Non-commuting n-tuples of operators and dilation theory

by

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For Lyn

Abstract

Every contractive n-tuple of operators has a minimal joint isometric dilation to isometries acting on a larger space. Each of these dilations decomposes into a *Cuntz* part and a *pure* part. The Cuntz part determines a representation of the Cuntz C*-algebra. When an n-tuple acts on finite dimensional space, its dilation is completely described in terms of the original n-tuple. This is accomplished by classifying the associated Cuntz representations. In fact, simple complete unitary invariants for the representations are obtained. The pure part of a dilation is determined by copies of the left regular representation of the free semigroup on n letters. The number of copies can be computed directly in terms of the original n-tuple. Davidson and Pitts have shown that the non-selfadjoint WOT-closed algebras generated by the pure isometries or 'left creation operators' are the appropriate non-commutative analytic Toeplitz algebras. Factorization problems in these algebras are investigated. Positive results are obtained when norm conditions are placed on possible factors; however, over the full algebra deep factorization pathologies are exposed. This leads to information on the left ideals in these algebras. Finally, non-commutative versions of Arveson's curvature invariant and Euler characteristic for a commuting n-tuple of operators are developed. They are sensitive enough to detect when an n-tuple is free. The curvature invariant is shown to be upper semi-continuous. A new class of examples is introduced and is used to obtain information on the ranges of the invariants.

Contents

Chapter 1

Introduction

The basic goal of this thesis is to understand the behaviour of non-commuting n-tuples of operators by using dilation theory. The dilation theory utilized here derives from a theorem of Frahzo, Bunce and Popescu. Every contractive n-tuple of operators acting on the same Hilbert space has a unique minimal joint isometric dilation to isometries with pairwise orthogonal ranges acting on a larger space. The general relationship between an n-tuple and its dilation is investigated. In particular, using non-selfadjoint operator algebra techniques, a complete characterization is obtained when the n-tuple acts on finite dimensional space. Recently, Davidson and Pitts showed that the n-tuples of isometries arising from the left regular representation of the free semigroup on n letters determine the appropriate non-commutative analytic Toeplitz algebras. The structure theory for these so called 'left creation operators' is considered. Finally, the completely positive map defined by an n-tuple is used to obtain non-commutative versions of Arveson's curvature invariant and Euler characteristic of a commuting n-tuple.

The dilation of an n-tuple is closely linked to the original n-tuple. Popescu's Wold decomposition shows that every n-tuple of isometries with pairwise orthogonal ranges decomposes into a direct sum of isometries which determine a representation of the Cuntz C*-algebra, together with copies of the left creation operators. Thus every contractive n-tuple determines a Cuntz representation and copies of the left regular representation through its dilation.

When the n-tuple acts on finite dimensional space, the WOT-closed non-selfadjoint algebra generated by the dilation is completely described in terms of the properties of the original n-tuple and the algebra it generates. This provides complete unitary invariants for the corresponding C*-representations. including a simple characterization of irreducibility. The algebra determined by the dilation is also shown to be hyper-reflexive, adding to the short list of algebras known to have this property. Further, the number of copies of the left regular representation in the dilation can in general be computed directly from the original n-tuple. This is the content of Chapter 2, and is joint work with Davidson and Shpigel from [**16**].

The non-commutative analytic Toeplitz algebra is the WOT-closed algebra generated by the left regular representation of the free semigroup on n letters. Chapter 3 contains a detailed analysis of operators in these algebras from the paper [**27**]. The structure theory of contractions is examined. Each is shown to have an H^∞ functional calculus. The isometries defined by words are shown to factor only as the words do over the unit ball of the algebra. This turns out to be false over the full algebra. The natural identification of WOT-closed left ideals with invariant subspaces of the algebra is shown to hold only for a proper subcollection of the

subspaces.

Finally, there is a completely positive contractive map determined by every contractive n-tuple of operators. Recently, Arveson used this map to introduce the notion of a curvature invariant and Euler characteristic for a commuting n-tuple of operators. Chapter 4 contains the development, from [28], of the non-commutative versions of these invariants. They possess some of the basic properties analogous to those from the commutative setting. Most importantly, the invariants are sensitive enough to determine if an n-tuple is free. Moreover, the curvature invariant is shown to be upper semi-continuous. A new class of examples is used to illustrate the differences encountered in the non-commutative setting. The examples provided yield information on the ranges of the invariants. In particular, the range of the curvature invariant is shown to be the entire positive real line.

The reader may find some slight overlaps in the preliminary sections of the different chapters since they are taken from three different papers. I have attempted to minimize these overlaps. A note on the major objects of study, the notation and the tools used in the thesis is included below. Finally, I would like to thank my advisor Ken Davidson for his assistance. Without his help this thesis would not have been possible.

1.1 Notation and Nomenclature

An *operator* refers to a continuous linear operator which acts on a finite or infinite dimensional Hilbert space $\mathcal{H}$. There are three important topologies which can be placed on the collection of operators $\mathcal{B}(\mathcal{H})$ acting on $\mathcal{H}$. The strongest of these is

the *norm topology* which is determined by convergence in the operator norm:

$$||A|| = \sup\{||Ax|| : ||x|| \leq 1,\ x \in \mathcal{H}\}.$$

Next is the *strong operator topology* (SOT), which is simply point-wise convergence. The weakest is the *weak operator topology* (WOT), which amounts to convergence on all of the inner product functionals (Ax, y), for x, y in $\mathcal{H}$.

This thesis is concerned with analyzing the behaviour of n-tuples of operators $A = (A_1, \ldots, A_n)$, with $A_i \in \mathcal{B}(\mathcal{H})$ where the A_i are pairwise non-commuting in general. Such an n-tuple can be thought of as a $1 \times n$ row matrix mapping $\mathcal{H}^{(n)}$ to $\mathcal{H}$. Hence its norm is given by

$$||A|| = ||AA^*||^{1/2} = ||\sum_{i=1}^{n} A_i A_i^*||^{1/2}.$$

The n-tuple is said to be *contractive* if $||A|| \leq 1$. Thus a contractive n-tuple really is the multi-variable analogue of a contraction.

Much of the theory is motivated by the situation for an n-tuple of isometries $S = (S_1, \ldots, S_n)$. Since S_i is an isometry, the operator $S_i S_i^*$ is the orthogonal projection onto the range of S_i. Hence S is contractive precisely when the S_i have pairwise orthogonal ranges. That is,

$$SS^* = \sum_{i=1}^{n} S_i S_i^* \leq I \quad \text{if and only if} \quad S_i S_j^* = \delta_{ij} I.$$

Recall that a C*-algebra is a norm closed self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$. For a

contractive n-tuple of isometries there are only two possible C*-algebras which the S_i can generate.

When the ranges of the S_i span the whole space ($\sum_{i=1}^n S_i S_i^* = I$), the C*-algebra generated by the S_i is the *Cuntz algebra* $\mathcal{O}_n$. This is the simple universal C*-algebra generated by such an n-tuple of isometries and examining its representation theory is important in several areas of mathematics. Namely, there has been recent interest in classifying subclasses of Cuntz representations because of a correspondence between these representations and endomorphisms of $\mathcal{B}(\mathcal{H})$ (see **[36, 29, 8, 9]**). There is also a connection with wavelet theory, as certain Cuntz representations give rise to wavelets **[11]**. The link with this thesis comes from the connection with dilation theory, which is discussed below.

The other possible C*-algebra generated by the S_i is when the ranges span a proper subspace ($\sum_{i=1}^n S_i S_i^* < I$). This is the *Cuntz-Toeplitz algebra* $\mathcal{E}_n$, the universal C*-algebra generated by such an n-tuple of isometries. It is the extension of the compact operators by $\mathcal{O}_n$. For, the projection $I - \sum_{i=1}^n S_i S_i^*$ is minimal in $\mathcal{E}_n$ and the ideal it generates determines a copy of the compacts. The prototypical example in this case comes from the left regular representation λ of the unital free semigroup $\mathcal{F}_n$ on n letters $\{1, \ldots, n\}$ which acts on n-variable Fock space $\mathcal{H}_n = \ell^2(\mathcal{F}_n)$. The *pure isometries* $L_i = \lambda(i)$ arise in theoretical physics as the *left creation operators*. They also form a contractive n-tuple of isometries and in fact

$$I - \sum_{i=1}^n L_i L_i^* = P_e,$$

where P_e is the rank one projection onto the span of the *vacuum vector*, the basis

vector corresponding to the unit or empty word in $\mathcal{F}_n$.

The WOT-closed unital (non-selfadjoint) algebras generated by the L_i form the appropriate *non-commutative analytic Toeplitz algebras* and are denoted by $\mathcal{L}_n$ (see [1, 17, 18, 19]). They are so named since for $n = 1$ one obtains the analytic Toeplitz algebra and for $n \geq 2$ there are analogues of Beurling's Theorem and inner-outer factorization in particular. In general it can be advantageous to consider the WOT-closed, non-selfadjoint setting. Indeed, using the WOT-closure can preserve a strong spatial link with the associated representation. Further, non-selfadjoint algebras generated by an n-tuple S often possess *wandering subspaces*. That is, subspaces $\mathcal{W}$ for which the ranges $S_v\mathcal{W}$ and $S_w\mathcal{W}$ are pairwise orthogonal for distinct words v, w (for $v \in \mathcal{F}_n$, S_v is the isometry $v(S_1, \ldots, S_n)$).

The basic idea of *dilation theory* is to view objects of concern as pieces of larger objects for which much more is known. One of the most well known operator-theoretic dilation theory results is Sz.-Nagy's unique minimal isometric dilation of a contraction [40]. Every contraction $A \in \mathcal{B}(\mathcal{H})$ is the compression of an isometry acting on a larger space to a co-invariant subspace. The minimal dilation is unique up to unitary equivalence fixing $\mathcal{H}$. The multi-variable analogue of this theorem has recently been developed. It derives from the work of Frahzo [22], Bunce [12] and Popescu [31]. Every contractive n-tuple $A = (A_1, \ldots, A_n)$ with $A_i \in \mathcal{B}(\mathcal{H})$ has a unique minimal joint isometric dilation to isometries $S = (S_1, \ldots, S_n)$ with pairwise orthogonal ranges acting on a larger space which contains $\mathcal{H}$ as a co-invariant subspace. Again, the uniqueness is up to unitary equivalence fixing $\mathcal{H}$.

One also has a Wold decomposition in this setting. Recall that the original

Wold decomposition shows that every isometry decomposes into an orthogonal direct sum of a unitary operator together with copies of the unilateral shift. Popescu's non-commutative multi-variable version is proved in the same way. It shows that an n-tuple of isometries with pairwise orthogonal ranges decomposes into a joint orthogonal direct sum of a Cuntz n-tuple and copies of the pure isometries. Equivalently, this says that every representation of $\mathcal{E}_n$ decomposes into the direct sum of a Cuntz representation and copies of λ. A key difference between the two settings is that there is a good structure theory for unitary operators from the Spectral Theorem; whereas relatively little is known about Cuntz representations. In any event, from the dilation theory and Wold decomposition, it follows that associated with every contractive n-tuple is a Cuntz representation and copies of the left regular representation through its minimal dilation.

Lastly, a particular completely positive map is important in the thesis. A mapping $\Psi : \mathcal{A} \to \mathcal{B}$ between C*-algebras is said to be *completely positive* if the natural amplifications $\Psi^{(k)} : \mathcal{M}_k(\mathcal{A}) \to \mathcal{M}_k(\mathcal{B})$ are positive for $k \geq 1$. This is a strong notion and one that is pervasive through much of operator theory and operator algebras. For an excellent introductory treatment see the text [**30**]. Let $A = (A_1, \ldots, A_n)$ be a contraction on $\mathcal{H}$. The completely positive map of concern here is defined for $X \in \mathcal{B}(\mathcal{H})$ by,

$$\Phi(X) = \sum_{i=1}^{n} A_i X A_i^* = A X^{(n)} A^*.$$

In fact this map is also *completely contractive* ($||\Phi^{(k)}|| \leq 1$ for $k \geq 1$) since $\Phi(I) \leq 1$. It turns out that the decreasing sequence of positive operators $\Phi^k(I)$ can yield

information on the determining n-tuple.

In this thesis, generally n will be taken to be a finite integer with $n \geq 2$. However, Popescu's version of the dilation theorem is valid for $n = \infty$, as are the results of [17, 18] on the structure of $\mathfrak{L}_n$ which are used. So the results of Chapters 2 and 3 go through for $n = \infty$ with only a few minor changes in notation, not in substance. On the other hand, in Chapter 4 the $n = \infty$ case cannot be considered by the very nature of the invariants defined there. For ease of presentation, the entire thesis has been written as though n were finite.

Chapter 2

Isometric dilations of finite rank n-tuples

In [17, 18], Davidson and Pitts studied a class of algebras which they called *free semigroup algebras*. These are the WOT-closed non-selfadjoint unital operator algebras generated by an n-tuple of isometries with pairwise orthogonal ranges. When these ranges span the whole space, the associated norm-closed self-adjoint algebra is a representation of the Cuntz algebra $\mathcal{O}_n$. This non-selfadjoint algebra can contain detailed information about the unitary invariants of the corresponding C*-algebra representation. Indeed, in [17] the set of atomic representations of the Cuntz algebra is completely classified. On the other hand, when the ranges span a proper subspace, a representation of the Cuntz-Toeplitz algebra $\mathcal{E}_n$ is obtained. Such a representation contains a multiple of the left regular representation of the free semigroup on n letters. The WOT-closed algebra determined by the left regular representation is called the *non-commutative analytic Toeplitz algebra*. This

terminology is justified by an analogue of Beurling's Theorem [**34, 1, 17**], hyperreflexivity [**17**] and the relationship [**18**] between its automorphism group and the group of conformal automorphisms of the ball in $\mathbb{C}^n$. The structure theory of these algebras is particularly useful in Chapters 3 and 4.

The connection with dilation theory derives from a theorem of Frahzo, Bunce and Popescu [**22, 12, 31**]. If $A = (A_1, \ldots, A_n)$ is an n-tuple of operators such that $AA^* = \sum_{i=1}^n A_i A_i^* \leq I$, then there is a unique minimal joint isometric dilation to isometries S_i on a larger space with pairwise orthogonal ranges. Popescu [**31**] establishes the analogue of Wold's decomposition which splits this into a direct sum of a multiple of the left regular representation and a representation of the Cuntz algebra. Moreover, Popescu [**33**] obtains the non-commutative analogue of von Neumann's inequality in this context.

On the other hand, representations of the Cuntz algebra correspond to endomorphisms of $\mathcal{B}(\mathcal{H})$ [**36, 29, 8, 9**]. This has created new interest in classifying these representations up to unitary equivalence. There is a theorem of Glimm [**25**] which shows that this classification is 'non-smooth' because $\mathcal{O}_n$ is anti-liminal (or NGCR). Nevertheless, interesting classes of representations do lend themselves to a complete analysis. In [**10**], Bratteli and Jorgensen introduced a class of representations which turned out to be a special case of the atomic representations classified in [**17**] using non-selfadjoint techniques. In [**9**] they introduce a different class associated to finitely correlated states. There are a lot of parallels between their results and those here, though the approach is quite different. In the end, they specialize to the subclass of diagonalizable shifts in order to obtain a classification theorem. In

this chapter, simple unitary invariants for the class of all of these finitely correlated representations are obtained.

This chapter has two broad goals. The first is to understand the structure of the free semigroup algebra generated by the dilation of an n-tuple A in terms of information obtained from the n-tuple itself (and the algebra it generates). In particular, unitary invariants for the associated C*-algebra representation are sought. Secondly, it is desirable to determine if these algebras are reflexive and even hyper-reflexive. In this chapter, the focus is on the case in which the n-tuple A acts on a finite dimensional space. A complete description of the algebra is obtained here. This enables one to decompose the associated representation as a direct sum of irreducible representations and leads to complete unitary invariants. These algebras all turn out to be hyper-reflexive.

2.1 Preliminaries

Let $\mathcal{F}_n$ denote the unital free semigroup on n letters $\{1, 2, \ldots, n\}$, and let $\mathcal{H}_n = \ell^2(\mathcal{F}_n)$ denote the Hilbert space with basis $\{\xi_w : w \in \mathcal{F}_n\}$, which is known as n-variable Fock space. The left regular representation λ of $\mathcal{F}_n$ is given by $\lambda(v)\xi_w := L_v\xi_w = \xi_{vw}$. In particular, the generators of $\mathcal{F}_n$ determine isometries L_i for $1 \leq i \leq n$ with orthogonal ranges such that $\sum_{i=1}^n L_i L_i^* = I - P_e$ where $P_e = \xi_e\xi_e^*$ is the rank one projection onto the basis vector for the empty word e, which is the identity of $\mathcal{F}_n$. The algebra $\mathfrak{L}_n$ is the WOT-closed algebra generated by the n-tuple $L = (L_1, \ldots, L_n)$ of 'left creation operators'. See **[17, 18, 19, 27, 32, 34]** for detailed information about this algebra.

More generally if S_i, for $1 \leq i \leq n$, are isometries with $\sum_{i=1}^n S_i S_i^* \leq I$, let $\mathfrak{S}$ denote the unital WOT-closed non-selfadjoint algebra generated by them. Let S_v be the isometry $v(S) := v(S_1, \ldots, S_n)$ for each $v \in \mathcal{F}_n$. A subspace $\mathcal{W}$ is said to be *wandering* for the n-tuple $S = (S_1, \ldots, S_n)$ provided that the subspaces $S_v\mathcal{W}$ are pairwise orthogonal for all $v \in \mathcal{F}_n$. Thus the smallest $\mathfrak{S}$-invariant subspace containing a wandering space $\mathcal{W}$ is $\mathfrak{S}[\mathcal{W}] = \sum_{v \in \mathcal{F}_n} \oplus S_v\mathcal{W}$. The restriction of $\mathfrak{S}$ to this subspace is evidently a multiple of the left regular representation algebra $\mathfrak{L}_n$, where the multiplicity is given by $\dim \mathcal{W}$. Popescu's Wold decomposition [**31**] works as follows: the subspace $\mathcal{W} = \mathrm{Ran}(I - \sum_{i=1}^n S_i S_i^*)$ is easily seen to be wandering. Moreover, the complement $\mathcal{H}_c = \mathfrak{S}[\mathcal{W}]^\perp$ is also invariant for $\mathfrak{S}$, and the restriction to $\mathcal{H}_c$ yields isometries $T_i = S_i|_{\mathcal{H}_c}$ satisfying $\sum_{i=1}^n T_i T_i^* = I_{\mathcal{H}_c}$.

Suppose $A = (A_1, \ldots, A_n)$ is an n-tuple of operators on a Hilbert space $\mathcal{V}$ such that $AA^* = \sum_{i=1}^n A_i A_i^* \leq I$. Frahzo [**22**] (for $n = 2$), Bunce [**12**] (for $n < \infty$) and Popescu [**31**] (for $n = \infty$) show that there is a joint dilation of the A_i to isometries S_i on a Hilbert space $\mathcal{H} = \mathcal{V} \oplus \mathcal{V}^\perp$ which have pairwise orthogonal ranges. Popescu observes that if this dilation is minimal in the sense that $\mathcal{H} = \mathrm{span}\{S_v\mathcal{V} : v \in \mathcal{F}_n\}$. then the dilation is unique (up to a unitary equivalence which fixes $\mathcal{V}$). This minimal isometric dilation will always be the dilation considered throughout the thesis.

Popescu also observes [**33**] that the norm-closed non-selfadjoint algebra $\mathcal{A}_n$ spanned by $\{L_w : w \in \mathcal{F}_n\}$ is the appropriate non-commutative analogue of the disk algebra for a version of von Neumann's inequality. Namely, if A is a contractive n-tuple as above, then $\|p(A)\| \leq \|p(L)\|$ for every non-commuting polynomial in n variables. This is immediate from the dilation theorem and the fact that there is

a contractive homomorphism of $\mathcal{E}_n$ onto $\mathcal{O}_n$, the two possible C*-algebras for the dilation. However, it turns out that this quotient map is completely isometric on $\mathcal{A}_n$. So this norm estimate is an equality for any contractive n-tuple of *isometries*. This shows that $\mathcal{O}_n$ is the C*-envelope of $\mathcal{A}_n$.

This presents a rather precise picture of the *norm-closed* algebra generated by an n-tuple of isometries with orthogonal ranges. However, the WOT-closed algebras can be quite different. They can detect the unitary invariants of the representation. The case $n = 1$ is familiar, where the WOT-closed algebra depends on the spectral invariants of the unitary part and the multiplicity of the shift, from the original Wold decomposition.

When $\sum_{i=1}^n S_i S_i^* = I$, the C*-algebra generated by the isometries S_i is the Cuntz algebra $\mathcal{O}_n$; and when $\sum_{i=1}^n S_i S_i^* < I$, this C*-algebra is $*$-isomorphic to the Cuntz–Toeplitz algebra $\mathcal{E}_n$ generated by the left regular representation λ. This algebra is an extension of the compact operators $\mathfrak{K}$ by $\mathcal{O}_n$. To each n-tuple S_i there is an associated representation σ of $\mathcal{E}_n$ given by $\sigma(s_i) = S_i$, where s_i are the canonical generators of $\mathcal{E}_n$. When $\sum_{i=1}^n S_i S_i^* = I$, this may be considered as a representation of $\mathcal{O}_n$ instead. Let $\mathfrak{S}_\sigma$ denote the WOT-closed non-selfadjoint algebra determined by the representation σ. One can view the Wold decomposition as the spatial view of the C*-algebra fact that every representation σ of $\mathcal{E}_n$ splits as a direct sum $\sigma = \lambda^{(\alpha)} \oplus \tau$ of a representation $\lambda^{(\alpha)}$, which is faithful on $\mathfrak{K}$ and thus is a multiple of the identity representation λ, and a representation τ which factors through $\mathcal{O}_n$.

A representation is called *atomic* if there is an orthonormal basis $\{\xi_j\}$ which

are permuted up to scalars by the generating isometries S_i. That is, for each i there is an endomorphism $\pi_i : \mathbb{N} \to \mathbb{N}$ and scalars $\lambda_{i,j}$ of modulus 1 such that $S_i \xi_j = \lambda_{i,j} \xi_{\pi_i(j)}$. These representations decompose as a direct integral of irreducible atomic representations [17], and these irreducible atomic representations are of three types. The first is just the left regular representation; which is the only one which does not factor through $\mathcal{O}_n$. The second type is a class of inductive limits of the left regular representation, and are classified by an infinite word (up to tail equivalence) that describes the imbeddings. The third type fits into the context of this chapter and is also important in Chapter 4, hence it is described here in more detail. See [17] for a complete description.

The third type is given by a word $u = i_1 i_2 \dots i_d$ in $\mathcal{F}_n$ and a scalar λ of modulus 1. A finite dimensional space $\mathcal{V}$ of dimension d is formed with a basis $e_1, \dots, e_d$. Operators A_j, for $1 \leq j \leq n$, are partial isometries defined by

$$A_j e_k = \delta_{j i_k} e_{k+1} \quad \text{for} \quad 1 \leq k < d$$

$$A_j e_d = \lambda \delta_{j i_d} e_1.$$

The minimal isometric dilation of this n-tuple yields isometries S_j acting on a space $\mathcal{H} = \mathcal{V} \oplus \mathcal{V}^\perp$. The isometry S_{i_k} maps e_k to e_{k+1} (or λe_1 when $k = d$) and the other $n-1$ isometries send e_k to pairwise orthogonal vectors which are all wandering vectors for $\mathfrak{S}$. Thus $\mathcal{V}^\perp$ is determined by a wandering space $\mathcal{W}$ of dimension $d(n-1)$, and therefore $\mathcal{V}^\perp = \mathfrak{S}[\mathcal{W}] \simeq \mathcal{H}_n^{(d(n-1))}$. The associated representation $\sigma_{u,\lambda}$ is irreducible precisely when the word u is primitive, meaning that it is not a power of a smaller word. In this case, $\mathfrak{S}$ can be completely described as the sum

of $\mathcal{B}(\mathcal{H})P_{\mathcal{V}}$ and a multiple of $\mathfrak{L}_n$ acting on $\mathcal{V}^{\perp}$ via its identification with $\mathcal{H}_n^{(d(n-1))}$. The invariant subspaces of this algebra are readily described, and it turns out to be hyper-reflexive (see below).

The algebra which occurs here is important to this chapter. Let $\mathfrak{B}_{n,d}$ denote the WOT-closed algebra on a Hilbert space $\mathcal{H} = \mathcal{V} \oplus \mathcal{H}_n^{(d(n-1))}$ where $\dim \mathcal{V} = d$ given by

$$\mathfrak{B}_{n,d} = \mathcal{B}(\mathcal{H})P_{\mathcal{V}} + \left(0_{\mathcal{V}} \oplus \mathfrak{L}_n^{(d(n-1))}\right).$$

Another class of representations which have been studied are the *finitely correlated* representations [9]. A representation of $\mathcal{O}_n$ is finitely correlated if there is a finite dimensional cyclic subspace $\mathcal{V}$ which is invariant for each S_i^*. Likewise, a *finitely correlated state* is a state φ such that in the GNS construction, the invariant subspace for the S_i^*'s generated by the cyclic vector ξ_φ is finite dimensional. It is evident that these representations are exactly those which will be studied here from the viewpoint of dilation theory. In this chapter, a complete classification of these representations up to unitary equivalence is obtained. A note will be made later on how this classification relates to the work of Bratteli and Jorgensen.

If $\mathfrak{A}$ is an algebra of operators, then $\operatorname{Lat}\mathfrak{A}$ denotes the lattice of all $\mathfrak{A}$-invariant subspaces. Further, if $\mathcal{L}$ is a lattice of subspaces, then $\operatorname{Alg}\mathcal{L}$ denotes the WOT-closed unital algebra of all operators which leave each element of $\mathcal{L}$ invariant. The algebra $\mathfrak{A}$ is reflexive if it equals $\operatorname{Alg}\operatorname{Lat}\mathfrak{A}$. For each reflexive algebra, there is a

quantitative measure of the distance to $\mathfrak{A}$ given by

$$\beta_{\mathfrak{A}}(T) = \sup_{L \in \operatorname{Lat}\mathfrak{A}} \|P_L^{\perp} T P_L\|.$$

The inequality $\beta_{\mathfrak{A}}(T) \leq \operatorname{dist}(T, \mathfrak{A})$ is always satisfied since

$$\|P_L^{\perp} T P_L\| = \|P_L^{\perp}(T - A)P_L\| \leq \|T - A\|.$$

for all $L \in \operatorname{Lat}\mathfrak{A}$ and $A \in \mathfrak{A}$. The algebra is called *hyper-reflexive* if in addition there is a constant C such that $\operatorname{dist}(T, \mathfrak{A}) \leq C\beta_{\mathfrak{A}}(T)$. The optimal C, if it is finite, is called the distance constant for $\mathfrak{A}$.

The list of algebras known to be hyper-reflexive is rather short. Arveson [2] showed that nest algebras have distance constant 1, so that equality is achieved. Christensen [13] showed that AF von Neumann algebras have distance constant at most 4. Concerning the algebras studied in this chapter, Davidson [15] showed that the analytic Toeplitz algebra has distance constant at most 19, while Davidson and Pitts [17] proved that all atomic free semigroup algebras have distance constant at most 51. The worst case for these estimates was the algebra $\mathfrak{L}_n$. However, a recent general result of Bercovici [6] applies to show that $\mathfrak{L}_n$ actually has a distance constant no greater than 3.

2.2 Main Results

Consider a contractive n-tuple $A = (A_1, \ldots, A_n)$ acting on a finite dimensional space $\mathcal{V}$ of dimension d; i.e. $AA^* = \sum_{i=1}^n A_i A_i^* \leq I$. The Frahzo–Bunce–Popescu minimal dilation yields isometries S_i acting on a larger space $\mathcal{H}$. Let $\mathfrak{A}$ denote the algebra generated by the A_i's, and let $\mathfrak{S}$ be the WOT-closed algebra generated by the S_i's. The associated completely positive contractive map on $\mathcal{B}(\mathcal{V})$ is given by

$$\Phi(X) = \sum_{i=1}^n A_i X A_i^* = AXA^*.$$

The operator $\Phi^\infty(I) := \lim_{k\to\infty} \Phi^k(I)$ will also be useful. This map is of course completely positive and contractive for any contractive n-tuple A, and it plays an integral role in Chapter 4.

The first fairly easy observation is that the dilation is of Cuntz type or $\sum_{i=1}^n S_i S_i^* = I$ if and only if $\sum_{i=1}^n A_i A_i^* = I$. This is equivalent to the condition $\Phi(I) = I$. In general, define the **pure rank** of $\mathfrak{S}$ (or of A) to be the multiplicity of the left regular representation in the Wold decomposition of $\mathfrak{S}$. This is the dimension of the wandering space $\mathcal{W} = \mathrm{Ran}(I - \sum_{i=1}^n S_i S_i^*)$. It is easy to see that this wandering space need not be contained in $\mathcal{V}$, and that even when the pure rank is one, the pure part may have large intersection with $\mathcal{V}$. Nevertheless, it turns out that this pure rank may be readily computed in terms of A as

$$\text{pure rank}(A) = \mathrm{rank}(I - \Phi(I)) = \mathrm{rank}\Big(I - \sum_{i=1}^n A_i A_i^*\Big).$$

The irreducible summands of Cuntz type are determined by the minimal $\mathfrak{A}^*$-

invariant subspaces $\mathcal{M}$ of $\mathcal{V}$ on which $\sum_{i=1}^{n} A_i A_i^*|_{\mathcal{M}} = I_{\mathcal{M}}$. Such a subspace generates an invariant subspace $\mathcal{H}_{\mathcal{M}} = \mathfrak{S}[\mathcal{M}]$ for $\mathfrak{S}$ which is necessarily reducing. The restriction $\mathfrak{S}|_{\mathcal{H}_{\mathcal{M}}}$ of $\mathfrak{S}$ to this subspace is isomorphic to the algebra $\mathfrak{B}_{n,m}$, where $m = \dim \mathcal{M}$, described in the Preliminary section. A crucial feature is that the projection $P_{\mathcal{M}}$ belongs to this algebra. This makes it possible to show that the restriction of the n-tuple A to $\mathcal{M}$ is a unitary invariant for the dilation.

The subspace $\widetilde{\mathcal{V}}$ spanned by all the minimal $\mathfrak{A}^*$-invariant subspaces of this type completely determines the Cuntz part of the dilation. The restriction of $\mathfrak{A}^*$ to $\widetilde{\mathcal{V}}$ is a finite dimensional C*-algebra. The standard invariants for a finite dimensional C*-algebra allow one to compute the multiplicities of each irreducible subrepresentation. In general, this information may be used to completely decompose the representation into a direct sum of finitely many irreducible representations of the types given above. This yields complete unitary invariants: the pure rank and the unitary equivalence class of the restriction of A^* to $\widetilde{\mathcal{V}}$.

For example, one can show that $\mathfrak{S}$ is irreducible if and only if either

(1) $\operatorname{rank}(I - \Phi(I)) = 1$ and $\Phi^{\infty}(I) = 0$, the pure case, or

(2) $\{X : \Phi(X) = X\} = \mathbb{C}I$, the Cuntz case.

The algebras $\mathfrak{L}_n$ and $\mathfrak{B}_{n,d}$ were shown to be hyper-reflexive in [17]. This analysis can be used to show that all of these algebras $\mathfrak{S}$ determined by a finite rank n-tuple are hyper-reflexive. The constant 51 of that paper may be improved to 5 using recent results of Bercovici [6] which show that the distance constant for $\mathfrak{L}_n$ is at most 3.

It is worthwhile to note there is further work in [16] which the author did

not participate in. For instance, a tight characterization of when two contractive n-tuples are similar is obtained.

2.3 Wandering Subspaces

Let $\mathcal{V}$ be a d-dimensional space (possibly infinite), and let $A_1, \ldots, A_n$ be an n-tuple of operators in $\mathcal{B}(\mathcal{V})$ such that $\sum_{i=1}^n A_i A_i^* \leq I$. The Frahzo–Bunce–Popescu minimal dilation yields isometries S_i on a larger space $\mathcal{H}$. Let $P_{\mathcal{V}}$ denote the projection of $\mathcal{H}$ onto $\mathcal{V}$. Further, let $\mathfrak{A}$ denote the algebra generated by the A_i's and $\mathfrak{S}$ be the WOT-closed algebra generated by the S_i's. First, the subspace $\mathcal{V}^\perp$ will be identified.

Lemma 2.3.1. *The subspace $\mathcal{W} = (\mathcal{V} + \sum_{i=1}^n S_i\mathcal{V}) \ominus \mathcal{V}$ is a wandering subspace for S, and $\sum_{v\in\mathcal{F}_n} \oplus S_v\mathcal{W} = \mathcal{V}^\perp$.*

Proof. $\mathcal{W}$ is contained in $\mathcal{V}^\perp$, which is invariant for S. Thus $S_u\mathcal{W}$ is orthogonal to $\mathcal{V}$ for every word $u \in \mathcal{F}_n$. Consequently, when $|u| \geq 1$, $S_u\mathcal{W}$ is also orthogonal to $S_j\mathcal{V}$, $1 \leq j \leq n$. It follows that $S_u\mathcal{W}$ is orthogonal to $\mathcal{V} + \sum_{i=1}^n S_i\mathcal{V}$, which contains $\mathcal{W}$. Therefore $\mathcal{W}$ is wandering. Minimality ensures that

$$\mathcal{H} = \text{span}\{S_u\mathcal{V} : u \in \mathcal{F}_n\} = \text{span}\{\mathcal{V}, S_u\mathcal{W} : u \in \mathcal{F}_n\}.$$

Since $\mathcal{W}$ lies in the invariant subspace $\mathcal{V}^\perp$, this can only occur because $\sum_{v\in\mathcal{F}_n} \oplus S_v\mathcal{W} = \mathcal{V}^\perp$. ∎

Thus $\mathcal{V}^\perp$ is unitarily equivalent to a multiple $\mathcal{H}_n^{(\alpha)}$ of Fock space, where $\alpha =$

$\dim \mathcal{W}$, and $S_i|_{\mathcal{V}^\perp} \simeq L_i^{(\alpha)}$. Hence decomposing $\mathcal{H} = \mathcal{V} \oplus \mathcal{V}^\perp$, one can write each S_i as a matrix $S_i = \begin{bmatrix} A_i & 0 \\ X_i & L_i^{(\alpha)} \end{bmatrix}$.

Remark 2.3.2. The range of $\sum_{i=1}^n S_i S_i^*$ includes $\sum_{i=1}^n S_i \mathcal{V}^\perp = (\mathcal{V} + \mathcal{W})^\perp$ as well as $\sum_{i=1}^n S_i \mathcal{V}$. Hence $\sum_{i=1}^n S_i S_i^* = I$ if and only if $\sum_{i=1}^n S_i \mathcal{V}$ contains $\mathcal{V}$. Since $\mathcal{V}$ is invariant for S_i^* and $S_i^*|_{\mathcal{V}} = A_i^*$,

$$\sum_{i=1}^n A_i A_i^* = \sum_{i=1}^n P_{\mathcal{V}} S_i P_{\mathcal{V}} S_i^*|_{\mathcal{V}} = P_{\mathcal{V}} \sum_{i=1}^n S_i S_i^*|_{\mathcal{V}}.$$

Therefore $\sum_{i=1}^n S_i S_i^* = I$ if and only if its range contains $\mathcal{V}$ if and only if $\sum_{i=1}^n A_i A_i^* = I_{\mathcal{V}}$.

Let $d = \dim \mathcal{V}$ be finite, and let $\alpha = \dim \mathcal{W}$. Then α can be as large as nd and as small as $(n-1)d$. This is easily seen since $\sum_{i=1}^n S_i \mathcal{V}$ is an orthogonal direct sum and thus has dimension nd, so that $\mathcal{W} = (\mathcal{V} + \sum_{i=1}^n S_i \mathcal{V}) \ominus \mathcal{V}$ can have no larger dimension than nd, and is at least $(n-1)d$.

When $\sum_{i=1}^n A_i A_i^* = I_{\mathcal{V}}$, it was shown above that $\sum_{i=1}^n S_i S_i^* = I$. Then

$$\mathcal{V} = \sum_{i=1}^n S_i S_i^* \mathcal{V} = \sum_{i=1}^n S_i A_i^* \mathcal{V} \subseteq \sum_{i=1}^n S_i \mathcal{V}.$$

Hence $\mathcal{W} = (\sum_{i=1}^n S_i \mathcal{V}) \ominus \mathcal{V}$ has dimension $(n-1)d$.

The case $\dim \mathcal{W} = nd$ occurs, for example, if $A_i = 0$ for $1 \le i \le n$. The minimal dilation is just $L_i^{(d)}$. Indeed, if $x, y \in \mathcal{V}$, then

$$(S_i x, y) = (x, S_i^* y) = (x, A_i^* y) = 0 \quad \text{for} \quad 1 \le i \le n.$$

Thus $\mathcal{V}$ is orthogonal to $\sum_{i=1}^n S_i\mathcal{V}$. Therefore $\mathcal{W} = \sum_{i=1}^n S_i\mathcal{V}$ has dimension nd.

It is easy to combine these examples to obtain any integer in between.

The n-tuple of isometries S is called **pure** if it is unitarily equivalent to a multiple of the left regular representation. A contractive n-tuple is pure if its dilation is as well. These contractions are investigated further in Chapter 4. Bunce [12] shows that whenever $\|A\| < 1$, the dilation S is pure. Popescu [31] shows that the dilation is pure if and only if $\operatorname{WOT-lim}_{k\to\infty} \Phi_A^k(I) = \operatorname{WOT-lim}_{k\to\infty} \sum_{|v|=k} A_v A_v^* = 0$.

Frequently *reducing* subspaces of $\mathfrak{S}$ will be constructed from $\mathfrak{A}^*$-invariant subspaces. This procedure preserves orthogonality as well.

Lemma 2.3.3. *Suppose that $\mathcal{V}$ contains an $\mathfrak{A}^*$-invariant subspace $\mathcal{V}_1$. Then $\mathcal{H}_1 = \mathfrak{S}[\mathcal{V}_1]$ reduces $\mathfrak{S}$.*

If $\mathcal{V}$ contains a pair of orthogonal $\mathfrak{A}^*$-invariant subspaces $\mathcal{V}_1$ and $\mathcal{V}_2$, then $\mathcal{H}_j = \mathfrak{S}[\mathcal{V}_j]$ for $j = 1, 2$ are mutually orthogonal.

If in addition $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2$, then $\mathcal{H}$ decomposes as $\mathcal{H}_1 \oplus \mathcal{H}_2$ and $\mathcal{H}_j \cap \mathcal{V} = \mathcal{V}_j$ for $j = 1, 2$.

Proof. Since $\mathcal{V}_1$ is invariant for A_i^*, it is also invariant for S_i^*. The $\mathfrak{S}$-invariant subspace $\mathcal{H}_1 = \mathfrak{S}[V_1]$ is spanned by vectors of the form $S_w x$ where $x \in \mathcal{V}_1$ and $w \in \mathcal{F}_n$. Notice that $S_i^* S_w x$ equals $S_{w'}x$ if $w = iw'$, 0 if $w = i'w'$ for some $i' \neq i$, and $S_i^* x$ if $w = e$. Since $\mathcal{V}_1$ is invariant for $\mathfrak{S}^*$, each of these possibilities belongs to $\mathcal{H}_1$. Thus $\mathcal{H}_1$ reduces $\mathfrak{S}$.

Likewise, if $\mathcal{V}_1$ and $\mathcal{V}_2$ are orthogonal $\mathfrak{A}^*$-invariant subspaces, it follows that $\mathcal{H}_1$ and $\mathcal{H}_2$ are orthogonal. For if $v_j \in \mathcal{V}_j$, the inner product $(S_u v_1, S_w v_2)$ can be reduced by cancellation of isometries until either u or w is the identity element.

Then, for example when $w = e$,

$$(S_u v_1, v_2) = (v_1, S_u^* v_2) = 0$$

by the $\mathfrak{A}^*$-invariance of $\mathcal{V}_2$ and orthogonality.

Now suppose that $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2$. Since $\mathcal{H}_1$ contains $\mathcal{V}_1$ and is orthogonal to $\mathcal{V}_2$, it follows that $\mathcal{H}_1 \cap \mathcal{V} = \mathcal{V}_1$. Finally, $\mathcal{H}_1 \oplus \mathcal{H}_2$ is an $\mathfrak{S}$-reducing subspace containing $\mathcal{V}$, so it is all of $\mathcal{H}$ by the minimality of the dilation. ■

2.4 Finite Dimensional n-tuples

This section specializes to the case when $\mathcal{V}$ is finite-dimensional. In general, the S_i decompose into a pure part and Cuntz part. Let $\mathcal{X}$ be the range of $I - \sum_{i=1}^{n} S_i S_i^*$, which is the wandering space for the reducing subspace $\mathcal{H}_p = \sum_{v \in \mathcal{F}_n} \oplus S_v \mathcal{X}$. The restriction of the S_i to this space yields a multiple of the left regular representation, where the multiplicity is $\dim \mathcal{X}$. This quantity will be called the **pure rank** of the representation. In section 2.6, it is shown how to compute the pure rank directly in terms of A. This invariant also plays a crucial role in Chapter 4. On the complement $\mathcal{H}_c = \mathcal{H}_p^{\perp}$, the restrictions of S_i yield a representation of the Cuntz algebra. Let P_p and P_c denote the projections onto $\mathcal{H}_p$ and $\mathcal{H}_c$ respectively. It is important to note that the projection P_p does not commute with $P_{\mathcal{V}}$ in general. An example of this phenomena is provided in section 2.6 which helps illustrate this point.

The key technical tool in the analysis shows that $\mathcal{H}_c$ is determined by $\mathcal{V}_c := \mathcal{H}_c \cap \mathcal{V}$. This is not the case for $\mathcal{H}_p$. Let R_k denote the projection onto $\sum_{|v|=k} \oplus S_v \mathcal{W}$,

where $\mathcal{W} = (\mathcal{V} + \sum_{i=1}^{n} S_i\mathcal{V}) \ominus \mathcal{V}$; and $Q_k = \sum_{j\geq k} R_j$. Notice that

$$Q_k = \sum_{|w|=k} S_w P_{\mathcal{V}}^{\perp} S_w^*.$$

On any $\mathfrak{S}$-invariant subspace $\mathcal{M}$ on which the restrictions T_i of S_i are pure, one has for every $x \in \mathcal{M}$

$$\lim_{k\to\infty} \sum_{|w|=k} \|P_{\mathcal{M}} S_w^* x\|^2 = \lim_{k\to\infty} \sum_{|w|=k} \|T_w^* x\|^2 = 0.$$

In particular, this applies to $\mathcal{H}_p$ and $\mathcal{V}^{\perp}$. While for $x \in \mathcal{H}_c$, one has

$$\sum_{|w|=k} \|S_w^* x\|^2 = \|x\|^2 \quad \text{for all} \quad k \geq 0.$$

Lemma 2.4.1. *Suppose that $\mathcal{H}_1$ is a reducing subspace for $\mathfrak{S}$ contained in $\mathcal{H}_c$. Let x be a vector such that $P_{\mathcal{H}_1} x \neq 0$. Then the subspace $\mathcal{M} = \mathfrak{S}^*[x]$ contains a vector v in $\mathcal{M} \cap \mathcal{V}_c$ with $P_{\mathcal{H}_1} v \neq 0$.*

Proof. Let P_1 denote the projection of $\mathcal{H}$ onto $\mathcal{H}_1$. Fix $\varepsilon > 0$; and let $x_1 = P_1 x$. By applying the preceding remarks to both $\mathcal{V}^{\perp}$ and $\mathcal{H}_p$, one may choose an integer k sufficiently large that

$$\sum_{|w|=k} \|P_{\mathcal{V}}^{\perp} S_w^* x\|^2 = \|Q_k x\|^2 < \varepsilon^2$$

$$\sum_{|w|=k} \|P_{\mathcal{V}}^{\perp} S_w^* x_1\|^2 = \|Q_k x_1\|^2 < \varepsilon^2$$

and

$$\sum_{|w|=k} \|P_p S_w^* x\|^2 = \sum_{|w|=k} \|S_w^* P_p x\|^2 < \varepsilon^2.$$

Since $\sum_{|w|=k} S_w S_w^* P_1 = P_1$,

$$\begin{aligned}\sum_{|w|=k} \|P_{\mathcal{V}} S_w^* x_1\|^2 &= \sum_{|w|=k} \left(\|S_w^* x_1\|^2 - \|P_{\mathcal{V}}^{\perp} S_w^* x_1\|^2\right) \\ &= \|x_1\|^2 - \|Q_k x_1\|^2 > \|x_1\|^2 - \varepsilon^2.\end{aligned}$$

Let $\mathcal{E}_1$ denote the set of words w of length k such that

$$\|P_{\mathcal{V}} S_w^* x_1\|^2 > \varepsilon^{-1} \|P_{\mathcal{V}}^{\perp} S_w^* x\|^2.$$

Likewise let $\mathcal{E}_2$ denote the set of words w of length k such that

$$\|P_{\mathcal{V}} S_w^* x_1\|^2 > \varepsilon^{-1} \|P_p S_w^* x\|^2.$$

The set $\mathcal{E}_1 \cap \mathcal{E}_2$ is relatively large in the sense that

$$\begin{aligned}\sum_{w \in \mathcal{E}_1 \cap \mathcal{E}_2} \|P_{\mathcal{V}} S_w^* x_1\|^2 &> \|x_1\|^2 - \varepsilon^2 - \sum_{w \notin \mathcal{E}_1} \|P_{\mathcal{V}} S_w^* x_1\|^2 - \sum_{w \notin \mathcal{E}_2} \|P_{\mathcal{V}} S_w^* x_1\|^2 \\ &> \|x_1\|^2 - \varepsilon^2 - \sum_{w \notin \mathcal{E}_1} \varepsilon^{-1} \|P_{\mathcal{V}}^{\perp} S_w^* x\|^2 - \sum_{w \notin \mathcal{E}_2} \varepsilon^{-1} \|P_p S_w^* x\|^2 \\ &> \|x_1\|^2 - \varepsilon^2 - \varepsilon - \varepsilon > \|x_1\|^2/4\end{aligned}$$

for small ε. Notice also that $\sum_{w\in\mathcal{E}_1\cap\mathcal{E}_2}\|P_{\mathcal{V}}S_w^*x\|^2 \le \|x\|^2$. Therefore there is a word w in $\mathcal{E}_1\cap\mathcal{E}_2$ for which

$$\|P_{\mathcal{V}}S_w^*x_1\| > \frac{\|x_1\|}{2\|x\|}\|P_{\mathcal{V}}S_w^*x\|.$$

In this way, construct a sequence of words w_k corresponding to $\varepsilon_k = 1/k$. Hence define unit vectors $y_k = S_{w_k}^*x/\|S_{w_k}^*x\|$ with the properties that

$$\lim_{k\to\infty}\|P_{\mathcal{V}}^{\perp}y_k\| \le \lim_{k\to\infty}\frac{1}{\sqrt{k}}\frac{\|P_{\mathcal{V}}S_{w_k}^*P_1x\|}{\|S_{w_k}^*x\|} = \lim_{k\to\infty}\frac{1}{\sqrt{k}}\frac{\|P_{\mathcal{V}}P_1S_{w_k}^*x\|}{\|S_{w_k}^*x\|} = 0.$$

Similarly, $\lim_{k\to\infty}\|P_py_k\| = 0$. Also

$$\begin{aligned}\|P_1y_k\| &= \|S_{w_k}^*x_1\|/\|S_{w_k}^*x\| \ge \|P_{\mathcal{V}}S_{w_k}^*x_1\|/\|S_{w_k}^*x\| \\ &> \frac{\|x_1\|}{2\|x\|}\frac{\|P_{\mathcal{V}}S_{w_k}^*x\|}{\|S_{w_k}^*x\|} = \frac{\|x_1\|}{2\|x\|}\|P_{\mathcal{V}}y_k\|.\end{aligned}$$

By the compactness of the unit ball in $\mathcal{V}$, there is a subsequence of the y_k's which converges to a unit vector v in $\mathcal{V}$. Clearly, $P_pv = 0$, and thus v belongs to $\mathcal{V}_c\cap\mathfrak{S}^*[x]$; whence this subspace is non-zero. By construction, $\|P_1v\| \ge \|x_1\|/2\|x\|$. and therefore is also non-zero. ■

Corollary 2.4.2. *Every non-zero subspace of $\mathcal{H}_c$ which is invariant for $\mathfrak{S}^*$ has non-zero intersection with $\mathcal{V}_c$. In particular $\mathcal{H}_c = \mathfrak{S}[\mathcal{V}_c]$.*

Proof. Let $\mathcal{M}$ be any non-zero $\mathfrak{S}^*$-invariant subspace contained in $\mathcal{H}_c$. If x is any non-zero vector in $\mathcal{M}$, the previous lemma applied to x and $\mathcal{H}_1 = \mathcal{H}_c$ shows that $\mathfrak{S}^*[x]$ intersects $\mathcal{V}_c$ non-trivially.

By Lemma 2.3.3, $\mathcal{N} = \mathfrak{S}[\mathcal{V}_c]$ reduces $\mathfrak{S}$. The claim is that $\mathcal{N} = \mathcal{H}_c$. For otherwise, let $\mathcal{H}_1 = \mathcal{H}_c \cap \mathcal{N}^\perp$. By the first paragraph, this reducing subspace for $\mathfrak{S}$ must intersect $\mathcal{V}_c$ non-trivially. So $\mathcal{H}_1$ is not orthogonal to $\mathcal{N}$, contrary to fact. Therefore $\mathcal{H}_1$ must be zero. ■

Corollary 2.4.3. *Suppose that* $\sum_{i=1}^n A_i A_i^* = I$ *and* $\mathfrak{A} = \mathcal{B}(\mathcal{V})$. *Then every invariant subspace of* $\mathfrak{S}^*$ *contains* $\mathcal{V}$.

Proof. Since $\mathcal{H} = \mathcal{H}_c$, any $\mathfrak{S}^*$-invariant subspace $\mathcal{M}$ intersects $\mathcal{V}$ in a non-trivial subspace. This subspace is invariant for $\mathfrak{S}^*|_{\mathcal{V}} = \mathfrak{A}^* = \mathcal{B}(\mathcal{V})$. Hence it is all of $\mathcal{V}$. ■

Let $\mathfrak{B}$ denote the WOT-closed operator algebra on $\mathcal{H} = \mathcal{V} \oplus \mathcal{H}_n^{(\alpha)}$ spanned by $\mathcal{B}(\mathcal{H})P_{\mathcal{V}}$ and $0_{\mathcal{V}} \ominus \mathfrak{L}_n^{(\alpha)}$.

Lemma 2.4.4. *Every weak-* continuous functional on* $\mathfrak{B}$ *is given by a trace class operator of rank at most* $d + 1$.

Proof. An element B of $\mathfrak{B}$ is determined by $BP_{\mathcal{V}}$ and $BP_{\mathcal{V}}^\perp$. If $e_1, \ldots, e_d$ is a basis for $\mathcal{V}$, the former is determined by the vectors Be_j. The latter term is unitarily equivalent to $A^{(\alpha)}$ for some $A \in \mathfrak{L}_n$. Any functional φ is thus determined by a functional φ_0 on $\mathfrak{L}_n^{(\alpha)}$ and by d functionals on $\mathcal{H}$ given by the Riesz Representation Theorem by a vector y_j. From [17], the functional on $\mathfrak{L}_n$ is given by a rank one functional $\varphi_0(A) = (A\eta, \zeta)$. Whence

$$\varphi(B) = \sum_{j=1}^{d} (Be_j, y_j) + (B\eta, \zeta).$$

■

Corollary 2.4.5. *The* WOT *and weak-* topologies coincide on* $\mathfrak{B}$*, and thus also on* $\mathfrak{S}$*. In particular, the weak-* closed algebra generated by the* S_i*'s coincides with* $\mathfrak{S}$*.*

2.5 The Cuntz Case

This section considers the Cuntz case: $\sum_{i=1}^{n} A_i A_i^* = I$, for which the isometric dilation yields a representation of the Cuntz algebra.

Example 2.5.1. The starting point is a description of the case in which $\mathcal{V}$ is one dimensional. A special case of a finite correlated state is a Cuntz state. This is determined by scalars $\eta = (\eta_1, \ldots, \eta_n)$ such that $\sum_{i=1}^{n} |\eta_i|^2 = 1$. The state is determined by

$$\varphi_\eta(s_{i_1} \ldots s_{i_k} s_{j_1}^* \ldots s_{j_l}^*) = \eta_{i_1} \ldots \eta_{i_k} \bar{\eta}_{j_1} \ldots \bar{\eta}_{j_l}.$$

The cyclic vector ξ_η from the GNS construction $(\mathcal{H}_\eta, \pi_\eta, \xi_\eta)$ spans a one-dimensional space invariant for every $\pi_\eta(S_i^*)$. Indeed,

$$\begin{aligned} \|\pi_\eta(S_i^*)\xi_\eta - \bar{\eta}_i \xi_\eta\|^2 &= \langle \pi_\eta(S_i^*)\xi_\eta, \pi_\eta(S_i^*)\xi_\eta \rangle - \eta_i \langle \pi_\eta(S_i^*)\xi_\eta, \xi_\eta \rangle \\ &\quad - \bar{\eta}_i \langle \xi_\eta, \pi_\eta(S_i^*)\xi_\eta \rangle + |\eta_i|^2 \\ &= \varphi_\eta(S_i S_i^*) - |\eta_i|^2 = |\eta_i|^2 - |\eta_i|^2 = 0. \end{aligned}$$

The restrictions $A_i^* = S_i^*|_{\mathrm{span}\{\xi_\eta\}} = \overline{\eta_i}$ satisfy $\sum_{i=1}^{n} A_i A_i^* = 1$. They may be dilated to their minimal isometric dilation, which is necessarily the original S_i since ξ_η is a

cyclic vector.

Specializing to the case of $\eta = (1, 0, \ldots, 0)$, one has $A_1 = 1$ and $A_i = 0$ for $2 \leq i \leq n$. This yields the atomic representation $\sigma_{1,1}$ mentioned in the Background section. In particular, the algebra $\mathfrak{S}$ is unitarily equivalent to $\mathfrak{B}_{n,1}$.

The various Cuntz states are related by the action of the gauge group $\mathcal{U}(n)$ which acts as an automorphism group on $\mathcal{O}_n$ and on the Cuntz–Toeplitz algebra $\mathcal{E}_n$. Indeed, if one writes Fock space $\mathcal{H}_n$ as a direct sum $\mathbb{C} \ominus \mathcal{K}_n \ominus \mathcal{K}_n^{\otimes 2} \ominus \mathcal{K}_n^{\otimes 3} \ominus \cdots$ where $\mathcal{K}_n$ is an n-dimensional Hilbert space, then each unitary matrix $U \in \mathcal{U}(n)$ determines a unitary operator $\widetilde{U} = I \ominus U \ominus U^{\otimes 2} \ominus U^{\otimes 3} \ominus \ldots$ on $\mathcal{H}_n$. Conjugation by $\widetilde{U}$ acts as an automorphism Θ_U of $\mathcal{E}_n$. Moreover, it maps the ideal of compact operators onto itself. So it also induces an automorphism θ_U of $\mathcal{O}_n$. If $U = [u_{ij}]$ is an $n \times n$ unitary matrix, this automorphism can also be seen to be given by

$$\Theta_U(L_j) = \sum_{i=1}^{n} u_{ij} L_i \quad \text{for} \quad 1 \leq j \leq n.$$

Given η, let U be any unitary with $u_{1j} = \eta_j$. Then it follows that

$$\varphi_\eta(A) = \varphi_{(1,0,\ldots,0)}(\theta_U(A)) \quad \text{for all} \quad A \in \mathcal{O}_n.$$

So the corresponding representations are equivalent up to this automorphism. In particular, the *algebras* $\mathfrak{S}_\eta$ generated by these representations are unitarily equivalent even though the representations are not.

A crucial step in the analysis of atomic representations was to show that certain projections lie in the algebra $\mathfrak{S}$. Indeed, this is a major advantage of $\mathfrak{S}$ over the

C*-algebra, which does not contain these projections, and over the von Neumann algebra it generates, which contains too many projections. As a case in point, the projection $P_\eta = \xi_\eta \xi_\eta^*$ belongs to $\mathfrak{S}_\eta$. In fact, it is the *only* non-trivial projection in the whole algebra $\mathfrak{S}_\eta$.

Integral to the analysis here is the identification of projections in $\mathfrak{S}$ in greater generality. The starting point is the irreducible case.

Theorem 2.5.2. *Assume that* $\sum_{i=1}^n A_i A_i^* = I$ *and* $\mathfrak{A} = \mathcal{B}(\mathcal{V})$. *Then* $\mathfrak{S}$ *contains the projection* $P_\mathcal{V}$.

Proof. Both $\mathfrak{S}$ and $P_\mathcal{V}$ belong to $\mathfrak{B}$. If $P_\mathcal{V}$ were not in $\mathfrak{S}$, Lemma 2.4.4 would provide a weak-* continuous functional φ which annihilates $\mathfrak{S}$ such that $\varphi(P_\mathcal{V}) = 1$. Represent φ as a functional of rank $d+1$ in the form $\varphi(B) = \sum_{j=0}^d (Bx_j, y_j)$. This then may be realized as a rank one functional on the $d+1$-fold ampliation of $\mathfrak{B}$. Indeed, form the vectors $x = (x_0, \ldots, x_d)$ and $y = (y_0, \ldots, y_d)$. Then $\varphi(B) = (B^{(d+1)}x, y)$.

Now the fact that φ annihilates $\mathfrak{S}$ means that x is orthogonal to the subspace $\mathcal{M} = \mathfrak{S}^{*(d+1)}[y]$. The algebra $\mathfrak{S}^{(d+1)}$ is generated by isometries $S_i^{(d+1)}$, which form the minimal dilation of the $A_i^{(d+1)}$'s. So Corollary 2.4.2 applies, and shows that $\mathcal{M}$ intersects $\mathcal{V}^{(d+1)}$ in a non-zero subspace $\mathcal{M}_0$ which is invariant for $\mathfrak{S}^{*(d+1)}$, and thus for $\mathfrak{A}^{*(d+1)}$.

By hypothesis, $\mathfrak{A}^{*(d+1)} = \mathcal{B}(\mathcal{V})^{(d+1)} \simeq \mathcal{B}(\mathcal{V}) \otimes \mathbb{C}^{d+1}$, which is a finite dimensional C*-algebra. The invariant subspace $\mathcal{M}_0$ is thus the range of a projection Q in the commutant $\mathbb{C}^d \otimes \mathfrak{M}_{d+1}$. Let $\tilde{Q}$ denote the operator in $\mathbb{C}I_\mathcal{H} \otimes \mathfrak{M}_{d+1}$ acting on

$\mathcal{H}^{(d+1)}$ with the same matrix coefficients as Q. That is, $\tilde{Q}$ is the unique operator in $(\mathcal{B}(\mathcal{H}) \otimes \mathbb{C}^{d+1})'$ such that $P_{\mathcal{V}}^{(d+1)}\tilde{Q} = Q$.

The projection $\tilde{Q}$ yields a decomposition of $\mathcal{H}^{(d+1)}$ into $\mathfrak{S}$-reducing subspaces $\mathcal{H}_1 \ominus \mathcal{H}_2$ where $\mathcal{H}_1 = \ker \tilde{Q}$ and $\mathcal{H}_2 = \operatorname{Ran} \tilde{Q}$; and likewise $\mathcal{V}^{(d+1)} = \mathcal{V}_1 \ominus \mathcal{V}_2$ where

$$\mathcal{V}_1 := \mathcal{H}_1 \cap \mathcal{V}^{(d+1)} = \ker Q \quad \text{and} \quad \mathcal{V}_2 := \mathcal{H}_2 \cap \mathcal{V}^{(d+1)} = \operatorname{Ran} Q.$$

Observe that $\mathcal{M}_0$ is contained in $\mathcal{H}_2$. For if there was a vector $x \in \mathcal{M}$ such that $P_{\mathcal{H}_1} x \neq 0$, then Lemma 2.4.1 implies that there would be a non-zero vector v in $\mathcal{M} \cap \mathcal{V}^{(d+1)} = \mathcal{M}_0$ such that $P_{\mathcal{H}_1} v \neq 0$. But by definition of Q and $\tilde{Q}$, $\mathcal{M}_0$ is orthogonal to $\mathcal{H}_1$, a contradiction.

In particular, as $y \in \mathcal{M}$, one has $y = \tilde{Q}y$. Thus

$$P_{\mathcal{V}}^{(d+1)} y = P_{\mathcal{V}}^{(d+1)} \tilde{Q} y = Q P_{\mathcal{V}}^{(d+1)} y$$

belongs to $Q\mathcal{V}^{(d+1)} = \mathcal{M}_0$. Since x is orthogonal to $\mathcal{M}$ and hence to $\mathcal{M}_0$, one sees that

$$\varphi(P_{\mathcal{V}}) = (P_{\mathcal{V}}^{(d+1)} x, y) = (x, P_{\mathcal{V}}^{(d+1)} y) = 0.$$

Consequently $P_{\mathcal{V}}$ belongs to $\mathfrak{S}$. ■

This immediately yields a structure theorem for $\mathfrak{S}$. Note that this does not classify the associated representations, as they depend on the specific generators, not just the algebra.

Corollary 2.5.3. *Assume that* $\sum_{i=1}^n A_i A_i^* = I$ *and* $\mathfrak{A} = \mathcal{B}(\mathcal{V})$. *Then*

$$\mathfrak{S} \simeq \mathcal{B}(\mathcal{H})P_{\mathcal{V}} + \left(0_{\mathcal{V}} \oplus \mathfrak{L}_n^{((n-1)d)}\right) \simeq \mathfrak{B}_{n,d}.$$

Proof. By Theorem 2.5.2, $P_{\mathcal{V}}$ belongs to $\mathfrak{S}$. Therefore, $\mathfrak{S}$ contains $P_{\mathcal{V}}\mathfrak{S} = \mathcal{B}(\mathcal{V})$. Moreover, it contains $S_i P_{\mathcal{V}}^{\perp} \simeq 0_{\mathcal{V}} \oplus L_i^{(\alpha)}$, where $\alpha = (n-1)d$. Thus $\mathfrak{S}$ contains the WOT-closed algebra that these operators generate, which is evidently $0_{\mathcal{V}} \oplus \mathfrak{L}_n^{(\alpha)}$. Finally, if v is any non-zero vector in $\mathcal{V}$, $\mathfrak{S}[v]$ contains $\mathcal{V}$ by hypothesis. So it is all of $\mathcal{H}$ by minimality of the dilation. Therefore for any $x \in \mathcal{H}$, there are operators $T_k \in \mathfrak{S}$ such that $T_k v$ converges to x. Thus $\mathfrak{S}$ contains $T_k v v^*$, which converge to the rank one operator xv^*. So $\mathcal{B}(\mathcal{H})P_{\mathcal{V}}$ belongs to $\mathfrak{S}$. This is the whole WOT-closed algebra which was denoted $\mathfrak{B}$, and which trivially contains $\mathfrak{S}$. It is clear that $\mathfrak{B}$ is unitarily equivalent to $\mathfrak{B}_{n,d}$. ∎

Now suppose that $\mathfrak{A}$ is a more general subalgebra of $\mathcal{B}(\mathcal{V})$. The next step is to determine the structure of $\mathfrak{S}$ from information about $\mathfrak{A}$.

Lemma 2.5.4. *Assume that* $\sum_{i=1}^n A_i A_i^* = I$. *Suppose that* $\mathcal{V}$ *contains a minimal* $\mathfrak{A}^*$*-invariant subspace* $\mathcal{V}_0$ *of dimension* d_0 *which is cyclic for* $\mathfrak{A}$. *Then* $\mathfrak{S}$ *contains* $\mathcal{B}(\mathcal{H})P_{\mathcal{V}_0}$, *and is unitarily equivalent to* $\mathfrak{B}_{n,d_0}$.

Proof. By Burnside's Theorem [**37**, Corollary 8.6], since $\mathfrak{A}^*|_{\mathcal{V}_0}$ has no proper invariant subspaces, it must equal all of $\mathcal{B}(\mathcal{V}_0)$. Let $\mathcal{H}_0 = \mathfrak{S}[\mathcal{V}_0]$. This is a reducing subspace for $\mathfrak{S}$ by Lemma 2.3.3. The point is that $\mathcal{H}_0 = \mathcal{H}$.

Indeed, suppose that x is a non-zero vector orthogonal to $\mathcal{H}_0$. By Corollary 2.4.2, $\mathfrak{S}^*[x] \cap \mathcal{V}$ contains a non-zero vector v. Moreover since $\mathcal{H}_0^{\perp}$ reduces $\mathfrak{S}$, v is orthog-

onal to $\mathcal{H}_0$. Therefore $\mathfrak{S}^*[v] = \mathfrak{A}^*[v]$ is an $\mathfrak{A}^*$-invariant subspace orthogonal to $\mathcal{V}_0$. Since $\mathfrak{A}\mathcal{V}_0 = \mathcal{V}$, there is an $A \in \mathfrak{A}$ and $v_0 \in \mathcal{V}_0$ such that $Av_0 = v$. Hence

$$\|v\|^2 = (Av_0, v) = (v_0, A^*v) = 0.$$

This contradiction establishes the claim.

Now consider the compressions $\widetilde{A}_i = P_{\mathcal{V}_0} A_i|_{\mathcal{V}_0} = (A_i^*|_{\mathcal{V}_0})^*$. Then $\sum_{i=1}^n \widetilde{A}_i \widetilde{A}_i^* = I_{\mathcal{V}_0}$ follows from the $\mathfrak{A}^*$-invariance of $\mathcal{V}_0$. Also by hypothesis, the algebra $\widetilde{\mathfrak{A}}$ generated by the $\widetilde{A}_i$'s is $\mathcal{B}(\mathcal{V}_0)$. The minimal dilation of this n-tuple must be precisely the restriction of S_i to $\mathfrak{S}[\mathcal{V}_0] = \mathcal{H}$, which is S_i. So by Corollary 2.5.3, it follows that $\mathfrak{S}$ is unitarily equivalent to $\mathfrak{B}_{n,d_0}$. ■

The following corollary is almost immediate from the structure of $\mathfrak{B}_{n,d_0}$. It is pointed out in order to obtain some non-trivial consequences.

Corollary 2.5.5. *Assume that $\sum_{i=1}^n A_i A_i^* = I$. If $\mathcal{V}$ contains a subspace $\mathcal{V}_0$ which is cyclic for $\mathfrak{A}$ and is a minimal invariant subspace for $\mathfrak{A}^*$, then $\mathcal{V}_0$ is the unique minimal $\mathfrak{A}^*$-invariant subspace.*

Proof. Recall that $\mathfrak{A}^* = \mathfrak{S}^*|_{\mathcal{V}}$. So by the previous lemma, $\mathfrak{A}^*$ contains $P_{\mathcal{V}_0}\mathcal{B}(\mathcal{V})$. Consequently, $\mathcal{V}_0$ is contained in every non-zero $\mathfrak{A}^*$-invariant subspace. ■

Remark 2.5.6. This puts constraints on which subalgebras $\mathfrak{A}$ of $\mathcal{B}(\mathcal{V})$ can be generated by A_i's which satisfy $\sum_{i=1}^n A_i A_i^* = I$. For example, the semisimple algebra of matrices of the form $\mathfrak{A}_t = \left[\begin{smallmatrix} a & 0 \\ (b-a)t & b \end{smallmatrix}\right]$ for a, b in $\mathbb{C}$ and a fixed $t \neq 0$ is similar to the 2×2 diagonal algebra. Note that $\mathfrak{A}_t$ has two independent vectors which are cyclic for $\mathfrak{A}_t$ and eigenvalues for $\mathfrak{A}_t^*$, namely e_1 and $f_2 = -\bar{t}e_1 + e_2$. By

the corollary above, this cannot equal the algebra $\mathfrak{A}$. For, if the generators of the algebra were $A_i = \begin{bmatrix} a_i & 0 \\ (b_i - a_i)t & b_i \end{bmatrix}$, then a computation would show that $\sum_{i=1}^{n} |a_i|^2 = 1$. Likewise considering the matrix with respect to an orthonormal basis $\{f_1, f_2\}$ would show that $\sum_{i=1}^{n} |b_i|^2 = 1$. This then forces $\sum_{i=1}^{n} |a_i - b_i|^2 |t|^2 = 0$. Since $t \neq 0$, this forces all the A_i's to be scalar, and hence they do not generate $\mathfrak{A}_t$.

Example 2.5.7. Consider a special case of the previous corollary: if $\mathfrak{A}$ has a cyclic vector e which is an eigenvalue for $\mathfrak{A}^*$. Then $\mathfrak{S}$ is unitarily equivalent to $\mathfrak{B}_{n,1}$. The algebra $\mathfrak{A}$ decomposes as $\mathfrak{A} = \mathcal{B}(\mathcal{V})P_e + J\mathfrak{A}_1 P_e^{\perp}$ where P_e is the orthogonal projection onto $\mathbb{C}e$, J is the injection of $\mathcal{V}_1 = \{e\}^{\perp}$ into $\mathcal{V}$, and $\mathfrak{A}_1$ is a unital subalgebra of $\mathcal{B}(\mathcal{V}_1)$. It is easy to see that

$$\operatorname{Lat} \mathfrak{A} = \{\mathcal{V}, JM : M \in \operatorname{Lat} \mathfrak{A}_1\}.$$

Hence if $\mathfrak{B}_1 = \operatorname{Alg} \operatorname{Lat} \mathfrak{A}_1$, then

$$\mathfrak{B} := \operatorname{Alg} \operatorname{Lat} \mathfrak{A} = \mathcal{B}(\mathcal{V})P_e + J\mathfrak{B}_1 P_e^{\perp}.$$

It follows that $\mathfrak{A}$ is reflexive if and only if $\mathfrak{A}_1$ is.

Thus if $\dim \mathcal{V}_1 > 1$, there are non-reflexive examples. For example, consider the non-reflexive algebra $\mathfrak{A}_1 = \{\left[\begin{smallmatrix} a & 0 \\ b & a \end{smallmatrix}\right] : a, b \in \mathbb{C}\}$. Take $n = 3$ and let

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 1/\sqrt{2} \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1/2 & 1/2 & 0 \end{bmatrix} \quad A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This can be seen to satisfy $\sum_{i=1}^{3} A_i A_i^* = I_3$ and to generate the algebra $\mathfrak{A} = \left\{ \begin{bmatrix} c & 0 & 0 \\ d & a & 0 \\ e & b & a \end{bmatrix} : a, b, c, d, e \in \mathbb{C} \right\}$. This is not reflexive.

Nevertheless, $\mathfrak{A}^*$ has a unique minimal invariant subspace, and thus $\mathfrak{S}$ is unitarily equivalent to $\mathfrak{B}_{3,1}$, which is hyper-reflexive. So there is no direct correspondence between the reflexivity of $\mathfrak{A}$ and $\mathfrak{S}$.

Lemma 2.5.8. *Let $A = (A_1, \ldots, A_n)$ be an n-tuple on a finite dimensional space $\mathcal{V}$ such that $\sum_{i=1}^{n} A_i A_i^* = I$. Let $\mathfrak{A}$ be the unital algebra that they generate. Let $S = (S_1, \ldots, S_n)$ be the minimal isometric dilation, and $\mathfrak{S}$ the* WOT*-closed algebra they generate. Then $\mathfrak{S}$ is irreducible if and only if $\mathfrak{A}^*$ has a unique minimal invariant subspace $\mathcal{V}_0$.*

Proof. If $\mathcal{V}_0$ is unique, then it must be cyclic for $\mathfrak{A}$ since $\mathcal{V} \ominus \mathfrak{A}[\mathcal{V}_0]$ is an invariant subspace of $\mathfrak{A}^*$ orthogonal to $\mathcal{V}_0$. So Lemma 2.5.4 applies. Since $\mathfrak{S}$ contains $\mathcal{B}(\mathcal{H})P_{\mathcal{V}_0}$, it is evidently irreducible.

Indeed, this conclusion follows if there is *any* minimal $\mathfrak{A}^*$-invariant subspace $\mathcal{V}_0$ which is cyclic for $\mathfrak{A}$. By Corollary 2.5.5, $\mathcal{V}_0$ is necessarily the unique minimal $\mathfrak{A}^*$-invariant subspace.

Finally suppose that there is a minimal $\mathfrak{A}^*$-invariant subspace $\mathcal{V}_0$ which is not cyclic. Then as in the first paragraph, $\mathcal{V} \ominus \mathfrak{A}[\mathcal{V}_0]$ is an invariant subspace of $\mathfrak{A}^*$ orthogonal to $\mathcal{V}_0$. Let $\mathcal{V}_1$ be a minimal $\mathfrak{A}^*$-invariant subspace contained therein. Notice that $\mathfrak{S}[\mathcal{V}_i]$ are pairwise orthogonal reducing subspaces for $\mathfrak{S}$ by Lemma 2.3.3. Hence $\mathcal{H}$ contains proper reducing subspaces, and so $\mathfrak{S}$ is reducible. ■

Now the case when more than one minimal $\mathfrak{A}^*$-invariant subspace is present will be addressed. In the following lemma, questions of uniqueness are not pertinent.

Lemma 2.5.9. *Assume that $\sum_{i=1}^{n} A_i A_i^* = I$. There is a family of minimal $\mathfrak{A}^*$-invariant subspaces $\mathcal{V}_j$ of $\mathcal{V}$, $1 \leq j \leq s$, such that $\mathcal{H}$ decomposes into an orthogonal direct sum of $\mathcal{H}_j = \mathfrak{S}[\mathcal{V}_j]$; and the algebras $\mathfrak{S}|_{\mathcal{H}_j}$ are irreducible.*

Proof. This is just a matter of choosing a maximal family of pairwise orthogonal minimal $\mathfrak{A}^*$-invariant subspaces, say $\mathcal{V}_j$ for $1 \leq j \leq s$. By Lemma 2.3.3, the subspaces $\mathcal{H}_j = \mathfrak{S}[\mathcal{V}_j]$ are pairwise orthogonal and reducing for $\mathfrak{S}$. Moreover a direct application of the previous lemma applied to $\mathcal{H}_j$ and $\mathcal{V}_j$ shows that $\mathfrak{S}|_{\mathcal{H}_j}$ is irreducible. Finally, it is required to show that $\sum_{j=1}^{s} \oplus \mathcal{H}_j = \mathcal{H}$. Take any vector x orthogonal to this sum. By Corollary 2.4.2, $\mathfrak{S}^*[x]$ intersects $\mathcal{V}$ in a non-zero $\mathfrak{A}^*$-invariant subspace orthogonal to all of the $\mathcal{H}_j$'s, and thus orthogonal to all of the $\mathcal{V}_j$'s. This is contrary to construction, and so yields a contradiction. ■

Given an n-tuple $A = (A_1, \ldots, A_n)$ such that $\sum_{i=1}^{n} A_i A_i^* = I$, choose a maximal family of mutually orthogonal minimal $\mathfrak{A}^*$-invariant subspaces $\mathcal{V}_j$ of $\mathcal{V}$, $1 \leq j \leq s$; and let $P_j = P_{\mathcal{V}_j}$. From the minimality of each $\mathcal{V}_j$ as an $\mathfrak{A}^*$-invariant subspace, we know that $P_j \mathfrak{A}^* P_j = \mathcal{B}(\mathcal{V}_j)$. Set $\widetilde{\mathcal{V}} = \sum_{j=1}^{s} \oplus \mathcal{V}_j$. Let $\widetilde{A}_i = P_{\widetilde{\mathcal{V}}} A_i|_{\widetilde{\mathcal{V}}} = (A_i^*|_{\widetilde{\mathcal{V}}})^*$ be the compression of A_i to $\widetilde{\mathcal{V}}$; and let $\widetilde{\mathfrak{A}}$ denote the algebra they generate in $\mathcal{B}(\widetilde{\mathcal{V}})$.

Notice that the minimal isometric dilation of $\widetilde{A} = (\widetilde{A}_1, \ldots, \widetilde{A}_n)$ is precisely S. It is evident that S is a joint isometric dilation of $\widetilde{A}$. To show that it is minimal, it suffices to show that $\mathfrak{S}[\widetilde{\mathcal{V}}] = \mathcal{H}$. But this is established above in Lemma 2.5.9.

The goal now is to show that $\widetilde{\mathfrak{A}}$ is a C*-algebra. For the moment, observe that it is semisimple. Note that $\widetilde{\mathfrak{A}}$ is contained in $\sum_{1 \leq j \leq s} \oplus \mathcal{B}(\mathcal{V}_j)$. Moreover the quotient map q_j of compression to $\mathcal{V}_j$ maps $\mathfrak{A}$ onto $\mathcal{B}(\mathcal{V}_j)$. Thus the kernel of this map is a maximal ideal. Since $\sum \oplus q_j = \mathrm{id}$ is faithful, the intersection of all maximal ideals

is $\{0\}$. Hence $\widetilde{\mathfrak{A}}$ is semisimple.

Indeed, there is a minimal family G so that $\sum_{g\in G}\oplus q_g$ is faithful. By the Wedderburn theory, the minimal ideal $\mathfrak{A}_g = \ker\sum_{h\in G\setminus\{g\}}\oplus q_h$ is isomorphic to $\mathcal{B}(\mathcal{V}_g)$. But this kernel will, in practice, be supported on several of the $\mathcal{V}_j$'s. This yields a partition $\widetilde{\mathcal{V}} = \sum_{g\in G}\oplus\mathcal{W}_g$ where $\mathcal{W}_g = \sum_{j\in G_g}\oplus\mathcal{V}_j$ is a sum of those $\mathcal{V}_j$'s equivalent to $\mathcal{V}_g$. Because $\mathcal{B}(\mathcal{V}_g)$ is simple, it follows that there is an algebra isomorphism σ_j of $\mathcal{B}(\mathcal{V}_g)$ onto $\mathcal{B}(\mathcal{V}_j)$ for each $j \in G_g$ such that

$$\widetilde{\mathfrak{A}}|_{\mathcal{W}_g} \simeq \Big\{\sum_{j\in G_g}\oplus\sigma_j(X) : X \in \mathcal{B}(\mathcal{V}_g)\Big\}.$$

It is well-known that every isomorphism between $\mathcal{B}(\mathcal{V}_g)$ and $\mathcal{B}(\mathcal{V}_j)$ is spatial: $\sigma_j(X) = T_jXT_j^{-1}$ for some invertible operator T_j, which is unique up to a scalar multiple.

There is also the associated unital completely positive map Φ on $\mathcal{B}(\widetilde{\mathcal{V}})$ given by

$$\Phi(X) = \sum_{i=1}^{n}\widetilde{A}_iX\widetilde{A}_i^*.$$

Suppose that two blocks $\mathcal{V}_1$ and $\mathcal{V}_2$ are related by a similarity as above. Let $B_i := P_{\mathcal{V}_1}A_i|_{\mathcal{V}_1}$ and $C_i := P_{\mathcal{V}_2}A_i|_{\mathcal{V}_2} = TB_iT^{-1}$. Since

$$\sum_{i=1}^{n}B_iB_i^* = I_{\mathcal{V}_1} \qquad \text{and} \qquad \sum_{i=1}^{n}C_iC_i^* = I_{\mathcal{V}_2},$$

a computation shows that

$$I_{\mathcal{V}_2} = \sum_{i=1}^{n}(TB_iT^{-1})(TB_iT^{-1})^* = T\Phi_1(T^{-1}T^{*-1})T^*,$$

where $\Phi_1(X) = \sum_{i=1}^n B_i X B_i^* = P_1 \Phi(P_1 X P_1)|_{\mathcal{V}_1}$. Therefore

$$\Phi_1(T^{-1}T^{*-1}) = T^{-1}T^{*-1}.$$

An analysis of this completely positive map yields information on the structure of $\widetilde{\mathfrak{A}}$.

Lemma 2.5.10. *Let $\Phi(X) = \sum_{i=1}^n A_i X A_i^*$ be a unital completely positive map on $\mathcal{B}(\mathcal{V})$, where $\mathcal{V}$ is finite dimensional. If there is a non-scalar operator X such that $\Phi(X) = X$, then $\mathfrak{A}^* = \mathrm{Alg}\{A_1^*, \ldots, A_n^*\}$ has two pairwise orthogonal minimal invariant subspaces.*

Proof. Since Φ is self-adjoint and unital, there is a positive non-scalar X such that $\Phi(X) = X$. Let $\|X\| = 1$ and let μ denote the smallest eigenvalue of X. Then $\mathcal{M} = \ker(X - I)$ and $\mathcal{N} = \ker(X - \mu I)$ are pairwise orthogonal non-zero subspaces. For any unit vector $x \in \mathcal{M}$,

$$\begin{aligned} \|x\|^2 = (\Phi(X)x, x) &= \sum_{i=1}^n (X A_i^* x, A_i^* x) \\ &\leq \sum_{i=1}^n (A_i^* x, A_i^* x) = \|x\|^2 \end{aligned}$$

This equality can only hold if each $A_i^* x$ belongs to $\mathcal{M}$. Hence $\mathcal{M}$ is invariant for $\mathfrak{A}^*$.

This argument worked because 1 is an extreme point in the spectrum of X. This is also the case for μ. Hence a similar argument shows that $\mathcal{N}$ is invariant for $\mathfrak{A}^*$. ∎

The following is a partial converse to the previous lemma.

Lemma 2.5.11. *Let $\Phi(X) = \sum_{i=1}^{n} A_i X A_i^*$ be a unital completely positive map on $\mathcal{B}(\mathcal{V})$, where $\mathcal{V}$ is finite dimensional. Suppose that $A_i = B_i \oplus C_i$ with respect to an orthogonal decomposition $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2$. Moreover, suppose that $\mathrm{Alg}\{B_i\} = \mathcal{B}(\mathcal{V}_1)$ and $\mathrm{Alg}\{C_i\} = \mathcal{B}(\mathcal{V}_2)$. If there is an operator X such that $\Phi(X) = X$ and $X_{21} := P_{\mathcal{V}_2} X P_{\mathcal{V}_1} \neq 0$, then there is a unitary operator W such that $C_i = W^* B_i W$. Moreover the fixed point set of Φ consists of all matrices of the form $\begin{bmatrix} a_{11} I_{\mathcal{V}_1} & a_{12} W^* \\ a_{21} W & a_{22} I_{\mathcal{V}_2} \end{bmatrix}$.*

Proof. Since Φ is self-adjoint, it can be assumed that $X = X^*$. Then normalize so that $\|X_{21}\| = 1$. Let $\mathcal{M} = \{v \in \mathcal{V}_1 : \|X_{21} v\| = \|v\|\}$. Also let $\mathcal{N} = X_{21}\mathcal{M}$ denote the corresponding subspace of $\mathcal{V}_2$. Write $B = \begin{bmatrix} B_1 & \dots & B_n \end{bmatrix}$ and $C = \begin{bmatrix} C_1 & \dots & C_n \end{bmatrix}$, so that

$$X_{21} v = \Phi(X_{21}) v = C X_{21}^{(n)} B^* v \quad \text{for} \quad v \in \mathcal{M}.$$

Since C and B^* are contractions, and X_{21} achieves its norm on v, it follows that $B^* v$ belongs to the subspace $\mathcal{M}^{(n)}$ on which $X_{21}^{(n)}$ achieves its norm. Consequently each B_i^* leaves $\mathcal{M}$ invariant. But as $\mathrm{Alg}\{B_i\} = \mathcal{B}(\mathcal{V}_1)$, this forces $\mathcal{M} = \mathcal{V}_1$. Similarly, consideration of $X_{12} = X_{21}^*$ shows that $\mathcal{N} = \mathcal{V}_2$. Thus X_{21} and X_{21}^* are isometries; so $W = X_{21}|_{\mathcal{V}_1}$ is a unitary map from $\mathcal{V}_1$ onto $\mathcal{V}_2$.

Further, the identity above now shows that $W = C W^{(n)} B^*$. Hence for all $v \in \mathcal{V}_1$

$$\|v\| = \|Wv\| = \|C W^{(n)} B^* v\| \leq \|W^{(n)} B^* v\| \leq \|v\|.$$

In particular, C acts as an isometry from the range of $W^{(n)} B^*$ onto the range

Ran $W = \mathcal{V}_2$. Since C is contractive, it must be zero on the orthogonal complement of Ran $W^{(n)}B^*$. This implies that C^* is an isometry of $\mathcal{V}_2$ onto Ran $W^{(n)}B^*$. Consequently, $C^*W = W^{(n)}B^*$; or equivalently, $C_i^* = WB_i^*W^*$ for $1 \leq i \leq n$.

Finally, if $Y \in \mathcal{B}(\mathcal{V}_1, \mathcal{V}_2)$ and $\left[\begin{smallmatrix} 0 & 0 \\ Y & 0 \end{smallmatrix}\right]$ is fixed by Φ, then

$$Y = \sum_{i=1}^{n} C_i Y B_i^* = \sum_{i=1}^{n} W B_i W^* Y B_i^* = W\Phi_1(W^*Y)$$

where $\Phi_1(X) = \sum_{i=1}^{n} B_i X B_i^*$ acts on $\mathcal{B}(\mathcal{V}_1)$. By Lemma 2.5.10, W^*Y is scalar; so Y is a multiple of W. A similar analysis works for the other coordinates. ■

Example 2.5.12. Let

$$A_1 = \begin{bmatrix} 1/\sqrt{2} & 0 & 0 \\ 1/2\sqrt{2} & 1/2 & 1/2\sqrt{2} \\ 0 & 0 & 1/\sqrt{2} \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 1/\sqrt{2} & 0 & 0 \\ -1/2\sqrt{2} & 1/2 & -1/2\sqrt{2} \\ 0 & 0 & 1/\sqrt{2} \end{bmatrix}.$$

Then the matrix $X = \left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right]$ satisfies $\Phi(X) = X$. A calculation shows that the fixed point set of Φ is the set of matrices $X = [x_{ij}]$ such that $x_{12} = x_{21} = x_{23} = x_{32} = 0$ and $x_{11} + x_{13} + x_{31} + x_{33} = 2x_{22}$. In particular, this is not an algebra. The algebra $\mathfrak{A}^*$ has two minimal invariant subspaces, $\mathbb{C}e_1$ and $\mathbb{C}e_3$. Note that the compression of $\mathfrak{A}$ to span$\{e_1, e_3\}$ consists of scalar matrices, and the fixed point set of the restricted completely positive map is the full 2×2 matrix algebra.

The detailed information about the map Φ can now be utilized to determine the algebra $\widetilde{\mathfrak{A}}$.

Theorem 2.5.13. *Let $\Phi(X) = \sum_{i=1}^{n} A_i X A_i^*$ be a unital completely positive map on $\mathcal{B}(\mathcal{V})$, where $\mathcal{V}$ is finite dimensional. Suppose that $\mathcal{V}$ is the orthogonal direct sum of minimal $\mathfrak{A}^*$-invariant subspaces. Then $\mathfrak{A}$ is a C^*-algebra and the fixed point set of Φ coincides with the commutant of $\mathfrak{A}$.*

Proof. Let $\mathcal{V} = \sum_j \oplus \mathcal{V}_j$ be an orthogonal decomposition into minimal $\mathfrak{A}^*$-invariant subspaces. The restriction of $\mathfrak{A}$ to $\mathcal{V}_j$ is all of $\mathcal{B}(\mathcal{V}_j)$ by Burnside's Theorem. Thus the restriction of Φ to $\mathcal{B}(\mathcal{V}_j)$ maps onto the scalars by Lemma 2.5.10. By the earlier analysis, $\mathfrak{A}$ splits into an algebraic direct sum of minimal ideals which are isomorphic to full matrix algebras. These are determined by certain spatial intertwining relations between some of the summands. If the restriction of A_i's to $\mathcal{V}_1$ and $\mathcal{V}_2$ are related by an intertwining operator T, then it was shown that $\Phi_1(T^{-1}T^{*-1}) = T^{-1}T^{*-1}$. But this is scalar by Lemma 2.5.10. So after scaling T, it becomes a unitary. It follows that $\mathfrak{A}$ is a C^*-algebra.

Evidently, Φ fixes the commutant of $\mathfrak{A} = \mathfrak{A}^*$. Suppose that $\Phi(X) = X$. If $\mathcal{V}_k$ and $\mathcal{V}_l$ are not related by a unitary intertwining map, then by Lemma 2.5.11, $P_k X P_l = 0$. While if they are related by a unitary W_{kl}, then $P_k X P_l = x_{kl} W_{kl}$ belongs to $\mathfrak{A}'$. It now follows that the fixed point set of Φ is precisely the commutant of $\mathfrak{A}$. ∎

Now it is possible to provide a complete description of the algebra $\mathfrak{S}$ in the Cuntz case.

Lemma 2.5.14. *Let P_g for $g \in G$ denote the minimal central projections in $\widetilde{\mathfrak{A}}$. These projections belong to $\mathfrak{S}$.*

Proof. The proof follows the lines of Theorem 2.5.2. The setting will be in the

algebra $\mathfrak{B} = \mathcal{B}(\mathcal{H})P_{\mathcal{V}} + \left(0_{\mathcal{V}} \oplus \mathfrak{L}_n^{(a)}\right)$ which contains $\mathfrak{S}$ and each projection P_g. If a central projection P of $\widetilde{\mathfrak{A}}$ were not in $\mathfrak{S}$, by Lemma 2.4.4 it could be separated from $\mathfrak{S}$ by a functional of rank $d+1$, which as before can be written as $\varphi(A) = (A^{(d+1)}x, y)$. Let $\mathcal{M} = \mathfrak{S}^{*(d+1)}[y]$ and $\mathcal{M}_0 = \widetilde{\mathcal{V}}^{(d+1)} \cap \mathcal{M}$. This subspace $\mathcal{M}_0$ is invariant for the C*-algebra $\widetilde{\mathfrak{A}}^{*(d+1)} = \widetilde{\mathfrak{A}}^{(d+1)}$, and thus is the range of a projection Q in its commutant.

Now $P^{(d+1)}$ lies in the centre of $\widetilde{\mathfrak{A}}^{(d+1)}$, and thus commutes with Q as well. Therefore $\widetilde{\mathcal{V}}^{(d+1)}$ decomposes as

$$P^{(d+1)}Q\widetilde{\mathcal{V}}^{(d+1)} \ominus P^{\perp(d+1)}Q\widetilde{\mathcal{V}}^{(d+1)} \ominus P^{(d+1)}Q^{\perp}\widetilde{\mathcal{V}}^{(d+1)} \ominus P^{\perp(d+1)}Q^{\perp}\widetilde{\mathcal{V}}^{(d+1)}$$
$$=: \mathcal{M}_{pq} \ominus \mathcal{M}_{p^{\perp}q} \ominus \mathcal{M}_{pq^{\perp}} \ominus \mathcal{M}_{p^{\perp}q^{\perp}}.$$

This determines an orthogonal decomposition of $\widetilde{\mathcal{V}}^{(d+1)}$ into four reducing subspaces for $\widetilde{\mathfrak{A}}^{(d+1)}$. Note that $\mathcal{M}_0$ is the sum of the first two. Recall the remarks following Lemma 2.5.9 that S is the minimal isometric dilation of $\widetilde{A}$. So by Lemma 2.3.3, $\mathcal{H}^{(d+1)}$ has an orthogonal decomposition into the four reducing subspaces for $\mathfrak{S}^{(d+1)}$ generated by these subspaces of $\widetilde{\mathcal{V}}^{(d+1)}$, say

$$\mathcal{H}^{(d+1)} = \mathcal{H}_{pq} \ominus \mathcal{H}_{p^{\perp}q} \ominus \mathcal{H}_{pq^{\perp}} \ominus \mathcal{H}_{p^{\perp}q^{\perp}}.$$

Moreover, Lemma 2.4.1 shows as in the proof of Theorem 2.5.2 that y belongs to $\mathcal{H}_{pq} \ominus \mathcal{H}_{p^{\perp}q} = \mathfrak{S}^{(d+1)}[\mathcal{M}_0]$.

It is evident from this construction that each of these four subspaces $\mathcal{H}_{ij}$ is mapped onto the corresponding $\mathcal{M}_{ij}$ by the orthogonal projection $P_{\widetilde{\mathcal{V}}}^{(d+1)}$ onto

$\widetilde{\mathcal{V}}^{(d+1)}$. Therefore, since $P^{(d+1)}$ is dominated by this projection, it is clear that it maps y into $\mathcal{M}_{pq}$, which is contained in $\mathcal{M}_0$. As before, it follows that x is orthogonal to $\mathcal{M}_0$, and therefore $\varphi(P) = 0$. Hence one concludes that P belongs to $\mathfrak{S}$. ■

Theorem 2.5.15. *Let $A_1, \ldots, A_n$ be operators on a finite dimensional space $\mathcal{V}$ such that $\sum_{i=1}^{n} A_i A_i^* = I$, and let $S_1, \ldots, S_n$ be their joint isometric dilation. Let $\widetilde{\mathcal{V}}$ be the subspace of $\mathcal{V}$ spanned by all minimal $\mathfrak{A}^*$-invariant subspaces. Then the compression $\widetilde{\mathfrak{A}}$ of $\mathfrak{A}$ to $\widetilde{\mathcal{V}}$ is a C^*-algebra. Let $\widetilde{\mathfrak{A}}$ be decomposed as $\sum_{g\in G} \oplus \mathfrak{M}_{d_g} \otimes \mathbb{C}^{m_g}$ with respect to a decomposition $\widetilde{\mathcal{V}} = \sum_{g\in G} \oplus \mathcal{V}_g^{(m_g)}$, where $\mathcal{V}_g$ has dimension d_g and multiplicity m_g. Let P_g denote the projection onto $\mathcal{V}_g$. Then the dilation acts on the space*

$$\mathcal{H} = \sum_{g\in G} \oplus \mathcal{H}_g^{(m_g)} = \widetilde{\mathcal{V}} \oplus \mathcal{H}_n^{(\alpha)}$$

where $\mathcal{H}_g = \mathcal{V}_g \ominus \mathcal{H}_n^{(\alpha_g)}$ and $\alpha_g = d_g(n-1)$ and

$$\alpha = \sum_{g\in G} \alpha_g m_g = (n-1) \sum_{g\in G} d_g m_g.$$

The algebra $\mathfrak{S}$ decomposes as

$$\mathfrak{S} \simeq \sum_{g\in G} \oplus \left(\mathcal{B}(\mathcal{H}_g) P_g\right)^{(m_g)} + \left(0_{\widetilde{\mathcal{V}}} \oplus \mathfrak{L}_n^{(\alpha)}\right).$$

Proof. This is now just a matter of putting the pieces together and clearing up some final details. Let $\mathcal{V}_g$, $1 \leq g \leq s$, be any maximal family of pairwise orthogonal minimal $\mathfrak{A}^*$-invariant subspaces. Let $\widetilde{\mathcal{V}} = \sum_{1\leq g\leq s} \oplus \mathcal{V}_g$. (Do not worry at this stage

about the uniqueness of the definition of $\widetilde{\mathcal{V}}$.) By Lemma 2.3.3, $\mathcal{H}_g = \mathfrak{S}[\mathcal{V}_g]$ are pairwise orthogonal reducing subspaces of $\mathfrak{S}$. Let $\mathcal{M} = \sum_{1 \le g \le s} \oplus \mathcal{H}_g$. The claim is that $\mathcal{M} = \mathcal{H}$. Indeed, if there was a non-zero vector in $\mathcal{M}^\perp$, then by Corollary 2.4.2, $\mathcal{M}^\perp \cap \mathcal{V}$ would be a non-zero $\mathfrak{A}^*$-invariant subspace orthogonal to $\widetilde{\mathcal{V}}$, contrary to fact.

It now follows as above that if each A_i is compressed to $\widetilde{A}_i$ on $\widetilde{\mathcal{V}}$, then this new n-tuple has the identical joint isometric dilation S_i, and it is the minimal dilation by the previous paragraph. By Theorem 2.5.13, the algebra $\widetilde{\mathfrak{A}}$ that they generate is self-adjoint. Then applying Lemma 2.5.14, one has that the projection onto $\widetilde{\mathcal{V}}$ belongs to $\mathfrak{S}$, and that $P_{\widetilde{\mathcal{V}}}\mathfrak{S} = \widetilde{\mathfrak{A}}$.

The restriction of $\mathfrak{S}$ to each reducing subspace $\mathcal{H}_g$ is isomorphic to $\mathfrak{B}_{n,d_g}$. Moreover, the restriction of $\mathfrak{S}$ to $\widetilde{\mathcal{V}}^\perp$ is canonically isomorphic to $\mathfrak{L}_n^{(\alpha)}$, where by canonical it is meant that $u(S)|_{\widetilde{\mathcal{V}}^\perp} \simeq L_u^{(\alpha)}$ when the natural identification of $\widetilde{\mathcal{V}}^\perp$ with $\mathcal{H}_n^{(\alpha)}$ is made as in Lemma 2.3.1.

Now the finite dimensional C*-algebra $\widetilde{\mathfrak{A}}$ may be decomposed as $\sum_{g \in G}^{\oplus} \mathfrak{M}_{d_g} \otimes \mathbb{C}^{m_g}$. The multiplicities reflect the fact that the restrictions of A_i^* to different $\mathcal{V}_g$'s may be unitarily equivalent. As before, choose a maximal subset G of pairwise inequivalent subspaces $\mathcal{V}_g$, and let $\mathcal{W}_g = \sum_{j \in G_g} \oplus \mathcal{V}_j$ be the sum of all subspaces equivalent to $\mathcal{V}_g$. Then $\mathcal{W}_g$ may be naturally identified with $\mathcal{V}_g \otimes \mathbb{C}^{m_g}$ so that $A_i^*|_{\mathcal{W}_g} \simeq \left(A_i^*|_{\mathcal{V}_g}\right)^{(m_g)}$. This identifies $\widetilde{\mathcal{V}}$ with $\sum_{g \in G} \oplus \mathcal{V}_g^{(m_g)}$.

By the uniqueness of the minimal isometric dilation, it also follows that there is a corresponding unitary equivalence between $\sum_{j \in G_g} \oplus \mathcal{H}_j$ and $\mathcal{H}_g \otimes \mathbb{C}^{(m_g)}$ so that the restriction of S_i is identified with $\left(S_i|_{\mathcal{H}_g}\right)^{(m_g)}$. By Lemma 2.5.14, the projection

$P_{\mathcal{W}_g} \simeq P_g^{(m_g)}$ belongs to $\mathfrak{S}$. Thus it is now apparent that $\mathfrak{S}P_{\widetilde{\mathcal{V}}}$ decomposes as $\sum_{g\in G} \oplus (\mathcal{B}(\mathcal{H}_g)P_g)^{(m_g)}$. Combining all of the pieces, the desired structure theory for $\mathfrak{S}$ is obtained.

It remains to establish the uniqueness of $\widetilde{\mathcal{V}}$. Notice that $P_{\widetilde{\mathcal{V}}}$ is the unique maximal finite rank projection in $\mathfrak{S}$. Indeed, every operator in $\mathfrak{S}$ has a lower triangular form with respect to the decomposition $\mathcal{H} = \widetilde{\mathcal{V}} \oplus \mathcal{H}_n^{(\alpha)}$. From [17], $\mathfrak{L}_n$ contains no proper projections. Therefore all finite rank projections are supported by $\widetilde{\mathcal{V}}$. Now suppose that $\mathcal{V}_0$ is any minimal $\mathfrak{A}^*$-invariant subspace. It may be extended to a maximal family of pairwise orthogonal minimal $\mathfrak{A}^*$-invariant subspaces, and the construction may proceed as above. The same subspace $\widetilde{\mathcal{V}}$ necessarily is obtained by the uniqueness of this maximal projection. In particular, $\widetilde{\mathcal{V}}$ must contain *every* minimal $\mathfrak{A}^*$-invariant subspace. Thus it is the span of all such subspaces. ■

2.6 The General Finite Dimensional Case

The problem posed in Section 2.4 will now be addressed. Starting with a contractive n-tuple $(A_1, \ldots, A_n)$ with minimal joint isometric dilation $(S_1, \ldots, S_n)$, the goal is to understand the structure of $\mathfrak{S} = \mathrm{Alg}\{S_1, \ldots, S_n\}$ in terms of the structure of the n-tuple A and the algebra $\mathfrak{A}$ that it generates.

Recall from the discussion in Section 2.4 that $\mathcal{H} = \mathcal{H}_p \oplus \mathcal{H}_c$, where $\mathcal{H}_p$ is the pure part determined by the wandering subspace of S, and $\mathcal{H}_c$ is the Cuntz part; and that P_p and P_c denote the orthogonal projections onto these subspaces. A method of getting information about this decomposition from A is required. Corollary 2.4.2 shows that $\mathcal{H}_c = \mathfrak{S}[\mathcal{V}_c]$, so $\mathcal{H}_c$ will be recovered if $\mathcal{V}_c$ can be computed.

Once again, consider the completely positive map

$$\Phi(X) = \sum_{i=1}^{n} A_i X A_i^* = AXA^*.$$

This is no longer unital, since $\Phi(I) = AA^* = \sum_{i=1}^{n} A_i A_i^* \leq I$. But it is completely contractive. Thus the sequence $\Phi^k(I)$ is a decreasing sequence of positive operators, and therefore converges in the strong operator topology to a limit which will be denoted as $\Phi^\infty(I)$.

Lemma 2.6.1. *$\Phi^\infty(I) = P_{\mathcal{V}} P_c P_{\mathcal{V}}$. Hence $\mathcal{V}_c = \ker(I - \Phi^\infty(I))$.*

Proof. If $x \in \mathcal{H}_c$, then

$$\sum_{|w|=k} \|S_w^* x\|^2 = \|x\|^2.$$

On the other hand, any vector x in $\mathcal{H}_p$ satisfies

$$\lim_{k\to\infty} \sum_{|w|=k} \|S_w^* x\|^2 = 0.$$

Thus if x is any vector in $\mathcal{H} = \mathcal{H}_c \ominus \mathcal{H}_p$,

$$\lim_{k\to\infty} \sum_{|w|=k} \|S_w^* x\|^2 = \|P_c x\|^2.$$

Write $A_w^* := w(A)^* = S_w^*|_{\mathcal{V}}$. Now if $v \in \mathcal{V}$,

$$\lim_{k\to\infty} \sum_{|w|=k} \|A_w^* v\|^2 = \lim_{k\to\infty} \sum_{|w|=k} \|S_w^* v\|^2 = \|P_c v\|^2.$$

It is evident that $\Phi^k(I) = \sum_{|w|=k} A_w A_w^*$ and thus

$$(\Phi^k(I)v, v) = \sum_{|w|=k} \|A_w^* v\|^2.$$

Therefore

$$(\Phi^\infty(I)v, v) = \|P_c v\|^2 = (P_\mathcal{V} P_c P_\mathcal{V} v, v).$$

It follows that $\Phi^\infty(I) = P_\mathcal{V} P_c P_\mathcal{V}$. In particular, $\ker(I - \Phi^\infty(I)) = \mathcal{V} \cap \mathcal{H}_c = \mathcal{V}_c$. ■

Now $\mathcal{H}_c = \mathfrak{S}[\mathcal{V}_c]$, and thus the restriction of the S_i's to $\mathcal{H}_c$ are the minimal joint isometric dilations of the compressions of the A_i's to $\mathcal{V}_c$. By the previous section, the algebra $\mathfrak{S}|_{\mathcal{H}_c}$ is determined by the restriction of $\mathfrak{A}$ to the span $\widetilde{\mathcal{V}}$ of all $\mathfrak{A}^*$-invariant subspaces contained in $\mathcal{V}_c$. It is desirable to give a definition that is somewhat independent of the definition of $\mathcal{V}_c$. *The space $\widetilde{\mathcal{V}}$ is the span of all $\mathfrak{A}^*$-invariant subspaces $\mathcal{W}$ on which $\sum_{i=1}^n A_i A_i^*|_\mathcal{W} = I_\mathcal{W}$.* Indeed, the condition

$$I_\mathcal{W} = \sum_{i=1}^n A_i A_i^*|_\mathcal{W} = \sum_{i=1}^n S_i S_i^*|_\mathcal{W}$$

implies that $\mathcal{W}$ is contained in $\mathcal{H}_c$, whence in $\mathcal{H}_c \cap \mathcal{V} = \mathcal{V}_c$. Thus $\mathcal{W}$ is contained in $\widetilde{\mathcal{V}}$ by Theorem 2.5.15. The converse follows from the description there of $\widetilde{\mathcal{V}}$.

Lemma 2.6.2. *The projection $P_{\widetilde{\mathcal{V}}}$ belongs to $\mathfrak{S}$.*

Proof. Assume that $P_{\widetilde{\mathcal{V}}} \neq 0$. Suppose to the contrary that φ is a WOT-continuous functional which separates $P_{\widetilde{\mathcal{V}}}$ from $\mathfrak{S}$. Then as before, represent $\varphi(X) = (X^{(d+1)}x, y)$ on an algebra $\mathfrak{B}$ containing $\mathfrak{S}$ and $P_{\widetilde{\mathcal{V}}}$.

Split $x = x_c \oplus x_p$ and $y = y_c \oplus y_p$ corresponding to the decomposition of $\mathcal{H}^{(d+1)} = \mathcal{H}_c^{(d+1)} \oplus H_p^{(d+1)}$. The functional $\varphi_p(X) = (X^{(d+1)}x_p, y_p)$ acts on the pure part $\mathfrak{S}|_{\mathcal{H}_p}$. Since the Cuntz part is non-zero and contains wandering subspaces on which the S_i's are unitarily equivalent to L_i, it is easy to find vectors x_0 and y_0 in $\mathcal{H}_c$ such that $\varphi_p(X) = (Xx_0, y_0)$. Thus, by setting $x' = x_c \ominus x_0$ and $y' = y_c \ominus y_0$ in $\mathcal{H}^{(d+2)}$, one obtains vectors in $\mathcal{H}_c^{(d+2)}$ for which $\varphi(X) = (X^{(d+2)}x', y')$.

Thus by restricting to $\mathcal{H}_c$, one obtains a WOT-continuous linear functional which separates $P_{\tilde{\mathcal{V}}}$ from $\mathfrak{S}|_{\mathcal{H}_c}$. This contradicts Lemma 2.5.14. Hence $P_{\tilde{\mathcal{V}}}$ must belong to $\mathfrak{S}$. ■

Next the pure rank of the dilation is computed.

Lemma 2.6.3. *The pure rank of $\mathcal{A}$ is computed as*

$$\text{pure rank}(\mathcal{A}) = \text{rank}(I - \Phi(I)) = \text{rank}\Big(I - \sum_{i=1}^{n} A_i A_i^*\Big).$$

Proof. The wandering space is $\mathcal{X} = \text{Ran}\big(I - \sum_{i=1}^{n} S_i S_i^*\big)$ and the pure rank of $\mathcal{A}$ equals $\dim \mathcal{X}$. The minimality of the dilation means that $\mathcal{X}$ does not intersect $\mathcal{V}^{\perp}$. Therefore $P_{\mathcal{V}} P_{\mathcal{X}} P_{\mathcal{V}}$ has the same rank as $P_{\mathcal{X}}$. However, it is easy to see that

$$\begin{aligned} P_{\mathcal{V}} P_{\mathcal{X}} P_{\mathcal{V}}|_{\mathcal{V}} &= P_{\mathcal{V}}\Big(I_{\mathcal{H}} - \sum_{i=1}^{n} S_i S_i^*\Big) P_{\mathcal{V}}|_{\mathcal{V}} \\ &= I_{\mathcal{V}} - \sum_{i=1}^{n} A_i A_i^* = I_{\mathcal{V}} - \Phi(I_{\mathcal{V}}). \end{aligned}$$

Thus $\text{pure rank}(\mathcal{A}) = \text{rank}\big(I - \Phi(I)\big)$. ■

Notice that this proof works for any n-tuple $\mathcal{A}$. The pure rank of a general

contractive n-tuple plays an important role in Chapter 4.

Example 2.6.4. The subtlety of the preceding lemma is due to fact that $\mathcal{X}$ is not, in general, contained in $\mathcal{V}$. To illustrate this, consider the following example. Let

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then

$$A_1 A_1^* + A_2 A_2^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It is clear that $\mathbb{C}e_1$ and $\mathbb{C}e_3$ are pairwise orthogonal minimal $\mathfrak{A}^*$-invariant subspaces. The vector e_1 generates the subspace $\mathcal{H}_1 = \overline{\mathfrak{S}e_1}$ on which the representation is equivalent to the atomic representation $\sigma_{1,1}$. Furthermore, e_3 is a wandering vector generating a copy of the left regular representation on $\mathcal{H}_3 = \overline{\mathfrak{S}e_3}$. However e_2 is not orthogonal to $\mathcal{H}_1 \ominus \mathcal{H}_3$. One can show that there is a second wandering vector $\zeta := e_2 - P_{\mathcal{V}}^{+} S_2(e_1 + e_3)$. The subspace $\mathcal{H}_2 = \overline{\mathfrak{S}\zeta}$ yields the decomposition $\mathcal{H} = \mathcal{H}_1 \ominus \mathcal{H}_2 \ominus \mathcal{H}_3$.

The point here is that this decomposition does not decompose $\mathcal{V}$ into orthogonal pieces. In fact, $\mathcal{H}_2$ has trivial intersection with $\mathcal{V}$; and the vector e_2 has components in all three pieces.

One can now completely describe the algebra $\mathfrak{S}$ determined by the joint iso-

metric dilation of a contractive n-tuple. There is nothing to do except combine the information in Theorem 2.5.15 with the preceding two lemmas.

Theorem 2.6.5. *Let $(A_1, \ldots, A_n)$ be a contractive n-tuple on a finite dimensional space $\mathcal{V}$ with joint minimal isometric dilation $(S_1, \ldots, S_n)$ on $\mathcal{H}$. The space $\mathcal{H}$ decomposes as $\mathcal{H}_p \oplus \mathcal{H}_c$ into its pure and Cuntz parts. The multiplicity of $\mathcal{H}_p$ is* $\text{pure rank}(\mathcal{A}) = \text{rank}\big(I - \sum_{i=1}^n A_i A_i^*\big)$. *The subspace $\widetilde{\mathcal{V}}$ spanned by all minimal $\mathfrak{A}^*$-invariant subspaces $\mathcal{W}$ on which $\sum_{i=1}^n A_i A_i^*|_{\mathcal{W}} = I_{\mathcal{W}}$ determines $\mathcal{H}_c = \mathfrak{S}[\widetilde{\mathcal{V}}]$.*

The compression $\widetilde{\mathfrak{A}}$ of $\mathfrak{A}$ to $\widetilde{\mathcal{V}}$ is a C^*-algebra. Let $\widetilde{\mathfrak{A}}$ be decomposed as $\sum_{g\in G} \ominus \mathfrak{M}_{d_g} \otimes \mathbb{C}^{m_g}$ with respect to a decomposition $\widetilde{\mathcal{V}} = \sum_{g\in G} \ominus \mathcal{V}_g^{(m_g)}$, where $\mathcal{V}_g$ has dimension d_g and multiplicity m_g; and let P_g denote the projection onto $\mathcal{V}_g$. Then the dilation acts on the space

$$\mathcal{H} = \sum_{g\in G} \oplus \mathcal{H}_g^{(m_g)} \oplus \mathcal{H}_p = \widetilde{\mathcal{V}} \oplus \mathcal{H}_n^{(\alpha)}$$

where $\mathcal{H}_g = \mathcal{V}_g \oplus \mathcal{H}_n^{(\alpha_g)}$, $\alpha_g = d_g(n-1)$ and

$$\begin{aligned} \alpha &= \sum_{g\in G} \alpha_g m_g + \text{pure rank}(\mathcal{A}) \\ &= (n-1)\sum_{g\in G} d_g m_g + \text{rank}\Big(I - \sum_{i=1}^n A_i A_i^*\Big). \end{aligned}$$

The algebra $\mathfrak{S}$ decomposes as

$$\mathfrak{S} \simeq \sum_{g\in G} \oplus \big(\mathcal{B}(\mathcal{H}_g)P_g\big)^{(m_g)} \;+\; \big(0_{\widetilde{\mathcal{V}}} \oplus \mathfrak{C}_n^{(\alpha)}\big).$$

Some of the consequences of this theorem will now be collected. First, simple conditions for determining when the dilation of A is irreducible are obtained.

Corollary 2.6.6. *The algebra $\mathfrak{S}$ determined by the joint isometric dilation of a contractive n-tuple A on a finite dimensional space $\mathcal{V}$ is irreducible if and only if either*

(1) $\mathrm{Ran}(I - \sum_{i=1}^{n} A_i A_i^*) = \mathbb{C}v \neq 0$ *and v is cyclic for $\mathfrak{A}$. In this case, $\mathfrak{S}$ is unitarily equivalent to $\mathfrak{L}_n$.*

or

(2) $\sum_{i=1}^{n} A_i A_i^* = I$ *and $\mathfrak{A}^*$ has a minimal invariant subspace $\mathcal{V}_0$ which is cyclic for $\mathfrak{A}$. In this case, $\mathfrak{S}$ is unitarily equivalent to $\mathfrak{B}_{n,d_0}$ where $d_0 = \dim \mathcal{V}_0$.*

which are respectively equivalent to

(1′) $\mathrm{rank}(I - \Phi(I)) = 1$ *and* $\Phi^{\infty}(I) = 0$.

or

(2′) $\{X : \Phi(X) = X\} = \mathbb{C}I$.

Proof. $\mathfrak{S}$ is irreducible if and only if either it is pure with pure rank 1, or it has pure rank 0 and, by Lemma 2.5.8, has a unique minimal $\mathfrak{A}^*$-invariant subspace.

By Lemma 2.6.3, the pure rank is 1 precisely when $\mathrm{rank}(I - \Phi(I)) = 1$, or equivalently that $\mathrm{Ran}(I - \sum_{i=1}^{n} A_i A_i^*)$ is a one-dimensional subspace $\mathbb{C}v$. Now $\mathfrak{S}$ is pure precisely when $\mathcal{H}_c = \{0\}$, which by Corollary 2.4.2 is equivalent to $\mathcal{V}_c = \{0\}$. By Lemma 2.6.1, this is equivalent to $\Phi^{\infty}(I) = 0$, which establishes the equivalence

with (1'). Now $\mathcal{V}_c$ is $\mathfrak{A}^*$-invariant and orthogonal to v, and therefore orthogonal to $\mathfrak{A}v$. So if v is $\mathfrak{A}$–cyclic, then $\mathfrak{A}[v] = \mathcal{V}$ and $\mathcal{V}_c = \{0\}$. Conversely, if $\mathfrak{A}[v]$ is proper, then $\mathcal{M} = \mathfrak{A}[v]^\perp$ is $\mathfrak{A}^*$-invariant. But $\sum_{i=1}^n A_i A_i^*|_{\mathcal{M}} = I_{\mathcal{M}}$ because of the condition on $\Phi(I)$. So $\mathcal{M}$ is contained in $\mathcal{H}_c$. This verifies the equivalence with (1).

The Cuntz case is synonymous with the condition $\sum_{i=1}^n A_i A_i^* = I$. If $\mathcal{M}$ is a minimal $\mathfrak{A}^*$-invariant subspace, then $\mathfrak{A}[\mathcal{M}]^\perp$ contains another. So if $\mathcal{M}$ is unique, it must be cyclic. Conversely, if it is not unique, then by Theorem 2.5.15, $\widetilde{\mathcal{V}}$ contains at least two pairwise orthogonal minimal $\mathfrak{A}^*$-invariant subspaces, one of which may be taken to be $\mathcal{M}$; call the other $\mathcal{M}'$. Then $\mathfrak{A}[\mathcal{M}]$ is orthogonal to $\mathcal{M}'$ and thus it is not cyclic for $\mathfrak{A}$. This establishes the equivalence with (2).

Condition (2') contains the fact that $\Phi(I) = I$, so this is the Cuntz case. If there were more than one minimal $\mathfrak{A}^*$-invariant subspace, then by Theorem 2.5.13 the fixed point algebra contains non-scalar operators. Conversely, if Φ has non-scalar fixed points, then Lemma 2.5.10 shows that there are two orthogonal $\mathfrak{A}^*$-invariant subspaces. So (2') is equivalent to irreducibility. ■

Corollary 2.6.7. *The minimal isometric dilation of a finite dimensional n-tuple* $A = (A_1, \ldots, A_n)$ *is pure if and only if* $\mathfrak{A}(I - \sum_{i=1}^n A_i A_i^*)\mathcal{V} = \mathcal{V}$ *or equivalently that* $\Phi^\infty(I) = 0$.

Proof. The dilation has a Cuntz part if and only if there is a $\mathfrak{A}^*$-invariant subspace $\mathcal{M}$ contained in $\ker(I - \sum_{i=1}^n A_i A_i^*)$. This is equivalent to having the proper $\mathfrak{A}$-invariant subspace $\mathcal{M}^\perp$ containing

$$\left(\ker(I - \sum_{i=1}^n A_i A_i^*)\right)^\perp = \mathrm{Ran}(I - \sum_{i=1}^n A_i A_i^*).$$

The minimal such subspace is clearly $\mathfrak{A}(I-\sum_{i=1}^{n} A_i A_i^*)\mathcal{V}$. Thus the dilation is pure precisely when $\mathfrak{A}(I - \sum_{i=1}^{n} A_i A_i^*)\mathcal{V} = \mathcal{V}$.

Evidently, if there is a Cuntz part, then

$$\Phi^{\infty}(I) \geq \Phi^{\infty}(P_{\widetilde{\mathcal{V}}}) = P_{\widetilde{\mathcal{V}}}.$$

Conversely, if A is pure, then $\text{SOT–lim}_{k\to\infty} \sum_{|w|=k} S_w S_w^* = 0$. The compression of $S_w S_w^*$ to $\mathcal{V}$ is $A_w A_w^*$, and thus

$$\sum_{|w|=k} P_{\mathcal{V}} S_w S_w^* |_{\mathcal{V}} = \sum_{|w|=k} A_w A_w^* = \Phi^k(I).$$

Since $\mathcal{V}$ is finite dimensional, this converges to 0 in norm. ■

This theorem also provides simple complete unitary invariants for the associated finitely correlated representations of $\mathcal{E}_n$ (or of $\mathcal{O}_n$ in the Cuntz case).

Theorem 2.6.8. *Let $A = (A_1, \ldots, A_n)$ and $B = (B_1, \ldots, B_n)$ be contractive n-tuples on finite dimensional spaces $\mathcal{V}_A$ and $\mathcal{V}_B$ respectively. Let $S = (S_1, \ldots, S_n)$ and $T = (T_1, \ldots, T_n)$ be their joint minimal isometric dilations on Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$; and let σ_A and σ_B be the induced representations of $\mathcal{E}_n$. Let $\widetilde{\mathcal{V}}_A$ be the subspace spanned by all minimal $\mathfrak{A}^*$-invariant subspaces $\mathcal{W}$ on which $\sum_{i=1}^{n} A_i A_i^*|_{\mathcal{W}} = I_{\mathcal{W}}$; and similarly define $\widetilde{\mathcal{V}}_B$. Then σ_A and σ_B are unitarily equivalent if and only if*

1. $\text{rank}(I_{\mathcal{V}_A} - \sum_{i=1}^{n} A_i A_i^*) = \text{rank}(I_{\mathcal{V}_B} - \sum_{i=1}^{n} B_i B_i^*)$; *and*

2. $A^*|_{\widetilde{\mathcal{V}}_A}$ *is unitarily equivalent to* $B^*|_{\widetilde{\mathcal{V}}_B}$.

Proof. The two representations are equivalent if and only if they have the same pure rank and the Cuntz parts are unitarily equivalent. By Theorem 2.6.5, the algebra $\mathfrak{S}$ contains the projection onto $\widetilde{\mathcal{V}}_A$. It is the unique maximal finite rank projection in $\mathfrak{S}$. Therefore the restriction $A^*|_{\widetilde{\mathcal{V}}_A}$ is a unitary invariant. Conversely, if these two conditions hold, then the unitary identifying $A^*|_{\widetilde{\mathcal{V}}_A}$ and $B^*|_{\widetilde{\mathcal{V}}_B}$ extends to a unitary equivalence between the dilations S_A of $\widetilde{A} := P_{\widetilde{\mathcal{V}}_A} A|_{\widetilde{\mathcal{V}}_A}$ and S_B of $\widetilde{B} := P_{\widetilde{\mathcal{V}}_B} B|_{\widetilde{\mathcal{V}}_B}$ because of the uniqueness of the minimal isometric dilation. This identifies the restriction of S_A to $\mathfrak{S}[\widetilde{\mathcal{V}}_A] = \mathcal{H}_{Ac}$, namely the Cuntz part of S_A, with the corresponding Cuntz part of S_B. The pure rank condition allows a unitary equivalence between the two pure parts. ∎

Bratteli and Jorgensen [9] give a detailed analysis of representations of the Cuntz algebra which has a lot in common with these results. They look somewhat different since they concentrate on the state and not on the restriction to the subspace $\mathcal{V}$. In particular, their contractions are not the same as those here. They point out the relationship in the discussion preceding their Theorem 5.3. They obtain Corollary 2.6.6 in the Cuntz case, and in particular recognize the role of the completely positive map Φ. Again however, their different normalization results in a different map. But they do not appear to classify these representations up to unitary equivalence. The reason they do not succeed is that they did not identify the subspace which here is denoted $\widetilde{\mathcal{V}}$, and instead work with a subspace they call $\mathcal{V}_k$ which is often strictly larger. The space $\widetilde{\mathcal{V}}$ does not occur in their hierarchy of invariant subspaces. Instead, they specialize in section $\bar{7}$ to a smaller class which they call diagonalizable shifts. These they do completely classify up to unitary

equivalence. It is not determined in this case how their special invariants relate to those here.

Corollary 2.6.9. *The algebra $\mathfrak{S}$ determined by the joint isometric dilation of a contractive n-tuple on a finite dimensional space is hyper-reflexive with distance constant at most 5.*

Proof. This follows immediately from [**17**, Theorem 3.14] since the algebra $\mathfrak{S}$ is unitarily equivalent to the algebra of certain atomic representations. Indeed, the projection $P = P_{\tilde{\mathcal{V}}}$ belongs to $\mathfrak{S}$ and $\mathfrak{S}P = \mathfrak{W}P$ where $\mathfrak{W}$ is a type I von Neumann algebra containing the projection P. Thus by Christensen's result [**13**] which shows that type I von Neumann algebras have distance constant at most 4, the same is obtained here. The upper bound for the distance constant of $\mathfrak{L}_n$ was improved by Bercovici [**6**] to 3 from the original 51. Arguing as in [**17**], one obtains a distance constant no larger than $(3^2 + 4^2)^{1/2} = 5$. ■

Chapter 3

Factoring in non-commutative analytic Toeplitz algebras

In [17] and [18], the algebraic and invariant subspace structures of the non-commutative analytic Toeplitz algebras were developed extensively. Several analogues of the analytic Toeplitz algebra were obtained. Many of these results came from a lucid characterization of the WOT-closed right ideals of these algebras. Although technical difficulties were encountered, a similar characterization of the left ideals was expected. In this chapter, it is shown that, although it holds for a subcollection, the analogous characterization of the WOT-closed left ideals fails. The reason for this failure is a deep factorization problem in these algebras. Reasonable factorization results can be obtained when norm conditions are placed on possible factors. Indeed, positive results concerning isometries in the unit ball are included. However, in the general setting it turns out that even seemingly obvious unique factorizations do not hold. The examples provided help illustrate the pathologies

of factorization involved. Many of these examples rely on an understanding of the structure theory of contractions in these algebras. The minimal isometric dilation of these contractions is determined. Further, each is shown to have an H^∞ functional calculus.

3.1 Preliminaries

The terminology and notation used in this chapter is consistent with Chapter 2. The left regular representation of the unital free semigroup $\mathcal{F}_n$ on n generators acts naturally on n-variable Fock space $\mathcal{H}_n = \ell_2(\mathcal{F}_n)$ by $\lambda(w)\xi_v = \xi_{wv}$, for v, w in $\mathcal{F}_n$. The non-commutative analytic Toeplitz algebra $\mathfrak{L}_n$ is the unital, WOT-closed algebra generated by the isometries $L_i = \lambda(i)$ for $1 \leq i \leq n$. The isometry determined by a word w is denoted by $L_w = \lambda(w)$. The case $n = 1$ is the analytic Toeplitz algebra. The algebra corresponding to the right regular representation is denoted by $\mathfrak{R}_n$. The generating isometries are defined by $\rho(w) = R_{w'}$ where $R_u\xi_v = \xi_{vu}$, and w' denotes the word w in reverse order. It is unitarily equivalent to $\mathfrak{L}_n$ and is precisely the commutant of $\mathfrak{L}_n$. See **[1, 17, 18, 19, 27, 34, 32]** for more detailed information on these algebras.

In the first section it is proved that the minimal isometric dilation of each non-unitary contraction in $\mathfrak{L}_n$ is a shift. This is accomplished by showing that the powers of the adjoint of such a contraction converge strongly to zero. All of these shifts have infinite multiplicity. Further, each of these contractions has an H^∞ functional calculus.

The second section contains positive factorization results for isometries over the

unit ball of $\mathfrak{L}_n$. The isometries L_w factor exactly as the words w. Hence, such isometries are irreducible over the unit ball exactly when the length of the word $|w| = 1$. A characterization of a broader class of irreducible isometries over the unit ball is also included. Surprisingly, these isometries are reducible over the full algebra. Particular factors are constructed using the structure of the orthogonal complement of their ranges.

The last section contains a discussion of the WOT-closed ideals of these algebras. Unlike right ideals, the left ideal generated by the isometry L_2 is not WOT-closed. The natural identification of subspaces in $\operatorname{Lat}\mathfrak{R}_n$ with the WOT-closed right and two-sided ideals is shown to only transfer over to a proper subcollection of $\operatorname{Lat}\mathfrak{L}_n$, when considering left ideals.

3.2 Contractions

The inner-outer factorization in $\mathfrak{L}_n$ (see [1], [17] and [32]) shows that much of the game in these algebras is played in the world of isometries (inner operators). Thus, it is worthwhile to examine the structure theory of these operators in the more general setting of contractions.

For notational purposes recall from [18] that given $k \geq 1$, every X in $\mathfrak{L}_n$ can be uniquely written as a sum

$$X = \sum_{|w|<k} x_w L_w + \sum_{|w|=k} L_w X_w, \tag{3.1}$$

where $x_w \in \mathbb{C}$ and $X_w \in \mathfrak{L}_n$ (for $n = \infty$, the first sum belongs to ℓ_2 and the

second sum is actually a WOT-limit). The scalars $\{x_w\}_{w\in\mathcal{F}_n}$ are called the Fourier coefficients of X. This notation is justified since they determine the operator. Indeed, if e is the identity in $\mathcal{F}_n$, then $X\xi_e = \sum_w x_w\xi_w$ and hence

$$X\xi_v = X(R_v\xi_e) = R_v(X\xi_e) = \sum_w x_w\xi_{wv}.$$

One writes $X \sim \sum_w x_w L_w$. The starting point is a simple result which is used several times throughout the chapter. In **[18]** it was shown that the only normal operators belonging to $\mathfrak{L}_n$ are the scalars. So the only unitaries in $\mathfrak{L}_n$ are scalar. The proof of the latter is actually quite elementary.

Proposition 3.2.1. *The collection of unitary operators in $\mathfrak{L}_n$ is the set $\mathbb{T}I$.*

Proof. Let U be unitary in $\mathfrak{L}_n$. Then there is a unit vector η in $\mathcal{H}_n$ with $U\eta = \xi_e$. The scalar $\lambda = (\eta, \xi_e)$ satisfies $|\lambda| \leq 1$. Consider the Fourier expansion, $U \sim \sum_w a_w L_w$. Evidently,

$$1 = \|U\xi_e\|^2 = \sum_w |a_w|^2 \geq |a_e|^2.$$

However, it is also true that

$$1 = (U\eta, \xi_e) = \sum_{w\in\mathcal{F}_n} a_w(L_w\eta, \xi_e) = a_e\lambda.$$

Whence, $|a_e| = |\lambda| = 1$. Thus $\eta = \lambda\xi_e$, so that

$$\overline{\lambda}\xi_e = \overline{\lambda}U\eta = U\xi_e.$$

Therefore $U = \overline{\lambda} I$, and the proof is finished. ■

In the seminal text [40], it is shown that understanding the behaviour of the powers of the adjoint of a contraction is a key issue. In particular, strong convergence to zero yields information on the minimal isometric dilation of the operator. This condition holds for every non-unitary contraction in $\mathfrak{L}_n$.

Lemma 3.2.2. *If L is a non-unitary contraction in $\mathfrak{L}_n$, then*

$$\lim_{k\to\infty} \|(L^*)^k \xi\| = 0,$$

for all ξ in $\mathcal{H}_n$.

Proof. The key is the unique decomposition of L. Using 3.1, write

$$L = \lambda I + \sum_{i=1}^{n} L_i A_i,$$

with each A_i in $\mathfrak{L}_n$. Then by the previous proposition, $|\lambda| < 1$ since L is not unitary. Hence if $A = \sum_{i=1}^{n} L_i A_i$, then $\|A\| < 2$.

The lemma will be proved for basis vectors corresponding to words. Suppose that w is a word of length l. For $k > l$,

$$\begin{aligned}
(L^*)^k \xi_w &= (\overline{\lambda} I + A^*)^k \xi_w \\
&= \left[\overline{\lambda}^k I + \begin{pmatrix} k \\ 1 \end{pmatrix} \overline{\lambda}^{k-1} A^* + \ldots + \begin{pmatrix} k \\ l \end{pmatrix} \overline{\lambda}^{k-l} (A^*)^l\right] \xi_w + 0 \\
&= \sum_{j=0}^{l} p_j(k)\, \overline{\lambda}^{k-j} (A^*)^j \xi_w,
\end{aligned}$$

where $p_j(k) = \frac{1}{j!}k(k-1)\cdots(k-j+1)$. However, $\lim_{k\to\infty}\frac{k^m}{\alpha^k} = 0$ for any real number m and $\alpha > 1$ (see [38] p.57). It follows that for $0 \leq j \leq l$,

$$\lim_{k\to\infty} p_j(k)\,|\lambda|^{k-j} = 0.$$

Now given $\varepsilon > 0$, choose $K > l$ such that $k \geq K$ implies $p_j(k)\,|\lambda|^{k-j} < \varepsilon$ for $0 \leq j \leq l$. Then for all sufficiently large k one has

$$\begin{aligned} \|(L^*)^k\xi_w\| &\leq \sum_{j=0}^{l} p_j(k)\,|\lambda|^{k-j}\|(A^*)^j\xi_w\| \\ &< \sum_{j=0}^{l} \varepsilon\, 2^j = (2^{l+1}-1)\,\varepsilon. \end{aligned}$$

Hence, $\lim_{k\to\infty}\|(L^*)^k\xi_w\| = 0$ for all words w.

That it is true in full generality follows from the boundedness of the sequence $\{(L^*)^k\}$. Indeed, any uniformly bounded sequence of operators $\{A_k\}$ on $\mathcal{H}_n$ which satisfies $\lim_{k\to\infty}\|A_k\xi_w\| = 0$ for all words w, must converge in the strong operator topology to zero. ■

The minimal isometric dilation of these operators can now be determined.

Theorem 3.2.3. *The minimal isometric dilation of any non-unitary contraction in $\mathfrak{L}_n$ is a unilateral shift.*

Proof. In [40] it is shown that every contraction has a minimal isometric dilation. By the Wold decomposition for isometries, every isometry is the orthogonal direct sum of a unitary and copies of the unilateral shift. It is further shown in [40] that the powers of the adjoint of the contraction converging strongly to zero (in other

words, belonging to the class $C_{\cdot 0}$) is equivalent to the unitary part of its minimal isometric dilation being vacuous. Hence the lemma yields the result. ■

Remark 3.2.4. This appears to be new for $n = 1$. The author could find no references, but it is probably known in this case.

The multiplicity of these shifts is always infinite. This is shown below. First, some consequences of the theorem will be pointed out. Recall that a contraction is called *completely non-unitary* provided its restriction to any non-zero reducing subspace is never unitary.

Corollary 3.2.5. *Every non-unitary contraction L in $\mathfrak{L}_n$ is completely non-unitary.*

Proof. Any non-zero reducing subspace for which the restriction of L to it is unitary would be contained in the unitary summand of the Wold decomposition for the minimal isometric dilation of L. However, by the theorem this space is vacuous. ■

Another important result which comes out of the Sz.-Nagy and Foiaş machinery is that every completely non-unitary contraction possesses an H^∞ functional calculus.

Corollary 3.2.6. *Every non-unitary contraction in $\mathfrak{L}_n$ has an H^∞ functional calculus.*

Given a non-unitary contraction L in $\mathfrak{L}_n$, the collection of operators defined by this H^∞ functional calculus is denoted $H^\infty(L)$. See [**40**] for properties of this functional calculus.

The cardinality of the shift in the Wold decomposition of an isometry V in $\mathcal{B}(\mathcal{H})$ is given by the dimension of the wandering space $\mathcal{H} \ominus V\mathcal{H}$. For isometries in the analytic Toeplitz algebra this cardinality can be both finite and infinite. In fact if φ belongs to H^∞, the dimension of $H^2 \ominus \varphi H^2$ is finite exactly when the analytic inner symbol is continuous. That is, when φ belongs to the disk algebra (see [**20**]). For the non-commutative algebras there is in general more room in the orthogonal complement, and this cardinality turns out to always be infinite. To prove this, first note the following result, the proof of which is actually contained in the proof of Theorem 1.7 from [**17**].

Theorem 3.2.7. *For $n \geq 2$, if A in $\mathfrak{L}_n$ has proper closed range, then $\overline{\mathrm{Ran}(A)}$ has infinite codimension.*

Remark 3.2.8. For isometries L in $\mathfrak{L}_n$, the infinite cardinality of $\mathcal{H}_n \ominus L\mathcal{H}_n$ is actually easy to see when $(L\xi_e, \xi_e) = 0$. Indeed, let P_k be the projection of $\mathcal{H}_n$ onto $\mathrm{span}\{\xi_w : |w| = k\}$. Then,

$$P_k\mathcal{H}_n \supseteq P_k(\mathrm{Ran}(L)) = P_k\left(\sum_{i=0}^{k-1} L(P_i\mathcal{H}_n)\right).$$

The former space has dimension n^k, the latter has dimension at most $\frac{n^k-1}{n-1}$. Summing over $k \geq 1$ proves the claim.

In any event, it follows that the range of any non-outer operator has infinite codimension. Recall that A in $\mathfrak{L}_n$ is **inner** if it is an isometry and **outer** if $\mathrm{Ran}(A)$ is dense in $\mathcal{H}_n$.

Corollary 3.2.9. *For $n \geq 2$, if A in $\mathfrak{L}_n$ is not outer, then $\overline{\mathrm{Ran}(A)}$ has infinite codimension.*

Proof. Since A is not outer, by the inner-outer factorization it can be written as $A = LB$ for some non-scalar isometry L and outer operator B, both in $\mathfrak{L}_n$. But then,

$$\overline{A\,\mathcal{H}_n} = \overline{LB\,\mathcal{H}_n} = L(\overline{B\,\mathcal{H}_n}) = L\,\mathcal{H}_n.$$

The latter space has infinite codimension by the theorem. ■

The general result can now be proved. The proof makes use of Fredholm theory in $\mathfrak{L}_n$.

Theorem 3.2.10. *Let L be a non-unitary contraction in $\mathfrak{L}_n$, for $n \geq 2$. If V in $\mathcal{B}(\mathcal{K})$ is its minimal isometric dilation, then* $\dim(\mathcal{K} \ominus V\mathcal{K}) = \infty$.

Proof. This multiplicity is given by the rank of the projection $I - VV^*$. From the construction of the minimal isometric dilation, $I - LL^*$ is the compression of $I - VV^*$ to $\mathcal{H}_n$. Hence, $I - VV^*$ has infinite rank when $I - LL^*$ does. Suppose this number is finite. Then L^* has an essential left inverse, and hence $\ker L^* = (\overline{\mathrm{Ran}\, L})^\perp$ is finite dimensional. Thus by the previous corollary, L must in fact be outer.

It now follows that the operators L^*L and LL^* are unitarily equivalent. For, the partial isometry in the polar decomposition of L is really invertible and acts as the intertwining unitary. Therefore,

$$\mathrm{rank}(I - L^*L) = \mathrm{rank}(I - LL^*) < \infty,$$

so that L is an essential unitary.

As a Fredholm operator, L has closed range and is thus surjective since it is outer. From [17], every operator in $\mathfrak{L}_n$ is injective. Hence L is invertible. However, it was also shown in [17] that the essential norm of every operator in $\mathfrak{L}_n$ is the same as its original norm. Since $\mathfrak{L}_n$ is inverse closed [17], this implies that

$$\|L^{-1}\| = \|L^{-1}\|_e = \|L^*\|_e = 1.$$

As an invertible isometry in $\mathfrak{L}_n$, The contraction L must be scalar. This contradiction completes the proof. ■

The investigation of the structure of $\mathcal{H}_n \ominus L\mathcal{H}_n$ will be revisited next section in the context of factoring. In many ways contractions satisfying $(L\xi_e, \xi_e) = 0$ are easier to deal with. It is thus helpful to finish off this section by observing that there is a large class of contractions in $\mathfrak{L}_n$ for which this inner product is non-zero, however these operators are unitarily equivalent to contractions in $\mathfrak{L}_n$ which have no scalar part. Observe that for any operator L in $\mathfrak{L}_n$, the space $\mathcal{H}_n \ominus \overline{L\mathcal{H}_n}$ belongs to $\operatorname{Lat}\mathfrak{R}_n^*$.

Theorem 3.2.11. *Suppose L is a contraction in $\mathfrak{L}_n$ for which $\mathcal{H}_n \ominus \overline{L\mathcal{H}_n}$ contains an eigenvector of $\mathfrak{R}_n^*$. Then L is unitarily equivalent to a contraction in $\mathfrak{L}_n$ which has no scalar part. In addition, this unitary implements an automorphism of $\mathfrak{L}_n$.*

Proof. In [17] the eigenvectors of $\mathfrak{R}_n^*$ are identified. Each scalar λ in the unit ball of n-dimensional Hilbert space defines an eigenvector v_λ. Further, for each such vector there is a unitary U_λ on $\mathcal{H}_n$ for which $\operatorname{Ad} U_\lambda$ determines an automorphism

of $\mathfrak{L}_n$ with $U_\lambda v_\lambda = \xi_e$.

Thus, suppose some v_λ belongs to $\mathcal{H}_n \ominus \overline{L\mathcal{H}_n}$. Then the operator $U_\lambda L U_\lambda^*$ is a contraction in $\mathfrak{L}_n$ and

$$(U_\lambda L U_\lambda^* \xi_e, \xi_e) = (Lv_\lambda, v_\lambda) = 0,$$

which proves the result. ■

3.3 Factoring

In the analytic Toeplitz algebra $\mathfrak{L}_1 = \mathfrak{R}_1 = \mathcal{T}(H^\infty)$, the associated function theory yields a good factorization theory over the full algebra (see **[20]** and **[26]** for example). When moving to several non-commutative variables, the strong link to the function theory is lost and factorization becomes much more difficult to deal with. Nonetheless, positive results such as inner-outer factorization can be obtained. Other factorization results can be obtained when norm restrictions are placed on possible factors.

Theorem 3.3.1. *Let $w \in \mathcal{F}_n$. Over the unit ball of $\mathfrak{L}_n$, the isometry L_w factors only in the same way as the word w, modulo scalars in $\mathbb{T}$.*

Proof. Suppose $L_w = BC$ with B and C belonging to $b_1(\mathfrak{L}_n)$. It is clear that C must in fact be an isometry. For each $k \geq 1$, consider the corresponding form of (3.1) for B and C.

Let $\{b_v\}$ be the scalars and let $\{B_v\}$ be the operators for B in this decomposition.

Use similar notation for C. If $w = e$, then comparing coefficients yields $I = b_e c_e I$. Whence, $|b_e| = |c_e| = 1$ and the operators B and C are scalar unitaries.

Otherwise, put $w = i_1 \cdots i_k$ and note that $b_e c_e = 0$. First suppose $b_e = 0$. Then $B = \sum_{i=1}^{n} L_i B_i$ and equating factorizations yields,

$$L_{i_1}(B_{i_1} C) = L_{i_1}(L_{i_2} \cdots L_{i_k}) \quad \text{and} \quad L_j(B_j C) = 0 \quad \text{for} \quad j \neq i_1.$$

In particular, $B_j C = 0$ for $j \neq i_1$. But every non-zero element of $\mathfrak{L}_n$ is injective [17], so that $B_j = 0$ for $j \neq i_1$. Further, B_{i_1} is a contraction and $B_{i_1} C = L_{i_2} \cdots L_{i_k}$. Hence, by induction one has $B_{i_1} = \lambda L_u$ and $C = \bar{\lambda} L_v$ where $uv = i_2 \cdots i_k$ and λ belongs to $\mathbb{T}$. Thus, $B = \lambda L_{i_1} L_u$ and $C = \bar{\lambda} L_v$.

Next suppose $b_e \neq 0$, so that $c_e = 0$. Note first that

$$\begin{aligned} (\xi_w, \xi_{i_1}) &= (BC\xi_e, \xi_{i_1}) \\ &= \sum_{v,u} b_u c_v (\xi_{uv}, \xi_{i_1}) \\ &= b_e c_{i_1} + b_{i_1} c_e \\ &= b_e c_{i_1} \end{aligned}$$

Hence if $w = i_1$, one would have $b_e c_{i_1} = 1$. As B and C are contractions it would follow that $|b_e| = |c_{i_1}| = 1$, and that B is a scalar unitary. Otherwise suppose $|w| > 1$. Inductively, one can show in this case that

$$0 = c_e = c_{i_k} = \ldots = c_{i_2 \cdots i_k}.$$

To see this, observe that as above $0 = (\xi_w, \xi_{i_k}) = b_e c_{i_k}$, and hence $c_{i_k} = 0$. Then suppose $0 = c_e = c_{i_k} = \ldots = c_{i_j \cdots i_k}$ for some $2 < j \leq k$. Equating Fourier coefficients of $L_w = B\,C$ shows that

$$\begin{aligned} 0 = (\xi_w, \xi_{i_{j-1} \cdots i_k}) &= (BC\xi_e, \xi_{i_{j-1} \cdots i_k}) \\ &= b_e c_{i_{j-1} \cdots i_k} + b_{i_{j-1}} c_{i_j \cdots i_k} + \ldots + b_{i_{j-1} \cdots i_k} c_e \\ &= b_e c_{i_{j-1} \cdots i_k}. \end{aligned}$$

Whence, $c_{i_{j-1} \cdots i_k} = 0$ as claimed. But then,

$$\begin{aligned} 1 = (\xi_w, \xi_w) &= (BC\xi_e, \xi_w) \\ &= b_e c_w + b_{i_1} c_{i_2 \cdots i_k} + \ldots + b_w c_e \\ &= b_e c_w. \end{aligned}$$

Thus, $|b_e| = |c_w| = 1$ and B is a scalar unitary. ■

It is immediate that the generating isometries are irreducible over the unit ball.

Corollary 3.3.2. *For $n \geq 2$, each L_i is irreducible over the unit ball of $\mathfrak{L}_n$.*

In fact, many more isometries are irreducible over the unit ball of $\mathfrak{L}_n$. Indeed, one can work harder to obtain the next result which includes a large collection of isometries. For example, the isometries $L = \frac{1}{\sqrt{2}}(L_1 + L_2)$ and $L = \frac{1}{\sqrt{2}} L_1 + \frac{1}{2} L_2^2 + \frac{1}{2} L_3^3$ are irreducible.

Theorem 3.3.3. *Suppose $L \sim \sum_{w \neq e} a_w L_w$ is an isometry in $\mathfrak{L}_n$ for which there is an i with $a_i \neq 0$ and $R_i R_i^*(L\xi_e) = a_i \xi_i$. Then L is irreducible over the unit ball of*

$\mathfrak{L}_n$.

Proof. Suppose $L = BC$ with B and C in $b_1(\mathfrak{L}_n)$. The operator C must be an isometry. As in the proof of the previous theorem, consider the expansions of B and C determined by 3.1. Recall that

$$B = b_e I + \sum_{j=1}^{n} L_j B_j \quad \text{and} \quad C = c_e I + \sum_{j=1}^{n} L_j C_j.$$

By equating unique factorizations of $L = BC$, one obtains $b_e c_e = 0$. By hypothesis.

$$L_i (a_i I) = L_i (b_e C_i + B_i C).$$

First suppose $b_e = 0$. Then, $B_i C = a_i I \neq 0$. Thus by the injectivity of all elements in $\mathfrak{L}_n$, the operator B_i must be invertible. Hence C is also invertible. Therefore, as an invertible isometry in $\mathfrak{L}_n$, C is a scalar unitary.

Next suppose $b_e \neq 0$ and so $c_e = 0$. This corresponds to the case when B is a scalar unitary. Indeed, note first from the Fourier expansions.

$$a_i = (L\xi_e, \xi_i) = (BC\xi_e, \xi_i) = b_e c_i + b_i c_e = b_e c_i.$$

As B is a contraction, $|c_i| \geq |a_i|$. By hypotheses one has

$$\begin{aligned} (L - a_i L_i)\,\xi_e &= \sum_{j=1}^{n} R_j R_j^{*} (L - a_i L_i)\,\xi_e \\ &= \sum_{j \neq i} R_j R_j^{*} (L\xi_e). \end{aligned}$$

Further, since $L = BC$,

$$\begin{aligned} B(C - c_i L_i)\,\xi_e &= \sum_{j=1}^{n} R_j R_j^*(L - c_i B L_i)\,\xi_e \\ &= \sum_{j \neq i} R_j R_j^*(L\,\xi_e) + R_i R_i^*(L - c_i B L_i)\,\xi_e. \end{aligned}$$

Evidently, $\|B\,(C - c_i L_i)\,\xi_e\| \geq \|(L - a_i L_i)\,\xi_e\|$. Thus the following is true:

$$\begin{aligned} 1 = \|C\xi_e\|^2 &= \|(C - c_i L_i)\,\xi_e\|^2 + |c_i|^2 \\ &\geq \|B(C - c_i L_i)\,\xi_e\|^2 + |c_i|^2 \\ &\geq \|(L - a_i L_i)\,\xi_e\|^2 + |a_i|^2 \\ &= \|L\xi_e\|^2 = 1. \end{aligned}$$

Therefore, $|a_i| = |c_i|$ and $|b_e| = 1$, which shows that B is a scalar unitary. ■

Remark 3.3.4. This proof can be perturbed to include broader classes of isometries. For example, any isometry L which satisfies,

$$R_i R_i^*(L\,\xi_e) = a_{wi}\xi_{wi},$$

for some i and word w is irreducible over the unit ball of $\mathfrak{L}_n$.

As it turns out, the unique factorizations over the unit ball of $\mathfrak{L}_n$ discussed above do not hold over the full algebra. Remarkably, even the operator L_2 has proper factorizations in $\mathfrak{L}_n$. This comes out of an interesting result from function theory.

Lemma 3.3.5. *The function $f(z)$ defined on the unit disk by $f(z) = \sum_{k\geq 0} \frac{z^k}{k+1}$ belongs to $H^2 \setminus H^\infty$. However, $1/f$ defines a function which lies in H^∞.*

Proof. Since f is analytic on the unit disk and the Fourier coefficients of f are ℓ_2-summable, the function belongs to H^2 with

$$\|f\|_2 = \left(\sum_{k\geq 0} \frac{1}{(k+1)^2}\right)^{\frac{1}{2}}.$$

For $|z| < 1$, $f(z)$ is defined by the formula

$$z\, f(z) = -\log(1-z) \tag{3.2}$$

using the principal branch of the logarithm. Given r such that $0 < r < 1$, let f_r be the function on $\mathbb{T}$ defined by $f_r(e^{i\theta}) = f(r\, e^{i\theta})$. Then the identity

$$\lim_{r\to 1^-} f(r) = \lim_{r\to 1^-} \frac{-\log(1-r)}{r} = \infty,$$

together with the continuity of f on the disk, shows that $\|f\|_{H^\infty} := \lim_{r\to 1} \|f_r\|_\infty = \infty$. Hence the function f is not in H^∞.

To prove that $1/f$ defines a function in H^∞, it is sufficient to show that $1/f$ defines an analytic function on the unit disk with $\|1/f\|_\infty < \infty$. Now, $1/f$ is analytic on the unit disk by 3.2. To see that f is bounded below first observe the

identity $1 - e^{i\theta} = (2\sin\frac{\theta}{2})\, e^{\frac{\theta-\pi}{2}i}$. Hence for $|\theta| \leq \pi$ with $\theta \neq 0$,

$$\begin{aligned} |f(e^{i\theta})|^2 &= \left|\frac{-\log(1-e^{i\theta})}{e^{i\theta}}\right|^2 \\ &= \left|\frac{\log|2\sin\frac{\theta}{2}| + i\,\left(\frac{\theta-\pi}{2}\right)}{e^{i\theta}}\right|^2 \\ &= \left(\log|2\sin\frac{\theta}{2}|\right)^2 + \left(\frac{\theta-\pi}{2}\right)^2. \end{aligned}$$

But $\left(\frac{\theta-\pi}{2}\right)^2 \geq \frac{\pi^2}{16}$ for $-\pi \leq \theta \leq \frac{\pi}{2}$ and $|2\sin\frac{\theta}{2}| \geq \sqrt{2}$ for $\frac{\pi}{2} \leq \theta \leq \pi$. Thus,

$$|f(e^{i\theta})|^2 \geq \min\left\{\frac{(\log 2)^2}{4}, \frac{\pi^2}{16}\right\} = \frac{(\log 2)^2}{4}$$

for $\theta \neq 0$. It now follows that $\|1/f\|_\infty < \infty$. ■

This function theoretic result allows one to construct explicit factorizations which are exclusive to the non-commutative setting.

Theorem 3.3.6. *Suppose L is an isometry in $\mathfrak{L}_n$ for which $\mathcal{H}_n \ominus L\,\mathcal{H}_n$ contains the range of an isometry X in $\mathfrak{L}_n$. Then L has proper factorizations in $\mathfrak{L}_n$.*

Proof. As the ranges of the isometries $X^k L$ are pairwise orthogonal for $k \geq 0$, the operator $A = \sum_{k\geq 0} \frac{1}{k+1} X^k L$ belongs to $\mathfrak{L}_n$ with

$$\|A\| = \left(\sum_{k\geq 0} \frac{1}{(k+1)^2}\right)^{\frac{1}{2}}.$$

Let $g = 1/f$ be the H^∞ function obtained in the previous lemma. By the H^∞ functional calculus for X (Corollary 3.2.6), $g(X)$ defines an operator in $\mathfrak{L}_n$. The

claim is that $g(X)A = L$. This comes as a result of a more general fact.

Note that given h in H^2, an operator $h(X)L$ can be naturally defined in $\mathfrak{L}_n$. Indeed, one can set

$$h(X)L = \sum_{k\geq 0} \hat{h}(k) X^k L.$$

where the $\hat{h}(k)$ are the Fourier coefficients for h. Clearly, the map from H^2 to $\mathcal{B}(\mathcal{H}_n)$ which sends h to $h(X)L$ is isometric. The key is that this map is also continuous from the topology of weak vector convergence in H^2 to the weak operator topology in $\mathfrak{L}_n$. To see this, suppose h_m converges weakly to h in H^2. Without loss of generality assume $h = 0$. Then $\hat{h}_m(k)$ converges to 0 for each k and

$$\sup_m \|h_m\|_2 = c < \infty,$$

for some constant c. Let x and y be unit vectors in $\mathcal{H}_n$ and let S_k be the orthogonal projection onto $\mathrm{Ran}(X^k L)$. Then

$$\begin{aligned} \sum_{k\geq 0} |(X^k Lx, y)|^2 &= \sum_{k\geq 0} |(X^k Lx, S_k y)|^2 \\ &\leq \sum_{k\geq 0} \|S_k y\|^2 \\ &\leq \|y\|^2 = 1. \end{aligned}$$

Thus, given $\varepsilon > 0$ one can choose $N = N(\varepsilon)$ for which the Nth ℓ_2 tail of the above

series is smaller than ε. Then for each m the Cauchy-Schwarz inequality shows that

$$\begin{aligned} |(h_m(X)\,Lx, y)| &= \left|\sum_{k\geq 0} \hat{h}_m(k)\,(X^k Lx, y)\right| \\ &\leq \sum_{0\leq k\leq N} |\hat{h}_m(k)| + \left(\sum_{k>N} |\hat{h}_m(k)|^2\right)^{\frac{1}{2}} \varepsilon \\ &\leq \sum_{0\leq k\leq N} |\hat{h}_m(k)| + c\varepsilon. \end{aligned}$$

As $\hat{h}_m(k)$ converges to 0 for each k, it follows that the operators $h_m(X)L$ converge weakly to zero.

Recall that the analytic trigonometric polynomials are weak* dense in H^∞ [**20**]. Let g_m be such a sequence converging weak* to g. From the definition of this weak* topology, the sequence g_m converges boundedly to g in H^∞. Thus, the sequence $g_m f$ converges boundedly to $gf = 1$ in H^2. Since each g_m is a polynomial, by the functional calculus for X one has

$$g_m(X)\,A = g_m(X)\,(f(X)L) = (g_m f)(X)\,L \xrightarrow{\text{WOT}} (g\,f)(X)\,L = L.$$

On the other hand, again by the H^∞ functional calculus for X, $g_m(X)$ converges in the weak operator topology to $g(X)$ [**40**]. Therefore, WOT-$\lim_m g_m(X)\,A = g(X)\,A$. Whence, $g(X)\,A = L$.

It remains to observe that $g(X)$ and A are both not invertible. The invertibility of $g(X)$ in $\mathfrak{L}_n$ would imply the invertibility of g in H^∞, contradicting the previous lemma. If A was invertible, it would be the scalar multiple of an invertible isometry in $\mathfrak{L}_n$, hence scalar itself by Proposition 3.2.1. This completes the proof. ■

Remark 3.3.7. The theorem really is exclusive to the non-commutative setting. The hypothesis of the theorem cannot be satisfied when $n = 1$. For if φ and ψ are inner functions in H^∞, then the function $\varphi\psi = \psi\varphi$ belongs to φH^2 and ψH^2.

As a surprising consequence of the theorem the reducibility of the generating isometries is revealed in the non-commutative setting.

Corollary 3.3.8. *For $n \geq 2$, each L_i has proper factorizations in $\mathfrak{L}_n$.*

Proof. The isometry X can be taken to be L_j for $j \neq i$. ■

In fact, there is a large collection of isometries which can be seen to be reducible in this manner. Note that by unique factorization, every operator L in $\mathfrak{L}_n$ with $(L\,\xi_e, \xi_e) = 0$ can be written as

$$L = \sum_{|w|=k} L_w A_w,$$

for some $k \geq 1$ and A_w in $\mathfrak{L}_n$.

Corollary 3.3.9. *Let L be an isometry in $\mathfrak{L}_n$ with $L = \sum_{|w|=k} L_w A_w$, for some $k \geq 1$ and A_w in $\mathfrak{L}_n$. Suppose either one of the following conditions holds:*

(*i*) *There is an A_w for which there exists a $B \neq 0$ in $\mathfrak{L}_n$ with the range of B orthogonal to that of A_w,*

or

(*ii*) *Some A_w is a scalar multiple of an isometry.*

Then L properly factors in $\mathfrak{L}_n$.

Proof. Let ξ and η be vectors belonging to $\mathcal{H}_n$ throughout the proof. To prove (i), set $A = L_w B$. Write the inner-outer factorization for A as $A = L_A C$. Since C is outer, there are vectors ζ_m such that $\eta = \lim_{m\to\infty} C\zeta_m$. It follows that

$$\begin{aligned}(L\xi, L_A\eta) = \lim_{m\to\infty}(L\xi, L_A C\zeta_m) &= \lim_{m\to\infty}(L\xi, L_w B\zeta_m)\\ &= \lim_{m\to\infty}(L_w A_w\xi, L_w B\zeta_m)\\ &= \lim_{m\to\infty}(A_w\xi, B\zeta_m) = 0.\end{aligned}$$

Thus, L_A is an isometry with range orthogonal to the range of L.

Lastly, suppose A_w is a scalar multiple of an isometry. Write L as $L = L_w A_w + A$. Let $A = L_A B$ be the inner-outer factorization of A. Since

$$\overline{A\mathcal{H}_n} = \overline{L_A B\mathcal{H}_n} = L_A\mathcal{H}_n,$$

the ranges of L_w and L_A are orthogonal. Further, one has

$$A_w^* A_w + B^* B = I.$$

So B is also a scalar multiple of an isometry, and it is therefore scalar since it is outer. Suppose that $A_w^* A_w = a^2 I$ and $B = \beta I$. Let $c = a^2|\beta|^{-2}$, and let X be the operator $X = L_w A_w - c\beta L_A$. Then X is an operator with range orthogonal to the

range of L. Indeed,

$$\begin{aligned}(L\,\xi, X\,\eta) &= (L_w A_w\,\xi, L_w A_w\,\eta) - (A\,\xi, c\beta\,L_A\,\eta) \\ &= a^2(\xi,\eta) - c\,|\beta|^2(\xi,\eta) \\ &= 0.\end{aligned}$$

As in the proof of (i), using the inner part of X yields a desired isometry. ■

It is worthwhile to point out a striking special case of the first condition in the corollary.

Corollary 3.3.10. *Let L be an isometry in $\mathfrak{L}_n$ with $L = \sum_{|w|=k} L_w A_w$, for some $k \geq 1$ and A_w in $\mathfrak{L}_n$. If any $A_w = 0$, then L properly factors in $\mathfrak{L}_n$.*

Remark 3.3.11. Obviously there are many isometries which satisfy the first condition. Further, all of the isometries shown to be irreducible over the unit ball of the algebra in Theorem 3.3.3 and Remark 3.3.4 are reducible over the full algebra since they satisfy the second condition. Other isometries which satisfy the second condition include the collection of all operators which are the sum of pairwise orthogonal words. For, in this case every non-zero A_w would necessarily be a scalar multiple of an isometry.

There are also other more specialized classes of isometries which can be factored using this method. As an example, let f and g belong to H^∞ with $|f|^2 + |g|^2 = 1$ on $\mathbb{T}$. Such functions can be found by using the logmodularity of H^∞ [26]. Then

$$L = L_1\,f\,(L_1) + L_2\,g\,(L_1)$$

is an isometry in $\mathfrak{L}_2$ which apparently does not satisfy the conditions in the corollary. Let $\alpha = f(0)$ and $\beta = g(0)$ and choose λ in $\mathbb{T}$ such that $\lambda(\alpha\overline{\beta}) = \overline{\alpha}\beta$. Then L and the isometry

$$X = \frac{1}{|\alpha|^2 + |\beta|^2}\left(\beta L_1 L_2 - \lambda \alpha L_2^2\right)$$

have orthogonal ranges. Indeed, for ξ and η in $\mathcal{H}_n$ one has

$$\begin{aligned}(|\alpha|^2 + |\beta|^2)(L\xi, X\eta) &= (f(L_1)\xi, \beta L_2\eta) - (g(L_1)\xi, \lambda\alpha L_2\eta) \\ &= \alpha\overline{\beta}(\xi, L_2\eta) - \beta\overline{\alpha}\,\overline{\lambda}(\xi, L_2\eta) \\ &= 0.\end{aligned}$$

The reducibility of this large collection of isometries, together with the fact that the orthogonal complement of the range is always infinite dimensional (Theorem 3.2.7), leads one to believe that perhaps the theorem can be applied to every isometry L with $(L\xi_e, \xi_e) = 0$. However, this is not the case. The trouble is that difficulties arise when the space $\mathcal{H}_n \ominus L\mathcal{H}_n$ is too 'thin' at each level of the $\mathcal{H}_n$ tree. That is, the dimension of $P_k(\mathcal{H}_n \ominus L\mathcal{H}_n)$ remains small as k increases (recall Remark 3.2.8).

Theorem 3.3.12. *There are isometries L in $\mathfrak{L}_n$ with $(L\xi_e, \xi_e) = 0$ for which $\mathcal{H}_n \ominus L\mathcal{H}_n$ does not contain the range of an isometry in $\mathfrak{L}_n$.*

Proof. For $k \geq 0$, put

$$x_k = R_1^k R_2 \sum_{|w|=k} R_w \xi_w,$$

and let x be the unit vector

$$x = \sum_{k \geq 0} 2^{\frac{-k-1}{2}} \|x_k\|^{-1} x_k.$$

Suppose y is in $\mathcal{H}_n$ with $(R_u^* x, y) = 0$ for all words u in $\mathcal{F}_n$. Now given $k \geq 0$, choose a word u with $|u| = k$. Then

$$\begin{aligned} 0 &= (2^{\frac{k+1}{2}} \|x_k\| R_u^* R_2^* (R_1^*)^k x, y) \\ &= (R_u^* (\sum_{|w|=k} R_w \xi_w), y) \\ &= (\xi_u, y). \end{aligned}$$

Therefore, $y = 0$.

Next, write x as $x = L\eta$, where L is an isometry in $\mathfrak{L}_n$ and η is an $\mathfrak{R}_n$-cyclic vector (every vector in $\mathcal{H}_n$ can be written in this form [17]). The claim is that L is the desired isometry. As $(\eta, \xi_e) \neq 0$ and $(x, \xi_e) = 0$, one has $(L\xi_e, \xi_e) = 0$. Suppose X is an isometry in $\mathfrak{L}_n$ with range contained in $\mathcal{H}_n \ominus L\mathcal{H}_n$. Then the vectors $X\xi_u = X(R_u \xi_e) = R_u(X\xi_e)$ are orthogonal to $L\eta = x$ for every u in $\mathcal{F}_n$. In other words, $(X\xi_e, R_u^* x) = 0$ for every word u. Thus, by the above argument one would have $X\xi_e = 0$, whence $X = 0$. This contradiction completes the proof. ■

So this method cannot be applied to all isometries in $\mathfrak{L}_n$. Nevertheless, with the

large body of examples it is still reasonable to make the guess that every isometry L in $\mathfrak{L}_n$ with $(L\xi_e, \xi_e) = 0$ properly factors over the full algebra.

3.4 Ideals and Invariant Subspaces

The characterization of the WOT-closed right and two sided ideals ($\mathrm{Id}_r(\mathfrak{L}_n)$ and $\mathrm{Id}(\mathfrak{L}_n)$) in [17] and [18] is complete. The main theorem from [18] is stated as follows.

Theorem 3.4.1. *Let $\mu : \mathrm{Id}_r(\mathfrak{L}_n) \to \mathrm{Lat}(\mathfrak{R}_n)$ be given by $\mu(\mathcal{I}) = \overline{\mathcal{I}\xi_e}$. Then μ is a complete lattice isomorphism. The restriction of μ to the set $\mathrm{Id}(\mathfrak{L}_n)$ is a complete lattice isomorphism onto $\mathrm{Lat}(\mathfrak{L}_n) \cap \mathrm{Lat}(\mathfrak{R}_n)$. The inverse map i sends a subspace $\mathcal{M}$ to*

$$i(\mathcal{M}) = \{J \in \mathfrak{L}_n : J\xi_e \in \mathcal{M}\}.$$

The maps μ and i are still well defined when considering $\mathrm{Id}_l(\mathfrak{L}_n)$ and $\mathrm{Lat}(\mathfrak{L}_n)$. Although technical difficulties were encountered by the authors, a similar characterization was expected for left ideals. The key observation for right and two-sided ideals is that the subspace $\mu(\mathcal{I})$ is the full range of the ideal $\mathcal{I}$. Indeed, $\overline{\mathcal{I}\xi_e} = \overline{\mathcal{I}\mathfrak{L}_n\xi_e} = \overline{\mathcal{I}\mathcal{H}_n}$. This is not true for left ideals, and is why the methods of the authors cannot be applied in this setting.

Towards the identification of right ideals it is first proved that $\mu i = \mathrm{id}$. It is then shown that this leads to the conclusion, $\overline{\mathcal{I}\xi} = \overline{i\mu(\mathcal{I})\xi}$ for every vector ξ in $\mathcal{H}_n$. The proof that $i\mu = \mathrm{id}$ exploits this fact together with the following more

general result about ideals in $\mathfrak{L}_n$.

Proposition 3.4.2. *Let $\mathcal{I}_1$ and $\mathcal{I}_2$ both be* WOT*-closed right, left or two-sided ideals in $\mathfrak{L}_n$. If $\overline{\mathcal{I}_1\xi} = \overline{\mathcal{I}_2\xi}$ for all ξ in $\mathcal{H}_n$, then $\mathcal{I}_1 = \mathcal{I}_2$.*

Proof. In [17] it was shown that the weak* and weak operator topologies on $\mathfrak{L}_n$ coincide. Suppose that φ is a WOT-continuous functional on $\mathfrak{L}_n$ which annihilates $\mathcal{I}_1$. Then again from [17], there are vectors ξ and η in $\mathcal{H}_n$ such that

$$\varphi(J) = (J\xi, \eta)$$

for J in $\mathfrak{L}_n$. But then η is orthogonal to $\overline{\mathcal{I}_1\xi} = \overline{\mathcal{I}_2\xi}$, and hence φ annihilates $\mathcal{I}_2$ as well. Repeating the argument by exchanging the roles of $\mathcal{I}_1$ and $\mathcal{I}_2$ shows that the two ideals are identical. ■

Remark 3.4.3. Now, let $\mathcal{I}_1$ and $\mathcal{I}_2$ be WOT-closed left ideals of $\mathfrak{L}_n$. Notice that, $\overline{\mathcal{I}_1\xi_e} = \overline{\mathcal{I}_2\xi_e}$ implies $\overline{\mathcal{I}_1\xi} = \overline{\mathcal{I}_2\xi}$ when $\xi = R\xi_e$ for some isometry R in $\mathfrak{R}_n$. Indeed, one would have

$$\overline{\mathcal{I}_1\xi} = \overline{\mathcal{I}_1 R\xi_e} = R\overline{\mathcal{I}_1\xi_e} = R\overline{\mathcal{I}_2\xi_e} = \overline{\mathcal{I}_2 R\xi_e} = \overline{\mathcal{I}_2\xi}.$$

These vectors form a dense collection of vectors in $\mathcal{H}_n$. Whether this implies the same is true for all vectors in $\mathcal{H}_n$ is unclear. As Remark 3.4.13 points out, this requires an understanding of unbounded WOT-convergence. Thus, it becomes apparent that there are difficulties encountered when considering left ideals.

Upon further investigation concrete differences become evident. In particular,

the WOT-closed right ideal generated by a finite collection of isometries with pairwise orthogonal ranges is exactly the algebraic right ideal they generate. For left and two-sided ideals the corresponding result turns out to be false even for one isometry with norm closure.

Theorem 3.4.4. *The algebraic two-sided ideal in $\mathfrak{L}_n$ generated by L_2 is not norm closed.*

Proof. The operator

$$A = \sum_{k\geq 0} \frac{1}{k+1} L_1^k L_2$$

clearly belongs to the norm closure of the algebraic two-sided (in fact left) ideal generated by L_2. Suppose that A could be written as

$$A = \sum_{i=1}^{p} B_i L_2 C_i$$

with each B_i and C_i in $\mathfrak{L}_n$. Consider the Fourier expansions $B_i \sim \sum_w b_w^i L_w$ and $C_i \sim \sum_w c_w^i L_w$. Then the unique factorization in $\mathfrak{L}_n$ shows for each $k \geq 0$,

$$\begin{aligned} \frac{1}{k+1} = (A\xi_e, \xi_{1^k 2}) &= \sum_{i=1}^{p} (B_i L_2 C_i \xi_e, \xi_{1^k 2}) \\ &= \sum_{i=1}^{p} b_{1^k}^i \, c_e^i. \end{aligned}$$

However, by compressing the operators B_i to the subspace $\operatorname{span}\{\xi_{1^k} : k \geq 0\}$, one

sees that each of the operators

$$h_i(L_1) = \sum_{k \geq 0} b_{1k}^i L_1^k \quad \text{for} \quad 1 \leq i \leq p$$

must be in $H^\infty(L_1) \simeq H^\infty$. Hence the function

$$\sum_{k \geq 0} \frac{z^k}{k+1} = \sum_{i=1}^{p} c_e^i h_i$$

would belong to H^∞, a contradiction (see Lemma 3.3.5). Therefore A does not belong to the algebraic two-sided ideal generated by L_2. ■

As an immediate corollary of the proof, the corresponding fact about left ideals is proved.

Corollary 3.4.5. *The algebraic left ideal $\mathfrak{L}_n L_2$ is not norm closed.*

Proof. Consider the same operator A. Simply use the proof of the theorem with $p = 1$ and $C_1 = I$. ■

Remark 3.4.6. It seems reasonable to expect that the norm and weak operator topology closures of the preceding ideals are distinct. It also becomes apparent that proving this would be quite subtle. Indeed, it is difficult to construct a bounded operator belonging to the weak closure of $\mathfrak{L}_n L_2$ without being in the norm closure of $\mathfrak{L}_n L_2$.

Even with these differences, it is still surprising that the analogous identification of left ideals does not hold. It turns out that the subspaces $\mathcal{M}$ in $\mathrm{Lat}(\mathfrak{L}_n)$ for which $\mu i(\mathcal{M}) = \mathcal{M}$ do not fill out the entire subspace lattice.

Theorem 3.4.7. *There exists $\mathcal{M} \neq \{0\}$ in $\mathrm{Lat}(\mathfrak{L}_n)$ for which the associated left ideal $i(\mathcal{M})$ is trivial.*

Proof. Define an isometry R in $\mathfrak{R}_n$ by

$$R = \sum_{k \geq 0} \lambda_k R_1^k R_2,$$

where the scalars λ_k satisfy $\sum_{k \geq 0} |\lambda_k|^2 = 1$ but $\sum_{k \geq 0} \lambda_k z^k$ is not in H^∞. For example,

$$\lambda_k = \frac{c}{k+1} \quad \text{where} \quad c = \left(\sum_{k \geq 1} \frac{1}{k^2} \right)^{-\frac{1}{2}}.$$

Let $\mathcal{M}$ be the subspace in $\mathrm{Lat}(\mathfrak{L}_n)$ given by $\mathcal{M} = R\mathcal{H}_n$. Actually, every cyclic $\mathfrak{L}_n$-invariant subspace is of this form for some isometry in $\mathfrak{R}_n$ (see [1], [17], [32] and [34]). Recall that

$$i(\mathcal{M}) = \{J \in \mathfrak{L}_n : J\xi_e \in R\mathcal{H}_n\}.$$

Suppose there is a non-zero J in $\mathfrak{L}_n$ and ξ in $\mathcal{H}_n$ for which $J\xi_e = R\xi$. Put $\xi = \sum_w a_w \xi_w$ and let v be a word of minimal length such that $a_v \neq 0$. Since $R\xi = \sum_w a_w L_w (R\xi_e)$ one has

$$J \sim \sum_{|w| \geq |v|,\ k \geq 0} a_w \lambda_k L_w L_2 L_1^k.$$

Let Q be the projection onto the subspace $\text{span}\{\xi_{1^k} : k \geq 0\}$. Evidently,

$$QL_2^* L_v^* JQ = a_v \sum_{k \geq 0} \lambda_k L_1^k Q,$$

incorrectly implying that J would be unbounded. Thus, it follows that $i(\mathcal{M}) = 0$.

■

There is still a strong relation between $\text{Id}_l(\mathfrak{L}_n)$ and $\text{Lat}(\mathfrak{L}_n)$. Essentially, it is determined by those isometries in $\mathfrak{R}_n$ which do not have the qualities of those used in the proof of the theorem.

Definition 3.4.8. Let R be an isometry in $\mathfrak{R}_n$. Then

(*i*) R is called a **flip** if there is a non-zero vector ξ in $\mathcal{H}_n$ and an operator J in $\mathfrak{L}_n$ with $R\xi = J\xi_e$.

and

(*ii*) R is called a **cyclic flip** if there are operators J_α in $\mathfrak{L}_n$ such that $J_\alpha \xi_e \in R\mathcal{H}_n$ and $R\xi_e = \lim_\alpha J_\alpha \xi_e$.

The motivation for these definitions is when $R\xi_e = J\xi_e$, for some J in $\mathfrak{L}_n$. This means that the Fourier coefficients of R can be 'flipped' into an element of $\mathfrak{L}_n$.

Proposition 3.4.9. *Let R be an isometry in $\mathfrak{R}_n$ with $\mathcal{M} = R\mathcal{H}_n$. The following are equivalent:*

(*i*) *R is a flip,*

(*ii*) $i(\mathcal{M}) \neq 0$.

Proof. This is straight from the definitions of $i(\mathcal{M})$ and flip isometries. ■

Remark 3.4.10. It should be noted that when R is a flip, the subspace $\mu\, i(\mathcal{M})$ is 'large'. Indeed, suppose $J \neq 0$ belongs to $i(\mathcal{M})$. Then there is a vector ξ in $\mathcal{H}_n$ with $R\xi = J\xi_e$. Write ξ as $\xi = S\eta$, where S is an isometry in $\mathfrak{R}_n$ and η is an $\mathfrak{L}_n$-cyclic vector (this can be done for any vector in $\mathcal{H}_n$ [17]). Note that the set $\mathfrak{L}_n J$ is contained in $i(\mathcal{M})$. Thus,

$$RS\mathcal{H}_n = \overline{\mathfrak{L}_n RS\eta} = \overline{\mathfrak{L}_n R\xi} = \overline{\mathfrak{L}_n J\xi_e} \subseteq \mu\, i(\mathcal{M}),$$

which shows that $\mu\, i(\mathcal{M})$ contains the range of the isometry RS.

Proposition 3.4.11. *Let R be an isometry in $\mathfrak{R}_n$ with $\mathcal{M} = R\mathcal{H}_n$. The following are equivalent:*

(i) R is a cyclic flip,

(ii) $\mu\, i(\mathcal{M}) = \mathcal{M}$.

Proof. It is always true that $\mu\, i(\mathcal{M}) \subseteq \mathcal{M}$. Suppose that R is a cyclic flip, so that $R\xi_e = \lim_\alpha J_\alpha \xi_e$ with each $J_\alpha \xi_e \in \mathcal{M} \cap \mathfrak{L}_n \xi_e$. Then $R\xi_e$ lies in $\mu\, i(\mathcal{M})$, and hence

$$\mathcal{M} = R\mathcal{H}_n = \overline{\mathfrak{L}_n R\xi_e} \subseteq \mu\, i(\mathcal{M}).$$

If this latter inclusion holds, then $R\xi_e$ is such a limit since it belongs to $\mu\, i(\mathcal{M})$. ■

This gives a good one vector characterization of cyclic subspaces $\mathcal{M}$ in $\mathrm{Lat}(\mathfrak{L}_n)$ for which $\mu\, i(\mathcal{M}) = \mathcal{M}$. The following corollary shows that the above condition is

satisfied for a wealth of examples. For instance, consider the situation below even for $\xi = \xi_e$.

Corollary 3.4.12. *Let $\mathcal{M} = R\mathcal{H}_n$, where R is an isometry in $\mathfrak{R}_n$. If there is an $\mathfrak{L}_n$-cyclic vector ξ such that $R\xi = J\xi_e$ for some J in $\mathfrak{L}_n$, then $\mu\, i\,(\mathcal{M}) = \mathcal{M}$. Further, if $J = L_w$ for some word w, then $i\,(\mathcal{M})$ is exactly the* WOT*-closed left ideal generated by L_w.*

Proof. The condition in the previous proposition is satisfied since,

$$R\xi_e \in R\mathcal{H}_n = \overline{\mathfrak{L}_n R\xi} = \overline{\mathfrak{L}_n J\xi_e},$$

which shows that R is a cyclic flip.

In general, given A in $\mathfrak{L}_n$,

$$AJ\xi_e = AR\xi = RA\xi \in R\mathcal{H}_n = \mathcal{M}.$$

Thus, the WOT-closed left ideal generated by J is contained in $i\,(\mathcal{M})$. The other inclusion is true for the words $J = L_w$. Indeed, in this case it is easy to see that every A in $i\,(\mathcal{M})$ has a Fourier expansion of the form

$$A \sim \sum_{v \in \mathcal{F}_n} a_{vw} L_{vw}.$$

Actually, an analogous claim can be made for the right and two-sided WOT-closed

ideals generated by L_w. However, in [17] it was shown that the Cesaro sums for A,

$$\Sigma_k(A) = \sum_{|v|<k} \left(1 - \frac{|v|}{k}\right) a_v L_v \in \mathfrak{L}_n L_w$$

converge in the strong* topology to A. Hence $i(\mathcal{M})$ is contained in the WOT-closed left ideal generated by L_w. ■

It has been mentioned that every vector $J\xi_e$ with J in $\mathfrak{L}_n$ factors as $J\xi_e = R\xi$ for some isometry R in $\mathfrak{R}_n$ and $\mathfrak{L}_n$-cyclic vector ξ. As the corollary observes, one has that $i(R\mathcal{H}_n)$ always contains the WOT-closed left ideal generated by J. The other inclusion holds for words $J = L_w$. Proving the other inclusion holds in full generality would require an understanding of unbounded WOT-convergence in these algebras. This is discussed further below.

Remark 3.4.13. The corollary shows that the image in $\mathrm{Id}_l(\mathfrak{L}_n)$ contains the left ideals generated by the words L_w. It is not clear whether the image in $\mathrm{Id}_l(\mathfrak{L}_n)$ is surjective. Given a WOT-closed left ideal $\mathcal{I}$, it is always true that $\mathcal{I} \subseteq i\,\mu(\mathcal{I})$. In general, the other inclusion requires an understanding of unbounded WOT-convergence. For example, suppose $\mathcal{I}$ belongs to $\mathrm{Id}_l(\mathfrak{L}_n)$ with $\mu(\mathcal{I}) = \mathcal{H}_n$. It is not even known if one must have $\mathcal{I} = \mathfrak{L}_n$. For, one would like to say that the identity I belongs to $\mathcal{I}$, but all that can be said is $\xi_e = \lim_\alpha J_\alpha \xi_e$ for some J_α in $\mathcal{I}$. For bounded nets, WOT-convergence amounts to strong convergence on the vector ξ_e [18]. However, this is not true for unbounded nets. Indeed, as an example consider the sequence

J_m of operators in $\mathfrak{L}_n$ given by $J_m = \sum_{k=0}^{m} \frac{1}{k+1} L_1^m$. It is clear that

$$\lim_{m\to\infty} J_m \xi_e = \sum_{k\geq 0} \frac{1}{k+1} \xi_{1^k},$$

but the latter vector does not represent the Fourier coefficients of any operator in $\mathfrak{L}_n$.

Nonetheless, it has been shown that the maps μ and i define a bijective correspondence between the cyclic subspaces of $\mathrm{Lat}(\mathfrak{L}_n)$ determined by cyclic flips on the one hand, and the image under i in $\mathrm{Id}_l(\mathfrak{L}_n)$ of these subspaces on the other. Concerning the lattice properties of these maps, it is not hard to show that μ sends closed spans to WOT-closed sums and i sends intersections to intersections. However, the behaviour of μ on intersections and i on sums again comes back to requiring an understanding of unbounded WOT-convergence.

Chapter 4

The curvature invariant of a non-commuting n-tuple

Recently the notions of a curvature invariant and Euler characteristic for a commuting n-tuple of operators were introduced by Arveson [4]. These invariants were developed from two different perspectives, both dependent on commutative methods. In the examples considered by Arveson, the two perspectives yield the same invariant and it is always an integer. The goal of this chapter is to develop non-commutative versions of these invariants. The non-commutative versions introduced here possess some of the basic properties of their commutative cousins; however, their are some fundamental differences. In particular, the two perspectives yield distinct but related invariants. A new class of examples is included which help illustrate these differences. Yet, as with Arveson's invariant, the non-commutative curvature invariant is sensitive enough to detect when the original n-tuple is free. Continuity properties of these functions are also investigated.

The first section contains requisite preliminary material. This includes a discussion of the completely positive map defined by every contractive n-tuple which determines the invariants. The related dilation theory is also recalled. In the second section, the existence of the non-commutative curvature invariant and Euler characteristic is proved. The connection with dilation theory is used to provide motivation and establish a hierarchy of the invariants.

The third section contains an analysis of how the invariants behave on pure contractions. The basic property is that the curvature invariant can detect when an n-tuple is free. That is, unitarily equivalent to copies of the left regular representation. This is analogous to Arveson's basic property; however, different methods must be used to prove the result here. For pure contractions, the Euler characteristic can provide a measure of the freeness of an n-tuple.

In the fourth section, the curvature invariant is shown to be upper semi-continuous with respect to the natural notions of convergence. The rigidity of the Euler characteristic prevents any non-trivial continuity results. Stability properties of the invariants are also considered.

In the last section, a new class of examples is introduced which illustrate the differences in the non-commutative setting. These examples are finite rank perturbations of a subclass of the atomic Cuntz representations originally considered by Davidson and Pitts in [**18**]. The curvature invariant and Euler characteristic are shown to be distinct in general. Two different collections of examples are used to prove that the range of the curvature invariant is the entire positive real line.

4.1 Preliminaries

To every contraction $A = (A_1, \ldots, A_n)$ of operators A_i acting on a Hilbert space $\mathcal{H}$, there is a corresponding completely positive map $\Phi(\cdot)$ on $\mathcal{B}(\mathcal{H})$ defined by

$$\Phi(X) = \sum_{i=1}^{n} A_i X A_i^* = AXA^*.$$

When there is more than one n-tuple involved, the associated map will be written $\Phi_A(\cdot)$. This map is also completely contractive since

$$\Phi(I) = \sum_{i=1}^{n} A_i A_i^* = AA^* \leq I.$$

Given a word $w = i_1 \cdots i_d$ in the unital free semigroup on n generators $\mathcal{F}_n$, define the contraction $A_w := w(A) = A_{i_1} \cdots A_{i_d}$. For $k \geq 1$, the sequence

$$\Phi^k(I) = \sum_{i=1}^{n} A_i \Phi^{k-1}(I) A_i^* = \sum_{|w|=k} A_w A_w^*$$

satisfies $\Phi(I) \geq \Phi^2(I) \geq \ldots \geq 0$. Hence this sequence has a strong operator topology limit which has been denoted by

$$\Phi^\infty(I) = \operatorname*{SOT-lim}_{k \to \infty} \Phi^k(I).$$

The two extreme cases are important.

Definition 4.1.1. Let $A = (A_1, \ldots, A_n)$ be a contractive n-tuple of operators acting on $\mathcal{H}$. Then

(i) A is a **pure** contraction if $\Phi^\infty(I) = 0$

and

(ii) A is a **Cuntz** contraction if $\Phi^\infty(I) = I$.

These definitions are motivated by the special case of an n-tuple of isometries. The isometries have pairwise orthogonal ranges exactly when the n-tuple is contractive. An important example is determined by the left regular representation λ of $\mathcal{F}_n$ on n-variable Fock space $\mathcal{H}_n = \ell^2(\mathcal{F}_n)$ which has orthonormal basis $\{\xi_w : w \in \mathcal{F}_n\}$. The isometries $L_i = \lambda(i)$ are defined by $L_i\xi_w = \xi_{iw}$, and are also known as the 'left creation operators'. Similarly, isometries determined by words w are denoted $\lambda(w) = L_w$. The n-tuple $L = (L_1, \ldots, L_n)$ satisfies

$$LL^* = \sum_{i=1}^{n} L_i L_i^* = I - \xi_e \xi_e^*,$$

where ξ_e is the vacuum vector determined by the identity e of $\mathcal{F}_n$ (which corresponds to the empty word). There has been extensive recent work related to this n-tuple. In particular, the WOT-closed non-selfadjoint algebras $\mathfrak{L}_n$ generated by the L_i have been shown to be the appropriate non-commutative analytic Toeplitz algebras [**1, 17, 18, 34**]. Also see [**19, 27, 32**] for more detailed information on these algebras. The WOT-closed algebras corresponding to the right regular representation are denoted by $\mathfrak{R}_n$. The generating isometries are defined by $\rho(w) = R_{w'}$ where $R_u\xi_v = \xi_{vu}$, and w' denotes the word w in reverse order. It is unitarily equivalent to $\mathfrak{L}_n$ and is precisely the commutant of $\mathfrak{L}_n$. Throughout this chapter, for $k \geq 1$ let Q_k be the orthogonal projection onto the subspace $\text{span}\{\xi_w : |w| < k\}$.

Notice that $\mathrm{rk}(Q_k) = \frac{n^k-1}{n-1}$. Evidently,

$$Q_k = I - \sum_{|w|=k} L_w L_w^* = I - \Phi_L^k(I).$$

For an n-tuple of isometries $S = (S_1, \ldots, S_n)$ with pairwise orthogonal ranges, the subspaces

$$\mathcal{H}_p := \{x \in \mathcal{H} : \lim_{k\to\infty} (\Phi^k(I)x, x) = 0\}$$

and

$$\mathcal{H}_c := \{x \in \mathcal{H} : \Phi^k(I)x = x, \text{ for } k \geq 1\}.$$

are easily seen to reduce the S_i and are orthogonal complements of each other. This fact leads to Popescu's Wold decomposition [**31**]. The S_i simultaneously decompose as a direct sum $S_i \simeq T_i \oplus L_i^{(\alpha)}$, where the $T_i = S_i|_{\mathcal{H}_c}$ are isometries which form a representation of the Cuntz C*-algebra $\mathcal{O}_n$ since $\sum_{i=1}^n T_i T_i^* = I_{\mathcal{H}_c}$. The multiplicity of the left regular representation is given by the rank of the projection $I - \sum_{i=1}^n S_i S_i^*$. Although Cuntz and pure contractions are important, there is no such decomposition for an arbitrary contraction. The point is that for a general contractive n-tuple A, the subspaces $\mathcal{H}_p$ and $\mathcal{H}_c$ will not be orthogonal complements as they can be skewed when the A_i are not isometries. An example of this phenomenon was provided in Chapter 2.

The connection with dilation theory is also important. Analogous to the minimal

isometric dilation of a contraction to an isometry proved by Sz.-Nagy [40], every contractive n-tuple of operators has recently been shown to have a minimal joint isometric dilation to an n-tuple of isometries with pairwise orthogonal ranges on a larger space. This is a theorem of Frahzo, Bunce and Popescu [22, 12, 31]. Let $A = (A_1, \ldots, A_n)$ be a contractive n-tuple on $\mathcal{H}$, with minimal isometric dilation $S = (S_1, \ldots, S_n)$ acting on $\mathcal{K}$. Then each A_i is the compression to $\mathcal{H}$ of S_i. Minimality means the closed span of the subspaces $S_w\mathcal{H}$ is all of $\mathcal{K}$. The condition of uniqueness is up to unitary equivalence fixing $\mathcal{H}$. In addition, from the construction of the dilation, the subspace $\mathcal{H}$ of $\mathcal{K}$ is co-invariant for S. Indeed, recall from Lemma 2.3.1 that upon decomposing $\mathcal{K} = \mathcal{H} \ominus \mathcal{H}^\perp$, each S_i may be written as a matrix $S_i = \begin{bmatrix} A_i & 0 \\ X_i & L_i^{(\alpha)} \end{bmatrix}$.

Popescu's Wold decomposition shows that every contractive n-tuple A determines a Cuntz representation and copies of the left regular representation through its minimal isometric dilation S. Every Cuntz contraction has a Cuntz representation as its dilation. For, it is not hard to see that $\sum_{i=1}^n A_i A_i^* = I$ if and only if $\sum_{i=1}^n S_i S_i^* = I$. When an n-tuple A acts on finite dimensional space, the associated Cuntz representations have been completely classified [16]. As an aside, there has been considerable interest recently in classifying Cuntz representations. For instance, see [3, 8, 9, 10, 16, 29, 36]. This is due to the correspondence between Cuntz representations and endomorphisms of $\mathcal{B}(\mathcal{H})$.

The strength of this chapter is in the information it yields for pure contractions. The multiplicity of the left regular representation in the dilation S of A is given by the rank of $I - \sum_{i=1}^n S_i S_i^*$. This quantity has been called the *pure rank* of A. In

Chapter 2 it was computed directly in terms of A as

$$\text{pure rank}(A) = \text{rk}\Big(I - \sum_{i=1}^{n} A_i A_i^*\Big).$$

The pure rank is a key invariant in this chapter.

A valuable property of pure contractions is that they are compressions of multiples of the left regular representation. This was first observed by Popescu [31] and is included here to illustrate the connection that the minimal dilation can have with the original n-tuple.

Proposition 4.1.2. *If $A = (A_1, \ldots, A_n)$ is a non-zero pure contraction, then the minimal isometric dilation of A is $L^\alpha := (L_1^{(\alpha)}, \ldots, L_n^{(\alpha)})$, where $\alpha = \text{pure rank}(A)$.*

Proof. Suppose A acts on $\mathcal{H}$ and its minimal dilation S acts on $\mathcal{K}$. It suffices to show that $\mathcal{K} = \mathcal{K}_p$. If this is the case, then $S \simeq L^\alpha$ where $\alpha = \text{pure rank}(A)$.

Consider a vector of the form $S_v x$ for some x in $\mathcal{H}$ and v in $\mathcal{F}_n$. Then for $k > |v|$,

$$\begin{aligned} ||\Phi_S^k(I) S_v x||^2 &= \sum_{|w|=k} ||S_w S_w^* S_v x||^2 \\ &= \sum_{|u|=k-|v|} ||S_u^* x||^2 \\ &= (\Phi_A^{k-|v|}(I)x, x), \end{aligned}$$

which converges to zero since A is pure. By minimality it follows that $\Phi_S^\infty(I) = 0$, so that $\mathcal{K} = \mathcal{K}_p$. ∎

In [4], Arveson considers commuting contractive n-tuples for which $I - \Phi(I)$ is finite rank (ie: pure rank$(A) < \infty$). The invariants introduced in this paper are

defined for any contractive n-tuple, however the greatest amount of information is obtained in the finite rank case. In this case, the operators

$$I - \Phi^k(I) = \sum_{i=0}^{k-1} \Phi^i(I - \Phi(I))$$

form an increasing sequence of finite rank positive operators. The idea is to use this sequence to obtain information on the associated n-tuple. Arveson defines two invariants from different perspectives. The curvature invariant is defined by integration of the trace of a certain defect operator which is defined on the range of $I - \Phi(I)$. If A is a commuting n-tuple acting on $\mathcal{H}$, then a natural commutative Hilbert A-module structure can be placed on $\mathcal{H}$. The A-submodule determined by the range of $I - \Phi(I)$ is finitely generated, hence from commutative module theory this module has a finite free resolution. The ranks of the free modules from the associated exact sequence are known as the 'Betti numbers'. The Euler characteristic is defined as the alternating sum of these ranks. An operator theoretic version of the Gauss-Bonnet-Chern theorem is obtained from this point of view.

The following asymptotic formulae were obtained for the commutative curvature invariant and Euler characteristic respectively:

$$n! \lim_{k\to\infty} \frac{\mathrm{tr}(I - \Phi^{k+1}(I))}{k^n} \quad \text{and} \quad n! \lim_{k\to\infty} \frac{\mathrm{rk}(I - \Phi^{k+1}(I))}{k^n}.$$

In the examples considered by Arveson (those which yield graded Hilbert modules), the invariants are always positive integers and equal. This is not the case for the non-commutative versions. To obtain these new invariants, the traces and ranks of

the operators $I - \Phi^k(I)$ must be re-normalized. The factors $k^n/n!$ turn out to be specific to the commutative setting.

As mentioned in the introduction, throughout this chapter n is taken to be a finite integer with $n \geq 2$. Although the results used from dilation theory and the non-commutative analytic Toeplitz algebras go through for $n = \infty$, the invariants considered here are not defined in this case. However, there may be analogous invariants in the $n = \infty$ setting.

4.2 The Non-commutative Invariants

In this section, the non-commutative versions of the curvature invariant and Euler characteristic are developed. The connection with dilation theory is utilized to provide motivation for the definitions and leads to a general hierarchy of the related invariants. The starting point is an elementary lemma.

Lemma 4.2.1. *If $\{a_k\}_{k\geq 1}$ and $\{s_k\}_{k\geq 1}$ are sequences of non-negative numbers with $a_{k+1} \leq a_k + s_k$ and $\sum_{k\geq 1} s_k < \infty$, then $\lim_{k\to\infty} a_k$ exists.*

Proof. Let $\alpha_1 = \liminf a_k$ and $\alpha_2 = \limsup a_k$. Suppose $0 \leq \alpha_1 < \alpha_2$, with $\alpha_2 - \alpha_1 \geq \delta > 0$. Choose positive integers $m_1 > n_1$ such that $a_{m_1} - a_{n_1} \geq \delta/2$. Then one has

$$\begin{aligned} \delta/2 &\leq a_{m_1} - a_{m_1-1} + a_{m_1-1} - \ldots + a_{n_1+1} - a_{n_1} \\ &\leq \sum_{n_1 \leq k \leq m_1-1} s_k. \end{aligned}$$

In a similar fashion obtain a sequence of positive integers $n_1 < m_1 < n_2 < m_2 < \dots$ for which $\sum_{n_j \le k \le m_j - 1} s_k \ge \delta/2$ for $j \ge 1$. This would contradict the summability of the sequence s_k. ■

Note that this lemma does not hold if the sequence s_k simply converges to zero. With this result in hand, the existence of the invariants in the general non-commutative setting can be proved. The subtlety of the proof is clarified below by considering n-tuples of isometries. The uniformity of the estimates obtained leads to continuity results in Section 4.4.

Theorem 4.2.2. *Let $A = (A_1, \dots, A_n)$ be a contractive n-tuple of operators acting on $\mathcal{H}$. Then the limits*

(i) $(n-1)\lim_{k\to\infty} \operatorname{tr}(I - \Phi^k(I))/n^k$

and

(ii) $(n-1)\lim_{k\to\infty} \operatorname{rk}(I - \Phi^k(I))/n^k$,

both exist.

Proof. The key identity used to prove the existence of both limits is

$$I - \Phi^{k+1}(I) \;=\; I - \Phi(I) + \Phi(I - \Phi^k(I)). \tag{4.1}$$

To prove (i), note that if X is a positive operator on $\mathcal{H}$, then

$$\operatorname{tr}(\Phi(X)) = \sum_{i=1}^{n} \operatorname{tr}(A_i X A_i^*) = \sum_{i=1}^{n} \operatorname{tr}(X^{1/2} A_i^* A_i X^{1\ 2}) \le n \operatorname{tr}(X).$$

Observe that since the sequence $I - \Phi^k(I)$ is increasing, the limit of the normalized traces is infinite when $\mathrm{tr}(I - \Phi(I)) = \infty$. Otherwise, equation (4.1) yields for $k \geq 1$,

$$0 \leq \frac{\mathrm{tr}(I - \Phi^{k+1}(I))}{n^{k+1}} \leq \frac{\mathrm{tr}(I - \Phi^k(I))}{n^k} + \frac{\mathrm{tr}(I - \Phi(I))}{n^{k+1}}.$$

Hence the lemma applies and the existence of the associated limit is proved.

The proof of (ii) proceeds in a similar manner. If

$$\mathrm{rk}(I - \Phi(I)) = \text{pure rank}(\mathcal{A}) = \infty,$$

it follows that the limit of the ranks is infinite. In the finite pure rank case, the subadditivity of rank on sums of operators yields as above

$$0 \leq \frac{\mathrm{rk}(I - \Phi^{k+1}(I))}{n^{k+1}} \leq \frac{\mathrm{rk}(I - \Phi^k(I))}{n^k} + \frac{\text{pure rank}(\mathcal{A})}{n^{k+1}}.$$

The previous lemma applies again. ■

Remark 4.2.3. These limits exist for any contractive n-tuple; however, this chapter focuses on when the defect operator $I - \Phi(I)$ is finite rank. These appear to be the cases for which the most information can be obtained. The naive motivation for considering these particular re-normalizations comes from an analysis of the words on n letters. Consider the words on n letters of length less than k for large k. The total number of words in n commuting letters is on the order of $k^n/n!$, while the number of words in n non-commuting letters is on the order of $n^k/(n-1)$. This motivation is clarified further below by considering the situation for an n-tuple of

isometries with pairwise orthogonal ranges.

In keeping with Arveson's nomenclature, the notation from the commutative setting will be kept.

Definition 4.2.4. Let $A = (A_1, \ldots, A_n)$ be a contractive n-tuple of operators.

(i) The **curvature invariant** of A is defined to be the limit

$$\mathrm{K}(A) := (n-1) \lim_{k\to\infty} \frac{\mathrm{tr}(I - \Phi_A^k(I))}{n^k},$$

and

(ii) The **Euler characteristic** of A is defined to be the limit

$$\chi(A) := (n-1) \lim_{k\to\infty} \frac{\mathrm{rk}(I - \Phi_A^k(I))}{n^k}.$$

Since the operators $I - \Phi^k(I)$ are positive contractions, the inequality $\mathrm{K}(A) \leq \chi(A)$ is apparent. Before considering n-tuples of isometries, a simple but helpful lemma is presented.

Lemma 4.2.5. *If A and B are contractive n-tuples, then $\mathrm{K}(A \ominus B) = \mathrm{K}(A) + \mathrm{K}(B)$ and $\chi(A \ominus B) = \chi(A) + \chi(B)$, where the direct sums are taken coordinate-wise.*

Proof. This is from the identity

$$\mathrm{rk}(I - \Phi_{A\oplus B}^k(I)) = \mathrm{rk}(I - \Phi_A^k(I)) + \mathrm{rk}(I - \Phi_B^k(I)).$$

The same is true for the traces. ∎

If a contractive n-tuple consists of isometries, then the invariants are both equal to the wandering dimension of the n-tuple.

Lemma 4.2.6. *If $S = (S_1, \ldots, S_n)$ is an n-tuple of isometries with pairwise orthogonal ranges, then*

$$\mathrm{K}(S) = \chi(S) = \text{pure rank}(S).$$

Proof. Recall from Popescu's Wold decomposition that the S_i are unitarily equivalent to $T_i \ominus L_i^{(\alpha)}$, where the T_i form a representation of the Cuntz algebra. The multiplicity α is equal to $\mathrm{rk}(I - \sum_{i=1}^n S_i S_i^*) = \text{pure rank}(S)$. The invariants are clearly stable under unitary equivalence. Thus, from the previous lemma, $\mathrm{K}(S) = \text{pure rank}(S)\,\mathrm{K}(L)$ and $\chi(S) = \text{pure rank}(S)\,\chi(L)$.

However, $Q_k = I - \Phi_L^k(I) = I - \sum_{|w|=k} L_w L_w^*$ is the orthogonal projection onto the subspace $\mathrm{span}\{\xi_w : |w| < k\}$. Hence,

$$\mathrm{tr}(Q_k) = \mathrm{rk}(Q_k) = 1 + n + \ldots + n^{k-1} = \frac{n^k - 1}{n - 1}.$$

Evidently, $\mathrm{K}(L) = \chi(L) = \lim_{k\to\infty} \frac{n^k-1}{n^k} = 1$, finishing the proof. ∎

The characterization of pure rank from Lemma 2.6.3, together with the previous lemma, yields an esthetically pleasing result.

Theorem 4.2.7. *Let A be a contractive n-tuple with minimal isometric dilation S. Then*

(i) $\mathrm{K}(A) \leq \mathrm{K}(S) = \text{pure rank}(A)$

and

(ii) $\chi(\mathcal{A}) \leq \chi(S) = \text{pure rank}(\mathcal{A})$.

Proof. If $\mathcal{A} = (A_1, \ldots, A_n)$ act on $\mathcal{H}$ and $S = (S_1, \ldots, S_n)$ act on a larger space $\mathcal{K}$, then each A_i is the compression to $\mathcal{H}$ of S_i. Recall from the construction of the dilation, $\mathcal{H}$ is co-invariant for the isometries S_i. Thus,

$$\begin{aligned} I_{\mathcal{H}} - \Phi_{\mathcal{A}}^k(I_{\mathcal{H}}) &= I_{\mathcal{H}} - \sum_{|w|=k} A_w A_w^* \\ &= P_{\mathcal{H}}\Big(I_{\mathcal{K}} - \sum_{|w|=k} S_w S_w^*\Big) \mid_{\mathcal{H}} \\ &= P_{\mathcal{H}}(I_{\mathcal{K}} - \Phi_S^k(I_{\mathcal{K}})) \mid_{\mathcal{H}}. \end{aligned}$$

which yields the inequalities. Further, from Lemma 2.6.3, the minimality of the dilation guarantees that

$$\begin{aligned} \text{pure rank}(S) &= \text{rk}(I_{\mathcal{K}} - \sum_{i=1}^n S_i S_i^*) \\ &= \text{rk}(I_{\mathcal{H}} - \sum_{i=1}^n A_i A_i^*) = \text{pure rank}(\mathcal{A}). \end{aligned}$$

An application of the previous lemma finishes the proof. ∎

The following immediate corollary establishes the hierarchy of these invariants.

Corollary 4.2.8. *Let $\mathcal{A}$ be a contractive n-tuple of operators. Then*

$$0 \leq \text{K}(\mathcal{A}) \leq \chi(\mathcal{A}) \leq \text{pure rank}(\mathcal{A}).$$

In Section 4.3, it is shown that these three numbers are equal precisely when the n-tuple is free.

4.3 Detection of Freeness

In the commutative setting, the invariants are sensitive enough to detect when an n-tuple is 'free'. This is what Arveson calls the basic property of these functions. The notion of freeness for non-commuting n-tuples can be expressed as proximity of an n-tuple to the left regular representation.

Definition 4.3.1. A contractive n-tuple $A = (A_1, \ldots, A_n)$ is called **free** if there is a positive integer α for which A is unitarily equivalent to $L^\alpha = (L_1^{(\alpha)}, \ldots, L_n^{(\alpha)})$.

The α above is of course necessarily the pure rank of A. It turns out that the non-commutative analogue of Arveson's result also holds. In fact, the non-commutative invariants can be thought of as measuring the freeness of the n-tuple. The key technical device in proving the detection of freeness is the lemma which follows. It depends on the existence of a limit related to the curvature invariant. If $\mathcal{M}$ is a subspace of $\mathcal{H}_n$ which is invariant for $\mathcal{L}_n$, then a contractive n-tuple $A = (A_1, \ldots, A_n)$ is defined by

$$A_i = P_{\mathcal{M}^\perp} L_i|_{\mathcal{M}^\perp} = (L_i^*|_{\mathcal{M}^\perp})^*.$$

From the structure of the Frahzo-Bunce-Popescu dilation, all pure contractive n-

tuples are direct sums of such n-tuples. By Theorem 4.2.2 and Lemma 4.2.6,

$$1 = \mathrm{K}(L) = \mathrm{K}(A) + (n-1)\lim_{k\to\infty} \frac{\mathrm{tr}(P_{\mathcal{M}} Q_k P_{\mathcal{M}})}{n^k}.$$

In particular, the latter limit exists.

Definition 4.3.2. Let $\mathcal{M}$ be a subspace of $\mathcal{H}_n$ which is invariant for $\mathfrak{L}_n$. Then define

$$\tilde{K}(\mathcal{M}) := (n-1)\lim_{k\to\infty} \frac{\mathrm{tr}(P_{\mathcal{M}} Q_k P_{\mathcal{M}})}{n^k}.$$

The lattice of $\mathfrak{L}_n$-invariant subspaces has been determined by Davidson and Pitts. These subspaces are infinite dimensional if they are non-zero. The important observation here is that the limit $\tilde{K}(\mathcal{M})$ is non-zero exactly when the subspace $\mathcal{M}$ is non-zero.

Lemma 4.3.3. *If $\mathcal{M}$ is a non-zero subspace of $\mathcal{H}_n$ which is invariant for each L_i, then $\tilde{K}(\mathcal{M}) > 0$.*

Proof. From the decomposition theory for non-zero $\mathfrak{L}_n$-invariant subspaces developed in [17], the subspace $\mathcal{M}$ has a unique decomposition into cyclic invariant subspaces, $\mathcal{M} = \sum_j \oplus R_{\zeta_j}\mathcal{H}_n$, where each R_{ζ_j} is an isometry in $\mathfrak{R}_n$ with $\mathfrak{L}_n$-wandering vector $R_{\zeta_j}\xi_e = \zeta_j$. Thus,

$$\mathrm{tr}(P_{\mathcal{M}} Q_k P_{\mathcal{M}}) = \sum_j \mathrm{tr}(P_j Q_k P_j).$$

where P_j is the projection onto $R_{\zeta_j}\mathcal{H}_n$. Without loss of generality assume $\mathcal{M} =$

$R\mathcal{H}_n$, for an isometry $R = R_\zeta$ in $\mathfrak{R}_n$. Then the vectors $R\xi_w = RL_w\xi_e = L_w\zeta$ form an orthonormal basis for $\mathcal{M}$.

Suppose $\zeta = \sum_{u\in\mathcal{F}_n} a_u\xi_u$ and let v be a word of minimal length $|v| = k_0$ such that $a_v = (\zeta, \xi_v) \neq 0$. Then for $k > k_0$ and words w with $|w| \geq k - k_0$, one has

$$Q_k R\xi_w = Q_k L_w\zeta = Q_k(I - Q_k)L_w\zeta = 0.$$

Whence, $(P_\mathcal{M}Q_kP_\mathcal{M}R\xi_w, R\xi_w) = 0$. Conversely, for words w with $|w| < k - k_0$,

$$\begin{aligned}(Q_kR\xi_w, R\xi_w) &= \sum_{|u|\geq k_0} a_u(Q_k\xi_{wu}, L_w\zeta)\\ &= \sum_{|u|\geq k_0,\ |w|+|u|<k} a_u(L_w\xi_u, L_w\zeta)\\ &= \sum_{|u|\geq k_0,\ |w|+|u|<k} |a_u|^2.\end{aligned}$$

In particular, for words w with $|w| < k - k_0$, the lower bound $(Q_kR\xi_w, R\xi_w) \geq |a_v|^2$ is obtained. Hence, computing the trace yields

$$\mathrm{tr}(P_\mathcal{M}Q_kP_\mathcal{M}) \geq |a_v|^2(1 + n + \ldots + n^{k-k_0-1}) = |a_v|^2\frac{n^{k-k_0} - 1}{n-1}.$$

Therefore, it follows that $\tilde{K}(\mathcal{M}) \geq \frac{|a_v|^2}{n^{k_0}} > 0$. ■

The freeness condition associated with these invariants can now be proved.

Theorem 4.3.4. *If $A = (A_1, \ldots, A_n)$ is a non-zero pure contraction with finite pure rank, then the following are equvalent:*

(*i*) *A is free*

(*ii*) $\mathrm{K}(\mathcal{A}) = \chi(\mathcal{A}) = \text{pure rank}(\mathcal{A})$.

Proof. If $\mathcal{A}$ is free, then the three invariants are equal by Lemma 4.2.6. Conversely, let $\alpha = \text{pure rank}(\mathcal{A})$ and suppose $\mathcal{A}$ acts on $\mathcal{H}$. Note that since $\mathcal{A}$ is pure, its minimal isometric dilation is $L^\alpha = (L_1^{(\alpha)}, \ldots, L_n^{(\alpha)})$ acting on $\mathcal{K} := \mathcal{H}_n^{(\alpha)}$, which will be regarded as containing $\mathcal{H}$. However, recall from the construction of the dilation, the space $\mathcal{H}$ is co-invariant for the isometries $L_i^{(\alpha)}$. Thus,

$$\begin{aligned} \mathrm{tr}(I_\mathcal{K} - \Phi_{L^\alpha}^k(I)) &= \mathrm{tr}(P_\mathcal{H}(I_\mathcal{K} - \Phi_{L^\alpha}^k(I))P_\mathcal{H}) + \mathrm{tr}(P_{\mathcal{H}^\perp}(I_\mathcal{K} - \Phi_{L^\alpha}^k(I))P_{\mathcal{H}^\perp}) \\ &= \mathrm{tr}(I_\mathcal{H} - \Phi_\mathcal{A}^k(I)) + \mathrm{tr}(P_{\mathcal{H}^\perp}(I_\mathcal{K} - \Phi_{L^\alpha}^k(I))P_{\mathcal{H}^\perp}). \end{aligned}$$

Co-invariance also means that $\mathcal{H}^\perp$ is invariant for the algebra $\mathfrak{L}_n^{(\alpha)}$. Hence $\mathcal{H}^\perp$ decomposes as a direct sum of α subspaces $\mathcal{V}_j$, each invariant for $\mathfrak{L}_n$. So the traces decompose as

$$\mathrm{tr}(I_\mathcal{K} - \Phi_{L^\alpha}^k(I)) = \mathrm{tr}(I_\mathcal{H} - \Phi_\mathcal{A}^k(I)) + \sum_{j=1}^{\alpha} \mathrm{tr}(P_{\mathcal{V}_j} Q_k P_{\mathcal{V}_j}).$$

If $\mathcal{H}^\perp \neq \{0\}$, then some $\mathcal{V}_i \neq \{0\}$. Hence by Lemma 4.3.3, one would have

$$\begin{aligned} \alpha = \mathrm{K}(L^\alpha) &\geq \mathrm{K}(\mathcal{A}) + \tilde{K}(\mathcal{V}_i) \\ &> \mathrm{K}(\mathcal{A}) = \alpha, \end{aligned}$$

an absurdity. This shows that $\mathcal{H} = \mathcal{K}$. Hence $\mathcal{A}$ is its own minimal isometric dilation L^α, and the result follows. ■

In general these numbers can all be distinct for non-commuting n-tuples. Exam-

ples are introduced in Section 4.5 which illustrate this. Nonetheless these invariants, and in particular the curvature invariant, can be used to detect when an n-tuple is free. A further investigation of the Euler characteristic allows one to view it as a measure of the freeness of an n-tuple in certain cases. In fact, it can be used to describe what a pure contractive n-tuple 'looks like' in a sense. Recall that such an n-tuple $A = (A_1, \ldots, A_n)$, with each $A_i \in \mathcal{B}(\mathcal{H})$, has $L^\alpha = (L_1^{(\alpha)}, \ldots, L_n^{(\alpha)})$ as its minimal isometric dilation where $\alpha = \text{pure rank}(A)$. The n-tuple L^α acts on $\mathcal{H}_n^{(\alpha)}$, which contains $\mathcal{H}$ as a co-invariant subspace. For the sake of brevity, in the following two theorems the pure rank$(A) = 1$ case will be considered. Further, the space $\mathcal{H}$ will be regarded as contained in $\mathcal{H}_n$, so that the A_i actually are the compressions of the L_i to $\mathcal{H}$. To begin with, a necessary condition on the size of the Euler characteristic is obtained when A possesses freeness.

Theorem 4.3.5. *Let $A = (A_1, \ldots, A_n)$ be a pure contraction which acts on $\mathcal{H}$ with* pure rank$(A) = 1$. *Suppose $\mathcal{H}^\perp$ is a cyclic $\mathfrak{L}_n$-invariant subspace of $\mathcal{H}_n$. If $\mathcal{H}$ contains* span$\{\xi_w : |w| \leq k\}$, *then*

$$\chi(A) \geq \mathrm{K}(A) > 1 - \frac{1}{n^k}.$$

Proof. The inequality is trivial if $\mathcal{H}$ is all of $\mathcal{H}_n$. Otherwise, the subspace $\mathcal{H}^\perp$ can be written as $\mathcal{H}^\perp = R_\zeta \mathcal{H}_n$ for an isometry R_ζ in $\mathfrak{R}_n$. The generating wandering vector is $R_\zeta \xi_e = \zeta$, and an orthonormal basis for $\mathcal{H}^\perp$ is given by $\{L_w \zeta = R_\zeta \xi_w : w \in \mathcal{F}_n\}$.

Let v be a word of minimal length $|v| = k_0$ such that $(\zeta, \xi_v) \neq 0$. The hypothesis guarantees that $P_{\mathcal{H}^\perp} \xi_w = P_{\mathcal{H}^\perp} Q_{k+1} \xi_w = 0$ for $|w| \leq k$, so that $k_0 > k$. For $l > k_0$

and words w with $|w| \geq l - k_0$, one has

$$Q_l R_\zeta \xi_w = Q_l L_w \zeta = Q_l (I - Q_{|w|+k_0}) L_w \zeta = 0.$$

Hence, the range of $P_{\mathcal{H}^\perp} Q_l P_{\mathcal{H}^\perp}$ for $l > k_0$ is given by

$$\mathrm{Ran}(P_{\mathcal{H}^\perp} Q_l P_{\mathcal{H}^\perp}) = \mathrm{span}\{P_{\mathcal{H}^\perp} Q_l R\xi_w : |w| < l - k_0\}.$$

Thus, the following upper bound is obtained:

$$\begin{aligned} \mathrm{tr}(P_{\mathcal{H}^\perp} Q_l P_{\mathcal{H}^\perp}) &\leq \mathrm{rk}(P_{\mathcal{H}^\perp} Q_l P_{\mathcal{H}^\perp}) \\ &\leq 1 + n + \ldots + n^{l-k_0-1} = \frac{n^{l-k_0} - 1}{n-1}. \end{aligned}$$

Therefore, $\tilde{K}(\mathcal{H}^\perp) \leq 1/n^{k_0}$. This finishes the proof since.

$$1 = \mathrm{K}(\mathcal{A}) + \tilde{K}(\mathcal{H}^\perp) \leq \mathrm{K}(\mathcal{A}) + \frac{1}{n^{k_0}} < \mathrm{K}(\mathcal{A}) + \frac{1}{n^k}. \qquad \blacksquare$$

This theorem does not hold when all subspaces $\mathcal{H}$ are considered. For example, when $\mathcal{H}$ is finite dimensional and contains $\mathrm{span}\{\xi_w : |w| \leq k\}$, both invariants are zero. Further, a converse of this theorem does not hold for the curvature invariant, since the traces can in general be spread over the entire space. However, the rigidity of the Euler characteristic can be used to derive a related converse with more work.

Theorem 4.3.6. *Let $\mathcal{A} = (A_1, \ldots, A_n)$ be a pure contraction which acts on $\mathcal{H}$ with* pure rank$(\mathcal{A}) = 1$. *Suppose that $Q_l \mathcal{H} \subseteq \mathcal{H}$ for all sufficiently large l. Then the*

subspace $\mathcal{H}$ *of* $\mathcal{H}_n$ *contains* $\text{span}\{\xi_w : |w| \leq k\}$ *when*

$$\chi(A) > 1 - \frac{1}{n^k}.$$

Proof. Suppose $\mathcal{H}^\perp$ decomposes as $\mathcal{H}^\perp = \sum_{i=1}^{a} \oplus R_{\zeta_i}\mathcal{H}_n$ and k_0 is minimal for which there is a word v with $|v| = k_0$ and a j such that $(\zeta_j, \xi_v) \neq 0$. To prove the result, it suffices to show that $k_0 > k$. If this is the case, then $P_{\mathcal{H}^\perp}\xi_w = 0$ for words w with $|w| \leq k$. Indeed, for such a word w and a typical basis vector $R_{\zeta_i}\xi_u$, one would have

$$\begin{aligned}(\xi_w, R_{\zeta_i}\xi_u) &= (\xi_w, L_u\zeta_i) \\ &= (Q_{k+1}\xi_w, (I - Q_{k_0})L_u\zeta_i) \\ &= 0.\end{aligned}$$

The point here is that for all sufficiently large $l > k_0$, the vectors $Q_l R\xi_w = Q_l L_w \zeta_j$ for $0 \leq |w| < l - k_0$ are linearly independent. To see this, suppose b_w are scalars for which the vector $x = \sum_{0 \leq |w| < l-k_0} b_w Q_l L_w \zeta_j = 0$. Since $l > k_0$, the vector ξ_v satisfies $Q_l\xi_v = \xi_v$. Thus, by the minimality of the word v.

$$\begin{aligned}0 = (x, \xi_v) &= \sum_{0 \leq |w| < l-k_0} b_w (L_w\zeta_j, \xi_v) \\ &= b_e(\zeta_j, \xi_v),\end{aligned}$$

so that $b_e = 0$. Now assume $b_w = 0$ for $0 \leq |w| < m < l - k_0$ and let w_0 be a word

of length m. Evidently, $Q_l \xi_{w_0 v} = \xi_{w_0 v}$. Hence, again by minimality

$$\begin{aligned} 0 = (x, \xi_{w_0 v}) &= \sum_{0 \leq |w| < l - k_0} b_w (L_w \zeta_j, \xi_{w_0 v}) \\ &= \sum_{|w| = m} b_w (L_w \zeta_j, \xi_{w_0 v}) \\ &= b_{w_0} (\zeta_j, \xi_v). \end{aligned}$$

Ergo, each $b_w = 0$. Thus for large l such that $Q_l \mathcal{H}^\perp$ is contained in $\mathcal{H}^\perp$, this shows that $\mathrm{rk}(P_{\mathcal{H}^\perp} Q_l P_{\mathcal{H}^\perp}) \geq \frac{n^{l-k_0}-1}{n-1}$. Since the projection Q_l reduces $\mathcal{H}$ for large l,

$$\frac{n^l - 1}{n-1} = \mathrm{rk}(Q_l) = \mathrm{rk}(P_{\mathcal{H}} Q_l P_{\mathcal{H}}) + \mathrm{rk}(P_{\mathcal{H}^\perp} Q_l P_{\mathcal{H}^\perp}).$$

It follows that

$$\begin{aligned} 1 = \chi(\mathcal{A}) + (n-1) \lim_{l \to \infty} \frac{\mathrm{rk}(P_{\mathcal{H}^\perp} Q_l P_{\mathcal{H}^\perp})}{n^l} &\geq \chi(\mathcal{A}) + \frac{1}{n^{k_0}} \\ &> 1 - \frac{1}{n^k} + \frac{1}{n^{k_0}}. \end{aligned}$$

Therefore, $k_0 > k$ as required. ∎

Analogous results can be derived for the general finite pure rank($\mathcal{A}$) case. Notice that the limiting case as k becomes arbitrarily large of the previous two theorems is the statement of Theorem 4.3.4.

Remark 4.3.7. There are many examples which satisfy the hypothesis of the theorem. Let R be an isometry in $\mathfrak{R}_n$ of the form $R = \sum_{|v|=k} a_v R_v$. Consider the cyclic $\mathfrak{L}_n$-invariant subspace $R\mathcal{H}_n$. If $\mathcal{H}^\perp$ is the orthogonal direct sum of the ranges of such

isometries, then the contraction $A = (A_1, \ldots, A_n)$ acting on $\mathcal{H}$ by $A_i = P_{\mathcal{H}} L_i|_{\mathcal{H}}$ fulfills the condition in the theorem. In fact, when the subspace $\mathcal{H}^\perp$ is cyclic both of the previous theorems apply. As an example, consider $\mathcal{H}^\perp = R\mathcal{H}_n$ for the isometry R above. Then for $l > k$ and basis vectors ξ_w one has

$$Q_l R\xi_w = \sum_{|v|=k} a_v Q_l \xi_{wv} = \begin{cases} R\xi_w & \text{if } l > |w| + k \\ 0 & \text{otherwise.} \end{cases}$$

In this case, the Euler characteristic can be computed directly as $\chi(A) = 1 - \frac{1}{n^k}$, and $\mathcal{H}$ contains the subspace $\text{span}\{\xi_w : |w| < k\}$.

At the other end of the spectrum one has commutative n-tuples and finite dimensional n-tuples.

Proposition 4.3.8. *Let $A = (A_1, \ldots, A_n)$ be a contractive n-tuple such that A acts on finite dimensional space or the A_i commute. Then $\mathrm{K}(A) = \chi(A) = 0$.*

Proof. If the A_i act on finite dimensional space, the traces and ranks of $I - \Phi^k(I)$ are bounded above. The invariants are trivial when the A_i commute simply because Arveson's invariant exists. For instance, for the curvature invariant,

$$\begin{aligned} \mathrm{K}(A) &= (n-1) \lim_{k\to\infty} \frac{\mathrm{tr}(I - \Phi^k(I))}{n^k} \\ &= (n-1) \lim_{k\to\infty} \frac{k^n}{n^k} \frac{\mathrm{tr}(I - \Phi^k(I))}{k^n} = 0. \end{aligned}$$

■

It would be interesting to know if there is an enlightening characterization of when these invariants are zero. For instance, it is not known if they are always zero

at the same time.

4.4 Continuity and Stability

Any continuity result for these invariants must involve some sort of limit exchange, hence uniform estimates are required. They are provided by the estimates obtained in Theorem 4.2.2. The abstract framework for proving semi-continuity comes in the form of the following elementary lemma.

Lemma 4.4.1. *Suppose* $\{s_k\}_{k\geq 1}$, $\{a_k\}_{k\geq 1}$ *and* $\{a_k^l\}_{k\geq 1}$ *for* $l \geq 1$ *are sequences of non-negative real numbers which satisfy*

$$a_{k+1} \leq a_k + s_k, \quad a_{k+1}^l \leq a_k^l + s_k \quad \text{for} \quad l \geq 1 \quad \text{and} \quad \sum_{k\geq 1} s_k < \infty.$$

If $\lim_{k\to\infty} a_k = a$, $\lim_{k\to\infty} a_k^l = a^l$ *for* $l \geq 1$ *and* $\lim_{l\to\infty} a_k^l = a_k$ *for* $k \geq 1$, *then*

$$\limsup_{l\geq 1} a^l \leq a.$$

Proof. The proof is by contradiction. Without loss of generality, by dropping to a subsequence it can be assumed that there is an $\varepsilon > 0$ such that for $l \geq 1$,

$$a^l - a \geq \varepsilon > 0.$$

Then choose an integer $K \geq 1$ for which $\sum_{m\geq K} s_m < \varepsilon/8$ and $|a_k - a| < \varepsilon/4$ for $k \geq K$. Now fix $l_0 \geq 1$ and $k_0 \geq K$. It will be shown that the sequence $\{a_{k_0}^l\}_{l\geq 1}$

cannot converge to a_{k_0}.

There are two cases to consider. First, if $a_{k_0}^{l_0} \geq a^{l_0}$, then

$$\begin{aligned} |a_{k_0}^{l_0} - a_{k_0}| &\geq |a_{k_0}^{l_0} - a| - |a_{k_0} - a| \\ &> (a_{k_0}^{l_0} - a^{l_0}) + (a^{l_0} - a) - \varepsilon/4 \geq 3\varepsilon/4. \end{aligned}$$

On the other hand, if $a_{k_0}^{l_0} < a^{l_0}$ one can choose $k > k_0$ for which $a_{k_0}^{l_0} \leq a_k^{l_0}$ and $|a_k^{l_0} - a^{l_0}| < \varepsilon/8$. From the uniform estimates provided by the s_k,

$$\begin{aligned} 0 \leq a_k^{l_0} - a_{k_0}^{l_0} &= a_k^{l_0} - a_{k-1}^{l_0} + \ldots + a_{k_0+1}^{l_0} - a_{k_0}^{l_0} \\ &\leq \sum_{k_0 \leq m \leq k-1} s_k < \varepsilon/8. \end{aligned}$$

Hence,

$$|a_{k_0}^{l_0} - a^{l_0}| \leq |a_{k_0}^{l_0} - a_k^{l_0}| + |a_k^{l_0} - a^{l_0}| < \varepsilon/4.$$

Thus the following estimate is obtained:

$$\begin{aligned} |a_{k_0}^{l_0} - a_{k_0}| &\geq |a_{k_0} - a^{l_0}| - |a_{k_0}^{l_0} - a^{l_0}| \\ &> |a - a^{l_0}| - |a_{k_0} - a| - \varepsilon/4 > \varepsilon/2. \end{aligned}$$

Therefore, $|a_{k_0}^{l_0} - a_{k_0}| > \varepsilon/2$ for $l_0 \geq 1$, and an absurdity is realized. ■

The notion of continuity here requires a spatial link with the associated n-tuples. Thus a natural setting for considering the continuity of these invariants is with pure

contractions, because of the strong spatial link provided by the dilations. Recall that such a contraction A is the compression of L^α to an $(\mathfrak{L}_n^*)^{(\alpha)}$-invariant subspace where $\alpha = \text{pure rank}(A)$. For a pure contraction, let P_A denote the projection onto this determining co-invariant subspace.

Theorem 4.4.2. *Suppose* $A = (A_1, \ldots, A_n)$ *and* $A_l = (A_{l,1}, \ldots, A_{l,n})$ *for* $l \geq 1$ *are pure contractions acting on the same space with*

$$\text{pure rank}(A), \text{pure rank}(A_l) \leq \alpha < \infty.$$

If WOT–$\lim_l P_{A_l} = P_A$, *then*

$$\limsup_{l \geq 1} \mathrm{K}(A_l) \leq \mathrm{K}(A).$$

Proof. The lemma is applied with $a_k^l = \text{tr}(I - \Phi_{A_l}^k(I))/n^k$, $a^l = \mathrm{K}(A_l)$, $a_k = \text{tr}(I - \Phi_A^k(I))/n^k$ and $a = \mathrm{K}(A)$. The estimates are from Theorem 4.2.2:

$$a_{k+1} \leq a_k + \alpha/n^{k+1} \quad \text{and} \quad a_{k+1}^l \leq a_k^l + \alpha/n^{k+1} \quad \text{for} \quad l \geq 1.$$

Thus it suffices to show that $\lim_{l \to \infty} \text{tr}(I - \Phi_{A_l}^k(I)) = \text{tr}(I - \Phi_A^k(I))$, for $k \geq 1$. This follows from weak convergence. For instance, when $\alpha = 1$ the operators can be thought of as acting on $\mathcal{H}_n$, so that

$$\text{tr}(I - \Phi_{A_l}^k(I)) = \text{tr}(P_{A_l} Q_k P_{A_l}) = \text{tr}(Q_k P_{A_l}),$$

for a fixed $k \geq 1$, and hence

$$\begin{aligned}\lim_{l\to\infty} \operatorname{tr}(I - \Phi_{\mathcal{A}_l}^k(I)) &= \lim_{l\to\infty} \sum_{|w|<k} (P_{\mathcal{A}_l}\xi_w, \xi_w) \\ &= \sum_{|w|<k} (P_{\mathcal{A}}\xi_w, \xi_w) = \operatorname{tr}((I - \Phi_{\mathcal{A}}^k(I)).\end{aligned}$$

This completes the proof. ■

Note 4.4.3. In the infinite pure rank case there are a couple of possibilities. When $\operatorname{tr}(I - \Phi_{\mathcal{A}}(I)) = \infty$ the theorem is trivial since $\mathrm{K}(\mathcal{A}) = \infty$. Otherwise, it is unclear at this point whether upper semi-continuity is satisfied.

The upper semi-continuity of the Euler characteristic is addressed below. Simple examples show that neither invariant is lower semi-continuous with respect to this convergence.

Example 4.4.4. For $l \geq 1$ define contractions $\mathcal{A}_l = (\mathcal{A}_{l,1}, \ldots, \mathcal{A}_{l,n})$ by $\mathcal{A}_{l,i} = Q_l L_i|_{Q_l\mathcal{H}_n} = (L_i^*|_{Q_l\mathcal{H}_n})^*$ for $1 \leq i \leq n$. Then certainly $\mathrm{WOT}\text{-}\lim_{l\to\infty} Q_l = I$, so the associated limit is $L = (L_1, \ldots, L_n)$. However, since $Q_l Q_k Q_l = Q_l$ for $k \geq l$ one has

$$\operatorname{rk}(I - \Phi_{\mathcal{A}_l}^k(I)) = \operatorname{rk}(Q_l Q_k Q_l) = \frac{n^l - l}{n - 1}.$$

Therefore,

$$\mathrm{K}(\mathcal{A}_l) \leq \chi(\mathcal{A}_l) = \lim_{k\to\infty} \frac{n^l - 1}{n^k} = 0.$$

Whereas the limit satisfies $\mathrm{K}(L) = \chi(L) = 1$.

More general continuity results can be obtained for the curvature invariant. It turns out to be upper semi-continuous with respect to coordinate-wise norm convergence. This result can be proved without focusing on the pure setting.

Theorem 4.4.5. *Suppose that* $A = (A_1, \ldots, A_n)$ *and* $A_l = (A_{l,1}, \ldots, A_{l,n})$ *for* $l \geq 1$ *are contractions acting on* $\mathcal{H}$ *with* pure rank(A), pure rank$(A_l) \leq \alpha < \infty$. *If* $\lim_{l\to\infty} ||A_{l,i} - A_i|| = 0$ *for* $1 \leq i \leq n$, *then*

$$\limsup_l \mathrm{K}(A_l) \leq \mathrm{K}(A).$$

Proof. As in Theorem 4.4.2, an application of Lemma 4.4.1 is the key here. In particular, it suffices to check that $\lim_{l\to\infty} \mathrm{tr}(I - \Phi_{A_l}^k(I)) = \mathrm{tr}(I - \Phi_A^k(I))$, for $k \geq 1$.

Fix $k \geq 1$. For ease of notation; let $B_l = I - \Phi_{A_l}^k(I)$, $B = (I - \Phi_A^k(I))$ and P be the projection onto $\mathrm{Ran}(B) = \ker(B)^\perp$. The hypothesis guarantees that B_l converges in norm to B. If $\{e_1, \ldots, e_r\}$ is an orthonormal basis for $\ker(B)^\perp$, then

$$\mathrm{tr}(B) = \sum_{i=1}^{r} (Be_i, e_i).$$

On the other hand, for $1 \leq i \leq r$ the sequence $B_l e_i$ converges to Be_i. Further,

$$\mathrm{tr}(B_l) = \sum_{i=1}^{r} (B_l e_i, e_i) + \mathrm{tr}(B_l P^\perp).$$

However, the ranks of the B_l are uniformly bounded above. Indeed, if Φ is one

of the associated completely positive maps, then

$$I - \Phi^k(I) = \sum_{i=0}^{k-1} \Phi^i(I - \Phi(I)).$$

Thus, $\mathrm{rk}(B_l) \leq \alpha + n\alpha + \ldots + n^{k-1}\alpha = \alpha\left(\frac{n^k-1}{n-1}\right)$, for $l \leq 1$. Hence, let $m = \sup_{l\geq 1} \mathrm{rk}(B_l) < \infty$. Let $\varepsilon > 0$ and choose $K \geq 1$ such that $l \geq K$ implies $||B_l P^\perp|| < \varepsilon/m$. Then since $P^\perp B_l P^\perp$ is a positive contraction of rank at most m,

$$\mathrm{tr}(B_l P^\perp) = \mathrm{tr}(P^\perp B_l P^\perp) < \varepsilon,$$

for $l \geq K$. It follows that $\lim_{l\to\infty} \mathrm{tr}(B_l) = \mathrm{tr}(B)$, and the lemma can be applied. ■

This theorem is used in Section 4.5 to obtain information on the range of the curvature invariant. Again, the validity of the infinite pure rank case in the theorem is unsettled at this point. Concerning the Euler characteristic, continuity at infinity in any natural topology is trivial. Indeed, if pure $\mathrm{rank}(\mathcal{A}) = \mathrm{rk}(I - \sum_{i=1}^n A_i A_i^*) = \infty$, then the ranks of the increasing sequence $I - \Phi^k(I)$ are infinite. However, the rigidity of the Euler characteristic prevents non-trivial continuity results.

Example 4.4.6. Consider the sequences $\{x_k\}_{k\geq 1}$ and $\{y_k\}_{k\geq 1}$ of $\mathfrak{L}_n$-wandering vectors belonging to $\mathcal{H}_3$ given by

$$x_k = \alpha_k \xi_1 + \beta_k \xi_2 \quad \text{and} \quad y_k = \beta_k \xi_2 + \alpha_k \xi_3.$$

where $\alpha_k, \beta_k > 0$, $\alpha_k^2 + \beta_k^2 = 1$ and $\lim_{k\to\infty} \beta_k = 1$. Let $\mathcal{M}_k$ be the $\mathfrak{L}_3^*$-invariant

subspace

$$\mathcal{M}_k = \overline{\mathfrak{L}_3 x_k} \bigvee \overline{\mathfrak{L}_3 y_k} \bigvee \text{span}\{\xi_e\},$$

with P_k the projection onto $\mathcal{M}_k$. If P is the projection onto $\text{span}\{\xi_e, \xi_{u2} : u \in \mathcal{F}_3\}$, the first claim is that $\text{SOT–lim}_{k\to\infty} P_k = P$.

To prove this, it suffices to check convergence on basis vectors ξ_w. Consider a vector of the form ξ_{w1}. Then ξ_{w1} is orthogonal to all of the determining vectors of $\mathcal{M}_k$ except $L_w x_k = \alpha_k \xi_{w1} + \beta_k \xi_{w2}$. Hence,

$$P_k \xi_{w1} = (\xi_{w1}, L_w x_k) L_w x_k = \alpha_k L_w x_k,$$

which converges to zero by hypothesis. Similarly, $\lim_{k\to\infty} P_k \xi_{w3} = \lim_{k\to\infty} \alpha_k L_w y_k = 0$ for all words w in $\mathcal{F}_3$. Next let ξ_{w2} be a basis vector in $P\mathcal{H}_3$. Then ξ_{w2} is perpendicular to all the determining vectors in $\mathcal{M}_k$ except $L_w x_k$ and $L_w y_k$. Let z_k be the unit vector orthogonal to x_k obtained from the Gram-Schmidt process for which $\text{span}\{x_k, y_k\} = \text{span}\{x_k, z_k\}$. It follows that $\{L_w x_k, L_w z_k\}$ forms an orthonormal basis for $\text{span}\{L_w x_k, L_w y_k\}$. A computation shows that

$$\begin{aligned} P_k \xi_{w2} &= (\xi_{w2}, L_w x_k) L_w x_k + (\xi_{w2}, L_w z_k) L_w z_k \\ &= \beta_k L_w x_k + \frac{\alpha_k}{\sqrt{2 - 1/\beta_k^2}} L_w z_k. \end{aligned}$$

Therefore, $\lim_{k\to\infty} P_k \xi_{w2} = \xi_{w2}$. The first claim follows.

Hence the contractions $A = (A_1, \ldots, A_n)$ and $A_k = (A_{k,1}, \ldots, A_{k,n})$ defined by

$$A_i = PL_i|_{P\mathcal{H}_3} \quad \text{and} \quad A_{k,i} = P_k L_i|_{P_k\mathcal{H}_3} \quad \text{for} \quad k \geq 1$$

satisfy SOT–$\lim_{k\to\infty} P_k = P$. The second claim is that $\lim_{k\to\infty} ||A_{k,i} - A_i|| = 0$ for $1 \leq i \leq n$. To see this let $d_k = \frac{\alpha_k}{\sqrt{2-1/\beta_k^2}}$ and suppose $z_k = a_k\xi_1 + b_k\xi_2 + c_k\xi_3$ for $k \geq 1$. Since z_k is a unit vector, each of a_k, b_k and c_k belong to the unit disk. If x is a unit vector in $\mathcal{H}_3$, then a (long) computation yields for $1 \leq i \leq n$

$$\begin{aligned} ||(A_{k,i} - A_i)x|| \quad \leq \quad & 2\alpha_k + |\beta_k\alpha_k + d_k a_k| + |\beta_k^2 + d_k b_k - 1| + |d_k c_k| \\ & \left(|\beta_k\alpha_k + d_k a_k|^2 + |\beta_k^2 + d_k b_k - 1|^2 + |d_k c_k|^2\right)^{1/2} \end{aligned}$$

In any event, this proves the claim since the upper bound converges to zero as k becomes arbitrarily large.

Thus this example satisfies the hypotheses of the previous two theorems. But it is easy to see that

$$\mathrm{rk}(I - \Phi^l_{A_k}(I)) = 1 + 2 + 2 \cdot 3 + \ldots + 2 \cdot 3^{l-2} = 3^{l-1}$$

and

$$\mathrm{rk}(I - \Phi^l_{A}(I)) = 1 + 1 + 3 + \ldots + 3^{l-2} = \frac{3^{l-1} + 1}{2}.$$

Therefore, $\chi(A_k) = 2/3$ for $k \geq 1$, while $\chi(A) = 1/3$.

This section finishes with a look at stability properties. It is obvious that the

invariants are stable under unitary equivalences. There is at least one other stability property.

Proposition 4.4.7. *Let $A = (A_1, \ldots, A_n)$ and $B = (B_1, \ldots, B_n)$ be contractive n-tuples of operators such that each A_i is the compression of B_i to a co-invariant subspace of finite co-dimension. Then $\mathrm{K}(A) = \mathrm{K}(B)$ and $\chi(A) = \chi(B)$.*

Proof. Suppose B acts on $\mathcal{H}$ and A acts on a subspace H_0 for which $\mathcal{H}_1 := \mathcal{H} \ominus \mathcal{H}_0$ is finite dimensional. By hypothesis, each $A_i = P_{\mathcal{H}_0} B_i|_{\mathcal{H}_0} = (B_i^*|_{\mathcal{H}_0})^*$. Let $B_k = I - \Phi_B^k(I)$. Then by co-invariance,

$$\mathrm{rk}(B_k) \leq \mathrm{rk}(I - \Phi_A^k(I)) + \mathrm{rk}(P_{\mathcal{H}_0} B_k P_{\mathcal{H}_1}) + \mathrm{rk}(P_{\mathcal{H}_1} B_k).$$

Thus, the associated Euler characteristics are evaluated as

$$\begin{aligned} |\chi(B) - \chi(A)| &= (n-1) \lim_{k\to\infty} \left| \frac{\mathrm{rk}(B_k)}{n^k} - \frac{\mathrm{rk}(I - \Phi_A^k(I))}{n^k} \right| \\ &\leq (n-1) \lim_{k\to\infty} \frac{2 \dim \mathcal{H}_1}{n^k} = 0. \end{aligned}$$

An easier computation works for the curvature invariant. ∎

This stability property is not true in general. not even if $\mathcal{H}_0$ is invariant for A.

Example 4.4.8. For $1 \leq i \leq n$, let A_i be the compression of L_i to the $\mathfrak{L}_n$-invariant subspace $\xi_e^\perp := \mathrm{span}\{\xi_w : |w| \geq 1\}$. Recall that $\mathrm{K}(L) = \chi(L) = 1$. However, the

contractive n-tuple $A = (A_1, \ldots, A_n)$ satisfies

$$\Phi_A^k(I)\xi_w = \sum_{|v|=k} A_v A_v^* \xi_w = \sum_{|v|=k} L_v P_{\xi_e^\perp} L_v^* \xi_w$$
$$= \begin{cases} 0 & \text{if } 1 \leq |w| \leq k \\ \xi_w & \text{if } |w| > k. \end{cases}$$

In particular, this shows that pure $\text{rank}(A) = \text{rk}(I - \Phi_A(I)) = n$. Further.

$$\text{K}(A) = \chi(A) = (n-1) \lim_{k \to \infty} \left[\frac{n + n^2 + \ldots + n^k}{n^k} \right] = n.$$

Thus by Theorem 4.3.4, $A \simeq L^{(n)}$. This can also be observed by noting that $\xi_e^\perp$ decomposes as the direct sum of n subspaces which reduce A. The compression of A to each of these subspaces is unitarily equivalent to L.

4.5 Examples

It is probably not reasonable to expect a tight characterization of pure contractive n-tuples, since they are the multi-variable analogues of completely non-unitary operators. Even in the $n = 1$ finite pure rank case, these operators are essential co-isometries. Nonetheless, this section contains a rich collection of examples for which $I - \Phi(I)$ is finite rank. In particular, new classes of examples are introduced which fill out the range of the curvature invariant and illustrate the fact that generally the two invariants are not equal in the non-commutative setting.

Example 4.5.1. In [17], Davidson and Pitts described a class of representations

of the Cuntz-Toeplitz C*-algebra $\mathcal{E}_n$ which they called atomic free semigroup representations. These representations decompose as a direct integral of irreducible atomic representations, which are of three types. The first is the left regular representation, and is the only one which does not factor through the Cuntz algebra $\mathcal{O}_n$. The second type is a class of inductive limits of the left regular representation which are classified by an infinite word up to tail equivalence. The third type can be called atomic ring representations. These representations have the shape of a benzene ring, with infinite trees leaving each node. The nodes correspond to basis vectors and each tree swept out is a copy of the left regular representation. The associated isometries with pairwise orthogonal ranges map ring basis vectors either to the next vector in the ring, allowing for modulus one multiples of the image vector, or to a top of the tree which lies below the original node.

These representations can be perturbed to obtain new examples which fit into the context of this chapter. The idea is to preserve the structure of the ring representations, with the proviso that the images of vectors lying in the ring are allowed to be strictly contractive multiples, instead of just modulus one multiples. These new representations can be thought of as possessing a certain decaying property as one moves around the ring.

The construction proceeds as follows: Suppose $u = i_1 i_2 \dots i_d$ is a word in $\mathcal{F}_n$. Let $\mathcal{H}_u$ be the Hilbert space with orthonormal basis,

$$\{\xi_{s,w} : 1 \leq s \leq d \text{ and } w \in \mathcal{F}_n \setminus \mathcal{F}_n i_s\}.$$

If $\vec{\lambda} = (\lambda_1, \dots, \lambda_d)$ is a d-tuple of complex scalars with each $|\lambda_i| \leq 1$, then define a

contractive n-tuple $A = (A_1, \ldots, A_n)$ acting on $\mathcal{H}_u$ by

$$\begin{aligned} A_i\xi_{s,e} &= \lambda_s\xi_{s+1,e} \quad \text{if} \quad i = i_s,\ 1 \leq s \leq d \\ A_i\xi_{s,e} &= \xi_{s,i} \quad \text{if} \quad i \neq i_s \\ A_i\xi_{s,w} &= \xi_{s,iw} \quad \text{if} \quad w \neq e \end{aligned}$$

So the ring vectors are given by $\xi_{s,e}$ for $1 \leq s \leq d$. The associated representation of $\mathcal{F}_n$ is denoted by $\sigma_{u,\vec{\lambda}}$. When some λ_s is on the open unit disk, there is actual decaying which occurs around the ring. Hence the associated representation will be called a **decaying atomic representation**. The dimension of the subspace determined by the central ring vectors is referred to as the dimension of the representation.

In general the pure rank of these representations is determined by the amount of decaying which occurs.

Proposition 4.5.2. *If $\sigma_{u,\vec{\lambda}}$ is the decaying atomic representation associated with the word $u = i_1 \cdots i_d$ and vector $\vec{\lambda} = (\lambda_1, \ldots, \lambda_d)$, then the rank of $I - \Phi(I)$ is equal to the cardinality of the set $\{s : |\lambda_s| < 1\}$.*

Proof. This is straight from a computation of $I - \Phi(I)$ on the determining basis for the representation. For ring basis vectors $\xi_{s,e}$ one has $A_i^*\xi_{s,e} = 0$ when $i \neq i_{s-1}$. Whence

$$(I - \Phi(I))\xi_{s,e} = (I - A_{i_{s-1}}A_{i_{s-1}}^*)\xi_{s,e} = (1 - |\lambda_{s-1}|^2)\xi_{s,e}.$$

On the other hand, every basis vector outside the ring is of the form $\xi_{s,wi}$ for some letter i_s in u and $i \neq i_s$. If $wi = j_0v$, where $1 \leq j_0 \leq n$ and v is a word in $\mathcal{F}_n$.

then $A_j^* \xi_{s,wi} = 0$ for $j \neq j_0$. Thus,

$$(I - \Phi(I))\xi_{s,wi} = (I - A_{j_0} A_{j_0}^*)\xi_{s,j_0 v} = 0,$$

and the result follows. ■

These examples form a tractable class of pure contractions. This also shows how a Cuntz n-tuple can be perturbed by a finite rank operator to become pure.

Theorem 4.5.3. *Let $A = (A_1, \ldots, A_n)$ be a decaying atomic n-tuple. Then A is a pure contraction.*

Proof. Suppose A is determined by a word $u = i_1 \cdots i_d$ and vector $\vec{\lambda} = (\lambda_1, \ldots, \lambda_d)$. It is required to show that $\Phi^\infty(I) = 0$. Equivalently, $\lim_{k\to\infty} \Phi^k(I)\xi_{s,w} = 0$, for all basis vectors $\xi_{s,w}$.

Consider a fixed basis vector $\xi_{s,w}$ where $1 \leq s \leq d$ and $w \in \mathcal{F}_n \setminus \mathcal{F}_n i_s$. For a given k, there is only one word v_k of length k for which $A_{v_k}^* \xi_{s,w} \neq 0$. For sufficiently large k, this word v_k will pull $\xi_{s,w}$ back toward the benzene ring, and then move around the ring. Such a word will be of the form

$$v_k = w i_{s-1} \cdots i_d u^{l_k} i_1 \cdots i_{m_k}, \text{ for some } 1 \leq m \leq d.$$

Obviously l_k becomes arbitrarily large as k does. Let $r = \max\{|\lambda_j| : |\lambda_j| < 1\}$.

Then,

$$\begin{aligned} ||\Phi^k(I)\xi_{s,w}|| &= ||\sum_{|v|=k} A_v A_v^* \xi_{s,w}|| \\ &= ||A_{v_k} A_{v_k}^* \xi_{s,w}|| \\ &= ||A_{v_k} A_{i_{m_k}}^* \cdots A_{i_1}^* (A_u^*)^{l_k} \xi_{d,e}|| \\ &\leq ||(A_u^*)^{l_k} \xi_{d,e}|| \\ &\leq r^{l_k}. \end{aligned}$$

Hence, $\lim_{k\to\infty} ||\Phi^k(I)\xi_{s,w}|| = 0$ as required. ■

The Euler characteristic and curvature invariant of these decaying atomic n-tuples can be computed directly. The general pure rank one proof of the Euler characteristic is included.

Lemma 4.5.4. *Let $A = (A_1, \ldots, A_n)$ be a d-dimensional decaying atomic n-tuple with* pure rank$(A) = 1$ *determined by a scalar λ with $0 \leq |\lambda| < 1$. Then*

$$\chi(A) = 1 - \frac{1}{n^d}.$$

Proof. Without loss of generality assume the n-tuple A is determined by the representation $\sigma_{u,\vec{\lambda}}$ where $u = i_1 \cdots i_d$ and $\vec{\lambda}$ is the d-tuple $\vec{\lambda} = (\lambda, 1, \ldots, 1)$. The associated orthonormal basis is

$$\{\xi_{s,w} : 1 \leq s \leq d \text{ and } w \in \mathcal{F}_n \setminus \mathcal{F}_n i_s\}.$$

Let $r = |\lambda|^2$ and let $R_k = I - \Phi^k(I) = I - \sum_{|v|=k} A_v A_v^*$. The action around the

ring is given by $A_{i_1}\xi_{1,e} = \lambda\xi_{2,e}$ and $A_{i_s}\xi_{s,e} = \xi_{s+1,e}$ for $2 \leq s \leq d$ (where $d+1$ is identified with 1).

Consider a typical basis vector $\xi_{s,e}$ with $2 \leq s \leq d+1$ and put $m = s-2 \geq 0$. Let $k \geq d$ be a positive integer and let w be a word in $\mathcal{F}_n$ with $|w| = k-s+1$. Then

$$\begin{aligned} R_k\xi_{s,wi} &= \xi_{s,wi} - A_{wii_{s-1}\cdots i_2}A_{i_2}^* \cdots A_{i_{s-1}}^* A_i^* A_w^* \xi_{s,wi} \\ &= \xi_{s,wi} - A_{wii_{s-1}\cdots i_2}\xi_{2,e} = 0. \end{aligned}$$

Similarly, $R_k\xi_{s,wi} = 0$ for $|w| \geq k-s+1$.

On the other hand let w be a word with $|w| = k-s$. Then

$$\begin{aligned} R_k\xi_{s,wi} &= \xi_{s,wi} - A_{wii_{s-1}\cdots i_1}(A_{wii_{s-1}\cdots i_1})^*\xi_{s,wi} \\ &= (1-r)\xi_{s,wi}. \end{aligned}$$

Analogously, for $k \geq d$, every basis vector $\xi_{s,wi}$ with $|w| \leq k-s$ will belong to the range of R_k since $R_k\xi_{s,wi} = (1-r^t)\xi_{s,wi}$, for some t depending on $|w|$ and d. The total number of such vectors is

$$1 + (n-1) + (n-1)n + \ldots + (n-1)n^{k-s} = n^{k-s+1}.$$

Therefore the rank of R_k for $k \geq d$ is computed as

$$\mathrm{rk}(R_k) = n^{k-1} + n^{k-2} + \ldots + n^{k-d} = \frac{n^k}{n-1}(1 - \frac{1}{n^d}).$$

which shows that $\chi(A) = 1 - \frac{1}{n^d}$, as desired. ■

The formula for the curvature invariant of the one-dimensional n-tuples is readily obtained.

Lemma 4.5.5. *Let $A = (A_1, \ldots, A_n)$ be a one-dimensional decaying atomic n-tuple determined by a scalar λ with $0 \leq |\lambda| < 1$ (hence* pure rank$(A) = 1$*). Then*

$$\mathrm{K}(A) = (n-1)\frac{1-|\lambda|^2}{n-|\lambda|^2}.$$

Proof. Without loss of generality assume $u = 1$ so that the central ring vector is $\xi_{1,e}$, and define r and R_k as above. Then the ring action is given by $A_1^*\xi_{1,e} = \bar{\lambda}\xi_{1,e}$ and $A_i^*\xi_{1,e} = 0$ for $i \neq 1$. Let $k \geq 2$ be a fixed positive integer. Then

$$\begin{aligned} R_k\xi_{1,e} &= \xi_{1,e} - A_1^k(A_1^*)^k\xi_{1,e} \\ &= (1-r^k)\xi_{1,e}. \end{aligned}$$

Further, if $|w| = k - l$ for some $2 \leq l \leq k$, then

$$\begin{aligned} R_k\xi_{1,wi} &= \xi_{1,wi} - A_wA_iA_1^{l-1}(A_1^*)^{l-1}A_i^*A_w^*\xi_{1,wi} \\ &= (1-r^{l-1})\xi_{1,wi}. \end{aligned}$$

However, if w is a word of length at least $k-1$, say $wi = uv$ with $|u| = k$, then

$$R_k\xi_{1,wi} = \xi_{1,wi} - A_uA_u^*\xi_{1,uv} = 0.$$

Therefore the traces can be evaluated as

$$\begin{aligned}
\mathrm{tr}(R_k) &= 1-r^k+(n-1)\left[(1-r^{k-1})+n(1-r^{k-2})+\ldots+n^{k-2}(1-r)\right] \\
&= n^{k-1}-r^k-(n-1)n^{k-1}\left[\frac{r}{n}\frac{(r/n)^k-1}{r/n-1}\right] \\
&= n^{k-1}-r^k-\frac{(n-1)r}{n(n-r)}(n^k-r^k).
\end{aligned}$$

Thus the curvature invariant is given by

$$\begin{aligned}
\mathrm{K}(\mathcal{A}) &= (n-1)\lim_{k\to\infty}\left[\frac{1}{n}-\frac{r^k}{n^k}-\frac{(n-1)r}{n(n-r)}(1-\frac{r^k}{n^k})\right] \\
&= \frac{n-1}{n}\left[1-\frac{(n-1)r}{n-r}\right] \\
&= (n-1)\frac{1-r}{n-r},
\end{aligned}$$

as claimed. ∎

Remark 4.5.6. For the commuting examples considered by Arveson in [4], the curvature invariant and Euler characteristic are always the same number and equal to an integer. This is not the case for the non-commutative versions. For instance, even consider the one dimensional decaying atomic 2-tuple associated with $\lambda = 1/\sqrt{2}$. The theorem tells us that $\mathrm{K}(\mathcal{A}) = 1/3$ and $\chi(\mathcal{A}) = 1/2$ in this case. At this point there does not appear to be a good general characterization of when the two invariants are equal. It is possible to say things in special cases. For instance, one can show for a decaying n-tuple $A = (A_1, \ldots, A_n)$ determined by the representation $\sigma_{u,\vec{\lambda}}$, the condition $\mathrm{K}(\mathcal{A}) = \chi(\mathcal{A})$ is satisfied exactly when one of the two extreme cases occurs. That is, the vector $\vec{\lambda}$ is either $\vec{\lambda} = (0, \ldots .0)$ or $\vec{\lambda} = (1, \ldots .1)$; in

other words, either there is full annihilation around the ring, or there is no decaying at all and therefore $\sigma_{u,\vec{\lambda}}$ is a Cuntz representation.

The variety of even the one-dimensional decaying 2-tuples turns out to be extensive enough to obtain the entire positive real line in the image of the curvature invariant.

Theorem 4.5.7. *For every $r \geq 0$, there is a contraction $\mathcal{A} = (A_1, A_2)$ for which* $\mathrm{K}(\mathcal{A}) = r$.

Proof. By Lemma 4.2.5, it is sufficient to obtain an interval in the range of $\mathrm{K}(\mathcal{A})$ which includes 0. This is just a matter of using the previous lemma and solving an identity. For positive numbers r with $0 \leq r \leq 1/2$, the number $s = \frac{1-2r}{1-r}$ belongs to the unit interval. The one dimensional decaying 2-tuple $\mathcal{A}$ determined by $\lambda = \sqrt{s}$ satisfies $\mathrm{K}(\mathcal{A}) = r$. ■

Remark 4.5.8. There is not as much information available on the range of the Euler characteristic. It is easy to construct examples $\mathcal{A}$ which satisfy $\chi(\mathcal{A}) = 1/n$. Indeed, defining $\mathcal{A}$ by $A_i = P_{\mathcal{M}} L_i|_{\mathcal{M}}$ where $\mathcal{M}^{\perp} = \sum_{i=2}^{n} \oplus R_i \mathcal{H}_n$ suffices. Thus, using direct sums, it follows that every positive rational number is in the range of the Euler characteristic. It seems reasonable to make the guess that every positive real is in the range of $\chi(\mathcal{A})$, it would be surprising if this were not the case.

For decaying n-tuples with higher dimensional central rings the formulae for the curvature invariant become particularly nasty. Nonetheless, the continuity results from the previous section can allow one to avoid these computations.

Theorem 4.5.9. *For every $\varepsilon > 0$ and integers $k, n \geq 1$, there is a contraction $A = (A_1, \ldots, A_n)$ for which* pure rank$(A) = k$,

$$k - \chi(A) < \varepsilon \quad \textit{and} \quad \mathrm{K}(A) < \varepsilon.$$

Proof. It is sufficient to prove the pure rank$(A) = 1$ case since direct sums can then be used in the general case. The pure rank one d-dimensional decaying atomic n-tuples provide the concrete examples here. The Euler characteristic is always $\chi(A) = 1 - \frac{1}{n^d}$, independent of the decaying factor λ. Hence by choosing large enough central rings, $\chi(A)$ asymptotically approaches 1.

Given a fixed word u in $\mathcal{F}_n$ of length d, consider the decaying atomic n-tuple $A_r = (A_{1,r}, \ldots, A_{n,r})$ acting on $\mathcal{H}_u$ which is determined by the d-tuple $\vec{r} = (r, 1, \ldots, 1)$ for $0 \leq r \leq 1$. When $r = 1$ the n-tuple forms a Cuntz representation, so that $\mathrm{K}(A_1) = 0$. However, observe that if $x = \sum_{s,w,i} a_{s,w,i}\xi_{s,wi}$ is a unit vector in $\mathcal{H}_u$, then

$$\begin{aligned} ||(A_{j,1} - A_{j,r})x|| &= ||\sum_{s,w,i} a_{s,w,i}(A_{j,1} - A_{j,r})\xi_{s,wi}|| \\ &= ||a_{1,e,e}(A_{j,1} - A_{j,r})\xi_{1,e}|| \\ &= \begin{cases} |a_{1,e,e}||1 - r| & \text{if } j = i_1 \\ 0 & \text{if } j \neq i_1 \end{cases} \end{aligned}$$

It follows that $\lim_{r \uparrow 1} ||A_{j,1} - A_{j,r}|| = 0$ for $1 \leq j \leq n$. Thus by the upper semi-continuity of K proved in Theorem 4.4.5, one has $\limsup_r \mathrm{K}(A_r) = \mathrm{K}(A_1) = 0$. This finishes the proof. ■

Remark 4.5.10. Thus the invariants can be asymptotically as far apart as possible. It would be interesting to know if the extreme case can be attained. In other words, is there an A with pure rank$(A) = 1$ such that $\mathrm{K}(A) = 0$ and $\chi(A) = 1$?

There is another class of examples which in a sense are pervasive. If $\mathcal{M}$ is a subspace of $\mathcal{H}_n$ which is co-invariant for $L = (L_1, \ldots, L_n)$, then a contractive n-tuple $A = (A_1, \ldots, A_n)$ is defined by $A_i = P_{\mathcal{M}} L_i|_{\mathcal{M}} = (L_i^*|_{\mathcal{M}})^*$ for $1 \leq i \leq n$. Co-invariance shows that

$$\begin{aligned} I - \Phi_A^k(I) &= I_{\mathcal{M}} - \sum_{|w|=k} A_w A_w^* \\ &= P_{\mathcal{M}}(I - \sum_{|w|=k} L_w L_w^*)|_{\mathcal{M}}. \end{aligned}$$

Thus, L is an isometric dilation of A which is minimal when ξ_e belongs to $\mathcal{M}$. Recall from the structure of the Frahzo-Bunce-Popescu dilation that all pure contractions can be obtained from direct sums of such n-tuples. It turns out that examples can be constructed from this point of view which fill out the range of the curvature invariant. The proof uses the invariant defined in Section 4.3.

Theorem 4.5.11. *For every $r \geq 0$, there are positive integers n and k and a subspace $\mathcal{M}$ in $\mathrm{Lat}(\mathfrak{L}_n^*)^{(k)}$ for which the contraction $A = (A_1, \ldots, A_n)$ defined by $A_i = P_{\mathcal{M}} L_i^{(k)}|_{\mathcal{M}}$ satisfies $\mathrm{K}(A) = r$.*

Proof. The case $\mathrm{K}(A) = 0$ is covered by $\mathcal{M} = \{0\}$. Recall from the remarks preceding Lemma 4.3.3 that for every $\mathfrak{L}_n^*$-invariant subspace $\mathcal{M}$,

$$1 = \mathrm{K}(L) = \mathrm{K}(A) + \tilde{K}(\mathcal{M}^\perp).$$

Thus it suffices to capture every positive real number in the range of $\tilde{K}$.

Consider $0 < r \leq 1/4$. Choose $n \geq 3$ such that $1/n^2 < r \leq 1/(n-1)^2$. A computation shows that $1 - nr > 0$ and $\frac{n}{n-1}(1-nr) < 1$. Let

$$a_2 = \sqrt{\frac{n}{n-1}(1-nr)} \quad \text{and} \quad a_1 = \sqrt{1-a_2^2}.$$

Define an isometry R in $\mathfrak{R}_n$ by $R = a_1 R_1 + a_2 R_2^2$. Let $\mathcal{M}^\perp = R\mathcal{H}$. Then for words w with $|w| \geq k-1$,

$$Q_k R\xi_w = Q_k L_w R\xi_1 = Q_k(a_1\xi_{w1} + a_2\xi_{w2^2}) = 0.$$

Thus the trace is computed as

$$\begin{aligned}
\mathrm{tr}(P_{\mathcal{M}^\perp} Q_k P_{\mathcal{M}^\perp}) &= \sum_{|w|\leq k-2} (Q_k R\xi_w, R\xi_w) \\
&= \sum_{|w|=k-2} (a_1\xi_{w1}, R\xi_w) + \sum_{|w|<k-2} (R\xi_w, R\xi_w) \\
&= a_1^2 n^{k-2} + \frac{n^{k-2}-1}{n-1}.
\end{aligned}$$

Another computation yields,

$$\tilde{K}(\mathcal{M}^\perp) = (n-1)\lim_{k\to\infty} \frac{\mathrm{tr}(P_{\mathcal{M}^\perp} Q_k P_{\mathcal{M}^\perp})}{n^k} = \frac{(n-1)a_1^2}{n^2} + \frac{1}{n^2} = r.$$

The examples constructed show that the interval $(1/9, 1/4]$ is obtained in the range of $\tilde{K}$ with $n = 3$ and pure rank$(\mathcal{A}) = 1$. It follows that intervals of the form $(k/9, k/4]$ for $k \geq 1$ can be obtained with $n = 3$ and pure rank$(\mathcal{A}) = k$. ■

These examples are not as satisfying numerically as the decaying n-tuples since arbitrarily large n and pure ranks must be used. However, in the pure rank one case the associated $\mathfrak{L}_n$-invariant subspace is always cyclic. They also show how the connection with dilation theory can be used to derive information on the ranges of the invariants.

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