On Convolution Squares of Singular Measures

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

We prove that if $1 > \alpha > 1/2$, then there exists a probability measure μ such that the Hausdorff dimension of its support is α and $\mu * \mu$ is a Lipschitz function of class $\alpha - 1/2$.

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Contents

Chapter 1

Preliminaries

Unless otherwise stated, we will be working on $\mathbb{T} = \mathbb{R}/\mathbb{Z} = [-1/2, 1/2]$, and only with positive, Borel, regular measures. We will use λ to denote the Lebesgue measure on $\mathbb{T}$ (and dx, dy, dt, and so forth to denote integration with respect to Lebesgue measure), and $|E| = \lambda(E)$. Measures that are absolutely continuous or singular are taken with respect to λ in the case of $\mathbb{T}$, and (left) Haar measure, should we require a more general setting. When speaking of measures living in an L^p space or having other properties associated to functions, we mean the measure is absolutely continuous and we consider its Radon-Nikodym derivative to have this property.

1.1 Introduction

The absolutely continuous measures form an ideal in the algebra of measures, so that for an absolutely continuous measure μ, $\mu * \mu$ is still absolutely continuous and so the Radon-Nikodym theorem gives rise to an associated function $f \in L^1$ such that $\mu * \mu = f\lambda$. For any σ-finite measure μ, Lebesgue's Decomposition theorem decomposes μ as $\mu = \mu_{ac} + \mu_s$, where μ_{ac} is absolutely continuous and μ_s is singular. It is thus natural to ask whether a similar situation holds with singular measures, that is, given a singular measure μ, does there exist a function $f \in L^1$ with $\mu * \mu = f\lambda$? This is easily seen to be false, for if

$\mu = \sum_n a_n \delta_{x_n}$ is a discrete measure, then $\mu * \mu = \sum_{n,m} a_n a_m \delta_{x_n + x_m}$ is also a discrete measure.

To progress further, it may be helpful to appeal to an extension of Lebesgue's Decomposition theorem, which gives for any σ-finite measure μ three unique measures μ_{ac}, μ_{cs} and μ_d which are absolutely continuous, continuous singular, and discrete, respectively, such that $\mu = \mu_{ac} + \mu_{cs} + \mu_d$. Then we ask if we might recover the nice property of $\mu * \mu$ being absolutely continuous in the case of μ being a continuous singular measure. This is still not always the case, as there exist continuous singular measures on $\mathbb{T}$ where even their nth convolution power is singular for any n; for such an example we will need a preliminary definition.

Definition 1.1. Let $-1/2 \leq a_k \leq 1/2$ and define the trigonometric polynomials on $\mathbb{T}$

$$P_N(t) = \prod_{k=1}^{N} (1 + 2a_k \cos 3^k t).$$

It is a standard result that $P_N \lambda$ are probability measures which converge weak-$*$ in $M(\mathbb{T})$ to a probability measure μ with

$$\widehat{\mu}(n) = \begin{cases} 1 & \text{if } n = 0, \\ \prod_{j=1}^{M} a_{k_j} & \text{if } n = \sum_{j=1}^{M} \epsilon_j 3^{k_j}, \epsilon_j = \pm 1, \\ 0 & \text{otherwise.} \end{cases}$$

This measure μ is called the *Riesz product measure associated to* $\{a_k\}$.

Lemma 1.2. *If μ_1 and μ_2 are Riesz product measures, then so is $\mu_1 * \mu_2$. Moreover, if μ_i is associated to $\{a_{k,i}\}$, then $\mu_1 * \mu_2$ is associated to $\{a_{k,1} a_{k,2}\}$.*

Proof. We calculate the Fourier coefficients of the convolution:

$$\widehat{\mu_1 * \mu_2}(n) = \widehat{\mu_1}(n)\widehat{\mu_2}(n) = \begin{cases} 1 & \text{if } n = 0, \\ \prod_{j=1}^{M} a_{k_j,1} a_{k_j,2} & \text{if } n = \sum_{j=1}^{M} \epsilon_j 3^{k_j}, \epsilon_j = \pm 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

Then we can define the trigonometric polynomials on $\mathbb{T}$

$$P_N(t) = \prod_{k=1}^{N}(1 + 2a_{k,1}a_{k,2}\cos 3^k t),$$

noticing that $-1/2 \leq -1/4 \leq a_{k,1}a_{k,2} \leq 1/4 \leq 1/2$. The weak-$*$ limit of $P_N\lambda$ produces a Riesz product measure whose coefficients satisfy (1.1), so we are finished by uniqueness. ∎

A standard result (see, for example, [6] Theorem 7.2.2) gives that a Riesz product measure μ is singular if and only if $\sum_{k=1}^{\infty} a_k^2 = \infty$. In particular, suppose μ is a Riesz product measure associated to the constant sequence $\{c\}$ for some constant $0 < c \leq 1/2$. Certainly $\sum_{k=1}^{\infty} c^2 = \infty$, so that μ is singular. Inductively applying Lemma 1.2, $\mu^n = \mu * \mu * \cdots * \mu$ (the nth convolution power of μ) is a Riesz product measure associated to $\{c^n\}$. Then $\sum_{k=1}^{\infty} c^{2n} = \infty$, and so μ^n remains singular. It is also not hard to show that Riesz product measures are continuous, in general.

It should be remarked that the choice of frequencies 3^k is not essential, and Riesz product measures may be generalized further. In addition, the concept of Riesz product measures as a whole and the conclusion that there exist continuous singular measures whose convolution powers always remain singular can be extended to any infinite, compact, abelian group (see [6]).

We now know it is impossible to hope the square convolution of a singular measure is absolutely continuous in general, even when restricting to the case of continuous singular measures. However, we may expect that are still some positive results; convolution behaves a smoothing operation, evidenced by the fact that both the continuous measures and the absolutely continuous measures form ideals in the space of measures. Indeed, a famous result due to Wiener and Wintner [16] produces a singular measure μ on $\mathbb{T}$ such that $\widehat{\mu}(n) = o(|n|^{-1/2+\varepsilon})$ as $n \to \infty$ for every $\varepsilon > 0$. By an application of the Hausdorff-Young inequality, such a measure can be shown to have the property that its square convolution $\mu * \mu$ is absolutely continuous, and in fact, $\mu * \mu \in L^p(\mathbb{T})$ for every $p \geq 1$.

Other authors have generalized this result; Hewitt and Zuckerman [9] constructed a singular measure μ on any non-discrete, locally compact, abelian group G such that $\mu*\mu \in L^p(G)$ for all $p \geq 1$. It is worth noting that the condition of non-discrete cannot be dropped, since otherwise all measures would be absolutely continuous. Using Rademacher-Riesz products, Karanikas and Koumandos [10] prove the existence of a singular measure with $\mu * \mu \in L^1(G)$ for any non-discrete, locally compact group G. Shortly thereafter, Dooley and Gupta [3] produced a measure μ with $\mu * \mu \in L^p(G)$ for every $p \geq 1$ in the case of compact, connected groups and compact Lie groups using the theory of compact Lie groups.

Saeki [15] took this concept further in a different direction by proving the existence of a singular measure μ on $\mathbb{T}$ with support having zero Lebesgue measure such that $\mu * \mu$ has a uniformly convergent Fourier series. This is an improvement on previous results; in this case the continuity of the partial Fourier sums for $\mu * \mu$ ensure that their limit, namely $\mu * \mu$, is continuous. Then $\mu * \mu \in L^\infty(\mathbb{T})$ and subsequently $\mu * \mu \in L^p(\mathbb{T})$ for $p \geq 1$ since $\mathbb{T}$ is compact. Generalizing this, Gupta and Hare [7] show such a measure exists (now using Haar measure in place of Lebesgue measure) when replacing $\mathbb{T}$ with any compact, connected group.

Körner [11] recently expanded on Saeki's work; to discuss how, we need the following definitions.

Definition 1.3. For $1 \geq \beta \geq 0$, we say a function $f : \mathbb{T} \to \mathbb{C}$ is *Lipschitz of class* β[1], or simply β-*Lipschitz* and write $f \in \Lambda_\beta$ if

$$\sup_{t\in\mathbb{T}} \sup_{h\neq 0} |h|^{-\beta} |f(t+h) - f(t)| < \infty. \tag{1.2}$$

Some references define a function to be Lipschitz of class β if there is a constant C with $|f(x) - f(y)| \leq C|x-y|^\beta$ for every $x, y \in \mathbb{T}$. These definitions are in fact equivalent, with a straightforward proof.

Lemma 1.4. *$f \in \Lambda_\beta$ if and only if $|f(x) - f(y)| \leq C|x-y|^\beta$ for a constant C, for every $x, y \in \mathbb{T}$.*

Proof. This is a simple matter of relabeling; use $t = y$ and $h = x - y$. ∎

[1] Functions with this property are also sometimes referred to as *Hölder of class* β.

It is useful to note that the Λ_β classes are nested downwards:

Lemma 1.5. *Suppose* $1 \geq \beta_i \geq 0$ *for* $i = 1, 2$. *If* $\beta_1 \leq \beta_2$, *then* $\Lambda_{\beta_1} \supseteq \Lambda_{\beta_2}$.

Proof. Since $1 \geq \beta_i \geq 0$, for $h \in \mathbb{T} = [-1/2, 1/2]$ we get that $|h|^{-\beta_1} \leq |h|^{-\beta_2}$. Thus if

$$\sup_{t\in\mathbb{T}} \sup_{h\neq 0} |h|^{-\beta_2} |f(t+h) - f(t)| < \infty,$$

we certainly have

$$\sup_{t\in\mathbb{T}} \sup_{h\neq 0} |h|^{-\beta_1} |f(t+h) - f(t)| < \infty,$$

implying $\Lambda_{\beta_1} \supseteq \Lambda_{\beta_2}$. ∎

We will also require the notion of Hausdorff dimension, which we recall here:

Definition 1.6. Fix $\alpha \geq 0$ and let $\delta > 0$. For a set E, define

$$\mathcal{H}^\alpha_\delta(E) = \inf \left\{ \sum_{i=1}^\infty |E_i|^\alpha : \bigcup_{i=1}^\infty E_i \supseteq E, |E_i| \leq \delta \right\},$$

where E_i can be taken to be intervals. Notice $\mathcal{H}^\alpha_\delta(E)$ is monotone increasing as δ decreases, since fewer permissible collections of sets are taken in the infimum. Then we take

$$\mathcal{H}^\alpha(E) = \lim_{\delta\to 0^+} \mathcal{H}^\alpha_\delta(E) = \sup_{\delta>0} \mathcal{H}^\alpha_\delta(E),$$

and we call $\mathcal{H}^\alpha(E)$ the *α-Hausdorff measure of E*; it is standard that this is indeed a measure (see [4], for example). Of special interest is the case $\alpha = 1$, in which case we recover the usual Lebesgue measure.

It is easy to show that there is a critical value for α such that $\mathcal{H}^\alpha(E) = \infty$ for α less than this critical value, and $\mathcal{H}^\alpha(E) = 0$ for α greater than this critical value. Then we define the *Hausdorff dimension of* E by

$$\begin{aligned} \dim_H(E) &= \sup\{\alpha : \mathcal{H}^\alpha(E) > 0\} = \sup\{\alpha : \mathcal{H}^\alpha(E) = \infty\} \\ &= \inf\{\alpha : \mathcal{H}^\alpha(E) < \infty\} = \inf\{\alpha : \mathcal{H}^\alpha(E) = 0\}. \end{aligned}$$

Heuristically, the Hausdorff dimension allows us to compare sets that are too sparse for the Lebesgue measure to be of use.

For our results, we will need to have a way of describing where a measure "lives"; we make this formal by introducing the concept of the support of a measure.

Definition 1.7. Let (X, T) be a topological space, and μ be a Borel measure on (X, T). Then the *support* of μ is defined to be the set of all points x in X for which every open neighborhood N_x of x has positive measure:

$$\operatorname{supp}\mu = \{x \in X : \mu(N_x) > 0, \forall x \in N_x \in T\}. \tag{1.3}$$

Among other things, Körner demonstrated the existence of a probability measure μ whose support has a prescribed Hausdorff dimension (between 1 and 1/2) such that $\mu * \mu$ is a Lipschitz function. Our goal will be to prove one of his main results, which is the following:

Theorem 4.19. If $1 > \alpha > 1/2$, then there exists a probability measure μ such that $\dim_H(\operatorname{supp}\mu) = \alpha$ and $\mu * \mu = f\lambda$ where $f \in \Lambda_{\alpha-1/2}$.

We will show that this theorem is indeed an extension of Saeki's work. Using Lemma 1.4, we can see that $f \in \Lambda_\beta$ yields a constant C such that

$$|f(x+h) - f(x)| \leq C|h|^\beta$$

for any x and h, implying $f(x+h) - f(x) = o(((\ln|h|^{-1})^{-1})$ (see Appendix B.1). By the Dini-Lipschitz test (see Appendix B.2), f has a uniformly convergent Fourier series. This says that the measure Körner produced has the same property as Saeki's, provided it were also singular. This fact will hold due to the Hausdorff dimension condition; indeed, the fact that $\dim_H(\operatorname{supp}\mu) < 1$ implies that $|\operatorname{supp}\mu| = 0$ by definition of the dimension and recalling $\mathcal{H}^1$ is simply Lebesgue measure. Since we always have $\mu((\operatorname{supp}\mu)^c) = 0$, this provides the necessary decomposition for μ to be singular.

Consider now the condition $1 > \alpha > 1/2$. As we have seen, this ensures the constructed measure will be singular since sets of Hausdorff dimension less than 1 have zero Lebesgue measure; however the converse is not true. Indeed, it is well known that there exists a set of Hausdorff dimension 1 with zero Lebesgue measure. Then the case $\alpha = 1$ could give still give rise to singular measures. Since we are working on $\mathbb{T}$ which has Hausdorff dimension 1, monotonicity of Hausdorff dimension (easily seen from monotonicity of the Hausdorff measure) ensures we need not consider the case $\alpha > 1$. We shall see soon that we also need $\alpha \geq 1/2$ for a positive result involving this Lipschitz condition, after proving a tightness condition.

1.2 A tightness condition

In this section, we aim to show that the result in Theorem 4.19 is nearly the best we can hope for. In particular, we will show that if the support of a measure μ has Hausdorff measure dimension α and $\mu * \mu \in \Lambda_\beta$, then $\alpha - 1/2 \geq \beta$. To begin, we mention a relationship between the Hausdorff dimension of the support of a measure and its Fourier coefficients.

Definition 1.8. We define the *s-energy* of a finite, compactly supported measure μ on $\mathbb{R}^n$, denoted $I_s(\mu)$, to be

$$I_s(\mu) = \iint \frac{d\mu(x)\, d\mu(y)}{|x-y|^s}.$$

Lemma 1.9. *Suppose μ is a non-zero measure, $0 < \eta < 1$, and*

$$\sum_{k \neq 0} \frac{|\widehat{\mu}|^2}{|k|^{1-\eta}} < \infty.$$

Then $\dim_H(\operatorname{supp} \mu) \geq \eta$.

Proof. By Theorem 2.2 in [8], with $d = 1$ and $0 < \eta < 1$, we have for some constant b that

$$I_\eta(\mu) \leq b \left(|\widehat{\mu}(0)|^2 + \sum_{k \neq 0} \frac{|\widehat{\mu}(k)|^2}{|k|^{1-\eta}} \right) < \infty.$$

Then by Theorem 4.13 in [5], $\dim_H(\operatorname{supp} \mu) \geq \eta$. ∎

Now, we develop a bound for portions of the ℓ^p norm of the Fourier coefficients of a β-Lipschitz function.

Lemma 1.10. *If $f \in \Lambda_\beta$ and $0 < p \leq 2$, then*

$$\sum_{n \leq |k| \leq 2n-1} |\widehat{f}(k)|^p \leq C_1 n^{1-p(\beta+\frac{1}{2})},$$

for some constant C_1.

Proof. Fix $h \in [-1/2, 1/2]$. Let $g(t) = f(t+h) - f(t-h)$. A simple calculation shows that the Fourier coefficients of g are given by

$$\widehat{g}(k) = 2i \sin(2\pi kh) \widehat{f}(k).$$

By Parseval's identity, $\sum_k |\widehat{g}(k)|^2 = \|g\|^2$, so that

$$4 \sum_k |\widehat{f}(k)|^2 \sin^2(2\pi kh) = \int |f(t+h) - f(t-h)|^2 \, dt.$$

Since $f \in \Lambda_\beta$, there is a constant C such that

$$|f(t+h) - f(t-h)| \leq C|(t+h) - (t-h)|^\beta = C|2h|^\beta,$$

for all $t \in \mathbb{T}$, so

$$4 \sum_k |\widehat{f}(k)|^2 \sin^2(2\pi kh) \leq \int (C|2h|^\beta)^2 \, dt = 4^\beta C^2 |h|^{2\beta} \leq 4C^2 |h|^{2\beta}.$$

Thus for any n,

$$\sum_{n \leq |k| \leq 2n-1} |\widehat{f}(k)|^2 \sin^2(2\pi kh) \leq C^2 |h|^{2\beta}.$$

As this holds for any $h \in [-1/2, 1/2]$, consider $h = 1/(8n)$. Notice $\sin^2(\frac{k\pi}{4n}) \geq 1/2$ for $n \leq |k| \leq 2n-1$, so

$$\sum_{n \leq |k| \leq 2n-1} |\widehat{f}(k)|^2 \leq 2C^2 |h|^{2\beta} = 2C^2 (\tfrac{1}{8})^{2\beta} n^{-2\beta} \leq 2C^2 (\tfrac{1}{8})^2 n^{-2\beta}. \tag{1.4}$$

If $p = 2$, we are done; taking $C_1 = 2C^2(\frac{1}{8})^2 = \frac{1}{32}C^2$, and (1.4) gives

$$\sum_{n \leq |k| \leq 2n-1} |\widehat{f}(k)|^2 \leq C_1 n^{-2\beta} = C_1 n^{1-2(\beta+\frac{1}{2})}.$$

Otherwise, $0 < p < 2$ and so $1 < 2/p, 1/(1-p/2) < \infty$; notice they are conjugate indices. By Hölder's inequality with counting measure on $\{|\widehat{f}(k)|^p\}_{n\le|k|\le 2n-1}$ and $\{1\}_{n\le|k|\le 2n-1}$, we have

$$\begin{aligned}
\sum_{n\le|k|\le 2n-1} |\widehat{f}(k)|^p &\le \left(\sum_{n\le|k|\le 2n-1} (|\widehat{f}(k)|^p)^{2/p}\right)^{p/2} \left(\sum_{n\le|k|\le 2n-1} (1)^{1/(1-p/2)}\right)^{1-p/2} \\
&= \left(\sum_{n\le|k|\le 2n-1} |\widehat{f}(k)|^2\right)^{p/2} (2n)^{1-p/2} \\
&\le (2C^2(\tfrac{1}{8})^2 n^{-2\beta})^{p/2}(2n)^{1-\frac{p}{2}} && \text{by (1.4)} \\
&= C_1 n^{1-p(\beta+1/2)}
\end{aligned}$$

as required, where $C_1 = 2(\frac{C}{8})^p$ is a constant. ∎

Lemma 1.11. *If μ is a measure with $\dim_H(\operatorname{supp}\mu) = \alpha$ and $\mu * \mu = f\lambda$ where $f \in \Lambda_\beta$, then $\alpha - 1/2 \ge \beta$.*

Proof. Since $f \in \Lambda_\beta$, taking $p = 1$ in Lemma 1.10 yields

$$\sum_{n\le|k|\le 2n-1} |\widehat{f}(k)| \le C_1 n^{1-1(\beta+1/2)} = C_1 n^{(1-2\beta)/2},$$

for some constant C_1 depending on f. Since $|\widehat{f}(k)| = |\widehat{\mu}(k)|^2$, we have

$$\sum_{n\le|k|\le 2n-1} |\widehat{\mu}(k)|^2 \le C_1 n^{1-1(\beta+1/2)} = C_1 n^{(1-2\beta)/2},$$

and so if $\eta > 0$, we have

$$\sum_{n\le|k|\le 2n-1} \frac{|\widehat{\mu}(k)|^2}{|k|^{1-\eta}} \le \sum_{n\le|k|\le 2n-1} \frac{|\widehat{\mu}(k)|^2}{n^{1-\eta}} \le \frac{C_1 n^{(1-2\beta)/2}}{n^{1-\eta}} = C_1 n^{-(1+2\beta-2\eta)/2}$$

for all $n \ge 1$. In particular, for $n = 2^j$ for each $j \ge 0$,

$$\sum_{2^j\le|k|\le 2^{j+1}-1} \frac{|\widehat{\mu}(k)|^2}{|k|^{1-\eta}} \le C_1 (2^j)^{-(1+2\beta-2\eta)/2},$$

so that

$$\sum_{k\neq 0}\frac{|\widehat{\mu}(k)|^2}{|k|^{1-\eta}} = \sum_{j=0}^{\infty}\left(\sum_{2^j\leq|k|\leq 2^{j+1}-1}\frac{|\widehat{\mu}(k)|^2}{|k|^{1-\eta}}\right) \leq \sum_{j=0}^{\infty} C_1(2^j)^{-(1+2\beta-2\eta)/2}.$$

Notice this converges whenever $-(1+2\beta-2\eta)/2 < 0$, that is, whenever $(1+2\beta)/2 > \eta$. Then by Lemma 1.9, $\dim_H(\operatorname{supp}\mu) \geq \eta$, for every η that satisfies $(1+2\beta)/2 > \eta$. Thus $\dim_H(\operatorname{supp}\mu) \geq (1+2\beta)/2$, that is, $\alpha - 1/2 \geq \beta$ as required. ∎

Lemma 1.11 now gives us a motivation for the lower bound on α; taking $\beta = 0$ we see that for $\mu * \mu \in \Lambda_0$, we must have $\dim_H(\operatorname{supp}\mu) \geq 1/2$. Of course, this still leaves open the question of what happens in the case of $\alpha = 1/2$; the boundary cases $\alpha = 1$ and $\alpha = 1/2$ are beyond the scope of this paper, as the proof relies on the strict inequalities.

In Chapter 2, we prove a key lemma that provides us with a discrete measure satisfying useful boundedness properties which we help in our construction. The key to this lemma will be using a probabilistic argument, which will span most of the chapter.

In Chapter 3, we utilize the key lemma from the previous chapter to construct an infinitely differentiable periodic function satisfying several key properties to be used in the main theorem.

In Chapter 4, the main theorem will be proved, first by developing some complete metric spaces and applying the key lemma from the previous chapter in a density argument. Employing the Baire Category theorem will bridge the final gap towards the proof of the main theorem.

Chapter 2

A Probabilistic Result

The key result that we will need is not easy to prove directly, so we will prove a probabilistic variant of it instead, which will imply the result we need. We proceed by proving the following result:

Lemma 2.1. *Suppose that* $0 < Np \leq 1$ *and* $m \geq 2$*. Then if* $Y_1, Y_2, \ldots, Y_N$ *are independent random variables with*

$$\mathrm{P}(Y_j = 1) = p, \mathrm{P}(Y_j = 0) = 1 - p,$$

it follows that

$$\mathrm{P}\left(\sum_{j=1}^{N} Y_j \geq m\right) \leq \frac{2(Np)^m}{m!}.$$

Proof. First note that $Y_j \in \{0, 1\}$ almost surely, so that their sum cannot exceed N with positive probability. That is, we may assume without loss of generality that $m \leq N$. For $1 \leq k \leq N$, define

$$u_k = \binom{N}{k} p^k = \frac{N!}{k!(N-k)!} p^k.$$

Since $0 < Np \leq 1$ and $k, p > 0$, we have $(N-k)p = Np - kp \leq 1$. Then as $k \geq 1$,

$$\frac{u_{k+1}}{u_k} = \frac{\frac{N!}{(k+1)!(N-k-1)!} p^{k+1}}{\frac{N!}{k!(N-k)!} p^k} = \frac{(N-k)p}{k+1} \leq \frac{1}{k+1} \leq \frac{1}{2}.$$

Since Y_j takes on only 0 or 1 almost surely,

$$\begin{aligned}
\mathrm{P}\left(\sum_{j=1}^{N} Y_j \geq m\right) &= \sum_{k=m}^{N} \mathrm{P}\left(\sum_{j=1}^{N} Y_j = k\right) \\
&= \sum_{k=m}^{N} \binom{N}{k} p^k (1-p)^{N-k} && \text{since } Y_j \text{ are independent} \\
&\leq \sum_{k=m}^{N} \binom{N}{k} p^k = \sum_{k=m}^{N} u_k \\
&\leq \sum_{k=m}^{N} \frac{u_m}{2^{k-m}} \leq 2u_m && \text{since } \frac{u_{k+1}}{u_k} \leq \frac{1}{2} \\
&= 2\frac{N!}{m!(N-m)!} p^m \leq \frac{2(Np)^m}{m!}.
\end{aligned}$$

∎

Definition 2.2. For a set $A \subseteq \mathbb{R}$ and a random variable X, we define the random variable $\delta_X(A)$ by

$$\delta_X(A)(x) = \begin{cases} 1 & \text{if } X(x) \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2.3. *If* $1 > \gamma > 0$ *and* $\varepsilon > 0$, *there exists an* $M(\gamma, \varepsilon) \geq 1$ *with the following property. Suppose that* $n \geq 2$, $n^\gamma \geq N$ *for sufficiently large* n, *and* $X_1, X_2, \ldots, X_N$ *are independent random variables each uniformly distributed on*

$$\Gamma_n = \{r/n : \ 1 \leq r \leq n\}.$$

Then with probability at least $1 - \varepsilon/n$,

$$\sum_{j=1}^{N} \delta_{X_j}(\{r/n\}) \leq M(\gamma, \varepsilon)$$

for all $1 \leq r \leq n$.

Proof. Since $1 - \gamma > 0$, there exists an integer $M(\gamma, \varepsilon) \geq 2$ such that

$$n^{2-M(\gamma,\varepsilon)(1-\gamma)} < \frac{\varepsilon}{2} \tag{2.1}$$

for all $n \geq 2$. Fix r and set $Y_j = \delta_{X_j}(\{r/n\})$. Note that Y_j are independent random variables, as X_j are. Since X_j are uniformly distributed on Γ_n, we get

$$\begin{aligned} \mathrm{P}(Y_j = 1) &= 1/n, \\ \mathrm{P}(Y_j = 0) &= 1 - 1/n. \end{aligned}$$

Since $1 > \gamma > 0$ and $n^\gamma \geq N$, we have $n \geq n^\gamma \geq N$, yielding $Nn^{-1} \leq 1$. Clearly $0 \leq Nn^{-1}$. Then with $p = n^{-1}$ and $m = M(\gamma, \varepsilon) \geq 2$, Lemma 2.1 tells us that

$$\begin{aligned} \mathrm{P}\left(\sum_{j=1}^{N} \delta_{X_j}(\{r/n\}) \geq M(\gamma,\varepsilon)\right) &= \mathrm{P}\left(\sum_{j=1}^{N} Y_j \geq M(\gamma,\varepsilon)\right) \\ &\leq \frac{2(Nn^{-1})^{M(\gamma,\varepsilon)}}{M(\gamma,\varepsilon)!} \\ &\leq \frac{2n^{\gamma M(\gamma,\varepsilon)} n^{-M(\gamma,\varepsilon)}}{M(\gamma,\varepsilon)!} \\ &\leq 2n^{-M(\gamma,\varepsilon)(1-\gamma)} < \frac{\varepsilon}{n^2}, \end{aligned}$$

where we used $n^\gamma \geq N$ in the second inequality, and the final inequality follows from (2.1). Allowing r to vary, we get immediately

$$\begin{aligned} &\mathrm{P}\left(\sum_{j=1}^{N} \delta_{X_j}(\{r/n\}) \geq M(\gamma,\varepsilon) \text{ for some } 1 \leq r \leq n\right) \\ &= \sum_{r=1}^{N} \mathrm{P}\left(\sum_{j=1}^{N} \delta_{X_j}(\{r/n\}) \geq M(\gamma,\varepsilon)\right) \\ &< N\frac{\varepsilon}{n^2} \leq \frac{\varepsilon}{n}, \end{aligned}$$

where we have used the fact that $N \leq n$. ∎

Definition 2.4. Suppose P is a probability measure, X is a random variable with respect to the σ-algebra $\mathcal{F}$, and $\mathcal{G} \subseteq \mathcal{F}$ is a sub-σ-algebra. If Y is a random variable with respect to the σ-algebra $\mathcal{G}$ such that for all $A \in \mathcal{G}$,

$$\int_A X \, dP = \int_A Y \, dP,$$

then we call Y the *conditional expectation of X with respect to $\mathcal{G}$*, and denote it by $E(X \mid \mathcal{G})$. It is a consequence of the Radon-Nikodym Theorem that such a random variable always exists, and is unique almost surely. If X_1 is some random variable (with respect to the Borel σ-algebra $\mathcal{B}$), we can define

$$E(X \mid X_1) := E(X \mid \sigma(X_1)).$$

Recall $\sigma(X_1) = \{X_1^{-1}(B) : B \in \mathcal{B}\}$ is the smallest σ-algebra such that X_1 is a random variable. More generally, if $X_1, \ldots, X_n$ are random variables, we define

$$E(X \mid X_1, \ldots, X_n) := E(X \mid \sigma(X_1, \ldots, X_n)).$$

We will require two simple facts about conditional expectation.

Proposition 2.5. (i) *If X is $\mathcal{G}$-measurable $E(X \mid \mathcal{G}) = X$.*

(ii) *For any X, $E(E(X \mid \mathcal{G})) = E(X)$.*

Proof. Both parts are immediate from the definition. ∎

Lemma 2.6. *Let $\delta > 0$ and for $j = 0, 1, \ldots, n$, let W_j be a random variable that is measurable with respect to $\sigma(X_1, \ldots, X_j)$. Define $Y_j = W_j - W_{j-1}$ and suppose that*

$$\mathrm{E}(e^{sY_j} \mid X_0, X_1, \ldots, X_{j-1}) \leq e^{a_j s^2/2}$$

for all $|s| < \delta$ and some $a_j \geq 0$. Suppose further that $A \geq \sum_{j=1}^N a_j$. Then provided that $0 \leq x < A\delta$, we have

$$\mathrm{P}(|W_N - W_0| \geq x) \leq 2\exp\left(\frac{-x^2}{2A}\right).$$

Proof. Suppose $-\delta < s < \delta$. By definition of W_j, we have $e^{s(W_N - W_0)} = e^{s(W_N - W_{N-1})} e^{s(W_{N-1} - W_0)}$. Since W_{N-1} and W_0 are $\sigma(X_1, \ldots, X_{N-1})$-measurable,

$$\begin{aligned}
\mathrm{E}(e^{s(W_N - W_0)} \mid X_0, X_1, \ldots, X_{N-1}) &= e^{s(W_{N-1} - W_0)} \mathrm{E}(e^{s(W_N - W_{N-1})} \mid X_0, X_1, \ldots, X_{N-1}) \\
&= e^{s(W_{N-1} - W_0)} \mathrm{E}(e^{sY_N} \mid X_0, X_1, \ldots, X_{N-1})
\end{aligned}$$

$$\leq e^{s(W_{N-1}-W_0)}e^{a_N s^2/2}.$$

By property of conditional expectation, taking expectation of both sides gives

$$\mathrm{E}(e^{s(W_N-W_0)}) \leq \mathrm{E}(e^{s(W_{N-1}-W_0)})e^{a_N s^2/2}.$$

By induction, we get

$$\mathrm{E}(e^{s(W_N-W_0)}) \leq \prod_{j=1}^{N} e^{a_j s^2/2} \leq e^{As^2/2},$$

since $A \geq \sum_{j=1}^{N} a_j$. Then by Markov's inequality (see Appendix C.1),

$$\mathrm{P}(W_N - W_0 \geq x) = \mathrm{P}(e^{s(W_N-W_0)} \geq e^{sx}) \leq \mathrm{E}(e^{s(W_N-W_0)})e^{-sx} \leq e^{As^2/2}e^{-sx}.$$

Setting $s = xA^{-1}$, $0 \leq x < A\delta$ ensures that $0 \leq xA^{-1} < \delta$, so that $|s| < \delta$. Then by the above,

$$\mathrm{P}(W_N - W_0 \geq x) \leq e^{As^2/2}e^{-sx} = e^{Ax^2A^{-2}/2}e^{-xA^{-1}x} = \exp\left(-\frac{x^2}{2A}\right).$$

The same argument applies to the sequence $-W_j$, so that we also have

$$\mathrm{P}(W_0 - W_N \geq x) \leq \exp\left(-\frac{x^2}{2A}\right).$$

The result follows. ∎

Lemma 2.7. *Suppose $\phi : \mathbb{N} \to \mathbb{R}$ is a sequence with $\phi(n)(\log n)^{1/2} \to \infty$ as $n \to \infty$, and for any $\delta > 0$, we have $\phi(n)n^{-\delta} \to 0$ as $n \to \infty$. Suppose $1 > \gamma > 1/2$, and the positive integer $N = N(n)$ satisfies $n^{\gamma} \geq N \geq n^{1/2+\eta}$ for some $\eta > 0$, for all sufficiently large n. If $\varepsilon > 0$, there exists an $M(\gamma)$ and an $n_0(\phi, \gamma, \varepsilon) \geq 1$ with the following property. Suppose that $n \geq n_0(\phi, \gamma, \varepsilon)$, and n is odd. Suppose further that $X_1, X_2, \ldots, X_N$ are independent random variables each uniformly distributed on*

$$\Gamma_n = \{r/n \in \mathbb{T} :\ 1 \leq r \leq n\}.$$

Then, if we write $\sigma = N^{-1}\sum_{j=1}^{N} \delta_{X_j}$, we have

$$|\sigma * \sigma(\{k/n\}) - n^{-1}| \leq \varepsilon \frac{\phi(n)(\log n)^{1/2}}{Nn^{1/2}},$$

and

$$\sigma(\{k/n\}) \leq \frac{M(\gamma)}{N}$$

for all $1 \leq k \leq n$*, with probability at least* $1/2$*.*

Proof. Set $M(\gamma) = M(\gamma, 1/4)$ coming from Lemma 2.3. Without loss of generality, assume $M(\gamma) \geq 3$. Fix r for now, and define $Y_1, Y_2, \ldots, Y_N$ as follows. If $\sum_{v=1}^{j-1} \delta_{X_v}(\{u/n\}) \leq M(\gamma)$ for all u with $1 \leq u \leq n$, set

$$Y_j = -\frac{2j-1}{n} + \delta_{2X_j}(\{r/n\}) + 2\sum_{v=1}^{j-1} \delta_{X_v+X_j}(\{r/n\}).$$

Otherwise, set $Y_j = 0$. Take $X_0 = W_0 = 0$ and $W_j = \sum_{m=1}^{j} Y_m$; notice $Y_j = W_j - W_{j-1}$.

Suppose first that $\sum_{v=1}^{j-1} \delta_{X_v}(\{u/n\}) \leq M(\gamma)$ for all u with $1 \leq u \leq n$. Observe that if s is a fixed integer, then $X_j + s/n$ and $2X_j$ are uniformly distributed over Γ_n (using addition modulo 1), since n is odd and X_j are independent. Then by uniform distribution,

$$\begin{aligned}
\mathrm{E}(Y_j) &= -\frac{2j-1}{n} + \mathrm{E}(\delta_{2X_j}(\{r/n\})) + 2\sum_{v=1}^{j-1} \mathrm{E}(\delta_{X_v+X_j}(\{r/n\})) \\
&= -\frac{2j-1}{n} + \frac{1}{n} + 2\sum_{v=1}^{j-1} \frac{1}{n} = -\frac{2j-2}{n} + 2\frac{j-1}{n} = 0.
\end{aligned}$$

On the other hand, if it is not the case that $\sum_{v=1}^{j-1} \delta_{X_v}(\{u/n\}) \leq M(\gamma)$ for all u with $1 \leq u \leq n$, then $Y_j = 0$ by construction so that $\mathrm{E}(Y_j) = 0$ automatically.

We wish to make use of Lemma 2.6, and so we bound $\mathrm{E}(e^{sY_j} \mid X_0, X_1, \ldots, X_{j-1})$.

Notice as $n^\gamma \geq N$ and $\gamma < 1$, we have $n > n^\gamma \geq N$. Then $(2N-1)/n \leq 2N/n \leq 2$. Suppose first that $\sum_{v=1}^{j-1} \delta_{X_v}(\{u/n\}) \leq M(\gamma)$ for all u with $1 \leq u \leq n$. Then

$$\begin{aligned}
|Y_j| &\leq \frac{2j-1}{n} + \delta_{2X_j}(\{r/n\}) + 2\sum_{v=1}^{j-1} \delta_{X_v+X_j}(\{r/n\}) \\
&\leq \frac{2N-1}{n} + 1 + 2M(\gamma) \leq 2 + 1 + 2M(\gamma) \leq 3M(\gamma),
\end{aligned}$$

where we have used the fact that $M(\gamma) \geq 3$. Setting $Z_j = Y_j + (2j-1)/n$, we notice that $Z_j \neq 0$ if any of $\delta_{X_v+X_j}(\{r/n\}) \neq 0$ for $1 \leq v \leq j$. As X_j are uniformly distributed and by subadditivity, we thus get $\mathrm{P}(Z_j \neq 0) \leq j/n$. Since $E(Y_j) = 0$,

$$\begin{aligned}
\mathrm{E}(e^{sY_j}) &= \mathrm{E}\left(\sum_{k=0}^{\infty} \frac{(sY_j)^k}{k!}\right) = \sum_{k=0}^{\infty} \frac{s^k}{k!}\mathrm{E}(Y_j^k) \\
&= 1 + \sum_{k=2}^{\infty} \frac{s^k}{k!}\mathrm{E}(Y_j^k) \leq 1 + \sum_{k=2}^{\infty} \frac{|s|^k}{k!}\mathrm{E}|Y_j|^k \\
&= 1 + \mathrm{P}(Z_j = 0)\sum_{k=2}^{\infty} \frac{|s|^k}{k!}\mathrm{E}(|Y_j|^k \mid Z_j = 0) + \mathrm{P}(Z_j \neq 0)\sum_{k=2}^{\infty} \frac{|s|^k}{k!}\mathrm{E}(|Y_j|^k \mid Z_j \neq 0) \\
&\leq 1 + \sum_{k=2}^{\infty} \frac{|s|^k}{k!}\mathrm{E}(|Y_j|^k \mid Z_j = 0) + \frac{j}{n}\sum_{k=2}^{\infty} \frac{|s|^k}{k!}\mathrm{E}(|Y_j|^k \mid Z_j \neq 0).
\end{aligned}$$

where the third line follows from the partition formula, and the final inequality follows since $\mathrm{P}(Z_j = 0) \leq 1$ and $\mathrm{P}(Z_j \neq 0) \leq j/n$. Since $Z_j = Y_j + (2j-1)/n$, in the case $Z_j = 0$ we have $|Y_j| = (2j-1)/n$. Otherwise, our earlier approximation gives $|Y_j| \leq 3M(\gamma)$. Thus,

$$\begin{aligned}
\mathrm{E}(e^{sY_j}) &\leq 1 + \sum_{k=2}^{\infty} \frac{(|s|\frac{2j-1}{n})^k}{k!} + \frac{j}{n}\sum_{k=2}^{\infty} \frac{(|s|3M(\gamma))^k}{k!} \\
&\leq 1 + \sum_{k=2}^{\infty} \frac{(|s|\frac{2N-1}{n})^k}{k!} + \frac{N}{n}\sum_{k=2}^{\infty} \frac{(|s|3M(\gamma))^k}{k!}
\end{aligned}$$

since $1 \leq j \leq N$. Notice if we have a constant $0 \leq a \leq 1$, then

$$\sum_{k=2}^{\infty} \frac{a^k}{k!} \leq \sum_{k=1}^{\infty} \frac{a^2}{2^k} = a^2.$$

For sufficiently large n, we have $0 \leq |s|\frac{2N-1}{n} \leq 1$. If $|s| < \frac{1}{3M(\gamma)}$, we also get $0 \leq |s|3M(\gamma) \leq 1$, so by our quick result above,

$$\begin{aligned}
\mathrm{E}(e^{sY_j}) &\leq 1 + \left(|s|\frac{2N-1}{n}\right)^2 + \frac{N}{n}(|s|3M(\gamma))^2 \leq 1 + 4\frac{N^2}{n^2}s^2 + 9M(\gamma)^2\frac{N}{n}s^2 \\
&\leq 1 + (4 + 9M(\gamma)^2)Nn^{-1}s^2 \leq 1 + 10M(\gamma)^2Nn^{-1}s^2 \\
&\leq \exp\left(20M(\gamma)^2Nn^{-1}s^2/2\right),
\end{aligned}$$

where we have used $M(\gamma)^2 \geq 9$ in the second to last inequality, and the fact that $1+x \leq e^x$ for every real x in the last inequality. Define $a_j = 20M(\gamma)^2 \frac{N}{n} > 0$.

If it is not the case that $\sum_{v=1}^{j-1} \delta_{X_v}(\{u/n\}) \leq M(\gamma)$ for all u with $1 \leq u \leq n$, then by definition of Y_j, we get automatically

$$\mathrm{E}(e^{sY_j}) = \mathrm{E}(e^0) = \mathrm{E}(1) = 1 \leq \exp\left(20M(\gamma)^2 N n^{-1} s^2/2\right).$$

Define

$$A = \sum_{j=1}^{N} a_j = \sum_{j=1}^{N} 20M(\gamma)^2 N n^{-1} = 20M(\gamma)^2 N^2 n^{-1}.$$

Set $\delta = \frac{1}{3M(\gamma)}$ and

$$x = \varepsilon\phi(n)(\log n)^{1/2} N n^{-1/2}.$$

To apply Lemma 2.6, we must check that $0 \leq x < A\delta$ for sufficiently large n. Indeed, $\phi(n)(\log n)^{1/2} \to \infty$ as $n \to \infty$, while $\varepsilon > 0$ and $Nn^{-1/2} > 0$ for every n, so that $x \geq 0$ for sufficiently large n. In order to show $x < A\delta$, we must have

$$\begin{aligned} &\varepsilon\phi(n)(\log n)^{1/2} N n^{-1/2} < 20M(\gamma)^2 N^2 n^{-1} \frac{1}{3M(\gamma)} \\ \iff \quad &\varepsilon \frac{3}{20M(\gamma)} \phi(n)(\log n)^{1/2} n^{1/2} < N. \end{aligned}$$

By hypothesis, N is such that there exists $\eta > 0$ with $N > n^{1/2+\eta}$ for all sufficiently large n. For this value of $\eta > 0$, we get $\phi(n)n^{-\eta/2} < \frac{20M(\gamma)}{3\varepsilon}$ for sufficiently large n by property of ϕ. Then we easily get that for sufficiently large n,

$$\varepsilon \frac{3}{20M(\gamma)} \phi(n)(\log n)^{1/2} n^{1/2} < n^{\eta/2}(\log n)^{1/2} n^{1/2} < n^{1/2+\eta} < N,$$

as required. Then by Lemma 2.6,

$$\begin{aligned} \mathrm{P}(|W_N - W_0| \geq x) \leq 2\exp\left(-\frac{x^2}{2A}\right) &= 2\exp\left(-\frac{\varepsilon^2\phi(n)^2(\log n)N^2 n^{-1}}{40M(\gamma)^2 N^2 n^{-1}}\right) \\ &= 2\exp\left(-\frac{\varepsilon^2}{40M(\gamma)^2}\phi(n)^2 \log n\right). \end{aligned}$$

Notice $-\frac{\varepsilon^2}{40M(\gamma)^2} < 0$ is a constant, and we are given $\phi(n)^2 \log n \to \infty$ as $n \to \infty$. Then, we may choose $n_0(\phi, \gamma, \varepsilon) \geq 1$ such that for all $n \geq n_0$,

$$\mathrm{P}(|W_N - W_0| \geq \varepsilon\phi(n)(\log n)^{1/2} N n^{-1/2}) \leq \frac{1}{4}. \tag{2.2}$$

For the rest of the proof, we assume n satisfies this condition.

Recall $M(\gamma) = M(\gamma, 1/4)$, as given by Lemma 2.3. By that lemma, we have that with probability at least $1 - 1/(4n)$,

$$\sum_{v=1}^{N} \delta_{X_v}(\{r/n\}) \leq M(\gamma), \tag{2.3}$$

for all $1 \leq r \leq n$. Then for every $1 \leq r \leq n$ and for every $1 \leq j \leq N$,

$$\sum_{v=1}^{j} \delta_{X_v}(\{r/n\}) \leq \sum_{v=1}^{N} \delta_{X_v}(\{r/n\}) \leq M(\gamma),$$

with probability at least $1 - 1/(4n)$. Then by construction,

$$Y_j = -\frac{2j-1}{n} + \delta_{2X_j}(\{r/n\}) + 2\sum_{v=1}^{j-1} \delta_{X_v+X_j}(\{r/n\}).$$

By definition of W_j,

$$W_N - W_0 = \sum_{j=1}^{N} \left(-\frac{2j-1}{n} + \delta_{2X_j}(\{r/n\}) + 2\sum_{v=1}^{j-1} \delta_{X_v+X_j}(\{r/n\}) \right).$$

Notice

$$\sum_{j=1}^{N} \left(-\frac{2j-1}{n} \right) = -\frac{N^2}{n},$$

while

$$\sum_{j=1}^{N} \left(\delta_{2X_j}(\{r/n\}) + 2\sum_{v=1}^{j-1} \delta_{X_v+X_j}(\{r/n\}) \right) = \sum_{v=1}^{N}\sum_{j=1}^{N} \delta_{X_v+X_j}(\{r/n\}). \tag{2.4}$$

Combining (2.2) and (2.4), we get that with probability at least $1 - 1/(4n) - 1/4 \geq 1/2$, we have

$$|W_N - W_0| = \left| \sum_{v=1}^{N}\sum_{j=1}^{N} \delta_{X_v+X_j}(\{r/n\}) - \frac{N^2}{n} \right| < \varepsilon\phi(n)(\log n)^{1/2} N n^{-1/2}.$$

Writing $\sigma = N^{-1}\sum_{j=1}^{N} \delta_{X_j}$, this becomes

$$|\sigma * \sigma(\{r/n\}) - n^{-1}| < \varepsilon \frac{\phi(n)(\log n)^{1/2}}{N n^{1/2}},$$

and (2.3) becomes

$$\sigma(\{r/n\}) \leq \frac{M(\gamma)}{N}.$$

Allowing r to take values from 1 to n, the result follows. ∎

Lemma 2.8. *Suppose ϕ, γ, and N are as in Lemma 2.7. If $\varepsilon > 0$, there exists an $M(\gamma)$ and an $n_0(\phi, \gamma, \varepsilon) \geq 1$ with the following property. Suppose that $n \geq n_0(\phi, \gamma, \varepsilon)$, and n is odd. Then we can find N points*

$$x_j \in \{r/n \in \mathbb{T} :\ 1 \leq r \leq n\}$$

(not necessarily distinct) such that, writing

$$\mu = N^{-1} \sum_{j=1}^{N} \delta_{x_j},$$

we have

$$|\mu * \mu(\{k/n\}) - n^{-1}| \leq \varepsilon \frac{\phi(n)(\log n)^{1/2}}{N n^{1/2}},$$

and

$$\mu(\{k/n\}) \leq \frac{M(\gamma)}{N}$$

for all $1 \leq k \leq n$.

Proof. This follows immediately from Lemma 2.7, since an event with positive probability must have at least one occurrence. ∎

Chapter 3

A Key Lemma

In this chapter, we will take steps to convert the discrete measure generated by Lemma 2.8 into an incredibly well-behaved function, being a periodic, positive, infinitely differentiable function satisfying several key properties; this procedure will be completed in four lemmas. We will write 1_A for the indicator function of the set A.

Lemma 3.1. *Suppose ϕ, γ, and N are as in Lemma 2.7. If $\varepsilon > 0$, there exists an $M(\gamma)$ and an $n_0(\phi, \gamma, \varepsilon) \geq 1$ with the following property. Suppose that $n \geq n_0(\phi, \gamma, \varepsilon)$, and n is odd. Then we can find N points*

$$x_j \in \{r/n \in \mathbb{T} :\ 1 \leq r \leq n\}$$

(not necessarily distinct) such that, writing

$$g = \frac{n}{N} \sum_{j=1}^{N} 1_{[x_j-(2n)^{-1}, x_j+(2n)^{-1})},$$

we have:

(i) $\|g * g - 1\|_\infty \leq \varepsilon \frac{\phi(n)(\log n)^{1/2} n^{1/2}}{N}$.

(ii) *For each $t \in \mathbb{T}$, if $0 < |h| < 1/n$, then*

$$|h|^{-1}|g * g(t+h) - g * g(t)| \leq 4\varepsilon \frac{\phi(n)(\log n)^{1/2} n^{3/2}}{N}.$$

(iii) $|g(t)| \leq \frac{nM(\gamma)}{N}$ *for all* $t \in \mathbb{T}$.

(iv) $\|g\|_1 = 1$.

Proof. By Lemma 2.8, we can find N points

$$x_j \in \{r/n \in \mathbb{T} : 1 \leq r \leq n\}$$

(not necessarily distinct) such that, writing

$$\mu = N^{-1} \sum_{j=1}^{N} \delta_{x_j},$$

we have

$$|\mu * \mu(\{k/n\}) - n^{-1}| \leq \varepsilon \frac{\phi(n)(\log n)^{1/2}}{N n^{1/2}},$$

and

$$\mu(\{k/n\}) \leq \frac{M(\gamma)}{N}$$

for all $1 \leq k \leq n$. Observe that as δ_{x_j} is supported on $\{x_j\}$, for any a, b and any $x \in \mathbb{T}$ we have

$$\delta_{x_j} * 1_{[a,b)}(x) = \int 1_{[a,b)}(x - y)\, d\delta_{x_j}(y) = 1_{[a,b)}(x - x_j) = 1_{[x_j+a, x_j+b)}(x).$$

As convolution respects addition,

$$n\mu * 1_{[-(2n)^{-1},(2n)^{-1})} = \frac{n}{N} \sum_{j=1}^{N} \delta_{x_j} * 1_{[-(2n)^{-1},(2n)^{-1})} = \frac{n}{N} \sum_{j=1}^{N} 1_{[x_j-(2n)^{-1},x_j+(2n)^{-1})} = g.$$

Since $1_{[-(2n)^{-1},(2n)^{-1})}$ is symmetric almost everywhere,

$$\begin{aligned} 1_{[-(2n)^{-1},(2n)^{-1})} * 1_{[-(2n)^{-1},(2n)^{-1})} &= \int 1_{[-(2n)^{-1},(2n)^{-1})}(t - x) 1_{[-(2n)^{-1},(2n)^{-1})}(t)\, dt \\ &= \int 1_{[x-(2n)^{-1},x+(2n)^{-1})}(t) 1_{[-(2n)^{-1},(2n)^{-1})}(t)\, dt \\ &= \begin{cases} 0 & \text{if } x < -1/n, \\ \int 1_{[-(2n)^{-1},x+(2n)^{-1})}(t)\, dt & \text{if } -1/n \leq x \leq 0, \\ \int 1_{[x-(2n)^{-1},(2n)^{-1})}(t)\, dt & \text{if } 0 \leq x \leq 1/n, \\ 0 & \text{if } x > 1/n, \end{cases} \end{aligned}$$

$$= \begin{cases} 0 & \text{if } |x| > 1/n, \\ 1/n - |x| & \text{if } |x| \leq 1/n, \end{cases}$$
$$= \max\{0, 1/n - |x|\}.$$

Writing

$$\Delta_n = \max\{0, 1 - n|x|\},$$

the previous calculations give

$$g * g = (n\mu * 1_{[-(2n)^{-1},(2n)^{-1})}) * (n\mu * 1_{[-(2n)^{-1},(2n)^{-1})}) = \mu * \mu * n\Delta_n.$$

In particular, we have

$$g * g(r/n) = (\mu * \mu) * n\Delta_n(r/n) = \int n \max\{0, 1 - n|(r/n) - t|\}\ d(\mu * \mu)(t).$$

Since μ is supported on $\{k/n\}$, $1 \leq k \leq n$, so is $\mu * \mu$. Then the above integral is zero except possibly when $t = r_0/n$ for $r_0 \in \mathbb{Z}$, *i.e.* $nt \in \mathbb{Z}$. Now, $\Delta_n((r/n) - t)$ is non-zero only if

$$1 - n|(r/n) - t| > 0,$$

yielding $1 > |r - nt|$. But $nt \in \mathbb{Z}$, implying $nt = r$. Therefore,

$$g * g(r/n) = n(1 - n|(r/n) - (r/n)|)\mu * \mu(\{r/n\}) = n\mu * \mu(\{r/n\}). \tag{3.1}$$

It is routine to check that for $x_j \in \{r/n \in \mathbb{T} :\ 1 \leq r \leq n\}$, we have

$$1_{[x_i-(2n)^{-1},x_i+(2n)^{-1})} * 1_{[x_j-(2n)^{-1},x_j+(2n)^{-1})}(x)$$
$$= \begin{cases} 0 & \text{if } x \notin [x_i + x_j - 1/n, x_i + x_j + 1/n], \\ x - (x_i + x_j) + 1/n & \text{if } x \in [x_i + x_j - 1/n, x_i + x_j], \\ -x + (x_i + x_j) + 1/n & \text{if if } x \in [x_i + x_j, x_i + x_j + 1/n]. \end{cases}$$

Notice $x_i + x_j$ is of the form r/n as well, so that this convolution is continuous and piecewise linear on each interval $[(r-1)/n, r/n]$, for $1 \leq r \leq n$. Then $g * g$, being a linear combination of such convolutions, will be continuous[1] and piecewise linear on each

[1] Continuity can also be obtained by noticing $g \in L^2(\mathbb{T})$, and $L^2(\mathbb{T}) * L^2(\mathbb{T}) \subseteq C(\mathbb{T})$.

interval $[(r-1)/n, r/n]$, for $1 \le r \le n$. Thus, $g * g$ attains a local maximum at points in $\{r/n \in \mathbb{T} : 1 \le r \le n\}$.

By property of the chosen μ, we have

$$|\mu * \mu(\{r/n\}) - n^{-1}| \le \varepsilon \frac{\phi(n)(\log n)^{1/2}}{N n^{1/2}}.$$

By (3.1), this becomes

$$|g * g(r/n) - 1| \le \varepsilon \frac{\phi(n)(\log n)^{1/2} n^{1/2}}{N}.$$

Since $g * g$ has local maxima on $\{r/n \in \mathbb{T} : 1 \le r \le n\}$, we get

$$\|g * g - 1\|_\infty = \max_{1 \le r \le n} |g * g(r/n) - 1| \le \varepsilon \frac{\phi(n)(\log n)^{1/2} n^{1/2}}{N},$$

verifying part (i).

To verify part (ii), we need to check the claimed inequality for each $t \in \mathbb{T}$. We have two cases.

<u>Case 1:</u> $t = r/n$, for some $1 \le r \le n$.

Suppose $0 < h < 1/n$. Since $g * g$ is linear on $[r/n, (r+1)/n]$, $g * g(r/n + h)$ can be expressed as the convex combination:

$$g * g\left(\tfrac{r}{n} + h\right) = (1 - nh)(g * g)\left(\tfrac{r}{n}\right) + nh(g * g)\left(\tfrac{r+1}{n}\right).$$

Then

$$\begin{aligned}
|h|^{-1}|g * g(t+h) - g * g(t)| &= |h|^{-1}|g * g(\tfrac{r}{n} + h) - g * g(\tfrac{r}{n})| \\
&= |h|^{-1}|(1 - nh)(g * g)(\tfrac{r}{n}) + nh(g * g)(\tfrac{r+1}{n}) - g * g(\tfrac{r}{n})| \\
&= |h|^{-1} n|h||g * g(\tfrac{r+1}{n}) - g * g(\tfrac{r}{n})| \\
&\le n(|g * g(\tfrac{r+1}{n}) - 1| + |g * g(\tfrac{r}{n}) - 1|) \\
&\le n\left(\varepsilon \frac{\phi(n)(\log n)^{1/2} n^{1/2}}{N} + \varepsilon \frac{\phi(n)(\log n)^{1/2} n^{1/2}}{N}\right) \\
&= 2\varepsilon \frac{\phi(n)(\log n)^{1/2} n^{3/2}}{N}.
\end{aligned}$$

The argument is similar if $-1/n < h < 0$.

Case 2: $t \neq r/n$, for any $1 \leq r \leq n$.

Suppose $0 < |h| < 1/n$. Then there exists r/n, for some $1 \leq r \leq n$, within a distance of $1/n$ of both t and $t+h$. By Case 1,

$$\begin{aligned}|h|^{-1}|g*g(t+h) - g*g(t)| &\leq |h|^{-1}(|g*g(t+h) - g*g(\tfrac{r}{n})| + |g*g(\tfrac{r}{n}) - g*g(t)|) \\ &\leq 4\varepsilon\frac{\phi(n)(\log n)^{1/2}n^{3/2}}{N}.\end{aligned}$$

Cases 1 and 2 together give part (ii).

Fix $t \in \mathbb{T}$. We have that $[t-(2n)^{-1}, t+(2n)^{-1})$ intersects $\{r/n \in \mathbb{T} : 1 \leq r \leq n\}$ exactly once, say at k/n. Then as μ is supported on $\{r/n\}$,

$$\begin{aligned}|g(t)| &= |n\mu * 1_{[-(2n)^{-1},(2n)^{-1})}(t)| \\ &= n\int 1_{[-(2n)^{-1},(2n)^{-1})}(t-y)\,d\mu(y) \\ &= n\mu(\{k/n\}) \leq \frac{nM(\gamma)}{N},\end{aligned}$$

by property of μ, completing part (iii).

By definition of g, it is easy to see that

$$\begin{aligned}\|g\|_1 &= \int \frac{n}{N}\sum_{j=1}^{N} 1_{[x_j-(2n)^{-1},x_j+(2n)^{-1})}(t)\,dt \\ &= \frac{n}{N}\sum_{j=1}^{N} |(x_j+(2n)^{-1}) - (x_j-(2n)^{-1})| \\ &= \frac{n}{N}\sum_{j=1}^{N}\frac{1}{n} = \frac{n}{N}\frac{N}{n} = 1,\end{aligned}$$

verifying (iv). ∎

We next smooth the function obtained in the previous lemma:

Lemma 3.2. *Suppose ϕ, γ, and N are as in Lemma 2.7. If $\varepsilon > 0$, there exists an $M(\gamma)$ and an $n_0(\phi, \gamma, \varepsilon) \geq 1$ with the following property. Suppose that $n \geq n_0(\phi, \gamma, \varepsilon)$, and n is odd. Then we can find a positive, infinitely differentiable function f such that:*

(i) $\|f * f - 1\|_\infty \leq \varepsilon \frac{\phi(n)(\log n)^{1/2} n^{1/2}}{N}$.

(ii) $\|(f * f)'\|_\infty \leq 4\varepsilon \frac{\phi(n)(\log n)^{1/2} n^{3/2}}{N}$.

(iii) $\|f\|_\infty \leq \frac{nM(\gamma)}{N}$.

(iv) $\|f'\|_\infty \leq \frac{8n^2 M(\gamma)}{N}$.

(v) $\int_{\mathbb{T}} f(t)\, dt = 1$.

(vi) $\operatorname{supp} f$ *can be covered by N intervals of length $2/n$.*

Proof. Take $M(\gamma)$, $n_0(\phi, \gamma, \varepsilon)$, and g as in Lemma 3.1. Let $K : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable positive function such that $K(x) = 0$ for $|x| \geq 1/2$, $|K'(x)| \leq 8$ for all x, and $\int_{\mathbb{R}} K(x)\, dx = 1$. (Such a function does exist: as an example, take

$$K(x) = \frac{k(x)}{\int_{\mathbb{R}} k(x)\, dx},$$

where

$$k(x) = \begin{cases} \exp\left(-\frac{1}{1-4x^2}\right) & \text{if } |x| < \frac{1}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

After some work, one can show that $K(x)$ is infinitely differentiable and $|K'(x)| \leq 7.2$ everywhere. It is trivial that it is non-negative and the support lies in $(-1/2, 1/2)$.) Define $K_n : \mathbb{T} \to \mathbb{R}$ by $K_n(t) = nK(nt)$ for $-1/2 \leq t < 1/2$. Then K_n and $K_n * K_n$ are also positive and lie in $L^1(\mathbb{T})$, so for each n, we get that

$$\begin{aligned} \|K_n * K_n\|_1 = \|K_n\|_1^2 &= \left(\int_{-1/2}^{1/2} nK(nt)\, dt\right)^2 \\ &= \left(\int_{-n/2}^{n/2} K(x)\, dx\right)^2 = \left(\int_{-1/2}^{1/2} K(x)\, dx\right)^2 = 1, \end{aligned}$$

since $K(x)$ is supported on $(-1/2, 1/2)$ and $n/2 \geq 1/2$ for every n, where equality holds since we have positive functions.

Set $f = g * K_n$. Then for $s \in \mathbb{T}$,

$$\begin{aligned}
|(f * f - 1)(s)| &= |(g * K_n) * (g * K_n)(s) - 1| \\
&= \left| \int (g * g)(s-t)(K_n * K_n)(t)\, dt - \int K_n * K_n(t)\, dt \right| \\
&\leq \int |(g * g)(s-t) - 1||(K_n * K_n)(t)|\, dt
\end{aligned}$$

and so

$$\begin{aligned}
\|f * f - 1\|_\infty &\leq \|g * g - 1\|_\infty \|K_n * K_n\|_1 \\
&\leq \varepsilon \frac{\phi(n)(\log n)^{1/2} n^{1/2}}{N},
\end{aligned}$$

verifying part (i).

Fix $t \in \mathbb{T}$. By Lemma 3.1, if $0 < |h| < 1/n$, then

$$|h|^{-1} |g * g(t+h) - g * g(t)| \leq 4\varepsilon \frac{\phi(n)(\log n)^{1/2} n^{3/2}}{N}.$$

Then

$$\begin{aligned}
&|h|^{-1}|f * f(t+h) - f * f(t)| \\
&\quad = |h|^{-1}|(g * g) * (K_n * K_n)(t+h) - (g * g) * (K_n * K_n)(t)| \\
&\quad = |h|^{-1} \left| \int (g * g)(t+h-y)(K_n * K_n)(y)\, dy - \int (g * g)(t-y)(K_n * K_n)(y)\, dy \right| \\
&\quad \leq \int |h|^{-1} |g * g(t+h-y) - g * g(t-y)|(K_n * K_n)(y)\, dy \\
&\quad \leq \int 4\varepsilon \frac{\phi(n)(\log n)^{1/2} n^{3/2}}{N} (K_n * K_n)(y)\, dy \\
&\quad = 4\varepsilon \frac{\phi(n)(\log n)^{1/2} n^{3/2}}{N}.
\end{aligned}$$

Since $g, K_n \in L^1$ and K_n is infinitely differentiable, $f = g * K_n$ is infinitely differentiable. Then $f * f$ is infinitely differentiable, and so the following limit exists: for each $t \in \mathbb{T}$,

$$|(f*f)'(t)| = \lim_{h \to 0} |h|^{-1} |f*f(t+h) - f*f(t)| \leq 4\varepsilon \frac{\phi(n)(\log n)^{1/2} n^{3/2}}{N},$$

and hence

$$\|(f*f)'\|_\infty \leq 4\varepsilon \frac{\phi(n)(\log n)^{1/2} n^{3/2}}{N},$$

verifying part (ii).

Recall $|g(t)| \leq \frac{nM(\gamma)}{N}$ for each $t \in \mathbb{T}$, so that for $s \in \mathbb{T}$,

$$|f(s)| = |g * K_n(s)| \leq \int |g(s-t)| K_n(t)\,dt \leq \int \frac{nM(\gamma)}{N} K_n(t)\,dt = \frac{nM(\gamma)}{N},$$

and hence

$$\|f\|_\infty \leq \frac{nM(\gamma)}{N},$$

verifying part (iii).

Since K_n is infinitely differentiable, $(g * K_n)' = g * K_n'$[2]. By choice of K, for $s \in \mathbb{T}$ we have

$$\begin{aligned} |f'(s)| = |(g*K_n)'(s)| = |g * K_n'(s)| &\leq \int |g(s-t)||K_n'(t)|\,dt \\ &= \int_{-1/2}^{1/2} |g(s-t)||n^2 K'(nt)|\,dt \leq \frac{nM(\gamma)}{N} \int_{-1/2}^{1/2} |n^2 K'(nt)|\,dt \\ &= \frac{n^2 M(\gamma)}{N} \int_{-1/2}^{1/2} |K'(x)|\,dx \leq \frac{n^2 M(\gamma)}{N} \int_{-1/2}^{1/2} 8\,dx = \frac{8n^2 M(\gamma)}{N}, \end{aligned}$$

verifying part (iv).

Since g and K_n are positive,

$$\int_{\mathbb{T}} f(t)\,dt = \int (g * K_n)(t)\,dt = \|g\|_1 \|K_n\|_1 = 1,$$

verifying part (v).

[2] Indeed, we may consider the Fourier transform of each function: for each $r \in \mathbb{Z}$ we have $((g*K_n)')\widehat{\ }(r) = ir\widehat{g * K_n}(r) = ir\widehat{g}(r)\widehat{K_n}(r) = \widehat{g}(r)\widehat{K_n'}(r) = (g * K_n')\widehat{\ }(r)$.

By construction, the support of g can be covered by N intervals of length $1/n$. K is supported on $(-1/2, 1/2)$, so that K_n is supported on $(-1/(2n), 1/(2n))$. Then the support of $f = g * K_n$ is covered the sum of these supports, so is covered by N intervals of length $2/n$, for part (vi). ∎

In our proof, we will not need such precise estimates on the bounds, so we modify the previous lemma and introduce a connection with the parameter α.

Lemma 3.3. *Suppose that* $1/2 > \alpha - 1/2 > \beta > 0$. *If* $\varepsilon > 0$, *there exists an* $n_1(\alpha, \beta, \varepsilon) \geq 1$ *with the following property. Suppose that* $n \geq n_0(\phi, \gamma, \varepsilon)$, *and* n *is odd. Then we can find a positive, infinitely differentiable function* f *such that:*

(i) $\|f * f - 1\|_\infty \leq \varepsilon n^{-\beta}/2$.

(ii) $\|(f * f)'\|_\infty \leq \varepsilon n^{1-\beta}$.

(iii) $\|f\|_\infty \leq \varepsilon n$.

(iv) $\|f'\|_\infty \leq \varepsilon n^2$.

(v) $\int_{\mathbb{T}} f(t)\, dt = 1$.

(vi) $\operatorname{supp} f$ *can be covered by less than* $\varepsilon n^\alpha/2$ *intervals of length* $2/n$.

(vii) $|\widehat{f}(r)| \leq \varepsilon$ *for all* $r \neq 0$.

Proof. Without loss of generality, assume $\varepsilon < 1$, for it only puts a more severe restriction on the bounds. Set $\kappa = (\alpha + \beta + 1/2)/2$, $\phi(n) = \log(n)$, and $N = \lfloor n^\kappa \rfloor$. Notice $\phi : \mathbb{N} \to \mathbb{R}$ is a sequence with $\phi(n)(\log n)^{1/2} = (\log n)^{3/2} \to \infty$ as $n \to \infty$ and for any $\eta > 0$, we have $\phi(n)n^{-\eta} = (\log n)n^{-\eta} \to 0$ as $n \to \infty$. We have $\alpha > \kappa > \beta + 1/2$, so since $1 > \alpha$ and $\beta > 0$, we have $1 > \kappa > 1/2$. N is a function of n taking values in the positive integers. Since $\kappa = (\alpha + \beta + 1/2)/2$, we define $\eta = (\alpha - (\beta + 1/2))/2 > 0$ so that $\kappa = \beta + 1/2 + \eta$ and $\kappa = \alpha - \eta$. For sufficiently large n we get

$$N(n) = \lfloor n^\kappa \rfloor \geq \lfloor n^{\beta + 1/2} \rfloor \geq n^{(\beta/2)+(1/2)}.$$

For every n, $n^\kappa \geq \lfloor n^\kappa \rfloor = N$. By Lemma 3.2, there exists an $M(\kappa)$ and an $n_0(\phi, \kappa, \varepsilon) \geq 1$ with the following property. If $n \geq n_0$ and n is odd, we can find a positive, infinitely differentiable function f such that:

(i)′ $\|f * f - 1\|_\infty \leq \varepsilon \frac{\phi(n)(\log n)^{1/2} n^{1/2}}{N}$.

(ii)′ $\|(f * f)'\|_\infty \leq 4\varepsilon \frac{\phi(n)(\log n)^{1/2} n^{3/2}}{N}$.

(iii)′ $\|f\|_\infty \leq \frac{nM(\kappa)}{N}$.

(iv)′ $\|f'\|_\infty \leq \frac{8n^2 M(\kappa)}{N}$.

(v)′ $\int_{\mathbb{T}} f(t)\, dt = 1$.

(vi)′ $\operatorname{supp} f$ can be covered by N intervals of length $2/n$.

This f is the function we want in the lemma. Choose $n_1(\alpha, \beta, \varepsilon) \geq n_0$ sufficiently large so that if $n \geq n_1$ is odd, then

$$4(\log n)^{3/2} n^{-\eta} < \varepsilon, \tag{3.2}$$

and

$$8M(\kappa)\lfloor n^\kappa \rfloor^{-1} < \varepsilon. \tag{3.3}$$

Using our particular values, $4(\log n)^{3/2} n^{-\eta} < \varepsilon \leq 1$ turns (i)′ into

$$\|f * f - 1\|_\infty \leq \varepsilon \frac{(\log n)^{3/2} n^{1/2}}{\lfloor n^\kappa \rfloor} \leq \varepsilon \frac{(\log n)^{3/2} n^{1/2}}{n^{\beta + \frac{1}{2} + \eta}} = \varepsilon n^{-\beta} \frac{(\log n)^{3/2}}{n^\eta} \leq \varepsilon n^{-\beta}/2.$$

We also get $\|f * f - 1\|_\infty < \varepsilon^2$, since $4(\log n)^{3/2} n^{-\eta} < \varepsilon$. This will be useful in proving part (vii) later.

(ii)′ becomes, after another application of (3.2),

$$\|(f * f)'\|_\infty \leq 4\varepsilon \frac{(\log n)^{3/2} n^{3/2}}{\lfloor n^\kappa \rfloor} \leq 4\varepsilon \frac{(\log n)^{3/2} n^{3/2}}{n^{\beta + \frac{1}{2} + \eta}} = \varepsilon n^{1-\beta} \frac{4(\log n)^{3/2}}{n^\eta} \leq \varepsilon n^{1-\beta}.$$

Now using (3.3), (iii)′ becomes

$$\|f\|_\infty \leq \frac{nM(\kappa)}{\lfloor n^\kappa \rfloor} \leq \varepsilon n,$$

and (iv)$'$ becomes

$$\|f'\|_\infty \le \frac{8n^2 M(\kappa)}{\lfloor n^\kappa \rfloor} \le \varepsilon n^2.$$

(v)$'$ gives what we need immediately.

(vi)$'$ tells us that $\operatorname{supp} f$ can be covered by $\lfloor n^\kappa \rfloor = \lfloor n^{\alpha-\eta} \rfloor$ intervals of length $2/n$. Our choice of n satisfying (3.2) will ensure that $n^{-\eta} < \varepsilon/2$ provided $n > \exp(1/4) \approx 1.28$, which we will assume. Thus, $\operatorname{supp} f$ can be covered by less than $n^{\alpha-\eta} < \varepsilon n^\alpha/2$ intervals of length $2/n$.

For $r \neq 0$,

$$|\widehat{f}(r)| = |\widehat{f * f}(r)|^{1/2} = |\widehat{f * f}(r) - \widehat{1}(r)|^{1/2} \le \|f * f - 1\|_\infty^{1/2} < \varepsilon,$$

verifying part (vii).

∎

For the final step of this chapter, we will impose a periodicity condition on the smooth function we generate.

Lemma 3.4. *Suppose that $1/2 > \alpha - 1/2 > \beta > 0$. There exists an integer $k(\alpha, \beta)$ such that given any $\varepsilon > 0$, there exists an $m_1(k, \alpha, \beta, \varepsilon) \ge 1$ with the following property. Suppose that $m > m_1(k, \alpha, \beta, \varepsilon)$, and m is odd. Then we can find a positive, infinitely differentiable, periodic function F with period $1/m$ with the following properties:*

(i) $\|F * F - 1\|_\infty \le \varepsilon$.

(ii) $\|(F * F)'\|_\infty \le \varepsilon m^{k(1-\beta)}$.

(iii) $\|F\|_\infty \le \varepsilon m^k$.

(iv) $\|F'\|_\infty \le \varepsilon m^{2k+1}$.

(v) $\int_{\mathbb{T}} F(t)\, dt = 1$.

(vi) *We can find a finite collection of intervals* $\mathcal{I}$ *such that*

$$\bigcup_{I\in\mathcal{I}} I \supseteq \operatorname{supp} F \text{ and } \sum_{I\in\mathcal{I}} |I|^\alpha < \varepsilon.$$

(vii) $|\widehat{F}(r)| \leq \varepsilon$ *for all* $r \neq 0$.

(viii) $|h|^{-\beta}|F * F(t+h) - F * F(t)| \leq \varepsilon$ *for all* $t, h \in \mathbb{T}$ *with* $h \neq 0$.

Proof. Since $1/2 > \alpha - 1/2 > \beta > 0$, we can find α_1 and β_1 such that $1/2 > \alpha - 1/2 > \alpha_1 - 1/2 > \beta_1 > \beta > 0$. For an explicit choice, let

$$\alpha_1 = \tfrac{3}{4}\alpha + \tfrac{1}{4}\left(\beta + \tfrac{1}{2}\right), \beta_1 = \tfrac{1}{4}\left(\alpha - \tfrac{1}{2}\right) + \tfrac{3}{4}\beta.$$

Since $1/2 > \alpha - 1/2 > \beta > 0$, clearly $\beta_1 > 0$ and

$$\alpha_1 - \tfrac{1}{2} = \beta_1 + \tfrac{1}{2}(\alpha - \tfrac{1}{2} - \beta) > \beta_1,$$

while

$$\alpha_1 - \tfrac{1}{2} = (\alpha - \tfrac{1}{2}) - \tfrac{1}{4}(\alpha - \tfrac{1}{2} - \beta) < \tfrac{1}{2}.$$

Moreover,

$$\beta_1 = \tfrac{1}{4}(\alpha - \tfrac{1}{2} - \beta) + \beta > \beta,$$

and

$$\alpha_1 = \alpha - \tfrac{1}{4}(\alpha - \tfrac{1}{2} - \beta) < \alpha.$$

Choose an integer k such that $k(\beta_1 - \beta) \geq 1$ (as $\beta_1 > \beta$) and $k(\alpha - \alpha_1) > 1 - \alpha$ (as $\alpha > \alpha_1$); then k is determined entirely by α and β alone. By Lemma 3.3, if $\varepsilon > 0$ there exists an $n_1(\alpha_1, \beta_1, \varepsilon) \geq 1$ such that if for some odd m, $n = m^k > n_1$ (note that n is odd since m is), then we can find a positive, infinitely differentiable function f such that:

(i)′ $\|f * f - 1\|_\infty \leq \varepsilon m^{-k\beta_1}/2$.

(ii)′ $\|(f * f)'\|_\infty \leq \varepsilon m^{k(1-\beta_1)}$.

(iii)′ $\|f\|_\infty \leq \varepsilon m^k$.

(iv)′ $\|f'\|_\infty \le \varepsilon m^{2k}$.

(v)′ $\int_{\mathbb{T}} f(t)\,dt = 1$.

(vi)′ $\operatorname{supp} f$ can be covered by less than $\varepsilon m^{k\alpha_1}/2$ intervals of length $2m^{-k}$.

(vii)′ $|\widehat{f}(r)| \le \varepsilon$ for all $r \ne 0$.

Extend f by periodicity to $[-m/2, m/2]$. Setting $F(t) = f(mt)$ for $t \in [-1/2, 1/2]$, we get that F is a positive, infinitely differentiable, periodic function with period $1/m$, since f is a function on $\mathbb{T} = [-1/2, 1/2]$ with endpoints identified. We have

$$\begin{aligned} F * F(s) &= \int F(s-t)F(t)\,dt = \int_{-1/2}^{1/2} f(m(s-t))f(mt)\,dt \\ &= \int_{-m/2}^{m/2} f(ms-u)f(u)\frac{1}{m}\,du = \int_{-1/2}^{1/2} f(ms-u)f(u)\,du = f * f(ms), \end{aligned}$$

by translation invariance of Lebesgue measure. Then F has the following properties:

(i)″ $\|F * F - 1\|_\infty \le \varepsilon m^{-k\beta_1}/2$.

(ii)″ $\|(F * F)'\|_\infty \le \varepsilon m^{k(1-\beta_1)+1}$.

(iii)″ $\|F\|_\infty \le \varepsilon m^{k}$.

(iv)″ $\|F'\|_\infty \le \varepsilon m^{2k+1}$.

(v)″ $\int_{\mathbb{T}} F(t)\,dt = 1$.

(vi)″ $\operatorname{supp} F$ can be covered by less than $\varepsilon m^{k\alpha_1+1}/2$ intervals of length $2m^{-k-1}$.

(vii)″ $|\widehat{F}(r)| \le \varepsilon$ for all $r \ne 0$.

As we chose k such that $k(\beta_1 - \beta) \ge 1$, we get $1 - k\beta_1 \le -k\beta$, and so (i)″ gives

$$\|F * F - 1\|_\infty \le \varepsilon m^{-k\beta_1}/2 \le \varepsilon m^{1-k\beta_1}/2 \le \varepsilon m^{-k\beta}/2 \le \varepsilon,$$

verifying (i). We have kept the inequality $\|F * F - 1\|_\infty \le \varepsilon m^{-k\beta}/2$ as it is instrumental in proving (viii) later.

Similarly, we have by choice of k that $k(1-\beta) \geq k(1-\beta_1)+1$, and so (ii)$''$ gives

$$\|(F*F)'\|_\infty \leq \varepsilon m^{k(1-\beta_1)+1} \leq \varepsilon m^{k(1-\beta)},$$

verifying (ii).

(vi)$''$ tells us that there is a finite collection of intervals $\mathcal{I}$ (in fact, with at most $\varepsilon m^{k\alpha_1+1}/2$ elements) such that

$$\bigcup_{I\in\mathcal{I}} I \supseteq \operatorname{supp} F,$$

and

$$\sum_{I\in\mathcal{I}} |I|^\alpha \leq \varepsilon \frac{m^{k\alpha_1+1}}{2}(2m^{-k-1})^\alpha = \varepsilon 2^{\alpha-1} m^{1-\alpha+k(\alpha_1-\alpha)}.$$

Since $\alpha_1 < \alpha$ and k is fixed satisfying $k(\alpha-\alpha_1) > 1-\alpha$, there exists an $m_1(k,\alpha,\beta,\varepsilon) \geq 1$ such that if $m \geq m_1$, then $2^{\alpha-1}m^{1-\alpha+k(\alpha_1-\alpha)} < 1$, so that

$$\sum_{I\in\mathcal{I}} |I|^\alpha < \varepsilon,$$

verifying (vi).

We have already proven part (ii), namely that $\|(F*F)'\|_\infty \leq \varepsilon m^{k(1-\beta)}$. Then since $F*F$ is smooth, the Mean Value theorem yields

$$|h|^{-1}|F*F(t+h) - F*F(t)| \leq |(F*F)'\|_\infty \leq \varepsilon m^{k(1-\beta)}.$$

Thus if $|h| \leq m^{-k}$,

$$\begin{aligned}|h|^{-\beta}|F*F(t+h) - F*F(t)| &= |h|^{1-\beta}|h|^{-1}|F*F(t+h) - F*F(t)| \\ &\leq \varepsilon|h|^{1-\beta}m^{k(1-\beta)} \leq \varepsilon,\end{aligned}$$

since $(1-\beta) > 0$. If $|h| \geq m^{-k}$, then the inequality we showed in the proof of (i) above gives

$$|h|^{-\beta}|F*F(t+h) - F*F(t)| \leq |h|^{-\beta}2\|F*F-1\|_\infty \leq \varepsilon|h|^{-\beta}m^{-k\beta} \leq \varepsilon,$$

since $-\beta < 0$. This verifies part (viii).

The remaining parts (iii), (iv), (v), and (vii) follow directly from their counterparts (iii)$''$, (iv)$''$, (v)$''$, and (vii)$''$. ∎

Chapter 4

The Main Theorem

Our method of proof will, in fact, result in a slightly more general version than what we need for Theorem 4.19. For the sake of generality, we will introduce the concept of ψ-Lipschitz functions in the following section, and develop a complete metric space to work in so that we may eventually apply a Baire Category argument.

4.1 ψ-Lipschitz functions

Lemma 4.1. (i) *Consider the space $\mathcal{F}$ of non-empty closed subsets of $\mathbb{T}$. If we set*

$$d_{\mathcal{F}}(E, F) = \max\left\{\sup_{e\in E}\inf_{f\in F}|e-f|, \sup_{f\in F}\inf_{e\in E}|e-f|\right\},$$

then $(\mathcal{F}, d_{\mathcal{F}})$ is a complete metric space. $d_{\mathcal{F}}$ is known as the Hausdorff metric.

(ii) *Consider the space $\mathcal{E}$ consisting of ordered pairs (E, μ) where $E \in \mathcal{F}$ and μ is a probability measure with $\operatorname{supp}\mu \subseteq E$ and $\widehat{\mu}(r) \to 0$ as $|r| \to \infty$. If we take*

$$d_{\mathcal{E}}((E,\mu),(F,\sigma)) = d_{\mathcal{F}}(E,F) + \sup_{r\in\mathbb{Z}}|\widehat{\mu}(r) - \widehat{\sigma}(r)|,$$

then $(\mathcal{E}, d_{\mathcal{E}})$ is a complete metric space.

(iii) *Consider the space* $\mathcal{G}$ *consisting of those* $(E, \mu) \in \mathcal{E}$ *such that* $\mu * \mu = f_\mu \lambda$ *with* f_μ *continuous. If we take*

$$d_{\mathcal{G}}((E, \mu), (F, \sigma)) = d_{\mathcal{E}}((E, \mu), (F, \sigma)) + \|f_\mu - f_\sigma\|_\infty,$$

then $(\mathcal{G}, d_{\mathcal{G}})$ *is a complete metric space.*

Proof. (i) It is a standard result that the Hausdorff metric defined with usual distance $|\cdot|$ is a complete metric (see Appendix D.3).

(ii) If $(E, \mu), (F, \sigma) \in \mathcal{E}$ with $E = F$ and $\mu = \sigma$, then it is clear that $d_{\mathcal{E}}((E, \mu), (F, \sigma)) = 0$. Conversely, if $d_{\mathcal{E}}((E, \mu), (F, \sigma)) = d_{\mathcal{F}}(E, F) + \sup_{r \in \mathbb{Z}} |\widehat{\mu}(r) - \widehat{\sigma}(r)| = 0$, then $E = F$ by (i) and $\mu = \sigma$ by the uniqueness theorem.

Since $d_{\mathcal{F}}$ is a metric, $d_{\mathcal{E}}((E, \mu), (F, \sigma)) = d_{\mathcal{E}}((F, \sigma), (E, \mu))$ for any $(E, \mu), (F, \sigma) \in \mathcal{E}$.

Triangle inequality follows directly from the usual triangle inequality on $|\cdot|$ and part (i). Thus, $(\mathcal{E}, d_{\mathcal{E}})$ is indeed a metric space.

To see completeness, let (E_n, μ_n) be a Cauchy sequence in $(\mathcal{E}, d_{\mathcal{E}})$. Then $\{E_n\}$ is Cauchy in $(\mathcal{F}, d_{\mathcal{F}})$, so by completeness, we can find a set $E \in \mathcal{F}$ such that $d_{\mathcal{F}}(E_n, E) \to 0$ as $n \to \infty$. Since the space of probability measures is weak-$*$ compact, there exists a convergent subsequence, say $\mu_{n_k} \to \mu$ w-$*$, μ a probability measure. Since $\mathbb{T}$ is compact, this implies $\widehat{\mu_{n_k}}(r) \to \widehat{\mu}(r)$ for every fixed r. By hypothesis, $\widehat{\mu_{n_k}}(r) \to 0$ as $|r| \to \infty$, and by the Cauchy condition,

$$\sup_{r \in \mathbb{Z}} |\widehat{\mu_{n_k}}(r) - \widehat{\mu_{n_l}}(r)| \to 0$$

as $k, l \to \infty$, and thus $\widehat{\mu}(r) \to 0$ as $|r| \to \infty$.

Suppose $x \in \operatorname{supp} \mu$. By definition, for every neighborhood U_x of x, we have $\mu(U_x) > 0$. Since $\mu_n \to \mu$ w-$*$, there exists an integer N_x sufficiently large that $n \geq N_x$ implies $\mu_n(U) \geq \mu(U)$ for every open set U. In particular, for every neighborhood U_x of x, this gives $\mu_n(U_x) \geq \mu(U_x) > 0$, so that $x \in \operatorname{supp} \mu_n$ if $n \geq N_x$.

Let $\varepsilon > 0$. By property of Hausdorff metric, (see Appendix D.4), we have that there exists N_ε such that $n \geq N_\varepsilon$ implies $E_n \subseteq E_\varepsilon$. Combined with the previous paragraph, this gives $x \in \operatorname{supp} \mu_n \subseteq E_n \subseteq E_\varepsilon$ for $n \geq \max\{N_x, N_\varepsilon\}$. Since this holds for arbitrary ε and E is closed, we have that $x \in E$, and so $\operatorname{supp} \mu \subseteq E$.

Weak convergence gives $\widehat{\mu_{n_k}}(r) \to \widehat{\mu}(r)$ for each fixed r, thus

$$d_{\mathcal{E}}((E_{n_k}, \mu_{n_k}), (E, \mu)) \to 0$$

as $k \to \infty$. Since a Cauchy sequence with a convergent subsequence must converge to the same limit as the subsequence,

$$d_{\mathcal{E}}((E_n, \mu_n), (E, \mu)) \to 0,$$

completing the proof.

(iii) That $(\mathcal{G}, d_{\mathcal{G}})$ is a metric follows a similar proof as in (ii).

To see completeness, let (E_n, μ_n) be a Cauchy sequence in $(\mathcal{G}, d_{\mathcal{G}})$. Then (E_n, μ_n) is Cauchy in $(\mathcal{E}, d_{\mathcal{E}})$, so by completeness, we can find $(E, \mu) \in \mathcal{E}$ such that $d_{\mathcal{E}}((E_n, \mu_n), (E, \mu)) \to 0$ as $n \to \infty$. Since f_{μ_n} is Cauchy in the uniform norm, f_{μ_n} converges uniformly to some continuous function f. By properties of weak convergence, $\mu * \mu = f\lambda$, so $(E, \mu) \in \mathcal{G}$ and $d_{\mathcal{G}}((E_n, \mu_n), (E, \mu)) \to 0$. ∎

Definition 4.2. Let $1 > \alpha > 1/2$ and suppose that $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a strictly increasing, continuous function with $\psi(t) \geq t^{\alpha - 1/2}$ for all $t \geq 0$, $\psi(0) = 0$. We call ψ a *generalized α-power function*. For our uses, ψ need only be defined on $[0, 1/2]$ since we are on the torus and will be using non-negative values.

Definition 4.3. Suppose ψ is a generalized α-power function. We say that $f : \mathbb{T} \to \mathbb{C}$ is *ψ-Lipschitz* if

$$\sup_{t,h \in \mathbb{T}, h \neq 0} \psi(|h|)^{-1} |f(t+h) - f(t)| < \infty.$$

In this case, we denote this value by $\omega_\psi(f)$; notice $\omega_\psi(f) \geq 0$. In the particular case of $\psi(t) = t^\beta$, this definition reduces to the usual definition of Lipschitz of class β, recall (1.2). We denote by Λ_ψ the collection of all ψ-Lipschitz functions. Analogously to Lipschitz functions (see Lemma 1.4), we have a result linking this definition to another form which may be more useable in certain circumstances, with the same proof.

Lemma 4.4. *$f \in \Lambda_\psi$ if and only if $|f(x) - f(y)| \leq C\psi(|x - y|)$ for a constant C, for every $x, y \in \mathbb{T}$.*

We develop some basic properties of operations under ω_ψ which will be used later.

Lemma 4.5. *Let ψ be a generalized α-power function. Suppose $f, g \in \Lambda_\psi$, c is a constant, and $F \in L^1(\mathbb{T})$. Then the following hold:*

(i) Λ_ψ-*translation invariance: $c + f \in \Lambda_\psi$ and $\omega_\psi(f + c) = \omega_\psi(f)$.*

(ii) Λ_ψ-*positive homogeneity: $cf \in \Lambda_\psi$ and $\omega_\psi(cf) = |c|\omega_\psi(f)$.*

(iii) Λ_ψ-*addition: $f + g \in \Lambda_\psi$ and $\omega_\psi(f + g) \leq \omega_\psi(f) + \omega_\psi(g)$.*

(iv) Λ_ψ-*multiplication: $fg \in \Lambda_\psi$ and $\omega_\psi(fg) \leq \omega_\psi(f)\|g\|_\infty + \omega_\psi(g)\|f\|_\infty$.*

(v) Λ_ψ-*convolution: $f * F \in \Lambda_\psi$ and $\omega_\psi(f * F) \leq \omega_\psi(f)\|F\|_1$.*

Proof. (i) and (ii) are immediate from the definition.

(iii) This follows by triangle inequality.

(iv) If $f, g \in \Lambda_\psi$, we have

$$\begin{aligned}
&\sup_{t,h\in\mathbb{T},h\neq 0} \psi(|h|)^{-1}|(fg)(t+h) - (fg)(t)| \\
&\quad \leq \sup_{t,h\in\mathbb{T},h\neq 0} \psi(|h|)^{-1}(|f(t+h) - f(t)||g(t+h)| + |f(t)||g(t+h) - g(t)|) \\
&\quad \leq \sup_{t,h\in\mathbb{T},h\neq 0} \psi(|h|)^{-1}|f(t+h) - f(t)||g(t+h)| + \sup_{t,h\in\mathbb{T},h\neq 0} \psi(|h|)^{-1}|f(t)||g(t+h) - g(t)| \\
&\quad \leq \sup_{t,h\in\mathbb{T},h\neq 0} \psi(|h|)^{-1}|f(t+h) - f(t)|\|g\|_\infty + \sup_{t,h\in\mathbb{T},h\neq 0} \psi(|h|)^{-1}\|f\|_\infty|g(t+h) - g(t)| \\
&\quad = \omega_\psi(f)\|g\|_\infty + \omega_\psi(g)\|f\|_\infty < \infty,
\end{aligned}$$

so that $fg \in \Lambda_\psi$ and

$$\omega_\psi(fg) \leq \omega_\psi(f)\|g\|_\infty + \omega_\psi(g)\|f\|_\infty.$$

(v) If $f \in \Lambda_\psi$ and $F \in L^1(\mathbb{T})$, we have for fixed $t, h \in \mathbb{T}, h \neq 0$:

$$\begin{aligned}
&\psi(|h|)^{-1}|(f * F)(t+h) - (f * F)(t)| \\
&\quad = \psi(|h|)^{-1}\left|\int f(t+h-y)F(y)\,dy - \int f(t-y)F(y)\,dy\right|
\end{aligned}$$

$$\leq \psi(|h|)^{-1} \int |f(t+h-y) - f(t-y)||F(y)|\,dy$$
$$\leq \int \omega_\psi(f)|F(y)|\,dy = \omega_\psi(f)\|F\|_1 < \infty,$$

so that taking the supremum over permissible t and h yields $f * F \in \Lambda_\psi$ and

$$\omega_\psi(f * F) \leq \omega_\psi(f)\|F\|_1.$$

∎

There is also a nice relationship between the ω_ψ value of a function and the magnitude of its derivative, provided f is continuously differentiable.

Lemma 4.6. *Let ψ be a generalized α-power function. If $f : \mathbb{T} \to \mathbb{C}$ has a continuous derivative, then $f \in \Lambda_\psi$ and $\omega_\psi(f) \leq \|f'\|_\infty$.*

Proof. If $f : \mathbb{T} \to \mathbb{C}$ has continuous derivative, then $|f'|$ has a maximum on $\mathbb{T}$, so $\|f'\|_\infty < \infty$. By the Mean Value theorem, for any $t, h \in \mathbb{T}, h \neq 0$, we have

$$|h|^{-1}|f(t+h) - f(t)| \leq \|f'\|_\infty,$$

so then $|h| \leq \psi(|h|)$ yields

$$\psi(|h|)^{-1}|f(t+h) - f(t)| \leq |h|^{-1}|f(t+h) - f(t)| \leq \|f'\|_\infty.$$

Taking the supremum over $t, h \in \mathbb{T}$ with $h \neq 0$, we get

$$\sup_{t,h\in\mathbb{T},h\neq 0} \psi(|h|)^{-1}|f(t+h) - f(t)| \leq \|f'\|_\infty < \infty$$

so that $f \in \Lambda_\psi$ and

$$\omega_\psi(f) \leq \|f'\|_\infty.$$

∎

Lemma 4.7. *Let $1 > \alpha > 1/2$ and suppose that ψ is a generalized α-power function. Consider the space $\mathcal{L}_\psi$ consisting of those $(E, \mu) \in \mathcal{G}$ such that $\mu * \mu = f_\mu \lambda$ with $f_\mu \in \Lambda_\psi$. If we take*

$$d_\psi((E, \mu), (F, \sigma)) = d_\mathcal{G}((E, \mu), (F, \sigma)) + \omega_\psi(f_\mu - f_\sigma),$$

then $(\mathcal{L}_\psi, d_\psi)$ is a complete metric space.

Proof. If $(E,\mu),(F,\sigma) \in \mathcal{L}_\psi$ with $E=F$ and $\mu=\sigma$, then it is clear that $d_\psi((E,\mu),(F,\sigma)) = 0$. Conversely, if $d_\psi((E,\mu),(F,\sigma)) = d_\mathcal{G}((E,\mu),(F,\sigma)) + \omega_\psi(f_\mu - f_\sigma) = 0$, then $E=F$ and $\mu=\sigma$ since $d_\mathcal{G}$ is a metric.

Since $d_\mathcal{G}$ is a metric and by Λ_ψ-positive homogeneity, $d_\psi((E,\mu),(F,\sigma)) = d_\psi((F,\sigma),(E,\mu))$ for any $(E,\mu),(F,\sigma) \in \mathcal{L}_\psi$.

Triangle inequality follows directly from Λ_ψ-addition and the fact that $d_\mathcal{G}$ is a metric. Thus, $(\mathcal{L}_\psi, d_\psi)$ is indeed a metric space.

If (E_n,μ_n) is a Cauchy sequence in $(\mathcal{L}_\psi, d_\psi)$, then it is Cauchy in $(\mathcal{G}, d_\mathcal{G})$. Thus there exists $(E,\mu) \in \mathcal{G}$ such that

$$d_\mathcal{G}((E_n,\mu_n),(E,\mu)) \to 0$$

as $n \to \infty$. By definition of $\mathcal{G}$, we may write $\mu_n * \mu_n = f_n\lambda$ and $\mu * \mu = f\lambda$ with f_n and f continuous. The condition with $d_\mathcal{G}$ above gives $f_n \to f$ uniformly. If $m \geq n$, we have for any $t,h \in \mathbb{T}, h \neq 0$ that

$$\begin{aligned}
&\psi(|h|)^{-1}|(f-f_n)(t+h)-(f-f_n)(t)| \\
&\quad \leq \psi(|h|)^{-1}|(f-f_m)(t+h)-(f-f_m)(t)| + \psi(|h|)^{-1}|(f_m-f_n)(t+h)-(f_m-f_n)(t)| \\
&\quad \leq \psi(|h|)^{-1}|(f-f_m)(t+h)-(f-f_m)(t)| + d_\psi((E_n,\mu_n),(E_m,\mu_m)) \\
&\quad \leq \psi(|h|)^{-1}|(f-f_m)(t+h)-(f-f_m)(t)| + \sup_{p,q\geq n} d_\psi((E_p,\mu_p),(E_q,\mu_q)).
\end{aligned}$$

Allowing $m \to \infty$, the first term limits to 0 and thus

$$\psi(|h|)^{-1}|(f-f_n)(t+h)-(f-f_n)(t)| \leq \sup_{p,q\geq n} d_\psi((E_p,\mu_p),(E_q,\mu_q))$$

for all $t,h \in \mathbb{T}, h \neq 0$. Thus $f - f_n \in \Lambda_\psi$ and so Λ_ψ-addition gives $f \in \Lambda_\psi$, implying $(E,\mu) \in \mathcal{L}_\psi$. Moreover, the above calculation gives

$$\omega_\psi(f-f_n) \leq \sup_{p,q\geq n} d_\psi((E_p,\mu_p),(E_q,\mu_q))$$

so that

$$d_\psi((E_n,\mu_n),(E,\mu)) \to 0$$

as $n \to \infty$, and hence $(\mathcal{L}_\psi, d_\psi)$ is a complete metric space. ∎

Lemma 4.8. *Let $1 > \alpha > 1/2$ and suppose that ψ is a generalized α-power function. Consider the set $\{(E, \mu) \in \mathcal{L}_\psi : \mu * \mu = f_\mu \lambda, f_\mu$ is infinitely differentiable$\}$. Let $\mathcal{M}_\psi$ denote the closure of this set with respect to the d_ψ metric. Then $(\mathcal{M}_\psi, d_\psi)$ is a complete metric space.*

Proof. By definition, $(\mathcal{M}_\psi, d_\psi)$ is a closed subspace of the complete metric space $(\mathcal{L}_\psi, d_\psi)$ (by Lemma 4.7), and hence is complete. ∎

4.2 Density results

Let $1 > \alpha > 1/2$ and suppose that ψ is a generalized α-power function. Let $\mathcal{H}_n$ be the subset of $(\mathcal{M}_\psi, d_\psi)$ consisting of those $(E, \mu) \in \mathcal{M}_\psi$ such that we can find a finite collection of closed intervals $\mathcal{I}$ with

$$\bigcup_{I \in \mathcal{I}} I \supseteq E \text{ and } \sum_{I \in \mathcal{I}} |I|^{\alpha + 1/n} < \frac{1}{n}.$$

Notice that $\mathcal{H}_n \supseteq \mathcal{H}_{n+1}$. Our goal will be to show that $\mathcal{H}_n$ are dense in $(\mathcal{M}_\psi, d_\psi)$ which will be Lemma 4.16.

Lemma 4.9. *Let $1 > \alpha > 1/2$ and suppose that ψ is a generalized α-power function. Suppose further that $n \geq 1$, $g : \mathbb{T} \to \mathbb{R}$ is a positive, infinitely differentiable function with*

$$\int_{\mathbb{T}} g(t)\, dt = 1$$

and H is a closed set with $H \supseteq \operatorname{supp} g$. Then, given $\varepsilon > 0$, we can find a positive, infinitely differentiable function $f : \mathbb{T} \to \mathbb{R}$ with

$$\int_{\mathbb{T}} f(t)\, dt = 1$$

and a closed set $E \supseteq \operatorname{supp} f$ such that $(E, f\lambda) \in \mathcal{H}_n$ and

$$d_\psi((E, f\lambda), (H, g\lambda)) < \varepsilon.$$

Proof. Since $\mathcal{H}_n \supseteq \mathcal{H}_{n+1}$ and $1 > \alpha > 1/2$, we may restrict ourselves to the case when $1 > \alpha + 1/n > 1/2$. Lemma 3.4 provides us with an integer $k = k(\alpha + 1/n, \alpha - 1/2) \geq 1$ with the property described in the next sentence. Fix $0 < \eta < 1/n$ for the time being; then there exists an integer $m_1(k, \alpha + 1/n, \alpha - 1/2, \eta) \geq 1$ such that if $2m+1 > m_1$, we can find a positive, infinitely differentiable function F_m which is periodic with period $1/(2m+1)$ with the following properties:

(i)$_m$ $\|F_m * F_m - 1\|_\infty \leq \eta$.

(ii)$_m$ $\|(F_m * F_m)'\|_\infty \leq \eta(2m+1)^{k(1-\alpha+1/2)}$.

(iii)$_m$ $\|F_m\|_\infty \leq \eta(2m+1)^k \leq (2(2m))^k = 4^k m^k$.

(iv)$_m$ $\|F_m'\|_\infty \leq \eta(2m+1)^{2k+1} \leq (2(2m))^{2k+1} = 4^{2k+1} m^{2k+1}$.

(v)$_m$ $\|F_m\|_1 = \int_{\mathbb{T}} F_m(t)\, dt = 1$.

(vi)$_m$ We can find a finite collection of intervals $\mathcal{I}_m$ such that

$$\bigcup_{I \in \mathcal{I}_m} I \supseteq \operatorname{supp} F_m \text{ and } \sum_{I \in \mathcal{I}_m} |I|^{\alpha + \frac{1}{n}} < \eta < \frac{1}{n}.$$

(vii)$_m$ $|\widehat{F_m}(r)| \leq \eta$ for all $r \neq 0$.

(viii)$_m$ $|h|^{-\alpha+1/2}|F_m * F_m(t+h) - F_m * F_m(t)| \leq \eta$ for all $t, h \in \mathbb{T}$ with $h \neq 0$.

Since ψ is a generalized α-power function, $\psi(t) \geq t^{\alpha - 1/2}$ for every $t \geq 0$. Then using (viii)$_m$ and taking the supremum over $t, h \in \mathbb{T}$ with $h \neq 0$,

$$\begin{aligned}
\omega_\psi(F_m * F_m) &= \sup_{t,h \in \mathbb{T}, h \neq 0} \psi(|h|)^{-1} |F_m * F_m(t+h) - F_m * F_m(t)| \\
&\leq \sup_{t,h \in \mathbb{T}, h \neq 0} |h|^{-\alpha+1/2} |F_m * F_m(t+h) - F_m * F_m(t)| \leq \eta.
\end{aligned}$$

We shall also call this property (viii)$_m$, as we will no longer need the previous version.

Since g is infinitely differentiable, all of its derivatives $g^{(j)}$ are continuous on the torus. In particular, $g^{(j)}(-1/2) = g^{(j)}(1/2)$ for every j. Then by integration by parts with $u = g^{(j)}(t)$ and $dv = e^{-irt}\,dt$, we have

$$\widehat{g^{(j)}}(r) = \left| \int_{-1/2}^{1/2} g^{(j)}(t) e^{-2\pi irt}\,dt \right| = \frac{1}{ir}\widehat{g^{(j+1)}}(r).$$

Then by induction, for every j we have

$$\widehat{g}(r) = \frac{1}{(ir)^j}\widehat{g^{(j)}}(r).$$

In particular, for $j = 2k+4$, we have that for $r \neq 0$

$$|\widehat{g}(r)| = \frac{1}{|r|^{2k+4}} \left| \int g^{(j)} e^{-2\pi irt}\,dt \right| \leq C_1 |r|^{-(2k+4)},$$

where $C_1 = \|g^{(j)}\|_1$ is a constant. By a straightforward application of the integral test,

$$\sum_{|r|\geq m} |r||\widehat{g}(r)| \leq C_1 \sum_{|r|\geq m} |r|^{-(2k+3)} \leq 2C_1 \left[\left(1 + \frac{1}{2k+2}\right) m^{-(2k+2)} \right],$$

since $m \geq 1$. Thus, we get that with $C = 2C_1 \frac{2k+3}{k+1}$ a constant,

$$\sum_{|r|\geq m} |r||\widehat{g}(r)| \leq Cm^{-(2k+2)}. \tag{4.1}$$

Since $\widehat{g}(0) = \int_{\mathbb{T}} g(t)\,dt = 1$, we have in particular that

$$\sum_{r=-\infty}^{\infty} |\widehat{g}(r)| \leq |\widehat{g}(0)| + \sum_{|r|\geq 1} |r||\widehat{g}(r)| \leq 1 + C. \tag{4.2}$$

Set $G_m(t) = g(t)F_m(t)$. Since F_m is periodic of period $1/(2m+1)$, we know that $\widehat{F_m}(r) = 0$ if $r \notin (2m+1)\mathbb{Z}$. Then since $\widehat{g}(0) = 1$ and $\widehat{F_m}(0) = \int_{\mathbb{T}} F_m(t)\,dt = 1$,

$$\begin{aligned}\widehat{G_m}(0) - 1 = \widehat{g} * \widehat{F_m}(0) - 1 &= \sum_{r=-\infty}^{\infty} \widehat{g}(r)\widehat{F_m}(-r) - 1 \\ &= \widehat{g}(0)\widehat{F_m}(0) - 1 + \sum_{j=-\infty, j\neq 0}^{\infty} \widehat{g}((2m+1)j)\widehat{F_m}(-(2m+1)j)\end{aligned}$$

$$= \sum_{|r|\geq 2m+1} \widehat{g}(r)\widehat{F_m}(-r).$$

Also, $|\widehat{F_m}(-r)| \leq \|F_m\|_1 = 1$, so that the previous bound of $\widehat{G_m}(0) - 1$ gives

$$|\widehat{G_m}(0) - 1| \leq \sum_{|r|\geq 2m+1} |r||\widehat{g}(r)| \leq C(2m+1)^{-(2k+2)} \leq Cm^{-(2k+2)}, \tag{4.3}$$

using (4.1). Since $\widehat{G_m}(0) = \int_{\mathbb{T}} G_m(t)\, dt$, (4.3) gives that for sufficiently large m, $\int_{\mathbb{T}} G_m(t)\, dt \neq 0$, so that we may define

$$f(t) = \left(\int_{\mathbb{T}} G_m(s)\, ds\right)^{-1} G_m(t).$$

This function f will be the function we want to approximate the given g, but we will need in addition a special closed set E that is close to H.

Notice

$$\operatorname{supp} f \subseteq \operatorname{supp} g \cap \operatorname{supp} F_m \subseteq H \cap \operatorname{supp} F_m \subseteq H.$$

In particular,

$$\sup_{e\in \operatorname{supp} f} \inf_{h\in H} |e - h| = 0.$$

$H \subseteq \mathbb{T}$ is a closed subset of a compact set, hence compact. Then as $(\mathbb{T}, |\cdot|)$ is a metric space, H is totally bounded. That is, (with $\varepsilon > 0$ given in the hypothesis) there exist finitely many balls of radius $r_i < \varepsilon/4$ centered at points a_i, say $B_1(a_1, r_1), \ldots, B(a_N, r_N)$, such that $H \subseteq \bigcup_{i=1}^N B(a_i, r_i)$. Consider the finite set $A = \bigcup_{i=1}^N \{a_i\}$, and fix $h_0 \in H$. Then $h_0 \in \bigcup_{i=1}^N B(a_i, r_i)$, so that there is some $1 \leq i \leq N$ with $h_0 \in B(a_i, r_i)$. In particular, we get an element $a_i \in A$ with $|h_0 - a_i| < \varepsilon/4$. Then

$$\inf_{a\in A} |h_0 - a| \leq |h_0 - a_i| < \frac{\varepsilon}{4}.$$

Taking the supremum over elements in H,

$$\sup_{h\in H} \inf_{a\in A} |h - a| \leq \frac{\varepsilon}{4}.$$

Define $E := A \cup \operatorname{supp} f$. Then the previous estimate remains unchanged when A is replaced by E, since for each fixed element h_0 in H, the infimum of distances to an element in E is still smaller than the distance between h_0 and a_i. E is closed, being the union of finitely many closed sets, so $E \in \mathcal{F}$. $f\lambda$ is a probability measure since by construction, $\int_{\mathbb{T}} f(t)\,dt = 1$, and clearly $\operatorname{supp} f \subseteq E$. $\widehat{f}(r) \to 0$ as $|r| \to \infty$ since f is infinitely differentiable, and so $(E, f\lambda) \in \mathcal{E}$. $f\lambda * f\lambda = (f * f)(\lambda * \lambda) = (f * f)\lambda$, and $f * f$ is continuous (in fact, infinitely differentiable) as f is, so that $(E, f\lambda) \in \mathcal{G}$. $f \in \Lambda_\psi$ as f is infinitely differentiable, and so $(E, f\lambda) \in \mathcal{M}_\psi$. The finite collection of intervals $\mathcal{I}$ consisting of those in $\mathcal{I}_m$ together with the degenerate closed intervals $[a_i, a_i] = \{a_i\}$ has the property that

$$\bigcup_{I \in \mathcal{I}} I \supseteq A \cup \operatorname{supp} F_m \supseteq A \cup \operatorname{supp} f \text{ and } \sum_{I \in \mathcal{I}} |I|^{\alpha+1/n} < \eta < \frac{1}{n},$$

so that $(E, f\lambda) \in \mathcal{H}_n$. Moreover,

$$d_{\mathcal{F}}(E, H) = \sup_{e \in E} \inf_{h \in H} |e - h| + \sup_{h \in H} \inf_{e \in E} |e - h| \le 0 + \frac{\varepsilon}{4} = \frac{\varepsilon}{4}. \tag{4.4}$$

Recall we are to show that

$$d_\psi((E, f\lambda), (H, g\lambda)) < \varepsilon.$$

By definition, the metric d_ψ is expanded as

$$\begin{aligned}
&d_\psi((E, f\lambda), (H, g\lambda)) \\
&\quad = d_{\mathcal{G}}((E, f\lambda), (H, g\lambda)) + \omega_\psi(f * f - g * g) \\
&\quad = d_{\mathcal{E}}((E, f\lambda), (H, g\lambda)) + \|f * f - g * g\|_\infty + \omega_\psi(f * f - g * g) \\
&\quad = d_{\mathcal{F}}(E, H) + \sup_{r \in \mathbb{Z}} |\widehat{f}(r) - \widehat{g}(r)| + \|f * f - g * g\|_\infty + \omega_\psi(f * f - g * g).
\end{aligned}$$

We have bounded the first term by $\varepsilon/4$, it remains to show the remaining three terms are also bounded by $\varepsilon/4$. In order to show these, we first will show that f is close to G_m for sufficiently large m, and appeal to a triangle inequality argument.

For ease of notation, write $C_m = Cm^{-(2k+2)} > 0$. We may assume $C_m < 1/2$ for all sufficiently large m. With this notation, (4.3) yields

$$1 - C_m \le \widehat{G_m}(0) \le 1 + C_m \le 2.$$

Then

$$|\widehat{G_m}(0)^{-2} - 1| = \frac{|1 + \widehat{G_m}(0)||1 - \widehat{G_m}(0)|}{|\widehat{G_m}(0)^2|} \le \frac{3C_m}{(1 - C_m)^2} \le \frac{3C_m}{(1 - 1/2)^2} = 12C_m,$$

since $C_m < 1/2$. As an immediate consequence, we have

$$|\widehat{G_m}(0)^{-1} - 1| \leq |\widehat{G_m}(0)^{-2} - 1| \leq 12Cm^{-(2k+2)}, \tag{4.5}$$

since $|x^{-1} - 1| \leq |x^{-2} - 1|$ for $x > 0$.

Claim 4.10.

$$\sup_{r \in \mathbb{Z}} |\widehat{f}(r) - \widehat{G_m}(r)| \leq \frac{\varepsilon}{8}.$$

Proof. Fix $r \in \mathbb{Z}$. Notice that as G_m is the product of two positive functions, it is itself positive and hence $\|G_m\|_1 = \int_{\mathbb{T}} G_m(t)\, dt = \widehat{G_m}(0) \leq 2$. As $|\widehat{G_m}(r)| \leq \|G_m\|_1$, by (4.5) we get

$$\begin{aligned} |\widehat{f}(r) - \widehat{G_m}(r)| &= \left| \left(\int_{\mathbb{T}} G_m(s)\, ds \right)^{-1} \widehat{G_m}(r) - \widehat{G_m}(r) \right| \\ &= |\widehat{G_m}(r)| \left| \left(\int_{\mathbb{T}} G_m(s)\, ds \right)^{-1} - 1 \right| \leq 24Cm^{-(2k+2)}. \end{aligned}$$

Since $24C$ is a constant, m can be taken sufficiently large that

$$|\widehat{f}(r) - \widehat{G_m}(r)| < \frac{\varepsilon}{8}.$$

Taking the supremum over $r \in \mathbb{Z}$ yields the desired result. ∎

Claim 4.11.

$$\|f * f - G_m * G_m\|_\infty < \frac{\varepsilon}{8}.$$

Proof. A direct computation gives, using (4.5),

$$\begin{aligned} \|f * f - G_m * G_m\|_\infty &= \left\| \left(\int_{\mathbb{T}} G_m(s)\, ds \right)^{-2} G_m * G_m - G_m * G_m \right\|_\infty \\ &\leq \|G_m\|_1 \|G_m\|_\infty \left| \left(\int_{\mathbb{T}} G_m(s)\, ds \right)^{-2} - 1 \right| \\ &\leq 2\|G_m\|_\infty 12Cm^{-(2k+2)}. \end{aligned}$$

We have $\|G_m\|_\infty = \|gF_m\|_\infty \le \|g\|_\infty \|F_m\|_\infty$, and our estimate from (iii)$_m$ gives

$$\|f * f - G_m * G_m\|_\infty \le 24C\|g\|_\infty 4^k m^k m^{-(2k+2)} = 24C\|g\|_\infty 4^k m^{-(k+2)}.$$

Since $24C\|g\|_\infty 4^k$ is a constant, m can be taken sufficiently large that

$$\|f * f - G_m * G_m\|_\infty < \frac{\varepsilon}{8}.$$

∎

Claim 4.12.

$$\omega_\psi(f * f - G_m * G_m) < \frac{\varepsilon}{8}.$$

Proof. We have

$$\begin{aligned}
\omega_\psi(G_m * G_m) &\le \omega_\psi(G_m)\|G_m\|_1 && \text{by } \Lambda_\psi\text{-convolution} \\
&= \omega_\psi(gF_m)\|G_m\|_1 \le 2\omega_\psi(gF_m) \\
&\le 2\left[\omega_\psi(g)\|F_m\|_\infty + \omega_\psi(F_m)\|g\|_\infty\right] && \text{by } \Lambda_\psi\text{-multiplication} \\
&\le 2\left[\|g'\|_\infty\|F_m\|_\infty + \|F_m'\|_\infty\|g\|_\infty\right] && \text{by Lemma 4.6} \\
&\le 2\left[\|g'\|_\infty 4^k m^k + 4^{2k+1}m^{2k+1}\|g\|_\infty\right] && \text{by (iii)}_m \text{ and (iv)}_m \\
&\le 2(\|g'\|_\infty + \|g\|_\infty)4^{2k+1}m^{2k+1}.
\end{aligned}$$

Then using Λ_ψ-positive homogeneity and (4.5),

$$\begin{aligned}
\omega_\psi(f * f - G_m * G_m) &= \omega_\psi\left(\left(\int_{\mathbb{T}} G_m(s)\,ds\right)^{-2} G_m * G_m - G_m * G_m\right) \\
&= \left|\left(\int_{\mathbb{T}} G_m(s)\,ds\right)^{-2} - 1\right| \omega_\psi(G_m * G_m) \\
&\le 12Cm^{-(2k+2)}2(\|g'\|_\infty + \|g\|_\infty)4^{2k+1}m^{2k+1} \\
&= 24C(\|g'\|_\infty + \|g\|_\infty)4^{2k+1}m^{-1}.
\end{aligned}$$

Since $24C(\|g'\|_\infty + \|g\|_\infty)4^{2k+1}$ is a constant, m can be taken sufficiently large that

$$\omega_\psi(f * f - G_m * G_m) < \frac{\varepsilon}{8}.$$

∎

We next need to show that G_m is close to g for sufficiently large m.

Claim 4.13.

$$\sup_{r\in\mathbb{Z}} |\widehat{g}(r) - \widehat{G_m}(r)| \le \frac{\varepsilon}{8}.$$

Proof. For a fixed $r \in \mathbb{Z}$, we have

$$\begin{aligned}
|\widehat{g}(r) - \widehat{G_m}(r)| &= \left|\widehat{g}(r) - \sum_{j=-\infty}^{\infty} \widehat{g}(r-j)\widehat{F_m}(j)\right| \\
&= \left|\sum_{j\neq 0} \widehat{g}(r-j)\widehat{F_m}(j)\right| && \text{since } \widehat{F_m}(0) = 1 \\
&\le \sum_{j\neq 0} |\widehat{g}(r-j)|\eta && \text{by } (\text{vii})_m \\
&\le \eta(1+C) && \text{by (4.2).}
\end{aligned}$$

We may choose $\eta < \varepsilon/(8(1+C))$, and so taking supremum over all $r \in \mathbb{Z}$ gives

$$\sup_{r\in\mathbb{Z}} |\widehat{g}(r) - \widehat{G_m}(r)| \le \frac{\varepsilon}{8}.$$

∎

We now fix η for the remainder of the proof, so that the previous claim holds and we have

$$\eta < (\|g\|_\infty + \omega_\psi(g*g) + 2)^{-2}\frac{\varepsilon}{24}, \tag{4.6}$$

but we leave m free for now, subject only to the constraint that previous claims remain true.

Claim 4.14.

$$\|g*g - G_m * G_m\|_\infty < \frac{\varepsilon}{8}.$$

Proof. Consider the mth partial Fourier sum,

$$S_m(t) = \sum_{|r|<m} \widehat{g}(r)e^{2\pi irt}.$$

Since g is infinitely differentiable on $\mathbb{T}$, the Fourier inversion theorem tells us that

$$g(t) = \sum_{r=-\infty}^{\infty} \widehat{g}(r)e^{2\pi irt},$$

and so

$$|(g - S_m)(t)| = \left|\sum_{|r|\geq m} \widehat{g}(r)e^{2\pi irt}\right| \leq \sum_{|r|\geq m} |\widehat{g}(r)| \leq \sum_{|r|\geq m} |r||\widehat{g}(r)|.$$

Then by (4.1),

$$\|g - S_m\|_\infty \leq Cm^{-(2k+2)}, \tag{4.7}$$

and hence we may choose m sufficiently large that

$$\|g - S_m\|_\infty \leq 1. \tag{4.8}$$

Similarly,

$$|(g - S_m)'(t)| = \left|\sum_{|r|\geq m} \widehat{g}(r)(2\pi ir)e^{2\pi irt}\right| \leq 2\pi \sum_{|r|\geq m} |r||\widehat{g}(r)|,$$

so by (4.1),

$$\|(g - S_m)'\|_\infty \leq 2\pi Cm^{-(2k+2)}. \tag{4.9}$$

Consider the Fourier coefficients of S_mF_m. For $u, v \in \mathbb{Z}$ with $0 \leq v \leq 2m$,

$$\widehat{S_mF_m}((2m+1)u + v) = \sum_{j=-\infty}^{\infty} \widehat{S_m}(j)\widehat{F_m}((2m+1)u + v - j).$$

Since F_m is periodic of period $1/(2m+1)$, $\widehat{F_m}((2m+1)u+v-j) = 0$ unless $(2m+1)u+v-j \in (2m+1)\mathbb{Z}$, that is, unless $v - j \in (2m+1)\mathbb{Z}$. On the other hand, S_m is a trigonometric polynomial of degree at most m, and hence $\widehat{S_m}(j) = 0$ unless $|j| < m$. Since $0 \leq v \leq 2m$, we only get non-zero terms if $j = v$. Thus we have

$$\widehat{S_mF_m}((2m+1)u + v) = \widehat{S_m}(v)\widehat{F_m}((2m+1)u).$$

Likewise, we can consider the Fourier coefficients of $(S_m * S_m)(F_m * F_m)$. Since $S_m * S_m$ is a trigonometric polynomial of degree at most m and $F_m * F_m$ is periodic of period $1/(2m+1)$, the above work shows

$$((S_m * S_m)(F_m * F_m))\widehat{\ }((2m+1)u+v) = (\widehat{S_m}(v))^2(\widehat{F_m}((2m+1)u))^2.$$

These last two identities give

$$\begin{aligned}((S_mF_m) * (S_mF_m))\widehat{\ }((2m+1)u+v) &= (\widehat{S_mF_m}((2m+1)u+v))^2 \\ &= (\widehat{S_m}(v))^2(\widehat{F_m}((2m+1)u))^2 \\ &= ((S_m * S_m)(F_m * F_m))\widehat{\ }((2m+1)u+v).\end{aligned}$$

Since $u, v \in \mathbb{Z}$ and $0 \le v \le 2m$, every Fourier coefficient of these functions agree, so the uniqueness theorem tells us that

$$(S_mF_m) * (S_mF_m)(t) = (S_m * S_m)(t)(F_m * F_m)(t), \tag{4.10}$$

for every $t \in \mathbb{T}$. Then

$$\begin{aligned}\|g * g - G_m * G_m\|_\infty &= \|g * g - (gF_m) * (gF_m)\|_\infty \\ &\le \|g * g - S_m * S_m\|_\infty + \|S_m * S_m - (S_mF_m) * (S_mF_m)\|_\infty \\ &\quad + \|(S_mF_m) * (S_mF_m) - (gF_m) * (gF_m)\|_\infty \\ &= \|g * g - S_m * S_m\|_\infty + \|S_m * S_m - (S_m * S_m)(F_m * F_m)\|_\infty \\ &\quad + \|(S_mF_m) * (S_mF_m) - (gF_m) * (gF_m)\|_\infty.\end{aligned}$$

Let us consider each of these three terms separately.

We rewrite the first term,

$$\begin{aligned}\|g * g - S_m * S_m\|_\infty &= \|(g + S_m) * (g - S_m)\|_\infty \le \|g + S_m\|_1 \|g - S_m\|_\infty \\ &\le (\|g\|_1 + \|S_m\|_1)\|g - S_m\|_\infty \le (1 + \|S_m\|_\infty)\|g - S_m\|_\infty \\ &\le (2 + \|g\|_\infty)(Cm^{-(2k+2)}),\end{aligned}$$

where in the last inequality we used (4.7) and (4.8). Since $(2 + \|g\|_\infty)C$ is a constant, we can take sufficiently large m that

$$\|g * g - S_m * S_m\|_\infty < \frac{\varepsilon}{24}. \tag{4.11}$$

For the second term,

$$\begin{aligned}\|S_m * S_m - (S_m * S_m)(F_m * F_m)\|_\infty &= \|(S_m * S_m)(1 - F_m * F_m)\|_\infty \\ &\le \|S_m * S_m\|_\infty \|1 - F_m * F_m\|_\infty \\ &\le \|S_m\|_\infty^2 \|1 - F_m * F_m\|_\infty \\ &\le (1 + \|g\|_\infty)^2 \eta,\end{aligned}$$

by (4.8) and (i)$_m$. Recall that η was chosen so that (4.6) holds, that is,

$$\eta < (\|g\|_\infty + \omega_\psi(g * g) + 2)^{-2} \frac{\varepsilon}{24} \le (\|g\|_\infty + 1)^{-2} \frac{\varepsilon}{24},$$

since $\omega_\psi(g * g) \ge 0$. Thus,

$$\|S_m * S_m - (S_m * S_m)(F_m * F_m)\|_\infty < \frac{\varepsilon}{24}. \tag{4.12}$$

For the third term, we proceed as in the first case,

$$\begin{aligned}&\|(S_m F_m) * (S_m F_m) - (g F_m) * (g F_m)\|_\infty \\ &\quad \le \|S_m F_m + g F_m\|_1 \|S_m F_m - g F_m\|_\infty \\ &\quad \le \|S_m + g\|_\infty \|F_m\|_1 \|S_m - g\|_\infty \|F_m\|_\infty \\ &\quad \le (1 + 2\|g\|_\infty) C m^{-(2k+2)} (4^k m^k) \\ &\quad = 4^k (1 + 2\|g\|_\infty) C m^{-(k+2)},\end{aligned}$$

where we have used (iii)$_m$, (v)$_m$, (4.7), and (4.8) in the last inequality. Since $4^k(1+2\|g\|_\infty)C$ is a constant, we can take sufficiently large m that

$$\|(S_m F_m) * (S_m F_m) - (g F_m) * (g F_m)\|_\infty < \frac{\varepsilon}{24}. \tag{4.13}$$

Equations (4.11), (4.12) and (4.13) together give

$$\|g * g - G_m * G_m\|_\infty < \frac{\varepsilon}{8}.$$

∎

Claim 4.15.

$$\omega_\psi(g * g - G_m * G_m) < \frac{\varepsilon}{8}.$$

Proof. This claim proceeds much like Claim 4.14. By Λ_ψ-addition, ω_ψ satisfies the triangle inequality:

$$\begin{aligned}\omega_\psi(g*g-G_m*G_m) &= \omega_\psi(g*g-(gF_m)*(gF_m)) \\ &\leq \omega_\psi(g*g-S_m*S_m)+\omega_\psi(S_m*S_m-(S_mF_m)*(S_mF_m)) \\ &\quad +\omega_\psi((S_mF_m)*(S_mF_m)-(gF_m)*(gF_m)) \\ &= \omega_\psi(g*g-S_m*S_m)+\omega_\psi(S_m*S_m-(S_m*S_m)(F_m*F_m)) \\ &\quad +\omega_\psi((S_mF_m)*(S_mF_m)-(gF_m)*(gF_m)),\end{aligned}$$

using (4.10) for the last equality. We consider each of the three terms separately.

For differentiable functions F and G we have

$$\begin{aligned}\omega_\psi(F*F-G*G) &= \omega_\psi((F+G)*(F-G)) \leq \|((F+G)*(F-G))'\|_\infty \\ &= \|(F+G)*(F-G)'\|_\infty \leq \|F+G\|_1\|(F-G)'\|_\infty \\ &\leq (\|F\|_1+\|G\|_1)\|(F-G)'\|_\infty. \end{aligned} \tag{4.14}$$

For the first term, we use (4.14) to get

$$\omega_\psi(g*g-S_m*S_m) \leq (\|g\|_1+\|S_m\|_1)\|(g-S_m)'\|_\infty \leq (2+\|g\|_\infty)2\pi C m^{-(2k+2)}$$

by (4.8) and (4.9). Since $(2+\|g\|_\infty)2\pi C$ is a constant, we can take sufficiently large m that

$$\omega_\psi(g*g-S_m*S_m) < \frac{\varepsilon}{24}. \tag{4.15}$$

For the second term,

$$\begin{aligned}&\omega_\psi(S_m*S_m-(S_m*S_m)(F_m*F_m)) \\ &\quad = \omega_\psi((S_m*S_m)(1-F_m*F_m)) \\ &\quad \leq \omega_\psi(S_m*S_m)\|1-F_m*F_m\|_\infty+\omega_\psi(1-F_m*F_m)\|S_m*S_m\|_\infty \\ &\quad \leq \omega_\psi(S_m*S_m)\|1-F_m*F_m\|_\infty+\omega_\psi(F_m*F_m)\|S_m*S_m\|_\infty \\ &\quad \leq \omega_\psi(S_m*S_m)\eta+\eta\|S_m*S_m\|_\infty,\end{aligned}$$

where the first inequality follows by Λ_ψ-multiplication, the second follows by Λ_ψ-translation invariance, and the final inequality is a consequence of $(\mathrm{i})_m$ and $(\mathrm{viii})_m$. To handle this expression, the proof of (4.15) gives

$$\omega_\psi(S_m * S_m) \leq \omega_\psi(g * g) + 1$$

and the proof of (4.11) gives

$$\|S_m * S_m\|_\infty \leq \|g\|_\infty^2 + 1.$$

Thus,

$$\omega_\psi(S_m * S_m - (S_m * S_m)(F_m * F_m)) \leq (\omega_\psi(g * g) + \|g\|_\infty^2 + 2)\eta < \frac{\varepsilon}{24}, \tag{4.16}$$

by our choice of η to satisfy (4.6).

For the third term, we use (4.14) to get

$$\begin{aligned}
&\omega_\psi((S_m F_m) * (S_m F_m) - (gF_m) * (gF_m)) \\
&\quad \leq (\|S_m F_m\|_1 + \|gF_m\|_1)\|(S_m F_m - gF_m)'\|_\infty \\
&\quad \leq (\|S_m\|_\infty \|F_m\|_1 + \|g\|_\infty \|F_m\|_1)\|((S_m - g)F_m)'\|_\infty \\
&\quad = (1 + 2\|g\|_\infty)\|((S_m - g)F_m)'\|_\infty,
\end{aligned}$$

by (4.8) and $(\mathrm{v})_m$. Now by the product rule,

$$\begin{aligned}
\|((S_m - g)F_m)'\|_\infty &\leq \|(S_m - g)'F_m\|_\infty + \|(S_m - g)F_m'\|_\infty \\
&\leq \|(S_m - g)'\|_\infty \|F_m\|_\infty + \|S_m - g\|_\infty \|F_m'\|_\infty \\
&\leq 2\pi C m^{-(2k+2)} 4^k m^k + C m^{-(2k+2)} 4^{2k+1} m^{2k+1} \\
&\leq C m^{-(2k+2)} 4^{2k+1} m^{2k+1} + C m^{-(2k+2)} 4^{2k+1} m^{2k+1} \\
&= 2 \cdot 4^{2k+1} C m^{-1},
\end{aligned}$$

where we have used (4.7), (4.9), $(\mathrm{iii})_m$, and $(\mathrm{iv})_m$ in the third inequality. Thus,

$$\omega_\psi((S_m F_m) * (S_m F_m) - (gF_m) * (gF_m)) \leq 2(1 + 2\|g\|_\infty) 4^{2k+1} C m^{-1}.$$

Then we can take sufficiently large m that

$$\omega_\psi((S_m F_m) * (S_m F_m) - (gF_m) * (gF_m)) < \frac{\varepsilon}{24}. \tag{4.17}$$

Equations (4.15), (4.16) and (4.17) together give

$$\omega_\psi(g * g - G_m * G_m) < \frac{\varepsilon}{8}.$$

■

Combining (4.4) with Claims 4.10 through 4.15, we get

$$\begin{aligned} &d_\psi((E, f\lambda), (H, g\lambda)) \\ &\quad = d_{\mathcal{F}}(E, H) + \sup_{r\in\mathbb{Z}} |\widehat{f}(r) - \widehat{g}(r)| + \|f * f - g * g\|_\infty + \omega_\psi(f * f - g * g) \\ &\quad < \frac{\varepsilon}{4} + \frac{\varepsilon}{8} + \frac{\varepsilon}{8} + \frac{\varepsilon}{8} + \frac{\varepsilon}{8} + \frac{\varepsilon}{8} + \frac{\varepsilon}{8} = \varepsilon, \end{aligned}$$

as required. ■

Lemma 4.16. *Let $1 > \alpha > 1/2$ and suppose that ψ is a generalized α-power function. Then $\mathcal{H}_n$ is dense in $(\mathcal{M}_\psi, d_\psi)$ for every n.*

Proof. Fix $\varepsilon > 0$ and let $(E, \mu) \in \mathcal{M}_\psi$. By definition of $\mathcal{M}_\psi$, we can find an infinitely differentiable function $g : \mathbb{T} \to \mathbb{R}$ and a closed set H such that $(H, g\lambda) \in \mathcal{M}_\psi$ and

$$d_\psi((E, \mu), (H, g\lambda)) < \frac{\varepsilon}{2}. \tag{4.18}$$

Notice $\int_{\mathbb{T}} g(t)\, dt = 1$ since $g\lambda$ is a probability measure. By Lemma 4.9, there exists a positive, infinitely differentiable function $f : \mathbb{T} \to \mathbb{R}$ such that $\int_{\mathbb{T}} f(t)\, dt = 1$ and a closed set $F \supseteq \operatorname{supp} f$ such that $(F, f\lambda) \in \mathcal{H}_n$ and

$$d_\psi((F, f\lambda), (H, g\lambda)) < \frac{\varepsilon}{2}. \tag{4.19}$$

Combining (4.18) and (4.19) yields the desired result. ■

Theorem 4.17. *Let $1 > \alpha > 1/2$ and suppose that ψ is a generalized α-power function. The complement of the set*

$$\mathcal{H} = \{(E, \mu) \in \mathcal{M}_\psi : \ \dim_H(E) \leq \alpha\}$$

is of first category in $(\mathcal{M}_\psi, d_\psi)$. In particular, $\mathcal{H}$ is dense in $(\mathcal{M}_\psi, d_\psi)$.

Proof. We claim $\mathcal{H}_n$ is open in $(\mathcal{M}_\psi, d_\psi)$. Suppose $(E, \mu) \in \mathcal{H}_n$. By definition of $\mathcal{H}_n$, we can find a finite collection of closed intervals $\mathcal{I}$ with

$$\bigcup_{I \in \mathcal{I}} I \supseteq E \text{ and } \sum_{I \in \mathcal{I}} |I|^{\alpha+1/n} < \frac{1}{n}.$$

Since $\mathcal{I}$ is finite, we can find an $\eta > 0$ such that

$$\sum_{I \in \mathcal{I}} (|I| + 2\eta)^{\alpha+1/n} < \frac{1}{n}.$$

Define $\widetilde{\mathcal{I}} = \{[a - \eta, b + \eta] : \ [a, b] \in \mathcal{I}\}$. If $(F, \sigma) \in \mathcal{M}_\psi$ with

$$d_\psi((E, \mu), (F, \sigma)) < \eta,$$

then automatically,

$$\bigcup_{I \in \widetilde{\mathcal{I}}} I \supseteq F \text{ and } \sum_{I \in \widetilde{\mathcal{I}}} |I|^{\alpha+1/n} < \frac{1}{n},$$

and so $(F, \sigma) \in \mathcal{H}_n$. Thus $\mathcal{H}_n$ is open in $(\mathcal{M}_\psi, d_\psi)$.

Lemma 4.16 tells us that $\mathcal{H}_n$ is dense in $(\mathcal{M}_\psi, d_\psi)$, so it follows that the complement of $\mathcal{H}_n$ is nowhere dense, being closed. Thus, the complement of $\bigcap_{n=1}^\infty \mathcal{H}_n$ is of first category in $(\mathcal{M}_\psi, d_\psi)$.

Suppose $(E, \mu) \in \bigcap_{n=1}^\infty \mathcal{H}_n$, then for each n we can find a finite collection $\mathcal{I}$ of closed intervals such that

$$\bigcup_{I \in \mathcal{I}} I \supseteq E \text{ and } \sum_{I \in \mathcal{I}} |I|^{\alpha+1/n} < \frac{1}{n}.$$

Assume $\dim_H(E) > \alpha$. Then there is an integer N such that $\dim_H(E) > \alpha + 1/N$. Fix $\delta > 0$ and take $n \geq N$ sufficiently large that $\alpha + 1/n < 1$ and $1/n < \delta$. Then

$$|I| \leq |I|^{\alpha+1/n} \leq \sum_{I \in \mathcal{I}} |I|^{\alpha+1/n} < \frac{1}{n} < \delta,$$

and $\bigcup_{I\in\mathcal{I}} I \supseteq E$. In particular, this gives a uniform bound

$$\mathcal{H}_\delta^{\alpha+1/N}(E) \leq \mathcal{H}_\delta^{\alpha+1/n}(E) \leq \sum_{I\in\mathcal{I}} |I|^{\alpha+1/n} < \frac{1}{n} \leq 1 < \infty,$$

for every $\delta > 0$. Taking the supremum over $\delta > 0$, we get that

$$\mathcal{H}^{\alpha+1/N}(E) \leq 1 < \infty,$$

and so $\dim_H(E) \leq \alpha + 1/N$, a contradiction. Therefore, $\dim_H(E) \leq \alpha$, and so $(E, \mu) \in \mathcal{H}$. This gives $\mathcal{H}^c \subseteq \left(\bigcap_{n=1}^\infty \mathcal{H}_n\right)^c$. Since any subset of a set of first category is of first category, this completes the first part of the proof. We will suppose $\mathcal{H}^c = \bigcup_{n=1}^\infty E_n$ for some nowhere dense sets E_n.

By Lemma 4.8, $(\mathcal{M}_\psi, d_\psi)$ is a complete metric space, so the Baire Category theorem tells us that this space is *Baire*, that is, the countable intersection of open dense sets is dense. The interior of the complement of a nowhere dense set is dense, so that $\bigcap_{n=1}^\infty \operatorname{int}(E_n^c)$ is dense as $\mathcal{M}_\psi$ is Baire. Then $\mathcal{H}$ contains a dense set, and thus is itself dense in $(\mathcal{M}_\psi, d_\psi)$. ■

We finally arrive at a somewhat generalized form of our main theorem:

Theorem 4.18. *Let $1 > \alpha > 1/2$ and ψ be a generalized α-power function. Then there exists a probability measure μ such that $\dim_H(\operatorname{supp}\mu) \leq \alpha$ and $\mu * \mu = f\lambda$ with $f \in \Lambda_\psi$.*

Proof. By Theorem 4.17, $\mathcal{H}$ is non-empty. Thus we can find $(E, \mu) \in \mathcal{M}_\psi$ with $\dim_H(E) \leq \alpha$, and so $\dim_H(\operatorname{supp}\mu) \leq \alpha$. By definition of $\mathcal{M}_\psi$, we also have that μ is a probability measure with $\mu * \mu = f\lambda$ with $f \in \Lambda_\psi$. ■

Theorem 4.19. *If $1 > \alpha > 1/2$, then there exists a probability measure μ such that $\dim_H(\operatorname{supp}\mu) = \alpha$ and $\mu * \mu = f\lambda$ where $f \in \Lambda_{\alpha-1/2}$.*

Proof. Taking $\psi(t) = t^{\alpha-1/2}$, Theorem 4.18 gives us a probability measure with $\dim_H(\operatorname{supp}\mu) \leq \alpha$ and $\mu * \mu = f\lambda$ with $f \in \Lambda_\psi = \Lambda_{\alpha-1/2}$. By Lemma 1.11,

$$\dim_H(\operatorname{supp}\mu) - \frac{1}{2} \geq \alpha - \frac{1}{2},$$

and hence $\dim_H(\operatorname{supp}\mu) = \alpha$. ■

APPENDICES

Appendix A

Measure Theory

Some of these results are applicable in the more general case of σ-finite measures, but for our purposes, we will only require finite measures. As such, we will make the assumption that all measures are finite. Measures will also be taken to be Borel and regular.

Definition A.1. Suppose (Ω, Σ) is a measure space and μ and ν are complex measures on (Ω, Σ). μ is said to be *absolutely continuous with respect* ν if whenever $\nu(E) = 0$ for a set $E \in \Sigma$, then we have $\mu(E) = 0$. In this case, we typically write $\mu \ll \nu$. If Ω is a topological group and ν is the Haar measure, we typically say μ is *absolutely continuous.*

Absolutely continuous measures are nice, due to their correspondence with L^1 functions.

Theorem A.2 [Radon-Nikodym theorem]**.** *Suppose* (Ω, Σ) *is a measure space, and* μ, ν *are measures on* (Ω, Σ)*. Then* $\mu \ll \nu$ *if and only if there exists a function* $f \in L^1(\mu)$ *such that*

$$\nu(E) = \int_E f \, d\mu$$

for any measurable set $E \in \Sigma$*.* f *is referred to as the* Radon-Nikodym derivative*.*

Such a correspondence allows us to treat absolutely continuous measures as functions, by taking their Radon-Nikodym derivatives.

Definition A.3. Suppose (Ω, Σ) is a measure space. A measure μ on (Ω, Σ) is said to be *singular* with respect to a measure ν on (Ω, Σ) if there exist two disjoint sets $E, F \in \Sigma$ with $E \cup F = \Omega$ such that μ is zero on every subset of E and ν is zero on every subset of F. In this case, we typically write $\mu \perp \nu$. As this definition is symmetric in μ and ν, we may sometimes simply say μ and ν are singular. Notice that in the case of positive measures, it suffices to have the condition $\mu(E) = 0 = \nu(F)$. If Ω is a topological group and ν is the Haar measure, we typically say μ is *singular*.

Theorem A.4 [Lebesgue's Decomposition theorem]**.** *Suppose (Ω, Σ) is a measure space, and μ, ν are measures on (Ω, Σ). Then there exist two measures μ_{ac} and μ_s such that*

(i) $\mu = \mu_{ac} + \mu_s$,

(ii) $\mu_{ac} \ll \nu$, *and*

(iii) $\mu_s \perp \nu$.

Furthermore, these two measures are uniquely determined.

Both Theorems A.2 and A.4 are standard results that can be found in any good measure theory (see, for example, [13] Theorem 6.9).

In the special case of ν being Lebesgue measure, Lebesgue's Decomposition theorem may be further extended to include the concepts of continuous measures and discrete measures.

Definition A.5. Suppose (Ω, Σ) is a measure space. A measure μ on (Ω, Σ) is said to be *continuous* if $\mu(\{x\}) = 0$ for every $x \in \Omega$. A measure μ is said to be *discrete* if it is concentrated on a countable set, that is, if it can be expressed as

$$\mu = \sum_{n=1}^{\infty} a_n \delta_{x_n},$$

where a_n are constants, $x_n \in \Omega$, and δ_x is the Dirac measure defined by

$$\delta_x(E) = \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \notin E. \end{cases}$$

Recall that we assume measures are finite, so this summation is finite.

The next lemma shows another decomposition of the space of measures, this time using continuous and discrete measures.

Lemma A.6. *Suppose (Ω, Σ) is a measure space, and μ is a measure on (Ω, Σ). Then there exist two measures μ_c and μ_d such that*

(i) $\mu = \mu_c + \mu_d$,

(ii) *μ_c is continuous, and*

(iii) *μ_d is discrete.*

Furthermore, these two measures are uniquely determined.

Proof. Let $E = \{x : \mu(\{x\}) \neq 0\}$. For each finite subset $F \subseteq E$, $\sum_{x \in F} |\mu(\{x\})| \leq \|\mu\|_{M(G)} < \infty$. Taking the supremum over all possible finite subsets $F \subseteq F$, we get $\sum_{x \in E} |\mu(\{x\})| \leq \|\mu\|_{M(G)} < \infty$. Thus for every $n \in \mathbb{N}$, there can only be finitely many $x \in E$ such that $\mu(\{x\}) \geq 1/n$, and so E must be countable, say $E = \{x_1, x_2, \ldots\}$. Define the discrete measure $\mu_d = \sum_{n=1}^{\infty} \mu(\{x_n\})\delta_{x_n}$, and define $\mu_c = \mu - \mu_d$. Then for every $x \in \Omega$,

$$\mu_c(\{x\}) = \begin{cases} 0 & \text{if } x \notin E, \\ \mu(\{x_n\}) - \mu_d(\{x_n\}) = 0 & \text{if } x = x_n \in E, \end{cases}$$

so that μ_c is continuous.

For uniqueness, suppose μ_c' is a continuous measure and μ_d' is a discrete measure, with $\mu = \mu_c' + \mu_d'$. Then $\mu_c + \mu_d = \mu_c' + \mu_d'$ yields $\mu_c - \mu_c' = \mu_d' - \mu_d$. It is clear from the definitions that both the continuous measures and discrete measures are closed under linear combinations, so we have the continuous measure $\mu_c - \mu_c'$ equal to the discrete measure $\mu_d' - \mu_d = \sum_n a_n \delta_{y_n}$. Since this is continuous, for each n the discrete measure must assign a weight of 0 to the point y_n, which implies $a_n = 0$. Therefore $\mu_c = \mu_c'$ and $\mu_d = \mu_d'$, so the decomposition is unique. ∎

It is clear from the definition of the support of a measure and a discrete measure that if $\mu_d = \sum_{n=1}^{\infty} a_n \delta_{x_n}$ is a discrete measure, then

$$\operatorname{supp} \mu_d = \{x_n : \, a_n \neq 0\}.$$

In particular, μ_d is supported on a countable set, and hence is easily seen to be singular with respect to Lebesgue measure. Then by Lemma A.6, the singular measures can be decomposed into a discrete part and a part that is both singular and continuous, which we call *continuous singular*. This proves the following extension of Lebesgue's Decomposition theorem:

Theorem A.7. *Suppose μ is a σ-finite measure on $\mathbb{R}^n$ or $\mathbb{T}^n$. Then there exist three σ-finite measures μ_{ac}, μ_{cs}, and μ_d such that*

(i) $\mu = \mu_{ac} + \mu_{cs} + \mu_d$,

(ii) μ_{ac} *is absolutely continuous,*

(iii) μ_{cs} *is continuous singular, and*

(iv) μ_d *is discrete.*

Furthermore, these three measures are uniquely determined.

Appendix B

Convergence of Fourier Series

Definition B.1. Suppose we have two functions, $f(x)$ and $g(x)$. We write

$$f(x) = o(g(x))$$

as $x \to 0$ if and only if $f(x)/g(x) \to 0$ as $x \to 0$. For brevity, we typically omit the condition $x \to 0$ and simply write $f(x) = o(g(x))$.

Theorem B.2 [Dini-Lipschitz test]. *As a function of h, if*

$$f(x+h) - f(x) = o\left((\ln |h|^{-1})^{-1}\right)$$

uniformly in x on $\mathbb{T}$, then the Fourier series of f converges uniformly on $\mathbb{T}$.

For a proof and more detailed discussion, refer to [1] Chapter IV §4.

Appendix C

Probability Theory

Theorem C.1 [Markov's inequality]**.** *If X is a non-negative random variable, then for all $\alpha > 0$,*

$$\mathrm{P}(X \geq \alpha) \leq \mathrm{E}(X)\alpha^{-1}.$$

Proof. Define a random variable Y by

$$Y(x) = \begin{cases} \alpha & \text{if } X(x) \geq \alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $Y \leq X$, so that $\mathrm{E}(Y) \leq \mathrm{E}(X)$ by monotonicity. On the other hand, we have by direct computation that $\mathrm{E}(Y) = \alpha \mathrm{P}(X \geq \alpha)$. The result follows immediately. ∎

Appendix D

The Hausdorff Metric

Definition D.1. Suppose (X, d) is a metric space, and let $\mathcal{F}$ be the family of all closed and bounded subsets of X. For two sets $E, F \in \mathcal{F}$, define

$$d_{\mathcal{F}}(E, F) = \max \left\{ \sup_{e \in E} \inf_{f \in F} d(e, f), \sup_{f \in F} \inf_{e \in E} d(e, f) \right\}.$$

We call $d_{\mathcal{F}}$ the *Hausdorff metric.*

It is worth noting that there is a natural reason to restrict to sets belonging to $\mathcal{F}$. Indeed, if we allow arbitrary sets to be measured in this way, then we shall see $d_{\mathcal{F}}(E, F) = 0$ if and only if E and F have the same closure, which does not necessarily mean $E = F$, violating the definition for a metric. As well, allowing unbounded sets will allow $d_{\mathcal{F}}$ to take on the value ∞, which is undesirable. We now show that the term "metric" is used here in a natural way, that is, we do in fact have a metric.

Theorem D.2. *$(\mathcal{F}, d_{\mathcal{F}})$ is a metric space.*

Proof. Clearly if $E, F \in \mathcal{F}$ with $E = F$, then $d_{\mathcal{F}}(E, F) = 0$. Conversely, if $d_{\mathcal{F}}(E, F) = 0$ then

$$\sup_{e \in E} \inf_{f \in F} d(e, f) = 0 = \sup_{f \in F} \inf_{e \in E} d(e, f).$$

Since these infima are non-negative, we get

$$\inf_{f\in F} d(e,f) = 0 = \inf_{e\in E} d(e,f).$$

As E and F are closed, this implies $E = F$.

Trivially, $d_{\mathcal{F}}(E,F) = d_{\mathcal{F}}(F,E)$ for any $E, F \in \mathcal{F}$.

Finally, suppose $E, F, G \in \mathcal{F}$ and choose $e_0 \in E$, $f_0 \in F$. By triangle inequality,

$$\begin{aligned}\inf_{g\in G} d(e_0,g) &\leq \inf_{g\in G}(d(e_0,f_0) + d(f_0,g)) \\ &= d(e_0,f_0) + \inf_{g\in G} d(f_0,g) \\ &\leq d(e_0,f_0) + \sup_{f\in F}\inf_{g\in G} d(f,g).\end{aligned}$$

Since this holds for any $f_0 \in F$, it holds in the infimum,

$$\begin{aligned}\inf_{g\in G} d(e_0,g) &\leq \inf_{f\in F} d(e_0,f) + \sup_{f\in F}\inf_{g\in G} d(f,g) \\ &\leq \sup_{e\in E}\inf_{f\in F} d(e,f) + \sup_{f\in F}\inf_{g\in G} d(f,g),\end{aligned}$$

so that taking supremum,

$$\sup_{e\in E}\inf_{g\in G} d(e,g) \leq \sup_{e\in E}\inf_{f\in F} d(e,f) + \sup_{f\in F}\inf_{g\in G} d(f,g).$$

By symmetry, we get

$$\sup_{g\in G}\inf_{e\in E} d(g,e) \leq \sup_{g\in G}\inf_{f\in F} d(g,f) + \sup_{f\in F}\inf_{e\in E} d(f,e).$$

Combining these equations together gives

$$d_{\mathcal{F}}(E,G) \leq d_{\mathcal{F}}(E,F) + d_{\mathcal{F}}(F,G).$$

Thus, $(\mathcal{F}, d_{\mathcal{F}})$ is indeed a metric space. ∎

A nice property of the Hausdorff metric is that it inherits completeness from the underlying metric.

Theorem D.3. *If (X, d) is a complete metric space, then $(\mathcal{F}, d_{\mathcal{F}})$ is a complete metric space.*

Proof. Theorem D.2 ensures $(\mathcal{F}, d_{\mathcal{F}})$ is a metric space. To see completeness, let $E_1, E_2, \ldots$ be a Cauchy sequence in $(\mathcal{F}, d_{\mathcal{F}})$. By taking a subsequence if necessary, we may assume $d_{\mathcal{F}}(E_n, E_{n+1}) < 2^{-n-1}$. Define

$$E = \{e = \lim_{n\to\infty} e_n : \ e_n \in E_n \text{ and } d(e_n, e_{n+1}) < 2^{-n}\}.$$

To see E is non-empty, we will inductively create a sequence satisfying the necessary property for its limit to lie in E. Since $E_1 \in \mathcal{F}$, it is non-empty, thus we can choose $e_1 \in E_1$. Inductively, having chosen $e_n \in E_n$, our assumption gives

$$\begin{aligned}\inf_{f\in E_{n+1}} d(e_n, f) &\leq \max\left\{\sup_{g\in E_n}\inf_{f\in E_{n+1}} d(g,f),\ \sup_{f\in E_{n+1}}\inf_{g\in E_n} d(g,f)\right\}\\ &= d_{\mathcal{F}}(E_n, E_{n+1}) < 2^{-n-1}.\end{aligned}$$

Since E_{n+1} is closed, the infimum is attained and so there exists $e_{n+1} \in E_{n+1}$ such that

$$d(e_n, e_{n+1}) = \inf_{f\in E_{n+1}} d(e_n, f) < 2^{-n-1}.$$

Thus $\{e_n\}$ forms a Cauchy sequence. Since X is complete with respect to d, e_n converges to some $e \in X$. That is, $e \in E$, so E is non-empty. Notice that by triangle inequality, we also get that for this e,

$$d(e_n, e) < 2^{-n+1}.$$

Let $\varepsilon > 0$. Let N be sufficiently large that $2^{-N+1} < \varepsilon$. Suppose $n \geq N$ and take $e_n \in E_n$. By the construction above, we can get $e \in E$ with $d(e_n, e) < 2^{-n+1}$. Then certainly

$$\inf_{f\in E} d(e_n, f) < 2^{-n+1}.$$

Since this holds for any $e_n \in E_n$,

$$\sup_{g\in E_n}\inf_{f\in E} d(g,f) \leq 2^{-n+1}.$$

Conversely, take $e \in E$. By definition, there is a sequence e_n with $e = \lim_{n\to\infty} e_n$, $e_n \in E_n$, and $d(e_n, e_{n+1}) < 2^{-n}$. Then by triangle inequality, $d(e_n, e) < 2^{-n+1}$, and so certainly

$$\inf_{g\in E_n} d(g,e) < 2^{-n+1}.$$

Since this holds for any $e \in E$,

$$\sup_{f\in E} \inf_{g\in E_n} d(g,f) \leq 2^{-n+1}.$$

The suprema and infima are unchanged if we replace E with its closure $\overline{E}$. Then together, these give

$$d_{\mathcal{F}}(E_n, \overline{E}) = \max\left\{ \sup_{g\in E_n} \inf_{f\in \overline{E}} d(g,f), \sup_{f\in \overline{E}} \inf_{g\in E_n} d(g,f) \right\} \leq 2^{-n+1} \leq 2^{-N+1} < \varepsilon.$$

Hence, E_n converges to $\overline{E} \in \mathcal{F}$ in the $d_{\mathcal{F}}$ metric, that is, $(\mathcal{F}, d_{\mathcal{F}})$ is a complete metric space. ∎

Two sets are close in the Hausdorff metric if they lie in a "thickened" version of the other set. To be precise, let (X, d) be a metric space. For a subset $E \subseteq X$ and $\varepsilon > 0$, define

$$E_\varepsilon = \bigcup_{e\in E} \{f \in X : d(e,f) \leq \varepsilon\}.$$

Then we have the following:

Lemma D.4. *If $d_{\mathcal{F}}(E,F) < \varepsilon$, then $E \subseteq F_\varepsilon$ and $F \subseteq E_\varepsilon$.*

Proof. Suppose $d_{\mathcal{F}}(E,F) < \varepsilon$. Then by definition, we have in particular that

$$\sup_{e\in E} \inf_{f\in F} d(e,f) < \varepsilon.$$

Fix $e_0 \in E$; then

$$\inf_{f\in F} d(e_0, f) \leq \sup_{e\in E} \inf_{f\in F} d(e,f) < \varepsilon.$$

By definition of infimum, there exists $f_0 \in F$ with $d(e_0, f_0) \leq \varepsilon$, so that $e_0 \in F_\varepsilon$. Hence $E \subseteq F_\varepsilon$, and the second identity follows by symmetry. ∎

It may be worth noting that the converse almost holds: a quick proof of the nature above shows that if $E \subseteq F_\varepsilon$ and $F \subseteq E_\varepsilon$, then $d_{\mathcal{F}}(E, F) < 2\varepsilon$. This provides a useful alternative viewpoint to sets converging in the Hausdorff metric, by examining containments in thickened sets.

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