Quantum Walks on Strongly Regular Graphs

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis.

This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

This thesis studies the transition matrix of a quantum walk on strongly regular graphs. It is proposed by Emms, Hancock, Severini and Wilson in 2006, that the spectrum of $S^+(U^3)$, a matrix based on the amplitudes of walks in the quantum walk, distinguishes strongly regular graphs.

We begin by finding the eigenvalues of $S^+(U)$ and $S^+(U^2)$ for regular graphs. We also show that if two graphs G and H are isomorphic, then the corresponding matrices $S^+(U^3)$ are cospectral. We then look at the entries of the cube of the transition matrix and find an expression for $S^+(U^3)$ in terms of the adjacency matrix and incidence matrices of the graph.

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Dedication

To my parents, Guang and Yuping

Contents

List of Figures

List of Tables

Introduction

A graph G and a graph H are *isomorphic* if there is a bijective mapping of the vertices of G onto the vertices of H which preserves adjacency. Isomorphic graphs are usually treated as the same graph and the set of graphs isomorphic to a given graph is called the *isomorphism class*. The *Graph Isomorphism Problem* is the problem of deciding whether or not two given graphs G and H belong to the same isomorphism class.

The *adjacency matrix* of a graph G is a matrix $A(G)$ whose rows and columns are indexed by the vertices of G and

$$(A(G))_{i,j} = \begin{cases} 1 & \text{if } i, j \text{ are adjacent,} \\ 0 & \text{otherwise.} \end{cases}$$

The *spectrum of a graph* is the set of the eigenvalues of its adjacency matrix. The following theorem is well-known and found in [9, p.164].

0.0.1 Theorem. *If two graphs G and H are isomorphic, then there exists a permutation matrix P such that*

$$A(G) = PA(H)P^{-1}.$$

From this, we can see that if two graphs are isomorphic, then they are *cospectral*, which is to say that their adjacency matrices have the same spectrum. However, it is not true that if two graphs are cospectral then they are isomorphic. Strongly regular graphs are a class of graphs for which any two strongly regular graphs with the same parameters are cospectral.

A graph is said to be *regular* with valency k if each vertex has k neighbours. A graph G on n vertices is said to be *strongly regular* with parameters (n, k, a, c) if G is regular with valency k, every pair of adjacent vertices of G have a common neighbours and every pair of non-adjacent vertices of G have c common neighbours, where a and c are constants and $0 < k < n - 1$.

With regard to complexity, the Graph Isomorphism Problem is NP, but is not known to be NP-complete; Schöning shows in [12] that the Graph Isomorphism Problem is not NP-complete unless the polynomial-time hierarchy collapses. There is also no known polynomial-time algorithm for Graph Isomorphism. This implies that Graph Isomorphism lies in the class of problems, along with factoring, where it is speculated that quantum algorithms can exceed classical algorithms in efficiency.

In [6, 5], Emms, Severini, Wilson and Hancock propose that the quantum walk transition matrix can be used to distinguish between non-isomorphic graphs. Given a matrix M, the *positive support* of M, denoted $S^+(M)$, is the matrix obtained from M as follows:

$$(S^+(M))_{i,j} = \begin{cases} 1 & \text{if } M_{i,j} > 0 \\ 0 & \text{otherwise.} \end{cases}$$

They conjecture that the spectrum of $S^+(U^3)$ is a graph invariant for strongly regular graphs, where U is the transition matrix of a quantum walk.

Conjecture. *[5] If G and H are strongly regular graphs, then $S^+(U(G)^3)$ and $S^+(U(H)^3)$ are cospectral if and only if G and H are isomorphic.*

Through experimental results, they see that this proposed invariant distinguishes many more graphs than does the spectrum of the adjacency matrix. In particular, no strongly regular graph is known to have a cospectral mate with respect to this invariant. In particular, the proposed procedure would imply a polynomial-time classical algorithm for testing graph isomorphism on strongly regular graphs.

In this thesis, we study the properties of $S^+(U^3)$. We show that $S^+(U(G)^3)$ and $S^+(U(H)^3)$ are cospectral if G and H are isomorphic. In [6, 5], the authors find the eigenvalues of $S^+(U)$ and $S^+(U^2)$ for strongly regular graphs. However their proof is incomplete; in particular, they mention that they are not able to fully specify the multiplicities of ± 1 as eigenvalues of $S^+(U)$ and the multiplicity of 2 as an eigenvalue of $S^+(U^2)$. Using the methods of [8], we find the eigenvalues of $S^+(U)$ for any regular graph of valency k where $k \geq 2$, and offer a complete proof in Chapter 2. We also resolve the eigenvalues of $S^+(U^2)$ by showing that

$$S^+(U^2) = S^+(U)^2 + I$$

for any regular graph of valency k, where $k \geq 2$.

In [5], the authors find the value of $(U^3)_{i,j}$ given arcs i and j through a case analysis for a strongly regular graph. Here, we present a more thorough case analysis and, in addition, we show that, under some restriction of the parameters of the strongly regular graph, for each case of the case analysis, there always exists a pair of arcs i and j belonging to the case. Thus, we show, in Chapter 3, that if G is a primitive strongly regular graph G with parameters (n, k, a, c), if $a \geq 2$ and $c \geq 2$, then the entries $U(G)^3$ are determined by (n, k, a, c).

The authors of [5, 6] suggest that $S^+(U^3)$ is a good candidate for a graph invariant because it seems that its eigenvalues do not depend on the eigenvalues of the adjacency matrix and it is difficult to write an expression for the eigenvalues of $S^+(U^3)$. In Chapter 4 We find $S^+(U^3)$ as a sum of matrices that are products of the adjacency matrix and other incidence matrices of G, in order to better consider the eigenvalues. Let D be the digraph of G and consider the following incidence matrices of D, both with rows indexed by the vertices of D and columns indexed by the arcs of D:

$$(D_h)_{i,j} = \begin{cases} 1 & \text{if } i \text{ is the head of arc } j \\ 0 & \text{otherwise} \end{cases}$$

and

$$(D_t)_{i,j} = \begin{cases} 1 & \text{if } i \text{ is the tail of arc } j \\ 0 & \text{otherwise.} \end{cases}$$

Following the case analysis to find the entries of U^3, we show that for a strongly regular graph G with parameters (n, k, a, c), if $a \geq 1$ and $c \geq 1$, $a \leq \frac{k}{2}$ and $c \leq \frac{k}{2}$, then

$$S^+(U^3) = J - D_t^T A D_t - D_h^T A D_h + (D_h^T A D_h) \circ (D_t^T A D_t) + D_h^T D_t - P$$

and, if $a > \frac{k}{2}$ and $c > \frac{k}{2}$, then

$$S^+(U^3) = J - (D_h^T A D_h) \circ (D_t^T A D_t) + D_h^T D_t - P.$$

Chapter 1

Strongly Regular Graphs

Recall that an isomorphism from graph G to graph H is an isomorphism from the vertex set of G to the vertex set of H which preserves the adjacency relations of G. It is clear that any isomorphism must take a vertex of G with k neighbours to a vertex of H that has k neighbours. Thus, some graphs can be distinguished by the multiset of the number of neighbours of its vertices. This naive approach fails in general and, in particular, in the case where every vertex of G and H have the same number of neighbours. A finer requirement for G and H to be isomorphic is that an isomorphism must map a pair of vertices u and v of G to a pair of vertices in H with the same number of common neighbours. Here again, we are able to find families of graphs such that this condition is not sufficient. For strongly regular graphs, in particular, every pair of vertices u, v has some constant number of common neighbours, where the constant only depends on the adjacency of u and v.

Graph isomorphism remains an interesting problem for strongly regular graphs because simple necessary conditions for isomorphisms fail to be sufficient. In this chapter, we will present the definitions and well-known facts about strongly regular graphs. Since we are interested in examining the graph isomorphism problem for strongly regular graphs, it is interesting and necessary to examine strongly regular graphs that not only have the same parameter set, but also result from the same construction method. Any two strongly regular graphs generated from generalized quadrangles of the same parameters have many common attributes. For example, the size of the maximum clique and the number of maximum cliques will be the same. In Section 1.2, we will look at constructing strongly regular graphs from generalized quadrangles as well as two constructions of generalized quadrangles.

1.1 Definitions and Basic Concepts

Recall that a graph is said to be *regular* with valency k if each vertex has k neighbours. A graph on n vertices, G, is said to be *strongly regular* with parameters (n, k, a, c) if G is regular with valency k, every pair of adjacent vertices of G have a common neighbours and every pair of non-adjacent vertices of G have c common neighbours, where a and c are constants and $0 < k < n - 1$. Note that the complete graph and its complement are not strongly regular graphs. The following facts about strongly regular graphs are well-known and can be found in any standard text on algebraic graph theory. See Godsil and Royle's *Algebraic Graph Theory* [9].

The complement of a graph G is a graph $\overline{G}$ with vertex set $V(G)$ and vertices u and v are adjacent in $\overline{G}$ if and only if u and v are not adjacent in G. If a graph is strongly regular with parameters (n, k, a, c), then its complement is strongly regular and has parameters

$$(n, n - k - 1, n - 2 - 2k + c, n - 2k + a).$$

If a strongly regular graph X and its complement are both connected, X is said to be *primitive*. Otherwise, X is said to be *imprimitive*. It can be easily seen that an imprimitive strongly regular graph is a disjoint union of complete graphs. The smallest example of a primitive strongly regular graph is the cycle on five vertices.

If G and H are both strongly regular graphs with the same parameter set, then G are H are said to be *coparametric*.

The *adjacency matrix* of a graph G is a matrix $A(G)$ whose rows and columns are indexed by the vertices of G and

$$A_{i,j} = \begin{cases} 1 & i, j \text{ are adjacent,} \\ 0 & \text{otherwise.} \end{cases}$$

We may write A for $A(G)$ if the graph is clear from the context. The spectrum of a matrix is the list of its eigenvalues. The *spectrum of a graph* is the spectrum of its adjacency matrix. If two graph G and H have the same spectrum then they are said to be *cospectral* and H is said to be a *cospectral mate* of G. We will see that, for strongly regular graphs, being coparametric implies being cospectral.

If A is the adjacency matrix of a strongly regular graph G with parameters (n, k, a, c), then A has three distinct eigenvalues k, θ and τ with multiplicities

1, m_θ and m_τ respectively, where

$$\theta = \frac{(a-c) + \sqrt{(a-c)^2 + 4(k-c)}}{2},$$

$$\tau = \frac{(a-c) - \sqrt{(a-c)^2 + 4(k-c)}}{2},$$

$$m_\theta = \frac{1}{2}\left((n-1) - \frac{2k + (n-1)(a-c)}{\sqrt{(a-c)^2 + 4(k-c)}}\right), \text{ and}$$

$$m_\tau = \frac{1}{2}\left((n-1) + \frac{2k + (n-1)(a-c)}{\sqrt{(a-c)^2 + 4(k-c)}}\right).$$

Thus, we see that the spectrum of a strongly regular graph depends only on its parameters; all co-parametric strongly regular graphs have the same spectrum. Further, we have the following lemma, found in [9, p.220], which details one of the fundamental algebraic properties of strongly regular graphs.

1.1.1 Lemma. *A connected, regular graph is strongly regular if and only if it has exactly three distinct eigenvalues.*

1.2 Generalized Quadrangles

Finding strongly regular graphs with a given parameter set is often a non-trivial task. Classes of strongly regular graphs with parameter sets (n, k, a, c) are well-understood when n is sufficiently small. However, the number of strongly regular graphs with a given parameter set is not known in general. The reader is referred to [2] for a list of parameter sets of strongly regular graphs.

Some known constructions for strongly regular graphs arise from combinatorial objects including orthogonal arrays, partial geometries and generalized quadrangles. We will look, in particular, at reducing the problem of generating strongly regular graphs to generating generalized quadrangles. We refer to [10] for the presented information about generalized quadrangles and to [15] for incidence structures.

1.2.1 Strongly Regular Graphs from Generalized Quadrangles

A generalized quadrangle is a special type of incidence structure, a general object commonly used in design theory. An *incidence structure* is a triple $S = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ where:

i) $\mathcal{P}$ is a set, whose elements are called *points*,

ii) $\mathcal{B}$ is a set, whose elements are called *blocks* and

iii) $\mathcal{I}$ is a subset of $\mathcal{P} \times \mathcal{B}$ and is called an *incidence relation*.

For $p \in \mathcal{P}$ and $\ell \in \mathcal{B}$, if $(p, \ell) \in \mathcal{I}$ then the point p and the block ℓ are said to be *incident*.

A *generalized quadrangle* is an incidence structure $S = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ such that:

i) each point is incident with $t + 1$ blocks and any two points are incident with at most one block,

ii) each block is incident with $s + 1$ points and any two blocks are incident with at most one point, and

iii) for every $x \in \mathcal{P}$ and L a block in $\mathcal{B}$ not incident with x, there exists a unique pair of point and block, say (y, M), where x is incident with M, M is incident with y and y is incident with L.

We say that (s, t) is the *order* of S. We often refer to the blocks of a generalized quadrangle and other incidence structures as *lines*. In general, if two points are incident with the same line, we say that they are *collinear*. If two lines are incident with the same point, we say that they are *intersecting lines*.

For a generalized quadrangle $S = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ of order (s, t), consider a line L. Then, L is incident with $s+1$ points. For each point, x, not incident with L, there is a unique line M intersecting L such that x and M are incident. There are $(s+1)t$ lines that intersect L, each containing s points not incident with L. Then, by counting the points incident and not incident to L, we get

$$|\mathcal{P}| = (s + 1) + (s + 1)ts = (s + 1)(st + 1)$$

By symmetry, we see that $|\mathcal{B}| = (t + 1)(st + 1)$.

The following theorem gives the connection between strongly regular graphs and generalized quadrangles and can be found in [10, p.291].

1.2.1 Theorem. *For a generalized quadrangle $S = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ of order (s,t), let G be the graph whose vertices are the points of S and two vertices are adjacent if the points are collinear in S. Then, G is a strongly regular graph with parameters*

$$((s+1)(st+1), s(t+1), s-1, t+1)$$

We may refer to the graph obtained in this way as the *point graph* of the generalized quadrangle. The *dual of a generalized quadrangle* $S = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ of order (s,t) is $T = (\mathcal{B}, \mathcal{P}, \mathcal{I})$. We can see that T has order (t,s).

Let N be the incidence matrix of S; that is, N has rows indexed by the points of S, columns indexed by the lines of S and

$$N_{i,j} = \begin{cases} 1 & \text{if } i,j \text{ are incident,} \\ 0 & \text{otherwise.} \end{cases}$$

We see that N^T is the incidence matrix of the dual of S. Further, we can consider NN^T. The (i,j)th entry of NN^T is the number of lines incident to both points i and j. Since S is a generalized quadrangle, each pair of distinct points is incident with at most one line. Then $(NN^T)_{i,j}$ is either 0 or 1 when $i \neq j$ and $(NN^T)_{i,i}$ is $t+1$, the number of lines incident to a given point. Let $I_{x \times x}$ be the $x \times x$ identity matrix. By the definition of point graph, we see that the adjacency matrix of the point graph of S is

$$NN^T - (t+1)I_{|\mathcal{P}| \times |\mathcal{P}|}$$

and the adjacency matrix of the point graph of the dual of S is

$$N^TN - (s+1)I_{|\mathcal{B}| \times |\mathcal{B}|}.$$

1.2.2 A Construction of Generalized Quadrangles

Theorem 1.2.1 gives the construction of a strongly regular graph from a generalized quadrangle. It would be useful to have a construction of generalized quadrangles. We give two constructions of generalized quadrangles. We will consider strongly regular graphs generated from these constructions in conjunction with quantum walks.

Let V be a vector space of dimension p over $GF(q)$. A *projective space* of projective dimension $p-1$ over $GF(q)$, denoted $PG(p-1,q)$, is an incidence structure where the points are the 1-dimensional subspaces of V, the blocks are the 2-dimensional subspaces of V and the incidence structure is as follows: a point p is incident to a block B if p is contained in B as a subspace. If q is even, an *hyperoval* in $PG(p-1, q)$ is a set of $q+2$ points of which no three points are collinear.

The following construction, the first that we will look at, is found in [10, p. 38].

Consider a projective space $P = PG(3, q)$, for some q, with $H = PG(2, q)$ embedded as a hyperplane of P. Let O be a hyperoval in H. Let $\mathcal{P}$ be the set of points of $P - H$ and $\mathcal{B}$ be the lines of P not contained in H and which meet O. Let $\mathcal{I}$ be the incidence relation on $\mathcal{P}$ and $\mathcal{B}$ inherited from the incidence relation of the points and lines of P. The incidence structure $(\mathcal{P}, \mathcal{B}, \mathcal{I})$ is denoted $T_2^*(O)$ in the literature. The proof the following theorem, which is not difficult, can be found in [10, p. 38].

1.2.2 Theorem. *If* $\{\mathcal{P}, \mathcal{B}, \mathcal{I}\}$ *are defined as above, then* $T_2^*(O) = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ *is a generalized quadrangle of order* $(q-1, q+1)$.

Similar to the isomorphism of graphs, two generalized quadrangles S and T are *isomorphic* if there is a bijective mapping between from the points and blocks of S to those of T which preserves the incidence relation. We have further that we can generate generalized quadrangles of the same parameters by this method using the following theorem, found in [1].

1.2.3 Theorem. *If* O_1 *and* O_2 *are both hyperovals in* $PG(2, q)$*, then* $T_2^*(O_2)$ *is isomorphic to* $T_2^*(O_1)$ *if and only if there is an isomorphism of* $PG(3, q)$ *which takes* O_1 *to* O_2.

The second construction is referred to as $W(q)$ in the literature. We will present the construction without proof, which can be found in [9, p. 83-84]. We consider $PG(3, q)$ and let V be the vector space of dimension 4 over $GF(q)$. The points and lines of $PG(3, q)$ are the 1-dimensional and 2-dimensional subspaces of V, respectively. Let H be as follows:

$$H = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

We say that a subspace S of V, and, by extension, a point or a line of $PG(3, q)$, is *totally isotropic* if $u^T H v = 0$ for all u, v in S. Let $W(q)$ be the incidence structure where the points are the totally isotropic points of $PG(3,q)$, the lines are the totally isotropic lines of $PG(3,q)$ and the incidence structure is inherited from $PG(3,q)$.

1.2.4 Theorem. *$W(q)$ is a generalized quadrangle with order (q,q).*

We note that the dual of $W(q)$ is isomorphic to $W(q)$ if and only if q is even. If q is odd, the dual of $W(q)$ is a generalized quadrangle with order (q,q) and its point graph is coparametric with the point graph of $W(q)$.

For a vertex v in the point graph of any generalized quadrangle of order (s,t), we see that the neighbourhood of v is a union of $t+1$ disjoint cliques of size $s+1$. This follows since from every point lies on $t+1$ blocks and each block has $s+1$ points and any two blocks can intersect in at most one point. Thus, every strongly regular graph constructed from a generalized quadrangle of order (s,t) has isomorphic neighbourhoods. This similarity in the structure of the point graphs of generalized quadrangles motivate looking at such graphs in the context of graph isomorphism.

Chapter 2

Quantum Walks

A quantum walk is a quantum process on a graph. In this chapter, we introduce the concept of a quantum walk and how it might be used to detect graph isomorphism. In Section 2.1, we present basic definitions about quantum walks. In Section 2.2, we give an overview of quantum walk algorithms for the Graph Isomorphism problem. Then, we present the quantum walk procedure of Emms, Severini, Wilson and Hancock, which is the main focus of this dissertation. In Section 2.3, we present the procedure and perform eigenvalue analysis in Sections 2.4.1 and 2.4.2.

2.1 Discrete-time Quantum Walks

For a graph G, the *digraph of G* is the digraph on $V(G)$ with arcs xy and yx for each edge $\{x, y\}$ in $E(G)$. A *discrete-time quantum walk* is a process on a graph G whose state space is the set of arcs in the digraph of G and, at each time t, the walk moves from arc uv to arc wx if $v = w$ with some amplitude. In other words, if $|\psi_t\rangle$ is the state vector at time t, then the state vector at time $t + 1$ is $|\psi_{t+1}\rangle = U|\psi_{t+1}\rangle$, where

$$U_{wx,uv} = \begin{cases} \frac{2}{d(v)} & \text{if } v = w \text{ and } u \neq x, \\ \frac{2}{d(v)} - 1 & \text{if } v = w \text{ and } u = x, \\ 0 & \text{otherwise.} \end{cases}$$

We say that U is the *transition matrix* of the quantum walk. We will write $U(G)$ when the context is unclear.

Following [8], we can arrive at another formulation for the transition matrix of a quantum walk on a graph G. We represent by $A(G)$ the adjacency matrix of G and, when there is no confusion, we write A for simplicity. Let D be the digraph of G and consider the following incidence matrices of D, both with rows indexed by the vertices of D and columns indexed by the arcs of D:

$$(D_h)_{i,j} = \begin{cases} 1 & \text{if } i \text{ is the head of arc } j \\ 0 & \text{otherwise} \end{cases}$$

and

$$(D_t)_{i,j} = \begin{cases} 1 & \text{if } i \text{ is the tail of arc } j \\ 0 & \text{otherwise.} \end{cases}$$

We see that

$$(D_h D_t^T)_{i,j} = \begin{cases} 1 & \text{if there exists arc}(i,j) \\ 0 & \text{otherwise.} \end{cases}$$

From this we see that $D_h D_t^T = A(G)$.

The *line digraph* of a graph G is the graph $L(G)$ whose vertices are the arcs of D, the digraph of G, where there is an edge (ab, xy) in L if ab and xy are arcs such that $b = x$. We see that $A(L) = D_t^T D_h$, in particular that

$$(D_t^T D_h)_{wx,uv} = \begin{cases} 1 & \text{if } v = w, \\ 0 & \text{otherwise.} \end{cases}$$

Let P be a permutation matrix with row and columns indexed by the arcs of D such that,

$$P_{wx,uv} = \begin{cases} 1 & \text{if } x = u \text{ is the tail of arc } w = v \\ 0 & \text{otherwise.} \end{cases}$$

In particular, if G is regular with valency k, we see that

$$U = \frac{2}{k} D_t^T D_h - P = \frac{2}{k} A(L) - P.$$

2.2 Quantum Walk Algorithms for Graph Isomorphism

The *Graph Isomorphism problem* is the problem of deciding whether or not two given graphs belong to the same isomorphism class. There has been recent interest in using the concept of quantum walks to develop classical algorithms. Such algorithms use some aspect of quantum walks on graphs and are implementable in polynomial time. A simple algorithm for the Graph Isomorphism Problem is to evolve a quantum walk on the given two graphs for some number of steps, then compare a permutation-invariant aspect of the states of the quantum walk on each graph. For example, one can compare the two sets of amplitudes resulting from the quantum walk on the two graphs. If there is an amplitude that occurs in one set and not the other, then the graphs are distinguished. An early algorithm of this type is that of Shiau, Joynt and Coppersmith, in [13].

Another classical algorithm based on the simple algorithm is the algorithm of Douglas and Wang in [4]. This algorithm compares the two sets of amplitudes resulting from the quantum walk on the two graphs, along with a phase factor. In each graph, a reference node is chosen and the vertices are partitioned into three sets; one set containing just the reference node, one set containing the neighbours of the reference node and one set containing the rest of the vertices for the graph. The phase factor is determined by two sets the quantum walk is moving between. Douglas and Wang propose this algorithm for general graphs, including strongly regular graphs.

In [7], Gamble, Friesen, Zhou and Joynt propose that quantum walks of interacting bosons distinguish non-isomorphic pairs of graphs. Jamie Smith, in [14], showed that this approach does not work for general graphs, using a relation on graphs relying on their association schemes to generate a class of counterexamples. The counterexample graphs are, however, not strongly regular.

The main algorithm that this dissertation is concerned with is the procedure of Emms, Severini, Wilson and Hancock, presented in [6, 5]. This approach compares a matrix obtained from the transition matrix of a quantum walk after evolving the walk for three steps, and will be presented in full detail in Section 2.3.

For all four algorithms mentioned here, there are experimental results that suggest that the approaches may distinguish strongly regular graph. There is no known pair of non-isomorphic strongly regular graphs which are not distinguished by any of the three algorithms.

2.3 Procedure of Emms, Severini, Wilson and Hancock

In [6, 5], Emms, Severini, Wilson and Hancock propose that the quantum walk transition matrix can be used to distinguish between non-isomorphic graphs. Given a matrix M, the *positive support* of M, denoted $S^+(M)$, is the matrix obtained from M as follows:

$$(S^+(M))_{i,j} = \begin{cases} 1 & \text{if } M_{i,j} > 0 \\ 0 & \text{otherwise.} \end{cases}$$

The following theorem is stated without proof in [6, 5].

2.3.1 Theorem. *If G and H are isomorphic regular graphs, then $S^+(U(G)^3)$ and $S^+(U(H)^3)$ are cospectral.*

Proof. Since G is isomorphic to H, then their line digraphs are also isomorphic. From [9, p.164], we know that directed graphs X and Y are isomorphic if and only if there is a permutation matrix R such that

$$R^T A(X) R = A(Y).$$

Since permutation matrices are orthogonal, $R^T = R^{-1}$. Let $L(G)$ and $L(H)$ be the adjacency matrices of the line graphs of G and H respectively. Then there exists a permutation matrix Q such that

$$L(G) = Q^{-1} L(H) Q.$$

But, from the definition of U, we see that

$$U(G) = Q^{-1} U(H) Q$$

and, cubing both sides, we obtain,

$$U(G)^3 = (Q^{-1} U(H) Q)^3 = Q^{-1} U(H)^3 Q$$

Multiplying a matrix by a permutation matrix does not change its set of entries. Thus, taking the positive support commutes with conjugation by a permutation matrix. Then,

$$S^+(U(G)^3) = S^+(Q U(H)^3 Q^{-1}) = Q S^+(U(H)^3) Q^{-1}$$

and hence $S^+(U(G)^3)$ and $S^+(U(H)^3)$ are similar matrices and hence cospectral. □

The authors of [5, 6] propose that the converse of Theorem 2.3.1 is also true; they conjecture that the spectrum of $S^+(U^3)$ is a graph invariant for some classes of graphs. Through experimental results, they see that this proposed invariant distinguishes many more graphs than does the spectrum of the adjacency matrix. In particular, no strongly regular graph is known to have a cospectral mate with respect to this invariant.

2.3.2 Conjecture. *[6] If G and H are strongly regular graphs, then $S^+(U(G)^3)$ and $S^+(U(H)^3)$ are cospectral if and only if G and H are isomorphic.*

If Conjecture 2.3.2 is true, it would yield a polynomial-time algorithm for the Graph Isomorphism Problem on strongly regular graphs.

2.4 Eigenvalues

In [6, 5], the authors compute the spectra of $S^+(U)$ and $S^+(U^2)$ for strongly regular graphs, showing that they are determined by the spectra of the adjacency matrix. Here we will arrive at the same conclusion through different means. We will follow the methods in [8] to find the spectrum of $S^+(U)$. Then, we will show that the spectrum of $S^+(U^2)$ is determined by the spectrum of $S^+(U)$ by showing that $S^+(U^2) = (S^+(U))^2 + I$, where I is the identity matrix of appropriate size. Since we are concerned with strongly regular graphs, it suffices to consider regular graphs with valency k for the rest of the section.

2.4.1 Spectrum of $S^+(U)$

If G is a regular graph with valency k on n vertices, then

$$U = \frac{2}{k} D_t^T D_h - P.$$

The only negative entries have values $\frac{2}{k} - 1$, for $k \geq 2$, so $S^+(U) = D_t^T D_h - P$. The only regular graphs that have valency 1 are matchings, so we may assume $k \geq 2$.

From the previous section, we see that

$$D_t D_h^T = D_h D_t^T = A(G)$$

and

$$D_t^T D_h = A(L)$$

where L is the line digraph of G. We can see further that $D_t D_t^T = kI$ and $D_h D_h^T = kI$. We can show the following theorem based on [8].

2.4.1 Theorem. *If G is a regular connected graph with valency $k \geq 2$ and n vertices, then $S^+(U(G))$ has eigenvalues as follows:*

i) $k-1$ with multiplicity 1,

ii) $\frac{\lambda \pm \sqrt{\lambda^2 - 4(k-1)}}{2}$ as λ ranges over the eigenvalues of A, the adjacency matrix of G, and $\lambda \neq k$,

iii) 1 with multiplicity $\frac{n(k-1)}{2} + 1$, and

iv) -1 with multiplicity $\frac{n(k-1)}{2}$.

Proof. For a matrix M, we write $\mathrm{col}(M)$ to denote the column space of M and $\ker(M)$ to denote the kernel of M. Let $K = \mathrm{col}(D_h^T) + \mathrm{col}(D_t^T)$ and let $L = \ker(D_h) \cup \ker(D_t)$. Observe that K and L are orthogonal complements of each other. Then $\mathbb{R}^{vk}$ is the direct sum of orthogonal subspaces K and L. We will proceed by considering eigenvectors of $S^+(U)$ in K and in L separately. For K, we will show that the eigenvectors of $S^+(U)$ in K lie in subspaces $C(\lambda)$ where λ ranges over the eigenvalues of A. The eigenspace $C(k)$ has dimension 1 while $C(\lambda)$ has dimension 2 for all $\lambda \neq k$. In L, we will show that all eigenvectors of $S^+(U)$ have eigenvalue ± 1 and we will find their multiplicities.

First, we show that K and L are $S^+(U)$-invariant. Since L is the orthogonal complement of K, it suffices to check that K is $S^+(U)$-invariant. We have that $S^+(U) = D_t^T D_h - P$ and obtain that:

$$S^+(U) D_h^T = (D_t^T D_h - P) D_h^T = k D_t^T - D_t^T = (k-1) D_t^T \tag{2.4.1}$$

and

$$S^+(U) D_t^T = (D_t^T D_h - P) D_t^T = D_t^T A - D_h^T. \tag{2.4.2}$$

Then, K is $S^+(U)$-invariant.

Now, we may consider eigenvectors of $S^+(U)$ in K. From equations (2.4.1) and (2.4.2), we obtain:

$$\begin{aligned} S^+(U)^2 D_t^T &= S^+(U)(D_t^T A - D_h^T) \\ &= S^+(U) D_t^T A - S^+(U) D_h^T \\ &= S^+(U) D_t^T A - (k-1) D_t^T \end{aligned} \tag{2.4.3}$$

Let $\mathbf{z}$ be an eigenvector of A with eigenvalue λ. Let $\mathbf{y} := D_t^T \mathbf{z}$. Then, applying $\mathbf{y}$ to equation (2.4.3), we obtain:

$$\begin{aligned} S^+(U)^2 \mathbf{y} &= S^+(U)^2 D_t^T \mathbf{z} \\ &= S^+(U) D_t^T A\mathbf{z} - (k-1) D_t^T \mathbf{z} \\ &= \lambda S^+(U) D_t^T \mathbf{z} - (k-1) D_t^T \mathbf{z} \\ &= \lambda S^+(U)\mathbf{y} - (k-1)\mathbf{y}. \end{aligned}$$

Rearranging and factoring out $\mathbf{y}$, we get

$$(S^+(U)^2 - \lambda S^+(U) + (k-1)I)\mathbf{y} = 0. \tag{2.4.4}$$

Let $C(\lambda) = \mathrm{Span}\{\mathbf{y}, S^+(U)\mathbf{y}\}$. From equation (2.4.4) we see that $C(\lambda)$ has dimension at most 2, is $S^+(U)$-invariant and is contained in K. If $C(\lambda)$ is 1-dimensional, then $\mathbf{y}$ is an eigenvector of $S^+(U)$. Let θ be the corresponding eigenvalue. Then

$$\begin{aligned} \theta\mathbf{y} &= S^+(U)\mathbf{y} \\ &= S^+(U) D_t^T \mathbf{z} \\ &= (D_t^T A - D_h^T)\mathbf{z} \\ &= \lambda D_t^T \mathbf{z} - D_h^T \mathbf{z} \\ &= \lambda \mathbf{y} - D_h^T \mathbf{z} \end{aligned}$$

Then $(\theta - \lambda)\mathbf{y} = -D_h^T \mathbf{z}$ and $\mathbf{z}$ is in $\mathrm{col}(D_h^T) \cap \mathrm{col}(D_t^T)$. Then $\mathbf{y}$ is constant on arcs with a given head and on arcs with a given tail. Then $\mathbf{y}$ is constant on arcs of any component of G. Since G is connected, $\mathbf{y}$ is the constant vector, which implies that $\mathbf{z}$ is a constant vector and $\lambda = k$. The eigenvalue of $S^+(U)$ corresponding to $\mathbf{y}$ is $k-1$.

Now suppose $C(\lambda)$ is 2-dimensional. Then, the minimum polynomial of $C(\lambda)$ is

$$t^2 - \lambda t + (k-1) = 0$$

from (2.4.4) and the eigenvalues are

$$\frac{\lambda \pm \sqrt{\lambda^2 - 4(k-1)}}{2}.$$

These subspaces $C(\lambda)$ account for $2n-1$ eigenvalues of $S^+(U)$. Since D_h^T and D_t^T are both $(nk) \times n$ matrices, K has dimension at most $2n$. But, $D_h^T\mathbf{j} = D_t^T\mathbf{j} = \mathbf{j}$, where $\mathbf{j}$ is the all ones vector, since each row of both D_h^T and D_t^T has exactly one entry with value 1 and all other entries have value 0. Then, K has dimension at most $2n-1$ and we have found all of the eigenvectors of $S^+(U)$ in K. We will now turn our attention to eigenvectors of $S^+(U)$ in L, where we hope to find the remaining $n(k-2)+1$ eigenvalues of $S^+(U)$. Let $\mathbf{y}$ be in L. Then

$$\begin{aligned} S^+(U)\mathbf{y} &= (D_t^T D_h - P)\mathbf{y} \\ &= D_t^T D_h \mathbf{y} - P\mathbf{y} \\ &= -P\mathbf{y}. \end{aligned}$$

If $\mathbf{y}$ is an eigenvector of $S^+(U)$ with eigenvalue λ and $\mathbf{y}$ is in L, then $\mathbf{y}$ is an eigenvector of P with eigenvalue $-\lambda$. Since P is a permutation matrix, $\lambda = \pm 1$.

To find the multiplicities we consider the sum of all the eigenvalues of $S^+(U)$, which is equal to the trace of $S^+(U)$. Since arc ij cannot be the reverse arc of itself, P is a traceless matrix. Then

$$\operatorname{tr}(S^+(U)) = \operatorname{tr}(D_t^T D_h - P) = \operatorname{tr}(D_t^T D_h) = \operatorname{tr}(D_h D_t^T) = \operatorname{tr}(A) = 0.$$

The sum over all eigenvalues of $S^+(U)$ should be 0. Let $sp(A)$ be the set of eigenvalues of A. Consider the sum over the eigenvalues of eigenvectors of K:

$$\begin{aligned} &(k-1) + \sum_{\lambda \in sp(A), \lambda \neq k} \frac{\lambda \pm \sqrt{\lambda^2 - 4(k-1)}}{2} \\ &= (k-1) + \sum_{\lambda \in sp(A), \lambda \neq k} \lambda \\ &= -1 + \sum_{\lambda \in sp(A)} \lambda \\ &= -1. \end{aligned}$$

Then, the sum of the eigenvalue of the eigenvectors over L is 1. So, 1 and -1 have multiplicities $\frac{n(k-2)}{2} + 1$ and $\frac{n(k-2)}{2}$, respectively. □

2.4.2 Spectrum of $S^+(U^2)$

We will show that $S^+(U^2) = (S^+(U))^2 + I$. Then, the eigenvalues of $S^+(U^2)$ are determined by the eigenvalues of $S^+(U)$. The proof of the theorem will proceed by an analysis of which pair of arcs give a negative entry in U^2, similar to the ideas which we will later use in Chapter 3.

2.4.2 Theorem. *For any regular graph with valency k, if $k > 2$ then $S^+(U^2) = S^+(U)^2 + I$.*

Proof. Since $D_t^T D_h$ is the adjacency matrix of the line digraph of G, then $(D_t^T D_h)^2$ has the property that its (j,i)th entry counts the number of length two, directed walks in the line digraph of G. Observe that there is such a walk from i to j in $L(G)$ if and only if the head of i is adjacent to the tail of j in G. In particular, if there is a walk of length two from i to j, there is only one such walk. Then, $(D_t^T D_h)^2$ is a 0-1 matrix and is the support of U^2. We will find the required expression for $S^+(U^2)$ by subtracting from $(D_t^T D_h)^2$ the entries which have negative value in U^2.

We then proceed to look at the possible arrangements of i and j such that there is a length two, directed walk in $L(G)$ from i to j, in Table 2.1.

We see that the only negative entries of U^2 occur for i, j in Cases 3 and 4, when $k > 2$. Then $(U^2)_{j.i}$ is negative when i and j share the same head but not the same tail and when i and j share the same tail but not the same head. Then,

$$\begin{aligned} S^+(U^2) &= (D_t^T D_h)^2 - (D_t^T D_t - I) - (D_h^T D_h - I) \\ &= (D_t^T D_h)^2 - D_t^T D_t - D_h^T D_h + I + I \\ &= (D_t^T D_h)^2 - (D_t^T D_h)P - P(D_t^T D_h) + P^2 + I \\ &= (D_t^T D_h - P)^2 + I \end{aligned}$$

From 2.4.1, we know that $S^+(U) = D_t^T D_h - P$. Then, we have that

$$S^+(U^2) = S^+(U)^2 + I.$$

□

The next theorem explicitly lists the eigenvalues of $S^+(U^2)$.

2.4.3 Theorem. *If G is a regular connected graph with valency $k \geq 2$ and n vertices, then $S^+(U(G)^2)$ has eigenvalues as follows:*

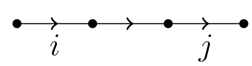
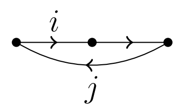
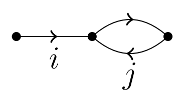
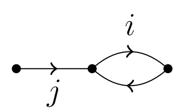
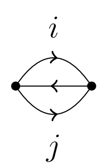
	Directed walk of length 3 from i to j	Value of $(U^2)_{i,j}$
Case 1.		$\left(\frac{2}{k}\right)^2$
Case 2.		$\left(\frac{2}{k}\right)^2$
Case 3.		$\left(\frac{2}{k}\right)\left(\frac{2}{k}-1\right)$
Case 4.		$\left(\frac{2}{k}-1\right)\left(\frac{2}{k}\right)$
Case 5.		$\left(\frac{2}{k}-1\right)^2$

Table 2.1: All possible pairs i, j such that there is a length 2 walk in $L(G)$

i) $k^2 - 2k + 2$ *with multiplicity 1,*

ii) $\frac{\lambda^2-2k+4}{2} \pm \frac{\lambda\sqrt{\lambda^2-4(k-1)}}{4}$ *as* λ *ranges over the eigenvalues of* A*, the adjacency matrix of* G*, and* $\lambda \neq k$ *and*

iii) 2 with multiplicity $n(k-1)+1$.

Proof. From Theorem 2.4.2, we get that $S^+(U^2) = (S^+(U))^2 + I$. Let $\mathbf{y}$ be an eigenvector of $S^+(U)$ with eigenvalues θ. Then,

$$\begin{aligned} S^+(U^2)\mathbf{y} &= ((S^+(U))^2 + I)\mathbf{y} \\ &= (S^+(U))^2\mathbf{y} + I\mathbf{y} \\ &= \theta^2\mathbf{y} + \mathbf{y} \\ &= (\theta^2+1)\mathbf{y}. \end{aligned}$$

Then, $\mathbf{y}$ is an eigenvector of $S^+(U^2)$ with eigenvalue θ^2+1. From Theorem 2.4.1, we know the eigenvalues of $S^+(U)$ are:

i) $k-1$ with multiplicity 1,

ii) $\frac{\lambda\pm\sqrt{\lambda^2-4(k-1)}}{2}$ as λ ranges over the eigenvalues of A, the adjacency matrix of G, and $\lambda \neq k$,

iii) 1 with multiplicity $\frac{n(k-1)}{2}+1$, and

iv) -1 with multiplicity $\frac{n(k-1)}{2}$.

Then, squaring each eigenvalue and adding 1, we obtain the eigenvalues of $S^+(U^2)$ as follows:

$$(k-1)^2 + 1 = k^2 - 2k + 2,$$

$$\left(\frac{\lambda \pm \sqrt{\lambda^2-4(k-1)}}{2}\right)^2 + 1 = \frac{\lambda^2-2k+4}{2} \pm \frac{\lambda\sqrt{\lambda^2-4(k-1)}}{4}$$

and $1^2 + 1 = (-1)^2 + 1 = 2$. The multiplicities are retained from the multiplicities of the eigenvalues of $S^+(U)$. □

2.5 Computations

We carried out computations of the spectrum of $S^+(U^3)$ for some graphs. In some cases, we confirm the results of [6, 5] and we also find new graphs that are distinguished by the procedure. We computed the spectrum of $S^+(U^3)$ for strongly regular graphs of the following parameter sets:

- $(16, 6, 2, 2)$,
- $(25, 12, 5, 6)$,
- $(26, 10, 3, 4)$,
- $(28, 12, 6, 4)$,
- $(29, 14, 6, 7)$,
- $(40, 12, 2, 4)$ and
- $(45, 12, 3, 3)$.

We find that all graphs in the listed parameter classes are distinguished by the spectrum of $S^+(U^3)$, confirming the results in [6, 5]. In addition, we looked at graphs with algebraic constructions which were not studied in [6, 5], including the Paley and Peisert graphs.

The *Paley graph*, denoted $P(q)$, has the elements of $GF(q)$ as vertices, where q is a prime power such that $q \equiv 1(\text{mod } 4)$. Two vertices are adjacent if their difference is a nonzero square in $GF(q)$. More information can be found in [9]. The *Peisert graph*, denoted $\mathcal{P}^*(q)$ is defined for $q = p^r$ where p is a prime and $p \equiv 3(\text{mod } 4)$ and r is even. Let a be a generator of $GF(q)$ and let M be a subset of $GF(q)$ as follows:

$$M = \{a^j : j \equiv 0, 1(\text{mod } 4)\}.$$

The vertices of $\mathcal{P}^*(q)$ are the elements of $GF(q)$ and two vertices are adjacent if their difference is in M. Peisert graphs were first defined by Peisert in [11]. The Paley graph $P(q)$ and the Peisert graph $\mathcal{P}^*(q)$ are strongly regular with parameters

$$\left(q, \frac{q-1}{2}, \frac{q-5}{4}, \frac{q-1}{4}\right).$$

It is interesting to compare the Paley graphs to the Peisert graphs because it is shown in [11] that the set of self-complementary, symmetric graphs consists of the Paley graphs, the Peisert graphs and one graph not belonging the two previous families of graphs.

- The Paley graph $P(49)$ and Peisert graph $\mathcal{P}^*(49)$ are non-isomorphic strongly regular graphs with parameters $(49, 24, 11, 12)$ and are distinguished by the spectrum of $S^+(U^3)$.
- The Paley graph $P(81)$ and Peisert graph $\mathcal{P}^*(81)$ are non-isomorphic strongly regular graphs with parameters $(81, 40, 19, 20)$ and are distinguished by the spectrum of $S^+(U^3)$.
- The point graph of $W(3)$ and the point graph of the dual of $W(3)$ non-isomorphic strongly regular graphs with parameters $(40, 12, 2, 4)$ and are distinguished by the spectrum of $S^+(U^3)$.
- The point graph of $W(5)$ and the point graph of the dual of $W(5)$ non-isomorphic strongly regular graphs with parameters $(156, 30, 4, 6)$ and are distinguished by the spectrum of $S^+(U^3)$.

It is worth remarking that, although the spectrum of $S^+(U^3)$ distinguished the point graphs of $W(q)$ and its dual for $q = 3$ and $q = 5$, the characteristic polynomials have a large common factor.

Chapter 3

Analysis of the Entries of U^3 for Strongly Regular Graphs

In this chapter, we wish to understand the structure of the cube of the transition matrix for strongly regular graphs. In order to understand the positive support of U^3, we will look at possible entries of U^3 in Section 3.1 and for which graphs these entries occur, in Section 3.2.

For the purposes of showing that $S^+(U^3)$ can be constructed directly from G, without first constructing the matrix U, the authors of [5] do a similar case analysis for finding the value of $(U^3)_{wx,uv}$ given arcs uv and wx. In each case, they find an expression for $(U^3)_{wx,uv}$ as a function of entries of A and the parameters of G. Here, we present a more refined case analysis and find $(U^3)_{wx,uv}$ as a function of the parameters of G. Further, we show that, for each entry ρ in the case analysis to find the value of $(U^3)_{wx,uv}$ given arcs uv and wx, there is a pair of arcs uv and wx such that $(U^3)_{wx,uv} = \rho$.

Using these observations, we will then consider writing U^3 as a sum of 01-matrices, where U is the transition matrix of a quantum walk on a strongly regular graph, in Section 3.3. This is motivated by the fact that $S^+(U^3)$ will be the sum of a subset of those 01-matrices. If G and H are coparametric strongly regular graphs, we will show that the coefficients of 01-matrices in the decompositions of the cubes of the transition matrices for G and H are the same. In order to do this, we will show that the possible values of $(U^3)_{wx,uv}$ and for which graphs these possible values occur depend only on the parameters of the strongly regular graph.

3.1 Possible Entries of U^3

For a graph G and arcs wx and uv in the digraph of G, the wx, uv entry of $U(G)^3$ is the amplitude with which a walk moves from uv to wx after three steps. If G is a strongly regular graph and uv and wx are given, then it is possible to calculate the value of $(U(G)^3)_{wx,uv}$ from the parameters of G. In order to show this, we will look first at all possible walks between two arcs after three steps and then analyze which of these walks exist for arcs uv and wx.

A walk starting from a given arc and progressing three steps is exactly a walk of length 4. It is not difficult to enumerate all such walks and they are given in Table 3.1.

By analyzing the walks of length 4 that can occur between uv and wx, we can show the following technical lemma:

3.1.1 Lemma. *If G is a strongly regular graph and uv and wx are arcs in the digraph of G, then the value of $(U(G)^3)_{wx,uv}$ depends only on the parameters of G.*

Proof. Given uv and wx, there are three cases with regard to the number of distinct elements of $\{u, v, w, x\}$. For each of the three cases, we will look at the possible induced subgraph formed by $\{u, v, w, x\}$ and count the walks of length 4 that occur. Let G have parameter set (n, k, a, c). The main cases are Case I, II and III where $\{u, v, w, x\}$ have four, three and two distinct elements, respectively. The sub-cases are enumerated with small roman ordinals and further enumerated with lower case letter, if necessary, to indicate a refinement by which of $\{u, v, w, x\}$ are equal and by adjacency of the elements of $\{u, v, w, x\}$, respectively.

Case I: $\{u, v, w, x\}$ has 4 distinct vertices.

In this case, there are a few possibilities for the subgraph induced by $\{u, v, w, x\}$ in G. We will say that W_i as defined in Table 3.1 is a *4-walk from uv to wx* if W_i is isomorphic to a subgraph of G and the isomorphism fixes u, v, w, and x. There may be more than one subgraph of G isomorphic to some W_i; we will call the number of such subgraphs the *multiplicity*.

We present a summary of the 4-walks from uv to wx where $\{u, v, w, x\}$ are distinct and their multiplicities in Table 3.2, followed by detailed case analysis in each of the cases.

Case I.a: u is adjacent to w and v is adjacent to w and x. See Figure 3.1.

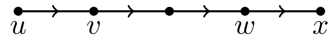
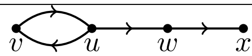
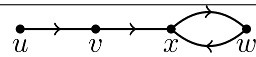
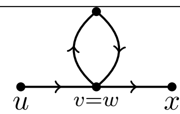
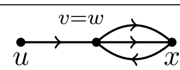
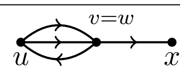
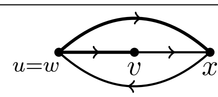
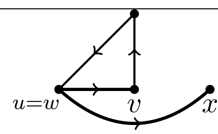
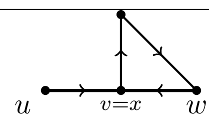
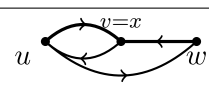
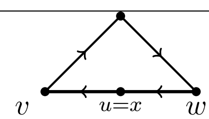
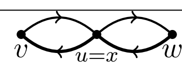
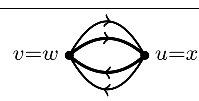
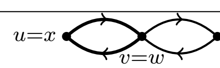
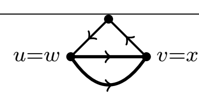
W_1	
W_2	
W_3	
W_4	
W_5	
W_6	
W_7	
W_8	
W_9	
W_{10}	
W_{11}	
W_{12}	
W_{13}	
W_{14}	
W_{15}	

Table 3.1: All directed walks of length 4 with uv as the first edge and wx as the last edge.

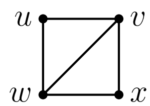
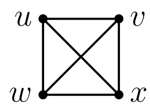
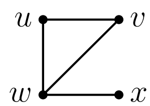
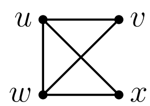
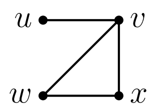
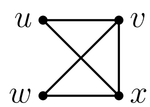
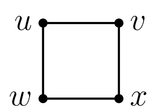
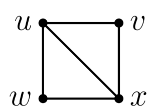
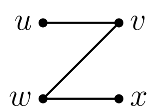
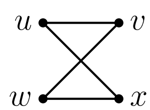
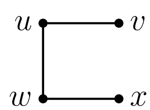
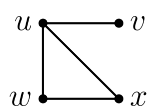
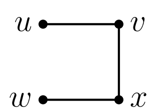
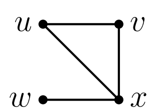
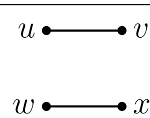
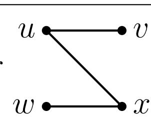
Case	Induced subgraph of $\{u, v, w, x\}$ in G	4-walks from uv to wx	Multiplicity of 4-walk
I.a.	 or 	W_1 W_2 W_3	$\max\{0, a-2\}$ 1 1
I.b.	 or 	W_1 W_2	$\max\{0, a-1\}$ 1
I.c.	 or 	W_1 W_3	$\max\{0, a-1\}$ 1
I.d.	 or 	W_1 W_2 W_3	$\max\{0, c-2\}$ 1 1
I.e.	 or 	W_1	a
I.f.	 or 	W_1 W_2	$\max\{0, c-1\}$ 1
I.g.	 or 	W_1 W_3	$\max\{0, c-1\}$ 1
I.h.	 or 	W_1	c

Table 3.2: Walks of length 4 from uv to wx where $\{u, v, w, x\}$ has 4 distinct elements.

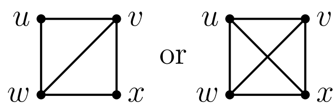


Figure 3.1: Induced subgraph of $\{u, v, w, x\}$ in G in Case I.a.

The directed walk defined by

$$\{u, uv, v, vy, y, yw, w, wx, x\}$$

for each y a common neighbour of v and w in G, is isomorphic to W_1. Since G is strongly regular, v and w are adjacent and hence have $a-2$ common neighbours other than u and x. Then G has $\max\{0, a-2\}$ distinct walks from uv to wx that are isomorphic to W_1. In addition

$$\{u, uv, v, vu, u, uw, w, wx, x\}$$

is a length 4 directed walk isomorphic to W_2 and

$$\{u, uv, v, vx, x, xw, w, wx, x\}$$

is a length 4 directed walk isomorphic to W_3. The only 4-walks from uv to wx are W_1, W_2, and W_3.

Since each step of W_1 visits a new vertex, each walk isomorphic to W_1 contributes $\left(\frac{2}{k}\right)^3$ to $(U^3)_{wx,uv}$. Both W_2 and W_3 have exactly one step that goes back along an arc to a vertex visited in the previous step, so each walk contributes $\left(\frac{2}{k}\right)^2\left(\frac{2}{k}-1\right)$ to $(U^3)_{wx,uv}$. Then,

$$\begin{aligned}
(U^3)_{wx,uv} &= \max\{0, a-2\}\left(\frac{2}{k}\right)^3 + \left(\frac{2}{k}\right)^2\left(\frac{2}{k}-1\right) + \left(\frac{2}{k}\right)^2\left(\frac{2}{k}-1\right) \\
&= \max\{0, a-2\}\frac{8}{k^3} + \frac{8}{k^2}\left(\frac{2-k}{k}\right) \\
&= \frac{8\max\{0, a-2\} + 16 - 8k}{k^3}.
\end{aligned}$$

Case I.b: u is adjacent to w and v is adjacent to w and not x. See Figure 3.2.

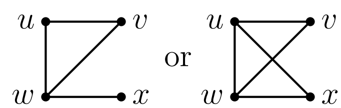


Figure 3.2: Induced subgraph of $\{u, v, w, x\}$ in G in Case I.b.

The directed walk defined by

$$\{u, uv, v, vy, y, yw, w, wx, x\}$$

for each y a common neighbour of v and w in G, is isomorphic to W_1. Since G is strongly regular, v and w are adjacent and hence have $a - 1$ common neighbours other than u. Then G has $\max\{0, a-1\}$ distinct walks from uv to wx that are isomorphic to W_1. In addition

$$\{u, uv, v, vu, u, uw, w, wx, x\}$$

is a length 4 directed walk isomorphic to W_2. The only 4-walks from uv to wx are W_1 and W_2.

As in the previous case, each walk isomorphic to W_1 contributes $\left(\frac{2}{k}\right)^3$ to $(U^3)_{wx,\,uv}$ and the walk isomorphic to W_2 contributes $\left(\frac{2}{k}\right)^2\left(\frac{2}{k} - 1\right)$ to $(U^3)_{wx,\,uv}$. Then,

$$\begin{aligned}(U^3)_{wx,\,uv} &= \max\{0, a-1\}\left(\frac{2}{k}\right)^3 + \left(\frac{2}{k}\right)^2\left(\frac{2}{k} - 1\right)\\ &= \max\{0, a-1\}\frac{8}{k^3} + \frac{4}{k^2}\left(\frac{2-k}{k}\right)\\ &= \frac{8\max\{0, a-1\} + 8 - 4k}{k^3}.\end{aligned}$$

Case I.c: u is not adjacent to w and v is adjacent to w and x. See Figure 3.3.

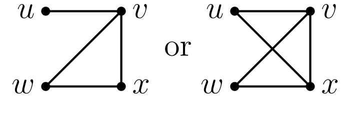


Figure 3.3: Induced subgraph of $\{u, v, w, x\}$ in G in Case I.c.

The directed walk defined by

$$\{u, uv, v, vy, y, yw, w, wx, x\}$$

for each y a common neighbour of v and w in G, is isomorphic to W_1. Since G is strongly regular, v and w are adjacent and hence have $a-1$ common neighbours other than x. Then G has $\max\{0, a-1\}$ distinct walks from uv to wx that are isomorphic to W_1. In addition

$$\{u, uv, v, vx, x, xw, w, wx, x\}$$

is a length 4 directed walk isomorphic to W_3. The only 4-walks from uv to wx are W_1 and W_3.

As in the previous cases, each walk isomorphic to W_1 contributes $\left(\frac{2}{k}\right)^3$ to $(U^3)_{wx,\,uv}$ and the walk isomorphic to W_3 contributes $\left(\frac{2}{k}\right)^2\left(\frac{2}{k}-1\right)$ to $(U^3)_{wx,\,uv}$. Then,

$$\begin{aligned}
(U^3)_{wx,\,uv} &= \max\{0, a-1\}\left(\frac{2}{k}\right)^3 + \left(\frac{2}{k}\right)^2\left(\frac{2}{k}-1\right) \\
&= \max\{0, a-1\}\frac{8}{k^3} + \frac{4}{k^2}\left(\frac{2-k}{k}\right) \\
&= \frac{8\max\{0, a-1\} + 8 - 4k}{k^3}.
\end{aligned}$$

Case I.d: u is adjacent to w and v is adjacent to x and not w. See Figure 3.4.

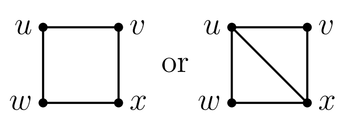


Figure 3.4: Induced subgraph of $\{u, v, w, x\}$ in G in Case I.d.

The directed walk defined by

$$\{u, uv, v, vy, y, yw, w, wx, x\}$$

for each y a common neighbour of v and w in G, is isomorphic to W_1. Since G is strongly regular, v and w are not adjacent and hence have $c-2$ common neighbours other than u and x. Then G has $\max\{0, c-2\}$ distinct walks from uv to wx that are isomorphic to W_1. In addition

$$\{u, uv, v, vu, u, uw, w, wx, x\}$$

is a length 4 directed walk isomorphic to W_2 and

$$\{u, uv, v, vx, x, xw, w, wx, x\}$$

is a length 4 directed walk isomorphic to W_3. The only 4-walks from uv to wx are W_1, W_2, and W_3.

As in the previous cases, each walk isomorphic to W_1 contributes $\left(\frac{2}{k}\right)^3$ to $(U^3)_{wx,uv}$ and the walks isomorphic to W_2 or W_3 contribute $\left(\frac{2}{k}\right)^2\left(\frac{2}{k}-1\right)$ to $(U^3)_{wx,uv}$. Then,

$$\begin{aligned}(U^3)_{wx,uv} &= \max\{0, c-2\}\left(\frac{2}{k}\right)^3 + \left(\frac{2}{k}\right)^2\left(\frac{2}{k}-1\right) + \left(\frac{2}{k}\right)^2\left(\frac{2}{k}-1\right) \\ &= \max\{0, c-2\}\frac{8}{k^3} + \frac{8}{k^2}\left(\frac{2-k}{k}\right) \\ &= \frac{8\max\{0, c-2\} + 16 - 8k}{k^3}.\end{aligned}$$

Case I.e: u is not adjacent to w and v is adjacent to w and not x. See Figure 3.5.

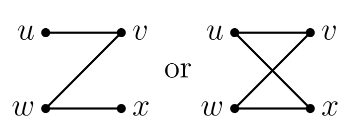


Figure 3.5: Induced subgraph of $\{u, v, w, x\}$ in G in Case I.e.

The directed walk defined by

$$\{u, uv, v, vy, y, yw, w, wx, x\}$$

for each y a common neighbour of v and w in G, is isomorphic to W_1. Since G is strongly regular, v and w are adjacent and hence have a common neighbours. Then G has a distinct walks from uv to wx that are isomorphic to W_1. There are no other 4-walks from uv to wx.

As in the previous cases, each walk isomorphic to W_1 contributes $\left(\frac{2}{k}\right)^3$ to $(U^3)_{wx,\,uv}$. Then,

$$\begin{aligned}(U^3)_{wx,\,uv} &= a\left(\frac{2}{k}\right)^3\\ &= \frac{8a}{k^3}.\end{aligned}$$

Case I.f: u is adjacent to w and v is not adjacent to both w and x. See Figure 3.6.

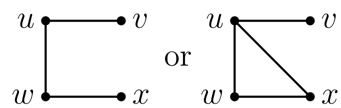


Figure 3.6: Induced subgraph of $\{u, v, w, x\}$ in G in Case I.f.

The directed walk defined by

$$\{u, uv, v, vy, y, yw, w, wx, x\}$$

for each y a common neighbour of v and w in G, is isomorphic to W_1. Since G is strongly regular, v and w are not adjacent and hence have $c-1$ common neighbours other than u. Then G has $\max\{0, c-1\}$ distinct walks from uv to wx that are isomorphic to W_1. In addition

$$\{u, uv, v, vu, u, uw, w, wx, x\}$$

is a length 4 directed walk isomorphic to W_2. The only 4-walks from uv to wx are W_1 and W_2.

As in the previous cases, each walk isomorphic to W_1 contributes $\left(\frac{2}{k}\right)^3$ to $(U^3)_{wx,\,uv}$ and the walk isomorphic to W_2 contributes $\left(\frac{2}{k}\right)^2\left(\frac{2}{k}-1\right)$ to $(U^3)_{wx,\,uv}$. Then,

$$\begin{aligned}(U^3)_{wx,\,uv} &= \max\{0, c-1\}\left(\frac{2}{k}\right)^3 + \left(\frac{2}{k}\right)^2\left(\frac{2}{k}-1\right)\\ &= \frac{8\max\{0, c-1\} + 8 - 4k}{k^3}.\end{aligned}$$

Case I.g: u is not adjacent to w and v is adjacent to x and not w. See Figure 3.3.

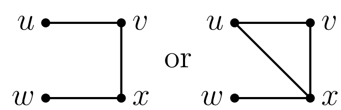


Figure 3.7: Induced subgraph of $\{u, v, w, x\}$ in G in Case I.g.

The directed walk defined by

$$\{u, uv, v, vy, y, yw, w, wx, x\}$$

for each y a common neighbour of v and w in G, is isomorphic to W_1. Since G is strongly regular, v and w are not adjacent and hence have $c-1$ common neighbours other than x. Then G has $\max\{0, c-1\}$ distinct walks from uv to wx that are isomorphic to W_1. In addition

$$\{u, uv, v, vx, x, xw, w, wx, x\}$$

is a length 4 directed walk isomorphic to W_3. The only 4-walks from uv to wx are W_1 and W_3.

As in the previous cases, each walk isomorphic to W_1 contributes $\left(\frac{2}{k}\right)^3$ to $(U^3)_{wx,\,uv}$ and the walk isomorphic to W_3 contributes $\left(\frac{2}{k}\right)^2\left(\frac{2}{k}-1\right)$ to $(U^3)_{wx,\,uv}$. Then,

$$\begin{aligned}(U^3)_{wx,\,uv} &= \max\{0, c-1\}\left(\frac{2}{k}\right)^3 + \left(\frac{2}{k}\right)^2\left(\frac{2}{k}-1\right)\\ &= \frac{8\max\{0, c-1\} + 8 - 4k}{k^3}.\end{aligned}$$

Case I.h: u is not adjacent to w and v is not adjacent to both w and x. See Figure 3.8.

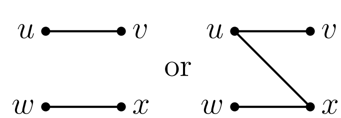


Figure 3.8: Induced subgraph of $\{u, v, w, x\}$ in G in Case I.h.

The directed walk defined by

$$\{u, uv, v, vy, y, yw, w, wx, x\}$$

for each y a common neighbour of v and w in G, is isomorphic to W_1. Since G is strongly regular, v and w are adjacent and hence have c common neighbours. Then G has c distinct walks from uv to wx that are isomorphic to W_1. There are no other 4-walks from uv to wx.

As in the previous cases, each walk isomorphic to W_1 contributes $\left(\frac{2}{k}\right)^3$ to $(U^3)_{wx,\,uv}$. Then,

$$(U^3)_{wx,\,uv} = c\left(\frac{2}{k}\right)^3 = \frac{8c}{k^3}.$$

Case II: $\{u, v, w, x\}$ has 3 distinct vertices.

In this case, we can have either $v = w$, $u = w$, $v = x$ or $u = x$, since uv and wx are edges and G is simple. In each case, the adjacency relation of the vertices in the symmetric difference of $\{u, v\}$ and $\{w, x\}$ (denoted $\{u, v\} \oplus \{w, x\}$) determines the induced subgraph of $\{u, v, w, x\}$.

Table 3.3 summarizes these cases and is followed by a detailed analysis of each case.

Case II.i: $v = w$. See Figure 3.9.

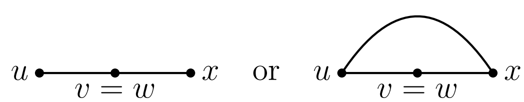


Figure 3.9: Induced subgraph of $\{u, v, w, x\}$ in G in Case II.i.

The directed walk defined by

$$\{u, uv, v, vy, y, yv, v, vx = wx, x\}$$

for each y a neighbour of v, different from u and x in G, is a isomorphic to W_4. Since G is strongly regular, v has $\max 0, k-2$ neighbours other than u and x. Then G has $\max\{0, k-2\}$ distinct walks from uv to wx that are isomorphic to W_4. In addition

$$\{u, uv, v, vx, x, xv, v, vx, x\}$$

Case		Adjacency Relations of $\{u,v\} and \{w,x\}$ in G	4-walks from uv to wx	Multiplicity of 4-walk
II.i.	$v = w$	$u \sim x$ or $u \nsim x$	W_4	$\max\{0, k-2\}$
			W_5	1
			W_6	1
II.ii.a	$u = w$	$v \sim x$	W_8	$\max\{0, a-1\}$
			W_7	1
II.ii.b	$u = w$	$v \nsim x$	W_8	a
II.iii.a	$v = x$	$u \sim w$	W_9	$\max\{0, a-1\}$
			W_{10}	1
II.iii.b	$v = x$	$u \nsim w$	W_9	a
II.iv.a	$u = x$	$v \sim w$	W_{11}	$\max\{0, a-1\}$
			W_{12}	1
II.iv.b	$u = x$	$v \nsim w$	W_{11}	$\max\{0, c-1\}$
			W_{12}	1

Table 3.3: Walks of length 4 from uv to wx where $\{u, v, w, x\}$ has 3 distinct elements.

is a length 4 directed walk isomorphic to W_5 and

$$\{u, uv, v, vu, u, uv, v, vx, x\}$$

is a length 4 directed walk isomorphic to W_6. The only 4-walks from uv to wx are W_4, W_5, and W_6.

Since W_4 visits an arc and then the reverse arc in consecutive sequence exactly once, each walk isomorphic to W_4 contributes $\left(\frac{2}{k}\right)^2\left(\frac{2}{k}-1\right)$ to $(U^3)_{wx,uv}$. Each of W_5 and W_6 visits an arc and then the reverse arc in consecutive sequence exactly twice, so they each contribute $\left(\frac{2}{k}\right)\left(\frac{2}{k}-1\right)^2$ to $(U^3)_{wx,uv}$. Then,

$$\begin{aligned}(U^3)_{wx,uv} &= \max\{0,k-2\}\left(\frac{2}{k}\right)^2\left(\frac{2}{k}-1\right)+\left(\frac{2}{k}\right)\left(\frac{2}{k}-1\right)^2+\left(\frac{2}{k}\right)\left(\frac{2}{k}-1\right)^2\\ &= \left(\frac{2}{k}\right)\left(\frac{2}{k}-1\right)\left(\frac{2\max\{0,k-2\}}{k}+2\left(\frac{2-k}{k}\right)\right).\end{aligned}$$

Then, $(U^3)_{wx,uv} = 0$ if $k \geq 2$. Since G is strongly regular, $k \neq 0$. If $k = 1$ then G is a direct sum of copies of K_2, and $(U^3)_{wx,uv} = 4$.

Case II.ii.a: $u = w$ and v is adjacent to x. See Figure 3.10.

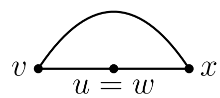


Figure 3.10: Induced subgraph of $\{u, v, w, x\}$ in G in Case II.ii.a.

The directed walk defined by

$$\{u, uv, v, vy, y, yu, u, ux, x\}$$

for each y a common neighbour of u and v, different from x in G, is isomorphic to W_8. Since G is strongly regular, u and v have $\max\{0, a-1\}$ such neighbours. Then G has $\max\{0, a-1\}$ distinct walks from uv to wx that are isomorphic to W_8. In addition

$$\{u, uv, v, vx, x, xu, u, ux, x\}$$

is a length 4 directed walk isomorphic to W_7. The only 4-walks from uv to wx are W_8, and W_7.

Since W_8 visits a new edge at every step, each walk isomorphic to W_8 contributes $\left(\frac{2}{k}\right)^3$ to $(U^3)_{wx,\,uv}$. Walk W_7 visits an arc and then the reverse arc in consecutive sequence exactly once and so contributes $\left(\frac{2}{k}\right)^2\left(\frac{2}{k}-1\right)$ to $(U^3)_{wx,\,uv}$ Then,

$$\begin{aligned}(U^3)_{wx,\,uv} &= \max\{0, a-1\}\left(\frac{2}{k}\right)^3 + \left(\frac{2}{k}\right)^2\left(\frac{2}{k}-1\right)\\ &= \frac{8\max\{0, a-1\} + 8 - 4k}{k^3}.\end{aligned}$$

Case II.ii.b: $u = w$ and v is not adjacent to x. See Figure 3.11.

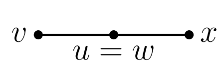


Figure 3.11: Induced subgraph of $\{u, v, w, x\}$ in G in Case II.ii.b.

The directed walk defined by

$$\{u, uv, v, vy, y, yu, u, ux, x\}$$

for each y a common neighbour of u and v, is isomorphic to W_8. Since G is strongly regular, u and v have a such neighbours. Then G has a distinct walks from uv to wx that are isomorphic to W_8, which is the only type of 4-walk from uv to wx. Then,

$$(U^3)_{wx,\, uv} = a\left(\frac{2}{k}\right)^3 = \frac{8a}{k^3}.$$

Case II.iii.a: $v = x$ and u is adjacent to w. See Figure 3.12.

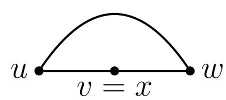


Figure 3.12: Induced subgraph of $\{u, v, w, x\}$ in G in Case II.iii.a.

The directed walk defined by

$$\{u, uv, v, vy, y, yw, w, wv, v\}$$

for each y a common neighbour of w and v, different from u in G, is isomorphic to W_9. Since G is strongly regular, w and v have $\max\{0, a-1\}$ such neighbours. Then G has $\max\{0, a-1\}$ distinct walks from uv to wx that are isomorphic to W_9. In addition

$$\{u, uv, v, vu, u, uw, w, wv, v\}$$

is a length 4 directed walk isomorphic to W_{10}. The only types of 4-walk from uv to wx are W_9, and W_{10}.

Since W_9 visits a new edge at every step, each walk isomorphic to W_9 contributes $\left(\frac{2}{k}\right)^3$ to $(U^3)_{wx,\, uv}$. Walk W_{10} visits uv and vu in consecutive sequence and so contributes $\left(\frac{2}{k}\right)^2\left(\frac{2}{k}-1\right)$ to $(U^3)_{wx,\, uv}$ Then,

$$\begin{aligned}(U^3)_{wx,\, uv} &= \max\{0, a-1\}\left(\frac{2}{k}\right)^3 + \left(\frac{2}{k}\right)^2\left(\frac{2}{k}-1\right)\\ &= \frac{8\max\{0, a-1\} + 8 - 4k}{k^3}.\end{aligned}$$

Case II.iii.b: $v = x$ and u is not adjacent to w. See Figure 3.13.

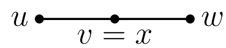


Figure 3.13: Induced subgraph of $\{u, v, w, x\}$ in G in Case II.iii.b.

The directed walk defined by

$$\{u, uv, v, vy, y, yw, w, wv, v\}$$

for each y a common neighbour of w and v is isomorphic to W_9. Since G is strongly regular, w and v have a such neighbours and hence G has a distinct walks from uv to wx that are isomorphic to W_9, which is the only type of 4-walk from uv to wx. Then,

$$(U^3)_{wx,\,uv} = a\left(\frac{2}{k}\right)^3 = \frac{8a}{k^3}.$$

Case II.iv.a: $u = x$ and v is adjacent to w. See Figure 3.14.

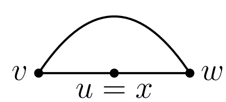


Figure 3.14: Induced subgraph of $\{u, v, w, x\}$ in G in Case II.iv.a.

The directed walk defined by

$$\{u, uv, v, vy, y, yw, w, wu, u\}$$

for each y a common neighbour of w and v, different from u in G, is isomorphic to W_{11}. Since G is strongly regular, w and v have $\max\{0, a-1\}$ such neighbours. Then G has $\max\{0, a-1\}$ distinct walks from uv to wx that are isomorphic to W_{11}. In addition

$$\{u, uv, v, vu, u, uw, w, wu, u\}$$

is a length 4 directed walk isomorphic to W_{12}. The only 4-walks from uv to wx are W_{11}, and W_{12}.

Since W_{11} visits a new edge at every step, each walk isomorphic to W_{11} contributes $\left(\frac{2}{k}\right)^3$ to $(U^3)_{wx,\,uv}$. Walk W_{12} visits uv and vu and uw and wu in consecutive sequence and so contributes $\left(\frac{2}{k}\right)\left(\frac{2}{k}-1\right)^2$ to $(U^3)_{wx,\,uv}$. Then,

$$\begin{aligned}(U^3)_{wx,\,uv} &= \max\{0, a-1\}\left(\frac{2}{k}\right)^3 + \left(\frac{2}{k}\right)\left(\frac{2}{k}-1\right)^2 \\ &= \frac{8\max\{0, a-1\} + (2-k)^2}{k^3} \\ &= \frac{8\max\{0, a-1\} + 8 - 8k + 2k^2}{k^3}.\end{aligned}$$

Case II.iv.b: $u = x$ and v is not adjacent to w. See Figure 3.15.

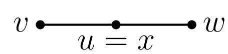


Figure 3.15: Induced subgraph of $\{u, v, w, x\}$ in G in Case II.iv.b.

The directed walk defined by

$$\{u, uv, v, vy, y, yw, w, wu, u\}$$

for each y a common neighbour of w and v, different from u in G, is isomorphic to W_{11}. Since G is strongly regular, w and v have $\max\{0, c-1\}$ such neighbours. Then G has $\max\{0, c-1\}$ distinct walks from uv to wx that are isomorphic to W_{11}. In addition

$$\{u, uv, v, vu, u, uw, w, wu, u\}$$

is a length 4 directed walk isomorphic to W_{12}. The only 4-walks from uv to wx are W_{11}, and W_{12}. Then, as in the previous case,

$$\begin{aligned}(U^3)_{wx,\,uv} &= \max\{0, c-1\}\left(\frac{2}{k}\right)^3 + \left(\frac{2}{k}\right)\left(\frac{2}{k}-1\right)^2 \\ &= \frac{8\max\{0, c-1\} + 8 - 8k + 2k^2}{k^3}.\end{aligned}$$

Case III: $\{u, v, w, x\}$ has 2 distinct vertices.

In this case, either $u = x$ and $v = w$ or $v = x$ and $u = w$. The 4-walks for these cases are summarized in Table 3.4 and will be followed by a detailed analysis of each case.

Case	4-walks from uv to wx	Multiplicity of 4-walk
III.i $u = x$, $v = w$	W_{13}	1
	W_{14}	$k - 1$
III.ii $v = x$, $u = w$	W_{15}	a

Table 3.4: Walks of length 4 from uv to wx where $\{u, v, w, x\}$ has 2 distinct elements.

Case III.i: $u = x$ and $v = w$. See Figure 3.16.

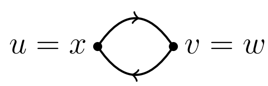


Figure 3.16: Arcs uv and wx in the digraph of G in Case III.i.

The directed walk

$$\{u, uv, v, vy, y, yv, v, vu, u\}$$

is a length 4 directed walk with uv as the first arc and wx as the last arc, where y is a neighbour of v other than u, and is isomorphic to W_{14}. In G, the vertex v has $k - 1$ such neighbours, so there are $k - 1$ such walks. In addition,

$$\{u, uv, v, vu, u, uv, v, vu, u\}$$

is a walk isomorphic to W_{13}. The only 4-walks from uv to wx are W_{13}, and W_{14}.

Since W_{14} visits vy and the yv, each walk isomorphic to W_{14} contributes $\left(\frac{2}{k}\right)^2 \left(\frac{2}{k} - 1\right)$ to $(U^3)_{wx,uv}$. Walk W_{13} visits an arc followed by its reverse arc at every step and so contributes $\left(\frac{2}{k}-1\right)^3$ to $(U^3)_{wx,\,uv}$. Then

$$\begin{aligned}(U^3)_{wx,\,uv} &= (k-1)\left(\frac{2}{k}\right)^2\left(\frac{2}{k}-1\right)+\left(\frac{2}{k}-1\right)^3\\ &= \left(\frac{2}{k}-1\right)\left((k-1)\left(\frac{2}{k}\right)^2+\left(\frac{2-k}{k}\right)^2\right)\\ &= \left(\frac{2}{k}-1\right)\left(\frac{4(k-1)+4-4k+k^2}{k^2}\right)\\ &= \left(\frac{2-k}{k}\right).\end{aligned}$$

Case III.ii: $v = x$ and $u = w$. See Figure 3.17.

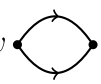


Figure 3.17: Arcs uv and wx in the digraph of G in Case III.ii.

The directed walk

$$\{u, uv, v, vy, y, yu, u, uv, v\}$$

is a length 4 directed walk with uv as the first arc and wx as the last arc, where y is a common neighbour of u and v, and is isomorphic to W_{15}. In G, vertices u and v are adjacent and hence have a such neighbours, so there are a such walks. The only 4-walks from uv to wx are of type W_{15}.

Since W_{15} never visits an arc followed immediately by its reverse arc, each walk isomorphic to W_{15} contributes $\left(\frac{2}{k}\right)^3$ to $(U^3)_{wx,\,uv}$. Then

$$(U^3)_{wx,\,uv} = a\left(\frac{2}{k}\right)^3 = \frac{8a}{k^3}.$$

From the analysis of the 4-walks from uv to wx and their amplitudes, we have calculated the entries of $(U(G)^3)_{wx,uv}$, which are collated in Table 3.5.

This concludes the proof. □

Case	Entry of $(U(G)^3)_{wx,uv}$
I.a	$\frac{8\max\{0,a-2\}+16-8k}{k^3}$
I.b	$\frac{8\max\{0,a-1\}+8-4k}{k^3}$
I.c	$\frac{8\max\{0,a-1\}+8-4k}{k^3}$
I.d	$\frac{8\max\{0,c-2\}+16-8k}{k^3}$
I.e	$\frac{8a}{k^3}$
I.f	$\frac{8\max\{0,c-1\}+8-4k}{k^3}$
I.g	$\frac{8\max\{0,c-1\}+8-4k}{k^3}$
I.h	$\frac{8c}{k^3}$
II.i	$\begin{cases} 4 & \text{if } k=1 \\ 0 & \text{otherwise} \end{cases}$
II.ii.a	$\frac{8\max\{0,a-1\}+8-4k}{k^3}$
II.ii.b	$\frac{8a}{k^3}$
II.iii.a	$\frac{8\max\{0,a-1\}+8-4k}{k^3}$
II.iii.b	$\frac{8a}{k^3}$
II.iv.a	$\frac{8\max\{0,a-1\}+8-8k+2k^2}{k^3}$
II.iv.b	$\frac{8\max\{0,c-1\}+8-8k+2k^2}{k^3}$
III.i	$\frac{2-k}{k}$
III.ii	$\frac{8a}{k^3}$

Table 3.5: Entries of $(U(G)^3)_{wx,uv}$ given uv and wx.

3.2 When Do These Entries Occur?

Lemma 3.1.1 shows the entry of $(U(G)^3)_{wx,uv}$ given a strongly regular graph G and two arcs uv and wx. However, to know whether one of the values given in Section 3.1 occurs in $(U(G)^3)_{wx,uv}$, it is also necessary to know when there exist two arcs uv and wx for each of the cases of the previous section.

Recall that a strongly regular graph G is primitive if G is connected and the complement of G is connected. Otherwise, G is said to be imprimitive. The next lemma, found in [9, p.218] characterizes the only class of imprimitive graphs.

3.2.1 Lemma. *[9] Let G be a strongly regular graph with parameters*

$$(n, k, a, c).$$

Then the following are equivalent:

i) G is not connected,

ii) $c = 0$,

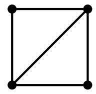
iii) $a = k - 1$, and

iv) G is the disjoint union of complete graphs on k vertices.

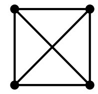
We will proceed to show that, up to some restrictions, a strongly regular graph must have certain induced subgraphs. It will be natural to sometimes only focus on primitive strongly regular graphs and strongly regular graphs where $a, c \geq 2$. Many parameter classes with $a, c < 2$ have unique graphs. For example, if $a = 0$ and $c = 1$, then the smallest known examples are the pentagon with parameters $(5, 2, 0, 1)$, the Petersen graph with parameters $(10, 3, 0, 1)$ and the Hoffman-Singleton graph with parameters $(50, 7, 0, 1)$. All three graphs are known to be the unique graphs with their parameter sets. Under these restrictions, we will show that the requirement of every case is satisfied by some pair of arcs of the digraph of G. Then, we will show that the values that occur as entries of $(U(G)^3)_{wx,uv}$ are completely determined by the parameters of G.

For convenience, we will write $a \sim b$ to say that a is adjacent to b, where a and b are vertices. We will also write $a \nsim b$ when a is not adjacent to b.

3.2.2 Lemma. *If G is a strongly regular graph with $a \geq 2$, then G contains either H_1 or H_2 as an induced subgraph.*



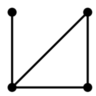
H_1



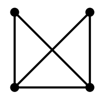
H_2

Proof. Let x, y be adjacent vertices of G. Since $a \geq 2$, we have that x and y have at least two adjacent neighbours, say p and q. Let H be induced subgraph of G on vertex set $\{x, y, p, q\}$. If $p \nsim q$, then H is isomorphic to H_1. Otherwise, H is isomorphic to H_2. □

3.2.3 Lemma. *If G is a primitive strongly regular graph with $a \geq 1$, then G contains either H_3 or H_4 as an induced subgraph.*



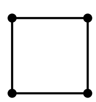
H_3



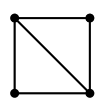
H_4

Proof. Let x and y be adjacent vertices of G. Since $a \geq 1$, there exists $z \in V(G)$ adjacent to both x and y. Consider the neighbourhoods of x, y and z. Suppose there is a vertex $p \notin \{x, y, z\}$ that lies in the neighbourhoods of two of $\{x, y, z\}$, but not all three. Then, the subgraph of G induced by vertex set $\{x, y, z, p\}$ is isomorphic to H_3, or H_4, when p is in the neighbourhood of exactly one, or two, of $\{x, y, z\}$, respectively. Otherwise, x and y have exactly $k-1$ neighbours in common, so $a = k-1$. Then G is imprimitive by Lemma 3.2.1, a contradiction. □

3.2.4 Lemma. *If G is a primitive strongly regular graph with $c \geq 2$, then G contains either H_5 or H_6 as an induced subgraph.*



H_5



H_6

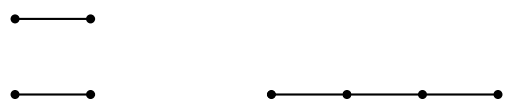
Proof. Since G is primitive, there exists two vertices of G, say x and y, such that $x \nsim y$. Since $c \geq 2$, we have that x and y have at least two common neighbours, say p and q. Let H be induced subgraph of G on vertex set $\{x, y, p, q\}$. If $p \nsim q$, then H is isomorphic to H_5. Otherwise, H is isomorphic to H_6. □

3.2.5 Lemma. *If G is a primitive strongly regular graph, then G contains either H_7 or H_8 as an induced subgraph.*

H_7 H_8

Proof. Since G is primitive, $c > 0$. Let x and y be nonadjacent vertices of G and z be a common neighbour of x and y. If $c = k$, then the complement of G, an $(n, n-k-1, n-2-2k+c, n-2k+1)$ strongly regular graph, is imprimitive by Lemma 3.2.1 and as is G, a contradiction. Then $c < k$, so x has a neighbour that is not a neighbour of y, say w. Let H be induced subgraph of G on vertex set $\{x, y, z, w\}$. If $z \nsim w$, then H is isomorphic to H_7. Otherwise, H is isomorphic to H_8. □

3.2.6 Lemma. *If G is a primitive strongly regular graph, then G contains either H_9 or H_7 as an induced subgraph.*



H_9 H_7

Proof. As in the proof of Lemma 3.2.5, let x and y be nonadjacent vertices of G, let w a neighbour of x that is not adjacent to y and z be a neighbour of y that is not adjacent to x. Let H be induced subgraph of G on vertex set $\{x, y, z, w\}$. If $z \nsim w$, then H is isomorphic to H_9. Otherwise, H is isomorphic to H_7. □

We can now conclude with the following theorem.

3.2.7 Theorem. *If G is a primitive, (n, k, a, c), strongly regular graph with $a \geq 2$ and $c \geq 2$, then the distinct entries of $U(G)^3$ are determined by the parameters of G.*

Proof. We showed in Lemma 3.1.1 that, given uv and wx, the entry

$$(U(G)^3)_{wx,uv}$$

depends only on the parameters of G. We will show that, if G is primitive with $a \geq 2$ and $c \geq 2$, then the digraph G will always have some pair of arcs that fall into each of the cases in the proof of Lemma 3.1.1. Let D be the digraph of G.

Lemma 3.2.2 shows that D always has some pair of arcs that fall into Case I.a in the proof of Lemma 3.1.1. Lemma 3.2.3 shows that D always has some pair of arcs that fall into Case I.b and Case I.c. Lemma 3.2.4 shows that D always has some pair of arcs that fall into Case I.d and Case I.e. Lemma 3.2.5 shows that D always has some pair of arcs that fall into Case I.f and Case I.g. Lemma 3.2.6 shows that D always has some pair of arcs that fall into Case I.h.

Since $a > 0$, any edge in G lies on a triangle. Since $c > 0$, any pair of nonadjacent vertices in G lies on an induced path of length 2. Then, the digraph of G has pairs of arcs that fall into all of the subcases of Case II. Since G is nonempty, the digraph G will have pairs of arcs that fall into both of the subcases of Case III. □

3.3 Decompositions of U^3 into 0-1 Matrices

Armed with Theorem 3.2.7, we can now examine decompositions of $U(G)^3$ into a summation of 01-matrices, for a strongly regular graph G. We can write,

$$U(G)^3 = \sum_{i=1}^{m} c_i A_i$$

where i is the number of distinct entries in $U(G)^3$ and

$$(A_i)_{x,y} = \begin{cases} 1 & \text{if } (U(G)^3)_{x,y} = c_i \\ 0 & \text{otherwise.} \end{cases}$$

If the hypotheses of Theorem 3.2.7 are fulfilled, then we know that the entries of $U(G)^3$ depend on the parameters. We have shown the following corollary.

3.3.1 Corollary. *Let G and H be two primitive strongly regular graphs with the same parameters (n, k, a, c), such that $a \geq 2$ and $c \geq 2$. Then there exists decompositions*

$$U(G)^3 = \sum_{i=1}^{m} c_i A_i$$

and

$$U(H)^3 = \sum_{i=1}^{\ell} c_i B_i$$

into summations of 0-1 matrices, where

$$(A_i)_{p,q} = \begin{cases} 1 & \textit{if } (U(G)^3)_{p,q} = c_i \\ 0 & \textit{otherwise} \end{cases}$$

and

$$(B_i)_{p,q} = \begin{cases} 1 & \textit{if } (U(H)^3)_{p,q} = c_i \\ 0 & \textit{otherwise.} \end{cases}$$

In Chapter 5, the above corollary and observations of the regular graphs not distinguished by the procedure will be used to give insight into how counterexamples for Conjecture 2.3.2 might arise.

3.4 Remarks

Strongly regular graphs with parameters (n, k, a, c) such that $c > 0$ have diameter 2. Thus, fixing a reference vertex v in a strongly regular graph G, we can consider its set of neighbours, say $B_1(v)$, and the set of vertices at distance two from v, which we will call $B_2(v)$. If we consider walks starting at arc i and ending at arc j, where v is the tail of i, then the head of i is in B_1. We can then consider where the ends of j lie with respect to the partition $\{\{v\}, B_1, B_2\}$ of $V(G)$. If the ends of j are in $\{v\} \cup B_1$, then $U^3_{i,j}$ is determined by the structure of the first neighbourhoods of vertices of G.

Since any two point graphs of generalized quadrangles of the same order have isomorphic neighbours, with respect to some reference vertex, it is interesting to consider such graphs. In particular, it is interesting to consider the point graphs of $T_2^*(O)$ and $T_2^*(O')$, where O and O' are hyperovals such that there is no isomorphism of the underlying space mapping O and O', as the graphs will have not only isomorphic first neighbourhoods $B_1(v)$, but similar structures in $B_2(v)$ as well, for any choice of v in the vertex sets.

Chapter 4

An Expression for $S^+(U^3)$

From Chapter 3, we can decompose U^3 as a sum of matrices with the entries of U^3 as the coefficients;

$$U(G)^3 = \sum_{i=1}^{m} c_i A_i$$

where $\{c_i\}$ is a multiset of entries in $U(G)^3$ and each A_i is a matrix with entries in $\{0,1\}$. From this, we can write $S^+(U)$ as a sum of a subset of the A_is. In particular, we would like to write $S^+(U^3)$ as a sum of matrices which are products of D_h, D_t and A, as this approach was successful in finding the eigenvalues of $S^+(U^2)$.

In this chapter, we will obtain an expression for $S^+(U)$ using the case analysis in Section 3.1, by summing the matrices corresponding to the positive entries of U^3.

4.1 Positive Entries of $S^+(U^3)$

Since we have found the entries of U^3 for a strongly regular graph G with parameters (n, k, a, c), it is also possible to find the cases when the entries of U^3 are positive. Table 4.1 is an altered version of Table 3.5 with added information about when each value is positive and excluding graphs where $k < 2$.

In particular, we see that if $a > 1$, $c > 1$ and $a, c \leq \frac{k}{2}$, then only uv and wx falling into cases I.e, I.h, II.ii.b, II.iii.b, II.iv.a., II.iv.b and III.ii will give positive entries in U. The requirements on a, c and k are met by many

Case	Entry of $(U(G)^3)_{wx,uv}$	$(U(G)^3)_{wx,uv} > 0$ if and only if:
I.a	$\frac{8\max\{0,a-2\}+16-8k}{k^3}$	never; $(U(G)^3)_{wx,uv} < 0$ for all strongly regular graphs
I.b	$\frac{8\max\{0,a-1\}+8-4k}{k^3}$	$a > 1$ and $2a > k$
I.c	$\frac{8\max\{0,a-1\}+8-4k}{k^3}$	$a > 1$ and $2a > k$
I.d	$\frac{8\max\{0,c-2\}+16-8k}{k^3}$	never; $(U(G)^3)_{wx,uv} < 0$ for all strongly regular graphs
I.e	$\frac{8a}{k^3}$	$a > 0$
I.f	$\frac{8\max\{0,c-1\}+8-4k}{k^3}$	$c > 1$ and $2c > k$
I.g	$\frac{8\max\{0,c-1\}+8-4k}{k^3}$	$c > 1$ and $2c > k$
I.h	$\frac{8c}{k^3}$	$c > 0$
II.i	$\begin{cases} 4 & \text{if } k=1 \\ 0 & \text{otherwise} \end{cases}$	$k = 1$
II.ii.a	$\frac{8\max\{0,a-1\}+8-4k}{k^3}$	$a > 1$ and $2a > k$
II.ii.b	$\frac{8a}{k^3}$	$a > 0$
II.iii.a	$\frac{8\max\{0,a-1\}+8-4k}{k^3}$	$a > 1$ and $2a > k$
II.iii.b	$\frac{8a}{k^3}$	$a > 0$
II.iv.a	$\frac{8\max\{0,a-1\}+8-8k+2k^2}{k^3}$	$a > 1$
II.iv.b	$\frac{8\max\{0,c-1\}+8-8k+2k^2}{k^3}$	$c > 1$
III.i	$\frac{2-k}{k}$	never; $(U(G)^3)_{wx,uv} < 0$ for all strongly regular graphs
III.ii	$\frac{8a}{k^3}$	$a > 0$

Table 4.1: Entries of $(U(G)^3)_{wx,uv}$ given uv and wx and positivity conditions.

strongly regular graphs, in particular by the point graph of a generalized quadrangle with order (s, t) if $s > 2$ and $t > 1$. We may then write an algebraic expression for $S^+(U^3)$ entry-wise, by finding an expression such that

$$(S^+(U^3))_{wx,uv} = \begin{cases} 1 & \text{if } uv \text{ and } wx \text{ fall into one of cases:} \\ & \text{I.e, I.h, II.ii.b, II.iii.b, II.iv.a., II.iv.b and III.ii} \\ 0 & \text{otherwise.} \end{cases}$$

Let A be the adjacency matrix of G. Then, the wx, uv entry of $S^+(U(G)^3)$ is:

$$\begin{aligned} S^+(U(G)^3)_{wx,uv} = & \Delta(1 - A_{uw})(1 - A_{vx}) - (1 - A_{vx})\delta_{uw} + (1 - A_{uw})\delta_{vx} - \\ & \delta_{vx}\delta_{uw} + \delta_{ux}(1 - \delta_{vw}) \end{aligned}$$

where δ is the Kronecker delta and

$$\Delta = (1 - \delta_{uw})(1 - \delta_{ux})(1 - \delta_{vw})(1 - \delta_{vx}).$$

This equation is, however, too complicated. Instead, we want to write $S^+(U^3)$ in terms of D_h and D_t as defined in Chapter 2 so as to express $S^+(U^3)$ in terms of the properties of G.

4.2 Incidence Matrices from Case Analysis

The cases as defined in Section 3.1 partition the pairs of arcs of G. Each case defines a relation; we can write that $uv \sim_{\text{I.a}} wx$ if (uv, wx) falls into case I.a in the analysis, which is to say that $\{u, v, w, x\}$ are all distinct and either $\{u, v, w, x\}$ are all mutually adjacent or $\{u, v, w, x\}$ are all mutually adjacent, except for u and x. We can write an incidence matrix for this relation; let $M_{\text{I.a}}$ be the matrix such that

$$(M_{\text{I.a}})_{wx,uv} = \begin{cases} 1 & \text{if } uv \sim_{\text{I.a}} wx \\ 0 & \text{otherwise.} \end{cases}$$

Since (uv, wx) falls into exactly one case, the incidence matrices of two different cases do not have ones in the same positions and the sum of the incidence matrices of all of the cases is the all ones matrix.

Then, if $a > 1$, $c > 1$ and $a, c \leq \frac{k}{2}$, we have

$$S^+(U^3) = M_{\text{I.e}} + M_{\text{I.h}} + M_{\text{II.ii.b}} + M_{\text{II.iii.b}} + M_{\text{II.iv.a}} + M_{\text{II.iv.b}} + M_{\text{III.ii}}.$$

It remains to find each of the matrices M_x for each case. Let A and $\overline{A}$ be the adjacency matrix of G and the complement of G respectively. Then

$$\overline{A} = J - I - A$$

where J is the all-ones matrix and I is the identity. Recall the following incidence matrices of D, the digraph of G, with rows indexed by the vertices of D and columns indexed by the arcs of D:

$$(D_h)_{i,j} = \begin{cases} 1 & \text{if } i \text{ is the head of arc } j \\ 0 & \text{otherwise} \end{cases}$$

and

$$(D_t)_{i,j} = \begin{cases} 1 & \text{if } i \text{ is the tail of arc } j \\ 0 & \text{otherwise.} \end{cases}$$

If B and C are both $m \times n$ matrices, then the *Schur product* of B and C is the entry-wise product of B and C, as follows:

$$(B \circ C)_{p,q} = B_{p,q} C_{p,q}.$$

4.2.1 Lemma. *If G is a strongly regular graph, then*

$$M_{I.e} = (D_h^T \overline{A} D_h) \circ (D_t^T \overline{A} D_t) \circ (D_t^T A D_h) \circ (J + D_h^T D_t).$$

Proof. In this case, $uv \sim_{\text{I.e}} wx$ if $\{u, v, w, x\}$ are all distinct, v is adjacent to w and not x, u is not adjacent to w.

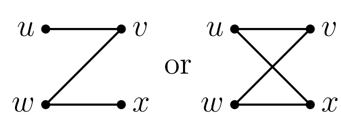


Figure 4.1: Possible subgraphs of G induced by $\{u, v, w, x\}$ such that $uv \sim_{\text{I.e}} wx$.

In other words, if $i \sim_{\text{I.e}} j$, then the head of i is adjacent with the tail of j, the head of i is not adjacent with the head of j, the tail of i is not adjacent with the tail of j, the tail of i is either adjacent or not adjacent with the head of j. Note that by specifying the adjacency relation between the heads and tails of i and j, we have also determined that i and j do not have any common endpoints. Observe that, letting N_ℓ be the ℓth column of matrix N, we obtain:

$$(D_h^T A D_h)_{j,i} = (D_h)_j^T A (D_h)_i = \begin{cases} 1 & \text{if head of } j \text{ is adjacent to the head of } i \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$M_{\text{I.e}} = (D_h^T \overline{A} D_h) \circ (D_t^T \overline{A} D_t) \circ (D_t^T A D_h) \circ (D_h^T \overline{A} D_t + D_h^T A D_t).$$

Each column of D_h and D_t has one position with an entry of 1 and all other positions have entry 0. Then $D_h\mathbf{j} = \mathbf{j}$ and $D_t\mathbf{j} = \mathbf{j}$, where $\mathbf{j}$ is the all ones vector. Then, we can observe that $D_h^T J D_t = J$ and obtain:

$$\begin{aligned} D_h^T \overline{A} D_t + D_h^T A D_t &= D_h^T(\overline{A} + A)D_t. \\ &= D_h^T(J - I - A + A)D_t \\ &= D_h^T J D_t + D_h^T I D_t \\ &= J + D_h^T D_t \end{aligned}$$

and

$$M_{\text{I.e}} = (D_h^T \overline{A} D_h) \circ (D_t^T \overline{A} D_t) \circ (D_t^T A D_h) \circ (J + D_h^T D_t).$$

□

4.2.2 Lemma. *If G is a strongly regular graph, then*

$$M_{I.h} = (D_h^T \overline{A} D_h) \circ (D_t^T \overline{A} D_t) \circ (D_t^T \overline{A} D_h) \circ (J + D_h^T D_t).$$

Proof. In this case, $uv \sim_{\text{I.h}} wx$ if $\{u, v, w, x\}$ are all distinct, v is not adjacent to w nor x, u is not adjacent to w.

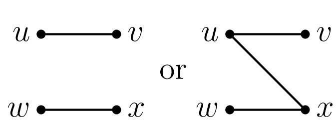


Figure 4.2: Possible subgraphs of G induced by $\{u, v, w, x\}$ such that $uv \sim_{\text{I.h}} wx$.

In other words, if $i \sim_{\text{I.h}} j$, then the head of i is not adjacent with the tail of j or with the head of j, the tail of i is not adjacent with the tail of j, the tail of i is either adjacent or not adjacent with the head of j. Then, similar to the previous case, we find

$$\begin{aligned} M_{\text{I.h}} &= (D_h^T \overline{A} D_h) \circ (D_t^T \overline{A} D_t) \circ (D_t^T \overline{A} D_h) \circ (D_h^T \overline{A} D_t + D_h^T A D_t) \\ &= (D_h^T \overline{A} D_h) \circ (D_t^T \overline{A} D_t) \circ (D_t^T \overline{A} D_h) \circ (J + D_h^T D_t) \end{aligned}.$$

□

4.2.3 Lemma. *If G is a strongly regular graph, then*

$$M_{\text{II.ii.b}} = D_t^T D_t - I - D_t^T D_t \circ D_h^T A D_h.$$

Proof. In this case, $uv \sim_{\text{II.ii.b}} wx$ if $\{u, v, x\}$ are all distinct and $u = w$ and v is not adjacent to x.

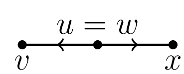


Figure 4.3: Subgraph of G induced by $\{u, v, w, x\}$ such that $uv \sim_{\text{II.ii.b}} wx$.

In other words, if $i \sim_{\text{II.ii.b}} j$, then i and j share the same tail and the heads of i and j are not adjacent. Then,

$$\begin{aligned} M_{\text{II.ii.b}} &= (D_h^T \overline{A} D_h) \circ (D_t^T D_t) \\ &= (J - D_h^T D_h - D_h^T A D_h) \circ (D_t^T D_t) \\ &= D_t^T D_t - I - D_t^T D_t \circ D_h^T A D_h \end{aligned}$$

noting that $D_t^T D_t \circ D_h^T D_h = I$ since if two arcs share the same head and tail, then the arcs are equal. □

4.2.4 Lemma. *If G is a strongly regular graph, then*

$$M_{\text{II.iii.b}} = D_h^T D_h - I - D_h^T D_h \circ D_t^T A D_t.$$

Proof. In this case, $uv \sim_{\text{II.iii.b}} wx$ if $\{u, v, w\}$ are all distinct and $v = x$ and u is not adjacent to w.

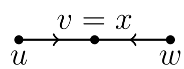


Figure 4.4: Subgraph of G induced by $\{u, v, w, x\}$ such that $uv \sim_{\text{II.iii.b}} wx$.

In other words, if $i \sim_{\text{II.iii.b}} j$, then i and j share the same head and the tails of i and j are not adjacent. Then,

$$\begin{aligned} M_{\text{II.iii.b}} &= (D_h^T D_h) \circ (D_t^T \overline{A} D_t) \\ &= (D_h^T D_h) \circ (J - D_t^T D_t - D_t^T A D_t). \\ &= D_h^T D_h - I - D_h^T D_h \circ D_t^T A D_t \end{aligned}$$

□

4.2.5 Lemma. *If G is a strongly regular graph, then*

$$M_{II.iv.a} = (D_h^T D_t) \circ (D_t^T A D_h).$$

Proof. In this case, $uv \sim_{\text{II.iv.a}} wx$ if $\{u, v, w\}$ are all distinct and $u = x$ and v is adjacent to w.

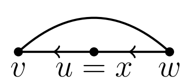


Figure 4.5: Subgraph of G induced by $\{u, v, w, x\}$ such that $uv \sim_{\text{II.iv.a}} wx$.

In other words, if $i \sim_{\text{II.iv.a}} j$, then the tail i is equal to the head of j and the head of i is adjacent to the tail of j. Then,

$$M_{\text{II.iv.a}} = (D_h^T D_t) \circ (D_t^T A D_h).$$

□

4.2.6 Lemma. *If G is a strongly regular graph, then*

$$M_{\text{II.iv.b}} = (D_h^T D_t) \circ (D_t^T \overline{A} D_h).$$

Proof. In this case, $uv \sim_{\text{II.iv.b}} wx$ if $\{u, v, w\}$ are all distinct and $u = x$ and v is not adjacent to w.

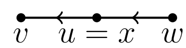


Figure 4.6: Subgraph of G induced by $\{u, v, w, x\}$ such that $uv \sim_{\text{II.iv.b}} wx$.

In other words, if $i \sim_{\text{II.iv.b}} j$, then the tail i is equal to the head of j and the head of i is not adjacent to the tail of j. Then,

$$M_{\text{II.iv.b}} = (D_h^T D_t) \circ (D_t^T \overline{A} D_h).$$

□

4.2.7 Lemma. *If G is a strongly regular graph, then*

$$M_{\text{III.ii}} = I$$

where I is the identity matrix on the number of arcs of G.

Proof. In this case, $uv \sim_{\text{III.ii}} wx$ if $u = w$ and $v = x$.

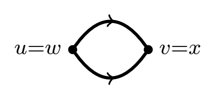


Figure 4.7: Subgraph of G induced by $\{u, v, w, x\}$ such that $uv \sim_{\text{III.ii}} wx$.

In other words, if $i \sim_{\text{III.ii}} j$, then the heads and tail of i and j both agree. Then, $M_{\text{III.ii}} = I$. □

We can now determine an expression for $S^+(U^3)$.

4.2.8 Theorem. *If G is a strongly regular graph with parameters (n, k, a, c) such that $a > 1$, $c > 1$ and $a, c, \leq \frac{k}{2}$, then*

$$S^+(U^3) = J - D_t^T A D_t - D_h^T A D_h + (D_h^T A D_h) \circ (D_t^T A D_t) + D_h^T D_t - P.$$

Proof. From Lemmas 4.2.1, 4.2.2, 4.2.3, 4.2.4, 4.2.5, 4.2.6 and 4.2.7, we have

$$\begin{aligned}
S^+(U^3) &= M_{\mathrm{I.e}} + M_{\mathrm{I.h}} + M_{\mathrm{II.ii.b}} + M_{\mathrm{II.iii.b}} + M_{\mathrm{II.iv.a}} + M_{\mathrm{II.iv.b}} + M_{\mathrm{III.ii}} \\
&= (D_h^T \overline{A} D_h) \circ (D_t^T \overline{A} D_t) \circ (D_t^T A D_h) \circ (J + D_h^T D_t) \\
&\quad + (D_h^T \overline{A} D_h) \circ (D_t^T \overline{A} D_t) \circ (D_t^T \overline{A} D_h) \circ (J + D_h^T D_t) \\
&\quad + D_t^T D_t - I - D_t^T D_t \circ D_h^T A D_h + D_h^T D_h - I - D_h^T D_h \circ D_t^T A D_t \\
&\quad + (D_h^T D_t) \circ (D_t^T A D_h) + (D_h^T D_t) \circ (D_t^T \overline{A} D_h) + I.
\end{aligned}$$

We need the following observations in order to simplify the expression. If arcs i and j have non-adjacent heads and non-adjacent tails, then i and j do not share head or tails. Then,

$$(D_h^T \overline{A} D_h) \circ (D_t^T \overline{A} D_t) \circ D_t^T D_h = (D_h^T \overline{A} D_h) \circ (D_t^T \overline{A} D_t) \circ D_h^T D_t = 0$$

and

$$(D_h^T \overline{A} D_h) \circ (D_t^T \overline{A} D_t) \circ P = 0$$

where P is the matrix of the permutation that takes each arc uv to its reverse arc vu. Then, we can simplify $M_{\mathrm{I.e}} + M_{\mathrm{I.h}}$ as follows:

$$\begin{aligned}
M_{\mathrm{I.e}} + M_{\mathrm{I.h}} &= (D_h^T \overline{A} D_h) \circ (D_t^T \overline{A} D_t) \circ (D_t^T A D_h) \circ (J + D_h^T D_t) \\
&\quad + (D_h^T \overline{A} D_h) \circ (D_t^T \overline{A} D_t) \circ (D_t^T \overline{A} D_h) \circ (J + D_h^T D_t) \\
&= (D_h^T \overline{A} D_h) \circ (D_t^T \overline{A} D_t) \circ (J + D_h^T D_t) \circ (D_t^T \overline{A} D_h + D_t^T A D_h) \\
&= (D_h^T \overline{A} D_h) \circ (D_t^T \overline{A} D_t) \circ (J + D_h^T D_t) \circ (J + D_t^T D_h) \\
&= (D_h^T \overline{A} D_h) \circ (D_t^T \overline{A} D_t) \circ (J + D_h^T D_t + D_t^T D_h + (D_h^T D_t) \circ (D_t^T D_h)) \\
&= (D_h^T \overline{A} D_h) \circ (D_t^T \overline{A} D_t) \circ (J + D_h^T D_t + D_t^T D_h + P) \\
&= (D_h^T \overline{A} D_h) \circ (D_t^T \overline{A} D_t)
\end{aligned}$$

Since we want the expression to be a sum of products of D_h, D_t and A. We substitute $\overline{A} = J - I - A$ in the expression for $M_{\mathrm{I.e}} + M_{\mathrm{I.h}}$ to obtain:

$$\begin{aligned}
M_{\mathrm{I.e}} + M_{\mathrm{I.h}} &= (D_h^T \overline{A} D_h) \circ (D_t^T \overline{A} D_t) \\
&= (D_h^T (J - I - A) D_h) \circ (D_t^T (J - I - A) D_t) \\
&= (J - D_h^T D_h - D_h^T A D_h) \circ (J - D_t^T D_t - D_t^T A D_t) \\
&= J + I - D_t^T D_t - D_h^T D_h - D_t^T A D_t - D_h^T A D_h \\
&\quad + (D_h^T D_h) \circ (D_t^T A D_t) + (D_t^T D_t) \circ (D_h^T A D_h) \\
&\quad + (D_h^T A D_h) \circ (D_t^T A D_t)
\end{aligned}$$

We can also simply $M_{\text{II.iv.a}} + M_{\text{II.iv.b}}$ as follows:

$$\begin{aligned}
M_{\text{II.iv.a}} + M_{\text{II.iv.b}} &= (D_h^T D_t) \circ (D_t^T A D_h) + (D_h^T D_t) \circ (D_t^T \overline{A} D_h) \\
&= (D_h^T D_t) \circ (D_t^T A D_h + D_t^T \overline{A} D_h) \\
&= (D_h^T D_t) \circ (D_t^T (A + \overline{A}) D_h) \\
&= (D_h^T D_t) \circ (D_t^T (J - I) D_h) \\
&= (D_h^T D_t) \circ (J - D_t^T D_h) \\
&= D_h^T D_t - P
\end{aligned}$$

From the above analysis, we obtain:

$$\begin{aligned}
S^+(U^3) = {} & M_{\text{I.e}} + M_{\text{I.h}} + M_{\text{II.ii.b}} + M_{\text{II.iii.b}} + M_{\text{II.iv.a}} + M_{\text{II.iv.b}} + M_{\text{III.ii}} \\
= {} & J + I - D_t^T D_t - D_h^T D_h - D_t^T A D_t - D_h^T A D_h \\
& + (D_h^T D_h) \circ (D_t^T A D_t) + (D_t^T D_t) \circ (D_h^T A D_h) \\
& + (D_h^T A D_h) \circ (D_t^T A D_t) + D_t^T D_t - I - (D_t^T D_t) \circ (D_h^T A D_h) \\
& + D_h^T D_h - I - (D_h^T D_h) \circ (D_t^T A D_t) + D_h^T D_t - P + I \\
= {} & J - D_t^T A D_t - D_h^T A D_h + (D_h^T A D_h) \circ (D_t^T A D_t) + D_h^T D_t - P
\end{aligned}$$

□

It must be noted that by finding the adjacency matrices M_x for the other cases, one can find the expression for $S^+(U^3)$ for other values of a and c. We will do one more case here.

4.2.9 Theorem. *If G is a strongly regular graph with parameters (n, k, a, c) such that $a, c > \frac{k}{2}$, then*

$$S^+(U^3) = J - (D_h^T A D_h) \circ (D_t^T A D_t) + D_h^T D_t - P.$$

Proof. Observe that if $a > \frac{k}{2}$ and $c > \frac{k}{2}$, then $(U^3)_{wx,uv}$ is positive for all pairs (uv, wx) in all cases of the proof of Lemma 3.1.1, except Case I.a, Case I.d, Case II.i, and Case III.i. We observe that, given two arcs i and j, if the heads of i and j are adjacent and the tails of i and j are adjacent, the (i, j) must fall into one of Cases I.a, I.d, II.i, II.iv.a, II.iv.b, and III.i. Then

$$(D_h^T A D_h) \circ (D_t^T A D_t) = M_{\text{I.a}} + M_{\text{I.d}} + M_{\text{II.i}} + M_{\text{II.iv.a}} + M_{\text{II.iv.b}} + M_{\text{III.i}}. \quad (4.2.1)$$

We then obtain

$$\begin{aligned} S^+(U^3) &= J - M_{\text{I.a}} - M_{\text{I.d}} - M_{\text{II.i}} - M_{\text{III.i}} \\ &= J - (M_{\text{I.a}} + M_{\text{I.d}} + M_{\text{II.i}} + M_{\text{II.iv.a}} + M_{\text{II.iv.b}} + M_{\text{III.i}}) \\ &\quad + (M_{\text{II.iv.a}} + M_{\text{II.iv.b}}) \end{aligned}$$

Then, by Equation (4.2.1) and from our earlier simplification in the proof of Theorem 4.2.8 that

$$M_{\text{II.iv.a}} + M_{\text{II.iv.b}} = D_h^T D_t - P,$$

we obtain:

$$S^+(U^3) = J - (D_h^T A D_h) \circ (D_t^T A D_t) + D_h^T D_t - P.$$

□

Chapter 5

Graphs Undistinguished by the Spectrum of $S^+(U^3)$

Emms, Severini, Wilson and Hancock only claim Conjecture 2.3.2 for strongly regular graphs; the procedure does not distinguish non-isomorphic graphs in general. In [5, 6], the authors give two regular graphs on 14 vertices that are not isomorphic but have the same spectrum with respect to $S^+(U^3)$. Gordon Royle provides two regular graphs on 16 vertices that are not isomorphic but have the same spectrum with respect to $S^+(U^3)$. Further, the construction of Cai, Fürer and Immerman in [3] also gives one pair of graphs which are not distinguished by the quantum walk procedure.

Although none of these graphs are strongly regular, it is still useful to look at these graphs to obtain intuition in constructing strongly regular graphs that are undistinguished by the spectrum of $S^+(U^3)$.

5.1 Regular Graphs with Cospectral Mates

Let G_1 and G_2 be defined as follows in Figure 5.1:

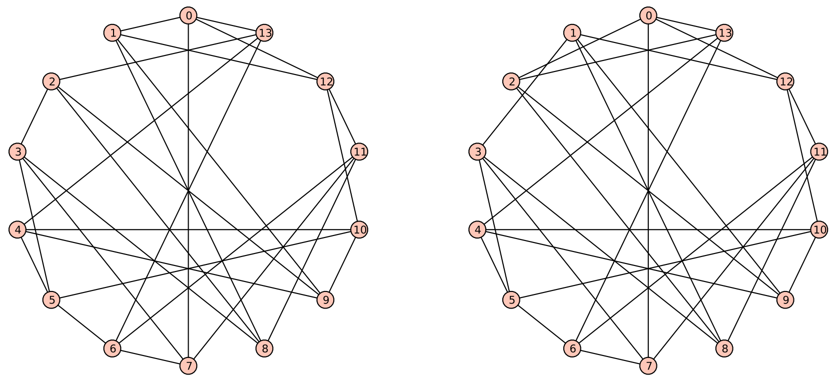


Figure 5.1: 14-vertex graphs G_1 and G_2

Both of G_1 and G_2 are regular graphs on 14 vertices with valency 4 and are found in [5, 6]. Observe that G_2 is obtained from G_1 by replacing the edges $\{0, 1\}$ and $\{2, 3\}$ with edges $\{0, 2\}$ and $\{1, 3\}$. It can be verified that G_1 is not isomorphic to G_2 but $S^+(U(G_1)^3)$ and $S^+(U(G_2)^3)$ have the same spectrum.

Let G_3 and G_4 be defined as follows, in Figure 5.2:

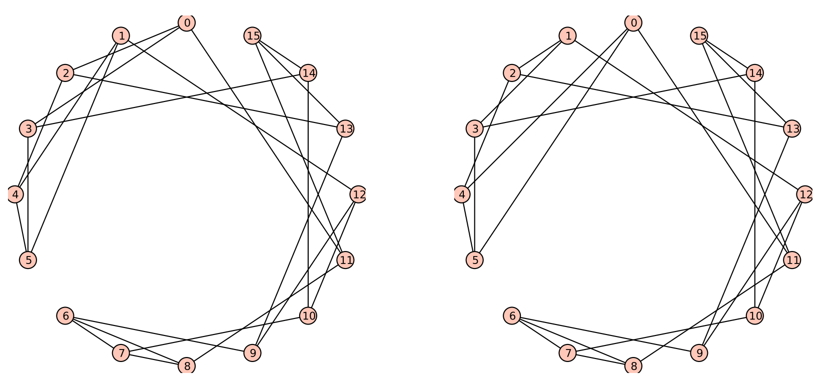


Figure 5.2: 16-vertex graphs G_3 and G_4

Both of G_3 and G_4 are regular graphs on 16 vertices with valency 3 and are found by Royle. Observe that G_4 is obtained from G_3 by changing the neighbours of vertices 0 and 1 in $\{2,3,4,5\}$ to non-neighbours and non-neighbours to neighbours. It can be verified that G_3 is not isomorphic to G_4 but $S^+(U(G_3)^3)$ and $S^+(U(G_4)^3)$ have the same spectrum.

5.2 Cai-Fürer-Immerman Graphs

In [3], the authors constructed graphs $X(G)$ from a given graph G, usually a low degree graph with linear size separators, and a switching operation as counterexamples for the symmetric power of a graph being a polynomial time graph invariant. We will refer to the graph $X(G)$ as defined in this section as the *Cai-Fürer-Immerman graph* of G.

For each positive integer d, we define a graph X_d as follows: X_d has vertex set V_d and edge set E_d. The vertex set consists of 3 disjoint sets, A_d, B_d and M_d such that A_d and B_d each contain d elements, indexed by $\{1,\dots,d\}$ and M_d contains one element indexed by each even subset of $\{1,\dots,d\}$. Vertices $a_i \in A_d$ and $m_S \in M_d$ are adjacent if $i \in S$ and vertices $b_j \in B_d$ and $m_S \in M_d$ are adjacent if $j \notin S$.

To obtain the Cai-Fürer-Immerman graph of G, we replace each vertex v of G with a graph $X(v)$ where $X(v) = X_d$ and d is the valency of v. In $X(v)$ we associate one pair of vertices $\{a_i, b_i\}$ with each neighbour w of v and write $a(v,w) = a_i$ and $b(v,w) = b_i$. For each edge v,w in G, we add the edges $\{a(v,w), a(w,v)\}$ and $\{b(v,w), b(w,v)\}$.

A *twist* of $X(G)$ is obtained by choosing an edge of G and replacing the edges $\{a(v,w), a(w,v)\}$ and $\{b(v,w), b(w,v)\}$ in $X(G)$ with $\{a(v,w), b(w,v)\}$ and $\{b(v,w), a(w,v)\}$.

If we take the complete graph on 4 vertices, then $X(K_4)$ has 40 vertices and is regular with valency 4, see Figure 5.3. Let $\widetilde{X}(K_4)$ be the twist of $X(K_4)$. It can be verified that $X(K_4)$ and $\widetilde{X}(K_4)$ are not isomorphic but $S^+\big(U\left(X(K_4)\right)^3\big)$ and $S^+\left(U\left(\widetilde{X}(K_4)\right)^3\right)$ have the same spectrum.

5.3 Observations

For all three pairs of graphs,

$$P_1 = \{G_1, G_2\},\ P_2 = \{G_3, G_4\}$$

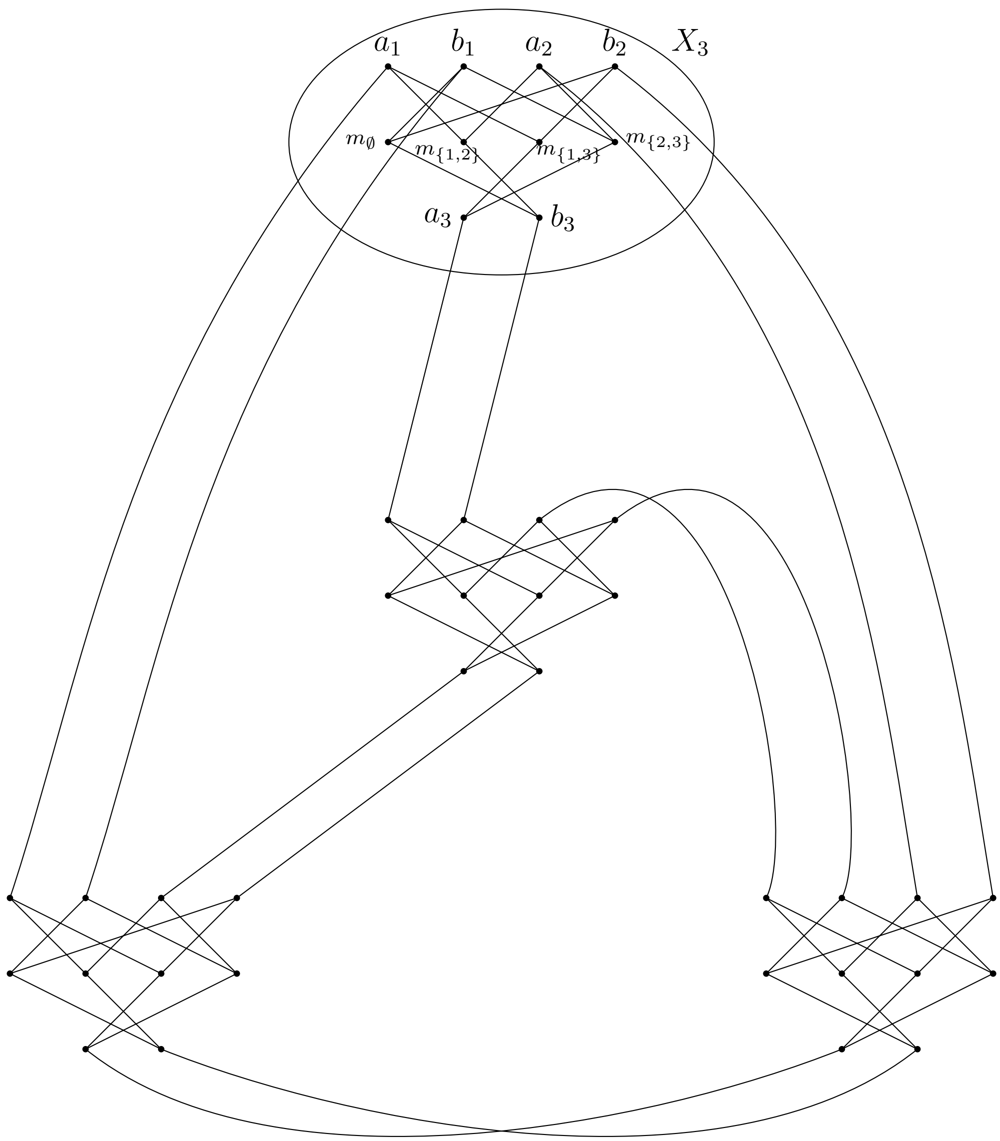


Figure 5.3: Cai-Fürer-Immerman graph of K_4, denoted $X(K_4)$

and

$$P_3 = \{X(K_4), \widetilde{X}(K_4)\},$$

we have some algebraic observations common to all three pairs.

For P_i, with $i = 1, 2, 3$, we notice that $U(\Gamma_1)^3$ and $U(\Gamma_2)^3$ for $\Gamma_1, \Gamma_2 \in P_i$ have the same set of distinct entries. Thus, if $\{c_j\}_{j=1}^m$ is the set of distinct entries of $U(\Gamma_1)^3$, we can decompose $U(\Gamma_1)^3$ and $U(\Gamma_2)^3$ into a sum of matrices with entries in $\{0, 1\}$, as follows:

$$U(\Gamma_1)^3 = \sum_{j=1}^{m} c_i A_i$$

and

$$U(\Gamma_2)^3 = \sum_{j=1}^{m} c_i B_i,$$

where

Then A_j and B_j are cospectral for each $j = 1, \ldots, m$.

Chapter 6

Conclusion

We proceed to summarize our results and propose questions for future research.

6.1 Summary of Results

We are mainly concerned with Conjecture 2.3.2 of Emms, Hancock, Severini and Wilson in [5, 6], which we will restate here.

Conjecture. *[5, 6] If G and H are strongly regular graphs, then $S^+(U(G)^3)$ and $S^+(U(H)^3)$ are cospectral if and only if G and H are isomorphic.*

Although we are unable to either prove or disprove the conjecture, we were able to prove a number of interesting properties.

- We resolved one direction of the conjecture in Theorem 2.3.1, showing that $S^+(U(G)^3)$ and $S^+(U(H)^3)$ are cospectral if G and H are isomorphic.
- Using the methods of [8], we find a complete proof that the eigenvalues of $S^+(U)$ are determined by the eigenvalues of the adjacency matrix, for any regular graph of valency k where $k \geq 2$.
- We show that $S^+(U^2) = S^+(U)^2 + I$ for any regular graph of valency k where $k \geq 2$.
- We show that a primitive strongly regular graph G with parameters (n, k, a, c), if $a \geq 2$ and $c \geq 2$, then the entries $U(G)^3$ are determined by (n, k, a, c).

- We show that for a strongly regular graph G with parameters (n, k, a, c), if $a \geq 1$ and $c \geq 1$, $a \leq \frac{k}{2}$ and $c \leq \frac{k}{2}$, then

  $$S^+(U^3) = J - D_t^T A D_t - D_h^T A D_h + (D_h^T A D_h) \circ (D_t^T A D_t) + D_h^T D_t - P$$

  and, if $a > \frac{k}{2}$ and $c > \frac{k}{2}$, then

  $$S^+(U^3) = J - (D_h^T A D_h) \circ (D_t^T A D_t) + D_h^T D_t - P.$$

6.2 Future Work

It is still an open problem to prove or disprove the remaining direction of Conjecture 2.3.2.

We have obtained some intuition in trying to construct counterexamples for the conjecture. In Chapter 5, we observed that for a pair $P = (\Gamma_1, \Gamma_2)$ of the known pairs of regular graphs not distinguished by the quantum walk procedure of [5, 6], that $U(\Gamma_1)^3$ and $U(\Gamma_2)^3$ have the same set of distinct entries and, even stronger, if $\{c_j\}_{j=1}^m$ is the set of distinct entries of $U(\Gamma_1)^3$, we can decompose $U(\Gamma_1)^3$ and $U(\Gamma_2)^3$ into a sum of matrices with entries in $\{0, 1\}$, as follows:

$$U(\Gamma_1)^3 = \sum_{j=1}^{m} c_i A_i$$

and

$$U(\Gamma_2)^3 = \sum_{j=1}^{m} c_i B_i,$$

where

$$(A_i)_{p,q} = \begin{cases} 1 & \text{if } (U(\Gamma_1)^3)_{p,q} = c_i \\ 0 & \text{otherwise} \end{cases}$$

and

$$(B_i)_{p,q} = \begin{cases} 1 & \text{if } (U(\Gamma_2)^3)_{p,q} = c_i \\ 0 & \text{otherwise.} \end{cases}$$

Then A_j and B_j are cospectral for $j = 1, \ldots, m$. By Chapter 3, we know that such a decomposition is also possible for any pair of strongly regular graphs with the same parameters. That is, given G and H coparametric strongly regular graphs and let $\{c_j\}_{j=1}^{\ell}$ be the set of distinct entries of $U(G)^3$, then a similar decomposition is possible. We then hope to find strongly regular graphs such that the corresponding matrices in the decomposition are cospectral, as in the case of the regular graphs undistinguished by the procedure.

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Index