Inflation derivatives pricing with a forward CPI model

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

The Zero-Coupon Inflation Indexed Swap (ZCIIS) is a derivative contract through which inflation expectations on the Consumer Price Index (CPI) are actively traded in the US. In this thesis we consider different ways to use the information from the ZCIIS market for modeling forward inflation in a risk-neutral framework. We choose to implement a model using a Monte Carlo methodology that simulates the evolution of the forward CPI ratio. We prefer this approach for its flexibility, ease of implementation, instant calibration to the ZCIIS market and intrinsic convexity adjustment on the inflation-linked payoff. Subsequently, we present a series of results we obtain when modeling a chain of consecutive CPI ratios for simulating the evolution of spot inflation. Furthermore, we use this for pricing inflation caplets and floorlets. Finally, we use the intuition gained from this exercise to analyse our results for pricing inflation caps.

Acknowledgements

I am grateful to Professor Ken Vetzal from the School of Accounting and Finance and Professor Adam Kolkiewicz from the Department of Statistics and Actuarial Science at the University of Waterloo. Their guidance was key to writing this thesis. Thank you to my readers Professor Tony S. Wirjanto and Professor Carole Bernard who helped me polishing this document through their helpful comments. Finally, I would also like to express my gratitude to Fred Nastos and Nick Neary for having inspired me to tackle this project.

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À ma famille.

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Nomenclature

$I(t)$ Price index at time t

$\mathcal{I}_i(t)$ Forward contract on the price index maturing at time $T_i > t$ and defined as $E_{T_i}[I(T_i) \mid \mathcal{F}_t]$

$\mathcal{Y}_i(t)$ Forward inflation contract over period $[T_{i-1}, T_i]$ and defined as $E_{T_i}[I(T_i)/I(T_{i-1}) - 1 \mid \mathcal{F}_t]$

$Y_i(t)$ Forward inflation contract over period $[T_{i-1}, T_i]$ and defined as $\mathcal{I}_i(t)/\mathcal{I}_{i-1}(t) - 1$

$P(t,T)$ Discount factor for the nominal interest rate over period $[t,T]$

$P_R(t,T)$ Discount factor for the real interest rate over period $[t,T]$

$P_I(t,T)$ Discount factor for the inflation rate over period $[t,T]$

$P_n(t,T)$ Discount factors in the nominal currency (Foreign Currency Analogy)

$P_r(t,T)$ Discount factors in the real currency (Foreign Currency Analogy)

Chapter 1

Introduction

Inflation is an old concept used as a broad measure for the increase of prices and wages over time. The idea is of tremendous importance to long term investors who care about the "real" value of money: its purchasing power. It can be measured in various ways depending on the methodology, seasonal adjustments and locations covered by the underlying price index. In the US, the most familiar measure is the Consumer Price Index which covers all urban cities (CPI-U) and is non seasonally adjusted (nsa). In Canada, the main index is the all items CPI (as opposed to the Core CPI which excludes the most volatile components like food and energy). A price index is made up of the prices of hundreds of goods and services ranging from basic items like bread to more recent products such as computers. Prices are usually sampled on a monthly basis over the covered region and are combined to produce the overall index of prices. The inflation rate is then simply the variation in the overall price of an average basket of goods denoted by the price index.

The main purpose of a price index is to allow central banks to track inflation in order to formulate sound economic policies. The same concept can be applied in other areas as well but requires some methodological adjustments. For example, many pension funds in the US have their benefits tied to the Urban Wage Earners and Clerical Workers index (CPI-W). This index covers only a certain demographic with at least 50% of the household income is coming from clerical or wage paying jobs and at least one of the household's earners having been employed for at least 70% of the year. In a similar fashion, the Association of American Railroad (AAR) compiles a price index that covers all of the costs of operating a railroad network which include for example wages, fuel costs and interest rates. The AAR also publishes the All-Inclusive Less Fuel index (AAR AII-LF) which tends to follow quite closely the CPI. Many railroad shipping contracts in the US include agreements where the client's bill is bound to a certain percentage of this index appreciation which often includes a cap and a floor.

To standardize the market and enhance liquidity, inflation-indexed bonds issued by governments only track one version of the country's CPI. However we can see from Figure 1.1 that the CPI-U, CPI-W and AII-LF have a lot in common.[1]

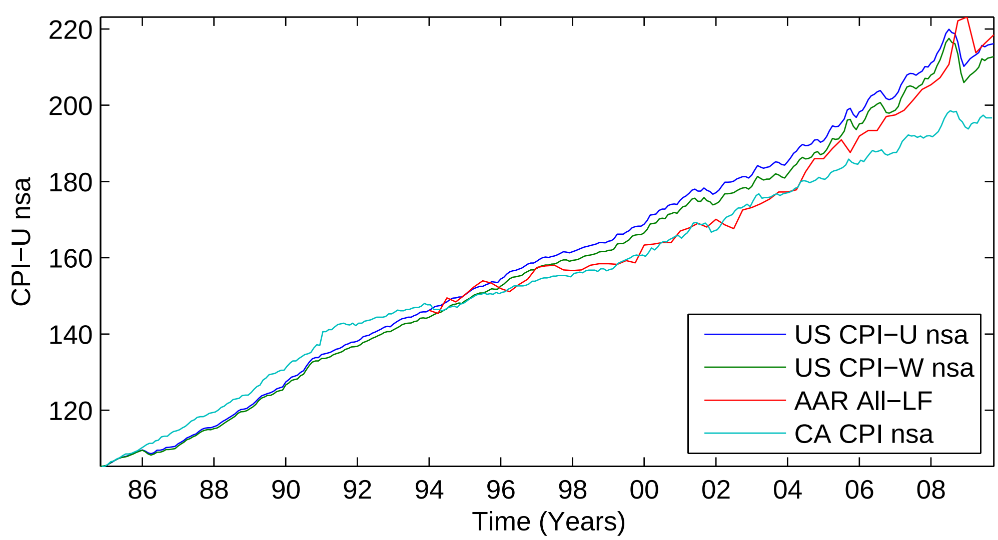

Figure 1.1: US CPI All Urban Average, US CPI Urban Wage Earners, AAR All-inclusive Less Fuel and the Canadian CPI.

Note that the different price indices have different degrees of similarity but only the CPI-U nsa is actively traded in the derivative market. This creates a basis risk for institutions that would like to hedge their liabilities if they are tied to a different index.

[1]The dataset AAR AII-LF is obtained from http://www.aar.org/ and the others are obtained from Datastream.

The behavior of consumers possesses some seasonal features such as a rise in consumption around Christmas time followed by a period of reduced spending in January. Climatic conditions can play a major role as well. For example, oranges can be purchased year-round but prices tend to be significantly higher in the summer months when the major sources of supply are between harvests. There exist seasonally adjusted publications of price indices and these are usually preferred in the formulation of economic policies. However, many collective bargaining contract agreements and pension plans tie compensation changes to a price index that is unadjusted for seasonal inflation. The seasonally unadjusted index is preferred for simplicity and to avoid the methodological bias of seasonal adjustments which can be subjective. Subsequently, it is no surprise that the underlying index of inflation derivatives is unadjusted as they are typically used to hedge contracts of the same nature. The following figure shows the auto-correlation of the US CPI All Urban Average over an increasing montly lag to outline the seasonality of the index.[2]

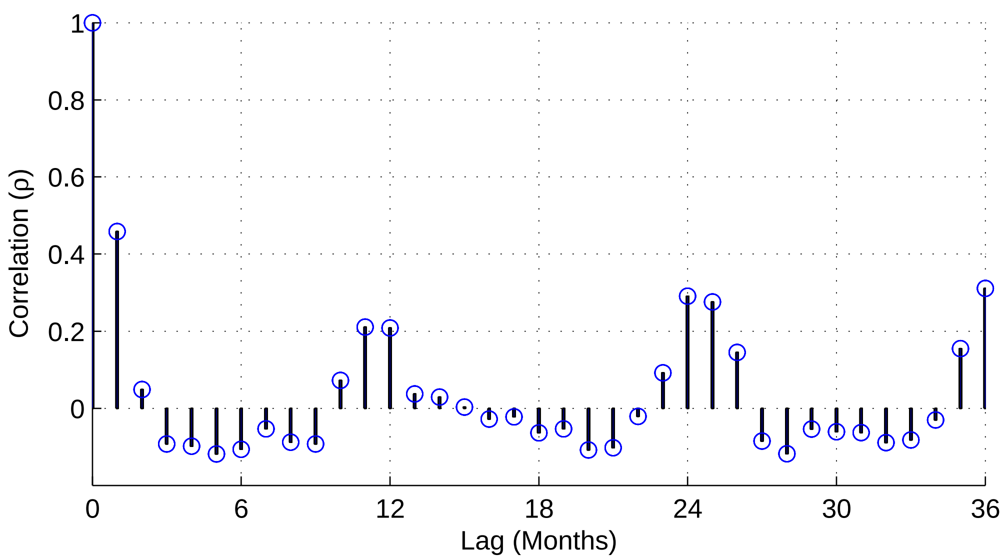

Figure 1.2: Inflation auto-correlation for different monthly lags in the time series.

[2]Data obtained from Datastream. Results were obtained using the Matlab cross-correlation function (see the xcorr function in the Signal Processing Toolbox) which was applied to a vector of monthly inflation minus the average monthly inflation.

Inflation has been well under control since the 1970's in North America and until recently there was a feeling that depression prevention was a problem that had been solved for all practical purposes.[3] The belief was that the business cycle had been tamed and economists should switch focus to things like long-term economic growth. This belief has now vanished as central banks scramble to stimulate their economies in an environment of low inflation and high unemployment. Target interest rates in the US and in Canada are as low as they can be, which forces central banks to resort to unconventional measures for stimulating their economies.

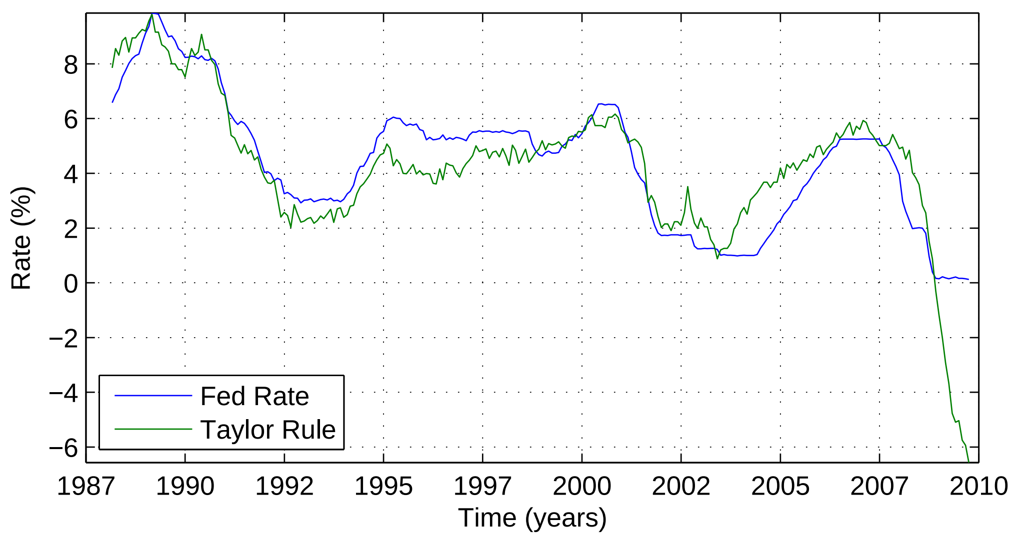

Figure 1.3: US Federal funds rate and a Taylor rule recommendation

Figure 1.3 shows a simple Taylor rule which is a rough rule of thumb for predicting target interest rates as a function of unemployment and inflation. It clearly shows the extent of the problem as it suggests that nominal interest rates should currently be deeply negative to fully account for the current economic environment.[4] However, the nominal rates are

[3] See Krugman [2009] for a concise review of past economic crises and similarities with the financial crisis of 2008.

[4] See Rudebusch [2009] for a detailed account of the Fed's response to the crisis. The data and an excel file that presents how the results from Figure 1.3 can be reproduced is located at http://www.frbsf.org/publications/economics/letter/2009/el2009-17.html. The Taylor rule is also available in Bloomberg under the ticker <TAYLOR>.

stuck against the zero barrier as people would much prefer to stick their money under a mattress rather then "collect" a negative interest rate. This means that the Federal Reserve has to resort to other more unconventional measures of quantitative easing to stimulate the economy. While it is clear that the current policies are inflationary in nature, the question is: are they effective and what impact will they have?

We can also ask ourselves if central banks might not be tempted to increase their inflation targets in the future given how limited the current monetary policy is. In a recent report from the IMF (Blanchard et al. [2010]), the authors point out that a higher average inflation and thus higher nominal interest rates to begin with would have given more room to central banks to cut interest rates more drastically during the crisis. The fact that nominal interest rates were quite low to begin with was a major limiting factor which could be prevented in the future by aiming for higher inflation. Another argument by which central banks may be tempted to increase target inflation is presented in Akerlof et al. [2000]. The authors argue that in an environment with very low inflation a significant number of workers should be taking wage cuts to get relative wages where they ought to be. They invoke the psychological literature on decision making and perception to propose that when inflation is low, wages and prices tend to respond less than proportionally to expected inflation. However, at a sufficiently high rate, anticipating inflation becomes a significant part of the industry's decision process which means that wages and prices respond fully to expected inflation. Their model even provides evidence that higher inflation could produce lower unemployment on a sustained basis.

The standard approach for modeling inflation is based on econometric models which attempt to forecast inflation given historical data. These can be used to gain intuition on how to relate the inflation rate to other macro economic variables such as the target rate and unemployment. However, in the context of option pricing we typically prefer using a risk neutral measure to ensure that the market is arbitrage free.

This thesis is divided three main chapters. In Chapter 2, we look at the different markets for trading inflation starting with the bond market and then we move on to common inflation derivatives. We highlight the connection between the two markets because any inflation derivative is ultimately hedged through the bond market. Then we discuss the main inflation derivatives that currently exist in the market. In Chapter 3, we provide a review the literature for what we believe to be the important milestones in inflation derivative pricing. Chapter 4 focuses on the implementation of a Monte Carlo simulation of the forward CPI ratio. Our contribution is mainly about how to succesfuly implement the model. We also present some empirical results from our parameter estimation. Furthermore, we provide some insights about the behavior of a portfolio of inflation options.

Chapter 2

Markets for trading inflation

Given the extent to which capital markets around the world have become integrated, one might expect a certain uniformity across inflation-indexed securities markets. However this is not quite the case for many reasons that are specific to each market. For example, measures of inflation vary quite a lot depending on the basket of goods and services we wish to track and the geographical region that is being covered. Some countries like the UK offer tax incentives on their inflation indexed-bonds which can induce a bias on the measures of inflation expectations. Issues such as liquidity, hedging costs and others are also major differentiating factors. For this reason we will also comment on how mature some markets have become depending on their geographical location[1]. We focus our attention mainly on the US, UK and finally Canada.

2.1 Inflation-indexed securities

Expected inflation varies over time which means that sovereign bonds that pay nominal yields are not safe under real terms for long term investors. There was a need for truly risk-less long-term investments and this is why different governments have started to issue inflation-indexed bonds. Another reason for issuing these bonds was to instil market trust

[1] See Deacon et al. [2004] for a detailed account of inflation-indexed securities markets by region.

in a country's macro economic policies. By issuing inflation-indexed debt governments were able to reduce their borrowing costs by sending out the message that they would be liable if inflation was allowed to slip out of control in the future.

It was in 1997 that the US Treasury started issuing Treasury Inflation Protected Securities (TIPS). The inflation index used for TIPS is the non seasonally adjusted US City Average All Items Consumer Price Index for all Urban Consumers (CPI-U). The principal on these securities is floored to par which means that the principal at maturity is protected against deflation but not the coupon payments. In other words, the floor is a European style option that only applies to the principal at maturity. Before maturity, coupons can be paid off a principal amount that is potentially less than the par amount. Since inflation is generally positive most TIPS that trade in the market have a floor that is quite deep out of the money and can be ignored for most practical purposes. However, for new issues the floor is at the money and should carefully be considered especially in a deflationary environment. The inflation escalation of the principal is taxable as an annual income in the US. For this reason, holding TIPS is rather unattractive for non-tax exempt investors since it creates a tax on unrealized gains resulting from the inflation-adjusted principal.

The UK Treasury has been issuing index-linked gilts (ILG) tied to the Retail Price Index (RPI) since 1981 and now these ILG make up roughly 30% of the UK bond market. These linkers tend to trade at a relatively rich level for two reasons. The first is a tight regulation in the pension fund arena in the UK that forces these institutions to match their inflation-linked liabilities through the linker market, thus creating a strong demand for ILG with long maturities. The second reason is that investors are taxed on their real return based on the yearly change in the RPI. Since most holders of ILG are long term investors (pension funds, insurance companies) that do not pay tax, the tax effect only has an impact on the short end of the real yield curve where private individuals are more involved in the market.

New issues of UK linkers use the Canadian model (discussed below) for the inflation index ratio but they co-exist in the market with an old style of linkers based on an 8-month RPI lag method. Infrastructure intensive businesses like railroad companies have also been big issuers of inflation-linked debt in the UK. These companies tend to have inflation-linked revenues and find that they can reduce their borrowing costs through inflation-linked debt issues.

In Canada, the Real Return Bond market (RRB) is based on the Non Seasonally Adjusted All Items Consumer Price Index (CPI All Items nsa). Domestic pension funds have been the main focus of RRBs and there have been only five issues so far. The year 2021 is the shortest maturity on the curve. Also, due to the buy and hold nature of pension funds, liquidity in the secondary market is limited. However, the demand from domestic pension funds has been growing quite fast and some are seeking an exposure through inflation swaps. Canadian residents will be taxed on the income from RRBs in a given year, while the inflation adjustment on the principal is also taxable but only as a capital gain when it is realized.

Most countries have adopted what is referred to as the Canadian model for the inflation index ratio needed to calculate de coupon payments. Initially implemented in Canada, this method was subsequently adopted by the US, UK and others as a way of reducing the indexation lag to three months and to better align with the methods already used in the bond market. Although the price index is usually compiled on a monthly basis, in practice there is a lag in the inflation index ratio used to calculate the coupon payments. The purpose of this lag is so that market participants know in advance the amount of the upcoming coupon which is needed for an appropriate level of accrual of the cash flow. An index ratio using the Canadian model is based on the daily linear interpolation of the 3-month lagging price index. We write the reference index $R(m,d)$ where m is an index on the monthly price index I and d is the day of the month where payment occurs:

$$R(m,d) = I(m-3) + \frac{d-1}{D_m}\left[I(m-2) - I(m-3)\right] \tag{2.1}$$

where D_m denotes the number of days in month m. The resulting index ratio used to compute the inflation-adjusted principal is then:

$$Index\ Ratio = \frac{R(m,d)}{R(m_0,d_0)} \tag{2.2}$$

where $R(m,d)$ denotes the reference index on the day of a coupon payment or maturity while $R(m_0,d_0)$ is the reference index at the issue date.

Inflation-indexed bonds come in various cash flow structures but the most common one is by far the *capital indexed bond*. For instance, these have been issued in the UK, US, France and Canada as well. In this structure, the coupon and the principal both increase in relation to the price index. As a general example, an annual interest payments would be denoted by $Nr\frac{R(m,d)}{R(m_0,d_0)}$ where N is the notional and r is the coupon rate. At maturity T, the bond would pay back the inflation-indexed notional $N\frac{R(m,d)}{R(m_0,d_0)}$.

General characteristics of the US, UK and Canadian market are summarized in the following table.[2]

	United States	United Kingdom	Canada
Generic name	Treasury Inflation Protected Securities (TIPS)	Index-Linked Gilts (ILG)	Real Return Bonds (RRB)
Linking Index	CPI All urban nsa	UK RPI	CPI All Items nsa
Indexation Lag	2-3 months	8 months or 2-3 months	2-3 months
Floor	Par Floor	No Floor	No Floor
Coupon Frequency	Semi-Annual	Semi-Annual	Semi-Annual
Tax on income	yes	yes	yes
Tax on capital gain	yes, on unrealized gain	no	yes, on realized gain only

Table 2.1: Details on US, UK and Canadian inflation-linked securities

[2]See Barclays [2008] for a complete version of this table.

2.2 Inflation derivatives

Inflation markets across the globe can be divided under four broad categories. In Deacon et al. [2004], what is referred to as a level one market is where there is no tradable instrument linked to inflation. Inflation swaps in this context are made only on the basis of matched trades. An example of this would be a case where an entity with inflation-linked revenues (such as a utility corporation) agrees to pay the rate of inflation in return for a fixed rate to a pension fund seeking to match its liabilities (in real terms). Spain is an example of a level one market.

A level two market is one where a few inflation-indexed instruments are tradable. Typically these are government bonds and can be used to gather a few data points on the real yield curve. However the number of instruments is limited and matched trades are still required for maturities that are not covered by this basic yield curve. Canada is an example of a level two market.

Level three encompasses markets where enough inflation-linked instruments exist to say that the market is somewhat close to being complete. Market participants can reasonably hedge inflation swaps through traded securities while others can enter cheap/rich trades and attempt to exploit anomalies in the term structure of inflation. Supply and demand for a given instrument takes the back seat as market participants become more concerned about swap reset risks, asset-swap rates and the convexity of the real yield curve. The UK and US are level three markets.

Finally, a level four market would be where the inflation-indexed swap market is so liquid that it drives the bond market. Such behaviors have already been reported in the Euro-zone market. Although a level four market might never fully be reached, the UK, US and Eurozone are certainly approaching it as their governments re-iterate their commitment to issuing inflation-indexed debt and as corporations become more accustomed to the benefits of managing inflation risk.

Market participants that supply inflation protection through the swap market in the US are levered investors such as hedge funds and banks proprietary trading desks. Recalling the broad classification of the US inflation swap market as a level three market, these investors typically hedge their inflation swap positions through long positions in TIPS and short positions in nominal Treasuries in the asset swap market.

2.2.1 Connecting the bond and swap markets

It should be clear by now that an inflation derivative market can only really take off in a level three or level four type of market according to the categories we have laid down in the previous section. The main building block for more complex inflation derivatives is the zero-coupon inflation swap (ZCIIS). Although this thesis will not focus on pricing the ZCIIS it is worth spending a little time to look at the mechanisms that connect this instrument to the bond market.

It is helpful to think of the inflation-indexed bond market in parallel to the usual nominal bond market. Taking this a step further, the asset swap market on these inflation-indexed securities is analogous to the standard asset swap market. Furthermore, one can conceptualize a real rate swap similar to a standard swap except that the fixed leg pays a fixed rate in real terms.

Inflation-linked asset swap

An asset swap on an inflation-linked security is the same thing as a nominal bond asset swap. It consists of the exchange of cash flows from a bond (inflation-linked or not) against a LIBOR flow (or a similar measure for floating interest rates) plus a spread. Asset swaps tie together the bond market and the swap market in both nominal and real terms.

The inflation-linked asset swap market is particularly developed in the UK and Eurozone. However, in the US the activity is concentrated in the bond market. This has consequences in modeling calibration which is best carried out using market data from liquid markets.

Real rate swaps

Real rate swaps are seldom used in developed markets because most investors choose to manage nominal interest rates and inflation exposure separately. However it is useful conceptually to make the bridge between inflation-linked asset swaps and zero-coupon inflation swaps (see figure 2.1 below). The real rate swap consists of exchanging a floating rate against a fixed nominal rate plus an inflation rate or; in other words, a fixed rate under real terms. This structure is most commonly found in emerging markets where inflation risk is substantial.

Connecting the different markets

The following figure illustrates how nominal and real interest rates are traded in parallel.[3] Real rate swaps is not very active but it is useful conceptually to make meaningful comparisons with the nominal swap market. As a first approximation to inflation expectations, one can compare the yield on an inflation-indexed bond to the yield on a standard bond paying a nominal interest from the same issuer and with a similar cashflow structure and maturity. Depending on the geographic region, the bond market or the asset swap market might be more appropriate for this comparision because of differences in liquidity. Hedging an inflation swap is more easily conceptualized as being long the floating leg of a real swap while also being short the floating leg of a nominal swap. In practice however, this kind of trade would probably be done through the bond market or the asset swap market because these are more liquid.

[3] See Deacon et al. [2004] for more insights on inflation trading practices. Also see Mirfendereski [2007] for a more up to date perspective.

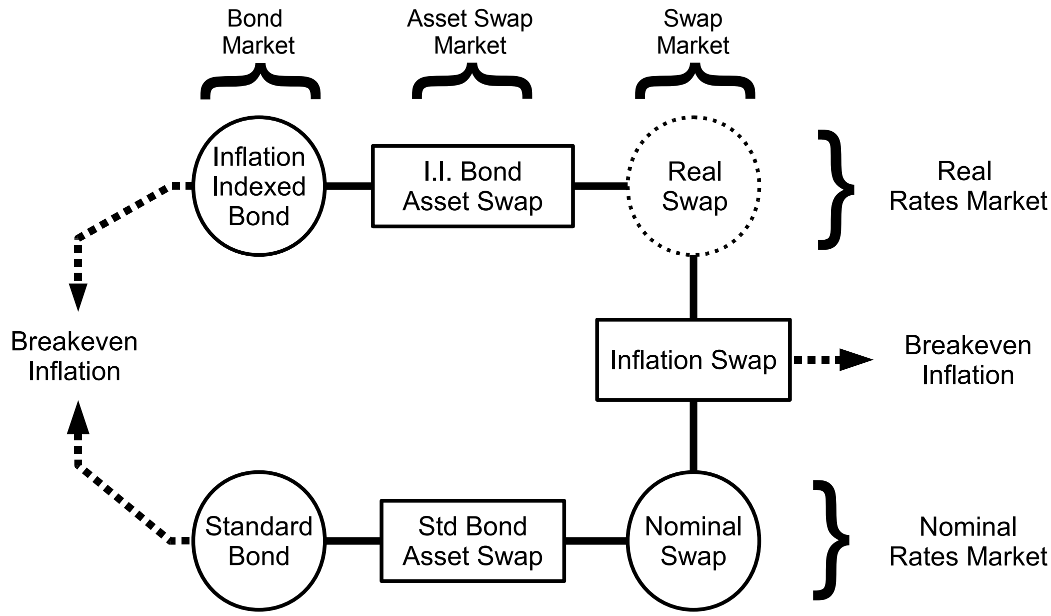

Figure 2.1: Inflation markets conceptual diagram.

2.2.2 Zero coupon inflation swaps

Most of the liquidity in the inflation derivatives market is concentrated in the *zero-coupon inflation-indexed swap* (ZCIIS). As the name suggests, this contract triggers only one cash flow which happens at maturity. The value of the floating leg will vary according to the inflation rate when the swap reaches the end of its period. The swap is quoted in terms of the fixed rate K for which the fixed leg yields a rate of return that is exactly the average (lagging) inflation rate that is needed for the swap to have a value of zero when the contract is initiated. For this reason, the fixed rate of the zero-coupon swap is analogous to the break even inflation rate from the inflation-indexed bond market but captures the idea in a more natural way. The fixed leg consists of receiving (paying) a fixed amount in T years of:

$$N\left[(1+K)^{T}-1\right] \tag{2.3}$$

where K is the quoted fixed rate and N is the notional value of the swap. The floating leg of the contract pays (receives) the real value of the notional which depends on the rate of inflation during the life of the swap:

$$N\left[\frac{I(m-L)}{I(m_0-L)}-1\right] \tag{2.4}$$

where m is the time index in months and L is the lag in the index (usually $L=2$ or $L=3$ months). So we have $I(m-L)$ as the lagged price index at maturity and $I(m_0-L)$ is the lagged price index at the start date. Zero-coupon swaps typically trade on the same underlying price index as the government bonds in the same market. This means that zero-coupon swaps always trade on non-seasonally adjusted price indices.

The following figure shows ZCIIS quotes in terms of the fixed rate K for various maturities. Note that before the 2008 crisis the term structure for the inflation rate was practically flat but became very steep during the crisis. The curve on November 26^{th} is roughly one of the steepest we have observed. However one should be careful when interpreting the economic meaning of these numbers. It is very unlikely that the market has ever really predicted 3% deflation. A more plausible explanation is that the TIPS market was suddenly overflowed during the crisis by some participants who were scrambling to liquidate their position in a market that is fairly illiquid. Since TIPS and their asset swap equivalent are the only vehicle for hedging the ZCIIS the liquidity shock would naturally be reflected in the derivatives market as well. In Chapter 4, we implement a model that uses the ZCIIS curve as one of its inputs. We choose to focus on the curve observed on April 23^{rd} because it exhibits some deflationary expectations and is presumably not so affected by the liquidity issues we have discussed. In Section 4.3.4 we also compare the delta of a cap observed on November 26^{th} and the same delta observed on April 23^{rd}. We conclude that the steepness of the inflation term structure can have a great impact on our ability to delta-hedge an inflation cap due to the difference in moneyness of each caplet that form the cap.

We will also see in Section 3.2 that there exists a one to one relationship between the

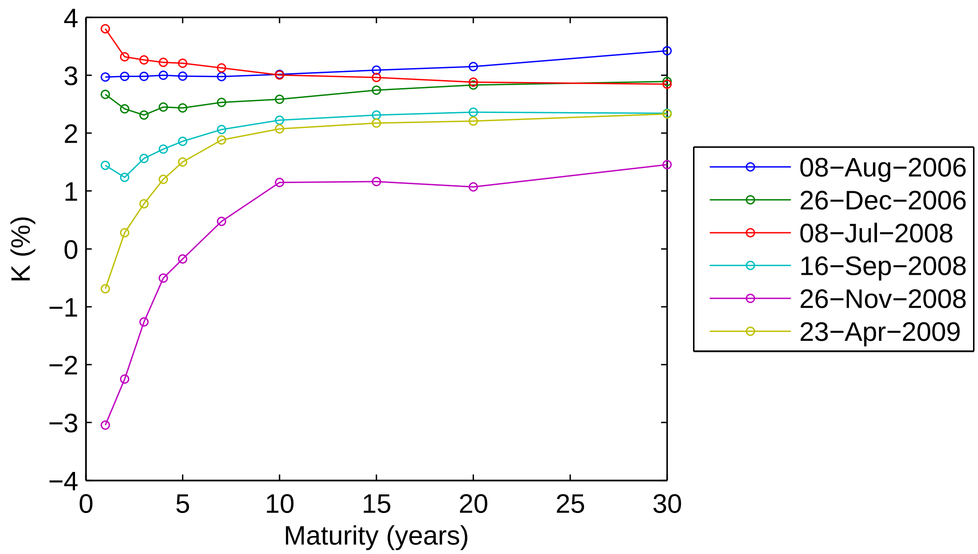

Figure 2.2: Term structure of expected inflation from zero-coupon inflation swaps

ZCIIS and a forward CPI contract denoted by $\mathcal{I}_i$:

$$\mathcal{I}_i(t) = I(t)(1 + K_i(t))^{T_i} \tag{2.5}$$

where $I(t)$ is the actual value of the CPI while K_i is a quote for the ZCIIS maturing at time T_i. In Section 3.5 we discuss how the forward CPI contract is a very useful notion as it allows us to define forward inflation as a forward CPI ratio in way that is analogous to how forward interest rates are defined through a ratio of discount factors.

Zero-coupon inflation indexed swap quotes can be obtained from Bloomberg under the ticker <USSWIT1 Curncy> for 1 year maturity, <USSWIT2 Curncy> for 2 year maturity and so on. Data is available for maturities of 1,2,3,4,5,7,10,15,20 and 30 years.

2.2.3 Year-on-year inflation swaps

The *year-on-year inflation-indexed swap* is the most common alternative to ZCIIS for trading inflation within the derivative arena. It involves one counterparty agreeing to receive an annual coupon determined by the year-on-year inflation rate against paying a fixed rate. In some cases the non-inflation leg of the contract could also be defined as a floating rate (LIBOR) plus a spread instead of a fixed rate.

At the end of each time interval $[T_{i-1}, T_i]$ cash flows from the fixed leg and floating leg are again exchanged in proportion to the nominal value N. At time T_i, the end of the period, the fixed leg pays

$$N\psi_i K \tag{2.6}$$

where ψ_i is the day count fraction for interval $[T_{i-1}, T_i]$. The floating leg pays

$$N\psi_i \left[\frac{I(T_i - L)}{I(T_{i-1} - L)} - 1 \right] \tag{2.7}$$

where L is the lag in the index. The contract's start date is $T_0 = 0$. Also note that the lag in the index means that we always know in advance the amount of the next payment. Later, when we strip the YYIIS into other instruments we will drop L to simplify the notation. Also note that it is not straightforward to reconciliate YYIIS and ZCIIS quotes because of the convexity adjustment that is built into the YYIIS payoffs. We will discuss this in further detail in the next chapter.

2.2.4 Inflation caps and floors

It is common in the US to issue TIPS where the principal is floored at par to protect the holder against deflation. However, the coupons are not protected and are still subject to a deflated principal. This structure effectively provides the holder of these TIPS with a "pure" inflation-linked strips with a floor on their inflation adjusted principal.

In the derivative world, banks sometimes trade inflation caps and floors over the counter with their institutional clients. These could also be referred to as *inflation-indexed year-on-year coupon floors* and *inflation-indexed year-on-year coupon caps*. Typical clients conduct a business that leads them to naturally have large cash flows that somehow fluctuate with inflation which they want to hedge. Thus, inflation caps and floors are usually valued once when they are initiated by both parties who will then hold their end of the contract until maturity. Floors with low strikes are the most active market for inflation options.

Inflation caps and floors quotes are given in terms of their strike K. At reset date T, these would pay:

$$N\left[\omega\left(\frac{I(T)}{I(0)} - 1 - K\right)\right]^{+} \tag{2.8}$$

where we let $\omega = 1$ for the cap, $\omega = -1$ for the floor and N is the notional. Thus, the present value of cap or floor is written as

$$\mathbf{IICapFloor}(t = 0, T, K, N, w) = N\sum_{i=1}^{M} P(t, T_i)E^{T_i}\left[\left[\omega\left(\frac{I(T_i)}{I(t)} - K\right)\right]^{+} \middle| \mathcal{F}_t\right] \tag{2.9}$$

where each expectation is taken under its own forward measure $\mathcal{Q}_{T_i}$.

Like the YYIIS, inflation options have a built in convexity adjustment in their payoff. For this reason, inflation options can easily be calibrated on the forward inflation curve through the YYIIS market. Unfortunately, most of the liquidity is located in the ZCIIS market where we will require a change of measure to mimic this convexity adjustment if we wish to calibrate on the ZCIIS forward inflation curve. We discuss this approach further in Section 3.4.

2.2.5 Inflation futures

In the U.S. there is currently no such thing as exchange traded futures on the CPI. In 2004, the Chicago Mercantile Exchange launched an electronically traded futures contract on the U.S. Consumer Price Index. This contract was later discontinued presumably because it failed to attract enough participants. However, futures contracts on the Eurozone price index, the Harmonized Index of Consumer Prices excluding tobacco (HICP), do trade on the CME. This instrument trades with an underlying of 10,000 euros times the HICP Futures index defined as 100 less the annual inflation rate. More specifically, the final settlement price is calculated as

$$100 - \left[100\left(\frac{HICP(t)}{HICP(t-12)} - 1\right)\right] \tag{2.10}$$

where t is the monthly time index on the HICP publications. At any given time, the available maturities run monthly up to one year ahead. Euro denominated futures on the HICP are also traded in a similar fashion on the Eurex exchange. However it is important to note that the volumes traded in this market are extremely low. Moreover, trading inflation over the next twelve months is not very interesting for long term investors or an institution with long term inflation-linked liabilities.

2.3 Canadian perspective

It has been reported that demand from Canadian institutional investors for inflation-linked instruments has grown proportionately to the levels of demand in the UK and US. Oddly, the market for inflation derivatives in Canada has not taken off while others are growing steadily as both countries reiterate their commitment to issue inflation-linked debt. The amount of inflation-linked bond issues for all G-7 countries more than tripled between 2000 and 2006 to reach US$1 trillion outstanding. The UK, US and Eurozone respectively had outstanding amounts of US$403, $260 and $257 billions at the end of this period. In comparison, Canada had only $36 billion outstanding which represented less than 10% of its total debt while the UK for example had more than a quarter of its debt as inflation-indexed issues. Note that it is not only the small size of the Canadian market that restrains the development of inflation derivatives in Canada but also the maturity profile of our government's inflation-linked debt. Canada has issued only four 30-year bonds which will mature in 2021, 2026, 2031 and 2036 meaning that there are very few possibilities for hedging inflation derivatives.

The lack of real return bonds with short maturities has been reported by economists as one of the missing links required for appropriately measuring inflation expectations through the calculation of break-even inflation rates. For this reason, it has been proposed that the Government of Canada start issuing one, two and five years real return bonds (see Smith [2009]). This could happen as soon as 2011 when the Bank of Canada and the federal government review the framework under which they manage inflation. Undoubtedly, a strong commitment from these two institutions to enhance the size and maturity profile of the real return bond market would allow dealers to better hedge inflation swaps and mark a big step in the evolution of this market in Canada.

Chapter 3

Inflation models

There are two main approaches in the inflation model literature for pricing inflation derivatives. The first that we will present is the foreign-currency analogy which models the price index as a currency between a nominal and real economy having their own respective term structure. This analogy is a natural one in the sense that a large body of literature already exists in the area of cross-currency term structure models. However, one downside of the foreign-currency analogy is the difficulty of estimating historically the real rate parameters. Most of the recent publications have moved away from the foreign-currency analogy in favor of market models which attempt to model directly the market observables. Thus, we will also present a series of market models which allow for a more realistic behavior of interest rates and inflation rates. An important point to note about these market models is that they rely on forward CPI or forward inflation contracts as their model primitive. It will be shown that these contracts are traded implicitly through zero-coupon inflation swaps and year-on-year inflation swaps.

In the next chapter where we implement a simulation based on Monte Carlo we choose to focus on modeling the forward CPI ratio presented in Section 3.5. Thus, a particular attention should be devoted to this section before proceeding to the next chapter.

3.1 The Foreign-Currency Analogy Framework

3.1.1 Pricing on multiple term structures

The first publication to ever tackle pricing of inflation-linked derivatives is based on a foreign currency analogy first developed in Jarrow and Turnbull [1998] which was subsequently applied to pricing inflation-indexed securities in Jarrow and Yildirim [2003]. The main argument developed in Jarrow and Turnbull [1998] consists of a unified approach for pricing contingent claims on multiple term structures. The authors show that for every traded contract delivering an asset i at some maturity T in the future, the term structure $v_i(t,T)$ representing the price at time t of such contracts can be written in terms of a similar normalized term structure $p_i(t,T) \equiv \dfrac{v_i(t,T)}{e_i(t)}$ where $e_i(t)$ is the spot price of the deliverable asset i:

$$v_i(t,T) \equiv p_i(t,T)e_i(t) \tag{3.1}$$

This decomposition assumes that the securities $v_i(t,T)$ are traded for all assets i and all intervals $[t,T]$. It also assumes that zero-coupon bonds are part of these traded securities and deliver a sure dollar at maturity T. However note that the assumption that the asset i can be stored is not required because no short selling is required.

The equation 3.1 implies that one can specify a stochastic process for modeling both the normalized term structure $p_i(t,T)$ and the spot price $e_i(t)$ to characterize the evolution of the term structure $v_i(t,T)$. This is an important result for which the foreign-currency analogy provides an interesting interpretation. We have

- $p_i(t,T)$ as a zero-coupon bond denominated in units of a foreign currency i
- $e_i(t)$ as the spot exchange rate of domestic currency per unit of currency i
- $v_i(t,T)$ as the value of the i^{th} foreign currency zero-coupon bond quoted in the domestic currency

We define implicitly the instantaneous forward rate $f_i(t,T)$ quoted in the foreign currency

i for the zero-coupon bonds $p_i(t,T)$:

$$p_i(t,T) = \exp\left\{-\int_t^T f_i(t,u)du\right\} \tag{3.2}$$

assuming $\dfrac{\partial \log p_i(t,T)}{\partial T}$ exists. Now, substituting 3.2 in 3.1 we obtain the desired result:

$$v_i(t,T) = \exp\left\{-\int_t^T f_i(t,u)du\right\} e_i(t) \tag{3.3}$$

In words, equation 3.3 allows us to model the stochastic structure of the instantaneous forward rates $f_i(t,T)$ and spot exchange rates $e_i(t)$ to characterize a foreign term structure in the domestic currency. The following section shows how this concept can be extended for modeling inflation-indexed securities.

3.1.2 JY model for Inflation-Indexed Securities

We now take the foreign-currency analogy a bit further to apply the same concept to inflation and real interest rates. The argument developed in the previous section underpins the idea that nominal rates, real rates and the price index can be modeled as three different processes without being worried about any redundancy. This is presented in Jarrow and Yildirim [2003] where the authors apply the idea to inflation-indexed securities pricing.

Imagine two currencies which we will call the nominal currency and the real currency. Like real currencies, these two are associated with an interest rate that prevails in their respective economies: nominal interest rates and real interest rates. Now imagine it would be possible to trade the nominal currency for the real currency. What would be a fair exchange rate between the two? Denote $I(0)$ as the price index on a reference basket of goods in the nominal economy and $I(T)$ as another price index on the same basket of goods taken within the real economy. Under the no-arbitrage assumption the logical conclusion is that the fair exchange rate to trade the nominal currency for the real currency is $I(T)/I(0)$.

Note that the exchange rate will depend on the interval $[0,T]$ and that we do not actually know the real interest rate or the value of the price index at time T. However, expectations regarding real interest rates are traded through inflation-linked securities. These form the basis of an expected term structure for real rates and one could effectively lock in the real rate for some interval $[0, T]$ through the inflation market.

The model proposed by Jarrow and Yildirim [2003] defines two instantaneous forward interest rate processes for the nominal and real economy as f_n and f_r respectively. Recalling the notation from equation 3.2, they can be expressed as follows

$$P_k(t,T) = \exp\left\{-\int_t^T f_k(t,u)du\right\} \ , \ k \in \{n,r\} \tag{3.4}$$

where $P_n(t,T)$ is the time t price of a T maturing nominal zero-coupon bond denominated in dollars (the "nominal currency") and $P_r(t,T)$ is the price of a real zero-coupon bond denominated in units of the underlying price index (the "foreign currency").

A three factor model is defined on the probability space (Ω, F, P) where we let the standard filtration $\{\mathcal{F}_t : t \in [0,T]\}$ be generated by three Brownian motions $(W_n(t), W_r(t), W_I(t) : t \in [0,T])$. The two forward interest rate processes are defined as solutions to the following diffusion processes:

$$\begin{aligned} df_n(t,T) &= \alpha_n(t,T)dt + \sigma_n(t,T)dW_n(t) \\ df_r(t,T) &= \alpha_r(t,T)dt + \sigma_r(t,T)dW_r(t) \end{aligned} \tag{3.5}$$

where we have $\rho_{n,r}$ as the correlation between the Brownian motions W_n and W_r. The price index $\mathcal{I}(t)$ is modeled as a log-normal process:

$$\frac{d\mathcal{I}(t)}{\mathcal{I}(t)} = \mu_I(t)dt + \sigma_I W_I(t) \tag{3.6}$$

where by $\rho_{n,I}$ and $\rho_{r,I}$ we denote the correlation coefficients between the Brownian motions W_I and W_n and W_r, respectively. The functions for the drifts $\alpha_n, \alpha_r and \mu_I$ are random and F_t-adapted. The volatilities must satisfy the boundedness condition $\int_0^T \sigma_x^2(v,T)dv < \infty$,

$k \in \{n, r, I\}$ almost surely. The deterministic hypothesis on all the volatility coefficients allows the authors to obtain closed formulas on the value of the floating leg of the inflation swap. Additionally, Black-Scholes formulas are obtained for inflation options valuation which essentially depend on the volatility coefficients and the factors correlations.

In Amin and Jarrow [1991] it is shown that these three processes in 3.5,3.5 and 3.6 are arbitrage-free if there exists a unique equivalent probability measure Q such that

$$\frac{P_n(t,T)}{B_n(t)} \; , \; \frac{I(t)P_r(t,T)}{B_n(t)} \text{ and } \frac{I(t)B_r(t)}{B_n(t)} \quad \text{are } Q\text{−martingales} \tag{3.7}$$

where $B_k(t) = exp\left\{\int_0^t r_k(v)dv\right\}$, $k \in \{n, r\}$ is the time t money market account value in the nominal and real currency and $r_k(t) \equiv f_k(t,t)$, $k \in \{n, r\}$. The authors state three conditions for this to hold among which we have the *Fisher equation* often used in macroeconomics. It relates the nominal interest rate to the real interest rate and the expected inflation rate:

$$\mu_I(t) = r_n(t) - r_r(t) - \sigma_I(t)\lambda_I(t) \tag{3.8}$$

where $\lambda_I(t)$ is the inflationary risk premium.

Recalling the equation 3.1, the authors define the price in dollars of a real zero-coupon bond $P_{TIPS}(t,T)$ (in reference to the US market) in terms of the inflation index and a real zero-coupon bond in units of the same inflation index:

$$P_{TIPS}(t,T) = I(t)P_r(t,T) \tag{3.9}$$

For such a zero-coupon bond $P_{TIPS}(t,T)$, they find that under the above martingale measure the evolution of the price can be expressed as

$$\frac{dP_{TIPS}(t,T)}{P_{TIPS}(t,T)} = r_n(t)dt + \sigma_I(t)dW_I(t) - \int_t^T \sigma_r(t,s)dsW_r(t) \tag{3.10}$$

One drawback of the Jarrow-Yildirim model is that it was initially intended for pricing inflation indexed-securities and it was not clear at the time how it could be extended to price derivative pricing beyond a simple European call on the price index. Such a contract is not very useful in practice since the price index itself is not traded and one probably wishes to hedge an inflation rate over a given notional rather than the price index itself. We describe in the following section how the foreign-currency analogy can be extended for pricing more useful contracts such as the zero-coupon inflation swap and the year-on-year inflation swap.

Note that the assumption in 3.7 that $\frac{I(t)B_r(t)}{B_n(t)}$ is a Q-martingale is rejected in Hinnerich [2008]. She argues that the real money market account is fictive and it cannot be considered as a traded asset and consequently, no economic argument justifies this assumption. Nonetheless, she offers a different proof that leads to the same result as Jarrow and Yildirim [2003] from which it can be obtained that $\frac{I(t)B_r(t)}{B_n(t)}$ is indeed a Q-martingale; so the JY model still holds.

3.1.3 Stripping Inflation Derivatives

Pricing inflation derivatives through the foreign currency analogy implies that we have the appropriate apparatus for obtaining quotes for real zero coupon bonds which is not ideal. However, the concepts developed in the following two sections may be useful in the sense that they allow us to strip real zero coupon bond prices from the quotes of zero-coupon and year-on-year inflation-indexed swaps. We first present those results and finally conclude on a note that prepares us to move away from the foreign-currency analogy in a consistent fashion in order to pave the way for the market models.

In the following discussion, we write the nominal and real instantaneous short rates, respectively, as

$$\begin{aligned} n(t) &= f_n(t,t) \\ r(t) &= f_r(t,t) \end{aligned} \tag{3.11}$$

More importantly, we denote by Q_n and Q_r the nominal and real risk-neutral measures and E_n and E_r the expectation to which they are associated.

Stripping the Zero-Coupon Inflation Swap

The foreign-currency analogy enables us to strip real zero-coupon bond prices from quoted prices of zero-coupon inflation-indexed swaps. This derivation is expressed nicely in Mercurio [2005] using a standard no-arbitrage pricing theory. Recall the value of the inflation indexed leg of the zero-coupon inflation-indexed swap (ZCIIS) from equation 2.4. At time t, the discounted payoff of the floating leg for a notional of one dollar is

$$\mathbf{ZCIIS}(t,T_0,T_M) = E_n\left[e^{-\int_t^{T_M} r(u)du}\left(\frac{I(T_M)}{I(T_0)}-1\right)\,\middle|\,\mathcal{F}_t\right] \tag{3.12}$$

where we have $T_0 < t < T_M$ and $\mathcal{F}_t$ is the σ-algebra generated by the relevant underlying processes up to time t. By the foreign-currency analogy and recalling the third measure Q in 3.7, we have

$$I(t)P_r(t,T) = I(t)E_r\left[e^{-\int_t^T r(u)du}\,\middle|\,\mathcal{F}_t\right] = E_n\left[e^{-\int_t^T n(u)du}I(T)\,\middle|\,\mathcal{F}_t\right] \tag{3.13}$$

In words, the nominal price of a real zero-coupon bond equals the nominal price of the contract paying off one unit of the price index at the bond maturity. Substituting the result from 3.13 in 3.12 we obtain

$$\mathbf{ZCIIS}(t,T_0,T) = \left(\frac{I(t)}{I(T_0)}P_r(t,T) - P_n(t,T)\right) \tag{3.14}$$

which simplifies to the following if we value the contract at the issue date $t = T_0$:

$$\mathbf{ZCIIS}(t,T) = P_r(t,T) - P_n(t,T) \tag{3.15}$$

Note that this last expression for the inflation-indexed leg of the ZCIIS is model-independent and is not based on specific assumptions on the evolution of the interest rate market. It simply follows from the absence of arbitrage. This result can be used for obtaining real zero-coupon bond prices from the quoted prices of zero-coupon inflation-indexed swaps. Recall that the ZCIIS quotes are given such that the floating leg and the fixed leg have the same present value. Thus, at $t = T_0$, we have:

$$\begin{aligned} P_r(t,T) - P_n(t,T) &= P_n(t,T)\left(\left[(1+K)^T - 1\right]\right) \\ P_r(t,T) &= P_n(t,T)(1+K)^T \end{aligned} \tag{3.16}$$

This result is important because we will see later that it is possible to obtain an expression for the forward CPI which is analogous to this one. Thus, it shows the consistency between the foreign currency analogy and the more recent market models.

Stripping the Year on Year Inflation Swap

We now outline various ways of stripping the year on year inflation swap (YYIIS) which are presented in Mercurio [2005]. We first look at how to express the YYIIS in terms of the zero coupon inflation indexed swap (ZCIIS). We follow with an expression in terms of the nominal and real zero coupon bonds and conclude with an expression that uses nominal and real forward rates which can be conveniently modeled as a lognormal process.

Recall expression 2.4 for the payoff of the YYIIS floating leg. To simplify the notation we use a notional of one dollar, we ignore the day count fraction and we ignore the lag in the index. Now, each payoff from the floating leg of the YYIIS can be discounted back as

ZCIIS in the following manner:

$$\begin{aligned}
\mathbf{YYIIS}(t, T_{i-1}, T_i) &= E_n\left[e^{-\int_t^{T_i} n(u)du}\left(\frac{I(T_i)}{I(T_{i-1})} - 1\right) \,\Big|\, \mathcal{F}_t\right] \\
&= E_n\left[e^{-\int_t^{T_{i-1}} n(u)du} E_n\left[e^{-\int_{T_{i-1}}^{T_i} n(u)du}\left(\frac{I(T_i)}{I(T_{i-1})} - 1\right) \,\Big|\, \mathcal{F}_{T_{i-1}}\right] \,\Big|\, \mathcal{F}_t\right] \\
&= E_n\left[e^{-\int_t^{T_{i-1}} n(u)du} \;\; \mathbf{ZCIIS}(t = T_{i-1}, T_{i-1}, T_i) \,\Big|\, \mathcal{F}_t\right]
\end{aligned} \tag{3.17}$$

Since $\mathbf{ZCIIS}(t = T_{i-1}, T_{i-1}, T_i)$ is the contract's value at issue date $t = T_{i-1}$, we can replace with expression 3.15 which allows us to value in terms of the discounted real zero coupon bond:

$$\begin{aligned}
\mathbf{YYIIS}(t, T_{i-1}, T_i) &= E_n\left[e^{-\int_t^{T_{i-1}} n(u)du}(P_r(T_{i-1}, T_i) - P_n(T_{i-1}, T_i)) \,\Big|\, \mathcal{F}_t\right] \\
&= P_n(t, T_{i-1}) E_n\left[(P_r(T_{i-1}, T_i) - P_n(T_{i-1}, T_i)) \mid \mathcal{F}_t\right] \\
&= P_n(t, T_{i-1}) E_n\left[P_r(T_{i-1}, T_i)) \mid \mathcal{F}_t\right] - P_n(t, T_{i-1}) P_n(T_{i-1}, T_i) \\
&= P_n(t, T_{i-1}) E_n^{T_{i-1}}\left[P_r(T_{i-1}, T_i) \mid \mathcal{F}_t\right] - P_n(t, T_i)
\end{aligned} \tag{3.18}$$

Finally, we can change the measure of the expectation in the last line above from $Q_n^{T_{i-1}}$ to $Q_n^{T_i}$:

$$E_n^{T_{i-1}}\left[P_r(T_{i-1}, T_i) \mid \mathcal{F}_t\right] = \frac{P_n(t, T_i)}{P_n(t, T_{i-1})} E_n^{T_i}\left[\frac{P_r(T_{i-1}, T_i)}{P_n(T_{i-1}, T_i)} \,\Big|\, \mathcal{F}_t\right] \tag{3.19}$$

This allows us to rewrite the expectation as follows

$$\mathbf{YYIIS}(t, T_{i-1}, T_i) = P_n(t, T_{i-1}) E_n^{T_i}\left[\frac{P_r(T_{i-1}, T_i)}{P_n(T_{i-1}, T_i)} \,\Big|\, \mathcal{F}_t\right] - P_n(t, T_i) \tag{3.20}$$

3.1.4 A market model for YYIIS

Recall from the foreign currency analogy that $P_r(t, T_i)$ is the price of an asset in the real economy while $I(t)P_r(t, T_i)$ is the price of the same asset quoted in the nominal currency. We formally introduce for the first time the definition for the forward CPI through the foreign currency analogy as follows:

$$\mathcal{I}_i^{FCA}(t) := I(t)\frac{P_r(t, T_i)}{P_n(t, T_i)} \tag{3.21}$$

where $I(t)P_r(t, T_i)$ is a tradable asset in the economy (inflation indexed securities). This definition will be used later to bridge the gap between the foreign currency analogy and the more recent market models. In the mean time, we quickly look at how inflation derivative pricing can be tackled using the above definition.

Assuming a lognormal dynamic for $\mathcal{I}_i^{FCA}(t)$ under the $Q_n^{T_i}$ measure we have

$$\frac{d\mathcal{I}_i^{FCA}(t)}{\mathcal{I}_i^{FCA}(t)} = \sigma_{I,i} dW_i^I(t) \tag{3.22}$$

We also assume a lognormal LIBOR market model for both nominal and real interest rates. Recall the foreign currency analogy and use the result from Schlogl [2002] to obtain the dynamics of the nominal and real forward rates, $F_n(t; T_{i-1}, T_i)$ and $F_r(t; T_{i-1}, T_i)$ where $\mathcal{I}_i(t)$ is the exchange rate between the nominal and real economies, under the $Q_n^{T_i}$ measure as follows:

$$\frac{dF_n(t; T_{i-1}, T_i)}{F_n(t; T_{i-1}, T_i)} = \sigma_{n,i} dW_i^n(t) \tag{3.23}$$

$$\frac{dF_r(t; T_{i-1}, T_i)}{F_r(t; T_{i-1}, T_i)} = -\rho_{I,r,i}\sigma_{I,i}\sigma_{r,I} dt + \sigma_{r,i} dW_i^r(t) \tag{3.24}$$

where $\sigma_{n,i}$ and $\sigma_{r,i}$ are positive constants while W_i^n and W_i^r are two Brownian motions with the usual corresponding correlations ($dW_i^n dW_i^r = \rho_{n,r,i} dt$, $dW_i^n dW_i^I = \rho_{n,I,i} dt$, $dW_i^n dW_i^r = \rho_{r,I,i} dt$).

Note that from 3.21 that we have

$$\begin{aligned}\frac{\mathcal{I}_i^{FCA}(t)}{\mathcal{I}_{i-1}^{FCA}(t)} &= \frac{P_r(t,T_i)P_n(t,T_{i-1})}{P_n(t,T_i)P_r(t,T_{i-1})} \\ &= \frac{P_n(T_{i-1},T_i)}{P_r(T_{i-1},T_i)} \end{aligned} \tag{3.25}$$

Furthermore, it was noted in Schlogl [2002] that there exists a precise relation between two consecutive forward CPIs and their respective nominal and real forward rates:

$$\frac{\mathcal{I}_i^{FCA}(t)}{\mathcal{I}_{i-1}^{FCA}(t)} = \frac{1+\tau_i F_n(T_{i-1};T_{i-1},T_i)}{1+\tau_i F_r(T_{i-1};T_{i-1},T_i)} \tag{3.26}$$

where $\tau_i = T_i - T_{i-1}$ is the tenor of the forward rate. This expression allows us to rewrite the expectation in 3.20 terms of the nominal and real forward rates as:

$$\mathbf{YYIIS}(t,T_{i-1},T_i) = P_n(t,T_{i-1})E_n^{T_i}\left[\frac{1+\tau_i F_n(T_{i-1};T_{i-1},T_i)}{1+\tau_i F_r(T_{i-1};T_{i-1},T_i)} \;\middle|\; \mathcal{F}_t\right] - P_n(t,T_i) \tag{3.27}$$

Under the above dynamics all the nominal and real forward rates can easily be modeled and used to calculate the expectation in 3.27. In Mercurio [2005], the author points out that the expectation in 3.27 can be rewritten as a numerical integration instead of subjecting it to evaluation by a Monte Carlo approach. This leads to an analytical formula for pricing YYIIS that depends on the volatilities of nominal and real forward rates as well as their correlations for each payment time. The main drawback of this approach stems from relying on the foreign currency analogy. It requires us to estimate historically the volatility of real rates which are not observed directly in the market. This is the main reason why a second type of market model, which does not rely on the foreign currency analogy, has been proposed in the literature.

3.2 Moving away from the Foreign-Currency Analogy

The following presents what are known as market models because they aim at modeling suitable market quantities that are directly observable in the market. The goal is to obtain more realistic models for which historical parameters are easier to observe and derive formulas for the most common inflation derivatives.

We will first present different model primitives that have been proposed as foundations for market models as a way to give an overview of the subject. Then we will present each model separately.

3.2.1 Overview of model primitives

The first model primitive for a market model that we will be discussing is the forward CPI. This is an approach taken by Kazziha [1999], Belgrade et al. [2004] and Mercurio [2005]. Denote the forward CPI for period $[t, T_i]$ by $\mathcal{I}_i(t, T_i, T_i + \delta_i)$. We have δ_i as the tenor of the price index typically equal to one month. To simplify notation we drop the tenor and assume that it is one month. Thus, we write $\mathcal{I}_i(t)$ to represent the forward price index that prevails for period $[T_i, T_i + \sigma_i]$.

As an alternative definition from equation 3.21 we can have the same quantity $\mathcal{I}_i(t)$ defined as the fixed amount to be exchanged for the price index at time T_i. This allows us to leave the foreign currency analogy behind which can help simplifying the model. No-arbitrage theory tells us that

$$\mathcal{I}_i(t) = E_{T_i}\left[I(T_i) \mid \mathcal{F}_t \right] \tag{3.28}$$

Using the fact that $P_n(T, T_i) = P_r(T, T_i) = 1$, we show that this is an equivalent definition to the forward CPI from equation 3.21:

$$\begin{aligned}\mathcal{I}_i(t) &= E_{T_i}\left[I(T_i) \mid \mathcal{F}_t\right] \\ &= E_{T_i}\left[I(T_i)\frac{P_r(T,T_i)}{P_n(T,T_i)} \;\middle|\; \mathcal{F}_t\right] \\ &= E_{T_i}\left[\mathcal{I}_i^{FCA}(T) \;\middle|\; \mathcal{F}_t\right] \\ &= \mathcal{I}_i^{FCA}(t) \end{aligned} \tag{3.29}$$

Also, substituting 3.21 in 3.16 we obtain

$$\mathcal{I}_i(t) = I(t)(1 + K_i(t))^{T_i} \tag{3.30}$$

This last expression allows us to obtain the forward CPI quotes directly from ZCIIS quotes in a way that does not require the foreign-currency analogy. It also shows once again that this approach is perfectly consistent with the foreign currency analogy since equations 3.30 and 3.16 are equivalent. We will discuss this in further detail in Section 3.3.

An inflation market model is not necessarily based on the previous definition. One could also model the joint evolution of consecutive forward inflation rates as presented in Mercurio [2008]. This is similar to the LIBOR market model where consecutive forward LIBOR rates are simulated. Thus, we define $\mathcal{Y}_i(t)$ as the forward inflation rate for the period $[T_{i-1}, T_i]$ in the following way:

$$\mathcal{Y}_i(t) = E_{T_i}\left[\frac{I(T_i)}{I(T_{i-1})} - 1 \;\middle|\; \mathcal{F}_t\right] \tag{3.31}$$

Essentially, $\mathcal{Y}_i(t)$ is the fixed rate for which the fixed leg of a swaplet is equal to the expected payoff of its floating leg paying the inflation rate. In this case the inputs for the model can be obtained by a bootstrapping method for extracting forward inflation rates from year-on-year inflation swap quotes. It is also worth mentioning at this point that YYIIS quotes are still hard to obtain while the market for ZCIIS is much more liquid.

Another approach for modeling forward inflation rates which is more friendly in terms of obtaining the model inputs is to define the quantity to be simulated as the ratio of the corresponding forward CPI minus one:

$$Y_i(t) := \frac{\mathcal{I}_i(t)}{\mathcal{I}_{i-1}(t)} - 1 \ , \quad t \leq T_{i-1} \tag{3.32}$$

This definition allows us to obtain the forward inflation rates through equation 3.30. The downside here is that $Y_i(t)$ is not a martingale under the T_i forward measure while $\mathcal{Y}_i(t)$ from equation 3.31 was a martingale under the T_i forward measure. Both approaches will be discussed in Section 3.4

Finally, we present another method for modeling inflation rates from Leung and Wu [2008] which attempts to build a framework that is more general than the previous ones. The authors also choose to leave the foreign currency analogy behind although they keep real zero coupon bond as a model primitive. Starting with an expression equivalent to the one obtained in 3.16 and without resorting to the foreign currency analogy they note that real zero coupon bond quotes can be obtained directly from ZCIIS quotes:

$$P_r(t,T) = E_n\left[e^{-\int_t^T r(u)du} \frac{I(T)}{I(t)} \;\middle|\; \mathcal{F}_t \right] = P_n(t,T)(1+K)^{T-t} \tag{3.33}$$

Note however that $P_r(t,T)$ does not refer to the foreign currency analogy and it is not a tradable asset. The previous expression is only true when doing the valuation of $P_r(t,T)$ at its issue date. This can easily be fixed but we refrain from doing so now for simplicity. The authors define a discount factor for inflation:

$$P_i(t,T) := \frac{P_n(t,T)}{P_r(t,T)} \tag{3.34}$$

which is quite similar to 3.25 with the exception that they use the price index ratio as an inflation discount factor. This allows us to factorize nominal discount factors into real and inflation discount factors:

$$P_n(t,T) = P_r(t,T)P_i(t,T) \tag{3.35}$$

which is convenient for the introduction of an instantaneous inflation forward rate. This quantity can be modeled in an extended HJM framework or in a market model if involving simple inflation forward rates. The main advantage of this approach compared to the foreign currency analogy is that it does not require any knowledge about real rates. It also allows nominal rates and inflation rates to be modeled in a consistent way which can be more convenient then modeling a forward CPI ratio as previously discussed. This third and last approach will be discussed in Section 3.6.

3.3 Modeling the forward CPI

As an alternative to the foreign currency analogy, a market model based on forward CPI modeling was first presented by Belgrade et al. [2004] and subsequently studied in Mercurio [2005]. In the previous section, we have seen how the forward CPI can be defined simply through a no-arbitrage argument and still be consistent with the foreign currency analogy. Now we will be looking at what kind of dynamic we might be giving to the forward CPI and give an example of how this might be used to price YYIIS.

We know that both $\mathcal{I}_i(t)$ and $\mathcal{I}_{i-1}(t)$ must be martingales under their own respective measures $Q_n^{T_i}$ and $Q_n^{T_{i-1}}$. The main issue is to obtain a dynamic for both quantities under the same measure in order to compute the expectation of the CPI ratio that is bound to occur in most inflation derivative payoff. Thus, for modeling $\mathcal{I}_i(t)$ we use the same dynamic as in 3.22 and we do a change of numeraire to express the dynamic of $\mathcal{I}_{i-1}(t)$ under the same measure $Q_n^{T_i}$:

$$\begin{aligned} \frac{d\mathcal{I}_i(t)}{\mathcal{I}_i(t)} &= \sigma_{I,i} dW_i(t) \\ \frac{d\mathcal{I}_{i-1}(t)}{\mathcal{I}_{i-1}(t)} &= -\sigma_{I,i-1} \frac{\tau_i \sigma_{n,i} F_n(t;T_{i-1},T_i)}{1+\tau_i F_n(t;T_{i-1},T_i)} \rho_{I,n,i} dt + \sigma_{I,i-1} dW_{i-1}^I(t) \end{aligned} \tag{3.36}$$

where $\sigma_{I,i}$ and $\sigma_{I,i-1}$ are positive constants while W_i and W_{i-1} are two Brownian motions with the usual corresponding correlation $dW_i dW_{i-1} = \rho_{I,i,i-1}$. The authors show that the above dynamic simplifies to the following when we freeze the drift of $\mathcal{I}_{i-1}(t)$ to its initial $t=0$ value:

$$E_n^{T_i}\left[\frac{\mathcal{I}_i(T_{i-1})}{\mathcal{I}_{i-1}(T_{i-1})}\,\middle|\,\mathcal{F}_t\right]=\frac{\mathcal{I}_i(t)}{\mathcal{I}_{i-1}(t)}e^{D_i(t)} \tag{3.37}$$

where

$$D_i(t)=\sigma_{I,i-1}\left[\frac{\tau_i\sigma_{n,i}F_n(t;T_{i-1},T_i)}{1+\tau_iF_n(t;T_{i-1},T_i)}\rho_{I,n,i}-\rho_{I,n,i}\sigma_{I,i}+\sigma_{I,i-1}\right](T_{i-1}-t) \tag{3.38}$$

Next we give an example of how this result might be applied to price YYIIS. We start with the usual reset payoff of the YYIIS floating leg from 2.7 and use the fact that the forward CPI is a martingale under the measure $Q_n^{T_i}$; so that $E_n^{T_i}\left[\mathcal{I}_i(T_i)\mid\mathcal{F}_t\right]=E_n^{T_i}\left[\mathcal{I}_i(T_{i-1})\mid\mathcal{F}_t\right]$ and we obtain:

$$\begin{aligned}\mathbf{YYIIS}(t,T_{i-1},T_i) &= P(t,T_i)E_n^{T_i}\left[\frac{I(T_i)}{I(T_{i-1})}-1\,\middle|\,\mathcal{F}_t\right]\\ &= P(t,T_i)E_n^{T_i}\left[\frac{\mathcal{I}_i(T_i)}{\mathcal{I}_{i-1}(T_{i-1})}-1\,\middle|\,\mathcal{F}_t\right]\\ &= P(t,T_i)E_n^{T_i}\left[\frac{\mathcal{I}_i(T_{i-1})}{\mathcal{I}_{i-1}(T_{i-1})}-1\,\middle|\,\mathcal{F}_t\right]\end{aligned} \tag{3.39}$$

which allows us to write the following by substituting in 3.39 equation 3.37 and 3.25:

$$\mathbf{YYIIS}(t,T_{i-1},T_i)=P(t,T_i)\left(\frac{\mathcal{I}_i(t)}{\mathcal{I}_{i-1}(t)}e^{D_i(t)}-1\right) \tag{3.40}$$

The simplicity of this equation illustrates how abandoning the foreign currency analogy, while facilitating parameter estimation, can also make the model more intuitive. In fact, it is easier to see now that $e^{D_i(t)}$ is more or less equivalent to the convexity adjustment often encountered in fixed income pricing. However when it comes to the Monte Carlo simulation, this approach has an inconvenient feature. We wish to price YYIIS, caps or floors with more than one cash flow which means that we will have to model every forward CPI under the same measure. This can be accomplished in a way that is similar to the

Lognormal Forward Model where forward interest rates are modeled under the terminal measure or the T_1-forward measure. The LFM approach is very common for modeling forward interest rates, but one can legitimately wonder if it would not be simpler to model the inflation *rate* rather than the CPI. We look at this alternative in the next section.

3.4 Modeling forward inflation

Many common inflation derivatives like inflation caps and floors revolve around the *inflation rate* which makes it a good candidate for modeling purposes. However note that zero-coupon and year-on-year rates do not provide the same information about forward inflation. This is because the YY rates contain an intrinsic convexity adjustment coming from the CPI ratio. Since most inflation derivatives are defined through the same CPI ratio it would be more convenient to model forward inflation rates obtained by bootstrapping the YY curve. We will first look at this possibility. However, liquidity is concentrated in ZC market rather than in the YY market and we are forced to look at an alternative model based on the CPI dynamic. Furthermore, we show in the second part of this section that a lognormal dynamic for the forward CPI implies a shifted lognormal dynamic for inflation rates. This is a convenient result because it allows us to diffuse forward inflation rates while using the ZC market to obtain the initial curve.

3.4.1 Modeling forward inflation rates

Modeling inflation has a certain similarity to interest rate market models in which we typically model the joint evolution of consecutive forward LIBOR rates. A similar approach is explored briefly in Mercurio [2008] where consecutive inflation rates are modeled jointly. We present this concept for completeness and to illustrate the difference between this and the second approach that we present in the next section.

Define the time t forward inflation rate $\mathcal{Y}_i(t)$ for the future interval $[T_{i-t}, T_i]$. Define also a swaplet for which, at time T_i, the fixed rate K is exchanged for the inflation rate over $[T_{i-t}, T_i]$. We obtain:

$$\mathcal{Y}_i(t) = K = E^{T_i}\left[\frac{I(T_i)}{I(T_{i-1})} - 1 \,\middle|\, \mathcal{F}_t\right] = E^{T_i}\left[\frac{\mathcal{I}_i(T_i)}{\mathcal{I}_i(T_{i-1})} - 1 \,\middle|\, \mathcal{F}_t\right] \tag{3.41}$$

The dynamic of the consecutive forward inflation rates $\mathcal{Y}_i(t)$ under their own respective measure Q_{T_i} is assumed to have a normal volatility:

$$d\mathcal{Y}_i(t) = \sigma_i(t) dZ_i(t) \quad i = 1, 2...M - 1 \tag{3.42}$$

where $M - 1$ denotes the last T_M maturing inflation forward in the joint series. We specify a covariance structure for the Brownian motions Z_i

$$\rho_{i,j} dt = dZ_i(t) dZ_j(t) \quad i, j = 1, 2...M - 1 \tag{3.43}$$

Since each $\mathcal{Y}_i(t)$ is a martingale under its own corresponding measure Q^{T_i}, Girsanov's Theorem can be applied repeatedly to derive the joint dynamics of each inflation forward under a common measure. Adopting a Gaussian SABR model, one can also obtain a closed-form formula for pricing caplets with an easy calibration to market data. The Gaussian SABR model is fairly common in interest rate derivatives. It is a stochastic volatility model which simulates volatility as a driftless geometric Brownian motion. The main undesirable feature of this model is that we have defined the forward inflation rate as the fixed rate K of a swaplet. Quotes for swaplets are obtained by stripping the year-on-year inflation swaps. This problematic because liquidity is located in the zero-coupon inflation swap market rather than in the year-on-year market.

3.5 Modeling the forward CPI ratio

In an attempt to focus modeling efforts on quantities that can easily be found in the market, one can define inflation forwards through their corresponding forward CPI ratio minus one as follows:

$$Y_i(t) := \frac{\mathcal{I}_i(t)}{\mathcal{I}_{i-1}(t)} - 1 \ , \quad t \leq T_{i-1} \tag{3.44}$$

The main desirable feature of $Y_i(t)$ over $\mathcal{Y}_i(t)$ is that the forward CPIs can be stripped from zero-coupon inflation swaps instead of relying on the year-on-year inflation swap.

However, note that unlike the previous expression $\mathcal{Y}_i(t)$ for forward inflation, this expression is not a martingale under the T_i forward measure because the forward CPIs within the ratio have different maturities. This is a problem because it makes it hard to find no-arbitrage dynamics to impose on the inflation rate. In Mercurio [2008] the author seeks to model the forward CPIs in a way that would imply a simple dynamic for $Y_i(t)$. He does so by first noticing the following relationship between the forward CPI and forward inflation:

$$\mathcal{I}_i(t) = I(0) \prod_{j=1}^{i} [1 + Y_j(t)] \tag{3.45}$$

which can guide us when it comes to choosing a convenient dynamic for the forward CPI ratios. Thus, we choose to assume a shifted-lognormal volatility for $Y_i(t)$ and we have:

$$dY_j(t) := \ldots dt + \sigma_j [1 + Y_j(t)] 1_{\{t<T_j\}} dW_j(t) \tag{3.46}$$

where we ignore the drift component for now. Note that this assumption is consistent with the findings of Leung and Wu [2008], who show that simply-compounded forward inflation rates follow a shifted-lognormal dynamic. Proceeding onto finding the appropriate forward

CPI dynamic we differentiate 3.45 using the dynamic 3.46 to obtain:

$$
\begin{aligned}
d\mathcal{I}_i(t) &= I(0)\sum_{j=1}^{i}\prod_{h=1,h\neq j}^{i}[1+Y_h(t)]dY_j(t) \\
&= \ldots dt + I(0)\sum_{j=1}^{i}\prod_{h=1,h\neq j}^{i}[1+Y_h(t)]\sigma_j[1+Y_j(t)]1_{\{t<T_j\}}dW_j^i(t) \\
&= \ldots dt + \sum_{j=1}^{i}\left(I(0)\prod_{h=1,h\neq j}^{i}[1+Y_h(t)][1+Y_j(t)]\right)\sigma_j 1_{\{t<T_j\}}dW_j^i(t) \\
&= \ldots dt + \sum_{j=1}^{i}\mathcal{I}_i(t)\sigma_j 1_{\{t<T_j\}}dW_j^i(t) \\
&= \ldots dt + \mathcal{I}_i(t)\sum_{j=\beta(t)}^{i}\sigma_j dW_j^i(t) \qquad (3.47)
\end{aligned}
$$

where $\beta(t)$ is the index of the first tenor date T_j strictly larger than t. We avoided the drift term thus far for simplicity because by definition $\mathcal{I}_i(t)$ is a martingale under measure Q^{T_i}. Thus, we define the forward CPI as having a zero drift under its corresponding forward measure Q^{T_i}:

$$
d\mathcal{I}_i(t) = \mathcal{I}_i(t)\sum_{j=\beta(t)}^{i}\sigma_j dW_j^i(t) \qquad (3.48)
$$

where $dW_h^i(t)dW_k^i(t) = \rho_{h,k}dt$ and σ_j is the volatility of of the factors $W_j(t)$.

We can do a change of measure to obtain the dynamic of $\mathcal{I}_{i-1}(t)$ under the forward measure Q^{T_i}. This allows us to apply Ito's lemma to 3.44 which leads to the following dynamic (see Mercurio and Moreni [2009] for the derivation):

$$
dY_i(t) = (1+Y_i(t))\left[C_i(t)dt + \sigma_i dW_i^i(t)\right] \qquad (3.49)
$$

where

$$C_i(t) = \sum_{j=\beta(t)}^{i-1} \sigma_j \left(\rho_{i,j}^{F,Y} \frac{\sigma_{F,i}\tau_i F_i(t)}{1+\tau_i F_i(t)} - \sigma_i \rho_{i,j} \right) \tag{3.50}$$

and $\sigma_{F,i}$ is the volatility (assumed to be constant) of $F_i(t)$ and $\rho_{i,j}^{F,Y}$ is the correlation between $F_i(t)$ and $Y_j(t)$. In Mercurio and Moreni [2009], the authors introduce stochastic volatility of the SABR type in the dynamic of $\mathcal{I}_i(t)$ (equation 3.48). This allows them to calibrate on cap/floor smiles to reproduce market values for caps/floors prices and imply forward CPIs volatility from quoted ZC options prices. We use this approach in our simulations in the next chapter but we choose not to use stochastic volatility because quotes for caps and floors are still quite hard to find in the market which means that smile modeling can seldom be implemented through calibrating the market smile. In our simulation we will be assuming volatility that is constant with respect to time to maturity.

3.6 Modeling through inflation discount factors

In Leung and Wu [2008], the authors derive a simple expression for inflation discount factors which they define in the following way:

$$P_I(t, T_0, T) := \frac{P(t,T)}{P_R(t,T_0,T)} \tag{3.51}$$

where $P(t,T)$ is the usual discount factor or zero-coupon bond and $P_R(t,T_0,T)$ is a synthetic (but tradable) real discount factor or, in other words, a real zero-coupon bond. We have changed the notation slightly for the discount factors to emphasize that we are not using the foreign -currency analogy anymore.

Before we go any further, it is worth looking at how real discount factors can be obtained. To be consistent with previous discussions we present how these can be obtained from ZCIIS quotes although it would certainly be possible to obtain these from the inflation-indexed bond market or its asset swap equivalent. First recall that the ZCIIS is quoted in terms of the fixed rate K_i such that the contract initially has a value of zero. At issue, we have the following relation for the ZCIIS maturing at T_i

$$\frac{I(T_i)}{I(t)} = (1 + K_i)^{T-t} \tag{3.52}$$

Denote the value of a tradable real zero-coupon bond with issue date $T_0 < t$ and which pays an inflation adjusted principal at maturity T_i by $P_R(t; T_0, T_i)$. Invoking a standard no-arbitrage argument and dropping the index i for clarity ($T_i = T$) we obtain:

$$\begin{aligned} P_R(t; T_0, T) &= E_Q\left[e^{-\int_t^T r(u)du} \frac{I(T)}{I(T_0)} \,\middle|\, \mathcal{F}_t\right] \\ &= \frac{I(t)}{I(T_0)} E_Q\left[e^{-\int_t^T r(u)du} \frac{I(T)}{I(t)} \,\middle|\, \mathcal{F}_t\right] \\ &= \frac{I(t)}{I(T_0)} E_Q\left[e^{-\int_t^T r(u)du} (1+K)^{T-t} \,\middle|\, \mathcal{F}_t\right] \\ &= \frac{I(t)}{I(T_0)} P(t,T)(1+K)^{T-t} \end{aligned} \tag{3.53}$$

which shows that $P_R(t; T_0, T)$ can be modeled as a tradable asset since quotes for this contract can be obtained at any point during its life. Going back to definition 3.51, we have shown that $P_I(t, T_0, T)$ is tradable asset through $P(t, T)$ and $P_R(t, T_0, T)$ which now opens the door for modeling inflation rates within the same framework as nominal interest rates. Two approaches for this will be discussed. The first one is an extended HJM type of model where instantaneous inflation rates are diffused while the second one is a market model where simple inflation rates are diffused.

3.6.1 Extended HJM model

Analogously to what is accomplished in the HJM model, the authors in Leung and Wu [2008] introduce the instantaneous inflation forward rate $f^I(t; T)$ through the inflation discount factor discussed in equation 3.51:

$$f^{(I)}(t; T) := -\frac{\partial \ln P_I(t; T_0, T)}{\partial T} \ , \forall T \geq t \tag{3.54}$$

which means that the inflation discount factor $P_I(t,T)$ can also be expressed as

$$P_I(t,T) = \exp\left(-\int_t^T f^{(I)}(t,s)ds\right) \tag{3.55}$$

Subsequently, the authors assume a standard HJM dynamic for $P(t,T)$ and similarly for the real zero-coupon bond $P_R(t,T_0,T)$:

$$\begin{aligned} dP(t,T) &= P(t,T)(r_t dt + \Sigma(t,T)d\mathbf{Z}_t) \\ dP_R(t,T_0,T) &= P_R(t;,T_0,T)(r_t dt + \Sigma_R(t,T)\cdot d\mathbf{Z}_t) \end{aligned} \tag{3.56}$$

where r_t is the instantaneous spot rate, $d\mathbf{Z}_t$ is a d-dimensional P-Wiener process and $\Sigma(t,T)$ and $\Sigma_R(t,T)$ are the volatilities of $P(t,T)$ and $P_R(t,T_0,T)$. Applying Ito's Lemma to definition 3.54 and using definition 3.51, we obtain the dynamic for $f^{(I)}(t,T)$ under the risk neutral measure $\mathcal{Q}$:

$$\begin{aligned} df^{(I)}(t,T) &= \left[(\sigma(t,T)-\sigma_I(t,T))\left(\int_t^T \sigma_I(t,s)ds - \Sigma_I(t,t)\right) + \sigma_I\int_t^T \sigma(t,s)ds\right]dt \\ &\quad +\sigma_I(t,T)dZ_t \end{aligned} \tag{3.57}$$

This model represents a powerful approach because it builds a sound HJM framework around the inflation rate. It allows for the pricing of inflation derivatives in a way that is similar to the nominal interest-rate derivatives under the classic HJM model. It can be used as a platform for further developments like volatility smiles or jumps in the driving dynamic.

3.6.2 Market model

From a practical perspective, it is quite desirable to have an analytically tractable model for simple inflation forward rates since most inflation derivative cash flows are of the same nature. Using the inflation discount factor defined in 3.51, a simple forward inflation rate

$f^I(t; T_1, T_2)$ over period $[T_1, T_2]$ is defined as:

$$df^I(t; T_1, T_2) := \frac{1}{T_2 - T_1}\left(\frac{P_I(t, T_0, T_1)}{P_I(t, T_0, T_2)} - 1\right) \tag{3.58}$$

If we let $T = T_2$ and $\Delta T = T_2 - T_1$, this can be expressed more generally as:

$$df^I(t; T - \Delta T, T) + \frac{1}{\Delta T} = \frac{1}{\Delta T}\frac{P_I(t, T_0, T - \Delta T)}{P_I(t, T_0, T)} \tag{3.59}$$

Note that this forward inflation rate can be replicated by trading the real zero coupon bonds. Thus it is the unique fair rate observed at time t for a T maturing contract. The dynamic of the inflation discount factors is derived from 3.56 and expressed as:

$$dP_I(t; T_0, T) = P_I(t; T_0, T)\Sigma_I(t, T)(dZ_t - \Sigma_R(t, T)dt) \tag{3.60}$$

which allows us to obtain the following dynamic for the forward simple inflation rates $f_j^{(I)}(t)$ in conjunction with a term structure model with simple compounding nominal forward rates $f_j(t)$. Here, we use the LIBOR Market Model for the dynamic of $f_j(t)$ and find $f_j^{(I)}(t)$ accordingly:

$$\begin{aligned} df_j(t) &= f_j(t)\gamma_j(t)\left(dZ_t - \Sigma_{j+1}(t)dt\right) \\ d\left(f_j^{(I)}(t) + \frac{1}{\Delta T_j}\right) &= \left(f_j^I(t) + \frac{1}{\Delta T_j}\right)\Big(\gamma_j^{(I)}(t)(dZ_t - \Sigma_j(t)dt) \\ &\quad + \Sigma_I(t, T_{j-1})(\gamma_j^{(I)}(t) - \Sigma_{j-1}(t) + \Sigma_j(t))dt\Big) \end{aligned} \tag{3.61}$$

where

$$\gamma^{(I)}(t, T) = \Sigma_I(t, T - \Delta T) - \Sigma_I(t, T) \tag{3.62}$$

The dynamics from 3.61 has two main desirable features. First, it allows inflation rates to take both positive and negative values to reflect an inflationary or deflationary environment. Second, it is analytically tractable for derivatives pricing. In addition, this market model lends itself for further extensions. We may include additional risk factors like jumps and stochastic volatilities in order to account for implied volatility smiles or skews. The resulting inflation rate model is superimposed on the popular LIBOR Market model. Both nominal forward rates and inflation forward rates are diffused according to a lognormal process which makes pricing YYIIS or caplets convenient.

Chapter 4

Simulation

Inflation derivatives are relatively new in the market place and most publications on the subject to this date have focused on building an analytical framework for pricing such derivatives. The usual objective is pricing through an analytical formula which is relatively easy to implement and fast to compute. However this approach makes it hard to modify the underlying assumptions of the model. In contrast, Monte Carlo simulations are computationally more intensive but allow us to blow a model wide open to study its intricacies. It also allows us to understand the potential costs of the simplifying assumptions needed to derive analytical solutions.

In this chapter we will apply a Monte Carlo method to the framework discussed in Section 3.5 for modeling the forward CPI ratio. There are three main reasons why we prefer this approach relative to the others that we have discussed.

1. It provides an instant calibration to the zero coupon inflation swap market which is the most commonly traded inflation derivative instrument.

2. Simulating a CPI ratio easily captures the convexity adjustment needed for pricing more interesting inflation contracts like caps and floors.

3. We find that modeling the CPI ratio allows us to define explicitly a more intuitive correlation structure for the factors that drive our model compared to the approach

offered by discount factors or the simple forward CPIs.

This chapter is divided in three sections. We will first present how the simulation was implemented along with a few additional assumptions and an example. We follow this with a section on parameters estimation for our model using historical data for forward interest rates and forward CPI. The third section presents our results when this method is applied to pricing caplets and floorlets which we then extend to the pricing of caps.

4.1 Simulating the forward CPI ratio with Monte Carlo

Recall equation 3.30 from which we can obtain quotes for the forward CPI $\mathcal{I}_i(t)$ where $K_i(t)$ are the quotes from the Zero Coupon Inflation Indexed Swaps (ZCIIS) traded in the market:

$$\mathcal{I}_i(t) = I(t)(1 + K_i(t))^{T_i} \tag{4.1}$$

If we assume that we can observe the price index $I(t)$ at time t then we immediately know the value of the CPIs in the future as implied by the ZCIIS market. Let i denote the time to maturity in years of the forward CPI contract. Then a series of forward CPI contracts as quoted in the market would have cash flows as shown in the following figure:

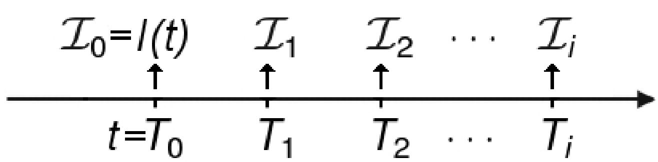

Figure 4.1: A series of forward CPI contracts.

Note that $\mathcal{I}_0(t)$ is not actually traded: instead it is simply the level of the price index at the time when these contracts are being quoted. One can immediately see how annual seasonality in the price index can play a strong role if we try to value these contracts for $T_{i-1} < t < T_i$. We ignore this problem for now by focusing on the case where $t = T_0$.

Now, recall definition 3.44 for the forward CPI ratio $Y_i(t)$:

$$Y_i(t) := \frac{\mathcal{I}_i(t)}{\mathcal{I}_{i-1}(t)} - 1 \ , \quad i = 1, 2, 3... \tag{4.2}$$

Analogous to the main market model for interest rates, we will be modeling the joint evolution of consecutive forward inflation rates in the manner presented below:

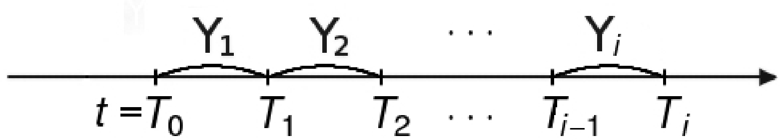

Figure 4.2: A chain of forward inflation contracts.

We defined the dynamic of $Y_i(t)$ in equation 3.49 presented in Mercurio and Moreni [2009] as

$$dY_i(t) = (1 + Y_i(t)) \left[C_i(t)dt + \sigma_i dW_i^i(t) \right] \tag{4.3}$$

where

$$C_i(t) = \sum_{j=\beta(t)}^{i-1} \sigma_j \left(\rho_{i,j}^{F,Y} \frac{\sigma_{F,i} \tau_i F_i(t)}{1 + \tau_i F_i(t)} - \sigma_i \rho_{i,j} \right) \tag{4.4}$$

and

$$\rho_{i,j} dt := dW_h^i(t) dW_k^i(t) \tag{4.5}$$

$$\rho_{i,j}^{F,Y} dt := dF_i(t) dY_j(t) \tag{4.6}$$

while $\sigma_{F,i}$ denotes the volatility of the forward interest rates, τ_i denotes the tenor of the forward rates (always equal to one year in our case) and σ_i denotes the volatility of the factors W_i. Furthermore, $\beta(t)$ is the index of the first time T_i strictly larger than t with i denoting the time to maturity in years. Thus, for $t = T_0 = 0$ we have $\beta(t) = 1$.

We apply a simple Euler scheme to discretize this process using a time step equal to one trading day:

$$Y_i(t + \Delta t) = Y_i(t) + [1 + Y_i(t)] [C_i(t)\Delta t + \sigma_i \Delta W_i] \tag{4.7}$$

where $\Delta t = \frac{1}{252}$ since there is roughly 252 trading days in a year. Furthermore, the paths for the Brownian Motions W_i are generated by drawing samples $\varepsilon_{i,n}$ from a standard normal distribution:

$$W_i(T_n) - W_i(T_{n-1}) = \varepsilon_{i,n}\sqrt{T_n - T_{n-1}} \quad \text{for} \quad n = 0, 1, 2, ..., i \tag{4.8}$$

Freezing the drift term is a technique often employed in interest rate models as a simplification to derive analytical formulas for pricing derivative. For simplicity, we also keep the drift fixed to focus on the problem at hand (modeling inflation), but this assumption can be relaxed if we use a standard Libor Market Model for simulating interest rate forwards along with the forward CPI ratios. This is a rather strong assumption because it implies that the forward rate curve stays the same throughout the life of the contract. We investigate this later. For now, we limit ourselves to pointing out that smaller values for $\rho_{i,j}^{F,Y}\sigma_{F,i}$ and larger values for $\sigma_i\rho_{i,j}$ have the effect of decreasing the sensitivity of $C_i(t)$ to the shape of the forward rate curve. From now on we will have

$$C_i(t) = C_i(0) = \sum_{j=\beta(t)}^{i-1} \sigma_j \left(\rho_{i,j}^{F,Y} \frac{\sigma_{F,i}\tau_i F_i(0)}{1 + \tau_i F_i(0)} - \sigma_i \rho_{i,j} \right) \tag{4.9}$$

where we use the input curves from Table 4.11, the volatilities from Table 4.2 and Table 4.5 and the correlation matrix from Table 4.6. These will be discussed shortly. Furthermore, we use the correlation matrix for dW_i from Table 4.6 to do a Cholesky decomposition of equation 4.7.

In our implementation of this model we choose to keep the evolution of volatility with respect to time as constant. One could consider using interpolation between σ_i and σ_j for $i = j + 1$ to come up with values of $\sigma_i(t)$ for $T_i > t > T_j$. This seems like a sensible thing to do although there is no way of validating this approach empirically since, unlike exchange traded futures contracts with predefined calendar maturities, the quotes available for forward CPI contracts are always given with a constant time to maturity. However, note that similarly to an HJM model, the drift and the volatility are indexed by time to maturity. This means that they will both change through the simulation as the shortest contract matures while the others get re-indexed with their new time to maturity.

To give the reader a better perspective on how this model is implemented we write here explicitly the dynamics we will be using:

$$\begin{aligned}
\Delta Y_1(t) &= (1+Y_1(t))\left[C_1(0)\Delta t + \sigma_1 \Delta W_1^1(t)\right] \\
\Delta Y_2(t) &= (1+Y_2(t))\left[C_2(0)\Delta t + \sigma_2 \Delta W_2^2(t)\right] \\
\Delta Y_3(t) &= (1+Y_3(t))\left[C_3(0)\Delta t + \sigma_3 \Delta W_3^3(t)\right] \\
& \ldots
\end{aligned}$$

where

$$\begin{aligned}
C_1(0) &= \sum_{j=1}^{0} \sigma_j \left(\rho_{1,j}^{F,Y} \frac{\sigma_{F,1}\tau_1 F_1(0)}{1+\tau_1 F_1(0)} - \sigma_1 \rho_{1,j} \right) \\
&= 0 \\
C_2(0) &= \sum_{j=1}^{1} \sigma_j \left(\rho_{2,j}^{F,Y} \frac{\sigma_{F,2}\tau_2 F_2(0)}{1+\tau_2 F_2(0)} - \sigma_2 \rho_{2,j} \right) \\
&= \sigma_1 \left(\rho_{2,1}^{F,Y} \frac{\sigma_{F,2}\tau_2 F_2(0)}{1+\tau_2 F_2(0)} - \sigma_2 \rho_{2,1} \right) \\
C_3(0) &= \sum_{j=1}^{2} \sigma_j \left(\rho_{3,j}^{F,Y} \frac{\sigma_{F,3}\tau_3 F_3(0)}{1+\tau_3 F_3(0)} - \sigma_3 \rho_{3,j} \right) \\
&= \sigma_1 \left(\rho_{3,1}^{F,Y} \frac{\sigma_{F,3}\tau_3 F_3(0)}{1+\tau_3 F_3(0)} - \sigma_3 \rho_{3,1} \right) + \sigma_2 \left(\rho_{3,2}^{F,Y} \frac{\sigma_{F,3}\tau_3 F_3(0)}{1+\tau_3 F_3(0)} - \sigma_3 \rho_{3,2} \right) \\
& \ldots
\end{aligned}$$

Note that we are pricing the contract at $t = 0$, so we used $\beta(t) = 1$. The details about how each parameter is estimated will be discussed shortly.

4.1.1 A closer look at the drift term

The next figure shows the evolution of the drift term in basis points to help us justify our approximation done by keeping the drift fixed. We are show the evolution of a drift term with a constant time to maturity. The latter arises from the fact that we have assumed constant historical volatility and constant historical correlation over time. The fact that $C_i(t)$ is practically constant over time shows that the evolution of the forward interest rates $F_i(t)$ have a very negligible impact on the drift. This means that keeping the drift fixed is a sensible approximation to avoid having to use an LFM model in parallel to our inflation model. This is especially true if our estimates for the factor volatilities are high.

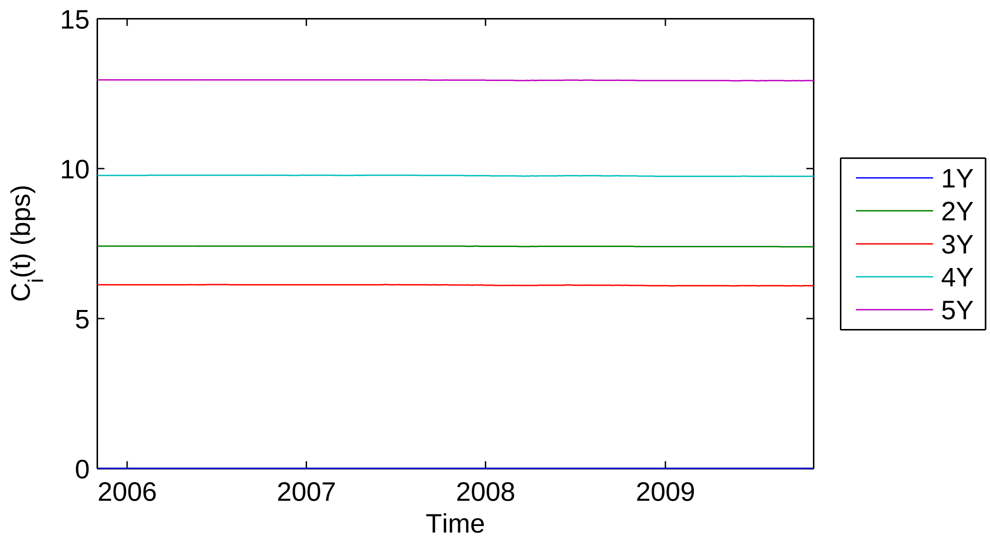

Figure 4.3: Evolution of the drift term $C_i(t)$ using historical forward rates.

The discrepancy between the drift terms for different time to maturities means that our method might benefit from indexing the volatility σ_i with a time to maturity that has a smaller time step. This would be of special importance if we are pricing a one year contract since the difference in drift between a one year and two year contract is the greatest.

4.1.2 A simplified example

The following table illustrates how the forward CPI ratios are modeled in way that is analogous to an HJM model. Here, each row represents the evolution through time of the forward CPI ratio Y_i with maturity T_i for $i = 1, 2, 3, 4, 5$. We show a time-step of half a year. The first column denotes the inflation curve we use as input into our model.

	0y	**0.5y**	**1y**	**1.5y**	**2.0y**	**2.5y**	**3.0y**	**3.5y**	**4.0y**	**4.5y**	**5.0y**
1y	-0.0069	-0.0093	-0.0106								
2y	0.0126	0.0101	0.0088	0.0064	0.0046						
3y	0.0185	0.0160	0.0147	0.0123	0.0105	0.0148	0.0167				
4y	0.0265	0.0240	0.0227	0.0203	0.0185	0.0228	0.0248	0.0242	0.0266		
5y	0.0258	0.0233	0.0221	0.0196	0.0178	0.0222	0.0242	0.0236	0.0260	0.0208	0.0213

Table 4.1: Example for a single diffusion of the forward CPI ratios Y_i.

The following figure illustrates the result of a Monte Carlo simulation for forward inflation rates. We have $Y_i(0)$ to denote the input curve while $Y_i(T_i)$ captures the evolution of spot inflation (1 year tenor) over many trials.

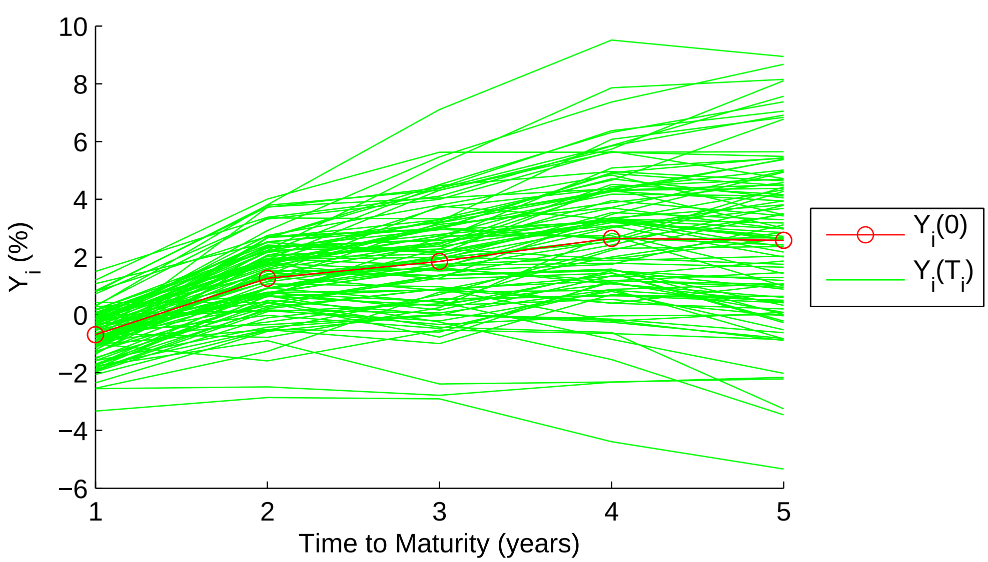

Figure 4.4: Example of a Monte Carlo simulation for the forward CPI ratios Y_i.

4.2 Parameters Estimation

The data we use for estimating the model parameters are two-fold. Zero-coupon bond quotes are obtained from Datastream to calculate the forward interest rates. The zero-coupon inflation indexed swap quotes are obtained from Bloomberg under the ticker <USSWIT1 Curncy> for 1 year maturity, <USSWIT2 Curncy> for 2 year maturity and so on. Data is available for maturities of 1,2,3,4,5,7,10,15,20 and 30 years. The ultimate goal of this project is to price caps with a one year tenor. This means that the market is clearly incomplete if we wish to hedge such a contract for maturities beyond 5 years. We choose to focus on the contracts with 1,2,3,4 and 5 years to keep things readable and avoid having to commit to one specific interpolation method.

On the next page, Figure 4.5 shows the evolution of the zero coupon curve which was used to obtain values for forward interest rates. Figure 4.6 shows the evolution of the zero coupon inflation curve as obtained from ZCIIS quotes (K). Both curves are essentially analogous in the sense that Figure 4.5 presents the rate at which money can be lended while 4.6 shows the rate one has to pay in order to protect the value of their money against inflation. The data has interesting features since it ranges from November 1$^{\text{st}}$ 2005 to October 30$^{\text{th}}$ 2009. In particular, it encompasses the pre 2008 crisis levels which were deceptively stable and the clear market dislocation due to lack of liquidity in the TIPS market (used for hedging ZCIIS) that happened in October 2008 followed by a partial recovery through 2009.

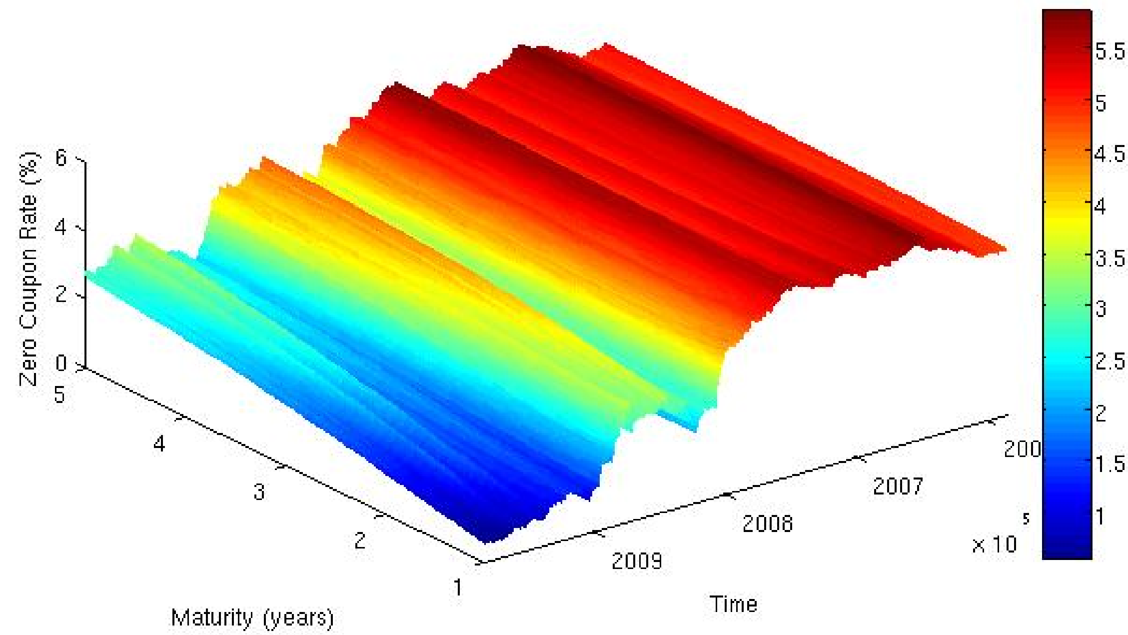

Figure 4.5: Historical zero coupon curve used to obtain forward interest rates F_i.

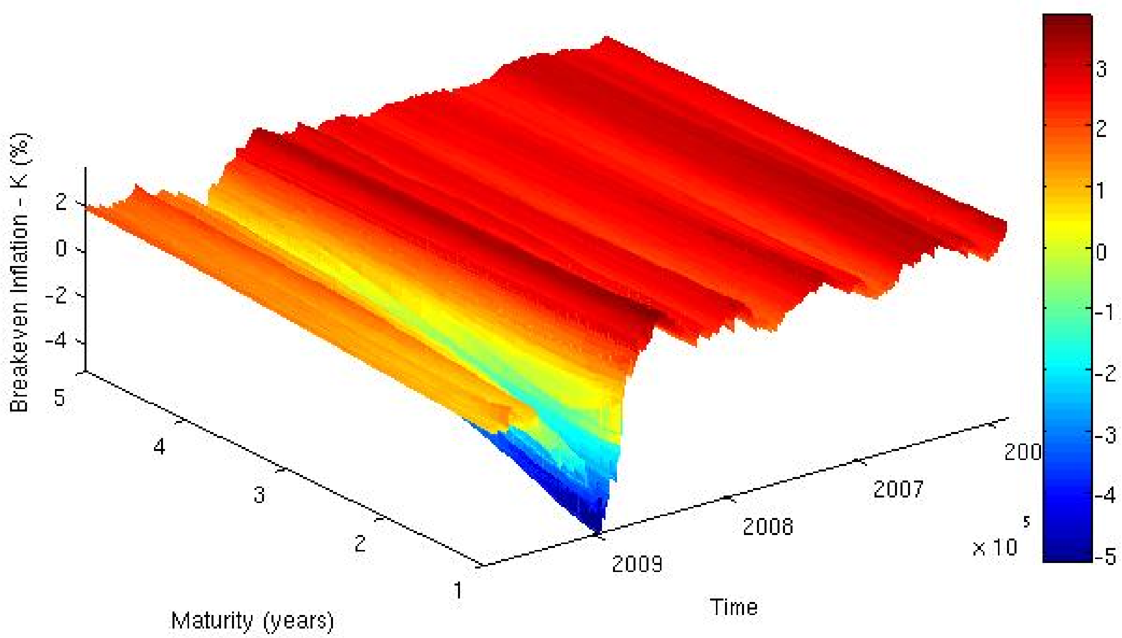

Figure 4.6: Historical zero coupon inflation (K) from ZCIIS quotes used to obtain forward CPI ratios Y_i

4.2.1 Forward inflation and forward interest correlations and volatilities

The rates $r_i(t)$ from zero coupon curves of Figure 4.5 are used to obtain the discount factors using the following equation:

$$P_i(t) = \left(\frac{1}{1 + r_i(t)} \right)^{T_i} \tag{4.10}$$

from which we compute the forward rates we need for our simulation. We use the relation

$$F_i(t) = \frac{1}{T_i - T_{i-1}} \left(\frac{P_{i-1}(t)}{P_i(t)} - 1 \right) \tag{4.11}$$

where T_{i-1} denotes the time to expiration of the forward contract and T_i denotes its time to maturity. It should be understood that by Y_1 we actually mean the spot annual rate while Y_i for $i = 2, 3, 4...$ are true forward rates.

The normalized forward CPI is obtained directly from the zero-coupon inflation quotes through an expression we have derived in Section 3.2.1 :

$$\frac{\mathcal{I}_i(t)}{I(t)} = (1 + K_i(t))^{T_i} \tag{4.12}$$

for $t = 0$. Equation 4.12 will be used later to estimate the parameters for our model's factors. We conclude by obtaining the forward CPI ratio which we have defined previously as:

$$Y_i(t) := \frac{\mathcal{I}_i(t)}{\mathcal{I}_{i-1}(t)} - 1 = \frac{(1 + K_i(t))^{T_i}}{(1 + K_{i-1}(t))^{T_{i-1}}} \ , \quad t \leq T_{i-1} \tag{4.13}$$

where it should be understood that for $Y_1(t)$ we have the denominator of the ratio equal to the current level of the price index, i.e. $\mathcal{I}_{i-1}(t) = \mathcal{I}_0(t) = I(t)$. Thus, we have $Y_1(t) = (1 + K_i(t))^{T_i}$. Figure 4.7 shows the historical evolution of the forward interest rates and forward CPI ratio.

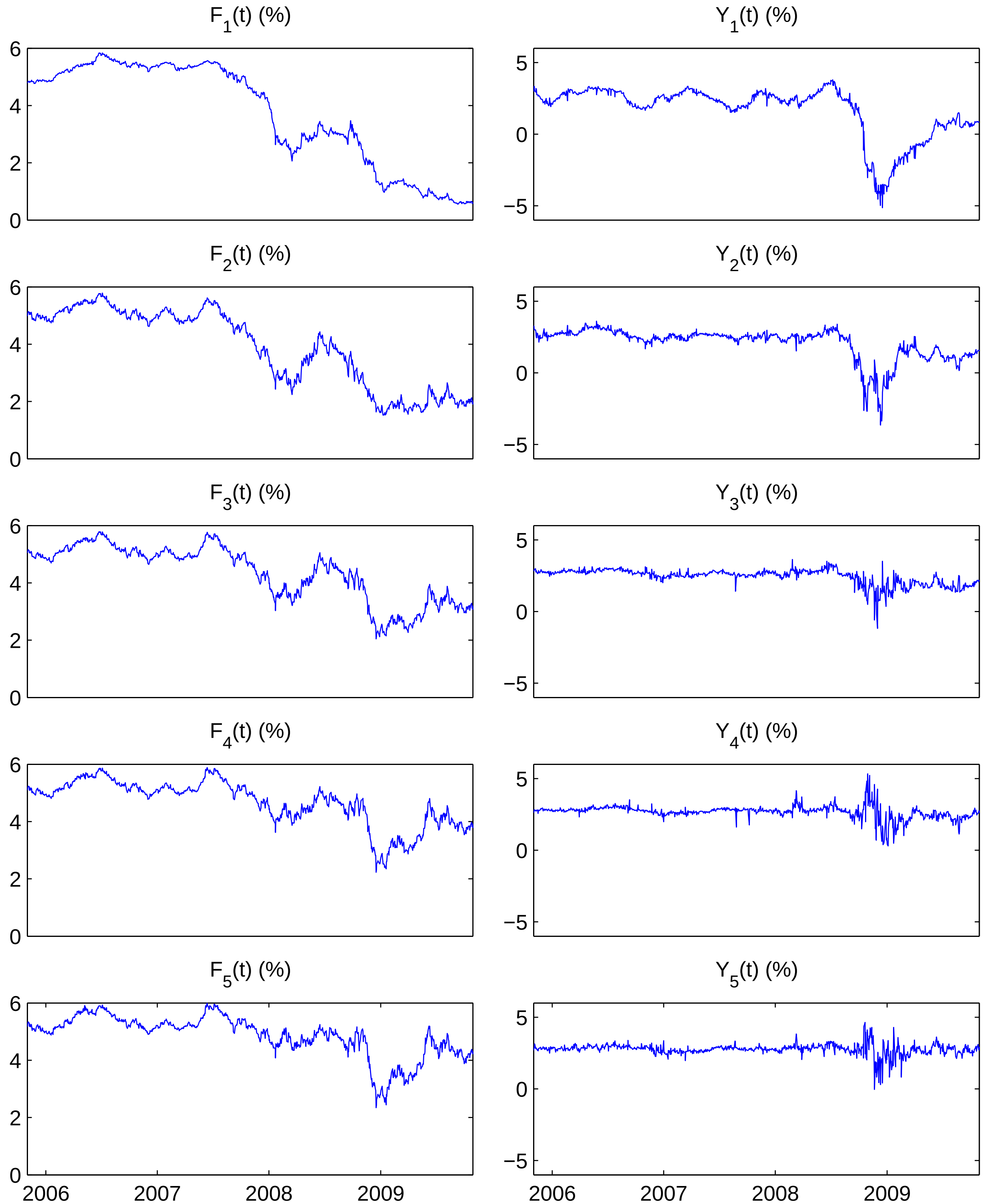

Figure 4.7: Historical forward interest rates F_i and forward CPI ratios Y_i.

The following table presents the forward interest rate volatilities we have estimated from historical forward interest rates. Notice that volatility tends to increase with time to maturity. This can be explained by the longer term inflation outlook.

	1y	**2y**	**3y**	**4y**	**5y**
$\sigma_{F,i}$ - **Stdev**$(\Delta F_i/\sqrt{\Delta t})$	0.0085	0.0140	0.0140	0.0144	0.0154

Table 4.2: Forward interest volatilities estimated from daily historical zero curve.

Table 4.3 contains the correlations between forward interest rates and the forward CPI ratio. It is hard to gain any firm intuition from these results. At most, we could claim that forward interest rates in general are slightly more sensitive to 1 year and 5 year forward inflation. Note that only the lower triangular part of the matrix will actually be used to calculate the drift. Table 4.4 shows a 95% confidence interval for the values presented in Table 4.3.

	ΔY_1	ΔY_2	ΔY_3	ΔY_4	ΔY_5
ΔF_1	0.11	0.05	0.04	-0.11	0.09
ΔF_2	0.14	0.07	0.08	-0.09	0.16
ΔF_3	0.12	0.10	0.07	-0.07	0.15
ΔF_4	0.16	0.05	0.04	-0.05	0.18
ΔF_5	0.08	0.07	0.09	-0.09	0.17

Table 4.3: Correlation matrix $\rho^{F,Y}$ between forward inflation rates with maturity T_i and forward interest rates with maturity T_j obtained from daily historical data.

	ΔY_1	ΔY_2	ΔY_3	ΔY_4	ΔY_5
ΔF_1	0.05, 0.17	-0.01, 0.11	-0.02, 0.10	-0.17,-0.05	0.03, 0.15
ΔF_2	0.08, 0.20	0.01, 0.13	0.02, 0.14	-0.15,-0.03	0.10, 0.22
ΔF_3	0.06, 0.18	0.04, 0.16	0.01, 0.13	-0.13,-0.01	0.09, 0.21
ΔF_4	0.10, 0.22	-0.01, 0.11	-0.02, 0.10	-0.12, 0.01	0.12, 0.24
ΔF_5	0.02, 0.14	0.00, 0.13	0.03, 0.15	-0.15,-0.03	0.12, 0.23

Table 4.4: Confidence interval of 95% for values of $\rho^{F,Y}$.

4.2.2 Factors volatility and correlations

Our model assumes the following CPI dynamic:

$$d\mathcal{I}_i(t) = \mathcal{I}_i(t) \sum_{j=\beta(t)}^{i} \sigma_j dZ_j^i(t) \tag{4.14}$$

where $\rho_{h,k}dt = dZ_h^i(t)dZ_k^i(t)$. To illustrate how parameters can be estimated we explicitly write an example for $i = 1, 2, 3...$:

$$\begin{aligned}
\Delta\mathcal{I}_1(t) &= \mathcal{I}_1(t) \sum_{j=1}^{1} \sigma_j \Delta W_j^1(t) \\
&= \mathcal{I}_1(t)\sigma_1 \Delta W_1^1(t) \\
\Delta\mathcal{I}_2(t) &= \mathcal{I}_2(t) \sum_{j=1}^{2} \sigma_j \Delta W_j^2(t) \\
&= \mathcal{I}_2(t) \left[\sigma_1 \Delta W_1^2(t) + \sigma_2 \Delta W_2^2(t)\right] \\
\Delta\mathcal{I}_3(t) &= \mathcal{I}_3(t) \sum_{j=1}^{3} \sigma_j \Delta W_j^3(t) \\
&= \mathcal{I}_3(t) \left[\sigma_1 \Delta W_1^3(t) + \sigma_2 \Delta W_2^3(t) + \sigma_3 \Delta W_3^3(t)\right] \\
& \quad ...
\end{aligned} \tag{4.15}$$

We expand the sum in the above equation to highlight the fact that we will have to estimate σ_i in a somewhat recursive manner. We start with $\Delta\mathcal{I}_1(t)$ to obtain the time series of the first factor. This will be needed to obtain a time series for the second factor. Then, $\Delta\mathcal{I}_2(t)$ which contains the information from the first two factors can be used to obtain a time series for the third factor and so on.

The next figure shows historical stability of the model factors. Observe that this is stable over time which is an important criterion for the empirical validity of our model.

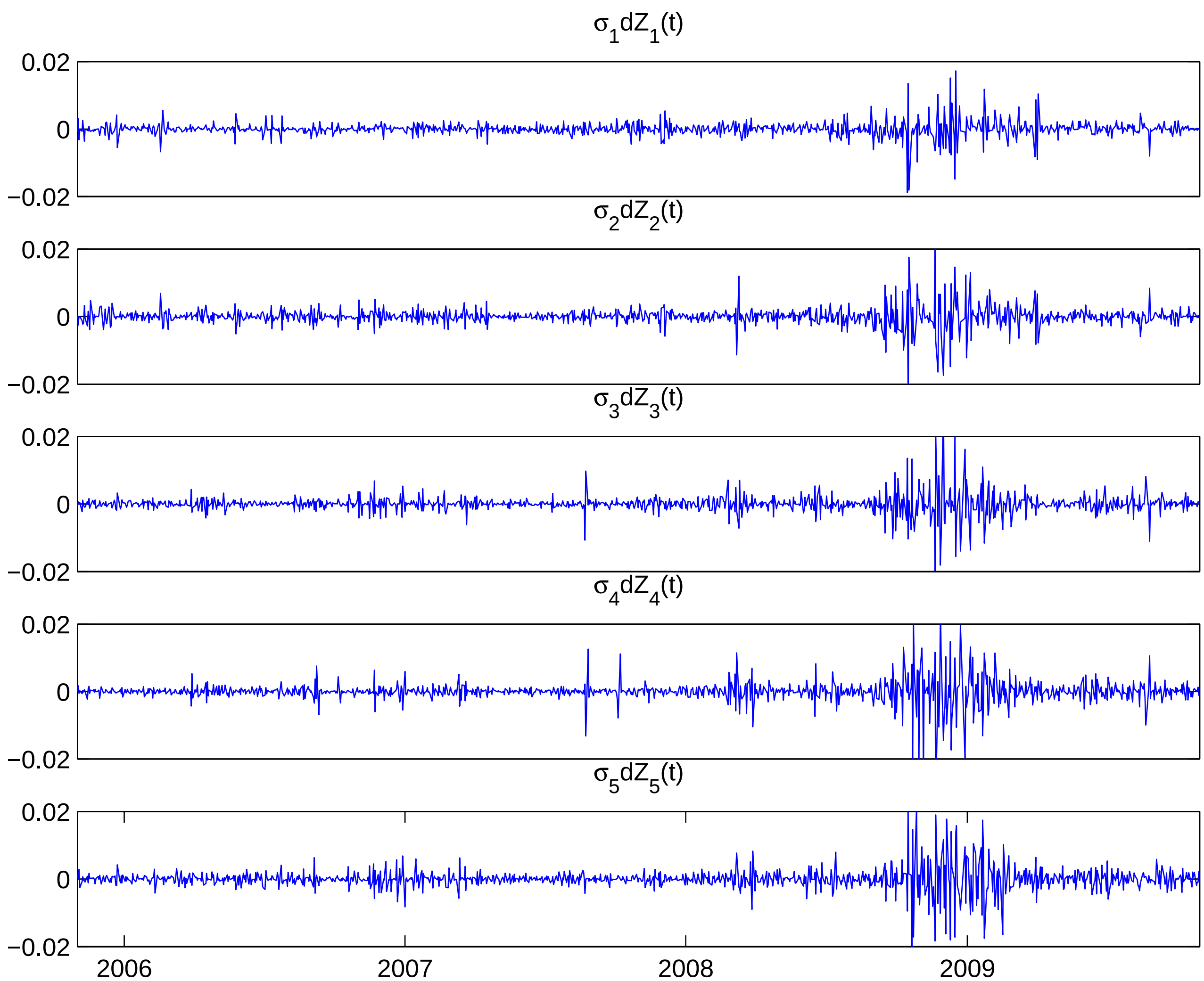

Figure 4.8: Dataset from which the factors volatility and correlation are obtained.

Next, we can estimate standard deviation as follows:

$$\begin{aligned}
\sigma_1 &\approx \frac{\text{stdev}\left(\sigma_1 \Delta W_1(t)\right)}{\sqrt{\Delta t}} = \text{stdev}\left(\frac{\Delta \mathcal{I}_1(t)}{\mathcal{I}_1(t)}\right)\frac{1}{\sqrt{\Delta t}} \\
\sigma_2 &\approx \frac{\text{stdev}\left(\sigma_2 \Delta W_2(t)\right)}{\sqrt{\Delta t}} = \text{stdev}\left(\frac{\Delta \mathcal{I}_2(t)}{\mathcal{I}_2(t)} - \frac{\Delta \mathcal{I}_1(t)}{\mathcal{I}_1(t)}\right)\frac{1}{\sqrt{\Delta t}} \\
\sigma_3 &\approx \frac{\text{stdev}\left(\sigma_3 \Delta W_3(t)\right)}{\sqrt{\Delta t}} = \text{stdev}\left(\frac{\Delta \mathcal{I}_3(t)}{\mathcal{I}_3(t)} - \frac{\Delta \mathcal{I}_2(t)}{\mathcal{I}_2(t)}\right)\frac{1}{\sqrt{\Delta t}} \\
&\ldots \\
\sigma_i &\approx \frac{\text{stdev}\left(\sigma_i \Delta W_i(t)\right)}{\sqrt{\Delta t}} = \text{stdev}\left(\frac{\Delta \mathcal{I}_i(t)}{\mathcal{I}_i(t)} - \frac{d\mathcal{I}_{i-1}(t)}{\mathcal{I}_{i-1}(t)}\right)\frac{1}{\sqrt{\Delta t}}
\end{aligned}$$

Table 4.5 shows the volatilities we obtained. Similar to what we observed with forward interest rates, the volatility of forward inflation tends to increase with time to maturity. In these results we feel that we have probably overestimated volatility. When we removed the the financial crisis from our dataset the volatility we estimated was about half of what we obtain with the full dataset. Perhaps the greatest limitation of our parameter estimation is that we are using a dataset that is too short. We use a dataset that is roughly 4 years of data for modeling spot inflation 5 years ahead in our simulations. In hindsight, this seems grossly insufficient. Furthermore, one can argue that the lack of liquidity during the financial crisis led the market to violate the following simple no arbitrage assumption:

$$\mathcal{I}_i(t) = E_{T_i}\left[I(T_i) \mid \mathcal{F}_t\right] \tag{4.16}$$

Consequently, we might want to adjust our dataset to remove the liquidity effect for future parameter estimation.

	1y	**2y**	**3y**	**4y**	**5y**
σ_i **- Volatility of** W_i	0.0357	0.0435	0.0456	0.0531	0.0528

Table 4.5: Factor volatilities estimated from the daily historical forward CPI curve.

Next we look at correlations between the factors. The approach to obtain the time series for the factors is the same that we used for the volatilities:

$$\begin{aligned}
\rho_{2,1} &= \operatorname{corr}(dW_2, dW_1) = \operatorname{corr}\left(\frac{d\mathcal{I}_2(t)}{\mathcal{I}_2(t)} - \frac{d\mathcal{I}_1(t)}{\mathcal{I}_1(t)}, \frac{d\mathcal{I}_1(t)}{\mathcal{I}_1(t)}\right) \\
\rho_{3,2} &= \operatorname{corr}(dW_3, dW_2) = \operatorname{corr}\left(\frac{d\mathcal{I}_3(t)}{\mathcal{I}_3(t)} - \frac{d\mathcal{I}_2(t)}{\mathcal{I}_2(t)}, \frac{d\mathcal{I}_2(t)}{\mathcal{I}_2(t)} - \frac{d\mathcal{I}_1(t)}{\mathcal{I}_1(t)}\right) \\
&\dots \\
\rho_{i,j} &= \operatorname{corr}(dW_i, dW_j) = \operatorname{corr}\left(\frac{d\mathcal{I}_i(t)}{\mathcal{I}_i(t)} - \frac{d\mathcal{I}_{i-1}(t)}{\mathcal{I}_{i-1}(t)}, \frac{d\mathcal{I}_j(t)}{\mathcal{I}_j(t)} - \frac{d\mathcal{I}_{j-1}(t)}{\mathcal{I}_{j-1}(t)}\right) \qquad (4.17)
\end{aligned}$$

Table 4.6 shows the results we obtained. Each factor W_i accounts for a variation at different time to maturities on the forward CPI curve. Note that the correlations $\rho_{2,1}$, $\rho_{3,2}$, $\rho_{3,4}$ and $\rho_{4,5}$ are all strongly negative which means that the variations at the short end of the forward CPI curve tend to be dampened in the longer end of the curve. Consider for example

$$\rho_{2,1} = \operatorname{corr}\left(\frac{d\mathcal{I}_2(t)}{\mathcal{I}_2(t)} - \frac{d\mathcal{I}_1(t)}{\mathcal{I}_1(t)}, \frac{d\mathcal{I}_1(t)}{\mathcal{I}_1(t)}\right) \qquad (4.18)$$

where a strong negative correlation tells us that changes in $\mathcal{I}_1(t)$ are only partially propagated to $\mathcal{I}_2(t)$. Table 4.7 shows a 95% confidence interval for the values presented in Table 4.6.

	ΔW_1	ΔW_2	ΔW_3	ΔW_4	ΔW_5
ΔW_1	1.00	-0.48	-0.11	0.03	0.08
ΔW_2	-0.48	1.00	-0.22	0.01	-0.13
ΔW_3	-0.11	-0.22	1.00	-0.44	0.11
ΔW_4	0.03	0.01	-0.44	1.00	-0.51
ΔW_5	0.08	-0.13	0.11	-0.51	1.00

Table 4.6: Factor correlation matrix $\rho_{i,j}$.

We use these results in two ways. First we use it for the calculation of the drift term. Second, we use the correlation matrix to do a Cholesky decomposition in order to simulate inflation rates from the dynamic in expression 4.7.

We have also performed a principal component analysis on the factors W_i to see if the number of factors used in our model could be reduced. The first table shows the factor

	ΔW_1	ΔW_2	ΔW_3	ΔW_4	ΔW_5
ΔW_1	1.00, 1.00	-0.52,-0.43	-0.17,-0.05	-0.03, 0.09	0.02, 0.14
ΔW_2	-0.52,-0.43	1.00, 1.00	-0.27,-0.16	-0.05, 0.08	-0.19,-0.07
ΔW_3	-0.17,-0.05	-0.27,-0.16	1.00, 1.00	-0.49,-0.39	0.05, 0.17
ΔW_4	-0.03, 0.09	-0.05, 0.08	-0.49,-0.39	1.00, 1.00	-0.55,-0.46
ΔW_5	0.02, 0.14	-0.19,-0.07	0.05, 0.17	-0.55,-0.46	1.00, 1.00

Table 4.7: Confidence interval of 95% for values of $\rho_{i,j}$.

coefficients (rows) for each principal component (columns). The columns are in order of decreasing component variance. The second table shows how much variance can be accounted for as we add more components. From the results, it is clear that anything less than three factors is not enough. Beyond three factors is probably a matter of preference.

	ΔZ_1	ΔZ_2	ΔZ_3	ΔZ_4	ΔZ_5
ΔW_1	0.03	-0.54	0.14	0.50	0.66
ΔW_2	-0.14	0.80	0.16	0.11	0.55
ΔW_3	0.37	-0.02	-0.75	-0.37	0.41
ΔW_4	-0.69	-0.24	0.12	-0.62	0.27
ΔW_5	0.61	-0.06	0.62	-0.47	0.15

Table 4.8: Construction of the principal components.

	ΔZ_1	ΔZ_2	ΔZ_3	ΔZ_4	ΔZ_5
% of Variance	42.85	65.05	84.78	94.35	100.00

Table 4.9: Cumulative variance we can account for as we increase the number of factors.

4.3 Results

4.3.1 Evolution of spot inflation

The next two tables outline further inputs that are required for our model. We choose $t = 0$ to be April 23$^{\text{rd}}$ 2009 because the Y_i curve (see Table 4.11) exhibits some deflationary expectations in the short end which emphasizes that this is not a limitation of the model. Since we choose to keep the drift fixed and we use constant volatility, there is no point in taking a smaller time step than one year as this would provide no additional precision. Thus, keeping the drift fixed greatly diminishes the amount of computational effort that we need do.

Date	23-Apr-2009
Number of Trials	500000
Timestep (years)	1

Table 4.10: Monte Carlo simulation parameters

	1y	**2y**	**3y**	**4y**	**5y**
$P_i(0)$ - **Discount Factors**	0.9882	0.9697	0.9436	0.9147	0.8842
$F_i(0)$ - **Forward Interest Rates (%)**	1.1986	1.9078	2.7582	3.1610	3.4476
$Y_i(0)$ - **Forward Inflation Rates (%)**	-0.6865	1.2630	1.8516	2.6501	2.5813

Table 4.11: Input curves observed at $t = 0$ used for initiating the simulation.

Table 4.12 summarizes the results of the simulation. Comparing the means of $Y_i(T_i)$ to the input curve $Y_i(0)$ from Table 4.11 one can get a good idea of the impact that the drift $C_i(0)$ has on our results. Notice that the standard errors of our estimates get higher as we increase the time horizon which is something that we would have expected. A simulation with 500 000 trials gives us a standard error below two basis points which is smaller than any typical bid/ask spread in interest rate derivatives.

	$Y_1(T_1)$	$Y_2(T_2)$	$Y_3(T_3)$	$Y_4(T_4)$	$Y_5(T_5)$
$C_i(0)$ **- Drift (%)**	0.0000	0.0741	0.0611	0.0977	0.1296
$Y_i(T_i)$ **- Mean (%)**	-0.6858	1.3288	1.9805	2.8826	2.9391
$Y_i(T_i)$ **- Standard Error (%)**	0.0050	0.0081	0.0105	0.0131	0.0152

Table 4.12: Results from Monte Carlo simulation of forward inflation.

4.3.2 Pricing Caplets and Floorlets

Once we have simulated the evolution of the spot CPI ratio, it is straightforward to price caplets and floorlets according to the following payoff:

$$\mathbf{IICplt}(t, T_{i-1}, T_i, K, N, w) = NP(t, T_i)E^{T_i}\left[[\omega(Y_i(T_i) - K)]^+ \;\middle|\; \mathcal{F}_t\right] \tag{4.19}$$

where we let $\omega = 1$ for the caplet, $\omega = -1$ for the floorlet and N is the notional. We only consider maturities up to five years to keep things simple and readable.

In Figure 4.9 and Figure 4.10 on the next page we show the price of the caplets and floorlets with respect to their moneyness. We simply express moneyness as the difference between $Y_i(T_i)$ and the strike K. The prices are expressed in basis points for a 1 dollar notional. Relative to their moneyness, the one year caplet and floorlet seem slightly different from the other curves. This is because the one year contracts experience no drift in our model which means that their moneyness stays unchanged on average. The contracts with higher maturities have a positive drift $C_i(t)$ which causes their expected payoff to be slightly in the money.

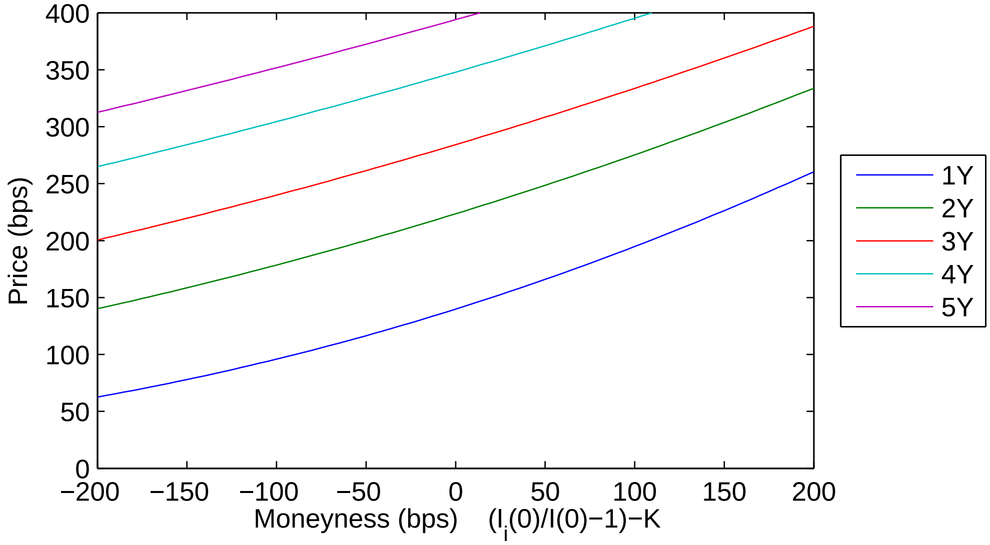

Figure 4.9: Price in basis points of caplets for different strikes K and maturities T_i.

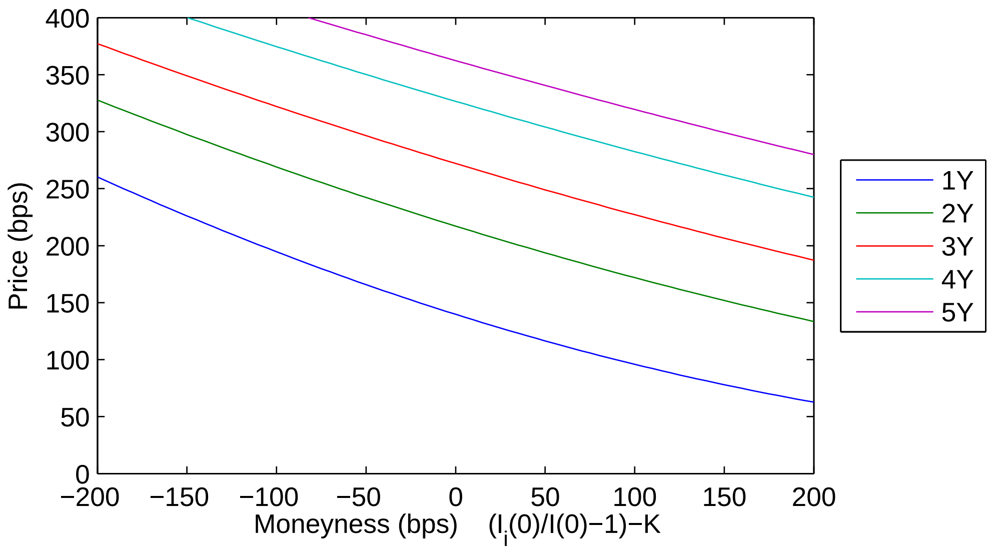

Figure 4.10: Price in basis points of floorlets for different strikes K and maturities T_i.

4.3.3 Greeks for Caplets and Floorlets

The next pages show the delta and gamma of the caplet in Figure 4.11 and 4.12 and the floorlet on the subsequent page with Figure 4.13 and 4.14. Note that we choose to present deltas and gammas with respect to the forward CPI rather than the forward CPI ratio. This is because the ratio does not actually trade in the market while the forward CPI implicitly trades through the Zero Coupon Inflation Swap.

The deltas and gammas are obtained using a central-difference estimator

$$\Delta(n,h) = \frac{Y_n(\theta + h) - Y_n(\theta - h)}{2h} \tag{4.20}$$

where $n = 1, 2...$ is an index for the independent prices obtained during the Monte Carlo simulation, θ is the model input for which we wish to calculate the derivative (in this case θ is the forward CPI, i.e. $\theta = \mathcal{I}_i(0)$). Finally, h is the spacing used for the central-difference approximation. We found $h = 0.005$ to give good results.[1]

On the next page, we have grouped the deltas and gammas with respect to the forward CPI in two ways. First they are grouped by the maturity of the caplet (see legend). Second, for each caplet we have seperated the greeks that were from the numerator forward CPI or the denominator forward CPI (upper and lower graph). These graphs lead to the following observations. The first one is that for an equivalent level of moneyness, the deltas of the caplet and floorlet are quite similar with the exception that the numerator has a positive delta while the denominator has a negative delta. This means that holding a portfolio of a joint series of caplets or floorlets generally has a fairly low delta. We consider this more closely in the next section when we start pricing the caps. Second, because our payoff is expressed in terms of a ratio of two securities, we should expect to have a high gamma. This is what we observe in the gamma plots for the floorlet and caplet. Note that gamma is positive for the numerator and denominator which means that the gamma exposure will stack up by holding a series a joint caplet.

[1]See Glasserman [2004] for more details about estimating sensitivies and optimal values of h with the central-difference estimator.

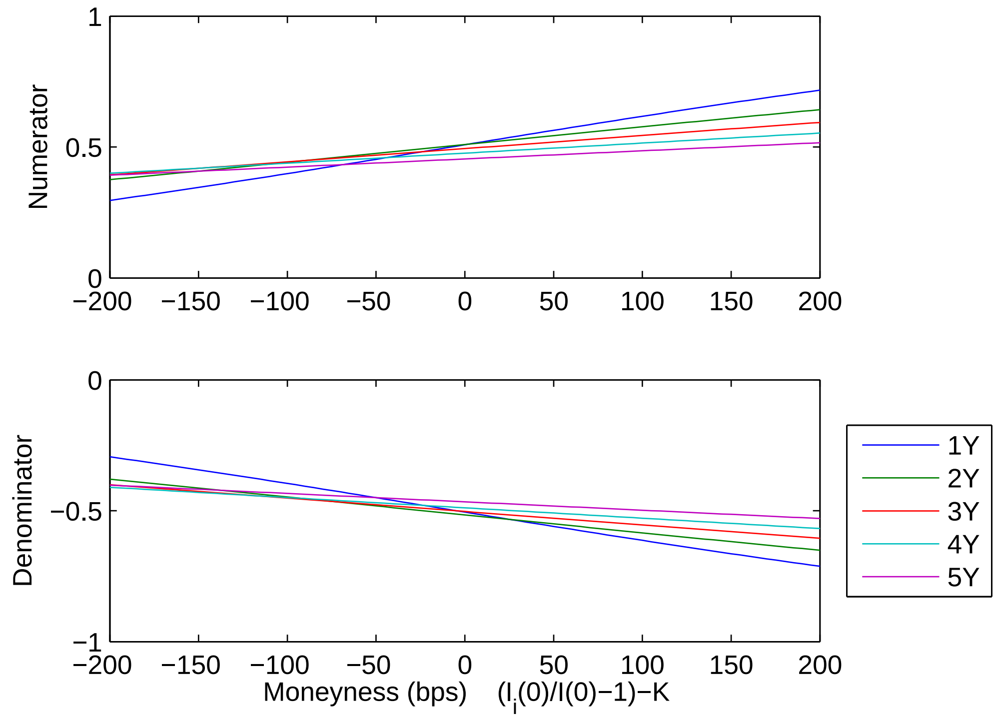

Figure 4.11: Delta of the caplet with respect to the forward CPIs.

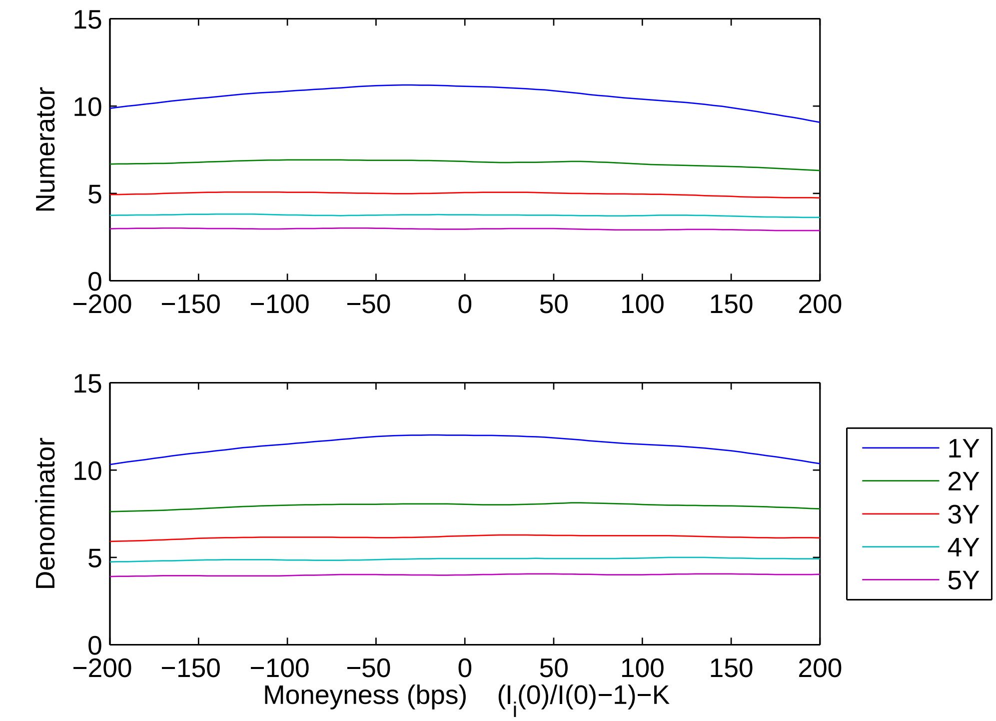

Figure 4.12: Gamma of the caplet with respect to the forward CPIs.

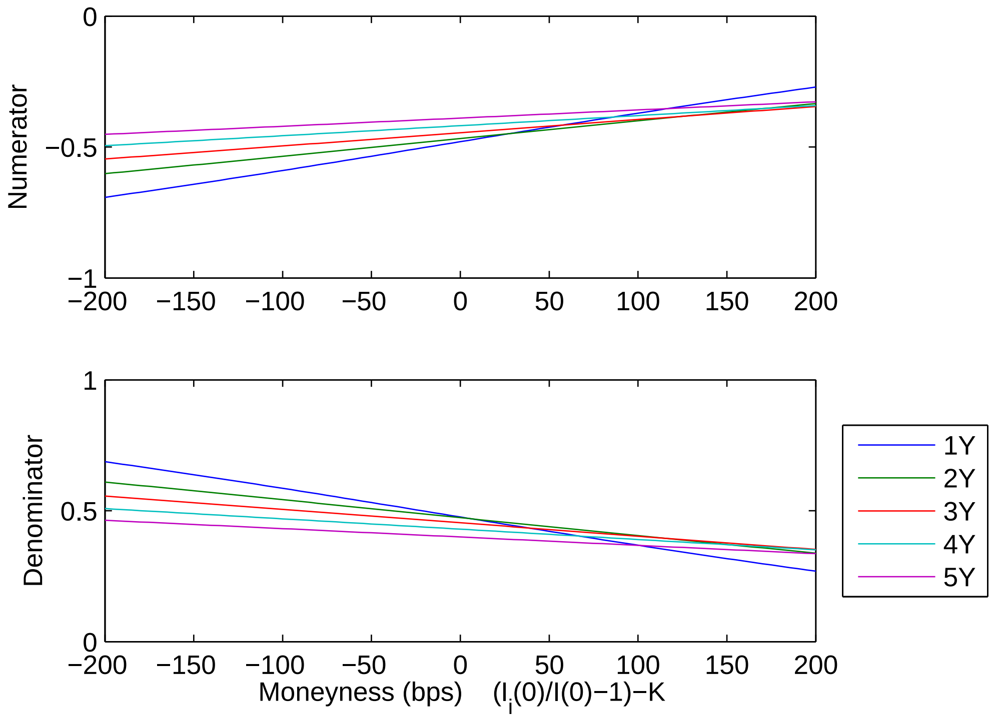

Figure 4.13: Delta of the floorlet with respect to the forward CPIs.

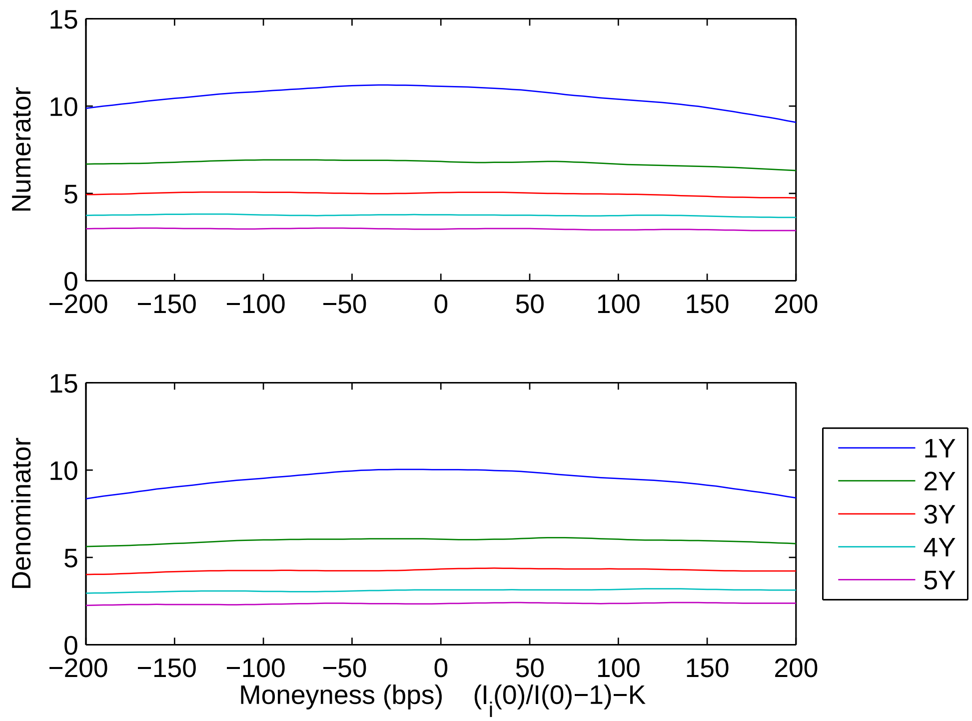

Figure 4.14: Gamma of the floorlet with respect to the forward CPIs.

4.3.4 Pricing Caps

Pricing caps is straightforward once we are able to price caplets since a cap is the sum of a series of joint caplets. We write the cap present value as

$$\mathbf{IICapFloor}(t, T, K, N, w) = N \sum_{i=1}^{M} P_n(0, T_i) E^{T_i} \left[[\omega(Y_i(T_i) - K)]^+ \;\middle|\; \mathcal{F}_t \right] \tag{4.21}$$

where we let $\omega = 1$ for the caplet, $\omega = -1$ for the floorlet and N is the notional.

Although each caplet within a cap does not necessarily have the same moneyness, they still all have the same strike. For this reason we replace the moneyness on the X axis from the previous plots by the strike of the cap which is a more convenient representation. We only include the figures for the caps for conciseness but the floors can easily be obtained as well by the same method. In the next set of figures, the deltas appear scrambled due to the different levels of moneyness for each caplet.

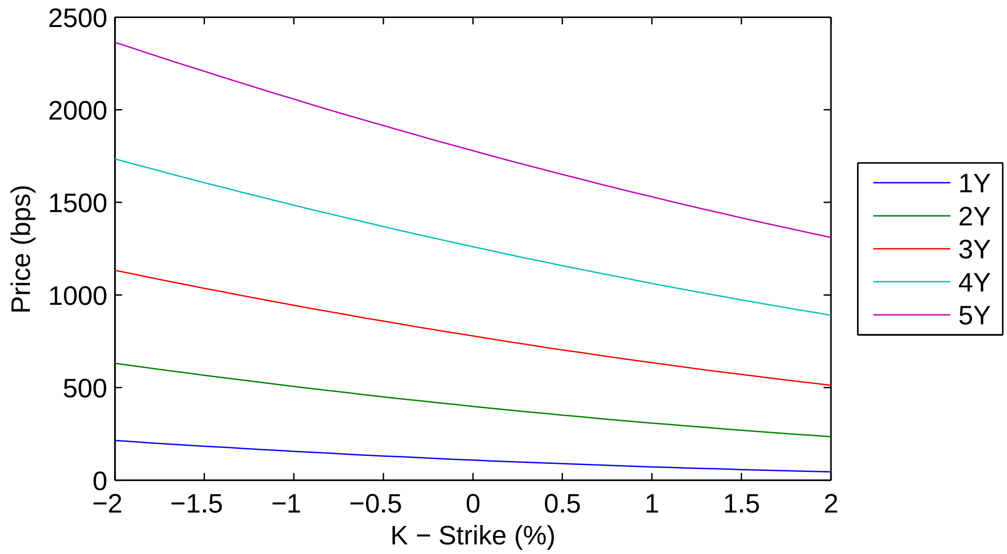

Figure 4.15: Price in basis points of caps for different strikes K and maturities T_i.

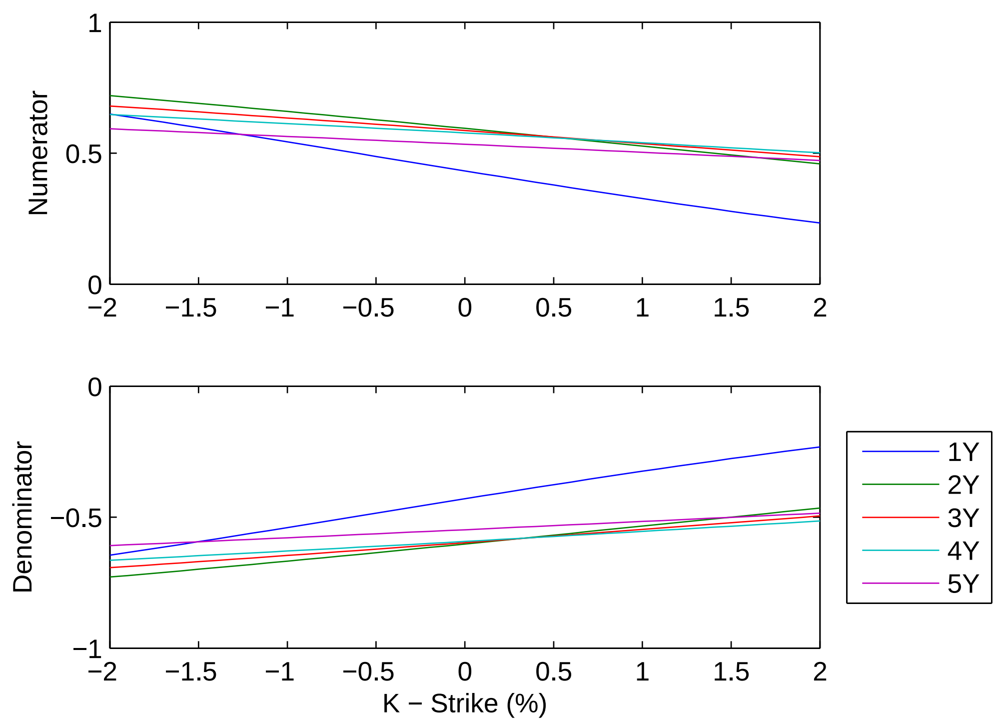

Figure 4.16: Delta of the cap with respect to the forward CPIs.

On the next pages, Figure 4.17 shows the deltas of the 5Y cap while Figure 4.18 shows the gammas. So far we have been showing deltas with respect to the numerator and the denominator of the CPI ratio within the payoff function to outline the consistent patterns in the model. However, from a practical perspective it is obviously more appropriate to consider the aggregate delta of a position with respect to each forward CPI which is what we are presenting in the next figures. We see that, since a cap is essentially a portfolio of joint caplets, there is already a fair amount of delta hedging that is naturally built into the portfolio. For example, the 2 year forward CPI delta in a 2 year caplet contract is very well hedged by a an equivalent position in a 3 year caplet with the same strike. Of course, the denominator within the very last caplet ratio that forms the joint caplet series cannot be hedged in that manner. This is why we observe a high delta with respect to the 5Y forward CPI contract. On the other hand, the gamma of the denominator and numerator will add up which means that the delta is quite sensible to changes in the underlying CPI ratio.

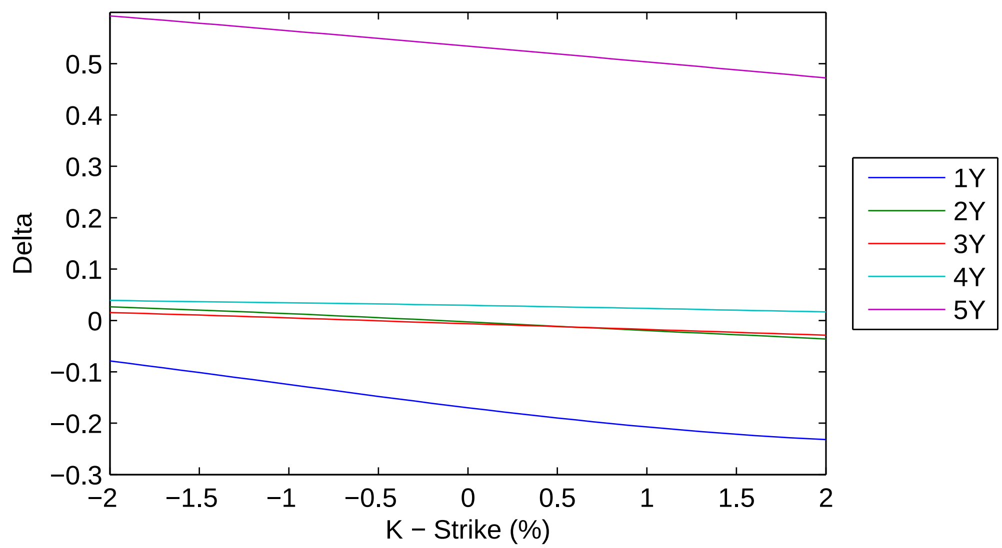

Figure 4.17: Delta of a 5Y cap as of April 23^{rd} 2009

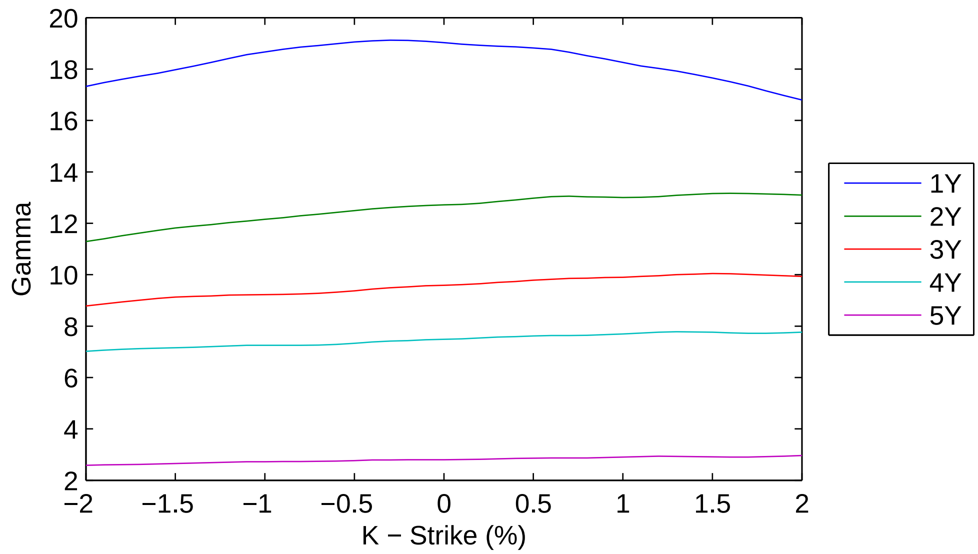

Figure 4.18: Gamma of a 5Y cap as of April 23^{rd} 2009

One might have noted in the previous figure that the delta with respect to the 1Y forward CPI is fairly high but negative. This is mainly due to the steepness in the short end of the forward inflation curve we have used as an input in our model. A steep forward inflation curve means that caplets that form a cap have a larger difference in moneyness which reduces the effectiveness of their mutual hedging within the portfolio. In the next figure we have used a practically flat forward inflation curve observed on August 8^{th} 2006. Notice that the delta for 1Y to 4Y forward CPI is very small for any strike K. This shows how the high gamma can have a big impact on the fluctuations of the delta, especially with respect to the shorter maturities.

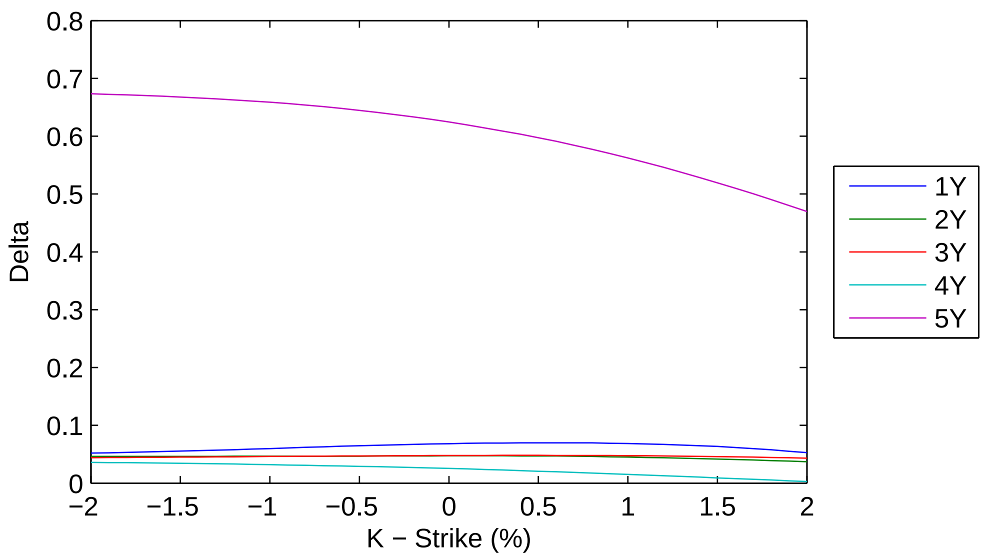

Figure 4.19: Delta of a 5Y cap observed on August 8^{th} 2006 when forward inflation expectations were flat (See Figure 2.2)

4.3.5 A close look at the evolution of the delta

We have seen in the previous section that in a portfolio of joint caplets with the same strike most of the hedging stems from two sources. The first and main source of delta with respect to the forward CPI is through the numerator within the payoff of the last caplet. The second source of forward CPI delta is due to the steepness of the forward inflation curve which causes caplets within the cap to have different levels of moneyness.

In the following figure, we go back to pricing a cap as of April 23^{rd} 2009 and we show the likely evolution of its delta throughout the life of the contract. Deltas were obtained at the end of each year when a caplet matures by applying the forward difference method over 30 000 nodes in our tree. We also show the standard deviations of our estimates denoted by the error bars. Notice that the error bars tend to increase quite a bit with maturity for the last delta. In some of our simulation paths the inflation can get very high causing the last caplet in the 5Y cap to be deep in the money. This is why the error bar indicates that in some instances the 5Y delta could be very high. On the other hand, we can see by looking at the small error bars from the 1Y to 4Y delta that delta hedging these caplets is very manageable. A delta hedging strategy where one wishes to hedge the forward CPI delta in a portfolio of caplets implies that we are trading the underlying forward CPI. This can accomplished through the Zero Coupon Inflation Swap since we have shown that there exist a one to one relationship between a ZCIIS quote and a forward CPI quote.

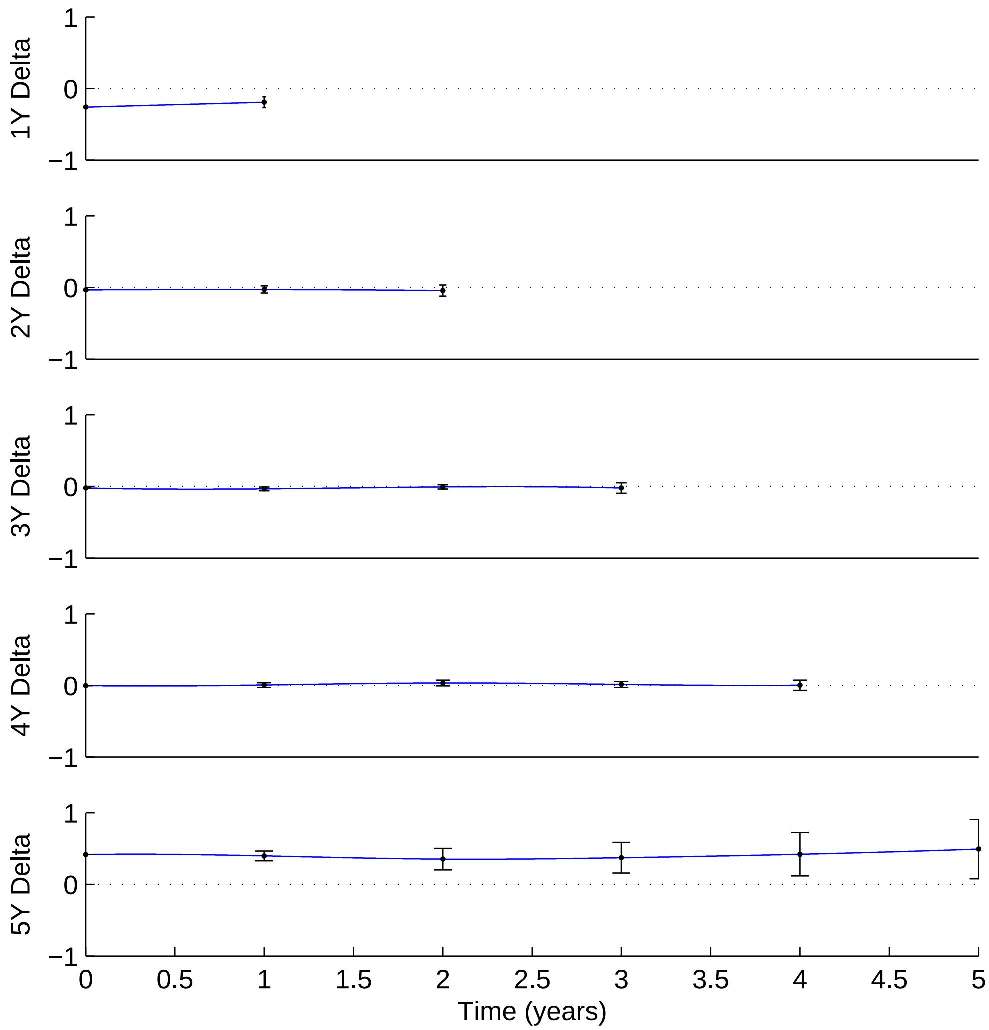

Figure 4.20: Evolution of the delta over the life of a 5Y inflation cap.

Chapter 5

Conclusion

Inflation can highly important to corporations and long term investors who care about the long term purchasing power of their investment. Hedging inflation risk is a sensible thing to do in cases where, due to the nature of its business, an institution has to deal with inflation-linked cash flows. Inflation derivatives provide them with a way to do this. The financial crisis of 2008 has renewed uncertainty regarding inflation. Following the massive quantitative easing programs from many central banks around the world, some are worried about hyperinflation while others argue that we are still in a liquidity trap and urge us to consider deflation as a potential threat. Meanwhile, core inflation still hovers around the normal levels in Canada and the U.S. leading some people to think that we might be doing just fine.

Diverging views on future inflation is a great opportunity for revisiting inflation derivatives. In a setting where the inflation-indexed securities market is fairly liquid, levered investors can enter cheap/rich trades to exploit anomalies in the term structure of inflation or hedge the inflation swaps that they sell to clients. We have seen that the most common inflation derivative, the Zero-Coupon Inflation Indexed Swap (ZCIIS), is in essence a contract through which one can trade the discount factors of a real economy versus a nominal economy within the foreign currency analogy framework. Adopting a different methodology, one can also view the ZCIIS as a way to trade the forward CPI. This leads us to the more recent developments of inflation derivative pricing through market models. Finally, we have discussed an extended HJM model where instantaneous inflation rates can be simulated in parallel to the instantaneous interest rates. This is a great unifying framework that potentially allows for pricing inflation derivatives while calibrating directly to the inflation term structure from the bond market.

In this thesis, we have mainly focused on pricing inflation derivatives that trade on the U.S. price index, the Consumer Price Index (CPI), because the market for trading inflation in Canada is inadequate for two reasons. First, the market for inflation-indexed securities in Canada is not very liquid. Second, its maturity profile is inadequate because no Real Return Bond is scheduled to mature before the year 2021. Nonetheless, we were able to obtain daily historical quotes from ZCIIS based on the U.S. CPI which is our main building block for pricing more complex derivatives.

In our simulations, we have implemented using a Monte Carlo methodology a market model for inflation that simulates the evolution of the forward CPI ratio. We chose the forward CPI ratio because it offers an instant calibration to the ZCIIS market and easily captures the convexity adjustment needed for pricing inflation derivatives that have their payoff tied to forward inflation rather than spot inflation. To obtain a pricing of instruments with multiple reset dates we have modeled the joint evolution of consecutive forward inflation rates in a way that is similar to the usual market model methodology in interest rate derivative pricing.

Using the historical data from the zero-coupon bond and the ZCIIS, we estimate the input parameters of our model. We estimate the volatility of forward interest rates and their correlation with forward inflation using the time series for forward interest rate and forward CPI ratio. Furthermore, we isolate the evolution of the factors that drive the forward CPI which allows us to obtain the volatility and the correlation structure needed for modeling the CPI ratio. Aside from the simple methodology we have used for parameter estimation, the most serious limitation of our estimation is probably that we have used a dataset that is too short. We had access to roughly four years of daily historical data but since we are trying to price derivatives with maturities up to five years this seems insufficient. Furthermore, the dataset encompasses the financial crisis of 2008 during which many liquidity issues were reported in the fixed income market. We believe that the no-arbitrage assumption of our model was violated during this period and that historical data should be adjusted for this period to remove the liquidity shock.

Using our market model, we simulated the evolution of spot inflation with a one year tenor. The main inputs are the forward interest rate curve and the forward inflation curve (forward CPI ratio) observed in the market. Finally, we have carried out the pricing of caplet and floorlets over a wide range of moneyness. The results were used to price caps and we emphasized that a portfolio of joint caplets has two main sources of delta. The first is the steepness of the forward inflation curve causing each caplet to have different levels of moneyness and thus different deltas. The second source of delta is with respect to the CPI in the numerator of the last caplet in the joint series which will never be hedged naturally within the portfolio.

Variance reduction techniques are very common to improve the convergence rate of Monte Carlo simulations and increase computation efficiency. Future work might involve the use of such techniques to allow for computationally more efficient simulations. The improved efficiency could, for example, allow us to fix the drift or reduce the simulation time step. In doing so, we could use interpolation to reduce the indexing of volatility from a yearly time to maturity to monthly time to maturity. Another area for future research which is critical for practical applications of inflation derivative pricing is to include seasonality in our simulations. Without modeling seasonality, the year to year contracts cannot be valued appropriately in-between reset dates. This could be an issue if an institution wants to unwind a position at a time that does not coincide with a reset date. Finally, it would also be interesting to look at the amount of basis risk one might be facing by using inflation derivatives on the CPI-u to hedge liabilities tied to a different price index.

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