On the Surplus Process of Ruin Theory When Perturbed by a Diffusion

by

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Abstract

This thesis studies in detail the expected discounted function of a penalty at ruin which involves the time of ruin, the surplus immediately prior to the time of ruin, and the deficit at the time of ruin, based on the surplus process of ruin theory containing an independent *Wiener (diffusion)* process.

First, main background for this thesis is reviewed in chapter 1, which contains the surplus process of ruin theory with and without a Wiener process, the defective renewal equations for some expected (discounted) functions, reliability-based classification and equilibrium distribution.

In chapter 2, we will derive the defective renewal equation and the asymptotic formula for the expected discounted function of a penalty at time of ruin, and propose the Tijms-type approximation for and an upper and a lower bounds on a compound geometric distribution function. Moreover, the reliability-based class implications for the associated claim size distribution are also given. When the claim size distribution is a combination of exponentials or a mixture of Erlangs, explicit analytical solutions to the compound geometric distribution function and to the expected discounted probability of ruin due to oscillation and a claim, respectively, can be obtained.

Moments are studied in chapter 3 include the (discounted) moment of the deficit at the time of ruin, the joint moment of the deficit at ruin and the time of ruin, and the moments of the time of ruin due to oscillation and caused by a claim, respectively.

In chapter 4, we give the explicit expressions for the (discounted) joint and marginal distribution functions of the surplus immediately before the time of ruin and the deficit at the time of ruin, and for the (discounted) distribution function of the amount of the claim causing ruin, Then the (discounted) probability density functions are obtained by differentiating the corresponding (discounted) distribution functions. In addition, the defective renewal equations for these (discounted) distribution functions and probability density functions, respectively, are also derived.

Finally, summary and future research are presented in chapter 5.

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Contents

List of Symbols and Corresponding Equation Numbers

$A(u)$	(1.27).
a	(1.26).
α	(2.201).
$\alpha_{-1}(u,\rho)$	(3.9), (3.42).
$\alpha_{0;-1}(u,\rho)$	(3.15).
$\alpha_{0;0}(u,\rho)$	(3.23).
$\alpha_{0;n}(u,\rho)$	(3.13), (3.14), (3.24).
$\alpha_{0;n+1}(u,\rho)$	(3.18).
$\alpha_0(u,\rho)$	(3.21), (3.43).
$\alpha_n(u,\rho)$	(3.7), (3.8), (3.22), (3.44).
$\alpha_{n+1}(u,\rho)$	(3.16), (3.17).
$B(u)$	(2.85), (2.98), (2.99), (2.114), (2.115), (2.122), (3.50), (3.66), (4.15), (4.17).
$B'(u)$	(4.16), (4.18).
$B_{d;2}(u)$	(3.125).

$B_{s;2}(u)$	(3.156).
$B_{\delta=0}(u)$	(3.73), (3.98)
$B_0(u)$	(2.116), (4.19).
$B_{0;2}(u)$	(3.159).
$B_{0,\delta=0}(u)$	(3.81).
$B_{0;1,w}(u)$	(3.82), (3.83).
$B_{1,w}(u)$	(3.74), (3.75), (3.86).
b	(1.25).
β	(2.35).
β_0	(2.36).
C	(2.199).
$\overline{C}_n$	(2.160) .
$C(z)$	(2.157).
$C(z,w)$	(2.163).
D_0	(2.132), (2.169).
D_j	(2.133), (2.145), (2.170), (2.179), (2.182).
$D_{j,\delta=0}$	(2.140), (2.176).
$F(x,y;\delta,D\|u)$	(4.14), (4.20).
$F(x,y;\delta,D\|0)$	(4.21).
$F(x,y;\delta,0\|u)$	(4.22).
$F(x,y;\delta,0\|0)$	(4.23).
$F(x,y;0,0\|u)$	(4.24).
$F(x,y;0,0\|0)$	(4.25).

$F_Z(z;\delta,D\|u)$	(4.94), (4.101).
$F_Z(z;\delta,D\|0)$	(4.95).
$F_Z(z;\delta,0\|u)$	(4.102).
$F_Z(z;\delta,0\|0)$	(4.96).
$F_Z(z;0,0\|u)$	(4.103).
$F_Z(z;0,0\|0)$	(4.97).
$F_Z(\infty;\delta,D\|u)$	(4.98).
$F_Z(\infty;\delta,0\|u)$	(4.99).
$F_Z(\infty;0,0\|u)$	(4.100).
$F_1(x;\delta,D\|u)$	(4.59), (4.73), (4.77), (4.82).
$F_1(x;\delta,D\|0)$	(4.60).
$F_1(x;\delta,0\|u)$	(4.62), (4.74), (4.83).
$F_1(x;\delta,0\|0)$	(4.63).
$F_1(x;0,0\|u)$	(4.65), (4.75), (4.81).
$F_1(x;0,0\|0)$	(4.66).
$F_1(\infty;\delta,D\|u)$	(4.61).
$F_1(\infty;\delta,0\|u)$	(4.64).
$F_1(\infty;0,0\|u)$	(4.67).
$F_2(y;\delta,D\|u)$	(4.32), (4.47), (4.51), (4.52).
$F_2(y;\delta,D\|0)$	(4.33).
$F_2(y;\delta,0\|u)$	(4.35), (4.48).
$F_2(y;\delta,0\|0)$	(4.36).
$F_2(y;0,0\|u)$	(4.38), (4.49).

$F_2(y;0,0\|0)$	(4.39).
$F_2(\infty;\delta,D\|u)$	(4.34).
$F_2(\infty;\delta,0\|u)$	(4.37).
$F_2(\infty;0,0\|u)$	(4.40).
$F_{2,u}(y;\delta,D)$	(4.56).
$F_{2,\infty}(y;\delta,D)$	(4.57), (4.58).
$f(x,y;\delta,D\|u)$	(4.1), (4.26).
$f(x,y;\delta,D\|0)$	(4.27).
$f(x,y;\delta,0\|u)$	(4.28).
$f(x,y;\delta,0\|0)$	(4.29).
$f(x,y;0,0\|u)$	(4.30).
$f(x,y;0,0\|0)$	(4.31).
$f_Z(z;\delta,D\|u)$	(4.86), (4.87), (4.104).
$f_Z(z;\delta,D\|0)$	(4.88).
$f_Z(z;\delta,0\|u)$	(4.89), (4.105).
$f_Z(z;\delta,0\|0)$	(4.90).
$f_Z(z;0,0\|u)$	(4.91), (4.106).
$f_Z(z;0,0\|0)$	(4.92).
$f_1(x;\delta,D\|u)$	(4.2), (4.68), (4.78).
$f_1(x;\delta,D\|0)$	(4.69).
$f_1(x;\delta,0\|u)$	(4.6), (4.70), (4.79).
$f_1(x;\delta,0\|0)$	(4.7).
$f_1(x;0,0\|u)$	(4.4), (4.71), (4.80).

$f_1(x;0,0\|0)$	(4.5).
$f_2(y;\delta,D\|u)$	(4.3), (4.41), (4.53).
$f_2(y;\delta,D\|0)$	(4.42).
$f_2(y;\delta,0\|u)$	(4.43), (4.54).
$f_2(y;\delta,0\|0)$	(4.44).
$f_2(y;0,0\|u)$	(4.45), (4.55)
$f_2(y;0,0\|0)$	(4.46).
$G(x)$	(2.40), (2.43).
$G'(x)$	(2.39), (2.42), (2.62), (2.63), (2.68).
$\overline{G}(x)$	(2.48), (2.49), (2.53), (2.54).
$\tilde{G}(s)$	(1.38).
$g(y)$	(1.28), (1.29), (1.30), (2.18), (2.20).
$g(x,y)$	(2.26), (2.29).
$\tilde{g}(\xi,\delta)$	(2.19).
$\tilde{g}'(\xi,\delta)$	(2.192).
$g_\omega(u)$	(2.11), (2.12), (2.13), (2.21), (2.23), (2.25), (2.30).
$\tilde{g}_\omega(\xi,\delta)$	(2.22).
$\Gamma'(x)$	(1.53), (2.59).
$\overline{\Gamma}(x)$	(1.49), (1.52), (2.50).
$\overline{\Gamma}_n(x)$	(1.57).
$\Gamma_n * H(u)$	(3.12).
$\overline{\Gamma}_n * H(u)$	(3.10), (3.11).
$\overline{\Gamma}_{n+1}(x)$	(1.58), (3.3).

$\psi_d(u)$	(1.5), (1.15), (2.152).
$\psi_{d;n}(u)$	(3.111), (3.112), (3.113), (3.117).
$\psi_{d;1}(u)$	(3.118), (3.119), (3.120), (3.122).
$\psi_{d;2}(u)$	(3.123), (3.124), (3.126).
$\psi_s(u)$	(1.6), (1.16), (2.154).
$\tilde{\psi}_s(\xi)$	(2.196).
$\psi_{s;n}(u)$	(3.129), (3.130), (3.131), (3.143).
$\psi_{s;1}(u)$	(3.144), (3.145), (3.146), (3.153), (3.162).
$\psi_{s;2}(u)$	(3.154), (3.155), (3.157).
$\psi_t(u)$	(1.7), (1.17), (2.150).
$\psi_0(u)$	(1.3), (2.92).
$\psi_{0;1}(u)$	(3.148), (3.149), (3.151).
$\psi_{0;1,n}(u)$	(3.95), (3.96), (3.97).
$\psi_{0;1,w}(u)$	(3.76), (3.77), (3.78).
$\psi_{0;1,1}(u)$	(3.102), (3.103), (3.104).
$\psi_{0;2}(u)$	(3.158), (3.160), (3.161).
$\psi_{0;n}(u)$	(3.133), (3.134), (3.136).
$\psi_{1,n}(u)$	(3.92), (3.93), (3.94).
$\psi_{1,w}(u)$	(3.68), (3.69), (3.70), (3.88).
$\psi_{1,1}(u)$	(3.99), (3.100), (3.101), (3.110).
$Q^*(z)$	(2.155).
$Q^{**}(z)$	(2.161).
q^*	(2.55), (2.64).

$q^*_{\delta=0}$	(2.56), (2.65).
q^*_k	(2.51), (2.60).
$q^*_{k,\delta=0}$	(2.52), (2.61).
q^{**}_k	(2.57), (2.66).
$q^{**}_{k,\delta=0}$	(2.58), (2.67).
$r_G(x)$	(2.208).
$r_G(\infty)$	(2.209).
$r_\Gamma(x)$	(2.203), (2.205).
$r_\Gamma(\infty)$	(2.206).
$r_P(x)$	(1.48).
$r_P(\infty)$	(2.202), (2.206).
$\tau(\xi)$	(2.186).
$\tau_0(u)$	(3.28), (3.45).
$\tau_{0;0}(u)$	(3.32).
$\tau_{0;1}(u)$	(3.40).
$\tau_{0;n}(u)$	(3.33), (3.34), (3.41).
$\tau_{0;n+1}(u)$	(3.37).
$\tau_1(u)$	(3.38), (3.46).
$\tau_n(u)$	(3.29), (3.30), (3.39), (3.47).
$\tau_n(0)$	(3.31).
$\tau_{n+1}(u)$	(3.36).
$U(t)$	(1.1), (1.4).
$w(x,z)$	(4.50).

$w(x_1, x_2)$	(4.12).
$w(z, y)$	(4.76).
$Z(u)$	(1.12), (1.46).

Chapter 1

Background

1.1 Classical risk model

In the classical continuous time risk model, the number of claims is assumed to follow a Poisson process $\{N(t) : t \geq 0\}$ with mean λ. The individual claim sizes $X_1, X_2, \ldots$, independent of $N(t)$, are positive, independent and identical random variables with common distribution function (df) $P(x) = Pr(X \leq x)$, moments $p_j = \int_0^\infty x^j dP(x)$ for $j = 0, 1, 2, \ldots$. The aggregate process $\{S(t) : t \geq 0\}$, where $S(t) = X_1 + X_2 + \cdots + X_{N(t)}$ (with $S(t) = 0$ if $N(t) = 0$) is the aggregate claims up to time t, is a compound Poisson process with parameter λ. The surplus of an insurer at time t is

$$U(t) = u + c\,t - S(t), \quad t \geq 0, \tag{1.1}$$

where $u = U(0)$ is the initial surplus, $c = \lambda p_1(1 + \theta)$ is the constant rate per unit time at which the premiums are received, and $\theta > 0$ is the relative security loading.

Let $T = inf\{t : U(t) < 0\}$ be the time of ruin (the first time that the surplus becomes negative). Two important nonnegative random variables in connection with the time of ruin T are $|U(T)|$, the deficit at the time of ruin, and $U(T-)$, the surplus immediately before the time of ruin, where $T-$ is the left limit of T. Another associated random variable is $\{U(T-) + |U(T)|\}$, the amount of the claim causing ruin.

There have been many papers discussing various issues, such as marginal and joint distributions of T, $U(T-)$ and $|U(T)|$, based on the model (1.1). See Gerber, Goovaerts and Kaas (1987) [23], Dufresne and Gerber (1988b) [15], Dickson (1992) [5] and (1993) [6], Dickson and Waters (1992) [13], Dickson, Egídio dos Reis and Waters (1995) [11], Dickson and Egídio dos Reis (1996) [9], Gerber and Shiu (1997) [28] and (1998a) [29], Willmot and Lin (1998) [47], Lin and Willmot (1999) [36], Picard and Lefèvre (1998) [38] and (1999) [39], and references therein.

In particular, Gerber and Shiu (1998a) [29] considered a function associated with a given penalty function w and the joint distributions of T, $U(T-)$ and $|U(T)|$ as follows: For $\delta \geq 0$, define

$$\phi_0(u) = E[e^{-\delta T} w(U(T-), |U(T)|) I(T < \infty) | U(0) = u], \quad u \geq 0, \tag{1.2}$$

where $w(x, y), 0 \leq x, y < \infty$, is a nonnegative function; $I(T < \infty) = 1, T < \infty$ and $I(T < \infty) = 0$ otherwise.

Equation (1.2) may be viewed as the Laplace transform of w with the argument δ, or as the expectation of the discounted penalty function with the force of interest

δ. Of course, the ruin probability is

$$\psi_0(u) = E[I(T < \infty)|U(0) = u] = Pr(T < \infty|U(0) = u), \quad u \geq 0, \tag{1.3}$$

a special case for $w(x, y) = 1$ and $\delta = 0$.

1.2 Diffusion process

Dufresne and Gerber (1991) [16] extended the classical risk model (1.1) by adding an independent diffusion process (or Wiener process) to (1.1) to form

$$U(t) = u + c\,t - S(t) + \sigma W(t), \quad t \geq 0, \tag{1.4}$$

where $\sigma > 0$ and $\{W(t) : t \geq 0\}$ is a standard Wiener process (that is, $W(t) \sim N(0, t)$ and then $\sigma W(t) \sim N(0, \sigma^2 t) = N(0, 2Dt)$, where $D = \sigma^2/2$) that is independent of the compound Poisson process $\{S(t) : t \geq 0\}$.

They studied $\psi_d(u)$, the probability of ruin caused by oscillation, $\psi_s(u)$, the probability of ruin caused by a claim, and $\psi_t(u)$, the probability of ruin caused either by oscillation or by a claim, where

$$\psi_d(u) = Pr(T < \infty, U(T) = 0|U(0) = u), \; u \geq 0, \tag{1.5}$$

$$\psi_s(u) = Pr(T < \infty, U(T) < 0|U(0) = u), \; u \geq 0, \tag{1.6}$$

and

$$\psi_t(u) = \psi_d(u) + \psi_s(u) = Pr(T < \infty|U(0) = u), \; u \geq 0. \tag{1.7}$$

Later, Gerber and Landry (1998) [24] generalized the discussion of Dufresne and Gerber(1991) [16] based on the same model (1.4) by considering a penalty scheme which is defined by a constant w_0 and a nonnegative function $w(-y), y > 0$. Then the penalty due at ruin is w_0 if ruin occurs by oscillation and $w(U(T))$ if ruin is caused by a jump. They declared that the expected discounted penalty $\phi(u)$, as a function of the initial surplus, is

$$\begin{aligned}\phi(u) &= w_0\, E[e^{-\delta T} I(T < \infty, U(T) = 0) | U(0) = u] \\ &+ E[e^{-\delta T} w(U(T)) I(T < \infty, U(T) < 0) | U(0) = u], \quad u \geq 0. \quad (1.8)\end{aligned}$$

In the special case that $w(-y) = 0$ and $w_0 = 1$, $\phi(u)$ becomes

$$\phi_d(u) = E[e^{-\delta T} I(T < \infty, U(T) = 0) | U(0) = u], \quad u \geq 0, \quad (1.9)$$

the defective Laplace transform or the expectation of the present value of the time of ruin T due to oscillation. Note that when $\delta = 0$, $\phi_d(u) = \psi_d(u)$.

For another special case with $w(-y) = 1$ and $w_0 = 0$, $\phi(u)$ turns out to be

$$\phi_s(u) = E[e^{-\delta T} I(T < \infty, U(T) < 0) | U(0) = u], \quad u \geq 0, \quad (1.10)$$

the defective Laplace transform or the expectation of the present value of the time of ruin T due to a claim. Similarly, when $\delta = 0$, $\phi_s(u) = \psi_s(u)$.

If we define

$$\phi_t(u) = \phi_d(u) + \phi_s(u)$$

$$\begin{aligned} &= E[e^{-\delta T} I(T < \infty, U(T) = 0) | U(0) = u] \\ &+ E[e^{-\delta T} I(T < \infty, U(T) < 0) | U(0) = u], \quad u \geq 0, \end{aligned} \tag{1.11}$$

then obviously, when $\delta = 0$, $\phi_t(u) = \psi_t(u)$.

1.3 Defective renewal equation

Function $Z(u)$ is said to satisfy a renewal equation if $Z(u)$ can be expressed as

$$Z(u) = \int_0^u Z(u-x)dF(x) + v(u), \ \ u \geq 0, \tag{1.12}$$

where F is a distribution function concentrated on $[0, \infty)$ with $F(0) = 0$. If F is a defective distribution function (that is, $F(\infty) < 1$), then the renewal equation above is defective.

Gerber and Shiu (1998a) [29] derived a defective renewal equation based on (1.1) for $\phi_0(u)$ in (1.2) as follows:

$$\phi_0(u) = \frac{\lambda}{c}\int_0^u \phi_0(u-x)\int_x^\infty e^{-\rho(y-x)}dP(y)dx + \frac{\lambda}{c}e^{\rho u}\int_u^\infty e^{-\rho x}\int_x^\infty w(x, y-x)dP(y)dx. \tag{1.13}$$

In (1.13), $\rho = \rho(\delta)$ is the unique nonnegative root of Lundberg's equation

$$\lambda\tilde{p}(\xi) = \lambda + \delta - c\xi, \tag{1.14}$$

where $\tilde{p}(s) = \int_0^\infty e^{-sx}dP(x)$ and $\rho(0) = 0$.

For the surplus process with a diffusion process, Dufresne and Gerber (1991) [16] derived defective renewal equations based on (1.4) for $\psi_d(u)$, $\psi_s(u)$ and $\psi_t(u)$, in (1.5), (1.6) and (1.7) respectively, as follows:

$$\psi_d(u) = \frac{1}{1+\theta}\int_0^u \psi_d(u-x)h_1 * h_2(x)dx + [1 - H_1(u)], \tag{1.15}$$

$$\psi_s(u) = \frac{1}{1+\theta}\int_0^u \psi_s(u-x)h_1 * h_2(x)dx + \frac{1}{1+\theta}[H_1(u) - H_1 * H_2(u)], \tag{1.16}$$

and

$$\psi_t(u) = \frac{1}{1+\theta}\int_0^u \psi_t(u-x)h_1 * h_2(x)dx + \frac{\theta}{1+\theta}[1 - H_1(u)] + \frac{1}{1+\theta}[1 - H_1 * H_2(u)] \tag{1.17}$$

where

$$\begin{aligned} H_1(u) &= 1 - e^{-\frac{c}{D}u}, \quad u \geq 0, & (1.18)\\ H_2(u) &= \frac{\int_0^u [1 - P(x)]dx}{p_1}, \quad u \geq 0, & (1.19) \end{aligned}$$

and

$$\begin{aligned} h_1(u) &= H_1'(u) = \frac{c}{D}e^{-\frac{c}{D}u}, \quad u \geq 0, & (1.20)\\ h_2(u) &= H_2'(u) = \frac{1 - P(u)}{p_1}, \quad u \geq 0. & (1.21) \end{aligned}$$

Note that $\psi_s(0) = 0$ and $\psi_t(0) = \psi_d(0) = 1$ by the oscillating nature of the sample paths.

With regard to the expected discounted penalty function, Gerber and Landry (1998) [24] first showed based on (1.4) that $\phi_d(u)$ in (1.9) satisfies the defective renewal equation

$$\phi_d(u) = \int_0^u \phi_d(u-y)g(y)dy + A(u), \quad u \geq 0, \tag{1.22}$$

where

$$\begin{aligned} h(s) &= \frac{c}{D}e^{-bs}, &(1.23)\\ \gamma(s) &= \frac{\lambda}{c}\int_s^\infty e^{-\rho(x-s)}dP(x), &(1.24)\\ b &= a-\rho = \frac{c}{D}+\rho, &(1.25)\\ a &= \frac{c}{D}+2\rho, &(1.26)\\ A(u) &= e^{-bu}, \quad u \geq 0, &(1.27) \end{aligned}$$

is the expected discounted value of a contingent payment of 1 that is due at ruin, provided that ruin occurs before the first record low (the first time where the surplus falls below the initial level) that is caused by a jump, and

$$\begin{aligned} g(y) &= h*\gamma(y) = \int_0^y h(y-s)\gamma(s)ds &(1.28)\\ &= \frac{\lambda}{D}\int_0^y e^{-b(y-s)}\int_s^\infty e^{-\rho(x-s)}dP(x)ds &(1.29)\\ &= \frac{\lambda}{D}e^{\rho y}\int_0^y e^{-a(y-s)}\int_s^\infty e^{-\rho x}dP(x)ds, &(1.30) \end{aligned}$$

is the discounted probability that the first record low is caused by a jump with

$$\int_0^\infty g(y)dy = \left(\int_0^\infty h(y)dy\right)\left(\int_0^\infty \gamma(y)dy\right)$$

$$= \left(\frac{c}{c+\rho D}\right)\left(\frac{D\rho^2 + c\rho - \delta}{c\rho}\right)$$
$$= \frac{D\rho^2 + c\rho - \delta}{D\rho^2 + c\rho} < 1. \quad (1.31)$$

In (1.24), (1.25), (1.29), (1.30) and (1.31), the quantity $\rho = \rho(\delta)$ is the unique nonnegative root of generalized Lundberg's equation

$$\lambda\tilde{p}(\xi) = \lambda + \delta - c\xi - D\xi^2 \quad (1.32)$$

with $\rho(0) = 0$.

Then they demonstrated by a probabilistic interpretation that $\phi(u)$ in (1.8) satisfies the defective renewal equation

$$\begin{aligned} \phi(u) &= \int_0^u \phi(u-y)g(y)dy + w_0\, A(u) \\ &+ \int_u^\infty w(u-y)g(y)dy - A(u)\int_0^\infty w(-y)g(y)dy, \quad u \geq 0, \end{aligned} \quad (1.33)$$

Clearly, from (1.22) and (1.33), $\phi_s(u)$ satisfies the defective renewal equation

$$\phi_s(u) = \int_0^u \phi_s(u-y)g(y)dy + \int_u^\infty g(y)dy - A(u)\int_0^\infty g(y)dy, \quad u \geq 0, \quad (1.34)$$

and $\phi_t(u)$ satisfies the defective renewal equation

$$\phi_t(u) = \int_0^u \phi_t(u-y)g(y)dy + \int_u^\infty g(y)dy + A(u)\left[1 - \int_0^\infty g(y)dy\right], \quad u \geq 0. \quad (1.35)$$

Next, consider the defective renewal equation as follows:

$$\phi(u) = \frac{1}{1+\beta}\int_0^u \phi(u-x)dG(x) + \frac{1}{1+\beta}B(u), \quad u \geq 0. \tag{1.36}$$

where $\beta > 0$, $G(x) = 1 - \overline{G}(x)$ is a distribution function, and $B(u)$ is of bounded variation. Define the associated compound geometric distribution function $K(u) = 1 - \overline{K}(u)$ by

$$\overline{K}(u) = \sum_{n=1}^{\infty} \frac{\beta}{1+\beta}(\frac{1}{1+\beta})^n \overline{G}^{*n}(u), \quad u \geq 0, \tag{1.37}$$

with $\overline{K}(0) = \frac{1}{1+\beta}$, and where $\overline{G}^{*n}(u)$ is the tail of the n-fold convolution of $G(u)$, i.e.

$$\int_0^{\infty} e^{-su}\overline{G}^{*n}(u)du = \frac{1}{s}\{1 - [\tilde{G}(s)]^n\}$$

with

$$\tilde{G}(s) = \int_0^{\infty} e^{-su}dG(u). \tag{1.38}$$

Then Lin and Willmot (1999) [36] showed that $\overline{K}(u)$ satisfies the defective renewal equation

$$\overline{K}(u) = \frac{1}{1+\beta}\int_0^u \overline{K}(u-x)dG(x) + \frac{1}{1+\beta}\overline{G}(u), \quad u \geq 0, \tag{1.39}$$

with

$$\int_0^{\infty} e^{-su}\overline{K}(u)du = \frac{1-\tilde{G}(s)}{s[1+\beta-\tilde{G}(s)]}, \tag{1.40}$$

or

$$\int_{0-}^{\infty} e^{-su} dK(u) = \frac{\beta}{1+\beta-\tilde{G}(s)}, \tag{1.41}$$

and the solution $\phi(u)$ to (1.36) may be expressed as

$$\phi(u) = -\frac{1}{\beta}\int_{0-}^{u} \overline{K}(u-x)dB(x) + \frac{1}{\beta}B(u) \tag{1.42}$$

$$= \frac{1}{\beta}\int_{0-}^{u} B(u-x)dK(x), \quad u \geq 0, \tag{1.43}$$

which, if $B(u)$ and $K(u)$ are differentiable with derivatives $B'(u)$ and $K'(u)$ for $u > 0$, respectively, may be expressed as

$$\phi(u) = -\frac{1}{\beta}\int_{0}^{u} \overline{K}(u-x)B'(x)dx + \frac{1}{\beta}B(u) - \frac{1}{\beta}B(0)\overline{K}(u) \tag{1.44}$$

$$= \frac{1}{\beta}\int_{0}^{u} B(u-x)K'(x)dx + \frac{1}{\beta}K(0)B(u), \quad u \geq 0. \tag{1.45}$$

They also showed the following theorem regarding the order between $\phi(u)$ and $\overline{K}(u)$:

Theorem 1.1 *If $B(u) \geq (\leq)\ c^*\overline{G}(u)$ where $c^* \in (0, \infty)$, then $\phi(u) \geq (\leq)\ c^*\overline{K}(u)$.*

Sometimes, we want to investigate the asymptotic behavior as $u \to \infty$ of the function $Z(u)$ satisfying the renewal equation (1.12). Feller (1971) [21] proposed a renewal theorem for this asymptotic behavior as follows:

Theorem 1.2 *Suppose that the distribution function $F(x)$ is non-arithmetic (that is, not concentrated on a set of points of the form $0, \pm h, \pm 2h, \ldots$), $\kappa > 0$ satisfies $\int_0^{\infty} e^{\kappa x} dF(x) = 1$, and $Z(u)$ satisfies the defective renewal equation (1.12). If*

$e^{\kappa x}v(x)$ is directly Riemann integrable on $(0, \infty)$, then

$$Z(u) \sim \frac{\int_0^\infty e^{\kappa x}v(x)dx}{\int_0^\infty xe^{\kappa x}dF(x)}e^{-\kappa u}, \quad as \ u \to \infty, \tag{1.46}$$

where the notation $a(x) \sim b(x)$, as $x \to \infty$, means $lim_{x\to\infty}a(x)/b(x) = 1$.

1.4 Reliability classification

Before introducing reliability-based classifications of distributions, we would like to define the failure rate and the mean residual lifetime of a distribution as follows:

If the distribution function $P(x)$ is absolutely continuous, the failure rate (hazard rate) of $P(x)$ is defined as

$$h_P(x) = -\frac{d}{dx}log\overline{P}(x) = \frac{P'(x)}{\overline{P}(x)}, \tag{1.47}$$

and the mean residual lifetime of the distribution function $P(x)$ (this does not require absolute continuity for its existence) is defined by

$$r_P(x) = \frac{\int_x^\infty \overline{P}(t)dt}{\overline{P}(x)} = \frac{\int_0^\infty \overline{P}(x+t)dt}{\overline{P}(x)}. \tag{1.48}$$

Now we briefly review various reliability-based classifications of distributions (see Fagiouli and Pellerey (1993) [18] and (1994) [19] for further details) as follows:

The distribution function $P(x)$ is:

DFR (IFR) or decreasing (increasing) failure rate if $h_P(x)$ is nonincreasing (nondecreasing) in x, or equivalently $\overline{P}(x+y)/\overline{P}(x)$ is nondecreasing (nonincreasing) in x for fixed $y \geq 0$;

IMRL (DMRL) or increasing (decreasing) mean residual lifetime

if $r_P(x)$ is nondecreasing (nonincreasing) in x, or equivalently $\overline{P}_1(x+y)/\overline{P}_1(x)$ is nondecreasing (nonincreasing) in x for fixed $y \geq 0$
(that is, $P_1(x)$ is DFR(IFR));

UWA (UBA) or used worse (better) than aged

if $r_P(x)$ satisfies $r_P(\infty) = \lim_{x\to\infty} r_P(x) \in (0,\infty)$ and $\overline{P}(x+y) \leq (\geq)$ $\overline{P}(y)e^{-x/r_P(\infty)}$ for all $x \geq 0$ and $y \geq 0$, or equivalently $h_P(x) \geq (\leq)\, h_P(\infty)$ where $h_P(\infty) \in (0,\infty)$;

UWAE (UBAE) or used worse (better) than aged in expectation

if $r_P(x)$ satisfies $r_P(x) \leq (\geq)\, r_P(\infty)$ where $r_P(\infty) \in (0,\infty)$;

NWU (NBU) or new worse (better) than used

if $\overline{P}(x+y) \geq (\leq)\, \overline{P}(x)\overline{P}(y)$ for all $x \geq 0$ and $y \geq 0$;

2-NWU (2-NBU) or 2-new worse (better) than used

if $\overline{P}_1(x+y) \geq (\leq)\, \overline{P}_1(x)\overline{P}_1(y)$ for all $x \geq 0$ and $y \geq 0$
(that is, $\overline{P}_1(x)$ is NWU (NBU));

NWUC (NBUC) or new worse (better) than used in convex ordering

if $\overline{P}_1(x+y) \geq (\leq)\, \overline{P}_1(x)\overline{P}(y)$ for all $x \geq 0$ and $y \geq 0$;

<u>**NWUE (NBUE)**</u> or <u>**new worse (better) than used in expectation**</u>

if $r_P(x) \geq (\leq)\, r_P(0)$, or equivalently $\overline{P}_1(x) \geq (\leq)\, \overline{P}(x)$ for $x \geq 0$.

The following diagram lists the implications of the classes of the distributions.

DFR (IFR)		$\Rightarrow$		NWU (NBU)
$\Downarrow$				$\Downarrow$
IMRL (DMRL)	$\Rightarrow$	2-NWU (2-NBU)	$\Rightarrow$	NWUC (NBUC)
$\Downarrow$				$\Downarrow$
UWA (UBA)	$\Rightarrow$	UWAE (UBAE)		NWUE (NBUE)

Note that the implications $IMRL(DMRL) \Rightarrow UWA(UBA) \Rightarrow UWAE(UBAE)$ hold provided the mean residual lifetime $r(\infty) \in (0, \infty)$. And the former implication was shown recently by Willmot and Cai (1999) [46].

Reliability-based classifications of distribution functions can be applied to deriving a lower or/and an upper bound on a function, and to proving preservation of classes for a function under some operations. Some results for risk theory have been found, such as Alzaid (1994) [1], Willmot (1997) [45], Willmot and Lin (1998) [47] and (1999) [48], Lin and Willmot (1999) [36], and Willmot and Cai (1999) [46].

In particular, Lin and Willmot (1999) [36] showed the following theorem concerning class implications:

Theorem 1.3 *The following class implications hold.*

(a) If $P(x)$ is IFR (DFR) then $\Gamma(x)$ is IFR (DFR).

(b) If $P(x)$ is DMRL (IMRL) then $\Gamma(x)$ is DMRL (IMRL).

(c) If $P(x)$ is 2-NBU (2-NWU) then $\Gamma(x)$ is NBUE (NWUE).

And Willmot and Cai (1999) [46] proved that

Theorem 1.4 *The following class implications hold.*

(a) If $P(x)$ is UBA (UWA) then $\Gamma(x)$ is UBA (UWA).

(b) If $P(x)$ is UBAE (UWAE) then $\Gamma(x)$ is UBAE (UWAE).

where $P(x)$ is the claim size distribution and

$$\overline{\Gamma}(x) = 1 - \Gamma(x) = \frac{\int_x^\infty \gamma(s)ds}{\int_0^\infty \gamma(s)ds} = \frac{\int_x^\infty e^{\rho s}\int_s^\infty e^{-\rho y}dP(y)ds}{\int_0^\infty e^{\rho s}\int_s^\infty e^{-\rho y}dP(y)ds}. \tag{1.49}$$

Reverse the order of integration and integrate by parts, we obtain

$$\int_x^\infty e^{\rho s}\int_s^\infty e^{-\rho y}p(y)dyds = \frac{1}{\rho}\left[\overline{P}(x) - e^{\rho x}\int_x^\infty e^{-\rho y}dP(y)\right] = e^{\rho x}\int_x^\infty e^{-\rho y}\overline{P}(y)dy. \tag{1.50}$$

Put $x = 0$ to get

$$\int_0^\infty e^{\rho s}\int_s^\infty e^{-\rho y}p(y)dyds = \frac{1-\tilde{p}(\rho)}{\rho} = \int_0^\infty e^{-\rho y}\overline{P}(y)dy. \tag{1.51}$$

Therefore, $\overline{\Gamma}(x)$ in (1.49) becomes

$$\overline{\Gamma}(x) = \frac{e^{\rho x}\int_x^\infty e^{-\rho y}\overline{P}(y)dy}{\int_0^\infty e^{-\rho y}\overline{P}(y)dy} = \frac{\int_0^\infty e^{-\rho y}\overline{P}(x+y)dy}{\int_0^\infty e^{-\rho y}\overline{P}(y)dy}, \tag{1.52}$$

and $\Gamma'(x)$ is a probability density function where

$$\Gamma'(x) = \frac{e^{\rho x}\int_x^\infty e^{-\rho y} dP(y)}{\int_0^\infty e^{-\rho y}\overline{P}(y)dy} \tag{1.53}$$

Clearly, when $\rho = 0$ which is the case when $\delta = 0$, from (1.53) we obtain $\Gamma'(x) = \overline{P}(x)/p_1 = P_1'(x)$, i.e. $\Gamma(x) = P_1(x)$.

1.5 Higher order equilibrium distributions

The equilibrium of the distribution $P_{n-1}(x)$ is defined by

$$P_n(x) = 1 - \overline{P}_n(x) = \frac{\int_0^x \overline{P}_{n-1}(y)dy}{\int_0^\infty \overline{P}_{n-1}(y)dy}, \quad n = 1, 2, 3, \ldots. \tag{1.54}$$

where $\overline{P}_0(x) = 1 - P_0(x) = \overline{P}(x)$, and $P_n(x)$ is called the n^{th} equilibrium distribution function of $P(x)$. Then it can be shown that for $n = 0, 1, 2, \ldots$,

$$\begin{aligned}
\overline{P}_n(x) &= \frac{\int_x^\infty \overline{P}_{n-1}(y)dy}{\int_0^\infty \overline{P}_{n-1}(y)dy} = \frac{\int_0^\infty \overline{P}_{n-1}(x+y)dy}{\int_0^\infty \overline{P}_{n-1}(y)dy} \\
&= \frac{\int_x^\infty (y-x)^n dP(y)}{\int_0^\infty y^n dP(y)} = \frac{\int_x^\infty (y-x)^n dP(y)}{p_n},
\end{aligned} \tag{1.55}$$

and

$$\int_0^\infty \overline{P}_n(x)dx = \frac{\int_0^\infty y^{n+1}dP(y)}{(n+1)\int_0^\infty y^n dP(y)} = \frac{p_{n+1}}{(n+1)p_n}. \tag{1.56}$$

See Hesselager, Wang, and Willmot (1998) [34] for more details.

Lin and Willmot (1999) [36] defined the distribution function $\Gamma_n(x) = 1 - \overline{\Gamma}_n(x)$ to be

$$\overline{\Gamma}_n(x) = \frac{e^{\rho x}\int_x^\infty e^{-\rho y}\overline{P}_n(y)dy}{\int_0^\infty e^{-\rho y}\overline{P}_n(y)dy}, \quad x \geq 0 \ \textit{and} \ n = 0, 1, 2, \ldots, \tag{1.57}$$

and showed that

$$\overline{\Gamma}_{n+1}(x) = \frac{\int_x^\infty \overline{\Gamma}_n(y)dy}{\int_0^\infty \overline{\Gamma}_n(y)dy} = \frac{\int_0^\infty \overline{\Gamma}_n(x+y)dy}{\int_0^\infty \overline{\Gamma}_n(y)dy}, \quad x \geq 0 \ \textit{and} \ n = 0, 1, 2, \ldots, \tag{1.58}$$

that is, the equilibrium distribution function of $\Gamma_n(x)$ is $\Gamma_{n+1}(x)$.

Since $\Gamma(x) = \Gamma_0(x) = 1 - \overline{\Gamma}_0(x)$ and $P(x) = P_0(x)$, (1.52) is (1.57) with $n = 0$. This means that $\Gamma_n(x)$ satisfying (1.57) is the n^{th} equilibrium distribution function of $\Gamma(x)$. Note that if $\delta = 0$, which implies $\rho = 0$, then $\overline{\Gamma}_n(x) = \overline{P}_{n+1}(x)$ by (1.54) and (1.57).

Chapter 2

Surplus process with a diffusion factor

In this chapter, the defective renewal equation for the expected discounted function of a penalty at the time of ruin based on the surplus process of the classical continuous time risk model containing an independent Wiener process is generalized. Then we propose the asymptotic formulas for the expected discounted penalty function. The Tijms-type approximation for and the upper and lower bounds on a compound geometric distribution function are also given if the claim size distribution function satisfies a certain condition. Besides, the reliability-based class implications for the associated claim size distribution are also given. Explicit analytical solutions to the compound geometric distribution function and to the expected discounted probability of ruin due to oscillation and a claim, respectively, can be obtained if the claim size distribution is a combination of exponentials or a mixture of Erlangs.

2.1 A generalized defective renewal equation

In this section, we are going to further generalize equation (1.8) based on the model (1.4) by involving both the random variables, $|U(T)|$, the deficit at the time of ruin, and $U(T-)$, the surplus immediately before the time of ruin, and derive the corresponding defective renewal equation.

$$\begin{aligned} \phi(u) &= w_0\, \phi_d(u) + \phi_w(u) \\ &= w_0\, E[e^{-\delta T} I(T < \infty, U(T) = 0)|U(0) = u] \\ &+ \quad E[e^{-\delta T} w(U(T-), |U(T)|) I(T < \infty, U(T) < 0)|U(0) = u]. \quad (2.1) \end{aligned}$$

where

$$\phi_w(u) = E[e^{-\delta T} w(U(T-), |U(T)|) I(T < \infty, U(T) < 0)|U(0) = u]. \quad (2.2)$$

Note that $\phi_w(0) = 0$ since $Pr(T < \infty, U(T) < 0)|U(0) = 0) = \psi_s(0) = 0$.

We first deal with a simpler case $\phi_w(u)$ where the penalty due at ruin is $w(U(T-), |U(T)|)$ if ruin is caused by a jump. To derive the defective renewal equation for (2.2), consider the infinitesimal time interval between 0 and dt. The discount factor for the interval [0 dt] is $e^{-\delta dt} \doteq 1 - \delta dt$. The process $\{S(t) : t \geq 0\}$ will either have exactly one claim with probability λdt or have no claim with probability $1 - \lambda dt$. By conditioning on this, the amount of the claim (if it occurs) and the value of $W(dt)$, we have that

$$\phi_w(u) = (1 - \lambda dt)(1 - \delta dt) E[\phi_w(u + cdt + \sigma W(dt))]$$

$$
\begin{aligned}
&+ \quad \lambda dt(1-\delta dt)\Big\{\int_0^{u+cdt+\sigma W(dt)} \phi_w(u+cdt+\sigma W(dt)-x)p(x)dx \\
&+ \quad \int_{u+cdt+\sigma W(dt)}^{\infty} w(u+cdt+\sigma W(dt), x-u-cdt-\sigma W(dt))p(x)dx\Big\}.
\end{aligned} \tag{2.3}
$$

First we expand $\phi_w(u+cdt+\sigma W(dt))$ to a Taylor's series about u to the term of $\phi''_w(u)$, and ignore the term containing $(dt)^2$ to get

$$
\begin{aligned}
& \phi_w(u+cdt+\sigma W(dt)) \\
\doteq \; & \phi_w(u)+\phi'_w(u)\Big[cdt+\sigma W(dt)\Big]+\frac{1}{2}\phi''_w(u)\Big[cdt+\sigma W(dt)\Big]^2 \\
\doteq \; & \phi_w(u)+c\phi'_w(u)dt+\sigma\phi'_w(u)W(dt)+\frac{\sigma^2}{2}\phi''_w(u)W^2(dt)+c\sigma\phi''_w(u)W(dt)dt.
\end{aligned}
$$

Then from the facts that $E[W(dt)] = 0$, $E[W^2(dt)] = Var[W(dt)] = dt$, and $D = \sigma^2/2$, we have

$$
E[\phi_w(u+cdt+\sigma W(dt))] \doteq \phi_w(u)+c\phi'_w(u)dt+D\phi''_w(u)dt. \tag{2.4}
$$

Now substitute (2.4) in (2.3) and let $dt \to 0$, we obtain

$$
D\phi''_w(u)+c\phi'_w(u)+\lambda\left[\int_0^u \phi_w(u-x)p(x)dx+\int_u^\infty w(u,x-u)p(x)dx\right] = (\lambda+\delta)\phi_w(u). \tag{2.5}
$$

Then we perform the Laplace transform on the both sides of (2.5) and get

$$
D\int_0^\infty e^{-\xi u}\phi''_w(u)du+c\int_0^\infty e^{-\xi u}\phi'_w(u)du+\lambda\tilde{\phi}_w(\xi)\tilde{p}(\xi)+\lambda\tilde{\omega}(\xi) = (\lambda+\delta)\tilde{\phi}_w(\xi). \tag{2.6}
$$

where $\tilde{\phi}_w(\xi) = \int_0^\infty e^{-\xi u}\phi_w(u)du$, $\tilde{\omega}(\xi) = \int_0^\infty e^{-\xi u}\omega(u)du$ and

$$\omega(x) = \int_x^\infty w(x, y-x)p(y)dy = \int_0^\infty w(x,y)p(x+y)dy. \tag{2.7}$$

Note that if $w(x,y) = 1$, then $\omega(x) = \overline{P}(x)$.

By intergration by parts, $\int_0^\infty e^{-\xi u}\phi_w'(u)du = e^{-\xi u}\phi_w(u)\Big|_0^\infty + \xi\tilde{\phi}_w(\xi) = \xi\tilde{\phi}_w(\xi)$ since $\phi_w(0) = 0$, and hence $\int_0^\infty e^{-\xi u}\phi_w''(u)du = e^{-\xi u}\phi_w'(u)\Big|_0^\infty + \xi\int_0^\infty e^{-\xi u}\phi_w'(u)du = -\phi_w'(0) + \xi^2\tilde{\phi}_w(\xi)$.

Then (2.6) can be simplified to

$$\Big[D\xi^2 + c\xi + \lambda\tilde{p}(\xi) - (\lambda+\delta)\Big]\tilde{\phi}_w(\xi) = D\phi_w'(0) - \lambda\tilde{\omega}(\xi). \tag{2.8}$$

Since ρ satisfies generalized Lundberg's equation (1.32), letting $\xi = \rho$ in (2.8) leads to $D\phi_w'(0) = \lambda\tilde{\omega}(\rho)$. Substitute $\lambda+\delta$ for $D\rho^2 + c\rho + \lambda\tilde{p}(\rho)$, (2.8) can be written as

$$\Big[D\xi^2 + c\xi + \lambda\tilde{p}(\xi) - D\rho^2 - c\rho - \lambda\tilde{p}(\rho)\Big]\tilde{\phi}_w(\xi) = \lambda\Big[\tilde{\omega}(\rho) - \tilde{\omega}(\xi)\Big]. \tag{2.9}$$

Since $D\xi^2 + c\xi - D\rho^2 - c\rho = D(\xi+\rho)(\xi-\rho) + c(\xi-\rho) = (D\xi + D\rho + c)(\xi-\rho) = (D\xi + Db)(\xi-\rho) = D(\xi+b)(\xi-\rho)$, Dividing (2.9) by $D\xi^2 + c\xi - D\rho^2 - c\rho$ gives

$$\left[1 - \frac{\lambda[\tilde{p}(\xi) - \tilde{p}(\rho)]}{D(b+\xi)(\rho-\xi)}\right]\tilde{\phi}_w(\xi) = \frac{\lambda[\tilde{\omega}(\xi) - \tilde{\omega}(\rho)]}{D(b+\xi)(\rho-\xi)}, \tag{2.10}$$

which is exactly the the Laplace transform of $\phi_w(u) = \int_0^u \phi_w(u-y)g(y)dy + g_\omega(u)$ (the Laplace transforms of both $g(y)$ and $g_\omega(u)$ are stated in (2.19) and (2.22), and the detailed proof can be referred to Lemma 2.1 and Corollary 2.1, respectively), where

$$g_\omega(u) = h * \gamma_\omega(u) = \int_0^u h(u-s)\gamma_\omega(s)ds, \tag{2.11}$$

$$= \frac{\lambda}{D}\int_0^u e^{-b(u-s)}\int_s^\infty e^{-\rho(x-s)}\omega(x)dxds, \tag{2.12}$$

$$= \frac{\lambda}{D}e^{\rho u}\int_0^u e^{-a(u-s)}\int_s^\infty e^{-\rho x}\omega(x)dxds \tag{2.13}$$

and

$$\gamma_\omega(s) = \frac{\lambda}{c}\int_s^\infty e^{-\rho(x-s)}\omega(x)dx. \tag{2.14}$$

The uniqueness of the Laplace transform gives

Theorem 2.1 *The function $\phi_w(u)$ in (2.2) satisfies the defective renewal equation*

$$\phi_w(u) = \int_0^u \phi_w(u-y)g(y)dy + g_\omega(u) = \phi_w * h * \gamma(u) + h * \gamma_\omega(u), \quad u \geq 0. \tag{2.15}$$

We remark that with the definitions of $\gamma(s)$ in (1.24) and $\gamma_\omega(s)$ in (2.14), equation (1.13) can be rewritten as

$$\phi_0(u) = \int_0^u \phi_0(u-y)\gamma(y)dy + \gamma_\omega(u) = \phi_0 * \gamma(u) + \gamma_\omega(u), \quad u \geq 0. \tag{2.16}$$

Similarly, combining (2.15) with (1.22), we have the following defective renewal equation (2.17) for (2.1), which is more general than (1.33) (that is, equation (26) in Theorem 3 of Gerber and Landry (1998) [24]), the defective renewal equation for (1.8).

Theorem 2.2 *The function $\phi(u)$ in (2.1) satisfies the defective renewal equation*

$$\phi(u) = \int_0^u \phi(u-y)g(y)dy + w_0\, A(u) + g_\omega(u), \quad u \geq 0, \tag{2.17}$$

with $A(u)$ given by (1.27).

Moreover, $g(y)$ and $g_\omega(u)$ can be reexpressed as follows for more and later applications, especially the limiting behavior of $\phi(u)$ in (2.17) as $D \to 0$.

Lemma 2.1 *The function $g(y)$ in (1.29) can be simplified as*

$$g(y) = \frac{\lambda}{c+2\rho D}\left[\int_0^y e^{-b(y-x)}p(x)dx + \int_y^\infty e^{-\rho(x-y)}p(x)dx - e^{-by}\int_0^\infty e^{-\rho x}p(x)dx\right], \tag{2.18}$$

and the Laplace transform of $g(y)$ is

$$\tilde{g}(\xi,\delta) = \int_0^\infty e^{-\xi y}g(y)dy = \frac{\lambda[\tilde{p}(\xi)-\tilde{p}(\rho)]}{(b+\xi)(\rho-\xi)D} = \frac{\lambda[\tilde{p}(\xi)-1]+D\rho^2+c\rho-\delta}{(b+\xi)(\rho-\xi)D}. \tag{2.19}$$

Therefore, for $y > 0$, when $D \to 0$,

$$g(y) = h * \gamma(y) \to \gamma(y) = \frac{\lambda}{c}\int_y^\infty e^{-\rho(x-y)}p(x)dx. \tag{2.20}$$

Proof: By changing the order of integration, $b = c/D+\rho$ and $a = c/D+2\rho = b+\rho$,

$$\begin{aligned} & g(y) \\ = \; & \frac{\lambda}{D}\int_0^y e^{-b(y-s)}\int_s^\infty e^{-\rho(x-s)}p(x)dxds \\ = \; & \frac{\lambda}{D}e^{-by}\left[\int_0^y e^{-\rho x}p(x)\int_0^x e^{as}dsdx + \int_y^\infty e^{-\rho x}p(x)\int_0^y e^{as}dsdx\right] \end{aligned}$$

$$
\begin{aligned}
&= \frac{\lambda}{aD}e^{-by}\left[\int_0^y e^{-\rho x}p(x)e^{ax}-1)dx+\int_y^\infty e^{-\rho x}p(x)(e^{ay}-1)dx\right]\\
&= \frac{\lambda}{aD}e^{-by}\left[\int_0^y e^{bx}p(x)dx+\int_y^\infty e^{ay}e^{-\rho x}p(x)dx-\int_0^\infty e^{-\rho x}p(x)dx\right]\\
&= \frac{\lambda}{c+2\rho D}\left[\int_0^y e^{-b(y-x)}p(x)dx+\int_y^\infty e^{-\rho(x-y)}p(x)dx-e^{-by}\int_0^\infty e^{-\rho x}p(x)dx\right]\\
&\to \frac{\lambda}{c}\int_y^\infty e^{-\rho(x-y)}p(x)dx=\gamma(y),
\end{aligned}
$$

since $D \to 0$ implies that $c + 2\rho D \to c$, $b = c/D + \rho \to \infty$, and the first and third terms in the bracket vanish.

Changing the order of integration gives

$$
\begin{aligned}
&\tilde{g}(\xi,\delta)\\
&= \int_0^\infty e^{-\xi y}g(y)dy\\
&= \frac{\lambda}{c+2\rho D}\Big[\int_0^\infty e^{-\xi y}\int_0^y e^{-b(y-x)}p(x)dxdy+\int_0^\infty e^{-\xi y}\int_y^\infty e^{-\rho(x-y)}p(x)dxdy-\\
&\qquad\Big(\int_0^\infty e^{-\xi y}e^{-by}dy\Big)\Big(\int_0^\infty e^{-\rho x}p(x)dx\Big)\Big]\\
&= \frac{\lambda}{aD}\left[\int_0^\infty e^{bx}p(x)\int_x^\infty e^{-(b+\xi)y}dydx+\int_0^\infty e^{-\rho x}p(x)\int_0^x e^{(\rho-\xi)y}dydx-\frac{\tilde{p}(\rho)}{b+\xi}\right]\\
&= \frac{\lambda}{aD}\left[\frac{1}{b+\xi}\int_0^\infty e^{bx}p(x)e^{-(b+\xi)x}dx+\frac{1}{\rho-\xi}\int_0^\infty e^{-\rho x}p(x)[e^{(\rho-\xi)x}-1]dx-\frac{\tilde{p}(\rho)}{b+\xi}\right]\\
&= \frac{\lambda}{aD}\left[\frac{1}{b+\xi}+\frac{1}{\rho-\xi}\right]\left[\tilde{p}(\xi)-\tilde{p}(\rho)\right]\\
&= \frac{\lambda[\tilde{p}(\xi)-\tilde{p}(\rho)]}{(b+\xi)(\rho-\xi)D}\\
&= \frac{\lambda[\tilde{p}(\xi)-1]+D\rho^2+c\rho-\delta}{(b+\xi)(\rho-\xi)D},
\end{aligned}
$$

since ρ satisfies (1.32). □

Since $g_\omega(u)$ (equation (2.12) or (2.13)) has the same form as $g(y)$ (equation

(1.29) or (1.30)) except that $g_\omega(u)$ and $g(y)$ have the different innermost integrands $\omega(x)$ and $p(x)$, respectively, we have

Corollary 2.1 *The function $g_\omega(u)$ in (2.12) can be simplified as*

$$g_\omega(u) = \frac{\lambda}{c+2\rho D}\left[\int_0^u e^{-b(u-x)}\omega(x)dx + \int_u^\infty e^{-\rho(x-u)}\omega(x)dx - e^{-bu}\int_0^\infty e^{-\rho x}\omega(x)dx\right] \tag{2.21}$$

and the Laplace transform of $g_\omega(u)$ is

$$\tilde{g}_\omega(\xi,\delta) = \int_0^\infty e^{-\xi u}g_\omega(u)du = \frac{\lambda[\tilde{\omega}(\xi)-\tilde{\omega}(\rho)]}{(b+\xi)(\rho-\xi)D}, \tag{2.22}$$

where $\tilde{\omega}(s) = \int_0^\infty e^{-sx}\omega(x)dx$ with $\omega(x)$ given in (2.7).
Therefore, for $u > 0$, when $D \to 0$,

$$g_\omega(u) = h * \gamma_\omega(u) \to \gamma_\omega(u) = \frac{\lambda}{c}\int_u^\infty e^{-\rho(x-u)}\omega(x)dx \tag{2.23}$$
$$= \frac{\lambda}{c}e^{\rho u}\int_u^\infty e^{-\rho x}\int_x^\infty w(x, y-x)p(y)dydx. \tag{2.24}$$

Intuitively, one can conceive that when $D \to 0$, the Wiener process $\{\sigma W(t) : t \geq 0\} \to$ a random variable degenerating at 0 since $\sigma W(t) \sim N(0, \sigma^2 t) = N(0, 2Dt)$. Therefore, as the surplus process (1.4) containing an independent diffusion (or Wiener) process approaches the classical surplus process (1.1), so does the corresponding defective renewal equation. That is, for the function $\phi_w(u)$ defined in (2.2), the defective renewal equation (2.15) based on the model (1.4) reduces to the one (2.16) for $\phi_0(u)$ defined in (1.2) based on the model (1.1) as $D \to 0$, which can be proved by (2.20) and (2.24).

Moreover, from (1.22) when $D \to 0$, $\phi_d(u) = \phi_d * (h*\gamma)(u) + A(u) \to \phi_d * \gamma(u)$. If we perform the Laplace transform on the both sides, then when $D \to 0$, $\tilde{\phi}_d(s) \to \tilde{\phi}_d(s)\tilde{\gamma}(s)$, i.e. $\tilde{\phi}_d(s)[1-\tilde{\gamma}(s)] \to 0$ for all $s > 0$. Since $\tilde{\gamma}(s) \neq 1$ for some s, we have when $D \to 0$, $\tilde{\phi}_d(s) \to 0$ for all $s > 0$, which implies $\phi_d(u) \to 0$. Therefore, we obtain the following theorem.

Theorem 2.3 *For $u > 0$, if $D \to 0$, then*

$$\begin{aligned}
\sigma W(t) = N(0, 2Dt) \quad &\to \quad \text{a r. v. degenerating at } 0;\\
U(t) = u + c\,t - S(t) + \sigma W(t) \quad &\to \quad U(t) = u + c\,t - S(t);\\
\lambda \tilde{p}(\xi) = \lambda + \delta - c\xi - D\xi^2 \quad &\to \quad \lambda \tilde{p}(\xi) = \lambda + \delta - c\xi, \text{ Lundberg's equation};\\
\phi_w(u) = \phi_w * (h*\gamma)(u) + h * \gamma_\omega(u) \quad &\to \quad \phi_0(u) = \phi_0 * \gamma(u) + \gamma_\omega(u);\\
\phi_d(u) = \phi_d * (h*\gamma)(u) + A(u) \quad &\to \quad 0; \text{ and}\\
\phi(u) = w_0\, \phi_d(u) + \phi_w(u) \quad &\to \quad \phi_0(u), \text{ independent of } w_0.
\end{aligned}$$

To connect (2.17) with (1.33), we first reexpress $g_\omega(u)$ in another form as follows:

Lemma 2.2

$$g_\omega(u) = \int_u^\infty g(x-u, x)dx - A(u)\int_0^\infty g(x,x)dx, \tag{2.25}$$

where

$$g(x,y) = \int_0^y h(y-s)\gamma(x,s)ds, \tag{2.26}$$

$$\gamma(y,s) = \frac{\lambda}{c}\int_s^\infty e^{-\rho(x-s)}w(x-y,y)p(x)dx, \tag{2.27}$$

with $h(s)$ and $A(u)$ given in (1.23) and (1.27), respectively.

Proof: By (2.7) and (2.12), changing the order of integration and replacing the variable z by $s = z + t$ yield,

$$\begin{aligned}
g_\omega(u) &= \frac{\lambda}{D}\int_0^u e^{-b(u-z)}\int_z^\infty e^{-\rho(x-z)}\int_0^\infty w(x,t)p(x+t)dtdxdz \\
&= \frac{\lambda}{D}\int_0^\infty\int_0^u e^{-b(u-z)}\int_z^\infty e^{-\rho(x-z)}w(x,t)p(x+t)dxdzdt \\
&= \frac{\lambda}{D}\int_0^\infty\int_0^u e^{-b(u-z)}\int_{t+z}^\infty e^{-\rho(x-t-z)}w(x-t,t)p(x)dxdzdt \\
&= \frac{\lambda}{D}\int_0^\infty\int_t^{t+u} e^{-b(t+u-s)}\int_s^\infty e^{-\rho(x-s)}w(x-t,t)p(x)dxdsdt \\
&= \frac{\lambda}{D}\int_0^\infty\int_0^{t+u} e^{-b(t+u-s)}\int_s^\infty e^{-\rho(x-s)}w(x-t,t)p(x)dxdsdt - \\
&\quad \frac{\lambda}{D}\int_0^\infty\int_0^{t} e^{-b(t+u-s)}\int_s^\infty e^{-\rho(x-s)}w(x-t,t)p(x)dxdsdt \\
&= \frac{\lambda}{D}\int_u^\infty\int_0^{t} e^{-b(t-s)}\int_s^\infty e^{-\rho(x-s)}w(x-t+u,t-u)p(x)dxdsdt - \\
&\quad \frac{\lambda}{D}e^{-bu}\int_0^\infty\int_0^{t} e^{-b(t-s)}\int_s^\infty e^{-\rho(x-s)}w(x-t,t)p(x)dxdsdt \\
&= \int_u^\infty\int_0^t h(t-s)\gamma(t-u,s)dsdt - e^{-bu}\int_0^\infty\int_0^t h(t-s)\gamma(t,s)dsdt \\
&= \int_u^\infty g(t-u,t)dt - A(u)\int_0^\infty g(t,t)dt,
\end{aligned}$$

which is (2.25). □

When $w(x,y) = w(y)$, $w(x,y)$ can be extracted from the integrand of (2.27), and hence (2.25), (2.26) and (2.27) can be simplied as follows:

Corollary 2.2 *If $w(x,y) = w(y)$, then*

$$\gamma(y,s) = w(y)\gamma(s), \tag{2.28}$$

$$g(x,y) = w(x)g(y), \tag{2.29}$$

$$g_\omega(u) \quad = \quad \int_u^\infty w(x-u)g(x)dx - A(u)\int_0^\infty w(x)g(x)dx, \tag{2.30}$$

with $\gamma(s)$ given in (1.24).

Proof: If $w(x,y)=w(y)$, (1.24) and (2.27) give $\gamma(y,s)=w(y)\frac{\lambda}{c}\int_s^\infty e^{-\rho(x-s)}p(x)dx$ $= w(y)\gamma(s)$; by (1.28) and (2.26), $g(x,y) = w(x)\int_0^y h(y-s)\gamma(s)ds = w(x)g(y)$; therefore from (2.25), $g_\omega(u) = \int_u^\infty w(x-u)g(x)dx - A(y)\int_0^\infty w(x)g(x)dx$. □

When $w(x,y) = w(-y)$, (2.30) turns out to be $g_\omega(u) = \int_u^\infty w(u-y)g(y)dy -$ $A(u)\int_0^\infty w(-y)g(y)dy$, which are exactly the third and fourth terms of the right side of (1.33). Therefore, the defective renewal equation (1.33) is a special case of (2.17).

2.2 The associated claim size distribution

In this section, we will demonstrate where each of (1.22), (2.15) and (2.17) is of the form (1.36), and we shall determine β, $G(x)$, and $B(u)$ for each of them.

Consider the general case

$$\phi(u) = \left[\int_0^\infty g(y)dy\right]\int_0^u \phi(u-x)\frac{g(x)}{\int_0^\infty g(y)dy}dx + V(u). \tag{2.31}$$

Equating (1.36) and (2.31) leads to $\int_0^\infty g(y)dy = \frac{1}{1+\beta}$, $G'(x) = \frac{g(x)}{\int_0^\infty g(y)dy}$, and $B(u) = \frac{V(u)}{\int_0^\infty g(y)dy}$. From (1.31), (1.32) and that

$$\int_0^\infty e^{-\rho y}\overline{P}(y)dy = \frac{1}{\rho}\left[1 - \int_0^\infty e^{-\rho y}p(y)dy\right] = \frac{1-\tilde{p}(\rho)}{\rho}, \tag{2.32}$$

we get

$$\begin{aligned} \frac{1}{1+\beta} &= \int_0^\infty g(y)dy \\ &= \frac{D\rho^2 + c\rho - \delta}{D\rho^2 + c\rho} = \frac{\lambda[1-\tilde{p}(\rho)]}{(c+D\rho)\rho} \qquad (2.33) \\ &= \frac{\lambda}{c+\rho D}\int_0^\infty e^{-\rho y}\overline{P}(y)dy = \frac{\lambda}{bD}\int_0^\infty e^{-\rho y}\overline{P}(y)dy, \qquad (2.34) \end{aligned}$$

or equivalently,

$$\beta = \frac{\delta}{D\rho^2 + c\rho - \delta} = \frac{c+\rho D}{\lambda \int_0^\infty e^{-\rho y}\overline{P}(y)dy} - 1 = \frac{bD}{\lambda \int_0^\infty e^{-\rho y}\overline{P}(y)dy} - 1. \tag{2.35}$$

Clearly, $\beta > 0$ since $\frac{1}{1+\beta} > 0$ from (2.34). Also, when $\rho = 0$, from (2.35) we have $\beta = \frac{c}{\lambda p_1} - 1 = \theta$. Moreover, we denote β_0 for the case $D = 0$, that is,

$$\beta_0 = \frac{\delta}{c\rho - \delta} = \frac{c}{\lambda \int_0^\infty e^{-\rho y}\overline{P}(y)dy} - 1, \tag{2.36}$$

or equivalently,

$$\frac{1}{1+\beta_0} = \frac{c\rho - \delta}{c\rho} = \frac{\lambda[1-\tilde{p}(\rho)]}{c\rho} = \frac{\lambda}{c}\int_0^\infty e^{-\rho y}\overline{P}(y)dy. \tag{2.37}$$

From (2.34) and (2.37), we get

$$1+\beta = \frac{bD}{c}(1+\beta_0) = \frac{c+\rho D}{c}(1+\beta_0). \tag{2.38}$$

Since $G'(x) = \dfrac{g(x)}{\int_0^\infty g(y)dy} = \dfrac{h * \gamma(x)}{\int_0^\infty g(y)dy}$, from (1.29) and (2.34) we see that

$$G'(x) = \frac{\int_0^x e^{-b(x-s)} \int_s^\infty e^{-\rho(y-s)} dP(y) ds}{\frac{1}{b}\int_0^\infty e^{-\rho y}\overline{P}(y)dy}. \tag{2.39}$$

Therefore, from (1.28) and (1.49) we have

$$\begin{aligned} G(x) &= \frac{\int_0^x g(y)dy}{\int_0^\infty g(y)dy} = \frac{\int_0^x \int_0^y h(y-s)\gamma(s)dsdy}{\int_0^\infty \int_0^y h(y-s)\gamma(s)dsdy} \\ &= \frac{\int_0^x \gamma(y)\int_0^{x-y} h(s)dsdy}{\int_0^\infty \gamma(y)dy \int_0^\infty h(y)dy} = \frac{\int_0^x \gamma(y)H(x-y)dy}{\int_0^\infty \gamma(y)dy} \\ &= \int_0^x H(x-y)d\Gamma(y) = H * \Gamma(x), \end{aligned} \tag{2.40}$$

where $G(x)$ is called the associated "claim size" distribution, and $H(x) = 1 - \overline{H}(x)$ is a distribution function with

$$\overline{H}(x) = \frac{\int_x^\infty h(s)ds}{\int_0^\infty h(s)ds} = e^{-bx} = \frac{H'(x)}{b} \tag{2.41}$$

($H'(x) = be^{-bx}$ is a probability density function). Hence,

$$G'(x) = \frac{g(x)}{\int_0^\infty g(y)dy} = H' * \Gamma(x) = b\overline{H} * \Gamma(x). \tag{2.42}$$

Note that when $D \to 0$, $g(y) \to \gamma(y)$ by (2.20), and hence

$$G(x) = \frac{\int_0^x g(y)dy}{\int_0^\infty g(y)dy} \to \frac{\int_0^x \gamma(y)dy}{\int_0^\infty \gamma(y)dy} = \Gamma(x), \quad x \geq 0. \tag{2.43}$$

From (1.52), (2.37) and (2.38), we also have

$$\begin{aligned} \rho\overline{\Gamma}(x) + \Gamma'(x) &= -e^{\rho x}[e^{-\rho x}\overline{\Gamma}(x)]' = \frac{\overline{P}(x)}{\int_0^\infty e^{-\rho y}\overline{P}(y)dy} \\ &= \frac{\lambda}{c}(1+\beta_0)\overline{P}(x) = \frac{\lambda}{bD}(1+\beta)\overline{P}(x). \end{aligned} \tag{2.44}$$

The Laplace transform of $\Gamma'(x)$ is easily obtained. From (1.51), (1.53) and integration by parts, we have

$$\int_0^\infty e^{-sx}d\Gamma(x) = \frac{\int_0^\infty e^{-(s-\rho)x}\int_x^\infty e^{-\rho y}dP(y)dx}{\int_0^\infty e^{-\rho y}\overline{P}(y)dy} = \frac{\rho}{\rho - s}\left[\frac{\tilde{p}(s) - \tilde{p}(\rho)}{1 - \tilde{p}(\rho)}\right]. \tag{2.45}$$

The Laplace transform of $G'(x)$ can be got from (2.19), (2.33) and (2.45) as follows:

$$\begin{aligned} \int_0^\infty e^{-sx}dG(x) &= \frac{\int_0^\infty e^{-sx}g(x)dx}{\int_0^\infty g(x)dx} \\ &= \frac{\tilde{p}(s) - \tilde{p}(\rho)}{(b+s)(\rho - s)}\left[\frac{b\rho}{1 - \tilde{p}(\rho)}\right] \end{aligned} \tag{2.46}$$

$$= \frac{b}{b+s}\int_0^\infty e^{-sx}d\Gamma(x). \tag{2.47}$$

Since $G(x) = H * \Gamma(x) = (1-\overline{H}) * \Gamma(x) = \Gamma(x) - \overline{H} * \Gamma(x)$, from (2.41), we get

$$\begin{aligned}\overline{G}(x) &= 1 - [\Gamma(x) - \overline{H} * \Gamma(x)] = \overline{\Gamma}(x) + \overline{H} * \Gamma(x) \\ &= \overline{\Gamma}(x) + \frac{H' * \Gamma(x)}{b} = \overline{\Gamma}(x) + \frac{G'(x)}{b}. \end{aligned} \tag{2.48}$$

Similarly, by (2.40) and the commutative property of convolution, $G(x) = \Gamma * H(x) = (1-\overline{\Gamma}) * H(x) = H(x) - \overline{\Gamma} * H(x)$ and we get the alternative form

$$\begin{aligned}\overline{G}(x) &= 1 - (H(x) - \overline{\Gamma} * H(x)) = \overline{H}(x) + \overline{\Gamma} * H(x) \\ &= e^{-bx} + b\int_0^x e^{-b(x-y)}\overline{\Gamma}(y)dy \\ &= e^{-bx} + \frac{\int_0^x e^{-b(x-y)}\int_y^\infty e^{-\rho(s-y)}\overline{P}(s)dsdy}{\frac{1}{b}\int_0^\infty e^{-\rho y}\overline{P}(y)dy}. \end{aligned} \tag{2.49}$$

Example 2.1 Combination of exponentials

Suppose $P'(x) = \sum_{k=1}^{r} q_k\mu_k e^{-\mu_k x}, x \geq 0$, where $q_1 + q_2 + \ldots + q_r = 1$. If $q_k > 0$ for all k then $P'(x)$ is a mixture of exponentials whereas if $q_k < 0$ for some k then $P'(x)$ is called a combination of exponentials (see page 79 of Everitt and Hand (1981) [17] for the respective sufficient and necessary conditions for $P'(x)$ to be a probability density function). Then we have $\overline{P}(x) = \sum_{k=1}^{r} q_k e^{-\mu_k x}, x \geq 0$, and

$$\int_x^\infty e^{-\rho y}\overline{P}(y)dy = \sum_{k=1}^{r} q_k \int_x^\infty e^{-(\rho+\mu_k)y}dy = \sum_{k=1}^{r}\frac{q_k}{\rho+\mu_k}e^{-(\rho+\mu_k)x}.$$

From (1.52),

$$\overline{\Gamma}(x) = \sum_{k=1}^{r} q_k^* e^{-\mu_k x}, \quad x \geq 0, \tag{2.50}$$

where

$$q_k^* = \frac{\frac{q_k}{\rho + \mu_k}}{\sum_{j=1}^{r} \frac{q_j}{\rho + \mu_j}}, \quad k = 1, 2, \ldots, r, \tag{2.51}$$

with

$$q_{k,\delta=0}^* = \frac{\frac{q_k}{\mu_k}}{\sum_{j=1}^{r} \frac{q_j}{\mu_j}}, \quad k = 1, 2, \ldots, r. \tag{2.52}$$

Hence, $\overline{\Gamma}(x)$ is also the tail of a mixture or combination of exponentials with new weights q_k^*s. Moreover, $\overline{G}(x) = \overline{H}(x) + \overline{\Gamma} * H(x)$, as shown below (assume $b \neq \mu_k, k = 1, 2, \ldots, r$),

$$\begin{aligned}
\overline{G}(x) &= e^{-bx} + b\sum_{k=1}^{r} q_k^* \int_0^x e^{-\mu_k(x-y)} e^{-by} dy \\
&= e^{-bx} + b\sum_{k=1}^{r} q_k^* e^{-\mu_k x} \frac{e^{(\mu_k - b)x} - 1}{\mu_k - b} \\
&= \sum_{k=1}^{r} q_k^* e^{-bx} + b\sum_{k=1}^{r} \frac{q_k^*}{\mu_k - b}\left[e^{-bx} - e^{-\mu_k x}\right] \\
&= \sum_{k=1}^{r} \frac{q_k^* \mu_k}{\mu_k - b} e^{-bx} + b\sum_{k=1}^{r} \frac{q_k^*}{b - \mu_k} e^{-\mu_k x} \\
&= \sum_{k=1}^{r} \frac{q_k^* \mu_k}{\mu_k - b} e^{-bx} + bq^* \sum_{k=1}^{r} q_k^{**} e^{-\mu_k x} &(2.53) \\
&= (1 - bq^*)e^{-bx} + bq^* \sum_{k=1}^{r} q_k^{**} e^{-\mu_k x} &(2.54)
\end{aligned}$$

is again the tail of a mixture or combination of exponentials, where

$$q^* = \sum_{j=1}^{r} \frac{q_j^*}{b - \mu_j} \tag{2.55}$$

with

$$q^*_{\delta=0} = \sum_{j=1}^{r} \frac{q^*_{j,\delta=0}}{c/D - \mu_j}, \tag{2.56}$$

and

$$q_k^{**} = \frac{\frac{q_k^*}{b-\mu_k}}{q^*} = \frac{\frac{q_k^*}{b-\mu_k}}{\sum_{j=1}^{r} \frac{q_j^*}{b-\mu_j}}, \quad k = 1, 2, \ldots, r, \tag{2.57}$$

with

$$q^{**}_{k,\delta=0} = \frac{\frac{q^*_{k,\delta=0}}{c/D-\mu_k}}{q^*_{\delta=0}} = \frac{\frac{q^*_{k,\delta=0}}{c/D-\mu_k}}{\sum_{j=1}^{r} \frac{q^*_{j,\delta=0}}{c/D-\mu_j}}, \quad k = 1, 2, \ldots, r. \tag{2.58}$$

Note that when $r = 1$, the distribution function is exponential with parameter μ, and we obtain $\overline{\Gamma}(x) = \overline{P}(x)$, and if $b \neq \mu$, $\overline{G}(x) = \dfrac{\mu e^{-bx} - be^{-\mu x}}{\mu - b}$, a combination of two exponentials. □

Example 2.2 Mixture of Erlangs

Suppose $P'(x) = \sum_{k=1}^{r} q_k \dfrac{\mu(\mu x)^{k-1} e^{-\mu x}}{(k-1)!}, x \geq 0$, where $\{q_1, q_2, \ldots, q_r\}$ is a probability distribution. Then

$$\begin{aligned}
\int_x^\infty e^{-\rho y} dP(y) &= \sum_{j=1}^{r} q_j \int_x^\infty e^{-\rho y} \frac{\mu(\mu y)^{j-1} e^{-\mu y}}{(j-1)!} dy \\
&= \sum_{j=1}^{r} q_j \left(\frac{\mu}{\mu+\rho}\right)^j \int_x^\infty \frac{(\mu+\rho)^j y^{j-1} e^{-(\mu+\rho)y}}{(j-1)!} dy
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{r} q_j \Big(\frac{\mu}{\mu+\rho}\Big)^j \sum_{k=1}^{j} \frac{[(\mu+\rho)x]^{k-1} e^{-(\mu+\rho)x}}{(k-1)!} \\
&= \frac{e^{-\rho x}}{\mu+\rho} \sum_{k=1}^{r} \Bigg[\sum_{j=k}^{r} q_j \Big(\frac{\mu}{\mu+\rho}\Big)^{j-k}\Bigg] \frac{\mu(\mu x)^{k-1} e^{-\mu x}}{(k-1)!}.
\end{aligned}
$$

When $x = 0$, we get (if $\rho > 0$) that

$$
\begin{aligned}
\int_0^\infty e^{-\rho y}\overline{P}(y)dy &= \frac{1}{\rho}\Big[1 - \int_0^\infty e^{-\rho y} dP(y)\Big] \\
&= \frac{1}{\rho}\Big[1 - \sum_{j=1}^{r} q_j \Big(\frac{\mu}{\mu+\rho}\Big)^j\Big] \\
&= \frac{1}{\mu+\rho} \sum_{j=1}^{r} q_j \frac{1-(\frac{\mu}{\mu+\rho})^j}{1-\frac{\mu}{\mu+\rho}} \\
&= \frac{1}{\mu+\rho} \sum_{j=1}^{r} q_j \sum_{i=0}^{j-1} \Big(\frac{\mu}{\mu+\rho}\Big)^i.
\end{aligned}
$$

From (1.53), we obtain

$$
\Gamma'(x) = \sum_{k=1}^{r} q_k^* \frac{\mu(\mu x)^{k-1} e^{-\mu x}}{(k-1)!}, \quad x \geq 0, \tag{2.59}
$$

where

$$
q_k^* = \frac{\sum_{j=k}^{r} q_j \Big(\frac{\mu}{\mu+\rho}\Big)^{j-k}}{\sum_{j=1}^{r} q_j \sum_{i=0}^{j-1} \Big(\frac{\mu}{\mu+\rho}\Big)^i}, \quad k = 1, 2, \ldots, r, \tag{2.60}
$$

with

$$
q_{k,\delta=0}^* = \frac{\sum_{j=k}^{r} q_j}{\sum_{j=1}^{r} j q_j}, \quad k = 1, 2, \ldots, r. \tag{2.61}
$$

Consequently, $\Gamma'(x)$ is also the probability density function of a mixture of Erlangs with new weights q_k^*s. However, $G'(x) = \int_0^x h(x-y)\Gamma'(y)dy$, as shown below with the help of the identity $\int_0^x y^n e^{-ty}dy = \frac{n!}{t^{n+1}}\Big[1 - e^{-tx}\sum_{j=1}^{n+1}\frac{(tx)^{j-1}}{(j-1)!}\Big]$, when $b \neq \mu$,

$$
\begin{aligned}
G'(x) &= \int_0^x be^{-b(x-y)}\sum_{k=1}^{r} q_k^* \frac{\mu(\mu y)^{k-1}e^{-\mu y}}{(k-1)!}dy \\
&= be^{-bx}\sum_{k=1}^{r} q_k^* \frac{\mu^k}{(k-1)!}\int_0^x y^{k-1}e^{-(\mu-b)y}dy \\
&= be^{-bx}\sum_{k=1}^{r} q_k^* \frac{\mu^k}{(k-1)!}\frac{(k-1)!}{(\mu-b)^k}\Big[1 - e^{-(\mu-b)x}\sum_{j=1}^{k}\frac{[(\mu-b)x]^{j-1}}{(j-1)!}\Big] \\
&= b\left[\sum_{k=1}^{r} q_k^*\Big(\frac{\mu}{\mu-b}\Big)^k\right]e^{-bx} - b\sum_{j=1}^{r}\left[\sum_{k=j}^{r} q_k^*\Big(\frac{\mu}{\mu-b}\Big)^k\right]\frac{[(\mu-b)x]^{j-1}}{(j-1)!}e^{-\mu x} \\
&= b\left[\sum_{k=1}^{r} q_k^*\Big(\frac{\mu}{\mu-b}\Big)^k\right]e^{-bx} - \frac{b}{\mu-b}\sum_{j=1}^{r}\left[\sum_{k=j}^{r} q_k^*\Big(\frac{\mu}{\mu-b}\Big)^{k-j}\right]\frac{\mu(\mu x)^{j-1}}{(j-1)!}e^{-\mu x} \\
&= b\left[\sum_{k=1}^{r} q_k^*\Big(\frac{\mu}{\mu-b}\Big)^k\right]e^{-bx} - \frac{b}{\mu-b}\sum_{k=1}^{r}\left[\sum_{j=k}^{r} q_j^*\Big(\frac{\mu}{\mu-b}\Big)^{j-k}\right]\frac{\mu(\mu x)^{k-1}}{(k-1)!}e^{-\mu x} \\
&= \left[\sum_{k=1}^{r} q_k^*\Big(\frac{\mu}{\mu-b}\Big)^k\right]be^{-bx} + \frac{bq^*}{b-\mu}\sum_{k=1}^{r} q_k^{**}\frac{\mu(\mu x)^{k-1}}{(k-1)!}e^{-\mu x} \qquad (2.62) \\
&= \Big[1 - \frac{bq^*}{b-\mu}\Big]be^{-bx} + \frac{bq^*}{b-\mu}\sum_{k=1}^{r} q_k^{**}\frac{\mu(\mu x)^{k-1}}{(k-1)!}e^{-\mu x} \qquad (2.63)
\end{aligned}
$$

is the probability density function (the function form is $\sum_{i=1}^{n}\sum_{k=1}^{r} q_{ik}\frac{\mu_i(\mu_i x)^{k-1}}{(k-1)!}e^{-\mu_i x}$ with $q_{ik} \geq 0$ and $\sum_{i=1}^{n}\sum_{k=1}^{r} q_{ik} = 1$) of a general Erlangs mixture if $\frac{bq^*}{b-\mu} \leq 1$ and $\frac{bq^*}{b-\mu}q_k^{**} \geq 0$, $k = 1, 2, \ldots, r$, where

$$q^* = \sum_{i=1}^{r}\sum_{j=i}^{r} q_j^*\Big(\frac{\mu}{\mu-b}\Big)^{j-i} = \sum_{j=1}^{r} q_j^*\sum_{i=1}^{j}\Big(\frac{\mu}{\mu-b}\Big)^{j-i} = \sum_{j=1}^{r} q_j^*\sum_{i=0}^{j-1}\Big(\frac{\mu}{\mu-b}\Big)^{i} \qquad (2.64)$$

with

$$q^*_{\delta=0} = \sum_{i=1}^{r}\sum_{j=i}^{r} q^*_{j,\delta=0}\left(\frac{\mu}{\mu - c/D}\right)^{j-i} = \sum_{j=1}^{r} q^*_{j,\delta=0}\sum_{i=0}^{j-1}\left(\frac{\mu}{\mu - c/D}\right)^{i}, \tag{2.65}$$

and

$$q^{**}_k = \frac{\sum_{j=k}^{r} q^*_j\left(\frac{\mu}{\mu - b}\right)^{j-k}}{q^*} = \frac{\sum_{j=k}^{r} q^*_j\left(\frac{\mu}{\mu - b}\right)^{j-k}}{\sum_{j=1}^{r} q^*_j\sum_{i=0}^{j-1}\left(\frac{\mu}{\mu - b}\right)^{i}}, \quad k = 1, 2, \ldots, r, \tag{2.66}$$

with

$$q^{**}_{k,\delta=0} = \frac{\sum_{j=k}^{r} q^*_{j,\delta=0}\left(\frac{\mu}{\mu - c/D}\right)^{j-k}}{q^*_{\delta=0}} = \frac{\sum_{j=k}^{r} q^*_{j,\delta=0}\left(\frac{\mu}{\mu - c/D}\right)^{j-k}}{\sum_{j=1}^{r} q^*_{j,\delta=0}\sum_{i=0}^{j-1}\left(\frac{\mu}{\mu - c/D}\right)^{i}}, \quad k = 1, 2, \ldots, r; \tag{2.67}$$

whereas when $b = \mu$

$$\begin{aligned} G'(x) &= \int_0^x \mu e^{-\mu(x-y)} \sum_{k=1}^{r} q^*_k \frac{\mu(\mu y)^{k-1} e^{-\mu y}}{(k-1)!} dy \\ &= \mu e^{-\mu x} \sum_{k=1}^{r} q^*_k \int_0^x \frac{\mu(\mu y)^{k-1}}{(k-1)!} dy \\ &= \sum_{k=1}^{r} q^*_k \frac{\mu(\mu x)^k}{k!} e^{-\mu x} \end{aligned} \tag{2.68}$$

is the probability density function of a mixture of Erlangs.

Note that when $r = 1$, the distribution function is exponential with parameter μ, and we obtain $\Gamma'(x) = P'(x)$ and $G'(x) = \frac{\mu}{\mu - b} b e^{-bx} - \frac{b}{\mu - b} \mu e^{-\mu x}$, a combination of two exponentials when $b \neq \mu$, whereas $G'(x) = \mu(\mu x)e^{-\mu x}$ when $b = \mu$. □

Tijms (1994) [44] (pp. 163-64) has shown that the probability distribution function of any positive random variable can be arbitrarily closely approximated by a mixture of Erlangian distributions with the same scale parameters, thus justifying its importance.

Lemma 2.3 *For n=0, 1, 2, ..., if $\rho > 0$ then the moments of $G(x)$ are given by*

$$\begin{aligned}\mu_{G,n}(\rho) &= \int_0^\infty x^n dG(x) \\ &= \frac{b}{b+\rho}\left\{\frac{n!}{(-\rho)^n}\left[1 + \frac{\sum_{j=1}^{n}\frac{(-\rho)^j}{j!}p_j}{1-\tilde{p}(\rho)}\right]\right\} + \frac{\rho}{b+\rho}\left\{\frac{n!}{b^n}\left[1 + \frac{\sum_{j=1}^{n}\frac{b^j}{j!}p_j}{1-\tilde{p}(\rho)}\right]\right\}.\end{aligned} \tag{2.69}$$

Proof: Lin and Willmot (1999) [36] showed that if $\rho > 0$ then the moments of $\Gamma(x)$ are given by

$$\mu_{\Gamma,n}(\rho) = \int_0^\infty x^n d\Gamma(x) = \frac{n!}{(-\rho)^n}\left\{1 + \frac{\sum_{j=1}^{n}\frac{(-\rho)^j}{j!}p_j}{1-\tilde{p}(\rho)}\right\}, \quad n = 0, 1, 2, \ldots. \tag{2.70}$$

And note the identity

$$\int_y^\infty x^n e^{-sx} dx = \frac{n!}{s^{n+1}} e^{-sy} \sum_{j=0}^{n} \frac{(sy)^j}{j!}, \tag{2.71}$$

or equivalently,

$$\int_0^y x^n e^{-sx} dx = \frac{n!}{s^{n+1}}\left[1 - e^{-sy} \sum_{j=0}^{n} \frac{(sy)^j}{j!}\right]. \tag{2.72}$$

Now from (2.18), (2.42), (1.51) and (1.53), and reverse the order of integration, we have

$$
\begin{aligned}
&\int_0^\infty x^n dG(x) \\
= &\frac{\int_0^\infty x^n g(x)dx}{\int_0^\infty g(x)dx} \\
= &\frac{\frac{1}{a}\int_0^\infty x^n\left\{e^{-bx}\int_0^x e^{by}p(y)dy + e^{\rho x}\int_x^\infty e^{-\rho y}p(y)dy - e^{-bx}\int_0^\infty e^{-\rho y}p(y)dy\right\}dx}{\frac{1}{b}\int_0^\infty e^{-\rho y}\overline{P}(y)dy} \\
= &\frac{b}{a}\int_0^\infty x^n\left\{\Gamma'(x) + \frac{e^{-bx}\int_0^x e^{by}p(y)dy - e^{-bx}\int_0^\infty e^{-\rho y}p(y)dy}{\int_0^\infty e^{-\rho y}\overline{P}(y)dy}\right\}dx \\
= &\frac{b}{a}\left\{\mu_{\Gamma,n}(\rho) + \frac{\int_0^\infty e^{by}p(y)\int_y^\infty x^n e^{-bx}dxdy - \left[\int_0^\infty x^n e^{-bx}dx\right]\tilde{p}(\rho)}{\frac{1}{\rho}[1-\tilde{p}(\rho)]}\right\} \\
= &\frac{b}{a}\left\{\mu_{\Gamma,n}(\rho) + \frac{\rho\left[\frac{n!}{b^{n+1}}\sum_{j=0}^{n}\int_0^\infty e^{by}p(y)e^{-by}\frac{(by)^j}{j!}dy - \frac{n!}{b^{n+1}}\tilde{p}(\rho)\right]}{1-\tilde{p}(\rho)}\right\} \\
= &\frac{b}{a}\mu_{\Gamma,n}(\rho) + \frac{\rho}{a}\frac{n!}{b^n}\left\{\frac{\sum_{j=0}^{n}\frac{b^j}{j!}\int_0^\infty y^j p(y)dy - \tilde{p}(\rho)}{1-\tilde{p}(\rho)}\right\} \\
= &\frac{b}{a}\mu_{\Gamma,n}(\rho) + \frac{\rho}{a}\frac{n!}{b^n}\left\{1 + \frac{\sum_{j=1}^{n}\frac{b^j}{j!}p_j}{1-\tilde{p}(\rho)}\right\} \qquad (2.73) \\
= &\frac{b}{b+\rho}\left\{\frac{n!}{(-\rho)^n}\left[1 + \frac{\sum_{j=1}^{n}\frac{(-\rho)^j}{j!}p_j}{1-\tilde{p}(\rho)}\right]\right\} + \frac{\rho}{b+\rho}\left\{\frac{n!}{b^n}\left[1 + \frac{\sum_{j=1}^{n}\frac{b^j}{j!}p_j}{1-\tilde{p}(\rho)}\right]\right\}.
\end{aligned}
$$

□

Note that when $D \to 0$, then $b/a \to 1$ and the second term of the right side of (2.73) vanishes; hence,

$$\mu_{G,n}(\rho) \to \mu_{\Gamma,n}(\rho), \quad n = 0, 1, 2, \ldots. \tag{2.74}$$

Corollary 2.3 *If $\rho > 0$ then the mean of $G(x)$ is*

$$\mu_{G,1}(\rho) = \frac{p_1}{1-\tilde{p}(\rho)} - \frac{b-\rho}{b\rho} = \lambda p_1\left[\frac{1}{D\rho^2 + c\rho - \delta} - \frac{1+\theta}{D\rho^2 + c\rho}\right] = \frac{\lambda p_1}{D\rho^2 + c\rho}[\beta - \theta]. \tag{2.75}$$

Proof: For $n = 1$, from (1.32), (2.33), (2.69) and $c = \lambda p_1(1+\theta)$ we have

$$\begin{aligned}
\mu_{G,1}(\rho) &= -\frac{b}{\rho(b+\rho)}\left[1 - \frac{\rho p_1}{1-\tilde{p}(\rho)}\right] + \frac{\rho}{b(b+\rho)}\left[1 + \frac{b p_1}{1-\tilde{p}(\rho)}\right] \\
&= \frac{p_1}{1-\tilde{p}(\rho)} - \frac{b-\rho}{b\rho} \\
&= \frac{\lambda p_1}{\lambda[1-\tilde{p}(\rho)]} - \frac{c}{b\rho D} \\
&= \frac{\lambda p_1}{D\rho^2 + c\rho - \delta} - \frac{\lambda p_1(1+\theta)}{D\rho^2 + c\rho} \\
&= \lambda p_1\left[\frac{1}{D\rho^2 + c\rho - \delta} - \frac{1+\theta}{D\rho^2 + c\rho}\right] \\
&= \frac{\lambda p_1}{D\rho^2 + c\rho}\left[\frac{D\rho^2 + c\rho}{D\rho^2 + c\rho - \delta} - (1+\theta)\right] \\
&= \frac{\lambda p_1}{D\rho^2 + c\rho}[\beta - \theta].
\end{aligned}$$

□

Corollary 2.4 *If $\rho = 0$ then the moments of $G(x)$ are given by*

$$\mu_{G,n}(0) = \frac{n!}{(c/D)^n}\left[1 + \sum_{j=1}^{n} \frac{(c/D)^j}{(j+1)!}\frac{p_{j+1}}{p_1}\right], \quad n = 0, 1, 2, \ldots. \tag{2.76}$$

Proof: Lin and Willmot (1999) [36] showed that if $\rho = 0$ then the moments of $\Gamma(x)$ are given by

$$\mu_{\Gamma,n}(0) = \frac{p_{n+1}}{(n+1)p_1}, \quad n = 0, 1, 2, \ldots. \tag{2.77}$$

Now if $\rho = 0$, then $a = b = c/D$, and (1.51) becomes
$\frac{1}{\rho}[1 - \tilde{p}(\rho)]|_{\rho=0} = \int_0^\infty e^{-\rho y}\overline{P}(y)dy|_{\rho=0} = p_1$. Therefore, from (2.73) we have

$$\begin{aligned}
\mu_{G,n}(0) &= \mu_{\Gamma,n}(0) + \frac{n!}{(c/D)^{n+1}} \sum_{j=1}^{n} \frac{(c/D)^j}{j!}\frac{p_j}{p_1} \\
&= \frac{p_{n+1}}{(n+1)p_1} + \frac{n!}{(c/D)^{n+1}}\left[\frac{c}{D} + \sum_{j=2}^{n} \frac{(c/D)^j}{j!}\frac{p_j}{p_1}\right] \\
&= \frac{n!}{(c/D)^{n+1}}\left[\frac{c}{D} + \sum_{j=2}^{n+1} \frac{(c/D)^j}{j!}\frac{p_j}{p_1}\right] \\
&= \frac{n!}{(c/D)^n}\left[1 + \sum_{j=1}^{n} \frac{(c/D)^j}{(j+1)!}\frac{p_{j+1}}{p_1}\right].
\end{aligned}$$

□

We remark that the relationship, associated with the equilibrium distribution functions of $P(x)$, between $\mu_{\Gamma,n}(\rho)$ and $\mu_{\Gamma,n}(0)$ was found by Lin and Willmot (1999) [36] as follows:

$$\mu_{\Gamma,n}(\rho) = \mu_{\Gamma,n}(0)\frac{\int_0^\infty e^{-\rho x}dP_{n+1}(x)}{\int_0^\infty e^{-\rho x}dP_1(x)}, \quad n = 0, 1, 2, \ldots. \tag{2.78}$$

Example 2.3 Combination of exponentials

As shown in example 2.1, if $\overline{P}(x) = \sum_{k=1}^{r} q_k e^{-\mu_k x}$, then $\overline{\Gamma}(x) = \sum_{k=1}^{r} q_k^* e^{-\mu_k x}, x \geq 0$, and $\overline{G}(x) = (1 - bq^*)e^{-bx} + bq^* \sum_{k=1}^{r} q_k^{**} e^{-\mu_k x}, x \geq 0$. Hence,

$$\mu_{\Gamma,n}(\rho) = \int_0^\infty x^n d\Gamma(x) = \sum_{k=1}^{r} q_k^* \mu_k \int_0^\infty x^n e^{-\mu_k x} dx = n! \sum_{k=1}^{r} \frac{q_k^*}{\mu_k^n}, \tag{2.79}$$

and

$$\begin{aligned} \mu_{G,n}(\rho) &= b(1 - bq^*) \int_0^\infty x^n e^{-bx} dx + bq^* \sum_{k=1}^{r} q_k^{**} \mu_k \int_0^\infty x^n e^{-\mu_k x} dx \\ &= (1 - bq^*)\frac{n!}{b^n} + bq^* n! \sum_{k=1}^{r} \frac{q_k^{**}}{\mu_k^n}. \end{aligned} \tag{2.80}$$

□

Example 2.4 Mixture of Erlangs

In example 2.2, if $P'(x) = \sum_{k=1}^{r} q_k \frac{\mu(\mu x)^{k-1} e^{-\mu x}}{(k-1)!}$, then $\Gamma'(x) = \sum_{k=1}^{r} q_k^* \frac{\mu(\mu x)^{k-1} e^{-\mu x}}{(k-1)!}$ and $G'(x) = \left[1 - \frac{bq^*}{b-\mu}\right] be^{-bx} + \frac{bq^*}{b-\mu} \sum_{k=1}^{r} q_k^{**} \frac{\mu(\mu x)^{k-1}}{(k-1)!} e^{-\mu x}, x \geq 0$. Thus,

$$\mu_{\Gamma,n}(\rho) = \sum_{k=1}^{r} q_k^* \frac{\mu^k}{(k-1)!} \int_0^\infty x^{n+k-1} e^{-\mu x} dx = \frac{n!}{\mu^n} \sum_{k=1}^{r} q_k^* \binom{n+k-1}{n} \tag{2.81}$$

and

$$\mu_{G,n}(\rho) = b\left[1 - \frac{bq^*}{b-\mu}\right] \int_0^\infty x^n e^{-bx} dx + \frac{bq^*}{b-\mu} \sum_{k=1}^{r} q_k^{**} \frac{\mu^k}{(k-1)!} \int_0^\infty x^{n+k-1} e^{-\mu x} dx$$

$$= \left[1 - \frac{bq^*}{b-\mu}\right]\frac{n!}{b^n} + \frac{bq^*}{b-\mu}\frac{n!}{\mu^n}\sum_{k=1}^{r} q_k^{**}\binom{n+k-1}{n}. \tag{2.82}$$

□

The computations of $\mu_{G,n}(\rho)$ and $\mu_{\Gamma,n}(\rho)$ seem complicated. However, much computational effort can be saved by the following recursive formulas.

Lemma 2.4 *For n=1, 2, 3,...,*

$$\mu_{G,n}(\rho) - \frac{n}{b}\mu_{G,n-1}(\rho) = \mu_{\Gamma,n}(\rho) = -\frac{n}{\rho}\mu_{\Gamma,n-1}(\rho) + \frac{p_n}{1-\tilde{p}(\rho)}. \tag{2.83}$$

Proof: From (2.70),

$$\mu_{\Gamma,n}(\rho) + \frac{n}{\rho}\mu_{\Gamma,n-1}(\rho) = \frac{n!}{(-\rho)^n}\frac{(-\rho)^n}{n!}\frac{p_n}{1-\tilde{p}(\rho)} = \frac{p_n}{1-\tilde{p}(\rho)}.$$

Similarly, from (2.73)

$$\begin{aligned}
\mu_{G,n}(\rho) - \frac{n}{b}\mu_{G,n-1}(\rho) &= \frac{b}{a}\mu_{\Gamma,n}(\rho) - \frac{n}{a}\mu_{\Gamma,n-1}(\rho) + \frac{\rho}{a}\frac{p_n}{1-\tilde{p}(\rho)} \\
&= \frac{b}{a}\left[-\frac{n}{\rho}\mu_{\Gamma,n-1}(\rho) + \frac{p_n}{1-\tilde{p}(\rho)}\right] - \frac{n}{a}\mu_{\Gamma,n-1}(\rho) + \frac{\rho}{a}\frac{p_n}{1-\tilde{p}(\rho)} \\
&= -\frac{n}{\rho}\mu_{\Gamma,n-1}(\rho) + \frac{p_n}{1-\tilde{p}(\rho)} \\
&= \mu_{\Gamma,n}(\rho).
\end{aligned}$$

□

With $n = 1$ in (2.83), we get the mean of $\Gamma(x)$, $\mu_{\Gamma,1}(\rho) = \frac{p_1}{1-\tilde{p}(\rho)} - \frac{1}{\rho}$, and the mean of $G(x)$, $\mu_{G,1}(\rho) = \mu_{\Gamma,1}(\rho) + \frac{1}{b}$, which can be verified from (2.48) too.

For the case $\rho = 0$, (2.76) and (2.77) easily lead to the following result.

Corollary 2.5 *For n=1, 2, 3,...,*

$$\mu_{G,n}(0) - \frac{nD}{c}\mu_{G,n-1}(0) = \mu_{\Gamma,n}(0) = \frac{p_{n+1}}{(n+1)p_1}. \tag{2.84}$$

2.3 Compound geometric distribution and discounted probabilities of ruin

In this section, we are going to discuss four cases based on different choices of $V(u)$ in (2.31). One of them can lead $\phi(u)$ to a compound geometric distribution function.

To see this, since $B(u) = \dfrac{V(u)}{\int_0^\infty g(y)dy} = (1+\beta)V(u)$,

<u>Case 1</u> : when $V(u) = w_0 A(u) + g_\omega(u)$ which is the case of (2.17), if $w_0 = \dfrac{1}{1+\beta}$ and $w(x,y) = 1$ then from (2.12), (2.34) and (2.49)

$$\begin{aligned} B(u) &= \overline{H}(u) + \frac{\int_0^u e^{-b(u-s)} \int_s^\infty e^{-\rho(x-s)} \int_x^\infty w(x, y-x)p(y)dydxds}{\frac{1}{b}\int_0^\infty e^{-\rho y}\overline{P}(y)dy} \\ &= \overline{H}(u) + \frac{\int_0^u e^{-b(u-s)} \int_s^\infty e^{-\rho(x-s)}\overline{P}(x)dxds}{\frac{1}{b}\int_0^\infty e^{-\rho y}\overline{P}(y)dy} \\ &= \overline{G}(u), \end{aligned} \tag{2.85}$$

and (2.17) becomes

$$\phi(u) = \frac{1}{1+\beta}\int_0^u \phi(u-x)G'(x)dx + \frac{1}{1+\beta}\overline{G}(u), \quad u \geq 0. \tag{2.86}$$

Thus, in this case, (1.36) and (1.39) imply that $\phi(u) = \overline{K}(u)$. Therefore, from (1.37) we obtain

$$\phi(u) = \overline{K}(u) = \sum_{n=1}^{\infty} \frac{\beta}{1+\beta}(\frac{1}{1+\beta})^n \overline{G^{*n}}(u), \quad u \geq 0, \tag{2.87}$$

a compound geometric distribution function, and (2.1) turns out to be

$$\begin{aligned}\overline{K}(u) &= \frac{1}{1+\beta}E[e^{-\delta T}I(T<\infty, U(T)=0)] + E[e^{-\delta T}I(T<\infty, U(T)<0)] \\ &= \frac{1}{1+\beta}\phi_d(u) + \phi_s(u), \quad u \geq 0. \end{aligned} \tag{2.88}$$

We remark that when $D \to 0$, $\beta \to \beta_0$ and $G(u) \to \Gamma(u)$ by (2.43), then we have

$$\overline{K}(u) \to \overline{K}_0(u) \equiv \sum_{n=1}^{\infty} \frac{\beta_0}{1+\beta_0}(\frac{1}{1+\beta_0})^n \overline{\Gamma^{*n}}(u), \quad u \geq 0. \tag{2.89}$$

Lin and Willmot (1999) [36] showed that

$$\overline{K}_0(u) \equiv E[e^{-\delta T}I(T<\infty)], \quad u \geq 0, \tag{2.90}$$

with $\overline{K}_0(0) = \frac{1}{1+\beta_0}$. Then (1.39) becomes

$$\overline{K}_0(u) = \frac{1}{1+\beta_0}\int_0^u \overline{K}_0(u-x)d\Gamma(x) + \frac{1}{1+\beta_0}\overline{\Gamma}(u), \quad u \geq 0. \tag{2.91}$$

In fact, since when $D \to 0$, $\overline{K}(u) \to \overline{K}_0(u)$ for $u \geq 0$, and $\phi_d(u) \to 0$ for $u > 0$

by theorem 2.3, we have that both $\phi_t(u)$ and $\phi_s(u) \to \overline{K}_0(u)$ for $u > 0$ when $D \to 0$.

In the case $\delta = 0$, when $D \to 0$,

$$\overline{K}_{\delta=0}(u) \to \overline{K}_{0,\delta=0}(u) = E[I(T < \infty)] = Pr(T < \infty) = \psi_0(u), \quad u \geq 0. \tag{2.92}$$

Moreover, if $\delta = 0$, then $\beta = \theta$, (2.88) reduces to

$$\overline{K}_{\delta=0}(u) = \frac{1}{1+\theta}\psi_d(u) + \psi_s(u), \quad u \geq 0. \tag{2.93}$$

Also from (1.15) and (1.16), $\overline{K}_{\delta=0}(u)$ satisfies the following defective renewal equation

$$\overline{K}_{\delta=0}(u) = \frac{1}{1+\theta}\int_0^u \overline{K}_{\delta=0}(u-x)dH_1 * H_2(x) + \frac{1}{1+\theta}\overline{H_1 * H_2}(u), \tag{2.94}$$

which agrees with that $\overline{K}_{\delta=0}(u)$ is a compound geometric distribution function.

The defective moments of the compound geometric distribution function $\overline{K}(u)$ can be expressed in terms of the moments of $G(x)$ by

$$\int_0^\infty u^n \overline{K}(u)du = (-1)^n \frac{d^n}{ds^n}\int_0^\infty e^{-su}\overline{K}(u)du\bigg|_{s=0}, n = 0, 1, 2, \ldots.$$

For example, from (1.38) and (1.40),

$$\int_0^\infty \overline{K}(u)du = -\frac{\tilde{G}'(0)}{\beta} = \frac{\int_0^\infty u dG(u)}{\beta} = \frac{\mu_{G,1}(\rho)}{\beta} = \frac{\mu_{\Gamma,1}(\rho)}{\beta} + \frac{1}{b\beta}, \tag{2.95}$$

and the defective mean of $\overline{K}(u)$ is

$$\begin{aligned}
&\int_0^\infty u\overline{K}(u)du \\
&= -\frac{d}{ds}\int_0^\infty e^{-su}\overline{K}(u)du\Big|_{s=0} \\
&= -\Big[\frac{1-\tilde{G}(s)}{s}\frac{\tilde{G}'(s)}{[1+\beta-\tilde{G}(s)]^2} - \frac{s\tilde{G}'(s)+1-\tilde{G}(s)}{s^2}\frac{1}{1+\beta-\tilde{G}(s)}\Big]\Big|_{s=0} \\
&= \frac{1}{\beta^2}[\tilde{G}'(0)]^2 + \frac{1}{\beta}\frac{s\tilde{G}''(s)+\tilde{G}'(s)-\tilde{G}'(s)}{2s}\Big|_{s=0} \\
&= \frac{1}{\beta^2}\mu_{G,1}^2(\rho) + \frac{1}{2\beta}\mu_{G,2}(\rho),
\end{aligned} \tag{2.96}$$

with the help of L'Hopital's rule. Therefore, the proper mean of $\overline{K}(u)$ is

$$\frac{\int_0^\infty u\overline{K}(u)du}{\int_0^\infty \overline{K}(x)dx} = \frac{1}{\beta}\mu_{G,1}(\rho) + \frac{1}{2}\frac{\mu_{G,2}(\rho)}{\mu_{G,1}(\rho)}. \tag{2.97}$$

<u>Case 2</u> : when $V(u) = w_0\, A(u)$ which is the case of (1.22) then

$$B(u) = w_0(1+\beta)e^{-bu} = w_0(1+\beta)\overline{H}(u). \tag{2.98}$$

If $w_0 = 1$ then

$$B(u) = (1+\beta)\overline{H}(u), \tag{2.99}$$

and $B'(u) = -(1+\beta)h(u)$. Equation (1.22) becomes the defective renewal equation for the *discounted* (with discount factor δ) probability of ruin caused by oscillation,

$$\phi_d(u) = \frac{1}{1+\beta}\int_0^u \phi_d(u-x)d\Gamma * H(x) + \overline{H}(u), \quad u \geq 0. \tag{2.100}$$

In addition, by (1.44), $\phi_d(u)$ can be expressed as

$$\begin{aligned}
\phi_d(u) &= \frac{1+\beta}{\beta}\Big[\int_0^u \overline{K}(u-x)h(x)dx + \overline{H}(u) - \overline{K}(u)\Big] \\
&= \frac{1+\beta}{\beta}\Big[\overline{K} * H(u) + \overline{H}(u) - \overline{K}(u)\Big] \\
&= \frac{1+\beta}{\beta}\Big[\overline{K * H}(u) - \overline{K}(u)\Big] \qquad (2.101)
\end{aligned}$$

where

$$\overline{K * H}(u) = 1 - K * H(u) = 1 - (1-\overline{K}) * H(u) = \overline{H}(u) + \overline{K} * H(u). \qquad (2.102)$$

With (2.88) and (2.101), we get

$$\overline{K * H}(u) = \frac{\beta}{1+\beta}\phi_d(u) + \overline{K}(u) = \phi_d(u) + \phi_s(u) = \phi_t(u), \qquad (2.103)$$

and

$$\begin{aligned}
\overline{H} * K(u) &= \int_0^u \overline{H}(u-x)dK(x) = K(u) - K * H(u) \\
&= \overline{K * H}(u) - \overline{K}(u) = \frac{\beta}{1+\beta}\phi_d(u). \qquad (2.104)
\end{aligned}$$

When $\delta = 0$, then $\beta = \theta$ and (2.103) becomes

$$\overline{K_{\delta=0} * H_1}(u) = \frac{\theta}{1+\theta}\psi_d(u) + \overline{K}_{\delta=0}(u) = \psi_d(u) + \psi_s(u) = \psi_t(u). \qquad (2.105)$$

Since $K * H(u) = \int_0^u K(x)H'(u-x)dx = b\int_0^u e^{-b(u-x)}K(x)dx$ and by (2.101), we have

$$-\frac{d\phi_t(u)}{du} = \frac{dK * H(u)}{du} = b\Big[K(u) - K * H(u)\Big] = b\Big[\overline{K * H}(u) - \overline{K}(u)\Big] = \frac{b\beta}{1+\beta}\phi_d(u). \tag{2.106}$$

Therefore,

$$\int_u^\infty \overline{K * H}(x)dx = \int_u^\infty \overline{K}(x)dx + \frac{1}{b}\int_u^\infty dK * H(x) = \int_u^\infty \overline{K}(x)dx + \frac{1}{b}\overline{K * H}(u), \tag{2.107}$$

or by (2.103),

$$\int_u^\infty \phi_t(x)dx = \int_u^\infty \overline{K}(x)dx + \frac{1}{b}\phi_t(u) \tag{2.108}$$

with

$$\int_0^\infty \phi_t(x)dx = \int_0^\infty \overline{K * H}(x)dx = \int_0^\infty \overline{K}(x)dx + \frac{1}{b} = \frac{\mu_{G,1}(\rho)}{\beta} + \frac{1}{b} = \frac{\mu_{\Gamma,1}(\rho)}{\beta} + \frac{1+\beta}{b\beta} \tag{2.109}$$

by (2.95). Also from (2.103) and (2.106), the defective tail discounted (with discount factor δ) probability of the time of ruin caused by oscillation is

$$\int_u^\infty \phi_d(x)dx = \frac{1+\beta}{b\beta}\phi_t(u) = \frac{1+\beta}{b\beta}\overline{K * H}(u) \tag{2.110}$$

with

$$\int_0^\infty \phi_d(x)dx = \frac{1+\beta}{b\beta} = \frac{1}{bK(0)}. \tag{2.111}$$

The defective moments of the function $\phi_d(u)$ can be also expressed in terms of the moments of $G(x)$ by

$$\int_0^\infty u^n \phi_d(u)du = (-1)^n \frac{d^n}{ds^n}\int_0^\infty e^{-su}\phi_d(u)du\Big|_{s=0} = (-1)^n \tilde{\phi}_d^{(n)}(0,\delta),$$

n=0, 1, 2, ..., like the defective moments of the compound geometric distribution function $\overline{K}(u)$ above. However, with (2.109) and (2.110), the defective mean of $\phi_d(u)$, $\int_0^\infty u\phi_d(u)du$ can be easily derived as follows:

$$\begin{aligned}\int_0^\infty u\phi_d(u)du &= \int_0^\infty \int_0^u \phi_d(u)dxdu = \int_0^\infty \int_x^\infty \phi_d(u)dudx \\ &= \frac{1+\beta}{b\beta}\int_0^\infty \phi_t(x)dx = \frac{1+\beta}{b\beta}\Big[\frac{\mu_{G,1}(\rho)}{\beta}+\frac{1}{b}\Big], \end{aligned} \tag{2.112}$$

and hence the proper mean of the function $\phi_d(u)$ is

$$\frac{\int_0^\infty u\phi_d(u)du}{\int_0^\infty \phi_d(x)dx} = \frac{\mu_{G,1}(\rho)}{\beta}+\frac{1}{b} = \int_0^\infty \phi_t(x)dx. \tag{2.113}$$

<u>Case 3</u> : when $V(u) = g_\omega(u)$ which is the case of (2.15), if $w(x,y)=1$ then from (2.12), (2.34), (2.40) and (2.49)

$$B(u) = \frac{\int_0^u e^{-b(u-s)}\int_s^\infty e^{-\rho(x-s)}\int_x^\infty w(x,y-x)p(y)dydxds}{\frac{1}{b}\int_0^\infty e^{-\rho y}\overline{P}(y)dy} \tag{2.114}$$

$$\begin{aligned} &= \frac{\int_0^u e^{-b(u-s)}\int_s^\infty e^{-\rho(x-s)}\overline{P}(x)dxds}{\frac{1}{b}\int_0^\infty e^{-\rho y}\overline{P}(y)dy} \\ &= \overline{G}(u) - e^{-bu} \\ &= \overline{\Gamma} * H(u). \end{aligned} \tag{2.115}$$

We remark that the function $B(u)$ based on (1.13) is

$$B_0(u) = \frac{\int_u^\infty e^{-\rho(x-u)} \int_x^\infty w(x, y-x)p(y)dydx}{\int_0^\infty e^{-\rho y}\overline{P}(y)dy}. \tag{2.116}$$

Equation (1.34) becomes the defective renewal equation for the ***discounted*** (with discount factor δ) probability of ruin caused by a claim,

$$\phi_s(u) = \frac{1}{1+\beta}\int_0^u \phi_s(u-x)d\Gamma * H(x) + \frac{1}{1+\beta}\overline{\Gamma} * H(u), \quad u \geq 0. \tag{2.117}$$

Also by (1.44), $\phi_s(u)$ can be expressed as

$$\phi_s(u) = -\frac{1}{\beta}\int_0^u \overline{K}(u-x)d\overline{\Gamma} * H(x) + \frac{1}{\beta}\overline{\Gamma} * H(u). \tag{2.118}$$

Since $\overline{K}(u) = \phi_s(u) + \frac{1}{1+\beta}\phi_d(u)$, from (2.95) and (2.111),

$$\int_0^\infty \phi_s(x)dx = \int_0^\infty \overline{K}(x)dx - \frac{1}{1+\beta}\int_0^\infty \phi_d(x)dx = \frac{\mu_{G,1}(\rho)}{\beta} - \frac{1}{b\beta} = \frac{\mu_{\Gamma,1}(\rho)}{\beta}. \tag{2.119}$$

Also by (2.96), (2.112) and $\mu_{G,2}(\rho) = \frac{2\mu_{G,1}(\rho)}{b} + \mu_{\Gamma,2}(\rho) = \frac{2}{b^2} + \frac{2\mu_{\Gamma,1}(\rho)}{b} + \mu_{\Gamma,2}(\rho)$ from (2.83), the defective mean of $\phi_s(u)$ is

$$\begin{aligned}
\int_0^\infty u\phi_s(u)du &= \int_0^\infty u\overline{K}(u)du - \frac{1}{1+\beta}\int_0^\infty u\phi_d(u)du \\
&= \frac{1}{\beta^2}\mu_{G,1}^2(\rho) + \frac{1}{2\beta}\mu_{G,2}(\rho) - \frac{1}{b\beta}\left[\frac{\mu_{G,1}(\rho)}{\beta} + \frac{1}{b}\right] \\
&= \frac{1}{\beta^2}\mu_{G,1}(\rho)\mu_{\Gamma,1}(\rho) + \frac{1}{\beta}\left[\frac{\mu_{G,2}(\rho)}{2} - \frac{1}{b^2}\right]
\end{aligned}$$

$$= \frac{\mu_{\Gamma,1}(\rho)}{\beta}\left[\frac{\mu_{G,1}(\rho)}{\beta}+\frac{1}{b}\right]+\frac{\mu_{\Gamma,2}(\rho)}{2\beta}, \tag{2.120}$$

and hence the proper mean of the function $\phi_s(u)$ is

$$\frac{\int_0^\infty u\phi_s(u)du}{\int_0^\infty \phi_s(x)dx} = \frac{\mu_{G,1}(\rho)}{\beta}+\frac{1}{b}+\frac{\mu_{\Gamma,2}(\rho)}{2\mu_{\Gamma,1}(\rho)}. \tag{2.121}$$

<u>Case 4</u> : when $V(u) = w_0 A(u) + g_\omega(u)$ which is the case of (2.17), if $w_0 = 1$ and $w(x,y) = 1$ then from (2.99) and (2.115)

$$B(u) = (1+\beta)\overline{H}(u)+H(u)-\Gamma * H(u) = \beta\overline{H}(u)+1-H*\Gamma(u) = \beta\overline{H}(u)+\overline{\Gamma * H}(u). \tag{2.122}$$

Equation (1.35) becomes the defective renewal equation for the <u>*discounted*</u> (with discount factor δ) probability of ruin caused by both oscillation and a claim,

$$\phi_t(u) = \frac{1}{1+\beta}\int_0^u \phi_t(u-x)d\Gamma * H(x) + \frac{\beta}{1+\beta}\overline{H}(u) + \frac{1}{1+\beta}\overline{\Gamma * H}(u), \quad u \geq 0. \tag{2.123}$$

Since $\phi_t(u) = \phi_d(u) + \phi_s(u)$, from (2.100), (2.101), (2.117) and (2.118), $\phi_t(u)$ can be written as

$$\begin{aligned}
&\phi_t(u)\\
&= \frac{1}{1+\beta}\int_0^u \Big[\phi_d(u-x)+\phi_s(u-x)\Big]d\Gamma * H(x) + \frac{\beta}{1+\beta}\overline{H}(u) + \frac{1}{1+\beta}\overline{\Gamma * H}(u)\\
&= -\frac{1}{\beta}\int_0^u \overline{K}(u-x)d\overline{\Gamma} * H(x) + \left[\frac{1}{\beta}-\frac{1}{1+\beta}\right]\overline{\Gamma} * H(u) + \phi_d(u) - \overline{H}(u) +\\
&\quad \frac{\beta}{1+\beta}\overline{H}(u) + \frac{1}{1+\beta}\Big[\overline{H}(u) + \overline{\Gamma} * H(u)\Big]
\end{aligned}$$

$$= \frac{1+\beta}{\beta}\left[\overline{K*H}(u) - \overline{K}(u)\right] - \frac{1}{\beta}\int_0^u \overline{K}(u-x)d\overline{\Gamma} * H(x) + \frac{1}{\beta}\overline{\Gamma} * H(u), \quad (2.124)$$

and can be simplified to

$$\phi_t(u) = (1+\beta)\overline{K}(u) + \int_0^u \overline{K}(u-x)d\overline{\Gamma} * H(x) - \overline{\Gamma} * H(u) \qquad (2.125)$$

by (2.103).

Note that when $\rho = 0$ which is the case when $\delta = 0$, then $\beta = \theta$ and $H(u)$ and $\Gamma(u)$ simplfy to $H_1(u)$ and $H_2(u)$ of Dufresne and Gerber (1991) [16], respectively; hence (2.100), (2.117) and (2.123) reduce to their (5.10), (5.16) and (5.17) which are stated in (1.15), (1.16) and (1.17) respectively.

Since $\phi_t(u) = \phi_s(u) + \psi_d(u)$, from (2.112) and (2.120), the defective mean of $\phi_t(u)$ is

$$\begin{aligned}
\int_0^\infty u\phi_t(u)du &= \int_0^\infty u\phi_d(u)du + \int_0^\infty u\phi_s(u)du \\
&= \frac{1+\beta}{b\beta}\left[\frac{\mu_{G,1}(\rho)}{\beta} + \frac{1}{b}\right] + \frac{\mu_{\Gamma,1}(\rho)}{\beta}\left[\frac{\mu_{G,1}(\rho)}{\beta} + \frac{1}{b}\right] + \frac{\mu_{\Gamma,2}(\rho)}{2\beta} \\
&= \left[\frac{\mu_{G,1}(\rho)}{\beta} + \frac{1}{b}\right]^2 + \frac{\mu_{\Gamma,2}(\rho)}{2\beta}. \qquad (2.126)
\end{aligned}$$

Combining this with (2.109), the proper mean of the function $\phi_t(u)$ is

$$\frac{\int_0^\infty u\phi_t(u)du}{\int_0^\infty \phi_t(x)dx} = \left[\frac{\mu_{G,1}(\rho)}{\beta} + \frac{1}{b}\right] + \frac{\mu_{\Gamma,2}(\rho)}{2\beta}\left[\frac{\mu_{G,1}(\rho)}{\beta} + \frac{1}{b}\right]^{-1}. \qquad (2.127)$$

Recall that Gerber and Landry (1998) [24] mentioned that $g(y)$ is the discounted probability that the first record low is caused by a jump. That is, $g(y)dy$ is the discounted probability that the surplus will ever fall below its initial level u, and will be between $u-y$ and $u-y-dy$ when it happens for the first time caused by a jump ($\frac{\lambda}{c}[1-P(y)]dy$ for the case of the classical continuous risk model (1.1) that $D=0$ and $\delta=0$). In fact, based on the fact that a compound Poisson process has independent and stationary increments, $g(y)$ is the discounted probability that the surplus process $\{U(t):t\geq 0\}$ attains a record low caused by a jump. Let T_n^j (with $T_0^j=0$) be the n^{th} time when the surplus process $\{U(t):t\geq 0\}$ attains a record low caused by a jump, and $L_n=U(T_{n-1}^j)-U(T_n^j)$ be the amount by which the resulting n^{th} record low caused by a jump is below the surplus at time T_{n-1}^j, $n=1,2,\ldots$. One can define $L(t)=L_1+L_2+\cdots+L_{\tilde{N}(t)}$ (with $L(t)=0$ if $\tilde{N}(t)=0$), the total amount up to time t by which the resulting $\tilde{N}(t)^{th}$ record low caused by a jump is below the initial level u, where $\tilde{N}(t)$ is the total number of record lows up to time t caused by a jump, which is independent of $L_1, L_2, \ldots$ and $Pr(\tilde{N}(t)=n)=\beta(1+\beta)^{-n-1}$, $n=0,1,2,\ldots$. Then $G'(y)=g(y)/\int_0^\infty g(y)dy$ is the common probability density function of the identical and independent random variables $L_1, L_2, \ldots$. And the function $\overline{K}(u)$ in (2.88), which satisfies the defective renewal equation (2.86) as well as (2.87), can be treated as $\overline{K}(u)=Pr(L(t)>u), u\geq 0$.

2.4 Explicit analytical solution : examples

Since $\phi_t(u)=\overline{K*H}(u)=\overline{H}(u)+\overline{K}*H(u)$, $\phi_d(u)=\frac{1+\beta}{\beta}\left[\overline{K*H}(u)-\overline{K}(u)\right]$ and $\phi_s(u)=\phi_t(u)-\phi_d(u)=\frac{1+\beta}{\beta}\overline{K}(u)-\frac{1}{\beta}\overline{K*H}(u)$, if the explicit analytical solution to $\overline{K}(u)$ given in (1.37) is available in some cases, then all $\phi_t(u)$, $\phi_d(u)$ and $\phi_s(u)$

will also have explicit analytical solutions in these cases. We will demonstrate these cases by examples.

Example 2.5 Combination of exponentials

As shown in example 2.1, if $\overline{P}(x) = \sum_{k=1}^{r} q_k e^{-\mu_k x}$, then $\overline{G}(x) = (1 - bq^*)e^{-bx} + bq^* \sum_{k=1}^{r} q_k^{**} e^{-\mu_k x}$ and $G'(x) = (1-bq^*)be^{-bx} + bq^* \sum_{k=1}^{r} q_k^{**}\mu_k e^{-\mu_k x}, x \geq 0$. The Laplace transform of $G(x)$ is $\tilde{G}(-s) = \int_0^\infty e^{sx}dG(x) = (1 - bq^*)\frac{b}{b-s} + bq^* \sum_{k=1}^{r} q_k^{**} \frac{\mu_k}{\mu_k - s}$. Then by (1.40),

$$\int_0^\infty e^{su}\overline{K}(u)du = -\frac{1 - (1-bq^*)\frac{b}{b-s} - bq^* \sum_{k=1}^{r} q_k^{**}\frac{\mu_k}{\mu_k - s}}{s\left[1 + \beta - (1-bq^*)\frac{b}{b-s} - bq^* \sum_{k=1}^{r} q_k^{**}\frac{\mu_k}{\mu_k - s}\right]}. \tag{2.128}$$

The roots of the denominator are $s_0 = 0$ and $s_1, s_2, \ldots, s_{r+1}$. That is, $s_1, s_2, \ldots, s_{r+1}$ satisfy

$$(1-bq^*)\frac{b}{b-s} + bq^* \sum_{k=1}^{r} q_k^{**}\frac{\mu_k}{\mu_k - s} = 1 + \beta. \tag{2.129}$$

Note that by (2.55) and (2.57), $1 + \beta - (1-bq^*)\frac{b}{b-s} - bq^* \sum_{k=1}^{r} q_k^{**}\frac{\mu_k}{\mu_k - s} = 1 + \beta - \frac{b}{b-s} + b\sum_{k=1}^{r} q^* q_k^{**}\left[\frac{b}{b-s} - \frac{\mu_k}{\mu_k - s}\right] = \beta - \frac{s}{b-s} - \sum_{k=1}^{r} \frac{q_k^* b}{b - \mu_k}\frac{s(b-\mu_k)}{(b-s)(\mu_k - s)} = \beta - \frac{s}{b-s}\left[1 + \sum_{k=1}^{r} \frac{q_k^* b}{\mu_k - s}\right]$, thus (2.129) is equivalent to

$$\beta - \frac{s}{b-s}\left[1 + \sum_{k=1}^{r} \frac{q_k^* b}{\mu_k - s}\right] = 0. \tag{2.130}$$

If all the q_k^*'s are positive, so are all $s_1, s_2, \ldots, s_{r+1}$ by (2.130). Dufresne and Gerber (1988a) [14] showed that complex roots are only possible if at least one of q_k^*'s is negative; in this case, these complex roots must be paired (conjugate each other) and have positive real part.

We assume these roots are distinct for simplicity. One of $s_1, s_2, \ldots, s_{r+1}$, say s_1, satisfies generalized Lundberg's equation (1.32), and $\kappa = s_1 > 0$ is called the adjustment coefficient. By the principle of partial fractions as mentioned in Dufresne and Gerber (1988b) [15], there exist coefficients $D_0, D_1, \ldots, D_{r+1}$ such that

$$\frac{D_0}{s} + \sum_{k=1}^{r+1} \frac{D_k}{s - s_k} = -\left\{ s\left[1 + \beta - (1 - bq^*)\frac{b}{b - s} - bq^* \sum_{k=1}^{r} q_k^{**} \frac{\mu_k}{\mu_k - s} \right] \right\}^{-1}. \quad (2.131)$$

To find these coefficients $D_0, D_1, \ldots, D_{r+1}$, multiply (2.131) by s and then set $s = 0$ to get

$$D_0 = -\left[1 + \beta - (1 - bq^*) - bq^* \sum_{k=1}^{r} q_k^{**} \right]^{-1} = -\frac{1}{\beta}. \quad (2.132)$$

If we multiply (2.131) by $s(s - s_j)$ and then let $s \to s_j$, then we obtain

$$\begin{aligned} D_j &= \left\{ s_j \left[(1 - bq^*)\frac{b}{(b - s_j)^2} + bq^* \sum_{k=1}^{r} q_k^{**} \frac{\mu_k}{(\mu_k - s_j)^2} \right] \right\}^{-1} \\ &= \left\{ s_j \left[\frac{1 + \beta}{b - s_j} + bq^* \sum_{k=1}^{r} q_k^{**} \left(\frac{\mu_k}{(\mu_k - s_j)^2} - \frac{\mu_k}{(\mu_k - s_j)(b - s_j)} \right) \right] \right\}^{-1} \\ &= \left\{ s_j \left[\frac{1 + \beta}{b - s_j} + b \sum_{k=1}^{r} \frac{q_k^*}{b - \mu_k} \frac{\mu_k (b - \mu_k)}{(\mu_k - s_j)^2 (b - s_j)} \right] \right\}^{-1} \\ &= \frac{b - s_j}{s_j \left[1 + \beta + b \sum_{k=1}^{r} \frac{q_k^* \mu_k}{(\mu_k - s_j)^2} \right]}, \quad j = 1, 2, \ldots, r + 1, \end{aligned} \quad (2.133)$$

with the help of L'Hopital's rule, (2.55), (2.57) and (2.129). In addition, if we let $s \to b$ and $s \to \mu_j$, $j = 1, 2, \ldots, r+1$, in (2.131), respectively, we have that

$$\frac{D_0}{b} + \sum_{k=1}^{r+1} \frac{D_k}{b - s_k} = \frac{D_0}{\mu_j} + \sum_{k=1}^{r+1} \frac{D_k}{\mu_j - s_k} = 0, \quad j = 1, 2, \ldots, r+1. \tag{2.134}$$

Thus, (2.128) becomes

$$\begin{aligned}
&\int_0^\infty e^{su}\overline{K}(u)du \\
= \ & \left[1 - (1 - bq^*)\frac{b}{b-s} - bq^* \sum_{j=1}^{r} q_j^{**} \frac{\mu_j}{\mu_j - s}\right]\left[\frac{D_0}{s} + \sum_{k=1}^{r+1} \frac{D_k}{s - s_k}\right] \\
= \ & \frac{D_0}{s} - \sum_{k=1}^{r+1} \frac{D_k}{s_k - s} - (1 - bq^*)\frac{D_0 b}{s(b-s)} - bq^* D_0 \sum_{j=1}^{r} q_j^{**} \frac{\mu_j}{s(\mu_j - s)} + \\
& (1 - bq^*)\frac{b}{b-s}\sum_{k=1}^{r+1} \frac{D_k}{s_k - s} + bq^* \sum_{k=1}^{r+1}\sum_{j=1}^{r} D_k q_j^{**} \frac{1}{s_k - s}\frac{\mu_j}{\mu_j - s} \\
= \ & \frac{D_0}{s} - \sum_{k=1}^{r+1} \frac{D_k}{s_k - s} - (1 - bq^*)D_0\left[\frac{1}{s} + \frac{1}{b-s}\right] - bq^* D_0 \sum_{j=1}^{r} q_j^{**}\left[\frac{1}{s} + \frac{1}{\mu_j - s}\right] + \\
& (1 - bq^*)\sum_{k=1}^{r+1} \frac{D_k b}{b - s_k}\left[\frac{1}{s_k - s} - \frac{1}{b-s}\right] + bq^* \sum_{k=1}^{r+1}\sum_{j=1}^{r} \frac{D_k q_j^{**}\mu_j}{\mu_j - s_k}\left[\frac{1}{s_k - s} - \frac{1}{\mu_j - s}\right] \\
= \ & -\sum_{k=1}^{r+1} \frac{D_k}{s_k - s} - D_0\left[(1 - bq^*)\frac{1}{b-s} + bq^* \sum_{j=1}^{r} q_j^{**}\frac{1}{\mu_j - s}\right] + \\
& (1 - bq^*)\sum_{k=1}^{r+1} \frac{D_k b}{b - s_k}\left[\frac{1}{s_k - s} - \frac{1}{b-s}\right] + bq^* \sum_{k=1}^{r+1}\sum_{j=1}^{r} \frac{D_k q_j^{**}\mu_j}{\mu_j - s_k}\left[\frac{1}{s_k - s} - \frac{1}{\mu_j - s}\right].
\end{aligned}$$

By the uniqueness of the Laplace transform,

$$\begin{aligned}
\overline{K}(u) \ = \ & -\sum_{k=1}^{r+1} D_k e^{-s_k u} - D_0\left[(1 - bq^*)e^{-bu} + bq^* \sum_{j=1}^{r} q_j^{**} e^{-\mu_j u}\right] + \\
& (1 - bq^*)\sum_{k=1}^{r+1} \frac{D_k b}{b - s_k}\left[e^{-s_k u} - e^{-bu}\right] + bq^* \sum_{k=1}^{r+1}\sum_{j=1}^{r} \frac{D_k q_j^{**}\mu_j}{\mu_j - s_k}\left[e^{-s_k u} - e^{-\mu_j u}\right]
\end{aligned}$$

$$= \sum_{k=1}^{r+1} D_k \left[(1-bq*)\frac{b}{b-s_k} + bq^* \sum_{j=1}^{r} q_j^{**} \frac{\mu_j}{\mu_j - s_k} - 1 \right] e^{-s_k u} -$$

$$(1-bq^*)\left[D_0 + \sum_{k=1}^{r+1} \frac{D_k b}{b-s_k} \right] e^{-bu} - bq^* \sum_{j=1}^{r} q_j^{**} \left[D_0 + \sum_{k=1}^{r+1} \frac{D_k \mu_j}{\mu_j - s_k} \right] e^{-\mu_j u}$$

$$= \beta \sum_{k=1}^{r+1} D_k e^{-s_k u}, \tag{2.135}$$

a combination of exponential functions, by (2.129), (2.132) and (2.134). Moreover, letting $u = 0$ in (2.135) leads to $\overline{K}(0) = \beta \sum_{k=1}^{r+1} D_k$. Therefore,

$$\sum_{k=1}^{r+1} D_k = \frac{1}{\beta}\overline{K}(0) = \frac{1}{\beta(1+\beta)}, \tag{2.136}$$

and

$$\sum_{k=0}^{r+1} D_k = -\frac{1}{1+\beta} = -\overline{K}(0). \tag{2.137}$$

When $\delta = 0$, $\beta = \theta$ and (2.130), (2.133), (2.135), (2.136) and (2.137) become

$$\overline{K}_{\delta=0}(u) = \theta \sum_{k=1}^{r+1} D_{k,\delta=0} e^{-s_{k,\delta=0} u} \tag{2.138}$$

where $s_{1,\delta=0}, s_{2,\delta=0}, \ldots, s_{r+1,\delta=0}$ satisfy

$$\theta - \frac{s}{c/D - s}\left[1 + \sum_{k=1}^{r} \frac{q^*_{k,\delta=0} c/D}{\mu_k - s} \right] = 0, \tag{2.139}$$

and

$$D_{j,\delta=0} = \frac{c/D - s_{j,\delta=0}}{s_{j,\delta=0}\left[1 + \theta + \frac{c}{D} \sum_{k=1}^{r} \frac{q^*_{k,\delta=0}\mu_k}{(\mu_k - s_{j,\delta=0})^2} \right]}, \quad j = 1, 2, \ldots, r+1, \tag{2.140}$$

with

$$\sum_{k=1}^{r+1} D_{k,\delta=0} = \frac{1}{\theta}\overline{K}_{\delta=0}(0) = \frac{1}{\theta(1+\theta)}, \tag{2.141}$$

and

$$\sum_{k=0}^{r+1} D_{k,\delta=0} = -\frac{1}{1+\theta} = -\overline{K}_{\delta=0}(0). \tag{2.142}$$

In the case $D = 0$, similar arguments show

$$\overline{K}_0(u) = \beta_0 \sum_{k=1}^{r} D_k e^{-s_k u}, \tag{2.143}$$

a combination of exponential functions, where $s_1, s_2, \ldots, s_r$ satisfy

$$\sum_{k=1}^{r} q_k^* \frac{\mu_k}{\mu_k - s} = 1 + \beta_0; \tag{2.144}$$

and

$$D_j = \left\{ s_j \sum_{k=1}^{r} q_k^* \frac{\mu_k}{(\mu_k - s_j)^2} \right\}^{-1}, \quad j = 1, 2, \ldots, r, \tag{2.145}$$

with

$$\sum_{k=1}^{r} D_k = \frac{1}{1+\beta_0}\overline{K}_0(0) = \frac{1}{\beta_0(1+\beta_0)} \tag{2.146}$$

and

$$\sum_{k=0}^{r} D_k = -\frac{1}{1+\beta_0} = -\overline{K}_0(0). \tag{2.147}$$

For the special case $r = 1$, that is, $\overline{P}(x) = e^{-\mu x}$, then $q_1^* = q_1^{**} = 1$ and $q^* = (b-\mu)^{-1}$. By (2.130), s_1 and s_2 satisfy $\beta - \frac{s}{b-s}\left[1 + \frac{b}{\mu - s}\right] = 0$ or $(1+\beta)(s-b)(s-\mu) = b\mu$, or equivalently,

$$s^2 - (b+\mu)s + \frac{\beta}{1+\beta}b\mu = 0.$$

Since $s_1+s_2 = b+\mu$, $s_1 s_2 = \frac{\beta}{1+\beta}b\mu$ and $(b+\mu)^2-4\frac{\beta}{1+\beta}b\mu = (b-\mu)^2+\frac{4b\mu}{1+\beta} > 0$, s_1 and s_2 both are positive real numbers. In this case, $\overline{K}(u)$ in (2.135) becomes

$$\overline{K}(u) = \beta\Big[D_1 e^{-s_1 u} + D_2 e^{-s_2 u}\Big], \tag{2.148}$$

where

$$\begin{aligned} D_1 &= \frac{b-s_1}{s_1\Big[1+\beta+\dfrac{b\mu}{(\mu-s_1)^2}\Big]} = \frac{b-s_1}{s_1(1+\beta)\Big[1+\dfrac{s_1-b}{s_1-\mu}\Big]} \\ &= -\frac{(s_1-b)(s_1-\mu)}{(1+\beta)s_1(s_1-s_2)} = -\frac{s_2}{\beta(1+\beta)(s_1-s_2)} \end{aligned}$$

and

$$D_2 = \frac{b-s_2}{s_2\Big[1+\beta+\dfrac{b\mu}{(\mu-s_2)^2}\Big]} = -\frac{(s_2-b)(s_2-\mu)}{(1+\beta)s_2(s_2-s_1)} = \frac{s_1}{\beta(1+\beta)(s_1-s_2)}$$

with the help of $s_1+s_2 = b+\mu$, $s_1 s_2 = \frac{\beta}{1+\beta}b\mu$ and $(s_i-b)(s_i-\mu) = \frac{b\mu}{1+\beta} = \frac{s_1 s_2}{\beta}$, $i = 1, 2$.

With the explicit analytical solution (2.135) to $\overline{K}(u)$, then $\overline{K} * H(u) = \int_0^u \overline{K}(u-x)h(x)dx = \beta\sum_{k=1}^{r+1} D_k \int_0^u e^{-s_k(u-x)}be^{-bx}dx = b\beta\sum_{k=1}^{r+1} D_k e^{-s_k u}\int_0^u e^{-(b-s_k)x}dx$

$= b\beta \sum_{k=1}^{r+1} \frac{D_k}{b - s_k}\left[e^{-s_k u} - e^{-bu}\right]$, and from (2.102) and (2.103)

$$\begin{aligned}\phi_t(u) &= \overline{K * H}(u) = b\beta \sum_{k=1}^{r+1} \frac{D_k}{b - s_k} e^{-s_k u} + \left[-b\beta \sum_{k=1}^{r+1} \frac{D_k}{b - s_k} + 1\right] e^{-bu} \\ &= b\beta \sum_{k=1}^{r+1} \frac{D_k}{b - s_k} e^{-s_k u} \qquad (2.149)\end{aligned}$$

by (2.132) and (2.134), with

$$\psi_t(u) = \overline{K_{\delta=0} * H_1}(u) = \frac{c\theta}{D} \sum_{k=1}^{r+1} \frac{D_{k,\delta=0}}{c/D - s_{k,\delta=0}} e^{-s_{k,\delta=0} u}. \qquad (2.150)$$

In addition, from (2.101) and (2.103) we have

$$\begin{aligned}\phi_d(u) &= \frac{1+\beta}{\beta}\left[\overline{K * H}(u) - \overline{K}(u)\right] = (1+\beta) \sum_{k=1}^{r+1} D_k \left[\frac{b}{b - s_k} - 1\right] e^{-s_k u} \\ &= (1+\beta) \sum_{k=1}^{r+1} \frac{D_k s_k}{b - s_k} e^{-s_k u} \qquad (2.151)\end{aligned}$$

with

$$\psi_d(u) = \frac{1+\theta}{\theta}\left[\overline{K_{\delta=0} * H_1}(u) - \overline{K}_{\delta=0}(u)\right] = (1+\theta) \sum_{k=1}^{r+1} \frac{D_{k,\delta=0} s_{k,\delta=0}}{c/D - s_{k,\delta=0}} e^{-s_{k,\delta=0} u}, \qquad (2.152)$$

and

$$\phi_s(u) = \overline{K}(u) - \frac{1}{1+\beta}\phi_d(u) = \sum_{k=1}^{r+1} D_k \left[\beta - \frac{s_k}{b - s_k}\right] e^{-s_k u} \qquad (2.153)$$

with

$$\psi_s(u) = \overline{K}_{\delta=0}(u) - \frac{1}{1+\theta}\psi_d(u) = \sum_{k=1}^{r+1} D_{k,\delta=0}\left[\theta - \frac{s_{k,\delta=0}}{c/D - s_{k,\delta=0}}\right]e^{-s_{k,\delta=0}u}. \quad (2.154)$$

We remark that $\phi_t(u)$, $\phi_d(u)$ and $\phi_s(u)$ as well as $\psi_t(u)$, $\psi_d(u)$ and $\psi_s(u)$ are all a combination of exponential functions if the claim size distribution is a combination/mixture of Exponentials. □

Example 2.6 Mixture of Erlangs

As shown in example 2.2, when $b = \mu$ then $G'(x) = \sum_{k=1}^{r} q_k^* \frac{\mu(\mu x)^k}{k!} e^{-\mu x}$, whereas when $b \neq \mu$, $G'(x) = \left[1 - \frac{bq^*}{b-\mu}\right] be^{-bx} + \frac{bq^*}{b-\mu}\sum_{k=1}^{r} q_k^{**}\frac{\mu(\mu x)^{k-1}}{(k-1)!}e^{-\mu x}$. The former has $\tilde{G}(s) = \int_0^\infty e^{-sx}dG(x) = \sum_{k=1}^{r} q_k^* \frac{\mu^{k+1}}{k!}\int_0^\infty e^{-(\mu+s)x}x^k dx = \sum_{k=1}^{r} q_k^*\left(\frac{\mu}{\mu+s}\right)^{k+1} = \frac{\mu}{\mu+s}Q^*\left(\frac{\mu}{\mu+s}\right)$ where

$$Q^*(z) = \sum_{k=1}^{r} q_k^* z^k. \quad (2.155)$$

Then follow the method of Willmot and Lin (1998) [47] and by (1.41) we have

$$\int_{0-}^{\infty} e^{-su}dK(u) = \frac{\beta}{1+\beta-\tilde{G}(s)} = \frac{\frac{\beta}{1+\beta}}{1 - \frac{1}{1+\beta}\frac{\mu}{\mu+s}Q^*\left(\frac{\mu}{\mu+s}\right)} = C\left(\frac{\mu}{\mu+s}\right) \quad (2.156)$$

where

$$C(z) = \sum_{n=0}^{\infty} c_n z^n = \frac{\frac{\beta}{1+\beta}}{1 - \frac{1}{1+\beta}zQ^*(z)} \quad (2.157)$$

is the probability generating function of a compound geometric distribution for some coefficients $c_0, c_1 \ldots$. Letting $z = 0$ in (2.157), and differentiating (2.157) with respect to z and then setting $z = 0$, respectively, lead to $c_0 = \frac{\beta}{1+\beta}$ and $c_1 = 0$.

Since $\int_{0-}^{\infty} e^{-su} dK(u) = K(0) + \int_0^{\infty} e^{-su} K'(u) du$ and $K(0) = \frac{\beta}{1+\beta}$, (2.156) becomes

$$\int_0^{\infty} e^{-su} K'(u) du = \sum_{n=1}^{\infty} c_n \left(\frac{\mu}{\mu + s}\right)^n = \int_0^{\infty} e^{-su} \sum_{n=1}^{\infty} c_n \frac{\mu(\mu u)^{n-1}}{(n-1)!} e^{-\mu u} du. \tag{2.158}$$

Thus $K'(u) = \sum_{n=1}^{\infty} c_n \frac{\mu(\mu u)^{n-1}}{(n-1)!} e^{-\mu u}$ and by (2.71)

$$\overline{K}(u) = \sum_{n=1}^{\infty} c_n \frac{\mu^n}{(n-1)!} \int_u^{\infty} x^{n-1} e^{-\mu x} dx = e^{-\mu u} \sum_{n=1}^{\infty} c_n \sum_{j=0}^{n-1} \frac{(\mu u)^j}{j!} = e^{-\mu u} \sum_{j=0}^{\infty} \overline{C}_j \frac{(\mu u)^j}{j!} \tag{2.159}$$

where $\overline{C}_j = \sum_{n=j+1}^{\infty} c_n, j = -1, 0, 1, \ldots$ with $\overline{C}_{-1} = 1$, by interchanging the order of summation in the last equation.

Note that the coefficients $\{\overline{C}_0, \overline{C}_1, \overline{C}_2, \ldots\}$ may be calculated recursively as follows for computational purpose. By Feller (1968) [20] (p. 265), the associated probability generating function is

$$\sum_{n=0}^{\infty} \overline{C}_n z^n = \frac{1 - C(z)}{1 - z} = \frac{1 - \frac{\beta}{1+\beta}\left[1 - \frac{1}{1+\beta} z Q^*(z)\right]^{-1}}{1 - z} = \frac{\frac{1}{1+\beta} \frac{1 - zQ^*(z)}{1 - z}}{1 - \frac{1}{1+\beta} z Q^*(z)}.$$

Rearrangement gives

$$\sum_{n=0}^{\infty} \overline{C}_n z^n = \frac{1}{1+\beta} zQ^*(z) \sum_{n=0}^{\infty} \overline{C}_n z^n + \frac{1}{1+\beta} \frac{1 - zQ^*(z)}{1-z}.$$

Equating coefficients of z^n leads to

$$\overline{C}_n = \frac{1}{1+\beta} \sum_{k=1}^{n-1} q_k^* \overline{C}_{n-k-1} + \frac{1}{1+\beta} \sum_{k=n}^{\infty} q_k^*, \quad n = 1, 2, 3, \ldots, \tag{2.160}$$

with $q_k^* = 0$ if $k > r$. Thus the coefficients $\overline{C}_n$ for $n = 1, 2, 3, \ldots$ may be calculated recursively by (2.160) with starting value $\overline{C}_0 = 1 - c_0 = (1+\beta)^{-1}$, and (2.159) can be used to compute $\overline{K}(u)$.

When $b \neq \mu$, $\tilde{G}(s) = \int_0^\infty e^{-sx} dG(x) = \left[1 - \frac{bq^*}{b-\mu}\right] b \int_0^\infty e^{-(b+s)x} dx + \frac{bq^*}{b-\mu} \sum_{k=1}^{r} q_k^{**} \frac{\mu^k}{(k-1)!} \int_0^\infty x^{k-1} e^{-(\mu+s)x} dx = \left[1 - \frac{bq^*}{b-\mu}\right] \frac{b}{b+s} + \frac{bq^*}{b-\mu} Q^{**}\left(\frac{\mu}{\mu+s}\right)$ where

$$Q^{**}(z) = \sum_{k=1}^{r} q_k^{**} z^k. \tag{2.161}$$

Similarly, by (1.41) we have

$$\begin{aligned} \int_{0-}^{\infty} e^{-su} dK(u) &= \frac{\frac{\beta}{1+\beta}}{1 - \frac{1}{1+\beta}\left[\left(1 - \frac{bq^*}{b-\mu}\right)\frac{b}{b+s} + \frac{bq^*}{b-\mu} Q^{**}\left(\frac{\mu}{\mu+s}\right)\right]} \\ &= C\left(\frac{b}{b+s}, \frac{\mu}{\mu+s}\right) \end{aligned} \tag{2.162}$$

where

$$C(z,w) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} c_{m,n} z^m w^n = \frac{\frac{\beta}{1+\beta}}{1 - \frac{1}{1+\beta}\left[\left(1 - \frac{bq^*}{b-\mu}\right)z + \frac{bq^*}{b-\mu}Q^{**}(w)\right]} \quad (2.163)$$

for some coefficients $c_{m,n}, m, n = 0, 1, 2, \ldots$. Letting $z = w = 0$ in (2.163) leads to $c_{0,0} = \frac{\beta}{1+\beta}$.

Since $\int_{0-}^{\infty} e^{-su} dK(u) = K(0) + \int_0^{\infty} e^{-su} K'(u) du$ and $K(0) = \frac{\beta}{1+\beta}$, (2.162) becomes

$$\int_0^{\infty} e^{-su} K'(u) du = c_{1,0}\frac{b}{b+s} + c_{0,1}\frac{\mu}{\mu+s} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_{m,n}\left(\frac{b}{b+s}\right)^m\left(\frac{\mu}{\mu+s}\right)^n. \quad (2.164)$$

By the uniqueness of the Laplace transform, we have $K'(x) = c_{1,0}be^{-bx} + c_{0,1}\mu e^{-\mu x} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_{m,n} \int_0^x \frac{b[b(x-y)]^{m-1}}{(m-1)!} e^{-b(x-y)} \frac{\mu(\mu y)^{n-1}}{(n-1)!} e^{-\mu y} dy$ (note that $\int_0^x \frac{b[b(x-y)]^{m-1}}{(m-1)!} e^{-b(x-y)} \frac{\mu(\mu y)^{n-1}}{(n-1)!} e^{-\mu y} dy$ is the convolution of two Erlangian probability density functions) and

$$\begin{aligned}\overline{K}(u) &= c_{1,0}e^{-bu} + c_{0,1}\mu e^{-\mu u} \\ &+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_{m,n}\frac{b^m\mu^n}{(m-1)!(n-1)!}\int_u^{\infty}\int_0^x (x-y)^{m-1}y^{n-1}e^{-b(x-y)}e^{-\mu y}dydx\end{aligned} \quad (2.165)$$

Further calculations show that $\int_u^{\infty}\int_0^x (x-y)^{m-1}y^{n-1}e^{-b(x-y)}e^{-\mu y}dydx$ and hence $\overline{K}(u)$ can be expressed in terms of $e^{-bu}u^m$ and $e^{-\mu u}u^n$, $m, n = 0, 1, 2, \ldots$. However, the associated coefficients of $e^{-bu}u^m$ and $e^{-\mu u}u^n$ are very cumbersome and have no recursive relation like (2.160). An alternative method based on the principle of partial fractions can be used to express $\overline{K}(u)$ in terms of finite number of exponential functions as in example 2.5. To see this, by (1.40),

$$\int_0^\infty e^{su}\overline{K}(u)du = -\frac{1-\left(1-\frac{bq^*}{b-\mu}\right)\frac{b}{b-s}-\frac{bq^*}{b-\mu}\sum_{k=1}^{r}q_k^{**}\left(\frac{\mu}{\mu-s}\right)^k}{s\left[1+\beta-\left(1-\frac{bq^*}{b-\mu}\right)\frac{b}{b-s}-\frac{bq^*}{b-\mu}\sum_{k=1}^{r}q_k^{**}\left(\frac{\mu}{\mu-s}\right)^k\right]}. \tag{2.166}$$

The roots of the denominator are $s_0 = 0$ and $s_1, s_2, \ldots, s_{r+1}$, that is, $s_1, s_2, \ldots, s_{r+1}$ satisfy

$$\left(1-\frac{bq^*}{b-\mu}\right)\frac{b}{b-s}+\frac{bq^*}{b-\mu}\sum_{k=1}^{r}q_k^{**}\left(\frac{\mu}{\mu-s}\right)^k = 1+\beta. \tag{2.167}$$

We assume these roots are distinct for simplicity. One of $s_1, s_2, \ldots, s_{r+1}$, say s_1, satisfies generalized Lundberg's equation (1.32), and $\kappa = s_1 > 0$ is called the adjustment coefficient. Argument similar to the one in Dufresne and Gerber (1988a) [14] shows that either $s_1, s_2, \ldots, s_{r+1}$ are all positive or some of them are paired (conjugate each other) complex roots with positive real part. By the principle of partial fractions as mentioned in Dufresne and Gerber (1988b) [15], there exist coefficients $D_0, D_1, \ldots, D_{r+1}$ such that

$$\frac{D_0}{s}+\sum_{k=1}^{r+1}\frac{D_k}{s-s_k} = -\left\{s\left[1+\beta-\left(1-\frac{bq^*}{b-\mu}\right)\frac{b}{b-s}-\frac{bq^*}{b-\mu}\sum_{k=1}^{r}q_k^{**}\left(\frac{\mu}{\mu-s}\right)^k\right]\right\}^{-1}. \tag{2.168}$$

To find these coefficients $D_0, D_1, \ldots, D_{r+1}$, multiply (2.168) by s and then let $s = 0$ to obtain

$$D_0 = -\Big[1+\beta-\Big(1-\frac{bq^*}{b-\mu}\Big)-\frac{bq^*}{b-\mu}\sum_{k=1}^{r} q_k^{**}\Big]^{-1} = -\frac{1}{\beta}. \tag{2.169}$$

If we multiply (2.168) by $s(s-s_j)$ and then let $s \to s_j$, then we obtain

$$D_j = \Big\{s_j\Big[\Big(1-\frac{bq^*}{b-\mu}\Big)\frac{b}{(b-s_j)^2}+\frac{bq^*}{b-\mu}\sum_{k=1}^{r} q_k^{**}\frac{k\mu^k}{(\mu-s_j)^{k+1}}\Big]\Big\}^{-1}, \quad j=1,2,\ldots,r+1, \tag{2.170}$$

with the help of L'Hopital's rule. In addition, if we let $s \to b$ and $s \to \mu$, in (2.168), respectively, we get that

$$\frac{D_0}{b}+\sum_{k=1}^{r+1}\frac{D_k}{b-s_k} = \frac{D_0}{\mu}+\sum_{k=1}^{r+1}\frac{D_k}{\mu-s_k} = 0. \tag{2.171}$$

Thus, (2.166) becomes

$$\begin{aligned}
&\int_0^\infty e^{su}\overline{K}(u)du \\
=\ & \Big[1-\Big(1-\frac{bq^*}{b-\mu}\Big)\frac{b}{b-s}-\frac{bq^*}{b-\mu}\sum_{j=1}^{r} q_j^{**}\Big(\frac{\mu}{\mu-s}\Big)^j\Big]\Big[\frac{D_0}{s}+\sum_{k=1}^{r+1}\frac{D_k}{s-s_k}\Big] \\
=\ & \frac{D_0}{s}-\sum_{k=1}^{r+1}\frac{D_k}{s_k-s}-\Big[1-\frac{bq^*}{b-\mu}\Big]\Big[\frac{D_0 b}{s(b-s)}-\frac{b}{b-s}\sum_{k=1}^{r+1}\frac{D_k}{s_k-s}\Big]- \\
& \frac{bq^*}{b-\mu}\Big[D_0\sum_{j=1}^{r} q_j^{**}\frac{1}{s}\Big(\frac{\mu}{\mu-s}\Big)^j-\sum_{k=1}^{r+1}\sum_{j=1}^{r} D_k q_j^{**}\frac{1}{s_k-s}\Big(\frac{\mu}{\mu-s}\Big)^j\Big] \\
=\ & \frac{D_0}{s}-\sum_{k=1}^{r+1}\frac{D_k}{s_k-s}- \\
& \Big[1-\frac{bq^*}{b-\mu}\Big]\Big[D_0\Big(\frac{1}{s}+\frac{1}{b-s}\Big)-\sum_{k=1}^{r+1}\frac{D_k b}{b-s_k}\Big(\frac{1}{s_k-s}-\frac{1}{b-s}\Big)\Big]- \\
& \frac{bq^*}{b-\mu}\Big[D_0\sum_{j=1}^{r} q_j^{**}\frac{1}{s}\Big(\frac{\mu}{\mu-s}\Big)^j-\sum_{k=1}^{r+1}\sum_{j=1}^{r} D_k q_j^{**}\frac{1}{s_k-s}\Big(\frac{\mu}{\mu-s}\Big)^j\Big]
\end{aligned}$$

$$= -\sum_{k=1}^{r+1} \frac{D_k}{s_k - s} - \frac{bq^*}{b-\mu}\frac{D_0}{(-s)} + \Big[1 - \frac{bq^*}{b-\mu}\Big] b \sum_{k=1}^{r+1} \frac{D_k}{(b-s_k)(s_k-s)} + \frac{bq^*}{b-\mu}\Big[D_0 \sum_{j=1}^{r} q_j^{**} \frac{1}{(-s)}\Big(\frac{\mu}{\mu-s}\Big)^j + \sum_{k=1}^{r+1}\sum_{j=1}^{r} D_k q_j^{**} \frac{1}{s_k - s}\Big(\frac{\mu}{\mu - s}\Big)^j\Big]$$

with the help of (2.171). By the uniqueness of the Laplace transform,

$$\begin{aligned}
&\overline{K}(u) \\
&= -\sum_{k=1}^{r+1} D_k e^{-s_k u} - \frac{bq^*}{b-\mu} D_0 + \Big[1 - \frac{bq^*}{b-\mu}\Big]\sum_{k=1}^{r+1} \frac{D_k b}{b - s_k} e^{-s_k u} + \frac{bq^*}{b-\mu}\Big[D_0 \sum_{j=1}^{r} q_j^{**} \\
&\quad \int_0^u \frac{\mu(\mu x)^{j-1}}{(j-1)!} e^{-\mu x} dx + \sum_{k=1}^{r+1}\sum_{j=1}^{r} D_k q_j^{**} \int_0^u e^{-s_k(u-x)} \frac{\mu(\mu x)^{j-1}}{(j-1)!} e^{-\mu x} dx\Big] \\
&= -\sum_{k=1}^{r+1} D_k e^{-s_k u} - \frac{bq^*}{b-\mu} D_0 + \Big[1 - \frac{bq^*}{b-\mu}\Big]\sum_{k=1}^{r+1} \frac{D_k b}{b - s_k} e^{-s_k u} + \\
&\quad \frac{bq^*}{b-\mu}\Big[D_0 \sum_{j=1}^{r} q_j^{**}\Big(1 - e^{-\mu u}\sum_{m=0}^{j-1} \frac{(\mu u)^m}{m!}\Big) + \\
&\qquad \sum_{k=1}^{r+1}\sum_{j=1}^{r} D_k q_j^{**}\Big(\frac{\mu}{\mu - s_k}\Big)^j \Big(e^{-s_k u} - e^{-\mu u}\sum_{m=0}^{j-1} \frac{[(\mu - s_k)u]^m}{m!}\Big)\Big] \\
&= -\sum_{k=1}^{r+1} D_k e^{-s_k u} + (1+\beta)\sum_{k=1}^{r+1} D_k e^{-s_k u} - \frac{bq^*}{b-\mu} e^{-\mu u} \\
&\quad \Big[D_0 \sum_{j=1}^{r} q_j^{**} \sum_{m=0}^{j-1} \frac{(\mu u)^m}{m!} + \sum_{k=1}^{r+1}\sum_{j=1}^{r} D_k q_j^{**}\Big(\frac{\mu}{\mu - s_k}\Big)^j \sum_{m=0}^{j-1} \frac{[(\mu - s_k)u]^m}{m!}\Big] \\
&= \beta \sum_{k=1}^{r+1} D_k e^{-s_k u} - \frac{bq^*}{b-\mu} e^{-\mu u} \sum_{j=1}^{r} q_j^{**} \sum_{m=0}^{j-1} \frac{\mu^j u^m}{m!}\Big[\frac{D_0}{\mu^{j-m}} + \sum_{k=1}^{r+1} \frac{D_k}{(\mu - s_k)^{j-m}}\Big]
\end{aligned}$$

by (2.72) and (2.167). We declare that

$$\frac{D_0}{\mu^{j-m}} + \sum_{k=1}^{r+1} \frac{D_k}{(\mu - s_k)^{j-m}} = 0 \qquad (2.172)$$

for each $j-m=1,2,\ldots,j$, since $m=0,1,\ldots,j-1$ and $j=1,2,\ldots,r$, which implies

$$\overline{K}(u)=\beta\sum_{k=1}^{r+1}D_k e^{-s_k u}, \tag{2.173}$$

a combination of exponential functions, which has the same expression as (2.135) for the combination of exponentials case. Thus, (2.136) and (2.137) apply. When $\delta=0$, (2.173) becomes

$$\overline{K}_{\delta=0}(u)=\theta\sum_{k=1}^{r+1}D_{k,\delta=0}e^{-s_{k,\delta=0}u} \tag{2.174}$$

where $s_{1,\delta=0}, s_{2,\delta=0},\ldots,s_{r+1,\delta=0}$ satisfy

$$\left(1-\frac{cq^*_{\delta=0}}{c-D\mu}\right)\frac{c}{c-Ds}+\frac{cq^*_{\delta=0}}{c-D\mu}\sum_{k=1}^{r}q^{**}_{k,\delta=0}\left(\frac{\mu}{\mu-s}\right)^k=1+\theta \tag{2.175}$$

by (2.167), and for $j=1,2,\ldots,r+1$, from (2.170)

$$D_{j,\delta=0}$$
$$=\left\{s_{j,\delta=0}\left[\left(1-\frac{cq^*_{\delta=0}}{c-D\mu}\right)\frac{cD}{(c-Ds_{j,\delta=0})^2}+\frac{cq^*_{\delta=0}}{c-D\mu}\sum_{k=1}^{r}q^{**}_{k,\delta=0}\frac{k\mu^k}{(\mu-s_{j,\delta=0})^{k+1}}\right]\right\}^{-1}. \tag{2.176}$$

Note that (2.141) and (2.142) also apply here.

To show (2.172), we can differentiate both sides of (2.168) $(j-m-1)$ times with respect to s and then let $s\to\mu$. The consequence for the left side is

$$\frac{d^{j-m-1}}{ds^{j-m-1}}\left[\frac{D_0}{s}+\sum_{k=1}^{r+1}\frac{D_k}{s-s_k}\right]\Bigg|_{s=\mu}$$

$$= (-1)^{j-m-1}(j-m-1)!\Big[\frac{D_0}{s^{j-m}} + \sum_{k=1}^{r+1} \frac{D_k}{(s-s_k)^{j-m}}\Big]\Big|_{s=\mu}$$
$$= (-1)^{j-m-1}(j-m-1)!\Big[\frac{D_0}{\mu^{j-m}} + \sum_{k=1}^{r+1} \frac{D_k}{(\mu-s_k)^{j-m}}\Big]$$

which is exactly $(-1)^{j-m-1}(j-m-1)!$ times the left side of (2.172). Thus, it is sufficient to show the $(j-m-1)^{th}$ derivative of the right side of (2.168) at μ equals to 0. Let $a(x) \sim_c b(x)$ as $x \to c$ denote $lim_{x\to c} a(x)/b(x) = 1$. Then the right side of (2.168), as $s \to \mu$,

$$-\frac{1}{s\Big[1+\beta-\Big(1-\frac{bq^*}{b-\mu}\Big)\frac{b}{b-s} - \frac{bq^*}{b-\mu}\sum_{k=1}^{r} q_k^{**}\Big(\frac{\mu}{\mu-s}\Big)^k\Big]} \sim_\mu \frac{1}{\frac{b\mu q^* q_r^{**}}{b-\mu}\Big(\frac{\mu}{\mu-s}\Big)^r}$$
$$= \frac{(b-\mu)(\mu-s)^r}{bq^*q_r^{**}\mu^{r+1}}.$$

Since $j-m-1 = 0, 1, \ldots, j-1 < r$, we have $\frac{d^{j-m-1}}{ds^{j-m-1}}(\mu-s)^r\Big|_{s=\mu} = 0$ which implies

$$-\frac{d^{j-m-1}}{ds^{j-m-1}}\Big\{s\Big[1+\beta-\Big(1-\frac{bq^*}{b-\mu}\Big)\frac{b}{b-s} - \frac{bq^*}{b-\mu}\sum_{k=1}^{r} q_k^{**}\Big(\frac{\mu}{\mu-s}\Big)^k\Big]\Big\}^{-1}\Big|_{s=\mu} = 0.$$

For the case $b = \mu$, similar arguments show

$$\overline{K}(u) = \beta \sum_{k=1}^{r} D_k e^{-s_k u}, \tag{2.177}$$

a combination of exponential functions, where $s_1, s_2, \ldots, s_r$ satisfy

$$\sum_{k=1}^{r} q_k^*\Big(\frac{\mu}{\mu-s}\Big)^{k+1} = 1+\beta \tag{2.178}$$

and

$$D_j = \left\{ s_j \sum_{k=1}^{r} q_k^* \frac{(k+1)\mu^{k+1}}{(\mu - s_j)^{k+2}} \right\}^{-1}, \quad j = 1, 2, \ldots, r, \tag{2.179}$$

with relationships (2.146) and (2.147) but β_0 and $\overline{K}_0(u)$ replaced by β and $\overline{K}(u)$, respectively.

When $D = 0$, similar derivations lead to

$$\overline{K}_0(u) = \beta_0 \sum_{k=1}^{r} D_k e^{-s_k u}, \tag{2.180}$$

a combination of exponential functions, where $s_1, s_2, \ldots, s_r$ satisfy

$$\sum_{k=1}^{r} q_k^* \left(\frac{\mu}{\mu - s} \right)^k = 1 + \beta_0 \tag{2.181}$$

and

$$D_j = \left\{ s_j \sum_{k=1}^{r} q_k^* \frac{k\mu^k}{(\mu - s_j)^{k+1}} \right\}^{-1}, \quad j = 1, 2, \ldots, r. \tag{2.182}$$

Note that (2.146) and (2.147) also hold in this case.

To get explicit analytical solutions for $\phi_t(u)$, $\phi_d(u)$ and $\phi_s(u)$, since (2.173) and (2.177) have the same expressions as (2.135), equations (2.149), (2.151) and (2.153) apply for $b \neq \mu$ case, whereas substitute r for $r+1$ in the upper index of summation in these three equations for $b = \mu$. Similarly, equations (2.150), (2.152) and (2.154) apply for $\psi_t(u)$, $\psi_d(u)$ and $\psi_s(u)$ by the argument above.

Alternative explicit analytical solutions for $\phi_t(u)$, $\phi_d(u)$ and $\phi_s(u)$ can be obtained as follows for $b = \mu$ case if (2.159), $\overline{K}(u) = e^{-\mu u} \sum_{n=0}^{\infty} \overline{C}_n \frac{(\mu u)^n}{n!}$, is used. First calculate $\overline{K} * H(u) = \int_0^u \overline{K}(x) \mu e^{-\mu(u-x)} dx = \sum_{n=0}^{\infty} \overline{C}_n \frac{\mu^{n+1}}{n!} \int_0^u e^{-\mu x} x^n e^{-\mu(u-x)} dx =$

$e^{-\mu u}\sum_{n=0}^{\infty}\overline{C}_n\frac{(\mu u)^{n+1}}{(n+1)!}$. Then from (2.102) and (2.103)

$$\phi_t(u) = \overline{K*H}(u) = e^{-\mu u}\left[\sum_{n=0}^{\infty}\overline{C}_n\frac{(\mu u)^{n+1}}{(n+1)!}+1\right] = e^{-\mu u}\sum_{n=0}^{\infty}\overline{C}_{n-1}\frac{(\mu u)^n}{n!} \tag{2.183}$$

with $\overline{C}_{-1} = 1$. Moreover, from (2.101) and (2.103) we have

$$\begin{aligned}\phi_d(u) &= \frac{1+\beta}{\beta}\left[\overline{K*H}(u) - \overline{K}(u)\right] \\ &= \frac{1+\beta}{\beta}e^{-\mu u}\sum_{n=0}^{\infty}\left[\overline{C}_{n-1} - \overline{C}_n\right]\frac{(\mu u)^n}{n!} = \frac{1+\beta}{\beta}e^{-\mu u}\sum_{n=0}^{\infty}c_n\frac{(\mu u)^n}{n!}\end{aligned} \tag{2.184}$$

and

$$\phi_s(u) = \overline{K}(u) - \frac{1}{1+\beta}\phi_d(u) = e^{-\mu u}\sum_{n=0}^{\infty}\left[\overline{C}_n - \frac{c_n}{\beta}\right]\frac{(\mu u)^n}{n!}. \tag{2.185}$$

□

2.5 Asymptotic formula and Tijms-type approximation

In this section, we will first study the adjustment coefficient and propose the asymptotic formula for the defective renewal equations $\phi(u)$ in (2.17), then give the Tijms-type approximation for the associated compound geometric distribution function $\overline{K}(u)$ in (2.87).

Recall from (1.32) that $\rho \geq 0$ satisfies generalized Lundberg's equation $\lambda\tilde{p}(\xi) = \lambda + \delta - c\xi - D\xi^2$. If we define

$$\tau(\xi) = \lambda + \delta - c\xi - D\xi^2 = -D(\xi + \frac{c}{2D})^2 + \frac{c^2}{4D} + \lambda + \delta, \qquad (2.186)$$

then $\lambda\tilde{p}(\xi) = \lambda + \delta - c\xi - D\xi^2$ is equivalent to

$$\lambda\tilde{p}(\xi) = \tau(\xi) \qquad (2.187)$$

Since $\lambda\tilde{p}'(\xi) = -\lambda\int_0^\infty xe^{-\xi x}p(x)dx < 0$ and $\lambda\tilde{p}''(\xi) = \lambda\int_0^\infty x^2e^{-\xi x}p(x)dx > 0$, we have that $\lambda\tilde{p}(\xi)$ is a decreasing convex function with $\lambda\tilde{p}(0) = \lambda$. Moreover, due to the negative coefficient$(-D)$ of the ξ^2 term in (2.186), $\tau(\xi)$ is a concave parabola with $\tau(0) = \lambda+\delta$ and the maximum $\frac{c^2}{4D}+\lambda+\delta$ at $\xi = -\frac{c}{2D}$. Therefore (2.187) has exactly two roots, one nonnegative, says ξ_1, which is $\rho(\delta)$ by (1.32), and another negative, say $\xi_2 = -\kappa(\delta)$, where $\kappa(\delta) > 0$.

Note that since $\tau(-b) = \tau(\rho(\delta)) < \tau(-\kappa(\delta))$, we have $b > \kappa(\delta)$. In addition, when $\delta \to 0$, $\xi_1 = \rho(\delta)$ is decreasing to 0 and $|\xi_2| = \kappa(\delta)$ is also decreasing. And if $\delta \to 0$, which implies $\rho \to 0$, then from (1.32),

$$\frac{\delta}{\rho} = \rho D + c - \frac{\lambda}{\rho}[1 - \tilde{p}(\rho)] = \rho D + c - \lambda\int_0^\infty e^{-\rho x}\overline{P}(x)dx \to c - \lambda p_1. \qquad (2.188)$$

Let $\tilde{\phi}_w(\xi,\delta) = \int_0^\infty e^{-\xi u}\phi_w(u)du$, we multiply (2.15) by $e^{-\xi u}$, and integration from $u = 0$ to $u = \infty$ yields

$$\tilde{\phi}_w(\xi,\delta) = \tilde{\phi}_w(\xi,\delta)\tilde{g}(\xi,\delta) + \tilde{g}_\omega(\xi,\delta). \qquad (2.189)$$

Therefore,

$$\tilde{\phi}_w(\xi,\delta) = \frac{\tilde{g}_\omega(\xi,\delta)}{1-\tilde{g}(\xi,\delta)} = \sum_{n=0}^{\infty} \tilde{g}_\omega(\xi,\delta)\tilde{g}^n(\xi,\delta), \tag{2.190}$$

which is the Laplace transform of

$$\phi_w(u) = \sum_{n=0}^{\infty} g_\omega * g^{*n}(u). \tag{2.191}$$

From (2.19), after some computations, we get

$$\tilde{g}'(\xi,\delta) = \frac{\lambda\tilde{p}'(\xi) + [c+2D\xi]\tilde{g}(\xi,\delta)}{(b+\xi)(\rho-\xi)D}, \tag{2.192}$$

and

$$\begin{aligned}\tilde{g}(\xi,\delta) - 1 &= \frac{\lambda\tilde{p}(\xi) + D\xi^2 + c\xi - \delta - \lambda}{(b+\xi)(\rho-\xi)D} \\ &= \frac{\lambda\tilde{p}(\xi) - \tau(\xi)}{(b+\xi)(\rho-\xi)D}.\end{aligned} \tag{2.193}$$

Then from (2.22), (2.190) and (2.193), we obtain

$$\tilde{\phi}_w(\xi,\delta) = \frac{\tilde{g}_\omega(\xi,\delta)}{1-\tilde{g}(\xi,\delta)} = \frac{\lambda[\tilde{\omega}(\xi) - \tilde{\omega}(\rho)]}{\tau(\xi) - \lambda\tilde{p}(\xi)}. \tag{2.194}$$

If $w(x,y) = 1$, then $\phi_w(u) = \phi_s(u)$, $\omega(x) = \overline{P}(x)$, $\tilde{\omega}(s) = \int_0^\infty e^{-sx}\overline{P}(x)dx = \frac{1-\tilde{p}(s)}{s}$ and

$$\tilde{\phi}_s(\xi,\delta) = \frac{\lambda}{\tau(\xi) - \lambda\tilde{p}(\xi)}\left[\frac{1-\tilde{p}(\xi)}{\xi} - \frac{1-\tilde{p}(\rho)}{\rho}\right]$$

$$= \frac{\lambda\rho[1-\tilde{p}(\xi)] + \xi(\delta - c\rho - D\rho^2)}{\xi[\lambda\rho(1-\tilde{p}(\xi)) + \rho(\delta - c\xi - D\xi^2)]} \tag{2.195}$$

where $\tilde{\phi}_s(\xi, \delta) = \int_0^\infty e^{-\xi u}\phi_s(u)du$. If we further let $\delta \to 0$, then from (2.188)

$$\tilde{\phi}_s(\xi,\delta) = \frac{\frac{\lambda}{\xi}[1-\tilde{p}(\xi)] + \frac{\delta}{\rho} - c - \rho D}{\lambda[1-\tilde{p}(\xi)] + \delta - c\xi - D\xi^2}$$

$$\to \quad \tilde{\psi}_s(\xi) = \frac{\frac{\lambda}{\xi}[1-\tilde{p}(\xi)] - \lambda p_1}{\lambda[1-\tilde{p}(\xi)] - c\xi - D\xi^2} = \frac{\lambda[1-p_1\xi - \tilde{p}(\xi)]}{\xi[\lambda(1-\tilde{p}(\xi)) - c\xi - D\xi^2]} \tag{2.196}$$

where $\tilde{\psi}_s(\xi) = \int_0^\infty e^{-\xi u}\psi_s(u)du$. Moreover, if $D \to 0$, then equations (2.195) and (2.196) reduce to (2.59) and (2.60) of Gerber and Shiu (1998a) [29], respectively.

Now to obtain the asymptotic formula for $\phi_w(u)$ satisfying (2.15), $\phi_w(u) = \phi_w * g(u) + g_\omega(u)$, we seek a $\kappa > 0$, called the adjustment coefficient, such that $\int_0^\infty e^{\kappa x} g(x)dx = \tilde{g}(-\kappa, \delta) = 1$. By (2.193), $\tilde{g}(-\kappa, \delta) = 1$ is equivalent to $\lambda\tilde{p}(-\kappa) = \tau(-\kappa)$. Therefore $-\kappa$ is exactly ξ_2, the unique negative root of (2.187), which has been mentioned by Gerber and Landry (1998) [24] too.

With the asymptotic formula (1.46) for a renewal equation, we can apply it to (2.17) and obtain the following theorem.

Theorem 2.4 *The asymptotic formula for $\phi(u)$ satisfying (2.17) is*

$$\phi(u) \sim \frac{\lambda\int_0^\infty (e^{\kappa x} - e^{-\rho x})\int_x^\infty w(x, y-x)p(y)dydx + w_0(\rho+\kappa)D}{-\lambda\tilde{p}'(-\kappa) - c + 2\kappa D} e^{-\kappa u}, \text{ as } u \to \infty, \tag{2.197}$$

where $-\tilde{p}'(-\kappa) = \int_0^\infty x e^{\kappa x} dP(x)$, $\kappa = -\xi_2$ and ξ_2 is the unique negative root of (2.187).

Proof: From (2.22), (2.192), $\tilde{g}(-\kappa,\delta) = 1$ and the fact that $b > \kappa$, we have as $u \to \infty$

$$\begin{aligned}
\phi(u) &\sim \frac{\int_0^\infty e^{\kappa x}\left[g_\omega(x) + w_0 e^{-bx}\right]dx}{\int_0^\infty x e^{\kappa x} g(x)dx} e^{-\kappa u} \\
&= \frac{\tilde{g}_\omega(-\kappa,\delta) + \frac{w_0}{b-\kappa}}{-\tilde{g}'(-\kappa,\delta)} e^{-\kappa u} \\
&= \frac{\lambda[\tilde{\omega}(-\kappa) - \tilde{\omega}(\rho)] + w_0(\rho+\kappa)D}{-\lambda\tilde{p}'(-\kappa) - c + 2\kappa D} e^{-\kappa u} \\
&= \frac{\lambda \int_0^\infty (e^{\kappa x} - e^{-\rho x}) \int_x^\infty w(x, y-x)p(y)dydx + w_0(\rho+\kappa)D}{-\lambda\tilde{p}'(-\kappa) - c + 2\kappa D} e^{-\kappa u}.
\end{aligned}$$

□

Note that when $D = 0$, (2.197) reduces to (4.10) of Gerber and Shiu (1998a) [29], which agrees with the fact that the defective renewal equation (1.33) is a special case of (2.17).

Corollary 2.6 *If $w(x,y) = 1$, then the asymptotic formula for $\phi(u)$ satisfying (2.17) is*

$$\phi(u) \sim \frac{[\delta + (w_0 - 1)\rho\kappa D](\rho+\kappa)}{\rho\kappa[-\lambda\tilde{p}'(-\kappa) - c + 2\kappa D]} e^{-\kappa u}, \quad \text{as } u \to \infty, \tag{2.198}$$

where $-\tilde{p}'(-\kappa) = \int_0^\infty x e^{\kappa x} dP(x)$, $\kappa = -\xi_2$ and ξ_2 is the unique negative root of (2.187).

Proof: If $w(x,y) = 1$, from (2.197), integration by parts and that both ρ and $-\kappa$ satisfy (2.187), we have as $u \to \infty$

$$
\begin{aligned}
\phi(u) &\sim \frac{\lambda\int_0^\infty (e^{\kappa x}-e^{-\rho x})\int_0^\infty p(x+y)dydx + w_0(\rho+\kappa)D}{-\lambda\tilde{p}'(-\kappa)-c+2\kappa D}e^{-\kappa u} \\
&= \frac{\lambda\int_0^\infty (e^{\kappa x}-e^{-\rho x})\overline{P}(x)dx + w_0(\rho+\kappa)D}{-\lambda\tilde{p}'(-\kappa)-c+2\kappa D}e^{-\kappa u} \\
&= \frac{\frac{\lambda[\tilde{p}(-\kappa)-1]}{\kappa}+\frac{\lambda[\tilde{p}(\rho)-1]}{\rho} + w_0(\rho+\kappa)D}{-\lambda\tilde{p}'(-\kappa)-c+2\kappa D}e^{-\kappa u} \\
&= \frac{\frac{\delta+c\kappa-D\kappa^2}{\kappa}+\frac{\delta-c\rho-D\rho^2}{\rho} + w_0(\rho+\kappa)D}{-\lambda\tilde{p}'(-\kappa)-c+2\kappa D}e^{-\kappa u} \\
&= \frac{[\delta+(w_0-1)\rho\kappa D](\rho+\kappa)}{\rho\kappa[-\lambda\tilde{p}'(-\kappa)-c+2\kappa D]}e^{-\kappa u}.
\end{aligned}
$$

□

Note that if $D=0$, let $\delta\to 0$, then $\frac{[\delta+(w_0-1)\rho\kappa D](\rho+\kappa)}{\rho\kappa} = \frac{\delta}{\kappa}+\frac{\delta}{\rho} \to c-\lambda p_1$ by (2.188), and (2.198) implies $\phi_0(u) \sim \frac{c-\lambda p_1}{-\lambda\tilde{p}'(-\kappa)-c}e^{-\kappa u}$, which is exactly the equation (4.15) of Gerber and Shiu (1998a) [29].

Recall that if $w_0 = \frac{1}{1+\beta}$ and $w(x,y)=1$, then $\phi(u)=\overline{K}(u)$. Now we have the asymptotic formula for $\overline{K}(u)$ as follows:

Corollary 2.7 *If $w_0 = \frac{1}{1+\beta}$ and $w(x,y)=1$, then the asymptotic formula for $\overline{K}(u)$ satisfying (1.39) is $\overline{K}(u)\sim Ce^{-\kappa u}$, as $u\to\infty$, where*

$$
C = \frac{\left[\delta - \frac{\beta}{1+\beta}\rho\kappa D\right](\rho+\kappa)}{\rho\kappa[-\lambda\tilde{p}'(-\kappa)-c+2\kappa D]} = \frac{\delta\left[1-\frac{\rho\kappa D}{D\rho^2+c\rho}\right](\rho+\kappa)}{\rho\kappa[-\lambda\tilde{p}'(-\kappa)-c+2\kappa D]}, \tag{2.199}
$$

$-\tilde{p}'(-\kappa) = \int_0^\infty xe^{\kappa x}dP(x)$, $\kappa=-\xi_2$ and ξ_2 is the unique negative root of (2.187).

Proof: Substitute $\frac{1}{1+\beta}$ for w_0 in (2.198), then from (2.35) we obtain (2.199). □

From both a numerical and an analytical viewpoint, Tijms (1986) [43] suggested approximating $\overline{K}(u)$ by

$$\overline{K}_T(u) = \left(\frac{1}{1+\beta} - C\right)e^{-\alpha u} + Ce^{-\kappa u} = \left(\frac{D\rho^2 + c\rho - \delta}{D\rho^2 + c\rho} - C\right)e^{-\alpha u} + Ce^{-\kappa u}, \quad u \geq 0, \tag{2.200}$$

where α is chosen so that the approximation preserves the mean $\int_0^\infty \overline{K}_T(u)du = \int_0^\infty \overline{K}(u)du$, i.e.

$$\alpha = \left[\frac{1}{1+\beta} - C\right]\left[\int_0^\infty \overline{K}(u)du - \frac{C}{\kappa}\right]^{-1}, \tag{2.201}$$

with $\int_0^\infty \overline{K}(u)du = \frac{\mu_{G,1}(\rho)}{\beta}$ by (2.95) and $\mu_{G,1}(\rho)$ given in (2.75).

Note that the approximation preserves the true value $\overline{K}(0) = (1+\beta)^{-1}$, the mean, and the asymptotic right tail behavior as $u \to \infty$ if $\alpha > \kappa$. Moreover, $\overline{K}_T(u)$ is exactly equal to $\overline{K}(u)$ if $\overline{P}(x)$ (or more generally, $\overline{K}(u)$) is the sum or mixture of two exponentials.

2.6 Reliability-based class implication and bound

In this section, we are going to propose reliability-based class implications between $P(x)$ and $G(x)$. A lower bound and an upper bound on the associated compound geometric distribution function $\overline{K}(u)$ in (2.87) can be developed provided the claim size distribution $P(x)$ is in some of the reliability-based classes.

Recall we have defined the mean residual lifetime and the failure rate of the claim size distribution function $P(x)$ in the last section of the previous chapter.

Since $\overline{P}_1(x) = \int_x^\infty \overline{P}(t)dt \Big/ \int_0^\infty \overline{P}(t)dt = \int_x^\infty \overline{P}(t)dt/p_1$ by (1.54) with $n = 1$, $r_P(x)$ in (1.48) becomes $r_P(x) = p_1\overline{P}_1(x)/\overline{P}(x)$. By L'Hopital's rule,

$$r_P(\infty) = \lim_{x\to\infty} r_P(x) = \lim_{x\to\infty} \frac{-\overline{P}(x)}{-P'(x)} = \lim_{x\to\infty} \frac{1}{h_P(x)} = \frac{1}{h_P(\infty)}, \tag{2.202}$$

provided $r_P(\infty)$ and $h_P(\infty)$ are well defined.

Similar to (1.48), the mean residual lifetime of the distribution $\Gamma(x)$ is defined by

$$r_\Gamma(x) = \frac{\int_x^\infty \overline{\Gamma}(t)dt}{\overline{\Gamma}(x)} = \frac{\int_0^\infty \overline{\Gamma}(x+t)dt}{\overline{\Gamma}(x)}; \tag{2.203}$$

and $r_\Gamma(x) = \mu_{\Gamma,1}(\rho)\overline{\Gamma}_1(x)/\overline{\Gamma}(x)$ by (1.58). From (1.52) and (1.55),

$$\mu_{\Gamma,1}(\rho) = \int_0^\infty \overline{\Gamma}(x)dx = \frac{\int_0^\infty e^{-\rho y}\int_0^\infty \overline{P}(x+y)dxdy}{\int_0^\infty e^{-\rho y}\overline{P}(y)dy} = \frac{p_1\int_0^\infty e^{-\rho y}\overline{P}_1(y)dy}{\int_0^\infty e^{-\rho y}\overline{P}(y)dy}. \tag{2.204}$$

Then by (1.57),

$$r_\Gamma(x) = \frac{p_1\overline{\Gamma}_1(x)\int_0^\infty e^{-\rho y}\overline{P}_1(y)dy}{\overline{\Gamma}(x)\int_0^\infty e^{-\rho y}\overline{P}(y)dy} = \frac{p_1\int_x^\infty e^{-\rho y}\overline{P}_1(y)dy}{\int_x^\infty e^{-\rho y}\overline{P}(y)dy} = \frac{\int_x^\infty e^{-\rho y}\overline{P}(y)r_P(y)dy}{\int_x^\infty e^{-\rho y}\overline{P}(y)dy}, \tag{2.205}$$

a weighed average of values of $r_P(x)$ with weights proportional to $e^{-\rho x}\overline{P}(x)$. By L'Hopital's rule,

$$r_\Gamma(\infty) = \lim_{x\to\infty} r_\Gamma(x) = \lim_{x\to\infty} \frac{-e^{-\rho x}\overline{P}(x)r_P(x)}{-e^{-\rho x}\overline{P}(x)} = \lim_{x\to\infty} r_P(x) = r_P(\infty) \tag{2.206}$$

provided $r_P(\infty)$ is well defined.

From (2.48), we have the failure rate of $G(x) = \Gamma * H(x)$,

$$h_G(x) = \frac{G'(x)}{\overline{G}(x)} = b\Big[1 - \frac{\overline{\Gamma}(x)}{\overline{G}(x)}\Big] \tag{2.207}$$

with $0 \leq h_G(x) \leq b = h_H(x)$ and $h_G(0) = 0$ since $G'(0) = 0$, and the mean residual lifetime of the distribution $G(x)$,

$$r_G(x) = \frac{\int_x^\infty \overline{G}(t)dt}{\overline{G}(x)} = \frac{1}{b} + \frac{\int_x^\infty \overline{\Gamma}(t)dt}{\overline{G}(x)} = \frac{1}{b} + \Big[1 - \frac{h_G(x)}{b}\Big] r_\Gamma(x) \tag{2.208}$$

with the help of (2.48), (2.203) and (2.207).

Note that $r_H(x) = \frac{1}{b} \leq r_G(x) \leq \frac{1}{b} + r_\Gamma(x) = r_H(x) + r_\Gamma(x)$ and $r_G(0) = \mu_{G,1}(\rho)$. In addition, Since $r_G(\infty) = 1/h_G(\infty)$, from (2.208) $r_G(\infty) = r_H(\infty) + \Big[1 - \frac{r_H(\infty)}{r_G(\infty)}\Big] r_\Gamma(\infty)$, or equivalently,

$$r_G(\infty) = r_H(\infty) = \frac{1}{b} \quad or \quad r_G(\infty) = r_\Gamma(\infty) = r_P(\infty) \tag{2.209}$$

provided $r_G(\infty)$, $r_\Gamma(\infty)$ and $r_P(\infty)$ are well defined.

Similar to Theorems 1.3 and 1.4 which are concerning class implications between $P(x)$ and $\Gamma(x)$, we have the following theorem regarding class implications between $P(x)$ and $G(x)$ too.

Theorem 2.5 *The following class implications hold.*

(a) If $P(x)$ is IFR then $G(x)$ is IFR.

(b) If $P(x)$ is DMRL then $G(x)$ is DMRL.

(c) If $P(x)$ is UBA then $G(x)$ is UBA.

(d) If $P(x)$ is UBAE then $G(x)$ is UBAE.

(e) If $P(x)$ is 2-NBU then $G(x)$ is NBUE.

Proof: If $P(x)$ is IFR (DFR) then $\Gamma(x)$ is IFR (DFR) by Theorem 1.3.(a). Based on the facts that convolution preserves the IFR property (that is, if X and Y are independent with IFR distribution functions, then X+Y has an IFR distribution function), since $G(x) = H * \Gamma(x)$ by (2.40), the convolution of distribution functions $H(x)$ and $\Gamma(x)$, and $H(x) = 1 - e^{-bx}$ is IFR, we have that if $P(x)$ is IFR, so are $\Gamma(x)$ and $G(x)$, proving (a).

From Theorem 1.3.(b), if $P(x)$ is DMRL (IMRL) then $\Gamma(x)$ is DMRL (IMRL). Since $H(x)$ is IFR, by Bondesson (1983) [2], $G(x) = \Gamma * H(x)$ is DMRL, proving (b).

By Theorem 1.4, if $P(x)$ is UWA (UBA) then $\Gamma(x)$ is UWA (UBA) and that if $P(x)$ is UWAE (UBAE) then $\Gamma(x)$ is UWAE (UBAE). Since $H(x)$ is both UBA and UBAE, and UBA and UBAE are closed under the convolution operation by Alzaid (1994) [1], we get that $G(x) = \Gamma * H(x)$ is UBA (UBAE) if $P(x)$ is UBA (UBAE), proving (c) and (d).

If $P(x)$ is 2-NBU (2-NWU) then $\Gamma(x)$ is NBUE (NWUE) from Theorem 1.3.(c), that is, $r_\Gamma(x) \le (\ge) r_\Gamma(0)$. From (2.208), $r_G(x) \le \frac{1}{b} + r_\Gamma(x) \le \frac{1}{b} + r_\Gamma(0) = r_G(0)$, that is, $G(x)$ is NBUE, proving (e). □

Lin (1996) [35] demonstrated that if the mean residual lifetime of $G(x)$ satisfies $0 \le r_1 \le r_G(x) \le r_2 < \infty$ then

$$(1 - \kappa r_2)e^{-\kappa u} \le \overline{K}(u) \le (1 - \kappa r_1)e^{-\kappa u}, \quad u \ge 0. \tag{2.210}$$

In addition, Lin and Willmot (1999) [36] showed that if $P(x)$ is 2-NWU (2-NBU) then $r_\Gamma(x) \geq (\leq) \mu_{\Gamma,1}(\rho)$, and that if $P(x)$ is NWUC (NBUC) then $r_\Gamma(x) \geq (\leq) 1/h_\Gamma(0)$.

Therefore, since $\frac{1}{b} \leq r_G(x) \leq \frac{1}{b} + r_\Gamma(x)$,
if $P(x)$ is 2-NBU then $\frac{1}{b} \leq r_G(x) \leq \frac{1}{b} + \mu_{\Gamma,1}(\rho) = \mu_{G,1}(\rho)$ and

$$[1 - \kappa\mu_{G,1}(\rho)]e^{-\kappa u} \leq \overline{K}(u) \leq [1 - \kappa/b]e^{-\kappa u}, \quad u \geq 0; \tag{2.211}$$

if $P(x)$ is NBUC then $\frac{1}{b} \leq r_G(x) \leq \frac{1}{b} + \frac{1}{h_\Gamma(0)}$ and

$$\left\{1 - \kappa\left[\frac{1}{b} + \frac{1}{h_\Gamma(0)}\right]\right\}e^{-\kappa u} \leq \overline{K}(u) \leq [1 - \kappa/b]e^{-\kappa u}, \quad u \geq 0; \tag{2.212}$$

if $P(x)$ is UWAE with $r_P(\infty) \in (0, \infty)$ then $\Gamma(x)$ is UWAE (that is, $r_\Gamma(x) \leq r_\Gamma(\infty)$) by Willmot and Cai (1999) [46]. Hence $\frac{1}{b} \leq r_G(x) \leq \frac{1}{b} + r_\Gamma(\infty) = \frac{1}{b} + r_P(\infty)$ by (2.206), and

$$\left\{1 - \kappa\left[\frac{1}{b} + r_P(\infty)\right]\right\}e^{-\kappa u} \leq \overline{K}(u) \leq [1 - \kappa/b]e^{-\kappa u}, \quad u \geq 0; \tag{2.213}$$

whereas if $P(x)$ is UBAE with $r_P(\infty) \in (0, \infty)$, Alzaid (1994) [1] showed that $r_G(x) = \frac{\int_0^\infty \overline{G}(x+y)dy}{\overline{G}(x)} \geq \frac{1}{min[h_\Gamma(\infty), h_H(\infty)]} = \frac{1}{min[h_P(\infty), h_H(\infty)]} = \frac{1}{min[1/r_P(\infty), 1/h_H(\infty)]} = max[r_P(\infty), r_H(\infty)] = max[r_P(\infty), 1/b]$ with the help of (2.202) and (2.206), and

$$\overline{K}(u) \leq \{1 - \kappa[max(r_P(\infty), 1/b)]\}e^{-\kappa u}, \quad u \geq 0, \tag{2.214}$$

a tighter upper bound than $[1-\kappa/b]e^{-\kappa u}$.

Note that no matter what class the distribution $P(x)$ is, we always have the upper bound $[1-\frac{\kappa}{b}]e^{-\kappa u}$ for $\overline{K}(u)$. Moreover, the bounds in (2.210) may be improved. For example, if $G(x)$ is IMRL (DMRL), then the factor $1-\kappa r_1$ $(1-\kappa r_2)$ may be replaced by $(1+\beta)^{-1}$. See Lin (1996) [35] and Willmot (1997) [45] for details. Since if $P(x)$ is IFR then $G(x)$ is IFR, which implies that $G(x)$ is DMRL, we have that $(1+\beta)^{-1}e^{-\kappa u} \leq \overline{K}(u)$.

Chapter 3

Moments

In this chapter, we first derive the expression for the (discounted) moments of deficit at the time of ruin. An upper bound is also given if the claim size distribution function satisfies a certain condition. Next, we will show that the joint moment of the penalty function and the time of ruin due to a claim satisfies a defective renewal equation, and has an explicit expression. Then the joint moment of the deficit at ruin and the time of ruin is just a special case by appropriate choice in the penalty function $w(x, y)$. Finally, the moments of the time of ruin due to oscillation and caused by a claim, respectively, are studied. We also find that these two kinds of moments of the time of ruin have the same recursive expressions. The expressions for these functions based on the classical risk model without a diffusion process have been proposed in Lin and Willmot (1999) [36], and Picard and Lefèvre (1998) [38] and (1999) [39].

3.1 A technical preliminary

Recall that $\overline{\Gamma}_n(x)$ is the n^{th} equilibrium tail of the distribution function $\Gamma(x) = \Gamma_0(x)$. If we define $\gamma_{-1}(\rho) = p_1$ and $\gamma_n(\rho) = \int_0^\infty \overline{\Gamma}_n(x)dx$ for $n = 0, 1, 2, \ldots$ (that is, $\gamma_n(\rho)$ is the mean of the n^{th} equilibrium distribution function $\overline{\Gamma}_n(x)$ of $\Gamma(x)$), then from (1.54), (1.56) and (2.78) we have

$$\begin{aligned}
\gamma_n(\rho) &= \int_0^\infty \overline{\Gamma}_n(x)dx = \frac{\int_0^\infty e^{\rho x}\int_x^\infty e^{-\rho y}\overline{P}_n(y)dydx}{\int_0^\infty e^{-\rho y}\overline{P}_n(y)dy} \\
&= \frac{\int_0^\infty\int_0^\infty e^{-\rho y}\overline{P}_n(x+y)dydx}{\int_0^\infty e^{-\rho y}\overline{P}_n(y)dy} = \frac{\int_0^\infty e^{-\rho y}\int_0^\infty \overline{P}_n(x+y)dxdy}{\int_0^\infty e^{-\rho y}\overline{P}_n(y)dy} \\
&= \frac{\int_0^\infty e^{-\rho y}\int_y^\infty \overline{P}_n(x)dxdy}{\int_0^\infty e^{-\rho y}\overline{P}_n(y)dy} = \frac{p_{n+1}}{(n+1)p_n}\frac{\int_0^\infty e^{-\rho y}\overline{P}_{n+1}(y)dy}{\int_0^\infty e^{-\rho y}\overline{P}_n(y)dy} \\
&= \frac{p_{n+2}}{(n+2)p_{n+1}}\frac{\int_0^\infty e^{-\rho y}dP_{n+2}(y)}{\int_0^\infty e^{-\rho y}dP_{n+1}(y)} = \frac{\mu_{\Gamma,n+1}(\rho)}{(n+1)\mu_{\Gamma,n}(\rho)}, \quad n = 0, 1, \ldots . \qquad (3.1)
\end{aligned}$$

Note that if $\delta = 0$, which implies $\rho = 0$, then $\mu_{\Gamma,n}(0) = \frac{p_{n+1}}{(n+1)p_1}$ by (2.77), and

$$\gamma_n(0) = \frac{p_{n+2}}{(n+2)p_{n+1}}. \qquad (3.2)$$

Therefore $\overline{\Gamma}_{n+1}(x)$ in (1.58) can be expressed as

$$\overline{\Gamma}_{n+1}(x) = \frac{\int_x^\infty \overline{\Gamma}_n(y)dy}{\gamma_n(\rho)} = \frac{(n+1)\mu_{\Gamma,n}(\rho)}{\mu_{\Gamma,n+1}(\rho)}\int_x^\infty \overline{\Gamma}_n(y)dy, \quad n = 0, 1, 2, \ldots, \qquad (3.3)$$

Since $\Gamma_{n+1} * H(u) = \int_0^u \Gamma_{n+1}(u-s)H'(s)ds = b\int_0^u \Gamma_{n+1}(u-s)e^{-bs}ds$, then by (3.3)

$$\frac{d\Gamma_{n+1} * H(u)}{du} = b\Gamma_{n+1}(0)e^{-bu} + \frac{b\int_0^u \overline{\Gamma}_n(u-s)e^{-bs}ds}{\gamma_n(\rho)} = \frac{\overline{\Gamma}_n * H(u)}{\gamma_n(\rho)}. \tag{3.4}$$

Hence, we get

$$\int_u^\infty \overline{\Gamma}_n * H(x)dx = \gamma_n(\rho)\int_u^\infty d\Gamma_{n+1} * H(x) = \gamma_n(\rho)\overline{\Gamma_{n+1} * H}(u), \tag{3.5}$$

where

$$\overline{\Gamma_{n+1} * H}(u) = 1 - \Gamma_{n+1} * H(u) = \overline{H}(u) + H(u) - \Gamma_{n+1} * H(u) = \overline{H}(u) + \overline{\Gamma}_{n+1} * H(u). \tag{3.6}$$

If we define

$$\alpha_n(u,\rho) = \gamma_n(\rho)\Big[\int_0^u \overline{K}(u-x)d\Gamma_{n+1} * H(x) + \overline{\Gamma_{n+1} * H}(u)\Big], \quad n = -1, 0, 1, 2, \ldots, \tag{3.7}$$

with $\alpha_n(0,\rho) = \gamma_n(\rho)$, then by (3.4) and (3.5),

$$\alpha_n(u,\rho) = \int_0^u \overline{K}(u-x)\overline{\Gamma}_n * H(x)dx + \int_u^\infty \overline{\Gamma}_n * H(x)dx, \quad n = 0, 1, 2, \ldots, \tag{3.8}$$

and since $\gamma_{-1}(\rho) = p_1$, $G(x) = \Gamma * H(x) = \Gamma_0 * H(x)$, by (1.39),

$$\alpha_{-1}(u,\rho) = p_1(1+\beta)\Big[\int_0^u \overline{K}(u-x)dG(x) + \overline{G}(u)\Big] = p_1(1+\beta)\overline{K}(u). \tag{3.9}$$

Since $\overline{\Gamma}_n * H(u) = b\int_0^u e^{-b(u-x)}\overline{\Gamma}_n(x)dx = \dfrac{b\int_0^u e^{-b(u-x)}\int_x^\infty e^{-\rho(y-x)}\overline{P}_n(y)dydx}{\int_0^\infty e^{-\rho y}\overline{P}_n(y)dy}$ has the same form as $g(u) = \frac{\lambda}{D}\int_0^u e^{-b(u-x)}\int_x^\infty e^{-\rho(y-x)}p(y)dydx$, by (2.18) in lemma 2.1, $\overline{\Gamma}_n * H(u)$ can be expressed as

$$\overline{\Gamma}_n * H(u) = \frac{b}{a}\frac{\int_0^u e^{-b(u-y)}\overline{P}_n(y)dy + \int_u^\infty e^{-\rho(y-u)}\overline{P}_n(y)dy - e^{-bu}\int_0^\infty e^{-\rho y}\overline{P}_n(y)dy}{\int_0^\infty e^{-\rho y}\overline{P}_n(y)dy}. \tag{3.10}$$

Therefore, when $D \to 0$, then $b/a \to 1$,

$$\overline{\Gamma}_n * H(u) \to \frac{\int_u^\infty e^{-\rho(y-u)}\overline{P}_n(y)dy}{\int_0^\infty e^{-\rho y}\overline{P}_n(y)dy} = \overline{\Gamma}_n(u), \ \ if\ u > 0, \tag{3.11}$$

and

$$\Gamma_n * H(u) = H(u) - \overline{\Gamma}_n * H(u) \to \Gamma_n(u), \ \ if\ u \geq 0. \tag{3.12}$$

We remark that for the case $\delta = 0$, (3.11) and (3.12) still hold by just replacing $\overline{\Gamma}_n(u)$ by $\overline{P}_{n+1}(u)$, and $H(u)$ by H_1.

Hence, in the case that $D \to 0$, by (2.89), (3.11) and (3.12), equations (3.7), (3.8) and (3.15) become

$$\begin{aligned} &\alpha_{0;n}(u,\rho) \\ = \ &\gamma_n(\rho)\Big[\int_0^u \overline{K}_0(u-x)d\Gamma_{n+1}(x) + \overline{\Gamma}_{n+1}(u)\Big], \ \ n = -1, 0, 1, \ldots, && (3.13) \\ = \ &\int_0^u \overline{K}_0(u-x)\overline{\Gamma}_n(x)dx + \int_u^\infty \overline{\Gamma}_n(x)dx, \ \ n = 0, 1, 2, \ldots, && (3.14) \end{aligned}$$

and

$$\alpha_{0;-1}(u,\rho) = p_1(1+\beta_0)\overline{K}_0(u). \tag{3.15}$$

There is a recursive equation for $\alpha_n(u,\rho)$ as follows:

Lemma 3.1 *For $u \geq 0$*

$$\begin{aligned} & \alpha_{n+1}(u,\rho) \\ = & \frac{1}{\gamma_n(\rho)}\int_u^\infty \alpha_n(t,\rho)dt - \int_u^\infty \overline{K}(t)dt - \frac{1}{b}\overline{K*H}(u) \quad (3.16) \\ = & \frac{1}{\gamma_n(\rho)}\int_u^\infty \alpha_n(t,\rho)dt - \int_u^\infty \overline{K*H}(t)dt, \quad n=-1,0,1,\ldots, \quad (3.17) \end{aligned}$$

where $\overline{K*H}(u)$ can be replaced with $\phi_t(u)$ by (2.103).

Moreover, when $D \to 0$, $\alpha_{n+1}(u,\rho)$ reduces to

$$\alpha_{0;n+1}(u,\rho) = \frac{1}{\gamma_n(\rho)}\int_u^\infty \alpha_{0;n}(t,\rho)dt - \int_u^\infty \overline{K}_0(t)dt, \quad n=-1,0,1,2,\ldots. \tag{3.18}$$

Proof: For n=-1, 0, 1, 2, ..., by interchanging the order of integration, and by integration by parts

$$\begin{aligned} & \int_u^\infty \int_0^t \overline{K}(t-x)d\Gamma_{n+1} * H(x)dt \\ = & \int_0^u \int_u^\infty \overline{K}(t-x)dtd\Gamma_{n+1} * H(x) + \int_u^\infty \int_x^\infty \overline{K}(t-x)dtd\Gamma_{n+1} * H(x) \\ = & \int_0^u \int_{u-x}^\infty \overline{K}(t)dtd\Gamma_{n+1} * H(x) + \left[\int_u^\infty d\Gamma_{n+1} * H(x)\right]\int_0^\infty \overline{K}(t)dt \\ = & \left[\Gamma_{n+1} * H(x)\right]\int_{u-x}^\infty \overline{K}(t)dt\Big|_0^u - \int_0^u \Gamma_{n+1} * H(x)\overline{K}(u-x)dx + \\ & \overline{\Gamma_{n+1} * H}(u)\int_0^\infty \overline{K}(t)dt \end{aligned}$$

$$
\begin{aligned}
&= \Big[\Gamma_{n+1} * H(u)\Big]\int_0^\infty \overline{K}(t)dt - \int_0^u \overline{K}(u-x)\Gamma_{n+1} * H(x)dx + \\
&\quad \overline{\Gamma_{n+1} * H}(u)\int_0^\infty \overline{K}(t)dt \\
&= \int_0^\infty \overline{K}(t)dt - \int_0^u \overline{K}(u-x)H(x)dx + \int_0^u \overline{K}(u-x)\overline{\Gamma}_{n+1} * H(x)dx,
\end{aligned}
$$

combining with $\int_u^\infty \overline{\Gamma_{n+1} * H}(t)dt = \int_u^\infty \overline{H}(t)dt + \int_u^\infty \overline{\Gamma}_{n+1} * H(t)dt$ by (3.6), then from (3.7) and (3.8) we obtain

$$
\begin{aligned}
&\int_u^\infty \alpha_n(t,\rho)dt \\
&= \gamma_n(\rho)\Big[\int_u^\infty\int_0^t \overline{K}(t-x)d\Gamma_{n+1} * H(x)dt + \int_u^\infty \overline{\Gamma_{n+1} * H}(t)dt\Big] \\
&= \gamma_n(\rho)\Big[\alpha_{n+1}(u,\rho) + \int_0^\infty \overline{K}(t)dt - \int_0^u \overline{K}(u-x)H(x)dx + \int_u^\infty \overline{H}(t)dt\Big] \\
&= \gamma_n(\rho)\Big[\alpha_{n+1}(u,\rho) + \Big(\int_u^\infty + \int_0^u\Big)\overline{K}(t)dt - \int_0^u \overline{K}(x)H(u-x)dx + \int_u^\infty \overline{H}(t)dt\Big] \\
&= \gamma_n(\rho)\Big[\alpha_{n+1}(u,\rho) + \int_u^\infty \overline{K}(t)dt + \int_0^u \overline{K}(x)\overline{H}(u-x)dx + \int_u^\infty \overline{H}(t)dt\Big] \\
&= \gamma_n(\rho)\Big[\alpha_{n+1}(u,\rho) + \int_u^\infty \overline{K}(t)dt + \frac{1}{b}\int_0^u \overline{K}(u-x)dH(x) + \frac{1}{b}\overline{H}(u)\Big] \\
&= \gamma_n(\rho)\Big[\alpha_{n+1}(u,\rho) + \int_u^\infty \overline{K}(t)dt + \frac{1}{b}\overline{K * H}(u)\Big],
\end{aligned}
$$

which implies (3.16) as well as (3.17) by (2.107).

When $D \to 0$, then $b \to \infty$, $K(t) \to K_0(t)$ by (2.89), and

$$
\frac{1}{b}\overline{K * H}(u) = \frac{1}{b}\Big[\int_0^u \overline{K}(u-x)dH(x) + \overline{H}(u)\Big] = \int_0^u \overline{K}(u-x)e^{-bx}dx + \frac{1}{b}e^{-bu} \to 0, \tag{3.19}
$$

or

$$
\overline{K * H}(u) = \overline{K}(u) + \overline{H} * K(u) \quad = \quad \overline{K}(u) + \int_0^u e^{-b(u-x)}K'(x)dx + K(0)\overline{H}(u)
$$

$$\to \overline{K}_0(u) \text{ or } 1, \tag{3.20}$$

depending on $u > 0$ or $u = 0$, either implies

$$\alpha_{n+1}(u,\rho)(u) \to \alpha_{0;n+1}(u,\rho)(u) = \frac{1}{\gamma_n(\rho)} \int_u^\infty \alpha_{0;n}(t,\rho)dt - \int_u^\infty \overline{K}_0(t)dt. \qquad \square$$

In addition to expressions (3.8) and (3.16), we also have the following alternative representation for $\alpha_n(u,\rho)$ from lemma 3.1.

Theorem 3.1 *For $u \geq 0$*

$$\alpha_0(u,\rho) = \beta \int_u^\infty \overline{K}(x)dx - \frac{1}{b}\overline{K*H}(u) = \beta \int_u^\infty \overline{K*H}(x)dx - \frac{1+\beta}{b}\overline{K*H}(u), \tag{3.21}$$

and for n=1, 2, 3, ...

$$\begin{aligned} \alpha_n(u,\rho) &= \frac{\beta}{\mu_{\Gamma,n}(\rho)} \int_u^\infty (x-u)^n \overline{K}(x)dx - \frac{n}{b\mu_{\Gamma,n}(\rho)} \int_u^\infty (x-u)^{n-1} \overline{K*H}(x)dx \\ &- \sum_{j=0}^{n-1} \binom{n}{j} \frac{\mu_{\Gamma,n-j}(\rho)}{\mu_{\Gamma,n}(\rho)} \int_u^\infty (x-u)^j \overline{K*H}(x)dx, \end{aligned} \tag{3.22}$$

where $\mu_{\Gamma,n}(\rho)$ is given in (2.70), and $\overline{K*H}(u)$ can be replaced with $\phi_t(u)$ by (2.103). Moreover, when $D \to 0$, (3.21) and (3.22) reduce to

$$\alpha_{0;0}(u,\rho) = \beta_0 \int_u^\infty \overline{K}_0(x)dx, \tag{3.23}$$

and for n=1, 2, 3, ...

$$\alpha_{0;n}(u,\rho) = \frac{\beta_0}{\mu_{\Gamma,n}(\rho)}\int_u^\infty (x-u)^n\overline{K}_0(x)dx - \sum_{j=0}^{n-1}\binom{n}{j}\frac{\mu_{\Gamma,n-j}(\rho)}{\mu_{\Gamma,n}(\rho)}\int_u^\infty (x-u)^j\overline{K}_0(x)dx. \tag{3.24}$$

Proof: Since $\gamma_{-1}(\rho) = p_1$ and $\alpha_{-1}(u,\rho) = p_1(1+\beta)\overline{K}(u)$ from (3.9), with $n = -1$ and by (2.107), equation (3.17) becomes $\alpha_0(u,\rho) = \frac{p_1(1+\beta)}{p_1}\int_u^\infty \overline{K}(x)dx - \int_u^\infty \overline{K*H}(x)dx = \beta\int_u^\infty \overline{K}(x)dx - \frac{1}{b}\overline{K*H}(u) =$
$\beta\int_u^\infty \overline{K*H}(x)dx - \left[\frac{\beta}{b}+\frac{1}{b}\right]\overline{K*H}(u) = \beta\int_u^\infty \overline{K*H}(x)dx - \frac{1+\beta}{b}\overline{K*H}(u).$

We would like to prove (3.22) by induction on n with the help of the following equation. For $j > -1$, by interchanging the order of integration we have

$$\int_u^\infty\int_t^\infty (x-t)^j\overline{K}(x)dxdt = \frac{1}{j+1}\int_u^\infty (x-u)^{j+1}\overline{K}(x)dx. \tag{3.25}$$

Note that (3.25) still holds if we replace $\overline{K}(x)$ by $\overline{K*H}(x)$.

Now by $\gamma_0(\rho) = \mu_{\Gamma,1}(\rho)$ from (3.1), equation (3.17) becomes with $n = 1$

$$\begin{aligned}
&\alpha_1(u,\rho)\\
=\ &\frac{1}{\gamma_0(\rho)}\int_u^\infty \alpha_0(t,\rho)dt - \int_u^\infty \overline{K*H}(t)dt\\
=\ &\frac{\beta}{\gamma_0(\rho)}\int_u^\infty\int_t^\infty \overline{K}(x)dxdt - \frac{1}{b\gamma_0(\rho)}\int_u^\infty \overline{K*H}(t)dt - \int_u^\infty \overline{K*H}(t)dt\\
=\ &\frac{\beta}{\mu_{\Gamma,1}(\rho)}\int_u^\infty (x-u)\overline{K}(x)dx - \frac{1}{b\mu_{\Gamma,1}(\rho)}\int_u^\infty \overline{K*H}(x)dx - \int_u^\infty \overline{K*H}(x)dx,
\end{aligned}$$

which shows (3.22) with $n = 1$.

Assume (3.22) holds for $n = m$, $m > 1$, then by (3.1), (3.17) yields with

$n = m + 1$

$$
\begin{aligned}
& \alpha_{m+1}(u,\rho) \\
= \;& \frac{1}{\gamma_m(\rho)} \int_u^\infty \alpha_m(t,\rho)dt - \int_u^\infty \overline{K * H}(t)dt \\
= \;& \frac{(m+1)\mu_{\Gamma,m}(\rho)}{\mu_{\Gamma,m+1}(\rho)} \frac{\beta}{\mu_{\Gamma,m}(\rho)} \int_u^\infty \int_t^\infty (x-t)^m \overline{K}(x)dxdt - \\
& \frac{(m+1)\mu_{\Gamma,m}(\rho)}{\mu_{\Gamma,m+1}(\rho)} \frac{m}{b\mu_{\Gamma,m}(\rho)} \int_u^\infty \int_t^\infty (x-t)^{m-1} \overline{K * H}(x)dxdt - \int_u^\infty \overline{K * H}(t)dt \\
& - \sum_{j=0}^{m-1} \binom{m}{j} \frac{(m+1)\mu_{\Gamma,m}(\rho)}{\mu_{\Gamma,m+1}(\rho)} \frac{\mu_{\Gamma,m-j}(\rho)}{\mu_{\Gamma,m}(\rho)} \int_u^\infty \int_t^\infty (x-t)^j \overline{K * H}(x)dxdt \\
= \;& \frac{\beta}{\mu_{\Gamma,m+1}(\rho)} \int_u^\infty (x-u)^{m+1} \overline{K}(x)dx - \frac{m+1}{b\mu_{\Gamma,m+1}(\rho)} \int_u^\infty (x-u)^m \overline{K * H}(x)dx - \\
& \sum_{j=0}^{m-1} \binom{m+1}{j+1} \frac{\mu_{\Gamma,m-j}(\rho)}{\mu_{\Gamma,m+1}(\rho)} \int_u^\infty (x-u)^{j+1} \overline{K * H}(x)dx - \int_u^\infty \overline{K * H}(x)dx \\
= \;& \frac{\beta}{\mu_{\Gamma,m+1}(\rho)} \int_u^\infty (x-u)^{m+1} \overline{K}(x)dx - \frac{m+1}{b\mu_{\Gamma,m+1}(\rho)} \int_u^\infty (x-u)^m \overline{K * H}(x)dx - \\
& \sum_{j=0}^{m} \binom{m+1}{j} \frac{\mu_{\Gamma,m+1-j}(\rho)}{\mu_{\Gamma,m+1}(\rho)} \int_u^\infty (x-u)^j \overline{K * H}(x)dx,
\end{aligned}
$$

which proves (3.22).

When $D \to 0$, $\beta \to \beta_0$, (3.21) $\to$ (3.23) by (2.89) and (3.19), and (3.22) $\to$ (3.24) by (2.89) and (3.20), and since the second term of the right side of (3.22) approaches to 0. □

From (3.21), the defective tail probability of $\overline{K}(u)$ is

$$
\int_u^\infty \overline{K}(x)dx = \frac{1}{\beta}\Big[\alpha_0(u,\rho) + \frac{1}{b}\overline{K * H}(u)\Big] = \frac{1}{\beta}\Big[\alpha_0(u,\rho) + \frac{1}{b}\phi_t(u)\Big]. \tag{3.26}
$$

Since $\overline{K}(u) = \frac{1}{1+\beta}\phi_d(u) + \phi_s(u)$, by (2.110) and (3.26), the defective tail discounted (with discount factor δ) probability of the time of ruin caused by a claim is

$$\int_u^\infty \phi_s(x)dx = \int_u^\infty \overline{K}(x)dx - \frac{1}{1+\beta}\int_u^\infty \phi_d(x)dx = \frac{\alpha_0(u,\rho)}{\beta} \tag{3.27}$$

where $\alpha_0(u,\rho)$ is explicitly given by (3.7) or (3.8) with $n = 0$.

In the special case $\delta = 0$, which implies $\rho = 0$, then $\beta = \theta$, $\overline{H}(u) = \overline{H}_1(u)$, $\overline{\Gamma}_n(x) = \overline{P}_{n+1}(x)$, $\gamma_n(0) = \frac{p_{n+2}}{(n+2)p_{n+1}}$ by (3.2), and $\overline{K}_{\delta=0}(u) = \frac{1}{1+\theta}\psi_d(u)+\psi_s(u)$ by (2.93). In addition, $\overline{K * H}(u) = \phi_t(u)$ turns out to be $\overline{K_{\delta=0} * H_1}(u) = \psi_t(u)$.

If we denote $\tau_n(u) = \alpha_{n-1}(u,0)$, $n = 0,1,2,\ldots$, then with $H(u) = H_1(u)$, (3.7), (3.8) and (3.9) become

$$\tau_0(u) = p_1(1+\theta)\overline{K}_{\delta=0}(u) = p_1\Big[\psi_d(u) + (1+\theta)\psi_s(u)\Big], \tag{3.28}$$

and for $n = 1,2,3,\ldots$,

$$\tau_n(u) = \frac{p_{n+1}}{(n+1)p_n}\left\{\int_0^u \overline{K}_{\delta=0}(u-x)dP_{n+1} * H_1(x) + \overline{P_{n+1} * H_1}(u)\right\}, \tag{3.29}$$

or equivalently,

$$\tau_n(u) = \int_0^u \overline{K}_{\delta=0}(u-x)\overline{P}_n * H_1(x)dx + \int_u^\infty \overline{P}_n * H_1(x)dx, \tag{3.30}$$

with $\overline{K}_{\delta=0}(u)$ given in (2.93) and

$$\tau_n(0) = \int_0^\infty \overline{P}_n * H_1(x)dx = \frac{p_{n+1}}{(n+1)p_n}. \tag{3.31}$$

Note that when $D \to 0$, $\overline{K}_{\delta=0}(u) \to \psi_0(u)$ by (2.92), and from (3.11) and (3.12), equations (3.28), (3.29) and (3.30) reduce to

$$\tau_{0;0}(u) = p_1(1+\theta)\psi_0(u), \quad u \geq 0, \tag{3.32}$$

and for $n = 1, 2, 3, \ldots$,

$$\begin{aligned} \tau_{0;n}(u) &= \frac{p_{n+1}}{(n+1)p_n}\left\{\int_0^u \psi_0(u-x)dP_{n+1}(x) + \overline{P}_{n+1}(u)\right\} && (3.33) \\ &= \int_0^u \psi_0(u-x)\overline{P}_n(x)dx + \int_u^\infty \overline{P}_n(x)dx, \quad u \geq 0. && (3.34) \end{aligned}$$

Furthermore, when $D \to 0$, by (3.20) with H replaced by H_1

$$\overline{K_{\delta=0} * H_1}(u) \to \overline{K}_{0,\delta=0}(u) = \psi_0(u) \text{ or } 1, \tag{3.35}$$

depending on $u > 0$ or $u = 0$.

In the special case $\delta = 0$ and by (2.77), (3.2) and (3.35), lemma 3.1 and theorem 3.1 respectively become

Corollary 3.1 *For $u \geq 0$*

$$\tau_{n+1}(u) = \frac{(n+1)p_n}{p_{n+1}}\int_u^\infty \tau_n(t)dt - \int_u^\infty \overline{K_{\delta=0} * H_1}(t)dt, \quad n = 0, 1, 2, \ldots, \tag{3.36}$$

where $\overline{K_{\delta=0} * H_1}(u)$ can be replaced with $\psi_t(u)$ by (2.105).

Moreover, when $D \to 0$, $\tau_{n+1}(u)$ reduces to

$$\tau_{0;n+1}(u) = \frac{(n+1)p_n}{p_{n+1}} \int_u^\infty \tau_{0;n}(t)dt - \int_u^\infty \psi_0(t)dt, \quad n = 0, 1, 2, \ldots. \tag{3.37}$$

Corollary 3.2 *For* $u \geq 0$

$$\begin{aligned} \tau_1(u) &= \theta \int_u^\infty \overline{K}_{\delta=0}(x)dx - \frac{D}{c}\overline{K_{\delta=0} * H_1}(u) \\ &= \theta \int_u^\infty \overline{K_{\delta=0} * H_1}(x)dx - \frac{D}{\lambda p_1}\overline{K_{\delta=0} * H_1}(u), \end{aligned} \tag{3.38}$$

and for n=2, 3, 4, ...

$$\begin{aligned} &\tau_n(u) \\ = \; & \frac{np_1\theta}{p_n} \int_u^\infty (x-u)^{n-1}\overline{K}_{\delta=0}(x)dx - \frac{n(n-1)Dp_1}{cp_n} \int_u^\infty (x-u)^{n-2}\overline{K_{\delta=0} * H_1}(x)dx \\ - \; & \sum_{j=0}^{n-2} \binom{n}{j} \frac{p_{n-j}}{p_n} \int_u^\infty (x-u)^j \overline{K_{\delta=0} * H_1}(x)dx, \end{aligned} \tag{3.39}$$

where $\overline{K}_{\delta=0}(u)$ *is given in (2.93), and* $\overline{K_{\delta=0} * H_1}(u)$ *can be replaced with* $\psi_t(u)$ *by (2.105).*

Moreover, when $D \to 0$*, (3.38) and (3.39) reduce to*

$$\tau_{0;1}(u) = \theta \int_u^\infty \psi_0(x)dx, \tag{3.40}$$

and for n=2, 3, 4,...

$$\tau_{0;n}(u) = \frac{np_1\theta}{p_n}\int_u^\infty (x-u)^{n-1}\psi_0(x)dx - \sum_{j=0}^{n-2}\binom{n}{j}\frac{p_{n-j}}{p_n}\int_u^\infty (x-u)^j\psi_0(x)dx. \quad (3.41)$$

Example 3.1 Combination of exponentials and mixture of Erlangs ($b \neq \mu$)

As shown in example 2.5 and example 2.6 that $\overline{K}(u) = \beta\sum_{k=1}^{r+1} D_k e^{-s_k u}$ and $\overline{K*H}(u) = \beta\sum_{k=1}^{r+1}\frac{D_k b}{b-s_k}e^{-s_k u}$. By (3.9) and (3.21),

$$\alpha_{-1}(u,\rho) = p_1(1+\beta)\overline{K}(u) = p_1\beta(1+\beta)\sum_{k=1}^{r+1} D_k e^{-s_k u} \quad (3.42)$$

and

$$\begin{aligned}\alpha_0(u,\rho) &= \beta\int_u^\infty \overline{K}(x)dx - \frac{1}{b}\overline{K*H}(u)\\ &= \beta^2\sum_{k=1}^{r+1} D_k\int_u^\infty e^{-s_k x}dx - \frac{\beta}{b}\sum_{k=1}^{r+1}\frac{D_k b}{b-s_k}e^{-s_k u}\\ &= \beta\sum_{k=1}^{r+1} D_k\left[\frac{\beta}{s_k} - \frac{1}{b-s_k}\right]e^{-s_k u}, \quad (3.43)\end{aligned}$$

a combination of exponential functions. To compute $\alpha_n(u,\rho)$ for $n = 1,2,3,\ldots$, first calculate

$$\begin{aligned}&\int_u^\infty (x-u)^n\overline{K}(x)dx\\ =\ &\beta\sum_{k=1}^{r+1} D_k\int_u^\infty (x-u)^n e^{-s_k x}dx = \beta\sum_{k=1}^{r+1} D_k e^{-s_k u}\int_0^\infty x^n e^{-s_k x}dx\end{aligned}$$

$$= \beta n! \sum_{k=1}^{r+1} \frac{D_k}{s_k^{n+1}} e^{-s_k u}$$

and

$$\begin{aligned}\int_u^\infty (x-u)^j \overline{K * H}(x)dx &= \beta \sum_{k=1}^{r+1} \frac{D_k b}{b-s_k} \int_u^\infty (x-u)^j e^{-s_k x} dx \\ &= \beta j! \sum_{k=1}^{r+1} \frac{D_k b}{(b-s_k)s_k^{j+1}} e^{-s_k u}.\end{aligned}$$

Then by $\mu_{\Gamma,n}(\rho) = n! \sum_{i=1}^{r} \frac{q_i^*}{\mu_i^n}$ in (2.79), (3.22) for $n = 1, 2, 3, \ldots$ becomes,

$$\begin{aligned}\alpha_n(u,\rho) &= \frac{1}{\mu_{\Gamma,n}(\rho)} \Bigg[\beta^2 n! \sum_{k=1}^{r+1} \frac{D_k}{s_k^{n+1}} e^{-s_k u} - \beta n! \sum_{k=1}^{r+1} \frac{D_k}{(b-s_k)s_k^n} e^{-s_k u} - \\ &\qquad \beta \sum_{j=0}^{n-1} \begin{pmatrix} n \\ j \end{pmatrix} \mu_{\Gamma,n-j}(\rho) j! \sum_{k=1}^{r+1} \frac{D_k b}{(b-s_k)s_k^{j+1}} e^{-s_k u}\Bigg] \\ &= \frac{\beta}{\sum_{i=1}^{r} \frac{q_i^*}{\mu_i^n}} \sum_{k=1}^{r+1} D_k \left[\frac{\beta}{s_k^{n+1}} - \frac{1}{(b-s_k)s_k^n} - b \sum_{j=0}^{n-1} \sum_{m=1}^{r} \frac{q_m^*}{\mu_m^{n-j}} \frac{1}{(b-s_k)s_k^{j+1}}\right] e^{-s_k u} \\ &= \frac{\beta}{\sum_{i=1}^{r} \frac{q_i^*}{\mu_i^n}} \sum_{k=1}^{r+1} \frac{D_k}{s_k^n} \left\{\frac{\beta}{s_k} - \frac{1}{b-s_k}\left[1 + b \sum_{m=1}^{r} \frac{q_m^*}{\mu_m} \sum_{j=0}^{n-1} \frac{s_k^{n-1-j}}{\mu_m^{n-1-j}}\right]\right\} e^{-s_k u} \\ &= \frac{\beta}{\sum_{i=1}^{r} \frac{q_i^*}{\mu_i^n}} \sum_{k=1}^{r+1} \frac{D_k}{s_k^n} \left\{\frac{\beta}{s_k} - \frac{1}{b-s_k}\left[1 + b \sum_{m=1}^{r} q_m^* \frac{1 - \left(\frac{s_k}{\mu_m}\right)^n}{\mu_m - s_k}\right]\right\} e^{-s_k u}, \quad (3.44)\end{aligned}$$

a combination of exponential functions.

Since $\tau_n(u) = \alpha_{n-1}(u,0)$ for $n = 0, 1, 2, \ldots$, from (3.42), (3.43) and (3.44)

$$\tau_0(u) = p_1(1+\theta)\overline{K}_{\delta=0}(u) = p_1\theta(1+\theta)\sum_{k=1}^{r+1} D_{k,\delta=0}e^{-s_{k,\delta=0}u}, \tag{3.45}$$

$$\tau_1(u) = \theta\sum_{k=1}^{r+1} D_{k,\delta=0}\left[\frac{\theta}{s_{k,\delta=0}} - \frac{1}{c/D - s_{k,\delta=0}}\right]e^{-s_{k,\delta=0}u}, \tag{3.46}$$

and for $n = 1, 2, 3, \ldots$

$$\begin{aligned}\tau_n(u) = & \frac{\theta}{\sum_{i=1}^{r}\frac{q^*_{i,\delta=0}}{\mu_i^{n-1}}}\sum_{k=1}^{r+1}\frac{D_{k,\delta=0}}{s_{k,\delta=0}^{n-1}} \\ & \left\{\frac{\theta}{s_{k,\delta=0}} - \frac{1}{c/D - s_{k,\delta=0}}\left[1 + \frac{c}{D}\sum_{m=1}^{r} q^*_{m,\delta=0}\frac{1-\left(\frac{s_{k,\delta=0}}{\mu_m}\right)^{n-1}}{\mu_m - s_{k,\delta=0}}\right]\right\}e^{-s_{k,\delta=0}u}.\end{aligned} \tag{3.47}$$

□

3.2 Discounted moment of the deficit

In this section, we are going to study the (discounted) moment of the deficit at the time of ruin caused by a claim. To see this, let's consider the special case that $w(x,y) = y^n$ where n is a positive integer, then a relatively simple expression exists for (2.2) as follows:

Theorem 3.2 *For $u \geq 0$ and $n = 1, 2, 3, \ldots,$*

$$E\Big\{e^{-\delta T}|U(T)|^n I(T < \infty, U(T) < 0)\Big\} = \frac{1}{\beta}\Big[n\mu_{\Gamma,n-1}(\rho)\alpha_{n-1}(u,\rho) - \mu_{\Gamma,n}(\rho)\overline{K * H}(u)\Big] \tag{3.48}$$

where $\alpha_n(u, \rho)$ is given in (3.22), and $\overline{K * H}(u)$ can be replaced with $\phi_t(u)$ by (2.103).

In addition, when $D \to 0$, equation (3.48) reduces to

$$E\Big\{e^{-\delta T}|U(T)|^n I(T < \infty)\Big\} = \frac{1}{\beta_0}\Big[n\mu_{\Gamma,n-1}(\rho)\alpha_{0;n-1}(u,\rho) - \mu_{\Gamma,n}(\rho)\overline{K}_0(u)\Big] \tag{3.49}$$

with $\alpha_{0;n}(u, \rho)$ given in (3.24).

Proof: From (1.54), (1.55), (1.56) and (1.57), $B(u)$ in (2.114) with $w(x, y) = y^n$ becomes

$$\begin{aligned}
B(u) &= \frac{e^{-bu}\int_0^u e^{as}\int_s^\infty e^{-\rho x}\int_x^\infty (y-x)^n dP(y)dxds}{\frac{1}{b}\int_0^\infty e^{-\rho y}\overline{P}(y)dy} \\
&= \frac{p_n e^{-bu}\int_0^u e^{bs}e^{\rho s}\int_s^\infty e^{-\rho x}\overline{P}_n(x)dxds}{\frac{1}{b}p_1\int_0^\infty e^{-\rho y}dP_1(y)} \\
&= \frac{p_n b e^{-bu}\int_0^u e^{bs}\overline{\Gamma}_n(s)ds}{p_1\int_0^\infty e^{-\rho y}dP_1(y)}\int_0^\infty e^{-\rho y}\overline{P}_n(y)dy \\
&= \frac{p_n}{p_1}\frac{p_{n+1}}{(n+1)p_n}\frac{\int_0^\infty e^{-\rho y}dP_{n+1}(y)}{\int_0^\infty e^{-\rho y}dP_1(y)}\Big[be^{-bu}\int_0^u e^{bs}\overline{\Gamma}_n(s)ds\Big].
\end{aligned}$$

Then by (2.77) and (2.78),

$$B(u) = \mu_{\Gamma,n}(\rho)\int_0^u \overline{\Gamma}_n(s)H'(u-s)ds = \mu_{\Gamma,n}(\rho)\overline{\Gamma}_n * H(u), \tag{3.50}$$

and from (3.1), (3.6) and (3.7), (1.44) becomes

$$\begin{aligned}
\phi_w(u) &= -\frac{1}{\beta}\int_0^u \overline{K}(u-x)dB(x) + \frac{1}{\beta}B(u) - \frac{1}{\beta}B(0)\overline{K}(u) \\
&= \frac{\mu_{\Gamma,n}(\rho)}{\beta}\left[-\int_0^u \overline{K}(u-x)d\overline{\Gamma}_n * H(x) + \overline{\Gamma}_n * H(u)\right] \\
&= \frac{\mu_{\Gamma,n}(\rho)}{\beta}\left[-\int_0^u \overline{K}(u-x)d[H(x) - \Gamma_n * H(x)] + \overline{\Gamma_n * \overline{H}}(u) - \overline{H}(u)\right] \\
&= \frac{\mu_{\Gamma,n}(\rho)}{\beta}\left[\int_0^u \overline{K}(u-x)d\Gamma_n * H(x) + \overline{\Gamma_n * \overline{H}}(u) - \overline{K} * H(u) - \overline{H}(u)\right] \\
&= \frac{\mu_{\Gamma,n}(\rho)}{\beta}\left[\frac{\alpha_{n-1}(u,\rho)}{\gamma_{n-1}(\rho)} - \overline{K * \overline{H}}(u)\right] \\
&= \frac{n\mu_{\Gamma,n-1}(\rho)}{\beta}\alpha_{n-1}(u,\rho) - \frac{\mu_{\Gamma,n}(\rho)}{\beta}\overline{K * \overline{H}}(u).
\end{aligned}$$

That is, $E\left\{e^{-\delta T}|U(T)|^n I(T<\infty, U(T)<0)\right\} = \frac{1}{\beta}\Big[n\mu_{\Gamma,n-1}(\rho)\alpha_{n-1}(u,\rho) - \mu_{\Gamma,n}(\rho)$ $\overline{K * \overline{H}}(u)\Big]$, which is (3.48). If $D \to 0$, then $\beta \to \beta_0$, and $\overline{K * \overline{H}}(u) \to \overline{K}_0(u)$ for $u > 0$ by (3.20), proving (3.49) for the case $u > 0$.

For the case $u = 0$, (3.49) with $u = 0$ becomes

$$\begin{aligned}
&\frac{1}{\beta_0}\left[n\mu_{\Gamma,n-1}(\rho)\alpha_{0;n-1}(0,\rho) - \mu_{\Gamma,n}(\rho)\overline{K}_0(0)\right] \\
&= \frac{1}{\beta_0}\left[n\mu_{\Gamma,n-1}(\rho)\int_0^\infty \overline{\Gamma}_{n-1}(x)dx - \overline{K}_0(0)\mu_{\Gamma,n}(\rho)\right] \\
&= \frac{1}{\beta_0}\left[\mu_{\Gamma,n}(\rho) - \frac{1}{1+\beta_0}\mu_{\Gamma,n}(\rho)\right] \\
&= \frac{\mu_{\Gamma,n}(\rho)}{(1+\beta_0)},
\end{aligned}$$

by (3.1) and (3.14). Moreover, from (1.44), (2.77), (2.116), (1.55), (1.56) and (2.78),

$$\begin{aligned}
\phi_0(0) &= \frac{1}{\beta_0}\Big[B_0(0) - B_0(0)\overline{K}_0(0)\Big] \\
&= \frac{1}{(1+\beta_0)} B_0(0) \\
&= \frac{1}{(1+\beta_0)} \frac{\int_0^\infty e^{-\rho x} \int_x^\infty (y-x)^n dP(y) dx}{\int_0^\infty e^{-\rho x} \overline{P}(x) dx} \\
&= \frac{1}{(1+\beta_0)} \frac{p_n \int_0^\infty e^{-\rho x} \overline{P}_n(x) dx}{\int_0^\infty e^{-\rho x} \overline{P}(x) dx} \\
&= \frac{1}{(1+\beta_0)} \frac{p_{n+1}}{(n+1)p_1} \frac{\int_0^\infty e^{-\rho x} dP_{n+1}(x)}{\int_0^\infty e^{-\rho x} dP_1(x)} \\
&= \frac{\mu_{\Gamma,n}(\rho)}{(1+\beta_0)},
\end{aligned}$$

which shows that (3.49) also holds for the case $u = 0$. □

When $n = 1$, by (3.21) and $\mu_{G,1}(\rho) = \mu_{\Gamma,1}(\rho) + \frac{1}{b}$, (3.48) turns out to be

$$\begin{aligned}
& E\Big\{e^{-\delta T}|U(T)|I(T<\infty, U(T)<0)|U(0)=u\Big\} \\
&= \frac{1}{\beta}\Big[\mu_{\Gamma,0}(\rho)\alpha_0(u,\rho) - \mu_{\Gamma,1}(\rho)\overline{K*H}(u)\Big] \\
&= \frac{1}{\beta}\Big[\beta\int_u^\infty \overline{K}(x)dx - \frac{1}{b}\overline{K*H}(u) - \mu_{\Gamma,1}(\rho)\overline{K*H}(u)\Big] \\
&= \int_u^\infty \overline{K}(x)dx - \frac{\mu_{G,1}(\rho)}{\beta}\overline{K*H}(u) \qquad (3.51)
\end{aligned}$$

with $\mu_{G,1}(\rho)$ given in (2.75). If further $D \to 0$, then by (3.23), (3.49) becomes

$$E\left\{e^{-\delta T}|U(T)|I(T<\infty)|U(0)=u\right\}=\int_u^\infty \overline{K}_0(x)dx-\frac{\mu_{\Gamma,1}(\rho)}{\beta_0}\overline{K}_0(u) \quad (3.52)$$

where $\mu_{\Gamma,1}(\rho)=\dfrac{p_1}{1-\tilde{p}(\rho)}-\dfrac{1}{\rho}$.

If in the special case that $\delta=0$, then $\beta=\theta$, $H(u)=H_1(u)$ and $\alpha_{n-1}(u,0)=\tau_n(u)$, and from (2.77) and (3.35), theorem 3.2 becomes

Corollary 3.3 *For $u\geq 0$ and $n=1,2,3,\ldots$,*

$$E\left\{|U(T)|^n I(T<\infty,U(T)<0)|U(0)=u\right\}=\frac{1}{p_1\theta}\left[p_n\tau_n(u)-\frac{p_{n+1}}{n+1}\overline{K_{\delta=0}*H}_1(u)\right] \quad (3.53)$$

where $\tau_n(u)$ is given in (3.38) and (3.39), and $\overline{K_{\delta=0}*H}_1(u)$ can be replaced with $\psi_t(u)$ by (2.105). In addition, when $D\to 0$, equation (3.53) reduces to

$$E\left\{|U(T)|^n I(T<\infty)|U(0)=u\right\}=\frac{1}{p_1\theta}\left[p_n\tau_{0;n}(u)-\frac{p_{n+1}}{n+1}\psi_0(u)\right] \quad (3.54)$$

with $\tau_n(u)$ given in (3.40) and (3.41).

When $n=1$, by (3.38), (3.53) turns out to be

$$\begin{aligned}&E\left\{|U(T)|I(T<\infty,U(T)<0)|U(0)=u\right\}\\ =&\int_u^\infty \overline{K}_{\delta=0}(x)dx-\frac{cp_2+2D}{2cp_1\theta}\overline{K_{\delta=0}*H}_1(u) \quad (3.55)\end{aligned}$$

where $\overline{K_{\delta=0} * H_1}(u)$ can be replaced with $\psi_t(u)$ by (2.105). If further $D \to 0$, then $\overline{K}_{\delta=0}(u) \to \psi_0(u)$, and by (3.35), (3.55) reduces to

$$E\Big\{|U(T)|I(T<\infty)|U(0)=u\Big\} = \int_u^\infty \psi_0(x)dx - \frac{p_2}{2p_1\theta}\psi_0(u). \tag{3.56}$$

The second moment, and hence the variance, of the deficit at the time of ruin caused by a claim can also be easily obtained from (3.53).

Example 3.2 Combination of exponentials and mixture of Erlangs ($b \neq \mu$)

As shown in example 3.1 that $\alpha_0(u,\rho) = \beta \sum_{k=1}^{r+1} D_k \Big[\frac{\beta}{s_k} - \frac{1}{b-s_k}\Big] e^{-s_k u}$ and for $n = 1, 2, 3, \ldots$, $\alpha_n(u,\rho) = \frac{\beta}{\sum_{i=1}^{r} \frac{q_i^*}{\mu_i^n}} \sum_{k=1}^{r+1} \frac{D_k}{s_k^n} \Big\{\frac{\beta}{s_k} - \frac{1}{b-s_k}\Big[1 + b\sum_{m=1}^{r} q_m^* \frac{1-(s_k/\mu_m)^n}{\mu_m - s_k}\Big]\Big\} e^{-s_k u}$.

Combine these with $\mu_{\Gamma,n}(\rho) = n! \sum_{m=1}^{r} \frac{q_m^*}{\mu_m^n}$ from (2.79) and $\overline{K * H}(u) = \beta \sum_{k=1}^{r+1} \frac{D_k b}{b-s_k} e^{-s_k u}$ from (2.149), (3.48) with $n = 1$ turns out to be

$$\begin{aligned}
& E\Big\{e^{-\delta T}|U(T)|I(T<\infty, U(T)<0)|U(0)=u\Big\} \\
= & \frac{1}{\beta}\Big[\alpha_0(u,\rho) - \mu_{\Gamma,1}(\rho)\overline{K * H}(u)\Big] \\
= & \frac{1}{\beta}\Big[\beta \sum_{k=1}^{r+1} D_k \Big(\frac{\beta}{s_k} - \frac{1}{b-s_k}\Big) e^{-s_k u} - \sum_{m=1}^{r} \frac{q_m^*}{\mu_m} \beta \sum_{k=1}^{r+1} \frac{D_k b}{b-s_k} e^{-s_k u}\Big] \\
= & \sum_{k=1}^{r+1} D_k \Big[\frac{\beta}{s_k} - \frac{1}{b-s_k}\Big(1 + b\sum_{m=1}^{r} \frac{q_m^*}{\mu_m}\Big)\Big] e^{-s_k u},
\end{aligned} \tag{3.57}$$

a combination of exponential functions, and for $n = 2, 3, 4, \ldots$,

$$
\begin{aligned}
&E\Big\{e^{-\delta T}|U(T)|^n I(T<\infty, U(T)<0)|U(0)=u\Big\} \\
= \;& n!\sum_{k=1}^{r+1}\frac{D_k}{s_k^{n-1}}\Big[\frac{\beta}{s_k}-\frac{1}{b-s_k}\Big(1+b\sum_{m=1}^{r}q_m^*\frac{1-(s_k/\mu_m)^{n-1}}{\mu_m-s_k}\Big)\Big]e^{-s_k u}- \\
& n!\sum_{m=1}^{r}\frac{q_m^*}{\mu_m^n}\sum_{k=1}^{r+1}\frac{D_k b}{b-s_k}e^{-s_k u} \\
= \;& n!\sum_{k=1}^{r+1}\frac{D_k}{s_k^{n-1}}\Big\{\frac{\beta}{s_k}-\frac{1}{b-s_k}\Big[1+b\sum_{m=1}^{r}\frac{q_m^*}{\mu_m^{n-1}}\Big(\frac{\mu_m^{n-1}-s_k^{n-1}}{\mu_m-s_k}+\frac{s_k^{n-1}}{\mu_m}\Big)\Big]\Big\}e^{-s_k u} \\
= \;& n!\sum_{k=1}^{r+1}\frac{D_k}{s_k^{n-1}}\Big[\frac{\beta}{s_k}-\frac{1}{b-s_k}\Big(1+b\sum_{m=1}^{r}q_m^*\frac{1-(s_k/\mu_m)^{n}}{\mu_m-s_k}\Big)\Big]e^{-s_k u}, \qquad (3.58)
\end{aligned}
$$

a combination of exponential functions. Note that (3.58) with $n = 1$ reduces to (3.57). Therefore, (3.58) holds for $n = 1, 2, 3, \ldots$. When $\delta = 0$, for $n = 1, 2, 3, \ldots$, (3.58) turns out to be

$$
\begin{aligned}
&E\Big\{|U(T)|^n I(T<\infty, U(T)<0)|U(0)=u\Big\} \\
= \;& n!\sum_{k=1}^{r+1}\frac{D_{k,\delta=0}}{s_{k,\delta=0}^{n-1}}\Big[\frac{\theta}{s_{k,\delta=0}}-\frac{D}{c-Ds_{k,\delta=0}} \\
&\Big(1+\frac{c}{D}\sum_{m=1}^{r}q_{m,\delta=0}^*\frac{1-(s_{k,\delta=0}/\mu_m)^{n}}{\mu_m-s_{k,\delta=0}}\Big)\Big]e^{-s_{k,\delta=0}u}. \qquad (3.59)
\end{aligned}
$$

□

Now, we have an upper bound for functions of the deficit at the time of ruin caused by a claim as follows:

Theorem 3.3 *If $P(x)$ is DMRL, then for $n = 1, 2, 3, \ldots$,*

$$E\Big\{e^{-\delta T}|U(T)|^n I(T < \infty, U(T) < 0)\Big\} \leq \mu_{\Gamma,n}(\rho)\overline{K}(u). \tag{3.60}$$

Proof: If $P(x)$ is IMRL(DMRL), by theorem 3.2 of Lin and Willmot (1999) [36], $\Gamma(x)$ is IMRL (DMRL) which implies (Fagiuoli and Pellerey, (1994) [19]) that the n^{th} equilibrium distribution function $\Gamma_n(x)$ is NWUE (NBUE), that is, $\overline{\Gamma}_{n+1}(x) \geq (\leq)\, \overline{\Gamma}_n(x)$. Then by (3.50), $B(u) = \mu_{\Gamma,n}(\rho)\overline{\Gamma}_n * H(u) \geq (\leq)\, \mu_{\Gamma,n}(\rho)\overline{\Gamma} * H(u) = \mu_{\Gamma,n}(\rho)[\overline{G}(u) - \overline{H}(u)]$.

Therefore, if $P(x)$ is DMRL, then $B(u) \leq \mu_{\Gamma,n}(\rho)[\overline{G}(u) - \overline{H}(u)] \leq \mu_{\Gamma,n}(\rho)\overline{G}(u)$ which implies $E\Big\{e^{-\delta T}|U(T)|^n I(T < \infty, U(T) < 0)\Big\} \leq \mu_{\Gamma,n}(\rho)\overline{K}(u)$ by theorem 1.1.
□

Corollary 3.4 *If $P(x)$ is DMRL, then for $n = 1, 2, 3, \ldots$,*

$$E\Big\{|U(T)|^n I(T < \infty, U(T) < 0)\Big\} \leq \frac{p_{n+1}}{(n+1)p_1}\overline{K}_{\delta=0}(u) = \frac{p_{n+1}}{(n+1)p_1}\Big[\psi_s(u) + \frac{\psi_d(u)}{1+\theta}\Big]. \tag{3.61}$$

Proof: When $\delta = 0$, $\mu_{\Gamma,n}(0) = \dfrac{p_{n+1}}{(n+1)p_1}$ by (2.77), and $\overline{K}_{\delta=0}(u) = \psi_s(u) + \dfrac{1}{1+\theta}\psi_d(u)$. □

When $D \to 0$, we can obtain both lower bound and upper bound for functions of the deficit at the time of ruin caused by a claim as follows:

Corollary 3.5 *If $P(x)$ is IMRL (DMRL), then for $n = 1, 2, 3, \ldots$,*

$$E\Big\{e^{-\delta T}|U(T)|^n I(T < \infty)\Big\} \geq (\leq)\, \mu_{\Gamma,n}(\rho)\overline{K}_0(u). \tag{3.62}$$

Proof: From the proof of theorem 3.3, if $P(x)$ is IMRL (DMRL), then $B(u) = \mu_{\Gamma,n}(\rho)\overline{\Gamma}_n * H(u) \geq (\leq)\, \mu_{\Gamma,n}(\rho)\overline{\Gamma} * H(u)$. When $D \to 0$, by (3.11) the inequality reduces to $B_0(u) = \mu_{\Gamma,n}(\rho)\overline{\Gamma}_n(u) \geq (\leq)\, \mu_{\Gamma,n}(\rho)\overline{\Gamma}(u)$, then by theorem 1.1 which becomes "$B_0(u) \geq (\leq)\, c^*\overline{\Gamma}(u)$ where $0 < c^* < \infty$, then $\phi_0(u) \geq (\leq)\, c^*\overline{K}_0(u)$" in the case that $D \to 0$, we get (3.62). □

To obtain the discounted joint moments of the surplus $U(T-)$ before the time of ruin and the deficit $|U(T)|$ at the time of ruin due to a jump, just set $w(x, y) = x^m y^n$, for $m, n = 0, 1, 2, \ldots$, in (2.2) to form

$$\phi_w(u) = E[e^{-\delta T} U(T-)^m |U(T)|^n I(T < \infty, U(T) < 0)], \quad u \geq 0. \tag{3.63}$$

Then by (1.55), $B(u)$ in (2.114) becomes

$$\begin{aligned} B(u) &= \frac{e^{-bu}\int_0^u e^{as}\int_s^\infty e^{-\rho x} x^m \int_x^\infty (y-x)^n dP(y)\,dx\,ds}{\frac{1}{b}\int_0^\infty e^{-\rho y}\overline{P}(y)dy} \\ &= \frac{bp_n e^{-bu}\int_0^u e^{as}\int_s^\infty e^{-\rho x} x^m \overline{P}_n(x)\,dx\,ds}{\int_0^\infty e^{-\rho y}\overline{P}(y)dy}. \end{aligned}$$

If we define

$$p_{m,n}(\rho) = \int_0^\infty e^{-\rho x} x^m \overline{P}_n(x)dx \tag{3.64}$$

and

$$\overline{\Gamma}_{m,n}(x;\rho) = \frac{e^{\rho x}\int_x^\infty e^{-\rho y} y^m \overline{P}_n(y)dy}{\int_0^\infty e^{-\rho y} y^m \overline{P}_n(y)dy} = \frac{e^{\rho x}\int_x^\infty e^{-\rho y} y^m \overline{P}_n(y)dy}{p_{m,n}(\rho)} \tag{3.65}$$

with $\overline{\Gamma}_{0,n}(x;\rho) = \overline{\Gamma}_n(x)$ and $\overline{\Gamma}_{0,n}(x;0) = \overline{P}_{n+1}(x)$, then $B(u)$ can be written as

$$B(u) = p_n \frac{p_{m,n}(\rho)}{p_{0,0}(\rho)} \int_0^u \overline{\Gamma}_{m,n}(s;\rho) b e^{-b(u-s)} ds = d_{m,n}(\rho)\overline{\Gamma}_{m,n}(\rho) * H(u) \tag{3.66}$$

where $d_{m,n}(\rho) = p_n \frac{p_{m,n}(\rho)}{p_{0,0}(\rho)}$. From (1.44), (3.63) turns out to be

$$\begin{aligned}
& E[e^{-\delta T} U(T-)^m |U(T)|^n I(T < \infty, U(T) < 0)] \\
= & -\frac{1}{\beta}\int_0^u \overline{K}(u-x) dB(x) + \frac{1}{\beta} B(u) - \frac{1}{\beta} B(0)\overline{K}(u) \\
= & \frac{d_{m,n}(\rho)}{\beta}\Big[-\int_0^u \overline{K}(u-x) d\overline{\Gamma}_{m,n}(\rho) * H(x) + \overline{\Gamma}_{m,n}(\rho) * H(u)\Big] \\
= & \frac{d_{m,n}(\rho)}{\beta}\Big[-\int_0^u \overline{K}(u-x) d[H(x) - \Gamma_{m,n}(\rho) * H(x)] + \overline{\Gamma_{m,n}(\rho) * H}(u) - \overline{H}(u)\Big] \\
= & \frac{d_{m,n}(\rho)}{\beta}\Big[\int_0^u \overline{K}(u-x) d\Gamma_{m,n}(\rho) * H(x) + \overline{\Gamma_{m,n}(\rho) * H}(u) - \overline{K} * H(u) - \overline{H}(u)\Big] \\
= & \frac{d_{m,n}(\rho)}{\beta}\Big[\int_0^u \overline{K}(u-x) d\Gamma_{m,n}(\rho) * H(x) + \overline{\Gamma_{m,n}(\rho) * H}(u) - \overline{K * H}(u)\Big].
\end{aligned} \tag{3.67}$$

In general, the integration calculation for $\overline{\Gamma}_{m,n}(x;\rho)$ (or $\Gamma_{m,n}(x;\rho)$) is very complicated, especially when m is large. It seems that we have no similar expression like (3.48), in which case m is equal to 0.

3.3 Joint moment of the deficit at and the time of ruin

In this section, we will study the joint moment of the deficit $|U(T)|^n$ at ruin and the time of ruin T due to a claim. First, consider $\psi_{1,w}(u)$, the joint moment of the time of ruin T caused by a claim, and the associated penalty function, $w(U(T-), |U(T)|)$, that is,

$$\psi_{1,w}(u) = E[Tw(U(T-), |U(T)|)I(T < \infty, U(T) < 0)|U(0) = u], \quad u \geq 0, \quad (3.68)$$

then we can differentiate (2.2) with respect to δ and set $\delta = 0$ to get $\psi_{1,w}(u)$, namely, $\psi_{1,w}(u) = -\frac{d}{d\delta}\phi_w(u)|_{\delta=0}$. Now we will show that $\psi_{1,w}(u)$ satisfies a defective renewal equation and has an explicit expression as follows:

Theorem 3.4 *For $u \geq 0$, $\psi_{1,w}(u)$ in (3.68) satisfies the defective renewal equation*

$$\psi_{1,w}(u) = \frac{1}{1+\theta}\int_0^u \psi_{1,w}(u-x)dP_1 * H_1(x) + \frac{1}{1+\theta}B_{1,w}(u), \quad (3.69)$$

and is given explicitly by

$$\begin{aligned}\psi_{1,w}(u) &= \frac{1}{\lambda p_1\theta}\int_0^u \overline{K}_{\delta=0} * H_1(u-x)\phi_{w,\delta=0}(x)dx \\ &+ \frac{1}{\lambda p_1\theta^2}\Big[\int_0^u \overline{K_{\delta=0} * H}_1(u-x)B_{\delta=0}(x)dx + \int_u^\infty B_{\delta=0}(x)dx - \\ &\qquad \overline{K_{\delta=0} * H}_1(u)\int_0^\infty B_{\delta=0}(x)dx\Big], \quad (3.70)\end{aligned}$$

where

$$\begin{aligned}\phi_{w,\delta=0}(u) &= E[w(U(T-), |U(T)|)I(T < \infty, U(T) < 0)|U(0) = u] \quad (3.71)\\ &= -\frac{1}{\theta}\int_0^u \overline{K}_{\delta=0}(u-x)B'_{\delta=0}(x)dx + \frac{1}{\theta}B_{\delta=0}(u), \quad (3.72)\\ B_{\delta=0}(u) &= \frac{1}{p_1}\int_0^u h_1(u-x)\int_x^\infty \omega(y)dydx, \quad (3.73)\end{aligned}$$

$$\begin{aligned} B_{1,w}(u) &= \frac{1}{\lambda p_1}\int_0^u h_1(u-x)\int_x^\infty \phi_{w,\delta=0}(y)dydx \qquad (3.74)\\ &= \frac{1}{\lambda p_1 \theta}\Big[\int_0^u \overline{K_{\delta=0} * H}_1(u-x)B_{\delta=0}(x)dx + \int_u^\infty B_{\delta=0}(x)dx - \\ &\qquad \overline{H}_1(u)\int_0^\infty B_{\delta=0}(x)dx\Big], \qquad (3.75)\end{aligned}$$

and $\overline{K_{\delta=0} * H}_1(u)$ can be replaced with $\psi_t(u)$ by (2.105).

In addition, when $D \to 0$, equation (3.69) reduces to

$$\psi_{0;1,w}(u) = \frac{1}{1+\theta}\int_0^u \psi_{0;1,w}(u-x)dP_1(x) + \frac{1}{1+\theta}B_{0;1,w}(u), \qquad (3.76)$$

and is given explicitly by

$$\begin{aligned} \psi_{0;1,w}(u) &= \frac{1}{\lambda p_1 \theta}\int_0^u \psi_0(u-x)\phi_{0,\delta=0}(x)dx \\ &+ \frac{1}{\lambda p_1 \theta^2}\Big[\int_0^u \psi_0(u-x)B_{0,\delta=0}(x)dx + \int_u^\infty B_{0,\delta=0}(x)dx - \\ &\qquad \psi_0(u)\int_0^\infty B_{0,\delta=0}(x)dx\Big], \qquad (3.77)\end{aligned}$$

where

$$\psi_{0;1,w}(u) = E[Tw(U(T-),|U(T)|)I(T<\infty)|U(0)=u] = -\frac{d}{d\delta}\phi_0(u)|_{\delta=0}, \quad (3.78)$$

$$\begin{aligned} \phi_{0,\delta=0}(u) &= E[w(U(T-),|U(T)|)I(T<\infty)|U(0)=u] \qquad (3.79)\\ &= -\frac{1}{\theta}\int_0^u \psi_0(u-x)B'_{0,\delta=0}(x)dx + \frac{1}{\theta}B_{0,\delta=0}(u), \qquad (3.80)\\ B_{0,\delta=0}(u) &= \frac{1}{p_1}\int_u^\infty \omega(x)dx, \qquad (3.81)\end{aligned}$$

and

$$B_{0;1,w}(u) = \frac{1}{\lambda p_1}\int_u^\infty \phi_{0,\delta=0}(x)dx \tag{3.82}$$

$$= \frac{1}{\lambda p_1 \theta}\Big[\int_0^u \psi_0(u-x)B_{0,\delta=0}(x)dx + \int_u^\infty B_{0,\delta=0}(x)dx\Big]. \tag{3.83}$$

Proof: Perform the Laplace transform on the both sides of (2.2), then by (2.19) and (2.22), equation (2.189) becomes

$$\tilde{\phi}_w(\xi,\delta) = \frac{\lambda[\tilde{p}(\xi)-1] + D\rho^2 + c\rho - \delta}{(b+\xi)(\rho-\xi)D}\tilde{\phi}_w(\xi,\delta) + \frac{\lambda[\tilde{\omega}(\xi)-\tilde{\omega}(\rho)]}{(b+\xi)(\rho-\xi)D}.$$

Since $\tilde{p}_1(\xi) = \frac{1-\tilde{p}(\xi)}{p_1\xi}$ and $[(b+\xi)(\rho-\xi)D] - \{\lambda[\tilde{p}(\xi)-1] + D\rho^2 + c\rho - \delta\} = [D(\rho^2-\xi^2)+c(\rho-\xi)] - [-\lambda p_1\xi\tilde{p}_1(\xi) + D\rho^2 + c\rho - \delta] = -\{D\xi^2 + c\xi[1-\frac{1}{1+\theta}\tilde{p}_1(\xi)] - \delta\}$, we have

$$\Big\{D\xi^2 + c\xi\Big[1-\frac{1}{1+\theta}\tilde{p}_1(\xi)\Big] - \delta\Big\}\tilde{\phi}_w(\xi,\delta) = \lambda\Big[\tilde{\omega}(\rho) - \tilde{\omega}(\xi)\Big]. \tag{3.84}$$

Since $\tilde{\psi}_{1,w}(\xi) = \int_0^\infty e^{-\xi u}\psi_{1,w}(u)du = -\frac{d}{d\delta}\int_0^\infty e^{-\xi u}\phi_w(u)du|_{\delta=0} = -\frac{d}{d\delta}\tilde{\phi}_w(\xi,\delta)|_{\delta=0}$, differentiating with respect to δ and then setting $\delta = 0$ lead to

$$\tilde{\phi}_w(\xi,0) + \Big\{D\xi^2 + c\xi\Big[1-\frac{1}{1+\theta}\tilde{p}_1(\xi)\Big]\Big\}\tilde{\psi}_{1,w}(\xi) = \rho'(0)\int_0^\infty x\omega(x)dx.$$

Further letting $\xi = 0$ gives $\tilde{\phi}_w(0,0) = \rho'(0)\int_0^\infty x\omega(x)dx$. Therefore,

$$\Big\{D\xi^2 + c\xi\Big[1-\frac{1}{1+\theta}\tilde{p}_1(\xi)\Big]\Big\}\tilde{\psi}_{1,w}(\xi) = \tilde{\phi}_w(0,0) - \tilde{\phi}_w(\xi,0).$$

Division by $(D\xi^2 + c\xi)$ yields

$$\left\{1 - \frac{1}{1+\theta}\frac{c/D}{c/D+\xi}\tilde{p}_1(\xi)\right\}\tilde{\psi}_{1,w}(\xi) = \frac{1}{c}\frac{c/D}{c/D+\xi}\frac{\tilde{\phi}_w(0,0) - \tilde{\phi}_w(\xi,0)}{t},$$

which is exactly the Laplace transform of (3.69) with $B_{1,w}(u)$ given in (3.74). The uniqueness of the Laplace transform gives (3.69).

From (2.15), $\phi_{w,\delta=0}(y)$ in (3.71) satisfies the defective renewal equation

$$\phi_{w,\delta=0}(y) = \frac{1}{1+\theta}\int_0^y \phi_{w,\delta=0}(y-t)h_1 * h_2(t)dt + \frac{1}{1+\theta}B_{\delta=0}(y),$$

with $B_{\delta=0}(y)$ given in (3.73). Then by (1.44) with $\delta = 0$, $\phi_{w,\delta=0}(y)$ can be expressed as

$$\phi_{w,\delta=0}(y) = -\frac{1}{\theta}\int_0^y \overline{K}_{\delta=0}(y-t)B'_{\delta=0}(t)dt + \frac{1}{\theta}B_{\delta=0}(y).$$

If $\frac{c}{D}\int_0^y e^{\frac{c}{D}t}\int_t^\infty \omega(x)dxdt \to \infty$, as $y \to \infty$, then by L'Hopital's rule, we have

$$\begin{aligned}\lim_{y\to\infty} B_{\delta=0}(y) &= \lim_{y\to\infty}\frac{1}{p_1}\frac{c}{D}\frac{\int_0^y e^{\frac{c}{D}t}\int_t^\infty \omega(x)dxdt}{e^{\frac{c}{D}y}} = \frac{1}{p_1}\frac{c}{D}\lim_{y\to\infty}\frac{e^{\frac{c}{D}y}\int_y^\infty \omega(x)dx}{\frac{c}{D}e^{\frac{c}{D}y}} \\ &= \frac{1}{p_1}\lim_{y\to\infty}\int_y^\infty \omega(x)dx = 0. \qquad (3.85)\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_x^\infty \int_0^y \overline{K}_{\delta=0}(y-t)B'_{\delta=0}(t)dtdy \\ = &\int_0^x \int_x^\infty \overline{K}_{\delta=0}(y-t)B'_{\delta=0}(t)dydt + \int_x^\infty \int_t^\infty \overline{K}_{\delta=0}(y-t)B'_{\delta=0}(t)dydt\end{aligned}$$

$$
\begin{aligned}
&= \int_0^x \int_{x-t}^{\infty} \overline{K}_{\delta=0}(y) B'_{\delta=0}(t)\,dy\,dt + \left[\int_x^{\infty} B'_{\delta=0}(t)\,dt\right]\left[\int_0^{\infty} \overline{K}_{\delta=0}(y)\,dy\right] \\
&= B_{\delta=0}(t)\int_{x-t}^{\infty} \overline{K}_{\delta=0}(y)\,dy\Big|_0^x - \int_0^x \overline{K}_{\delta=0}(x-t) B_{\delta=0}(t)\,dt - \\
&\quad B_{\delta=0}(x)\int_0^{\infty} \overline{K}_{\delta=0}(y)\,dy \\
&= -\int_0^x \overline{K}_{\delta=0}(x-y) B_{\delta=0}(y)\,dy.
\end{aligned}
$$

Integrating (3.72) for $\phi_{w,\delta=0}(y)$ from $y = x$ to $y = \infty$ yields

$$
\begin{aligned}
\int_x^{\infty} \phi_{w,\delta=0}(y)\,dy &= -\frac{1}{\theta}\int_x^{\infty}\int_0^{y} \overline{K}_{\delta=0}(y-t) B'_{\delta=0}(t)\,dt\,dy + \frac{1}{\theta}\int_x^{\infty} B_{\delta=0}(y)\,dy \\
&= \frac{1}{\theta}\int_0^{x} \overline{K}_{\delta=0}(x-y) B_{\delta=0}(y)\,dy + \frac{1}{\theta}\int_x^{\infty} B_{\delta=0}(y)\,dy.
\end{aligned}
$$

Then by the associative property of convolution, integration by parts, and $\overline{K_{\delta=0} * H_1}(u) = \overline{K}_{\delta=0} * H_1(u) + \overline{H}_1(u)$, equation (3.74) becomes

$$
\begin{aligned}
&B_{1,w}(u) \\
&= \frac{1}{\lambda p_1}\int_0^u h_1(u-x)\int_x^{\infty} \phi_{w,\delta=0}(y)\,dy\,dx \\
&= \frac{1}{\lambda p_1 \theta}\left[\int_0^u h_1(u-x)\int_0^x \overline{K}_{\delta=0}(x-y) B_{\delta=0}(y)\,dy\,dx + \right. \\
&\qquad \left. \int_0^u h_1(u-x)\int_x^{\infty} B_{\delta=0}(y)\,dy\,dx\right] \qquad (3.86) \\
&= \frac{1}{\lambda p_1 \theta}\left[\int_0^u \overline{K}_{\delta=0} * H_1(u-x) B_{\delta=0}(x)\,dx + \int_0^u\int_x^{\infty} B_{\delta=0}(y)\,dy\,d\overline{H}_1(u-x)\right] \\
&= \frac{1}{\lambda p_1 \theta}\left[\int_0^u \overline{K}_{\delta=0} * H_1(u-x) B_{\delta=0}(x)\,dx + \int_u^{\infty} B_{\delta=0}(y)\,dy - \right. \\
&\qquad \left. \overline{H}_1(u)\int_0^{\infty} B_{\delta=0}(y)\,dy + \int_0^u \overline{H}_1(u-x) B_{\delta=0}(x)\,dx\right],
\end{aligned}
$$

which is (3.75).

Since $B_{1,w}(u) = \frac{1}{\lambda p_1}\int_0^u h_1(x)\int_{u-x}^{\infty}\phi_{w,\delta=0}(y)dydx$, and from (3.72)

$$\begin{aligned}\int_0^{\infty}\phi_{w,\delta=0}(u)du &= -\frac{1}{\theta}\int_0^{\infty}B'_{\delta=0}(x)dx\int_0^{\infty}\overline{K}_{\delta=0}(u)du + \frac{1}{\theta}\int_0^{\infty}B_{\delta=0}(u)du \\ &= \frac{1}{\theta}\int_0^{\infty}B_{\delta=0}(u)du \qquad (3.87)\end{aligned}$$

by $B_{\delta=0}(0) = 0$ and (3.85), then we have

$$\begin{aligned}B'_{1,w}(u) &= \frac{1}{\lambda p_1}\left[h_1(u)\int_0^{\infty}\phi_{w,\delta=0}(y)dy - \int_0^u h_1(x)\phi_{w,\delta=0}(u-x)dx\right] \\ &= \frac{1}{\lambda p_1\theta}h_1(u)\int_0^{\infty}B_{\delta=0}(y)dy - \frac{1}{\lambda p_1}\int_0^u h_1(u-x)\phi_{w,\delta=0}(x)dx.\end{aligned}$$

Apply (1.44) again with $\delta = 0$ and by the associative property of convolution,

$$\begin{aligned}\psi_{1,w}(u) &= -\frac{1}{\theta}\int_0^u \overline{K}_{\delta=0}(u-x)B'_{1,w}(x)dx + \frac{1}{\theta}B_{1,w}(u) \\ &= \frac{1}{\lambda p_1\theta}\int_0^u \overline{K}_{\delta=0}(u-x)\int_0^x h_1(x-y)\phi_{w,\delta=0}(y)dydx + \frac{1}{\theta}B_{1,w}(u) - \\ &\quad \frac{1}{\lambda p_1\theta^2}\int_0^u \overline{K}_{\delta=0}(u-x)h_1(x)dx\int_0^{\infty}B_{\delta=0}(y)dy \\ &= \frac{1}{\lambda p_1\theta}\int_0^u \overline{K}_{\delta=0} * H_1(u-x)\phi_{w,\delta=0}(x)dx + \frac{1}{\theta}B_{1,w}(u) - \\ &\quad \frac{1}{\lambda p_1\theta^2}\overline{K}_{\delta=0} * H_1(u)\int_0^{\infty}B_{\delta=0}(y)dy, \qquad (3.88)\end{aligned}$$

which is (3.70), by (3.75) and $\overline{K_{\delta=0} * H}_1(u) = \overline{K}_{\delta=0} * H_1(u) + \overline{H}_1(u)$.

When $D \to 0$, $dP_1 * H_1(x) = h_2 * h_1(x)dx \to h_2(x)dx = dP_1(x)$ by (1.19) and (2.20), and

$$\int_0^u h_1(u-x)\int_x^{\infty}\phi_w(y)dydx$$

$$
\begin{aligned}
&= \int_0^u \int_x^\infty \phi_w(y)dyd\overline{H}_1(u-x) \\
&= \overline{H}_1(u-x)\int_x^\infty \phi_w(y)dy\Big|_0^u + \int_0^u \overline{H}_1(u-x)\phi_w(x)dx \\
&= \int_u^\infty \phi_w(x)dx - \overline{H}_1(u)\int_0^\infty \phi_w(x)dx + \int_0^u \overline{H}_1(u-x)\phi_w(x)dx \\
&\to \int_u^\infty \phi_0(x)dx, \ \ if \ \ u>0,
\end{aligned}
$$

which prove equations (3.76) and (3.82) for $u > 0$ from (3.69) and (3.74).

Similarly, when $D \to 0$, $B_{\delta=0}(u) = \frac{1}{p_1}\int_0^u h_1(u-x)\int_x^\infty \omega(y)dydx \to \frac{1}{p_1}\int_u^\infty \omega(x)dx = B_{0,\delta=0}(u)$, and all $\overline{K}_{\delta=0}(u)$, $\overline{K}_{\delta=0} * H_1(u)$ and $\overline{K_{\delta=0} * H_1}(u)$ approach to $\psi_0(u)$ for $u > 0$, which imply (3.73) $\to$ (3.81), (3.72) $\to$ (3.80), (3.75) $\to$ (3.83) and (3.70) $\to$ (3.77) for $u > 0$. In fact, equations (3.76)-(3.83) can be shown for both cases $u > 0$ and $u = 0$ by similar derivations based on the traditional risk model (1.1). □

We remark that $\int_u^\infty B_{\delta=0}(x)dx$ and $\int_0^\infty B_{\delta=0}(x)dx$ on the right side of (3.70) and (3.75) can be written as follows:

$$
\begin{aligned}
&\int_x^\infty B_{\delta=0}(y)dy \\
&= \frac{1}{p_1}\int_x^\infty \int_0^y h_1(y-t)\int_t^\infty \omega(s)dsdtdy \\
&= \frac{1}{p_1}\int_0^x \int_x^\infty h_1(y-t)\int_t^\infty \omega(s)dsdydt + \frac{1}{p_1}\int_x^\infty \int_t^\infty h_1(y-t)\int_t^\infty \omega(s)dsdydt \\
&= \frac{1}{p_1}\int_0^x \int_{x-t}^\infty h_1(y)\int_t^\infty \omega(s)dsdydt + \frac{1}{p_1}\left[\int_0^\infty h_1(y)dy\right]\int_x^\infty \int_t^\infty \omega(s)dsdt \\
&= \frac{1}{p_1}\int_0^x \overline{H}_1(x-t)\int_t^\infty \omega(s)dsdt + \frac{1}{p_1}\int_x^\infty \int_x^s \omega(s)dtds \\
&= \frac{1}{p_1}\int_0^x \overline{H}_1(x-t)\int_t^\infty \omega(s)dsdt + \frac{1}{p_1}\int_x^\infty (s-x)\omega(s)ds \qquad (3.89) \\
&\to \frac{1}{p_1}\int_x^\infty (s-x)\omega(s)ds = \int_x^\infty B_{0,\delta=0}(y)dy, \ \ when \ \ D \to 0, \qquad (3.90)
\end{aligned}
$$

and

$$\int_0^\infty B_{\delta=0}(y)dy = \frac{1}{p_1}\int_0^\infty s\omega(s)ds = \int_0^\infty B_{0,\delta=0}(y)dy. \tag{3.91}$$

Theorem 3.4 provides the explicit expression and the defective renewal equation for $\psi_{1,w}(u)$, the joint moment of the time of ruin T due to a claim, and the associated penalty function $w(U(T-), |U(T)|)$. If we consider the important special case $w(U(T-), |U(T)|) = |U(T)|^n$, and define

$$\psi_{1,n}(u) = E[T|U(T)|^n I(T < \infty, U(T) < 0)|U(0) = u], \ \ u \geq 0, \ \ n = 0, 1, 2, \ldots, \tag{3.92}$$

the joint moment of the time of ruin T caused by a claim, and the deficit to the n^{th} at ruin, then we have the following result.

Theorem 3.5 *For $u \geq 0$ and $n = 0, 1, 2, \ldots$, $\psi_{1,n}(u)$ in (3.92) satisfies the defective renewal equation*

$$\psi_{1,n}(u) = \frac{1}{1+\theta}\int_0^u \psi_{1,n}(u-x)dP_1 * H_1(x) + \frac{p_{n+1}}{(n+1)cp_1\theta}\tau_{n+1} * h_1(u), \tag{3.93}$$

and is given explicitly by

$$\begin{aligned} &\psi_{1,n}(u) \\ = \ & \frac{p_n}{\lambda p_1^2\theta^2}\int_0^u \overline{K_{\delta=0} * H}_1(u-x)\tau_n(x)dx - \frac{p_{n+2}}{(n+1)(n+2)\lambda p_1^2\theta^2}\overline{K_{\delta=0} * H}_1(u) \\ + \ & \frac{p_{n+1}}{(n+1)\lambda p_1^2\theta^2}\left[\tau_{n+1}(u) - \int_0^u \overline{K_{\delta=0} * H}_1(u-x)\overline{K_{\delta=0} * H}_1(x)dx\right] \end{aligned} \tag{3.94}$$

where $\tau_{n+1} * h_1(u) = \int_0^u h_1(u-x)\tau_{n+1}(x)dx$, $\tau_0(u) = p_1(1+\theta)\overline{K}_{\delta=0}(u)$, $\tau_1(u)$ *is given by (3.38),* $\tau_n(u)$ *is given by (3.39) for* $n = 2, 3, 4, \ldots$, *and* $\overline{K_{\delta=0} * H_1}(u)$ *can be replaced with* $\psi_t(u)$ *by (2.105).*

In addition, when $D \to 0$, *equation (3.93) reduces to*

$$\psi_{0;1,n}(u) = \frac{1}{1+\theta}\int_0^u \psi_{0;1,n}(u-x)dP_1(x) + \frac{p_{n+1}}{(n+1)cp_1\theta}\tau_{0;n+1}(u), \tag{3.95}$$

and is given explicitly by

$$\begin{aligned}\psi_{0;1,n}(u) &= \frac{p_n}{\lambda p_1^2\theta^2}\int_0^u \psi_0(u-x)\tau_{0;n}(x)dx - \frac{p_{n+2}}{(n+1)(n+2)\lambda p_1^2\theta^2}\psi_0(u)\\ &+ \frac{p_{n+1}}{(n+1)\lambda p_1^2\theta^2}\left[\tau_{0;n+1}(u) - \int_0^u \psi_0(u-x)\psi_0(x)dx\right]\end{aligned} \tag{3.96}$$

where $\tau_{0;0}(u) = p_1(1+\theta)\psi_0(u)$, $\tau_{0;1}(u)$ *is given by (3.40),* $\tau_{0;n}(u)$ *is given by (3.41) for* $n = 2, 3, 4, \ldots$, *and*

$$\psi_{0;1,n}(u) = E[T|U(T)|^n I(T<\infty)|U(0)=u], \ ,u \ge 0, \ ,n = 0, 1, 2, \ldots. \tag{3.97}$$

Proof: When $w(x,y) = y^n$ and $\delta = 0$, $\overline{\Gamma}_n(x) = \overline{P}_{n+1}(x)$, and from (2.77) and (3.50),

$$B_{\delta=0}(u) = \frac{p_{n+1}}{(n+1)p_1}\overline{P}_{n+1} * H_1(u). \tag{3.98}$$

By (3.30) and (3.86),

$$\frac{1}{1+\theta}B_{1,w}(u)$$

$$\begin{aligned}
&= \frac{1}{\lambda p_1\theta(1+\theta)}\frac{p_{n+1}}{(n+1)p_1}\left[\int_0^u h_1(u-x)\int_0^x \overline{K}_{\delta=0}(x-y)\overline{P}_{n+1} * H_1(y)dydx + \right. \\
&\qquad \left. \int_0^u h_1(u-x)\int_x^\infty \overline{P}_{n+1} * H_1(y)dydx\right] \\
&= \frac{p_{n+1}}{(n+1)cp_1\theta}\int_0^u h_1(u-x)\left[\int_0^x \overline{K}_{\delta=0}(x-y)\overline{P}_{n+1} * H_1(y)dy + \right. \\
&\qquad \left. \int_x^\infty \overline{P}_{n+1} * H_1(y)dy\right]dx \\
&= \frac{p_{n+1}}{(n+1)cp_1\theta}\int_0^u h_1(u-x)\tau_{n+1}(x)dx \\
&= \frac{p_{n+1}}{(n+1)cp_1\theta}\tau_{n+1} * h_1(u),
\end{aligned}$$

which is the second term of the right side of (3.93). And hence the second term of the right side of (3.88),

$$\begin{aligned}
&\frac{1}{\theta}B_{1,w}(u) \\
&= \frac{p_{n+1}}{(n+1)\lambda p_1^2\theta^2}\tau_{n+1} * h_1(u) \\
&= \frac{p_{n+1}}{(n+1)\lambda p_1^2\theta^2}\int_0^u \tau_{n+1}(x)d\overline{H}_1(u-x) \\
&= \frac{p_{n+1}}{(n+1)\lambda p_1^2\theta^2}\left[\tau_{n+1}(x)\overline{H}_1(u-x)\Big|_0^u - \int_0^u \overline{H}_1(u-x)\tau'_{n+1}(x)dx\right] \\
&= \frac{p_{n+1}}{(n+1)\lambda p_1^2\theta^2}\left[\tau_{n+1}(u) - \tau_{n+1}(0)\overline{H}_1(u) + \frac{(n+1)p_n}{p_{n+1}}\int_0^u \overline{H}_1(u-x)\tau_n(x)dx - \right. \\
&\qquad \left. \int_0^u \overline{H}_1(u-x)\overline{K_{\delta=0} * H}_1(x)dx\right] \\
&= \frac{p_{n+1}}{(n+1)\lambda p_1^2\theta^2}\tau_{n+1}(u) - \frac{p_{n+2}}{(n+1)(n+2)\lambda p_1^2\theta^2}\overline{H}_1(u) + \\
&\quad \frac{p_n}{\lambda p_1^2\theta^2}\int_0^u \overline{H}_1(u-x)\tau_n(x)dx - \frac{p_{n+1}}{(n+1)\lambda p_1^2\theta^2}\int_0^u \overline{H}_1(u-x)\overline{K_{\delta=0} * H}_1(x)dx,
\end{aligned}$$

by (3.31) and (3.36).

Next, using (3.53), $\phi_{w,\delta=0}(u)$ in (3.71) becomes

$$\phi_{w,\delta=0}(u) = \frac{1}{p_1\theta}\Big[p_n\tau_n(u) - \frac{p_{n+1}}{n+1}\overline{K_{\delta=0} * H}_1(u)\Big]$$

Therefore, the first term of the right side of (3.88) turns out to be

$$\begin{aligned}
&\frac{1}{\lambda p_1^2\theta^2}\int_0^u \overline{K}_{\delta=0} * H_1(u-x)\Big[p_n\tau_n(x) - \frac{p_{n+1}}{n+1}\overline{K_{\delta=0} * H}_1(x)\Big]dx \\
= \; &\frac{p_n}{\lambda p_1^2\theta^2}\int_0^u \overline{K}_{\delta=0} * H_1(u-x)\tau_n(x)dx - \\
&\frac{p_{n+1}}{(n+1)\lambda p_1^2\theta^2}\int_0^u \overline{K}_{\delta=0} * H_1(u-x)\overline{K_{\delta=0} * H}_1(x)dx.
\end{aligned}$$

Finally, from (3.31), the third term of the right side of (3.88) is simplied to

$$\begin{aligned}
&-\frac{1}{\lambda p_1\theta^2}\frac{p_{n+1}}{(n+1)p_1}\overline{K}_{\delta=0} * H_1(u)\int_0^\infty \overline{P}_{n+1} * H_1(y)dy \\
= \; &-\frac{p_{n+1}}{(n+1)\lambda p_1^2\theta^2}\frac{p_{n+2}}{(n+2)p_{n+1}}\overline{K}_{\delta=0} * H_1(u) \\
= \; &-\frac{p_{n+2}}{(n+1)(n+2)\lambda p_1^2\theta^2}\overline{K}_{\delta=0} * H_1(u).
\end{aligned}$$

Adding up together these three alternative expressions above for the right side of (3.88) gives (3.94) using $\overline{K_{\delta=0} * H}_1(u) = \overline{K}_{\delta=0} * H_1(u) + \overline{H}_1(u)$.

As regarding (3.95) and (3.96), they can be shown either by similar arguments above from (3.76) and (3.77) or by letting $D \to 0$, $\tau_{n+1} * h_1(u) \to \tau_{0;n+1}(u)$, $\tau_n(u) \to \tau_{0;n}(u)$ and $\overline{K_{\delta=0} * H}_1(u) \to \psi_0(u)$ in (3.93) and (3.94). The latter approach, however, can show only for $u > 0$. □

Corollary 3.6

$$\psi_{1,1}(u) = E[T|U(T)|I(T < \infty, U(T) < 0)|U(0) = u], \;\; u \geq 0, \tag{3.99}$$

satisfies the defective renewal equation

$$\begin{aligned}
&\psi_{1,1}(u) \\
= \;& \frac{1}{1+\theta}\int_0^u \psi_{1,1}(u-x)dP_1 * H_1(x) + \frac{1}{c}\int_0^u h_1(u-x)\int_x^\infty (y-x)\overline{K}_{\delta=0}(y)dydx \\
- \;& \frac{2Dp_1 + cp_2}{2c^2 p_1\theta}\int_0^u h_1(u-x)\int_x^\infty \overline{K_{\delta=0} * H}_1(y)dydx
\end{aligned} \tag{3.100}$$

and is given explicitly by

$$\begin{aligned}
&\psi_{1,1}(u) \\
= \;& \frac{1}{\lambda p_1\theta}\Big[\int_0^u \overline{K_{\delta=0} * H}_1(u-x)\int_x^\infty \overline{K_{\delta=0} * H}_1(y)dydx + \int_0^u (u-x)\overline{K}_{\delta=0}(x)dx\Big] \\
- \;& \frac{2D + \lambda p_2}{2\lambda^2 p_1^2\theta^2}\int_0^u \overline{K_{\delta=0} * H}_1(u-x)\overline{K_{\delta=0} * H}_1(x)dx \\
- \;& \frac{2Dp_1 + cp_2}{2c\lambda p_1^2\theta^2}\int_u^\infty \overline{K_{\delta=0} * H}_1(x)dx - \frac{p_3}{6\lambda p_1^2\theta^2}\overline{K_{\delta=0} * H}_1(u)
\end{aligned} \tag{3.101}$$

where $\overline{K_{\delta=0} * H}_1(u)$ *can be replaced with* $\psi_t(u)$ *by (2.105).*

In addition, when $D \to 0$*, equation (3.100) reduces to*

$$\psi_{0;1,1}(u) = \frac{1}{1+\theta}\int_0^u \psi_{0;1,1}(u-x)dP_1(x) + \frac{1}{c}\int_u^\infty (x-u)\psi_0(x)dx - \frac{p_2}{2cp_1\theta}\int_u^\infty \psi_0(x)dx \tag{3.102}$$

and is given explicitly by

$$\begin{aligned}
&\psi_{0;1,1}(u) \\
= \;& \frac{1}{\lambda p_1 \theta}\Big[\int_0^u \psi_0(u-x)\int_x^\infty \psi_0(y)dydx + \int_u^\infty (x-u)\psi_0(x)dx\Big] \\
- \;& \frac{p_2}{2\lambda p_1^2\theta^2}\Big[\int_0^u \psi_0(u-x)\psi_0(x)dx + \int_u^\infty \psi_0(x)dx\Big] - \frac{p_3}{6\lambda p_1^2\theta^2}\psi_0(u) \quad (3.103)
\end{aligned}$$

where

$$\psi_{0;1,1}(u) = E[T|U(T)|I(T<\infty)|U(0)=u], \;\; u \geq 0. \qquad (3.104)$$

Proof: When $n=1$, $\tau_{n+1}(x)$ in (3.93) becomes $\tau_2(x) = \frac{2p_1\theta}{p_2}\int_x^\infty (y-x)\overline{K}_{\delta=0}(y)dy - \frac{2Dp_1+cp_2}{cp_2}\int_x^\infty \overline{K_{\delta=0} * H_1}(y)dy$ by (3.39) with $n=2$, which leads to (3.100).

Combining the expression for $\tau_2(u)$ with $\tau_1(x) = \theta\int_x^\infty \overline{K_{\delta=0} * H_1}(y)dy - \frac{D}{\lambda p_1}\overline{K_{\delta=0} * H_1}(x)$ by (3.38), equation (3.94) with $n=1$ becomes

$$\begin{aligned}
&\psi_{1,1}(u) \\
= \;& \frac{1}{\lambda p_1\theta^2}\int_0^u \overline{K_{\delta=0} * H_1}(u-x)\Big[\theta\int_x^\infty \overline{K_{\delta=0} * H_1}(y)dy - \frac{D}{\lambda p_1}\overline{K_{\delta=0} * H_1}(x)\Big]dx \\
+ \;& \frac{p_2}{2\lambda p_1^2\theta^2}\Big[\frac{2p_1\theta}{p_2}\int_u^\infty (x-u)\overline{K}_{\delta=0}(x)dx - \frac{2Dp_1+cp_2}{cp_2}\int_u^\infty \overline{K_{\delta=0} * H_1}(x)dx - \\
& \int_0^u \overline{K_{\delta=0} * H_1}(u-x)\overline{K_{\delta=0} * H_1}(x)dx\Big] - \frac{p_3}{6\lambda p_1^2\theta^2}\overline{K_{\delta=0} * H_1}(u),
\end{aligned}$$

which is (3.101) after some rearrangements.

(3.102) and (3.103) can be shown either by similar arguments from (3.95) and (3.96) with $\tau_{0;1}(x) = \theta\int_x^\infty \psi_0(y)dy$ and $\tau_{0;2}(x) = \frac{2p_1\theta}{p_2}\int_x^\infty (y-x)\psi_0(y)dy - \int_x^\infty \psi_0(y)dy$ by (3.40) and (3.41), or by letting $D \to 0$, both $\overline{K}_{\delta=0}(u)$ and $\overline{K_{\delta=0} * H_1}(u) \to \psi_0(u)$ in (3.100) and (3.101). The latter approach, however, can show only for $u > 0$. □

Example 3.3 Combination of exponentials and mixture of Erlangs ($b \neq \mu$)

As shown in example 2.5 and example 2.6 that $\overline{K_{\delta=0} * H_1}(u) = \frac{c\theta}{D}\sum_{k=1}^{r+1}\frac{D_{k,\delta=0}}{c/D - s_{k,\delta=0}}e^{-s_{k,\delta=0}u}$ and $\overline{K}_{\delta=0}(u) = \theta\sum_{k=1}^{r+1} D_{k,\delta=0}e^{-s_{k,\delta=0}u}$. Then

$$\int_u^\infty \overline{K_{\delta=0} * H_1}(x)dx = \frac{c\theta}{D}\sum_{k=1}^{r+1}\frac{D_{k,\delta=0}}{(c/D - s_{k,\delta=0})s_{k,\delta=0}}e^{-s_{k,\delta=0}u}. \tag{3.105}$$

The integrations $\int_0^u (u-x)e^{-s_{k,\delta=0}x}dx = -\frac{1}{s_{k,\delta=0}}\left[(u-x)e^{-s_{k,\delta=0}x}\Big|_0^u + \int_0^u e^{-s_{k,\delta=0}x}dx\right]$ $= \frac{u}{s_{k,\delta=0}} + \frac{e^{-s_{k,\delta=0}u}-1}{s^2_{k,\delta=0}}$ and

$$\int_0^u e^{-s_{k,\delta=0}(u-x)}e^{-s_{j,\delta=0}x}dx = \begin{cases} ue^{-s_{k,\delta=0}u}, & \text{if } j = k, \\ \dfrac{e^{-s_{j,\delta=0}u} - e^{-s_{k,\delta=0}u}}{s_{k,\delta=0} - s_{j,\delta=0}}, & \text{if } j \neq k \end{cases} \tag{3.106}$$

lead to

$$\begin{aligned}\int_0^u (u-x)\overline{K}_{\delta=0}(x)dx &= \theta\sum_{k=1}^{r+1} D_{k,\delta=0}\int_0^u (u-x)e^{-s_{k,\delta=0}x}dx \\ &= \theta\sum_{k=1}^{r+1} D_{k,\delta=0}\left[\frac{u}{s_{k,\delta=0}} + \frac{e^{-s_{k,\delta=0}u}-1}{s^2_{k,\delta=0}}\right],\end{aligned} \tag{3.107}$$

$$\begin{aligned}&\int_0^u \overline{K_{\delta=0} * H_1}(u-x)\int_x^\infty \overline{K_{\delta=0} * H_1}(y)dydx \\ = &\frac{c^2\theta^2}{D^2}\sum_{k=1}^{r+1}\sum_{j=1}^{r+1}\frac{D_{k,\delta=0}D_{j,\delta=0}}{(c/D - s_{k,\delta=0})(c/D - s_{j,\delta=0})s_{j,\delta=0}}\int_0^u e^{-s_{k,\delta=0}(u-x)}e^{-s_{j,\delta=0}x}dx\end{aligned}$$

$$= \frac{c^2\theta^2}{D^2}\sum_{k=1}^{r+1}\sum_{j=1,j\neq k}^{r+1}\frac{D_{k,\delta=0}D_{j,\delta=0}}{(c/D-s_{k,\delta=0})(c/D-s_{j,\delta=0})(s_{k,\delta=0}-s_{j,\delta=0})s_{j,\delta=0}} \left[e^{-s_{j,\delta=0}u}-e^{-s_{k,\delta=0}u}\right]+\frac{c^2\theta^2 u}{D^2}\sum_{k=1}^{r+1}\left(\frac{D_{k,\delta=0}}{c/D-s_{k,\delta=0}}\right)^2\frac{1}{s_{k,\delta=0}}e^{-s_{k,\delta=0}u} \quad (3.108)$$

and

$$\begin{aligned}
&\int_0^u \overline{K_{\delta=0}*H}_1(u-x)\overline{K_{\delta=0}*H}_1(x)dx \\
&= \frac{c^2\theta^2}{D^2}\sum_{k=1}^{r+1}\sum_{j=1}^{r+1}\frac{D_{k,\delta=0}D_{j,\delta=0}}{(c/D-s_{k,\delta=0})(c/D-s_{j,\delta=0})}\int_0^u e^{-s_{k,\delta=0}(u-x)}e^{-s_{j,\delta=0}x}dx \\
&= \frac{c^2\theta^2}{D^2}\sum_{k=1}^{r+1}\sum_{j=1,j\neq k}^{r+1}\frac{D_{k,\delta=0}D_{j,\delta=0}}{(c/D-s_{k,\delta=0})(c/D-s_{j,\delta=0})(s_{k,\delta=0}-s_{j,\delta=0})} \\
&\quad \left[e^{-s_{j,\delta=0}u}-e^{-s_{k,\delta=0}u}\right]+\frac{c^2\theta^2 u}{D^2}\sum_{k=1}^{r+1}\left(\frac{D_{k,\delta=0}}{c/D-s_{k,\delta=0}}\right)^2 e^{-s_{k,\delta=0}u}. \quad (3.109)
\end{aligned}$$

Therefore, (3.101) becomes

$$\begin{aligned}
\psi_{1,1}(u) &= c\theta(1+\theta)\sum_{k=1}^{r+1}\sum_{j=1,j\neq k}^{r+1}\left[\frac{1}{s_{j,\delta=0}}-\frac{2D+\lambda p_2}{2\lambda p_1\theta}\right] \\
&\quad \frac{D_{k,\delta=0}D_{j,\delta=0}\left[e^{-s_{j,\delta=0}u}-e^{-s_{k,\delta=0}u}\right]}{(c-Ds_{k,\delta=0})(c-Ds_{j,\delta=0})(s_{k,\delta=0}-s_{j,\delta=0})}+ \\
&\quad c\theta(1+\theta)u\sum_{k=1}^{r+1}\left[\frac{1}{s_{k,\delta=0}}-\frac{2D+\lambda p_2}{2\lambda p_1\theta}\right]\left(\frac{D_{k,\delta=0}}{c-Ds_{k,\delta=0}}\right)^2 e^{-s_{k,\delta=0}u}- \\
&\quad \frac{2Dp_1+cp_2}{2\lambda p_1^2\theta}\sum_{k=1}^{r+1}\frac{D_{k,\delta=0}}{(c-Ds_{k,\delta=0})s_{k,\delta=0}}e^{-s_{k,\delta=0}u}- \\
&\quad \frac{cp_3}{6\lambda p_1^2\theta}\sum_{k=1}^{r+1}\frac{D_{k,\delta=0}}{c-Ds_{k,\delta=0}}e^{-s_{k,\delta=0}u}+ \\
&\quad \frac{1}{\lambda p_1}\left[u\sum_{k=1}^{r+1}\frac{D_{k,\delta=0}}{s_{k,\delta=0}}-\sum_{k=1}^{r+1}\frac{D_{k,\delta=0}}{s_{k,\delta=0}^2}+\sum_{k=1}^{r+1}\frac{D_{k,\delta=0}}{s_{k,\delta=0}^2}e^{-s_{k,\delta=0}u}\right]. \quad (3.110)
\end{aligned}$$

after some rearrangements. □

3.4 Moment of the time of ruin due to oscillation

In this and the next sections, we are going to derive the n^{th} moment of the time of ruin caused by oscillation, and the n^{th} moment of the time of ruin due to a claim. We will see that these two kinds of n^{th} moments have the same recursive expressions.

Theorem 3.6 *For $u \geq 0$ and $n = 1, 2, 3, \ldots$, the n^{th} moment of the time of ruin due to oscillation, if this kind of ruin occurs,*

$$\psi_{d;n}(u) = E[T^n I(T < \infty, U(T) = 0) | U(0) = u], \tag{3.111}$$

satisfies a sequence of integro-difference equations

$$\psi_{d;n}(u) = \frac{1}{1+\theta} \int_0^u \psi_{d;n}(u-x) dP_1 * H_1(x) + \frac{n}{c} \int_0^u h_1(u-x) \int_x^\infty \psi_{d;n-1}(y) dy dx \tag{3.112}$$

and is given recursively by

$$\begin{aligned} \psi_{d;n}(u) &= \frac{n}{\lambda p_1 \theta} \Big[\int_0^u \overline{K_{\delta=0} * H}_1(u-x) \psi_{d;n-1}(x) dx + \int_u^\infty \psi_{d;n-1}(x) dx - \\ & \qquad \overline{K_{\delta=0} * H}_1(u) \int_0^\infty \psi_{d;n-1}(x) dx \Big] \end{aligned} \tag{3.113}$$

in terms of $\psi_{d;n-1}(u)$ with $\psi_{d;0}(u) = \psi_d(u)$ and

$$\int_0^\infty \psi_{d;n}(u)du = \frac{n}{\lambda p_1 \theta}\int_0^\infty u\psi_{d;n-1}(u)du, \tag{3.114}$$

where $\overline{K_{\delta=0} * H}_1(u)$ can be replaced with $\psi_t(u)$ by (2.105).

In addition, when $D \to 0$, $\psi_{d;n}(u) \to 0$, $n = 0, 1, 2, \ldots$

Proof: Perform the Laplace transform on the both sides of (1.22), we get

$$\tilde{\phi}_d(\xi,\delta) = \frac{\lambda[\tilde{p}(\xi) - 1] + D\rho^2 + c\rho - \delta}{(b+\xi)(\rho-\xi)D}\tilde{\phi}_d(\xi,\delta) + \frac{1}{b+\xi}$$

by (2.19), where $\tilde{\phi}_d(\xi,\delta) = \int_0^\infty e^{-\xi u}\phi_d(u)du$. With $\tilde{p}_1(\xi) = \dfrac{1-\tilde{p}(\xi)}{p_1\xi}$ and $[(b+\xi)(\rho-\xi)D] - \{\lambda[\tilde{p}(\xi)-1] + D\rho^2 + c\rho - \delta\} = [D(\rho^2-\xi^2) + c(\rho-\xi)] - [-\lambda p_1 \xi \tilde{p}_1(\xi) + D\rho^2 + c\rho - \delta] = -\{D\xi^2 + c\xi[1 - \frac{1}{1+\theta}\tilde{p}_1(\xi)] - \delta\}$, the equation above can be written as

$$\Big\{D\xi^2 + c\xi\Big[1 - \frac{1}{1+\theta}\tilde{p}_1(\xi)\Big] - \delta\Big\}\tilde{\phi}_d(\xi,\delta) = D[\xi - \rho]. \tag{3.115}$$

Since $\phi_d(u) = E[e^{-\delta T}I(T<\infty, U(T)=0)]$, $\psi_{d;n}(u) = (-1)^n \frac{d^n}{d\delta^n}\phi_d(u)\Big|_{\delta=0}$, $\tilde{\psi}_{d;n}(\xi) = \int_0^\infty e^{-\xi u}\psi_{d;n}(u)du = \int_0^\infty e^{-\xi u}(-1)^n \frac{d^n}{d\delta^n}\phi_d(u)\Big|_{\delta=0} du =$ $(-1)^n \frac{d^n}{d\delta^n}\tilde{\phi}_d(\xi,\delta)\Big|_{\delta=0} = (-1)^n \tilde{\phi}_d^{(n)}(\xi, 0)$, $n = 0, 1, 2, \ldots$, and $\rho(0) = 0$, differentiating (3.115) both sides n ($n \geq 1$) times with respect to δ and then setting $\delta = 0$ lead to

$$\begin{aligned}
&\Big\{D\xi^2 + c\xi\Big[1 - \frac{1}{1+\theta}\tilde{p}_1(\xi)\Big]\Big\}\frac{d^n}{d\delta^n}\tilde{\phi}_d(\xi,\delta)\Big|_{\delta=0} - \frac{d^n}{d\delta^n}\Big[\delta\tilde{\phi}_d(\xi,\delta) - D\rho(\delta)\Big]\Big|_{\delta=0} \\
= \;& (-1)^n\Big\{D\xi^2 + c\xi\Big[1 - \frac{1}{1+\theta}\tilde{p}_1(\xi)\Big]\Big\}\tilde{\psi}_{d;n}(\xi) - n(-1)^{n-1}\tilde{\psi}_{d;n-1}(\xi) + D\rho^{(n)}(0) \\
= \;& 0.
\end{aligned}$$

Letting $\xi = 0$ gives $n(-1)^{n-1}\tilde{\psi}_{d;n-1}(0) = D\rho^{(n)}(0)$. Thus, equation (3.115) becomes

$$(-1)^n\left\{D\xi^2 + c\xi\left[1 - \frac{1}{1+\theta}\tilde{p}_1(\xi)\right]\right\}\tilde{\psi}_{d;n}(\xi) = n(-1)^{n-1}\left[\tilde{\psi}_{d;n-1}(\xi) - \tilde{\psi}_{d;n-1}(0)\right],$$

or equivalently,

$$\left\{D\xi^2 + c\xi\left[1 - \frac{1}{1+\theta}\tilde{p}_1(\xi)\right]\right\}\tilde{\psi}_{d;n}(\xi) = n\left[\tilde{\psi}_{d;n-1}(0) - \tilde{\psi}_{d;n-1}(\xi)\right]. \tag{3.116}$$

Dividing (3.116) by $D\xi^2 + c\xi$ leads to

$$\left[1 - \frac{1}{1+\theta}\frac{c/D}{\xi + c/D}\tilde{p}_1(\xi)\right]\tilde{\psi}_{d;n}(\xi) = \frac{n}{c}\frac{c/D}{\xi + c/D}\frac{\tilde{\psi}_{d;n-1}(0) - \tilde{\psi}_{d;n-1}(\xi)}{\xi},$$

which is the Laplace transform of (3.112). The uniqueness of the Laplace transform gives (3.112).

Now apply (1.44) with $B(u) = \frac{n(1+\theta)}{c}\int_0^u h_1(x)\int_{u-x}^\infty \psi_{d;n-1}(y)dydx$ and $\delta = 0$, then

$$B'(u) = \frac{n}{\lambda p_1}\left[h_1(u)\int_0^\infty \psi_{d;n-1}(x)dx - \int_0^u h_1(x)\psi_{d;n-1}(u-x)dx\right],$$

and hence by the associative property of convolution, $\psi_{d;n}(u)$ can be written as

$$\begin{aligned}
&\psi_{d;n}(u) \\
= \;& \frac{n}{\lambda p_1\theta}\left[\int_0^u \overline{K}_{\delta=0} * H_1(u-x)\psi_{d;n-1}(x)dx + \int_0^u h_1(u-x)\int_x^\infty \psi_{d;n-1}(y)dydx - \right. \\
&\left. \overline{K}_{\delta=0} * H_1(u)\int_0^\infty \psi_{d;n-1}(x)dx\right].
\end{aligned} \tag{3.117}$$

Next, using integration by parts,

$$\begin{aligned}
&\int_0^u h_1(u-x)\int_x^\infty \psi_{d;n-1}(y)dydx \\
=\ &\int_0^u\int_x^\infty \psi_{d;n-1}(y)dyd\overline{H}_1(u-x) \\
=\ &\overline{H}_1(u-x)\int_x^\infty \psi_{d;n-1}(y)dy\Big|_0^u + \int_0^u \overline{H}_1(u-x)\psi_{d;n-1}(x)dx \\
=\ &\int_u^\infty \psi_{d;n-1}(y)dy - \overline{H}_1(u)\int_0^\infty \psi_{d;n-1}(y)dy + \int_0^u \overline{H}_1(u-x)\psi_{d;n-1}(x)dx.
\end{aligned}$$

Combining this with $\overline{K_{\delta=0} * H_1(u)} = \overline{K}_{\delta=0} * H_1(u) + \overline{H}_1(u)$ gives (3.113) from (3.117).

Since

$$\begin{aligned}
\int_0^\infty\int_0^u \psi_{d;n}(u-x)dP_1 * H_1(x)du &= \left[\int_0^\infty \psi_{d;n}(u)du\right]\left[\int_0^\infty dP_1 * H_1(x)\right] \\
&= \int_0^\infty \psi_{d;n}(u)du
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^\infty\int_0^u h_1(u-x)\int_x^\infty \psi_{d;n-1}(y)dydxdu \\
=\ &\int_0^\infty\int_x^\infty h_1(u-x)\int_x^\infty \psi_{d;n-1}(y)dydudx \\
=\ &\left[\int_0^\infty h_1(u)du\right]\left[\int_0^\infty\int_x^\infty \psi_{d;n-1}(y)dydx\right] \\
=\ &\int_0^\infty\int_0^y \psi_{d;n-1}(y)dxdy \\
=\ &\int_0^\infty y\psi_{d;n-1}(y)dy,
\end{aligned}$$

integrating (3.112) from $u = 0$ to $u = \infty$ gives (3.114).

From (3.113), we know that $\psi_{d;n}(u)$ is expressed recursively in terms of $\psi_{d;n-1}(u)$, $n = 1,2,3,\ldots$, and hence is expressed in terms of $\psi_{d;0}(u) = \psi_d(u)$. Since when $D \to 0$, $\psi_d(u) \to 0$ by theorem 2.3, we have $\psi_{d;n}(u) \to 0$ for $n = 0,1,2,\ldots$ □

The mean of the time of ruin caused by oscillation can be obtained by letting $n = 1$ in (3.112) and (3.113).

Corollary 3.7 *The mean time to ruin due to oscillation, if this kind of ruin occurs,*

$$\psi_{d;1}(u) = E[TI(T < \infty, U(T) = 0)|U(0) = u], \tag{3.118}$$

satisfies the defective renewal equation

$$\psi_{d;1}(u) = \frac{1}{1+\theta}\int_0^u \psi_{d;1}(u-x)dP_1 * H_1(x) + \frac{1}{\lambda p_1 \theta}\int_0^u \overline{K_{\delta=0} * H}_1(u-x)\overline{H}_1(x)dx \tag{3.119}$$

and is given explicitly by

$$\psi_{d;1}(u) = \frac{1+\theta}{\lambda p_1 \theta^2}\int_0^u \overline{K_{\delta=0} * H}_1(u-x)\overline{H}_1 * K_{\delta=0}(x)dx \tag{3.120}$$

with

$$\int_0^\infty \psi_{d;1}(u)du = \frac{1}{\lambda p_1 \theta}\int_0^\infty u\psi_d(u)du = \frac{\lambda D p_2 + 2D^2}{2\lambda^3 p_1^3 \theta^3}, \tag{3.121}$$

where $\overline{H}_1 * K_{\delta=0}(u) = \overline{K_{\delta=0} * H}_1(u) - \overline{K}_{\delta=0}(u)$ *can be replaced with* $\frac{\theta}{1+\theta}\psi_d(u)$ *by (2.104) with* $\delta = 0$, *and* $\overline{K_{\delta=0} * H}_1(u)$ *can be replaced with* $\psi_t(u)$ *by (2.105).*

Proof: Since $\psi_d(x) = \frac{1+\theta}{\theta}\left[\overline{K_{\delta=0} * H}_1(x) - \overline{K}_{\delta=0}(x)\right] = \frac{1+\theta}{\theta}\overline{H}_1 * K_{\delta=0}(x)$ by (2.101) and (2.104) with $\delta = 0$, $\int_u^\infty \psi_d(x)dx = \frac{D(1+\theta)}{c\theta}\overline{K_{\delta=0} * H}_1(u)$ by (2.110) with $\delta = 0$, and $\int_0^\infty \psi_d(x)dx = \frac{D(1+\theta)}{c\theta}$ by (2.111) with $\delta = 0$, we have from (3.113) with $n = 1$

$$\begin{aligned}\psi_{d;1}(u) &= \frac{1}{\lambda p_1\theta}\Big[\int_0^u \overline{K_{\delta=0} * H_1}(u-x)\psi_d(x)dx + \int_u^\infty \psi_d(x)dx - \\ &\qquad \overline{K_{\delta=0} * H_1}(u)\int_0^\infty \psi_d(x)dx\Big] \\ &= \frac{1}{\lambda p_1\theta}\Big[\int_0^u \overline{K_{\delta=0} * H_1}(u-x)\frac{1+\theta}{\theta}\overline{H}_1 * K_{\delta=0}(x)dx + \\ &\qquad \frac{D(1+\theta)}{c\theta}\overline{K_{\delta=0} * H_1}(u) - \frac{D(1+\theta)}{c\theta}\overline{K_{\delta=0} * H_1}(u)\Big],\end{aligned}$$

which is (3.120). Again, using $\int_x^\infty \psi_d(y)dy = \frac{D(1+\theta)}{c\theta}\overline{K_{\delta=0} * H_1}(x)$ by (2.110) with $\delta = 0$, the second term of (3.112) with $n = 1$ becomes $\frac{1}{c}\int_0^u h_1(u-x)\frac{D(1+\theta)}{c\theta}\overline{K_{\delta=0} * H_1}(x)dx = \frac{1}{\lambda p_1\theta}\int_0^u \overline{K_{\delta=0} * H_1}(u-x)\overline{H}_1(x)dx$, which is the second term of (3.119).

Clearly, letting $n = 1$ in (3.114) gives $\int_0^\infty \psi_{d;1}(u)du = \frac{1}{\lambda p_1\theta}\int_0^\infty u\psi_d(u)du = \frac{1}{\lambda p_1\theta}\frac{D(1+\theta)}{c\theta}\Big[\frac{\mu_{G,1}(0)}{\theta} + \frac{D}{c}\Big] = \frac{D}{\lambda^2 p_1^2\theta^2}\Big[\frac{p_2}{2p_1\theta} + \frac{D}{c\theta} + \frac{D}{c}\Big] = \frac{\lambda D p_2 + 2D^2}{2\lambda^3 p_1^3\theta^3}$ by (2.76) and (2.112) with $\delta = 0$. □

Example 3.4 Combination of exponentials and mixture of Erlangs ($b \neq \mu$)

As shown in examples 2.5 and 2.6 that
$\overline{K_{\delta=0} * H_1}(u) = \psi_t(u) = \frac{c\theta}{D}\sum_{k=1}^{r+1}\frac{D_{k,\delta=0}}{c/D - s_{k,\delta=0}}e^{-s_{k,\delta=0}u}$ and
$\psi_d(u) = \frac{1+\theta}{\theta}\Big[\overline{K_{\delta=0} * H_1}(u) - \overline{K}_{\delta=0}(u)\Big] = (1+\theta)\sum_{k=1}^{r+1}\frac{D_{k,\delta=0}s_{k,\delta=0}}{c/D - s_{k,\delta=0}}e^{-s_{k,\delta=0}u}$. Then (3.120) becomes

$$
\begin{aligned}
&\psi_{d;1}(u) \\
&= \frac{c(1+\theta)}{\lambda p_1 D}\sum_{k=1}^{r+1}\sum_{j=1}^{r+1}\frac{D_{k,\delta=0}D_{j,\delta=0}s_{j,\delta=0}}{(c/D-s_{k,\delta=0})(c/D-s_{j,\delta=0})}\int_0^u e^{-s_{k,\delta=0}(u-x)}e^{-s_{j,\delta=0}x}dx \\
&= D(1+\theta)^2\sum_{k=1}^{r+1}\sum_{j=1,j\neq k}^{r+1}\frac{D_{k,\delta=0}D_{j,\delta=0}s_{j,\delta=0}}{(c-Ds_{k,\delta=0})(c-Ds_{j,\delta=0})(s_{k,\delta=0}-s_{j,\delta=0})} \\
&\quad \left[e^{-s_{j,\delta=0}u}-e^{-s_{k,\delta=0}u}\right]+D(1+\theta)^2 u\sum_{k=1}^{r+1}\left(\frac{D_{k,\delta=0}}{c-Ds_{k,\delta=0}}\right)^2 s_{k,\delta=0}e^{-s_{k,\delta=0}u} \quad (3.122)
\end{aligned}
$$

with the help of (3.106), which is a combination of exponential functions. □

Corollary 3.8 *For* $u \geq 0$*, the second moment of the time of ruin due to oscillation, if this kind of ruin occurs,*

$$\psi_{d;2}(u) = E[T^2 I(T<\infty, U(T)=0)|U(0)=u], \tag{3.123}$$

satisfies the defective renewal equation

$$\psi_{d;2}(u) = \frac{1}{1+\theta}\int_0^u \psi_{d;2}(u-x)dP_1 * H_1(x) + \frac{1}{1+\theta}B_{d;2}(u) \tag{3.124}$$

with

$$
\begin{aligned}
B_{d;2}(u) \;=\; & \frac{2D}{\lambda^3 p_1^3\theta^2}\Big[\int_0^u h_1(u-x)\int_0^x \overline{K_{\delta=0} * H}_1(x-y)\overline{K_{\delta=0} * H}_1(y)dydx + \\
& \int_0^u h_1(u-x)\int_x^\infty \overline{K_{\delta=0} * H}_1(y)dydx\Big] \qquad (3.125)
\end{aligned}
$$

and is given explicitly by

$$\begin{aligned}\psi_{d;2}(u) &= \frac{2(1+\theta)}{\lambda^2 p_1^2\theta^3}\int_0^u \overline{K_{\delta=0}*H}_1(u-x)\int_0^x \overline{K_{\delta=0}*H}_1(x-y)\overline{H}_1 * K_{\delta=0}(y)dydx \\ &+ \frac{2D}{\lambda^3 p_1^3\theta^3}\Big[\int_0^u \overline{K_{\delta=0}*H}_1(u-x)\overline{K_{\delta=0}*H}_1(x)dx + \int_u^\infty \overline{K_{\delta=0}*H}_1(x)dx\Big] \\ &- \frac{\lambda D p_2 + 2D^2}{\lambda^4 p_1^4\theta^4}\overline{K_{\delta=0}*H}_1(u) \qquad (3.126)\end{aligned}$$

where $\overline{H}_1 * K_{\delta=0}(u) = \overline{K_{\delta=0}*H}_1(u) - \overline{K}_{\delta=0}(u)$ *can be replaced with* $\frac{\theta}{1+\theta}\psi_d(u)$ *by (2.104) with* $\delta = 0$, *and* $\overline{K_{\delta=0}*H}_1(u)$ *can be replaced with* $\psi_t(u)$ *by (2.105).*

Proof: First, for any two functions X() and Y(),

$$\begin{aligned}&\int_x^\infty\int_0^y X(y-t)Y(t)dtdy \\ &= \int_0^x\int_x^\infty X(y-t)Y(t)dydt + \int_x^\infty\int_t^\infty X(y-t)Y(t)dydt \\ &= \int_0^x\int_{x-t}^\infty X(y)Y(t)dydt + \int_x^\infty Y(t)dt\int_0^\infty X(y)dy \\ &= \int_0^x Y(x-t)\int_t^\infty X(y)dydt + \int_x^\infty Y(t)dt\int_0^\infty X(y)dy. \qquad (3.127)\end{aligned}$$

Then from (2.104), (2.106) and (2.107) with $\delta = 0$, and by (3.127), integrating (3.120) for $\psi_{d;1}(y)$ from $y = x$ to $y = \infty$ gives

$$\begin{aligned}&\int_x^\infty \psi_{d;1}(y)dy \\ &= \frac{1+\theta}{\lambda p_1\theta^2}\int_x^\infty\int_0^y \overline{K_{\delta=0}*H}_1(y-t)\overline{H}_1 * K_{\delta=0}(t)dtdy \\ &= \frac{1+\theta}{\lambda p_1\theta^2}\Big[\int_0^x \overline{H}_1 * K_{\delta=0}(x-t)\int_t^\infty \overline{K_{\delta=0}*H}_1(y)dydt + \\ &\qquad \int_x^\infty \overline{H}_1 * K_{\delta=0}(t)dt\int_0^\infty \overline{K_{\delta=0}*H}_1(y)dy\Big]\end{aligned}$$

$$
\begin{aligned}
&= \frac{1+\theta}{\lambda p_1 \theta^2}\frac{D}{c}\Big[\int_0^x \int_t^\infty \overline{K_{\delta=0} * H}_1(y)dyd\overline{K_{\delta=0} * H}_1(x-t) + \\
&\qquad \overline{K_{\delta=0} * H}_1(x)\int_0^\infty \overline{K_{\delta=0} * H}_1(y)dy\Big] \\
&= \frac{D}{\lambda^2 p_1^2 \theta^2}\Big[\overline{K_{\delta=0} * H}_1(x-t)\int_t^\infty \overline{K_{\delta=0} * H}_1(y)dy\Big|_0^x + \\
&\qquad \int_0^x \overline{K_{\delta=0} * H}_1(x-t)\overline{K_{\delta=0} * H}_1(t)dt + \\
&\qquad \overline{K_{\delta=0} * H}_1(x)\int_0^\infty \overline{K_{\delta=0} * H}_1(y)dy\Big] \\
&= \frac{D}{\lambda^2 p_1^2 \theta^2}\Big[\int_x^\infty \overline{K_{\delta=0} * H}_1(y)dy + \int_0^x \overline{K_{\delta=0} * H}_1(x-t)\overline{K_{\delta=0} * H}_1(t)dt\Big].
\end{aligned}
\tag{3.128}
$$

With this, (3.124) and (3.125) can be obtained directly from (3.112) with $n = 2$. Using (3.120), (3.121) and (3.128), equation (3.113) with $n = 2$ leads to (3.126). □

We remark that the variance of the time of ruin caused by oscillation can be obtained explicitly by (3.120) and (3.126).

3.5 Moment of the time of ruin caused by a claim

The next theorem shows that the moments of the time of ruin caused by a claim have the same recursive expression as the moments of the time of ruin caused by oscillation given in theorem 3.6.

Theorem 3.7 *For $u \geq 0$ and $n = 1, 2, 3, \ldots$, the n^{th} moment of the time of ruin due to a claim, if this kind of ruin occurs,*

$$\psi_{s;n}(u) = E[T^n I(T < \infty, U(T) < 0)|U(0) = u], \tag{3.129}$$

satisfies a sequence of integro-difference equations

$$\psi_{s;n}(u) = \frac{1}{1+\theta}\int_0^u \psi_{s;n}(u-x)dP_1 * H_1(x) + \frac{n}{c}\int_0^u h_1(u-x)\int_x^\infty \psi_{s;n-1}(y)dydx \tag{3.130}$$

and is given recursively by

$$\begin{aligned}\psi_{s;n}(u) \;=\; & \frac{n}{\lambda p_1\theta}\Big[\int_0^u \overline{K_{\delta=0} * H}_1(u-x)\psi_{s;n-1}(x)dx + \int_u^\infty \psi_{s;n-1}(x)dx - \\ & \overline{K_{\delta=0} * H}_1(u)\int_0^\infty \psi_{s;n-1}(x)dx\Big]\end{aligned} \tag{3.131}$$

in terms of $\psi_{s;n-1}(u)$ *with* $\psi_{s;0}(u) = \psi_s(u)$ *and*

$$\int_0^\infty \psi_{s;n}(u)du = \frac{n}{\lambda p_1\theta}\int_0^\infty u\psi_{s;n-1}(u)du. \tag{3.132}$$

where $\overline{K_{\delta=0} * H}_1(u)$ *can be replaced with* $\psi_t(u)$ *by (2.105).*

In addition, when $D \to 0$*, equation (3.130) reduces to*

$$\psi_{0;n}(u) = \frac{1}{1+\theta}\int_0^u \psi_{0;n}(u-x)dP_1(x) + \frac{n}{c}\int_u^\infty \psi_{0;n-1}(x)dx \tag{3.133}$$

and is given recursively by

$$\psi_{0;n}(u) = \frac{n}{\lambda p_1\theta}\Big[\int_0^u \psi_0(u-x)\psi_{0;n-1}(x)dx + \int_u^\infty \psi_{0;n-1}(x)dx - \psi_0(u)\int_0^\infty \psi_{0;n-1}(x)dx\Big] \tag{3.134}$$

in terms of $\psi_{0;n-1}(u)$ *with* $\psi_{0;0}(u) = \psi_0(u)$ *and*

$$\int_0^\infty \psi_{0;n}(u)du = \frac{n}{\lambda p_1\theta}\int_0^\infty u\psi_{0;n-1}(u)du. \tag{3.135}$$

where

$$\psi_{0;n}(u) = E[T^n I(T < \infty)|U(0) = u], \;\; u \geq 0. \tag{3.136}$$

Proof: When $w(x,y) = 1$, $\omega(x) = \overline{P}(x)$ by (2.7), and $\tilde{\omega}(\xi) = \int_0^\infty e^{-\xi x}\omega(x)dx = \frac{1-\tilde{p}(\xi)}{\xi} = p_1\tilde{p}_1(\xi)$. Therefore, the right side of (3.84), $\lambda\Big[\tilde{\omega}(\rho) - \tilde{\omega}(\xi)\Big] = \lambda\Big[\frac{1-\tilde{p}(\rho)}{\rho} - p_1\tilde{p}_1(\xi)\Big] = \frac{D\rho^2 + c\rho - \delta}{\rho} - \lambda p_1\tilde{p}_1(\xi) = \frac{1}{\rho}\Big\{D\rho^2 + c\rho\Big[1 - \frac{1}{1+\theta}\tilde{p}_1(\xi)\Big] - \delta\Big\}$ since ρ satisfies generalized Lundberg's equation (1.32), and then (3.84) with $\phi_w(u) = \phi_s(u)$ in the case $w(x,y) = 1$ can be written as

$$\Big\{D\xi^2 + c\xi\Big[1 - \frac{1}{1+\theta}\tilde{p}_1(\xi)\Big] - \delta\Big\}\rho(\delta)\tilde{\phi}_s(\xi,\delta) = D\rho(\delta)^2 + c\rho(\delta)\Big[1 - \frac{1}{1+\theta}\tilde{p}_1(\xi)\Big] - \delta. \tag{3.137}$$

Since $\rho(0) = 0$, differentiating (3.137) both sides n+1 ($n \geq 1$) times with respect to δ and then setting $\delta = 0$ lead the right side of (3.137) to

$$\begin{aligned} & D\frac{d^{n+1}}{d\delta^{n+1}}\rho(\delta)^2\Big|_{\delta=0} + c\Big[1 - \frac{1}{1+\theta}\tilde{p}_1(\xi)\Big]\frac{d^{n+1}}{d\delta^{n+1}}\rho(\delta)\Big|_{\delta=0} - \frac{d^{n+1}}{d\delta^{n+1}}\delta\Big|_{\delta=0} \\ = \;& D\sum_{k=1}^{n}\binom{n+1}{k}\rho^{(n+1-k)}(0)\rho^{(k)}(0) + c\Big[1 - \frac{1}{1+\theta}\tilde{p}_1(\xi)\Big]\rho^{n+1}(0) \end{aligned} \tag{3.138}$$

where $\rho^{(n)}$ is the n^{th} derivative of ρ, while the left side is

$$\begin{aligned} & \Big\{D\xi^2 + c\xi[1 - \frac{1}{1+\theta}\tilde{p}_1(\xi)]\Big\}\frac{d^{n+1}}{d\delta^{n+1}}\Big[\rho(\delta)\tilde{\phi}_s(\xi,\delta)\Big]\Big|_{\delta=0} - \frac{d^{n+1}}{d\delta^{n+1}}\Big[\delta\rho(\delta)\tilde{\phi}_s(\xi,\delta)\Big]\Big|_{\delta=0} \\ = \;& \Big\{D\xi^2 + c\xi[1 - \frac{1}{1+\theta}\tilde{p}_1(\xi)]\Big\}\frac{d^{n+1}}{d\delta^{n+1}}\Big[\rho(\delta)\tilde{\phi}_s(\xi,\delta)\Big]\Big|_{\delta=0} - \\ & (n+1)\frac{d^n}{d\delta^n}\Big[\rho(\delta)\tilde{\phi}_s(\xi,\delta)\Big]\Big|_{\delta=0} \end{aligned} \tag{3.139}$$

Moreover, since $\psi_{s;n}(u) = (-1)^n \frac{d^n}{d\delta^n}\phi_s(u)\Big|_{\delta=0}$ and $\tilde{\psi}_{s;n}(\xi) = \int_0^\infty e^{-\xi u}\psi_{s;n}(u)du$ $= \int_0^\infty e^{-\xi u}(-1)^n \frac{d^n}{d\delta^n}\phi_s(u)\Big|_{\delta=0} du = (-1)^n \frac{d^n}{d\delta^n}\tilde{\phi}_s(\xi,\delta)\Big|_{\delta=0} = (-1)^n \tilde{\phi}_s^{(n)}(\xi,0)$, $n = 0, 1, 2, \ldots$, we have

$$\begin{aligned}\frac{d^n}{d\delta^n}\Big[\rho(\delta)\tilde{\phi}_s(\xi,\delta)\Big]\Big|_{\delta=0} &= \sum_{k=0}^{n-1}\binom{n}{k}\rho^{(n-k)}(0)\tilde{\phi}_s^{(k)}(\xi,0)\\ &= \sum_{k=0}^{n-1}(-1)^k\binom{n}{k}\rho^{(n-k)}(0)\tilde{\psi}_{s;k}(\xi).\end{aligned}$$

Equating (3.138) and (3.139) gives

$$\begin{aligned}&\Big\{D\xi^2 + c\xi\Big[1 - \frac{1}{1+\theta}\tilde{p}_1(\xi)\Big]\Big\}\sum_{k=0}^{n}(-1)^k\binom{n+1}{k}\rho^{(n+1-k)}(0)\tilde{\psi}_{s;k}(\xi)\\ &= (n+1)\sum_{k=0}^{n-1}(-1)^k\binom{n}{k}\rho^{(n-k)}(0)\tilde{\psi}_{s;k}(\xi) + D\sum_{k=1}^{n}\binom{n+1}{k}\rho^{(n+1-k)}(0)\rho^{(k)}(0)\\ &+ c\Big[1 - \frac{1}{1+\theta}\tilde{p}_1(\xi)\Big]\rho^{n+1}(0). \qquad (3.140)\end{aligned}$$

Setting $\xi = 0$ in (3.140) leads to

$$\begin{aligned}&D\sum_{k=1}^{n}\binom{n+1}{k}\rho^{(n+1-k)}(0)\rho^{(k)}(0)\\ &= -(n+1)\sum_{k=0}^{n-1}(-1)^k\binom{n}{k}\rho^{(n-k)}(0)\tilde{\psi}_{s;k}(0) - c\Big[1 - \frac{1}{1+\theta}\Big]\rho^{n+1}(0).\end{aligned}$$

Since when $\delta \to 0$, $\dfrac{\delta}{\rho} \to c - \lambda p_1$ by (2.188); in the case (3.137) turns out to be

$$\Big\{D\xi^2+c\xi\Big[1-\frac{1}{1+\theta}\tilde{p}_1(\xi)\Big]\Big\}\tilde{\phi}_s(\xi,0) = c\Big[1-\frac{1}{1+\theta}\tilde{p}_1(\xi)\Big]-\Big[c-\lambda p_1\Big]= \frac{c}{1+\theta}\Big[1-\tilde{p}_1(\xi)\Big].$$

Combining these above with $\tilde{\psi}_{s;0}(\xi) = \tilde{\phi}_s(\xi,0)$, for $n \geq 1$, (3.140) can be simplied to

$$\begin{aligned}
&-(n+1)\sum_{k=0}^{n-1}(-1)^k \binom{n}{k} \rho^{(n-k)}(0)\Big[\tilde{\psi}_{s;k}(0) - \tilde{\psi}_{s;k}(\xi)\Big] \\
=&\ \Big\{D\xi^2 + c\xi\Big[1 - \frac{1}{1+\theta}\tilde{p}_1(\xi)\Big]\Big\}\sum_{k=1}^{n}(-1)^k \binom{n+1}{k} \rho^{(n+1-k)}(0)\tilde{\psi}_{s;k}(\xi) \\
=&\ \Big\{D\xi^2 + c\xi\Big[1 - \frac{1}{1+\theta}\tilde{p}_1(\xi)\Big]\Big\}\sum_{k=0}^{n-1}(-1)^{k+1} \binom{n+1}{k+1} \rho^{(n-k)}(0)\tilde{\psi}_{s;k+1}(\xi) \\
=&\ -(n+1)\Big\{D\xi^2 + c\xi\Big[1 - \frac{1}{1+\theta}\tilde{p}_1(\xi)\Big]\Big\}\sum_{k=0}^{n-1}(-1)^k \binom{n}{k} \rho^{(n-k)}(0)\frac{\tilde{\psi}_{s;k+1}(\xi)}{k+1}.
\end{aligned} \tag{3.141}$$

We declare that for $n = 1, 2, 3, \ldots$,

$$\Big\{D\xi^2 + c\xi\Big[1 - \frac{1}{1+\theta}\tilde{p}_1(\xi)\Big]\Big\}\tilde{\psi}_{s;n}(\xi) = n\Big[\tilde{\psi}_{s;n-1}(0) - \tilde{\psi}_{s;n-1}(\xi)\Big], \tag{3.142}$$

which is exactly the same form as the expression (3.116). Therefore, equations (3.130), (3.131) and (3.132) can be shown just like (3.112), (3.113) and (3.114). An alternative expression for $\psi_{s;n}(u)$ as (3.117) for $\psi_{d;n}(u)$ is

$$\begin{aligned}&\psi_{s;n}(u)\\ =\ &\frac{n}{\lambda p_1\theta}\Big[\int_0^u \overline{K}_{\delta=0} * H_1(u-x)\psi_{s;n-1}(x)dx + \int_0^u h_1(u-x)\int_x^\infty \psi_{s;n-1}(y)dydx - \\ &\overline{K}_{\delta=0} * H_1(u)\int_0^\infty \psi_{s;n-1}(x)dx\Big]. \qquad (3.143)\end{aligned}$$

Equation (3.142) can be proved by induction on n. Letting $n = 1$ in (3.141) shows the case $n = 1$ in (3.142). Assume that for $n = 1, \dots, m-1$, identity (3.142) holds. Now let $n = m$ in (3.141), then these terms but the last terms on both sides of (3.141) are canceled out. Thus (3.141) reduces to

$$\begin{aligned}&(-1)^{m-1}m\rho^{(1)}(0)\Big[\tilde{\psi}_{s;m-1}(0) - \tilde{\psi}_{s;m-1}(\xi)\Big]\\ =\ &\Big\{D\xi^2 + c\xi\Big[1 - \frac{1}{1+\theta}\tilde{p}_1(\xi)\Big]\Big\}(-1)^{m-1}\rho^{(1)}(0)\tilde{\psi}_{s;m}(\xi),\end{aligned}$$

which is equal to (3.142) for $n = m$.

Since when $D \to 0$, $\psi_{s;n}(u) \to \psi_{0;n}(u)$, which implies (3.132) $\to$ (3.135). As in the proof of theorem 3.4, (3.133) and (3.134) can be shown either by similar arguments based on traditional risk model (1.1) or by letting $D \to 0$ in (3.130) and (3.131). The latter approach, however, can show only for $u > 0$. □

To get the mean of the time of ruin because of a claim, just set $n = 1$ in (3.130) and (3.131).

Corollary 3.9 *The mean time to ruin due to a claim, if this kind of ruin occurs,*

$$\psi_{s;1}(u) = E[TI(T < \infty, U(T) < 0)|U(0) = u], \ \ u \geq 0, \qquad (3.144)$$

satisfies the defective renewal equation

$$\begin{aligned}\psi_{s;1}(u) &= \frac{1}{1+\theta}\int_0^u \psi_{s;1}(u-x)dP_1 * H_1(x) + \frac{1}{c}\int_u^\infty \overline{K_{\delta=0} * H_1}(x)dx \\ &- \frac{1}{c\theta}\int_0^u \overline{H}_1(u-x)\overline{K_{\delta=0} * H_1}(x)dx - \frac{\lambda p_2 + 2D}{2c\lambda p_1\theta}\overline{H}_1(u) \qquad (3.145)\end{aligned}$$

and is given explicitly by

$$\begin{aligned}\psi_{s;1}(u) &= \frac{1}{\lambda p_1\theta^2}\Big[(1+\theta)\int_0^u \overline{K_{\delta=0} * H_1}(u-x)\overline{K}_{\delta=0}(x)dx - \\ &\qquad \int_0^u \overline{K_{\delta=0} * H_1}(u-x)\overline{K_{\delta=0} * H_1}(x)dx\Big] \\ &+ \frac{1}{\lambda p_1\theta}\int_u^\infty \overline{K_{\delta=0} * H_1}(x)dx - \frac{\lambda p_2 + 2D}{2\lambda^2 p_1^2\theta^2}\overline{K_{\delta=0} * H_1}(u) \qquad (3.146)\end{aligned}$$

with

$$\int_0^\infty \psi_{s;1}(u)du = \frac{1}{\lambda p_1\theta}\int_0^\infty u\psi_s(u)du = \frac{3\lambda p_2^2 + 2\lambda p_1 p_3\theta + 6Dp_2}{12\lambda^2 p_1^3\theta^3}, \qquad (3.147)$$

where $\overline{K_{\delta=0} * H_1}(u)$ *can be replaced with* $\psi_t(u)$ *by (2.105).*

In addition, when $D \to 0$*, equation or (3.145) reduces to*

$$\psi_{0;1}(u) = \frac{1}{1+\theta}\int_0^u \psi_{0;1}(u-x)dP_1(x) + \frac{1}{c}\int_u^\infty \psi_0(x)dx \qquad (3.148)$$

and is given explicitly by

$$\psi_{0;1}(u) = \frac{1}{\lambda p_1\theta}\Big[\int_0^u \psi_0(u-x)\psi_0(x)dx + \int_u^\infty \psi_0(x)dx - \frac{p_2}{2p_1\theta}\psi_0(u)\Big] \qquad (3.149)$$

with

$$\int_0^\infty \psi_{0;1}(u)du = \frac{1}{\lambda p_1\theta}\int_0^\infty u\psi_0(u)du = \frac{3p_2^2+2p_1p_3\theta}{12\lambda p_1^3\theta^3}, \tag{3.150}$$

where

$$\psi_{0;1}(u) = E[TI(T<\infty)|U(0)=u],\ u\geq 0. \tag{3.151}$$

Proof: Since $\int_x^\infty \psi_s(y)dy = \frac{\alpha_0(x,0)}{\theta} = \frac{\tau_1(x)}{\theta} = \int_x^\infty \overline{K_{\delta=0}*H_1}(y)dy - \frac{D}{\lambda p_1\theta}\overline{K_{\delta=0}*H_1}(x)$ from (3.27) with $\delta=0$ and (3.38), the second term of the right side of equation (3.130) with $n=1$ become

$$\begin{aligned}
&\frac{1}{c}\int_0^u h_1(u-x)\int_x^\infty \overline{K_{\delta=0}*H_1}(y)dydx - \frac{D}{c\lambda p_1\theta}\int_0^u h_1(u-x)\overline{K_{\delta=0}*H_1}(x)dx\\
=\ &\frac{1}{c}\int_0^u\int_x^\infty \overline{K_{\delta=0}*H_1}(y)dyd\overline{H}_1(u-x) - \frac{1}{\lambda p_1\theta}\int_0^u \overline{H}_1(u-x)\overline{K_{\delta=0}*H_1}(x)dx\\
=\ &\frac{1}{c}\int_u^\infty \overline{K_{\delta=0}*H_1}(y)dy - \frac{1}{c}\overline{H}_1(u)\int_0^\infty \overline{K_{\delta=0}*H_1}(y)dy +\\
&\frac{1}{c}\int_0^u \overline{H}_1(u-x)\overline{K_{\delta=0}*H_1}(x)dx - \frac{1}{\lambda p_1\theta}\int_0^u \overline{H}_1(u-x)\overline{K_{\delta=0}*H_1}(x)dx,
\end{aligned}$$

which shows (3.145) using $\int_0^\infty \overline{K_{\delta=0}*H_1}(y)dy = \frac{\mu_{\Gamma,1}(0)}{\theta} + \frac{D(1+\theta)}{c\theta} = \frac{p_2}{2p_1\theta} + \frac{D}{\lambda p_1\theta} = \frac{\lambda p_2+2D}{2\lambda p_1\theta}$ by (2.77) and (2.109) with $\delta=0$.

In addition, $\psi_s(x) = \frac{1}{\theta}\Big[\psi_d(x)+(1+\theta)\psi_s(x)-\psi_t(x)\Big] = \frac{1+\theta}{\theta}\overline{K}_{\delta=0}(x) - \frac{1}{\theta}\overline{K_{\delta=0}*H_1}(x)$ by (1.7), (2.93) and (2.105), and $\int_0^\infty \psi_s(y)dy = \frac{\mu_{\Gamma,1}(0)}{\theta} = \frac{p_2}{2p_1\theta}$ by (2.77) and (2.119) with $\rho=0$ and $\beta=\theta$, equation (3.131) with $n=1$ leads to (3.146).

Clearly, (3.132) with $n=1$ becomes $\int_0^\infty \psi_{s;1}(u)du = \frac{1}{\lambda p_1\theta}\int_0^\infty u\psi_s(u)du = \frac{1}{\lambda p_1\theta}\Big\{\frac{\mu_{\Gamma,1}(0)}{\theta}\Big[\frac{\mu_{G,1}(0)}{\theta}+\frac{D}{c}\Big]+\frac{\mu_{\Gamma,2}(0)}{2\theta}\Big\} = \frac{1}{\lambda p_1\theta}\Big\{\frac{p_2}{2p_1\theta}\Big[\frac{p_2}{2p_1\theta}+\frac{D}{c\theta}+\frac{D}{c}\Big]+\frac{p_3}{6p_1\theta}\Big\} =$

$\dfrac{3\lambda p_2^2 + 2\lambda p_1 p_3 \theta + 6Dp_2}{12\lambda^2 p_1^3 \theta^3}$ by (2.76), (2.77) and (2.120) with $\delta = 0$.

Letting $D = 0$ in (3.147) leads to (3.150). Equations (3.148) and (3.149) can be shown directly from (3.133) and (3.134) with $n = 1$, and $\int_0^\infty \psi_0(x)dx = \dfrac{p_2}{2p_1\theta}$ by (2.77) and (2.119) with $\rho = 0$ and $\beta = \theta$. □

Note that the covariance of T, the time of ruin due to a jump, and the associated penalty function, $w(U(T-), |U(T)|)$, can be obtained from (3.70), (3.72) and (3.146). Especially, when $w(U(T-), |U(T)|) = |U(T)|^n$, the covariance of T (due to a jump) and $|U(T)|^n$ follows from (3.53), (3.94) and (3.146).

Example 3.5 Combination of exponentials and mixture of Erlangs ($b \neq \mu$)

As shown in example 2.5 and example 2.6 that $\overline{K_{\delta=0} * H_1}(u) = \dfrac{c\theta}{D}\sum_{k=1}^{r+1} \dfrac{D_{k,\delta=0}}{c/D - s_{k,\delta=0}} e^{-s_{k,\delta=0}u}$ and $\overline{K}_{\delta=0}(u) = \theta \sum_{k=1}^{r+1} D_{k,\delta=0} e^{-s_{k,\delta=0}u}$. Then

$$\begin{aligned}
&\int_0^u \overline{K_{\delta=0} * H_1}(u-x)\overline{K}_{\delta=0}(x)dx \\
= \; & \frac{c\theta^2}{D}\sum_{k=1}^{r+1}\sum_{j=1}^{r+1} \frac{D_{k,\delta=0}D_{j,\delta=0}}{c/D - s_{k,\delta=0}} \int_0^u e^{-s_{k,\delta=0}(u-x)} e^{-s_{j,\delta=0}x} dx \\
= \; & c\theta^2 \sum_{k=1}^{r+1} \sum_{j=1, j\neq k}^{r+1} \frac{D_{k,\delta=0}D_{j,\delta=0}}{(c - Ds_{k,\delta=0})(s_{k,\delta=0} - s_{j,\delta=0})}\left[e^{-s_{j,\delta=0}u} - e^{-s_{k,\delta=0}u}\right] + \\
& c\theta^2 u \sum_{k=1}^{r+1} \frac{D_{k,\delta=0}^2}{c - Ds_{k,\delta=0}} e^{-s_{k,\delta=0}u}
\end{aligned} \tag{3.152}$$

with the help of (3.106). Combining this with (3.105) and (3.109), (3.146) turns out to be

$$\begin{aligned}
&\psi_{s;1}(u) \\
= \;& (1+\theta)\sum_{k=1}^{r+1}\sum_{j=1,j\neq k}^{r+1}\left[(1+\theta)-\frac{c}{c-Ds_{j,\delta=0}}\right]\frac{D_{k,\delta=0}D_{j,\delta=0}\left[e^{-s_{j,\delta=0}u}-e^{-s_{k,\delta=0}u}\right]}{(c-Ds_{k,\delta=0})(s_{k,\delta=0}-s_{j,\delta=0})}+ \\
& (1+\theta)u\sum_{k=1}^{r+1}\left[(1+\theta)-\frac{c}{c-Ds_{k,\delta=0}}\right]\frac{D^2_{k,\delta=0}}{c-Ds_{k,\delta=0}}e^{-s_{k,\delta=0}u}+ \\
& (1+\theta)\sum_{k=1}^{r+1}\frac{D_{k,\delta=0}}{(c-Ds_{k,\delta=0})s_{k,\delta=0}}e^{-s_{k,\delta=0}u}- \\
& \frac{(\lambda p_2+2D)(1+\theta)}{2\lambda p_1\theta}\sum_{k=1}^{r+1}\frac{D_{k,\delta=0}}{c-Ds_{k,\delta=0}}e^{-s_{k,\delta=0}u} \qquad (3.153)
\end{aligned}$$

after some rearrangements, which is a combination of exponential functions. □

Corollary 3.10 *For $u \geq 0$, the second moment of the time of ruin due to a claim, if this kind of ruin occurs,*

$$\psi_{s;2}(u) = E[T^2 I(T<\infty, U(T)<0)|U(0)=u], \qquad (3.154)$$

satisfies the defective renewal equation

$$\psi_{s;2}(u) = \frac{1}{1+\theta}\int_0^u \psi_{s;2}(u-x)dP_1 * H_1(x) + \frac{1}{1+\theta}B_{s;2}(u) \qquad (3.155)$$

with

$$\begin{aligned}
B_{s;2}(u) \;=\; & \frac{2}{\lambda^2 p_1^2\theta}\left[\int_0^u h_1(u-x)\int_0^x \overline{K}_{\delta=0}(x-y)\int_y^\infty \overline{K_{\delta=0} * H}_1(t)dtdydx+\right. \\
& \left.\int_0^u h_1(u-x)\int_x^\infty (y-x)\overline{K_{\delta=0} * H}_1(y)dydx\right]
\end{aligned}$$

$$
\begin{aligned}
&- \frac{2D}{c\lambda^2 p_1^2 \theta^2}\Big[\int_0^u h_1(u-x)\int_x^\infty \overline{K_{\delta=0} * H}_1(y)dydx + \\
&\qquad \int_0^u h_1(u-x)\int_0^x \overline{K_{\delta=0} * H}_1(x-y)\overline{K_{\delta=0} * H}_1(y)dydx\Big] \\
&- \frac{\lambda D p_2 + 2D^2}{c\lambda^3 p_1^3 \theta^2}\int_0^u h_1(u-x)\overline{K_{\delta=0} * H}_1(x)dx
\end{aligned}
\tag{3.156}
$$

and is given explicitly by

$$
\begin{aligned}
&\psi_{s;2}(u) \\
= {} & \frac{2}{\lambda^2 p_1^2 \theta^3}\Big[(1+\theta)\int_0^u \overline{K_{\delta=0} * H}_1(u-x)\int_0^x \overline{K_{\delta=0} * H}_1(x-y)\overline{K}_{\delta=0}(y)dydx - \\
&\qquad \int_0^u \overline{K_{\delta=0} * H}_1(u-x)\int_0^x \overline{K_{\delta=0} * H}_1(x-y)\overline{K_{\delta=0} * H}_1(y)dydx\Big] \\
+ {} & \frac{2}{\lambda^2 p_1^2 \theta^2}\Big[\int_0^u \overline{K_{\delta=0} * H}_1(u-x)\int_x^\infty \overline{K_{\delta=0} * H}_1(y)dydx + \\
&\qquad \int_0^u \overline{K}_{\delta=0}(u-x)\int_x^\infty \overline{K_{\delta=0} * H}_1(y)dydx + \\
&\qquad \int_u^\infty (x-u)\overline{K_{\delta=0} * H}_1(x)dx\Big] \\
- {} & \left[\frac{p_2}{\lambda^2 p_1^3 \theta^3} + \frac{2D(2+\theta)}{c\lambda^2 p_1^2 \theta^3}\right]\int_0^u \overline{K_{\delta=0} * H}_1(u-x)\overline{K_{\delta=0} * H}_1(x)dx \\
- {} & \left[\frac{2p_1 p_3 \theta + 3p_2^2}{6\lambda^2 p_1^4 \theta^4} + \frac{D\lambda p_2(2+\theta) + 2D^2\theta}{c\lambda^3 p_1^3 \theta^4}\right]\overline{K_{\delta=0} * H}_1(u) \\
- {} & \frac{2D}{c\lambda^2 p_1^2 \theta^3}\int_u^\infty \overline{K_{\delta=0} * H}_1(x)dx
\end{aligned}
\tag{3.157}
$$

where $\overline{K_{\delta=0} * H}_1(u)$ *can be replaced with* $\psi_t(u)$ *by (2.105).*

In addition, when $D \to 0$*, equation (3.155) reduces to*

$$
\psi_{0;2}(u) = \frac{1}{1+\theta}\int_0^u \psi_{0;2}(u-x)dP_1(x) + \frac{1}{1+\theta}B_{0;2}(u) \tag{3.158}
$$

with

$$B_{0;2}(u) = \frac{2}{\lambda^2 p_1^2 \theta}\Big[\int_0^u \psi_0(u-x)\int_x^\infty \psi_0(y)dydx + \int_u^\infty (x-u)\psi_0(x)dx\Big] \tag{3.159}$$

and is given explicitly by

$$\begin{aligned}\psi_{0;2}(u) &= \frac{2}{\lambda^2 p_1^2 \theta^2}\int_0^u \psi_0(u-x)\int_0^x \psi_0(x-y)\psi_0(y)dydx \\ &+ \frac{4}{\lambda^2 p_1^2 \theta^2}\int_0^u \psi_0(u-x)\int_x^\infty \psi_0(y)dydx + \frac{2}{\lambda^2 p_1^2 \theta^2}\int_u^\infty (x-u)\psi_0(x)dx \\ &- \frac{p_2}{\lambda^2 p_1^3 \theta^3}\int_0^u \psi_0(u-x)\psi_0(x)dx - \frac{2p_1p_3\theta + 3p_2^2}{6\lambda^2 p_1^4 \theta^4}\psi_0(u) \end{aligned} \tag{3.160}$$

where

$$\psi_{0;2}(u) = E[T^2 I(T<\infty)|U(0)=u], \ \ u \geq 0. \tag{3.161}$$

Proof: Equations (3.120) and (3.146) imply that

$$\begin{aligned}&\psi_{s;1}(u) \\ &= -\frac{1}{1+\theta}\psi_{d;1}(u) - \frac{\lambda p_2 + 2D}{2\lambda^2 p_1^2 \theta^2}\overline{K_{\delta=0} * H_1}(u) \\ &+ \frac{1}{\lambda p_1 \theta}\Big[\int_0^u \overline{K_{\delta=0} * H_1}(u-x)\overline{K}_{\delta=0}(x)dx + \int_u^\infty \overline{K_{\delta=0} * H_1}(x)dx\Big]. \end{aligned} \tag{3.162}$$

By changing the order of integration, $\int_x^\infty \int_y^\infty \overline{K_{\delta=0} * H_1}(t)dtdy = \int_x^\infty \int_x^t \overline{K_{\delta=0} * H_1}(t)dydt = \int_x^\infty (t-x)\overline{K_{\delta=0} * H_1}(t)dt$. Also using (3.127), we have $\int_x^\infty \int_0^y \overline{K_{\delta=0} * H_1}(y-t)\overline{K}_{\delta=0}(t)dtdy = \int_0^x \overline{K}_{\delta=0}(x-t)$ $\int_x^\infty \overline{K_{\delta=0} * H_1}(y)dydt + \int_x^\infty \overline{K}_{\delta=0}(t)dt \int_0^\infty \overline{K_{\delta=0} * H_1}(y)dy$. With the help of (3.128) for $\int_x^\infty \psi_{d;1}(y)dy$, (2.107) with $\delta = 0$ for $\int_x^\infty \overline{K_{\delta=0} * H_1}(y)dy = \int_x^\infty \overline{K}_{\delta=0}(y)dy +$

$\frac{D}{c}\overline{K_{\delta=0} * H_1}(x)$, and (2.109) with $\delta = 0$ for $\int_0^\infty \overline{K_{\delta=0} * H_1}(y)dy = \frac{\lambda p_2 + 2D}{2\lambda p_1 \theta}$, we obtain

$$\begin{aligned}
&\int_x^\infty \psi_{s;1}(y)dy \\
= &\frac{1}{\lambda p_1 \theta}\Big[\int_0^x \overline{K}_{\delta=0}(x-y)\int_y^\infty \overline{K_{\delta=0} * H_1}(t)dtdy + \int_x^\infty (y-x)\overline{K_{\delta=0} * H_1}(y)dy\Big] \\
- &\frac{D}{c\lambda p_1 \theta^2}\Big[\int_x^\infty \overline{K_{\delta=0} * H_1}(y)dy + \int_0^x \overline{K_{\delta=0} * H_1}(x-y)\overline{K_{\delta=0} * H_1}(y)dy \\
- &\frac{\lambda D p_2 + 2D^2}{2c\lambda^2 p_1^2 \theta^2}\overline{K_{\delta=0} * H_1}(x). \qquad (3.163)
\end{aligned}$$

Letting $n = 2$ in (3.130) leads to (3.155) and (3.156). Now use (3.146), (3.147) and (3.163), and apply (3.131) with $n = 2$, we have (3.157) after some rearrangements.

Note that when $D \to 0$, $\int_x^\infty \psi_{s;1}(y)dy$ in (3.163) reduces to

$$\int_u^\infty \psi_{0;1}(x)dx = \frac{1}{\lambda p_1 \theta}\Big[\int_0^u \psi_0(u-x)\int_x^\infty \psi_0(y)dydx + \int_u^\infty (x-u)\psi_0(x)dx\Big]. \quad (3.164)$$

Equations (3.158), (3.159) and (3.160) can be shown by similar arguments from (3.133), (3.134), (3.149), (3.150) and (3.164). Or they can be proved by letting $D \to 0$ in (3.155), (3.156) and (3.157). The latter approach, however, can show only for $u > 0$. □

We remark that the variance of the time of ruin caused by a claim can be obtained explicitly by (3.146) and (3.157).

Chapter 4

Discounted distribution and probability density functions

In this chapter, we are going to derive the explicit expressions based on (1.4) for $F(x, y; \delta, D|u)$, $F_1(x; \delta, D|u)$ and $F_2(y; \delta, D|u)$, the discounted joint and marginal distribution functions of $U(T-)$, the surplus immediately prior to the time of ruin, and $|U(T)|$, the deficit at the time of ruin, and for $F_Z(z; \delta, D|u)$, the discounted distribution function of the amount of the claim causing ruin, $\{U(T-) + |U(T)|\}$. Then the discounted probability density functions, $f(x, y; \delta, D|u)$, $f_1(x; \delta, D|u)$, $f_2(y; \delta, D|u)$ and $f_Z(z; \delta, D|u)$ are obtained by differentiating the corresponding discounted distribution functions. We will show that $F_1(x; \delta, D|u)$, $F_2(y; \delta, D|u)$ and $F_Z(z; \delta, D|u)$ also satisfy a defective renewal equation, respectively. Besides, the explicit expressions, which have been derived by Lin and Willmot (1999) [36], and defective renewal equations based on (1.1) can be easily got by letting $D \to 0$.

4.1 Introduction

When ruin occurs due to a claim, let $f(x, y, t; D|u)$ denote the defective joint probability density function of $U(T-)$, $|U(T)|$ and T, and define the discounted defective marginal probability density functions with the discount factor $\delta \geq 0$ as follows:

$$f(x, y; \delta, D|u) = \int_0^\infty e^{-\delta t} f(x, y, t; D|u)dt, \tag{4.1}$$
$$f_1(x; \delta, D|u) = \int_0^\infty f(x, y; \delta, D|u)dy = \int_0^\infty \int_0^\infty e^{-\delta t} f(x, y, t; D|u)dtdy, \tag{4.2}$$
$$f_2(y; \delta, D|u) = \int_0^\infty f(x, y; \delta, D|u)dx = \int_0^\infty \int_0^\infty e^{-\delta t} f(x, y, t; D|u)dtdx. \tag{4.3}$$

Dickson (1992) [5] derived the defective probability density function of $U(T-)$, the surplus before the time of ruin, for the case $\delta = 0$ based on the classical surplus process (1.1) as follows:

$$f_1(x; 0, 0|u) = \begin{cases} \dfrac{\lambda}{c}\overline{P}(x)\,\dfrac{1-\psi_0(u)}{1-\psi_0(0)}, & \text{if } 0 \leq u < x, \\ \\ \dfrac{\lambda}{c}\overline{P}(x)\,\dfrac{\psi_0(u-x)-\psi_0(u)}{1-\psi_0(0)}, & \text{if } 0 < x \leq u, \end{cases} \tag{4.4}$$

with $\psi_0(u)$ given in (1.3)

$$f_1(x; 0, 0|0) = \frac{\lambda}{c}\overline{P}(x). \tag{4.5}$$

Later, Gerber and Shiu (1998a) [29] generalized Dickson's formula for the case $\delta \geq 0$ to get the discounted defective probability density function of $U(T-)$

$$f_1(x;\delta,0|u)=\begin{cases}\dfrac{\lambda}{c}e^{-\rho x}\overline{P}(x)\,\dfrac{e^{\rho u}-\overline{K}_\rho(u)}{1-\overline{K}_\rho(0)}, & \text{if } 0\le u<x,\\[2ex] \dfrac{\lambda}{c}e^{-\rho x}\overline{P}(x)\,\dfrac{e^{\rho x}\overline{K}_\rho(u-x)-\overline{K}_\rho(u)}{1-\overline{K}_\rho(0)}, & \text{if } 0<x\le u,\end{cases} \tag{4.6}$$

with

$$f_1(x;\delta,0|0)=\frac{\lambda}{c}e^{-\rho x}\overline{P}(x), \tag{4.7}$$

and

$$\overline{K}_\rho(u)=E[e^{-\delta T-\rho|U(T)|}I(T<\infty)|U(0)=u],\quad u\ge 0. \tag{4.8}$$

Set $w(x,y)=e^{-\rho y}$ in (1.2), then from (1.13) $\overline{K}_\rho(u)$ satisfies the defective renewal equation as follows:

$$\begin{aligned}&\overline{K}_\rho(u)\\ &= \frac{\lambda}{c}\int_0^u \overline{K}_\rho(u-x)\int_x^\infty e^{-\rho(y-u)}dP(y)dx+\frac{\lambda}{c}\int_u^\infty e^{-\rho(y-u)}\int_y^\infty e^{-\rho(x-y)}p(x)dxdy\\ &= \frac{\lambda}{c}\int_0^u \overline{K}_\rho(u-x)\int_x^\infty e^{-\rho(y-u)}dP(y)dx+\frac{\lambda}{c}\int_u^\infty\int_u^x e^{-\rho(x-u)}p(x)dydx\\ &= \frac{\lambda}{c}\int_0^u \overline{K}_\rho(u-x)\int_x^\infty e^{-\rho(y-u)}dP(y)dx+\frac{\lambda}{c}\int_u^\infty (x-u)e^{-\rho(x-u)}p(x)dx \qquad (4.9)\\ &= \frac{\lambda}{c}\int_0^u \overline{K}_\rho(u-x)\int_x^\infty e^{-\rho(y-u)}dP(y)dx+\\ &\quad \frac{\lambda}{c}\int_u^\infty e^{-\rho(x-u)}\overline{P}(x)dx-\frac{\lambda}{c}\rho\int_u^\infty (x-u)e^{-\rho(x-u)}\overline{P}(x)dx,\quad u\ge 0, \qquad (4.10)\end{aligned}$$

with $\overline{K}_\rho(u)|_{\delta=0}=\psi_0(u)$.

Since $\rho(\delta)$ is a root of (1.14), we have $\lambda\tilde{p}(\rho)=\lambda+\delta-c\rho$. If we differentiate with respect to δ, then $c\rho'(\delta) - 1 = \lambda\rho'(\delta)\int_0^\infty xe^{-\rho x}p(x)dx$. Therefore, from (4.9) we obtain

$$\overline{K}_\rho(0) = \frac{\lambda}{c}\int_0^\infty xe^{-\rho x}p(x)dx = 1 - \frac{1}{c\rho'(\delta)}, \tag{4.11}$$

with $\overline{K}_\rho(0)|_{\delta=0} = \lambda p_1/c = 1/(1+\theta) = \psi_0(0)$.

Later we are going to derive the more general discounted defective probability density function of $U(T-)$, the surplus before the time of ruin, for the discount factor $\delta \geq 0$ based on the surplus process (1.4) with an independent Wiener process.

4.2 Discounted joint distribution and probability density functions of $|U(T)|$ and $U(T-)$

First of all, by appropriate choice in the penalty function $w(x, y)$, we have that the discounted defective joint distribution function of $U(T-)$ and $|U(T)|$ is equal to $\phi_w(u)$. Then the explicit expression for the discounted defective joint distribution function of $U(T-)$ and $|U(T)|$ can be obtained by (1.36) and (1.44). To see this, for any fixed x and y, let

$$w(x_1, x_2) = \begin{cases} 1, & \text{if } x_1 \leq x, x_2 \leq y, \\ 0, & \text{otherwise.} \end{cases} \tag{4.12}$$

Then by (4.1), $\phi_w(u)$ in (2.2) becomes

$$\phi_w(u) \quad = \quad E[e^{-\delta T}w(U(T-), |U(T)|)I(T < \infty, U(T) < 0)|U(0) = u]$$

$$\begin{aligned}
&= \int_0^\infty \int_0^\infty \int_0^\infty e^{-\delta t} w(x_1, x_2) f(x_1, x_2, t; D|u) dt dx_1 dx_2 \\
&= \int_0^x \int_0^y \int_0^\infty e^{-\delta t} f(x_1, x_2, t; D|u) dt dx_1 dx_2 \\
&= \int_0^x \int_0^y f(x_1, x_2; \delta, D|u) dx_1 dx_2 \qquad (4.13) \\
&= F(x, y; \delta, D|u), \qquad (4.14)
\end{aligned}$$

the discounted joint distribution function of $U(T-)$ and $|U(T)|$. And the function $B(u)$ in (2.114) can be written by (2.34) and (1.52) as follows:

if $0 \le u < x$,

$$\begin{aligned}
& B(u) \\
= \; & \frac{\lambda}{D}(1+\beta) \int_0^u e^{-b(u-s)} \int_s^\infty e^{-\rho(x_1 - s)} \int_{x_1}^\infty w(x_1, x_2 - x_1) p(x_2) dx_2 dx_1 ds \\
= \; & \frac{\lambda}{D}(1+\beta) \int_0^u e^{-b(u-s)} \int_s^x e^{-\rho(x_1 - s)} \int_{x_1}^{x_1+y} p(x_2) dx_2 dx_1 ds \\
= \; & \frac{\lambda}{D}(1+\beta) \int_0^u e^{-b(u-s)} \int_s^x e^{-\rho(x_1 - s)} \Big[\overline{P}(x_1) - \overline{P}(x_1 + y)\Big] dx_1 ds \\
= \; & \frac{\lambda}{D}(1+\beta) \int_0^u e^{-b(u-s)} \Big\{ \int_s^\infty e^{-\rho(x_1 - s)} \overline{P}(x_1) dx_1 - \int_{s+y}^\infty e^{-\rho(x_1 - s - y)} \overline{P}(x_1) dx_1 - \\
& \qquad e^{-\rho(x-s)} \Big[\int_x^\infty e^{-\rho(x_1 - x)} \overline{P}(x_1) dx_1 - \int_{x+y}^\infty e^{-\rho(x_1 - x - y)} \overline{P}(x_1) dx_1 \Big] \Big\} ds \\
= \; & b \int_0^u e^{-b(u-s)} \Big\{ \overline{\Gamma}(s) - \overline{\Gamma}(s+y) - e^{-\rho(x-s)} \Big[\overline{\Gamma}(x) - \overline{\Gamma}(x+y) \Big] \Big\} ds \\
= \; & b \int_0^u e^{-b(u-s)} \Big[\Gamma(s+y) - \Gamma(s) \Big] ds - b e^{-\rho x} e^{-bu} \Big[\overline{\Gamma}(x) - \overline{\Gamma}(x+y) \Big] \int_0^u e^{as} ds \\
= \; & b \int_y^{u+y} e^{-b(u+y-s)} \Gamma(s) ds - \int_0^u \Gamma(s) H'(u-s) ds - \\
& \frac{b}{a} e^{-\rho x} e^{-bu} (e^{au} - 1) [\overline{\Gamma}(x) - \overline{\Gamma}(x+y)] \\
= \; & \overline{G}(u) - \overline{G}(u+y) - \overline{H}(u) G(y) - \frac{b}{a} e^{-\rho x} (e^{\rho u} - e^{-bu}) [\overline{\Gamma}(x) - \overline{\Gamma}(x+y)], \qquad (4.15)
\end{aligned}$$

and for $u > 0$

$$B'(u) = G'(u+y)-G'(u)+H'(u)G(y)-\frac{b}{a}e^{-\rho x}(\rho e^{\rho u}+be^{-bu})[\overline{\Gamma}(x)-\overline{\Gamma}(x+y)]; \quad (4.16)$$

if $0 < x \leq u$,

$$\begin{aligned}
B(u) &= \frac{\lambda}{D}(1+\beta)\int_0^u e^{-b(u-s)}\int_s^\infty e^{-\rho(x_1-s)}\int_{x_1}^\infty w(x_1, x_2-x_1)p(x_2)dx_2dx_1ds \\
&= \frac{\lambda}{D}(1+\beta)\left[\int_0^x + \int_x^u\right]e^{-b(u-s)}\int_s^x e^{-\rho(x_1-s)}\int_{x_1}^{x_1+y} p(x_2)dx_2dx_1ds \\
&= \frac{\lambda}{D}(1+\beta)\int_0^x e^{-b(u-s)}\int_s^x e^{-\rho(x_1-s)}\int_{x_1}^{x_1+y} p(x_2)dx_2dx_1ds \\
&= b\int_0^x e^{-b(u-s)}\left\{\overline{\Gamma}(s)-\overline{\Gamma}(s+y)-e^{-\rho(x-s)}\left[\overline{\Gamma}(x)-\overline{\Gamma}(x+y)\right]\right\}ds \\
&= b\int_0^x e^{-b(u-s)}\left[\Gamma(s+y)-\Gamma(s)\right]ds - be^{-\rho x}e^{-bu}\left[\overline{\Gamma}(x)-\overline{\Gamma}(x+y)\right]\int_0^x e^{as}ds \\
&= \int_y^{x+y} be^{-b(u+y-s)}\Gamma(s)ds - e^{-b(u-x)}\int_0^x be^{-b(x-s)}\Gamma(s)ds - \\
&\quad \frac{b}{a}e^{-\rho x}e^{-bu}(e^{ax}-1)[\overline{\Gamma}(x)-\overline{\Gamma}(x+y)] \\
&= e^{-b(u-x)}\int_0^{x+y} be^{-b(x+y-s)}\Gamma(s)ds - e^{-bu}\int_0^y be^{-b(y-s)}\Gamma(s)ds - \\
&\quad e^{-b(u-x)}\int_0^x be^{-b(x-s)}\Gamma(s)ds - \frac{b}{a}e^{-bu}(e^{bx}-e^{-\rho x})[\overline{\Gamma}(x)-\overline{\Gamma}(x+y)] \\
&= \overline{H}(u-x)[\overline{G}(x)-\overline{G}(x+y)] - \overline{H}(u)G(y) - \\
&\quad \frac{b}{a}e^{-bu}(e^{bx}-e^{-\rho x})[\overline{\Gamma}(x)-\overline{\Gamma}(x+y)], \qquad (4.17)
\end{aligned}$$

and

$$\begin{aligned}
B'(u) &= -H'(u-x)[\overline{G}(x)-\overline{G}(x+y)] + H'(u)G(y) + \\
&\quad \frac{b}{a}H'(u)(e^{bx}-e^{-\rho x})[\overline{\Gamma}(x)-\overline{\Gamma}(x+y)]. \qquad (4.18)
\end{aligned}$$

When $D \to 0$, $\frac{b}{a} \to 1$ and $G(x) \to \Gamma(x)$; in this case $B(u)$ becomes

$$B_0(u) = \begin{cases} [\overline{\Gamma}(u) - \overline{\Gamma}(u+y)] - e^{-\rho(x-u)}[\overline{\Gamma}(x) - \overline{\Gamma}(x+y)], & \text{if } 0 \le u < x, \\ 0, & \text{if } 0 < x \le u. \end{cases} \tag{4.19}$$

We remark that the expression above for $B_0(u)$, $0 \le u < x$, also holds when $u = 0$, which can not be obtained by $D \to 0$, but by derivations similar to (4.15) from $B_0(u) = \frac{\lambda}{c}(1+\beta_0)\int_u^\infty e^{-\rho(x_1-u)}\int_{x_1}^\infty w(x_1, x_2 - x_1)p(x_2)dx_2dx_1$.

Theorem 4.1 *The discounted defective joint distribution function of $U(T-)$ and $|U(T)|$ is*

$$F(x,y;\delta,D|u)$$
$$= \begin{cases} \frac{1+\beta}{\beta}\Big[\overline{K}(u) - \overline{K}(u+y)\Big] - \frac{1}{\beta}G(y)\overline{K*H}(u) \\ \quad + \frac{1}{\beta}\frac{b}{a}e^{-\rho x}\Big[\overline{\Gamma}(x) - \overline{\Gamma}(x+y)\Big] \\ \quad\quad \Big[\rho\int_0^u e^{\rho t}\overline{K}(u-t)dt + \overline{K*H}(u) - e^{\rho u}\Big] \\ \quad + \frac{1}{\beta}\int_0^y \overline{K}(u+y-t)G'(t)dt, & \text{if } 0 \le u < x, \\ \\ \frac{1}{\beta}\int_0^x \overline{K}(u-t)\Big[G'(t) - G'(y+t)\Big]dt - \frac{1}{\beta}G(y)\overline{K*H}(u) \\ \quad + \frac{1}{\beta}\frac{b}{a}e^{-\rho x}\Big[\overline{\Gamma}(x) - \overline{\Gamma}(x+y)\Big]\Big[\rho\int_0^x e^{\rho t}\overline{K}(u-t)dt + \overline{K*H}(u)\Big] \\ \quad + \frac{1}{\beta}\Big[\Big(\overline{G}(x) - \overline{G}(x+y)\Big) - \frac{b}{a}\Big(\overline{\Gamma}(x) - \overline{\Gamma}(x+y)\Big)\Big]\overline{K*H}(u-x), \\ & \text{if } 0 < x \le u, \end{cases} \tag{4.20}$$

with

$$F(x,y;\delta,D|0)=0, \tag{4.21}$$

where $\overline{K*H}(u)$ *can be replaced with* $\phi_t(u)$ *by (2.103).*

When $D\to 0$,

$$F(x,y;\delta,0|u)$$
$$=\begin{cases} \frac{1+\beta_0}{\beta_0}\Big[\overline{K}_0(u)-\overline{K}_0(u+y)\Big]-\frac{1}{\beta_0}\Gamma(y)\overline{K}_0(u) \\ \quad +\frac{1}{\beta_0}e^{-\rho x}\Big[\overline{\Gamma}(x)-\overline{\Gamma}(x+y)\Big]\Big[\rho\int_0^u e^{\rho t}\overline{K}_0(u-t)dt+\overline{K}_0(u)-e^{\rho u}\Big] \\ \quad +\frac{1}{\beta_0}\int_0^y\overline{K}_0(u+y-t)\Gamma'(t)dt, \qquad \text{if } 0\le u<x, \\ \\ \frac{1}{\beta_0}\int_0^x\overline{K}_0(u-t)\Big[\Gamma'(t)-\Gamma'(y+t)\Big]dt-\frac{1}{\beta_0}\Gamma(y)\overline{K}_0(u) \\ \quad +\frac{1}{\beta_0}e^{-\rho x}\Big[\overline{\Gamma}(x)-\overline{\Gamma}(x+y)\Big]\Big[\rho\int_0^x e^{\rho t}\overline{K}_0(u-t)dt+\overline{K}_0(u)\Big], \\ \qquad \text{if } 0<x\le u, \end{cases} \tag{4.22}$$

with

$$F(x,y;\delta,0|0)=\frac{1}{1+\beta_0}\Big[e^{-\rho x}\Gamma(x)+\Gamma(y)-e^{-\rho x}\Gamma(x+y)\Big]. \tag{4.23}$$

If further let $\delta=0$,

$$F(x,y;0,0|u)$$
$$= \begin{cases} \frac{1+\theta}{\theta}\Big[\psi_0(u)-\psi_0(u+y)\Big]+\frac{1}{\theta}\Big[P_1(x+y)-P_1(x)-P_1(y)\Big]\psi_0(u) \\ \quad -\frac{1}{\theta}\Big[P_1(x+y)-P_1(x)\Big]+\frac{1}{\theta p_1}\int_0^y \psi_0(u+y-t)\overline{P}(t)dt, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \textit{if } 0\le u<x, \\ \frac{1}{\theta p_1}\int_0^x \psi_0(u-t)\Big[\overline{P}(t)-\overline{P}(y+t)\Big]dt \\ \quad +\frac{1}{\theta}\Big[P_1(x+y)-P_1(x)-P_1(y)\Big]\psi_0(u), \qquad \textit{if } 0<x\le u, \end{cases} \tag{4.24}$$

with

$$F(x,y;0,0|0)=\frac{1}{1+\theta}\Big[P_1(x)+P_1(y)-P_1(x+y)\Big]. \tag{4.25}$$

Proof: If $0\le u<x$, from (1.39), (4.14), (4.15) and (4.16), equation (1.44) becomes

$$\begin{aligned}
& F(x,y;\delta,D|u) \\
= \;& -\frac{1}{\beta}\int_0^u \overline{K}(u-t)B'(t)dt+\frac{1}{\beta}B(u) \\
= \;& \frac{1}{\beta}\int_0^u \overline{K}(u-t)\Big[G'(t)-G'(t+y)\Big]dt-\frac{1}{\beta}G(y)\int_0^u \overline{K}(u-t)H'(t)dt+ \\
& \frac{1}{\beta}\frac{b}{a}e^{-\rho x}\Big[\overline{\Gamma}(x)-\overline{\Gamma}(x+y)\Big]\int_0^u \overline{K}(u-t)\Big[\rho e^{\rho t}+be^{-bt}\Big]dt+ \\
& \frac{1}{\beta}\Big[\overline{G}(u)-\overline{G}(u+y)-\overline{H}(u)G(y)-\frac{b}{a}e^{-\rho x}[e^{\rho u}-e^{-bu}][\overline{\Gamma}(x)-\overline{\Gamma}(x+y)]\Big] \\
= \;& \frac{1+\beta}{\beta}\overline{K}(u)-\frac{1}{\beta}G(y)\Big[\overline{K}*H(u)+\overline{H}(u)\Big]+ \\
& \frac{1}{\beta}\frac{b}{a}e^{-\rho x}\Big[\overline{\Gamma}(x)-\overline{\Gamma}(x+y)\Big]\Big[\rho\int_0^u e^{\rho t}\overline{K}(u-t)dt+\overline{K}*H(u)+\overline{H}(u)-e^{\rho u}\Big]- \\
& \frac{1}{\beta}\int_y^{u+y}\overline{K}(u+y-t)G'(t)dt-\frac{1}{\beta}\overline{G}(u+y)
\end{aligned}$$

$$= \frac{1+\beta}{\beta}\Big[\overline{K}(u) - \overline{K}(u+y)\Big] - \frac{1}{\beta}G(y)\overline{K * H}(u) + \frac{1}{\beta}\int_0^y \overline{K}(u+y-t)G'(t)dt + \frac{1}{\beta}\frac{b}{a}e^{-\rho x}\Big[\overline{\Gamma}(x) - \overline{\Gamma}(x+y)\Big]\Big[\rho\int_0^u e^{\rho t}\overline{K}(u-t)dt + \overline{K * H}(u) - e^{\rho u}\Big].$$

For $u = 0$, by (1.39),

$$\begin{aligned} F(x,y;\delta,D|0) &= \frac{1+\beta}{\beta}\Big[\overline{K}(0) - \overline{K}(y)\Big] - \frac{1}{\beta}G(y) + \frac{1}{\beta}\int_0^y \overline{K}(y-t)G'(t)dt \\ &= -\frac{1+\beta}{\beta}\overline{K}(y) + \frac{1}{\beta}\int_0^y \overline{K}(y-t)dG(t) + \frac{1}{\beta}\overline{G}(y) \\ &= 0, \end{aligned}$$

showing (4.21).

Similarly, if $0 < x \le u$, from (1.39), (4.14), (4.17) and (4.18), equation (1.44) turns out to be

$$\begin{aligned} &F(x,y;\delta,D|u) \\ =\ & -\frac{1}{\beta}\int_0^x \overline{K}(u-t)B'(t)dt - \frac{1}{\beta}\int_x^u \overline{K}(u-t)B'(t)dt + \frac{1}{\beta}B(u) \\ =\ & \frac{1}{\beta}\int_0^x \overline{K}(u-t)\Big[G'(t) - G'(t+y)\Big]dt - \frac{1}{\beta}G(y)\int_0^x \overline{K}(u-t)H'(t)dt + \\ & \frac{1}{\beta}\frac{b}{a}e^{-\rho x}\Big[\overline{\Gamma}(x) - \overline{\Gamma}(x+y)\Big]\int_0^x \overline{K}(u-t)\Big[\rho e^{\rho t} + be^{-bt}\Big]dt + \\ & \frac{1}{\beta}\Big[G(x+y) - G(x)\Big]\int_x^u \overline{K}(u-t)H'(t-x)dt - \frac{1}{\beta}G(y)\int_x^u \overline{K}(u-t)H'(t)dt - \\ & \frac{1}{\beta}\frac{b}{a}\Big[e^{bx} - e^{-\rho x}\Big]\Big[\overline{\Gamma}(x) - \overline{\Gamma}(x+y)\Big]\int_x^u \overline{K}(u-t)H'(t)dt + \\ & \frac{1}{\beta}\Big[\overline{H}(u-x)[\overline{G}(x) - \overline{G}(x+y)] - \frac{1}{\beta}\overline{H}(u)G(y) - \\ & \frac{1}{\beta}\frac{b}{a}e^{-bu}(e^{bx} - e^{-\rho x})[\overline{\Gamma}(x) - \overline{\Gamma}(x+y)]\Big] \end{aligned}$$

$$
\begin{aligned}
= \; & \frac{1}{\beta}\int_0^x \overline{K}(u-t)\Big[G'(t)-G'(t+y)\Big]dt - \frac{1}{\beta}G(y)\Big[\overline{K}*H(u)+\overline{H}(u)\Big]+ \\
& \frac{1}{\beta}\frac{b}{a}e^{-\rho x}\Big[\overline{\Gamma}(x)-\overline{\Gamma}(x+y)\Big]\Big[\rho\int_0^x e^{\rho t}\overline{K}(u-t)dt+\overline{K}*H(u)+\overline{H}(u)\Big]+ \\
& \frac{1}{\beta}\Big[\overline{G}(x)-\overline{G}(x+y)\Big]\Big[\int_x^u \overline{K}(u-t)H'(t-x)dt+\overline{H}(u-x)\Big]- \\
& \frac{1}{\beta}\frac{b}{a}\Big[\overline{\Gamma}(x)-\overline{\Gamma}(x+y)\Big]\Big[\int_x^u \overline{K}(u-t)H'(t-x)dt+\overline{H}(u-x)\Big] \\
= \; & \frac{1}{\beta}\int_0^x \overline{K}(u-t)\Big[G'(t)-G'(t+y)\Big]dt - \frac{1}{\beta}G(y)\overline{K*H}(u)+ \\
& \frac{1}{\beta}\frac{b}{a}e^{-\rho x}\Big[\overline{\Gamma}(x)-\overline{\Gamma}(x+y)\Big]\Big[\rho\int_0^x e^{\rho t}\overline{K}(u-t)dt+\overline{K*H}(u)\Big]+ \\
& \frac{1}{\beta}\Big[[\overline{G}(x)-\overline{G}(x+y)]-\frac{b}{a}[\overline{\Gamma}(x)-\overline{\Gamma}(x+y)]\Big] \\
& \quad \Big[\int_0^{u-x} \overline{K}(u-x-s)H'(s)ds+\overline{H}(u-x)\Big] \\
= \; & \frac{1}{\beta}\int_0^x \overline{K}(u-t)\Big[G'(t)-G'(t+y)\Big]dt - \frac{1}{\beta}G(y)\overline{K*H}(u)+ \\
& \frac{1}{\beta}\frac{b}{a}e^{-\rho x}\Big[\overline{\Gamma}(x)-\overline{\Gamma}(x+y)\Big]\Big[\rho\int_0^x e^{\rho t}\overline{K}(u-t)dt+\overline{K*H}(u)\Big]+ \\
& \frac{1}{\beta}\Big[[\overline{G}(x)-\overline{G}(x+y)]-\frac{b}{a}[\overline{\Gamma}(x)-\overline{\Gamma}(x+y)]\Big]\overline{K*H}(u-x),
\end{aligned}
$$

proving (4.20).

When $D \to 0$, $\beta \to \beta_0$, $\frac{b}{a} \to 1$, $G(y) \to \Gamma(y)$, $\overline{K}(u) \to \overline{K}_0(u)$, and by (3.20), $\overline{K*H}(u) \to \overline{K}_0(u)$ or 1 depending on $u > 0$ or $u = 0$. Therefore, (4.20) $\to$ (4.22) when $D \to 0$.

Note that the expression in (4.22) for $F(x, y; \delta, 0|u)$, $0 \le u < x$, also holds for $u = 0$, which can not be obtained by $D \to 0$ from (4.20), but by derivations similar to (4.20) from $F(x, y; \delta, 0|u) = -\frac{1}{\beta_0}\int_0^u \overline{K}_0(u-t)B_0'(t)dt + \frac{1}{\beta_0}B_0(u) - \frac{1}{\beta_0}B_0(0)\overline{K}_0(u)$ with $B_0(u)$ given in (4.19). When $u = 0$, by (2.91), (4.22) reduces to

$$\begin{aligned}
F(x,y;\delta,0|0) &= \frac{1+\beta_0}{\beta_0}\Big[\frac{1}{1+\beta_0}-\overline{K}_0(y)\Big]-\frac{1}{\beta_0(1+\beta_0)}\Gamma(y)+\\
&\quad \frac{1}{\beta_0}e^{-\rho x}\Big[\overline{\Gamma}(x)-\overline{\Gamma}(x+y)\Big]\Big[\frac{1}{1+\beta_0}-1\Big]+\frac{1}{\beta_0}\int_0^y \overline{K}_0(y-t)d\Gamma(t)\\
&= \frac{1}{\beta_0}-\frac{1+\beta_0}{\beta_0}\overline{K}_0(y)-\Big[\frac{1}{\beta_0}-\frac{1}{1+\beta_0}\Big]\Gamma(y)+\\
&\quad \frac{1}{1+\beta_0}e^{-\rho x}\Big[\Gamma(x)-\Gamma(x+y)\Big]+\frac{1}{\beta_0}\Big[(1+\beta_0)\overline{K}_0(y)-\overline{\Gamma}(y)\Big]\\
&= \frac{1}{1+\beta_0}\Big[e^{-\rho x}\Gamma(x)+\Gamma(y)-e^{-\rho x}\Gamma(x+y)\Big],
\end{aligned}$$

which is (4.23).

If further $\delta=0$, then $\beta_0=\theta$, $\overline{K}_{0,\delta=0}(u)=\psi_0(u)$, $\Gamma(x)=P_1(x)$ and $\Gamma'(x)=\dfrac{\overline{P}(x)}{p_1}$. In this case, (4.24) can be easily obtained from (4.22). When $u=0$, similar arguments show (4.25) from (4.24) by (2.91) with $\delta=0$. Alternatively, when $\delta=0$, $\rho=0$, $\beta_0=\theta$ and $\Gamma(x)=P_1(x)$, (4.25) is easily obtained from (4.23). □

Corollary 4.1 *The discounted defective joint probability density function of $U(T-)$ and $|U(T)|$ is*

$$f(x,y;\delta,D|u) = \begin{cases} \dfrac{\lambda e^{-\rho x}p(x+y)}{c+2\rho D}\,\dfrac{e^{\rho u}-\overline{K*H}(u)-\rho\int_0^u e^{\rho t}\overline{K}(u-t)dt}{1-\overline{K}(0)}, & \text{if } 0\le u<x,\\[2ex] \dfrac{\lambda e^{-\rho x}p(x+y)}{c+2\rho D}\,\dfrac{e^{\rho x}\overline{K*H}(u-x)-\overline{K*H}(u)-\rho\int_0^x e^{\rho t}\overline{K}(u-t)dt}{1-\overline{K}(0)}, & \text{if } 0<x\le u,\end{cases} \tag{4.26}$$

with

$$f(x,y;\delta,D|0)=0, \tag{4.27}$$

where $\overline{K*H}(u)$ *can be replaced with* $\phi_t(u)$ *by (2.103).*

When $D\to 0$,

$$f(x,y;\delta,0|u) = \begin{cases} \dfrac{\lambda}{c}e^{-\rho x}p(x+y)\,\dfrac{e^{\rho u}-\overline{K}_0(u)-\rho\int_0^u e^{\rho t}\overline{K}_0(u-t)dt}{1-\overline{K}_0(0)}, & \textit{if } 0\le u<x, \\ \dfrac{\lambda}{c}e^{-\rho x}p(x+y)\,\dfrac{e^{\rho x}\overline{K}_0(u-x)-\overline{K}_0(u)-\rho\int_0^x e^{\rho t}\overline{K}_0(u-t)dt}{1-\overline{K}_0(0)}, & \textit{if } 0<x\le u, \end{cases} \tag{4.28}$$

with

$$f(x,y;\delta,0|0)=\frac{\lambda}{c}e^{-\rho x}p(x+y). \tag{4.29}$$

If further let $\delta=0$,

$$f(x,y;0,0|u) = \begin{cases} \dfrac{\lambda}{c}p(x+y)\,\dfrac{1-\psi_0(u)}{1-\psi_0(0)}, & \textit{if } 0\le u<x, \\ \dfrac{\lambda}{c}p(x+y)\,\dfrac{\psi_0(u-x)-\psi_0(u)}{1-\psi_0(0)}, & \textit{if } 0<x\le u, \end{cases} \tag{4.30}$$

with

$$f(x,y;0,0|0)=\frac{\lambda}{c}p(x+y). \tag{4.31}$$

Proof: Since $f(x,y;\delta,D|u) = \dfrac{\partial^2 F(x,y;\delta,D|u)}{\partial x\partial y}$, from (4.20)

if $0 \le u < x$,

$$\begin{aligned}
\frac{\partial F(x,y;\delta,D|u)}{\partial y} &= -\frac{1+\beta}{\beta}\overline{K}'(u+y) - \frac{1}{\beta}G'(y)\overline{K*H}(u) \\
&\quad -\frac{1}{\beta}\frac{b}{a}e^{-\rho x}\overline{\Gamma}'(x+y)\Big[\rho\int_0^u e^{\rho t}\overline{K}(u-t)dt + \overline{K*H}(u) - e^{\rho u}\Big] \\
&\quad +\frac{1}{\beta}G'(y)\overline{K}(u) + \frac{1}{\beta}\int_0^y \overline{K}'(u+y-t)G'(t)dt,
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\partial^2 F(x,y;\delta,D|u)}{\partial x\partial y} \\
&= \frac{1}{\beta}\frac{b}{a}\Big[\rho e^{-\rho x}\overline{\Gamma}'(x+y) + e^{-\rho x}\Gamma''(x+y)\Big]\Big[\rho\int_0^u e^{\rho t}\overline{K}(u-t)dt + \overline{K*H}(u) - e^{\rho u}\Big] \\
&= \frac{1}{\beta}\frac{b}{a}e^{-\rho x}\Big[\rho\overline{\Gamma}(x+y) + \Gamma'(x+y)\Big]'\Big[\rho\int_0^u e^{\rho t}\overline{K}(u-t)dt + \overline{K*H}(u) - e^{\rho u}\Big] \\
&= \frac{1}{\beta}\frac{b}{a}e^{-\rho x}\frac{\lambda}{bD}(1+\beta)\overline{P}'(x+y)\Big[\rho\int_0^u e^{\rho t}\overline{K}(u-t)dt + \overline{K*H}(u) - e^{\rho u}\Big] \\
&= \frac{\lambda}{c+2\rho D}e^{-\rho x}p(x+y)\ \frac{e^{\rho u} - \overline{K*H}(u) - \rho\int_0^u e^{\rho t}\overline{K}(u-t)dt}{1-\overline{K}(0)},
\end{aligned}$$

with the help of (2.44);

if $0 < x \le u$,

$$\begin{aligned}
\frac{\partial F(x,y;\delta,D|u)}{\partial y} &= -\frac{1}{\beta}\int_0^x \overline{K}(u-t)G''(y+t)dt - \frac{1}{\beta}G'(y)\overline{K*H}(u) \\
&\quad -\frac{1}{\beta}\frac{b}{a}e^{-\rho x}\overline{\Gamma}'(x+y)\Big[\rho\int_0^x e^{\rho t}\overline{K}(u-t)dt + \overline{K*H}(u)\Big] \\
&\quad -\frac{1}{\beta}\Big[\overline{G}'(x+y) - \frac{b}{a}\overline{\Gamma}'(x+y)\Big]\overline{K*H}(u-x),
\end{aligned}$$

and

$$
\begin{aligned}
&\frac{\partial^2 F(x,y;\delta,D|u)}{\partial x \partial y} \\
= \;& -\frac{1}{\beta}G''(x+y)\overline{K}(u-x) - \frac{1}{\beta}\frac{b}{a}e^{-\rho x}\overline{\Gamma}'(x+y)\rho e^{\rho x}\overline{K}(u-x) \\
&+\frac{1}{\beta}\frac{b}{a}\Big[\rho e^{-\rho x}\overline{\Gamma}'(x+y) + e^{-\rho x}\Gamma''(x+y)\Big]\Big[\rho\int_0^x e^{\rho t}\overline{K}(u-t)dt + \overline{K*H}(u)\Big] \\
&-\frac{1}{\beta}\Big[\overline{G}''(x+y) - \frac{b}{a}\overline{\Gamma}''(x+y)\Big]\overline{K*H}(u-x) \\
&-\frac{b}{\beta}\Big[\overline{G}'(x+y) - \frac{b}{a}\overline{\Gamma}'(x+y)\Big]\Big[\overline{K*H}(u-x) - \overline{K}(u-x)\Big] \\
= \;& -\frac{1}{\beta}\Big[G''(x+y) + \frac{b\rho}{a}\overline{\Gamma}'(x+y) - b\overline{G}'(x+y) + \frac{b^2}{a}\overline{\Gamma}'(x+y)\Big]\overline{K}(u-x) \\
&-\frac{1}{\beta}\Big[\overline{G}''(x+y) - \frac{b}{a}\overline{\Gamma}''(x+y) + b\overline{G}'(x+y) - \frac{b^2}{a}\overline{\Gamma}'(x+y)\Big]\overline{K*H}(u-x) \\
&-\frac{\lambda}{c+2\rho D}e^{-\rho x}p(x+y)\,\frac{\overline{K*H}(u) + \rho\int_0^x e^{\rho t}\overline{K}(u-t)dt}{1-\overline{K}(0)} \\
= \;& -\frac{1}{\beta}\Big[\frac{b\rho}{a} - b + \frac{b^2}{a}\Big]\overline{\Gamma}'(x+y)\overline{K}(u-x) \\
&-\frac{1}{\beta}\Big[(b - \frac{b^2}{a})\overline{\Gamma}'(x+y) + \frac{b}{a}\Gamma''(x+y)\Big]\overline{K*H}(u-x) \\
&-\frac{\lambda}{c+2\rho D}e^{-\rho x}p(x+y)\,\frac{\overline{K*H}(u) + \rho\int_0^x e^{\rho t}\overline{K}(u-t)dt}{1-\overline{K}(0)} \\
= \;& -\frac{1}{\beta}\frac{b}{a}\frac{\lambda}{bD}(1+\beta)\overline{P}'(x+y)\overline{K*H}(u-x) \\
&-\frac{\lambda}{c+2\rho D}e^{-\rho x}p(x+y)\,\frac{\overline{K*H}(u) + \rho\int_0^x e^{\rho t}\overline{K}(u-t)dt}{1-\overline{K}(0)} \\
= \;& \frac{\lambda}{c+2\rho D}e^{-\rho x}p(x+y)\,\frac{e^{\rho x}\overline{K*H}(u-x) - \overline{K*H}(u) - \rho\int_0^x e^{\rho t}\overline{K}(u-t)dt}{1-\overline{K}(0)},
\end{aligned}
$$

with the help of (2.44), (2.48) and (2.106).

Similar arguments show (4.28) from (4.22). If further $\delta = 0$, then $\overline{K}_{0,\delta=0}(u) = \psi_0(u)$, and (4.28) reduces to (4.30). Letting $u = 0$ in (4.26), (4.28) and (4.30) easily lead to (4.27), (4.29) and (4.31), respectively. □

4.3 Discounted distribution and probability density functions of $|U(T)|$

The discounted distribution function of $|U(T)|$, $F_2(y;\delta, D|u)$, now is easily obtained from $F(x, y; \delta, D|u)$ by letting $x \to \infty$

Corollary 4.2 *The discounted defective marginal distribution function of $|U(T)|$ is*

$$F_2(y;\delta,D|u) = \frac{1+\beta}{\beta}\Big[\overline{K}(u)-\overline{K}(u+y)\Big]-\frac{1}{\beta}G(y)\overline{K*H}(u)+\frac{1}{\beta}\int_0^y \overline{K}(u+y-t)G'(t)dt \tag{4.32}$$

with

$$F_2(y;\delta,D|0) = 0 \tag{4.33}$$

and

$$F_2(\infty;\delta,D|u) = \lim_{y\to\infty} F_2(y;\delta,D|u) = \frac{1+\beta}{\beta}\overline{K}(u) - \frac{1}{\beta}\overline{K*H}(u) = \phi_s(u), \tag{4.34}$$

where $\overline{K*H}(u)$ can be replaced with $\phi_t(u)$ by (2.103).

When $D \to 0$,

$$F_2(y;\delta,0|u) = \frac{1+\beta_0}{\beta_0}\Big[\overline{K}_0(u) - \overline{K}_0(u+y)\Big] - \frac{1}{\beta_0}\Gamma(y)\overline{K}_0(u) + \frac{1}{\beta_0}\int_0^y \overline{K}_0(u+y-t)\Gamma'(t)dt \tag{4.35}$$

with

$$F_2(y;\delta,0|0) = \frac{1}{1+\beta_0}\Gamma(y) \tag{4.36}$$

and

$$F_2(\infty;\delta,0|u) = \lim_{y\to\infty} F_2(y;\delta,0|u) = \overline{K}_0(u). \tag{4.37}$$

If further let $\delta = 0$,

$$F_2(y;0,0|u) = \frac{1+\theta}{\theta}\Big[\psi_0(u) - \psi_0(u+y)\Big] - \frac{1}{\theta}P_1(y)\psi_0(u) + \frac{1}{\theta p_1}\int_0^y \psi_0(u+y-t)\overline{P}(t)dt. \tag{4.38}$$

with

$$F_2(y;0,0|0) = \frac{1}{1+\theta}P_1(y) = \frac{\lambda}{c}\int_0^y \overline{P}(t)dt \tag{4.39}$$

and

$$F_2(\infty;0,0|u) = \lim_{y\to\infty} F_2(y;0,0|u) = \psi_0(u). \tag{4.40}$$

Proof: By letting $x \to \infty$ in the case $0 \le u < x$ (since u is fixed) of (4.20), which implies both $\overline{\Gamma}(x)$ and $\overline{\Gamma}(x+y) \to 0$, we easiiy get (4.32). Equations (4.35) and (4.38) can be proved from (4.22) and (4.24), respectively, by similar arguments, whereas (4.33), (4.36) and (4.39) can be shown from (4.21), (4.23) and (4.25), respectively.

When $y \to \infty$, $\overline{K}(u+y) \to 0$ and $\int_0^y \overline{K}(u+y-t)G'(t)dt \le \int_0^\infty \overline{K}(u+y-t)$ $G'(t)dt \to 0$, which imply $F_2(y;\delta,D|u) \to \frac{1+\beta}{\beta}\overline{K}(u) - \frac{1}{\beta}\overline{K * H}(u) = \overline{K}(u) - \frac{1}{1+\beta}\phi_d(u) = \phi_s(u)$ by (2.101) and (2.103), proving (4.34). Similar arguments show (4.37) and (4.40). □

Corollary 4.3 *The discounted defective probability density function of $|U(T)|$ is*

$$f_2(y;\delta,D|u) = -\frac{1+\beta}{\beta}\overline{K}'(u+y) - \frac{1}{\beta}G'(y)\overline{H} * K(u) + \frac{1}{\beta}\int_0^y \overline{K}'(u+y-t)G'(t)dt \tag{4.41}$$

with

$$f_2(y;\delta,D|0) = 0, \tag{4.42}$$

where $\overline{H} * K(u) = \overline{K * H}(u) - \overline{K}(u)$ can be replaced with $\frac{\beta}{1+\beta}\phi_d(u)$ by (2.104). When $D \to 0$,

$$f_2(y;\delta,0|u) = -\frac{1+\beta_0}{\beta_0}\overline{K}_0'(u+y) + \frac{1}{\beta_0}\int_0^y \overline{K}_0'(u+y-t)\Gamma'(t)dt \tag{4.43}$$

with

$$f_2(y;\delta,0|0) = \frac{1}{1+\beta_0}\Gamma'(y). \tag{4.44}$$

If further let $\delta = 0$,

$$f_2(y;0,0|u) = -\frac{1+\theta}{\theta}\psi'(u+y) + \frac{1}{\theta p_1}\int_0^y \psi'(u+y-t)\overline{P}(t)dt \tag{4.45}$$

with

$$f_2(y;0,0|0) = \frac{1}{1+\theta}P_1'(y) = \frac{1}{(1+\theta)p_1}\overline{P}(y) = \frac{\lambda}{c}\overline{P}(y). \tag{4.46}$$

Proof: By differentiating (4.32) with respect to y, we get

$$\begin{aligned} f_2(y;\delta,D|u) &= -\frac{1+\beta}{\beta}\overline{K}'(u+y)-\frac{1}{\beta}G'(y)\Big[\overline{K*H}(u)-\overline{K}(u)\Big]+ \\ &\quad \frac{1}{\beta}\int_0^y \overline{K}'(u+y-t)G'(t)dt. \end{aligned}$$

Combining this with $\overline{K*H}(u)-\overline{K}(u)=\overline{H}*K(u)$ by (2.104), we obtain (4.41).

Similarly, differentiating (4.35) and (4.38) give (4.43) and (4.45), respectively, whereas differentiating (4.33), (4.36) and (4.39) give (4.42), (4.44) and (4.46), respectively. Alternatively, when $u=0$, (4.41) becomes

$$\begin{aligned} & f_2(y;\delta,D|0) \\ =& -\frac{1+\beta}{\beta}\overline{K}'(y)-\frac{1}{\beta}G'(y)\Big[1-\frac{1}{1+\beta}\Big]+\frac{1}{\beta}\int_0^y \overline{K}'(y-t)G'(t)dt \\ =& -\frac{1+\beta}{\beta}\overline{K}'(y)-\frac{1}{1+\beta}G'(y)+\frac{1}{\beta}\Big[\frac{\partial}{\partial y}\int_0^y \overline{K}(y-t)dG(t)-\overline{K}(0)G'(y)\Big] \\ =& -\frac{1+\beta}{\beta}\overline{K}'(y)-\frac{1}{1+\beta}G'(y)+\frac{1}{\beta}\Big[(1+\beta)\overline{K}'(y)-\overline{G}'(y)-\overline{K}(0)G'(y)\Big] \\ =& \ 0 \end{aligned}$$

with the help of (1.39), proving (4.42). Similar arguments show (4.44) and (4.46) from (4.43) and (4.45), respectively. □

Theorem 4.2 *The discounted defective distribution function of* $|U(T)|$ *satisfies the defective renewal equation*

$$\begin{aligned} F_2(y;\delta,D|u) &= \frac{1}{1+\beta}\int_0^u F_2(y;\delta,D|u-x)dG(x)+ \\ &\quad \frac{1}{1+\beta}\Big[\overline{G}(u)-\overline{G}(u+y)-G(y)\overline{H}(u)\Big]. \qquad (4.47) \end{aligned}$$

When $D \to 0$,

$$F_2(y;\delta,0|u) = \frac{1}{1+\beta_0}\int_0^u F_2(y;\delta,0|u-x)d\Gamma(x) + \frac{1}{1+\beta_0}\Big[\overline{\Gamma}(u) - \overline{\Gamma}(u+y)\Big]. \quad (4.48)$$

If further let $\delta = 0$,

$$F_2(y;0,0|u) = \frac{1}{1+\theta}\int_0^u F_2(y;0,0|u-x)dP_1(x) + \frac{1}{1+\theta}\Big[\overline{P}_1(u) - \overline{P}_1(u+y)\Big], \quad (4.49)$$

which is (5) of Gerber, Goovaerts, and Kaas (1987) [23].

Proof: For any fixed y, let

$$w(x,z) = \begin{cases} 1, & \text{if } z \le y, \\ 0, & \text{otherwise.} \end{cases} \quad (4.50)$$

Then by (4.1) and (4.3), $\phi_w(u)$ in (2.2) becomes

$$\begin{aligned} \phi_w(u) &= E[e^{-\delta T}w(U(T-),|U(T)|)I(T<\infty,U(T)<0)|U(0)=u] \\ &= \int_0^\infty\int_0^\infty\int_0^\infty e^{-\delta t}w(x,z)f(x,z,t;D|u)dtdxdz \\ &= \int_0^y\int_0^\infty\int_0^\infty e^{-\delta t}f(x,z,t;D|u)dtdxdz \\ &= \int_0^y\int_0^\infty f(x,z;\delta,D|u)dxdz \\ &= \int_0^y f_2(z;\delta,D|u)dz \\ &= F_2(y;\delta,D|u), \end{aligned} \quad (4.51)$$

the discounted distribution function of $|U(T)|$.

By (2.15), $F_2(y;\delta,D|u)$ satisfies the defective renewal equation $F_2(y;\delta,D|u) = \int_0^u F_2(y;\delta,D|u-x)g(x)dx + g_\omega(u)$ where $g_\omega(u) = h * \gamma_\omega(u)$. From (2.7) and (2.14),

$$\begin{aligned}
\gamma_\omega(s) &= \frac{\lambda}{c}\int_s^\infty e^{-\rho(x-s)}\int_0^\infty w(x,z)p(x+z)dzdx \\
&= \frac{\lambda}{c}\int_s^\infty e^{-\rho(x-s)}\int_0^y p(x+z)dzdx \\
&= \frac{\lambda}{c}\int_s^\infty e^{-\rho(x-s)}\Big[\overline{P}(x)-\overline{P}(x+y)\Big]dx \\
&= \frac{\lambda}{c}\int_0^\infty e^{-\rho x}\Big[\overline{P}(x+s)-\overline{P}(x+s+y)\Big]dx,
\end{aligned}$$

and by (1.49) and (1.52)

$$\frac{\gamma_\omega(s)}{\int_0^\infty \gamma(y)dy} = \frac{\gamma_\omega(s)}{\frac{\lambda}{c}\int_0^\infty e^{-\rho y}\overline{P}(y)dy} = \overline{\Gamma}(s)-\overline{\Gamma}(s+y) = \Gamma(s+y)-\Gamma(s).$$

This implies

$$\begin{aligned}
\frac{g_\omega(u)}{\int_0^\infty g(z)dz} &= \frac{\int_0^u h(u-s)\gamma_\omega(s)ds}{\int_0^\infty h(x)dx\int_0^\infty \gamma(y)dy} \\
&= \int_0^u H'(u-s)\Big[\Gamma(s+y)-\Gamma(s)\Big]ds \\
&= \int_y^{u+y} H'(u+y-t)\Gamma(t)dt - G(u) \\
&= G(u+y) - \int_0^y H'(u+y-t)\Gamma(t)dt - G(u) \\
&= \overline{G}(u)-\overline{G}(u+y) - e^{-bu}\int_0^y H'(y-t)\Gamma(t)dt \\
&= \overline{G}(u)-\overline{G}(u+y) - G(y)\overline{H}(u).
\end{aligned}$$

Since $\int_0^\infty g(z)dz = \frac{1}{1+\beta}$ and $G'(x) = \frac{g(x)}{\int_0^\infty g(z)dz}$,

$$\begin{aligned}
&F_2(y;\delta,D|u)\\
=\ &\int_0^u F_2(y;\delta,D|u-x)g(x)dx + g_\omega(u)\\
=\ &\frac{1}{1+\beta}\int_0^u F_2(y;\delta,D|u-x)\frac{g(x)}{\int_0^\infty g(z)dz}dx + \frac{1}{1+\beta}\frac{g_\omega(u)}{\int_0^\infty g(z)dz}\\
=\ &\frac{1}{1+\beta}\int_0^u F_2(y;\delta,D|u-x)G'(x)dx + \frac{1}{1+\beta}\Big[\overline{G}(u) - \overline{G}(u+y) - G(y)\overline{H}(u)\Big].
\end{aligned}$$

(4.48) and (4.49) can be shown by similar arguments. Alternatively, when $D \to 0$, $\beta \to \beta_0$, $\overline{H}(u) \to 0$ for $u > 0$ and $G(x) \to \Gamma(x)$, (4.47) $\to$ (4.48) for $u > 0$. If $\delta = 0$ then $\beta_0 = \theta$ and $\Gamma(x) = P_1(x)$, (4.48) reduces to (4.49). □

We remark that (4.47) can be shown from a probabilistic viewpoint. Since $g(x)$ defined in (1.30) is the discounted probability that the first record low (the first time where the surplus falls below the initial level) is caused by a jump where x is the amount by which the resulting first record low caused by a claim is below the initial surplus u, and $\overline{H}(u) = e^{-bu}$ is the expected discounted value of a contingent payment of 1 that is due at ruin, provided that ruin occurs before the first record low that is caused by a jump, by conditioning on the time and amount of the first record low caused by a claim, we obtain

$$F_2(y;\delta,D|u) = \int_0^u F_2(y;\delta,D|u-x)g(x)dx + \int_u^{u+y} g(x)dx - \overline{H}(u)\int_0^y g(x)dx. \quad (4.52)$$

Note that the second term of the right side includes an unwanted contribution for the situation where ruin occurs by oscillation prior to the first record low caused by a jump; the third term is the corresponding offset.

Since $\int_0^\infty g(z)dz = \frac{1}{1+\beta}$ and $G'(x) = \frac{g(x)}{\int_0^\infty g(z)dz}$, (4.52) can be written as

$$\begin{aligned}
&F_2(y;\delta, D|u) \\
= \; & \frac{1}{1+\beta}\int_0^u F_2(y;\delta, D|u-x)G'(x)dx + \frac{1}{1+\beta}\int_u^{u+y} G'(x)dx - \\
& \frac{1}{1+\beta}\overline{H}(u)\int_0^y G'(x)dx \\
= \; & \frac{1}{1+\beta}\int_0^u F_2(y;\delta, D|u-x)dG(x) + \frac{1}{1+\beta}\Big[\overline{G}(u) - \overline{G}(u+y) - G(y)\overline{H}(u)\Big],
\end{aligned}$$

which is exactly equation (4.47).

Corollary 4.4 *The discounted defective probability density function of $|U(T)|$ satisfies the defective renewal equation*

$$f_2(y;\delta, D|u) = \frac{1}{1+\beta}\int_0^u f_2(y;\delta, D|u-x)dG(x) + \frac{1}{1+\beta}\Big[G'(u+y) - G'(y)\overline{H}(u)\Big]. \tag{4.53}$$

When $D \to 0$,

$$f_2(y;\delta, 0|u) = \frac{1}{1+\beta_0}\int_0^u f_2(y;\delta, 0|u-x)d\Gamma(x) + \frac{1}{1+\beta_0}\Gamma'(u+y). \tag{4.54}$$

If further let $\delta = 0$,

$$f_2(y;0, 0|u) = \frac{1}{1+\theta}\int_0^u f_2(y;0, 0|u-x)dP_1(x) + \frac{1}{1+\theta}\frac{\overline{P}(u+y)}{p_1}. \tag{4.55}$$

Proof: Differentiating (4.47), (4.48) and (4.49) with respect to y easily yield (4.53), (4.54) and (4.55), respectively. □

Since $F_2(\infty;\delta,D|u) = \lim_{y\to\infty} F_2(y;\delta,D|u) = \phi_s(u)$, $F_2(y;\delta,D|u)$ is a discounted defective distribution function. It is convenient to define the discounted proper distribution function as follows:

$$F_{2,u}(y;\delta,D) = 1 - \overline{F}_{2,u}(y;\delta,D) = \frac{F_2(y;\delta,D|u)}{\phi_s(u)}. \tag{4.56}$$

Then we have the following result for
$F_{2,\infty}(y;\delta,D) = 1 - \overline{F}_{2,\infty}(y;\delta,D) = \lim_{u\to\infty} F_{2,u}(y;\delta,D)$.

Theorem 4.3 *$F_{2,\infty}(y;\delta,D) = \lim_{u\to\infty} F_{2,u}(y;\delta,D)$ satisfies*

$$F_{2,\infty}(y;\delta,D) = \frac{\int_0^\infty e^{\kappa x}\left[\overline{G}(x) - \overline{G}(x+y) - G(y)\overline{H}(x)\right]dx}{\int_0^\infty e^{\kappa x}\left[\overline{G}(x) - \overline{H}(x)\right]dx} \tag{4.57}$$

$$= \frac{\frac{\beta}{\kappa} - \frac{1}{b-\kappa}\overline{G}(y) - e^{-\kappa y}\int_y^\infty e^{\kappa t}\overline{G}(t)dt}{\frac{\beta}{\kappa} - \frac{1}{b-\kappa}}, \quad y \geq 0, \tag{4.58}$$

where κ satisfies $\int_0^\infty e^{\kappa x} dG(x) = 1 + \beta$, or equivalently, $\kappa = -\xi_2$ and ξ_2 is the unique negative root of (2.187).

Proof: Since $\phi_s(u)$ satisfies (2.117), $\phi_s(u) = \frac{1}{1+\beta}\int_0^u \phi_s(u-x)dG(x) + \frac{1}{1+\beta}\overline{\Gamma} * H(u)$, and $F_2(y;\delta,D|u)$ satisfies (4.47), $F_2(y;\delta,D|u) = \frac{1}{1+\beta}\int_0^u F_2(y;\delta,D|u-x)\, dG(x) + \frac{1}{1+\beta}\Big[\overline{G}(u) - \overline{G}(u+y) - G(y)\overline{H}(u)\Big]$, by theorem 1.2,

$$\phi_s(u) \sim \frac{\int_0^\infty e^{\kappa x}\overline{\Gamma} * H(x)dx}{\int_0^\infty xe^{\kappa x}dG(x)}e^{-\kappa u} = \frac{\int_0^\infty e^{\kappa x}\Big[\overline{G}(x) - \overline{H}(x)\Big]dx}{\int_0^\infty xe^{\kappa x}dG(x)}e^{-\kappa u}, \quad as\ u \to \infty,$$

and

$$F_2(y;\delta,D|u) \sim \frac{\int_0^\infty e^{\kappa x}\Big[\overline{G}(x) - \overline{G}(x+y) - G(y)\overline{H}(x)\Big]dx}{\int_0^\infty xe^{\kappa x}dG(x)}e^{-\kappa u}, \quad as\ u \to \infty,$$

where κ is positive and satisfies $\int_0^\infty e^{\kappa x}dG(x) = 1+\beta$. Therefore,

$$\begin{aligned} F_{2,\infty}(y;\delta,D) &= \lim_{u\to\infty} F_{2,u}(y;\delta,D) = \lim_{u\to\infty}\frac{e^{\kappa u}F_2(y;\delta,D|u)}{e^{\kappa u}\phi_s(u)} \\ &= \frac{\int_0^\infty e^{\kappa x}\Big[\overline{G}(x) - \overline{G}(x+y) - G(y)\overline{H}(x)\Big]dx}{\int_0^\infty e^{\kappa x}\Big[\overline{G}(x) - \overline{H}(x)\Big]dx}, \end{aligned}$$

which is (4.57).

Since $\int_0^\infty e^{\kappa x}\overline{G}(x+y)dx = e^{-\kappa y}\int_y^\infty e^{\kappa t}\overline{G}(t)dt$, $\int_0^\infty e^{\kappa x}\overline{H}(x)dx = \int_0^\infty e^{-(b-\kappa)x}dx = \frac{1}{b-\kappa}$ by $b > \kappa$, and $\int_0^\infty e^{\kappa x}\overline{G}(x)dx = \frac{1}{\kappa}\int_0^\infty \overline{G}(x)de^{\kappa x} = \frac{1}{\kappa}\Big[e^{\kappa x}\overline{G}(x)\Big|_0^\infty + \int_0^\infty e^{\kappa x}dG(x)\Big] = \frac{\beta}{\kappa}$, (4.57) leads to (4.58). □

4.4 Discounted distribution and probability density functions of $U(T-)$

The discounted distribution function of $U(T-)$, $F_1(x;\delta,D|u)$, is directly obtained from $F(x,y;\delta,D|u)$ by letting $y\to\infty$

Corollary 4.5 *The discounted defective marginal distribution function of $U(T-)$ is*

$$F_1(x;\delta,D|u) = \begin{cases} \frac{1+\beta}{\beta}\overline{K}(u) - \frac{1}{\beta}\overline{K*H}(u) \\ \quad +\frac{1}{\beta}\frac{b}{a}e^{-\rho x}\overline{\Gamma}(x)\left[\rho\int_0^u e^{\rho t}\overline{K}(u-t)dt + \overline{K*H}(u) - e^{\rho u}\right], \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } 0\le u<x, \\ \frac{1}{\beta}\int_0^x \overline{K}(u-t)G'(t)dt - \frac{1}{\beta}\overline{K*H}(u) \\ \quad +\frac{1}{\beta}\frac{b}{a}e^{-\rho x}\overline{\Gamma}(x)\left[\rho\int_0^x e^{\rho t}\overline{K}(u-t)dt + \overline{K*H}(u)\right] \\ \quad +\frac{1}{\beta}\left[\overline{G}(x) - \frac{b}{a}\overline{\Gamma}(x)\right]\overline{K*H}(u-x), \qquad \text{if } 0<x\le u, \end{cases} \tag{4.59}$$

with

$$F_1(x;\delta,D|0) = 0 \tag{4.60}$$

and

$$F_1(\infty;\delta,D|u) = \lim_{x\to\infty} F_1(x;\delta,D|u) = \frac{1+\beta}{\beta}\overline{K}(u) - \frac{1}{\beta}\overline{K*H}(u) = \phi_s(u), \tag{4.61}$$

where $\overline{K*H}(u)$ *can be replaced with* $\phi_t(u)$ *by (2.103).*

When $D \to 0$,

$$F_1(x;\delta,0|u) = \begin{cases} \overline{K}_0(u) + \frac{1}{\beta_0}e^{-\rho x}\overline{\Gamma}(x)\Big[\rho\int_0^u e^{\rho t}\overline{K}_0(u-t)dt + \overline{K}_0(u) - e^{\rho u}\Big], & \text{if } 0 \le u < x, \\ \frac{1}{\beta_0}\int_0^x \overline{K}_0(u-t)\Gamma'(t)dt - \frac{1}{\beta_0}\overline{K}_0(u) \\ \quad + \frac{1}{\beta_0}e^{-\rho x}\overline{\Gamma}(x)\Big[\rho\int_0^x e^{\rho t}\overline{K}_0(u-t)dt + \overline{K}_0(u)\Big], & \text{if } 0 < x \le u, \end{cases} \tag{4.62}$$

with

$$F_1(x;\delta,0|0) = \frac{1}{1+\beta_0}\Big[1 - e^{-\rho x}\overline{\Gamma}(x)\Big] \tag{4.63}$$

and

$$F_1(\infty;\delta,0|u) = \lim_{x\to\infty} F_1(x;\delta,0|u) = \overline{K}_0(u). \tag{4.64}$$

If further let $\delta = 0$,

$$F_1(x;0,0|u) = \begin{cases} \Big[1 + \frac{\overline{P}_1(x)}{\theta}\Big]\psi_0(u) - \frac{\overline{P}_1(x)}{\theta}, & \text{if } 0 \le u < x, \\ \frac{1}{\theta p_1}\int_0^x \psi_0(u-t)\overline{P}(t)dt - \frac{1}{\theta}P_1(x)\psi_0(u), & \text{if } 0 < x \le u, \end{cases} \tag{4.65}$$

with

$$F_1(x;0,0|0) = \frac{1}{1+\theta}P_1(x) \tag{4.66}$$

and

$$F_1(\infty;0,0|u) = \lim_{x\to\infty} F_1(x;0,0|u) = \psi_0(u). \tag{4.67}$$

Proof: By letting $y \to \infty$ in (4.20), which implies $\overline{K}(u+y)$, $\overline{\Gamma}(x+y)$, $\overline{G}(x+y)$ and $G'(y+t) \to 0$, $G(y) \to 1$, and $\int_0^y \overline{K}(u+y-t)G'(t)dt \leq \int_0^\infty \overline{K}(u+y-t)G'(t)dt \to 0$, we obtain (4.59). Equations (4.62) and (4.65) can be shown from (4.22) and (4.24), respectively, by similar arguments, whereas (4.60), (4.63) and (4.66) can be shown from (4.21), (4.23) and (4.25), respectively.

When $x \to \infty$, $F_1(x;\delta,D|u) \to \frac{1+\beta}{\beta}\overline{K}(u) - \frac{1}{\beta}\overline{K*H}(u) = \overline{K}(u) - \frac{1}{1+\beta}\phi_d(u)$ $= \phi_s(u)$ by (2.101) and (2.103), proving (4.61). (4.64) and (4.67) can be shown directly. □

Corollary 4.6 *The discounted defective probability density function of $U(T-)$ is*

$$f_1(x;\delta,D|u) = \begin{cases} \dfrac{\lambda e^{-\rho x}\overline{P}(x)}{c+2\rho D}\,\dfrac{e^{\rho u} - \overline{K*H}(u) - \rho\int_0^u e^{\rho t}\overline{K}(u-t)dt}{1-\overline{K}(0)}, & \text{if } 0 \leq u < x, \\[2ex] \dfrac{\lambda e^{-\rho x}\overline{P}(x)}{c+2\rho D}\,\dfrac{e^{\rho x}\overline{K*H}(u-x) - \overline{K*H}(u) - \rho\int_0^x e^{\rho t}\overline{K}(u-t)dt}{1-\overline{K}(0)}, & \text{if } 0 < x \leq u, \end{cases} \tag{4.68}$$

with

$$f_1(x;\delta,D|0) = 0, \tag{4.69}$$

where $\overline{K*H}(u)$ can be replaced with $\phi_t(u)$ by (2.103).

When $D \to 0$,

$$f_1(x;\delta,0|u) = \begin{cases} \dfrac{\lambda}{c}e^{-\rho x}\overline{P}(x)\,\dfrac{e^{\rho u} - \overline{K}_0(u) - \rho\int_0^u e^{\rho t}\overline{K}_0(u-t)dt}{1-\overline{K}_0(0)}, & \textit{if } 0 \le u < x, \\[2ex] \dfrac{\lambda}{c}e^{-\rho x}\overline{P}(x)\,\dfrac{e^{\rho x}\overline{K}_0(u-x) - \overline{K}_0(u) - \rho\int_0^x e^{\rho t}\overline{K}_0(u-t)dt}{1-\overline{K}_0(0)}, & \textit{if } 0 < x \le u, \end{cases} \tag{4.70}$$

with $f_1(x;\delta,0|0)$ *given (4.7).*

If further let $\delta = 0$,

$$f_1(x;0,0|u) = \begin{cases} \dfrac{\lambda}{c}\overline{P}(x)\,\dfrac{1-\psi_0(u)}{1-\psi_0(0)}, & \textit{if } 0 \le u < x, \\[2ex] \dfrac{\lambda}{c}\overline{P}(x)\,\dfrac{\psi_0(u-x)-\psi_0(u)}{1-\psi_0(0)}, & \textit{if } 0 < x \le u, \end{cases} \tag{4.71}$$

with $f_1(x;0,0|0)$ *given (4.5).*

Proof: By (2.44) and $\overline{K}(0) = \dfrac{1}{1+\beta}$, differentiating (4.59) with respect to x leads to that

if $0 \le u < x$,

$$\begin{aligned} f_1(x;\delta,D|u) &= \frac{1}{\beta}\frac{b}{a}\Big[e^{-\rho x}\overline{\Gamma}(x)\Big]'\Big[\rho\int_0^u e^{\rho t}\overline{K}(u-t)dt + \overline{K*H}(u) - e^{\rho u}\Big] \\ &= -\frac{1}{\beta}\frac{b}{a}\frac{\lambda}{bD}(1+\beta)e^{-\rho x}\overline{P}(x)\Big[\rho\int_0^u e^{\rho t}\overline{K}(u-t)dt + \overline{K*H}(u) - e^{\rho u}\Big] \end{aligned}$$

$$= \frac{\lambda}{c+2\rho D} e^{-\rho x}\overline{P}(x) \frac{e^{\rho u} - \overline{K*H}(u) - \rho\int_0^u e^{\rho t}\overline{K}(u-t)dt}{1-\overline{K}(0)};$$

if $0 < x \le u$, with the help of (2.48) and (2.106),

$$\begin{aligned}
& f_1(x;\delta, D|u) \\
= \; & \frac{1}{\beta}\overline{K}(u-x)G'(x) + \frac{1}{\beta}\frac{b}{a}\Big[e^{-\rho x}\overline{\Gamma}(x)\Big]'\Big[\rho\int_0^x e^{\rho t}\overline{K}(u-t)dt + \overline{K*H}(u)\Big] + \\
& \frac{1}{\beta}\frac{b}{a}e^{-\rho x}\overline{\Gamma}(x)\rho e^{\rho x}\overline{K}(u-x) + \frac{1}{\beta}\Big[\overline{G}'(x) - \frac{b}{a}\overline{\Gamma}'(x)\Big]\overline{K*H}(u-x) + \\
& \frac{1}{\beta}\Big[\overline{G}(x) - \frac{b}{a}\overline{\Gamma}(x)\Big]\overline{K*H}'(u-x) \\
= \; & \frac{1}{\beta}\Big[G'(x) + \frac{b\rho}{a}\overline{\Gamma}(x)\Big]\overline{K}(u-x) + \frac{1}{\beta}\Big[\overline{G}'(x) - \frac{b}{a}\overline{\Gamma}'(x)\Big]\overline{K*H}(u-x) + \\
& \frac{b}{\beta}\Big[\overline{G}(x) - \frac{b}{a}\overline{\Gamma}(x)\Big]\Big[\overline{K*H}(u-x) - \overline{K}(u-x)\Big] - \\
& \frac{\lambda}{c+2\rho D}e^{-\rho x}\overline{P}(x)\frac{\overline{K*H}(u) + \rho\int_0^x e^{\rho t}\overline{K}(u-t)dt}{1-\overline{K}(0)} \\
= \; & \frac{1}{\beta}\Big[G'(x) + \frac{b\rho}{a}\overline{\Gamma}(x) - b\overline{G}(x) + \frac{b^2}{a}\overline{\Gamma}(x)\Big]\overline{K}(u-x) - \\
& \frac{1}{\beta}\Big[G'(x) - \frac{b}{a}\Gamma'(x) - b\overline{G}(x) + \frac{b^2}{a}\overline{\Gamma}(x)\Big]\overline{K*H}(u-x) - \\
& \frac{\lambda}{c+2\rho D}e^{-\rho x}\overline{P}(x)\frac{\overline{K*H}(u) + \rho\int_0^x e^{\rho t}\overline{K}(u-t)dt}{1-\overline{K}(0)} \\
= \; & \frac{1}{\beta}\Big[\frac{b\rho}{a} - b + \frac{b^2}{a}\Big]\overline{\Gamma}(x)\overline{K}(u-x) - \\
& \frac{1}{\beta}\Big[(\frac{b^2}{a} - b)\overline{\Gamma}(x) - \frac{b}{a}\Gamma'(x)\Big]\overline{K*H}(u-x) - \\
& \frac{\lambda}{c+2\rho D}e^{-\rho x}\overline{P}(x)\frac{\overline{K*H}(u) + \rho\int_0^x e^{\rho t}\overline{K}(u-t)dt}{1-\overline{K}(0)} \\
= \; & \frac{1}{\beta}\frac{b}{a}\Big[\rho\overline{\Gamma}(x) + \Gamma'(x)\Big]\overline{K*H}(u-x) -
\end{aligned}$$

$$\begin{aligned}
&\frac{\lambda}{c+2\rho D}e^{-\rho x}\overline{P}(x)\,\frac{\overline{K*H}(u)+\rho\int_0^x e^{\rho t}\overline{K}(u-t)dt}{1-\overline{K}(0)}\\
=\;&\frac{\lambda}{c+2\rho D}e^{-\rho x}\overline{P}(x)\,\frac{e^{\rho x}\overline{K*H}(u-x)-\overline{K*H}(u)-\rho\int_0^x e^{\rho t}\overline{K}(u-t)dt}{1-\overline{K}(0)}.
\end{aligned}$$

Similar arguments derive (4.70) from (4.62). If further $\delta=0$, then $\overline{K}_{0,\delta=0}(u)=\psi_0(u)$, and (4.70) reduces to (4.71). Setting $u=0$ in (4.68), (4.70) and (4.71) easily give (4.69), (4.7) and (4.5), respectively. □

Now equate (4.6) and (4.70), we get

$$\frac{e^{\rho u}-\overline{K}_\rho(u)}{1-\overline{K}_\rho(0)}=\frac{e^{\rho u}-\overline{K}_0(u)-\rho\int_0^u e^{\rho t}\overline{K}_0(u-t)dt}{1-\overline{K}_0(0)},$$

or equivalently,

$$\overline{K}_\rho(u)=e^{\rho u}-\frac{1-\overline{K}_\rho(0)}{1-\overline{K}_0(0)}\Big[e^{\rho u}-\overline{K}_0(u)-\rho\int_0^u e^{\rho t}\overline{K}_0(u-t)dt\Big], \tag{4.72}$$

with $\overline{K}_{\rho,\delta=0}(u)=\overline{K}_{0,\delta=0}(u)=\psi_0(u)$.

Theorem 4.4 *For $0\le u<x$, the discounted defective distribution function of $U(T-)$ satisfies the defective renewal equation*

$$\begin{aligned}
F_1(x;\delta,D|u) \;=\;& \frac{1}{1+\beta}\int_0^u F_1(x;\delta,D|u-y)dG(y)+\\
&\frac{1}{1+\beta}\Big\{\overline{\Gamma}*H(u)-\frac{b}{a}e^{-\rho x}\overline{\Gamma}(x)\Big[e^{\rho u}-e^{-bu}\Big]\Big\}. \qquad (4.73)
\end{aligned}$$

When $D\to 0$,

$$F_1(x;\delta,0|u) = \frac{1}{1+\beta_0}\int_0^u F_1(x;\delta,0|u-y)d\Gamma(y) + \frac{1}{1+\beta_0}\Big[\overline{\Gamma}(u) - e^{-\rho(x-u)}\overline{\Gamma}(x)\Big]. \tag{4.74}$$

If further let $\delta = 0$,

$$F_1(x;0,0|u) = \frac{1}{1+\theta}\int_0^u F_1(x;0,0|u-y)dP_1(y) + \frac{1}{1+\theta}\Big[\overline{P}_1(u) - \overline{P}_1(x)\Big]. \tag{4.75}$$

Proof: For any fixed x, let

$$w(z,y) = \begin{cases} 1, & \text{if } z \le x, \\ 0, & \text{otherwise.} \end{cases} \tag{4.76}$$

Then by (4.1) and (4.2), $\phi_w(u)$ in (2.2) becomes

$$\begin{aligned} \phi_w(u) &= E[e^{-\delta T}w(U(T-),|U(T)|)I(T<\infty, U(T)<0)|U(0)=u] \\ &= \int_0^\infty\int_0^\infty\int_0^\infty e^{-\delta t}w(z,y)f(z,y,t;D|u)dtdydz \\ &= \int_0^x\int_0^\infty\int_0^\infty e^{-\delta t}f(z,y,t;D|u)dtdydz \\ &= \int_0^x\int_0^\infty f(z,y;\delta,D|u)dydz \\ &= \int_0^x f_1(z;\delta,D|u)dz \\ &= F_1(x;\delta,D|u), \end{aligned} \tag{4.77}$$

the discounted distribution function of $U(T-)$.

By (2.15), $F_1(x;\delta,D|u)$ satisfies the defective renewal equation $F_1(x;\delta,D|u) = \int_0^u F_1(x;\delta,D|u-y)g(y)dy + g_\omega(u)$ where $g_\omega(u) = h * \gamma_\omega(u)$. For $0 \le u < x$, from (2.7) and (2.14),

$$\begin{aligned}
\gamma_\omega(s) &= \frac{\lambda}{c}\int_s^\infty e^{-\rho(z-s)}\int_0^\infty w(z,y)p(z+y)dydz \\
&= \frac{\lambda}{c}\int_s^x e^{-\rho(z-s)}\int_0^\infty p(z+y)dydz \\
&= \frac{\lambda}{c}\Big[\int_s^\infty e^{-\rho(z-s)}\overline{P}(z)dz - \int_x^\infty e^{-\rho(z-s)}\overline{P}(z)dz\Big] \\
&= \frac{\lambda}{c}\Big[\int_s^\infty e^{-\rho(z-s)}\overline{P}(z)dz - e^{-\rho(x-s)}\int_x^\infty e^{-\rho(z-x)}\overline{P}(z)dz\Big]
\end{aligned}$$

and by (1.49) and (1.52)

$$\frac{\gamma_\omega(s)}{\int_0^\infty \gamma(y)dy} = \frac{\gamma_\omega(s)}{\frac{\lambda}{c}\int_0^\infty e^{-\rho y}\overline{P}(y)dy} = \overline{\Gamma}(s) - e^{-\rho(x-s)}\overline{\Gamma}(x).$$

This implies

$$\begin{aligned}
\frac{g_\omega(u)}{\int_0^\infty g(z)dz} &= \frac{\int_0^u h(u-s)\gamma_\omega(s)ds}{\int_0^\infty h(x)dx\int_0^\infty \gamma(y)dy} \\
&= \int_0^u H'(u-s)\Big[\overline{\Gamma}(s) - e^{-\rho(x-s)}\overline{\Gamma}(x)\Big]ds \\
&= \overline{\Gamma} * H(u) - e^{-\rho x}\overline{\Gamma}(x)be^{-bu}\int_0^u e^{(b+\rho)s}ds \\
&= \overline{\Gamma} * H(u) - e^{-\rho x}\overline{\Gamma}(x)\frac{b}{b+\rho}e^{-bu}\Big[e^{(b+\rho)u} - 1\Big] \\
&= \overline{\Gamma} * H(u) - \frac{b}{a}e^{-\rho x}\overline{\Gamma}(x)\Big[e^{\rho u} - e^{-bu}\Big].
\end{aligned}$$

Since $\int_0^\infty g(z)dz = \frac{1}{1+\beta}$ and $G'(x) = \frac{g(x)}{\int_0^\infty g(z)dz}$,

$$\begin{aligned}
& F_1(x;\delta, D|u) \\
= & \int_0^u F_1(x;\delta, D|u-y)g(y)dy + g_\omega(u) \\
= & \frac{1}{1+\beta}\int_0^u F_1(x;\delta, D|u-y)\frac{g(y)}{\int_0^\infty g(z)dz}dy + \frac{1}{1+\beta}\frac{g_\omega(u)}{\int_0^\infty g(z)dz} \\
= & \frac{1}{1+\beta}\int_0^u F_1(x;\delta, D|u-y)G'(y)dy + \\
& \frac{1}{1+\beta}\left\{\overline{\Gamma} * H(u) - \frac{b}{a}e^{-\rho x}\overline{\Gamma}(x)\left[e^{\rho u} - e^{-bu}\right]\right\}.
\end{aligned}$$

(4.74) and (4.75) can be shown by similar arguments. Alternatively, when $D \to 0$, $\beta \to \beta_0$, $b/a \to 1$, $\overline{\Gamma} * H(u) \to \overline{\Gamma}(u)$ and $\overline{H}(u) = e^{-bu} \to 0$ for $u > 0$, (4.73) $\to$ (4.74) for $u > 0$. If $\delta = 0$ then $\beta_0 = \theta$ and $\Gamma(x) = P_1(x)$, (4.74) reduces to (4.75). □

Corollary 4.7 *For* $0 \le u < x$, *the discounted defective probability density function of* $U(T-)$ *satisfies the defective renewal equation*

$$f_1(x;\delta, D|u) = \frac{1}{1+\beta}\int_0^u f_1(x;\delta, D|u-y)dG(y) + \frac{\lambda}{c+2\rho D}e^{-\rho x}\overline{P}(x)\left[e^{\rho u} - e^{-bu}\right]. \tag{4.78}$$

When $D \to 0$,

$$f_1(x;\delta, 0|u) = \frac{1}{1+\beta_0}\int_0^u f_1(x;\delta, 0|u-y)d\Gamma(y) + \frac{\lambda}{c}e^{-\rho(x-u)}\overline{P}(x). \tag{4.79}$$

If further let $\delta = 0$,

$$f_1(x;0,0|u) = \frac{1}{1+\theta}\int_0^u f_1(x;0,0|u-y)dP_1(y) + \frac{\lambda}{c}\overline{P}(x). \tag{4.80}$$

Proof: Differentiating (4.73), (4.74) and (4.75) with respect to x lead to (4.78), (4.79) and (4.80), respectively, with the help of (2.44). □

Dickson (1992) [5] proposed a relationship between $F_1(x;0,0|u)$, the probability that ruin with initial surplus u and the surplus immediately prior to ruin caused by a claim is at most x, and $F_2(y;0,0|u)$, the probability of ruin with initial surplus u and the deficit immediately after the claim causing ruin is at most y, as follows:

$$F_1(x;0,0|u) = F_2(x;0,0|u-x) - \left[1+\frac{\overline{P_1(x)}}{\theta}\right]\left[\psi_0(u-x)-\psi_0(u)\right], \quad 0 < x \leq u. \tag{4.81}$$

We also have corresponding relationships between $F_1(x;\delta,D|u)$ and $F_2(y;\delta,D|u)$ and between $F_1(x;\delta,0|u)$ and $F_2(y;\delta,0|u)$ as follows:

Lemma 4.1 *For* $0 < x \leq u$,

$$\begin{aligned}
&F_1(x;\delta,D|u)\\
= \; & F_2(x;\delta,D|u-x) - \left[\overline{K}(u-x) - \overline{K}(u)\right] - \frac{1}{\beta}\frac{b}{a}e^{-\rho x}\overline{\Gamma}(x)\int_0^x e^{\rho t}d\overline{K}(u-t) - \\
&\frac{1}{\beta}\left[1-\frac{b}{a}\overline{\Gamma}(x)\right]\left[\overline{K}(u-x) - \overline{K*H}(u-x)\right] + \\
&\frac{1}{\beta}\left[1-\frac{b}{a}e^{-\rho x}\overline{\Gamma}(x)\right]\left[\overline{K}(u) - \overline{K*H}(u)\right].
\end{aligned} \tag{4.82}$$

When $D \to 0$,

$$F_1(x;\delta,0|u) = F_2(x;\delta,0|u-x) - \left[\overline{K}_0(u-x) - \overline{K}_0(u)\right] - \frac{1}{\beta_0}e^{-\rho x}\overline{\Gamma}(x)\int_0^x e^{\rho t}d\overline{K}_0(u-t). \tag{4.83}$$

If further let $\delta = 0$, (4.83) simplifies to (4.81).

Proof: By (4.32), $F_2(x;\delta,D|u-x) = \frac{1+\beta}{\beta}\left[\overline{K}(u-x) - \overline{K}(u)\right] - \frac{1}{\beta}G(y)\overline{K*H}(u-x) + \frac{1}{\beta}\int_0^x \overline{K}(u-t)dG(t)$. Deducting this from (4.59) (the expression for $0 < x \le u$) gives

$$\begin{aligned}
& F_1(x;\delta,D|u) - F_2(x;\delta,D|u-x) \\
= \ & -\frac{1+\beta}{\beta}\left[\overline{K}(u-x) - \overline{K}(u)\right] + \frac{1}{\beta}\left[\overline{K*H}(u-x) - \overline{K*H}(u)\right] - \\
& \frac{1}{\beta}\frac{b}{a}\overline{\Gamma}(x)\overline{K*H}(u-x) + \frac{1}{\beta}\frac{b}{a}e^{-\rho x}\overline{\Gamma}(x)\left[\rho\int_0^x e^{\rho t}\overline{K}(u-t)dt + \overline{K*H}(u)\right] \\
= \ & -\frac{1+\beta}{\beta}\left[\overline{K}(u-x) - \overline{K}(u)\right] + \frac{1}{\beta}\left[\overline{K*H}(u-x) - \overline{K*H}(u)\right] - \\
& \frac{1}{\beta}\frac{b}{a}\overline{\Gamma}(x)\overline{K*H}(u-x) + \\
& \frac{1}{\beta}\frac{b}{a}e^{-\rho x}\overline{\Gamma}(x)\left[e^{\rho x}\overline{K}(u-x) - \overline{K}(u) - \int_0^x e^{\rho t}d\overline{K}(u-t) + \overline{K*H}(u)\right] \\
= \ & -\left[\overline{K}(u-x) - \overline{K}(u)\right] - \frac{1}{\beta}\frac{b}{a}e^{-\rho x}\overline{\Gamma}(x)\int_0^x e^{\rho t}d\overline{K}(u-t) - \\
& \frac{1}{\beta}\left[1 - \frac{b}{a}\overline{\Gamma}(x)\right]\left[\overline{K}(u-x) - \overline{K*H}(u-x)\right] + \\
& \frac{1}{\beta}\left[1 - \frac{b}{a}e^{-\rho x}\overline{\Gamma}(x)\right]\left[\overline{K}(u) - \overline{K*H}(u)\right]
\end{aligned}$$

after some arrangements.

When $D \to 0$, $b/a \to 1$, $\beta \to \beta_0$ and both $\overline{K}(u)$ and $\overline{K*H}(u) \to \overline{K}_0(u)$, implying that (4.82) $\to$ (4.83). If $\delta = 0$ then $\beta_0 = \theta$, $\overline{\Gamma}(x) = \overline{P}_1(x)$ and $\overline{K}_0(u) = \psi_0(u)$, (4.83) reduces to (4.81). □

4.5 Discounted distribution and probability density function of $\{|U(T)| + U(T-)\}$

When ruin occurs due to a claim, $\{|U(T)|+U(T-)\}$ is the amount of the claim causing ruin. If we observe corollaries 4.1 and 4.6, we find some relations as follows:

$$\frac{f(x,y;\delta,D|u)}{f_1(x;\delta,D|u)} = \frac{f(x,y;\delta,0|u)}{f_1(x;\delta,0|u)} = \frac{f(x,y;0,0|u)}{f_1(x;0,0|u)} = \frac{p(x+y)}{\overline{P}(x)}, \tag{4.84}$$

independent of u, where $x+y$ is the amount of the claim causing ruin. Though Dickson and Egídio dos Reis (1994) [8] first showed the relation above for the case $D = 0$ and $\delta = 0$, their proof seemed complicated. Later, Gerber and Shiu (1997) [28] gave a easier proof.

If we let $w(x,y) = x+y$, then by (4.1), $\phi_w(u)$ in (2.2) becomes

$$\begin{aligned}
\phi_w(u) &= E[e^{-\delta T}(U(T-)+|U(T)|)I(T<\infty, U(T)<0)|U(0)=u] \\
&= \int_0^\infty\int_0^\infty\int_0^\infty e^{-\delta t}(x+y)f(x,y,t;D|u)dtdxdy \\
&= \int_0^\infty\int_0^\infty (x+y)f(x,y;\delta,D|u)dxdy \\
&= \int_0^\infty z\int_0^z f(x,z-x;\delta,D|u)dxdz \\
&= \int_0^\infty z f_Z(z;\delta,D|u)dz,
\end{aligned} \tag{4.85}$$

the defective Laplace transform or the expectation of the present value of the amount of the claim causing ruin, where

$$f_Z(z;\delta,D|u) = \int_0^z f(x,z-x;\delta,D|u)dx = p(z)\int_0^z \frac{f_1(x;\delta,D|u)}{\overline{P}(x)}dx \tag{4.86}$$

by (4.84), is the discounted defective probability density function of the amount of the claim causing ruin. (4.86) provides alternative formulas for obtaining the expression for $f_Z(z;\delta,D|u)$ since the explicit expressions for $f(x,y;\delta,D|u)$ and $f_1(x;\delta,D|u)$ are available from (4.26) and (4.68), respectively.

Theorem 4.5 *The discounted defective probability density function of* $\{U(T-)+|U(T)|\}$ *is*

$$f_Z(z;\delta,D|u)$$
$$= \begin{cases} \dfrac{\lambda}{(c+2\rho D)\rho}\dfrac{e^{-\rho z}p(z)}{1-\overline{K}(0)}\Big\{\rho e^{\rho z}\Big[\int_0^u \overline{K*H}(u-t)dt - \int_0^u \overline{K}(u-t)dt\Big]+ \\ [e^{\rho z}-e^{\rho u}] - [e^{\rho z}-1]\overline{K*H}(u) + \rho\int_0^u e^{\rho t}\overline{K}(u-t)dt\Big\}, \quad if\ 0 \le u < z, \\ \\ \dfrac{\lambda}{(c+2\rho D)\rho}\dfrac{e^{-\rho z}p(z)}{1-\overline{K}(0)}\Big\{\rho e^{\rho z}\Big[\int_0^z \overline{K*H}(u-t)dt - \int_0^z \overline{K}(u-t)dt\Big]- \\ [e^{\rho z}-1]\overline{K*H}(u) + \rho\int_0^z e^{\rho t}\overline{K}(u-t)dt\Big\}, \quad if\ 0 < z \le u, \end{cases} \tag{4.87}$$

with

$$f_Z(z;\delta,D|0) = 0, \tag{4.88}$$

where $\overline{K*H}(u)$ *can be replaced with* $\phi_t(u)$ *by (2.103).*

When $D \to 0$,

$$f_Z(z;\delta,0|u) = \begin{cases} \dfrac{\lambda}{c\rho}e^{-\rho z}p(z)\,\dfrac{[e^{\rho z}-e^{\rho u}]-[e^{\rho z}-1]\overline{K}_0(u)+\rho\int_0^u e^{\rho t}\overline{K}_0(u-t)dt}{1-\overline{K}_0(0)}, \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } 0 \le u < z, \\ \dfrac{\lambda}{c\rho}e^{-\rho z}p(z)\,\dfrac{-[e^{\rho z}-1]\overline{K}_0(u)+\rho\int_0^z e^{\rho t}\overline{K}_0(u-t)dt}{1-\overline{K}_0(0)}, \quad \text{if } 0 < z \le u, \end{cases} \tag{4.89}$$

with

$$f_Z(z;\delta,0|0) = \frac{\lambda}{c}\frac{1-e^{-\rho z}}{\rho}p(z). \tag{4.90}$$

If further let $\delta = 0$,

$$f_Z(z;0,0|u) = \begin{cases} \dfrac{\lambda}{c}p(z)\,\dfrac{[z-u]-\psi_0(u)z+\int_0^u \psi_0(u-t)dt}{1-\psi_0(0)}, & \text{if } 0 \le u < z, \\ \dfrac{\lambda}{c}p(z)\,\dfrac{-\psi_0(u)z+\int_0^z \psi_0(u-t)dt}{1-\psi_0(0)}, & \text{if } 0 < z \le u, \end{cases} \tag{4.91}$$

with

$$f_Z(z;0,0|0) = \frac{\lambda}{c}zp(z). \tag{4.92}$$

Proof: By (4.68), if $0 < z \le u$, (4.86) becomes

$$
\begin{aligned}
&f_Z(z;\delta,D|u)\\
=\;& \frac{\lambda}{c+2\rho D}\frac{p(z)}{1-\overline{K}(0)}\Big\{\int_0^z \overline{K*H}(u-x)dx - \overline{K*H}(u)\int_0^z e^{-\rho x}dx - \\
&\qquad \rho\int_0^z e^{-\rho x}\int_0^x e^{\rho t}\overline{K}(u-t)dtdx\Big\}\\
=\;& \frac{\lambda}{c+2\rho D}\frac{p(z)}{1-\overline{K}(0)}\Big\{\int_0^z \overline{K*H}(u-x)dx - \frac{1-e^{-\rho z}}{\rho}\overline{K*H}(u) - \\
&\qquad \rho\int_0^z\int_t^z e^{-\rho(x-t)}\overline{K}(u-t)dxdt\Big\}\\
=\;& \frac{\lambda}{c+2\rho D}\frac{p(z)}{1-\overline{K}(0)}\Big\{\int_0^z \overline{K*H}(u-x)dx - \frac{1-e^{-\rho z}}{\rho}\overline{K*H}(u) - \\
&\qquad \int_0^z\Big[1-e^{-\rho(z-t)}\Big]\overline{K}(u-t)dt\Big\} \qquad (4.93)\\
=\;& \frac{\lambda}{(c+2\rho D)\rho}\frac{e^{-\rho z}p(z)}{1-\overline{K}(0)}\Big\{\rho e^{\rho z}\Big[\int_0^z \overline{K*H}(u-t)dt - \int_0^z \overline{K}(u-t)dt\Big] - \\
&\qquad [e^{\rho z}-1]\overline{K*H}(u) + \rho\int_0^z e^{\rho t}\overline{K}(u-t)dt\Big\};
\end{aligned}
$$

whereas if $0 \le u < z$, decompose $\int_0^z$ into $\int_0^u$ and $\int_u^z$. Then from (4.93)

$$
\begin{aligned}
&p(z)\int_0^u \frac{f_1(x;\delta,D|u)}{\overline{P}(x)}dx\\
=\;& \frac{\lambda}{c+2\rho D}\frac{p(z)}{1-\overline{K}(0)}\Big\{\int_0^u \overline{K*H}(u-t)dt - \frac{1-e^{-\rho u}}{\rho}\overline{K*H}(u) - \\
&\qquad \int_0^u\Big[1-e^{-\rho(u-t)}\Big]\overline{K}(u-t)dt\Big\},
\end{aligned}
$$

and from (4.68)

$$
\begin{aligned}
& p(z)\int_u^z \frac{f_1(x;\delta,D|u)}{\overline{P}(x)}dx \\
= \; & \frac{\lambda}{c+2\rho D}\,\frac{p(z)}{1-\overline{K}(0)}\left\{e^{\rho u}-\overline{K*H}(u)-\rho\int_0^u e^{\rho t}\overline{K}(u-t)dt\right\}\int_u^z e^{-\rho x}dx \\
= \; & \frac{\lambda}{c+2\rho D}\,\frac{p(z)}{1-\overline{K}(0)}\left\{e^{\rho u}-\overline{K*H}(u)-\rho\int_0^u e^{\rho t}\overline{K}(u-t)dt\right\}\frac{e^{-\rho u}-e^{-\rho z}}{\rho} \\
= \; & \frac{\lambda}{c+2\rho D}\,\frac{p(z)}{1-\overline{K}(0)}\left\{\frac{1-e^{-\rho(z-u)}}{\rho}-\frac{e^{-\rho u}-e^{-\rho z}}{\rho}\overline{K*H}(u)-\right. \\
& \qquad \left.\int_0^u\left[e^{-\rho(u-t)}-e^{-\rho(z-t)}\right]\overline{K}(u-t)dt\right\}.
\end{aligned}
$$

Therefore, for $0 \le u < z$,

$$
\begin{aligned}
& f_Z(z;\delta,D|u) \\
= \; & \frac{\lambda}{c+2\rho D}\,\frac{p(z)}{1-\overline{K}(0)}\left\{\frac{1-e^{-\rho(z-u)}}{\rho}-\frac{1-e^{-\rho z}}{\rho}\overline{K*H}(u)+\right. \\
& \qquad \left.\int_0^u \overline{K*H}(u-t)dt-\int_0^u\left[1-e^{-\rho(z-t)}\right]\overline{K}(u-t)dt\right\} \\
= \; & \frac{\lambda}{(c+2\rho D)\rho}\,\frac{e^{-\rho z}p(z)}{1-\overline{K}(0)}\left\{\rho e^{\rho z}\left[\int_0^u \overline{K*H}(u-t)dt-\int_0^u \overline{K}(u-t)dt\right]+\right. \\
& \qquad \left.[e^{\rho z}-e^{\rho u}]-[e^{\rho z}-1]\overline{K*H}(u)+\rho\int_0^u e^{\rho t}\overline{K}(u-t)dt\right\}.
\end{aligned}
$$

Similar arguments show (4.89) and (4.91) from (4.70), (4.71) and (4.86). Alternatively, when $D \to 0$, both $\overline{K*H}(u)$ and $\overline{K}(u) \to \overline{K}_0(u)$ for $u > 0$, which implies (4.87) $\to$ (4.89) for $u > 0$. When $\delta \to 0$, $\rho \to 0$ and $\overline{K}_0(u) \to \psi_0(u)$, it is easy to see that (4.89) $\to$ (4.91).

When $u = 0$, (4.88), (4.90) and (4.92) are easily obtained from (4.87), (4.89) and (4.91), respectively. □

The expressions for $F_Z(z;\delta,D|u)$, $F_Z(z;\delta,0|u)$ and $F_Z(z;0,0|u)$, the discounted defective distribution functions of the amount of the claim causing ruin, can be obtained from (4.86) as follows:

$$\begin{aligned}
F_Z(z;\delta,D|u) &= \int_0^z f_Z(y;\delta,D|u)dy \\
&= \int_0^z p(y)\int_0^y \frac{f_1(x;\delta,D|u)}{\overline{P}(x)}dxdy \\
&= -\int_0^z \int_0^y \frac{f_1(x;\delta,D|u)}{\overline{P}(x)}dxd\overline{P}(y) \\
&= -\overline{P}(y)\int_0^y \frac{f_1(x;\delta,D|u)}{\overline{P}(x)}dx\Big|_0^z + \int_0^z f_1(y;\delta,D|u)dy \\
&= F_1(z;\delta,D|u) - \overline{P}(z)\int_0^z \frac{f_1(x;\delta,D|u)}{\overline{P}(x)}dx. \qquad (4.94)
\end{aligned}$$

That is, subtractions of (4.87), (4.89) and (4.91) with $p(z)$ replaced with $\overline{P}(z)$ from (4.59), (4.62) and (4.65) with x replaced with z give $F_Z(z;\delta,D|u)$, $F_Z(z;\delta,0|u)$ and $F_Z(z;0,0|u)$, respectively, since $f_Z(z;\delta,D|u) = p(z)\int_0^z \frac{f_1(x;\delta,D|u)}{\overline{P}(x)}dx$.

In addition, integrations of $f_Z(t;\delta,D|0)$, $f_Z(t;\delta,0|0)$ and $f_Z(t;0,0|0)$ from $t=0$ to $t=z$ in (4.88), (4.90) and (4.92), respectively, yield

$$F_Z(z;\delta,D|0) = 0, \qquad (4.95)$$

$$F_Z(z;\delta,0|0) = \frac{\lambda}{c}\Big[\int_0^z e^{-\rho t}\overline{P}(t)dt - \frac{1-e^{-\rho z}}{\rho}\overline{P}(z)\Big] \qquad (4.96)$$

and

$$F_Z(z;0,0|0) = \frac{\lambda}{c}\Big[p_1 P_1(z) - z\overline{P}(z)\Big]. \qquad (4.97)$$

We remark that

$$\begin{aligned} F_Z(\infty;\delta,D|u) &= \lim_{z\to\infty} F_Z(z;\delta,D|u) = \phi_s(u) \\ &= \lim_{x\to\infty} F_1(x;\delta,D|u) = \lim_{y\to\infty} F_2(y;\delta,D|u), \end{aligned} \tag{4.98}$$

$$\begin{aligned} F_Z(\infty;\delta,0|u) &= \lim_{z\to\infty} F_Z(z;\delta,0|u) = \overline{K}_0(u) \\ &= \lim_{x\to\infty} F_1(x;\delta,0|u) = \lim_{y\to\infty} F_2(y;\delta,0|u) \end{aligned} \tag{4.99}$$

and

$$\begin{aligned} F_Z(\infty;0,0|u) &= \lim_{z\to\infty} F_Z(z;0,0|u) = \psi_0(u) \\ &= \lim_{x\to\infty} F_1(x;0,0|u) = \lim_{y\to\infty} F_2(y;0,0|u). \end{aligned} \tag{4.100}$$

Theorem 4.6 *For $0 \le u < z$, the discounted defective distribution function of $\{U(T-)+|U(T)|\}$ satisfies the defective renewal equation*

$$\begin{aligned} &F_Z(z;\delta,D|u) \\ =\ & \frac{1}{1+\beta}\int_0^u F_Z(z;\delta,D|u-y)dG(y) + \frac{1}{1+\beta}\Big\{\overline{\Gamma} * H(u) - \\ & \frac{b}{a}e^{-\rho z}\overline{\Gamma}(z)\Big[e^{\rho u} - e^{-bu}\Big]\Big\} - \frac{\lambda}{c+2\rho D}\frac{1-e^{-\rho z}}{\rho}\overline{P}(z)\Big[e^{\rho u} - e^{-bu}\Big]. \end{aligned} \tag{4.101}$$

When $D \to 0$,

$$\begin{aligned} F_Z(z;\delta,0|u) &= \frac{1}{1+\beta_0}\int_0^u F_Z(z;\delta,0|u-y)d\Gamma(y) + \\ & \frac{1}{1+\beta_0}\Big[\overline{\Gamma}(u) - e^{-\rho(z-u)}\overline{\Gamma}(z)\Big] - \frac{\lambda}{c}\frac{e^{\rho u}(1-e^{-\rho z})}{\rho}\overline{P}(z). \quad (4.102) \end{aligned}$$

If further let $\delta = 0$,

$$F_Z(z;0,0|u) = \frac{1}{1+\theta}\int_0^u F_Z(z;0,0|u-y)dP_1(y) + \frac{1}{1+\theta}\Big[\overline{P}_1(u) - \overline{P}_1(z) - \frac{z}{p_1}\overline{P}(z)\Big]. \quad (4.103)$$

Proof: From (4.94), $\frac{1}{1+\beta}\int_0^u F_Z(z;\delta,D|u-y)dG(y) =$
$\frac{1}{1+\beta}\int_0^u F_1(z;\delta,D|u-y)dG(y) - \frac{\overline{P}(z)}{1+\beta}\int_0^u\int_0^z \frac{f_1(x;\delta,D|u-y)}{\overline{P}(x)}dxdG(y).$
Subtraction of this equation from (4.94) and by (4.73) and (4.78) yields

$$\begin{aligned} & F_Z(z;\delta,D|u) - \frac{1}{1+\beta}\int_0^u F_Z(z;\delta,D|u-y)dG(y) \\ =\; & F_1(z;\delta,D|u) - \frac{1}{1+\beta}\int_0^u F_1(z;\delta,D|u-y)dG(y) - \\ & \overline{P}(z)\int_0^z \frac{1}{\overline{P}(x)}\Big[f_1(x;\delta,D|u) - \frac{1}{1+\beta}\int_0^u f_1(x;\delta,D|u-y)dG(y)\Big]dx \\ =\; & \frac{1}{1+\beta}\Big\{\overline{\Gamma} * H(u) - \frac{b}{a}e^{-\rho z}\overline{\Gamma}(z)\Big[e^{\rho u} - e^{-bu}\Big]\Big\} - \\ & \frac{\lambda}{c+2\rho D}\overline{P}(z)\Big[e^{\rho u} - e^{-bu}\Big]\int_0^z e^{-\rho x}dx \\ =\; & \frac{1}{1+\beta}\Big\{\overline{\Gamma} * H(u) - \frac{b}{a}e^{-\rho z}\overline{\Gamma}(z)\Big[e^{\rho u} - e^{-bu}\Big]\Big\} - \\ =\; & \frac{\lambda}{c+2\rho D}\frac{1-e^{-\rho z}}{\rho}\overline{P}(z)\Big[e^{\rho u} - e^{-bu}\Big], \end{aligned}$$

proving (4.101). (4.102) and (4.103) can be shown by similar arguments. Alternatively, when $D \to 0$, $\beta \to \beta_0$, $b/a \to 1$, $\overline{\Gamma} * H(u) \to \overline{\Gamma}(u)$ and $\overline{H}(u) = e^{-bu} \to 0$ for $u > 0$, (4.101) $\to$ (4.102) for $u > 0$. If $\delta = 0$ then $\beta_0 = \theta$ and $\Gamma(x) = P_1(x)$, (4.102) reduces to (4.103). □

Corollary 4.8 *For $0 \le u < z$, the discounted defective probability density function of $\{U(T-) + |U(T)|\}$ satisfies the defective renewal equation*

$$f_Z(z;\delta,D|u) = \frac{1}{1+\beta}\int_0^u f_Z(z;\delta,D|u-y)dG(y) + \frac{\lambda}{c+2\rho D}\frac{1-e^{-\rho z}}{\rho}p(z)\Big[e^{\rho u} - e^{-bu}\Big]. \tag{4.104}$$

When $D \to 0$,

$$f_Z(z;\delta,0|u) = \frac{1}{1+\beta_0}\int_0^u f_Z(z;\delta,0|u-y)d\Gamma(y) + \frac{\lambda}{c}\frac{e^{\rho u}(1-e^{-\rho z})}{\rho}p(z). \tag{4.105}$$

If further let $\delta = 0$,

$$f_Z(z;0,0|u) = \frac{1}{1+\theta}\int_0^u f_Z(z;0,0|u-y)dP_1(y) + \frac{\lambda}{c}zp(z). \tag{4.106}$$

Proof: Differentiating (4.101), (4.102) and (4.103) with respect to z give (4.104), (4.105) and (4.106), respectively, with the help of (2.44). □

Since (4.82) gives the expression for $F_1(z;\delta,D|u) - F_2(z;\delta,D|u-z)$ for $0 < z \le u$, combining this with (4.94), $F_Z(z;\delta,D|u) = F_1(z;\delta,D|u) - \overline{P}(z)\int_0^z \frac{f_1(x;\delta,D|u)}{\overline{P}(x)}dx$, we can obtain the expression for $F_Z(z;\delta,D|u) - F_2(z;\delta,D|u-z)$ for $0 < z \le u$.

Note that since each of the expressions for the discounted probability distribution functions $F(x,y;\delta,D|u)$, $F_1(x;\delta,D|u)$, $F_2(y;\delta,D|u)$ and $F_Z(z;\delta,D|u)$, and for the discounted probability density functions $f(x, y; \delta, D|u)$, $f_1(x; \delta, D|u)$, $f_2(y; \delta, D|u)$ and $f_Z(z; \delta, D|u)$ involves $\overline{G}(u)$, $\overline{\Gamma}(u)$, $\overline{K}(u)$ or/and $\overline{K * H}(u)$ which have explicit analytical solutions in (2.50), (2.54), (2.59), (2.63), (2.135) and (2.149) if $P(x)$ is a combination of exponentials or a mixture of Erlangs, each of these expressions for the discounted distribution functions and probability density functions can be obtained explicitly if $P(x)$ is a combination of exponentials or a mixture of Erlangs.

Chapter 5

Summary and future research

5.1 Summary

A defective renewal equation for the more general expected discounted function of a penalty at ruin which involves the time of ruin, the surplus immediately before the time of ruin, and the deficit at the time of ruin, based on the surplus process of ruin theory with an independent diffusion process, has been derived. When the variance of the distribution of the diffusion process (with the mean of zero) goes to zero, the defective renewal equation reduces to the one based on the surplus process of classical risk model. In addition, the asymptotic formula for this expected discounted function of a penalty at time of ruin is proposed by applying Feller's renewal theorem to the corresponding defective renewal equation.

Given the claim size distribution function $P(x)$, we can construct the classical distribution function $\Gamma(x)$ and the associated claim size distribution $G(x)$ which is just the convolution of $\Gamma(x)$ and an exponential distribution function $H(x)$. Then the associated compound geometric distribution function $\overline{K}(u)$ can be expressed in terms of $G(u)$. A Tijms-type approximation (a combination of two exponential functions) is given for $\overline{K}(u)$. When $P(x)$ is a combination of exponentials or a mixture of Erlangs, an explicit analytical solution (a combination of several exponential functions) to $\overline{K}(u)$ is achieved. Once $\overline{K}(u)$ has explicit analytical solution or Tijms-type approximation, so do $\phi_d(u)$ and $\phi_s(u)$, the discounted probabilities of ruin due to oscillation and a claim, respectively, since both $\phi_d(u)$ and $\phi_s(u)$ can be written in terms of $\overline{K}(u)$.

Moreover, when $P(x)$ satisfies a certain reliability-based class condition, not only upper and lower bounds on the compound geometric distribution function $\overline{K}(u)$ are obtained, but also $\Gamma(x)$ and $G(x)$ satisfy the same reliability-based class condition.

The (discounted) moment of the deficit at the time of ruin, the joint moment of the deficit at ruin and the time of ruin, and the moments of the time of ruin due to oscillation and caused by a claim, respectively, are also studied in detail, including the corresponding defective renewal equations and explicit expressions. The explicit expressions for the covariance of the deficit at ruin and the time of ruin, and for the variances of the time of ruin due to oscillation and caused by a claim, respectively, can be obtained from these (joint) moments. We also find that the moment of the time of ruin caused by a claim has the same recursive expression as the moment of the time of ruin caused by oscillation. When $P(x)$ is DMRL (decreasing mean residual lifetime), the discounted moment of the deficit at the time of ruin is bounded above by a constant multiplied by $\overline{K}(u)$.

The explicit expressions and defective renewal equations for the (discounted) joint and marginal distribution functions of the surplus immediately before the time of ruin and the deficit at the time of ruin, $F(x,y;\delta,D|u)$, $F_1(x;\delta,D|u)$ and $F_2(y;\delta,D|u)$, respectively, and for the (discounted) distribution function of the amount of the claim causing ruin, $F_Z(z;\delta,D|u)$, are derived. Then the (discounted) probability density functions are obtained by differentiating the corresponding (discounted) distribution functions. Besides, the relationships between $F_1(x;\delta,D|u)$ and $F_2(x;\delta,D|u-x)$ and between $F_1(z;\delta,D|u)$ and $F_Z(z;\delta,D|u)$ are also given.

Since each of the discounted distribution functions, $F(x,y;\delta,D|u)$, $F_1(x;\delta,D|u)$, $F_2(y;\delta,D|u)$ and $F_Z(z;\delta,D|u)$, and each of the discounted probability density functions, $f(x,y;\delta,D|u)$, $f_1(x;\delta,D|u)$, $f_2(y;\delta,D|u)$ and $f_Z(z;\delta,D|u)$ can be expressed in terms of $\overline{P}(u)$, $\overline{\Gamma}(u)$, $\overline{G}(u)$ and $\overline{K}(u)$, if $P(x)$ is a combination of exponentials or a mixture of Erlangs, each of these discounted distribution functions and probability density functions has an explicit analytical solution.

5.2 Future research

In the case where no explicit analytical solutions are available, some numerical algorithms (for example, Dickson and Waters (1991) [12], and Dickson, Egídio dos Reis and Waters (1995) [11]) can be applied to compute the value of probability of ruin if the specific initial surplus u is given.

In this thesis, the number of claims is assumed to follow a Poisson distribution, which is equivalent to that the inter-arrival time between successive claims is exponential distributed. There are some other assumptions about the interarrival time, like Erlang-2 (the probability density function is $k(t) = \gamma^2 te^{-\gamma t}$ for $t > 0$) and Coxian-2 (see pp. 360-361 of Tijms (1994) [44]) distributions. In particular, Dickson (1998) [7] and Dickson and Hipp (1998) [10] have proposed some results based on the Erlang-2 assumption.

Since our surplus process contains a diffusion factor, some applications to the pricing of financial securities, such as certain (American) perpetual options, are feasible. See Gerber and Shiu (1994) [25], (1996a) [26], (1996b) [27], (1998b) [30] and (1999) [31], and Gerber and Landry (1998) [24] for more details.

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