{12}$
Tr2: $f(x) = (x - \hat{d})_+$	$\hat{d} = 4,787.492$	$\hat{d} = 4,787.492$	$\hat{d} = 4,787.492$
Tr3: $f(x) = (x - d_\alpha)_+ \wedge l_\alpha$	$l_\alpha = 1,791.760$	$l_\alpha = 182.322$	$l_\alpha = 87.011$
Tr4: $f(x) = cx$, $c = 1/120$	CTE=5,568.460	CTE=3,972.435	CTE=3,285.064
Tr5: $f(x) = c(x - 1,000)_+$, $c = 0.01132617$	CTE=5,568.674	CTE=3,977.465	CTE=3,292.169
Tr6: $f(x) = c(x - 1,500)_+$, $c = 0.01493896$	CTE=5,553.843	CTE=3,968.449	CTE=3,285.656
Tr7: $f(x) = c(x - 2,000)_+$, $c = 0.02052516$	CTE=5,530.911	CTE=3,954.507	CTE=3,275.587

Table 4.1: CTE of some typical reinsurance treaties with $\pi = 10 < \pi_\alpha$.

The remaining four treaties take the form: $f(x) = c(x-d)_+$, *for* $d = 0$, 1000, 1500 *and* 2000 *and these treaties are labeled, respectively, as Tr4-Tr7. For the reinsurance treaty to be well-defined, we have yet to specify its parameter value (such as c in Tr1,* $\hat{d}$ *in Tr2, ...) The required parameter is determined in such a way that the loaded reinsurance premium coincides exactly with the reinsurance premium budget; i.e.*

$$(1+\theta)E[f(X)] = \pi.$$

Applying Theorem 4.1, we see that treaties Tr1, Tr2 and Tr3 are all optimal reinsurance treaties, while Tr4–Tr7 may not be. We have reported the CTE value of the resulting total loss in presence of each treaty in Table 4.1. From this table, we see that the values of CTE for these optimal treaties Tr1, Tr2* and Tr3* are less than those for the treaties Tr4—Tr7. This implies that the treaties Tr4—Tr7 are* *not optimal reinsurance solutions according to the CTE minimization criterion.*

Example 4.2 *In this part of the exercise, we use the same setup as in the previous example except that we increase the reinsurance premium budget to* $\pi = 400$*. This is to ensure that Theorem 4.2 applies with* $\pi_\alpha < \pi < \pi_\theta$*.*

Assume X *is also exponential distributed with mean* $\mu = 1000$ *and the loading factor* $\theta = 0.2$ *as in the previous example, and consider the three confidence levels of* $\alpha = 1\%$*,* 5% *and* 10% *as well. With such a setting, we have* $d_\alpha = S_X^{-1}(1/(1+\theta)) = \mu\ln(1+\theta)$*, and hence* $\pi_\theta = (1+\theta)E[(X-d_\theta)_+] = \mu = 1000$*. Note that in the previous example, we have derived the values of* π_α *as shown in the Table 4.1, which we also report in Table 4.2 for each* α*.*

In order to apply the results of Theorem 4.2, the reinsurance budget π *must satisfy the condition* $\pi_\alpha \le \pi \le \pi_\theta$*. Hence, we take* $\pi = 400$*. By Theorem 4.2, the stop-loss reinsurance* $f(x) = (x - d^*)$ *satisfying* $(1+\theta)E[(X-d^*)_+] = \pi$ *is one of the optimal treaties. We reported the value of retention* d^* *in Table 4.2, as well as the minimal CTE of the insurer's total loss in presence of this stop-loss reinsurance treaty. Moreover, we also considered other eight insurance treaties Tr1—Tr8 (as shown in Table 4.2) with coefficients such that the reinsurance premium for them are all 400. The CTE of insurer's total loss in presence of these treaties are also reported in Table 4.2. Clearly, the CTE value of these treaties Tr1—Tr8 are higher than that of the stop-loss treaty. Moreover, with the retention d in treaties Tr1—Tr8 increasing to the retorsion for the optimal stop-loss, the CTE decreasing to a value close the CTE for the stop-loss treaty Tr1*.*

Example 4.3 *Let* X *be a Pareto random loss variable with survival function* $S_X(x) = \left(\frac{2{,}000}{x+2{,}000}\right)^3$ *for* $x \ge 0$ *so that its mean* $E[X] = 1{,}000$ *is the same as the previous two examples. Assume the loading factor* $\theta = 0.2$ *and confident level* $1-\alpha = 95\%$*. Thus,*

$$d_\alpha = 2{,}000(\alpha^{-1/3} - 1) = 3{,}428.8352, \quad \pi_\alpha = (1+\theta)E[(X-d_\alpha)_+] = 162.8651,$$

	$\alpha = 1\%$	$\alpha = 5\%$	$\alpha = 10\%$
d_α π_α	4,605.170 12	2,995.732 60	2,302.585 120
Tr1*: $f(x) = (x - d^*)_+$, $d^* = 1,098.612$	CTE=1,498.612	CTE=1,498.612	CTE=1,498.612
Tr2: $f(x) = cx$, $c = 0.3333333$	CTE=4,136.780	CTE=3,063.822	CTE=2,601.723
Tr3: $f(x) = c(x - 200)_+$, $c = 0.4071343$	CTE=3,804.540	CTE=2,850.360	CTE=2,439.416
Tr4: $f(x) = c(x - 400)_+$, $c = 0.4972749$	CTE=3,416.770	CTE=2,607.665	CTE=2,259.202
Tr5: $f(x) = c(x - 600)_+$, $c = 0.6073729$	CTE=2,965.165	CTE=2,333.256	CTE=2,061.108
Tr6: $f(x) = c(x - 800)_+$, $c = 0.7418470$	CTE=2,440.469	CTE=2,024.988	CTE=1,846.050
Tr7: $f(x) = c(x - 1,000)_+$, $c = 0.9060939$	CTE=1,832.453	CTE=1,681.317	CTE=1,616.227
Tr8: $f(x) = (x - d)_+ \wedge l$, $d = 1,000, l = 2,365.460$	CTE=3,639.71	CTE=2,090.922	CTE=3,090.922

Table 4.2: CTE of some typical reinsurance treaties with $\pi_\alpha \le \pi = 400 \le \pi_\theta$.

and

$$d_\theta = 2,000\left[\left(\frac{1}{1+\theta}\right)^{-1/3} - 1\right] = 125.3171, \;\; \pi_\theta = (1+\theta)E[(X-d^*_\theta)_+] = 1,062.659.$$

By Theorem 4.2, we know that the stop-loss treaty $f^*(x) = (x - d^*)$ *with* d^* *determined by* $(1+\theta)E[f^*] = \pi$ *is optimal provided the reinsurance premium budget* $\pi \in [162.8651, \;\; 1,062.6586]$*. However, if the reinsurance premium budget* $\pi \geq 1,062.6586$*, it follows from Theorem 4.3 the stop-loss reinsurance* $f^*(x) = (x-d^*)_+$ *with retention* d^* *determined by* $(1+\theta)E[(X-d^*)_+] = 1,062.6586$ *is optimal. In fact, in this case, it is impossible for stop-loss treaties with reinsurance premium larger than* π_θ *to be optimal. This is because the stop-loss treaty will have a retention* d *less than* d_θ *and the total loss* $T_f = X \wedge d + (1+\theta)E[(X-d)_+]$*, which implies*

$$CTE_\alpha(T_f) = d + (1+\theta)\int_d^\infty S_X(x)dx.$$

Clearly, by the above expression, $CTE_\alpha(T_f)$*, as a function of* d*, is decreasing on* $[0, d_\theta]$ *while increasing on* $[d_\theta, \infty)$*.*

4.3 Optimal Reinsurance Model: Binding Case

To complete our analysis on the optimal reinsurance model, this section focuses on the following optimization problem:

$$(\mathbf{Q_0}) \quad \begin{cases} \min_{f\in\mathcal{Q}_f} & \text{CTE}_\alpha(T_f), \\ \text{s.t.} & (1+\theta)\mathsf{E}[f] = \pi. \end{cases} \tag{4.3.40}$$

As in the previously considered optimal reinsurance models, the notation $\mathcal{Q}_f$ denotes the set of feasible ceded loss functions, i.e. $\mathcal{Q}_f = \{f \in \mathcal{L}^2 : 0 \leq f(x) \leq x \text{ for } x \geq 0\}$, and π is an exogenous variable representing the reinsurance premium budget. The only difference between problem $(\mathbf{P_0})$ defined in (4.1.5) and problem ($\mathbf{Q_0}$) lies on how we interpret the reinsurance premium budget π. In the former case, the insurer is willing to spend up to π while in the latter case, the constraint is binding in that the reinsurance premium of the optimal ceded loss function is strictly equal to π. Hence problem ($\mathbf{Q_0}$) is more restrictive and we are interested in its solution for $\pi \in (0, \pi_X)$ where $\pi_X = (1+\theta)\mathsf{E}[X]$.

One motivation for considering the above optimization problem ($\mathbf{Q_0}$) is that it allows us to address more explicitly the tradeoff between risk and reward. To see this, we will first focus on the objective function in model ($\mathbf{Q_0}$) and then on its constraint condition. Recall that the notation $\Gamma(f)$ was introduced at the beginning of the chapter to denote an insurer's net risk of insuring risk X which takes into consideration the premium that the insurer receives from the policyholders and the reinsurance premium the insurer is obligated to pay for reinsuring its risk X; i.e. $\Gamma(f) = T_f - p_0 = X - f(X) + (1+\theta)\mathsf{E}[f] - p_0$. Because of the translation invariance property, we have $\mathrm{CTE}_\alpha(\Gamma(f)) = \mathrm{CTE}_\alpha(T_f) - p_0$. Since p_0 is a constant for a given X, this implies that if f^* is a minimizer of $\mathrm{CTE}_\alpha(T_f)$, it is also a minimizer of $\mathrm{CTE}_\alpha(\Gamma(f))$. In other words, using CTE as the relevant measure of risk, if f^* minimizes the CTE of the insurer's total risk T_f, then it also minimizes the corresponding CTE of the insurer's net risk.

We now shift our attention to the constraint condition in model ($\mathbf{Q_0}$). The term $b(f) \equiv -\mathsf{E}[\Gamma(f)] = p_0 - \mathsf{E}[X] - \theta\mathsf{E}[f]$ captures the insurer's expected net profit in the presence of reinsurance. Note that the insurer's expected net profit depends on the choice of the ceded loss function. Furthermore, the constraint $\mathsf{E}[f] = \frac{\pi}{1+\theta}$, where $\pi = \frac{1+\theta}{\theta}(p_0 - \mathsf{E}[X] - b(f))$, can be interpreted as the profitability requirement in that once the condition is attained, the resulting optimal ceded loss function f^* ensures a certain prescribed level of expected net profit $b(f^*)$. Consequently, f^* that solves model ($\mathbf{Q_0}$) represents the insurer's least risk exposure (as measured by the CTE) for a given level of expected profitability. Hence if model ($\mathbf{Q_0}$) is solved repeatedly for each $\pi \in (0, \pi_X)$, where $\pi_X = (1+\theta)\mathsf{E}[X]$, then we trace out pairs of $\big(\mathrm{CTE}_\alpha(\Gamma(f^*)), b(f^*)\big)$ that give the best possible risk and reward tradeoff. This is analogous to the efficient frontier of the Markowitz portfolio mean-variance analysis. For this reason, we refer the curve represented by $\big(\mathrm{CTE}_\alpha(\Gamma(f^*)), b(f^*)\big)$ as the insurer's reinsurance efficient frontier. Depending on the risk tolerance of an insurer, the reinsurance efficient frontier facilitates the insurer on its optimal selection of ceded loss function.

The mathematical technique used to solve problem $(\mathbf{P_0})$ can similarly be used to derive the optimal solution for problem $(\mathbf{Q_0})$. This entails reformulating problem $(\mathbf{Q_0})$ as

$$(\mathbf{Q_T})\begin{cases} \min\limits_{(\xi,f)\in\mathbb{R}\times\mathcal{Q}_f} & G_\alpha(\xi,f) \equiv \xi + \dfrac{1}{\alpha}\mathsf{E}\left[\Big(X - f + (1+\theta)\mathsf{E}[f] - \xi\Big)_+\right] \\ \text{s.t.} & (1+\theta)\mathsf{E}[f] = \pi. \end{cases} \quad (4.3.41)$$

If (ξ^*, f^*) are the optimal solutions to $(\mathbf{Q_T})$, then f^* is also the optimal solution to $(\mathbf{Q_0})$ (see Lemma 4.1). Moreover, the problem $(\mathbf{Q_T})$ is convex and thus a ceded loss function f^* is a solution to $(\mathbf{Q_T})$ if and only if there exist constants ξ^* and r, and the random variable $\lambda \in \mathcal{L}^2$ such that the three optimality conditions **C1**, **C2** and **C3** in Proposition 4.1 are satisfied except with the binding condition $(1+\theta)\mathsf{E}[f] = \pi$ in defining the set $\mathcal{Q}_\pi$. To avoid any confusion, we will define $\mathcal{Q}'_\pi$ as $\mathcal{Q}'_\pi = \{f \in \mathcal{L}^2 : (1+\theta)\mathsf{E}[f] = \pi\}$ while we reserve $\mathcal{Q}_\pi$ for $\mathcal{Q}_\pi = \{f \in \mathcal{L}^2 : (1+\theta)\mathsf{E}[f] = \pi\}$.

Remark 4.7 *The results in Theorems 4.1 and 4.2 indicate that for any given reinsurance premium budget $\pi \in (0, \pi_\theta]$, the pure stop-loss treaty $f^*(x) = (X - d^*)_+$, where $(1+\theta)E[f^*] = \pi$, is an optimal reinsurance solution to problem $(\mathbf{P_0})$. Note that the optimal retention d^* is determined such that the resulting reinsurance premium coincides with the reinsurance premium budget π. In other words, the optimal ceded loss function is attained at the reinsurance premium budget. Theorem 4.3, on the other hand, reinforces that even if an insurer is willing to spend $\pi \geq \pi_\theta$, the stop-loss treaty is still one possible optimal reinsurance treaty, except that the* *solution is no longer binding. More specifically for $\pi \geq \pi_\theta$, the optimal retention d^* is d_θ and the optimal reinsurance budget is $\pi_\theta \leq \pi$ even though the insurer is willing to spend more. In view of this result, it is therefore never rational for an insurer to spend more than π_θ to reinsure its risk. Nevertheless, it is of theoretical interest to examine the solution to our optimal reinsurance model under the binding reinsurance premium budget constraint, as we establish in the following theorem for $\pi \in (0, \pi_X)$.*

Theorem 4.4 *Assume $\alpha(1+\theta) \leq 1$ and there exists a constant d^* such that $(1+\theta)\mathsf{E}[(X-d^*)_+] = \pi$ for each $\pi \in (0, \pi_X]$. Then the stop-loss treaty $f^*(x) = (x-d^*)_+$ is an optimal solution to problem $(\mathbf{Q_0})$.*

Proof. In view of Remark 4.7, we only need to consider the case with $\pi \in (\pi_\theta, \pi_X]$. Moreover, it suffices to demonstrate that $f^*(x) = (x - d^*)_+$ satisfies the three optimality conditions **C1**, **C2** and **C3** in Proposition 4.1 with $\mathcal{Q}_\pi$ replaced by $\mathcal{Q}'_\pi = \{f \in \mathcal{L}^2 : (1+\theta)\mathsf{E}[f] = \pi\}$.

Before we proceed, we fix a $\pi \geq \pi_\theta$ and denote $\delta = \Pr\{X \geq d^*\}$. Then,

$$(1+\theta)\mathsf{E}[(X-d^*)_+] \equiv \pi > \pi_\theta \equiv (1+\theta)\mathsf{E}[(X-d_\theta)_+]$$

implies $d^* \leq d_\theta$, and hence $\delta \geq \Pr\{X \geq d_\theta\}$, which, together with the fact $\Pr\{X \geq d_\theta\} \geq \frac{1}{1+\theta}$ and the assumption $\alpha(1+\theta) \leq 1$, further implies that $\delta \geq \alpha$.

Note that we only need to verify conditions **C1** and **C2** since under the binding condition $f \in \mathcal{Q}'_\pi$, condition **C3** holds trivially for any constant $r \in \mathbb{R}$. By choosing $\xi^* = d^* + (1+\theta)\mathsf{E}[f^*]$, we have

$$g^*(x) = \begin{cases} x - d^*, & x < d^*; \\ 0, & x \geq d^*. \end{cases} \tag{4.3.42}$$

This implies $\{g^* > 0\}$ is an empty set and $\{g^* = 0\} = \{X \geq d^*\}$. If we further set

$$r = \frac{1}{\delta(1+\theta)} - 1 \text{ and } \lambda = -\frac{1}{\delta}\mathbf{1}_{\{X<d^*\}}, \tag{4.3.43}$$

then for any $f \in \mathcal{Q}'_f$,

$$\delta\mathsf{E}[\lambda(f-f^*)] = -\mathsf{E}\left[(f-f^*)\,\mathbf{1}_{\{X<d^*\}}\right] = -\mathsf{E}\left[f\mathbf{1}_{\{X<d^*\}}\right] \le 0,$$

where the second equality is due to the fact that $f^*(x) = 0$ for $x \le d_\alpha$. Thus, condition **C2** is satisfied with the chosen constants $\xi*$ and r, as well as the random variable λ.

To verify condition **C1**, let us first note that the condition $\delta \ge \alpha$ implies

$$\mathsf{E}[V_+\mathbf{1}_{\{g^*=0\}}] \ge \frac{\alpha}{\delta}\mathsf{E}\left[V_+\mathbf{1}_{\{g^*=0\}}\right] \ge \frac{\alpha}{\delta}\mathsf{E}[V\mathbf{1}_{\{X\ge d^*\}}] = \frac{\alpha}{\delta}\mathsf{E}[V(1+\delta\lambda)].$$

The above result, in turn, leads to

$$\begin{aligned} A(\xi,f) &\ge \alpha\big[\xi + r(1+\theta)\mathsf{E}[f] + \mathsf{E}[\lambda f]\big] + \frac{\alpha}{\delta}\mathsf{E}[V(1+\delta\lambda)] \\ &= \alpha\big[\xi + r(1+\theta)\mathsf{E}[f] + \mathsf{E}[\lambda f]\big] + \frac{\alpha}{\delta}\mathsf{E}\Big[\Big((1+\theta)\mathsf{E}[f] - f - \xi\Big)(1+\delta\lambda)\Big] \\ &= \xi\left(\alpha - \frac{\alpha}{\delta}\mathsf{E}\left[1+\delta\lambda\right]\right) + \alpha\mathsf{E}[\lambda f] - \frac{\alpha}{\delta}\mathsf{E}\left[f\left(1+\delta\lambda\right)\right] \\ &\quad + \mathsf{E}[f]\left(\alpha r(1+\theta) + \frac{\alpha}{\delta}(1+\theta)\mathsf{E}\left[1+\delta\lambda\right]\right). \end{aligned} \tag{4.3.44}$$

Moreover, by (4.3.43) we have $\mathsf{E}\left[(1+\delta\lambda)\right] = \delta$. Thus, (4.3.44) implies that

$$A(\xi,f) \ge \xi(\alpha-\alpha) + \mathsf{E}[f]\Big(\alpha r(1+\theta) + \alpha(1+\theta) - \frac{\alpha}{\delta}\Big) = 0,$$

and hence condition **C1** is also satisfied. This completes the proof. □

Remark 4.8 *From the above theorem, the ceded loss function* $f^*(x) = (x-d^*)_+$ *with* $(1+\theta)E[f^*] = \pi$ *solves model* $\mathbf{Q_0}$ *if* $\alpha(1+\theta) \le 1$ *and the underlying risk* X *is a continuous random variable. Thus the reinsurance efficient frontier is given by*

$$\Big\{\big(CTE_\alpha(\Gamma(f^*)), b(f^*)\big) : f^* = (X-d^*)_+, (1+\theta)E[f^*] = \pi, \ \ and \ \ \pi \in (0,\pi_X]\Big\},$$

where $\Gamma(f^*) = (X \wedge d^*) + \pi - p_0$*, and* $b(f^*) = p_0 - E[X] - \frac{\theta}{1+\theta}\pi$.

Example 4.4 *As in Example 4.1, we similarly assume that the underlying risk X is exponentially distributed with mean $\mu = 1000$. The expectation premium principle is adopted by both the insurer and reinsurer in setting the insurance premium with respective safety loading factor $\eta = 0.1$ and $\theta = 0.2$. We are interested in the reinsurance efficient frontier $\alpha = 5\%$.*

Based on the above setting, we have $d_\alpha = 2995.73, d_\theta = 182.32, \pi_\alpha = 60, \pi_\theta = 1000$ and $\pi_x = (1+\theta)E[X] = 1200$. Theorem 4.4 asserts that to obtain the optimal ceded loss function, we merely need to determine the retention level d^* that satisfies $(1+\theta)E[(X-d^*)_+] = \pi$ for each $\pi \in (0, \pi_X]$. Under the exponential distribution with mean μ, it is easy to show that

$$d^* = \mu \ln\left(\frac{\mu(1+\theta)}{\pi}\right).$$

Furthermore, it is clear that $CTE_\alpha(X \wedge d) = d$ for $d \le d_\alpha \equiv -\mu \ln \alpha$, or equivalently $\pi \ge \pi_\alpha$ For $d \ge d_\alpha$, i.e., $\pi \le \pi_\alpha$,

$$\begin{aligned} CTE_\alpha(X \wedge d) &= d_\alpha + \frac{1}{\alpha}\int_{d_\alpha}^{\infty} \Pr\{X \wedge d > x\} dx \\ &= d_\alpha + \frac{\mu}{\alpha}\left[e^{-d_\alpha/\mu} - e^{-d/\mu}\right] \\ &= \mu(1-\ln\alpha) - \frac{\pi}{\alpha(1+\theta)}. \end{aligned}$$

Thus the reinsurance efficient frontier, $\big(CTE_\alpha(\Gamma(f^*)), b(f^*)\big)$, is given by

$$\begin{aligned} CTE_\alpha(\Gamma(f^*)) &= CTE_\alpha(X \wedge d^*) + \pi - p_0 \\ &= \begin{cases} \mu(1-\ln\alpha) - p_0 + \pi\left[1 - \dfrac{1}{\alpha(1+\theta)}\right], & \pi \le \pi_\alpha, \\ \mu \ln\left(\dfrac{\mu(1+\theta)}{\pi}\right) + \pi - p_0, & \pi \ge \pi_\alpha, \end{cases} \\ &= \begin{cases} -\dfrac{47}{3}\pi + 2895.732, & \pi \le 60, \\ 1000 \ln\left(\dfrac{1200}{\pi}\right) + \pi - 1100, & \pi \ge 60, \end{cases} \end{aligned} \tag{4.3.45}$$

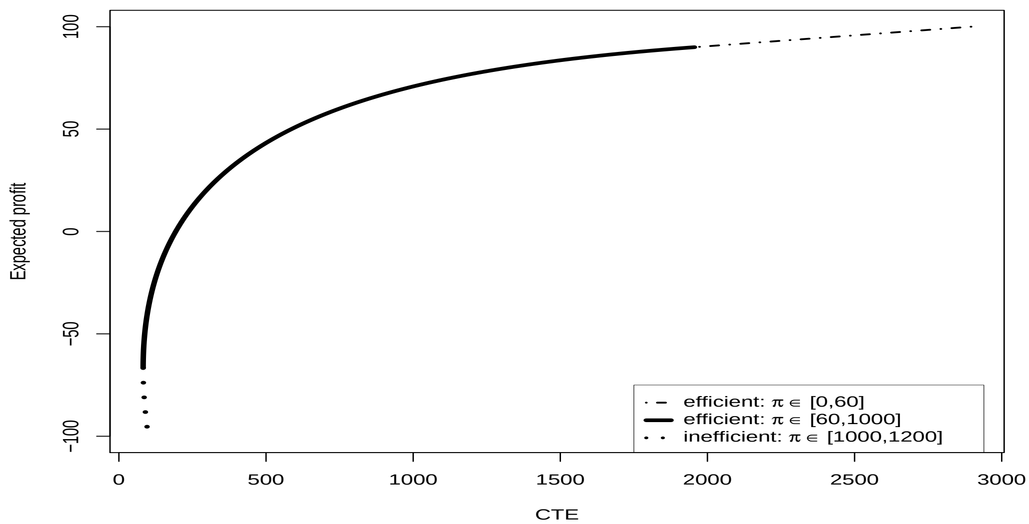


Figure 4.2: Risk reward under optimal reinsurance arrangement.

and

$$b(f^*) = p_0 - E[X] - \frac{\theta}{1+\theta}\pi = 100 - \frac{1}{6}\pi. \tag{4.3.46}$$

Figure 4.2 plots the resulting reinsurance efficient frontier for $\pi \in [0, \pi_X]$. *We now conclude the example with the following remarks:*

(i) It is striking to note that the reinsurance efficient frontier has a tremendous resemblance to the classical Markowitz mean-variance efficient frontier even though the risk in our reinsurance model is captured by the CTE.

(ii) Without reinsurance, the insurer retains the entire amount of the insurance premium and hence its expected profit margin is 100.[2] *This is not surprising since we have assumed that the insurer's loading factor is* $\eta = 10\%$. *Moreover, the insurer's risk exposure in term of CTE reaches its peak at 2895.72.*

[2] In practice, the profit margin will be less than 100 since this amount also includes expenses, adminstration charges, in addition to profits. In our analysis, we ignore these charges for simplicity.

These values can be obtained by setting $\pi = 0$ *in (4.3.45) and (4.3.46). However, as the insurer becomes more risk averse and is willing to spend more on purchasing reinsurance, its expected profit declines but its CTE risk exposure also decreases. This is the classical risk and reward tradeoff. More precisely, as the reinsurance premium budget* π *increases from 0 to 60, both the expected profit and the CTE declines linearly at the rate of* $\frac{1}{6}$ *and* $15\frac{2}{3}$ *from 100 and 2895.72, respectively. The dot dashed line in Figure 4.2 depicts the tradeoff for* $\pi \in [0, 60]$.

(iii) *When the reinsurance premium budget increases beyond 60, the insurer's expected profit continues to drop linearly. The CTE, on the other hand, continue to decrease but it reaches its minimum at* $\pi = \pi_\theta = 1000$. *When* $\pi > 1000$, *the CTE actually increases even though the expected profit is still declining. Consequently, it is never rational to spend more than* 1000 *in reinsuring its risk as already noted in Theorem 4.3. To distinguish these two parts of the frontier, we denote the portion with* $\pi \leq 1000$ *as the efficient frontier while the portion with* $\pi > 1000$ *as the inefficient frontier, in analogous to the Markowitz model. The efficient and inefficient reinsurance frontiers are depicted in Figure 4.2.*

(iv) *We point out that while* $\pi \leq 1000$ *yields a reinsurance frontier that is efficient, we also note that for* $\pi > 600$, *the expected profit of the insurer is negative (see (4.3.46)). Hence under ordinary circumstances, the insurer will not be spending more than 600 on reinsurance, otherwise it would be prudent of not insuring the risk at all.*

4.4 Appendix: Mathematical Background and Optimality Conditions

The main objective of this appendix is to formally prove Proposition 4.1, i.e., to establish the optimality conditions for optimization problem ($\mathbf{P_T}$). The technical details are described in Subsections 4.4.3 and 4.4.4. For completeness, we also collect some key concepts and results associated with convex analysis and directional derivatives in Subsections 4.4.1 and 4.4.2. For a comprehensive review on these aspects, we refer to Bonnans and Shapiro (2000).

4.4.1 Directional Differentiability

Throughout this subsection, let $\mathcal{E}$ and $\mathcal{F}$ denote two vector linear normed spaces and consider a mapping $g : \mathcal{E} \mapsto \mathcal{F}$.

Definition 4.1 *(1) g is said to be directionally differentiable at a point $x \in \mathcal{E}$ in a direction $h \in \mathcal{E}$ if the limit*

$$g'(x)[h] := \lim_{t \to 0^+} \frac{g(x + th) - g(x)}{t}$$

exists, and in this case, $g'(x)[h]$ is called the directional derivative of g at point x in direction h.

(2) If g is directionally differentiable at x in every direction $h \in \mathcal{E}$, then g is said to be directionally differentiable at x.

(3) g is said to be Gâteaux differentiable at x if g is directionally differentiable at x and the directional derivative $g'(x)[h]$ is linear and continuous in h.

Remark 4.9 *By the above definition, g is directionally differentiable at x in a direction h if and only if*

$$g(x + th) = g(x) + tw + o^+(t)$$

for $t \geq 0$, where $o^+(t)$ denotes a function such that $o^+(t)/t \to 0$ as $t \to 0^+$, and w is a vector in $\mathcal{E}$, which indeed is identified as the corresponding directional derivative.

4.4.2 Optimization in Banach Spaces

Throughout this subsection, we assume $\mathcal{E}$ and $\mathcal{F}$ to be two Banach spaces with dual spaces $\mathcal{E}^*$ and $\mathcal{F}^*$. Note that the dual space consists of all the linear and continuous operator which mapping the Banach space into the real line. For any operator $L \in \mathcal{E}^*$ (or $\mathcal{F}^*$), we use $< L, x >$ to denote $L(x)$ for $x \in \mathcal{E}$ (correspondingly $\mathcal{F}$).

Now let Q and K denote, respectively, two nonempty subsets of $\mathcal{E}$ and $\mathcal{F}$, and consider the following program:

$$(P) \quad \begin{cases} \min_{x\in Q} & H_0(x), \\ \text{s.t.} & H_1(x) \in K, \end{cases}$$

where H_0 and H_1 are two mappings such that $H_0 : \mathcal{E} \to \overline{\mathbb{R}}$ and $H_1 : \mathcal{E} \to \mathcal{F}$. Note that the feasible set of Problem (P) is

$$\Phi := \{x \in Q : H_1(x) \in K\} = Q \cap H_1^{-1}(K),$$

where $H_1^{-1}(K) = \{x \in \mathcal{E} : H_1(x) \in K\}$.

Definition 4.2 *A mapping $\Psi : \mathcal{E} \to 2^{\mathcal{F}}$ is called a multifunction, where $2^{\mathcal{F}}$ denotes the power set of $\mathcal{F}$, i.e., the collection of all the subsets of $\mathcal{F}$. Its graph is defined as*

$$gph(\Psi) := \{(x, y) \in \mathcal{E} \times \mathcal{F} : y \in \Psi(x)\}, x \in \mathcal{E},$$

and its (graph) inverse $\Psi^{-1} : \mathcal{F} \to 2^{\mathcal{E}}$ is defined as

$$\Psi^{-1}(y) := \{x \in \mathcal{E} : y \in \Psi(x)\}.$$

Definition 4.3 *(1) A multifunction $\Psi : \mathcal{E} \to 2^{\mathcal{F}}$ is said to be convex, if its graph grh(Ψ) is a convex subset of $\mathcal{E} \times \mathcal{F}$, or equivalently*

$$t\Psi(x_1) + (1-t)\Psi(x_2) \subset \Psi(tx_1 + (1-t)x_2)$$

for any $x_1, x_2 \in \mathcal{E}$ and $t \in [0,1]$.

(2)We say that a mapping $H_1 : \mathcal{E} \to \mathcal{F}$ is convex with respect to a convex closed set $C \subset \mathcal{F}$, or simply that H_1 is C-convex, if the corresponding multifunction $M_{H_1}(x) = H_1(x) + C$ is convex, where $H_1(x) + C$ denotes $\{H_1(x) + y : y \in C\}$.

Remark 4.10 *By the above definition, if $H_0(x)$ is linear then it is convex with respect to any convex subset of $\mathcal{F}$.*

Definition 4.4 *Problem (P) is called convex, if it satisfies these three conditions: (i) $H_0(x)$ is convex, (ii) $H_1(x)$ is convex with respect to the set $(-K)$, and (iii) both Q and K are convex and closed subsets.*

Definition 4.5 *(1) A function $x^* \in \mathcal{E}^*$ is said to be a subgradient of a function $H_0 : \mathcal{E} \to \overline{R}$ at a point $x \in \mathcal{E}$, if $H_0(x)$ is finite and*

$$H_0(y) - H_0(x) \geq < x^*, y - x > \quad \text{for all } y \in \mathcal{E}.$$

(2) The collection of all subgradients of H_0 at x, denoted as $\partial H_0(x)$, is called the subdifferential of H_0 at x, i.e.,

$$\partial H_0(x) = \{x^* \in \mathcal{E}^* : H_0(y) - H_0(x) \geq < x^*, y - x > \quad \text{holds for all } y \in \mathcal{E}\}.$$

(3) H_0 is said to be subdifferentiable at x if $H_0(x)$ is finite and $\partial H_0(x) \neq \varnothing$.

Definition 4.6 *The normal cone of the closed convex subset K of $\mathcal{F}$ at point y_0, denoted as $N_K(y_0)$, is defined as the set $\{\lambda \in \mathcal{F}^* :< \lambda, y - y_0) > \leq 0,$ holds for all $y \in K\}$.*

Lemma 4.2 *Assume that problem (P) is convex. Then one sufficient and necessary condition for a feasible point x_0 to solve problem (P) is as follows: There exists $\lambda \in \mathcal{F}^*$ such that*

$$0 \in \partial_x L(x_0, \lambda) + N_Q(x_0), \quad \text{and } \lambda \in N_K(H_1(x_0)). \tag{4.4.47}$$

Here, $L(x, \lambda)$ denotes the Lagrangian function of problem (P), which is defined as

$$L(x, \lambda) = H_0(x) + < \lambda, H_1(x) >, \quad (x, \lambda) \in \mathcal{E} \times \mathcal{F}^*;$$

$N_K(H_1(x_0))$ and $N_Q(x_0)$ respectively denote the normal cones of the closed convex sets K and Q at corresponding point $H_1(x_0)$ or x_0.

Proof. See Bonnans and Shapiro (2000, p148).

Lemma 4.3 *Suppose $\mathcal{X}$ is a linear vector space, and let f be a convex functional from $\mathcal{X}$ to the extended real line $\overline{\mathbb{R}}$ taking a finite value at a point $x \in \mathcal{X}$, and let $\psi(\cdot)$ denote the directional derivative $f'(x)[\cdot]$ of f. Then $\partial f(x) = \partial \psi(0)$.*

Proof. See Proposition 2.15 of Bonnans and Shapiro (2000, p86). □

4.4.3 Proof of Proposition 4.1 (Optimality Conditions)

The goal function $G_\alpha(\xi, f)$ in problem $(\mathbf{P_T})$ is a functional defined on the product space $H := \mathbb{R} \times \mathcal{L}^2$. It is clear that H is a Hilbert space if we equip it with the inner product $\ll \cdot, \cdot \gg$ defined by $\ll u_1, u_2 \gg = \mathsf{E}[\xi_1 \xi_2 + f_1 f_2] = \xi_1 \xi_2 + \mathsf{E}[f_1 f_2]$ for $u_i = (\xi_i, f_i) \in H$ and $i = 1, 2$. Therefore, we can discuss our problem $(\mathbf{P_T})$ as an optimization problem over the Hilbert space $(H, \ \ll \cdot, \cdot, \gg)$. Recall that we have summarized some key results about optimization on Banach space in Subsection 4.4.2 and note that a Hilbert space is a special Banach space. We will show that $(\mathbf{P_T})$ is a convex problem and then complete the proof by applying Lemma 4.2.

To show the convexity of the problem $(\mathbf{P_T})^3$, we first note that the feasible set $\mathcal{Q} \equiv \mathcal{Q}_f \bigcap \mathcal{Q}_\pi$ of the problem is clearly a closed convex subset of H. Moreover, for any $u_1 = (\xi_1, f_1)$ and $u_2 = (\xi_2, f_2)$ from $\mathcal{Q}$, and any scalar $b \in [0, 1]$, we have

$$
\begin{aligned}
& bG_\alpha(\xi_1, f_1) + (1-b)G_\alpha(\xi_2, f_2) \\
&= b\xi_1 + b\frac{1}{\alpha}\mathsf{E}\left[\Big(X - f_1 + (1+\theta)\mathsf{E}[f_1] - \xi_1\Big)_+\right] \\
&\quad +(1-b)\xi_2 + (1-b)\frac{1}{\alpha}\mathsf{E}\left[\Big(X - f_2 + (1+\theta)\mathsf{E}[f_2] - \xi_2\Big)_+\right] \\
&\geq [b\xi_1 + (1-b)\xi_2] + \frac{1}{\alpha}\mathsf{E}\Big\{\Big[b\Big(X - f_1 + (1+\theta)\mathsf{E}[f_1] - \xi_1\Big) \\
&\quad +(1-b)\Big(X - f_2 + (1+\theta)\mathsf{E}[f_2] - \xi_2\Big)\Big]_+\Big\} \\
&= G_\alpha\Big(b\xi_1 + (1-b)\xi_2, bf_1 + (1-b)f_2\Big), \qquad (4.4.48)
\end{aligned}
$$

which implies the convexity of the functional $G_\alpha(\xi, f)$. Finally, $\mathsf{E}(f)$ is clearly linear as a functional mapping $\mathcal{L}^2$ into $\mathbb{R}$, and hence, in light of Remark 4.10 in Subsection 4.4.2, it is clearly convex with respect to the interval $[0, \pi]$. Therefore, $(\mathbf{P_T})$ is a convex problem.

To proceed, it is worth emphasizing a fact about the Hilbert space H resulted from the Riez representation theorem: For any linear mapping $M \in H^{*4}$, there exists a unique element $(r, \lambda) \in H$ such that

$$
\langle M, (\xi, f)\rangle = \ll (r, \lambda), (\xi, f) \gg \equiv r\xi + \mathsf{E}[\lambda f]
$$

for all $(\xi, f) \in H$. Therefore, the Lagrangian function for Problem $(\mathbf{P_T})$ takes the following form:

$$
L(\xi, f; r) = G_\alpha(\xi, f) + r(1+\theta)\mathsf{E}[f], \qquad (4.4.49)
$$

[3]Refer to Definition 4.4 in Subsection 4.4.2 for the definition of a convex optimization problem on a Banach space.

[4]Here, H^* denotes the dual space of the Hilbert space H. The dual space consists of all the bounded linear functional defined on the Hilbert space H.

for $\xi \in \mathbb{R}$, $f \in \mathcal{L}^2$ and $r \in \mathbb{R}$. Denote $\mathcal{K}_\pi = [0, \pi]$. Then, applying Lemma 4.2, we see that the optimality conditions for $u^* \equiv (\xi^*, f^*)$ to solve problem $(\mathbf{P_T})$ are as follows: There exists constant $r \in \mathbb{R}$ such that

$$r \in N_{\mathcal{K}_\pi}(\mathsf{E}[f^*]) \tag{4.4.50}$$

and

$$0 \in \partial_{(\xi,f)} L(\xi^*, f^*; r) + N_{\mathbb{R}\times\mathcal{Q}_f}(\xi^*, f^*) \tag{4.4.51}$$

are satisfied. Here, $\partial_{(\xi,f)} L(\xi^*, f^*; r)$ denotes the subdifferential of $L(\xi, f; r)$ at point (ξ^*, f^*), $N_{\mathcal{K}_\pi}(\mathsf{E}[f^*])$ is the normal cone to the convex set $\mathcal{K}_\pi$ at $\mathsf{E}[f^*]$, and $N_{\mathbb{R}\times\mathcal{Q}_f}(\xi^*, f^*)$ denotes the normal cone to $\mathbb{R} \times \mathcal{Q}_f$ at point (ξ^*, f^*).

Clearly, it follows from the definition of the normal cone that (4.4.50) is equivalent to $r(m - \mathsf{E}[f^*]) \leq 0$ for all $m \in \mathcal{K}_\pi$, or equivalently

$$r(\mathsf{E}[f] - \mathsf{E}[f^*]) \leq 0 \text{ for all } f \in \mathcal{Q}_f, \tag{4.4.52}$$

which is condition **C3** in the Proposition. To analyze condition (4.4.51), let us investigate $N_{\mathbb{R}\times\mathcal{Q}_f}(\xi^*, f^*)$ first. Suppose $(\zeta, \lambda) \in N_{\mathbb{R}\times\mathcal{Q}_f}(\xi^*, f^*)$, then by the definition of $N_{\mathbb{R}\times\mathcal{Q}_f}(\xi^*, f^*)$, we see

$$\zeta(\xi - \xi^*) \leq 0, \text{ and } \mathsf{E}[\lambda(f - f^*)] \leq 0 \text{ for all } (\xi, f) \in \mathbb{R} \times \mathcal{Q}_f;$$

thus $\zeta = 0$, and (4.4.51) is equivalent to the condition that there exists a random variable $\lambda \in \mathcal{L}^2$ such that

$$\mathsf{E}[\lambda(f - f^*)] \leq 0 \text{ for all } f \in \mathcal{Q}_f, \tag{4.4.53}$$

and

$$(0, -\lambda) \in \partial_{(\xi,f)} L(\xi^*, f^*; r). \tag{4.4.54}$$

Note that (4.4.53) is exactly condition C2 in the proposition, and therefore the proof will be complete if we would show the equivalence between (4.4.54) and condition **C1** in the proposition. For this purpose, we derive the directional derivative of $L(\cdot\ ,\cdot\ ; r)$ as follows:

$$\begin{aligned}\Psi(\xi, f) &:= L'(\xi^*, f^*; r)[\xi, f] \\ &= \xi + \frac{1}{\alpha}\left\{\mathsf{E}[V\mathbf{1}_{\{g^*>0\}}] + \mathsf{E}[V_+\mathbf{1}_{\{g^*=0\}}]\right\} + r(1+\theta)\mathsf{E}[f], \quad (4.4.55)\end{aligned}$$

where g^* and V are defined, respectively, in (4.2.11) and (4.2.12). It is worth noting that it is quite non-trivial in establishing the directional derivative of $L(\cdot\ ,\cdot\ ; r)$. Subsection 4.4.4 provides the details of deriving the directional derivative $\Psi(\xi, f)$, where some approximation results in probability are employed.

Finally, by applying Lemma 4.3 to the established directional derivative Ψ, it is clear that condition (4.4.54) is equivalent to

$$(0, -\lambda) \in \partial\Psi(0, 0),$$

which is further equivalent to condition C1 in the proposition since $\Psi(0, 0) = 0$. Therefore, the proof of Proposition 4.1 is complete.

4.4.4 Directional Derivative of the Lagrangian Function

In this subsection, we shall discuss the directional derivative of the lagrangian function $L(\xi, f; r) : \mathbb{R} \times \mathcal{L}^2 \times \mathbb{R} \mapsto \mathbb{R}$ such that

$$L(\xi, f; r) = G_\alpha(\xi, f) + r\Big[(1+\theta)\mathsf{E}[f] - \pi\Big], \quad (4.4.56)$$

which is defined in (4.4.49) with

$$G_\alpha(\xi, f) = \xi + \frac{1}{\alpha}\mathsf{E}\left[\Big(X - f + (1+\theta)\mathsf{E}[f] - \xi\Big)_+\right]. \quad (4.4.57)$$

Let $g(\xi, f) = X - f + (1+\theta)\mathsf{E}[f] - \xi$, $h(Y) = Y_+$, and $e(Y) = \mathsf{E}[h(Y)]$, then $G_\alpha(\xi, f) = \xi + \frac{1}{\alpha}(e \circ g)(\xi, f)$, where "$e \circ g$" denotes the composition of functions g and e. We shall take several steps to obtain the directional directive of $\mathsf{E}[h(Y)]$ at $Y \in \mathcal{L}^2$ in a direction $Z \in \mathcal{L}^2$, and then obtain the corresponding result for function G_α. Also, recall that the sample space $\Omega = [0, \infty)$ for all random variables in this chapter.

Step 1: Let Z be an indicator random variable such that $Z(x) = a\mathbf{1}_{[\delta_l, \delta_r)}(x)$ for all $x \in \Omega$, where a, δ_l and δ_r are nonnegative constants and $\delta_l < \delta_r$. Then, it follows from the definition of the directional derivative that

$$\begin{aligned} h'(Y)[Z](x) &= \lim_{t \to 0^+} \frac{1}{t}\left[(Y+tZ)_+ - Y_+\right](x) \\ &= \begin{cases} a \cdot \mathbf{1}_{[\delta_l, \delta_r)}(x), & Y(x) \geq 0, \\ 0, & \text{otherwise.} \end{cases} \\ &= Z(x) \cdot \mathbf{1}_{\{Y \geq 0\}}(x). \end{aligned} \tag{4.4.58}$$

Step 2: Suppose Z is a nonnegative simple random variable such that $Z(x) = \sum_{i=1}^n a_i Z_i(x)$ for all $x \in \Omega$, where n is some positive integer, $\{Z_i, i = 1, 2, \cdots, n\}$ are indicator random variables of the form $\mathbf{1}_{[\delta_l, \delta_r)}$ with disjoint domains $[\delta_l, \delta_r)$, and $\{a_i\}_{i=1}^n$ is a sequence of positive real numbers. Then,

$$\begin{aligned} h'(Y)[Z] &= \lim_{t \to 0^+} \frac{1}{t}\left\{(Y+tZ)_+ - Y_+\right\} \\ &= \lim_{t \to 0^+} \frac{1}{t}\left\{\left[Y + t\sum_{i=1}^n a_i Z_i\right]_+ - Y_+\right\} \\ &= \lim_{t \to 0^+} \frac{1}{t}\left\{\sum_{i=1}^n \left[(Y + ta_i Z_i)_+ - Y_+\right]\right\} \\ &= \sum_{i=1}^n \lim_{t \to 0^+} \frac{1}{t}\left\{\left[(Y + ta_i Z_i)_+ - Y_+\right]\right\} \\ &= \sum_{i=1}^n a_i Z_i \cdot \mathbf{1}_{\{Y \geq 0\}} \\ &= Z \cdot \mathbf{1}_{\{Y \geq 0\}}, \end{aligned} \tag{4.4.59}$$

where the third equality is resulted from the assumption that $Z_i, i = 1, 2, \cdots, n$ have disjoint domains, and the last second equality is due to the result obtained in Step 1.

Step 3: Assume Z is a general nonnegative random variable from $\mathcal{L}^2$ and consider $e'(Y)[Z]$, the derivative of $e(\cdot)$ at Y in direction Z, which is defined as

$$e'(Y)[Z] = \lim_{t\to 0^+} \mathsf{E}\left\{\frac{1}{t}\left[(Y+tZ)_+ - Y_+\right]\right\}.$$

Clearly, on $\{x : Y(x) < 0\}$ we have

$$\left|\frac{1}{t}\left[(Y+tZ)_+ - Y_+\right]\right| = \left|\left(\frac{Y}{t}+Z\right)_+ - \left(\frac{Y}{t}\right)_+\right| \leq Z,$$

and on $\{x : Y(x) \geq 0\}$ we have

$$\left|\frac{1}{t}\left[(Y+tZ)_+ - Y_+\right]\right| = \left|\left(\frac{Y}{t}+Z\right) - \left(\frac{Y}{t}\right)\right| = Z.$$

Combining the above, we know that $\left|\frac{1}{t}\left[(Y+tZ)_+ - Y_+\right]\right|$ is uniformly dominated by the integrable random variable Z, and hence it follows from the dominated convergence theorem that

$$e'(Y)[Z] = \mathsf{E}\left\{\lim_{t\to 0^+} \frac{1}{t}\left[(Y+tZ)_+ - Y_+\right]\right\}.$$

On the other hand, it is well known that there exists a nondecreasing sequence of nonnegative simple random variable $\{Z_n, n \geq 1\}$ such that $Z_n \to Z$ almost surely; thus

$$e'(Y)[Z] \;\; = \;\; \mathsf{E}\left\{\lim_{t\to 0^+}\lim_{n\to\infty} \frac{1}{t}\left[(Y+tZ_n)_+ - Y_+\right]\right\}.$$

To proceed, we denote $M(t,n) = \frac{1}{t}\left[(Y+tZ_n)_+ - Y_+\right]$ as a function of the variables t and n. Clearly we know that $M(t,n)$ is nondecreasing in n for any fixed $t > 0$. Now fix n and consider the monotonicity of $M(t,n)$ as a function of t. Note that Z_n is a nonnegative random variable. Thus, on $\{x : Y(x) \geq 0\}$, $M(t,n)$ is uniformly equal to Z_n for all $t \geq 0$. On $\{x : Y(x) < 0\}$, for $0 < t \leq -Y/Z_n$, $M(t,n) = 0-(Y/t)_+ = 0$ and for $t \geq -Y/Z_n$, $M(t,n) = Y/t + Z_n$, which is monotonically decreasing to 0 as t decreases to $-Y/Z_n$. Therefore, for any sample point in Ω and any fixed n, $M(t,n)$ is decreasing as t decreases to 0. This implies that the two limits in the above expression of $e'(Y)[Z]$ are exchangeable and thus

$$\begin{aligned} e'(Y)[Z] &= \mathsf{E}\left\{\lim_{t\to 0^+}\lim_{n\to\infty}\frac{1}{t}\left[(Y+tZ_n)_+ - Y_+\right]\right\} \\ &= \mathsf{E}\left\{\lim_{n\to\infty}\lim_{t\to 0^+}\frac{1}{t}\left[(Y+tZ_n)_+ - Y_+\right]\right\} \\ &= \mathsf{E}\left[\lim_{n\to\infty} Z_n \cdot \mathbf{1}_{Y\geq 0}\right] \\ &= \mathsf{E}\left[Z \cdot \mathbf{1}_{Y\geq 0}\right], \end{aligned} \tag{4.4.60}$$

where the third equality follows from the result obtained in Step 2.

Step 4: Suppose $Z(x) = -b\mathbf{1}_{[\delta_l,\delta_r)}(x)$ for all $x \in \Omega$, where b, δ_l and δ_r are nonnegative real numbers such that $\delta_l < \delta_r$. Clearly, we have

$$\begin{aligned} h'(Y)[Z](x) &= \lim_{t\to 0^+}\frac{1}{t}\left[(Y+tZ)_+ - Y_+\right](x) \\ &= \lim_{t\to 0^+}\frac{1}{t}\{(Y-t(-Z))_+ - Y_+\}(x) \\ &= -\lim_{s\to 0^-}\frac{1}{s}\{(Y+s(-Z))_+ - Y_+\}(x) \\ &= \begin{cases} -b\cdot\mathbf{1}_{[\delta_l,\delta_r)}(x), & Y(x) > 0, \\ 0, & \text{otherwise.} \end{cases} \\ &= Z(x)\cdot\mathbf{1}_{\{Y>0\}}(x). \end{aligned} \tag{4.4.61}$$

Step 5: Suppose Z is a nonnegative simple random variable such that $Z(x) = -\sum_{i=1}^{n} b_i Z_i(x)$ for all $x \in \Omega$, where n is some positive integer, $\{Z_i, i = 1, 2, \cdots, n\}$ are indicator random variables of the form $\mathbf{1}_{[\delta_l,\delta_r)}$ with disjoint domains $[\delta_l, \delta_r)$, and

$\{b_i\}_{i=1}^n$ is a sequence of positive real numbers. Then,

$$
\begin{aligned}
h'(Y)[Z] &= \lim_{t\to 0^+} \frac{1}{t}\Big[(Y+tZ)_+ - Y_+\Big] \\
&= \lim_{t\to 0^+} \frac{1}{t}\Big\{\Big[Y - t\sum_{i=1}^n b_i Z_i\Big]_+ - Y_+\Big\} \\
&= \lim_{t\to 0^+} \frac{1}{t}\left\{\sum_{i=1}^n \Big[(Y - tb_iZ_i)_+ - Y_+\Big]\right\} \\
&= \sum_{i=1}^n \lim_{t\to 0^+} \frac{1}{t}[(Y - tb_iZ_i)_+ - Y_+] \\
&= \sum_{i=1}^n [-b_iZ_i \cdot \mathbf{1}_{\{Y>0\}}] \\
&= Z\cdot \mathbf{1}_{\{Y>0\}}, \qquad (4.4.62)
\end{aligned}
$$

where the third equality is resulted from the assumption that $Z_i, i = 1, 2, \cdots, n$ have disjoint domains, and the last second equality is due to the result obtained in Step 4.

Step 6: Assume Z is a general negative random variable from $\mathcal{L}^2$ and consider $e'(Y)[Z]$, the derivative of $e(\cdot)$ at Y in direction Z, which is defined as

$$e'(Y)[Z] = \lim_{t\to 0^+} \mathsf{E}\left\{\frac{1}{t}\left[(Y+tZ)_+ - Y_+\right]\right\}.$$

Clearly, on $\{x : Y(x) < 0\}$,

$$\left|\frac{1}{t}\left[(Y+tZ)_+ - Y_+\right]\right| = \left|\left(\frac{Y}{t}+Z\right)_+ - \left(\frac{Y}{t}\right)_+\right| = 0,$$

and on $\{x : Y(x) \geq 0\}$,

$$\left|\frac{1}{t}\left[(Y+tZ)_+ - Y_+\right]\right| = \left|\left(\frac{Y}{t}+Z\right)_+ - \left(\frac{Y}{t}\right)\right| \leq -Z.$$

Combining the above, we know that $\frac{1}{t}\left[(Y+tZ)_+ - Y_+\right]$ is uniformly dominated by the integrable random variable $-Z$, and hence it follows from the dominated convergence theorem that

$$e'(Y)[Z] = \mathsf{E}\left\{\lim_{t\to 0^+} \frac{1}{t}\left[(Y+tZ)_+ - Y_+\right]\right\}.$$

On the other hand, it is well known that there exists a decreasing sequence of negative simple random variable $\{Z_n, n \geq 1\}$ such that $Z_n \to Z$ almost surely; thus

$$e'(Y)[Z] \;=\; \mathsf{E}\left\{\lim_{t\to 0^+}\lim_{n\to\infty}\frac{1}{t}\left[(Y+tZ_n)_+ - Y_+\right]\right\}.$$

To proceed, we denote $\overline{M}(t,n) = \dfrac{1}{t}\left[(Y+tZ_n)_+ - Y_+\right]$ as a function of the variables t and n. It is clear that $\overline{M}(t,n)$ is decreasing in n for any fixed $t > 0$. Now fix n and consider the monotonicity of $\overline{M}(t,n)$ as a function of t. Note that Z_n is a negative random variable. Thus, on $\{x : Y(x) < 0\}$, $\overline{M}(t,n) = 0$ uniformly for all $t \geq 0$. On $\{x : Y(x) \geq 0\}$, for $0 < t \leq -Y/Z_n$, $\overline{M}(t,n) = (Y/t + Z_n) - (Y/t)_+ = Z_n$ and for $t \geq -Y/Z_n$, $\overline{M}(t,n) = 0 - Y/t$, which is monotonically decreasing to Z_n as t decreases to $-Y/Z_n$. Therefore, for any sample path in Ω and any fixed n, $\overline{M}(t,n)$ is decreasing as t decreases to 0. This implies that the two limits in the above expression of $e'(Y)[Z]$ are exchangeable and thus

$$\begin{aligned} e'(Y)[Z] &= \mathsf{E}\left\{\lim_{t\to 0^+}\lim_{n\to\infty}\frac{1}{t}\left[(Y+tZ_n)_+ - Y_+\right]\right\} \\ &= \mathsf{E}\left\{\lim_{n\to\infty}\lim_{t\to 0^+}\frac{1}{t}\left[(Y+tZ_n)_+ - Y_+\right]\right\} \\ &= \mathsf{E}\left[\lim_{n\to\infty} Z_n \cdot \mathbf{1}_{Y>0}\right] \\ &= \mathsf{E}\left[Z \cdot \mathbf{1}_{Y>0}\right], \end{aligned} \tag{4.4.63}$$

where the third equality follows from the result obtained in Step 5.

Step 7: Now consider the directional derivative of $e(Y) \equiv \mathsf{E}[h(Y)]$ in the direction of a general random variable $Z \in \mathcal{L}^2$.

Denote $N = \{x : Z(x) < 0\}$, $\overline{N} = \{x : Z(x) \geq 0\}$ and $Z_- = \max\{0, -Z\}$, then we have

$$
\begin{aligned}
e'(Y)[Z] &= \lim_{t\to 0^+} \frac{1}{t}\mathsf{E}\left[(Y+tZ)_+ - Y_+\right] \\
&= \lim_{t\to 0^+} \frac{1}{t}\mathsf{E}\left[(Y\mathbf{1}_{\overline{N}} + Y\mathbf{1}_N + tZ_+\mathbf{1}_{\overline{N}} - tZ_-\mathbf{1}_N)_+ - (Y_+\mathbf{1}_{\overline{N}} + Y_+\mathbf{1}_N)\right] \\
&= \lim_{t\to 0^+} \frac{1}{t}\mathsf{E}\left\{\mathbf{1}_{\overline{N}}\left[(Y+tZ_+)_+ - Y_+\right]\right\} \\
&\quad + \lim_{t\to 0^+} \frac{1}{t}\mathsf{E}\left\{\mathbf{1}_N\left[(Y-tZ_-)_+ - Y_+\right]\right\} \\
&= \lim_{t\to 0^+} \frac{1}{t}\mathsf{E}\left\{\left[(Y+tZ_+)_+ - Y_+\right]\right\} + \lim_{t\to 0^+} \frac{1}{t}\mathsf{E}\left\{\left[(Y-tZ_-)_+ - Y_+\right]\right\} \\
&= e'(Y)[Z_+] + e'(Y)[-Z_-]. \qquad (4.4.64)
\end{aligned}
$$

Applying the results (4.4.60) and (4.4.63) derived in Step 3 and Step 4 respectively, we can immediately obtain the derivative of function $e(\cdot)$ at Y in a direction Z as follows.

$$
e'(Y)[Z] = \mathsf{E}[\mathbf{1}_{\{Y\geq 0\}}Z_+] - \mathsf{E}[\mathbf{1}_{\{Y>0\}}Z_-]. \qquad (4.4.65)
$$

Step 8: Now let us consider the directional derivative of the Lagrangian function $L(\cdot,\cdot;r_1,r_2)$ at $u_0 := (\xi_0, f_0) \in \mathbb{R}\times\mathcal{L}^2$ in direction $u := (\xi, f) \in \mathbb{R}\times\mathcal{L}^2$. Within this step, all the equalities can be understood as $t\to 0^+$ if necessary. Recall that $g(\xi, f) = X - f + (1+\theta)\mathsf{E}[f] - \xi$. Hence,

$$
\begin{aligned}
g(\xi_0 + t\xi, f_0 + tf) &= X - (f_0 + tf) + (1+\theta)\mathsf{E}[f_0 + tf] - (\xi_0 + t\xi) \\
&= g(\xi_0, f_0) + t\Big[(1+\theta)\mathsf{E}[f] - f - \xi\Big].
\end{aligned}
$$

Thus, in light of the directional derivative of e obtained in (4.4.65), we have

$$
\begin{aligned}
&(e\circ g)(u_0 + tu) \\
={}& e\Big(g(\xi_0, f_0) + t\Big[(1+\theta)\mathsf{E}[f] - f - \xi\Big]\Big) \\
={}& \mathsf{E}\Big\{\Big(g(\xi_0, f_0) + t\Big[(1+\theta)\mathsf{E}[f] - f - \xi\Big]\Big)_+\Big\} \\
={}& \mathsf{E}\Big\{\Big[g(\xi_0, f_0)\Big]_+\Big\} + t\mathsf{E}\Big\{(Z_0)_+\mathbf{1}_{\{g_0\geq 0\}} - (Z_0)_-\mathbf{1}_{\{g_0>0\}}\Big\} + o^+(t),
\end{aligned}
$$

where $g_0 = X - f_0 + (1+\theta)\mathsf{E}[f_0] - \xi_0$, $Z_0 = (1+\theta)\mathsf{E}[f] - f - \xi$, and $o^+(t)$ denotes a function such that $o^+(t)/t \to 0$ as $t \to o^+$. Hence, by the definition of directional derivative, we obtain

$$\begin{aligned}\Big(e \circ g\Big)'(\xi_0, f_0)[(\xi, f)] &= \mathsf{E}[(Z_0)_+ \mathbf{1}_{\{g_0 \geq 0\}} - (Z_0)_- \mathbf{1}_{\{g_0 > 0\}}] \\ &= \mathsf{E}[Z_0 \mathbf{1}_{\{g_0 > 0\}} + (Z_0)_+ \mathbf{1}_{\{g_0 = 0\}}],\end{aligned}$$

and therefore,

$$L'(\xi_0, f_0)[(\xi, f)] = \frac{1}{\alpha}\Big[\mathsf{E}\{Z_0 \mathbf{1}_{\{g_0>0\}} + (Z_0)_+ \mathbf{1}_{\{g_0=0\}}\}\Big] + \xi + r(1+\theta)\mathsf{E}[f].$$

Chapter 5

Empirical-based Reinsurance Models

5.1 Introduction

In the last few chapters, we have been focusing on a number of reinsurance models with various degrees of generality. The primary aim of those chapters was to derive analytically the optimal ceded loss functions for the proposed models. We have observed that the tractability of the optimization models highly depends on the model specifications and assumptions and it is non-trivial to derive analytical solutions in many of these cases. In this chapter, we propose a new approach of analyzing optimal reinsurance by explicitly exploiting the loss data that is experienced by the insurance company. Because our proposed optimal reinsurance model is based directly on the empirically observed data, we term this model as the empirical-based reinsurance model, or simply the empirical reinsurance model. We will argue shortly that there is a number of advantages associated with our proposed empirical reinsurance model including its simplicity and tractability.

Recall that a general formulation of an optimal reinsurance model can be expressed as follows:

$$\begin{cases} \min_f & \rho(X,f) \\ \text{s.t.} & 0 \le f(x) \le x, \\ & \Pi(f(X)) \le \pi, \end{cases} \tag{5.1.1}$$

where $\rho(X, f)$ is an appropriately chosen risk measure, the first constraint is the conventional assumption on the ceded loss function f and the second constraint represents the premium budget. Note that $\rho(X, f)$ depends on the assumed loss distribution X and the ceded loss function f. Furthermore, the optimization is carried over all possible functions f so that the above problem is an infinite dimensional optimization problem. Unless additional simplifying assumptions are imposed (such as confining f to a class of increasing convex function and Π is an expectation premium principle), it can be extremely challenging to deriving the analytic solutions to model (5.1.1).

In considering model (5.1.1), an implicit assumption is the availability of the loss distribution X. In practice this is estimated empirically from the observed data. The estimated distribution of X is then incorporated into model (5.1.1) to derive the desired solutions. Instead of using such a two-step process, a natural question to ask is that if we can reformulate an optimal reinsurance model which directly exploits the observed empirical data. If this is possible, then such a model will be of greater interest. It will be intuitively appealing and practical in that it provides a direct linkage between the optimal ceded loss functions and the loss data experienced by an insurer. More importantly, we do not need to make any explicit assumption on the underlying risk. In order to distinguish between the reinsurance model (5.1.1) and our proposed empirical-based reinsurance model, we refer the former model as the *theoretical model* while the latter model as the *empirical model*. Analogously, the optimal solution to the former model is referred as the *theoretical solution* while the optimal solution to the latter model as the *empirical solution*.

The rest of the chapter is organized as follows. Section 5.2 describes the general formulation of our proposed empirical-based reinsurance model. Section 5.3 provides a brief introduction to the second-order cone (SOC) programming. It turns out that many of the empirical reinsurance models can be cast as the SOC programming as we establish in Section 5.4 for the variance minimization, CTE minimization and VaR minimization reinsurance models. Sections 5.5, 5.6 and 5.7, respectively, discuss the solutions to the three empirical models introduced in Section 5.4. Finally, Section 5.8 concludes the chapter by commenting on the pros and cons of the empirical-based reinsurance model.

5.2 General Empirical Reinsurance Models

In this section, we describe our proposed empirical-based reinsurance model. We begin by denoting x_i as the i-th loss (or claim amount) data (before any application of reinsurance) empirically observed by the insurance company. Let $\mathbf{x}^T := (x_1, x_2, \cdots, x_N)$ be a vector which collects all the N empirical data. Based on these experienced data, our objective is to determine an optimal reinsurance coverage f_i corresponds to each loss observation x_i. Obviously f_i is the decision variable and we use the vector $\mathbf{f}^T := (f_1, f_2, \cdots, f_N)$ to represent all the N optimization variables. The principle underling the empirical-based reinsurance model is to formulate the optimization model involving both $\mathbf{x}$ and $\mathbf{f}$ directly. More specifically, corresponding to the theoretical model (5.1.1), our proposed empirical reinsurance model can be formulated in the following symbolic form:

$$\begin{cases} \min\limits_{\mathbf{f}} & \widehat{\rho(\mathbf{x}, \mathbf{f})} \\ \text{s.t.} & 0 \leq f_i \leq x_i, \;\; i = 1, 2, \cdots, N, \\ & \widehat{\Pi(\mathbf{f})} \leq \pi. \end{cases} \tag{5.2.2}$$

Comparing to model (5.1.1), the above objective value $\widehat{\rho(\mathbf{x}, \mathbf{f})}$ depends explicitly on $\mathbf{x}$ and $\mathbf{f}$ and hence can be interpreted as the empirical estimate of $\rho(X, f)$. Similarly, $\widehat{\Pi(\mathbf{f})}$ can be interpreted as the empirical estimate of $\Pi(f(X))$ given the decision vector $\mathbf{f}$.

By construction, the above empirical model (5.2.2) is an N-dimensional optimization problem in that it requires to optimally determine f_i for each empirical data x_i, $i = 1, \cdots, N$. Let $\mathbf{f}^{*T} = (f_1^*, \cdots, f_N^*)$ denote the resulting empirical solutions to model (5.2.2). Note that the empirical reinsurance model effectively transforms an infinite dimensional optimization model (5.1.1) into a reinsurance optimization model that is of N-dimension. Furthermore, the optimal ceded loss function is now represented by a set of finite points (x_i, f_i^*), $i = 1, \cdots, N$, instead of a smooth ceded loss function $f(x)$ in terms of x. However, some standard smoothing techniques such as spline interpolation can always be used if we were interested in a smooth ceded loss function. In our examples to be discussed in later sections, we will represent the solution by simply showing the scatter plots of the pairs $\{(x_i, f_i^*), i = 1, 2, \ldots, N\}$ and then inferring the shape of the optimal ceded loss function. As we will shortly discover, the scatter plots reveal that the optimal ceded functions admit some interesting shapes depending on the actual specification of the empirical models.

To conclude this section, we point out that when implementing our proposed empirical-based reinsurance models, the optimal solutions can depend on how we formally define the empirical estimators of the objective function and the constraints in the empirical models. In our numerical examples, we estimate these quantities using the empirical distribution; i.e., assigning equal probability $1/N$ to each pair (x_i, f_i) for $i = 1, 2, \cdots, N$.

5.3 Second-Order Cone (SOC) Programming

In the last section, we argue that one of the advantages of the proposed empirical-based approach is that it transforms an infinite-dimensional optimization problem into a finite-dimensional problem. However, we have yet to address how to determine the optimal solution $\mathbf{f}^*$ of the resulting empirical models. It turns out that many of our proposed empirical reinsurance models can be cast as a SOC programming and hence numerical techniques associated with solving SOC programming can be used to derive the solutions to our proposed empirical models. Detailed discussion on SOC programming can be found in Alizadeh and Goldfarb (2003), Ben-Tal and Nemirovski (2001), and Lobo, et al. (1998). Below we provide a brief introduction to SOC programming and we draw most of these materials from the references mentioned above.

The most explicit form of a SOC programming is as follows:

$$\begin{cases} \min\limits_{\mathbf{x}\in\mathbb{R}^n} & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} & \|\mathbf{D}_i\mathbf{x}-\mathbf{d}_i\| \le \mathbf{p}_i^T\mathbf{x}-q_i, \quad i=1,2,\cdots,k, \end{cases} \tag{5.3.3}$$

where $\mathbf{x}\in\mathbb{R}^n$ is the optimization variable, and the problem parameters are $\mathbf{c}\in\mathbb{R}^n$, $\mathbf{D_i}\in\mathbb{R}^{(n_i-1)\times n}$, $\mathbf{d}_i\in\mathbb{R}^{n_i-1}$, $\mathbf{p}_i\in\mathbb{R}^n$, and $q_i\in\mathbb{R}$. The norm appearing in the constraints is the standard Euclidean norm, i.e., $\|\mathbf{u}\| = (\mathbf{u}^T\mathbf{u})^{1/2}$. The constraint

$$\|\mathbf{D}_i\mathbf{x}-\mathbf{d}_i\| \le \mathbf{p}_i^T\mathbf{x}-q_i$$

is called a second-order cone constraint (of dimension n). This is because the standard or unit second-order (convex) cone of dimension n is defined as

$$\mathcal{E}_n = \left\{ \begin{pmatrix} \mathbf{u} \\ t \end{pmatrix} : \mathbf{u}\in\mathbb{R}^{n-1}, \ t\in\mathbb{R}, \ \|\mathbf{u}\|\le t \right\}.$$

The above set $\mathcal{E}_n$ is also called the quadratic, ice-cream, or Lorentz cone. Some literature call the SOC programming as the “Conic Quadratic Programming” and accordingly the corresponding constraints as the "Conic Quadratic Constraints". The set of points satisfying a second-order cone constraint is the inverse image of the unit second-order cone under an affine mapping:

$$\|\mathbf{D}_i\mathbf{x} - \mathbf{d}_i\| \leq \mathbf{p}_i^T\mathbf{x} - q_i \Longleftrightarrow \begin{pmatrix} \mathbf{D}_i \\ \mathbf{p}_i^T \end{pmatrix} \mathbf{x} - \begin{pmatrix} \mathbf{d}_i \\ q_i \end{pmatrix} \in \mathcal{E}_{n_i}$$

and hence is convex. Thus, the SOC programming (5.3.3) is a convex programming problem since the objective is a convex function and the constraints define a convex set. Indeed, SOC programming includes a wide class of common convex optimization problems. Linear programs, convex quadratic programs and quadratically constrained convex quadratic programs can all be regarded as special cases of SOC programming problems, as can many other problems that do not fall into these categories. The SOC programming problems can be solved efficiently using several available solvers based on the interior-point method. These softwares include SeDuMi (see Sturm(1999)), SDPT3 (see Tütüncü et al. (2003)) and CVX (see Grant and Boyd (2008)). In this thesis, we will use CVX, which is available for free download on the author's homepage and it is a Matlab based package.

To discuss which type of problems can be cast as the SOC programming problem, it is helpful to introduce the concepts of second-order cone representable (abbreviated SOC-representable) sets and functions. We say a convex set $\mathcal{S} \subset \mathbb{R}^n$ SOC-representable if it can be represented by finitely many second-order cone constraints, possibly after introducing some auxiliary variables, i.e., there exists $\mathbf{D}_i \in \mathbb{R}^{(n_i-1)\times(n+m)}$, $\mathbf{d}_i \in \mathbb{R}^{n_i-1}$, $\mathbf{p}_i \in \mathbb{R}^{n+m}$ and scalar $q_i \in \mathbb{R}$ such that

$$\mathbf{x} \in \mathcal{S} \iff \exists \text{ vector } \mathbf{u} \in \mathbb{R}^m \text{ such that } \left\| \mathbf{D}_i \begin{pmatrix} \mathbf{x} \\ \mathbf{u} \end{pmatrix} - \mathbf{d}_i \right\| \leq \mathbf{p}_i^T \begin{pmatrix} \mathbf{x} \\ \mathbf{u} \end{pmatrix} - q_i, \quad i = 1, 2, \cdots, k.$$

We say a function $g(\cdot)$ is SOC-representable if its graph $\{(x,t) : g(x) \leq t\}$ is a SOC-representable set. Alizadeh and Goldfarb (2003), Ben-Tal and Nemirovski

(2001) and Lobo, et al. (1998) summarized many important SOC-representable sets/functions, as well as many operations under which the SOC-representability preserves for these sets/functions.

Below are some examples of the SOC-representable functions which will be used extensively throughout this chapter:

(A1) Affine function $g(\mathbf{x}) = \mathbf{a}^T\mathbf{x} + b$;

(A2) Convex quadratic function $g(\mathbf{x}) = \mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{q}^T\mathbf{x} + r$, where $\mathbf{Q}$ is a positive semidefinite matrix;

(A3) The L^p-norm $g(\mathbf{x}) = \|\mathbf{x}\|_p = \left(\sum_{i=1}^n x_i^p\right)^{1/p}$ ($p \geq 1$ a rational number).

Note that if $g(\mathbf{x})$ is SOC-representable, then the constraint $g(\mathbf{x}) \leq \mathbf{a}^T\mathbf{x} + b$ is also SOC-representable for any vector $\mathbf{a}$ with appropriate dimension and scalar b. This can be explained by the following equivalence:

$$g(\mathbf{x}) \leq \mathbf{a}^T\mathbf{x} + b \quad \Longleftrightarrow \quad \begin{cases} g(\mathbf{x}) \leq t \\ t \leq \mathbf{a}^T\mathbf{x} + b \end{cases}$$

of which both inequalities on the right hand side are SOC-representable constraints.

A consequence of the SOC-representability of functions $g(x)$, $h(x)$ and set $\mathcal{S}$ is that the optimization problem

$$\begin{cases} \min_x & g(x) \\ \text{s.t.} & x \in \mathcal{S} \\ & h(x) \leq 0 \end{cases}$$

can be cast as a SOC programming.

To discuss what kinds of empirical reinsurance models can be cast as the SOC programming problems, we focus on the general empirical model (5.2.2) and consider two cases. If the goal function $\widehat{\rho(\mathbf{x}, \mathbf{f})}$ is linear, then the resulting empirical model is SOC programming provided that the reinsurance premium budget constraint $\widehat{\Pi(\mathbf{f})} \leq \pi$ is SOC-representable since the conventional constraints

$0 \le f_i \le x_i$, $i = 1, 2, \cdots, N$ are linear. On the other hand, if the goal function is nonlinear, then we can reformulate (5.2.2) into an equivalent optimization problem of the following:

$$\begin{cases} \min_{\mathbf{f},t} & t \\ \text{s.t.} & 0 \le f_i \le x_i, \;\; i = 1, 2, \cdot, N, \\ & \widehat{\Pi(\mathbf{f})} \le \pi, \\ & \widehat{\rho(\mathbf{x}, \mathbf{f})} \le t. \end{cases} \tag{5.3.4}$$

Clearly, $(\mathbf{f}^*, t^*)$ solves (5.3.4) if and only if $\mathbf{f}^*$ solves model (5.2.2). Thus, both $\widehat{\Pi(\mathbf{f})} \le \pi$ and $\widehat{\rho(\mathbf{x}, \mathbf{f})} \le t$ are required to be SOC-representable in order that the empirical reinsurance model is a SOC programming problem.

5.4 SOC Programming and Empirical Reinsurance Models

In this section, we discuss the connection between the empirical reinsurance models and the SOC programming. In particular, Subsection 5.4.1 first shows that the empirical reinsurance premium budget constraint $\widehat{\Pi(\mathbf{f})} \le \pi$ is SOC-representable for as many as ten reinsurance premium principles. Then in subsequent subsections 5.4.2-5.4.4, we consider three specific empirical reinsurance models, namely the variance minimization model, the CTE minimization model, and the VaR minimization model. For each model, we will demonstrate how to reformulate the optimization problem as the SOC programming so that the solutions to these problems can obtained efficiently.

5.4.1 SOC-Representable Reinsurance Premium Constraint

In this subsection, we analyze one by one the premium principle for which the empirical reinsurance premium budget constraint $\widehat{\Pi(\mathbf{f})} \leq \pi$ is SOC-representable. The following notation is used extensively in the subsequent discussions.

- $\mathbf{e}$ denotes an N-dimensional vector with all elements equal to 1.
- $\mathbf{E}$ denotes an $N \times N$ matrix with all elements equal to 1.
- For given vectors $\mathbf{x}^T = (x_1, \ldots, x_N)$ and $\mathbf{f}^T = (f_1, \ldots, f_N)$, we let $\bar{x}$ and $\bar{f}$ be their respective average; i.e. $\bar{x} = \frac{1}{N}\sum_{i=1}^N x_i = \frac{1}{N}\mathbf{e}^T\mathbf{x}$ and $\bar{f} = \frac{1}{N}\sum_{i=1}^N f_i = \frac{1}{N}\mathbf{e}^T\mathbf{f}$.
- $\mathbf{Q}$ is an $N \times N$ matrix with the following specification:

$$\mathbf{Q} = \begin{pmatrix} \mathbf{q}_1^T \\ \vdots \\ \mathbf{q}_N^T \end{pmatrix} = \begin{pmatrix} 1-\frac{1}{N} & -\frac{1}{N} & \cdots & -\frac{1}{N} \\ -\frac{1}{N} & 1-\frac{1}{N} & \cdots & -\frac{1}{N} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{N} & -\frac{1}{N} & \cdots & 1-\frac{1}{N} \end{pmatrix}. \tag{5.4.5}$$

  In the above definition of $\mathbf{Q}$, $\mathbf{q}_i$ is an N-dimensional vector with its i-th element equals to $1 - \frac{1}{N}$ and the remaining entries assign to $-1/N$. It is also easy to verify the following relationship:

$$\sum_{i=1}^N (f_i - \bar{f})^2 = \|\mathbf{f} - \bar{f}\mathbf{e}\|^2 = \mathbf{f}^T\mathbf{Q}\mathbf{f}.$$

In the following discussion of the SOC-representability of the constraint $\widehat{\Pi(\mathbf{f})} \leq \pi$, we recall that the empirical estimator will be constructed based on the empirical distribution.

Q1. Expectation principle: $\Pi(f) = (1+\theta)\mathsf{E}[f]$ with $\theta > 0$.

$$\widehat{\Pi(\mathbf{f})} \leq \pi \iff (1+\theta)\bar{f} \leq \pi \iff \mathbf{e}^T\mathbf{f} \leq \frac{1+\theta}{N}\pi,$$

which is a linear constraint and hence is SOC-representable.

Q2. Standard deviation principle : $\Pi(f) = \mathsf{E}[f] + \beta\sqrt{\mathrm{D}[f]}$, where $\beta > 0$.

$$\begin{aligned} \widehat{\Pi(\mathbf{f})} \leq \pi \quad &\iff \quad \bar{f} + \beta\frac{1}{\sqrt{N}}\|\mathbf{f} - \bar{f}\mathbf{e}\| \leq \pi \\ &\iff \quad \|\mathbf{Q}\mathbf{f}\| \leq -\frac{1}{\beta\sqrt{N}}\mathbf{e}^T\mathbf{f} + \frac{\sqrt{N}}{\beta}\pi, \end{aligned}$$

which is clearly a second-order cone constraint.

Q3. Mixed principle: $\Pi(f) = \mathsf{E}[f] + \beta\mathrm{D}[f]/\mathsf{E}[f]$, where $\beta > 0$.

$$\begin{aligned} \widehat{\Pi(\mathbf{f})} \leq \pi \quad &\iff \quad \bar{f} + \frac{\beta}{\bar{f}}\frac{\|\mathbf{f} - \bar{f}\mathbf{e}\|^2}{N} \leq \pi \\ &\iff \quad \left(\bar{f}\right)^2 + \beta\frac{\|\mathbf{f} - \bar{f}\mathbf{e}\|^2}{N} \leq \pi\bar{f} \\ &\iff \quad \mathbf{f}^T\left(\frac{1}{N}\mathbf{E} + \beta\mathbf{Q}\right)\mathbf{f} \leq \pi\mathbf{e}^T\mathbf{f}, \end{aligned}$$

which is a convex quadratic constraint, since the matrix $\left(\frac{1}{N}\mathbf{E} + \beta\mathbf{Q}\right)$ is positive semi-definite. Thus, $\widehat{\Pi(\mathbf{f})} \leq \pi$ is SOC-representable for mixed principle Π.

Q4. Modified variation principle: $\Pi(f) = \mathsf{E}[f] + \beta\sqrt{\mathrm{D}[f]} + \gamma \mathrm{D}[f]/\mathsf{E}[f]$ with $\gamma, \beta > 0$.

$$\widehat{\Pi(\mathbf{f})} \le \pi \iff \bar{f} + \beta \frac{1}{\sqrt{N}} \|\mathbf{f} - \bar{f}\mathbf{e}\| + \gamma \frac{\|f - \bar{f}\mathbf{e}\|^2}{N\bar{f}} \le \pi$$

$$\iff \begin{cases} \bar{f} + \dfrac{\beta}{\sqrt{N}} \|\mathbf{f} - \mathbf{e}\bar{f}\| \le t_1 \\ \dfrac{\gamma}{N} \|\mathbf{f} - \mathbf{e}\bar{f}\|^2 / \bar{f} \le t_2 \\ t_1 + t_2 \le \pi \end{cases}$$

$$\iff \begin{cases} \|\mathbf{Q}\mathbf{f}\| \le -\frac{1}{\beta\sqrt{N}} \mathbf{e}^T \mathbf{f} + \frac{\sqrt{N}}{\beta} t_1 \\ \left\| \begin{pmatrix} \mathbf{Q}\mathbf{f} \\ \frac{N}{\gamma} \cdot \frac{\bar{f} - t_2}{2} \end{pmatrix} \right\| \le \frac{N}{\gamma} \frac{\bar{f} + t_2}{2} \\ t_2 \ge 0 \\ t_1 + t_2 \le \pi \end{cases}$$

which are second-order cone constraints with two auxiliary decision variables t_1 and t_2.

Q5. p-mean value principle: $\Pi(f) = (\mathsf{E}[f^p])^{1/p}$, where $p > 1$, a rational number.

$$\widehat{\Pi(\mathbf{f})} \le \pi \iff \left(\frac{1}{N}\right)^{1/p} \|\mathbf{f}\|_p \le \pi,$$

which is a second-order cone constraint.

Q6. Semi-deviation principle: $\Pi(f) = \mathsf{E}[f] + \beta \left\{ \mathsf{E}(f - \mathsf{E}[f])_+^2 \right\}^{1/2}$ with $0 < \beta < 1$.

$$\widehat{\Pi(\mathbf{f})} \le \pi \iff \bar{f} + \frac{\beta}{\sqrt{N}} \left(\sum_{i=1}^{N} (f_i - \bar{f})_+^2 \right)^{1/2} \le \pi$$

$$\iff \begin{cases} \left\| (y_1, \cdots, y_N)^T \right\| \le -\dfrac{1}{\beta\sqrt{N}} \mathbf{e}^T \mathbf{f} + \dfrac{\sqrt{N}}{\beta} \pi, \\ y_i \ge 0, \;\; y_i \ge f_i - \dfrac{1}{N} \mathbf{e}^T \mathbf{f}, \;\; i = 1, 2, \cdots, N, \end{cases}$$

which, by definition, are second-order cone constraints with auxiliary variables $y_1, \cdots, y_N$.

Q7. Dutch principle: $\Pi(f) = \mathsf{E}[f] + \beta \mathsf{E}(f - \mathsf{E}[f])_+$ with $0 < \beta \leq 1$.

$$\widehat{\Pi(\mathbf{f})} \leq \pi \iff \bar{f} + \frac{\beta}{N}\sum_{i=1}^{N}\left(f_i - \bar{f}\right)_+ \leq \pi$$
$$\iff \begin{cases} \bar{f} + \dfrac{\beta}{N}\displaystyle\sum_{i=1}^{N} u_i \leq \pi, \\ u_i \geq 0, u_i \geq f_i - \bar{f}, i = 1, 2, \cdots, N, \end{cases}$$

which are linear constraints and hence second-order cone constraints.

Q8. Variance principle: $\Pi(f) = \mathsf{E}[f] + \beta \mathrm{D}[f]$ with $\beta > 0$

$$\widehat{\Pi(\mathbf{f})} \leq \pi \iff \bar{f} + \frac{\beta}{N}\|\mathbf{f} - \mathbf{e}\bar{f}\|^2 \leq \pi$$
$$\iff \mathbf{f}^T\mathbf{Q}\mathbf{f} + \frac{1}{\beta}\mathbf{e}^T\mathbf{f} - \frac{N}{\beta}\pi \leq 0,$$

which is a convex quadratic constraint and hence a second-order cone constraint.

Q9. Semi-variance principle: $\Pi(f) = \mathsf{E}[f] + \beta \mathsf{E}\left(f - \mathsf{E}[f]\right)_+^2$ with $\beta > 0$.

$$\widehat{\Pi(\mathbf{f})} \leq \pi \iff \bar{f} + \frac{\beta}{N}\sum_{i=1}^{N}(f_i - \bar{f})_+^2 \leq \pi$$
$$\iff \begin{cases} \bar{f} + \dfrac{\beta}{N}\displaystyle\sum_{i=1}^{N} y_i^2 \leq \pi, \\ y_i \geq 0, \quad y_i \geq f_i - \bar{f}, i = 1, 2, \cdots, N \end{cases}$$

which, by definition, are clearly second-order cone constraints with auxiliary decision variables $y_1, \cdots, y_N$.

Q10. Quadratic utility principle: $\Pi(f) = \mathsf{E}[f] + \gamma - \sqrt{\gamma^2 - \mathrm{D}[f]}$ with $\gamma > 0$ and

$\gamma^2 \geq \mathrm{D}[f]$.

$$
\begin{aligned}
\widehat{\Pi(\mathbf{f})} \leq \pi \quad &\Longleftrightarrow \quad \bar{f} + \gamma - \sqrt{\gamma^2 - \frac{1}{N}\mathbf{f}^T\mathbf{Q}\mathbf{f}} \leq \pi \\
&\Longleftrightarrow \quad \begin{cases} \frac{1}{N}\mathbf{f}^T\mathbf{Q}\mathbf{f} \leq \gamma^2 \\ \bar{f} + \gamma - \pi \leq 0 \end{cases} \\
&\quad\text{or} \quad \begin{cases} \frac{1}{N}\mathbf{f}^T\mathbf{Q}\mathbf{f} \leq \gamma^2 \\ \bar{f} + \gamma - \pi \geq 0 \\ \left(\bar{f} + \gamma - \pi\right)^2 \leq \gamma^2 - \frac{1}{N}\mathbf{f}^T\mathbf{Q}\mathbf{f} \end{cases} \\
&\Longleftrightarrow \quad \begin{cases} \mathbf{f}^T\mathbf{Q}\mathbf{f} \leq N\gamma^2 \\ \mathbf{e}^T\mathbf{f} \leq N(\pi - \gamma) \end{cases} \\
&\quad\text{or} \quad \begin{cases} \mathbf{f}^T\mathbf{Q}\mathbf{f} \leq N\gamma^2 \\ \mathbf{e}^T\mathbf{f} \geq N(\pi - \gamma) \\ \mathbf{f}^T\left(\frac{1}{N^2}\mathbf{E} + \frac{1}{N}\mathbf{Q}\right)\mathbf{f} + 2\frac{\gamma - \pi}{N}\mathbf{e}^T\mathbf{f} + \pi^2 - 2\gamma\pi \leq 0 \end{cases}
\end{aligned}
$$

The constraints in the above two systems are either linear or convex quadratic. Hence, $\widehat{\Pi(\mathbf{f})} \leq \pi$ can be cast as the union of two SOC-representable sets.

5.4.2 Empirical Reinsurance Model: Variance Minimization

Recall that for the variance minimization model, the objective is to minimize the variance of the insurer's retained loss (or equivalently the total loss). More formally, the theoretical formulation of the reinsurance model can be described as:

$$
\begin{cases} \min_f & \mathsf{Var}(R_f) = \mathsf{Var}(X - f(X)) \\ \text{s.t.} & 0 \leq f(x) \leq x, \\ & \Pi(f) \leq \pi. \end{cases} \tag{5.4.6}
$$

To consider its empirical counterpart, first note that given $\mathbf{x}$ and $\mathbf{f}$, a sample estimate of the objective function $\mathsf{Var}(X - f(X))$ is given by[1]

$$\widehat{\mathsf{Var}(R_f)} = \frac{1}{N}\sum_{i=1}^{N}\left[(x_i - f_i) - (\bar{x} - \bar{f})\right]^2, \tag{5.4.7}$$

where $\bar{x}$ and $\bar{f}$ denotes the sample mean of the observed data $\mathbf{x}$ and the decision variable $\mathbf{f}$, respectively. Similarly, the empirical analog of the constraints are

$$0 \le f_i \le x_i, \;\; i = 1, 2, \cdots, N, \;\; \text{and} \;\; \widehat{\Pi(\mathbf{x})} \le \pi.$$

Consequently, we obtain the following empirical variance minimization model:

$$\begin{cases} \min\limits_{\mathbf{f}\in\mathbb{R}^N} & \widehat{\mathsf{Var}(R_f)} = \frac{1}{N}\sum_{i=1}^{N}\left[(x_i - f_i) - (\bar{x} - \bar{f})\right]^2 \\ \text{s.t.} & 0 \le f_i \le x_i, \;\; i = 1, 2, \cdots, N, \;\; \text{and} \;\; \widehat{\Pi(\mathbf{x})} \le \pi. \end{cases} \tag{5.4.8}$$

Furthermore, by rewriting (5.4.7) as

$$\begin{aligned} \widehat{\mathsf{Var}(R_f)} &= \frac{1}{N}(\mathbf{x} - \mathbf{f})^T\mathbf{Q}(\mathbf{x} - \mathbf{f}) \\ &= \frac{1}{N}\left[\mathbf{f}^T\mathbf{Q}\mathbf{f} - 2\mathbf{x}^T\mathbf{Q}\mathbf{f} + \mathbf{x}^T\mathbf{Q}\mathbf{x}\right]. \end{aligned}$$

The empirical reinsurance model (5.4.8) can be equivalently reformulated as

$$\begin{cases} \min\limits_{\mathbf{f}\in\mathbb{R}^N} & \mathbf{f}^T\mathbf{Q}\mathbf{f} - 2\mathbf{x}^T\mathbf{Q}\mathbf{f} \\ \text{s.t.} & 0 \le f_i \le x_i, \;\; i = 1, 2, \cdots, N, \\ & \widehat{\Pi(\mathbf{f})} \le \pi. \end{cases} \tag{5.4.9}$$

Note that the goal function is a convex quadratic function of $\mathbf{f}$ and hence is SOC-representable. Furthermore, we have demonstrated in Subsection 5.4.1 that the

[1] Alternatively, we could have used the unbiased estimator $\frac{1}{N-1}\sum_{i=1}^{N}\left[(x_i - f_i) - (\bar{x} - \bar{f})\right]^2$ for $\widehat{\mathsf{Var}(R_f)}$. Here we continue to use (5.4.7) in order to be consistent with all other estimators that are based on empirical distribution.

empirical reinsurance premium constraint $\widehat{\Pi(\mathbf{f})} \le \pi$ is SOC-representative for as many as ten premium principles. This implies that the above reinsurance model (5.4.9) can be cast as a SOC programming for the premium principles discussed in Subsection 5.4.1. Section 5.5 will provide a numerical example demonstrating how to solve the re-formulated SOC programming via the CVX software.

5.4.3 Empirical Reinsurance Model: CTE Minimization

Recall that the theoretical CTE minimization model takes the following form:

$$\begin{cases} \min_f \quad \mathrm{CTE}_\alpha(T_f) = \mathrm{CTE}_\alpha\Big(X - f(X) + \Pi[f(X)]\Big) \\ \text{s.t.} \quad\; 0 \le f(x) \le x, \;\; \Pi[f(X)] \le \pi, \end{cases} \tag{5.4.10}$$

where Π is the reinsurance premium principle, $T_f \equiv X - f(X) + \Pi[f(X)]$ denotes the total loss of the insurer in the presence of the reinsurance with a ceded loss function f, and π is a preset reinsurance premium budget. As we argued in Chapter 4, instead of considering the reinsurance model (5.4.10), it is more tractable to consider the following equivalent optimization model:

$$\begin{cases} \min\limits_{(\xi,f)} \quad G_\alpha(\xi, f) = \xi + \dfrac{1}{\alpha}\mathsf{E}\left[\Big(X - f(X) + \Pi(f(X)) - \xi\Big)_+\right] \\ \text{s.t.} \quad 0 \le f(x) \le x, \;\; \Pi(f(X)) \le \pi. \end{cases} \tag{5.4.11}$$

These two models (5.4.10) and (5.4.11) are equivalent in the sense that (ξ^*, f^*) solves (5.4.11) if and only if f^* solves (5.4.10) and ξ^* minimizes $G_\alpha(\xi, f^*)$. This result is due to Rockafellar and Uryasev (2002, Theorem 14).

Because of the tractability of latter model, we similarly focus on the empirical version of model (5.4.11), instead of model (5.4.10). Consequently, the empirical counterpart of model (5.4.11) is simply given by

$$\begin{cases} \min\limits_{(\xi,\mathbf{f})} \quad \widehat{G_\alpha(\xi, \mathbf{f})} = \xi + \dfrac{1}{\alpha N}\sum\limits_{i=1}^{N}\left[\Big(x_i - f_i + \widehat{\Pi(\mathbf{f})} - \xi\Big)_+\right] \\ \text{s.t.} \quad \widehat{\Pi(\mathbf{f})} \le \pi, \;\; \text{and} \;\; 0 \le f_i \le x_i \;\; \text{for} \;\; i = 1, 2, \cdots, N, \end{cases} \tag{5.4.12}$$

where we have taken the "sample average" as the estimator for the expectation $\mathsf{E}\left[\left(X - f(X) + \Pi(f(X)) - \xi\right)_+\right]$ in the theoretical model (5.4.11).

In order to demonstrate the linkage between the above model (5.4.12) and the SOC programming, it is convenient to introduce the auxiliary decision vector $\mathbf{z} = (z_1, \cdots, z_N)^T$, and reformulate model (5.4.12) as follows:

$$\begin{cases} \min\limits_{(\xi,\mathbf{f},\mathbf{z})} & \xi + \dfrac{1}{\alpha N}\displaystyle\sum_{i=1}^{N} z_i \\ \text{s.t.} & \widehat{\Pi(\mathbf{f})} \leq \pi, \\ & 0 \leq f_i \leq x_i, i = 1, 2, \cdots, N, \\ & z_i \geq 0 \ \text{ and } \ z_i \geq x_i - f_i + \widehat{\Pi(\mathbf{f})} - \xi, \ \ i = 1, 2, \cdots, N. \end{cases} \tag{5.4.13}$$

Clearly, model (5.4.13) is equivalent to (5.4.12) in the sense such that $(\xi^*, \mathbf{f}^*)$ solves model (5.4.12) if and only if $(\xi^*, \mathbf{f}^*, \mathbf{z}^*)$ solves (5.4.13) with an appropriately chosen constant vector $\mathbf{z}^*$. The remaining task is to verify that model (5.4.13) can be cast as a SOC programming. To ensure that the above optimization model is SOC programming, we only need to verify both constraints $z_i \geq x_i - f_i + \widehat{\Pi(\mathbf{f})} - \xi$ and $\widehat{\Pi(\mathbf{f})} \leq \pi$ are SOC-representable since the remaining constraints as well as the objective function are linear in the optimization variables. In Subsection 5.4.1 we have already established that there are at least ten premium principles for which the reinsurance premium budget $\widehat{\Pi(\mathbf{f})} \leq \pi$ is SOC-representable. Hence, it remains to verify the SOC-representability of $z_i \geq x_i - f_i + \widehat{\Pi(\mathbf{f})} - \xi$. To do this, it is convenient to first denote $g_i(\xi, \mathbf{f}, \mathbf{z}) = \widehat{\Pi(f)} - f_i - \xi - z_i + x_i$ for $i = 1, 2, \cdots, N$. Then we need to show $\{g_i(\xi, \mathbf{f}, \mathbf{z}) \leq 0, i = 1, 2, \cdots, N\}$ are SOC-representable under each premium principle. The discussion below confirms that these constraints indeed attain SOC-representability for the same set of ten premium principles.

Q1. Expectation principle: $\Pi(f) = (1+\theta)\mathsf{E}[f]$ with $\theta > 0$.

$$\begin{aligned} g_i(\xi, \mathbf{f}, \mathbf{z}) \leq 0 \quad &\Longleftrightarrow \quad (1+\theta)\bar{f} - f_i - \xi - z_i + x_i \leq 0 \\ &\Longleftrightarrow \quad \frac{1+\theta}{N}\mathbf{e}^T\mathbf{f} - f_i - \xi - z_i + x_i \leq 0, \end{aligned}$$

which is a linear constraint and hence is SOC-representable.

Q2. Standard deviation principle : $\Pi(f) = \mathsf{E}[f] + \beta\sqrt{\mathrm{D}[f]}$, where $\beta > 0$.

$$
\begin{aligned}
g_i(\xi, \mathbf{f}, \mathbf{z}) \leq 0 \quad &\Longleftrightarrow \quad \bar{f} + \frac{\beta}{\sqrt{N}}\|\mathbf{f} - \bar{f}\mathbf{e}\| - f_i - \xi - z_i + x_i \leq 0 \\
&\Longleftrightarrow \quad \|\mathbf{Qf}\| \leq -\frac{1}{\beta\sqrt{N}}\mathbf{e}^T\mathbf{f} + \frac{\sqrt{N}}{\beta}(f_i + \xi + z_i - x_i),
\end{aligned}
$$

which, by definition, is clearly a second-order cone constraint and SOC-representable.

Q3. Mixed principle: $\Pi(f) = \mathsf{E}[f] + \beta\mathrm{D}[f]/\mathsf{E}[f]$, where $\beta > 0$.

$$
\begin{aligned}
g_i(\xi, \mathbf{f}, \mathbf{z}) \leq 0 \quad &\Longleftrightarrow \quad \bar{f} + \frac{\beta}{\bar{f}}\frac{\|\mathbf{f} - \bar{f}\mathbf{e}\|^2}{N} - f_i - \xi - z_i + x_i \leq 0 \\
&\Longleftrightarrow \quad \frac{\|\mathbf{Qf}\|^2}{\bar{f}} \leq \frac{N}{\beta}(f_i + \xi + z_i - x_i - \bar{f}) \\
&\Longleftrightarrow \quad \begin{cases} \displaystyle\sum_{j=1}^{N} u_j \leq \frac{N}{\beta}(f_i + \xi + z_i - x_i - \bar{f}) \\ w_j^2 \leq u_j v_j, \;\; j = 1, 2 \cdots, N \\ v_j = \bar{f} \geq 0, \;\; j = 1, 2 \cdots, N \\ w_j = \mathbf{q}_j^T\mathbf{f}, \;\; j = 1, 2 \cdots, N, \end{cases}
\end{aligned}
$$

where $\mathbf{q}_j$ denotes the jth row in the matrix $\mathbf{Q}$. It is clear that all the above constraints are linear except $w_j^2 \leq u_i v_i$, which can be cast as a second-order cone constraint such that $\left\|\left(w_j, \frac{u_j - v_j}{2}\right)^T\right\| \leq \frac{u_j + v_j}{2}$. Thus, $g_i(\xi, \mathbf{f}, \mathbf{z}) \leq 0$ are SOC-representable.

Q4. Modified variation principle: $\Pi(f) = \mathsf{E}[f] + \beta\sqrt{\mathrm{D}[f]} + \gamma\mathrm{D}[f]/\mathsf{E}[f]$, where constants $\gamma, \beta > 0$.

$$
\begin{aligned}
g_i(\xi, \mathbf{f}, \mathbf{z}) \le 0 \quad & \Longleftrightarrow \quad \bar{f} + \frac{\beta}{\sqrt{N}}\|\mathbf{f} - \bar{f}\mathbf{e}\| + \gamma\frac{\|f - \bar{f}\mathbf{e}\|^2}{N\bar{f}} - f_i - \xi - z_i + x_i \le 0 \\
& \Longleftrightarrow \quad \begin{cases} \bar{f} + \beta\frac{1}{\sqrt{N}}\|\mathbf{Q}\mathbf{f}\| \le t_1 \\ \|\mathbf{Q}\mathbf{f}\|^2/\bar{f} \le \frac{N}{\gamma}t_2 \\ t_1 + t_2 \le f_i + \xi + z_i - x_i, \end{cases}
\end{aligned}
$$

$$
\Longleftrightarrow \quad \begin{cases} \|\mathbf{Q}\mathbf{f}\| \le \frac{\sqrt{N}}{\beta}t_1 - \frac{1}{\beta\sqrt{N}}\mathbf{e}^T\mathbf{f} \\ \sum_{j=1}^{N} u_j \le \frac{N-1}{\gamma}t_2 \\ w_j^2 \le u_j v_j, \quad j = 1, 2, \cdots, N \\ v_j = \bar{f} \ge 0, \quad j = 1, 2, \cdots, N \\ w_j = \mathbf{q}_j^T\mathbf{f}, \quad j = 1, 2, \cdots, N \\ t_1 + t_2 \le f_i + \xi + z_i - x_i, \end{cases}
$$

which are second-order cone constraints with two auxiliary decision variables t_1 and t_2.

Q5. p-mean value principle: $\Pi(f) = (\mathsf{E}[f^p])^{1/p}$, where $p > 1$ a rational number.

$$
g_i(\xi, \mathbf{f}, \mathbf{z}) \le 0 \iff \left(\frac{1}{N}\right)^{1/p} \|\mathbf{f}\|_p - f_i - \xi - z_i + x_i \le 0,
$$

which is clearly a second-order cone constraint.

Q6. Semi-deviation principle: $\Pi(f) = \mathsf{E}[f] + \beta\left\{\mathsf{E}(f - \mathsf{E}[f])_+^2\right\}^{1/2}$ with $0 < \beta < 1$.

$$
\begin{aligned}
g_i(\xi, \mathbf{f}, \mathbf{z}) \le 0 \quad & \Longleftrightarrow \quad \bar{f} + \frac{\beta}{\sqrt{N}}\left(\sum_{j=1}^{N}(f_j - \bar{f})_+^2\right)^{1/2} - f_i - \xi - z_i + x_i \le 0 \\
& \Longleftrightarrow \quad \begin{cases} \left\|(y_1, \cdots, y_N)^T\right\| \le \frac{\sqrt{N}}{\beta}(f_i + \xi + z_i - x_i - \bar{f}) \\ y_j \ge 0, y_j \ge f_j - \frac{1}{N}\mathbf{e}^T\mathbf{f}, \quad j = 1, 2, \cdots, N, \end{cases}
\end{aligned}
$$

which are second-order cone constraints.

Q7. Dutch principle: $\Pi(f) = \mathsf{E}[f] + \beta \mathsf{E}([f - \mathsf{E}[f])_+$ with $0 < \beta \leq 1$.

$$g_i(\xi, \mathbf{f}, \mathbf{z}) \leq 0 \iff \bar{f} + \frac{\beta}{N} \sum_{i=1}^{N} \left(f_i - \bar{f}\right)_+ - f_i - \xi - z_i + x_i \leq 0$$

$$\iff \begin{cases} \bar{f} + \dfrac{\beta}{N} \displaystyle\sum_{j=1}^{N} u_j - f_i - \xi - z_i + x_i \leq 0 \\ u_j \geq 0, \;\; u_j \geq f_j - \bar{f}, \;\; j = 1, 2, \cdots, N, \end{cases}$$

which are linear constraints.

Q8. Variance principle: $\Pi(f) = \mathsf{E}[f] + \beta \mathrm{D}[f]$ with $\beta > 0$

$$g_i(\xi, \mathbf{f}, \mathbf{z}) \leq 0 \iff \bar{f} + \frac{\beta}{N} \|\mathbf{f} - \mathbf{e}\bar{f}\|^2 - f_i - \xi - z_i + x_i \leq 0$$

$$\iff \mathbf{f}^T \mathbf{Q} \mathbf{f} + \frac{1}{\beta} \mathbf{e}^T \mathbf{f} - \frac{N}{\beta}(f_i + \xi + z_i - x_i) \leq 0,$$

which is a convex quadratic constraint and hence a second-order cone constraint.

Q9. Semi-variance principle: $\Pi(f) = \mathsf{E}[f] + \beta \mathsf{E}\left(f - \mathsf{E}[f]\right)_+^2$ with $\beta > 0$.

$$g_i(\xi, \mathbf{f}, \mathbf{z}) \leq 0 \iff \bar{f} + \frac{\beta}{N} \sum_{j=1}^{N} [f_j - \bar{f}]_+^2 - f_i - \xi - z_i + x_i \leq 0$$

$$\iff \begin{cases} \bar{f} + \frac{\beta}{N} \sum_{j=1}^{N} y_j^2 - f_i - \xi - z_i + x_i \leq 0 \\ y_j \geq 0, \;\; y_j \geq f_j - \bar{f}, \;\; j = 1, 2, \cdots, N, \end{cases}$$

which, by definition, are clearly second-order cone constraints with auxiliary decision variables $y_1, \cdots, y_N$.

Q10. Quadratic utility principle: $\Pi(f) = \mathsf{E}[f] + \gamma - \sqrt{\gamma^2 - \mathrm{D}[f]}$ with $\gamma > 0$ and $\gamma^2 \geq \mathrm{D}[f]$.

$$
\begin{aligned}
g_i(\xi, \mathbf{f}, \mathbf{z}) \leq 0 \quad &\Longleftrightarrow \quad \bar{f} + \gamma - \sqrt{\gamma^2 - \frac{1}{N}\mathbf{f}^T\mathbf{Q}\mathbf{f}} - f_i - \xi - z_i + x_i \leq 0 \\
&\Longleftrightarrow \quad \begin{cases} \frac{1}{N}\mathbf{f}^T\mathbf{Q}\mathbf{f} \leq \gamma^2 \\ \bar{f} + \gamma - f_i - \xi - z_i + x_i \leq 0, \end{cases} \\
&\quad\ \text{or} \quad\ \begin{cases} \frac{1}{N}\mathbf{f}^T\mathbf{Q}\mathbf{f} \leq \gamma^2 \\ \bar{f} + \gamma - f_i - \xi - z_i + x_i \geq 0 \\ \left(\bar{f} + \gamma - f_i - \xi - z_i + x_i\right)^2 \leq \gamma^2 - \frac{1}{N}\mathbf{f}^T\mathbf{Q}\mathbf{f}. \end{cases}
\end{aligned}
$$

The constraints in the above two systems are either linear or convex quadratic. Hence, $g_i(\xi, \mathbf{f}, \mathbf{z}) \leq 0$ can be cast as the union of two SOC-representable sets.

5.4.4 Empirical Reinsurance Model: VaR Minimization

Recall that if the objective were to minimize the VaR of the insurer's total risk in the presence of reinsurance, then the optimal reinsurance model can be written as:

$$
\begin{cases} \min\limits_{f} & \mathrm{VaR}_\alpha(T_f) = \mathrm{VaR}_\alpha(X - f(X)) + \Pi(f) \\ \text{s.t.} & 0 \leq f(x) \leq x, \ \ \Pi[f(X)] \leq \pi. \end{cases} \tag{5.4.14}
$$

To construct the empirical version of the above VaR minimization model (5.4.14), first note that the quantity $\mathrm{VaR}_\alpha(X - f(X))$ with ceded loss function f is defined as

$$
\mathrm{VaR}_\alpha(X - f(X)) = \min\left\{\xi \in \mathbb{R} : \Pr\left(X - f(X) \leq \xi\right) \geq \alpha\right\},
$$

and thus its empirical estimate is given by

$$
\mathrm{VaR}_\alpha(\mathbf{f}) = \max_{1 \leq i \leq N}{}^{\lfloor \alpha N \rfloor + 1}(x_i - f_i),
$$

where $\lfloor\cdot\rfloor$ denotes the integer part and $\max\limits_{1\le i\le N}{}^{k}$ denotes the kth biggest element. Consequently, we obtain the following empirical VaR minimization model

$$\begin{cases} \min\limits_{\mathbf{f}} & \max_{1\le j\le N}{}^{\lfloor\alpha N\rfloor+1}\ x_i - f_i + \widehat{\Pi(\mathbf{f})} \\ \text{s.t.} & 0 \le f_i \le x_i,\ \ i = 1, 2, \cdots, N, \\ & \widehat{\Pi(\mathbf{f})} \le \pi. \end{cases} \tag{5.4.15}$$

It should be emphasized that the above empirical VaR minimization model is no longer a convex optimization problem since generally VaR is not convex. Despite the lack of convexity and hence its posing additional challenge in obtaining the optimal solution, Section 5.7 will demonstrate how a solution can be obtained by using some heuristic algorithms.

5.5 Empirical Solutions to the Variance Minimization Model

We have illustrated how to develop the empirical model based on the theoretical variance minimization model in the last section. The resulting empirical model is shown in (5.4.9) where a convex quadratic function is minimized. Thus, model (5.4.9) can be cast as a SOC programming for premium principles Q1-Q10 discussed in Subsection 5.4.1. In this section, we provide some numerical illustrations of our proposed empirical variance minimization model by focusing on two reinsurance premium principles; i.e., Q1 (expectation principle) and Q2 (standard deviation principle). We also demonstrate that the solutions can be obtained efficiently using some existing SOC programming softwares such as the CVX.

5.5.1 Expectation Principle

Under the expectation premium principle, the reinsurance premium budget constraint becomes $(1+\theta)\bar{f} \leq \pi$. Clearly, this is a linear constraint and hence the empirical variance minimization model remains to be a convex quadratic programming (it is also a SOC programming) even if we were to replace the budget constraint with a binding one, i.e., $(1+\theta)\bar{f} = \pi$. With such modification, the empirical model revises to

$$\begin{cases} \min\limits_{\mathbf{f}\in\mathbb{R}^N} & \mathbf{f}^T\mathbf{Q}\mathbf{f} - 2\mathbf{x}^T\mathbf{Q}\mathbf{f} \\ \text{s.t.} & 0 \leq f_i \leq x_i, \quad i = 1, 2, \cdots, N, \\ & (1+\theta)\dfrac{1}{N}\mathbf{e}^T\mathbf{f} = \pi. \end{cases} \tag{5.5.16}$$

The above reinsurance model is of theoretical interest in that it can be considered as the empirical version of the classical variance minimization reinsurance model. Recall that the classical variance minimization reinsurance model seeks optimal reinsurance by minimizing the insurer's total loss subject to the binding insurance premium. It is also well-known that the optimal reinsurance design for the classical model is the stop-loss reinsurance. See, for example, Bowers, et al. (1997), Kaas, et al. (2001) and Gerber (1979). Such theoretic result, therefore, can be used to benchmark against the optimal solution $\mathbf{f}^*$ obtained from the empirical model (5.5.16). This explains why we focus on the empirical model with a binding, instead of unbinding, reinsurance premium constraint.

We now present two numerical examples to illustrate the applicability of our proposed empirical-based variance minimization reinsurance models. The numerical illustrations consist of the following steps:

Step 1: Simulate random samples $x_i, i = 1, \ldots, N$ from an appropriately chosen loss distribution. The simulated N samples are assumed to be the empirically observed loss data. In our examples, we simulate $N = 300$ random samples from exponential and Pareto distributions restrictively.

Step 2: Predetermine the loading factor θ and the reinsurance premium budget π. In our examples, we set $\theta = 0.2$ and $\pi = 400$, 600, 800, and 1000.

Step 3: Use CVX to obtain solutions $\mathbf{f}^* = (f_1^*, \ldots, f_N^*)$.

Step 4: Depict the resulting solutions using scatter plots for (x_i, f_i^*), $i = 1, 2, \cdots, N$.

As pointed out earlier that the optimal ceded loss function is a stop-loss for the theoretical variance minimization model. This implies that the scatter plot produced from Step 4 of the above empirical solutions should mimic the shape of a stop-loss reinsurance.

Example 5.1 Variance Minimization Model with the Expectation Principle and Exponential Loss Distribution

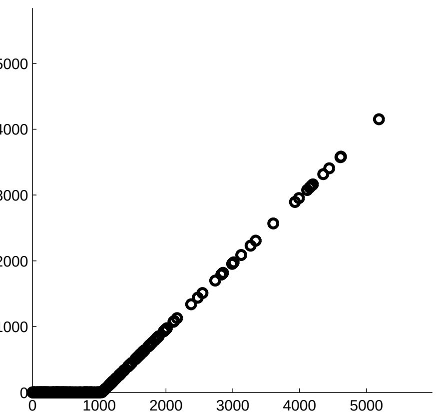
In this example, we carry out the above Steps 1-4 by drawing random samples from an exponential distribution with mean μ = 1,000, i.e,

$$F_X(x) = 1 - e^{-\frac{x}{\mu}}, \;\; x \geq 0.$$

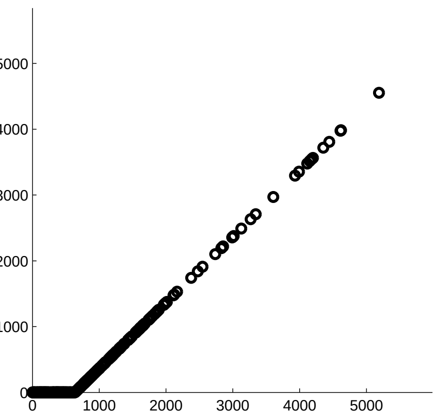
The scatter plots of the solutions for $\pi = 400$, 600, 800, and 1000 are shown in Figure 5.1. It is reassuring that the resulting shape of the empirical solutions looks like a stop-loss function, which is consistent with the classical result. Furthermore, as the insurer is willing to spend more on the reinsurance premium, more risk is transferred to a reinsurer as indicated by the lower stop-loss retention with higher π.

Example 5.2 Variance Minimization Model with the Expectation Principle and Pareto Loss Distribution

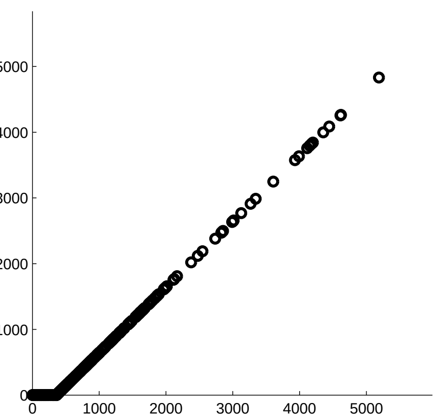
This example is similar to the last example except that the empirical samples are



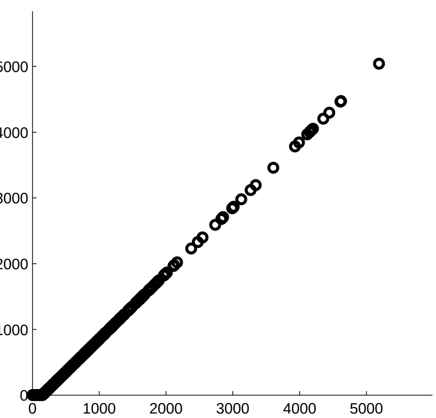
1) $\pi = 400$



2) $\pi = 600$



3) $\pi = 800$



4) $\pi = 1,000$

Figure 5.1: Empirical solutions to the variance minimization model with expectation principle and exponential loss distribution.

drawn from a Pareto distribution with mean $\mu = 1,000$*, i.e.,* $F_X(x) = 1 - \left(\frac{2,000}{x+2,000}\right)^3$, $x \geq 0$*. The solutions are presented in Figure 5.2 and the scatter plots are also consistently revealing that the shape of the optimal ceded loss function behave like a stop-loss, in accordance with the theoretical results.*

5.5.2 Standard Deviation Premium Principle

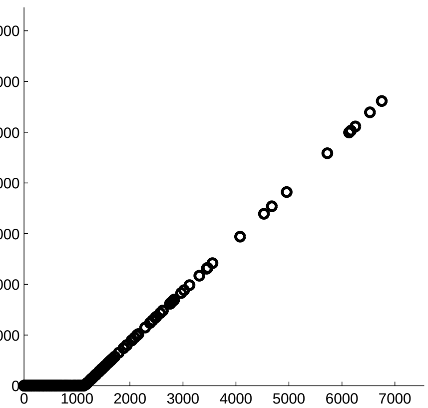
As we have established in Subsection 5.4.1, the reinsurance premium budget under the standard deviation principle is equivalent to $\|\mathbf{Qf}\| \leq -\frac{1}{\beta\sqrt{N}}\mathbf{e}^T\mathbf{f} + \frac{\sqrt{N}}{\beta}\pi$. This suggests that the empirical variance minimization model reduces to

$$\begin{cases} \min\limits_{\mathbf{f}\in\mathbb{R}^N} & \mathbf{f}^T\mathbf{Qf} - 2\mathbf{x}^T\mathbf{Qf} \\ \text{s.t.} & 0 \leq f_i \leq x_i, \;\; i = 1, 2, \cdots, N, \\ & \|\mathbf{Qf}\| \leq -\frac{1}{\beta\sqrt{N}}\mathbf{e}^T\mathbf{f} + \frac{\sqrt{N}}{\beta}\pi. \end{cases} \tag{5.5.17}$$

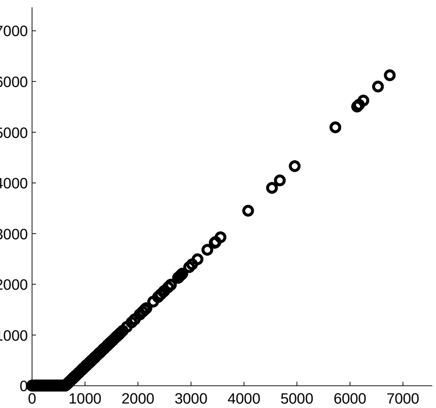
We study the empirical solutions of the above reinsurance model through the following two numerical examples 5.3 and 5.4.

Example 5.3 Variance Minimization Model with the Standard Deviation Principle and Exponential Loss Distribution

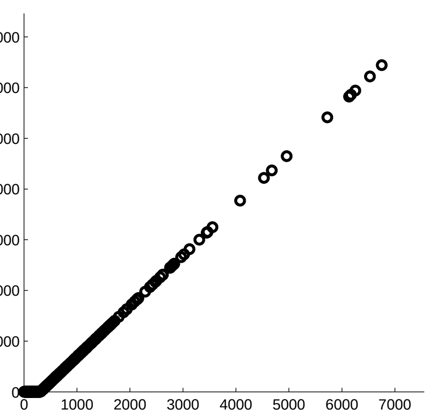
The setup of this example is similar to Example 5.1 in that the same exponential loss distribution is used to generate the empirical loss data and the CVX is also used to solve the resulting SOC programming. The only difference is that the reinsurance premium is determined by the standard deviation premium principle with loading factor $\beta = 0.2$*. The four scatter plots in Figure 5.3 depict the optimal reinsurance treaty* $\mathbf{f}^*$ *for different levels of reinsurance premium budget. These scatter plots still indicate that the stop-loss reinsurance treaty is optimal even when we modify the premium principle from the expectation principle to the standard deviation principle.*



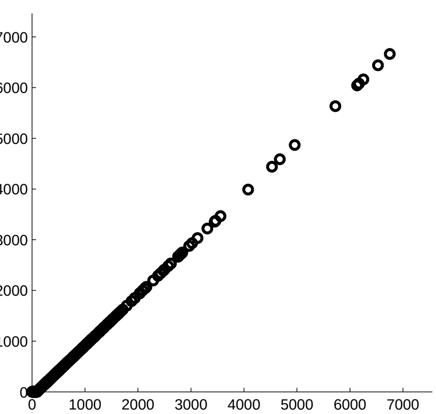
1) $\pi = 400$



2) $\pi = 600$

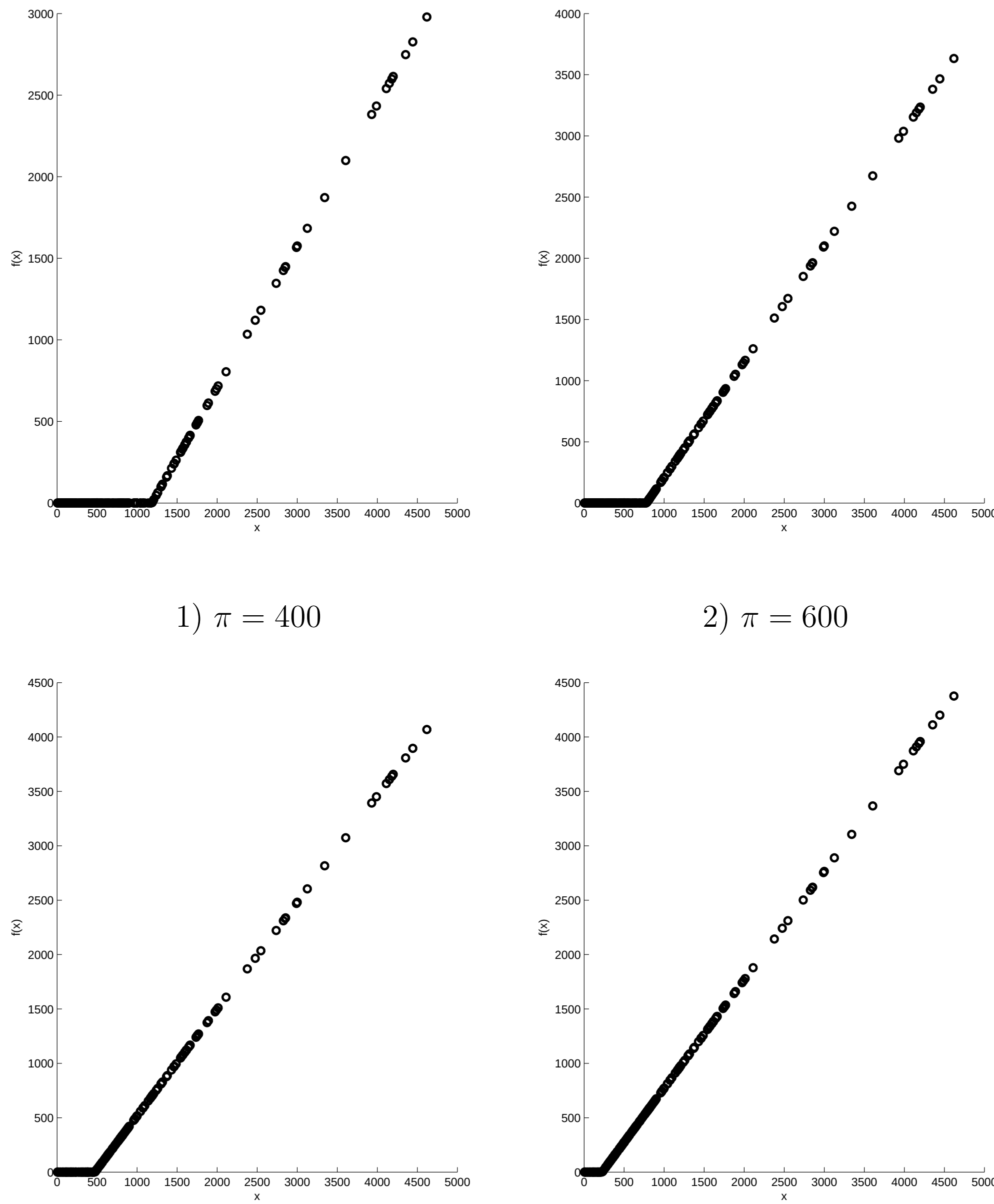


3) $\pi = 800$



4) $\pi = 1,000$

Figure 5.2: Empirical solutions to the variance minimization model with expectation principle and Pareto loss distribution.



1) $\pi = 400$ 2) $\pi = 600$

3) $\pi = 800$ 4) $\pi = 1,000$

Figure 5.3: Empirical solutions to the variance minimization model with standard deviation principle and exponential loss distribution.

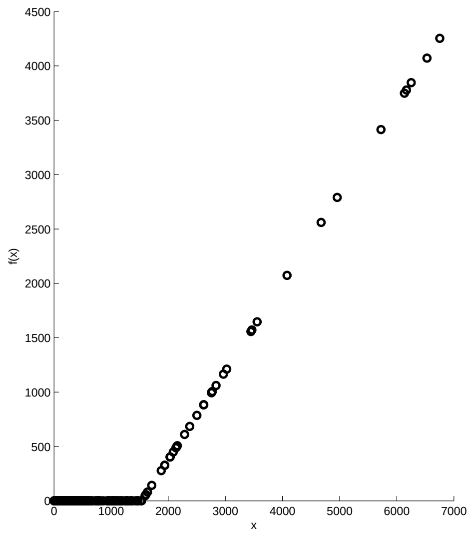
Example 5.4 Variance Minimization Model with the Standard Deviation Principle and Pareto Loss Distribution
This example is again similar to Example 5.3 except that Pareto distribution is employed. The results obtained by using CVX to the resulting programming problem are presented by the four scatter plots in Figure 5.4. Similar to the exponentially distributed case, the scatter plots suggest that stop-loss reinsurance is optimal.

5.6 Empirical Solutions to the CTE Minimization Model

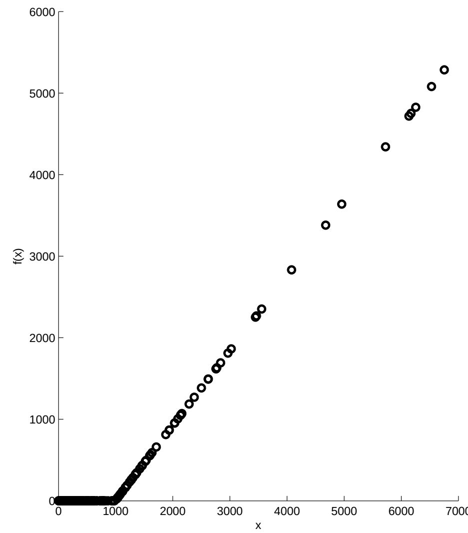
Recall that in Subsection 5.4.3 we have established for many of the premium principles that are of interest to us, that the resulting empirical CTE minimization model can be cast as a SOC programming. The aim of this section is to provide an additional insight to our empirical CTE minimization model by confining to two specific premium principles: Q1 (the expectation principle) and Q2 (the standard deviation principle). This is elaborated in greater details in the following two subsections.

5.6.1 Expectation Reinsurance Premium Principle

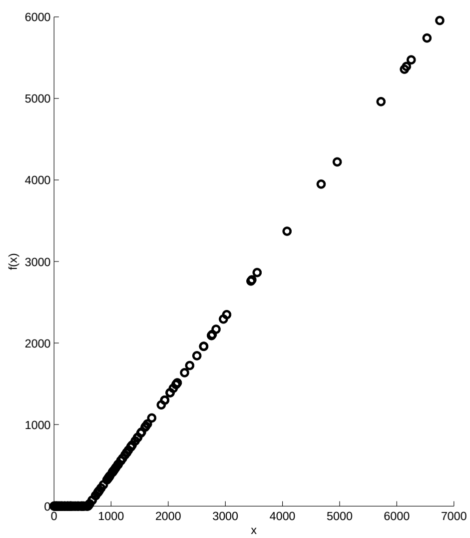
In this subsection, we assume that the reinsurance premium is calculated according to the expectation principle with a safety loading $\theta > 0$ and we shall discuss the optimal reinsurance treaties to the empirical CTE minimization model (5.4.13). Before doing so, let us first recall the result we established in the previous chapter, which states that a stop-loss reinsurance solves the theoretical model (5.4.10) under the expectation reinsurance premium principle. Thus, for the empirical model (5.4.13), we should also expect to derive a solution consistent to the stop-loss treaty.



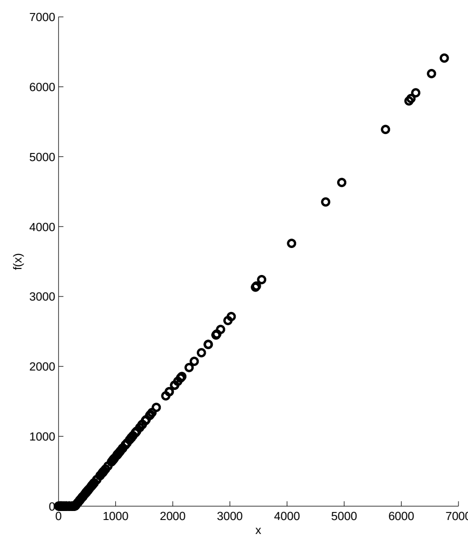
1) $\pi = 400$



2) $\pi = 600$



3) $\pi = 800$



4) $\pi = 1,000$

Figure 5.4: Empirical solutions to the variance minimization model with standard deviation principle and Pareto loss distribution.

By consistency, we mean that the empirical solutions have the same functional shape as the theoretical solutions. We will further discuss the consistency of the empirical solutions in the next chapter.

With the expectation principle and a safety loading factor θ, the empirical version of $\Pi(f(X))$ becomes $\widehat{\Pi(\mathbf{f})} = (1+\theta)\bar{f}$ and thus model (5.4.13) reduces to

$$\begin{cases} \min_{(\xi,\mathbf{f},\mathbf{z})} & \xi + \frac{1}{\alpha N}\sum_{i=1}^{N} z_i, \\ \text{s.t.} & \bar{f} \leq \pi/(1+\theta), \\ & 0 \leq f_i \leq x_i, \;\; i = 1, 2, \cdots, N \\ & z_i \geq 0, \;\; z_i \geq x_i - f_i + (1+\theta)\bar{f} - \xi, \;\; i = 1, 2, \cdots, N, \end{cases} \tag{5.6.18}$$

which is obviously a linear programming problem and thus it can be solved by simplex method or interior-point method in polynomial time. Parallel to Examples 5.1 and 5.2 in Subsection 5.5.1 for the variance minimization model, Examples 5.5 and 5.6 repeat the same analysis except for the empirical CTE minimization model (5.4.13). These examples suggest that the stop-loss reinsurance treaties are optimal for these models as we demonstrate below.

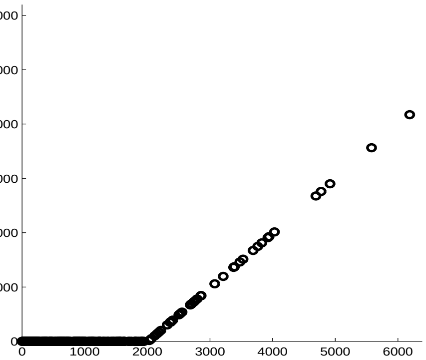
Example 5.5 CTE Minimization Model with the Expectation Principle and Exponential Loss Distribution

Similar to the previous examples, we create $N = 300$ empirical loss data by first sampling from an exponential distribution with mean $\mu = 1,000$. Then together with parameter values $\alpha = 5\%$ and $\theta = 0.2$, the CVX is used to solve the resulting empirical CTE minimization model (5.6.18) over various reinsurance premium budgets: $\pi \in \{200, 400, 600, 800, 1000, 1500\}$. The scatter plots of the solutions are shown in Figure 5.5. It is first interesting to note that the stop-loss reinsurance can still be optimal even when we change the optimality objective from minimizing variance to minimizing CTE. Second, when an insurer is willing to increase its premium budget on reinsuring its risk, more risks can be transferred to a reinsurer as indicated by the lower levels of retention. Third, as the reinsurance premium budget increases *beyond 1000, the stop-loss retention seems to remain unchanged. This phenomenon in fact is consistent with the theoretical results established in the last chapter. In particular, Remark 4.5 asserts that the stop-loss treaty* $f^*(x) = (x - d^*)_+$ *with the retention* d^* *satisfying* $(1+\theta)E[(x-d^*)_+] = \min\{\pi, \pi_\theta\}$ *is an optimal solution to the theoretical CTE minimization model. In our numerical setting, it is easy to verify that* $\pi_\theta = 1000$ *which suggests the optimal retention will not change for any premium budget greater than 1000. Hence our empirical solutions appear to be aligned with the theoretical results. In the next chapter, we will provide an in-depth analysis addressing the stability and consistency of the empirical solutions.*

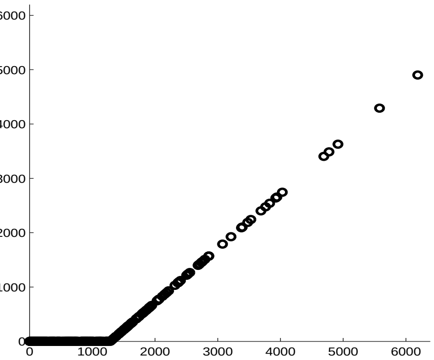
Example 5.6 CTE Minimization Model with the Expectation Principle and Pareto Loss Distribution
The setup for this example is identical to the last example except that we use the Pareto distribution with mean μ *= 1,000 and we assume* $\pi \in \{200, 400, 600, 800, 1062, 1500\}$*. The solutions in Figure 5.6 again suggest that the optimal ceded loss function has the same structure as the exponential case. Hence the observations and conclusion that we made for the exponential case are similarly applied to the Pareto case. Note however that in this example,* $\pi_\theta = 1062$ *and this explains why we consider this particular reinsurance premium budget in this example.*

5.6.2 Standard Deviation Reinsurance Premium Principle

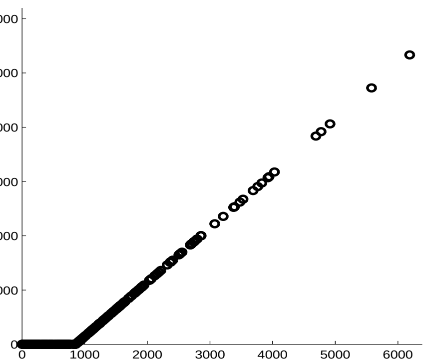
Let us now assume that the reinsurance premium is determined by the standard deviation principle. Under this special case, the empirical reinsurance premium budget constraint $\widehat{\Pi(\mathbf{f})} \leq \pi$ reduces to $\|\mathbf{Qf}\| \leq -\frac{1}{\beta\sqrt{N}}\mathbf{e}^T\mathbf{f} + \frac{\sqrt{N}}{\beta}\pi$ (see Subsection 5.4.1), and the constraint $\widehat{\Pi(f)} - f_i - \xi - z_i + x_i \leq 0$ becomes $\|\mathbf{Qf}\| \leq -\frac{1}{\beta\sqrt{N}}\mathbf{e}^T\mathbf{f} + \frac{\sqrt{N}}{\beta}(f_i + \xi + z_i - x_i)$ for $i = 1, \ldots, N$ (see Subsection 5.4.3). Furthermore, the



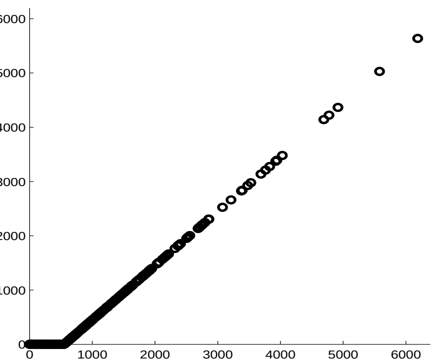
1) $\pi = 200$



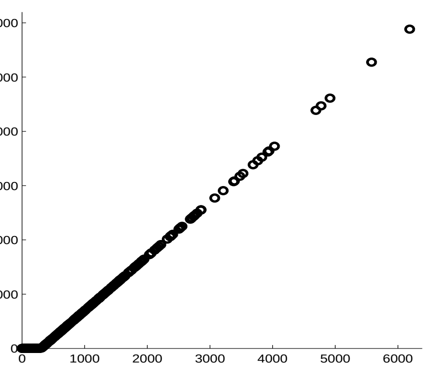
2) $\pi = 400$



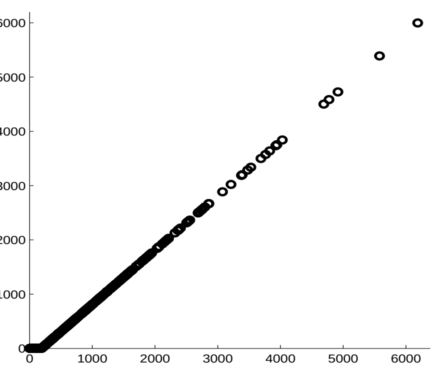
3) $\pi = 600$



4) $\pi = 800$

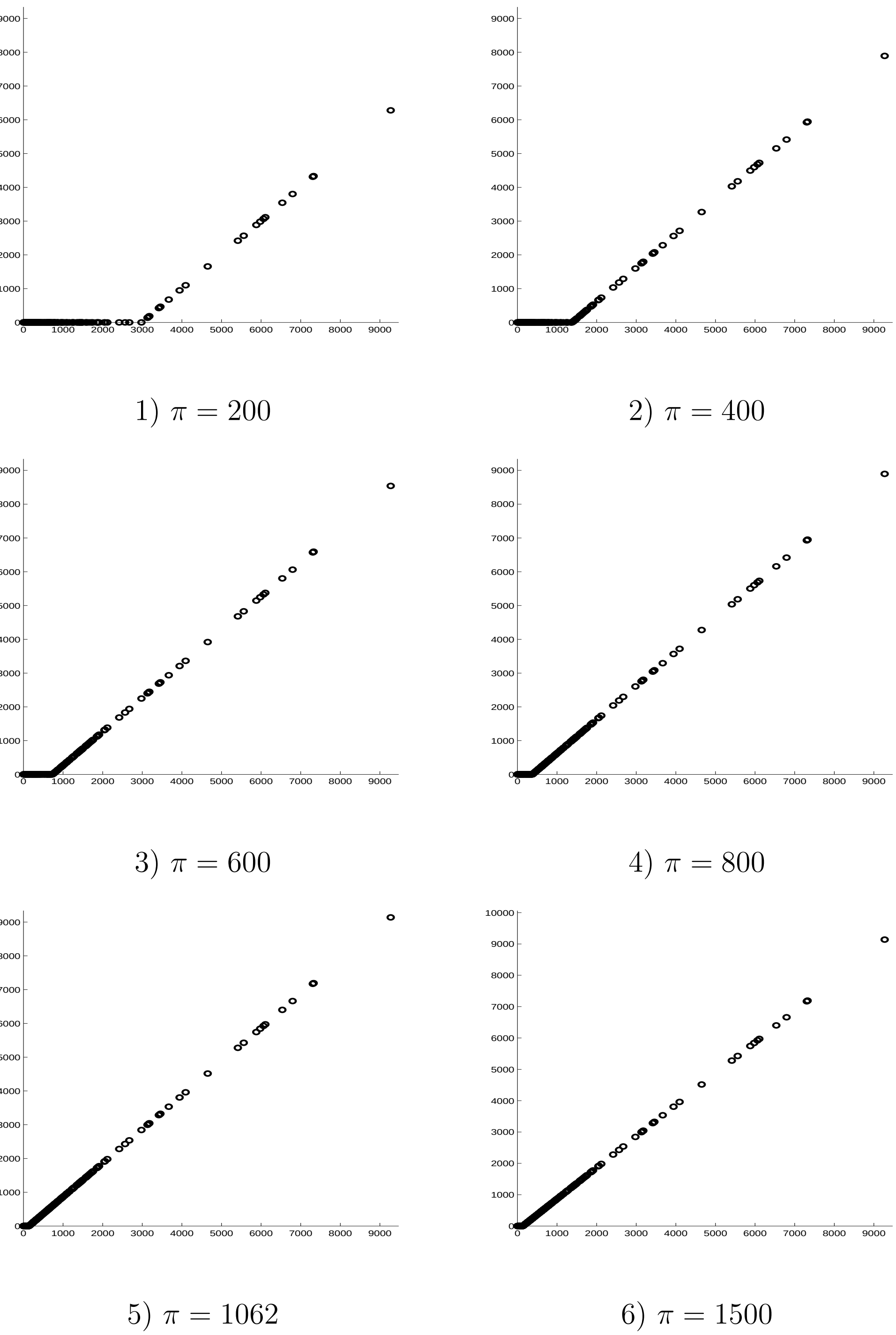


5) $\pi = 1000$



6) $\pi = 1500$

Figure 5.5: Empirical solutions to the CTE minimization model with expectation principle and exponential loss distribution.



1) $\pi = 200$ 2) $\pi = 400$

3) $\pi = 600$ 4) $\pi = 800$

5) $\pi = 1062$ 6) $\pi = 1500$

Figure 5.6: Empirical solutions to the CTE minimization model with expectation principle and Pareto loss distribution.

empirical CTE minimization model (5.4.13 is formulated as

$$\begin{cases} \min_{(\xi,\mathbf{f},\mathbf{z})} & \xi + \frac{1}{\alpha N}\sum_{i=1}^{N} z_i, \\ \text{s.t.} & 0 \le f_i \le x_i, \;\; i = 1, 2, \cdots, N \\ & \|\mathbf{Qf}\| \le -\frac{1}{\beta\sqrt{N}}\mathbf{e}^T\mathbf{f} + \frac{\sqrt{N}}{\beta}\pi \\ & z_i \ge 0, \;\; i = 1, 2, \cdots, N \\ & \|\mathbf{Qf}\| \le -\frac{1}{\beta\sqrt{N}}\mathbf{e}^T\mathbf{f} + \frac{\sqrt{N}}{\beta}(f_i + \xi + z_i - x_i), \;\; i = 1, 2, \cdots, N. \end{cases} \tag{5.6.19}$$

The solutions to the above model will be explored in the following two examples.

Example 5.7 CTE Minimization Model with the Standard Deviation Principle and Exponential Loss Distribution
In this example, we use loss data simulated from the same distribution as in Example 5.5 to solve model (5.6.19) with $\beta = 0.2$ *and for reinsurance premium* π *ranging from as low as 50 to as high as 2000. For this particular reinsurance model specification, the scatter plots of the solutions shown in Figure 5.8 reveal some interesting structures of the optimal reinsurance treaties. For instance, for higher reinsurance premium budget (say* $\pi \ge 120$*) the optimal treaty is a typical stop-loss reinsurance. On the other hand, for lower reinsurance premium (say* $\pi \le 100$*), the optimal treaty becomes a capped stop-loss reinsurance implying that the insurer no longer has an unlimited coverage from the reinsurer. In these situations, the reinsurer has a maximum capped payout and hence the insurer assumes any residual risk exposure for any loss exceeding the upper limit.*

Example 5.8 *Consider the optimal solutions to model (5.6.19) with* $\beta = 0.2$ *and using loss data simulated from the same loss distribution as in Example 5.6. The scatter plots in Figure 5.8 present the solutions for twelve different levels of reinsurance premium budget* π *in the range* $[50, 2000]$*. Based on these scatter plots, a similar conclusion can be obtained as in Example 5.7. The optimal reinsurance treaty is the capped stop-loss reinsurance for reinsurance budget* $\pi \le 200$ *while the optimal reinsurance is a stop-loss treaty for* $\pi \ge 200$.

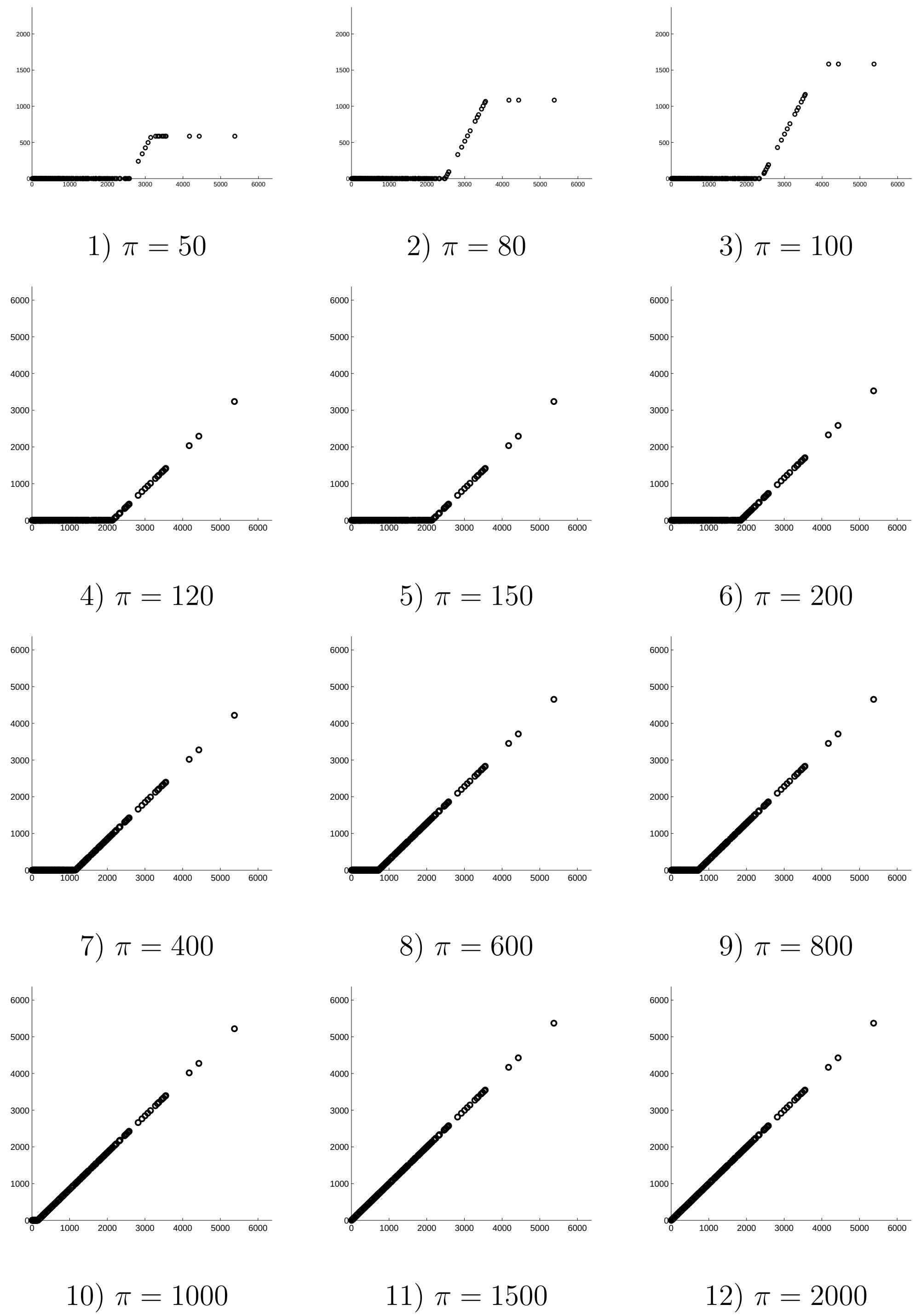
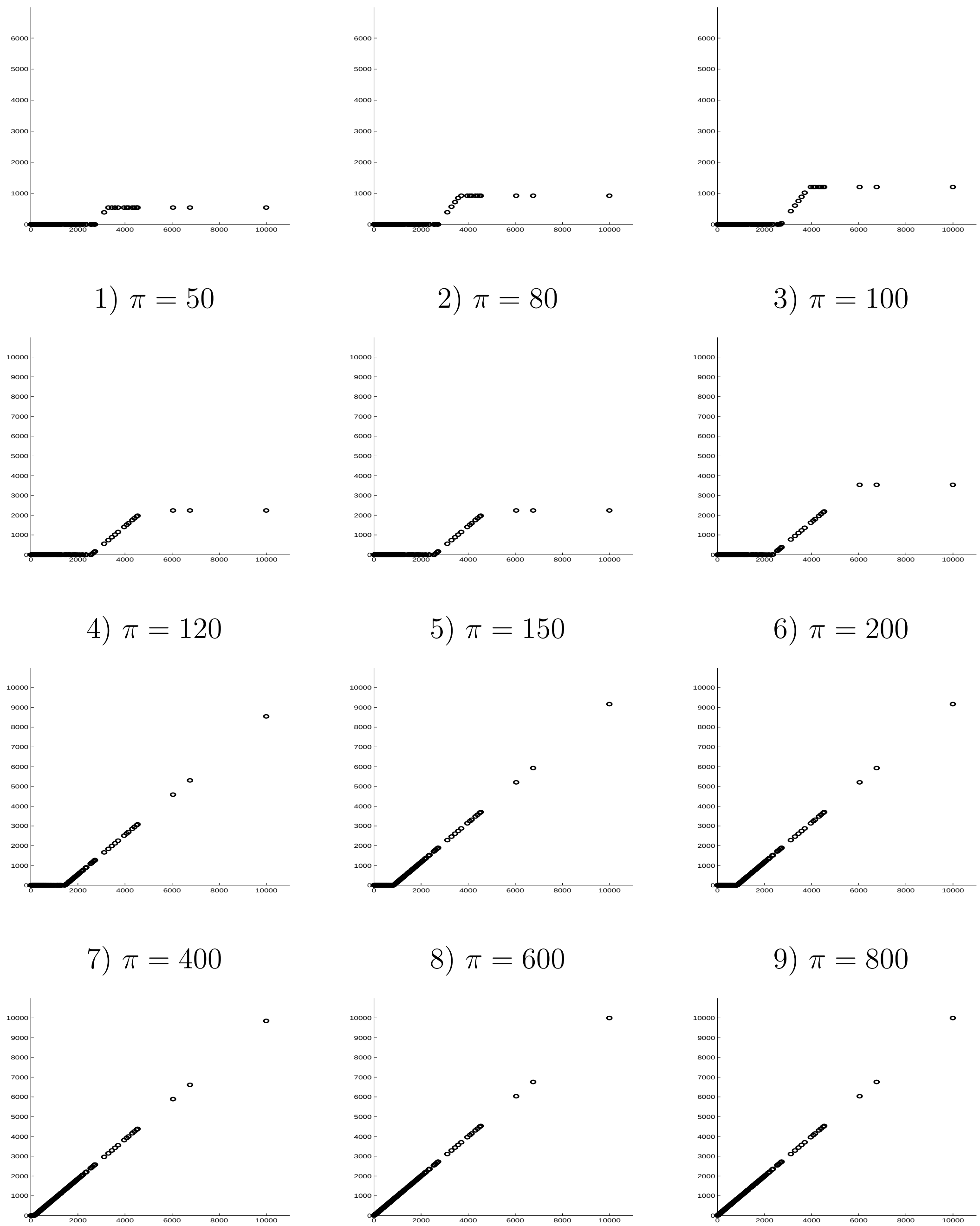


Figure 5.7: Empirical solutions to the CTE minimization model with standard deviation principle and exponential loss distribution.



1) $\pi = 50$ 2) $\pi = 80$ 3) $\pi = 100$

4) $\pi = 120$ 5) $\pi = 150$ 6) $\pi = 200$

7) $\pi = 400$ 8) $\pi = 600$ 9) $\pi = 800$

10) $\pi = 1000$ 11) $\pi = 1500$ 12) $\pi = 1500$

Figure 5.8: Empirical solutions to the CTE minimization model with standard deviation principle and Pareto loss distribution.

5.7 Empirical Solutions to the VaR Minimization Model

In this section, we shall investigate the solutions to the empirical VaR minimization model (5.4.15). In contrast to the variance minimization and CTE minimization models, the VaR minimization model is no longer a convex programming problem even under the simplest expectation reinsurance principle. For this reason it is considerably more challenging to deduce the optimal solution for the VaR minimization model. Nevertheless, we will demonstrate that an empirical solution to the VaR minimization model can still be deduced using some heuristic approaches such as those in Larsen et al. (2002) and Gaivoronski and Pflug (Winter 2004-2005). In these two papers, the authors proposed several algorithms for the VaR optimization problems in the context of the portfolio selection.

In this section, we will use the approach proposed by Larsen et al. (2002) to obtain the empirical solution to our empirical-based VaR reinsurance model. The procedure is summarized in Algorithm A below. The general idea underlying this type of algorithms is to construct upper bounds for VaR and then minimize these bounds. The following Algorithm A adopts the CTE as the upper bound to be minimized, and then split the scenarios into two groups (represented by the set H_i and its complement) depending on whether the losses exceed VaR_α, and "discard" the upper portion of these scenarios. The number of scenarios that are discarded is determined by a preset parameter ξ (e.g., if ξ is equal to 0.5, then the upper half is discarded). Then, a new α_1 is calculated in such a way that the CTE_{α_1} calculated based on the remaining losses is an upper bound for VaR_α of the original problem. Then, we minimize this upper bound, and so on. To summarize, we would construct a series of upper bounds and minimize them until we do not have anymore scenarios to discard.

Algorithm A

Step 0. Initialization

i) Set $\alpha_0 = \alpha$, $i = 0$, $H_0 = \{j : j = 1, 2, \cdots, N\}$, $v_0 = 1/(\alpha N)$

ii) Assign a value for the constant ζ, $0 < \zeta < 1$, say $\zeta = 0.75$

Step 1. Optimization subproblem

i) Minimize CTE_{α_i}

$$\begin{cases} \min_{(\gamma,\xi,\mathbf{z},\mathbf{f})} & \xi + v_i \sum_{j \in H_i} z_j \\ \text{s.t.} & 0 \le f_j \le x_j, \;\; j = 1, 2, \cdots, N, \\ & \widehat{\Pi(\mathbf{f})} \le \pi, \\ & z_j \ge 0 \text{ and } z_j \ge x_j - f_j + \widehat{\Pi(\mathbf{f})} - \xi, \\ & x_j - f_j + \widehat{\Pi(\mathbf{f})} \le \gamma, \;\; j \in H_i, \\ & x_j - f_j + \widehat{\Pi(\mathbf{f})} \ge \gamma, \;\; j \notin H_i, \end{cases}$$

Let $(\gamma_i^*, \xi_i^*, \mathbf{z}^*, \mathbf{f}^*)$ denote the solution to the above CTE_{α_i} minimization problem, where $\mathbf{z}^* = \{z_1^*, z_2^*, \cdots, z_N^*\}$ and $\mathbf{f}^* = \{f_1^*, f_2^*, \cdots, f_N^*\}$.

ii) Rearrange $\{x_j - f_j^* + \widehat{\Pi(\mathbf{f}^*)}, j = 1, 2, \cdots, N\}$ in an ascending order, and denote the ordered scenarios by j_l, $l = 1, 2, \cdots, N$.

Step 2. Estimate VaR

$$V_i = x_{j_{l(\alpha)}} - f_{j_{l(\alpha)}} + \widehat{\Pi(\mathbf{f}^*)}, \;\text{ where }\; l(\alpha) = \min\{l : l/N \ge 1 - \alpha\}$$

Step 3. Stopping the algorithm

If $H_i = H_{i-1}$, then stop the algorithm and $\mathbf{f}^*$ is the estimate of the optimal solution and set the minimal VaR as V_i.

Step 4. Re-initialization

i) $i = i + 1$

ii) $b_i = (1-\alpha) + \alpha(1-\zeta)^i$ and $1 - \alpha_i = (1-\alpha)/b_i$

iii) $H_i = \{j_l \in H_{i-1} : l/N \le b_i\}$

iv) $v_i = (\alpha_i \times \text{the number of elements in } H_i)^{-1}$

v) Go back to step 1.

In Example 5.9 below, we will consider the empirical VaR minimization model (5.4.15) with the expectation reinsurance premium principle and obtain the solutions by using Algorithm A. Before we proceed, let us recall some related theoretical results established by Wang, et al. (2005) and Bernard and Tian (2009). In Wang, et al. (2005), the authors discussed the problem in the context of optimal insurance, but their results are applicable to the optimal reinsurance design. Here, we rephrase their results to tailor to the reinsurance context. In their paper, the optimal reinsurance treaties are explored by maximizing the insurer's expected final wealth resulting from the reinsurance treaty starting from an initial wealth W_0 and a loss X on which the reinsurance is applied. They considered a reinsurance model with a solvency constraint as follows:

$$\begin{cases} \max_f & \overline{W} = W_0 - P - \mathsf{E}[X] + \mathsf{E}[f(X)], \\ \text{s.t.} & \Pr\{W \ge \overline{W} - v\} \ge 1-\alpha, \ \text{ and } \ P = (1+\theta)\mathsf{E}[f(X)], \\ & 0 \le f(x) \le x \ \text{ for all } \ x \ge 0, \end{cases} \tag{5.7.20}$$

where P is the reinsurance premium calculated according to the expectation principle with a loading factor $\theta > 0$, $W = W_0 - P - X + f(X)$ is the insurer's final wealth resulted from the reinsurance treaty, and v is an exogenously preset positive constant representing the VaR level. Wang, et al. (2005) proved that, under some mild conditions, the optimal treaty to model (5.7.20) is of the following form with two appropriately chosen constants $d, l > 0$:

$$f(x) = \begin{cases} 0, & x \leq d; \\ x - d, & d < x < l; \\ 0, & x \geq l. \end{cases} \tag{5.7.21}$$

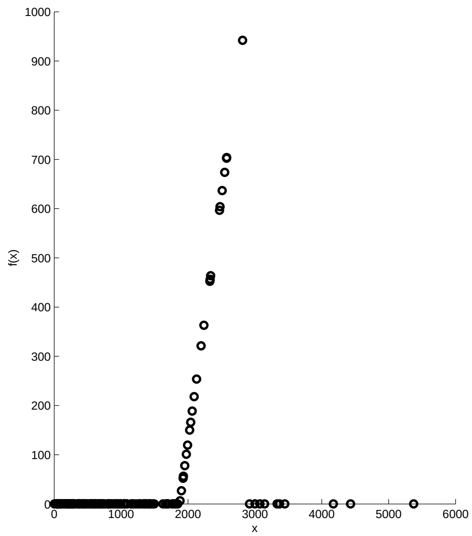
Clearly, the constraint $\Pr\{W \geq \overline{W} - v\} \geq 1 - \alpha$ in model (5.7.20) can be viewed as a VaR constraint $\mathrm{VaR}_\alpha(\overline{W} - W) \leq v$, and hence loosely speaking, the model can be regarded as the dual problem to our VaR minimization model (5.4.14).

While Wang, et al. (2005) just take $\Pr\{W \geq \overline{W} - v\} \geq 1 - \alpha$ as a constraint in their reinsurance model, Bernard and Tian (2009) consider a reinsurance model which directly minimizes the ruin probability $\Pr\{W \geq \overline{W} - v\}$ subject to the other two constraints as in model (5.7.20). The optimal ceded loss function obtained by them has the same functional form as the one in (5.7.21). In light of these two results, it is reasonable for us to expect that the empirical solutions to model (5.4.15) should have a deductable at certain loss level, then increase linearly, and eventually reduce to zero for loss exceeding a higher threshold level. The results demonstrated in Example 5.9 highly support such an assertion.

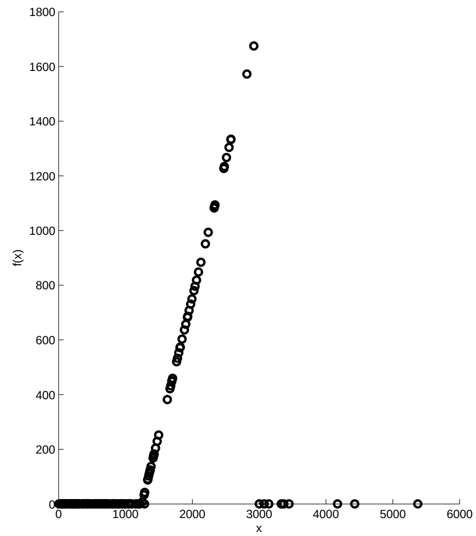
Example 5.9 VaR Minimization Model with the Expectation Principle and Exponential Loss Distribution

By setting $\alpha = 0.05$, using the same set of exponentially simulated empirical data, and applying Algorithm A in CVX, Figures 5.9 and 5.10 display the solutions to model (5.4.15) for various levels of reinsurance premium. These results are consistent with the solutions obtained in Wang, et al. (2005) and Bernard and Tian (2009); i.e. they have the same functional form as in (5.7.21).

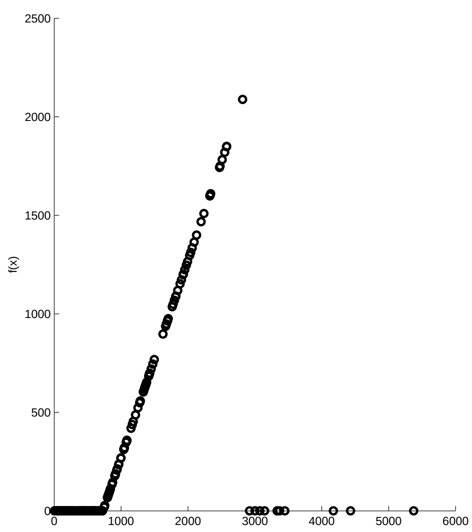
Remark 5.1 *It is worthy noting that Algorithm A might not be efficient if we were to change the expectation premium principle to other premium principles. This is because in the subproblem embedded in the algorithm involves the constraints*



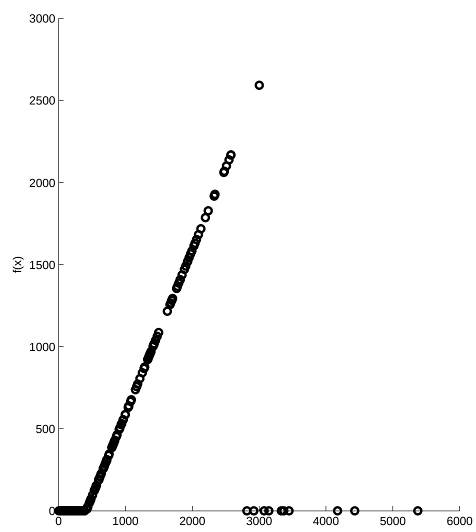
1) $\pi = 50$



2) $\pi = 200$

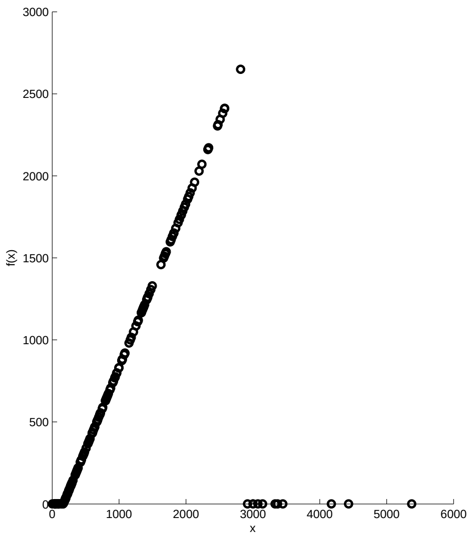


3) $\pi = 400$

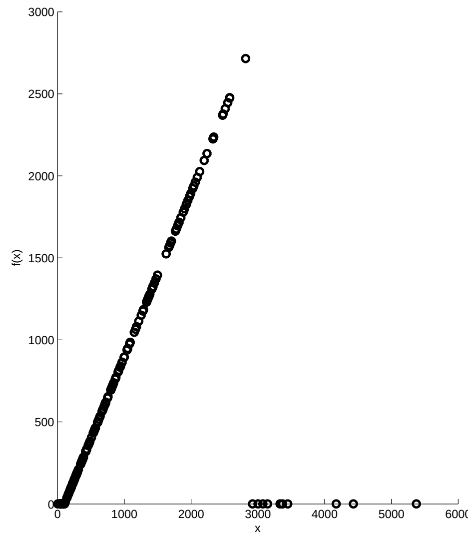


4) $\pi = 600$

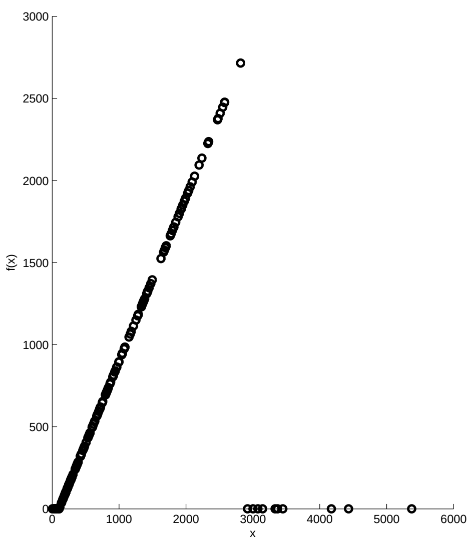
Figure 5.9: Empirical solutions to the VaR minimization model with expectation principle and exponential loss distribution (1).



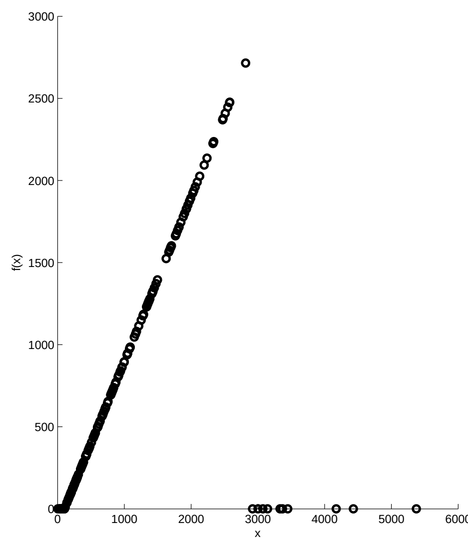
5) $\pi = 800$



6) $\pi = 1000$



7) $\pi = 1500$



8) $\pi = 2000$

Figure 5.10: Empirical solutions to the VaR minimization model with expectation principle and exponential loss distribution (2).

$x_j - f_j + \widehat{\Pi(\mathbf{f})} \geq \gamma, \;\; j \notin H_i$, which might not be SOC-representable for a nonlinear premium principle Π. Thus, for VaR minimization model, other algorithms are demanded in order to obtain the solutions for more reinsurance premium principles.

5.8 Conclusion

Due to the inherent infinite dimension nature of the optimization problem, most reinsurance models turn to be too mathematically challenging to be solved efficiently, and thus the insurer is tremendously restricted in exploring the optimal reinsurance treaties in their decision-making. To overcome such a restriction, this chapter proposes an innovative method to address the optimal solutions——the empirical approach. By experimenting with the variance minimization model, the CTE minimization model, and the VaR minimization model, this chapter shows that our empirical approach is strongly effective in the sense that the empirical solutions derived by the empirical approach are highly consistent to the theoretical solutions whenever they exist.

Next, let us remark on the pros and cons of our proposed empirical approach. Focus on its advantages first. The empirical models are completely empirical data based, and hence using this empirical approach we need not make any explicit assumption on the distribution of the underlying risk. The empirical models are finite dimensional optimization problems and hence they are much more tractable than their theoretical counterparts, which are usually infinite dimensional problems. Therefore, the empirical approach allows much more flexibility of the optimality objective as well as the reinsurance premium principle in the reinsurance models, compared with the theoretical models. For examples, with the CTE minimization criterion, we are unable to derive the solutions to the theoretical models if the reinsurance premiums principles are other than the expectation principle for which we discussed in the last chapter. However, using the empirical approach we derived the solutions to the theoretical models for the standard deviation principle. Following the same procedure as we did for the standard deviation principle, we can derive the solutions to the empirical CTE minimization model for all the other principles involved in this chapter with labels Q1-Q10. Finally, recall that in order to find the solutions to the theoretical CTE minimization model, we first establish some sufficient and necessary conditions which an optimal solution must satisfy and then identify the solutions by trying some candidates with these conditions. The choice of the candidates for this purpose can be quite non-trivial. Hence, once we derive the empirical solutions, they will provide valuable insights on how to choose an appropriate candidate in searching for the theoretical optimal solutions.

One obvious limitation of the empirical approach lies on the fact that the empirical model will turn out to be a large scale programming when the sample size is extremely large, and hence issues such as computational time and requirement for a substantial computer's memory will arise. Another issue on the empirical approach is that the general theoretical relationship between the empirical solutions and the theoretical solutions are unknown and it demands future research. It seems that we have to establish the uniform convergence of the goal function in the empirical models to the goal function in the theoretical model over all feasible ceded loss functions.

Chapter 6

Additional Analysis on the Empirical-based Reinsurance Models

6.1 Introduction

In the last chapter, we developed an empirical-based approach for optimal reinsurance design. This approach has several appealing features relative to the theoretical reinsurance model. First, the proposed approach is very intuitive. It determines the optimal reinsurance by directly exploiting the empirically observed loss data. Second, we do not need to make any explicit assumption on the underlying distribution of losses. Third, the resulting empirical model is of finite dimensions. Lastly, the proposed approach is much more versatile, practical and tractable. By resorting to SOC programming, optimal solutions can be obtained in a wide range of reinsurance models.

Let us now recall the empirical-based CTE minimization model. This particular reinsurance model is formulated as follows:

$$\begin{cases} \min\limits_{(\xi,\mathbf{f},\mathbf{z})} & \xi + \frac{1}{\alpha N}\sum_{i=1}^{N} z_i, \\ \text{s.t.} & \widehat{\Pi(\mathbf{f})} \leq \pi, \\ & 0 \leq f_i \leq x_i, i = 1, 2, \cdots, N \\ & z_i \geq 0, z_i \geq x_i - f_i + \widehat{\Pi(\mathbf{f})} - \xi, \;\; i = 1, 2, \cdots, N, \end{cases} \tag{6.1.1}$$

where $\mathbf{x} := \{x_1, \cdots, x_N\}$ denotes the empirical loss data of size N, $\mathbf{f} := \{f_1, \cdots, f_N\}$ stands for the reinsurance coverage corresponding to the empirical data, $\widehat{\Pi(\mathbf{f})}$ is the empirical version of the reinsurance premium, and π represents the preset level of the reinsurance premium budget. Parallel to the empirical model, the standard theoretical model is formulated as

$$\begin{cases} \min_f & \text{CTE}_\alpha(T_f) = \text{CTE}_\alpha\Big(X - f(X) + \Pi[f(X)]\Big) \\ \text{s.t.} & 0 \leq f(x) \leq x, \;\; \Pi[f(X)] \leq \pi, \end{cases} \tag{6.1.2}$$

where X is the underlying loss random variable with nonnegative support, f is the ceded loss function, and Π is the adopted reinsurance premium principle.

The main objective of this chapter is to provide an in-depth analysis of the solutions generated from the empirical reinsurance models. Recall that the empirical reinsurance model produces optimal ceded loss value for each data point. More specifically, when we solve model (6.1.1) based upon an input data $\mathbf{x} := \{x_1, \cdots, x_N\}$ consisting of N sample points, the output will be N corresponding optimal ceded values $\mathbf{f}^* := \{f_1^*, \cdots, f_N^*\}$. This is also the reason for displaying the empirical solutions in the form of scatter plots for the examples considered in the last chapter.

In this chapter, we are interested in the following issue: the stability and consistency of the empirical solutions. By stability, we mean that the empirical solutions always generate the same functional form of the optimal ceded loss function for independent random samples from the same loss distribution and over the same set of parameter values. By consistency, we mean that the empirical optimal ceded loss function converges to the theoretically true optimal ceded loss function as we increase the sample size N. While it is challenging to provide a formal analysis on the stability and consistency of our proposed empirical reinsurance models, we address these issues by resorting to some numerical experiments on the CTE minimization model. The numerical studies also allow us to gain important insights on the behavior of our proposed empirical solutions, particularly for small sample size.

The remaining chapter is organized as follows. Section 6.2 discusses the stability and consistency issues under the assumption of the expectation premium principle. This example is useful since in this special case, we know analytically the optimal ceded loss function based on the results developed in Chapter 4. Hence the analytic solution can be used as a benchmark against the solutions generated from the empirical models. Section 6.3 considers an example with the standard deviation premium principle. This example is even more interesting in that its optimal ceded loss function is unknown. Section 6.4 concludes the chapter.

6.2 Expectation Premium Principle Example

In this section, we consider an example with the following characteristics:

(i) reinsurance premium is determined by the expectation premium principle with loading factor $\theta = 0.2$;

(ii) the reinsurance premium budget $\pi = 300$;

(iii) $\alpha = 0.05$;

(iv) the loss random variable X has an exponential distribution or a Pareto distribution with mean $\mu = 1,000$.

Note that the exponential distribution are usually thought of as a light-tailed distribution while the Pareto distribution is often regraded as a heavy-tailed distribution. The reason we include both of them in the numerical example is that we hope to gain certain insights regarding the stability and consistency for both the light and heavy tailed loss distributions.

Let us further remark that under the setup of the above example, the solution to theoretical model (6.1.2) can be solved analytically. In particular, we can resort to Theorem 4.2 in Chapter 4 to determine an optimal ceded loss function f^*. Consider the exponential loss distribution first. In this case, we have

$$d_\alpha = \inf\{d : \Pr[X > d] \leq \alpha\} = -\mu \ln(0.05) = 2,995.73$$

and

$$d_\theta = \inf\left\{d :, \Pr[X > d] \leq \frac{1}{1+\theta}\right\} = -\mu \ln\left(\frac{1}{1+0.2}\right) = 1,823.22.$$

This in turn leads to $\pi_\alpha = (1+\theta)\mathsf{E}[(X - d_\alpha)_+] = 60$ and $\pi_\theta = (1+\theta)\mathsf{E}[(X - d_\theta)_+ = 1,000$ so that under reinsurance premium budget of $\pi = 300$, the condition $\pi_\alpha < \pi < \pi_\theta$ is satisfied. Hence Theorem 4.2 in Chapter 4 asserts that the stop-loss treaty $f = (x-d^*)_+$ with retention d^* determined through $(1+\theta)\mathsf{E}[(x-d^*)_+] = 300$ is an optimal solution to problem (6.1.2). It is then easy to derive the corresponding optimal retention value; i.e. $d^* = \mu \ln(4) = 1,386.29$.

If the loss random variable X is Pareto distributed with mean 1,000, we have

$$d_\alpha = 2,000\left(\alpha^{-1/3} - 1\right) = 3,428.84$$

and

$$d_\theta = \inf\left\{d :, \Pr[X > d] \leq \frac{1}{1+\theta}\right\} = -\mu \ln\left(\frac{1}{1+0.2}\right) = 125.32.$$

This in turn leads to $\pi_\alpha = (1+\theta)\mathsf{E}[(X - d_\alpha)_+] = 162.86$ and $\pi_\theta = (1+\theta)\mathsf{E}[(X - d_\theta)_+ = 1,062.66$ so that under reinsurance premium budget of $\pi = 300$, the condition $\pi_\alpha < \pi < \pi_\theta$ is satisfied. Hence Theorem 4.2 in Chapter 4 is also applicable, and again the stop-loss treaty $f = (x - d^*)_+$ is an optimal solution to problem (6.1.2) with d^* determined through $(1+\theta)\mathsf{E}[(x-d^*)_+] = 300$, i.e., $d^* = 2,000\left(\sqrt{4}-1\right) = 2,000$.

In our present setting, we are interested in analyzing the solutions to model (6.1.1). Our numerical experiment involves first drawing samples from the assumed loss distribution and then applying the random samples as input to model (6.1.1) to determine the shape of the optimal ceded loss function. Since model (6.1.2) yields an analytical solution of the ceded loss function, this implies that its solution can be used as a benchmark in numerically assessing the accuracy of the empirical solution of model (6.1.1). It is reassuring that the numerical evidence to be presented shortly indicates that the solutions to model (6.1.1) is in concordance with that to model (6.1.2), even for relatively small sample size.

Recall that we have observed in Examples 5.5 and 5.6 that for both exponential loss distribution and Pareto loss distribution, the scatter plots of the optimal ceded values resemble the shape of stop-loss functions. For this reason, we consider to fit the empirical solutions with the form of the more general change-loss function $f(x) = c(x-d)_+$, where parameters c and d are obtained by fitting to $\{(x_1, f_1^*), (x_2, f_2^*), \cdots, (x_N, f_N^*)\}$. Here $\mathbf{f}^* = (f_1^*, f_2^*, \cdots, f_N^*)$ corresponds to the solutions derived by solving model (6.1.1). If the empirical solution is converging to the analytical solution, then we expect that the fitted value of c and d, denoted by $\hat{c}$ and $\hat{d}$, respectively, converge to 1 and d^*, as we increase the sample size N. Recall that d^* denotes the retention in the theoretical solutions, which is equal to $1,386.29$ for the exponential case and $2,000$ for the Pareto case.

There exists a number of ways of fitting $c(x-d)_+$ to the optimal ceded loss values. The key of our fitting algorithm is first to determine (approximately) the retention d and then fit $c(x-d)$ to those data points that exceed the determined value for d. Let ε be the error tolerance parameter of our fitting algorithm, then our procedure can be described as follows:

[F1]. Sort the pairs $\{(x_1, f_1),\ (x_2, f_2),\ \cdots,\ (x_N, f_N)\}$ in an ascending order in x_i, and relabel the ordered pairs as $\{(x_{(1)}, f_{(1)}),\ (x_{(2)}, f_{(2)}),\ \cdots,\ (x_{(N)}, f_{(N)})\}$.

[F2]. Set n_0 to be the smallest i among $\{1, \ldots, N\}$ such that $|f_{(i)}| \geq \varepsilon$.[1]

[F3]. Fit $f(x) = c(x - d)$ to the subset of the data $\{(x_{(n_0)}, f_{(n_0)}),\ (x_{(n_0)+1}, f_{(n_0)+1}),\ \cdots,\ (x_{(N)}, f_{(N)})\}$ to obtain the fitted $\hat{c}$ and $\hat{d}$ using the ordinary least squares.

In summary, our numerical experiment consists of the following three steps. In Step 1, we generate a random sample $\mathbf{x} := \{x_1, \cdots, x_N\}$ from the underlying loss distribution. Then in Step 2, we solve model (6.1.1) to obtain the empirical solutions $\mathbf{f} := \{f_1, \cdots, f_N\}$. Finally in Step 3, we fit $c(x-d)_+$ to $\{\mathbf{x}, \mathbf{f}\}$ to deduce the fitted $\hat{c}$ and $\hat{d}$. This implies that for each independent sample of $\mathbf{x}$, we obtain a fitted pair $\hat{c}$ and $\hat{d}$. Furthermore, we also distinguish if the fitted pair $\hat{c}$ and $\hat{d}$ is *admissible* or *inadmissible*. The fitted pair $\hat{c}$ and $\hat{d}$ is said to be admissible if the following conditions are satisfied:

$$\begin{cases} |f_{(i)}| < \varepsilon & \text{for } i = 1, \ldots, n_0 - 1, \\ \left| f_{(i)} - \hat{c}\left(x_{(i)} - \hat{d}\right) \right| < \varepsilon & \text{for } i = n_0, \ldots, N. \end{cases} \tag{6.2.3}$$

If any of the above conditions is violated, then we refer the resulting fitted pair $\hat{c}$ and $\hat{d}$ as inadmissible. The admissibility criterion enforces the goodness of fit by ensuring that the residual values (i.e. difference between the fitted ceded loss value and the optimal empirical ceded loss value) at all data points are less than the error tolerance ε. Consequently, for an admissible solution, the smaller the ε, the better the fit.

In our numerical studies, we consider nine different sample sizes, ranging from $N = 150$ to $N = 390$ in multiple of 30. Note that we have intentionally chosen a rather small sample size in order to have a better understanding of the performance

[1] If $|f_{(i)}| < \varepsilon$ for all $i = 1, 2 \cdots, N$, then it is reasonable to take $f^*(x) = 0$ for all $x \geq 0$. We ignore this trivial case.

N	Admissibility	mean of $\hat{c}$	mean of $\hat{d}$	mean of $\mathbf{x}$
150	100%	1.00 (≈ 0)	1,386.68 (7.04)	999.77 (2.66)
180	100%	1.00 (≈ 0)	1,386.73 (6.10)	1,000.50 (2.31)
210	100%	1.00 (≈ 0)	1,379.63 (5.59)	996.60 (2.18)
240	100%	1.00 (≈ 0)	1,381.87 (5.34)	997.04 (2.06)
270	100%	1.00 (≈ 0)	1,384.98 (5.11)	999.12 (1.94)
300	100%	1.00 (≈ 0)	1,391.44 (4.65)	1,003.14 (1.76)
330	100%	1.00 (≈ 0)	1,384.16 (4.62)	998.87 (1.75)
360	100%	1.00 (≈ 0)	1,392.36 (4.40)	1,001.15 (1.64)
390	100%	1.00 (≈ 0)	1,385.55 (4.12)	1,000.58 (1.57)

Table 6.1: Empirical solutions based on 1,000 independent replications of an exponential loss distribution for the expectation premium principle. Column 1 gives the sample size of each replication. Column 2 gives the proportion of the solutions that are admissible. Columns 3, 4 and 5 tabulate the average of the fitted $\hat{c}$, fitted $\hat{d}$, and simulated random samples, respectively, over all admissible solutions. The standard errors of the estimates are given in parentheses.

of the proposed empirical solution. For each sample size N, we replicate the random samples for $M = 1,000$ times independently to obtain 1,000 independent estimates of $\hat{c}$ and $\hat{d}$ using $\varepsilon = 0.1$. Of the fitted pair $\hat{c}$ and $\hat{d}$, we also keep track of the proportion of admissibility and we report only the mean and standard errors of the admissible fitted $\hat{c}$ and $\hat{d}$. We analyze the results for both the exponential loss distribution and the Pareto loss distribution. The results for the exponential loss distribution are summarized in Table 6.1 and the boxplots in Figures 6.1 (for $\hat{c}$) and 6.2 (for $\hat{d}$). The corresponding results for the Pareto loss distribution are reported in Table 6.2 and the boxplots in Figures 6.3 and 6.4. Based on these results, we

N	Admissibility	mean of $\hat{c}$	mean of $\hat{d}$	mean of $\mathbf{x}$
150	93.8%	1.00 (≈ 0)	1,981.75 (23.71)	1,003.08 (4.52)
180	93.2%	1.00 (≈ 0)	1,945.92 (20.85)	1,001.70 (3.97)
210	95.0%	1.00 (≈ 0)	1,948.37 (19.96)	994.39 (3.59)
240	94.7%	1.00 (≈ 0)	2,005.88 (19.83)	1,003.36 (3.48)
270	95.0%	1.00 (≈ 0)	1,982.12 (18.17)	1,001.89 (3.30)
300	96.1%	1.00 (≈ 0)	1,983.13 (18.31)	997.00 (3.11)
330	96.2%	1.00 (≈ 0)	1,995.64 (17.43)	998.98 (3.03)
360	96.9%	1.00 (≈ 0)	1,994.08 (16.49)	998.04 (2.74)
390	97.6%	1.00 (≈ 0)	1,990.33 (14.99)	999.84 (2.62)

Table 6.2: Empirical solutions based on 1,000 independent replications of a Pareto loss distribution for the expectation premium principle.

draw the following remarks:

- Regarding the exponential loss distribution, the simulated results suggest that the empirical solutions are strongly stable and consistent. We now elaborate further on these observations. First, the empirical solutions are very stable with respect to the shape of the stop-loss function. This can be deduced from the 100% admissibility and the fact that all the fitted values of $\hat{c}$ are almost one with negligible standard error (see also its boxplot in Figure 6.1). This implies that all the empirical solutions consistently yield the shape of the stop-loss function to within 0.1 error tolerance level, even for sample size as small as $N = 150$. Second, the empirical solutions are consistent with the analytical solution. This can be concluded by the fact that both the theoretical solutions and empirical solutions take the same form as a stop-loss function. Moreover, the fitted values of $\hat{d}$ also demonstrate a strong concordance with the retention in the theoretical solution. As the sample size N increases, the fitted $\hat{d}$ appears to be converging consistently to $d^* = 1,386.29$ with a decreasing standard error. For example, even with only a sample size $N = 390$, the average of $\hat{d} = 1,385.55$ is very close to d^* with a tiny standard error 4.12.

- For the heavy-tailed Pareto distribution, the simulated results also imply that the empirical solutions are stable and consistent with the theoretically optimal stop-loss solution, although the results are not as perfect as that for the light-tailed exponential distribution. As shown in Table 6.2, the admissibility is more than 93% for all the considered sample size and in particular it reaches as high as 97.6% when sample size $N = 390$. Moreover, of those admissible solutions, all the fitted values of $\hat{c}$ are almost one with negligible standard error (see also its boxplot in Figures 6.3) and the average of the fitted values of $\hat{d}$ is close to the theoretical retention $d^* = 2,000$.

- For comparison, the last column of Tables 6.1 and 6.2 demonstrate the quality of the random samples by reporting the average of the randomly generated samples (together with its standard error). The reported values are consistent with the true value of $\mu = 1,000$ for both exponential loss distribution and Pareto loss distribution.

6.3 Standard Deviation Premium Principle Example

In this section, we use the same numerical setup as in the last section except that the reinsurance principle is the standard deviation premium principle with loading factor $\beta = 0.2$. In other words, the reinsurance premium constraint $\widehat{\Pi(\mathbf{f})} \leq \pi$

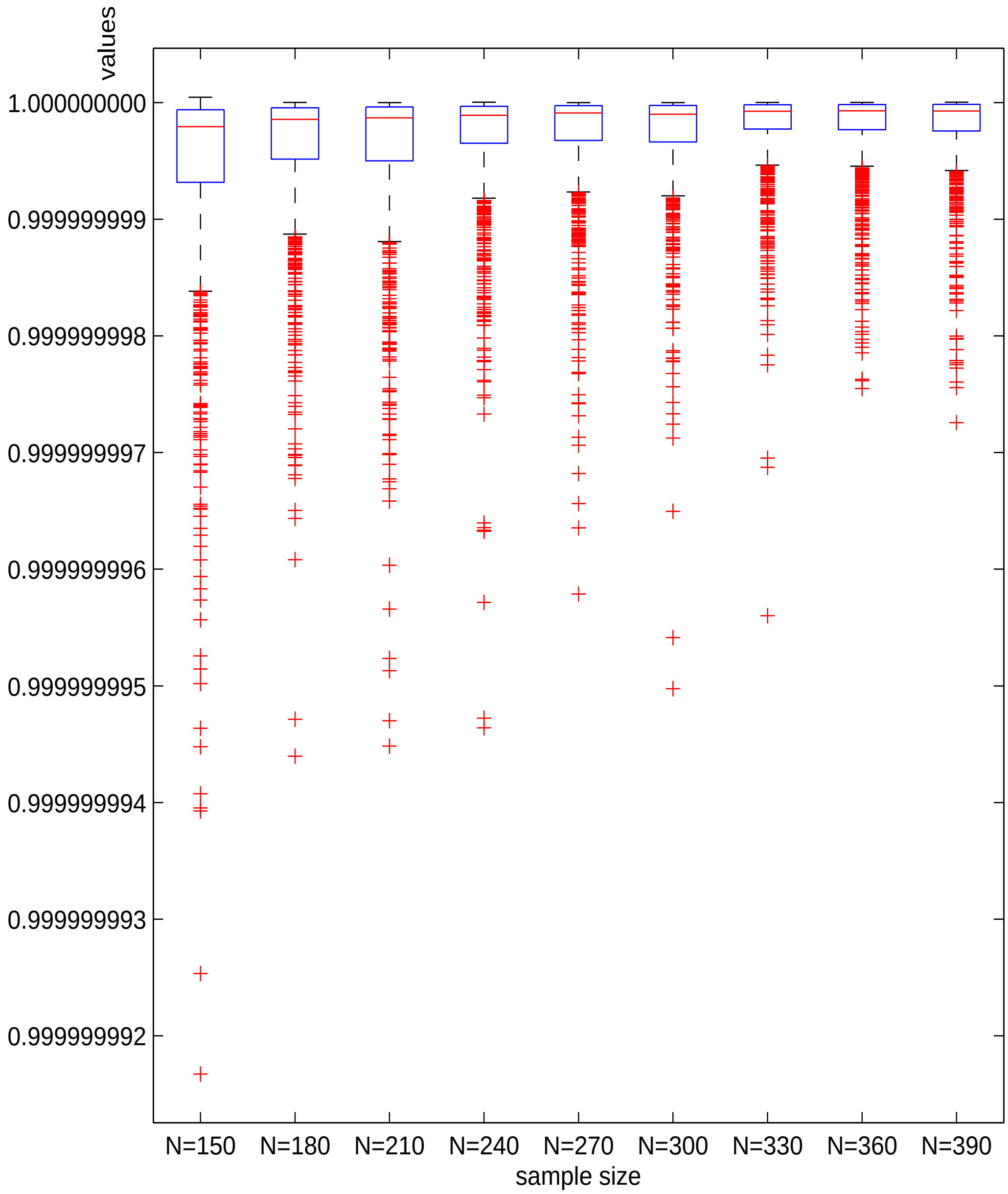


Figure 6.1: Boxplot of the admissible $\hat{c}$ under expectation premium principle and exponential loss distribution.

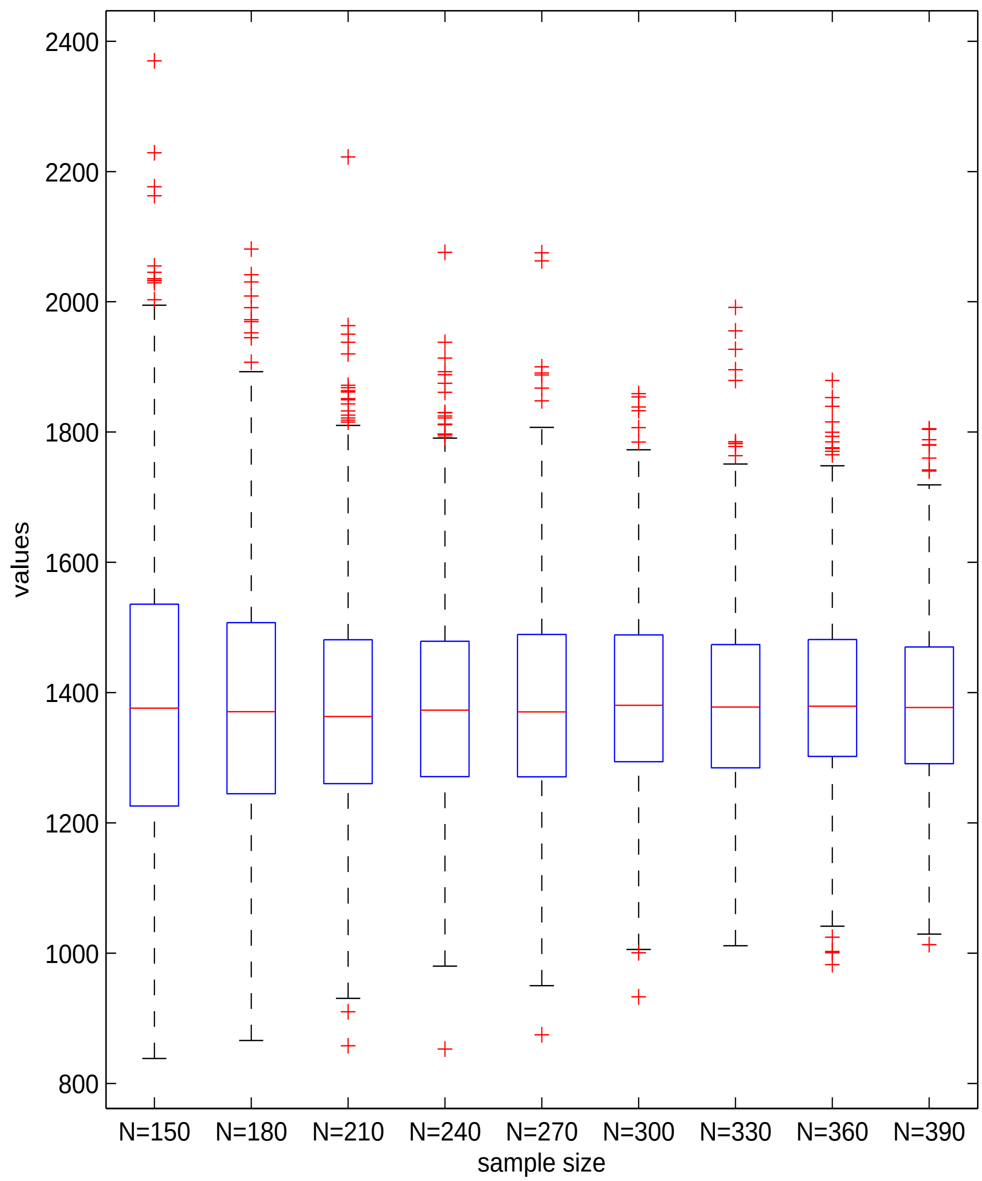


Figure 6.2: Boxplot of the admissible retention $\hat{d}$ under expectation premium principle and exponential loss distribution.

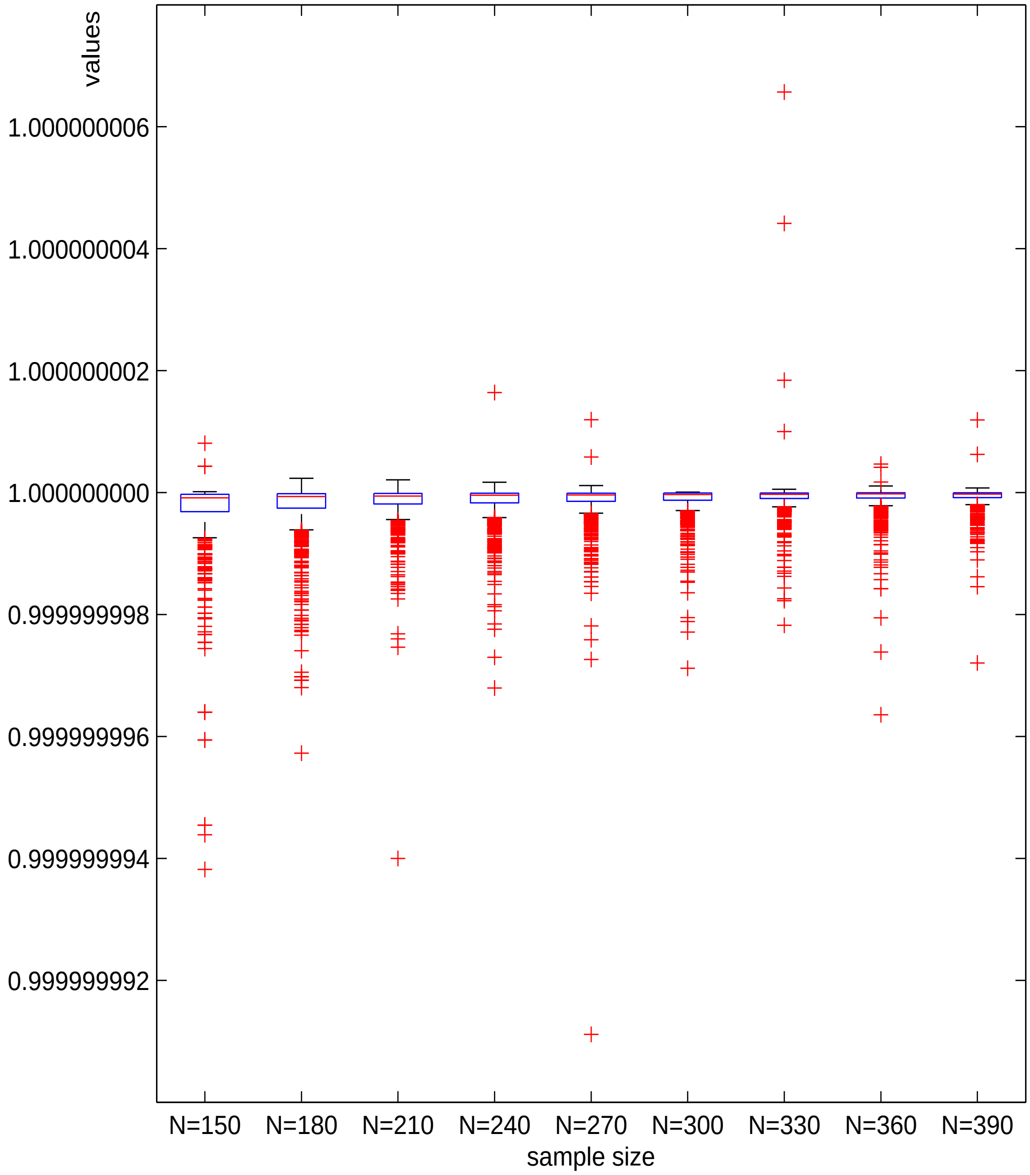


Figure 6.3: Boxplot of the admissible $\hat{c}$ under expectation premium principle and Pareto loss distribution.

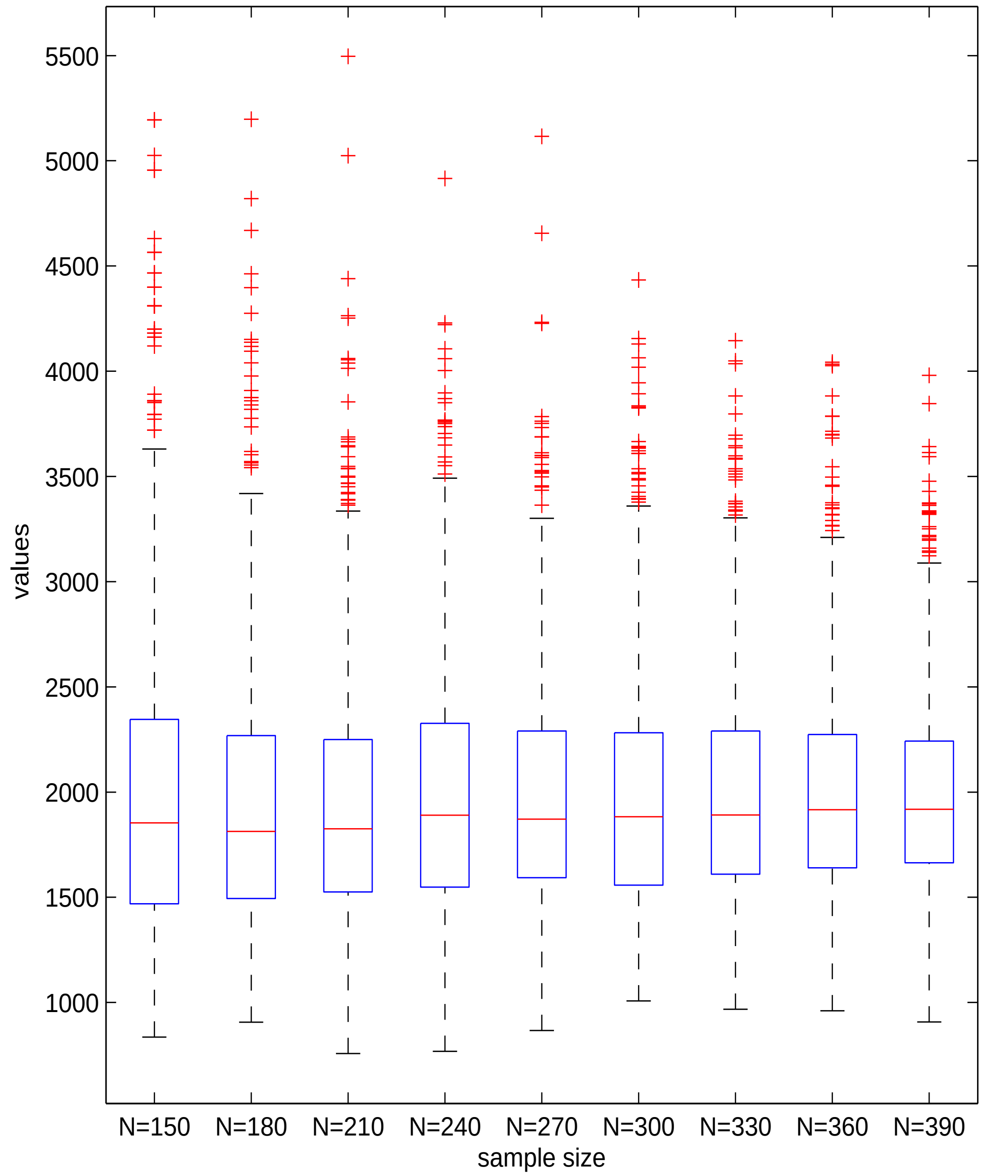


Figure 6.4: Boxplot of the admissible retention $\hat{d}$ under expectation premium principle and Pareto loss distribution.

becomes

$$\|\mathbf{Qf}\| \le -\frac{1}{\beta\sqrt{N}}\mathbf{e}^T\mathbf{f} + \frac{\sqrt{N}}{\beta}\pi.$$

Furthermore, the reinsurance premium budget is revised to $\pi = 100$, instead of $\pi = 300$ as in the previous case.

This example is considerably more challenging for two reasons. First, unlike the expectation premium principle, analytical solution under the standard deviation premium principle is no longer available. Second, the scatter plots in Examples 5.7 and 5.8 suggest that the functional form of the optimal ceded loss function depends on the magnitude of the reinsurance premium budget π. In particular when π is high, the scatter plots reveal that the stop-loss functions are optimal. On the other hand, as we reduce the budget, say to $\pi = 100$, the optimal ceded loss function changes from a stop-loss function to a capped stop-loss function. Henceforth, we assume that the optimal ceded loss function is of the form capped change-loss function for which the capped stop-loss function is a special case. The same numerical analysis as we have conducted in the last section is used to assess the stability of our proposed empirical method.

The general form of a capped change-loss function involves three parameters c, d, and m with the representation $f(x) = \min\{c(x-d)_+, m\}$. We now describe in details the fitting of this function to the optimal ceded loss values derived from the empirical model. As we did in the last section, we begin with ordered pairs $\{(x_{(1)}, f_{(1)}), (x_{(2)}, f_{(2)}), \cdots, (x_{(N)}, f_{(N)})\}$ (arrange in ascending order in x_i). The key of fitting these points to a capped change-loss function is, for a prespecified error tolerance ε, to identify the subset of the ordered pairs $\{(x_{(n_l)}, f_{(n_l)}), (x_{(n_l+1)}, f_{(n_l+1)}),$ $\cdots, (x_{(n_u)}, f_{(n_u)})\}$ that will be fitted to $c(x-d)$. Here both n_l and n_u are integers satisfying $n_u - n_l > 1$. Furthermore, n_l corresponds to the smallest integer i such that $|f_{(i)}| \geq \varepsilon$ while n_u is the largest k such that[2]

$$\left| f_{(k)} - \frac{1}{N-k+1} \sum_{j=k}^{N} f_{(j)} \right| \geq \varepsilon.$$

Accordingly, a reasonable estimate of the upper limit m, denoted by $\hat{m}$, is given by the average $\frac{1}{N-n_u} \sum_{j=n_u+1}^{N} f_{(j)}$. Also, the linear function $f(x) = c(x-d)$ can now be fitted to the ordered pairs $\{(x_{(n_l)}, f_{(n_l)}),\ (x_{(n_l+1)}, f_{(n_l+1)}),\ \cdots,\ (x_{(n_u)}, f_{(n_u)})\}$ to determine the fitted $\hat{c}$ and $\hat{d}$. Similar to the example in the previous section, the fitted values $\hat{c}$, $\hat{d}$, and $\hat{m}$ are said to be inadmissible if at least one of the following conditions is satisfied:

$$n_u - n_l \leq 1, \quad (6.3.4)$$

$$\left| f_{(i)} - \hat{c}\left(x_{(i)} - \hat{d}\right) \right| \geq \varepsilon \text{ for at least one } i \in \{n_l, \cdots, n_u\}, \quad (6.3.5)$$

$$\max\{f_{(i)} : i = n_l, \ldots, n_u\} \geq \hat{m}, \quad (6.3.6)$$

$$\left| f_{(N)} - \hat{c}\left(x_{(N)} - \hat{d}\right) \right| \geq \varepsilon. \quad (6.3.7)$$

Otherwise, the fitted values $\hat{c}$, $\hat{d}$, and $\hat{m}$ are admissible. Condition (6.3.7) precludes the possibility that an ordinary change-loss function (i.e. without a cap) is incorrectly identified as a capped change-loss function.

As in the previous section, we consider both the exponential and Pareto distributions with tolerance error $\varepsilon = 0.1$ and nine sample sizes $N = 130$ to 390 in multiple of 30. Tables 6.3 and 6.4 depict, respectively, for the exponential and Pareto distribution cases, the results of the fits for the standard deviation premium principle examples with 1,000 independent replications. The boxplots for all the admissible $\hat{c}, \hat{d}$, and $\hat{m}$ are also produced. Figures 6.5, 6.6 and 6.7 are, respectively, the boxplots of $\hat{c}$, $\hat{d}$ and $\hat{m}$ for the exponential distribution while Figures 6.8, 6.9 and 6.10 are the results for the Pareto distribution. Based on these results, we make the following remarks:

[2] If $\left| f_{(k)} - \frac{1}{N-k+1} \sum_{j=k}^{N} f_{(j)} \right| < \varepsilon$ for all $k = 1, 2 \cdots, N$, then we set $n_u = 0$ and we ignore this trivial case.

N	Admissibility	mean of $\hat{c}$	mean of $\hat{d}$	mean of $\hat{m}$	mean of $\mathbf{x}$
150	89.7%	1.00 (≈ 0)	2,694.13 (11.45)	1,502.85 (6.90)	1,002.62 (2.75)
180	93.8%	1.00 (≈ 0)	2,686.45 (10.04)	1,505.36 (6.51)	1,001.31 (2.42)
210	95.8%	1.00 (≈ 0)	2,686.67 (9.17)	1,509.18 (5.99)	1,001.31 (2.21)
240	97.1%	1.00 (≈ 0)	2,679.99 (8.51)	1,509.82 (5.94)	999.59 (2.06)
270	97.9%	1.00 (≈ 0)	2,677.78 (7.97)	1,507.24 (5.71)	999.08 (1.95)
300	98.8%	1.00 (≈ 0)	2,676.32 (7.58)	1,507.23 (5.36)	998.94 (1.84)
330	98.9%	1.00 (≈ 0)	2,673.58 (7.17)	1,499.80 (4.97)	999.17 (1.76)
360	99.8%	1.00 (≈ 0)	2,672.89 (6.61)	1,504.78 (4.71)	1,000.20 (1.65)
390	99.9%	1.00 (≈ 0)	2,677.89 (6.55)	1,501.04 (4.80)	999.56 (1.60)

Table 6.3: Empirical-based solutions based on 1000 independent replications of an exponential distribution for the standard deviation premium principle.

- Under our admissibility criteria specified above, not all of the fitted values are considered as admissible in both cases of the exponential and Pareto loss distributions. However, it is still reassuring that the empirical solutions are stable in that predominantly high proportion of the fitted values are identified as admissible. The worst case, which corresponds to $N = 150$ in the exponential distribution case, still suggests that 89.7% of the solutions are identified as the capped change-loss function. In the Pareto distribution case, the admissibility reaches as high as 97.1% in the worst case and it attains 100% when the sample size N is larger than 270.

- Of the admissible solutions, the fitted $\hat{c}$ is virtually equal to 1 (with negligible standard errors[3]) for both loss distributions. This strongly suggests that the optimal ceded loss function is a capped stop-loss function, rather than a capped change-loss function.

- The fitted values of $\hat{d}$, and $\hat{m}$ appear to be reasonable in that their standard

[3]The estimated standard errors are in the order of 10^{-8}.

N	Admissibility	mean of $\hat{c}$	mean of $\hat{d}$	mean of $\hat{m}$	mean of $\mathbf{x}$
150	97.1%	1.00 (≈ 0)	3,211.22 (23.19)	1,295.76 (4.65)	1,003.52 (4.58)
180	98.5%	1.00 (≈ 0)	3,168.34 (18.72)	1,291.44 (4.23)	1,000.20 (3.91)
210	99.5%	1.00 (≈ 0)	3,188.90 (17.09)	1,284.84 (3.69)	999.37 (3.67)
240	99.6%	1.00 (≈ 0)	3,155.74 (14.75)	1,281.42 (3.48)	993.46 (3.42)
270	99.7%	1.00 (≈ 0)	3,181.36 (15.40)	1,281.70 (3.27)	999.26 (3.28)
300	100%	1.00 (≈ 0)	3,161.11 (13.94)	1,278.89 (3.10)	997.82 (3.06)
330	100%	1.00 (≈ 0)	3,162.77 (13.82)	1,270.64 (2.77)	1,000.89 (3.06)
360	100%	1.00 (≈ 0)	3,157.03 (12.27)	1,270.78 (2.76)	998.10 (2.77)
390	100%	1.00 (≈ 0)	3,153.93 (12.09)	1,269.64 (2.59)	997.59 (2.97)

Table 6.4: Empirical-based solutions based on 1000 independent replications of a Pareto distribution for the standard deviation premium principle.

errors decrease as we increase the sample size. This is also supported by the boxplots in Figures 6.6, 6.7, 6.9 and 6.10.

- The last column of Tables 6.3 and 6.4 tabulates the sample mean of the simulated random samples. Again these estimates are consistent with the true value, which is 1,000.

6.4 Conclusion

In this chapter, an extensive numerical studies have been provided in addressing the stability and consistency of our proposed empirical reinsurance models. The focus is on the small sample size. For the examples where we know the analytic solutions, we observe that the empirical solutions are very stable and converge quickly to the theoretically true solution. For the examples where we do not know the theoretical solution, our results are still very encouraging even for small sample

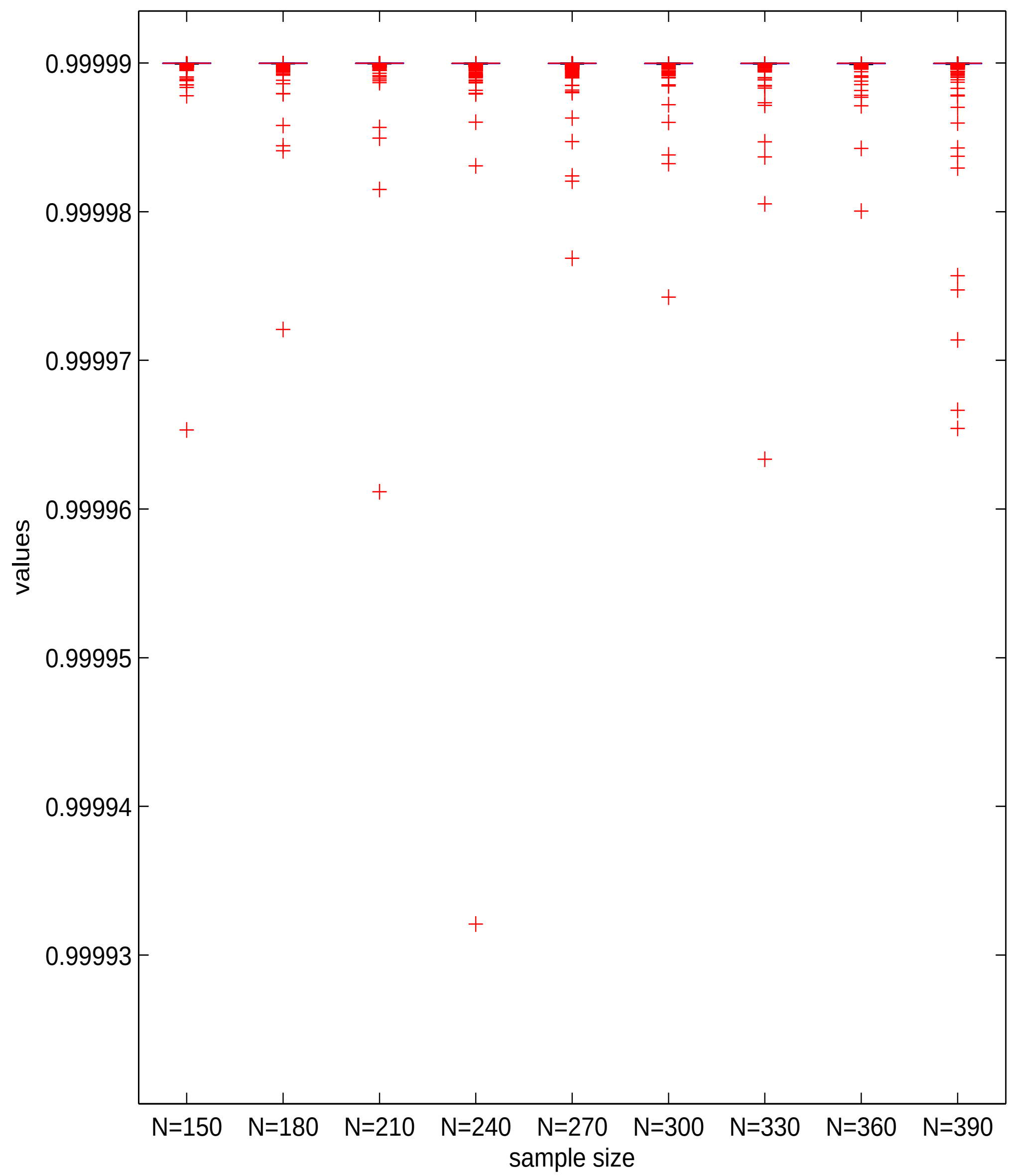


Figure 6.5: Boxplot of the admissible $\hat{c}$ under standard deviation premium principle and exponential loss distribution.

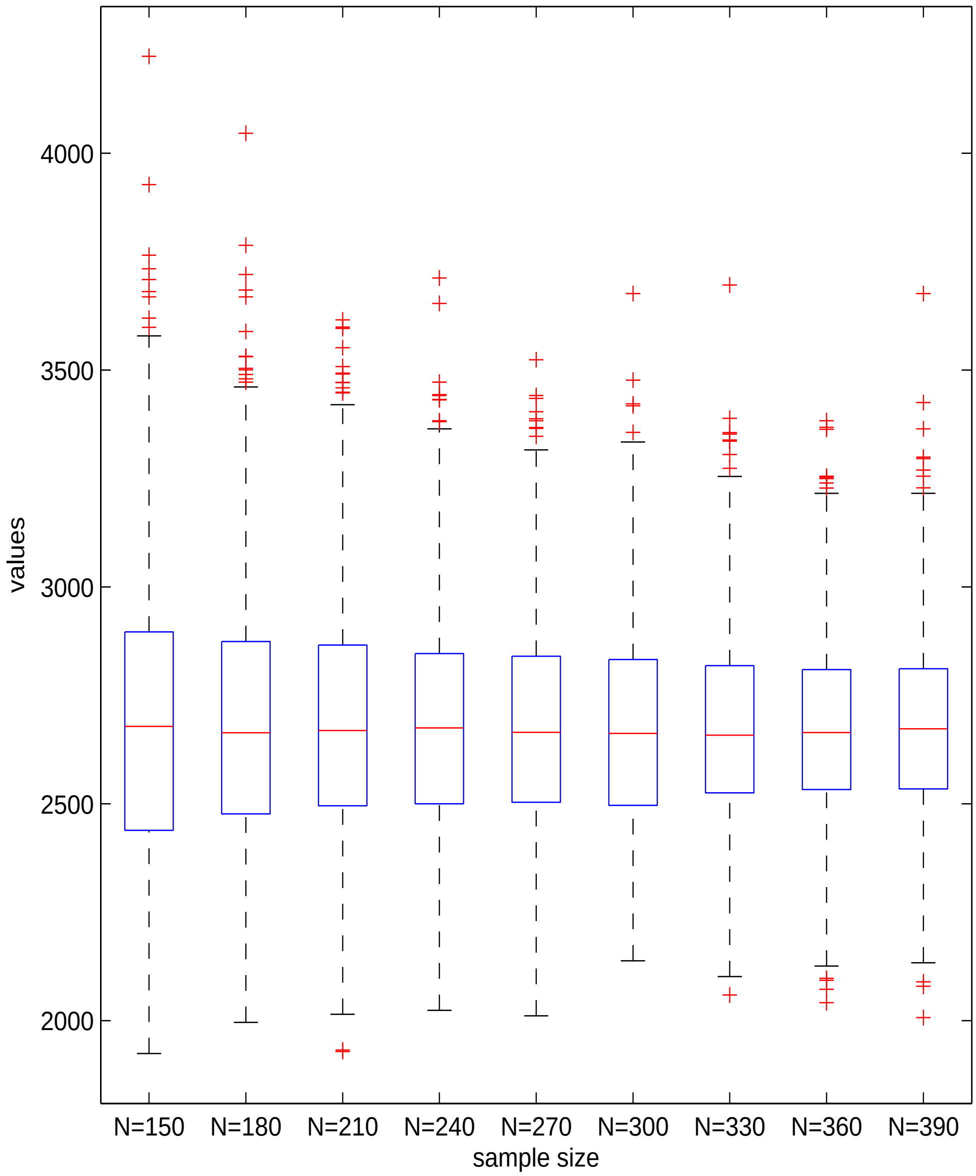


Figure 6.6: Boxplot of the admissible $\hat{d}$ under standard deviation premium principle and exponential loss distribution.

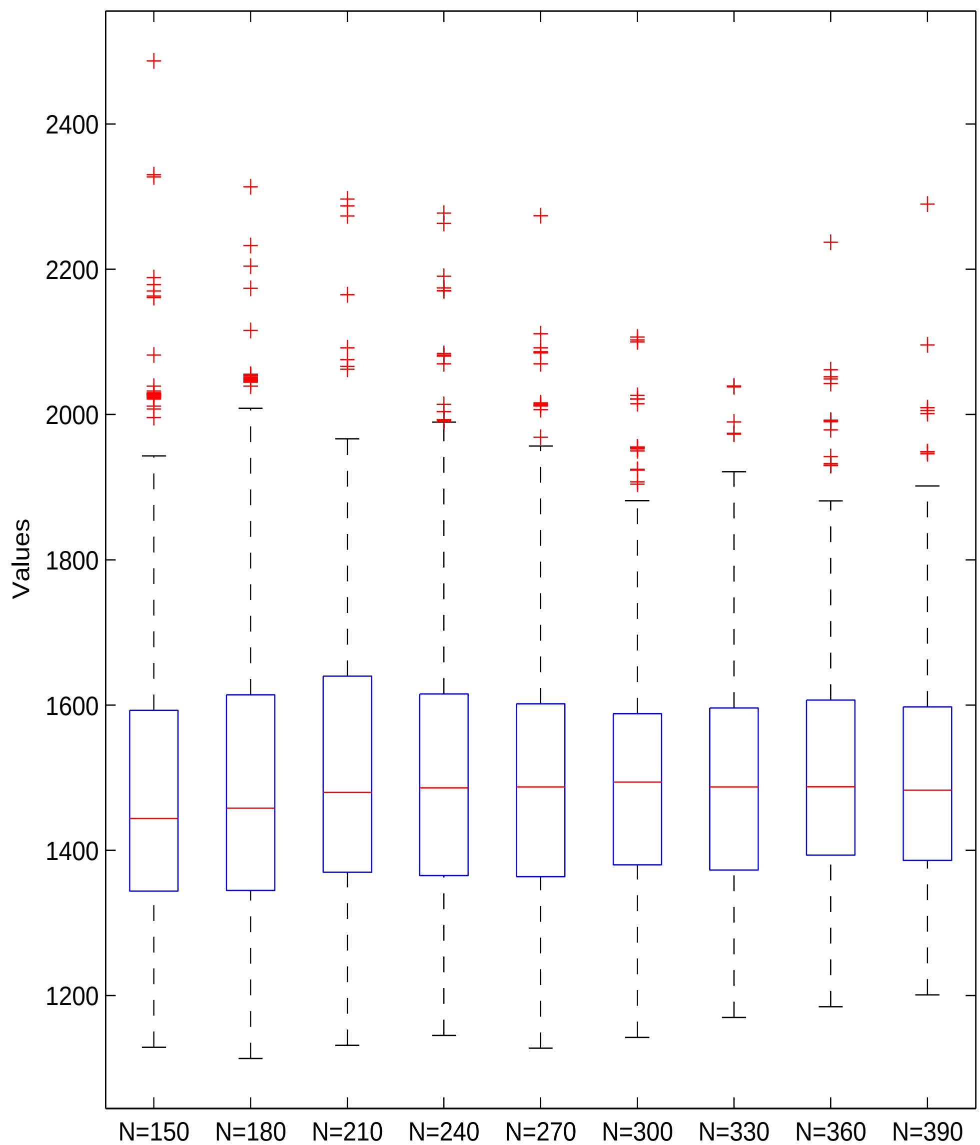


Figure 6.7: Boxplot of the admissible $\hat{m}$ under standard deviation premium principle and exponential loss distribution.

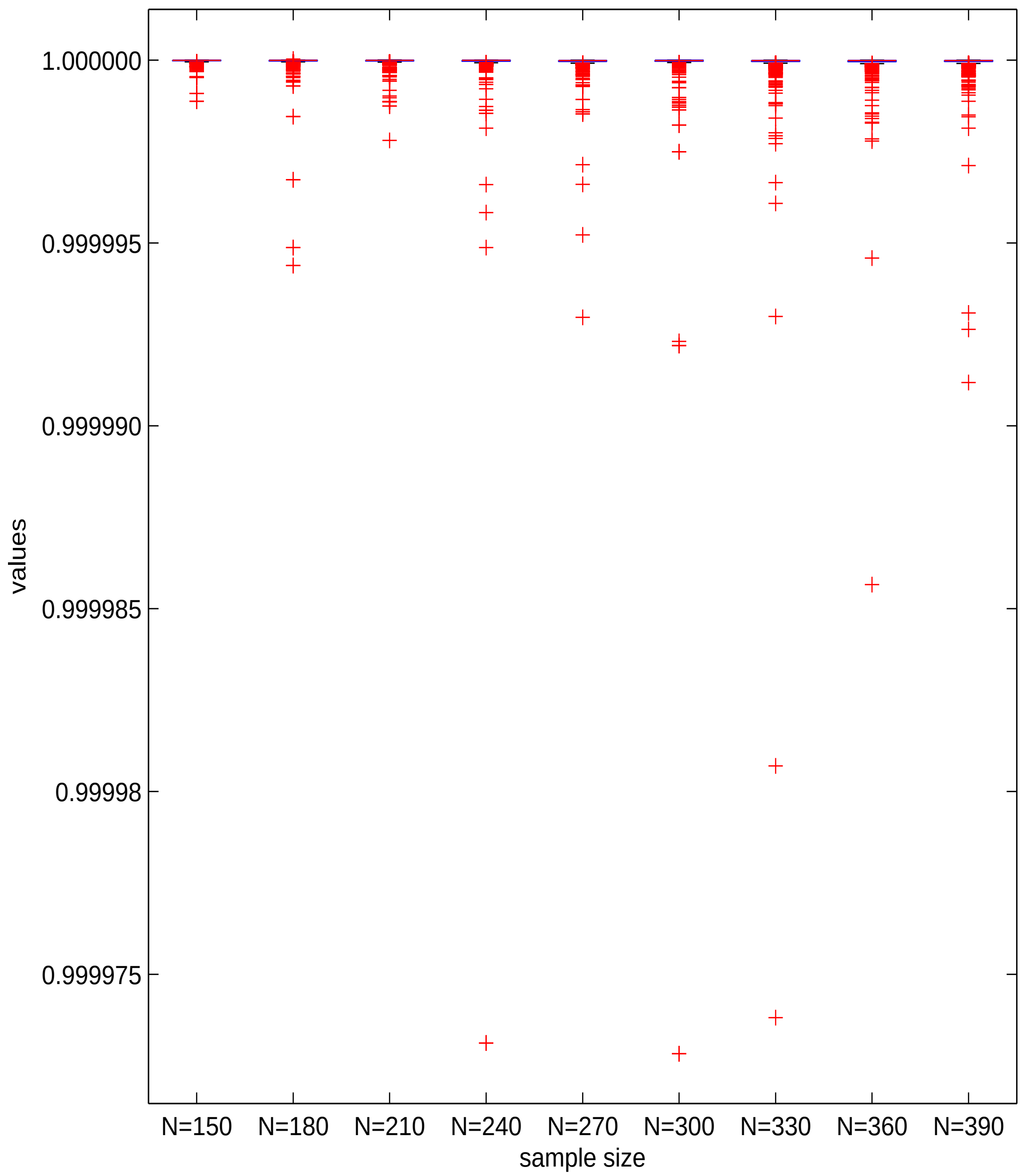


Figure 6.8: Boxplot of the admissible $\hat{c}$ under standard deviation premium principle and Pareto loss distribution.

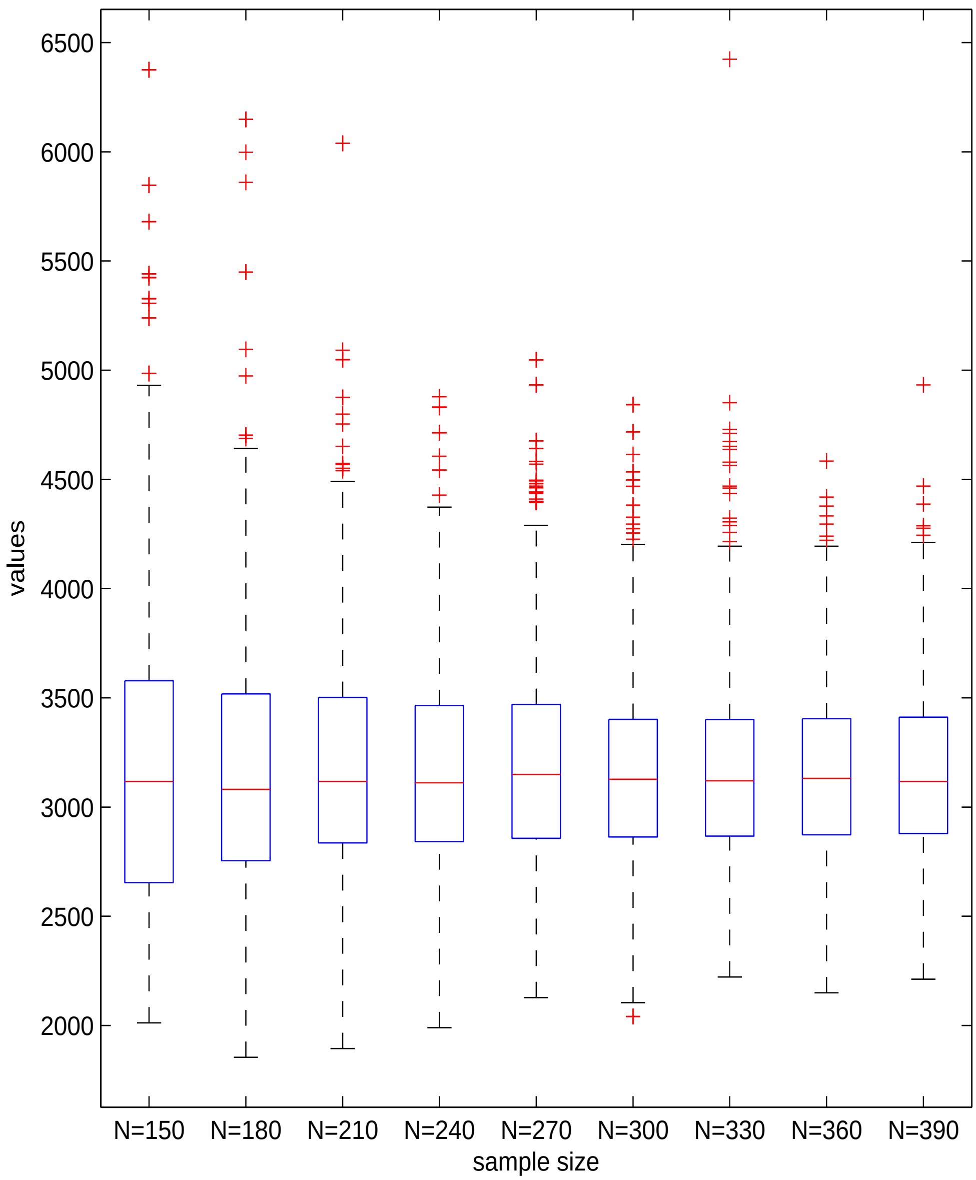


Figure 6.9: Boxplot of the admissible $\hat{d}$ under standard deviation premium principle and Pareto loss distribution.

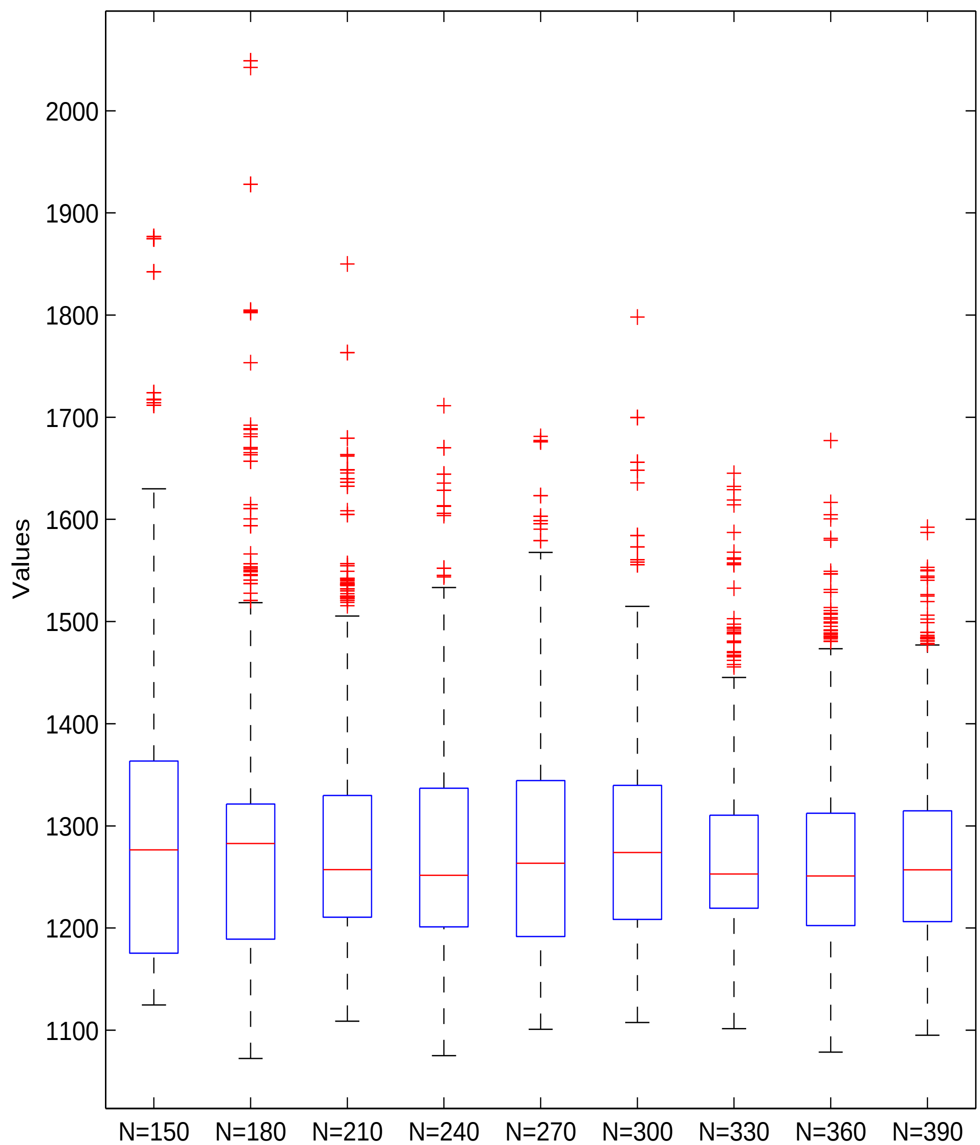


Figure 6.10: Boxplot of the admissible $\hat{m}$ under standard deviation premium principle and Pareto loss distribution.

size with respect to the stability of the empirical solutions. We emphasize that the ultimate advantage of our proposed empirical reinsurance models lies in its flexibility. It can be used to derive optimal ceded loss functions over a variety of premium principles where analytic solutions are typically not available.

Chapter 7

Concluding Remarks and Further Research

7.1 Achievements of the Thesis

In this thesis, I have established a series of reinsurance models and analyzed their optimal solutions. These reinsurance models have the following characteristics. First, the optimality objectives exploit commonly used risk measures such as VaR, CTE, and Variance. Second, the considered reinsurance premium principles encompass the expectation principle, the standard deviation principle and many others. Third, the feasible ceded loss functions are assumed to have different degree of generality. In some models, we confined our analysis to some specific functional forms of the ceded loss functions (such as quota-share, stop-loss or class of increasing convex functions) while in other models, the ceded loss function can be very general. Fourth, the incorporated constraints can be interpreted as either the insurer's profitability guarantee or the insurer's reinsurance premium budget. Finally, depending on the specific type of formulation, the optimal reinsurance designs can be of the form: quota-share, stop-loss, change-loss, or even non-convex type (such as stop-loss with an upper limit).

While the entire thesis was concerned with issues related to optimal reinsurance, we could further classify our results into two broad areas. First few chapters of the thesis focused on analytic results. We observed that in some reinsurance models it was relatively easy to analyze while in other cases the mathematical tools used to derive the optimal solutions can be quite tedious. In the last two chapters (i.e. Chapters 5 and 6), the reinsurance models are formulated directly based on the empirical data. We pointed out that there are a number of advantages associated with this approach. In particular, this method is intuitively appealing, distribution free, and flexible. The proposed reinsurance models can be formulated as SOC programming problems, which in turns facilitate us in obtaining the optimal solutions over a wide range of cases. Hence this approach is more practical and more tractable.

7.2 Future Research

Because of the research that we have been conducting on this thesis, we are very familiar with the tools and methodologies used on problems related to optimal reinsurance. Based upon our knowledge and experience, here we produce a list of possible research topics for future exploration:

- Extend the results of the VaR/CTE minimization criteria to other optimality criteria. For example, the primary goal of reinsurance for many insurers is to maintain, at an acceptable level, the random fluctuations of the business operation of the insurers. Motivated by this argument, an alternate formulation of the reinsurance model is to minimize the *earning volatilities* of the insurer via some reasonable measures. As another example, by noticing the tradeoff between the risk and reward, it is also of interest to consider the reinsurance designs by minimizing the objectives such as *return on risk-adjusted capital (RORAC)*.

- Extend the results for the expectation principle to other premium principles. Recall that the explicit solutions are derived only for the expectation principle when we were analyzing the parametric models without assuming a specific functional form for the feasible ceded loss functions; see Chapters 3 and 4. In Chapters 5 and 6 on the empirical approach to optimal reinsurance, we only analyzed the optimal solutions (to the VaR/CTE/Variance minimization models) for the expectation principle and the standard deviation principle, although we have shown that the resulting empirical reinsurance models can be equivalently cast as the SOC programming for as many as ten principles. Thus, further exploration of the optimal reinsurance under principles other than the expectation and standard deviation is a a natural future area of research. In particular, it will be great interesting to have a better understanding on how the shape of the optimal reinsurance would be affected on more elaborate premium principles. Recall that in the CTE minimization model, the optimal reinsurance changes from a stop-loss to a capped stop-loss when the premium principle changes from the expectation to the standard deviation for low reinsurance premium budget.

- Apply the empirical approach to real data. The tractability of the empirical models enables us to analyze the optimal reinsurance strategy from various angles in terms of different model formulations. The analysis based on the real data will make our research results more practical and applicable.

- Establish convergence results of the empirical solutions derived by the empirical approach. Recall that in Chapter 6, we conducted an analysis on the consistency and stability of the empirical solutions, where we concluded a strong stability and consistency by some numerical experiments on the CTE minimization model. We have not achieved any theoretical conclusion on the consistency and stability of the empirical solutions. Thus, it is theoretically significant to explore the convergence issues of the empirical solutions.

- Analyze the optimal reinsurance under the local models. Recall that in a local model, the reinsurance is applied to individual risk, instead of the overall aggregate risk as in the global model which are the main focus in my thesis. Due to multiple lines of products and the multiple risk exposures, it may be more prudent for an insurer to have the reinsurance coverage on each of these lines of products or each of these risks, instead of just reinsuring the risk in aggregate. Research on this topic is very limited. Kaluszka(2004b) and Cheung (2007) are two of the few related research papers. Since in a local model the reinsurance is applied to individual risk, the dependence structure of these risks therefore take a critical role in determining the optimal reinsurance. Thus, it is of interest to incorporate the copula method or stochastic ordering approach to describe the dependence among the individual risks or to compare the resulting risks in the presence of different reinsurance contracts.

Bibliography

[1] Aase, K. K., 2002. Perspectives of Risk Sharing. Scandinavian Actuarial Journal 2, 73-128.

[2] Alizadeh, F., and Goldfarb, D., 2003, Second-order Cone Programming. Mathematical Programming, Series B 95(1), 3-51.

[3] Arrow, K.J., 1974. Optimal Insurance and Generalized Deductibles. Scandinavian Actuarial Journal 1-42.

[4] Artzner, P., Delbaen, F., Eber, J.M., and Heath, D., 1999. Coherent Measures of Risks. Mathematical Finance 9, 203-228.

[5] Ben-Tal, A., and Nemirovski, A., 2001. Lectures on Modern Convex Optimization: Analysis, Algorithms, and Engineering Applications, SIAM-MPS Series in Optimizaton, SIAM Publications, Philadelphia.

[6] Bernard, C., and Tian, W., 2009. Optimal Reinsurance Arrangements under Tail Risk Measures. Journal of Risk and Insurance, forthcoming.

[7] Bonnans, J.F., and Shapiro, A., 2000. Perturbation Analysis of Optimization Problems. Springer Verlag.

[8] Borch, K., 1960. The Safety Loading of Reinsurance Premiums. Skandinavisk Aktuarietidshrift, 163-184.

[9] Bowers, N.J., Gerber, H. U., Hickman, J. C., Jones, D. A., and Nesbitt, C. J., 1997. Actuarial Mathematics. Second Edition. The Society of Actuaries, Schaumburg.

[10] Cai, J., and Tan, K. S., 2007. Optimal Retention for a Stop-loss Reinsurance under the VaR and CTE Risk Measure. The ASTIN Bulletin 37, 93-112.

[11] Cai, J., Tan, K. S., Weng, C., and Zhang, Y., 2008. Optimal Reinsurance under VaR and CTE Risk Measures. Insurance: Mathematics & Economics 43, 185-196.

[12] Cardin, M., and Pacelli, G., 2007. On characterization of convex premium principles, in Mathematical and Statistical Methods in Insurance and Finance, 53-60. Springer Verlag, New York.

[13] Centeno, M.L., 1985. On Combining Quota-share and Excess of loss. The ASTIN Bulletin 15, 49-63.

[14] Centeno, M.L., 1986. Some Mathematical Aspects of Combining Proportional and Non-proportional Reinsurance. In Insurance and Risk Theory, 247-266, edited by Goovaerts, M., de Vyldere, F., and Haezendonck, J., Reidel D. Publishing Company, Holland.

[15] Cheung, K.C., 2007. Optimal allocation of policy limits and deductibles. Insurance: Mathematics & Econonics 41, 382-391.

[16] Deprez, O., and Gerber, H.U., 1985. On Convex Principles of Premium Calculation. Insurance: Mathematics & Econonics 4, 179-189.

[17] Dhaene, J., Denuit, M., Goovaerts, M.J., Kaas, R., and Vyncke, D., 2002. The Concept of Comonotonicity in Actuarial Science and Finance: Theory. Insurance: Mathematics & Economics 31 (1), 3-33.

[18] Dhaene J., Vanduffel, S., Goovaerts, M.J., Kaas, R., Tang, Q., and Vyncke, D., 2006. Risk Measures and Comotonicity: A Review. Stochastic Models 22, 573-606.

[19] Dickson, D.C.M., and Waters, H.R., 2006. Optimal Dynamic Reinsurance. The ASTIN Bulletin 36, 415-432.

[20] Föllmer, H., and Schied, A., 2002. Convex Measures of Risk and Trading Constraints. Finance and Stochastics 6, 429-447.

[21] Gaivoronski, A.A., and Pflug, G., Winter 2004-2005. Value-at-Risk in Portfolio Optimizaiton: Properties and Computational Approach. Journal of Risk 7(2), 1-31.

[22] Gajek, L., and Zagrodny, D., 2000. Insurer's Optimal Reinsurance Strategies. Insurance: Mathematics & Econonics 27, 105-112.

[23] Gajek, L., and Zagrodny, D., 2004. Optimal Reinsurance under General Risk Measures. Insurance: Mathematics & Econonics 34, 227-240.

[24] Gerber, H.U., 1979. An Introduction to Mathematical Risk Theory. S.S. Huebner Foundation for Insurance Education, Wharton School, University of Pennsylvania, Philadelphia.

[25] Gerber, H.U., and Pafumi, G., 1998. Utility Functions: From Risk Theory to Finance. North American Actuarial Journal 2, 74-100.

[26] Grant, M., and Boyd, S., December 2008. CVX: MATLAB Software for Disciplined Convex Programming, (web page and software), version 1.2. http://stanford.edu/~boyd/cvx.

[27] Hald, M., and Schmidli, H., 2004. On the Maximization of the Adjustment Coefficient under Proportional Reinsurance. The ASTIN Bulletin 34, 75-83.

[28] Heerwaarden van, A.E., Kaas, R., and Goovaerts, M.J., 1989. Optimal Reinsurance in Relation to Ordering of Risks. Insurance: Mathematics & Econonics 8, 261-67.

[29] Hipp, C., and Vogt, M., 2003. Optimal Dynamic XL Reinsurance. The ASTIN Bulletin 33, 193-207.

[30] Kahn, P. M., 1961. Some Remarks on a Recent Paper by Borch. The ASTIN Bulletin 1, 265-272.

[31] Kaishev, V.K., and Dimitrova, D. S., 2006. Excess of Loss Reinsurance under Joint Survival Optimality. Insurance: Mathematics & Economics 39, 376-389.

[32] Kass, R., Goovaerts, M., Dhaene, J., and Denuit, M., 2001. Modern Actuarial Risk Theory. Kluwer Academic Publishers, Boston.

[33] Kaluszka, M., 2001. Optimal Reinsurance under Mean-variance Premium Principles. Insurance: Mathematics & Econonics 28, 61-67.

[34] Kaluszka, M., 2004a. An Extension of Arrow's Result on Optimality of a Stop Loss Contract. Insurance: Mathematics & Econonics 35, 527-536.

[35] Kaluszka, M., 2004b. Mean-variance Optimal Reinsurance Arrangements. Scandinavian Actuarial Journal 1, 28-41.

[36] Kaluszka, M., 2005. Optimal Reinsurance under Convex Principles of Premium Calculation. Insurance: Mathematics & Economics 36, 375-398.

[37] Larsen, N., Mausser, H., and Uryasev, S., 2002. Algorithms for Optimization of Value-at-Risk. in P. Pardalos and V.K. Tsitsiringos, (Eds.), Financial Engineering, e-Commerce and Supply Chain, Kluwer Academic Publishers, 129-157.

[38] Lobo, M.S., Vandenberghe, L., Boyd, S., and Lebret, H., 1998. Applications of Second Order Cone Programming. Linear Algebra and Its Applications 284, 193-228.

[39] McNeil, A. J., Frey, R., and Embrechts, P., 2005. Quantitative Risk Management: Concepts, Techniques, and Tools. Princeton University Press.

[40] Müller, A., and Stoyan, D., 2002. Comparison Methods for Stochastic Models and Risks. Willey Series in Probability and Statistics.

[41] Ohlin, J., 1969. On a Class of Measures of Dispersion with Application to Optimal Reinsurance", The ASTIN Bulletin 5, 249-266.

[42] Promislow, S.D., and Young, V. R., 2005. Unifying Framework for Optimal Insurance. Insurance: Mathematics & Econonics 36, 347-364.

[43] Rockafellar, R.T., and Uryasev, S., 2002. Conditional Value-at-Risk for General Loss Distributions. Journal of Banking & Finance 26, 1443-1471.

[44] Schied, A., 2006. Risk Measures and Robust Optimizaiton Problems. Stochastic Models 22, 753-831.

[45] Schmidli, H., 2001. Optimal Proportional Reinsurance Policies in a Dynamic Setting. Scandinavian Actuarial Journal 1, 55-68.

[46] Sturm, J.F., 1999. Using sedumi 1.02, a Matlab Toolbox for Optimization over Symmetric Cones. Optimization Methods and Software 11-12, 625-653.

[47] Tasche, D., 2002. Expected Shortfall and Beyond. Journal of Banking and Finance 26, 1519-1533.

[48] Tütüncü, R.H., Toh, K.C., and Todd, M.J., 2003. Solving Semidefinite-quadratic-linear Programs Using SDPT3. Mathematical Programming 95, 189-217.

[49] Verlaak, R., and Beirlant, J., 2003. Optimal Reinsurance Programs: An Optimal Combination of Several Reinsurance Protections on a Heterogeneous Insurance Portfolio. Insurance: Mathematics & Econonics 33, 381-403.

[50] Wang, C.P., Shyu, D., and Huang H.H., 2005. Optimal Insurance Design under a Value-at-Risk Framework. The Geneva Risk and Insurance Review 30, 161-179.

[51] Wang, S., Young, V., and Panjer, H, 1997. Axiomatic Characterization of Insurance Prices. Insurance: Mathematics and Economics 21, 173-183.

[52] Wirch J. W., and Hardy M. R., 1999. A Synthesis of Risk Measures for Capital Adequacy. Insurance: Mathematics & Economics 25, 337-347.

[53] Young, V.R., 1999. Optimal Insurance under Wang's Premium Principle. Insurance: Mathematics & Econonics 25, 109-122.

[54] Zagrodny, D., 2003. An optimality of the Change Loss type Strategies. Optimization 52, 757-772.