# Harmonic analysis of Rajchman algebras.

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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# Abstract

Abstract harmonic analysis is mainly concerned with the study of locally compact groups, their unitary representations, and the function spaces associated with them. The Fourier and Fourier-Stieltjes algebras are two of the most important function spaces associated with a locally compact group.

The Rajchman algebra associated with a locally compact group is defined to be the set of all elements of the Fourier-Stieltjes algebra which vanish at infinity. This is a closed, complemented ideal in the Fourier-Stieltjes algebra that contains the Fourier algebra. In the Abelian case, the Rajchman algebras can be identified with the algebra of Rajchman measures on the dual group. Such measures have been widely studied in the classical harmonic analysis. In contrast, for non-commutative locally compact groups little is known about these interesting algebras.

In this thesis, we investigate certain Banach algebra properties of Rajchman algebras associated with locally compact groups. In particular, we study various amenability properties of Rajchman algebras, and observe their diverse characteristics for different classes of locally compact groups. We prove that amenability of the Rajchman algebra of a group is equivalent to the group being compact and almost Abelian, a property that is shared by the Fourier-Stieltjes algebra. In contrast, we also present examples of large classes of locally compact groups, such as non-compact Abelian groups and infinite solvable groups, for which Rajchman algebras are not even operator weakly amenable. Moreover, we establish various extension theorems that allow us to generalize the previous result to all non-compact connected SIN-groups.

Finally, we investigate the spectral behavior of Rajchman algebras associated with Abelian locally compact groups, and construct point derivations at certain elements of their spectrum using Varopoulos' decompositions for Rajchman algebras. Having constructed similar decompositions, we obtain analytic discs around certain idempotent characters of Rajchman algebras. These results, and others that we obtain, illustrate the inherent distinction between the Rajchman algebra and the Fourier algebra of many locally compact groups.

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# Dedication

Dedicated to my family:

My mother Mahboobeh,

my sisters Mahsa, Maryam, and Mahta,

and my husband Hamed.

# Contents

# Chapter 1

# Introduction

Amenability of a group is a fundamental notion in analysis that was originally introduced by von Neumann in 1929. This remarkable property has many equivalent definitions and various interpretations. For instance, one can think of amenability as a translation-invariant averaging condition for a locally compact group.

In 1972, Johnson defined amenable Banach algebras as those satisfying a certain cohomological property. The choice of terminology was inspired by Johnson's well-known theorem demonstrating the equivalence of amenability for a locally compact group and its convolution algebra [Joh72].

Since many important Banach algebras in harmonic analysis, e.g. the Fourier-Stieltjes algebras, are operator spaces as well, it is natural to also define the notion of operator amenability in order to take the operator space structure into account. The concept of (operator) amenability turned out to be extremely fruitful in the theory of (completely contractive) Banach algebras. For example, Connes [Con78] and

Haagerup [Haa83] showed that for $C^*$-algebras amenability and nuclearity coincide.

In his influential work, Eymard [Eym64] defined the Fourier and Fourier-Stieltjes algebras of locally compact groups, and studied many of their properties. For a locally compact group $G$, let $C^*(G)$ denote its group $C^*$-algebra. The Fourier-Stieltjes algebra of $G$, denoted by $B(G)$, is defined to be the Banach space dual of $C^*(G)$. One can show that $B(G)$ is in fact a subalgebra of the algebra of bounded continuous functions $C_b(G)$. Moreover the Fourier-Stieltjes algebra together with its norm as a dual space turns out to be a Banach algebra. The Fourier algebra is defined to be the closed subalgebra of the Fourier-Stieltjes algebra generated by its compactly supported elements, and is denoted by $A(G)$. The Fourier algebra is in turn a subalgebra of $C_0(G)$, the algebra of all continuous functions on $G$ which vanish at infinity. In the special case of locally compact Abelian groups, one can identify the Fourier and Fourier-Stieltjes algebras with the $L^1$-algebra and the measure algebra of the dual group.

In addition to the Fourier and Fourier-Stieltjes algebras, one can define the *Rajchman algebra* associated with a locally compact group $G$, denoted by $B_0(G)$, to be the set of elements of the Fourier-Stieltjes algebra which vanish at infinity. It is easy to see that the Rajchman algebra is indeed a Banach subalgebra of the Fourier-Stieltjes algebra.

We recall that a measure $\mu$ in the measure algebra of a locally compact Abelian group is called a Rajchman measure if

$$\lim_{|n|\to 0} \hat{\mu}(n) = 0.$$

Consequently, the Rajchman algebra on a locally compact Abelian group can be identified with the algebra of Rajchman measures on the dual group, denoted by $M_0(\hat{G})$.

The importance of Rajchman measures first became apparent in the study of uniqueness of trigonometric series. A subset $E$ of $\mathbb{T}$ is a *set of uniqueness* (or a $\mathcal{U}$-set) if the trivial series is the only trigonometric series which converges to 0 on every element outside $E$. Otherwise it is a *set of multiplicity.* The classical Cantor $\frac{1}{3}$-set is an example of a $\mathcal{U}$-set.

Sets of uniqueness are typically small. In fact, every Borel $\mathcal{U}$-set has Lebesgue measure 0. However the converse is not true. In 1916, Menshov showed that there are closed sets of Lebesgue measure zero which are not sets of uniqueness [Men16]. In his proof, Menshov constructs a probability measure $\mu$ supported in a set of Lebesgue measure zero whose Fourier transform vanishes at infinity. This is one of the earliest examples of measures in $M_0(\mathbb{T})$ which do not belong to $L^1(\mathbb{T})$. Hewitt and Zuckerman generalized this result for all non-discrete locally compact Abelian groups [HZ66].

In the case of non-Abelian locally compact groups, understanding the asymptotic behavior of unitary representations turns out to be important due to its applications in other areas of mathematics such as the theory of automorphic forms and ergodic properties of flows on homogeneous spaces (e.g. see [HM79], [Moo66], and [Shi68]).

The Fourier and Fourier-Stieltjes algebras are two of the most important algebras associated with a locally compact group. The study of the structure and properties of these algebras have become an essential part of abstract harmonic analysis. For instance, a major trend in noncommutative harmonic analysis concerns with deep investigation of various amenability properties of the Fourier and Fourier-Stieltjes algebras. Combining the famous theorems of Johnson [Joh72] and Ruan [Rua95], one observes that for a locally compact group, the (weak) amenability of the $L^1$-algebra and the operator (weak) amenability of the Fourier algebra are equivalent. This fact leads one to suspect the analogous relation between measure algebras and Fourier-Stieltjes algebras.

For a locally compact group, it has been shown that the measure algebra is amenable if and only if the group is discrete and amenable [DGH02]. Since compactness is the dual notion to discreteness, it is natural to conjecture that the operator amenability of the Fourier-Stieltjes algebra is equivalent to the compactness of the group. In 2007, Runde and Spronk [RS07] found surprising examples of noncompact operator amenable Fell groups. These examples disproved the conjecture, and left the characterization of the operator amenability of Fourier-Stieltjes algebras wide open. In the case of non-Abelian locally compact groups, Rajchman algebras of many locally compact groups seem to have as rich a structure as their Fourier-Stieltjes algebras, and can be used as a crucial stepping stone in the study of the Fourier-Stieltjes algebras.

The purpose of this thesis is to investigate $B_0(G)$ as a Banach algebra. In particular, we study its various amenability properties. We show that Rajchman algebras behave widely in terms of amenability. We first characterize locally compact groups whose Rajchman algebras are amenable. In fact, we prove that amenability of the

Rajchman algebra of a group is equivalent to the group being compact and almost Abelian. On the other hand, we present examples of groups, such as non-compact Abelian groups and infinite solvable groups, for which Rajchman algebras are not even (operator) weakly amenable. We then extend these results to all non-compact connected SIN-groups. A locally compact group is called a SIN-group if it has a neighborhood basis of the identity consisting of pre-compact neighborhoods which are invariant under inner automorphisms. This is a very natural class of groups which contains all Abelian, all compact and all discrete groups.

Our main tool to prove the above-mentioned results is a deep theorem of Varopoulos [Var66a], where he obtains a direct decomposition of the measure algebra $M(G)$ of a non-discrete locally compact Abelian group $G$ into an $L$-subalgebra and $L$-ideal. Varopoulos constructs the decomposition based on a given compact perfect metrisable strongly independent subset $P$ of $G$. A set $P$ is a *strongly independent* subset of $G$ if for any positive integer $N$, any family $\{p_j\}_{j=1}^N$ of distinct elements of $P$, and any family of integers $\{n_j\}_{j=1}^N$, the equality $\sum_{j=1}^N n_j p_j = 0_G$ implies that $n_j p = 0$ for every $p$ in $P$ and $1 \leq j \leq N$. The following theorem is an application of the decomposition theorem:

**Theorem Varopoulos.** For any non-discrete locally compact Abelian group $G$,

(i) $M_c(G)/\overline{M_c^2(G)}$ is a non-separable Banach space.

(ii) $M_0(G)/\overline{M_0^2(G)}$ is an infinite-dimensional Banach space.

Note that this theorem implies that if $G$ is a non-compact locally compact Abelian group then $B_0(G)$ cannot be (operator) weak amenable. We also adopt Varopoulos' method to obtain similar decompositions for $M_0(G)$ using appropriate strongly independent subsets $P$ of $G$. These decompositions are used to study the spectral behaviors of $B_0(G)$.

One of the important and fundamental questions in the theory of Banach algebras is the existence and construction of derivations for certain classes of Banach algebras. In the particular case of the Fourier-Stieltjes algebras, the derivation problem is of great importance, as it sheds substantial light on the structure of algebra, and then in turn on the underlying group.

Amongst all derivations, point derivations play a particularly important role. However, examples of point derivations are rare, and except in a few basic instances we do not know how to construct them. In this thesis, we investigate the spectral behavior of the Rajchman algebra associated with an Abelian locally compact group, and construct derivations at certain points of the spectrum.

In contrast to the generally complex nature of the spectrum of the Rajchman algebra, the spectrum of the Fourier algebra is well-understood. In fact Eymard showed that the spectrum of the Fourier algebra is the group itself [Eym64]. From a result of Spronk [Spr02] and independently Samei [Sam05], it is also clear that the Fourier algebra does not admit any point derivations at the elements of its spectrum. These results illustrate the inherent distinction between the Rajchman algebra and the Fourier algebra of many locally compact groups.

As a natural continuation of the above discussion, we investigate the spectral structure of Rajchman algebras and illustrate aspects of the residual analytic structure of their maximal ideal space. The Rajchman algebra associated with a locally compact Abelian group is a commutative convolution measure algebra, i.e. it has a natural lattice structure which is compatible with its Banach algebra structure. Taylor [Tay65] showed that one can construct analytic discs around certain non-idempotent elements of the spectrum of a convolution measure algebra. It is now interesting to study the possibilities for elements of the spectrum whose modulus are idempotents. For the special case of the measure algebra of a locally compact group, Brown and Moran [BM76] constructed nontrivial continuous point derivations at the discrete augmentation character. In a subsequent paper, they used a method of Varopoulos to construct analytic discs around the discrete augmentation character [BM78a]. Having constructed similar decompositions for $M_0(G)$, we have been able to obtain analytic discs around certain idempotent characters of Rajchman algebras.

The rest of this thesis is organized as follows: In Chapter 2, we provide the necessary background, and review some basics of harmonic analysis. We finish this chapter by a brief discussion on induced representations.

In Chapter 3, we introduce the Rajchman algebra associated with a locally compact group, and briefly discuss its relationship with the Fourier algebra. We then study the functorial properties of the Rajchman algebras. In particular, we show that if $G$ is a SIN-group with a closed subgroup $H$, then the restriction map from $B_0(G)$ to $B_0(H)$ is surjective (Theorem 3.2.2).

In Chapter 4, we demonstrate a theorem of Varopoulos regarding certain decompositions of the measure algebra of a non-discrete locally compact Abelian group. We then find similar decompositions of Rajchman algebras associated with such groups, which will be used to construct nonzero point derivations on $M_0(G)$.

Chapter 5 investigates various amenability properties of Rajchman algebras using the results of the two preceding chapters. In this chapter, we prove that amenability of the Rajchman algebra of a group is equivalent to the group being compact and almost Abelian. We also present examples of large classes of locally compact groups, such as non-compact connected SIN-groups and infinite solvable groups, for which Rajchman algebras are not even (operator) weakly amenable.

The final chapter of the thesis studies the Rajchman algebra of the group $\mathrm{SL}_2(\mathbb{R})$. Using Kunze-Stein phenomena we show that $B_0(\mathrm{SL}_2(\mathbb{R}))$ has no nonzero continuous point derivation. On the other hand, we use the results of Repka [Rep78] and Pukánszky [Puk61] regarding the decomposition of tensor products of unitary representations of $\mathrm{SL}_2(\mathbb{R})$ to observe that $B_0(\mathrm{SL}_2(\mathbb{R}))$ is not (operator) weakly amenable.

# Chapter 2

# Background and literature

The present chapter contains the background necessary for this thesis. Here we introduce notations and provide some tools used in the following chapters. In Section 2.1, we review the basic properties of locally compact groups and their Haar measures. We then define various Banach algebras associated with locally compact groups such as the Fourier and Fourier-Stieltjes algebras in Section 2.2. In the final two sections, we overview the procedure of inducing representations from subgroups of locally compact groups. One can refer to [HR79], [Fol95] and [Eym64] for more details.

## 2.1 Locally compact groups

Let $X$ be a locally compact Hausdorff space. A *Radon measure* on $X$ is a positive Borel measure $\mu$ which is finite on compact sets, and satisfies

$$\mu(E) = \inf\{\mu(U) : E \subseteq U \text{ and } U \text{ open}\}$$

and

$$\mu(U) = \sup\{\mu(K) : K \subseteq U \text{ and } K \text{ compact}\},$$

for any Borel subset $E$ and open subset $U$ of $X$. A *locally compact group* is a group $G$ equipped with a locally compact Hausdorff topology which is compatible with the group structure, i.e. the group product is a jointly continuous map from $G \times G$ to $G$, and the inverse is a continuous map from the group to itself. A Borel measure $\mu$ on a locally compact group $G$ is called left-invariant if $\mu(xE) = \mu(E)$ for any $x$ in $G$ and Borel subset $E$ of $G$. The following theorem states a fundamental property of locally compact groups.

**Theorem 2.1.1.** *Let $G$ be a locally compact group. There exists a left-invariant Radon measure $\mu$ on $G$ which attains positive values on nonempty open sets. Moreover, if $\nu$ is another left-invariant Radon measure on $G$ with positive values on nonempty open sets, then there exists $c > 0$ such that $\nu = c\mu$. That is, the measure $\mu$ is unique up to multiplication by a positive constant.*

For a locally compact group $G$, we fix once and for all, a measure $\mu_G$ as in Theorem 2.1.1. Particularly, if $G$ is a compact group or a discrete group then we scale $\mu_G$ to be a probability measure or a counting measure respectively. The measure $\mu_G$ is called the *left Haar measure* of $G$. For $f$ in $C_c(G)$, let $\int_G f(x)dx$ denote its integral with respect to $\mu_G$. By the left-invariance of the Haar measure,

$$\int_G f(yx)dx = \int_G f(x)dx,$$

for every $y$ in $G$ and function $f$ in $C_c(G)$. It is important to note that the left Haar measure on $G$ need not to be right-invariant in general. However, there exists a multiplicative $\mathbb{R}^+$-valued function $\Delta_G$ on $G$ such that

$$\int_G f(xy)dx = \frac{1}{\Delta_G(y)} \int_G f(x)dx,$$

and

$$\int_G f(x^{-1})dx = \int_G f(x)\Delta_G(x^{-1})dx,$$

for every $y$ in $G$ and $\mu_G$-integrable function $f$ on $G$. The function $\Delta_G$ is called the modular function of $G$. The group $G$ is called *unimodular* if $\Delta_G \equiv 1$. Abelian, compact and discrete groups are examples of unimodular groups. On the other hand, the group $ax + b$ of affine transformations of the real line is not unimodular. The following lemma will be used in the proof of Proposition 2.4.1.

**Lemma 2.1.2.** *Let $G$ be a locally compact group with the left Haar measure $\mu$, and $\phi : G \to G$ be a topological group isomorphism. Define the measure $\mu^\phi$ on $G$ by $\mu^\phi(E) = \mu(\phi(E))$ for every Borel subset $E$ of $G$. Then $\mu^\phi$ is a constant multiple of $\mu$.*

**Proof.** The measure $\mu^\phi$ is a Radon measure with positive values on nonempty open sets, since $\phi$ is a topological isomorphism. Moreover, for any $y$ in $G$ and Borel subset $E$ of $G$, we have,

$$\mu^{\phi}(yE) = \mu(\phi(yE)) = \mu(\phi(y)\phi(E)) = \mu(\phi(E)) = \mu^{\phi}(E). \tag{2.1}$$

Hence $\mu^{\phi}$ is left-invariant as well. Therefore by uniqueness of the Haar measure, there exists a positive constant $c_{\phi}$ such that $\mu^{\phi} = c_{\phi}\mu$. $\square$

Let $\mathrm{Aut}(G)$ denote the set of all topological isomorphisms of $G$. The function $\Delta$ defined as $\Delta(\phi) = c_{\phi}$ is a homomorphism of $\mathrm{Aut}(G)$ to the multiplicative group of positive real numbers. In addition $\Delta(\gamma_x) = \Delta_G(x)$ where $\gamma_x$ is the inner automorphism on $G$ defined as $\gamma_x(s) = x^{-1}sx$.

## 2.2 Banach algebras associated with locally compact groups

Let $G$ be a locally compact group with the Haar measure $\lambda$. Let the group algebra of $G$, denoted by $L^1(G)$, be the Lebesgue space $L^1(G, \lambda)$. Recall that $L^1(G)$ equipped with pointwise addition and convolution is a Banach algebra. In fact, $L^1(G)$ is a Banach $*$-algebra with involution defined as

$$f^*(x) = \Delta(x^{-1})\overline{f(x^{-1})}.$$

Let $M(G)$ be the space of complex-valued Radon measures on $G$. We define the convolution of two measures $\mu$ and $\nu$ in $M(G)$ to be

$$\int_G f(z)d(\mu * \nu)(z) = \int_G \int_G f(xy)d\mu(x)d\nu(y),$$

for every $f$ in $C_c(G)$, the set of compactly supported continuous functions on $G$. The measure algebra $M(G)$ equipped with the total variation norm is in fact a Banach algebra, which contains the $L^1$-algebra as a closed ideal.

Let $\mathcal{H}$ be a Hilbert space, and $\mathcal{U}(\mathcal{H})$ denote the group of unitary operators on $\mathcal{H}$. A *continuous unitary representation* of $G$ on $\mathcal{H}$ is a group homomorphism $\pi : G \to \mathcal{U}(\mathcal{H})$ which is WOT-continuous, i.e. for every vector $\xi$ and $\eta$ in $\mathcal{H}$, the function

$$\xi *_\pi \eta : G \to \mathbb{C}, \quad g \mapsto \langle \pi(g)\xi, \eta \rangle$$

is continuous. Functions of the form $\xi *_\pi \eta$, for vectors $\xi$ and $\eta$ in $\mathcal{H}$, are called the *coefficient functions* of $G$ associated with the representation $\pi$. One can extend $\pi$ to a non-degenerate norm-decreasing $*$-representation of the Banach $*$-algebra $L^1(G)$ to $\mathcal{B}(\mathcal{H})$ via

$$\langle \pi(f)\xi, \eta \rangle = \int_G f(x)\langle \pi(x)\xi, \eta \rangle dx,$$

for every $f$ in $L^1(G)$ and vectors $\xi$ and $\eta$ in $\mathcal{H}$. We use the same symbol $\pi$ to denote the $*$-representation extension as well. Let $\pi_1$ and $\pi_2$ be unitary representations of $G$ on the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. $\pi_1$ and $\pi_2$ are *unitarily equivalent* if there exists a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$U\pi_1(x) = \pi_2(x)U,$$

for all $x$ in $G$.

For a locally compact group $G$, the *Fourier-Stieltjes algebra* of $G$ is the set of all the coefficient functions of $G$, and is denoted by $B(G)$. Clearly $B(G)$ is a subalgebra of $C_b(G)$, the algebra of bounded continuous functions on $G$. Recall that the group $C^*$-algebra $C^*(G)$ is the enveloping $C^*$-algebra of $L^1(G)$, i.e.

$$C^*(G) = \overline{L^1(G)}^{\|\cdot\|_{C^*(G)}},$$

where for each $L^1$-function $f$,

$$\|f\|_{C^*} = \sup\{\|\pi(f)\| : \pi \text{ is a continuous unitary representation of } G\}.$$

Eymard [Eym64] proved that $B(G)$ can be identified with the Banach space dual of $C^*(G)$ as following. For $u$ in $B(G)$ and $f$ in $L^1(G)$,

$$\langle f, u\rangle = \int_G u(x)f(x)dx.$$

Moreover, the Fourier-Stieltjes algebra together with the norm from the above duality turns out to be a Banach algebra. The *Fourier algebra* of $G$, denoted by $A(G)$, is the closed subalgebra of the Fourier-Stieltjes algebra generated by its compactly supported elements. Clearly, the Fourier algebra is a subalgebra of

$C_0(G)$, the algebra of continuous functions on $G$ which vanish at infinity. In the special case of locally compact Abelian groups, one can identify the Fourier and Fourier-Stieltjes algebras with the $L^1$-algebra and the measure algebra of the dual group respectively.

Let $\pi$ be a continuous unitary representation of $G$ on a Hilbert space $\mathcal{H}_\pi$. Let $A_\pi(G)$ denote the closed subspace of $B(G)$ generated by the coefficient functions of $G$ associated with $\pi$, i.e.

$$A_\pi = \overline{\operatorname{span}_{\mathbb{C}}\{\xi *_\pi \eta : \xi, \eta \in \mathcal{H}_\pi\}}^{\|\cdot\|_{B(G)}}.$$

It is easy to see that $A_\pi(G)$ is a left and right translation-invariant closed subspace of $B(G)$. Conversely, by Theorem (3.17) of [Ars76], any closed subspace of $B(G)$ which is left and right translation-invariant, is of the form $A_\pi(G)$ for some continuous unitary representation $\pi$.

Let $\lambda$ denote the left regular representation of $G$ on $L^2(G)$, i.e. for $x$ in $G$ and $f$ in $L^2(G)$,

$$\lambda(x)f(y) = f(x^{-1}y) \qquad \forall y \in G.$$

For a unitary representation $\pi$, let $\mathrm{VN}_\pi(G)$ denote the von Neumann algebra generated by $\pi(G)$ in $\mathcal{B}(\mathcal{H}_\pi)$. Note that by Theorem 2.2.1, $A_\pi(G)$ is the image in $L^\infty(G)$ of the projective tensor product $\mathcal{H}_\pi \otimes_\gamma \mathcal{H}_\pi$ under the continuous sesquilinear form taking $\xi \otimes \eta$ to $\xi *_\pi \eta$. Eymard [Eym64] proved that $A_\lambda(G)$ is just the Fourier algebra $A(G)$, and can be identified with the unique predual of $\mathrm{VN}_\lambda(G)$. The following theorem is a generalization of this result:

**Theorem 2.2.1.** *[Ars76]*

*(i) The dual of the Banach space $A_\pi(G)$ can be identified with $\mathrm{VN}_\pi$ in the following manner. For $u$ in $A_\pi(G)$ and $f$ in $L^1(G)$,*

$$\langle u, \pi(f)\rangle = \int_G f(x)u(x)dx.$$

*Moreover, $A_\pi(G)$ is the unique predual of $\mathrm{VN}_\pi(G)$.*

*(ii) The Banach space $A_\pi(G)$ is the subset of elements $u$ in $B(G)$ which are of the form*

$$u = \sum_{i=1}^{\infty} \xi_n *_\pi \eta_n,$$

*where $\xi_n$ and $\eta_n$ belong to $\mathcal{H}_\pi$ and $\sum_{i=1}^{\infty} \|\xi_i\|\|\eta_i\| < \infty$.*

*(iii) For every $u$ in $A_\pi(G)$,*

$$\|u\|_{B(G)} = \inf\{\sum_{i=1}^{\infty} \|\xi_i\|\|\eta_i\| : u \textit{ represented as above}\},$$

*and the infimum is attained.*

Recall that every unitary representation $\pi$ of $G$ extends to a non-degenerate norm-decreasing $*$-representation of $L^1(G)$, and in turn $C^*(G)$. By slight abuse of notation, we denote all of the above representations by $\pi$. Let $\mathrm{Ker}(\pi)$ and $\mathrm{Ker}_{C^*}(\pi)$ denote the kernel of the unitary representation $\pi$ of $G$ and the kernel of the $*$-representation $\pi$ of $C^*(G)$ respectively. The following lemma is due to Fell [Fel60].

**Lemma 2.2.2.** *Let $G$ be a locally compact group with unitary representations $\pi$ and $\sigma$. Then the following are equivalent:*

*(i)* $\mathrm{Ker}_{C^*}(\pi) \subseteq \mathrm{Ker}_{C^*}(\sigma)$

*(ii)* $\|\sigma(u)\| \leq \|\pi(u)\|$ *for* $u \in L^1(G)$.

*(iii) For every* $\eta \in \mathcal{H}_\sigma$, *the positive definite function* $\eta *_\sigma \eta$ *can be uniformly approximated on compacta by functions of the form* $\xi *_\pi \xi$ *with* $\xi \in \mathcal{H}_\pi$.

*(iv) Every function* $u$ *in* $A_\sigma(G)$ *can be uniformly approximated on compacta by functions* $v$ *in* $A_\pi(G)$ *with* $\|v\|_{A_\sigma} \leq \|u\|_{A_\pi}$.

If any (therefore all) of the above conditions hold, we say that $\sigma$ is *weakly contained* in $\pi$.

## 2.3 Induced representations

The most important method for producing representations is to induce representations for $G$ from representations of its subgroups $H$. The resulting representation is called an *induced representation.*

### 2.3.1 When $G/H$ admits an invariant measure

Let $H$ be a closed subgroup of a locally compact group $G$, and $q$ be the quotient map from $G$ to $G/H$. Assume that the quotient space $G/H$ admits a $G$-invariant measure $\mu$. Then from a unitary representation $\pi : H \to \mathcal{U}(\mathcal{H}_\pi)$ of $H$, we derive a unitary representation $\mathrm{Ind}_H^G\pi : G \to \mathcal{U}(\mathcal{F})$ of $G$ in the following way.

- We first define the new Hilbert space $\mathcal{F}$ as follows.
  - $\mathcal{F}_0 := \{f \in C(G, \mathcal{H}_\pi) : q(\mathrm{supp} f) \text{ is compact } \& f(xh) = \pi(h^{-1})f(x)\ \forall x \in G, h \in H\}$.
  - For $f, g \in \mathcal{F}_0$, define $\langle f, g\rangle_{\mathcal{F}_0} := \int_{G/H} \langle f(x), g(x)\rangle_{\mathcal{H}_\pi} d\mu(xH)$ to be the inner product.
  - For each $f \in \mathcal{F}_0$, we have $\|f\|^2_{\mathcal{F}_0} = \int_{G/H} \|f(x)\|^2_{\mathcal{H}_\pi} d\mu(xH)$.
  - $\mathcal{F} := \overline{\mathcal{F}_0}^{\|\cdot\|_{\mathcal{F}_0}}$.
- For $x$ in $G$, define the bounded operator

$$\mathrm{Ind}_H^G\pi(x) : \mathcal{F}_0 \to \mathcal{F}_0, \quad f \mapsto {}_xf,$$

  where ${}_xf(y) = f(x^{-1}y)$ for every $y$ in $G$. Since $\mu$ is a $G$-invariant measure, $\mathrm{Ind}_H^G\pi(x)$ is an isometry on $\mathcal{F}_0$, and can be extended to a unitary in $\mathcal{B}(\mathcal{F})$.
- The map $\mathrm{Ind}_H^G\pi : G \to \mathcal{U}(\mathcal{F})$, $g \mapsto \mathrm{Ind}_H^G\pi(g)$ is a unitary representation of $G$, called the *representation induced from* $\pi$.

Let $H$ be a closed subgroup of a locally compact group $G$. Let $\Delta_G$ and $\Delta_H$ denote the modular functions of $G$ and $H$ respectively. Then the quotient space $G/H$ admits a nonzero positive invariant measure if and only if $\Delta_G|_H = \Delta_H$. If this is the case, then the positive invariant measure is unique up to multiplication by a positive constant. Moreover, one can normalize the invariant measure $\mu$ on $G/H$ such that for every $f$ in $C_c(G)$,

$$\int_{G/H}\int_H f(xh)dhd\mu(xH) = \int_G f(x)dx, \tag{2.2}$$

where $dx$ and $dh$ denote the Haar measures of $G$ and $H$ respectively.

**Remark.** Let $G$, $H$, and $\pi$ be as above. Let $C_c(G, \mathcal{H}_\pi)$ be the set of continuous compactly supported $\mathcal{H}_\pi$-valued functions on $G$. Then the mapping

$$\mathcal{P} : C_c(G, \mathcal{H}_\pi) \to C(G, \mathcal{H}_\pi), \qquad (\mathcal{P}f)(x) = \int_H \pi(h)f(xh)dh$$

is well-defined, and $\mathcal{P}(C_c(G, \mathcal{H}_\pi)) = \mathcal{F}_0$. Moreover, every element of $\mathcal{F}_0$ is uniformly continuous.

**Remark.** Let $G$, $H$, and $\pi$ be as above. For any $\xi$ in $\mathcal{H}_\pi$ and $v$ in $C_c(G)$, we define the compactly supported function $f_{v,\xi} : G \to \mathcal{H}_\pi$, $x \mapsto v(x)\xi$. Let $\eta$ and $w$ be elements of $\mathcal{H}_\pi$ and $C_c(G)$ respectively, and compute the coefficient function of

$\mathrm{Ind}_H^G\pi$ corresponding to $\mathcal{P}f_{v,\xi}$ and $\mathcal{P}f_{w,\eta}$.

$$\begin{aligned}
\langle \mathrm{Ind}_\pi(x)\mathcal{P}f_{v,\xi}, \mathcal{P}f_{w,\eta}\rangle_{\mathcal{F}_0} &= \int_{G/H} \langle \mathrm{Ind}_\pi(x)\mathcal{P}f_{v,\xi}(g), \mathcal{P}f_{w,\eta}(g)\rangle_{\mathcal{H}_\pi} d\mu(gH) \\
&= \int_{G/H} \langle \int_H \pi(h)(v(x^{-1}gh)\xi)dh, \int_H \pi(h')(w(gh')\eta)dh'\rangle_{\mathcal{H}_\pi} d\mu(gH) \\
&= \int_{G/H}\int_H\int_H v(x^{-1}gh)w(gh')\langle \pi(h)\xi, \pi(h')\eta\rangle_{\mathcal{H}_\pi} dhdh'd\mu(gH) \\
&= \int_{G/H}\int_H\int_H v(x^{-1}gh)w(gh')\langle \pi(h'^{-1}h)\xi, \eta\rangle_{\mathcal{H}_\pi} dhdh'd\mu(gH) \\
&= \int_{G/H}\int_H\int_H v(x^{-1}gh'h)w(gh')\langle \pi(h)\xi, \eta\rangle_{\mathcal{H}_\pi} dhdh'd\mu(gH) \\
&= \int_G\int_H v(x^{-1}gh)w(g)\langle \pi(h)\xi, \eta\rangle_{\mathcal{H}_\pi} dhdg, \qquad (2.3)
\end{aligned}$$

where in the last equality, we used the normalized relation stated in Equation (2.2).

### 2.3.2 General case

**Realization I:** Let $H$ be a closed subgroup of a locally compact group $G$, and $\pi : H \to \mathcal{U}(\mathcal{H}_\pi)$ be a continuous unitary representation. Let $q$ be the quotient map from $G$ to $G/H$. We define a linear map $P : C_c(G) \to C_c(G/H)$ by

$$Pf(xH) = \int_H f(xh)dh,$$

for $f$ in $C_c(G)$. It is easy to see that $P$ is surjective, and maps $C_c^+(G)$ onto $C_c^+(G/H)$.

- To define the new Hilbert space, let:

- $\mathcal{F}_{0,I} := \left\{ f : \begin{array}{c} f \in C(G, \mathcal{H}_\pi), \quad q(\mathrm{supp} f) \text{ is compact,} \\ f(xh) = \sqrt{\frac{\Delta_H(h)}{\Delta_G(h)}}\pi(h^{-1})f(x) \ \forall x \in G, h \in H \end{array} \right\}.$
- For each $f, g \in \mathcal{F}_{0,I}$, define the inner product

$$\langle f, g \rangle_{\mathcal{F}_{0,I}} := \int_G \psi(x) \langle f(x), g(x) \rangle_{\mathcal{H}_\pi} dx,$$

  where $\psi$ is an element of $C_c(G)$ such that

$$P\psi(w) = 1 \quad \forall w \in q(\mathrm{supp} f) \cup q(\mathrm{supp} g).$$

  This inner product defines the norm $\| \cdot \|_{\mathcal{F}_{0,I}}$ on $\mathcal{F}_{0,I}$.
- $\mathcal{F}_I := \overline{\mathcal{F}_{0,I}}^{\| \cdot \|_{\mathcal{F}_{0,I}}}$.

- For each $x$ in $G$, define the bounded operator

$$\mathrm{Ind}_H^G \pi(x) : \mathcal{F}_{0,I} \to \mathcal{F}_{0,I}, \quad f \mapsto {}_x f,$$

  where ${}_x f(y) = f(x^{-1}y)$. It is easy to show that $\mathrm{Ind}_H^G \pi(x)$ is an isometry on $\mathcal{F}_{0,I}$, and can be extended to a unitary in $\mathcal{B}(\mathcal{F}_I)$.

- The map $\mathrm{Ind}_H^G \pi : G \to \mathcal{U}(\mathcal{F}_I)$, $g \mapsto \mathrm{Ind}_H^G \pi(g)$ is a unitary representation of $G$, called the *representation induced from* $\pi$.

**Remark.** Let $G$, $H$ and $\pi$ be as above. Then the linear map

$$\mathcal{P}_I : C_c(G, \mathcal{H}_\pi) \to C(G, \mathcal{H}_\pi), \qquad (\mathcal{P}_I f)(x) = \int_H \sqrt{\frac{\Delta_G(h)}{\Delta_H(h)}} \pi(h) f(xh) dh$$

is well-defined, and $\mathcal{P}_I(C_c(G,\mathcal{H}_\pi)) = \mathcal{F}_{0,I}$. Moreover, every element of $\mathcal{F}_{0,I}$ is uniformly continuous.

For $\alpha$ in $C_c(G)$ and $\xi$ in $\mathcal{H}_\pi$, we define $f_{\alpha,\xi}$ to be

$$f_{\alpha,\xi}(x) = \alpha(x)\xi \quad \forall x \in G.$$

Clearly $f_{\alpha,\xi}$ is a compactly supported $\mathcal{H}_\pi$-valued function. Let $\mathcal{D}$ be a total subset of $\mathcal{H}_\pi$. Then

$$\mathcal{F}_{\mathcal{D}}^I = \{\mathcal{P}_I(f_{\alpha,\xi}) : \alpha \in C_c(G), \xi \in \mathcal{D}\}$$

is total in $\mathcal{F}_I$.

**Realization II:** Let $H$ be a closed subgroup of a locally compact group $G$, and $\pi$ be a unitary representation of $H$ on the Hilbert space $\mathcal{H}_\pi$. One can use the above method to construct a representation for $G$ induced from $\pi$ on the Hilbert space $\mathcal{F}_I$. However, it is often useful to modify the Hilbert space $\mathcal{F}_I$ such that its inner product is given by integration over $G/H$ against a strongly quasi-invariant measure. A regular Borel measure $\mu$ on $G/H$ is called *quasi-invariant* if the measures $\mu$ and $\mu_x = x \cdot \mu$ are mutually absolutely continuous for all $x$ in $G$. Recall that $x \cdot \mu(E) = \mu(xE)$ for Borel subsets $E$ of $G/H$. A quasi-invariant measure $\mu$ on $G/H$ is *strongly quasi-invariant* if there exists a continuous $\mathbb{R}^+$-valued function $\lambda$ on $G \times G/H$ such that

$$d\mu_x(p) = \lambda(x,p)d\mu(p)$$

for all $p$ in $G/H$. Strongly quasi-invariant measures on $G/H$ are closely related to rho-functions on $G$. A real-valued function $\rho$ on $G$ is a *rho-function* for $(G,H)$ if it is positive, continuous, and satisfies

$$\rho(xh) = \frac{\Delta_H(h)}{\Delta_G(h)}\rho(x)$$

for all $x$ in $G$ and $h$ in $H$. The existence of strongly quasi-invariant measures is guaranteed by the following theorem.

**Theorem 2.3.1.** *Let $H$ be a closed subgroup of a locally compact group $G$, and $q : G \to G/H$ be the quotient map. Then*

*(i) There exists a rho-function $\rho$ for $(G, H)$ on $G$.*

*(ii) Given any rho-function $\rho$ for $(G, H)$, there exists a strongly quasi-invariant measure $\mu_\rho$ on $G/H$ such that*

$$\int_{G/H} Pf(xH)d\mu_\rho(xH) = \int_G f(x)\rho(x)dx$$

*for all $f$ in $C_c(G)$. Moreover $\mu_\rho$ satisfies*

$$\left(\frac{d(x \cdot \mu_\rho)}{d\mu_\rho}\right)(yH) = \frac{\rho(xy)}{\rho(y)},$$

*where $\frac{d(x\cdot\mu_\rho)}{d\mu_\rho}$ denotes the Radon-Nikodym derivative of $\mu_\rho$.*

*(iii) Every strongly quasi-invariant measure on $G/H$ arises from a rho-function as in (ii).*

*(iv) If $\mu$ and $\nu$ are two strongly quasi-invariant measures on $G/H$ then they are*

*strongly equivalent, i.e.* $\mu$ *and* $\nu$ *are mutually absolutely continuous with continuous derivations.*

Fix a strongly quasi-invariant measure $\mu$ as in Theorem 2.3.1. Let $\rho$ be the corresponding rho-function.

- To define the new Hilbert space $\mathcal{F}_{II}^{\mu}$ we proceed as follows.
  - $\mathcal{F}_{0,II}^{\mu} := \{f \in C(G, \mathcal{H}_\pi) : q(\text{supp} f) \text{ is compact } \& f(xh) = \pi(h^{-1})f(x)\, \forall x \in G, h \in H\}$.
  - For $f, g \in \mathcal{F}_{0,II}^{\mu}$, define $\langle f, g\rangle_{\mathcal{F}_{0,II}} := \int_{G/H} \langle f(x), g(x)\rangle_{\mathcal{H}_\pi} d\mu(xH)$ to be their inner product.
  - $\mathcal{F}_{II}^{\mu} := \overline{\mathcal{F}_{0,II}^{\mu}}^{\|\cdot\|_{\mathcal{F}_{0,II}}}$. Using a standard measure theory argument, one can identify $\mathcal{F}_{II}^{\mu}$ with the Hilbert space of (equivalence classes of) measurable functions $\eta : G \to \mathcal{H}_\pi$ such that $\eta(xh) = \pi(h^{-1})\eta(x)$ for all $h$ in $H$ and almost all $x$ in $G$, and $\int_{G/H} \|\eta(x)\|_{\mathcal{H}_\pi}^2 d\mu(xH) < \infty$.
- For each $x$ in $G$, define $\text{Ind}_{H,\mu}^{G}\pi(x)$ in $\mathcal{B}(\mathcal{F}_{0,II}^{\mu})$ to be

$$(\text{Ind}_{H,\mu}^{G}\pi(x)f)(y) = \sqrt{\frac{\rho(x^{-1}y)}{\rho(y)}}f(x^{-1}y).$$

  It is easy to see that $\text{Ind}_{H,\mu}^{G}\pi(x)$ is an isometry on $\mathcal{F}_{0,II}^{\mu}$, and extends to a unitary in $\mathcal{B}(\mathcal{F}_{II}^{\mu})$.
- The map $\text{Ind}_{H,\mu}^{G}\pi : G \to \mathcal{U}(\mathcal{F}_{II}^{\mu})$ is a unitary representation of $G$, called the induced representation.

The multiplication operator $M_{\sqrt{\rho}}$ extends to a linear isomorphism from $\mathcal{F}_{II}^{\mu}$ to $\mathcal{F}_I$, and provides a unitary equivalence between $\mathrm{Ind}_{H,\mu}^{G}\pi$ (from the second realization) and $\mathrm{Ind}_H^G\pi$ (from the first realization). Therefore, a different choice of a strongly quasi-invariant measure for $G/H$ will result in a new unitary representation for $G$ induced from $\pi$, which is unitarily equivalent to $\mathrm{Ind}_{H,\mu}^{G}\pi$. Moreover, if $\Delta_G|_H = \Delta_H$, then all three methods explained above will be identified. In other words, the equivalence class of the representation induced from $\pi$ is independent from the method of construction.

The notation $\mathrm{Ind}_H^G\pi$ denotes the representation of $G$ induced from the representation $\pi$ of the closed subgroup $H$ using any of the above methods. One can use the simpler notation $\mathrm{Ind}_\pi$ if by omitting $G$ and $H$ no confusion will arise.

### 2.3.3 Basic properties of induced representations

Let $H$ be a closed subgroup of a locally compact group $G$. In the following, we list some basic properties of the induction process from $H$ to $G$.

**Conjugate representation:** Let $\mathcal{H}$ be a Hilbert space. The conjugate of $\mathcal{H}$, denoted by $\overline{\mathcal{H}}$, is a new Hilbert space defined to be the vector space $\mathcal{H}$ together with the inner product

$$\langle \overline{v}, \overline{w} \rangle_{\overline{\mathcal{H}}} = \overline{\langle v, w \rangle_{\mathcal{H}}},$$

where $\overline{v}$ and $\overline{w}$ in $\overline{\mathcal{H}}$ are the corresponding elements to $v$ and $w$ in $\mathcal{H}$. Let $\pi$ be a unitary representation of $G$ on $\mathcal{H}$. Define the conjugate of $\pi$, denoted by $\overline{\pi}$, by

$$\overline{\pi} : G \to \mathcal{U}(\overline{\mathcal{H}}), \quad \overline{\pi}(x)(\overline{v}) = \overline{\pi(x)(v)},$$

for $x$ in $G$ and $v$ in $\mathcal{H}$. Clearly $\overline{\pi}$ is a unitary representation of $G$.

**Proposition 2.3.2.** *Let $H$ be a closed subgroup of a locally compact group $G$, and $\pi$ be a unitary representation of $H$. Then*

$$\mathrm{Ind}_H^G\overline{\pi} = \overline{\mathrm{Ind}_H^G\pi}.$$

**Quotient:** Let $N$ be a closed normal subgroup of $G$, and $q_N$ be the quotient map from $G$ to $G/N$. Let $H$ be a closed subgroup of $G$ which contains $N$, and $\pi$ be a unitary representation of $H/N$. Then $\tilde{\pi} = \pi \circ q_N|_H$ is a unitary representation of $H$, and

$$\mathrm{Ind}_H^G\tilde{\pi} \sim (\mathrm{Ind}_{H/N}^{G/N}\pi) \circ q_N.$$

**Direct sum:** Let $\{\pi_\gamma\}_{\gamma\in\Gamma}$ be a family of unitary representations of $H$. Then

$$\oplus_\gamma \mathrm{Ind}_H^G\pi_\gamma = \mathrm{Ind}_H^G(\oplus_\gamma\pi_\gamma).$$

**Induction in stages:** Let $K$ and $H$ be closed subgroups of a locally compact group $G$ with $K \subseteq H$, and $\pi$ be a unitary representation of $K$. Then

$$\mathrm{Ind}_H^G(\mathrm{Ind}_K^H\pi) \sim \mathrm{Ind}_K^G\pi.$$

**Tensor product:** Let $H_1$ and $H_2$ be closed subgroups of locally compact groups $G_1$ and $G_2$, and $\pi_1$ and $\pi_2$ be unitary representations of $H_1$ and $H_2$ respectively. Then

$$\mathrm{Ind}_{H_1}^{G_1}\pi_1 \otimes \mathrm{Ind}_{H_2}^{G_2}\pi_2 \sim \mathrm{Ind}_{H_1\times H_2}^{G_1\times G_2}(\pi_1 \otimes \pi_2).$$

## 2.4 Induced representations in special case:

### 2.4.1 Semi-direct product of locally compact groups

Let $H$ and $N$ be locally compact groups with identities $e_H$ and $e_N$ respectively. By $\mathrm{Aut}(N)$ we denote the group of automorphisms of $N$, i.e. the set of all topological group isomorphisms of $N$ to itself with composition as the group action. Let $\alpha : H \to \mathrm{Aut}(N)$ be a group homomorphism such that the map

$$\psi_\alpha : N \times H \to N, \quad (n, h) \mapsto \alpha(h)(n)$$

is continuous. Define the locally compact group $N \rtimes_\alpha H$ to be the set $N \times H$ equipped with the product topology for which the group actions are defined as

$$(n_1, h_1) \cdot (n_2, h_2) = (n_1 \alpha(h_1)(n_2), h_1 h_2),$$

and

$$(n, h)^{-1} = (\alpha(h^{-1})(n^{-1}), h^{-1}).$$

Clearly, $(e_N, e_H)$ is the identity element of $N \rtimes_\alpha H$. It is easy to see that the group operations of $N \rtimes_\alpha H$ are continuous with respect to the product topology. The locally compact group $N \rtimes_\alpha H$ is called the *semidirect product* of $N$ and $H$ over $\alpha$. The following proposition states some properties of the semidirect product of two groups.

**Proposition 2.4.1.** *Let $N$, $H$ and $\alpha : H \to \mathrm{Aut}(N)$ be as above. Let $\mu_N$ (or $dn$)*

*denote the Haar measure of $N$, and $\mu_H$ (or $dh$) denote the Haar measure of $H$.*

*(i) Let $\delta : H \to \mathbb{R}^{>0}$ be defined as $\delta(h) = c_{\alpha(h)}$ where $c_{\alpha(h)}$ denotes the constant obtained in Lemma 2.1.2 with $\mu_N^{\alpha(h)} = c_{\alpha(h)}\mu_N$. Then $\delta$ is a continuous homomorphism.*

*(ii) The measure $d\mu := \frac{1}{\delta(h)} d\mu_N d\mu_H$ is the Haar measure of $N \rtimes_\alpha H$.*

*(iii) Let $\Delta_N$ and $\Delta_H$ denote the modular functions of $N$ and $H$ respectively. Then the modular function of $N \rtimes H$ is $\Delta(n,h) = \frac{\Delta_N(n)\Delta_H(h)}{\delta(h)}$.*

**Proof. (i)** Note that $\mu_N^{\alpha(e_H)} = \mu_N$ which implies that $\delta(e_H) = 1$. For $h_1$ and $h_2$ in $H$ and a Borel subset $E$ of $N$ we have,

$$\begin{aligned}
\delta(h_1h_2)\mu_N(E) &= \mu_N^{\alpha(h_1h_2)}(E) = \mu_N(\alpha(h_1h_2)(E)) = \mu_N(\alpha(h_1)(\alpha(h_2)E)) \\
&= \mu_N^{\alpha(h_1)}(\alpha(h_2)E) = \delta(h_1)\mu_N(\alpha(h_2)E) = \delta(h_1)\mu_N^{\alpha(h_2)}(E) \\
&= \delta(h_1)\delta(h_2)\mu_N(E),
\end{aligned}$$

hence $\delta$ is a homomorphism. It remains to show that $\delta$ is continuous. Note that

$$\delta(h) = \frac{\int_N f(\alpha(h^{-1})(n))dn}{\int_N f(n)dn},$$

where $f$ is any positive continuous compactly supported function on $N$. Without loss of generality we can assume that $f(e_N) = 1$. Given $\epsilon > 0$, there exists an open subset $e_N \in U$ of $N$ such that $|f(x) - f(y)| < \epsilon$ for all $x$ and $y$ in $N$ with $y^{-1}x \in U$. By continuity of $\psi_\alpha$, there exist open neighborhoods $e_H \in V$ of $H$ and $e_N \in W$ of

$N$ such that $\alpha(h^{-1})(W) \subseteq U$ for all $h \in V$. In particular $W$ is a subset of $U$. Let $x \in N$, $w \in W$ and $h \in V$ be arbitrary. Then

$$
\begin{aligned}
& |f(\alpha(h^{-1})(xw)) - f(\alpha(e_H)(xw)| = |f(\alpha(h^{-1})(x)\alpha(h^{-1})(w)) - f(xw)| \\
\leq\ & |f(\alpha(h^{-1})(x)\alpha(h^{-1})(w)) - f(\alpha(h^{-1})(x))| + |f(\alpha(h^{-1})(x)) - f(x)| + |f(x) - f(xw)| \\
\leq\ & 2\epsilon + |f(\alpha(h^{-1})(x)) - f(x)|.
\end{aligned}
$$

Since $f$ is compactly supported, there exists a finite set $\{x_1, \ldots, x_n\}$ in $N$ with

$$\mathrm{supp}(f) \subseteq x_1 W \cup \ldots \cup x_n W.$$

Now for each $1 \leq i \leq n$, by continuity of $\psi_\alpha$ at $(e_H, x_i)$, there exist neighborhoods $e_H \in V_i \subseteq H$ and $x_i \in W_i \subseteq N$ such that

$$\alpha(h^{-1})(W_i) \subseteq x_i W \quad \forall h \in V_i,$$

in particular $|f(\alpha(h^{-1})(x_i)) - f(x_i)| \leq \epsilon$. Let $V' = V \cap \bigcap_{i=1}^n V_i$. Then for each $1 \leq i \leq n$, $w \in W$ and $h \in V'$,

$$|f(\alpha(h^{-1})(x_i w)) - f(\alpha(e_H)(x_i w))| \leq 3\epsilon.$$

Hence $\delta$ is continuous.

**(ii):** Define the positive linear form $I$ on $C_c(N \rtimes_\alpha H)$ as

$$I(f) = \int_N \int_H f(n,h) \frac{1}{\delta(h)} dh dn,$$

for all $f$ in $C_c(N \rtimes H)$. By Riesz representation theorem there exists a unique Radon measure $\mu$ on $N \rtimes H$ such that

$$\int_{N \rtimes H} f(n,h) d\mu = \int_N \int_H f(n,h) \frac{1}{\delta(h)} dh dn,$$

for all compactly supported continuous functions $f$. Let $f \in C_c(N \rtimes H)$ and $(n_1, h_1) \in N \rtimes H$ be arbitrary. Then,

$$\begin{aligned}
&\int_{N \rtimes H} f((n_1,h_1)\cdot(n,h)) d\mu(n,h) = \int_N \int_H f((n_1,h_1)\cdot(n,h)) \frac{1}{\delta(h)} dh dn \\
= &\int_N \int_H f(n_1\alpha(h_1)(n), h) \frac{1}{\delta(h_1^{-1}h)} dh dn = \int_H \int_N f(n_1\alpha(h_1)(n), h) \frac{\delta(h_1)}{\delta(h)} dn dh \\
= &\int_H \int_N f(n_1 n, h) \frac{1}{\delta(h)} dn dh = \int_H \int_N f(n,h) \frac{1}{\delta(h)} dn dh = \int_{N \rtimes H} f d\mu,
\end{aligned}$$

which proves that $\mu$ is left-invariant.

**(iii)** For arbitrary $f$ in $C_c(N \rtimes H)$ and $(n_1, h_1)$ in $N \rtimes H$, we have

$$
\begin{aligned}
& \int_{N\rtimes H} f((n,h)\cdot(n_1,h_1))d\mu(n,h) = \int_N \int_H f(n\alpha(h)(n_1), hh_1)\frac{1}{\delta(h)}dhdn \\
= & \int_H \int_N f(\alpha(h)(\alpha(h^{-1})(n)n_1), hh_1)\delta(h^{-1})dndh = \int_H \int_N f(\alpha(h)(nn_1), hh_1)dndh \\
= & \frac{1}{\Delta_N(n_1)} \int_H \int_N f(\alpha(h)(n), hh_1)dndh = \frac{1}{\Delta_N(n_1)} \int_N \int_H f(n, hh_1)\delta(h^{-1})dhdn \\
= & \frac{1}{\Delta_N(n_1)\Delta_H(h_1)} \int_N \int_H f(n,h)\delta(h^{-1})\delta(h_1)dhdn = \frac{\delta(h_1)}{\Delta_N(n_1)\Delta_H(h_1)} \int_{N\rtimes H} f d\mu,
\end{aligned}
$$

i.e. $\Delta_{N\rtimes H}(n,h) = \frac{\Delta_N(n)\Delta_H(h)}{\delta(h)}$. □

### 2.4.2 Mackey machine

Let $G$ be a locally compact group and $N$ be a nontrivial Abelian closed normal subgroup of $G$. Then $G$ acts on $N$ by conjugation. Suppose that $H$ is a closed subgroup of $G$ such that $G = N \rtimes H$, where $\alpha : H \to \mathrm{Aut}(N)$ is defined as $\alpha(h)(n) = h^{-1}nh$. The conjugation action of $G$ on $N$ induces an action of $G$ on the dual group $\hat{N}$ via $\langle n, x\cdot\nu\rangle = \langle x^{-1}nx, \nu\rangle$ for $n \in N$, $x \in G$ and $\nu \in \hat{N}$. Let $G_\nu$ and $O_\nu$ denote the stabilizer and orbit of $\nu$ respectively, i.e.

$$G_\nu = \{x \in G : x\cdot\nu = \nu\} \text{ and } O_\nu = \{x\cdot\nu : x \in G\}.$$

We say $G$ acts *regularly* on $\hat{N}$ if the following two conditions hold.

(R1) There exists a countable family $\{E_i\}_{i\in\mathbb{N}}$ of Borel sets in $\hat{N}$ which are $G$-invariant and for each $\nu$ in $\hat{N}$, we have $O_\nu = \cap_{O_\nu \subseteq E_j} E_j$.

(R2) For each $\nu$ in $\hat{N}$, the natural map $G/G_\nu \to O_\nu$ defined as $xG_\nu \mapsto x \cdot \nu$ forms a homeomorphism.

For each $\nu$ in $\hat{N}$ define the *little group* $H_\nu$ to be $H_\nu = G_\nu \cap H$. It is easy to show that $G_\nu = N \rtimes H_\nu$. Let $\nu \in \hat{N}$, and $\rho : H_\nu \to \mathcal{U}(\mathcal{H}_\rho)$ be an irreducible representation. Then the tensor product representation $\nu \otimes \rho$ forms an irreducible representation of $N \times H_\nu$. Note that $\nu \otimes \rho$ can be viewed as a representation of $N \rtimes H_\nu$ by the definition of $H_\nu$.

**Theorem 2.4.2.** *Suppose $G = N \rtimes H$ with $N$ and $H$ as above. Suppose that $G$ acts regularly on $\hat{N}$. Let $\nu \in \hat{N}$ and $\rho$ be an irreducible unitary representation of $H_\nu$. Then $\mathrm{Ind}_{G_\nu}^G(\nu \otimes \rho)$ is an irreducible representation of $G$. Conversely, every irreducible representation of $G$ is equivalent to one of this form. Moreover, two representations $\mathrm{Ind}_{G_\nu}^G(\nu \otimes \rho)$ and $\mathrm{Ind}_{G_{\nu'}}^G(\nu' \otimes \rho')$ are unitarily equivalent if and only if there exists $x$ in $G$ such that $\nu' = x \cdot \nu$ and the representations $\rho : h \mapsto \rho(h)$ and $\rho'' : h \mapsto \rho'(x^{-1}hx)$ of $H_\nu$ are unitarily equivalent.*

# Chapter 3

# Functorial properties of $B_0(G)$

Let $G$ be a locally compact group. The *Rajchman algebra* associated with $G$, denoted by $B_0(G)$, is the set of elements of the Fourier-Stieltjes algebra which vanish at infinity, that is

$$B_0(G) = B(G) \cap C_0(G).$$

Note that $B_0(G)$ is a subalgebra of $B(G)$, since both $C_0(G)$ and $B(G)$ are algebras. It is easy to see that the Rajchman algebra is indeed a Banach subalgebra of the Fourier-Stieltjes algebra which contains the Fourier algebra as a closed ideal. In the case of Abelian groups, Rajchman algebras can be identified with the algebra of Rajchman measures on the dual group. A measure $\mu$ in $M(G)$ is called a Rajchman measure if

$$\lim_{|n|\to 0} \hat{\mu}(n) = 0.$$

Rajchman was the first who studied the behaviors of these measures in a systematic manner. Due to their close relation to the question of uniqueness of trigonometric series, Rajchman measures have been widely studied in the classical harmonic analysis (e.g. see [Kah64]). On the other hand, Rajchman algebras of many locally compact non-Abelian groups have as complicated structure as their Fourier-Stieltjes algebras, and can be used to illustrate the structure of the Fourier-Stieltjes algebras. In addition, the study of asymptotic behaviors of unitary representations turns out to be important in other areas of mathematics such as the theory of automorphic forms, and ergodic properties of flows on homogeneous spaces (e.g. see [HM79], [Moo66], and [Shi68]).

In the present chapter, we review some basic properties of Rajchman algebras. Particularly, we illustrate the relations between the Rajchman algebra of a locally compact group and such algebras associated with its subgroups and quotients. We show that if $H$ is a closed subgroup of a SIN-group $G$ then the restriction map from $B_0(G)$ to $B_0(H)$ is surjective. For a general locally compact group such restriction maps are not necessarily onto. However, for certain subgroups such as open subgroups, the connected component of the identity, and the center of a locally compact group the restriction map is surjective.

## 3.1 Properties of $B_0(G)$

Let $G$ be a locally compact group. Recall that a linear space $\mathcal{A}$ of functions on $G$ is called translation-invariant if for every function $f$ in $\mathcal{A}$ and $x$ in $G$, the left and right translations of $f$ by $x$ belong to $\mathcal{A}$.

**Proposition 3.1.1.** *The algebra $B_0(G)$ is a left and right translation-invariant closed subspace of $B(G)$.*

**Proof.** First note that $B_0(G)$ is translation-invariant since both $B(G)$ and $C_0(G)$ are translation-invariant. We only need to show that $B_0(G)$ is a closed subspace of $B(G)$. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in $B_0(G)$ converging to an element $f$ in $B(G)$, i.e.

$$\|f_n - f\|_{B(G)} \to 0 \quad \text{as } n \to \infty.$$

Recall that $\|\cdot\|_\infty$ on $B(G)$ is bounded above by $\|\cdot\|_{B(G)}$, in particular,

$$\|f_n - f\|_\infty \leq \|f_n - f\|_{B(G)}.$$

Therefore the sequence $\{f_n\}_{n\in\mathbb{N}}$ converges to $f$ in $C_0(G)$ as well. Now by completeness of $C_0(G)$, we conclude that $f$ vanishes as infinity. Hence $B_0(G)$ is a closed subspace of $B(G)$. $\square$

Recall that any closed subspace of $B(G)$ which is left and right translation-invariant, is of the form $A_\pi(G)$ for some continuous unitary representation $\pi$. Therefore by Proposition 3.1.1, the algebra $B_0(G)$ admits such a form too.

## 3.2 Extension problem

Let $G$ be a locally compact group and $H$ be a closed subgroup of $G$. Then the set of restrictions $B_0(G)|_H$ is a subspace of $B_0(H)$, which we will show is also closed.

The extension problem asks whether every function in $B_0(H)$ has an extension in $B_0(G)$.

It has been proved that for every closed subgroup $H$ of a locally compact group $G$, one has $A(G)|_H = A(H)$ (see [TT72] or [Her70]). In fact, every function in the Fourier algebra of $H$ can be extended to a function of the same norm in the Fourier algebra of $G$. Unfortunately, the analogue of this result does not hold in general for the Fourier-Stieltjes algebra. However, for a locally compact group $G$ and a closed subgroup $H$, it has been shown that $B(H) = B(G)|_H$ if $G$ is Abelian, or if $H$ is open, or compact, or the connected component of the identity or the center of $G$. Moreover, Cowling and Rodway [CR79] answered the extension problem of the Fourier-Stieltjes algebras in affirmative for the case of SIN-groups. In this section, we present the following two theorems which are analogues of results in [CR79]. The proofs herein are motivated by those of Cowling and Rodway [CR79].

**Theorem 3.2.1.** *Let $N$ be a closed normal subgroup of a locally compact group $G$. Then*

$$B_0(G)|_N = \{x \in B_0(N) : \|x^g - x\|_{B_0(N)} \to 0 \text{ as } g \to e\}, \tag{3.1}$$

*where $x^g(k) = x(g^{-1}kg)$ for each $g$ in $G$ and $x$ in $B_0(G)$. If $x$ is an element of $B_0(G)|_N$ then*

$$\|x\|_{B_0(N)} = \inf\{\|u\|_{B_0(G)} : u \in B_0(G) \text{ and } u|_N = x\}.$$

**Theorem 3.2.2.** *Let $H$ be a closed subgroup of a SIN-group $G$. Then*

$$B_0(G)|_H = B_0(H), \tag{3.2}$$

*and for each $x$ in $B_0(H)$,*

$$\|x\|_{B_0(H)} = \inf\{\|u\|_{B_0(G)} : u \in B_0(G) \text{ and } u|_H = x\}.$$

Before proving Theorem 3.2.1 and Theorem 3.2.2, let us observe examples of groups for which the restriction map is not onto.

**Proposition 3.2.3.** *The restriction map $r : B_0(G) \to B_0(H)$ is not surjective in each of the following cases.*

*(i)* $G = ax + b$ *and* $H = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{R} \right\} \simeq \mathbb{R}$ *as its closed subgroup.*

*(ii)* $G = \mathrm{SL}_2(\mathbb{R})$ *and* $H = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{R} \right\} \simeq \mathbb{R}$ *as its closed subgroup.*

**Proof. (i)** Suppose not, i.e. $B_0(G)|_H = B_0(H)$. Khalil [Kha74] showed that $B_0(G) = A(G)$. Hence

$$B_0(H) = B_0(G)|_H = A(G)|_H = A(H),$$

which is a contradiction with the fact that $B_0(\mathbb{R}) \neq A(\mathbb{R})$.

**(ii)** In Theorem 4.6.2 and Proposition 6.4.1, we will show that $B_0(\mathbb{R})$ has a nonzero continuous point derivation, but $B_0(\mathrm{SL}_2(\mathbb{R}))$ does not have any. Suppose that the restriction map $r$ from $B_0(\mathrm{SL}_2(\mathbb{R}))$ to $B_0(H)$ is surjective. Let $d$ be a nonzero continuous point derivation of $B_0(H)$ at a character $\phi$. By Lemma 4.6.3, $d \circ r$ is a nonzero continuous point derivation of $B_0(\mathrm{SL}_2(\mathbb{R}))$ at the character $\phi \circ r$, which is a contradiction. $\square$

We now review the definition and basic properties of SIN-groups. We also present Lemma 3.2.6 which will be used in the proof of Theorem 3.2.1 and Theorem 3.2.2.

**Definition 3.2.4.** *Let $G$ be a locally compact group.*

- *A function $\nu : G \to \mathbb{C}$ is called* central *if*

$$\nu(gg') = \nu(g'g) \quad \forall g, g' \in G.$$

- *A locally compact group $G$ is a SIN-group if it has a basis of compact neighborhoods $\{\mathcal{U}_\alpha\}_{\alpha \in I}$ of the identity in $G$ with central characteristic functions.*

Let [SIN] denote the class of locally compact SIN-groups.

**Lemma 3.2.5.** *Let $G \in$ [SIN]. Then*

*(i) $G$ is unimodular.*

*(ii) For every neighborhood $\mathcal{V}$ of $e$ in $G$, there exists a non-negative central function $v$ in $C_c(G)$ with* $\mathrm{supp}(v) \subseteq \mathcal{V}$.

*(iii) If $H$ is a closed subgroup of $G$ then $H$ is a SIN-group as well.*

**Proof.**

(i) Let $U$ be a compact open neighborhood of the identity in $G$ which is invariant under inner automorphisms. Fix $g'$ in $G$. Then,

$$\Delta(g') = \frac{\mu(Ug')}{\mu(U)} = \frac{\mu(g'U)}{\mu(U)} = \frac{\mu(U)}{\mu(U)}.$$

Hence $\Delta(g') = 1$ for all $g'$ in $G$.

(ii) Since $G$ is a SIN-group, there exist relatively compact open neighborhoods $\mathcal{U}$ and $\mathcal{W}$ of the identity which are invariant under inner automorphisms and satisfy $\overline{\mathcal{U}\mathcal{U}^{-1}} \subseteq \mathcal{W} \subseteq \mathcal{V}$. Let $\phi_{\mathcal{U}}$ be the function on $G$ defined as

$$\phi_{\mathcal{U}}(g) = \int_G \chi_{\mathcal{U}}(x)\chi_{\mathcal{U}}(gx)dx.$$

Clearly $\phi_{\mathcal{U}}$ is supported in $\mathcal{W}$. For elements $g$ and $h$ in $G$, we have:

$$\begin{aligned}
\phi_{\mathcal{U}}(h^{-1}gh) &= \int_G \chi_{\mathcal{U}}(x)\chi_{\mathcal{U}}(h^{-1}ghx)dx = \int_G \chi_{\mathcal{U}}(h^{-1}x)\chi_{\mathcal{U}}(h^{-1}gx)dx \\
&= \int_G \chi_{\mathcal{U}}(xh^{-1})\chi_{\mathcal{U}}(gxh^{-1})dx = \int_G \chi_{\mathcal{U}}(x)\chi_{\mathcal{U}}(gx)dx \\
&= \phi_{\mathcal{U}}(g),
\end{aligned}$$

where we used part (i) in the last equality. Finally note that $\phi_{\mathcal{U}} = \check{\chi}_{\mathcal{U}} *_\lambda \chi_{\mathcal{U}}$ belongs to the Fourier algebra, hence it is continuous.

(iii) Let $\{\mathcal{U}_\alpha\}_{\alpha\in I}$ be a family of neighborhoods of the identity in $G$ as in the definition of a SIN-group. Then $\{\mathcal{U}_\alpha \cap H\}_{\alpha\in I}$ is such a family of neighborhoods of $e$ in $H$.

$\square$

**Lemma 3.2.6.** *Let $G$ be a locally compact group. For an element $g$ in $G$ and a function $u$ in $B(G)$, define the function $u^g$ in $B(G)$ as in Theorem 3.2.1.*

*(i) For each $g$ in $G$, the map $\phi_g : B_0(G) \to B_0(G)$, $x \mapsto x^g$, is an isometric isomorphism of Banach algebras.*

*(ii) $B_0(G)|_H$ is a closed subspace of $B_0(H)$, and for each $u$ in $B_0(G)$,*

$$\|u|_H\|_{B_0(H)} \leq \|u\|_{B_0(G)}.$$

*(iii) Fix $x$ in $B_0(G)$. Then the map $G \to B_0(G)$, $g \mapsto x^g$ is continues.*

**Proof.**

(i) The map $\phi_g$ is clearly an algebra homomorphism. Let $x(k) = \langle \pi(k)\xi, \eta\rangle$ be an element of $B_0(G)$ with $\|x\|_{B(G)} = \|\xi\|\|\eta\|$. Then for each $g$ in $G$,

$$x^g(k) = x(g^{-1}kg) = \langle \pi(k)\pi(g)\xi, \pi(g)\eta\rangle,$$

which implies that $x^g$ belongs to $B(G)$ and

$$\|x^g\|_{B(G)} \leq \|\pi(g)\xi\|\|\pi(g)\eta\| \leq \|\xi\|\|\eta\| = \|x\|_{B(G)}.$$

Hence,

$$\|x\|_{B(G)} = \|(x^g)^{g^{-1}}\|_{B(G)} \leq \|x^g\|_{B(G)} \leq \|x\|_{B(G)}.$$

Recall that for each $g$ in $G$ and compact subset $K$ of $G$, the set $gKg^{-1}$ is compact. Therefore $x^g$ vanishes at infinity if $x$ does, and the map $\phi_g$ is well-defined.

(ii) Note that any representation of $G$ restricts to a representation of $H$. Therefore $B_0(G)|_H$ is clearly a subspace of $B_0(H)$. To show that $B_0(G)|_H$ is closed in $B_0(H)$, it is enough to note that $B_0(G)$ is a translation-invariant closed subspace of $B(G)$. Therefore there exists a unitary representation $\pi$ of $G$ such that $B_0(G) = A_\pi(G)$. We now use the fact that $A_\pi(G)|_H = A_{\pi|_H}(H)$ which is a corollary of Theorem 2.2.1 (ii). Indeed, let $u$ be an element of $B_0(G)$. Then by Theorem 2.2.1 (ii)

$$u = \sum_{i=1}^{\infty} \xi_n *_\pi \eta_n,$$

where $\xi_n$ and $\eta_n$ belong to $\mathcal{H}_\pi$ and $\sum_{i=1}^{\infty} \|\xi_i\|\|\eta_i\| < \infty$. Therefore

$$u|_H = \sum_{i=1}^{\infty} \xi_n *_{\pi|_H} \eta_n,$$

where $\pi|_H$ is the restriction of the representation $\pi$ from $G$ to $H$. This implies that $u|_H$ belongs to $A_{\pi|_H}(H)$. On the other hand, let $v$ be an element of $A_{\pi|_H}(H)$. Applying Theorem 2.2.1 (ii) again, we get

$$v = \sum_{i=1}^{\infty} \xi'_n *_{\pi|_H} \eta'_n,$$

where $\xi'_n$ and $\eta'_n$ belong to $\mathcal{H}_\pi$ and $\sum_{i=1}^\infty \|\xi'_i\|\|\eta'_i\| < \infty$. Define

$$w = \sum_{i=1}^\infty \xi'_n *_\pi \eta'_n.$$

Then $w$ belongs to $A_\pi(G)$ and $w|_H = v$. Hence $A_\pi(G)|_H = A_{\pi|_H}(H)$, and the latter is a closed subspace of $B(G)$ by definition.

Finally, for every $u$ in $B(G)$, we can find a representation $u(x) = \langle \pi(x)\xi, \eta\rangle$ such that $\|u\|_{B(G)} = \|\xi\|\|\eta\|$. Then $u|_H(h) = \langle \pi|_H(h)\xi, \eta\rangle$, and $\|u|_H\|_{B(H)} \leq \|\xi\|\|\eta\| = \|u\|_{B(G)}$.

(iii) Fix $x$ in $B_0(G)$, and let $\{g_\alpha\}_\alpha$ be a net in $G$ converging to $g$. Let $x(k) = \langle \pi(k)\xi, \eta\rangle$ be a representation of $x$. Then

$$\begin{aligned}
\|x^{g_\alpha} - x^g\|_{B_0(G)} &= \|\langle \pi(\cdot)(\pi(g_\alpha)\xi), \pi(g_\alpha)\eta\rangle - \langle \pi(\cdot)(\pi(g)\xi), \pi(g)\eta\rangle\|_{B_0(G)} \\
&\leq \|\langle \pi(\cdot)(\pi(g_\alpha)\xi), \pi(g_\alpha)\eta\rangle - \langle \pi(\cdot)(\pi(g)\xi), \pi(g_\alpha)\eta\rangle\|_{B_0(G)} \\
&\quad + \|\langle \pi(\cdot)(\pi(g)\xi), \pi(g_\alpha)\eta\rangle - \langle \pi(\cdot)(\pi(g)\xi), \pi(g)\eta\rangle\|_{B_0(G)} \\
&\leq \|(\pi(g_\alpha) - \pi(g))\xi\|\|\pi(g_\alpha)\eta\| + \|(\pi(g_\alpha) - \pi(g))\eta\|\|\pi(g)\xi\| \\
&\leq \|(\pi(g_\alpha) - \pi(g))\xi\|\|\eta\| + \|(\pi(g_\alpha) - \pi(g))\eta\|\|\xi\|,
\end{aligned}$$

where in the last inequality we used the fact that $\pi$ is a unitary representation. Moreover note that $\|(\pi(g_\alpha)-\pi(g))\xi\|\|\eta\|+\|(\pi(g_\alpha)-\pi(g))\eta\|\|\xi\|$ tends to zero as $g_\alpha$ converges to $g$ by strong operator continuity of $\pi$, and we are done.

$\square$

In the proof of Theorems 3.2.1 and 3.2.2, we use the following lemma which is closely related to the open mapping theorem.

**Lemma 3.2.7.** *Let $X$ and $Y$ be normed spaces and $X$ be complete. Then for every $T$ in $\mathcal{B}(X,Y)$, if $D_s(Y) \subseteq \overline{T(D_r(X))}$ then $D_s(Y) \subseteq T(D_r(X))$, where $D_r(X)$ is the closed ball in $X$ centered at 0 with radius $r$.*

### 3.2.1 Proof of Theorem 3.2.1

**Proof. (of Theorem 3.2.1)** Define the set

$$\mathcal{A} = \{x \in B_0(N) : \|x^g - x\|_{B_0(N)} \to 0 \text{ as } g \to e\}.$$

Throughout the proof, let $dg$, $d\dot{g}$, and $dn$ be the Haar measures of $G$, $G/N$, and $N$ respectively, normalized so that

$$\int_{G/N} \int_N \omega(gn) dn d\dot{g} = \int_G \omega(g) dg \quad \forall \omega \in C_c(G). \tag{3.3}$$

By Lemma 3.2.6, the inclusion '$\subseteq$' of (3.1) is clear. To prove '$\supseteq$', by Lemma 3.2.7, it is enough to show the following:

$\forall x \in \mathcal{A}$ and $\forall \epsilon > 0$, $\exists u \in B_0(G)$ s.t. $\|u|_N - x\|_{B_0(N)} < \epsilon$ and $\|u\|_{B_0(G)} \leq \|x\|_{B_0(N)}$.

Given such $x$ and $\epsilon$, there exist a neighborhood $\mathcal{U}$ of the identity in $G$, and a neighborhood $\mathcal{H}$ of the identity in $N$ such that:

$$\|x^g - x\|_{B_0(N)} < \frac{\epsilon}{2} \quad \forall g \in \mathcal{U} \quad \text{and} \tag{3.4}$$

$$\|\lambda(h^{-1})x - x\|_{B_0(N)} < \frac{\epsilon}{2} \quad \forall h \in \mathcal{H}, \tag{3.5}$$

where $\lambda(h)$ is the left translation operator by $h^{-1}$. Now let $\mathcal{V}$ be a relatively compact neighborhood of identity such that

$$\mathcal{V} \subseteq \mathcal{U} \ \text{ and } \ \mathcal{V}^{-1} \cdot \mathcal{V} \cap N \subseteq \mathcal{H}, \tag{3.6}$$

and $v$ be a continuous $\mathbb{R}^{\geq 0}$-valued function on $G$ with $\text{supp}(v) \subseteq \mathcal{V}$ that satisfies

$$\int_{G/N} [\int_N v(gn)dn]^2 d\dot{g} = 1.$$

Note that

$$\begin{aligned} 1 &= \int_{G/N} \left[\int_N v(gn)dn\right]^2 d\dot{g} &\quad (3.7)\\ &= \int_{G/N} \left[\int_N v(gn)dn \int_N v(gn')dn'\right] d\dot{g} \\ &= \int_{G/N} \left[\int_N v(gn'n)dn \int_N v(gn')dn'\right] d\dot{g} \\ &= \int_{G/N} \int_N \int_N v(gn'n)v(gn')dndn'd\dot{g} \\ &= \int_G \int_N v(gn)v(g)dndg, &\quad (3.8) \end{aligned}$$

where we used Equation (3.3) in the last equality. Next, we define the function $u$

by

$$u(g') = \int_G \int_N v(g'g)v(gn)x(n)dndg.$$

We will check the following claims:

**Claim 3.2.8.** *The function $u$ belongs to $B_0(G)$. Moreover $\|u\|_{B_0(G)} \leq \|x\|_{B_0(N)}$.*

**Proof.** Let us first show that $x \in C_0(N)$ implies that $u \in C_0(G)$. Let $\epsilon > 0$ be given, and define $\epsilon_1 = \frac{\epsilon}{(\mu(\mathcal{V})\|v\|_\infty)^2}$. There exists a compact subset $K$ of $N$ such that $|x(n)| < \epsilon_1$ for any $n$ in $N \setminus K$. Let $K_1 = \overline{\mathcal{V}}K\overline{\mathcal{V}}^{-1}$, and note that since $\overline{\mathcal{V}}$ and $K$ are compact, $K_1$ is compact as well. If $g' \in G \setminus K_1$ then $v(g'g)v(gn) \neq 0$ implies that $g \notin \mathcal{V}K^{-1}$ and $n \notin K$. Hence,

$$\begin{aligned} |u(g')| &= |\int_G \int_N v(g'g)v(gn)x(n)dndg| \leq \int_G \int_N v(g'g)v(gn)|x(n)|dndg \\ &\leq \epsilon \int_G \int_N v(g'g)v(gn)dndg = \epsilon, \end{aligned}$$

which implies that $u$ vanishes at infinity.

Next, we will show that $u$ belongs to $B(G)$. Since $x$ is in $B(N)$, there exists a unitary representation $\pi$ of $N$ and vectors $\xi$ and $\eta$ in $\mathcal{H}_\pi$ such that $x(n) = \langle \pi(n)\xi, \eta \rangle$ with $\|x\|_{B_0(N)} = \|\xi\|\|\eta\|$. Note that

$$\begin{aligned} u(g') &= \int_G \int_N v(g'g)v(gn)x(n)dndg \\ &= \int_G \int_N v(g'g)v(gn)\langle \pi(n)\xi, \eta \rangle dndg \\ &= \int_G \int_N v(g)v(g'^{-1}gn)\langle \pi(n)\xi, \eta \rangle_{\mathcal{H}_\pi} dndg, \end{aligned}$$

which belongs to $B(G)$ as shown in (2.3). Moreover,

$$\begin{aligned}
\|u\|_{B_0(G)} &\leq \|\mathcal{P}f_{v,\xi}\|_{\mathcal{F}}\|\mathcal{P}f_{v,\eta}\|_{\mathcal{F}} = \left(\int_{G/N} \|\mathcal{P}f_{v,\xi}(x)\|^2 d(xN)\right)^{\frac{1}{2}} \left(\int_{G/N} \|\mathcal{P}f_{v,\eta}(x)\|^2 d(xN)\right)^{\frac{1}{2}} \\
&= \left(\int_{G/N} \| \int_N \pi(h)(v(xh)\xi)dh\|^2 d(xN)\right)^{\frac{1}{2}} \left(\int_{G/N} \| \int_N \pi(h)(v(xh)\eta)dh\|^2 d(xN)\right)^{\frac{1}{2}} \\
&\leq \left(\int_{G/N} \|\xi\|^2 \left[\int_N v(xh)dh\right]^2 d(xN)\right)^{\frac{1}{2}} \left(\int_{G/N} \|\eta\|^2 \left[\int_N v(xh)dh\right]^2 d(xN)\right)^{\frac{1}{2}} \\
&= \|\xi\|\|\eta\| \int_{G/N} \left[\int_N v(xh)dh\right]^2 d(xN) \\
&= \|\xi\|\|\eta\| = \|x\|_{B_0(N)},
\end{aligned}$$

where we used Equation (3.7) in the last equality.

**Claim 3.2.9.** *$u|_N \in B_0(N)$ and $\|u|_N - x\|_{B_0(N)} \leq \epsilon$.*

**Proof.** By Claim 3.2.8, the function $u$ belongs to $B_0(G)$. Therefore the restriction $u|_N$ belongs to $B_0(N)$. For $n'$ in $N$, we have

$$\begin{aligned}
u(n') &= \int_G \int_N v(n'g)v(gn)x(n)dndg \\
&= \int_G \int_N v(g)v(n'^{-1}gn)x(n)dndg \\
&= \int_G \int_N v(g)v(g(g^{-1}n'^{-1}g)n)x(n)dndg \\
&= \int_G \int_N v(g)v(gn)[\lambda(n^{-1})x]^{gn}(n')dndg.
\end{aligned}$$

The map from $G\times N$ to $B_0(N)$ defined as $(g,n) \mapsto v(g)v(gn)[\lambda(n^{-1})x]^{gn}$ is a continuous compactly supported map, so the vector-valued integral $\int_G \int_N v(g)v(gn)[\lambda(n^{-1})x]^{gn}dndg$

is well-defined and equal to $u|_N$. Moreover from (3.7), we have

$$x = \int_G \int_N v(g)v(gn)x,$$

and therefore,

$$\begin{aligned}
\|u|_N - x\|_{B_0(N)} &\leq \int_G \int_N v(g)v(gn)\|[\lambda(n^{-1})x]^{gn} - x\|_{B_0(N)} dndg \\
&\leq \int_G \int_N v(g)v(gn)(\|[\lambda(n^{-1})x]^{gn} - x^{gn}\|_{B_0(N)} + \|x^{gn} - x\|_{B_0(N)})dndg \\
&= \int_G \int_N v(g)v(gn)(\|[\lambda(n^{-1})x] - x\|_{B_0(N)} + \|x^{gn} - x\|_{B_0(N)})dndg.
\end{aligned}$$

To get an estimate, note that $v(g)v(gn) \neq 0$ implies that $g \in \mathcal{V}$ and $n \in \mathcal{V}^{-1} \cdot \mathcal{V} \cap N$. Hence by (3.4), (3.5) and (3.6), we have:

$$\int_G \int_N v(g)v(gn)(\|[\lambda(n^{-1})x] - x\|_{B_0(N)} + \|x^{gn} - x\|_{B_0(N)})dndg \leq \epsilon \int_G \int_N v(g)v(gn)dndg = \epsilon,$$

which finishes the proof of the claim.

Having Claim 3.2.9 and Claim 3.2.8, the proof of Theorem 3.2.1 is complete. $\square$

### 3.2.2 Proof of Theorem 3.2.2

**Proof. (of Theorem 3.2.2)** Let $dg$ and $dh$ denote the Haar measures of $G$ and $H$ respectively. Note that $G/H$ admits a $G$-invariant measure $d\dot{g}$, since $G$ is a SIN-group and therefore $G$ and $H$ are both unimodular by Lemma 3.2.5. Moreover assume that these measures are normalized so that

$$\int_{G/H} \int_H \omega(gh) dh d\dot{g} = \int_G \omega(g) dg \quad \forall \omega \in C_c(G). \tag{3.9}$$

By Lemma 3.2.6, the inclusion '$\subseteq$' of (3.2) is clear. To prove '$\supseteq$', by Lemma 3.2.7, it is enough to show the following:

$\forall x \in B_0(H)$ and $\forall \epsilon > 0$, $\exists u_\epsilon \in B_0(G)$ s.t. $\|u_\epsilon|_H - x\|_{B_0(H)} < \epsilon$ and $\|u_\epsilon\|_{B_0(G)} \leq \|x\|_{B_0(H)}$.

Let $x$ and $\epsilon$ be given as above. Let $V_\epsilon$ be a compact neighborhood of identity in $G$ such that

$$\|\lambda(h^{-1})x - x\|_{B_0(H)} < \epsilon, \ \forall h \in V_\epsilon^{-1} V_\epsilon \cap H, \tag{3.10}$$

and let $v_\epsilon$ be a nonnegative continuous central function on $G$ such that

$$\mathrm{supp}(v_\epsilon) \subseteq V_\epsilon \qquad \text{and} \tag{3.11}$$

$$\int_{G/H} \left[ \int_H v_\epsilon(gh) dh \right]^2 d\dot{g} = 1. \tag{3.12}$$

We now define the function $u_\epsilon$ on $G$ to be

$$u_\epsilon(g') = \int_G \int_H v_\epsilon(g'g) v_\epsilon(gh) x(h) dh dg \tag{3.13}$$

We then verify the following claims.

**Claim 3.2.10.** $\|u_\epsilon|_H - x\|_{B_0(H)} \leq \epsilon$.

**Proof.** Note that as in (3.7),

$$1 = \int_{G/H} [\int_H v_\epsilon(gh)dh]^2 d\dot{g} = \int_G \int_H v_\epsilon(gh)v_\epsilon(g)dhdg.$$

Moreover, for $h' \in H$, we have

$$\begin{aligned} u(h') &= \int_G \int_H v_\epsilon(h'g)v_\epsilon(gh)x(h)dhdg \\ &= \int_G \int_H v_\epsilon(g)v_\epsilon(h'^{-1}gh)x(h)dhdg \\ &= \int_G \int_H v_\epsilon(g)v_\epsilon(ghh'^{-1})x(h)dhdg \\ &= \int_G \int_H v_\epsilon(g)v_\epsilon(gh)[\lambda(h^{-1})x](h')dhdg, \end{aligned}$$

since $v_\epsilon$ is central and $H$ is unimodular. Using the same argument as in proof of Claim 3.2.9, we have

$$\begin{aligned} u_\epsilon &= \int_G \int_H v_\epsilon(g)v_\epsilon(gh)[\lambda(h^{-1})x]dhdg, \\ x &= \int_G \int_H v_\epsilon(g)v_\epsilon(gh)xdhdg, \end{aligned}$$

which easily imply that $\|u_\epsilon|_H - x\|_{B_0(H)} \leq \int_G \int_H v_\epsilon(g)v_\epsilon(gh)\|\lambda(h^{-1})x - x\|_{B_0(H)} \leq \epsilon$, using the fact that $v_\epsilon(g)v_\epsilon(gh) \neq 0$ implies that $h \in V_\epsilon^{-1}V_\epsilon \cap H$. □

**Claim 3.2.11.** *$u_\epsilon \in B_0(G)$ and $\|u_\epsilon\|_{B_0(G)} \leq \|x\|_{B_0(N)}$.*

The proof of Claim 3.2.11 is identical to Claim 3.2.8, and we are done. □

**Corollary 3.2.12.** *Let $H$ be a closed subgroup of a locally compact SIN-group $G$, and $\{u_\alpha\} \subseteq B_0(G)$ be a bounded approximate identity for $B_0(G)$. Then $\{u_\alpha|_H\} \subseteq B_0(H)$ is a bounded approximate identity for $B_0(H)$.*

**Proof.** Note that by Theorem 3.2.2, restriction map is a surjective contraction. Hence $\{u_\alpha|_H\} \subseteq B_0(H)$ is a bounded net. Moreover, for any $y$ in $B_0(H)$ there exists $x$ in $B_0(G)$ such that $x|_H = y$. Hence,

$$\begin{aligned}\lim_\alpha \|yu_\alpha|_H - y\|_{B_0(H)} = \lim_\alpha \|x|_H u_\alpha|_H - x|_H\|_{B_0(H)} &= \lim_\alpha \|(xu_\alpha - x)|_H\|_{B_0(H)} \\ &\leq \lim_\alpha \|(xu_\alpha - x)\|_{B_0(G)} = 0.\end{aligned}$$

Therefore $\{u_\alpha|_H\}$ is a bounded approximate identity for $B_0(H)$. □

## 3.3 Quotient

**Proposition 3.3.1.** *Let $N$ be a compact normal subgroup of a locally compact group $G$. Then*

$$B_0(G/N) = B_0(G : N),$$

*where $B_0(G : N) = \{u \in B_0(G) : u$ is constant on each coset of $N\}$.*

**Proof.** Let $q_N$ be the quotient map from $G$ to $G/N$. By Corollary (2.26) of [Eym64], the map

$$\iota : B(G/N) \to B(G : N), \quad f \mapsto \iota(f) = f \circ q_N$$

is an isometric Banach algebra isomorphism. Therefore, we only need to show that

(i) For each $g \in C_0(G:N)$, there exists $f \in C_0(G/N)$ such that $g = \iota(f)$.

(ii) If $f \in C_0(G/N)$ then $\iota(f) \in C_0(G:N)$.

Note that (i) is clear, because $q_N$ is continuous and maps compact subsets of $G$ to compact subsets of $G/N$. Now for $g$ in $C_0(G:N)$, the map $f : G/N \to \mathbb{C}$ defined as $f(xN) = g(x)$ vanishes at infinity. To prove part (ii), let $\epsilon > 0$ be given. Since $f$ belongs to $C_0(G/N)$, there exists a compact subset $K$ of $G/N$ such that $|f(xN)| < \epsilon$ for all $xN$ in $K^c$. In order to show that $\iota(f)$ vanishes at infinity, it is enough to prove that $q_N^{-1}(K)$ is a compact subset of $G$. Recall that since $K$ is compact, there exists a compact subset $L$ of $G$ such that $\sigma(L) = K$, hence $\sigma^{-1}(K) = LN$ is compact as well. $\square$

Note that the assumption of $N$ being compact is essential. For instance, let $G = \mathbb{R} \times \mathbb{T}$ and $N = \mathbb{R}$. Then $G/N = \mathbb{T}$, and $B_0(\mathbb{T}) = B(\mathbb{T}) = B(G:\mathbb{R})$, but $B_0(G:\mathbb{R}) = \{0\}$. For $G$ and $N$ as above, let $P : C_0(G) \to C_0(G:N)$ be defined as

$$(Pf)(x) = \int_N f(xn)dn.$$

It is well-known that $P$ is a projection of $C_0(G)$ onto $C_0(G:N)$. If we assume that the Haar measure on $N$ is normalized, we also have that $\|P\| = 1$.

**Lemma 3.3.2.** *The map $P$ defines a well-defined contractive projection from $B_0(G)$ onto $B_0(G:N)$ which maps positive definite functions to positive definite functions.*

**Proof.** Let $f$ be an element in $B_0(G)$. Then there exists a unitary representation $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$, and vectors $\xi$ and $\eta$ in $\mathcal{H}_\pi$ such that $f = \xi *_\pi \eta$ and $\|f\|_{B(G)} =$

$\|\xi\|\|\eta\|$. Note that by the above comment, $Pf$ belongs to $C_0(G:N)$, and if $P: B_0(G) \to B_0(G:N)$ is well-defined then it is a projection. Moreover,

$$\begin{aligned}
(Pf)(x) &= \int_N f(xn)dn = \int_N \langle \pi(xn)\xi, \eta\rangle dn = \int_N \langle \pi(n)\xi, \pi(x^{-1})\eta\rangle dn \\
&= \int_N \langle \pi|_N(n)\xi, \pi(x^{-1})\eta\rangle \chi_N(n)dn = \langle \pi|_N(\chi_N)\xi, \pi(x^{-1})\eta\rangle = \langle \pi(x)(\pi|_N(\chi_N)\xi), \eta\rangle.
\end{aligned}$$

Hence, $Pf$ belongs to $B(G)$. In addition

$$\|Pf\|_{B(G)} \leq \|\pi|_N(\chi_N)\xi)\|\|\eta\| \leq \|\pi|_N(\chi_N)\|\|\xi\|\|\eta\| \leq \|\chi_N\|_1\|\xi\|\|\eta\| = \|\xi\|\|\eta\| = \|f\|_{B(G)},$$

which implies that $P: B_0(G) \to B_0(G:N)$ is a contraction. Note that $B_0(G:N) \subseteq B_0(G)$ together with $P^2 = P$ gives the surjectivity. Finally, assume that $f$ is a positive definite element of $B_0(G)$, and let $f = \xi *_\pi \xi$ be a representation for $f$. By Lemma 2.1.2 and compactness of $N$, we have

$$\begin{aligned}
Pf(x) &= P^2f(x) = \int_N Pf(xn)dn = \int_N Pf(xnx^{-1}x)dn = \int_N Pf(nx)dn \\
&= \int_N \langle \pi(nx)(\pi|_N(\chi_N)\xi), \xi\rangle dn = \int_N \langle \pi(x)(\pi|_N(\chi_N)\xi), \pi(n^{-1})\xi\rangle dn \\
&= \int_N \langle \pi(x)(\pi|_N(\chi_N)\xi), \pi(n)\xi\rangle dn = \langle \pi(x)(\pi|_N(\chi_N)\xi), (\pi|_N(\chi_N)\xi)\rangle.
\end{aligned}$$

Hence $Pf$ is positive definite.

□

**Proposition 3.3.3.** *Let $P: B_0(G) \to B_0(G:N)$ be defined as in Lemma 3.3.2, and suppose $B_0(G)$ admits a bounded approximate identity $\{u_\alpha\}$. Then $\{Pu_\alpha\}$ is a* *bounded approximate identity for $B_0(G:N)$.*

**Proof.** Clearly $\{Pu_\alpha\}$ is a bounded net. Let $f$ be an arbitrary element in $B_0(G:N)$. Then,

$$(fPu_\alpha - f)(x) = f(x)\int_N u_\alpha(xn)dn - f(x) = \int_N (f(xn)u_\alpha(xn) - f(xn))dn = P(fu_\alpha - f),$$

where we used the facts that $f$ is constant on each conjugacy class of $N$, and the Haar measure on $N$ is normalized so that $\mu(N) = 1$. Therefore,

$$\lim_\alpha \|fPu_\alpha - f\|_{B(G)} = \lim_\alpha \|P(fu_\alpha - f)\|_{B(G)} \leq \lim_\alpha \|fu_\alpha - f\|_{B(G)} = 0,$$

hence $\{Pu_\alpha\}$ is a bounded approximate identity for $B_0(G:N)$. □

### 3.3.1 Open subgroups, center, connected component of the identity

For a general locally compact group, the restriction map from $B(G)$ to $B(H)$ is surjective if $H$ is open, or the connected component of the identity of $G$, or the center of $G$ [LM75]. In Theorem 3.3.5, we show that for the above-mentioned cases, the restriction map from $B_0(G)$ to $B_0(H)$ is surjective as well. The proofs herein are adopted from those of Liukkonen and Mislove [LM75].

Let us begin with the following proposition.

**Proposition 3.3.4.** *Let $K$ be a compact normal subgroup of a locally compact group $G$, and $\pi$ be a representation of $G$ on the Hilbert space $\mathcal{H}_\pi$. Let $dk$ denote the Haar* *measure of $K$, and define the operator $Q : \mathcal{H}_\pi \to \mathcal{H}_\pi$ to be*

$$\langle Q\xi, \eta\rangle = \int_K \langle \pi(k)\xi, \eta\rangle dk$$

*for $\xi$ and $\eta$ in $\mathcal{H}_\pi$.*

*(i) $Q$ is a projection.*

*(ii) For each $x$ in $G$, $Q\pi(x) = \pi(x)Q$.*

*(iii) Let $P$ be the map from $B(G)$ to $B(G : K)$ defined in Lemma 3.3.2. For vectors $\xi$ and $\eta$ in $\mathcal{H}_\pi$, we have*

$$P(\xi *_\pi \xi) = Q\xi *_\pi Q\xi.$$

*(iv) For each vector $\xi$ in $\mathcal{H}_\pi$,*

$$\xi *_\pi \xi = Q\xi *_\pi Q\xi + (I - Q)\xi *_\pi (I - Q)\xi.$$

**Proof. (i)** Since $K$ is compact, clearly $Q$ is a bounded linear map. We need to show that $Q^* = Q^2 = Q$. For $\xi, \eta$ in $\mathcal{H}_\pi$,

$$\begin{aligned}\langle Q^*\xi, \eta\rangle &= \langle \xi, Q\eta\rangle = \overline{\langle Q\eta, \xi\rangle} = \int_K \overline{\langle \pi(k)\eta, \xi\rangle} dk = \int_K \langle \xi, \pi(k)\eta\rangle dk \\ &= \int_K \langle \pi(k^{-1})\xi, \eta\rangle dk = \int_K \langle \pi(k)\xi, \eta\rangle dk = \langle Q\xi, \eta\rangle,\end{aligned}$$

where we used the fact that the Haar measure of $K$ is unimodular. Hence $Q^* = Q$.

Moreover,

$$\begin{aligned}\langle Q^2\xi,\eta\rangle &= \int_K \langle \pi(k)Q\xi,\eta\rangle dk = \int_K\int_K \langle \pi(k)\pi(t)\xi,\eta\rangle dtdk = \int_K\int_K \langle \pi(kt)\xi,\eta\rangle dtdk \\ &= \int_K\int_K \langle \pi(t)\xi,\eta\rangle dtdk = \langle Q\xi,\eta\rangle,\end{aligned}$$

using the fact that the Haar measure of $K$ is normalized. Therefore $Q^2 = Q$.

**(ii)** Let $x$ be an element of $G$. Since $N$ is compact and normal, by Lemma 2.1.2 the Haar measure is invariant under the inner automorphisms. Therefore, for $\xi$ and $\eta$ in $\mathcal{H}_\pi$,

$$\langle \pi(x)Q\xi,\eta\rangle = \int_N \langle \pi(x)\pi(n)\xi,\eta\rangle dn = \int_N \langle \pi(nx)\xi,\eta\rangle dn = \langle Q\pi(x)\xi,\eta\rangle.$$

**(iii)** For each $x$ in $G$,

$$\begin{aligned}P(\xi *_\pi \xi)(x) &= \int_N (\xi *_\pi \xi)(xn)dn = \int_N \langle \pi(xn)\xi,\xi\rangle dn = \int_N \langle \pi(n)\xi,\pi(x^{-1})\xi\rangle dn \\ &= \langle \pi(x)Q\xi,\xi\rangle = \langle Q\pi(x)Q\xi,\xi\rangle = \langle \pi(x)Q\xi,Q\xi\rangle = (Q\xi *_\pi Q\xi)(x).\end{aligned}$$

**(iv)** It is enough to show that for each vector $\xi$, the map $Q\xi *_\pi (I-Q)\xi = 0$. Indeed,

$$(Q\xi *_\pi (I-Q)\xi)(x) = \langle \pi(x)Q\xi,(I-Q)\xi\rangle = \langle Q\pi(x)\xi,(I-Q)\xi\rangle = \langle \pi(x)\xi, Q(I-Q)\xi\rangle = 0.$$

□

**Theorem 3.3.5.** *Let $G$ be a locally compact group, $H$ an open subgroup, $G_0$ the* *connected component of the identity in $G$ and $Z(G)$ the center of $G$. Then,*

1. *The restriction map $r : B_0(G) \to B_0(H)$ is surjective.*

2. *The restriction map $r : B_0(G) \to B_0(Z(G))$ is surjective.*

3. *The restriction map $r : B_0(G) \to B_0(G_0)$ is surjective.*

**Proof. 1.** Since $H$ is an open subgroup of $G$, the restriction map $r : C^*(G) \to C^*(H)$ is norm-decreasing (see [Rie74]). Moreover, it is very easy to see that for an open subgroup $H$, the inclusion map $i : C^*(H) \to C^*(G)$, $f \mapsto f^\circ$, is norm-decreasing, where for $f$ in $L^1(H)$, we define $f^\circ$ in $L^1(G)$ as

$$f^\circ(x) = \begin{cases} f(x) & x \in H \\ 0 & x \notin H \end{cases} .$$

However, since $r \circ i = id_{L^1(H)}$, $i$ is an isometry and $r$ is a surjection. Taking the dual map of $r$, we get the isometric $*$-homomorphism $\theta : B(H) \to B(G)$, $\phi \mapsto \phi^\circ$, which restricts to an isometric $*$-homomorphism from $B_0(H)$ to $B_0(G)$. Therefore, in the case of an open subgroup, we can consider $B_0(H)$ as a subalgebra of $B_0(G)$, which implies that the restriction map $r : B_0(G) \to B_0(H)$ is surjective.

**2.** First note that $Z(G)$ is a closed normal subgroup of $G$. Moreover for every $f$ in $B_0(Z(G))$ , $g$ in $G$, and $z$ in $Z(G)$, we have $f^g(z) = f(g^{-1}zg) = f(g^{-1}gz) = f(z)$; therefore $f^g = f$. Now by Theorem 3.2.1, $B_0(G)|_{Z(G)} = B_0(Z(G))$, hence the restriction map $r : B_0(G) \to B_0(Z(G))$ is surjective.

**3.** Since $G_0$ is the connected component of the identity, $G/G_0$ is totally disconnected, therefore, it contains a compact open subgroup $H/G_0$. Note that $H$ is an open subgroup of $G$, hence by part (1), $B_0(H) \subseteq B_0(G)$. It is now enough to prove that $r : B_0(H) \to B_0(G_0)$ is onto. So without loss of generality, we can assume that $G$ is almost connected. Therefore there exists a net $\{K_i\}_{i\in\mathbb{N}}$ of compact normal subgroups of $G$ such that $G_i = G/K_i$ is an almost connected Lie group for each $i$, and $G = \varprojlim G/K_i$ where $\varprojlim$ denotes the projective limit of groups.

Let $\phi$ be a positive definite function in $B_0^+(G_0)$, and $\epsilon > 0$ be fixed. Let $\pi$ be a representation of $G_0$, and $\xi$ be a vector in $\mathcal{H}_\pi$ such that $\phi = \xi *_\pi \xi$. For each $i$, let $\omega_i$ denote the Haar measure of $G_0 \cap K_i$. Since $\phi$ is continuous at $e_G$, there exists an index $i$ such that $|\phi(e_G) - \phi * \omega_i(e_G)| < \epsilon$. Note that $G_0 \cap K_i$ is compact and normal, hence $\Delta_{G_0}|_{G_0\cap K_i}$ is identically 1. Therefore,

$$\begin{aligned}(\phi * \omega_i)(x) &= \int_{G_0} \phi(xy^{-1})\Delta(y^{-1})d\omega_i(y) = \int_{G_0\cap K_i} \phi(xy^{-1})d\omega_i(y)\\ &= \int_{G_0\cap K_i} \phi(xy)d\omega_i(y) = P\phi(x(G_0\cap K_i)).\end{aligned}$$

Therefore by Lemma 3.3.2 and Proposition 3.3.1, the function $\phi * \omega_i$ can be viewed as a positive definite function on $G_0/(G_0 \cap K_i) \simeq G_0K_i/K_i$. Moreover $G_0K_i/K_i$ is open in $G/K_i$, so we can extend $\phi * \omega_i$ to a positive definite function $\psi$ in $B_0(G/K_i)$ by part (1). Let $\tilde{\phi} = \psi \circ q_{K_i}$ where $q_{K_i}$ is the quotient map from $G$ to $G/K_i$. By Proposition 3.3.1, $\tilde{\phi}$ can be viewed as a positive definite function in $B_0(G)$. In addition,

$$\|\tilde{\phi}|_{G_0} - \phi\|_{B(G_0)} = \|\phi * \omega_i - \phi\|_{B(G_0)} = |(\phi * \omega_i - \phi)(e_G)| < \epsilon,$$

where we used Proposition 3.3.4. Hence $\phi$ is a limit point of the closed set $B_0(G)|_{G_0}$, i.e. $\phi$ belongs to $B_0(G)|_{G_0}$. Therefore $B_0(G)|_{G_0}$ is a closed translation-invariant subspace of $B_0(G_0)$ which contains each element of $B_0^+(G_0)$, hence must be $B_0(G_0)$ itself.

$\square$

## 3.4 When is $B_0(G) = A(G)$?

One of the most natural questions about $B_0(G)$ is to characterize the groups $G$ for which the Rajchman algebra properly contains the Fourier algebra. In 1916, Menshov [Men16] constructed a probability measure $\mu$ supported in a set of Lebesgue measure zero whose Fourier-Stieltjes transform vanishes at infinity. This is one of the earliest examples of measures in $M_0(\mathbb{T})$ which do not belong to $L^1(\mathbb{T})$. Hewitt and Zuckerman [HZ66] proved that the inclusion of $A(G)$ in $B_0(G)$ is proper for every non-compact locally compact Abelian group $G$. On the other hand, in his study of the representations of $ax+b$ group, Khalil [Kha74] proved that the Rajchman algebra and the Fourier algebra coincide in this case. The question is open in general.

A locally compact group $G$ is called an AR-group if the left regular representation of $G$ decomposes into a direct sum of irreducible representations. Clearly $\mathbb{R}$ is not an AR-group. On the other hand, compact groups and $ax+b$ group are examples of AR-groups. Figà-Talamanca proved that if $G$ is a unimodular non-compact locally compact group for which $A(G) = B_0(G)$, then $G$ is an AR-group ([FT77] and [FT77]). In [BT79], Baggett and Taylor showed that the above result holds even without the unimodularity condition. This result together with Theorem 3.1 of [MM00] implies that $B_0(G)$ is larger than $A(G)$ for any non-compact IN-group $G$.

In this section, we prove that for the special case of non-compact connected SIN-groups, the Rajchman algebra contains the Fourier algebra properly. Our approach is completely different from [FT77]. In fact, our proof is a concrete application of the extension result obtained in Theorem 3.2.2. We begin with the following lemma.

**Lemma 3.4.1.** *Let $H$ be a closed subgroup of a locally compact group $G$ with $\Delta_G|_H = \Delta_H$, and $\pi : H \to \mathcal{U}(\mathcal{H}_\pi)$ be a unitary representation of $H$. If $A_\pi(H) \subseteq C_0(H)$ then $A_{\mathrm{Ind}_\pi}(G) \subseteq C_0(G)$.*

**Proof.** Suppose $H$, $G$ and $\pi$ are as above. Let $dh$ and $dx$ denote the Haar measures of $H$ and $G$ respectively. Since $\Delta_G|_H = \Delta_H$, the quotient space $G/H$ admits a nonzero positive left-invariant measure $\mu$ such that for every $f$ in $C_c(G)$,

$$\int_{G/H} \int_H f(xh) dh d\mu(xH) = \int_G f(x) dx.$$

Let $\mathrm{Ind}_\pi$ be the unitary representation of $G$ on the Hilbert space $\mathcal{F}$ induced from $\pi$. Recall that the set

$$\mathcal{F}^0 := \{x \mapsto \int_H \alpha(xh)\pi(h)\xi dh : \alpha \in C_c(G), \xi \in \mathcal{H}_\pi\}$$

is a total subset of $\mathcal{F}$. To prove $A_{\mathrm{Ind}_\pi}(G) \subseteq C_0(G)$, it is enough to show that for arbitrary vectors $\phi$ and $\psi$ in $\mathcal{F}^0$, the coefficient function $\phi *_{\mathrm{Ind}_\pi} \psi$ vanishes at infinity. Let $\alpha$ and $\beta$ be functions in $C_c(G)$, and $\xi$ and $\eta$ be vectors in $\mathcal{H}_\pi$. Define the $\mathcal{H}_\pi$-valued functions $\mathcal{P}f_{\alpha,\xi}$ and $\mathcal{P}f_{\beta,\eta}$ on $G$ to be

$$\mathcal{P}f_{\alpha,\xi}(x) = \int_H \alpha(xh)\pi(h)\xi dh, \qquad \text{and} \qquad \mathcal{P}f_{\beta,\eta}(x) = \int_H \beta(xh)\pi(h)\eta dh.$$

We now compute the coefficient function of $\mathrm{Ind}_\pi$ associated with $\mathcal{P}f_{\alpha,\xi}$ and $\mathcal{P}f_{\beta,\eta}$. For $g$ in $G$,

$$\begin{aligned}
\mathcal{P}f_{\alpha,\xi} *_{\mathrm{Ind}_\pi} \mathcal{P}f_{\beta,\eta}(g) &= \langle \mathrm{Ind}_\pi(g)\mathcal{P}f_{\alpha,\xi}, \mathcal{P}f_{\beta,\eta}\rangle \\
&= \int_{G/H} \langle \mathcal{P}f_{\alpha,\xi}(g^{-1}x), \mathcal{P}f_{\beta,\eta}(x)\rangle_{\mathcal{H}_\pi} d\mu(xH) \\
&= \int_{G/H} \langle \int_H \alpha(g^{-1}xh)\pi(h)\xi dh, \int_H \beta(xh')\pi(h')\eta dh'\rangle_{\mathcal{H}_\pi} d\mu(xH) \\
&= \int_{G/H}\int_H\int_H \alpha(g^{-1}xh)\beta(xh')\langle \pi(h'^{-1}h)\xi, \eta\rangle_{\mathcal{H}_\pi} dhdh'd\mu(xH) \\
&= \int_{G/H}\int_H\int_H \alpha(g^{-1}xh'h)\beta(xh')\langle \pi(h)\xi, \eta\rangle_{\mathcal{H}_\pi} dhdh'd\mu(xH) \\
&= \int_G\int_H \alpha(g^{-1}xh)\beta(x)\langle \pi(h)\xi, \eta\rangle_{\mathcal{H}_\pi} dhdx \\
&= \int_G \beta(gx)\int_H \alpha(xh)\langle \pi(h)\xi, \eta\rangle_{\mathcal{H}_\pi} dhdx.
\end{aligned}$$

Note that by the inclusion $A_\pi(H) \subseteq C_0(H)$, there exists a sequence $\{\gamma_n\}_{n\in\mathbb{N}}$ of compactly supported continuous functions on $H$ such that

$$\|\xi *_\pi \eta - \gamma_n\|_\infty \to 0 \quad \text{when } n \to \infty.$$

For each $n$ in $\mathbb{N}$, define the function $\Gamma_n$ to be

$$\Gamma_n : G \to \mathbb{C}, \qquad \Gamma_n(g) = \int_G \beta(gx) \int_H \alpha(xh)\gamma_n(h) dh dx.$$

It is easy to see that $\Gamma_n$ is compactly supported and continuous for each $n$. Moreover for $g$ in $G$,

$$\begin{aligned}
& |\mathcal{P}f_{\alpha,\xi} *_{\mathrm{Ind}_\pi} \mathcal{P}f_{\beta,\eta}(g) - \Gamma_n(g)| \\
= & \left| \int_G \beta(gx) \int_H \alpha(xh) \langle \pi(h)\xi, \eta\rangle_{\mathcal{H}_\pi} dh dx - \int_G \beta(gx) \int_H \alpha(xh)\gamma_n(h) dh dx \right| \\
\leq & \int_G \int_H |\beta(gx)\alpha(xh)| \cdot |\langle \pi(h)\xi, \eta\rangle_{\mathcal{H}_\pi} - \gamma_n(h)| dh dx \\
\leq & \|\pi_{\xi,\eta} - \gamma_n\|_\infty \int_G \int_H |\beta(gx)\alpha(xh)| dh dx \\
\leq & M_1 M_2 \mu_G(K_1)\mu_G(K_2) \|\pi_{\xi,\eta} - \gamma_n\|_\infty,
\end{aligned}$$

where $M_1$ and $M_2$ are the maximum values, and $K_1$ and $K_2$ are supports of $\alpha$ and $\beta$ respectively. Therefore $\|\mathcal{P}f_{\alpha,\xi} *_{\mathrm{Ind}_\pi} \mathcal{P}f_{\beta,\eta} - \Gamma_n\|_\infty \to 0$ as $n \to \infty$, which implies that $\mathcal{P}f_{\alpha,\xi} *_{\mathrm{Ind}_\pi} \mathcal{P}f_{\beta,\eta}$ belongs to $C_0(G)$. □

**Corollary 3.4.2.** *If $G$ is a connected non-compact SIN-group then $B_0(G) \neq A(G)$.*

**Proof.** By contradiction assume that $G$ is a connected non-compact SIN-group with $A(G) = B_0(G)$. Then $G$ has a non-compact Abelian closed subgroup $H$. By Theorem 3.2.2, the restriction map from the Rajchman algebra of a SIN-group to the Rajchman algebra of its closed subgroup is surjective. Hence

$$A(H) = A(G)|_H = B_0(G)|_H = B_0(H),$$

where we used the fact that for every locally compact group $G$ and its closed subgroup $H$, the restriction map from $A(G)$ to $A(H)$ is surjective. This contradicts with the fact that for any non-compact locally compact Abelian group $H$, $A(H) \neq B_0(H)$ (see [HZ66]). Hence $A(G) \subseteq B_0(G)$ is proper. $\square$

# Chapter 4

# A decomposition of $M(G)$ and its applications

Throughout this chapter, let $G$ denote a locally compact Abelian group, and $M(G)$ denote the Banach algebra of complex bounded Radon measures on $G$. Let $M_c(G)$ denote the subset of all continuous measures in $M(G)$, i.e. the set of all complex bounded Radon measures $\mu$ on $G$ such that $\mu(\{x\}) = 0$ for every element $x$ in $G$. Let $M_d(G)$ denote the algebra of discrete measures, i.e.

$$M_d(G) = \{\mu = \sum_{s\in G} \alpha_s \delta_s : \|\mu\| = \sum_{s\in G} |\alpha_s| < \infty\}.$$

Let $\Delta(G)$ denote $M_c(G)^{\perp}$. Note that $\Delta(G)$ is in fact the algebra of discrete measures $M_d(G)$. Recall that $M_0(G)$ is the set of all measures in $M(G)$ whose Fourier-Stieltjes transforms vanish at infinity. Clearly $M_c(G)$ and $M_0(G)$ are closed ideals of $M(G)$.

In [Var66a], Varopoulos obtains a direct decomposition of the algebra of continuous measures $M_c(G)$, and hence the measure algebra $M(G)$, of a non-discrete locally compact Abelian group $G$ into a subalgebra and an ideal. The following strong theorem has been mentioned in [Var66a] as an application of the decomposition theorem.

**Theorem 4.0.3.** *[Var66a] For any non-discrete locally compact Abelian group $G$,*

*(i) $M_c(G)/\overline{M_c^2(G)}$ is a non-separable Banach space.*

*(ii) $M_0(G)/\overline{M_0^2(G)}$ is an infinite-dimensional Banach space.*

In the present chapter, we give a detailed exposition of the proof of Varopoulos' Theorem which we need in Chapter 5 in order to study the cohomological properties of $B_0(G)$.

We begin this chapter by definition and basic properties of an $L$-space in Section 4.1. We then review strongly independent sets in Section 4.2. Next, we overview definitions and proofs from [Var66a] that are necessary tools for the subsequent sections.

Section 4.4 presents Varopoulos's construction of decompositions of $M(G)$ using suitable strongly independent subsets of $G$. We then obtain similar decompositions for $M_0(G)$ in the next section.

Section 4.6 provides us with examples of groups for which $B_0(G)$ has nonzero continuous point derivations. In fact, we show that if $G$ is a non-discrete locally compact Abelian group then $M_0(G)$ has nonzero continuous point derivations. Finally, we conclude this chapter with a brief discussion on analytic discs in the spectrum of $M_0(G)$.

## 4.1 $L$-spaces

**Definition 4.1.1.** *A subspace $B$ of $M(G)$ is called an* $L$-space *if it satisfies the following conditions.*

1. *$B$ is a closed subspace of $M(G)$.*

2. *If $\mu, \nu \in M(G)$, $\nu \in B$, and $\mu \ll \nu$, then $\mu \in B$.*

The following lemma shows that one can replace the second condition of Definition 4.1.1 with Condition (2'):

$$\text{If } \mu, \nu \in M(G), \nu \in B, \text{ and } |\mu| \leq |\nu|, \text{ then } \mu \in B. \tag{2'}$$

**Lemma 4.1.2.** *Let $B$ be a closed subspace of $M(G)$. Then $B$ is an $L$-space if and only if it satisfies Condition (2').*

**Proof.** First assume that $B$ is an $L$-space. Note that for measures $\mu$ and $\nu$ in $M(G)$, the inequality $|\mu| \leq |\nu|$ implies $\mu \ll \nu$. Therefore $B$ clearly satisfies (2') as well.

Conversely, assume that $B$ is a closed subspace of $M(G)$ that satisfies Condition (2'). Let $\mu$ and $\nu$ be measures in $M(G)$ such that $\nu$ belongs to $B$. By Condition (2'), $|\nu|$ belongs to $B$ as well. Now assume that $\mu \ll \nu$, i.e. $|\mu| \ll |\nu|$. By Radon-Nikodym Theorem $|\mu| \ll |\nu|$ implies that $|\mu| = f|\nu|$, where $f$ is a non-negative

Borel integrable function. For each $n \in \mathbb{N}$, let $f_n$ be defined by

$$f_n(x) = \begin{cases} f(x) & \text{if } f(x) \le n \\ n & \text{otherwise} \end{cases}.$$

Note that $f_n|\nu| \le n|\nu|$, which implies that $f_n|\nu|$ belongs to $B$. Therefore $f|\nu|$, being the limit of $f_n|\nu|$'s, belongs to $B$ as well. $\square$

It is known that $M_0(G)$ is a translation invariant $L$-subspace of $M(G)$ (for example see [Gra71]). In the following lemma, we use properties of $L$-spaces to prove the well-known fact that $M_0(G)$ is a subspace of continuous measures on $G$.

**Lemma 4.1.3.** *For a locally compact Abelian group $G$, $M_0(G) \subseteq M_c(G)$.*

**Proof.** Suppose $M_0(G) \nsubseteq M_c(G)$ and let $\mu \in M_0(G) \setminus M_c(G)$. Note that $\mathcal{R}\mu$ and $\mathcal{I}\mu$ belong to $M_0(G)$ as well, since $M_0(G)$ is an $L$-space. Moreover, at least one of $\mathcal{R}\mu$ or $\mathcal{I}\mu$ is not continuous. Hence without loss of generality, we can assume that $\mu$ is a real measure. Let $\mu = \mu_1 + \mu_2$ be the orthogonal decomposition of $\mu$ with $\mu_1$ in $M_c(G)$ and $0 \neq \mu_2$ in $\Delta(G)$. Then $\mu_2 \ll \mu$ implies that $\mu_2$ belongs to $M_0(G)$, which in turn implies that $\delta_g$ belongs to $M_0(G)$ for some $g$ in $G$. But $|\hat{\delta_g}(\chi)| = |\overline{\chi(g)}| = 1$, which is a contradiction. $\square$

**Remark.** Definition 4.1.1 of an $L$-space is equivalent to the definition of a band, which has been used by Varopoulos in [Var66a].

## 4.2 Strongly independent sets

Let $G$ be a locally compact Abelian group, and $P$ be a subset of $G$. Let $k(P)$ denote the smallest positive integer $k$ such that $\{kx : x \in P\} = \{0_G\}$, if such an integer exists. Otherwise, set $k(P) = \infty$. The integer $k(P)$ is called the torsion of $P$. The set $P$ is called *strongly independent* if for any positive integer $N$, any family $\{p_j\}_{j=1}^N$ of distinct elements of $P$, and any family of integers $\{n_j\}_{j=1}^N$, the equality $\sum_{j=1}^N n_j p_j = 0_G$ implies that $n_j$ is a multiple of $k(P)$ for each $1 \leq j \leq N$, unless $k(P) = \infty$, in which case $n_j = 0$ for each $1 \leq j \leq N$.

Note that if $G$ is a non-discrete locally compact Abelian group then $G$ has a perfect metrisable subset $P$ which is strongly independent [Var66b]. Recall that a subset $P$ of an Abelian group $G$ is called an *independent set* if for any positive integer $N$, any family $\{p_j\}_{j=1}^N$ of distinct elements of $P$, and any family of integers $\{n_j\}_{j=1}^N$, the equality $\sum_{j=1}^N n_j p_j = 0_G$ implies that $n_j p_j = 0$ for every $1 \leq j \leq N$. It is clear that the notions of strong independence and independence are equivalent in the case of a torsion-free group. In [Rud58], Rudin showed that every torsion-free locally compact Abelian group contains an independent set $P$ homeomorphic to Cantor's ternary set, called an independent Cantor set. For instance, if $G$ is the additive group of real numbers then one can proceed as follows. First note that for any positive integer $k$ and any family of $k$ integers $\{n_i\}_{i=1}^k$, the hyperplane

$$H_{n_1,\ldots,n_k} = \left\{ (x_1, \ldots, x_k) \in \mathbb{R}^k : \sum_{i=1}^k n_i x_i = 0 \right\}$$

is a closed subset of $\mathbb{R}^k$ with empty interior. We now define a collection of compact neighborhoods inductively. First, let $V_1 = (1,2)$. To construct $V_2$, observe that the set

$$\bigcup_{(n_1,n_2)\neq(0,0)\&|n_i|\leq 1} H_{n_1,n_2}$$

is closed and of empty interior. Therefore one can find disjoint compact neighborhoods $V_2^{(1)}$ and $V_2^{(2)}$ of diameters less than $\frac{1}{2}$ such that

$$V_2^{(1)} \times V_2^{(2)} \subseteq V_1^2 \setminus \bigcup_{(n_1,n_2)\neq(0,0)\&|n_i|\leq 1} H_{n_1,n_2}.$$

Let $V_2 = V_2^{(1)} \cup V_2^{(2)}$. For an integer $i$ in $\mathbb{N}$, suppose $V_i = V_i^{(1)} \cup \ldots \cup V_i^{(r_i)}$ is the disjoint union of $r_i$ compact neighborhoods of diameters less than $\frac{1}{r_i}$, where $r_i = 2^{i-1}$ for each $i$. To construct $V_{i+1}$, we use a similar argument to find disjoint compact neighborhoods $\{V_{i+1}^{(j)}\}_{j=1}^{r_{i+1}}$ of diameters less than $\frac{1}{r_{i+1}}$ such that

$$V_{i+1}^{(1)} \times \ldots \times V_{i+1}^{(r_{i+1})} \subseteq V_i^{(1)} \times V_i^{(1)} \times \ldots \times V_i^{(r_i)} \times V_i^{(r_i)} \setminus \bigcup_{(n_1,\ldots,n_{r_{i+1}})\neq(0,\ldots,0)\&|n_j|\leq i} H_{n_1,\ldots,n_{r_{i+1}}}.$$

Now define

$$V_{i+1} = V_{i+1}^{(1)} \cup \ldots \cup V_{i+1}^{(r_{i+1})}.$$

It is easy to see that for arbitrary elements $x_j$ in $V_i^{(j)}$, and any family of integers $\{n_j\}$ whose modulus are bounded by $i+1$, we have

$$\sum_{j=1}^{r_{i+1}} n_j x_j \neq 0.$$

This easily implies that the set $P$ defined as

$$P = \bigcap_{i \in \mathbb{N}} V_i$$

is a strongly independent Cantor set.

The proof of Theorem 4.0.3 is based on Theorem 4.2.1 of [Var66b] which proves the existence of certain strongly independent sets. One can refer to [Var66b] for the proof of Theorem 4.2.1.

**Theorem 4.2.1.** *[Var66b] Let $G$ be a non-discrete metrisable locally compact Abelian group. Then there exists a perfect strongly independent subset $P$ of $G$ such that $M_0^+(P) \neq \{0\}$, i.e. there exists a nonzero positive measure $\mu$ in $M_0(G)$ which is supported in $P$.*

The proof of the above theorem is rather difficult and technical. In fact, the argument in [Var66b] relies on structural theorems and treatment of some special groups. In what follows, we sketch a proof of Rudin for the special case of $\mathbb{T}$.

**Theorem 4.2.2.** *[Rud60] There exists an independent compact perfect subset $P$ of $\mathbb{T}$ such that $M_0^+(P) \neq \{0\}$.*

**Sketch of proof.** Let $\{\xi_k\}_{k \in \mathbb{N}}$ be a sequence of real numbers in $(0, \frac{1}{2})$. We first construct a compact perfect subset $Q$ of $[0, 2\pi]$ using the usual Cantor procedure. First we divide the interval $Q_1 = [0, 2\pi]$ into three intervals $Q_2^{(1)}$, $M_2^{(1)}$, and $Q_2^{(2)}$ of lengths proportional to $\xi_1$, $1 - 2\xi_1$, and $\xi_1$ respectively. Let

$$Q_2 = Q_2^{(1)} \cup Q_2^{(2)}.$$

Next, we split each interval $Q_2^{(1)}$ and $Q_2^{(2)}$ to three intervals of lengths proportional to $\xi_2$, $1-2\xi_2$, and $\xi_2$ respectively. Let $Q_3^{(1)}$, $M_3^{(1)}$, and $Q_3^{(2)}$ denote the intervals splitting $Q_2^{(1)}$, and $Q_3^{(3)}$, $M_3^{(2)}$, and $Q_3^{(4)}$ denote the intervals splitting $Q_2^{(2)}$. Define

$$Q_3 = Q_3^{(1)} \cup Q_3^{(2)} \cup Q_3^{(3)} \cup Q_3^{(4)}.$$

Repeating the above procedure, we construct a family $\{Q_i\}_{i\in\mathbb{N}}$ of subset of $[0, 2\pi]$. Note that for each positive integer $i$, $Q_i$ is written as a disjoint union of intervals

$$Q_i = Q_i^{(1)} \cup \ldots \cup Q_i^{(2^{i-1})},$$

where each $Q_i^{(j)}$ is of length $2\pi\xi_1 \ldots \xi_{i-1}$. Let

$$Q = \bigcap_{i\in\mathbb{N}} Q_i.$$

Clearly $Q$ is a compact perfect subset of $[0, 2\pi]$. Let $f$ be the classical Cantor-Lebesgue function associated with $Q$, i.e. $f$ is the uniform limit of the family $\{f_k\}_{k\in\mathbb{N}}$ of functions defined in the following way. For each positive integer $k$, let $f_k$ be the continuous function such that

$$f_k(t) = \frac{j}{2^{k-1}} \quad \text{for } t \in Q_k^{(j)},$$

and $f_k$ is linear on each interval off $Q_k$. Let $\mu$ be the first distributional derivative of $f$, i.e. for every $\phi$ in $C(\mathbb{T})$,

$$\langle \phi, \mu \rangle = \int_0^{2\pi} \phi(t) df(t).$$

Clearly $\mu$ is a singular probability measure supported in $Q$.

In [Sal42], Salem proved that there are sequences $\{\xi_n\}_{n\in\mathbb{N}}$ for which the associated set $Q$ is of measure zero, and the corresponding measure $\mu$ belongs to $M_0(Q)$. Rudin then constructed certain deformations which transform $Q$ to an independent set $P$. Furthermore, he showed that the measure $\mu$ is mapped to an element of $M_0(P)$ via such deformations. $\square$

The following lemma will be used in Theorem 4.6.2 to construct nonzero continuous point derivations on $M_0(G)$.

**Lemma 4.2.3.** *Let $G$ be a non-discrete metrisable locally compact Abelian group. Then there exists a compact perfect strongly independent subset $P$ of $G$ such that $M_0^+(P) \neq \{0\}$.*

**Proof.** By Theorem 4.2.1 there exists a perfect metrisable strongly independent subset $P'$ of $G$ which supports a nonzero Rajchman measure $\mu_0$. It is known that $M_0(G)$ is an $L$-space [Gra71]. Therefore, without loss of generality we can assume that $\mu_0$ is a positive measure. Note that $\mu_0(P') > 0$ and $\mu_0$ is a Radon measure, therefore there exists a compact subset $K$ of $P$ with $\mu_0(K) > 0$. But $\mu_0|_K$ belongs to $M_0(K) = M_0(G) \cap M(K)$, because it is a positive measure supported in $K$ and dominated by $\mu_0$. Note that $\text{supp}(\mu_0)$ is still a perfect set, because $\mu_0$ is a continuous measure by Lemma 4.1.3. Let $P = \text{supp}(\mu_0)$. Clearly $P$ is a strongly independent set, since it is a subset of the strongly independent set $P'$. Hence $P$ is a compact perfect strongly independent subset of $G$ with $M_0(P) \neq \{0\}$. $\square$

## 4.3 Geometric and measure theoretic results on independent sets

Let $G$ be a non-discrete locally compact Abelian group. Recall that the convolution of two measures $\mu$ and $\nu$ in $M(G)$ is defined as

$$\mu * \nu(E) = \int_G \mu(-y + E) d\nu(y),$$

and

$$\int_G f(z) d\mu * \nu(z) = \int_G \int_G f(x + y) d\mu(x) d\nu(y),$$

where $E$ is a measurable subset of $G$, and $f$ is an integrable function. It is easy to see that if $\mu$ and $\nu$ are elements of $M(G)$ with $\mathrm{supp}(\mu) \subseteq E$ and $\mathrm{supp}(\nu) \subseteq F$, then $\mathrm{supp}(\mu * \nu) \subseteq E + F$.

Let $\mu$ and $\nu$ be measures in $M(G)$. Then $\mu$ and $\nu$ are *mutually singular*, denoted by $\mu \perp \nu$, if there exists a partition $A \cup B$ of $G$ such that $\mu$ is concentrated in $A$ and $\nu$ is concentrated in $B$. We say $\mu$ is *absolutely continuous with respect to* $\nu$, denoted by $\mu \ll \nu$, if for every measurable set $A$, the following condition is satisfied.

$$|\nu|(A) = 0 \Rightarrow |\mu|(A) = 0.$$

For subsets $P$ and $Q$ of $G$, and an integer $n$ in $\mathbb{N}$, we recall the following notations:

- $P + Q = \{x + y : x \in P, y \in Q\}$.
- $nP = \{\sum_{i=1}^{n} x_i : x_i \in P\}$.
- $-nP = \{-\sum_{i=1}^{n} x_i : x_i \in P\}$.
- $\mathrm{Gp}(P)$ is the subgroup generated by $P$ in $G$.

For a positive integer $m$, define

$$\omega_m : P^m \to G, \quad \omega_m((p_j)_{j=1}^m) = \sum_{j=1}^{m} p_j.$$

Let $\breve{\omega}_m : M(P^m) \to M(G)$ be the map induced from $\omega_m$, i.e.

$$\breve{\omega}_m(\overline{\mu})(E) = \overline{\mu}(\omega_m^{-1}(E)) \quad \text{and} \quad \int_G f(x) d\breve{\omega}_m(\overline{\mu})(x) = \int_{P^m} (f \circ \omega_m)(y) d\overline{\mu}(y),$$

where $\overline{\mu}$ is a measure in $M(P^m)$, $E$ is a subset of $G$, and $f$ is a measurable function on $G$. Note that if $\overline{\mu}$ and $\overline{\nu}$ are measures in $M(P^m)$ and $M(P^n)$ respectively such that $\breve{\omega}_m(\overline{\mu}) = \mu$ and $\breve{\omega}_n(\overline{\nu}) = \nu$, then $\breve{\omega}_{m+n}(\overline{\mu} \otimes \overline{\nu}) = \mu * \nu$.

**Lemma 4.3.1.** *If $P$ is a strongly independent perfect metrisable subset of a locally compact Abelian group $G$, then $\breve{\omega}_m$ maps $M(P^m)$ onto $M(mP)$.*

**Proof.** First note that the subset $P$ is a metrisable perfect (hence closed) subset of a locally compact space $G$. Therefore both $P$ and $P^m$ are Polish spaces. Let

$\sim$ be the equivalence relation defined as $(p_1, \ldots, p_m) \sim (q_1, \ldots, q_m)$ if and only if $p_1 + \ldots + p_m = q_1 + \ldots + q_m$. For a permutation $s$ in $S_m$, let $\sigma_s : P^m \to P^m$ be defined as $\sigma_s((p_1, \ldots, p_m)) = (p_{s(1)}, \ldots p_{s(m)})$. It is clear that $(p_1, \ldots, p_m) \sim (q_1, \ldots, q_m)$ if and only if there exists a permutation $s$ such that $\sigma_s((p_1, \ldots, p_m)) = (q_1, \ldots, q_m)$. It is now easy to see that the map $Q : P^m \to P^m / \sim$ is a closed map, because $Q^{-1}Q(E) = \cup_{s \in S_m} \sigma_s(E)$ and each $\sigma_s$ is a homeomorphism of topological spaces. Hence the Polish space $P^m$ contains a Borel set $E_0$ which meets each equivalence class in exactly one point (see [Par05], Theorem I.4.2.). Now for a measure $\mu$ in $M(mP)$ define $\nu$ to be

$$\nu(B) = \mu(\omega_m(B \cap E_0)),$$

for Borel subsets $B$ of $P^m$. It is easy to check that $\breve{\omega}_m(\nu) = \mu$, hence $\breve{\omega}_m$ is onto. $\square$

A *reduced sum* on a strongly independent subset $P$ of torsion $k(P) = k$ is a formal expression $\sum_{i \in I} \dot{n}_i p_i$, where $I$ is a possibly empty finite index set, $p_i$'s are distinct elements of $P$, and

$$0 \neq \dot{n}_i \in \mathbb{Z}(\text{mod } k).$$

Two reduced sums are said to be equivalent if one can be obtained from a permutation of the other. Let $P$ be a subset of $G$. For $m$ and $k$ in $\mathbb{N}$, and $g$ in $G$, define the following sets.

$$D_m^k(g) = \{\omega = (p_j)_{j=1}^m \in P^m : p_k = g\}.$$

$$R_m^k = \bigcup_{1\leq l_1<l_2<\ldots<l_k\leq m} \{\omega = (p_j)_{j=1}^m \in P^m : p_{l_1} = \ldots = p_{l_k}\}.$$

The following two lemmas illustrate interesting properties of strongly independent sets which will be used in future. Lemma 4.3.3 is in fact a corollary of Lemma 4.3.2 which in turn has a straightforward proof. One can refer to [Var66a] to see the details.

**Lemma 4.3.2.** *[Var66a] Let $P$ be a strongly independent subset of an Abelian group $G$. Let $m, n \in \mathbb{Z}$, $m \geq 1$, and $m \geq n \geq 0$.*

1. *Every $x$ in $\mathrm{Gp}(P)$ can be expressed uniquely (up to equivalence) as a reduced sum.*
2. *If $g \in G \setminus \mathrm{Gr}(P)$ then $mP \cap (g + nP) = \emptyset$.*
3. *If $0 \neq g \in \mathrm{Gp}(P)$ and $g = \sum_{i\in I} \dot{n}_i p_i$ is the reduced sum expression of $g$ then:*
   - *(i) If $k > m > n$ then $mP \cap nP = \emptyset$, and in particular $\omega_m^{-1}(mP \cap nP) = \emptyset$.*
   - *(ii) If $m > n$ and $m \geq k$ then $\omega_m^{-1}(mP \cap nP) \subseteq R_m^k$.*
   - *(iii) If $k > m$ then $\omega_m^{-1}(mP \cap g + nP) \subseteq \bigcup_{i\in I} \bigcup_{1\leq j\leq m} D_m^j(p_i)$.*
   - *(iv) If $m \geq k$ then $\omega_m^{-1}(mP \cap g + nP) \subseteq R_m^k \cup \bigcup_{i\in I} \bigcup_{1\leq j\leq m} D_m^j(p_i)$.*

**Lemma 4.3.3.** *[Var66a] Let $P$ be a strongly independent perfect metrisable subset of a non-discrete locally compact Abelian group $G$. Let $\mu$ and $\nu$ be measures in $M_c^+(G)$ that satisfy conditions (i) to (iv) listed below.*

*(i)* $\mathrm{supp}(\mu) \subseteq mP$.

*(ii)* *For every* $g \in G$ *and* $0 \leq m' < m$, *the set* $g + m'P$ *is* $\mu$*-null.*

*(iii)* $\mathrm{supp}(\nu) \subseteq nP$.

*(iv)* *For every* $g \in G$ *and* $0 \leq n' < n$, *the set* $g + n'P$ *is* $\nu$*-null.*

*Then for every* $g \in G$ *and* $0 \leq r \leq m + n$ *that satisfy* $(g, r) \neq (0_G, m + n)$, *the set* $g + rP$ *is* $\mu * \nu$*-null.*

Let $s$ be a permutation in the symmetric group $S_m$ on $m$ elements. We define the *symmetric operation* associated with $s$ as

$$\sigma_s : P^m \to P^m, \quad \sigma_s[(p_j)_{j=1}^n] = (p_{s(j)})_{j=1}^n,$$

and we denote the set of all such symmetric operators on $P^m$ by $\Sigma_m$. Recall that $\sigma_s$ induces a map $\check{\sigma}_s$ on the measure algebra of $P^m$. An $L$-subspace $B$ of $M(P^m)$ is called *symmetric* if for all $\sigma$ in $\Sigma_m$, $\check{\sigma}(B)$ is contained in $B$. Let $B^\Sigma$ denote the smallest symmetric $L$-space which contains $B$, i.e.

$$B^\Sigma = \cap\{S : B \subseteq S, S \text{ is a symmetric } L\text{-space }\}.$$

Note that there is a natural one-to-one correspondence between $\Sigma_m$ and $S_m$ which preserves multiplication. Finally for a measure $\mu$ in $M(P^m)$, we define the measure $\mu^\Sigma = \sum_{s \in S_m} \check{\sigma}_s(\mu)$.

Let $\Omega$ be a measurable subset of $G$. Let $B(\Omega)$ be defined as

$$B(\Omega) = \{\mu \in M(G) : |\mu|(G \setminus \Omega) = 0\}.$$

It is easy to check that the space $B(\Omega)$ is an $L$-space. Recall that $M(\Omega)$ denotes the subspace of $M(G)$ whose measures are supported in $\Omega$. Note that $B(\Omega)$ and $M(\Omega)$ are different. For example, for any continuous measure $\mu$ in $M_c(G)$ and any $x$ in $\mathrm{supp}(\mu)$, we have $\mu \in B(G \setminus \{x\})$.

**Lemma 4.3.4.** *[Var66a] Let $P$ be a strongly independent perfect metrisable subset of a non-discrete locally compact Abelian group $G$. Let $B$ be an $L$-subspace of $M(P^m)$, and $m \geq 2$.*

*(a) If $\mu$ belongs to $M(mP) \cap \breve{\omega}_m(B)$ then $\mathcal{R}\mu$ belongs to $\breve{\omega}_m(B)$ as well.*

*(b) $\breve{\omega}_m^{-1}(\breve{\omega}_m[B(R_m^2)]) \cap M^+(P^m) \subseteq B(R_m^2)$ and*
*$\breve{\omega}_m^{-1}(\breve{\omega}_m[P^m \setminus B(R_m^2)]) \cap M^+(P^m) \subseteq B(P^m \setminus R_m^2)$.*

*(c) If $\mu, \nu \in B(P^m \setminus R_m^2) \cap M^+(P^m)$ and $\breve{\omega}_m(\mu) \ll \breve{\omega}_m(\nu)$ then $\mu \in B^\Sigma[\nu]$, where $B^\Sigma[\nu]$ is the symmetric $L$-space generated by $\nu$.*

*(d) Let $\{\gamma_\alpha\}_{\alpha\in\Gamma}$ be a family of measures in $B(P^m \setminus R_m^2)$ such that for each index $\alpha$, $\breve{\omega}_m(\gamma_\alpha) \geq 0$. Then there exists a family $\{\delta_\alpha\}_{\alpha\in\Gamma}$ in $M^+(P^m) \cap B(P^m \setminus R_m^2)$ that satisfies the following properties.*

- *$\delta_\alpha \in B^\Sigma[\gamma_\alpha]$ for all $\alpha$ in $\Gamma$.*
- *$\breve{\omega}_m(\delta_\alpha) = \breve{\omega}_m(\gamma_\alpha)$ for all $\alpha$ in $\Gamma$.*
- *For all $\alpha$ and $\beta$ in $\Gamma$, if $\breve{\omega}_m(\gamma_\alpha) \geq \breve{\omega}_m(\gamma_\beta)$ then $\delta_\alpha \geq \delta_\beta$.*

*(e) If $B \subseteq B(P^m \setminus R_m^2)$ is a symmetric $L$-space then $\breve{\omega}_m(B)$ is an $L$-space of $M(mP)$.*

**Proof.**

(a) Let $\overline{\mu}$ be a measure in $B$ such that $\breve{\omega}_m(\overline{\mu}) = \mu$. Note that $\breve{\omega}_m$ maps positive (respectively real) measures to positive (respectively real) measures. Now consider the decomposition $\overline{\mu} = \overline{\mu}_1 + i\overline{\mu}_2$, where $\overline{\mu_1}$ and $\overline{\mu_2}$ are real measures (the real and imaginary parts of $\overline{\mu}$). Then $\breve{\omega}_m(\overline{\mu}) = \breve{\omega}_m(\overline{\mu}_1) + i\breve{\omega}_m(\overline{\mu}_2)$, where $\breve{\omega}_m(\overline{\mu}_1)$ and $\breve{\omega}_m(\overline{\mu}_2)$ are real measures. Hence

$$\mathcal{R}(\mu) = \mathcal{R}(\breve{\omega}_m(\overline{\mu})) = \breve{\omega}_m(\overline{\mu}_1) = \breve{\omega}_m(\mathcal{R}(\overline{\mu})).$$

Moreover by the definition of $L$-space, $\mathcal{R}(\overline{\mu})$ belongs to $B$, which proves (a).

(b) It is clear that $R^2_m$ and $G \setminus R^2_m$ are "symmetric sets" i.e.

$$\forall (p_i)_{i=1}^m \in R^2_m \ \forall \pi \in S_m, (p_{\pi(i)})_{i=1}^m \in R^2_m,$$

and

$$\forall (p_i)_{i=1}^m \in G \setminus R^2_m \ \forall \pi \in S_m, (p_{\pi(i)})_{i=1}^m \in G \setminus R^2_m.$$

Therefore $\omega_m^{-1}(\omega_m(R^2_m)) = R^2_m$ and $\omega_m^{-1}(\omega_m(G \setminus R^2_m)) = G \setminus R^2_m$. Let $\mu$ in $M^+(P^m)$ and $\nu$ in $B(R^2_m)$ be such that $\breve{\omega}_m(\mu) = \breve{\omega}_m(\nu)$, i.e. $\mu(\omega_m^{-1}(E)) = \nu(\omega_m^{-1}(E))$ for every Borel subset $E$ of $mP$. Hence

$$\mu(G \setminus R^2_m) = \mu(\omega_m^{-1}(\omega_m(G \setminus R^2_m))) = \nu(\omega_m^{-1}(\omega_m(G \setminus R^2_m))) = \nu(G \setminus R^2_m) = 0.$$

This, together with positivity of $\mu$, implies that $\mu$ belongs to $B(R^2_m)$. Hence $\breve{\omega}_m^{-1}(\breve{\omega}_m[B(R^2_m)]) \cap M^+(P^m)$ is a subset of $B(R^2_m)$. The proof of the second claim is identical.

(c) The open subspace $P^m \setminus R_m^2$ of $P^m$ is a Polish space, and $\sim$ is a closed equivalence relation on $P^m \setminus R_m^2$. Hence the conditions of Borel cross-section theorem are satisfied, and $P^m \setminus R_m^2$ contains a Borel subset $A$ that meets each equivalence class in exactly one point. For $s$ in $S_m$, let $A_s$ denote the Borel set $\sigma_s(A)$. It is easy to see that for permutations $s$ and $t$ in $S_m$,

- $P^m \setminus R_m^2 = \cup_{s \in S_m} A_s$.
- $A_s \cap A_t = \emptyset$ if $s \neq t$.
- $\sigma_s(A_t) = A_{ts}$.

To a measure $\alpha$ in $M(P^m \setminus R_m^2)$, we associate the following orthogonal (Riesz-Lebesgue) decomposition:

$$\alpha = \sum_{s \in S_m} \alpha_s \qquad \text{where} \qquad \alpha_s(E) = \alpha(E \cap A_s).$$

Clearly $\alpha_s \ll \alpha$ for each $s$ in $S_m$. Let $E$ and $F$ be Borel subsets of $P^m$ and $mP$ respectively. For $\alpha$ in $M(P^m \setminus R_m^2)$, and permutations $s$ and $t$ in $S_m$, we have

$$[\check{\sigma}_s(\alpha)]_t(E) = \alpha(\sigma_s^{-1}(A_t \cap E)) = \alpha(\sigma_s^{-1}(E) \cap A_{ts^{-1}}) = \alpha_{ts^{-1}}(\sigma_{s^{-1}}(E)).$$

Moreover, observe that $\omega_m^{-1}(F)$ is a symmetric set, and $\sigma_s(\omega_m^{-1}(F)) = \omega_m^{-1}(F)$.

Therefore,

$$\check{\omega}_m([\check{\sigma}_s(\alpha)]_t)(F) = [\check{\sigma}_s(\alpha)]_t(\omega_m^{-1}(F)) = \alpha_{ts^{-1}}(\sigma_{s^{-1}}(\omega_m^{-1}(F))) = \alpha_{ts^{-1}}(\omega_m^{-1}(F)),$$

which implies that

$$\check{\omega}_m([\alpha^\Sigma]_r)(F) = \sum_{s\in S_m} \check{\omega}_m([\check{\sigma}_s(\alpha)]_r)(F) = \sum_{s\in S_m} \check{\omega}_m(\alpha_{rs^{-1}})(F) = \check{\omega}_m(\alpha)(F).$$

Let $\mu$ and $\nu$ be measures as described in (c), $r$ be a permutation in $S_m$, and $E$ be a Borel subset of $P^m$. Then

$$\begin{aligned}[\nu^\Sigma]_r(E) &= \sum_{s\in S_m} \check{\sigma}_s(\nu)(E\cap A_r) = \sum_{s\in S_m} \nu(\sigma_s^{-1}(E\cap A_r)) = \nu(\bigcup_{s\in S_m} \sigma_s^{-1}(E\cap A_r)) \\ &= \nu(\omega_m^{-1}(\omega_m(E\cap A_r))) = \check{\omega}_m(\nu)(\omega_m(E\cap A_r)), \qquad (4.1)\end{aligned}$$

where we used the fact that for distinct permutations $s$ and $t$ in $S_m$, the sets $\sigma_s^{-1}(E\cap A_r)$ and $\sigma_t^{-1}(E\cap A_r)$ are disjoint. Now $[\nu^\Sigma]_r(E) = 0$ implies that $\check{\omega}_m(\nu)(\omega_m(E\cap A_r)) = 0$ which in turn implies that

$$[\mu^\Sigma]_r(E) = \check{\omega}_m(\mu)(\omega_m(E\cap A_r)) = 0.$$

Hence $[\mu^\Sigma]_r \ll [\nu^\Sigma]_r$ for each $r \in S_m$. Therefore $\mu \ll \mu^\Sigma \ll \nu^\Sigma$, and $\mu$ belongs to the symmetric $L$-space generated by $\nu$.

(d) Let $\{\gamma_\alpha\}_{\alpha\in\Gamma}$ be a family as in (d), and $r$ in $S_m$ be a fixed permutation. For each $\alpha$ in $\Gamma$, define

$$\delta_\alpha = [\gamma_\alpha^\Sigma]_r.$$

Note that $\breve{\omega}_m(\delta_\alpha) = \breve{\omega}_m([\gamma_\alpha^\Sigma]_r) = \breve{\omega}_m(\gamma_\alpha)$. Let $E$ be a Borel subset of $P^m$. By Equation (4.1), we have $\delta_\alpha(E) = [\gamma_\alpha^\Sigma]_r(E \cap A_r) = \breve{\omega}_m(\gamma_\alpha)(\omega_m(E \cap A_r))$, therefore $\delta_\alpha$ belongs to $M^+(P^m)$. Moreover,

$$|\delta_\alpha|(R_m^2) = \delta_\alpha(R_m^2) = \gamma_\alpha(\omega_m^{-1}(\omega_m(R_m^2 \cap A_r)) = 0,$$

hence $\delta_\alpha$ belongs to $B(P^m \setminus R_m^2)$. Fix $\alpha$ and $\beta$ in $\Gamma$, and note that $\delta_\alpha(E) = \breve{\omega}_m(\gamma_\alpha)(\omega_m(E \cap A_r))$. Therefore $\breve{\omega}_m(\gamma_\alpha) \geq \breve{\omega}_m(\gamma_\beta)$ implies that $\delta_\alpha \geq \delta_\beta$. Finally, using part (c) and Lemma 4.1.2, we have $\delta_\alpha \in B^\Sigma[|\gamma_\alpha|] = B^\Sigma[\gamma_\alpha]$, since $\breve{\omega}_m(\delta_\alpha) = \breve{\omega}_m(\gamma_\alpha) \ll \breve{\omega}_m(|\gamma_\alpha|)$.

(e) This part follows easily from parts (a), (b) and (d).

□

## 4.4 A direct decomposition of $M(G)$

Fix a strongly independent perfect metrisable subset $P$ of $G$, and let

$$T_1 = M_c(P) = \{\mu \in M_c(G) : \mathrm{supp}(\mu) \subseteq P\}.$$

For a positive integer $n$, let

$$T_n = T_1 \otimes^{\gamma} \ldots \otimes^{\gamma} T_1$$

denote the tensor product of $n$ copies of $T_1$, and define

$$T = \ell^1\text{-} \oplus_{n \geq 1} T_n.$$

We equip $T$ with the multiplication defined as

$$t_m \cdot t_n = t_m \otimes t_n \in T_{m+n}$$

for $t_m$ in $T_m$ and $t_n$ in $T_n$, and extend it to $T$ by linearity and continuity. Let $\theta$ be a continuous function in $C_b(P)$ viewed as an element of the dual space $T_1^*$. Let $\theta^n$ denote the element $\theta \otimes \ldots \otimes \theta$ of $T_n^*$, and define

$$S_n = T_n / \bigcap_{\theta \in C_b(P) \subseteq T_1^*} \mathrm{Ker}(\theta^{\mathrm{n}}).$$

Let $S = \ell^1\text{-} \oplus_{n \geq 1} S_n$, and $p : T \to S$ be the natural projection. It is easy to see that $\mathrm{Ker}(p)$ is an ideal of $T$, therefore one can define a multiplication on $S$ using the multiplication on $T$. Indeed, for $t_m$ in $T_m$ and $t_n$ in $T_n$, let

$$p(t_m) \cdot p(t_n) = p(t_m \otimes t_n) \in S_{m+n},$$

and extend it to $S$ by linearity and continuity. These multiplications turn $T$ and $S$ into Banach algebras, and $p$ becomes a surjective algebra homomorphism. Let $\tau_1$ be the inclusion map from $T_1 = M_c(P)$ to $M(G)$. The map $\tau_1$ induces $\tau_n : T_n \to M(G)$,

$$\tau_n(\mu_1 \otimes \ldots \otimes \mu_n) = \mu_1 * \ldots * \mu_n, \quad \mu_1, \ldots, \mu_n \in M_c(P).$$

Let $\tau = \ell^1\text{-} \oplus_{n\geq 1} \tau_n : T \to M(G)$. Clearly $\tau$ is a Banach algebra homomorphism. Finally, let $i : \Delta(G) \to M(G)$ be the inclusion map, and define $\pi = i \otimes \tau$ from $\Delta \otimes T$ to $M(G)$ to be the linear extension of,

$$\pi(\delta_g \otimes (\mu_1 \otimes \ldots \otimes \mu_n)) = \delta_g * \mu_1 * \ldots * \mu_n.$$

Then $\pi$ is a Banach algebra homomorphism as well. In Lemma 4.4.2, we show that $\tau_n(T_n)$ is the $L$-space generated by products of $n$ elements of $T_1$, i.e.

$$\{\mu \in M(G) : \mu \ll \mu_1 * \ldots * \mu_n \text{ for some } \mu_1, \ldots, \mu_n \in T_1\}.$$

Denote $\pi_n^g = \pi|_{\delta_g \mathbb{C} \otimes T_n}$, and $\pi_n = \pi|_{T_n}$.

**Observation 4.4.1.** *Let $\phi_m : T_m \to M(P^m)$ be the map defined by*

$$\int_{P^m} f(x_1, \ldots, x_m) d\phi_m(\mu_1 \otimes \ldots \otimes \mu_m) = \int_P \ldots \int_P f(x_1, \ldots, x_m) d\mu_1 \ldots d\mu_m.$$

*Then,*

*(a) $\phi_m$ is an isometric injection. Moreover, $\pi_m = \check{\omega}_m \phi_m$.*

*(b)* $\phi_m(T_m)$ *is a symmetric L-subspace of* $M(P^m)$*.*

*(c) Let g be an element of G, and* $1 \leq l \leq m$*. Then for* $t_m$ *in* $T^m$*, we have*

$$|\phi_m(t_m)|(R^l_m) = 0 \text{ and } |\phi_m(t_m)|(D^l_m(g)) = 0.$$

*(d) For all g in G,* $\mathrm{Im}\pi^g_n = \delta_g * \mathrm{Im}\pi_n$*.*

**Proof.**

(a) Fix an element $x$ in $T_m$, and $\epsilon > 0$. There exists a representation of $x$

$$x = \sum_{i \in \mathbb{N}} \mu^i_1 \otimes \ldots \otimes \mu^i_m$$

with $\sum_{i \in \mathbb{N}} \|\mu^i_1\| < \|x\| + \epsilon$ and $\|\mu^i_2\| = \ldots = \|\mu^i_m\| = 1$ for each $i$ in $\mathbb{N}$. Fix an integer $1 \leq j \leq m$. The set $\{|\mu^i_j|\}_{i \in \mathbb{N}}$ is bounded, and $M_c(P)$ is an $L$-subspace of $M(G)$. Therefore $\nu_j = \sup\{|\mu^i_j|\}_{i \in \mathbb{N}}$ belongs to $M^+_c(P)$. By Radon-Nikodym Theorem

$$x \in \hat{\otimes}_{1 \leq j \leq m} L^1(P; \nu_j) = L^1(P^m; \otimes_{1 \leq j \leq m} \nu_j) \subseteq M(P^m),$$

where the last inclusion is an isometric injection. Moreover, for an integrable function $f$, and $\mu_1, \ldots, \mu_m$ in $T_1$,

$$\begin{aligned} \int_G f(x) d\check{\omega}_m \phi_m(\mu_1 \otimes \ldots \otimes \mu_m)(x) &= \int_{P^m} f \circ \omega_m(X) d\phi_m(\mu_1 \otimes \ldots \otimes \mu_m)(X) \\ &= \int_{P^m} f(\sum_{i=1}^m x_i) d\mu_1(x_1) \ldots d\mu_m(x_m), \end{aligned}$$

which finishes the proof.

(b) We observed that for $x$ in $T_m$ that for each integer $j$ in $\{1 \ldots m\}$, there exist $\nu_j$ in $M_c^+(P)$ such that

$$x \in \hat{\otimes}_{1 \leq j \leq m} L^1(P; \nu_j) \subseteq T_m.$$

Hence $\phi_m(x)$ belongs to $L^1(P^m; \otimes_{1 \leq m \leq m} \nu_j)$ which is a subset of $\phi_m(T_m)$, and $\phi_m(T_m)$ is an $L$-subspace of $M(P^m)$. Moreover,

$$\check{\sigma}_r \phi_m(\mu_1 \otimes \ldots \otimes \mu_m) = \phi_m(\mu_{r(1)} \otimes \ldots \otimes \mu_{r(m)})$$

for every $r$ in $S_m$, which implies that $\phi_m$ is a symmetric $L$-space.

(c) It follows from Fubini's theorem.

(d) It is trivial.

$\square$

**Lemma 4.4.2.** *[Var66a] Let* $\Pi = \mathrm{Im}\pi$ *and* $I = \Pi^{\perp} \cap M_c(G)$.

*(a) For* $g$ *in* $G$ *and* $n \geq 1$, $\mathrm{Im}(\pi_n^g)$ *is an* $L$*-subspace of* $M(-g + mP)$.

*(b) Let* $g_1$ *and* $g_2$ *be elements of* $G$, *and* $n_1$ *and* $n_2$ *be in* $\mathbb{Z}$ *such that* $(g_1, n_1) \neq (g_2, n_2)$. *Then* $\mathrm{Im}(\pi_{n_1}^{g_1}) \perp \mathrm{Im}(\pi_{n_2}^{g_2})$.

*(c)* $\Pi$ *is a translation invariant* $L$*-subspace of* $M_c(G)$.

*(d)* $I$ *is a translation invariant ideal of* $M(G)$.

**Remark.** Notice that to build $\Pi$, it is necessary to use components of the form

$$\mathrm{Im}(\pi_n) = \{\mu \in M(G) : \mu \ll \mu_1 * \ldots * \mu_n \text{ for some } \mu_1, \ldots, \mu_n \in M_c(P)\}$$

rather than using all of $M_c(nP)$. In fact, it is not even true that "$M_c(g_1 + nP) \perp M_c(g_2 + mP)$ for $(g_1, n) \neq (g_2, m)$". For instance, if $q$ is an element of $P$ then $q + P \subseteq 2P$ and $M_c(q + P) \subseteq M_c(2P)$.

**Proof.**

(a) It is very easy to see that $\mathrm{Im}(\pi_n^g) \subseteq M(-g + mP)$. The map $\mu \mapsto \delta_g * \mu$ is an invertible isometric linear map on $M(G)$ which takes positive measures to positive ones. So $B$ is an $L$-space if and only if $\delta_g * B$ is one. Hence it is enough to show that $\mathrm{Im}\pi_n$ is an $L$-space. By Observation 4.4.1, $\phi_m(T_m)$ is a symmetric $L$-subspace of $B(P^m \setminus R_m^2)$. Therefore by Lemma 4.3.4, $\pi_n(T_n) = \check{\omega}_n \circ \phi_n(T_n)$ is an $L$-space in $M(G)$ as well.

(b) Without loss of generality, we can assume that $g_1 = o_G$ and $n_1 \geq n_2$. Let $x$ and $y$ be elements of $\mathrm{Im}(\pi_{n_1}^{g_1})$ and $\mathrm{Im}(\pi_{n_2}^{g_2})$ respectively. By Lemma 4.3.2 and Observation 4.4.1,

$$\begin{aligned} |\pi_{n_1}(x)|(g_2 + n_2 P) &= |\check{\omega}_{n_1}\phi_{n_1}(x)|(g_2 + n_2 P) \leq \check{\omega}_{n_1}|\phi_{n_1}(x)|(g_2 + n_2 P) \\ &= |\phi_{n_1}(x)|(\omega_{n_1}^{-1}(n_1 P \cap g_2 + n_2 P)) \\ &\leq |\phi_{n_1}(x)|(R_{m_1}^2) + \sum_{r \in \Gamma} \sum_{j=1}^{n_1} |\phi_{n_1}(x)|(D_{n_1}^j(g_r)) = 0, \end{aligned}$$

where $g = \sum_{r \in \Gamma} \gamma_r g_r$ is the reduced sum expansion of $g$. Now $x \perp y$ follows from $\mathrm{supp}(y) \subseteq g_2 + n_2 P$.

(c) $\Pi = \mathrm{Im}(\pi)$ is an $L$-space since each $\pi_n^g$ is an $L$-space. Using the above argument with $n_2 = 0$, we obtain $\mathrm{Im}\pi_n^g \perp \Delta$.

(d) By Observation 4.4.1, $\Pi$ is translation invariant. Hence $I$ is translation invariant as well, and it is enough to show that $I$ is an ideal of $M_c(G)$, i.e.

$$\mu, \nu \in M_c^+(G),\ \mu \perp \Pi \Rightarrow \mu * \nu \perp \Pi. \tag{4.2}$$

For $\mu$ in $M^+(G)$, we say that $\mu$ has property (A) if

$$\forall g \in G\ \forall m \geq 0,\ \mu(g + mP) = 0. \tag{A}$$

**Case 1:** Assume that $\mu$ and $\nu$ are elements of $M_c^+(G)$ such that $\mu \perp \Pi$ and $\mu$ has property (A). Then for $g_1$ in $G$ and $m_1 \geq 0$,

$$\begin{aligned} \mu * \nu(g_1 + m_1 P) &= \int_G \int_G \chi_{g_1 + mP}(x + y) d\mu(x) d\nu(y) \\ &= \int_G \int_G \chi_{g_1 - y + mP}(x) d\mu(x) d\nu(y) = 0, \end{aligned}$$

which implies that $\mu * \nu \perp \Pi$.

**Case 2:** Now assume that $\mu$ in $M_c^+(G)$ does not have property (A). Then there exist $g$ in $G$ and $m > 0$ such that $\mu(g + mP) > 0$. Let $m_1$ be the smallest integer such that $\mu(g_1+m_1P) > 0$ for some $g_1$ in $G$. Let $\mu_1 = \mu|_{g_1+m_1P}$. Then

$$\mathrm{supp}(\mu_1) \subseteq g_1 + m_1P \quad \text{and} \quad \mu_1(g+m'P) = 0 \text{ for all } m' < m, g \in G. \tag{B}$$

A positive measure $\mu$ in $M_c(G)$ has property (B) if it satisfies Condition (B) for some $m_1$ and $g_1$. Note that $\mu - \mu_1 \in M_c^+(G)$ and $\mu - \mu_1 \perp \mu_1$. By repeating the above process, we can find measures $\{\mu_\alpha\}_{\alpha \in I}$ and $\nu$ in $M_c^+(G)$ such that

$$\mu = \sum_{\alpha \in I} \mu_\alpha + \nu,$$

where each $\mu_\alpha$ satisfies property (B), $\nu$ has property (A), and $\mu_\alpha \perp \mu_\beta$ for $\alpha \neq \beta$.

Note that the index set $I$ should be countable since the measures are orthogonal and $\mu$ is a finite measure. Using translation invariance of $I$, it is enough to show (4.2) with the further assumption that $\mu$ and $\nu$ satisfy property (B'), i.e.

$$\begin{aligned} &\exists m \geq 1 \text{ s.t. } \mu \text{ satisfies property (B) with } (m, 0_G). \\ &\exists n \geq 1 \text{ s.t. } \nu \text{ satisfies property (B) with } (n, 0_G). \end{aligned} \tag{B'}$$

**Case 3:** Let $\mu$ and $\nu$ be elements of $M_c^+(G)$ which satisfy property (B') as above, and $\mu \perp \Pi$.

(i) If $g \in G$ and $r > m+n$ then $\mu * \nu \perp \mathrm{Im}\pi_r^g$. Indeed, as we observed in (c) the set $(m+n)P$ is a null set for every element of $\mathrm{Im}\pi_r^g$, but $\mathrm{supp}(\mu * \nu) \subseteq (m+n)P$.

(ii) If $g \in G$ and $r < m+n$ then by Lemma 4.3.3 and property (B') $\mu *$

$\nu \perp \mathrm{Im}\pi_r^g$.

(iii) If $g \neq 0_G$ then by Lemma 4.3.3 and property (B') $\mu * \nu \perp \mathrm{Im}\pi_{m+n}^g$.

(iv) It only remains to show that $\mu * \nu \perp \mathrm{Im}\pi_{m+n}$. Let $\overline{\mu} \in M^+(P^m)$ and $\overline{\nu} \in M^+(P^n)$ be such that $\breve{\omega}_m(\overline{\mu}) = \mu$ and $\breve{\omega}_m(\overline{\nu}) = \nu$. Note that

$$\breve{\omega}_{n+m}(\overline{\mu} \otimes \overline{\nu}) = \mu * \nu.$$

**Claim:** $\overline{\mu} \perp \phi_m(T_m)$.

Let $p : P^{m+n} \to P^m$ be the projection of $P^{m+n}$ to its first $m$ entries. Define

$$\iota : M(P^m) \to M(P^{m+n}), \quad \iota(x)(E) = x(p(E))$$

for every measurable subset $E$ of $P^{m+n}$. Clearly $\iota$ identifies $M(P^m)$ isometrically as a subset of $M(P^{m+n})$. By the hypothesis, we have $\mu \perp \mathrm{Im}\pi_m^g$, i.e. for each $x$ in $T_m$ there are disjoint sets $A$ and $B$ partitioning $mP$ such that

$$\mu(A) = \pi_m(x)(B) = 0.$$

Hence $\breve{\omega}_m(\overline{\mu})(A) = \overline{\mu}(\omega_m^{-1}(A)) = 0$ and $\breve{\omega}_m(\phi_m(x))(B) = \phi_m(x)(\omega_m^{-1}(B)) = 0$, which implies that $\overline{\mu} \perp \phi_m(T_m)$.

Note that $\overline{\mu} \perp \phi_m(T_m)$ implies that $\overline{\mu} \otimes \overline{\nu} \perp \phi_m(T_m) \otimes \phi_n(T_n) = \phi_{m+n}(T_{m+n})$. Fix an element $x$ in $\phi_{m+n}(T_{m+n}) \cap M^+(P^{m+n})$. Note that $\mu \otimes \nu \perp x^\Sigma$ since $\phi_{m+n}(T_{m+n})$ is a symmetric $L$-space. Therefore, there exists a partition

$P^{m+n} = A \cup B$ such that

$$x^{\Sigma}(B) = (\mu \otimes \nu)(A) = 0.$$

Hence,

$$x^{\Sigma}(B) = x(\omega_{m+n}^{-1}(\omega_{m+n}(B))) = \check{\omega}_{m+n}(x)(\omega_{m+n}(B)) = 0,$$

and

$$\check{\omega}_{m+n}(\overline{\mu} \otimes \overline{\nu})((m+n)P \setminus \omega_{m+n}(B)) = 0.$$

Hence $\mu * \nu \perp \mathrm{Im}(\pi_{m+n})$, using the fact that $\check{\omega}_{m+n}(\overline{\mu} \otimes \overline{\nu}) = \mu * \nu$.

$\square$

We are now ready to state the decomposition theorem of [Var66a]. Recall that

$$S = \ell_1\text{-}\bigoplus_{n \geq 1} \left[ M_c(P)^{\hat{\otimes} n} / \bigcap_{\theta \in C_b(P) \subseteq M_c(P)^*} \mathrm{Ker}\theta^n \right].$$

Recall that $\Pi$ and $I$ are defined in Lemma 4.4.2.

**Theorem 4.4.3.** *[Var66a] Let $P$ be a perfect metrisable strongly independent subset of $G$. Then one can decompose $M_c(G)$ in the following way:*

1. $M_c(G) = \Pi \oplus I$ *(direct and orthogonal decomposition)*

2. $\Pi$ *is a closed subalgebra of* $M_c(G)$.

3. $\Pi$ *is an* $L$*-space of* $M(G)$.

*4. $I$ is an ideal and $L$-subspace of $M(G)$.*

*5.* $\mathrm{Ker}\tau = \mathrm{Ker}p \subseteq T$. *Therefore* $\Pi \simeq \Delta(G)\hat{\otimes}S$ *(topological and algebraic identification of Banach algebras)*

*6. Let* $j : \Delta(G)\hat{\bigotimes}\left(\ell_1\text{-}\bigoplus_{n\geq 1} M_c(P)^{\hat{\otimes}n}/\bigcap_{\theta\in C_b(P)\subseteq M_c(P)^*}\mathrm{Ker}\theta^n\right) \longrightarrow \Pi$ *be the identification map of part (5). Then*

$$j\left(\delta_{g_1}\otimes\left(M_c(P)^{\hat{\otimes}n}/\bigcap_{\theta\in C_b(P)}\mathrm{Ker}\theta^n\right)\right)\perp j\left(\delta_{g_2}\otimes\left(M_c(P)^{\hat{\otimes}m}/\bigcap_{\theta\in C_b(P)}\mathrm{Ker}\theta^m\right)\right),$$

*if* $(g_1, n) \neq (g_2, m)$.

Note that one can decompose $M(G)$ in a similar fashion as

$$M(G) = (\Delta(G)\oplus\Pi)\oplus I.$$

**Proof.** We only need to prove (5). By Lemma 4.4.2 (b), we just need to show that for every positive integer $n$,

$$\mathrm{Ker}\pi_n = \bigcap_{\theta\in C_b(P)\subseteq M_c(P)^*}\mathrm{Ker}\theta^n. \tag{4.3}$$

To prove "$\supseteq$" of (4.3), let $\alpha$ be an arbitrary element of $\bigcap_{\theta\in C_b(P)\subseteq M_c(P)^*}\mathrm{Ker}\theta^n$. For a character $\chi$ on $G$, define the following bounded continuous function on $P$:

$$f_\chi : P\to\mathbb{T}, \quad , t\mapsto \chi(t).$$

By Observation 4.4.1 we have,

$$
\begin{aligned}
0 = \langle \alpha, f_\chi^n \rangle &= \int_{P^n} f_\chi(x_1)\ldots f_\chi(x_n) d\phi_n(\alpha)(x_1,\ldots,x_n) \\
&= \int_{P^n} \chi(x_1)\ldots\chi(x_n) d\phi_n(\alpha)(x_1,\ldots,x_n) \\
&= \int_{P^n} \chi(x_1+\ldots+x_n) d\phi_n(\alpha)(x_1,\ldots,x_n) \\
&= \int_{P^n} \chi(\omega_n(x_1,\ldots,x_n)) d\phi_n(\alpha)(x_1,\ldots,x_n) \\
&= \int_{nP} \chi(x) d\check{\omega_n}\phi_n(\alpha)(x) \\
&= \langle \chi, \pi_n(\alpha) \rangle,
\end{aligned}
$$

where we used $\check{\omega_n}\phi_n = \pi_n$ in the last equality. This implies that $\pi_n(\alpha) = 0$, since $\chi$ is an arbitrary element of $\hat{G}$.

Conversely, let $\alpha$ be an element of $\mathrm{Ker}\pi_n$, and $\theta$ be a bounded continuous function on $P$. Then the function $\theta^n$ defined as

$$\theta^n(x_1,\ldots,x_n) = \theta(x_1)\ldots\theta(x_n)$$

is a bounded continuous function on $P^n$ which is symmetric under permutations, i.e. for every permutation $s$ in the symmetric group $S_n$,

$$\theta^n(x_1,\ldots,x_n) = \theta^n(x_{s(1)},\ldots,x_{s(n)}).$$

By the proof of Lemma 4.3.1, there exists a Borel subset $E_0$ of $P^n$ which is homeomorphic to $nP$. Therefore there exists a bounded Borel function $f_\theta$ on $nP$ such that $\theta^n = f_\theta \circ \omega_n$. Hence

$$\begin{aligned}\langle \alpha, \theta^n \rangle &= \int_{P^n} \theta^n(x_1, \ldots, x_n) d\phi_n(\alpha)(x_1, \ldots, x_n) \\ &= \int_{P^n} f_\theta(\omega_n(x_1, \ldots, x_n)) d\phi_n(\alpha)(x_1, \ldots, x_n) \\ &= \int_{nP} f_\theta(x) d\check{\omega}_n \phi_n(\alpha)(x) \\ &= \langle f_\theta, \pi_n(\alpha) \rangle = 0,\end{aligned}$$

which finishes the proof. $\square$

Theorem 4.0.3 is an important corollary of Theorem 4.4.3. Recall that in [Var66b], Varopoulos showed that if $G$ is a non-discrete locally compact Abelian group then there exists a perfect metrisable strongly independent subset $P$ of $G$. Moreover, if $G$ is metrisable as well then we can assume that the above-mentioned subset $P$ satisfies the additional condition

$$M_0(P) = \{\mu \in M_0(G) : \mathrm{supp}(\mu) \subseteq P\} \neq \{0\}.$$

**Proof of Theorem 4.0.3. (i)** Let $G$, $P$, $\Pi$ and $I$ be as in Theorem 4.4.3. Then

$$M_c(G)^2 = (\Pi \oplus I)^2 \subseteq \Pi^2 \oplus I.$$

By the construction of $\Pi$, it is easy to see that $M_c(P) \subseteq M_c(G)/\overline{M_c(G)^2}$. This implies that $M_c(G)/\overline{M_c(G)^2}$ is a non-separable Banach space, since $M_c(P)$ is one.

**(ii)** First assume that $G$ is metrisable, and let $P$ be a perfect metrisable strongly independent subset of $G$ such that $M_0(P) \neq \{0\}$. Since $M_0(G)$ is an $L$-space, one can easily show that

$$M_0(G) = (\Pi \cap M_0(G)) \oplus (I \cap M_0(G))$$

is a nontrivial decomposition of $M_0(G)$ to the subalgebra $\Pi \cap M_0(G)$ and the ideal $I \cap M_0(G)$ (see the proof of Theorem 4.5.1 for more details). Note that $M_0(P) \subseteq \Pi \cap M_0(G)$. Therefore

$$L^1(P) \subseteq M_0(P) \subseteq M_0(G)/\overline{M_0(G)^2},$$

which implies that $M_0(G)/\overline{M_0(G)^2}$ is infinite dimensional.

For a general non-discrete locally compact Abelian group $G$, let $H$ be a compact subgroup of $G$ such that $G/H$ is metrisable and non-discrete. Let $p$ denote the quotient map from $G$ to $G/H$. The map $p$ induces a Banach algebra homomorphism $\check{p}$ from $M(G)$ to $M(G/H)$. Moreover, since $H$ is compact, we have

$$\check{p}(M(G)) = M(G/H).$$

Therefore $M(G)/\overline{M(G)^2}$ is infinite dimensional, because its image under $\check{p}$, i.e.

$$M(G/H)/\overline{M(G/H)^2},$$

is infinite dimensional by part (1). $\square$

## 4.5 A direct decomposition of $M_0(G)$

In this section, we obtain decompositions for $M_0(G)$ similar to those of $M(G)$ discussed in Theorem 4.4.3. Our proofs are based on the results of Varopoulos in [Var66a].

**Theorem 4.5.1.** *1. For a non-discrete locally compact Abelian group $G$ and a subset $P$ as in Theorem 4.4.3, we have the orthogonal decomposition*

$$M_0(G) = \Pi_0 \oplus I_0,$$

*where $\Pi_0 = \Pi \cap M_0(G)$ is a closed subalgebra and $I_0 = I \cap M_0(G)$ is an ideal of the Banach algebra $M_0(G)$. In addition, both $\Pi_0$ and $I_0$ are $L$-subspaces of $M(G)$.*

*2. If $G$ is metrisable as well, there exists a subset $P$ such that the above decomposition is non-trivial, i.e. $\Pi_0 \neq \{0\}$ and $I_0 \neq \{0\}$.*

**Proof.**

1. Let $\mu$ be an element of $M_0(G)$. Since $M_0(G)$ is a subset of $M_c(G)$, we can orthogonally decompose $\mu$ to

$$\mu = \mu_1 + \mu_2,$$

with $\mu_1$ in $\Pi$ and $\mu_2$ in $I$. Note that $|\mu_1| \ll |\mu|$ and $|\mu_2| \ll |\mu|$. Therefore $\mu_1$ and $\mu_2$ belong to $M_0(G)$, since $M_0(G)$ is an $L$-space.

2. Let $G$ be a non-discrete metrisable locally compact Abelian group. Then there exists a perfect metrisable strongly independent subset $P$ of $G$ such that

$$M_0(P) = \{\mu \in M_0(G) : \mathrm{supp}(\mu) \subset P\} \neq \{0\}.$$

Hence

$$\{0\} \neq M_0(P) = M_0(P) \cap M_c(P) \subseteq \Pi_0,$$

which implies that $\Pi_0 \neq \{0\}$.

Moreover, $I_0 = I \cap M_0(G) \supseteq IM_0(G)$. Now let $\mu$ in $I$ and $\nu$ in $M_0(G)$ be nonzero positive measures with $\mu(E) > 0$ and $\nu(F) > 0$ for compact subsets $E$ and $F$ of $G$. Then

$$\begin{aligned} \mu * \nu(E+F) &= \int_G \int_G \chi_{E+F}(x+y) d\mu(x) d\nu(y) \\ &\geq \int_G \int_G \chi_E(x) \chi_F(y) d\mu(x) d\nu(y) = \mu(E)\nu(F) > 0. \end{aligned}$$

Hence $\mu * \nu \neq 0$ and $IM_0(G) \neq \{0\}$. To finish the proof, we just need to show that such $\mu$ and $\nu$ exist. Note that $M_0(G)$ and $I$ are non-trivial $L$-spaces, therefore contain positive measures.

$\square$

## 4.6 Point derivations on $M_0(G)$

Let $G$ be an Abelian locally compact group. To construct point derivations on $M_0(G)$, we use the decomposition of $M_0(G)$ presented in Theorem 4.5.1. We begin with the following lemma.

**Lemma 4.6.1.** *Let $G$ be a non-discrete locally compact Abelian group, and $P$ be a perfect metrisable strongly independent subset of $G$. Then*

1. *For each $\mu$ in $M_c(G)$, we have $\sum_{x\in G}\mu(x+P)<\infty$.*

2. *If $\mu,\nu\in M_c(G)$ then $(\mu*\nu)(P)=0$.*

**Proof.**

1. First note that if $x$ and $y$ are distinct elements of $G$ then $|(x+P)\cap(y+P)|\le 2$. Indeed, assume that there exist distinct elements $z_1$ and $z_2$ in $(x+P)\cap(y+P)$. Then there are $p_1,p_2,p_1'$, and $p_2'$ in $P$ such that

$$z_1=x+p_1=y+p_1' \text{ and } z_2=x+p_2=y+p_2',$$

which imply that $x-y=p_1'-p_1=p_2'-p_2$. Therefore $x-y$ should be an element of $P-P$. Note that since $z_1\ne z_2$ and $x\ne y$, we have

$$p_1\ne p_2,\quad p_1'\ne p_2',\quad p_1\ne p_1',\quad p_2\ne p_2'.$$

By Lemma 4.3.2, the element $x-y$ in $P-P$ can be expressed uniquely (up to permutation) as a reduced sum on $P$, i.e. one of the following cases happens:

**Case 1:** $p_1'=p_2'$ and $p_1=p_2$, which is a contradiction with $x\ne y$.

**Case 2:** $p_1'=-p_2$ and $p_2'=-p_1$, and $x-y=-p_1-p_2$ is the unique representation of $x-y$ in $P-P$. Taking permutations into account, there are at most two possibilities for $p_1$ and $p_2$, which implies that

$$|(x+P)\cap(y+P)| \leq 2.$$

Since $\mu$ is a continuous measure on $G$, it treats the sets $x+P$ as disjoint sets, i.e. $\mu((x+P)\cap(y+P)) = 0$ for distinct elements $x$ and $y$ in $G$. Hence for any finite number of points $x_1, \ldots, x_n$ in $G$,

$$\sum_{i=1}^{n} |\mu(x_i+P)| \leq |\mu|(\cup_{i=1}^{n}(x_i+P)) \leq |\mu|(G) < \infty.$$

Finally,

$$\sum_{x\in G} |\mu(x+P)| = \sup_{I\subset G, |I|<\infty} \sum_{x\in I} |\mu(x+P)| \leq |\mu|(G) < \infty.$$

2. Convergence of the sum in part 1 implies that only for countably many $x$ in $G$, $\mu(x+P)$ is nonzero. Therefore the function $x \mapsto \mu(x+P)$ is equal to 0 $\nu$-a.e. and the result follows.

$\square$

In [BM76], Brown and Moran constructed a nonzero continuous point derivation on the measure algebra $M(G)$ of a non-discrete locally compact Abelian group $G$. Their construction is based on the decomposition of the measure algebra of a locally compact group to its discrete and continuous parts. In Theorem 4.6.2, we prove a similar result for the algebra of Rajchman measures on a non-discrete locally compact Abelian group using the decomposition of $M_0(G)$ obtained in Theorem 4.5.1.

Our construction here is motivated by [BM76].

**Theorem 4.6.2.** *If $G$ is a non-discrete locally compact Abelian group, then $M_0(G)$ has a nonzero continuous point derivation.*

**Proof.** First assume that $G$ is metrisable. By Lemma 4.2.3, there exists a compact perfect metrisable strongly independent subset $P$ of $G$ which supports a nonzero Rajchman measure $\mu_0$. Using Theorem 4.5.1, we obtain a nontrivial decomposition $M_0(G) = \Pi_0 \oplus I_0$ with $\{0\} \neq M_0(P) \subseteq \Pi_0$. For each $\mu$ in $M_0(G)$, let $\mu = \mu_{\Pi_0} \oplus \mu_{I_0}$ denote its decomposition accordingly. Define the linear functionals $\chi$ and $d$ to be

$$\chi : M_0(G) \to \mathbb{C},\ \mu \mapsto \mu_{\Pi_0}(G),$$

and

$$d : M_0(G) \to \mathbb{C},\ \mu \mapsto \sum_{x \in G} \mu_{I_0}(x + P).$$

First, observe that $\chi$ is a nonzero character of $M_0(G)$. Indeed, it is clear that $\chi$ is a continuous linear map, and $\chi(\mu_0) = \mu_{0\Pi_0}(G) = \mu_0(G) \neq 0$. Let $\mu$ and $\nu$ be elements of $M_0(G)$. Then $(\mu * \nu)_{\Pi_0} = \mu_{\Pi_0} * \nu_{\Pi_0}$, since $I_0$ is an ideal and $\Pi_0$ is a subalgebra of $M_0(G)$. Therefore

$$\chi(\mu * \nu) = (\mu * \nu)_{\Pi_0}(G) = (\mu_{\Pi_0} * \nu_{\Pi_0})(G) = \mu_{\Pi_0}(G)\nu_{\Pi_0}(G) = \chi(\mu)\chi(\nu),$$

i.e. $\chi$ is a nonzero character. Next by Lemma 4.6.1, $d$ is well-defined and vanishes on $I_0^2$. Moreover, $d$ is clearly a nonzero linear map which vanishes on $\Pi_0$. Fix arbitrary elements $\mu$ in $\Pi_0$ and $\nu$ in $I_0$. Then

$$\begin{aligned}
d(\mu * \nu) &= \sum_{x\in G}(\mu * \nu)(x+P) \\
&= \sum_{x\in G}\int_G \nu(-y+x+P)d\mu(y) \\
&= \int_G \sum_{x\in G} \nu(-y+x+P)d\mu(y) \\
&= (\sum_{z\in G} \nu(z+P)) \int_G d\mu(y) \\
&= d(\nu)\chi(\mu).
\end{aligned}$$

We are now able to prove that $d$ is a point derivation of $M_0(G)$ at the character $\chi$. Let $\mu$ and $\nu$ be measures in $M_0(G)$. Then

$$\begin{aligned}
d(\mu * \nu) &= d(\mu_{\Pi_0} * \nu_{\Pi_0} + \mu_{\Pi_0} * \nu_{I_0} + \mu_{I_0} * \nu_{\Pi_0} + \mu_{I_0} * \nu_{I_0}) = d(\mu_{\Pi_0} * \nu_{I_0} + \mu_{I_0} * \nu_{\Pi_0}) \\
&= \chi(\mu_{\Pi_0})d(\nu_{I_0}) + \chi(\nu_{\Pi_0})d(\mu_{I_0}) = \chi(\mu)d(\nu) + \chi(\nu)d(\nu),
\end{aligned}$$

which finishes the proof for the metrisable case.

For the general case, let $G$ be a non-discrete locally compact Abelian group, and $H$ be a compact subgroup of $G$ such that $G/H$ is metrisable and non-discrete. Let $p$ be the quotient map from $G$ to $G/H$, and $\check{p}$ be the surjective Banach algebra homomorphism from $M_0(G)$ to $M_0(G/H)$ induced by $p$. By the above argument, $M_0(G/H)$ has a nonzero continuous point derivation. Hence by Lemma 4.6.3, $M_0(G)$ has a nonzero continuous point derivation as well. □

Let us remark that choosing a different perfect compact strongly independent subset $P$ may result in a different decomposition for $M_0(G)$. In fact, let $P$ and $\mu_0$ be as in Theorem 4.6.2. Let $P_1$ and $P_2$ be disjoint perfect subsets of $P$ such that $\mu_0$ restricts to nonzero measures on $P_1$ and $P_2$ respectively. Then for each $x$ and $y$ in $G$ and integers $m$ and $n$, the set $(x + mP_1) \cap (y + nP_2)$ is finite. Therefore $M_c(x+mP_1)$ and $M_c(y+nP_2)$ are orthogonal subsets of $M_c(G)$. This implies that the decomposition of $M_0(G)$ based on $P_1$ is different from the one that is based on $P_2$. We can now apply Theorem 4.6.2 to each decomposition and obtain distinct nonzero continuous point derivations for $M_0(G)$.

One can extend Theorem 4.6.2 to non-compact connected SIN-groups using the following lemma.

**Lemma 4.6.3.** *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras, and $\phi : \mathcal{A} \to \mathcal{B}$ be a Banach algebra homomorphism with dense range. If $\mathcal{B}$ has a nonzero continuous point derivation then $\mathcal{A}$ has one as well.*

**Proof.** Let $d : \mathcal{B} \to \mathbb{C}$ be a nonzero continuous derivation at the character $\chi : \mathcal{B} \to \mathbb{C}$. Then $D = d \circ \phi$ is a nonzero continuous derivation of $\mathcal{A}$ at the character $\theta = \chi \circ \phi$. Indeed, the function $\theta$ is a multiplicative linear map, since it is the composition of two multiplicative linear maps. Moreover, $\chi$ is nonzero and $\phi$ has dense range, therefore $\chi \circ \phi$ is nonzero as well. Similarly $D$ is a nonzero linear map, and for elements $x$ and $y$ in $\mathcal{A}$, we have:

$$\begin{aligned} D(xy) &= d(\phi(xy)) = d(\phi(x)\phi(y)) = d(\phi(x))\chi(\phi(y)) + d(\phi(y))\chi(\phi(x)) \\ &= D(x)\theta(y) + D(y)\theta(x). \end{aligned}$$

Hence $D$ is a nonzero continuous point derivation of $\mathcal{A}$ at the character $\theta$. $\square$

**Theorem 4.6.4.** *Let $G$ be a non-compact connected SIN group. Then $B_0(G)$ has a nonzero continuous point derivation.*

**Proof.** Any non-compact connected SIN group has a copy of $\mathbb{R}^n$ as a closed subgroup for some $n \geq 1$. Recall that the restriction map $r : B_0(G) \to B_0(\mathbb{R}^n)$ is a surjective homomorphism. By Theorem 4.6.2, $B_0(\mathbb{R}^n)$ has a nonzero continuous point derivation, and by Lemma 4.6.3 $B_0(G)$ also has one. $\square$

## 4.7 Analytic discs in the spectrum of $M_0(G)$

Let $G$ be a non-discrete locally compact Abelian group. Let $L^1(G)$ and $M(G)$ denote the group algebra and the measure algebra of $G$ respectively. The maximal ideal space of $L^1(G)$ can be identified with the character group of $G$. In analogy with this result, Taylor [Tay65] described the maximal ideal space of $M(G)$ as the set $\hat{S}$ of all semicharacters on a compact topological semigroup $S$. Moreover, he showed that for an element $\phi$ in $\hat{S}$, if $|\phi|$ is not an idempotent then there exists an analytic disc around $\phi$, and therefore there is a nontrivial continuous point derivation at $\phi$. By an analytic disc in the maximal ideal space $\Delta$, we mean an injection $\psi$ of the open unit disc in $\mathbb{C}$ into $\Delta$ such that $\hat{\mu} \circ \psi$ is holomorphic for each $\mu$ in $\mathcal{M}$. This method is applicable to a large class of convolution measure algebras including $M_0(G)$.

A *convolution measure algebra* is a closed subalgebra of $M(G)$ which is an $L$-space as well. Recall that $M_0(G)$ is a commutative convolution measure algebra.

Taking the above remark into account, it remains to study the possibilities for elements $\phi$ in $\hat{S}$ whose modulus are idempotents. For the special case of $M(G)$ and the discrete augmentation character $h$, Brown and Moran [BM76] have constructed nontrivial continuous point derivations at $h$. Later on, they used a method of Varopoulos to construct analytic discs around $h$ in the maximal ideal space of $M(G)$.

Having constructed certain decompositions for $M_0(G)$, we will show that similar results can be obtained for the Rajchman algebra as well. Especially, we construct analytic discs around idempotent characters of $M_0(G)$ associated with such decompositions. Such results will serve as a tool to determine whether those characters are strong boundary points. Let us recall some definitions and results for convolution measure algebras.

**Definition 4.7.1.** *Let $S$ be a topological semigroup. A* semicharacter *on $S$ is a nonzero continuous function of norm not bigger than 1 such that*

$$f(st) = f(s)f(t)$$

*for every $s$ and $t$ in $S$. The collection of semicharacters on $S$ is denoted by $\hat{S}$.*

**Theorem 4.7.2.** *[Tay65] Let $\mathcal{M}$ be a commutative convolution measure algebra with maximal ideal space $\Delta$. Then there exists a compact Abelian topological semigroup $S$ and a map*

$$\iota : \hat{S} \to \Delta$$

*such that $\iota$ is a bijection, and $\hat{S}$ separates the points of $S$.*

The semigroup $S$ of Theorem 4.7.2 is called the structure semigroup of $\mathcal{M}$. Let $r \geq 0$ be an element of $\hat{S}$, and $z$ be a complex number with strictly positive real part. Then $r^z$ belongs to $\hat{S}$. In fact the map $z \mapsto r^z$ is a vector valued analytic function from $\{z \in \mathbb{C} : \mathrm{Re} z > 0\}$ into $\hat{S}$. Let $f$ be an element of $\hat{S}$. Clearly the map $|f|$ belongs to $\hat{S}$ as well. In [Tay65], it has been shown that there exists a unique $h$ in $\hat{S}$ such that $f = |f|h$, $\mathrm{supp}(f) = \mathrm{supp}(h)$ and $|h|$ is an idempotent. If $\phi$ is a semicharacter such that $|\phi|$ is not an idempotent, then there exists an analytic disc around $|\phi|$. Indeed, let $\phi = |\phi|h_\phi$ be the polar decomposition of $\phi$. Then the map $z \mapsto |\phi|^z h_\phi$ is a vector-valued analytic map from $\{z \in \mathbb{C} : \mathrm{Re} z > 0\}$ to $\hat{S}$.

**Corollary 4.7.3.** *Let $\phi$ be an element of $\hat{S}$ such that $|\phi|$ is not an idempotent. Then $\mathcal{M}$ admits a point derivation at $\phi$.*

**Proof.** Note that for each $\mu$ in $\mathcal{M}$, the map $z \mapsto \langle \mu, |\phi|^z h_\phi \rangle$ is an analytic map from $\{z \in \mathbb{C} : \mathrm{Re} z > 0\}$ to $\mathbb{C}$. We then define

$$D : \mathcal{M} \to \mathbb{C}, \quad D(\mu) = \frac{d}{dz}(\langle \mu, |\phi|^z h_\phi \rangle)|_{z=1}.$$

It is easy to check that $D$ is a continuous point derivation. Moreover, using the polynomial expansion of $z \mapsto |\phi|^z h_\phi$ around $z = 1$ and the Gelfand representation of $\mathcal{M}$, we see that $D$ is nonzero. $\square$

To construct analytic discs in the spectrum of $M_0(G)$, we use the following construction which is due to Brown and Moran in the case of measure algebras [BM78a]. Let $M_0(G) = I \oplus A$ be a decomposition of $M_0(G)$ where $I$ is an $L$-ideal and $A$ is an $L$-subalgebra. Clearly

$$h(\mu) = \begin{cases} 0 & \mu \in I \\ 1 & \mu \in A \end{cases}$$

is a character on $M_0(G)$. Suppose that there exist mutually orthogonal $L$-subspaces $A = B_0, B_1, B_2, \ldots$ of $M_0(G)$ such that

- $B_1 \neq \{0\}$.
- If $\mu \in B_n$ and $\nu \in B_m$ then $\mu * \nu \in B_{m+n}$ for all positive integers $m, n$.
- $(\oplus_{n=0}^{\infty} B_n)^{\perp}$ is an $L$-ideal of $M(G)$.

For $z$ in $\mathbb{D}$ and $\mu$ in $M_0(G)$, define

$$\langle \mu, \phi(z) \rangle = \begin{cases} \int_G z^n d\mu & \mu \in B_n \\ 0 & \mu \in (\oplus_{n=0}^{\infty} B_n)^{\perp} \end{cases}.$$

One can easily verify that $\phi(z)$ is an element of the maximal ideal space of $M_0(G)$, and $\phi(0) = h$. Hence $\phi$ is an analytic disc around $h$.

**Proposition 4.7.4.** *Let $G$ be a metrisable locally compact Abelian group. Then one can construct an analytic disc in the maximal ideal space of $M_0(G)$.*

**Proof.** By the above argument from [BM78b], we only need to find a nontrivial decomposition $M_0(G) = A \oplus I$ and $L$-subspaces $B_0$, $B_1, \ldots$ as described above. Note that in a metrisable space, every perfect strongly independent compact set

$K$ is totally disconnected, and is homeomorphic to a standard Cantor set. Hence we can decompose $K$ into $K_1$ and $K_2$ such that each of them are compact, perfect, and strongly independent. Note that by Lemma 4.4.2, $M_c(nK_1)$ and $M_c(mK_2)$ and each of their translations are orthogonal for positive integers $m$ and $n$.

Now we can proceed similar to [BM78b] to construct analytic discs. Let $K_1$ and $K_2$ be perfect metrisable strongly independent compact subsets of $G$ constructed as above, such that $M_0(K_1)$ and $M_0(K_2)$ are nontrivial. By Theorem 4.5.1, we can decompose $M_0$ as $M_0(G) = A \oplus I$, where $A$ is constructed using the set $K_1$. Now let $B_1$ be the translation-invariant $L$-space generated by $M_0(K_2)$. For each $n$, let $B_n$ be the translation-invariant $L$-space generated by $\{\mu_1 * \ldots * \mu_n : \mu_1, \ldots, \mu_n \in M_0(K_2)\}$. Then the $L$-spaces $B_0, B_1, \ldots$ satisfy the desired properties, and we are done. $\square$

# Chapter 5

# Amenability properties of $B_0(G)$

In this chapter, we consider the problem of characterizing the groups $G$ for which $B_0(G)$ is (operator) [weakly] amenable. We can assume that our groups are non-compact. Indeed, if $G$ is compact then $B_0(G) = B(G) = A(G)$. Hence $B_0(G)$ is always operator weakly amenable, and it is weakly amenable if and only if the connected component of the identity in $G$ is Abelian.

In the present chapter, we prove extreme cases for amenability properties of $B_0(G)$. We first characterize locally compact groups for which their Rajchman algebras are amenable. In fact, we show that the Rajchman algebra of a locally compact group is amenable if and only if the group is compact and almost Abelian. On the other extreme, we present many examples of locally compact groups $G$ for which $B_0(G)$ fail to be even operator weakly amenable, hence fail to be weakly amenable or operator amenable. In particular, in Section 5.2 we show that the Rajchman algebra of a connected non-compact SIN-group cannot be (operator) weakly amenable. Our proofs are derived from the theorem of Varopoulous which we presented in Chapter 4.

For certain groups such as Fell groups and the $ax + b$ group, the associated Rajchman algebras are non-amenable, but they are operator amenable. This begs the question, to which we do not know the answer, if there are any (operator) weakly amenable examples which are not (operator) amenable.

## 5.1 Amenability of $B_0(G)$

Let $G$ be a locally compact group. Recall that the Rajchman algebra $B_0(G)$ is a translation-invariant closed subspace of $B(G)$. Therefore there exists a unitary representation $\pi$ of $G$ such that $B_0(G) = A_\pi(G)$, and $B_0(G)$ is a complemented ideal in $B(G)$ [Ars76]. (Complemented and weakly complemented ideals play an important role in the hereditary properties of amenable Banach algebras).

Let $\mathcal{A}$ be a Banach algebra, and $X$ be a Banach space. The space $X$ is a Banach $\mathcal{A}$-bimodule if it is an $\mathcal{A}$-bimodule whose module actions are continuous, i.e. there exists a positive constant $K$ such that

$$\|a \cdot x\| \leq K\|a\|\|x\| \text{ and } \|x \cdot a\| \leq K\|x\|\|a\|,$$

for every $x$ in $X$ and $a$ in $\mathcal{A}$. Note that $\mathcal{A}$ can be considered an $\mathcal{A}$-bimodule with usual multiplication as its module actions. For any $\mathcal{A}$-bimodule $X$, one can equip the dual space $X^*$ with the following module actions. For $f$ in $X^*$ and $a$ in $\mathcal{A}$,

$$f \cdot a(x) = f(a \cdot x) \ \text{ and } \ a \cdot f(x) = f(x \cdot a).$$

Then $X^*$ is an $\mathcal{A}$-bimodule, called a *dual bimodule.* A bounded linear map $D$ from $\mathcal{A}$ to an $\mathcal{A}$-bimodule $X$ is called a *derivation* if for all $a$ and $b$ in $\mathcal{A}$,

$$D(ab) = D(a) \cdot b + a \cdot D(b).$$

Let $x$ be an element of $X$, and define

$$D : \mathcal{A} \to X, \ D(a) = a \cdot x - x \cdot a.$$

The map $D$ is a derivation called the "inner derivation" associated with $x$. A Banach algebra $\mathcal{A}$ is *amenable* if every continuous derivation $D$ from $\mathcal{A}$ to a dual $\mathcal{A}$-bimodule $X^*$ is inner.

Johnson introduced the concept of amenability for Banach algebras, and showed that $L^1(G)$ is amenable as a Banach algebra if and only if $G$ is amenable [Joh72]. Later, Connes [Con78] and Haagerup [Haa83] showed that for $C^*$-algebras amenability and nuclearity coincide. The concept of amenability turned out to be very important in the study of Banach algebras. One can refer to [Run02] for a detailed discussion of amenability of Banach algebras.

**Theorem 5.1.1. (Hereditary properties)** *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras.*

*(i) Let $\phi$ be a surjective homomorphism from $\mathcal{A}$ to $\mathcal{B}$ . If $\mathcal{A}$ is amenable then $\mathcal{B}$*

*is amenable as well.*

(ii) *Let $I$ be a closed ideal of $\mathcal{A}$. If $\mathcal{A}$ is amenable then the following are equivalent.*

   – *$I$ is amenable.*

   – *$I$ has a bounded approximate identity.*

   – *$I$ is weakly complemented.*

**Theorem 5.1.2.** *Let $\mathcal{A}$ be a closed subalgebra of $B(G)$ which contains $B_0(G)$. Then $\mathcal{A}$ is amenable if and only if $G$ is compact and has an Abelian subgroup of finite index.*

**Proof.** Suppose $G$ is compact and has an Abelian subgroup of finite index. Then $B_0(G) = \mathcal{A} = B(G)$, and it is amenable by Corollary 4.2 of [LLW96].

Conversely, suppose that $\mathcal{A}$ is amenable. Since $B_0(G)$ and $A(G)$ are complemented ideals of $\mathcal{A}$, they are amenable as well. Hence, by the characterization of amenable Fourier algebras by Forrest and Runde [FR05], $G$ is almost Abelian, i.e. it has an Abelian subgroup $H$ of finite index. Note that $H$ is clearly an open subgroup. Hence the restriction map $r : B_0(G) \to B_0(H)$ is surjective, which implies that $B_0(H)$ is amenable as well. Since $H$ is Abelian, by Corollary 5.2.5 the amenability of $B_0(H)$ implies that $H$ is compact. Therefore $G$ is compact as well. $\square$

## 5.2 Weak amenability of $B_0(G)$

A Banach algebra $\mathcal{A}$ is called *weakly amenable* if every bounded derivation $D$ from $\mathcal{A}$ to $\mathcal{A}^*$ is inner. If $\mathcal{A}$ is a commutative Banach algebra, then $\mathcal{A}$ is weakly amenable if and only if every bounded derivation $D$ from $\mathcal{A}$ to $\mathcal{A}^*$ is identically 0. For a completely contractive Banach algebra $\mathcal{A}$, one can define operator weak amenability to be the analogue of weak amenability for Banach algebras.

A Banach algebra $\mathcal{A}$ is called a *completely contractive Banach algebra* if $\mathcal{A}$ has an operator space structure for which the multiplication map $m : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is a completely contractive bilinear map; equivalently if $m$ extends to a completely contractive map from $\mathcal{A}\hat{\otimes}\mathcal{A}$ to $\mathcal{A}$. Let $\mathcal{A}$ be a completely contractive Banach algebra. An operator space $X$ is called a *completely contractive $\mathcal{A}$-bimodule* if $X$ is an $\mathcal{A}$-bimodule, and the left and right module actions extend to completely contractive maps on $\mathcal{A}\hat{\otimes}X$ and $X\hat{\otimes}\mathcal{A}$ respectively. Note that if $\mathcal{A}$ is a completely contractive Banach algebra, then the usual multiplication gives $\mathcal{A}$ the structure of a completely contractive $\mathcal{A}$-module. It is also easy to see that this module action determines a completely contractive $\mathcal{A}$-module structure on $\mathcal{A}^*$.

**Definition 5.2.1.** *Let $\mathcal{A}$ be a completely contractive Banach algebra. Then $\mathcal{A}$ is operator weakly amenable if every completely bounded derivation $D$ from $\mathcal{A}$ to $\mathcal{A}^*$ is inner.*

One can refer to [ER00] for more information on operator spaces. The following lemma shows that weak amenability and operator weak amenability imply factorization.

**Lemma 5.2.2.** *For a Banach algebra $\mathcal{A}$, let $\mathcal{A}^2 = \mathrm{span}\{ab : a, b \in \mathcal{A}\}$.*

1. *If $\mathcal{A}$ is weakly amenable, then $\mathcal{A} = \overline{\mathcal{A}^2}$.*
2. *If $\mathcal{A}$ is a completely contractive Banach algebra which is operator weakly amenable then $\mathcal{A} = \overline{\mathcal{A}^2}$.*

**Proof. 1.** Let $\phi$ be an arbitrary element of $\mathcal{A}^*$ such that $\phi|_{\mathcal{A}^2} = 0$. In order to show $\mathcal{A} = \overline{\mathcal{A}^2}$, it is enough to prove that $\phi$ is identically 0. Indeed, let $D$ be defined as

$$D : \mathcal{A} \to \mathcal{A}^* \quad a \mapsto \phi(a)\phi.$$

It is easy to see that $\phi|_{\mathcal{A}^2} = 0$ implies that $D$ is a bounded linear derivation on $\mathcal{A}$. Since $\mathcal{A}$ is weakly amenable, $D$ should be inner. Therefore, there exists an element $f$ in $\mathcal{A}^*$ such that for every $a$ in $\mathcal{A}$,

$$\phi(a)\phi = D(a) = a \cdot f - f \cdot a.$$

Applying the above functions to $a$, we get $\phi(a)\phi(a) = (a \cdot f - f \cdot a)(a) = f(a^2 - a^2) = 0$. Hence $\phi$ is identically zero.

**2.** In this case, we only need to check that the derivation $D$ defined as above is a completely bounded map. The rest of proof is identical to part (1). Let $n$ be a positive integer, and consider the $n$th amplification of $D$:

$$D^{(n)} : M_n(\mathcal{A}) \to M_n(\mathcal{A}^*), \qquad [a_{i,j}] \mapsto [\phi(a_{i,j})\phi].$$

Then

$$\|[\phi(a_{i,j})\phi]\| = \|[\phi(a_{i,j})]I_\phi\| \leq \|[\phi(a_{i,j})]\|\|\phi\| \leq \|[a_{i,j}]\|\|\phi\|^2,$$

where $I_\phi$ is the $n \times n$ matrix in $M_n(\mathcal{A}^*)$ which has $\phi$ on the diagonal and zero elsewhere. Note that in the last inequality we have used Smith's Lemma saying that any bounded linear functional is a completely bounded map. □

Recall that the continuous homomorphic image of an amenable Banach algebra is amenable. It is also known that the above fails for weak amenability. However, in the case of commutative Banach algebras, we have the following result.

**Lemma 5.2.3.** *Let $A$ and $B$ be commutative Banach algebras, and $\phi : A \to B$ be a bounded homomorphism with dense range. Then weak amenability of $A$ implies weak amenability of $B$.*

**Proof.** Let $D$ be a bounded derivation from $B$ to $B^*$. Then $\phi^* \circ D \circ \phi$ is a bounded derivation from $A$ to $A^*$. Hence $\phi^* \circ D \circ \phi$ is inner by weak amenability of $A$, i.e. there exists $f$ in $A^*$ such that

$$(\phi^* \circ D \circ \phi)(a) = a \cdot f - f \cdot a \qquad \forall a \in A.$$

Hence for an arbitrary $a'$ in $A$,

$$\langle D(\phi(a)), \phi(a')\rangle = \langle(\phi^* \circ D \circ \phi)(a), a'\rangle = \langle a \cdot f - f \cdot a, a'\rangle = f(a'a - aa') = 0.$$

Therefore by the density of $\phi(A)$ in $B$ and continuity of $D$, we have $D = 0$. Hence

$B$ is weakly amenable. $\square$

**Lemma 5.2.4.** *Let $\mathcal{A}$ and $\mathcal{B}$ be commutative completely contractive Banach algebras, and $\phi : \mathcal{A} \to \mathcal{B}$ be a completely bounded homomorphism with dense range. Then operator weak amenability of $\mathcal{A}$ implies operator weak amenability of $\mathcal{B}$.*

**Proof.** First note that since $\phi$ is a completely bounded map, its dual $\phi^* : \mathcal{B}^* \to \mathcal{A}^*$ is completely bounded as well. Suppose $D$ is a completely bounded derivation from $\mathcal{B}$ to $\mathcal{B}^*$. Then $\phi^* \circ D \circ \phi$ is a completely bounded derivation from $\mathcal{A}$ to $\mathcal{A}^*$ as well. By operator weak amenability of $\mathcal{A}$, $\phi^* \circ D \circ \phi$ is inner, and by density of the range of $\phi$, we have $D = 0$. $\square$

### 5.2.1 Examples of groups with non-weakly amenable Rajchman algebras

Let us recall the important theorem of Varopoulos [Var66a] which we presented in the previous Chapter. For any non-discrete locally compact Abelian group $G$, the quotient $M_c/\overline{M_c^2}$ is a non-separable Banach space. Moreover, $M_0/\overline{M_0^2}$ is an infinite-dimensional Banach space. Note that for an Abelian group $G$, the algebras $B_0(G)$ and $M_0(\hat{G})$ are isometrically isomorphic via the Fourier-Stieltjes transform. The following facts are immediate corollaries of the above non-factorization theorem.

**Corollary 5.2.5.** *Let $G$ be an Abelian non-compact group. Then, the Rajchman algebra associated with $G$ is not (operator) weakly amenable. In addition, $B_0(G)$ does not have any bounded approximate identity.*

**Proof.** Let $G$ be an Abelian non-compact group. Then the dual group $\hat{G}$ is non-discrete. Hence applying Theorem 4.0.3 together with Lemma 5.2.2, we get the desired result. Moreover, suppose $B_0(G)$ has a bounded approximate identity. Then by Cohen factorization Theorem, $B_0(G)^2 = B_0(G)$, which contradicts the non-factorization theorem of Varopoulos. □

**Proposition 5.2.6.** *Let $G$ be a non-compact connected SIN-group. Then,*

1. *$B_0(G)$ is not weakly amenable.*
2. *$B_0(G)$ is not operator weakly amenable.*
3. *$B_0(G)$ does not have a bounded approximate identity.*

**Proof. 1.** Since $G$ is a non-compact connected SIN-group, it is of the form $G = \mathbb{R}^n \times K$, where $K$ is a compact subgroup. Hence $\mathbb{R}^n$ is a closed subgroup of the SIN-group $G$, and by Theorem 3.2.2, the restriction map $r : B_0(G) \to B_0(\mathbb{R}^n)$ is a surjective bounded algebra homomorphism between two commutative Banach algebras. Now suppose that $B_0(G)$ is weakly amenable. Then by Lemma 5.2.3, $B_0(\mathbb{R}^n)$ is also weakly amenable, which contradicts Corollary 5.2.5.

**2.** Note that the restriction map is a completely bounded surjective homomorphism. Moreover $B_0(\mathbb{R}^n)$ is not operator weakly amenable, so we can proceed exactly as in part (1) to conclude that $B_0(G)$ is not operator weakly amenable either.

**3.** Corollary 3.2.12 and the fact that $B_0(\mathbb{R}^n)$ does not have a bounded approximate identity imply part (3). □

**Proposition 5.2.7.** *Let $G$ be a discrete group which has an infinite Abelian subgroup $H$. Then, $B_0(G)$ is not (operator) weakly amenable. In particular, for a positive integer $n$, the free group $\mathbb{F}_n$ with $n$ generators is not (operator) weakly amenable. In addition, $B_0(G)$ does not have a bounded approximate identity.*

**Proof.** Discrete groups are SIN-groups, and any subgroup of a discrete group is closed. By Theorem 3.2.2, the restriction map $r : B_0(G) \to B_0(H)$ is a surjective completely contractive homomorphism. Assume that $B_0(G)$ is (operator) weakly amenable. Then by Lemma 5.2.3 and Lemma 5.2.4 $B_0(H)$ is (operator) weakly amenable as well, which contradicts Corollary 5.2.5, since an infinite discrete group is non-compact.

Now assume by contradiction that $B_0(G)$ has a bounded approximate identity, and let $\{u_\alpha\}$ be a bounded approximate identity of $B_0(G)$. Then by Corollary 3.2.12 $\{u_\alpha|_H\}$ is a bounded approximate identity for $B_0(H)$ which is a contradiction with Corollary 5.2.5. $\square$

Let $G$ be a discrete group such that $B_0(G)$ is (operator) weakly amenable. Then by Proposition 5.2.7, $G$ cannot have any infinite Abelian subgroup. In particular, every element of $G$ has finite order, i.e. $G$ is a periodic group.

**Definition 5.2.8.** *Let $G$ be a discrete group. Then*

- *The group $G$ is called* periodic *if for every element $g$ of $G$, there exists a positive integer $n(g)$ such that $g^{n(g)} = e$.*
- *The group $G$ is called* locally finite *if every finite subset of $G$ generates a finite subgroup of $G$.*

- *The group $G$ is called* $\mathrm{F}_2$ *if every two elements of $G$ generate a finite subgroup of $G$.*

Clearly the class of locally finite groups is contained in the class of $\mathrm{F}_2$ groups, which in turn is contained in the class of periodic groups. It has been shown in [HK64] that every infinite locally finite group contains an infinite Abelian subgroup. More generally, every infinite $\mathrm{F}_2$ group contains an infinite Abelian subgroup (see [Str66]). We then have the following corollary.

**Corollary 5.2.9.** *Let $G$ be a discrete group such that $B_0(G)$ is (operator) weakly amenable. Then*

1. *$G$ is periodic.*
2. *If $G$ is locally finite, then $G$ is finite.*
3. *If $G$ is* $\mathrm{F}_2$*, then $G$ is finite.*

### 5.2.2 Center and the connected component of the identity

In Theorem 3.3.5 of Chapter 3, we showed that for a general locally compact group, the restriction map from $B_0(G)$ to $B_0(H)$ is surjective for specific subgroups such as open subgroups, the center, and the connected component of the identity. The following proposition is a corollary of Theorem 3.3.5 and Lemma 5.2.3.

**Proposition 5.2.10.** *Let $G$ be a locally compact group, and $H$ be an open subgroup. Suppose $B_0(G)$ is (operator) weakly amenable. Then $B_0(H)$, $B_0(G_0)$ and $B_0(Z(G))$ are (operator) weakly amenable as well.*

**Corollary 5.2.11.** *Let $G$ be a locally compact group. If $B_0(G)$ is (operator) weakly amenable then $Z(G)$ is compact.*

**Proof.** By Proposition 5.2.10 $B_0(Z(G))$ is (operator) weakly amenable. In addition, $Z(G)$ is Abelian. Hence by Corollary 5.2.5, it should be compact. $\square$

As an application to the above corollary, one can note that the centers of $GL_n(\mathbb{C})$ and the Heisenberg group can be identified with the complex numbers and the real numbers respectively. Hence their Rajchman algebras are not (operator) weakly amenable. For the case of a SIN-group, one can study the structure of its connected component of the identity using the characterization in Proposition 5.2.6 for connected SIN-groups.

**Proposition 5.2.12.** *Let $G$ be a locally compact SIN-group such that $B_0(G)$ is (operator) weakly amenable.*

1. *The connected component of the identity $G_0$ is compact. In addition, if $B_0(G)$ is weakly amenable then $G_0$ is compact and Abelian.*

2. *If $G$ is a central group (that is $G/Z(G)$ is compact) then $G$ is compact.*

**Proof. 1.** By Proposition 5.2.10, $B_0(G_0)$ is (operator) weakly amenable. The group $G_0$ is a connected SIN-group. Hence by Proposition 5.2.6, $G_0$ is compact. In addition, if $B_0(G)$ is weakly amenable then $G_0$ is compact and $B_0(G_0) = A(G_0)$. Now using the characterization of connected SIN-groups with weakly amenable Fourier algebra [FSS09], we have that $G_0$ should be Abelian as well.

**2.** By Proposition 5.2.10, $B_0(Z(G))$ is (operator) weakly amenable, hence $Z(G)$ is compact since it is an Abelian group. Therefore $G$ is compact, because $G/Z(G)$ and $Z(G)$ are both compact. □

### 5.2.3 Solvable groups

A locally compact group $G$ is *solvable* if it has a finite series of closed subgroups

$$\{e\} = G_0 \lhd G_1 \lhd \ldots \lhd G_n = G$$

where each subgroup is a normal subgroup of its predecessor, and $G_{i+1}/G_i$ is Abelian for $0 \leq i \leq n-1$.

**Theorem 5.2.13.** *Let $G$ be a solvable discrete group such that $B_0(G)$ is weakly amenable. Then $G$ is finite.*

**Proof.** Suppose $G$ is solvable, i.e. it has a series $\{e\} = G_0 \lhd G_1 \lhd \ldots \lhd G_k = G$ such that $G_i$ is normal in $G_{i+1}$ and the quotient $G_{i+1}/G_i$ is Abelian for $i = 0, \ldots, k-1$. we proceed by induction on the length of the subnormal series:

Case 1: If $k = 1$, then $G$ is Abelian and we are done. So we start with $k = 2$, and assume that $\{e\} = G_0 \lhd G_1 \lhd G_2 = G$ is a subnormal series such that $G_1$ and $G/G_1$ are Abelian. By functorial properties for $B_0$, we have that $B_0(G_1)$ is weakly amenable as well. Hence $G_1$ is finite by Corollary 5.2.5. Now let $g_1, g_2$ be two elements in the group $G$, and let $w = g_1^{\alpha_1} g_2^{\beta_1} \ldots g_1^{\alpha_n} g_2^{\beta_n}$ be a word in the group generated by $g_1$ and $g_2$. Then

$$\begin{aligned} g_1^{\alpha_1} g_2^{\beta_1} \ldots g_1^{\alpha_n} g_2^{\beta_n} G_1 &= (g_1^{\alpha_1} G_1)(g_2^{\beta_1} G_1) \ldots (g_1^{\alpha_n} G_1)(g_2^{\beta_n} G_1) \\ &= (g_1^{\alpha_1} G_1) \ldots (g_1^{\alpha_n} G_1) \times (g_2^{\beta_1} G_1) \ldots (g_2^{\beta_n} G_1) \\ &= g_1^{\sum \alpha_i} g_2^{\sum \beta_i} G_1, \end{aligned}$$

therefore every word in $\langle g_1, g_2 \rangle$ is of the form $g_1^{\alpha} g_2^{\beta} z$ for some $z$ in $G_1$. Moreover $g_1$ and $g_2$ are periodic since the group has weakly amenable Rajchman algebra. Therefore $\langle g_1, g_2 \rangle$ is finite, i.e. $G$ is $F_2$. Recall that infinite $F_2$ groups always have infinite Abelian subgroups, hence their Rajchman algebras are not weakly amenable. Therefore $G$ is finite.

Case 2: First note that the group is periodic. Suppose that for periodic solvable groups of subnormal series of length less than $n$, if $B_0(G)$ is weakly amenable then $G$ is finite (induction hypothesis). Let $G$ be a periodic solvable group with the subnormal series $\{1\} = G_0 \leq G_1 \leq \ldots \leq G_n = G$. Then by functorial properties and induction hypothesis, $G_{n-1}$ is finite. Repeating the same argument as in Case 1, we get that $G$ is finite as well. $\square$

# Chapter 6

# The group $\mathrm{SL}_2(\mathbb{R})$

In the present chapter, we study the group $\mathrm{SL}_2(\mathbb{R})$ as an example of a locally compact group whose Rajchman algebra has no nonzero continuous point derivation. Using the Kunze-Stein phenomena, we show that the Rajchman algebra of $\mathrm{SL}_2(\mathbb{R})$, and more generally any connected semisimple Lie group with finite center, has simple spectrum and admits no nonzero continuous point derivations. Note that $\mathrm{SL}_2(\mathbb{R})$ is a nontrivial example of such groups. As a trivial example, one can consider the $n$'th rigid $p$-adic motion group, where the Rajchman algebra is the Fourier algebra itself.

Let us recall the definition of the $n$'th rigid $p$-adic motion group. Let $p$ be a prime number, and define the $p$-adic absolute value on $\mathbb{Q}$ as follows: Let $x$ be a nonzero rational number. Then there exists a unique integer $n$ such that $x = p^n \frac{a}{b}$, where neither of the integers $a$ and $b$ is divisible by $p$. We define $|x|_p = p^{-n}$ if $x \neq 0$, and $|0|_p = 0$. Let $d_p$ be the metric defined by the $p$-adic absolute value on $\mathbb{Q}$, and define the $p$-adic numbers $\mathbb{Q}_p$ to be the completion of $(\mathbb{Q}, d_p)$, which is both a totally disconnected complete metric space and a field. The $p$-adic absolute value is a multiplicative non-Archimedean evaluation on $\mathbb{Q}_p$, i.e. $|rs|_p = |r|_p|s|_p$ and $|r+s|_p \leq \max\{|r|_p, |s|_p\}$. It can be shown that every element $x$ in $\mathbb{Q}_p$ may be uniquely represented as $\sum_{i=k}^{\infty} a_i p^i$ where $k \in \mathbb{Z}$ and $a_i \in \{0, \ldots, p-1\}$. This series converges to $x$ with respect to $d_p$. We also define the $p$-adic integers $\mathbb{O}_p$ and the multiplicative group $\mathbb{T}_p$ to be

$$\mathbb{O}_p := \{r \in \mathbb{Q}_p : |r|_p \leq 1\} \text{ and } \mathbb{T}_p := \{r \in \mathbb{Q}_p : |r|_p = 1\}.$$

For an integer $n$ and a prime $p$, we define *the $n$'th rigid $p$-adic motion group* $G_{p,n}$ to be

$$G_{p,n} := GL(n, \mathbb{O}_p) \ltimes \mathbb{Q}_p^n,$$

where $GL(n, \mathbb{O}_p)$ denotes the multiplicative group of $n \times n$ matrices with entries in $\mathbb{O}_p$ and determinant of $p$-adic absolute value 1, which act on the vector space $\mathbb{Q}_p^n$ by matrix multiplication. Note that $\mathbb{O}_p$, and therefore $GL(n, \mathbb{O}_p)$, are compact. Each group $G_{p,n}$ is of the form $G_{p,n} = K_{p,n} \ltimes A_{p,n}$ where $K_{p,n}$ is a compact group acting on a noncompact Abelian group $A_{p,n}$. It has been shown that $B(G_{p,n}) = A(K_{p,n}) \circ q \oplus_{\ell^1} A(G_{p,n})$ (see [RS05]). Therefore $B_0(G_{p,n}) = A(G_{p,n})$, which implies that $B(G_{p,n})$ does not admit any point derivation.

Although both $B_0(\mathrm{SL}_2(\mathbb{R}))$ and $B_0(G_{p,n})$ admit no nonzero continuous point derivations, they behave differently as Banach algebras. For instance, we will later observe that $B_0(\mathrm{SL}_2(\mathbb{R}))$ is not (operator) weakly amenable. However $B_0(G_{p,n})$ is operator weakly amenable, since it is just the Fourier algebra of $G_{p,n}$. Taking

Proposition 6.1.1 into account, it is clear that $B_0(\mathrm{SL}_2(\mathbb{R}))$ is an interesting example regarding its amenability behaviors.

## 6.1 Point derivations and weak amenability

**Proposition 6.1.1.** *Let $\mathcal{A}$ be a (completely contractive) Banach algebra. If $\mathcal{A}$ has a nonzero continuous point derivation, then $\mathcal{A}$ is not even (operator) weakly amenable.*

**Proof.** Let $d : \mathcal{A} \to \mathbb{C}$ be a continuous nonzero point derivation at the character $\phi : \mathcal{A} \to \mathbb{C}$. Suppose by contradiction that $\mathcal{A}$ is (operator) weakly amenable. Then by Lemma 5.2.2, $\overline{\mathcal{A}^2} = \mathcal{A}$. Note that by Smith's lemma $d$ is completely bounded. Define the linear map $D$ on $\mathcal{A}$ to be

$$D : \mathcal{A} \to \mathcal{A}^*, \ a \mapsto d(a)\phi.$$

For elements $a, b$, and $x$ in $\mathcal{A}$, we have

$$\begin{aligned} D(ab)(x) &= d(ab)\phi(x) = (d(a)\phi(b) + d(b)\phi(a))\phi(x) \\ &= d(a)\phi(bx) + d(b)\phi(xa) \\ &= d(a)(\phi \cdot b)(x) + d(b)(a \cdot \phi)(x) \\ &= (D(a) \cdot b + a \cdot D(b))(x), \end{aligned}$$

hence $D$ is a derivation. Moreover note that the map $d$ is nonzero, therefore $D$ is a nonzero derivation as well. Next, we observe that $D$ is a completely bounded map.

Indeed for any $m$ in $\mathbb{N}$ and $[a_{i,j}]$ in $M_m(\mathcal{A})$, we have:

$$\|D^{(m)}[a_{i,j}]\| = \|[d(a_{i,j})\phi]\| = \|[d(a_{i,j})](\phi \cdot I)\| \leq \|[d(a_{i,j})]\|\|\phi\| \leq \|d\|\|\phi\|\|[a_{i,j}]\|.$$

Since we assumed $\mathcal{A}$ to be (operator) weakly amenable, the derivation $D$ should be inner, i.e. there exists an element $\psi$ in $\mathcal{A}^*$ such that $D = ad_\psi$. Now for every $a$ and $b$ in $\mathcal{A}$,

$$d(a)\phi(b) = D(a)(b) = ad_\psi(a)(b) = (a \cdot \psi - \psi \cdot a)(b) = \psi(ba - ab).$$

Hence

$$d(ab) = d(a)\phi(b) + d(b)\phi(a) = \psi(ba - ab) + \psi(ab - ba) = 0.$$

Therefore $d$ vanishes on $\mathcal{A}^2$ which is a dense subset of $\mathcal{A}$. This forces $d$ to be identically zero, which is a contradiction. Hence $\mathcal{A}$ is not (operator) weakly amenable. $\square$

Let us now remark that for any locally compact group $G$, its Fourier algebra has no nonzero continuous point derivation. In fact, Spronk [Spr02] and independently Samei [Sam06] showed that the Fourier algebra of a locally compact group is always operator weakly amenable, and hence has no nonzero continuous point derivations. Proposition 6.1.2 proves a similar result for certain closed subalgebras of $B(G)$. Examples of such algebras are provided in Proposition 6.1.3.

**Proposition 6.1.2.** *Let $G$ be a locally compact group and $\mathcal{A}$ be a closed subalgebra of $B(G)$ which contains $A(G)$. If $\sigma_{\mathcal{A}}$ is just the set of the point evaluations* *with elements of $G$ (denoted by $\sigma_{\mathcal{A}} \sim G$) then $\mathcal{A}$ has no nonzero continuous point derivation.*

**Proof.** Let $D$ be a continuous point derivation on $\mathcal{A}$ at the character $\phi$. By our assumption, there exists an element $g$ in $G$ such that $\phi$ is the point evaluation at $g$. Hence $\phi|_{A(G)}$ is a character for $A(G)$, and $D|_{A(G)}$ is a continuous point derivation of $A(G)$ at the character $\phi|_{A(G)}$. Therefore $D|_{A(G)}$ is identically zero, since $A(G)$ has no nonzero continuous point derivation. Fix an element $h$ in $A(G)$ with $\phi(h) = h(g) = 1$. For every $u$ in $\mathcal{A}$, we have

$$0 = D(uh) = D(u)\phi(h) + D(h)\phi(u) = D(u).$$

Hence $D$ is identically zero, and $\mathcal{A}$ has no nonzero continuous point derivation. $\square$

**Proposition 6.1.3.** *Let $G$ be a locally compact group. Let $\mathcal{A}$ be a closed subalgebra of $B(G)$ which contains $A(G)$. If the set $\mathcal{A}_0 = \{f \in \mathcal{A} : \exists n_f \in \mathbb{N}$ s.t. $f^{n_f} \in A(G)\}$ is dense in $\mathcal{A}$ then $\sigma_{\mathcal{A}} \sim G$.*

**Proof.** Let $\sigma : \mathcal{A} \to \mathbb{C}$ be a nonzero multiplicative linear functional on $\mathcal{A}$. Note that $\sigma|_{A(G)} \neq 0$. Indeed, assume $\sigma$ vanishes on $A(G)$, and let $f$ in $\mathcal{A}$ be an element such that $f^n$ belongs to $A(G)$ for some positive integer $n$. Then $|\sigma(f)| = |\sigma(f^n)|^{\frac{1}{n}} = 0$, and by density of such elements in $\mathcal{A}$, the function $\sigma$ is forced to be zero everywhere. Therefore $\sigma|_{A(G)}$ is a nonzero element of the spectrum of $A(G)$. By Theorem 3.34 of [Eym64], there exists an element $g$ in $G$ such that for every $f$ in $A(G)$,

$$\sigma(f) = f(g).$$

Now fix an element $h$ in $A(G)$ for which $\sigma(h) = h(g) = 1$. For any $u$ in $\mathcal{A}$,

$$\sigma(u) = \frac{\sigma(uh)}{\sigma(h)} = uh(g) = u(g)h(g) = u(g),$$

since $A(G)$ is an ideal in $\mathcal{A}$. Therefore $\sigma$ is a point evaluation, and $\sigma_{\mathcal{A}} \sim G$. $\square$

## 6.2 $B_0(\mathrm{SL}_2(\mathbb{R}))$ is not (operator) weakly amenable.

In this section, we use the results of Repka [Rep78] and Pukánszky [Puk61] regarding the decomposition of tensor products of unitary representations of $\mathrm{SL}_2(\mathbb{R})$ to observe that $B_0(\mathrm{SL}_2(\mathbb{R}))$ is not (operator) weakly amenable. The author would like to thank Viktor Losert for pointing her attention to the above-mentioned results. We begin with a brief overview of the theory of direct integrals. The reader may refer to [Fol95], [Dix69] and [Ars76] for more details.

Let $\{\mathcal{H}_\alpha\}_{\alpha\in A}$ be a family of nonzero separable Hilbert spaces, and $\mu$ be a measure on the index set $A$. For each Hilbert space $\mathcal{H}_\alpha$, let $\langle\cdot,\cdot\rangle_\alpha$ and $\|\cdot\|_\alpha$ denote its inner product and norm respectively. To define the direct integral of Hilbert spaces $\mathcal{H}_\alpha$, we need to assume a certain measurability condition on the family $\{\mathcal{H}_\alpha\}_{\alpha\in A}$. Indeed, we assume that there exists a countable subset $\{e_j\}_{j=1}^\infty$ of $\prod_{\alpha\in A}\mathcal{H}_\alpha$ with the following properties:

(i) The functions $\alpha \mapsto \langle e_j(\alpha), e_k(\alpha)\rangle_\alpha$ are measurable for all $j$ and $k$.

(ii) The linear span of $\{e_j(\alpha)\}_1^\infty$ is dense in $\mathcal{H}_\alpha$ for each $\alpha$.

An element $f$ in $\prod_{\alpha\in A}\mathcal{H}_\alpha$ is called measurable if the function

$$\alpha \mapsto \langle f(\alpha), e_j(\alpha)\rangle_\alpha$$

is a measurable function on $A$ for each index $j$. The direct integral of the family $\{\mathcal{H}_\alpha\}_{\alpha\in A}$, denoted by $\int^\oplus \mathcal{H}_\alpha d\mu(\alpha)$, is the space of measurable elements $f$ in $\prod_{\alpha\in A}\mathcal{H}_\alpha$ such that

$$\|f\|^2 = \int \|f(\alpha)\|_\alpha^2 d\mu(\alpha) < \infty.$$

It is not hard to show that $\int^\oplus \mathcal{H}_\alpha d\mu(\alpha)$ is a Hilbert space with inner product defined as

$$\langle f, g\rangle = \int \langle f(\alpha), g(\alpha)\rangle d\mu(\alpha).$$

We now define the direct integral of operators. Let $\{\mathcal{H}_\alpha\}_{\alpha\in A}$ and $\{e_j\}$ be as above. An element $T$ in $\prod_{\alpha\in A}\mathcal{B}(\mathcal{H}_\alpha)$ is called measurable if for all indices $j$ and $k$, the map

$$\alpha \mapsto \langle T(\alpha)e_j(\alpha), e_k(\alpha)\rangle_\alpha$$

is a measurable function on $A$. Suppose that $T$ is measurable, and satisfies

$$\|T\|_\infty = \text{ess sup}_{\alpha\in A}\|T(\alpha)\| < \infty.$$

Then $T$ defines the bounded operator $\int^\oplus T(\alpha)d\mu(\alpha)$ on the Hilbert space $\int^\oplus \mathcal{H}_\alpha d\mu(\alpha)$ in the following way:

$$\left[\left(\int^\oplus T(\alpha)d\mu(\alpha)\right) f\right](\alpha) = T(\alpha)f(\alpha).$$

Moreover one can see that $\| \int^{\oplus} T(\alpha) d\mu(\alpha)\| = \|T\|_{\infty} = \text{ess sup}_{\alpha\in A}\|T(\alpha)\|$.

Let $G$ be a locally compact group. The net $\{\pi_\alpha\}_{\alpha\in A}$ is called a measurable net of unitary representations of $G$ on Hilbert spaces $\{\mathcal{H}_\alpha\}_{\alpha\in A}$ if for every $x$ in $G$, the map $\alpha \mapsto \pi_\alpha(x)$ is measurable as an element of $\prod_{\alpha\in A} \mathcal{B}(\mathcal{H}_\alpha)$. For every $x$ in $G$, form the direct integral

$$\pi(x) = \int^{\oplus} \pi_\alpha(x) d\mu(\alpha).$$

Then $\pi$ is a unitary representation of $G$ on $\int^{\oplus} \mathcal{H}_\alpha d\mu(\alpha)$, called the direct integral of representations $\pi_\alpha$.

From now on, we assume that $G$ is a second countable locally compact unimodular group which is of type I. This assumption ensures that $\hat{G}$ admits a standard Borel structure induced from the Fell topology (see Theorem 7.6 of [Fol95]). Let $\mu$ be a positive Borel measure on $\hat{G}$, and $\{\mathcal{H}_\pi\}_{\pi\in\hat{G}}$ be the family of Hilbert spaces associated with elements of $\hat{G}$. By $L^1(\hat{G},\mu)^{\oplus}$, we denote the set of all the measurable elements $\{T_\pi\}_{\pi\in\hat{G}}$ of $\prod_{\pi\in\hat{G}} \text{Tr}(\mathcal{H}_\pi)$ that satisfy

$$\int_{\hat{G}} \|T_\pi\|_1 d\mu(\pi) < \infty.$$

Let $L^\infty(\hat{G},\mu)^{\oplus}$ denote the set of all the measurable elements $\{U_\pi\}_{\pi\in\hat{G}}$ of $\prod_{\pi\in\hat{G}} \mathcal{B}(\mathcal{H}_\pi)$ such that

$$\text{ess sup}\|U_\pi\| < \infty.$$

Arsac proved that if $\sigma = \int_{\hat{G}}^{\oplus} \pi d\mu(\pi)$ is a unitary representation of $G$ defined by $\mu$, then the Banach spaces $A_\sigma$ and $L^1(\hat{G},\mu)^{\oplus}$ are isometric (see Theorem 3.53 of

[Ars76]). In particular, every $u$ in $A_\sigma$ can be represented uniquely as

$$u(s) = \int_{\hat{G}} \operatorname{Tr}(\pi(s)T_\pi) d\mu(\pi),$$

where $\{T_\pi\}_{\pi\in\hat{G}}$ belongs to $L^1(\hat{G},\mu)^\oplus$ and satisfies

$$\|u\|_{B(G)} = \int_{\hat{G}} \|T_\pi\|_1 d\mu(\pi).$$

Similarly, every $S$ in $\mathrm{VN}_\sigma$ can be isometrically identified with an element $\{U_\pi\}_{\pi\in\hat{G}}$ in $L^\infty(\hat{G},\mu)^\oplus$ such that

$$\langle u, S\rangle = \int_{\hat{G}} \operatorname{Tr}(T_\pi U_\pi) d\mu(\pi).$$

**Proposition 6.2.1.** *Let $\mu$ and $\nu$ be positive Borel measures on $\hat{G}$ defining unitary representations $\tau$ and $\sigma$ of $G$ as direct integrals:*

$$\tau = \int_{\hat{G}}^\oplus \pi d\mu(\pi) \quad \textit{and} \quad \sigma = \int_{\hat{G}}^\oplus \pi d\nu(\pi). \tag{6.1}$$

*If $\mu$ is absolutely continuous with respect to $\nu$ then the matrix space $A_\tau$ is a subset of $A_\sigma$.*

**Proof.** Suppose that $\mu \ll \nu$, i.e. there exists a $\nu$-measurable function $f$ on $\hat{G}$ such that $\mu = f\nu$. Let $\xi = \{\xi_\pi\}$ and $\eta = \{\eta_\pi\}$ be vectors in $\int^\oplus \mathcal{H}_\pi d\mu(\pi)$. Note that

$$\|\xi\|^2 = \int_{\hat{G}} \|\xi_\pi\|^2_{\mathcal{H}_\pi} d\mu(\pi) = \int_{\hat{G}} f(\pi)\|\xi_\pi\|^2_{\mathcal{H}_\pi} d\nu(\pi),$$

hence $\{\sqrt{f(\pi)}\xi_\pi\}$, and similarly $\{\sqrt{f(\pi)}\eta_\pi\}$, belongs to $\int^\oplus \mathcal{H}_\pi d\nu(\pi)$. Now for an element $x$ in $G$, we have

$$\begin{aligned}
\xi *_\tau \eta(x) &= \langle \tau(x)\xi, \eta\rangle \\
&= \int_{\hat{G}} \langle \pi(x)\xi_\pi, \eta_\pi\rangle_{\mathcal{H}_\pi} d\mu(\pi) \\
&= \int_{\hat{G}} f(\pi)\langle \pi(x)\xi_\pi, \eta_\pi\rangle_{\mathcal{H}_\pi} d\nu(\pi) \\
&= \sqrt{f}\xi *_\sigma \sqrt{f}\eta(x),
\end{aligned}$$

which implies that $A_\tau$ is a subset of $A_\sigma$.

□

Let us consider the case $G = \mathrm{SL}_2(\mathbb{R})$. We use the notations from [Fol95] and parametrize the dual space $\widehat{\mathrm{SL}_2(\mathbb{R})}$ through its identification with the following family of representations:

trivial representation: $\iota$,

principal continuous series: $\{\pi_{it}^+ :\ t \geq 0\} \cup \{\pi_{it}^- :\ t > 0\}$,

discrete series: $\{\delta_{\pm n} :\ n \geq 2\}$,

mock discrete series: $\delta_{\pm 1}$,

complementary series: $\{\kappa_s :\ 0 < s < 1\}$.

**Theorem 6.2.2.** *If $G$ is the group $\mathrm{SL}_2(\mathbb{R})$ then $B_0(G)$ is not square-dense, i.e.*

$$\overline{B_0(G)^2} \neq B_0(G).$$

**Proof.** Let $\mu$ denote the Plancherel measure on $\widehat{\mathrm{SL}_2(\mathbb{R})}$. Recall that the Plancherel measure of the complementary series, mock discrete series, and the trivial representation is zero. Moreover, by Harish-Chandra's trace formula the Plancherel measure on the principal and discrete series is defined as

$$\begin{aligned} d\mu(\pi_{it}^{+}) &= \frac{t}{2}\tanh\frac{\pi t}{2}\, dt, \\ d\mu(\pi_{it}^{-}) &= \frac{t}{2}\coth\frac{\pi t}{2}\, dt, \\ \mu(\{\delta_{\pm n}\}) &= n-1. \end{aligned}$$

Therefore, by Proposition 8.4.4 of [Dix69], the left regular representation $\lambda$ is quasi-equivalent with the representation

$$\int_{(o,\infty)}^{\oplus} \pi_{it}^{+} dt \oplus \int_{(o,\infty)}^{\oplus} \pi_{it}^{-} dt \oplus \bigoplus_{n=2}^{\infty} (\delta_n \oplus \delta_{-n}). \tag{6.2}$$

Let $m_{\hat{G}}$ denote the renormalised Plancherel measure given in (6.2). Define the new representations

$$\Pi_0^{+} = \int_{(o,\infty)}^{\oplus} \pi_{it}^{+} dt \oplus \bigoplus_{k=1}^{\infty} (\delta_{2k} \oplus \delta_{-2k}),$$

and

$$\Pi_0^{-} = \int_{(o,\infty)}^{\oplus} \pi_{it}^{-} dt \oplus \bigoplus_{k=1}^{\infty} (\delta_{2k+1} \oplus \delta_{-2k-1}),$$

and note that by Proposition 6.2.1, the matrix coefficients $A_{\Pi_0^+}$ and $A_{\Pi_0^-}$ are contained in $A(G)$. Note that these representations are used in the direct integral decomposition of tensor products of irreducible unitary representations of $\mathrm{SL}_2(\mathbb{R})$. In fact, Repka [Rep78] proved that if $\pi$ and $\pi'$ are irreducible unitary representations of $\mathrm{SL}_2(\mathbb{R})$ then

$$\pi \otimes \pi' \simeq_q \begin{cases} \Pi_0^+ \oplus \kappa_{r+s-1} & \text{if } \{\pi, \pi'\} = \{\kappa_r, \kappa_s\} \text{ and } r+s \geq 1 \\ \Pi & \text{otherwise,} \end{cases}$$

where $\Pi$ is a subrepresentation of $\Pi_0^+$ or $\Pi_0^-$, and $\simeq_q$ denotes the quasi-equivalence of representations.

For irreducible unitary representations $\pi$ and $\pi'$ of $G$, let $m_{\pi,\pi'}$ denote the measure on $\hat{G}$ which appears in the direct integral decomposition of $\pi \otimes \pi'$. By [Rep78], $m_{\pi,\pi'}$ is absolutely continuous with respect to the Plancherel measure $m_{\hat{G}}$ on $\hat{G}_r$, and $\mathrm{supp}(m_{\pi,\pi'})$ contains at most one element from the complementary series. Now let $u$ and $u'$ be elements of the coefficient spaces $A_\pi$ and $A_{\pi'}$ respectively, with trace operators $T_\pi$ and $T_{\pi'}$ such that

$$u = \mathrm{Tr}(\pi(\cdot)T_\pi) \quad \text{and} \quad u' = \mathrm{Tr}(\pi'(\cdot)T_{\pi'}).$$

Then

$$uu' = \mathrm{Tr}(\pi \otimes \pi'(\cdot)T_\pi \otimes T_{\pi'}) = \int_{\hat{G}} \mathrm{Tr}(\pi''(\cdot)T_{\pi,\pi';\pi''})dm_{\pi,\pi'}(\pi''). \tag{6.3}$$

Finally let $u$ and $u'$ be elements of $B_0(G)$. By Corollary 3.55 of [Ars76], there exist positive measures $\mu$ and $\mu'$ on $\hat{G}$ such that

$$u = \int_{\hat{G}} \mathrm{Tr}(\pi(\cdot)T_\pi)d\mu(\pi) \qquad \text{and} \qquad u' = \int_{\hat{G}} \mathrm{Tr}(\pi(\cdot)T'_\pi)d\mu'(\pi),$$

where $\{T_\pi\}_{\pi\in\hat{G}}$ and $\{T'_\pi\}_{\pi\in\hat{G}}$ are elements of $L^1(\hat{G},\mu)^\oplus$ and $L^1(\hat{G},\mu')^\oplus$ respectively. Therefore by (6.3) we have,

$$\begin{aligned} uu(\cdot) &= \int_{\hat{G}\times\hat{G}} \mathrm{Tr}(\pi\otimes\pi'(\cdot)T_\pi\otimes T'_{\pi'})d\mu(\pi,\pi') \\ &= \int_{\hat{G}\times\hat{G}}\int_{\hat{G}} \mathrm{Tr}(\pi''(\cdot)T_{\pi,\pi';\pi''})dm_{\pi,\pi'}(\pi'')d\mu(\pi,\pi'). \end{aligned} \tag{6.4}$$

For a unitary representation $\pi$ of $G$, let $\tilde{\pi}$ denote the surjective map generated by $\pi$ from $\mathrm{VN}_\omega(G)$ to $\mathrm{VN}_\pi(G)$, where $\omega$ is the universal representation of $G$. Note that every unitary representation $\pi$ of $G$ extends to a nondegenerate norm-decreasing $*$-representation of $C^*$-algebras from $C^*(G)$ to $C^*_\pi(G)$, which identifies $C^*_\pi(G)$ with a quotient of $C^*(G)$. Then the dual map $\pi^*$ identifies $B_\pi(G)$ with a subset of $B(G)$, and we have

$$\tilde{\pi} = (\pi^*|_{A_\pi})^*.$$

Hence for every $S$ in $\mathrm{VN}_\omega(G)$, we have

$$\tilde{\pi}(S) = S|_{A_\pi}.$$

Now fix a positive real number $t$. Then $\pi^+_{it}$ and $\oplus_{\pi\in\hat{G}\setminus\{\pi^+_{it}\}}\pi$ are disjoint unitary representations of $\mathrm{SL}_2(\mathbb{R})$, and by Proposition 3.12 of [Ars76], $A_{\pi^+_{it}}$ and $A_{\oplus_{\pi\in\hat{G}\setminus\{\pi^+_{it}\}}\pi}$ intersect trivially. Therefore by the Hahn Banach theorem, there exists an element

$S$ in $\mathrm{VN}_\omega(G)$ such that $\widetilde{\pi_{it}^+}(S) \neq 0$ and $\tilde{\pi}(S) = 0$ for every other representation $\pi$ in $\hat{G}$. Hence by Equation (6.4),

$$\langle uu', S\rangle = \int_{\hat{G}\times\hat{G}} \left[\int_{\hat{G}} \mathrm{Tr}(\pi''(S)T_{\pi,\pi';\pi''})dm_{\pi,\pi'}(\pi'')\right] d(\mu\times\mu')(\pi,\pi') = 0,$$

where we used the fact that $m_{\pi,\pi'}$ is continuous on the principal continuous series. Therefore $S$ vanishes on $B_0(G)^2$ but does not vanish on $A_{\pi_{it}^+}$. Moreover, it is known that $A_{\pi_{it}^+}$ is a subset of $B_0(G)$ (e.g. an easy consequence of Kunze-Stein phenomena). Thus we conclude that $B_0(G)$ is not square-dense. □

The following corollary is a natural consequence of Theorem 6.2.2 and Lemma 5.2.2.

**Corollary 6.2.3.** *Let $G$ denote the group $\mathrm{SL}_2(\mathbb{R})$. Then $B_0(G)$ is not (operator) weakly amenable.*

## 6.3 On Kunze-Stein phenomena

This section contains a summary of the Kunze-Stein phenomena for $\mathrm{SL}_2(\mathbb{R})$. The reader may refer to [KS60] for more proofs and details. Note that using the Kunze-Stein phenomena for $\mathrm{SL}_2(\mathbb{R})$, one observes that the elements of $B_0(\mathrm{SL}_2(\mathbb{R}))$ which are nilpotent modula $A(\mathrm{SL}_2(\mathbb{R}))$ form a dense subset. Throughout this section, we let $G = \mathrm{SL}_2(\mathbb{R})$.

**Definition 6.3.1.** *Let $\pi$ be a unitary (not necessarily irreducible) representation of $\mathrm{SL}_2(\mathbb{R})$, and $p \geq 1$ be a fixed number. We say $\pi$ is* extendable *to $L_p(\mathrm{SL}_2(\mathbb{R}))$ if* *there exists a constant $A$ such that for every $f$ in $L_1 \cap L_p(\mathrm{SL}_2(\mathbb{R}))$, the inequality $\|\pi(f)\| \le A\|f\|_p$ holds.*

The following lemma, due to Kunze and Stein [KS60], presents equivalent conditions for extendability of a representation of a general locally compact group.

**Lemma 6.3.2.** *Let $G$ be a locally compact group, and $\pi$ be a unitary representation of $G$ on the Hilbert space $\mathcal{H}_\pi$. Let $p \ge 1$ and $q$ be its conjugate. Then the following are equivalent:*

*(i) $\pi$ is extendable to $L_p(G)$.*

*(ii) $\xi *_\pi \eta \in L_q(G)$ for all $\xi, \eta \in \mathcal{H}_\pi$.*

*(iii) $A_\pi \subseteq L_q(G)$.*

*(iv) There is a constant $C$ such that*

*(a) $\|\xi *_\pi \eta\|_q \le C\|\xi\|\|\eta\|$ for any $\xi$, $\eta \in \mathcal{H}_\pi$.*

*(b) $\|\pi(u)\| \le C\|u\|_p$, $u \in L_1 \cap L_p(G)$.*

**Proof. (i) $\Rightarrow$ (iv)** Suppose that $\pi$ is extendable to $L_p(G)$ with the constant factor $A$. Let $\xi, \eta \in \mathcal{H}_\pi$. Since $L_1 \cap L_p$ is dense in $L_p$,

$$\begin{aligned}\|\xi *_\pi \eta\|_q &= \sup_{f\in b_1(L_p)\cap L_1} \left|\int_G \langle \pi(x)\xi, \eta\rangle f(x)dx\right| = \sup_{f\in b_1(L_p)\cap L_1} |\langle \pi(f)\xi, \eta\rangle| \\ &\le \sup_{f\in b_1(L_p)\cap L_1} \|\pi(f)\|\|\xi\|\|\eta\| \le \sup_{f\in b_1(L_p)\cap L_1} A\|f\|_p\|\xi\|\|\eta\| \le A\|\xi\|\|\eta\|.\end{aligned}$$

Letting $C = A$, we get (iv).

**(iv) $\Rightarrow$ (iii)** Assume (iv) holds. Let $u$ be an arbitrary element of $A_\pi(G)$. By [Ars76], there exists sequences $\{\xi_i\}_{i=1}^\infty$ and $\{\eta_i\}_{i=1}^\infty$ in $\mathcal{H}_\pi$ such that

$$u = \sum_{i=1}^{\infty} \xi_i *_\pi \eta_i \text{ and } \|u\|_{A_\pi} = \sum_{i=1}^{\infty} \|\xi_i\|\|\eta_i\|.$$

For each $N \in \mathbb{N}$ define $u_N := \sum_{i=1}^N |\xi_i *_\pi \eta_i|$. Then $u_N \to \sum_{i=1}^\infty |\xi_i *_\pi \eta_i|$ pointwise as $N$ tends to infinity, and

$$\|u_N\|_q = \|\sum_{i=1}^{N} |\xi_i *_\pi \eta_i|\| \leq \sum_{i=1}^{N} \|\xi_i *_\pi \eta_i\|_q \leq \sum_{i=1}^{N} C\|\xi_i\|\|\eta_i\| \leq C\|u\|_{A_\pi}.$$

Hence by Lebesgue monotone convergence theorem,

$$\|u\|_q = \||u|\|_q \leq \|\sum_{i=1}^{\infty} |\xi_i *_\pi \eta_i|\|q \leq C\|u\|_{A_\pi}.$$

**(iii)$\Rightarrow$ (ii)** Clear.

**(ii) $\Rightarrow$ (i)** We use the closed graph theorem to prove this direction. Fix an element $\eta_0$ in $\mathcal{H}_\pi$. The map

$$\phi_{\eta_0} : \mathcal{H}_\pi \to L_q(G), \quad \xi \mapsto \xi *_\pi \eta_0$$

is an everywhere defined linear map from the Banach space $\mathcal{H}_\pi$ to the Banach space $L_q(G)$. Let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{H}_\pi$ which converges to $\xi_0$, and assume that the sequence $(\xi_n *_\pi \eta_0)_{n\in\mathbb{N}}$ converges to $f$ in $L_q$. We want to show that $\phi_{\eta_0}$ has a closed graph, i.e. $f = \xi_0 *_\pi \eta_0$. Note that since $\|\xi_n - \xi_0\|_{\mathcal{H}_\pi}$ converges to zero, the sequence $\xi_n *_\pi \eta_0$ converges pointwise to $\xi_0 *_\pi \eta_0$. Therefore $f$ and $\xi_0 *_\pi \eta_0$ are pointwise limits of the sequence $(\xi_n *_\pi \eta_0)_{n\in\mathbb{N}}$. Hence $\xi_0 *_\pi \eta_0 = f$, and $\phi_{\eta_0}$ is a closed map. Therefore, by closed graph theorem, $\phi_{\eta_0}$ is bounded, i.e. there exists a constant $A_{\eta_0}$ such that $\|\xi *_\pi \eta_0\|_q \leq A_{\eta_0}\|\xi\|$ for every $\xi$ in $\mathcal{H}_\pi$. Similarly $\|\xi_0 *_\pi \eta\|_q \leq A_{\xi_0}\|\eta\|$ for every $\eta$ in $\mathcal{H}_\pi$.

The family $\{\phi_\eta\}_{\eta\in b_1(\mathcal{H}_\pi)}$ of bounded operators are uniformly bounded. To see this, fix an element $\xi$ in $\mathcal{H}_\pi$, and note that

$$\|\phi_\eta(\xi)\|_q = \|\xi *_\pi \eta\|_q \leq A_\xi\|\eta\| \leq A_\xi < \infty$$

Hence by uniform boundedness principle, there exists a constant $A$ such that for each $\eta$ in $b_1(\mathcal{H}_\pi)$, we have $\|\phi_\eta\| \leq A$. Now for any $\xi, \eta \in \mathcal{H}_\pi$, we have

$$\|\xi *_\pi \eta\|_q = \|\eta\|\|\xi *_\pi \frac{\eta}{\|\eta\|}\|_q \leq A\|\eta\|\|\xi\|.$$

Finally for $f \in (L_1 \cap L_p)(G)$,

$$\begin{aligned} \|\pi(f)\| = \sup_{\xi,\eta\in b_1(\mathcal{H}_\pi)}|\langle\pi(f)\xi,\eta\rangle| &= \sup_{\xi,\eta\in b_1(\mathcal{H}_\pi)}|\int_G \langle\pi(x)\xi,\eta\rangle f(x)dx| \\ &\leq \sup_{\xi,\eta\in b_1(\mathcal{H}_\pi)}\|f\|_p\|\xi *_\pi \eta\|_q \leq A\|f\|_p. \end{aligned}$$

□

**Theorem 6.3.3. (Kunze-Stein phenomena)** *Let $\pi$ be a nontrivial irreducible unitary representation of* $\mathrm{SL}_2(\mathbb{R})$.

*(a) The following are equivalent:*

- *$\pi$ is unitarily equivalent to an element of the discrete series.*

- *$\pi$ is extendable to $L_2(\mathrm{SL}_2(\mathbb{R}))$.*

- *$\xi *_\pi \eta \in L_2(\mathrm{SL}_2(\mathbb{R}))$ for each $\xi, \eta \in \mathcal{H}_\pi$.*

- *$A_\pi \subseteq L_2(\mathrm{SL}_2(\mathbb{R}))$.*

*(b) The following are equivalent:*

- *$\pi$ is unitarily equivalent to an element of the continuous principal series.*

- *$\pi$ is extendable to $L_p(\mathrm{SL}_2(\mathbb{R}))$ for every $1 \leq p < 2$ but not to $L_2(\mathrm{SL}_2(\mathbb{R}))$.*

- *"$\xi *_\pi \eta \in L_q(\mathrm{SL}_2(\mathbb{R}))$ for all $\xi, \eta \in \mathcal{H}_\pi$" holds for all $2 < q$ but not for $q = 2$.*

- *$A_\pi \subseteq L_q(\mathrm{SL}_2(\mathbb{R}))$ for all $q > 2$ and $A_\pi \not\subseteq L_2(\mathrm{SL}_2(\mathbb{R}))$.*

*(c) The following are equivalent:*

- *$\pi$ is unitarily equivalent to an element of the complementary series indexed by $\sigma$, $0 < \sigma < \frac{1}{2}$.*

- *$\pi$ is extendable to $L_p(\mathrm{SL}_2(\mathbb{R}))$ for $1 \leq p < \frac{1}{1-\delta}$ but not to $L_{\frac{1}{1-\delta}}$.*

- *"$\xi *_\pi \eta \in L_q(\mathrm{SL}_2(\mathbb{R}))$ for each $\xi, \eta \in \mathcal{H}_\pi$" holds for all $q > \frac{1}{\delta}$ but not for $q = \frac{1}{\delta}$.*

- *$A_\pi \subseteq L_q(\mathrm{SL}_2(\mathbb{R}))$ for all $q > \frac{1}{\delta}$ and $A_\pi \not\subseteq L_{\frac{1}{\delta}}(\mathrm{SL}_2(\mathbb{R}))$.*

Let $\pi$ be a unitary (not necessarily irreducible) representation of $\mathrm{SL}_2(\mathbb{R})$ on a separable Hilbert space $\mathcal{H}$. We can find a direct integral decomposition for $\mathcal{H} = \int^{\oplus} \mathcal{H}^\lambda d\sigma(\lambda)$, such that in the corresponding direct integral decomposition for $\pi = \int^{\oplus} \pi^\lambda d\sigma(\lambda)$, the representation $\pi^\lambda$ is an irreducible unitary representation for almost every $\lambda$. Let $T \in \mathcal{B}(\mathcal{H})$ be an operator that can be decomposed with respect to the above decomposition of $\mathcal{H}$. We then write $T = (T^\lambda)$. Recall that $\|T\|_\infty = \text{esssup}_\lambda \|T^\lambda\|_\infty$. The following theorem extends Theorem 6.3.3 for some non-irreducible representations. The proof is based on the independence of the constant $C$, introduced in part (iv) of Lemma 6.3.2, from representations in the continuous or discrete series.

**Theorem 6.3.4.** *Let $\pi$ be a unitary representation (not necessarily irreducible) of $\mathrm{SL}_2(\mathbb{R})$ on a Hilbert space $\mathcal{H}$, and $\pi = \int^\oplus \pi^\lambda d\sigma(\lambda)$ be its decomposition into a direct integral of irreducible unitary representations $\pi^\lambda$. Then the following are equivalent:*

*(i) For $\sigma$-almost every $\lambda$, the representation $\pi^\lambda$ is unitarily equivalent to a representation in the discrete or continuous principal series.*

*(ii) The representation $\pi$ is extendable to $L_p(\mathrm{SL}_2(\mathbb{R}))$ for every $1 \leq p < 2$.*

*(iii) $A_\pi \subseteq L_q$ for every $2 < q$.*

*(iv) Every coefficient function of $\pi$ belongs to $L_q$ for every $2 < q$.*

**Remark.** Let $G = \mathrm{SL}_2(\mathbb{R})$, and $\hat{G}$ denote the set of all the (equivalence classes of) irreducible unitary representations of $G$. Let $\pi$ be an irreducible unitary representation of $G$. Cowling [Cow78] observed that there exist a constant $C$ independent of $\pi$ and a positive integer $q$ such that

$$\|\xi *_\pi \eta\|_{2q} \leq C\|\xi\|\|\eta\| \text{ for all } \xi, \eta \in \mathcal{H}_\pi. \tag{6.5}$$

Furthermore, for each positive integer $q$, the set $\hat{G}_q$ of all the (equivalence classes of) irreducible unitary representations $\pi$ of $G$ that satisfy (6.5) forms a closed subset of $\hat{G}$ in the Fell topology.

## 6.4 $B_0(\mathrm{SL}_2(\mathbb{R}))$ has no point derivations

**Proposition 6.4.1.** *Let $G = \mathrm{SL}_2(\mathbb{R})$. Then*

*(i) The elements of $B_0(G)$ which are nilpotent modulo $A(G)$ are dense.*

*(ii) $\sigma_{B_0(G)} \sim G$.*

*(iii) $B_0(G)$ has no nonzero point derivations.*

**Proof. (i)** By Remark 6.3, $\hat{G}$ is an increasing union of closed subsets $\hat{G}_q$ for positive integers $q$. Let $f$ be an element of $B_0(G)$, and write a direct integral decomposition

$$f = \int_{\hat{G}} \sum_{k=1}^{j(\pi)} \xi_\pi^k *_\pi \eta_\pi^k d\mu(\pi)$$

that satisfies

$$\int_{\hat{G}} \sum_{k=1}^{j(\pi)} \|\xi_\pi^k\| \|\eta_\pi^k\| d\mu(\pi) < \infty.$$

Let $\epsilon > 0$ be given. Since $\mu$ is a regular Borel measure on $\hat{G}$, one can use Remark 6.3 to find $q_0$ in $\mathbb{N}$ such that

$$\int_{\hat{G}\backslash\hat{G}_{q_0}} \sum_{k=1}^{j(\pi)} \|\xi_\pi^k\| \|\eta_\pi^k\| d\mu(\pi) < \epsilon.$$

Define $f_\epsilon$ to be $f_\epsilon := \int_{\hat{G}_{q_0}} \sum_{k=1}^{j(\pi)} \xi_\pi^k *_\pi \eta_\pi^k d\mu(\pi)$. Clearly $f_\epsilon$ lies within $\epsilon$-distance of $f$ in $B(G)$. Moreover by the definition of $\hat{G}_{q_0}$, the function $f_\epsilon$ belongs to $L^{2q_0}(G)$. Therefore $f^{q_0}$ belongs to $B(G) \cap L^2(G) \subseteq A(G)$, which proves (i).

**(ii)** This follows from Proposition 6.1.3 and part (i).

**(iii)** This follows from Proposition 6.1.2. $\square$

Note that $B_0(\mathrm{SL}_2(\mathbb{R}))$ is a subalgebra of $B(\mathrm{SL}_2(\mathbb{R}))$ of codimension one [Cho80]. The following corollary is a natural consequence of Proposition 6.4.1.

**Corollary 6.4.2.** *For every $g \in \mathrm{SL}_2(\mathbb{R})$, let $\phi_g$ denote the character on $B(\mathrm{SL}_2(\mathbb{R}))$ which acts by evaluating at $g$. Let $\phi_0$ denote the unique (nonzero) character on $B(\mathrm{SL}_2(\mathbb{R}))$ that vanishes on $B_0(\mathrm{SL}_2(\mathbb{R}))$. Then*

*(i)* $\sigma_{B(\mathrm{SL}_2(\mathbb{R}))} = \{\phi_g : g \in \mathrm{SL}_2(\mathbb{R})\} \cup \{\phi_0\}$ *as a set.*

*(ii) For $g$ in $\mathrm{SL}_2(\mathbb{R})$, $B(\mathrm{SL}_2(\mathbb{R}))$ has no nonzero continuous point derivation at the character $\phi_g$.*

*(iii)* $B(\mathrm{SL}_2(\mathbb{R}))$ *has nonzero continuous point derivations at $\phi_0$.*

**Proof.** **(i)** Let $\sigma$ be a nonzero multiplicative linear functional on $B(\mathrm{SL}_2(\mathbb{R}))$. Recall that $B(\mathrm{SL}_2(\mathbb{R})) = B_0(\mathrm{SL}_2(\mathbb{R})) \oplus_{\ell^1} \mathbb{C}1$. Clearly $\sigma(1) = 1$, since $\sigma$ is multiplicative and nonzero. If $\sigma|_{B_0(\mathrm{SL}_2(\mathbb{R}))} \neq 0$ then by Proposition 6.4.1, there exists an element $g$ in $\mathrm{SL}_2(\mathbb{R})$ such that $\sigma(u) = u(g)$ for every $u$ in $B_0(\mathrm{SL}_2(\mathbb{R}))$. Note that $1(g) = 1$. Hence $\sigma$ is the point evaluation at $g$ on $B(\mathrm{SL}_2(\mathbb{R}))$. On the other hand, if $\sigma|_{B_0(\mathrm{SL}_2(\mathbb{R}))} = 0$ then $\sigma$ is the unique character satisfying $\sigma(1) = 1$ and $\sigma(f) = 0$ for all $f$ in $B_0(\mathrm{SL}_2(\mathbb{R}))$. Hence $\sigma_{B(\mathrm{SL}_2(\mathbb{R}))} = \{\phi_g : g \in \mathrm{SL}_2(\mathbb{R})\} \cup \{\phi_0\}$ as a set.

**(ii)** Let $g$ be an element of $\mathrm{SL}_2(\mathbb{R})$, and suppose that $D$ is a nonzero continuous point derivation of $B(\mathrm{SL}_2(\mathbb{R}))$ at the character $\phi_g$. Note that $\phi_g(1) = 1$, hence $D(1) = D(1 \times 1) = 2D(1)\phi_g(1) = 2D(1)$. Therefore $D(1) = 0$. Since $D$ is nonzero, the restriction $D|_{B_0(\mathrm{SL}_2(\mathbb{R}))}$ is a nonzero continuous point derivation of $B_0(\mathrm{SL}_2(\mathbb{R}))$ at the character $\phi|_{B_0(\mathrm{SL}_2(\mathbb{R}))}$, which contradicts with Proposition 6.4.1.

**(iii)** Let $\phi_0$ be the character of $B(\mathrm{SL}_2(\mathbb{R}))$ defined by $\phi_0(1) = 1$ and $\phi_0|_{B_0(\mathrm{SL}_2(\mathbb{R}))} = 0$. Recall that by Theorem 6.2.2, $\overline{B_0(\mathrm{SL}_2(\mathbb{R}))^2} \neq B_0(\mathrm{SL}_2(\mathbb{R}))$. Let $d$ be a nonzero continuous functional on $B_0(\mathrm{SL}_2(\mathbb{R}))$ that vanishes on $\overline{B_0(\mathrm{SL}_2(\mathbb{R}))^2}$. For an element $f$ in $B(\mathrm{SL}_2(\mathbb{R}))$, let $f = f_0 + \lambda_f \cdot 1$ denote its decomposition with respect to $B(\mathrm{SL}_2(\mathbb{R})) = B_0(\mathrm{SL}_2(\mathbb{R})) \oplus_{\ell^1} \mathbb{C}$. Define

$$\tilde{d} : B(\mathrm{SL}_2(\mathbb{R})) \to \mathbb{C}, \quad f \mapsto d(f_0).$$

Then $\tilde{d}$ is a nonzero continuous point derivation of $B(\mathrm{SL}_2(\mathbb{R}))$ at $\phi_0$. In fact, it is very easy to see that $\tilde{d}$ is nonzero and continuous. Let $f, g \in B(\mathrm{SL}_2(\mathbb{R}))$. Then

$$\begin{aligned} \tilde{d}(fg) &= \tilde{d}((f_0 + \lambda_f \cdot 1)(g_0 + \lambda_g \cdot 1)) = \tilde{d}(f_0 g_0 + \lambda_f g_0 + \lambda_g f_0 + \lambda_f \lambda_g) \\ &= d(f_0 g_0 + \lambda_f g_0 + \lambda_g f_0) = \lambda_f d(g_0) + \lambda_g d(f_0) = \phi_0(f)\tilde{d}(g) + \phi_0(g)\tilde{d}(f), \end{aligned}$$

where we used the fact that $d|_{B_0(\mathrm{SL}_2(\mathbb{R}))^2} = 0$. Hence $\tilde{d}$ is a point derivation of $B(\mathrm{SL}_2(\mathbb{R}))$ at $\phi_0$. □

## 6.5 Connected semisimple Lie group with finite center

**Proposition 6.5.1.** *Let $G$ be a semi-simple connected Lie group with finite center.*

*(i) The elements of $B_0(G)$ which are nilpotent modulo $A(G)$ are dense.*

*(ii) $\sigma_{B_0(G)} \sim G$.*

*(iii) $B_0(G)$ has no nonzero point derivations.*

**Proof.** (ii) and (iii) follow from Propositions 6.1.3 and 6.1.2. We only need to prove (i). Let $G$ be a connected semisimple Lie group with finite center. Then $G$ has a finite covering group $G^\circ$ of the form

$$G^\circ = H_0^\circ \times H_1^\circ \times \ldots \times H_m^\circ,$$

where $H_0^\circ$ is compact, and for each $1 \leq j \leq m$, the subgroup $H_j^\circ$ is noncompact and simple. Let $\pi$ be an irreducible unitary representation of $G$. Then $\pi$ can be lifted to an irreducible representation of $G^\circ$, which in turn is the external tensor product of irreducible representations $\pi_0, \pi_1, \ldots, \pi_m$ of the subgroups $H_0^\circ, H_1^\circ, \ldots, H_m^\circ$ respectively. Using the results in [Cow79b], one can observe that for each $1 \leq j \leq m$, either $\pi_j$ is the trivial representation of $H_j^\circ$ or there exists a positive integer $p_j$ with $A_{\pi_j}(H_j^\circ) \subseteq L^{p_j}(H_j^\circ)$. Suppose that the first case happens, i.e. there exists an index $j_0$ such that $\pi_{j_0}$ is the trivial representation. This implies that every nonzero coefficient function of $\pi_{j_0}$ is constant on the equivalence classes of $H_{j_0}^\circ$, and therefore does not vanish at infinity. Hence for an irreducible $C_0$-representation $\pi$, there exists a positive integer $p$ such that $A_\pi(G) \subseteq L^p(G)$. Moreover, by [Cow78], there exists a positive integer $q$ and a constant $C$ independent from $\pi$ such that

$$\|\xi *_\pi \eta\|_{2q} \leq C\|\xi\|\|\eta\| \text{ for each } \xi, \eta \in \mathcal{H}_\pi. \tag{6.6}$$

Let $\mathcal{S}$ be the finite family of subgroups $S$ of $G$ defined in [Cow79a]. Recall that the only compact subgroup in the family $\mathcal{S}$ is the trivial subgroup $S_0 = \{e_G\}$. For each $S$ in $\mathcal{S}$, let $q_S$ denote the quotient map from $G$ to $G/S$. For each $q \in \mathbb{N}$, define $\hat{G}_{S,q}$ to be the set of all (equivalence classes of) irreducible unitary representations $\pi$ of $G$ that are trivial on $S$ and each coefficient function of $\pi$ satisfy (6.6) as a function on $G/S$. Let $u$ be an element of $B(G)$. Recall that any unitary representation of $G$ on a separable Hilbert space can be written as a direct integral of irreducible representations. Then we can decompose $u$ as

$$u = \sum_{S \in \mathcal{S}} u_S,$$

with $u_S \in B(G) \cap (C_0(G/S) \circ q_S)$. Each $u_S$ can be written as a direct integral of irreducible representations in $\hat{G}_S := \bigcup_{q \in \mathbb{N}} \hat{G}_{S,q}$. Clearly if $u$ belongs to $B_0(G)$ then

$$u = \int_{\hat{G}_{S_0}} \sum_{k=1}^{j(\pi)} \xi_\pi^k *_\pi \eta_\pi^k d\mu(\pi),$$

with $\int_{\hat{G}_{S_0}} \sum_{k=1}^{j(\pi)} \|\xi_\pi^k\|\|\eta_\pi^k\| d\mu(\pi) < \infty$. Now using an argument identical to the proof of Proposition 6.1.2, we obtain (i). $\square$

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