Well-Posedness of Boundary Control Systems

by

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Abstract

Boundary control systems are an important class of infinite dimensional control systems. A key question is whether the mappings from input/state, input/output, state/input and initial state/final state are well-defined bounded linear maps. When all four mappings are well-defined and bounded, the problem is said to be *well-posed.* This thesis examines boundedness of the input/output map.

Continuity of the input/output map for a boundary control system is shown through the system transfer function. Our approach transforms the question of boundedness of the input/output map of a boundary control system into boundedness of the solution to a related elliptic problem. Boundedness is shown for a class of boundary control systems with Dirichlet, Neumann or Robin boundary control. Use of the transfer function in approximations is also demonstrated.

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Contents

Notation

Symbol	Meaning
$D' = (D_1, D_2, \ldots, D_n)$	differential operator
D^α	$(D_1^{\alpha_1}, D_2^{\alpha_2}, \ldots, D_n^{\alpha_n})$, $\alpha_i \geq 0$
$(\mathcal{T}, \mathcal{B}, \mathcal{C}, \mathcal{G})$	well-posed system, the mapping $\mathcal{T}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{G}$ are all linear
(A, B, C, D)	state space realization
(Δ, Γ, K)	boundary control system
(Δ, Γ)	boundary control system without output equation
$(\Delta, \Gamma)_e$	abstract elliptic problem corresponding to (Δ, Γ)
$D(A)$	domain of A
$\mathcal{V}^*$	dual space of $\mathcal{V}$
$\mathcal{W} \hookrightarrow \mathcal{H}$	$\mathcal{W}$ has a continuous, dense injection to $\mathcal{H}$
$(\mathcal{V}, \mathcal{H}, \mathcal{V}^*)$	Gelfand triples, $\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}^*$
$\mathcal{L}(\mathcal{U}, \mathcal{Y})$	bounded linear operators from $\mathcal{U}$ to $\mathcal{Y}$
$L^p([0,T], \mathcal{H})$, $\mathcal{H}$ omitted if scalar valued	class of H-valued functions such that $\int_0^T \|f(x)\|_{\mathcal{H}}^p \, dx < \infty$
$H^m([0,T], \mathcal{H})$, m a positive integer. $\mathcal{H}$ omitted if scalar valued	class of $\mathcal{H}$-valued functions such that $\left\{ f \in L^2[0,T] \,\middle|\, f, \frac{d^{m-1}f}{dx^{m-1}} \text{ are absolutely continuous on } [0,T] \text{ with } \frac{d^m f}{dx^m} \in L^2[0,T] \right\}$
$H^{\frac{1}{2}}(\partial\Omega)$	the totality of the traces of functions belonging to $H^1(\Omega)$
$\hat{y}$	Laplace transform of y

Chapter 1

Introduction

Many control systems involve control on the boundaries. We shall refer to these systems as boundary control systems. For instance many structures are supported by cables, which are relatively light in weight with respect to the whole structure, hence they tend to vibrate. Outside influences (such as wind) further magnify this vibration. Thus one might wish to apply some control at the end of the cable (i.e. the boundary) to counter this effect. Another example is noise reduction in commercial aircraft. For both safety and comforts of passengers and crews, it is important to keep the noise to a minimum.

A key question is whether the state and output is continuously dependent on its initial state and input. Hence we wish to determine when are the linear mappings from input/state, input/output, state/output and initial state/final state are well-defined and continuous. (When all four mappings are well-defined and bounded, the system is said to be *well-posed.*) This is important since not every system arriving from a physical model is well-posed. Thus it is important for application purposes to distinguish between them. The concept of well-posedness under an abstract setting was unified in [Salamon, 1987]. In a later paper [Salamon, 1989], he showed that boundedness of the input/output map implies well-posedness of the boundary control system with respect to some state space. Thus showing boundedness of the input/output map of a boundary control system is important. This is the topic of discussion in this thesis.

In Chapter 2, we give some mathematical preliminaries and a detailed introduction to the concept of well-posedness. Examples are given throughout for clarification. In Chapter 3, we show that boundedness of the input/output map for a boundary control system can be shown through the system transfer function. This approach allows us to show boundedness of the input/output map for a large class of boundary control systems. Finally in Chapter 4, we examine the use of direct transfer function approximations for controller design as opposed to state-space approximations. This may be useful in practice since state-space approximations for multi-dimensional systems are usually of very high order while the number of inputs and outputs is relatively low.

Chapter 2

Infinite Dimensional System Theory

The theory of linear ordinary differential equations allows one to examine the solution to finite-dimensional time-invariant equations of the form

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0 \in \Re^n \tag{2.1}$$

where $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$ are constant matrices with underlying state space $\Re^n$. Assuming that $u(t)$ is sufficiently smooth, say Lebesgue measurable, then the above equation has solution

$$x(t) = \exp(At)x_0 + \int_0^t \exp(A(t-s))Bu(s)\, ds. \tag{2.2}$$

An infinite-dimensional time-invariant system may also be represented as equation (2.1). However, the operators A and B are now on an infinite dimensional state space. Moreover, they need not be bounded operators on the underlying state space.

It is tempting to claim that the solution is still given by equation (2.2), but this is not quite correct. Let $\mathcal{H}$ be the underlying state space and suppose B is unbounded with respect to $\mathcal{H}$.

Then the integral above is not necessarily well-defined in $\mathcal{H}$ hence $x(t)$ as given by equation (2.2) need not be in $\mathcal{H}$. Immediately, we can conclude that certain 'unboundedness' restrictions must be placed on B. This problem shall be the subject of discussions in Section 2.2.

For the remainder of this section we shall consider the solution to the homogeneous initial value problem,

$$\dot{x}(t) = Ax(t) \qquad x(0) = x_0 \in \mathcal{H}. \tag{2.3}$$

Is the solution given by

$$x(t) = \exp(At)x_0 \tag{2.4}$$

even when A is an operator on a possibly infinite-dimensional state space $\mathcal{H}$? There is a difficulty in expressing $\exp(At)$ since the traditional power series representation is no longer valid in cases where A is an unbounded operator.

Instead, we seek a solution of the form

$$x(t) = \mathcal{T}(t)x_0, \tag{2.5}$$

where $\mathcal{T}(t)$ is a bounded linear operator on $\mathcal{H}$. Thus we may view $\mathcal{T}(t)$ as a generalization of $\exp(At)$.

For this solution to be valid, it must satisfy the initial condition $x(0) = x_0$. Thus $\mathcal{T}(0) = I$. Next, the system at any time t is time-invariant. It can be shown that this condition translates to $\mathcal{T}$ satisfying $\mathcal{T}(t+s) = \mathcal{T}(t)\mathcal{T}(s)$ for all $t, s \geq 0$. Finally, we note that in the study of elementary differential equations (i.e. when $\mathcal{H} = \Re^n$), we seek solutions $x(t)$ continuously dependent on initial conditions. Thus it is natural to seek continuous solutions $x(t)$ even in this general setting. This can be accomplished if we impose certain continuity assumption on $\mathcal{T}(t)$. Since equation (2.5) must also describe the solution to equation (2.3) in the finite-dimensional case and $\lim_{t\to 0^+} \|\exp(At) - I\|$ is uniform so one might try to impose such a condition on $\mathcal{T}(t)$ as well. However it can be shown (e.g. [Pazy, 1983, Theorem 1.2]) that this holds if and only if A is bounded. Hence for unbounded operators A, we cannot make such assumption. Instead we weaken the uniform continuity assumption to strongly continuous. That is. $\lim_{t\to 0^+} \|\mathcal{T}(t)x - x\| = 0$ for all $x \in \mathcal{H}$.

Families of linear operators $\mathcal{T}(t)$ that satisfy the above conditions are known as strongly continuous semigroups. We shall look at some of their basic properties in the next section. The materials presented can be found in [Pazy. 1983] or [Curtain and Zwart. 1995].

2.1 Semigroup Theory

The theory of semigroups is an important tool in the study of infinite-dimensional systems. It allows one to describe the solution to abstract partial differential equations in a rigorous manner. In such a context. one can regard it as a generalization to the exponential function.

Definition 2.1.1: Let $\mathcal{H}$ be a Hilbert space. A family. $\{\mathcal{T}(t). t \geq 0\}$. of bounded linear operators in $\mathcal{H}$ is called a *strongly continuous semigroup* or *C_0-semigroup*. if it satisfies the following properties:

1. $\mathcal{T}(0) = I$.
2. $\|\mathcal{T}(t)x_0 - x_0\| \to 0$ as $t \to 0^+$ $\forall x_0 \in \mathcal{H}$.
3. $\mathcal{T}(t+s) = \mathcal{T}(t)\mathcal{T}(s)$ $t, s \geq 0$.

Example 2.1.2: [EXPONENTIAL FUNCTION] Let A be a bounded linear operator on some Hilbert space $\mathcal{H}$. Set $\exp(At) = \sum_{n=0}^{\infty} \frac{1}{n!}(At)^n$. We claim that $\mathcal{T}(t) = \exp(At)$ is a C_0-semigroup on $\mathcal{H}$. For each fixed t. $\mathcal{T}(t)$ is linear and bounded on $\mathcal{H}$. Moreover $\mathcal{T}(0) = I$. For any $x_0 \in \mathcal{H}$.

$$\left\|e^{At}x_0 - x_0\right\| = \left\|\sum_{n=1}^{\infty} \frac{(At)^n}{n!}x_0\right\| \leq \sum_{n=1}^{\infty} \frac{\|A\|^n t^n}{n!}\|x_0\| = \left(e^{\|A\|t} - 1\right)\|x_0\|.$$

Since $e^{\|A\|t}$ is continuous. $\lim_{t\to 0^+} \left\|e^{At}x_0 - x_0\right\| \longrightarrow 0$. Finally for any $t, s \geq 0$ we have

$$\sum_{n=0}^{\infty} \frac{1}{n!}[A(t+s)]^n = \sum_{n=0}^{\infty} \frac{1}{n!}A^n(t+s)^n$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{1}{n!} A^n \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} t^{n-k} s^k \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} A^n s^n + \sum_{n=1}^{\infty} \frac{1}{n!} A^n \frac{n!}{(n-1)!} t s^{n-1} + \sum_{n=2}^{\infty} \frac{1}{n!} A^n \frac{n!}{2!(n-2)!} t s^{n-2} + \cdots \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} A^n s^n + At \sum_{n=0}^{\infty} \frac{1}{n!} A^n s^n + \frac{1}{2!} A^2 t^2 \sum_{n=0}^{\infty} \frac{1}{n!} A^n s^n + \cdots \\
&= \left(\sum_{n=0}^{\infty} \frac{1}{n!} A^n t^n \right) \left(\sum_{n=0}^{\infty} \frac{1}{n!} A^n s^n \right).
\end{aligned}
$$

Thus $\mathcal{T}(t+s) = \mathcal{T}(t)\mathcal{T}(s)$. By definition, $\mathcal{T}(t) = \sum_{n=0}^{\infty} \frac{1}{n!}(At)^n$ is a C_0-semigroup. ■

Example 2.1.3: [LEFT SHIFT OPERATOR] Let $\mathcal{H} = L^2(0, \infty)$ and consider the shift operator on $\mathcal{H}$

$$(\mathcal{T}(t)h)(\tau) = h(t+\tau), \qquad t, \tau \geq 0.$$

Clearly $\mathcal{T}(0) = I$. Also,

$$(\mathcal{T}(t+s)h)(\tau) = h(\tau + t + s) = (\mathcal{T}(t)h)(\tau + s) = \big[(\mathcal{T}(t)\mathcal{T}(s))h\big](\tau).$$

For each fixed t, $\mathcal{T}(t)$ is a bounded linear operator on $\mathcal{H}$. It remains to show that $\lim_{t \to 0^+} \mathcal{T}(t)z = z$ for all $z \in \mathcal{H}$.

Let $\mathcal{W}$ denote the set of continuous function with compact support. Let $h \in \mathcal{W}$, then have for any $\epsilon > 0$, there exists $\delta > 0$ such that if $0 < t < \delta$,

$$\|\mathcal{T}(t)h - h\|_2 = \left(\int_0^{\infty} |h(\tau + t) - h(\tau)|^2 \, dx \right)^{\frac{1}{2}} < \epsilon.$$

But $\mathcal{W}$ is dense in $\mathcal{H}$, so for any $z \in \mathcal{H}$, there exists $h \in \mathcal{W}$ such that $\|z - h\|_2 \leq \frac{\epsilon}{3}$. Hence

$$
\begin{aligned}
\|\mathcal{T}(t)z - z\|_2 &= \|\mathcal{T}(t)(z-h) + \mathcal{T}(t)h - h + h - z\|_2 \\
&\leq \|z - h\|_2 + \|\mathcal{T}(t)h - h\|_2 + \|z - h\|_2 \qquad (\text{since } \|\mathcal{T}(t)\| \leq 1)
\end{aligned}
$$

$$\leq \epsilon.$$

Thus $\mathcal{T}(t)$ is a C_0-semigroup on $\mathcal{H}$. ■

Example 2.1.4: [1-D HEAT EQUATION] One of the simplest examples of a well-posed boundary control system is the problem of regulating heat flow in a one-dimensional rod of length 1, through one end of the rod.

$$\begin{aligned} \frac{\partial z}{\partial t} &= \frac{\partial^2 z}{\partial x^2}, & x \in [0,1] \\ z(x,0) &= 0, & x \in [0,1] \\ \frac{\partial z}{\partial x}(0,t) &= 0, & t > 0 \\ \frac{\partial z}{\partial x}(1,t) &= 0, & t > 0 \end{aligned}$$

Let $\mathcal{H} = L^2(0,1)$. Define $\lambda_n = -n^2\pi^2$ and $e_n = \sqrt{2}\cos n\pi x$ $n = 0,1,2,\ldots$. Then λ_n are the eigenvalues to the eigenvalue problem

$$\begin{aligned} \frac{d^2 z}{dx^2} &= \lambda z, \\ z'(0) &= 0, \\ z'(1) &= 0, \end{aligned}$$

with corresponding normalized eigenfunctions e_n. Moreover $\{e_n\}$ forms an orthonormal basis for $\mathcal{H}$.

Define the linear operator $\mathcal{T}(t)z = \sum_{n=0}^{\infty} \exp(\lambda_n t)\langle z, e_n\rangle\, e_n$ for all $z \in \mathcal{H}$ and $t \geq 0$. Then clearly $\mathcal{T}(0) = I$, moreover for any $t, s \geq 0$ and $z \in \mathcal{H}$,

$$\begin{aligned} \mathcal{T}(t)\mathcal{T}(s)z &= \sum_{n=0}^{\infty} \exp(\lambda_n t)\langle \mathcal{T}(s)z, e_n\rangle\, e_n \\ &= \sum_{n=0}^{\infty} \exp(\lambda_n t)\left\langle \sum_{m=0}^{\infty} \exp(\lambda_m s)\langle z, e_m\rangle\, e_m, e_n \right\rangle e_n \\ &= \sum_{n=0}^{\infty} \exp(\lambda_n (t+s))\langle z, e_n\rangle\, e_n \end{aligned}$$

$$= \mathcal{T}(t+s)z.$$

It remains to show that $\mathcal{T}$ satisfies the strong continuity assumption.

Let $\epsilon > 0$, and $T > 0$ be fixed but arbitrary. Define

$$M := \sup_{\substack{n \geq 1 \\ t \in [0,T]}} (\exp(\lambda_n t) - 1)^2.$$

Thus $M = 1$. For each $z \in \mathcal{H}$ there exists $N \in \mathbb{N}$ such that for all $k \geq N$, $\sum_{n=k}^{\infty} |\langle z, e_n\rangle|^2 \leq \frac{\epsilon}{2}$. For this N, we can choose $t_1 \in [0,T]$ such that $(\exp(\lambda_n t_1) - 1)^2 \leq \frac{\epsilon}{2\|z\|^2}$ for all λ_n, $1 \leq n \leq N$. Thus if $0 < t < t_1$,

$$\begin{aligned}
\|\mathcal{T}(t)z - z\|^2 &= \left\| \sum_{n=0}^{\infty} (\exp(\lambda_n t) - 1)\langle z, e_n\rangle\, e_n \right\|^2 \\
&= \sum_{n=0}^{N} (\exp(\lambda_n t) - 1)^2 |\langle z, e_n\rangle|^2 + \sum_{n=N+1}^{\infty} (\exp(\lambda_n t) - 1)^2 |\langle z, e_n\rangle|^2 \\
&\leq \sum_{n=0}^{N} (1 - \exp(\lambda_n t_1))^2 |\langle z, e_n\rangle|^2 + \sum_{n=N+1}^{\infty} |\langle z, e_n\rangle|^2 \\
&\leq \frac{\epsilon}{2\|z\|^2}\|z\|^2 + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Hence $\mathcal{T}$ generates a C_0-semigroup on $\mathcal{H}$. ■

Since our motive is to use the C_0-semigroup $\mathcal{T}(t)$ to solve a homogeneous initial value problem, we must relate it to the operator A that appears in the differential equation (2.3). This relationship is established via the following definition:

Definition 2.1.5: Let $\{\mathcal{T}(t), t \geq 0\}$ be a C_0-semigroup on a Hilbert space $\mathcal{H}$. *The infinitesimal* *generator of the semigroup is the operator A* defined by

$$Ax = \lim_{h \to 0^+} \frac{T(h)x - x}{h},$$

and the domain of A, $D(A)$, is the set of all vectors $x \in \mathcal{H}$ for which this limit exists.

To compute the infinitesimal generator of a C_0-semigroup, we often need series representations of the semigroup and state space elements. First, let's recall the following theorem.

Theorem 2.1.6: [FOURIER SERIES THEOREM] *Let $\{x_n\}$ be an orthonormal basis in a Hilbert space $\mathcal{H}$. Then any $x \in \mathcal{H}$ can be represented by*

$$x = \sum_n \langle x, x_n \rangle x_n. \blacksquare \tag{2.6}$$

A similar representation is possible if the eigenvectors form a *Riesz Basis* in the space $\mathcal{H}$.

Definition 2.1.7: A sequence of vectors $\{\phi_n, n \geq 1\}$ in a Hilbert space $\mathcal{H}$ forms a *Riesz Basis* for $\mathcal{H}$ if the following two conditions hold:

1. $\overline{\operatorname{span}_{n \geq 1}} \{\phi_n\} = \mathcal{H}$;
2. There exist positive constants m and M such that for arbitrary $K \in \mathbb{N}$ and arbitrary scalars $\alpha_k, k \in \{1, 2, \ldots, K\}$,

$$m \sum_{n=1}^{K} |\alpha_n|^2 \leq \left\| \sum_{n=1}^{K} \alpha_n \phi_n \right\|^2 \leq M \sum_{n=1}^{K} |\alpha_n|^2.$$

Definition 2.1.8: Two sequences $\{\phi_n\}, \{\psi_n\}, n \geq 1$ are said to be *biorthogonal* if $\langle \phi_n, \psi_m \rangle = \delta_{mn}$.

Theorem 2.1.9: [REPRESENTATION THEOREM] *Suppose that the closed, linear operator A on the Hilbert space $\mathcal{H}$ has simple eigenvalues $\{\lambda_n, n \geq 1\}$ and that its corresponding eigenvectors $\{\phi_n, n \geq 1\}$ form a Riesz basis in $\mathcal{H}$.*

1. *If $\{\psi_n, n \geq 1\}$ are the eigenvectors of A^* corresponding to the eigenvalues $\{\overline{\lambda_n}, n \geq 1\}$, then the $\{\psi_n\}$ can be suitably scaled so that $\{\phi_n\}, \{\psi_n\}$ are biorthogonal.*

2. *Every $x \in \mathcal{H}$ can be represented uniquely by*

$$x = \sum_{n=1}^{\infty} \langle x, \psi_n \rangle \phi_n. \tag{2.7}$$

3. *For all $x \in D(A)$, A has the representation*

$$Ax = \sum_{n=1}^{\infty} \lambda_n \langle x, \psi_n \rangle \phi_n. \tag{2.8}$$

4. *The C_0-semigroup$\mathcal{T}(t)$ generated by A is given by*

$$\mathcal{T}(t) = \sum_{n=1}^{\infty} \exp(\lambda_n t) \langle \cdot, \psi_n \rangle \phi_n. \tag{2.9}$$

■

Example 2.1.10: [EXPONENTIAL FUNCTION CONTINUED]
Let $\mathcal{T}(t) = \exp(Ct)$, where C is a $n \times n$ constant matrix. Using Definition 2.1.5 we obtain

$$Ax = \lim_{h \to 0^+} \frac{\exp(Ch)x - x}{h} = Cx,$$

for all $x \in \Re^n$. Thus as expected, the infinitesimal generator of $\exp(Ct)$ is C. ■

Example 2.1.11: [SHIFT OPERATOR CONTINUED] Consider the shift operator given in Example 2.1.3. Using Definition 2.1.5 we have

$$Ah = \lim_{t \to 0^+} \frac{(\mathcal{T}(t)h)(\tau) - h(\tau)}{t} = \lim_{t \to 0^+} \frac{h(\tau + t) - h(\tau)}{t} = \frac{dh}{dt}$$

for all differentiable h. Thus $A = \frac{d}{dt}$ and $D(A) = \{ h \in L^2[0, T] \mid h' \in L^2[0, T], h(T) = 0 \}$.■

Example 2.1.12: [1-D HEAT EQUATION CONTINUED] Consider the C_0-semigroup in

Example (2.1.4). By Theorem (2.1.9) we can write

$$T(t)z = \sum_{n=0}^{\infty} \exp(\lambda_n t)\langle z, e_n\rangle e_n \qquad \forall z \in L^2(0,1).$$

Using Definition 2.1.5 we have

$$\begin{aligned} Az &= \lim_{t\to 0^+} \frac{T(t)z - z}{t} \\ &= \lim_{t\to 0^+} \frac{\sum_{n=0}^{\infty}(\exp(\lambda_n t)\langle z, e_n\rangle e_n - \langle z, e_n\rangle e_n)}{t} \\ &= \lim_{t\to 0^+} \sum_{n=0}^{\infty} \lambda_n \exp(\lambda_n t)\langle z, e_n\rangle e_n \\ &= \sum_{n=0}^{\infty} \lambda_n \langle z, e_n\rangle e_n \\ &= \frac{d^2}{dx^2}\left(\sum_{n=0}^{\infty}\langle z, e_n\rangle e_n\right) \\ &= \frac{d^2 z}{dx^2} \end{aligned}$$

for all $z \in H^2(0,1)$. Thus $A = \dfrac{d^2}{dx^2}$ and $D(A) = \{\, h \in L^2[0,T] \mid h', h'' \in L^2[0,T] \,\}$. Of course, we could also have used Theorem (2.1.9) to obtain

$$Az = \sum_{n=0}^{\infty} \lambda_n \langle z, e_n\rangle e_n \qquad \forall z \in D(A).$$

■

The next theorem gives a series of elementary properties of a C_0-semigroup and its infinitesimal generator. These properties justify the solution representation of equation (2.3) given in (2.5).

Theorem 2.1.13: [ELEMENTARY PROPERTIES] *Let A be the infinitesimal generator of the strongly continuous semigroup $T(t)$ on a Hilbert space $\mathcal{H}$. Then the following hold:*

(i) For $x_0 \in D(A)$, and all $t \geq 0$,

$$T(t)x_0 \in D(A).$$

(ii) For $x_0 \in D(A)$ and all $t > 0$,

$$\frac{d}{dt}(\mathcal{T}(t)x_0) = A\mathcal{T}(t)x_0 = \mathcal{T}(t)Ax_0.$$

(iii) For $x \in \mathcal{H}$,

$$\int_0^t \mathcal{T}(s)x\, ds \in D(A),$$

and

$$A\int_0^t \mathcal{T}(s)x\, ds = \mathcal{T}(t)x - x.$$

(iv) $D(A)$ is dense in $\mathcal{H}$.

(v) A is a closed linear operator.

(vi) If $\omega_0 = \inf_{t>0} \frac{1}{t} \log \|\mathcal{T}(t)\|$, then $\omega_0 = \lim_{t\to\infty} \left(\frac{1}{t} \log \|\mathcal{T}(t)\|\right) < \infty$. The constant ω_0 is called *the growth bound of the semigroup.*

(vii) For all $\omega > \omega_0$, there exists a constant M_ω such that $\forall t \geq 0$, $\left\|\mathcal{T}(t)\right\| \leq M_\omega \exp(\omega t)$. ■

Given any C_0-semigroup $\mathcal{T}$ with infinitesimal generator A, parts (i) and (ii) imply equation (2.5) is the solution to the homogeneous IVP:

$$\dot{x}(t) = Ax(t) \qquad x(0) = x_0 \in \mathcal{H}. \tag{2.10}$$

More importantly, given the homogeneous IVP (2.10), the solution exists and is given by (2.5) provided that A is the infinitesimal generator of a C_0-semigroup. Part (iv) and (v) of Theorem 2.1.13 state necessary conditions on A to be an infinitesimal generator of a C_0-semigroup.

Existence of a solution to (2.3) relies on whether A generates a C_0-semigroup on $\mathcal{H}$. The Hille-Yosida Theorem below gives conditions for this in terms of the resolvent operator defined by $R(\lambda, A) = (\lambda I - A)^{-1}$. Before stating the theorem, we first give some properties of the resolvent operator.

Lemma 2.1.14: (e.g. [Curtain and Zwart, 1995, Lemma 2.1.11]) *Let $\mathcal{T}(t)$ be a C_0-semigroup* *with infinitesimal generator A and growth bound μ. Then for all $z \in \mathcal{H}$ and $Re(\lambda) > \omega > \mu$ the following results hold:*

a. $\lambda \in \rho(A)$ *and*

b. $\left\| R(\lambda, A) \right\| \leq \frac{M}{\text{Re}(\lambda) - \omega}$. ■

Theorem 2.1.15: [HILLE-YOSIDA]*(e.g. [*Curtain and Zwart, 1995, Lemma 2.12*]) A necessary and sufficient condition for a closed, densely defined, linear operator A on a Banach space Z to be the infinitesimal generator of a C_0-semigroup is that there exist real numbers M, ω such that for all λ with $Re(\lambda) > \omega$ we have*

$$\|R(\lambda, A)^n\| \leq \frac{M}{(\text{Re}(\lambda) - \omega)^n} \quad \forall n \geq 1. \quad ■ \tag{2.11}$$

The beauty of the Hille-Yosida theorem is that no assumption is made on the boundedness of A. However in general, equation (2.11) is non-trivial to justify. For practical purposes, the following corollary is much more useful.

Corollary 2.1.16: [LUMER-PHILLIPS] *Sufficient conditions for a closed, densely defined operator on a Hilbert space to be the infinitesimal generator of a C_0-semigroup satisfying $\|\mathcal{T}(t)\| \leq \exp(\omega t)$ are:*

$$\text{Re}\langle A\phi, \phi\rangle \leq \omega\|\phi\|^2 \quad \forall \phi \in D(A), \tag{2.12}$$

$$\text{Re}\langle A^* v, v\rangle \leq \omega\|v\|^2 \quad \forall v \in D(A^*). \tag{2.13}$$

■

Example 2.1.17: [EXPONENTIAL FUNCTION CONTINUED] Let $A \in \Re^{n \times n}$ be a constant matrix with $D(A) = \{\phi \in \Re^n\}$. The adjoint operator is simply A^T with $D(A^T) = \{v \in \Re^n\}$.

Let $\mathcal{H} = \Re^n$ with $\|\cdot\|_{\mathcal{H}} = \|\cdot\|_2$. Then

$$\begin{aligned} \mathrm{Re}\,\langle A\phi, \phi\rangle &\leq \sigma_{\max}(A^T A)\,\|\phi\|_2^2, \\ \mathrm{Re}\,\langle A^T v, v\rangle &\leq \sigma_{\max}(A^T A)\,\|v\|_2^2, \end{aligned}$$

where $\sigma_{\max}$ denotes the largest singular value. ∎

Example 2.1.18: [SHIFT OPERATOR] Consider the first derivative operator $A = \frac{d}{dt}$ on $\mathcal{H} = L^2[0, T]$ with domain

$$D(A) = \left\{ \phi \in L^2[0, T] \;\middle|\; \frac{d\phi}{dt} \in L^2[0, T], \phi(T) = 0 \right\}.$$

One can easily show that the adjoint operator A^* and its corresponding domain are given by

$$A^* = -\frac{d}{dt}, \quad D(A^*) = \left\{ v \in L^2[0, T] \;\middle|\; \frac{dv}{dt} \in L^2[0, T], v(0) = 0 \right\}.$$

Clearly $D(A)$ is dense in $\mathcal{H}$. Consider $\{\phi_n\} \in D(A)$ such that $\phi_n \to \phi$ and $A\phi_n \to y$. Let $\tilde{\phi}(t) = -\int_t^T y(\tau)\, d\tau$. Thus $\tilde{\phi} \in D(A)$ and $A\tilde{\phi} = y$.
We shall show that $\tilde{\phi} = \phi$.

$$\begin{aligned} \|\tilde{\phi} - \phi_n\|_{\mathcal{H}}^2 &= \int_0^T \left| -\int_t^T y(\tau)\, d\tau - \phi_n(t) \right|^2 dt \\ &= \int_0^T \left| -\int_t^T y(\tau) + \phi_n'(\tau)\, d\tau \right|^2 dt \\ &\leq \int_0^T \int_0^T |y(\tau) - \phi_n'(\tau)|^2\, d\tau\, dt \\ &= \sqrt{T} \int_0^T \|A\phi_n - y\|_{\mathcal{H}}^2\, dt \end{aligned}$$

Thus $\phi_n \to \tilde{\phi}$. Hence $\tilde{\phi} = \phi$. So A is closed. Finally consider $\phi \in D(A)$ and

$\psi \in D(A^*)$. Then

$$\begin{aligned} \langle A\phi, \phi\rangle &= -|\phi(0)|^2 - \overline{\langle A\phi, \phi\rangle}. \\ \langle A^*\psi, \psi\rangle &= -|\psi(T)|^2 - \overline{\langle A^*\psi, \psi\rangle}. \end{aligned}$$

Hence $\mathrm{Re}\langle A\phi, \phi\rangle = -\frac{1}{2}|\phi(0)|^2 \leq 0$ and $\mathrm{Re}\langle A^*\psi, \psi\rangle = -\frac{1}{2}|\psi(T)|^2 \leq 0$. Corollary 2.1.16 is satisfied with $\omega = 0$. Hence A is the infinitesimal generator of a C_0-semigroup on $\mathcal{H}$. From Example 2.1.11, we see that the corresponding C_0-semigroup is the left shift operator. ∎

Example 2.1.19: [1-D HEAT EQUATION CONTINUED]
Let $\mathcal{H} = L^2(0, 1)$, and set $A = \dfrac{d^2}{dx^2}$ with

$$D(A) = \{\ \phi \in L^2(0, 1) \mid \phi', \phi'' \in L^2(0, 1),\ \phi'(0) = 0, \phi'(1) = 0\ \}.$$

In this case, $A^* = A$ and $D(A^*) = D(A)$. As in the previous example, we can show that A is a closed operator with $\overline{D(A)} = \mathcal{H}$. Let $\phi \in D(A)$ and $\psi \in D(A^*)$ then

$$\begin{aligned} \langle A\phi, \psi\rangle &= \langle \phi_{xx}, \psi\rangle \\ &= \phi_x\psi|_0^1 - \langle \phi_x, \psi_x\rangle \\ &= -\phi\psi_x|_0^1 + \langle \phi, \psi_{xx}\rangle \\ &= \langle \phi, \psi_{xx}\rangle \\ &= \langle \phi, A\psi\rangle. \end{aligned}$$

Now for any $z \in D(A)$ we have $\langle Az, z\rangle = \overline{\langle Az, z\rangle}$ so $\langle Az, z\rangle$ is real. Also

$$\langle Az, z\rangle = -\langle z_x, z_x\rangle = -\|z_x\|^2 \leq 0.$$

Hence $\mathrm{Re}\langle Az, z\rangle = \mathrm{Re}\langle A^*z, z\rangle \leq 0$. By Corollary 2.1.16, A generates a C_0-semigroup on $\mathcal{H}$. ■

Thus far, we see that if A is an infinitesimal generator of a C_0-semigroup, then the homogeneous IVP has a unique solution. Next, we wish to examine the solution to the inhomogeneous IVP

$$\dot{x}(t) = Ax(t) + f(t), \qquad x(0) = x_0 \in \mathcal{H}, \tag{2.14}$$

where A is assumed to be the infinitesimal generator of a C_0-semigroup in $\mathcal{H}$.

The nature and existence of solutions to equation (2.14) depends on the smoothness of $f(t)$. In elementary differential equations, we seek a function $x \in C^1([0,T],\Re^n)$ such that $x(t) \in \Re^n$ for all $t \geq 0$ and $x(t)$ satisfies equation (2.14). This solution is referred to as the *classical solution.* Suppose such a solution, $x(t)$, exists and let $f \in L^1([0,T],\mathcal{H})$. For $0 < s < t$

$$\begin{aligned}
\frac{d}{ds}\Big(\mathcal{T}(t-s)x(s)\Big) &= -A\mathcal{T}(t-s)x(s) + \mathcal{T}(t-s)x'(s) \\
&= -A\mathcal{T}(t-s)x(s) + \mathcal{T}(t-s)\left(Ax(s) + f(s)\right) \\
&= \mathcal{T}(t-s)f(s). \qquad \text{(Property (ii) of Theorem 2.1.13)}
\end{aligned}$$

Since $f \in L^1([0,T],\mathcal{H})$, we can integrate the above equation from 0 to t giving

$$x(t) = \mathcal{T}(t)x_0 + \int_0^t \mathcal{T}(t-s)f(s)\, ds. \tag{2.15}$$

Any $x(t)$ defined by (2.15) is in $C([0,T],\mathcal{H})$. If $x_0 \notin D(A)$ then $\mathcal{T}(t)x_0$ need not be in $D(A)$ so it may or may not be differentiable. If $x_0 \in D(A)$ then $\mathcal{T}(t)x_0 \in D(A)$, hence it is differentiable. Thus in this case, whether (2.15) is differentiable or not on $[0,T]$ rests upon the differentiability of $\int_0^t \mathcal{T}(t-s)f(s)\, ds$ and whether $\int_0^t \mathcal{T}(t-s)f(s)\, ds \in D(A)$ for all $0 < t < T$. In the case $x(t)$ is not differentiable on $[0,T]$, it is natural to consider solution (2.15) as a generalized solution to the inhomogeneous IVP (2.14).

Definition 2.1.20: Let A be the infinitesimal generator of a C_0-semigroup $\mathcal{T}(t)$ and $f \in L^1([0,T],\mathcal{H})$.

Then (2.15) is a *mild solution* to the initial value problem (2.14) on $[0, T]$.

The concept of mild solution is the same as the concept of a weak solution in the study of partial differential equations.

Definition 2.1.21: Let $f \in L^1([0, T], \mathcal{H})$. We say that $x(t)$ is a *weak solution* of (2.14) on $[0, T]$ if $x(t)$ is continuous on $[0, T]$ and for all $\phi \in C([0, T], \mathcal{H})$

$$\int_0^T \langle x(t), \phi(t)\rangle \, dt + \int_0^T \langle f(t), g(t)\rangle \, dt + \langle x(0), g(0)\rangle = 0. \tag{2.16}$$

where $g(t) = -\int_t^T \mathcal{T}^*(s-t)\phi(s)\, ds$.

Substituting the expression of $g(t)$ into equation (2.16) we have

$$\begin{aligned}
0 &= \int_0^T \langle x(t), \phi(t)\rangle \, dt - \int_0^T \left\langle f(t), \int_t^T \mathcal{T}^*(s-t)\phi(s)\, ds \right\rangle dt - \left\langle x_0, \int_0^T \mathcal{T}^*(s)\phi(s)\, ds \right\rangle \\
&= \int_0^T \langle x(t), \phi(t)\rangle \, dt - \int_0^T \int_t^T \langle \mathcal{T}(s-t)f(t), \phi(s)\rangle \, ds\, dt - \int_0^T \langle \mathcal{T}(s)x_0, \phi(s)\rangle \, ds. \\
&= \int_0^T \langle x(t), \phi(t)\rangle \, dt - \int_0^T \int_s^T \langle \mathcal{T}(t-s)f(s), \phi(t)\rangle \, dt\, ds - \int_0^T \langle \mathcal{T}(t)x_0, \phi(t)\rangle \, dt
\end{aligned}$$

where the last line was obtained by interchanging the dummy variables s and t in the second term and replacing the dummy variable s by t in the third term. Changing the order of integration of the second we have

$$\begin{aligned}
0 &= \int_0^T \langle x(t), \phi(t)\rangle \, dt - \int_0^T \int_0^t \langle \mathcal{T}(t-s)f(s), \phi(t)\rangle \, ds\, dt - \int_0^T \langle \mathcal{T}(t)x_0, \phi(t)\rangle \, dt \\
&= \int_0^T \left\langle \left(x(t) - \mathcal{T}(t)x_0 - \int_0^t \mathcal{T}(t-s)f(s)\, ds \right), g(t) \right\rangle dt.
\end{aligned}$$

Hence the mild solution given by (2.15) is a weak solution and vice versa.

2.2 Well-Posed Systems

A system is said to be *well-posed* when both the state and output depend continuously on the initial state x_0 and the input u. This statement will be made precise below. We shall describe well-posedness of a linear time-invariant control system under three formulations: well-posed system, state space realization (SSR) and boundary control system (BCS).

A well-posed system is described through four linear maps, $(\mathcal{T}, \mathcal{B}, \mathcal{C}, \mathcal{G})$:

$$x(t) = \mathcal{T}(t)x_0 + \mathcal{B}(t)u(\cdot), \tag{2.17a}$$

$$y(t) = \mathcal{C}(t)x_0 + \mathcal{G}(t)u(\cdot). \tag{2.17b}$$

Any linear time-invariant control system can be described through equation (2.17). Let $\mathcal{H}$, $\mathcal{U}$ and $\mathcal{Y}$ denote the state, input and output space respectively. Then mathematically, well-posed means that for any $t \geq 0$, there exist constants $b_t, c_t > 0$ such that for all $x_0 \in \mathcal{H}$ and $u \in \mathcal{U}$,

$$\|x(t)\|_{\mathcal{H}}^2 \leq b_t \left(\|x_0\|_{\mathcal{H}}^2 + \int_0^t \|u(s)\|_{\mathcal{U}}^2 \, ds \right). \tag{2.18}$$

$$\int_0^t \|y(s)\|_{\mathcal{Y}}^2 \, ds \leq c_t \left(\|x_0\|_{\mathcal{H}}^2 + \int_0^t \|u(s)\|_{\mathcal{U}}^2 \, ds \right). \tag{2.19}$$

We now give the formal definition of a well-posed system with respect to the four linear maps $(\mathcal{T}, \mathcal{B}, \mathcal{C}, \mathcal{G})$.

Definition 2.2.1 : Let $\mathcal{H}$ be a Hilbert space. Suppose $u, v \in L^2([0, \infty), \mathcal{H})$ and let $\tau \geq 0$. We define the *τ-concatenation* of u and v by

$$\left(u \underset{\tau}{\diamond} v \right)(s) = \begin{cases} u(s) & s \in [0, \tau) \\ v(s - \tau) & s \geq \tau. \end{cases}$$

Definition 2.2.2 : Let the input, state and output Hilbert spaces be denoted by $\mathcal{U}$, $\mathcal{H}$, $\mathcal{Y}$. A *well-posed system* on $L^2([0, \infty), \mathcal{U})$, $\mathcal{H}$ and $L^2([0, \infty), \mathcal{Y})$ is a quadruple $\Sigma = (\mathcal{T}, \mathcal{B}, \mathcal{C}, \mathcal{G})$, where

[W0] $\mathcal{T}$ is a C_0-semigroup of bounded linear operators on $\mathcal{H}$.

[W1] $\mathcal{B}$ is a family of bounded linear operators from $L^2([0,\infty),\mathcal{U})$ to $\mathcal{H}$ such that for any $u, v \in L^2([0,\infty),\mathcal{U})$ and $\tau, t \geq 0$

$$\mathcal{B}_{\tau+t}(u \underset{\tau}{\diamond} v) = \mathcal{T}_t\mathcal{B}_\tau u + \mathcal{B}_t v. \tag{2.20}$$

[W2] $\mathcal{C}$ is a family of bounded linear operators from $\mathcal{H}$ to $L^2([0,\infty),\mathcal{Y})$ such that for any $x \in \mathcal{H}$ and $\tau, t \geq 0$

$$\mathcal{C}_{\tau+t}x = \mathcal{C}_\tau x \underset{\tau}{\diamond} \mathcal{C}_t\mathcal{T}_\tau x, \quad \mathcal{C}_0 = 0. \tag{2.21}$$

[W3] $\mathcal{G}$ is a family of bounded linear operators from $L^2([0,\infty),\mathcal{U})$ to $L^2([0,\infty),\mathcal{Y})$ such that for any $u, v \in L^2([0,\infty),\mathcal{U})$ and $\tau, t \geq 0$

$$\mathcal{G}_{\tau+t}(u \underset{\tau}{\diamond} v) = \mathcal{G}_\tau u \underset{\tau}{\diamond} (\mathcal{C}_t\mathcal{B}_\tau u + \mathcal{G}_t v), \quad \mathcal{G}_0 = 0. \tag{2.22}$$

A state space realization for an infinite-dimensional system (on some appropriate spaces to be made precise later) is described through four linear operators , (A, B, C, D):

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \qquad x(0) = x_0, & (2.23a)\\ y(t) &= Cx(t) + Du(t). & (2.23b) \end{aligned}$$

A boundary control system (on some appropriate spaces to be made precise later) is described through three linear operators, (Δ, Γ, K)[1]:

$$\begin{aligned} \dot{z}(t) &= \Delta z(t), \qquad z(0) = z_0, & (2.24a)\\ \Gamma z(t) &= u(t), & (2.24b)\\ y(t) &= Kz(t). & (2.24c) \end{aligned}$$

We shall see that under certain conditions a boundary control system (BCS) can be reformulated as

[1] When no output equation is given, we denote the boundary control system by the pair (Δ, Γ).

a state space realization on an infinite-dimensional state space of a well-posed system. Moreover, any well-posed system has a state-space realization.

Justification of well-posededness is non-trivial even in simple cases. We shall illustrate this point through the one dimensional heat equation with Neumann boundary control. Moreover, we will rewrite it as a boundary control system and give a state-space representation. Finally we are going to show the equivalence of these representations by transforming it from a BCS to a SSR.

For a finite-dimensional system in state space form, we have $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{r\times n}$ and $D \in \Re^{r\times m}$. The input, output and state space are $\Re^m$, $\Re^r$ and $\Re^n$ respectively. The solution to (2.23) is

$$\begin{aligned} x(t) &= \exp(At)x_0 + \int_0^t \exp(A(t-s))Bu(s)\,ds \\ y(t) &= C\left(\exp(At)x_0 + \int_0^t \exp(A(t-s))Bu(s)\,ds\right) + Du(t). \end{aligned}$$

Clearly, the state $x(t)$ and output $y(t)$ depend continuously on the initial state x_0 and the input $u(t)$. For $x_0 \in \Re^n$ and $u(t) = 0$,

$$\begin{aligned} \|x(t)\|_{\Re^n} &\leq \exp(\|A\|t)\|x_0\|_{\Re^n}, \\ \|y(\cdot)\|_{L^2([0,t];\Re^r)} &\leq \|C\|\exp(\|A\|t)\|x_0\|_{\Re^n}. \end{aligned}$$

Thus the mappings from initial state/final state and state/output are bounded. Similarly, for $x_0 = 0$ and $u \in L^2([0,t];\Re^m)$

$$\begin{aligned} \left\|\int_0^t \exp(A(t-s))Bu(s)\,ds\right\|_{\Re^n} &\leq \|B\|\left(\int_0^t \exp(A(t-s)\,ds\right)^{\frac{1}{2}}\|u(\cdot)\|_{L^2([0,t];\Re^m)} \\ &= b_t\|u(\cdot)\|_{L^2([0,t];\Re^m)}. \end{aligned}$$

Set $c_t = \|C\|b_t + \|D\|$ then

$$\|y(\cdot)\|_{L^2([0,t];\Re^r)} \leq c_t\|u(\cdot)\|_{L^2([0,t];\Re^m)}.$$

Thus the input/state map and input/output map are also bounded. Hence any linear time-invariant system on a finite-dimensional state space is well-posed.

For a linear-time invariant system on an infinite-dimensional state space, A, B and C need not be bounded operators in the underlying state space. What conditions are necessary on these operators to ensure that the system is well-posed? For the moment, we shall concentrate on the state equation (2.23a) only.

Let $\mathcal{U}$ and $\mathcal{H}$ denote the input and state space, both assumed to be Hilbert spaces. For $x_0 \in \mathcal{H}$ and $u(t) = 0$, equation (2.23a) is simply a homogeneous IVP. We know from Section 2.1 that if A is the infinitesimal generator of a C_0-semigroup $\mathcal{T}(t)$, then it has solution $x(t) = \mathcal{T}(t)x_0$. Since $\mathcal{T}(t)$ is a bounded linear operator on $\mathcal{H}$, we have

$$\|x(t)\|_{\mathcal{H}} \leq \|\mathcal{T}(t)\|_{\mathcal{H}} \|x_0\|_{\mathcal{H}}.$$

Hence the mapping from initial state to final state is well-defined and bounded if A generates a C_0-semigroup on $\mathcal{H}$. If $B \in \mathcal{L}(\mathcal{U}, \mathcal{H})$ we can write the weak solution to (2.23a) as

$$x(t) = \mathcal{T}(t)x_0 + \int_0^t \mathcal{T}(t-s)Bu(s)\,ds. \tag{2.25}$$

Since $B \in \mathcal{L}(\mathcal{U}, \mathcal{H})$, $\|B\| \leq M_B$ for some constant M_B. Let ω_0 denote the growth bound of $\mathcal{T}(t)$. Choose any positive $\omega > \omega_0$, then there exists some constant M_ω such that $\left\|\mathcal{T}(t)\right\| \leq M_\omega \exp(\omega t)$ for all $t \geq 0$. Setting $x_0 = 0$, we see that for all $u \in L^2([0,t], \mathcal{U})$ we have

$$\begin{aligned}
\left\| \int_0^t \mathcal{T}(t-s)Bu(s)\,ds \right\|_{\mathcal{H}} &\leq \int_0^t M_\omega M_B \exp(\omega(t-s)) \|u(s)\|_{\mathcal{U}}\,ds \\
&\leq M_\omega M_B \left(\int_0^t \exp(2\omega(t-s))\,ds \right)^{\frac{1}{2}} \left(\int_0^t \|u(s)\|_{\mathcal{U}}^2\,ds \right)^{\frac{1}{2}} \\
&\leq b_t \left\| u(\cdot) \right\|_{L^2([0,t],\mathcal{U})},
\end{aligned}$$

for some constant b_t. Hence the input/state map is well-defined and bounded. But we are most interested in problems with boundary control. This generally leads to a state-space realization with $B \notin \mathcal{L}(\mathcal{U}, \mathcal{H})$. So for any $u \in \mathcal{U}$, Bu is not necessarily in $\mathcal{H}$ hence the term $\mathcal{T}(t-s)Bu(s)$ in (2.25) is no longer necessarily well-defined and certainly $\int_0^t \mathcal{T}(t-s)Bu(s)\,ds$ need not be in $\mathcal{H}$. To resolve the first problem, we define an extension of the operator $\mathcal{T}(t)$ to some larger Hilbert space $\mathcal{V}$ so that the term $\mathcal{T}(t-s)Bu(s)$ is now well-defined on $\mathcal{V}$. This extension must be a C_0-semigroup on $\mathcal{V}$. To distinguish between $\mathcal{T}(t)$ and its extension we shall denote the C_0-semigroup and its infinitesimal generator on $\mathcal{H}$ by $\mathcal{T}_{\mathcal{H}}(t)$ and $A_{\mathcal{H}}$. The extended C_0-semigroup and its infinitesimal generator is denoted by $\mathcal{T}_{\mathcal{V}}(t)$ and $A_{\mathcal{V}}$.

Let $\mathcal{V}$ be a Hilbert space such that $B \in \mathcal{L}(\mathcal{U}, \mathcal{V})$ and $\mathcal{H} \hookrightarrow \mathcal{V}$. That is, there exists a bounded linear and injective map $\iota_{\mathcal{V}} : \mathcal{H} \to \mathcal{V}$ and $\mathcal{H}$ is dense in $\mathcal{V}$. The inverse map $i_{\mathcal{V}}$: Range $i_{\mathcal{V}} \to \mathcal{H}$ exists since $i_{\mathcal{V}}$ is injective. So for $x \in \mathcal{H}$, we have

$$\iota_{\mathcal{V}} x = x,$$

where on the right hand side x is viewed as an element in $\mathcal{V}$ and on the left hand side it is seen as an element in $\mathcal{H}$. For clarification, when $x \in \mathcal{H}$ but we wish to view it as an element in $\mathcal{V}$, we will write $\iota_{\mathcal{V}} x$. The C_0-semigroups $\mathcal{T}_{\mathcal{H}}(t), \mathcal{T}_{\mathcal{V}}(t)$ and infinitesimal generators $A_{\mathcal{H}}, A_{\mathcal{V}}$ can now be related as follows:

For $x \in \mathcal{H}$ we have,

$$\iota_{\mathcal{V}} \mathcal{T}_{\mathcal{H}}(t) x = \mathcal{T}_{\mathcal{V}}(t) \iota_{\mathcal{V}} x. \tag{2.26}$$

And for $x \in D(A_{\mathcal{H}})$

$$\iota_{\mathcal{V}} A_{\mathcal{H}} x = A_{\mathcal{V}} \iota_{\mathcal{V}} x. \tag{2.27}$$

By introducing the space $\mathcal{V}$, we can now describe the solution to (2.23a) meaningfully. In particular,

$$x(t) = \mathcal{T}_{\mathcal{V}}(t) \iota_{\mathcal{V}} x_0 + \int_0^t \mathcal{T}_{\mathcal{V}}(t-s) Bu(s)\,ds \tag{2.28}$$

is well-defined in $\mathcal{V}$ and satisfies

$$\dot{x}(t) = A_{\mathcal{V}} x(t) + Bu(t) \qquad x(0) = x_0 \in \mathcal{H}$$

almost everywhere for all $t \geq 0$ in $\mathcal{V}$. However our desired solution state space is $\mathcal{H}$, hence we need the range of $\int_0^t \mathcal{T}_\mathcal{V}(t-s)Bu(s)\,ds$ to lie in $i_\mathcal{V}\mathcal{H}$. If so, then

$$\begin{aligned} x(t) &= \iota_\mathcal{V}^{-1}\iota_\mathcal{V} x(t) \\ &= \iota_\mathcal{V}^{-1}\mathcal{T}_\mathcal{V}(t)\iota_\mathcal{V} x_0 + \iota_\mathcal{V}^{-1}\int_0^t \mathcal{T}_\mathcal{V}(t-s)Bu(s)\,ds \\ &= \mathcal{T}_\mathcal{H}(t)x_0 + \iota_\mathcal{V}^{-1}\int_0^t \mathcal{T}_\mathcal{V}(t-s)Bu(s)\,ds \qquad \text{(by (2.26))}. \end{aligned}$$

If the integral does have range in $\iota_\mathcal{V}\mathcal{H}$ then we usually write

$$x(t) = \mathcal{T}_\mathcal{H}(t)x_0 + \int_0^t \mathcal{T}_\mathcal{V}(t-s)Bu(s)\,ds.$$

Certainly this range condition is not satisfied for all $B \in \mathcal{L}(\mathcal{U},\mathcal{V})$. Thus we have the following definition.

Definition 2.2.3: [Admissible Control Operators] Let $\mathcal{U}$, $\mathcal{H}$ and $\mathcal{V}$ be Hilbert spaces satisfying $\mathcal{H} \hookrightarrow \mathcal{V}$ and $\mathcal{T}(t)$ be a C_0-semigroup on $\mathcal{H}$. The control operator $B \in \mathcal{L}(\mathcal{U},\mathcal{V})$ is said to be an *admissible control operator for* $\mathcal{T}(t)$ if for some (and hence any) $t > 0$ the operator $\mathcal{B}_t : L^2([0,\infty),\mathcal{U}) \to \mathcal{V}$ defined by

$$B_t u := \int_0^t \mathcal{T}_\mathcal{V}(t-s)Bu(s)\,ds$$

has its range in $\mathcal{H}$.

With an admissible B we can assign boundedness assumption on the input/state map:
[S1] $B \in \mathcal{L}(\mathcal{U},\mathcal{V})$ and there exists a constant $b_t > 0$ such that for all $u \in H^1([0,t],\mathcal{U})$:

$$\left\| \int_0^t \mathcal{T}(t-s)Bu(s)\,ds \right\|_\mathcal{H} \leq b_t \left\| u(\cdot) \right\|_{L^2([0,t],\mathcal{U})}. \tag{2.29}$$

We shall now rewrite the heat example 2.1.4 with Neumann boundary control at $x = 1$ (i.e. $z_x(1,t) = u(t)$) in state-space form, and show that condition **[S1]** is satisfied.

Example 2.2.4: [1-D HEAT EQUATION] Consider

$$\begin{aligned}
\frac{\partial z}{\partial t} &= \frac{\partial^2 z}{\partial x^2}, & x &\in [0,1]\\
z(x,0) &= 0, & x &\in [0,1]\\
\frac{\partial z}{\partial x}(0,t) &= 0, & t &> 0\\
\frac{\partial z}{\partial x}(1,t) &= u(t), & t &> 0
\end{aligned}$$

Let $\mathcal{H} = L^2(0,1)$ as before. Define $A = \dfrac{d^2}{dx^2}$ with

$$D(A) = \{\ \phi \in \mathcal{H} \mid \phi', \phi'' \in \mathcal{H}, \phi'(0) = 0, \phi'(1) = 0\ \}.$$

Then A generates a C_0-semigroup on $\mathcal{H}$. (Example (2.1.19)) Moreover, A is self-adjoint. Let $\phi \in D(A^*) = D(A)$ and z be a solution. Then

$$\begin{aligned}
\langle z_t, \phi\rangle &= \langle z_{xx}, \phi\rangle\\
&= z_x\phi\big|_0^1 - \langle z_x, \phi'\rangle\\
&= \phi(1)u(t) - z\phi'\big|_0^1 + \langle z, \phi_{xx}\rangle\\
&= \langle \delta(x-1)u, \phi\rangle + \langle z, A\phi\rangle.\\
&= \langle \delta(x-1)u, \phi\rangle + \langle Az, \phi\rangle.
\end{aligned}$$

holds true for all $\phi \in D(A)$ in the sense of a distribution. If we set $B\phi = \phi(1)$, then $B \notin \mathcal{L}(\mathcal{U}, \mathcal{H})$. Let $[D(A)]$ denote the $D(A)$ equipped with the graph norm

$$\|x\|_{\mathcal{V}}^2 = \|x\|_H^2 + \|Ax\|_H^2.$$

and define $\mathcal{V} = [D(A)]^*$ to be the dual space of $[D(A)]$. Since $[D(A)] \hookrightarrow \mathcal{H}$ this implies that $\mathcal{H} \hookrightarrow \mathcal{V}$. (e.g. [Curtain and Zwart, 1995, Lemma A.3.33]) Thus B is in $\mathcal{L}(\mathcal{U}, \mathcal{V})$

although Bu is not an element of $\mathcal{H}$. The differential equation

$$\dot{z}(t) = A_{\mathcal{V}} z + Bu,$$

is only satisfied on $\mathcal{V}$. The operator $A_{\mathcal{V}}$ takes an element $z \in \mathcal{H}$ and maps it into a linear functional $z' \in \mathcal{V}$. That is, $z' : D(A) \longrightarrow \mathbb{C}$.

We now show that B is an admissible control operator. From Example (2.1.12), the C_0-semigroup can be represented as

$$\mathcal{T}_{\mathcal{H}}(t)z = \sum_{n=0}^{\infty} \exp(\lambda_n t)\langle z, e_n\rangle e_n \quad \forall z \in \mathcal{H}$$

where

$$\begin{aligned} e_n &= \sqrt{2}\cos n\pi x, \\ \lambda_n &= -n^2\pi^2. \end{aligned}$$

The extension of $\mathcal{T}_{\mathcal{H}}$ into $\mathcal{V}$, denoted by $\mathcal{T}_{\mathcal{V}}$, is the C_0-semigroup generated by $A_{\mathcal{V}}$ and it satisfies equation (2.26). Thus

$$\begin{aligned} & \left\| \int_0^t \mathcal{T}_{\mathcal{V}}(t-s)Bu(s)\, ds \right\|_{\mathcal{H}}^2 \\ = & \left\| \int_0^t \sum_{n=0}^{\infty} \exp(\lambda_n(t-s))\langle Bu, e_n\rangle e_n \right\|_{\mathcal{H}}^2 \\ = & \left\langle \int_0^t \sum_{n=0}^{\infty} \exp(\lambda_n(t-s))\langle Bu, e_n\rangle \sqrt{2}\cos \pi x\, ds, \int_0^t \sum_{n=0}^{\infty} \exp(\lambda_n(t-s))\langle Bu, e_n\rangle \sqrt{2}\cos \pi x\, ds \right\rangle_{\mathcal{H}} \\ = & \; 2\sum_{n=0}^{\infty} \left| \int_0^t \exp(\lambda_n(t-s))\langle Bu, e_n\rangle\, ds \right|^2 \\ = & \; 2\sum_{n=0}^{\infty} \left| \int_0^t \exp(\lambda_n(t-s))u(s)\, ds \right|^2 \\ \leq & \; 2\sum_{n=0}^{\infty} \left(\int_0^t \mid \exp(\lambda_n(t-s))u(s) \mid ds \right)^2 \end{aligned}$$

$$
\begin{aligned}
&\leq 2\sum_{n=0}^{\infty}\left\{\left(\int_0^t \mid \exp(\lambda_n(t-s)) \mid^2 ds\right)^{\frac{1}{2}}\left(\int_0^t \mid u(s) \mid^2 ds\right)^{\frac{1}{2}}\right\}^2 \\
&= 2\|u(\cdot)\|^2_{L^2([0,t];\Re)}\sum_{n=0}^{\infty}\int_0^t \mid \exp(\lambda_n(t-s)) \mid^2 ds \\
&\leq 2\|u(\cdot)\|^2_{L^2([0,\infty);\Re)}\sum_{n=0}^{\infty}\int_0^t \exp(2\lambda_n(t-s))\, ds \\
&= 2\|u(\cdot)\|^2_{L^2([0,\infty);\Re)}\sum_{n=0}^{\infty}\frac{-1}{2\lambda_n}\exp(2\lambda_n(t-s))\Big|_0^t \\
&= \|u(\cdot)\|^2_{L^2([0,\infty);\Re)}\sum_{n=0}^{\infty}\frac{(\exp(2\lambda_n t)-1)}{\lambda_n} \\
&= K_t\|u(\cdot)\|^2_{L^2([0,\infty);\Re)},
\end{aligned}
$$

for some constant K_t. So $\int_0^t \mathcal{T}(t-s)Bu(s)\, ds$ is bounded from $L^2([0,\infty),\mathcal{U})$ into $\mathcal{H}$. Hence we can define for all $t \geq 0$, the input/state map $\mathcal{B}_t$ as

$$\mathcal{B}_t u = \int_0^t \mathcal{T}(t-s)Bu(s)\, ds.$$

Again, we must remind ourselves that although the range of $\mathcal{B}_t$ is in $\mathcal{H}$, the integral above is carried out over the space $\mathcal{V}$. ∎

Now let's consider the output equation (2.23b) with $u(t) = 0$ and output space $\mathcal{Y}$. If $C \in \mathcal{L}(\mathcal{H},\mathcal{Y})$ then

$$y(t) = C\mathcal{T}_{\mathcal{H}}(t)x_0$$

is well-defined for all $x_0 \in \mathcal{H}$. Moreover, there exists constant c_t such that for all $x_0 \in \mathcal{H}$,

$$\left\|C\mathcal{T}(\cdot)x\right\|_{L^2([0,t],\mathcal{Y})} \leq c_t\left\|x\right\|_{\mathcal{H}}.$$

For many problems $C \notin \mathcal{L}(\mathcal{H},\mathcal{Y})$, so $C\mathcal{T}_{\mathcal{H}}(t)x_0$ is not necessarily well-defined. To resolve this problem, we define a restriction of the operator $\mathcal{T}_{\mathcal{H}}(t)$, denoted by $\mathcal{T}_{\mathcal{W}}(t)$, to some smaller Hilbert space $\mathcal{W}$ so that $C\mathcal{T}_{\mathcal{W}}(t)x_0$ is well-defined. Hence the restriction operator should also be a C_0-

semigroup on $\mathcal{W}$ and its infinitesimal generator is denoted by $A_{\mathcal{W}}$.

Let $\mathcal{W}$ be a Hilbert space such that $C \in \mathcal{L}(\mathcal{W}, \mathcal{Y})$ and $\mathcal{W} \hookrightarrow \mathcal{H}$. That is, there exists a bounded linear and injective map $\iota_{\mathcal{H}} : \mathcal{W} \to \mathcal{H}$ and $\mathcal{W}$ is dense in $\mathcal{H}$. So for $x \in \mathcal{W}$ we have

$$\iota_{\mathcal{H}} \mathcal{T}_{\mathcal{W}}(t)x = \mathcal{T}_{\mathcal{H}}(t)\iota_{\mathcal{H}}x. \tag{2.30}$$

And for $x \in D(A_{\mathcal{W}})$ we have

$$\iota_{\mathcal{H}} A_{\mathcal{W}} x = A_{\mathcal{H}} \iota_{\mathcal{H}} x. \tag{2.31}$$

For $x \in \mathcal{W}$, we can thus describe the output by

$$y(t) = C\mathcal{T}_{\mathcal{W}}(t)x_0.$$

But we need to be able express the output for any $x_0 \in \mathcal{H}$. This is possible if the following operator $\mathcal{C}_T : \mathcal{W} \to L^2([0,T), \mathcal{Y})$

$$(\mathcal{C}_T x)(t) = \begin{cases} C\mathcal{T}_{\mathcal{W}}(t)x, & t \in [0,T) \\ 0, & t \geq T. \end{cases} \tag{2.32}$$

has a continuous extension to $\mathcal{H}$. This certainly is not satisfied for all C, we thus have the following definition:

Definition 2.2.5: [ADMISSIBLE OBSERVATION OPERATORS] Let $\mathcal{Y}$, $\mathcal{W}$ and $\mathcal{H}$ be Hilbert spaces satisfying $\mathcal{W} \hookrightarrow \mathcal{H}$ and $\mathcal{T}(t)$ be a C_0-semigroup on $\mathcal{H}$. The observation operator $C \in \mathcal{L}(\mathcal{W}, \mathcal{Y})$ is said to be an *admissible observation operator for* $\mathcal{T}(t)$ if for some (and hence any) $t > 0$, the operator $\mathcal{C}_T$ defined by (2.32) has a continuous extension to all of $\mathcal{H}$.

With an admissible C we can assign boundedness assumption on the state/output map:
[S2] $C \in \mathcal{L}(\mathcal{W}, \mathcal{Y})$ and there exists a constant $c_t > 0$ such that for all $x \in \mathcal{W}$:

$$\left\|C\mathcal{T}(\cdot)x\right\|_{L^2([0,t],\mathcal{Y})} \leq c_t \left\|x\right\|_{\mathcal{H}}. \tag{2.33}$$

We now show that Example (2.2.4) with point observation satisfies **[S2]**.

Example 2.2.6: [1-D HEAT EQUATION WITH NEUMANN BOUNDARY CONTROL CONTINUED] Let $\mathcal{H}$, A, e_n and λ_n be as defined in Example (2.2.4). Given $x_1 \in (0,1)$, set $y(t) = z(x_1, t)$, then $C\phi = \phi(x_1)$. Recall that

$$\mathcal{T}_{\mathcal{H}}(t)z = \sum_{n=0}^{\infty} \exp(\lambda_n t)\langle z, e_n\rangle e_n \quad \forall z \in \mathcal{H}$$

and $\mathcal{T}_{\mathcal{W}}(t) = \mathcal{T}_{\mathcal{H}}(t)\Big|_{\mathcal{W}}$.

$$\begin{aligned}
& \|C\mathcal{T}_{\mathcal{W}}(\cdot)w\|^2_{L^2([0,T];\mathcal{Y})} \\
= \;& \langle C\mathcal{T}_{\mathcal{W}}(\cdot)w, C\mathcal{T}_{\mathcal{W}}(\cdot)w\rangle_{L^2([0,T];\mathcal{Y})} \\
= \;& \int_0^T |\, C\mathcal{T}_{\mathcal{W}}(t)w\,|^2\, dt \\
= \;& \int_0^T \left| \sum_{n=0}^{\infty} C\exp(\lambda_n t)\langle w(x), e_n(x)\rangle e_n(x) \right|^2 dt \\
\le \;& 2\int_0^T \left| \sum_{n=0}^{\infty} \exp(\lambda_n t)\langle w(x), e_n(x)\rangle \right|^2 dt \\
\le \;& 2\int_0^T \left(\sum_{n=0}^{\infty} \exp(2\lambda_n t) \right) \left(\sum_{n=0}^{\infty} |\langle w(x) e_n(x)\rangle|^2 \right) dt \\
\le \;& \|w\|^2_{\mathcal{H}} \sum_{n=0}^{\infty} \frac{(\exp(2\lambda_n T) - 1)}{\lambda_n} \\
= \;& K_T \|w\|^2_{\mathcal{H}}
\end{aligned}$$

for some constant K_T. Thus, C is an admissible observation operator. ■

Since $Cx(t)$ is not well-defined for all $x(t)$ equation (2.23b) does not describe the input/output map appropriately. A logical extension to the output equation was given in [Salamon, 1984] as follow: Let $\mu \in \rho(A)$, then

$$x(t) \;=\; (\mu - A)^{-1}(\mu - A)x(t)$$

$$= (\mu - A)^{-1}(\mu x(t) - \dot{x}(t)) + (\mu - A)^{-1}Bu(t).$$

For sufficiently smooth $x(t)$, the first term is in $D(A)$ thus we may apply the operator C to the first term. However $Bu(t) \notin \mathcal{H}$ in general thus we have that $C(\mu - A)^{-1}Bu(t)$ is not well-defined. This suggests defining

$$y(t) = C(\mu - A)^{-1}(\mu x(t) - \dot{x}(t)) + G_\mu u(t), \tag{2.34}$$

for some $G_\mu \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$.

We now state the formal definition of well-posedness with respect to the operators A, B, C and G_μ. This concept was first introduced in [Salamon, 1987].

Definition 2.2.7: [Well-Posedness of State-Space Realizations] Let $\mathcal{W}, \mathcal{H}, \mathcal{V}$ be Hilbert spaces so that $\mathcal{W} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}$. Consider $A \in \mathcal{L}(\mathcal{W}, \mathcal{H})$, $B \in \mathcal{L}(\mathcal{U}, \mathcal{V})$, $C \in \mathcal{L}(\mathcal{W}, \mathcal{Y})$ and $G_\mu \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$. The state-space realization (2.23) on $\mathcal{H}$ is said to be *well-posed* if the following four assumptions hold for all $t \geq 0$:

[S0] A generates a strongly continuous semigroup on all three spaces $\mathcal{W}$, $\mathcal{H}$ and $\mathcal{V}$.

[S1] There exists a constant $b_t > 0$ such that for all $u \in H^1([0, t], \mathcal{U})$:

$$\left\| \int_0^t \mathcal{T}(t-s)Bu(s)\, ds \right\|_{\mathcal{H}} \leq b_t \|u(\cdot)\|_{L^2([0,t],\mathcal{U})}. \tag{2.35}$$

[S2] There exists a constant $c_t > 0$ such that for all $x \in \mathcal{W}$:

$$\|C\mathcal{T}(\cdot)x\|_{L^2([0,t],\mathcal{Y})} \leq c_t \|x\|_{\mathcal{H}}. \tag{2.36}$$

[S3] There exists a constant $g_t > 0$ such that for all $u \in H^2([0, t], \mathcal{U})$ with $u(0) = 0$:

$$\|y(\cdot)\|_{L^2([0,t],\mathcal{Y})} \leq g_t \|u(\cdot)\|_{L^2([0,t],\mathcal{U})}. \tag{2.37}$$

Any well-posed system $(\mathcal{T}, \mathcal{B}, \mathcal{C}, \mathcal{G})$ has a state space realization (A, B, C, G_μ) [Salamon, 1989;

Weiss, 1989a; Weiss, 1989b]. Also, (A, B, C) is a state-space realization of a well-posed system if the realization (A, B, C, G_u) is well-posed.

Aside from **[S0]** there are no restrictions on the choice of $\mathcal{W}$ and $\mathcal{V}$. However it was shown in[Weiss, 1989a] that if **[S1]** holds, then B has an extension to a bounded operator from $\mathcal{U}$ to $\mathcal{V}$. Here $\mathcal{V}^* := [D(A^*_{\mathcal{H}})]$ is the domain of $A^*_{\mathcal{H}}$ equipped with the graph norm, i.e.

$$\|x\|_{D(A^*_{\mathcal{H}})} = \|x\|_{\mathcal{H}} + \|A^*_{\mathcal{H}}x\|_{\mathcal{H}}.$$

$[D(A^*_{\mathcal{H}})]^*$. So may we choose $\mathcal{V} = [D(A^*_{\mathcal{H}})]^*$. In [Weiss, 1989b], it was shown that if **[S2]** holds, then C has a restriction to a bounded operator from $D(A)$ to Y, hence we can set $W = D(A)$ equipped with the graph norm.

The Hilbert spaces $\mathcal{W}$ and $\mathcal{V}$ satisfy $\mathcal{W} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}$. The restriction and extension of $\mathcal{T}_{\mathcal{H}}(t)$ and $A_{\mathcal{H}}$ are defined as follow: Let $\mathcal{T}_{\mathcal{H}}(t)$ be a C_0-semigroup on $\mathcal{H}$ with generator $A_{\mathcal{H}} : D(A_{\mathcal{H}}) \to \mathcal{H}$. Then $\mathcal{T}^*_{\mathcal{H}}(t)$ is a C_0-semigroup on $\mathcal{H}$ with generator $A^*_{\mathcal{H}} : D(A^*_{\mathcal{H}}) \to \mathcal{H}$. Then $\mathcal{V}^* \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}$ and $A^*_{\mathcal{H}}$ is a bounded linear operator from $\mathcal{V}^*$ into $\mathcal{H}$ where $\mathcal{V}^* = [D(A^*_{\mathcal{H}})]$. If we identify the dual of $\mathcal{H}$ with itself, then by duality, $(A^*_{\mathcal{H}})^*$ is a bounded linear operator from $\mathcal{H}$ into $\mathcal{V}$. This dual is identified as an extension of $A_{\mathcal{H}}$ and it is denoted by $A_{\mathcal{V}}$. The C_0-semigroup generated by $A_{\mathcal{V}}$ on $\mathcal{V}$ is denoted by $\mathcal{T}_{\mathcal{V}}(t)$ and is identified as the extension of $\mathcal{T}_{\mathcal{H}}(t)$.

Define $\mathcal{W} = D(A_{\mathcal{H}})$, for $x \in \mathcal{W}, \mathcal{T}_{\mathcal{H}}(t)x \in \mathcal{W}$. We thus define $\mathcal{T}_{\mathcal{W}}(t) = \mathcal{T}_{\mathcal{H}}(t)\Big|_{\mathcal{W}}$. The infinitesimal generator of $\mathcal{T}_{\mathcal{W}}(t)$ is denoted by $\mathcal{A}_{\mathcal{W}}$. So far, we have discussed well-posedness conditions **[S0]**-**[S2]**. Condition **[S3]** which corresponds to boundedness of the input/output map will be discussed in Chapter 3.

2.3 Boundary Control Systems

In this section, we discuss well-posedness of a boundary control system. That is, a system that involves control on the boundaries. The state equation is described through a partial differential equation which can be rewritten into the form $\dot{z}(t) = \Delta z$ for appropriate choice of Δ. The operator Γ can be defined naturally as the operator that maps $z(t)$ to where the boundary condition is applied. Similarly, the operator K can be defined naturally as the operator that maps $z(t)$ to where the output is to be observed. Hence any boundary control system can be described by the triple (Δ, Γ, K). That is,

$$\begin{aligned} \dot{z}(t) &= \Delta z(t), \qquad z(0) = z_0, \\ \Gamma z(t) &= u(t), \\ y(t) &= K z(t). \end{aligned}$$

We denote the state, input and output spaces by $\mathcal{H}$, $\mathcal{U}$ and $\mathcal{Y}$ respectively. It is assumed that $\Delta \in \mathcal{L}(\mathcal{Z}, \mathcal{H})$, $\Gamma \in \mathcal{L}(\mathcal{Z}, \mathcal{U})$ and $K \in \mathcal{L}(\mathcal{Z}, \mathcal{Y})$ where $\mathcal{Z}, \mathcal{H}, \mathcal{U}, \mathcal{Y}$ are Hilbert spaces. We assume that $\mathcal{Z} \subset \mathcal{H}$ with continuous dense injection and the operators Γ and Δ satisfy the following additional assumption:

[B+] Γ is onto, ker Γ is dense in $\mathcal{H}$, there exists a $\mu \in \Re$ such that $\ker(\mu I - \Delta) \cap \ker \Gamma = \{0\}$ and $(\mu I - \Delta)$ is onto.

Later on, we shall show that under certain conditions a boundary control system can be reformulated as a state-space realization and vice versa. We begin by rewriting the 1-D heat example (2.1.4) with Neumann boundary control (Example (2.2.4)) and point observation (Example (2.2.6)) as a boundary control system.

Example 2.3.1: [1-D HEAT EQUATION WITH NEUMANN BOUNDARY CONTROL CONTINUED] Let $\mathcal{H} = L^2(0,1)$, $\mathcal{U} = \mathcal{Y} = \Re$ and set

$$\mathcal{Z} = \{\, \phi \in \mathcal{H} \mid \phi', \phi'' \in \mathcal{H},\ \phi'(0) = 0 \,\},$$

with norm

$$\langle \phi, \phi \rangle_{\mathcal{Z}} = \langle \phi, \phi \rangle_{\mathcal{H}} + \langle \phi', \phi' \rangle_{\mathcal{H}} + \langle \phi'', \phi'' \rangle_{\mathcal{H}}.$$

For $\phi \in \mathcal{Z}$, define $\Delta \phi = \dfrac{d^2\phi}{dx^2}$, $\Gamma \phi = \phi'(1)$ and $K\phi = \phi(x_1)$. Note that $\mathcal{Z} \subset H^2(0,1)$.

Thus,

$$\begin{aligned}
\|\Delta\| &= \sup_{\|\phi\|_{\mathcal{Z}}=1} \|\Delta\phi\|_{\mathcal{H}} = \sup_{\|\phi\|_{\mathcal{Z}}=1} \|\phi''\|_{\mathcal{H}} < \infty, \\
\|\Gamma\| &= \sup_{\|\phi\|_{\mathcal{Z}}=1} \|\Gamma\phi\|_{\Re} = \sup_{\|\phi\|_{\mathcal{Z}}=1} \|\phi'(1)\| < \infty, \\
\|K\| &= \sup_{\|\phi\|_{\mathcal{Z}}=1} \|K\phi\|_{\Re} = \sup_{\|\phi\|_{\mathcal{Z}}=1} \|\phi(1)\| < \infty.
\end{aligned}$$

Hence Δ, Γ and K are bounded linear operators on the desired spaces. ∎

Well-posedness of a boundary control system is defined with respect to the triple (Δ, Γ, K). This concept was introduced by Salamon in [Salamon, 1987].

Definition 2.3.2: [WELL-POSEDNESS OF BCS] The boundary control system (2.24) is said to be *well-posed* if the following set of hypotheses are satisfied.

[B0] For every $z_0 \in \mathcal{Z}$ with $\Gamma z_0 = 0$ there exists a unique solution, $z(t) \in C([0,T],\mathcal{Z}) \cap C'([0,T],\mathcal{H})$, of (2.24) depending continuously on z_0.

[B1] For all $z_0 \in \mathcal{Z}$, and $u(\cdot) \in H^1([0,T],\mathcal{U})$ with $\Gamma z_0 = u(0)$, there exists a unique solution, $z(t) \in C([0,T],\mathcal{Z}) \cap C'([0,T],\mathcal{H})$, of (2.24) depending continuously on z_0 and $u(\cdot)$.

[B2] Assumption **[B0]** is satisfied and there exists a constant $c > 0$ such that

$$\int_0^T \|Kz(t;z_0,0)\|_{\mathcal{Y}}^2\, dt \le c\|z_0\|_{\mathcal{H}}^2,$$

for all $z_0 \in W$ with $\Gamma z_0 = 0$.

[B3] Assumption **[B0]** is satisfied and there exists a constant $c > 0$ such that

$$\int_0^T \|y(t;0,u)\|_{\mathcal{Y}}^2\, dt \le c\int_0^T \|u(t)\|_{\mathcal{U}}^2\, dt,$$

for all $u(\cdot) \in H^2([0,t];\mathcal{U})$.

One sees that hypothesis **[B2]** ensures the state/output map is bounded while **[B3]** ensures the boundedness of the input/output map. Thus they are equivalent to **[W2]** and **[W3]**. Condition **[B0]** assumes uniqueness of solution of the homogeneous IVP and is equivalent to **[W0]**. Finally **[B1]** is equivalent to **[W1]**.

In Section 2.2 and Section 2.3, we have formulated the heat example as a boundary control system and as a state space realization. We know that if the state-space realization is well-posed then it is the realization of some well-posed system $(\mathcal{T}(t), \mathcal{B}(t), \mathcal{C}(t), \mathcal{G}(t))$. It is natural to ask when a boundary control system is representable as a state space realization of a well-posed system and vice versa. The following two theorems show how this can be accomplished.

The first theorem states that if a boundary control system (Δ, Γ, K) satisfies **[B+]**, then we may rewrite the boundary control system as a state-space realization. The second theorem states that the reverse can also be accomplished if an additional condition is satisfied.

Theorem 2.3.3: ([Salamon, 1987])

Suppose the triple (Δ, Γ, K) with Hilbert spaces $\mathcal{Z}$, $\mathcal{H}$, $\mathcal{U}$, $\mathcal{Y}$ defines a boundary control system. Assume that the operators Δ, Γ and K satisfy **[B+]**. *Then the space $\mathcal{W}$ and the operators $A \in \mathcal{L}(\mathcal{W}, \mathcal{H})$ and $C \in \mathcal{L}(\mathcal{W}, \mathcal{Y})$ are given by*

$$\mathcal{W} = \{x \in \mathcal{Z} \mid \Gamma x = 0\} \tag{2.38}$$

$$Ax = \Delta\iota x. \tag{2.39}$$

$$Cx = K\iota x. \tag{2.40}$$

where ι denotes the canonical injection from $\mathcal{W}$ to $\mathcal{Z}$. Furthermore, if we let $\mathcal{V}$ be the dual space of $\mathcal{V}^* = [D(A^*)]$. Then the operator $B \in \mathcal{L}(\mathcal{U}, \mathcal{V})$ can be obtained as follow:

For any given $u \in \mathcal{U}$, choose $x \in \mathcal{Z}$ such that $\Gamma x = u$. (x exists since Γ is onto.) Define

$$Bu = \iota\Delta x - A_{\mathcal{V}}x.$$

Finally, for $\mu \in \rho(A)$ the function $G_\mu \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ is given by

$$G_\mu = K(\mu I - A)^{-1}B. \quad \blacksquare$$

Theorem 2.3.4: ([Salamon, 1987]) *Let the operators (A, B, C, G_μ) with Hilbert spaces $\mathcal{W}$, $\mathcal{H}$, $\mathcal{V}$, $\mathcal{U}$, $\mathcal{Y}$ define a state-space realization of a well-posed system. Assume there exists $\mu \in \Re$ such that $\mu I - A : \mathcal{W} \to \mathcal{H}$ is boundedly invertible. Moreover, assume that B is injective and that*

$$\text{Range}\{B\} \cap \mathcal{H} = \{0\}.$$

Define the Hilbert space

$$\mathcal{Z} = \{x \in \mathcal{H} \mid A_V x \in \mathcal{H} + \text{range } B\}$$

with norm

$$||x||^2_{\mathcal{Z}} = ||x||^2_{\mathcal{H}} + ||u||^2_{\mathcal{U}} + ||Ax + Bu||^2_{\mathcal{H}}$$

where $u \in \mathcal{U}$ is the unique input vector with $Ax + Bu \in \mathcal{H}$. For $x \in \mathcal{Z}$ and $u \in \mathcal{U}$ with $Ax + Bu \in \mathcal{H}$, the boundary control system operators $\Delta \in \mathcal{L}(\mathcal{Z}, \mathcal{H})$, $\Gamma \in \mathcal{L}(\mathcal{Z}, \mathcal{U})$ and $K \in \mathcal{L}(\mathcal{Z}, \mathcal{Y})$ can be obtained as follow:

$$\begin{aligned} \Delta x &= Ax + Bu \\ \Gamma x &= u, \\ Kx &= C(\mu I - A)^{-1}(\mu x - \Delta x) + G_\mu \Gamma x. \end{aligned}$$

$\blacksquare$

Example 2.3.5: [HEAT EQUATION WITH NEUMANN BOUNDARY CONTROL CONTINUED] Let $\Delta, \Gamma, K, \mathcal{H}, \mathcal{Z}$ be as defined in Example 2.3.1. Then by the above theorem

$$\mathcal{W} = \{v \in \mathcal{Z} \mid v'(0) = 0, \quad v'(1) = 0\}.$$

Moreover $A = \dfrac{d^2}{dx^2}$ with $D(A) = W$, and $C\phi = \phi(1)$. It remains to construct the operator B. Let $\phi \in \mathcal{Z}$ such that $\Gamma\phi = u(t)$. Then for any $v \in D(A^*) = D(A)$ we have

$$\begin{aligned}
\langle Bu, v\rangle &= \langle \Delta\phi - A_V\phi, v\rangle \\
&= \langle \phi'', v\rangle - \langle v'', \phi\rangle \\
&= \phi' v\big|_0^1 - v'\phi\big|_0^1 \\
&= \phi'(1)\phi(1) \\
&= u\phi(1) \\
&= \langle \delta(x-1)u, \phi\rangle.
\end{aligned}$$

Hence $B\phi = \phi(1)$. We note that although $Bu \notin \mathcal{H}$, $A_V x + Bu = \Delta x \in \mathcal{H}$. ∎

Theorem 2.3.3 shows that any well-posed boundary control system satisfying **[B+]** can be written as a state-space realization of a well-posed system. Theorem 2.3.4 shows that a state-space realization of a well-posed system can be written as a boundary control system provided the control operator is injective and strictly unbounded. Thus under these additional hypotheses, the well-posedness assumptions **[Bi]** must be equivalent to **[Si]** (**i=0,⋯ 4**). This is summarize in the following theorem.

Theorem 2.3.6: ([Salamon, 1987]) *Let (Δ, K, Γ) be a boundary control system with corresponding equivalent state-space realization (A, B, C, G_μ). Then the boundary control system satisfies hypothesis* **[Bi]** *if and only if the state-space realization (A, B, C, G_μ) satisfies hypothesis* **[Si]** (**i=0,⋯ 4**). ∎

Chapter 3

Input/Output Maps

In Chapter 2 we introduced the concept of well-posedness for infinite-dimensional systems. We demonstrated through a one dimensional heat equation that justification of well-posedness is a non-trivial matter. The input/output map describes the relationship between the inputs and the outputs in the time domain. It is also possible to describe the relationship using the Laplace transform.

Definition 3.0.7: Suppose that an input-output map is given by the convolution of $f(t)$ and $u(t)$, denoted by $y(t) = f(t) * u(t)$. We define the *system transfer function* to be the Laplace transform of $f(t)$, denoted by $F(s)$, when it exists.

The following theorem shows that if the input/output map of a system is bounded then it can be realized by a well-posed system in state-space form, thus we concentrate on the study of boundedness of this map. In particular, we develop sufficient conditions for boundedness of the input/output map for several classes of boundary control systems.

Theorem 3.0.8: ([Salamon, 1989]) *Every bounded, time invariant, causal, linear input-output operator has a well-posed state space realization.* ■

REMARK 3.0.9: *Salamon's proof involves giving an explicit representation for each of the four* *operators A, B, C and G_μ. An alternative proof was given in [*Jacob and Zwart, 1998*]. Their proof makes use of frequency domain analysis.*

The following theorem [Curtain and Weiss, 1989] shows that the boundedness of the input/output map can be examined through the system transfer function.

Theorem 3.0.10: ([Curtain and Weiss, 1989]) *If $\mathcal{U}, \mathcal{H}, \mathcal{Y}$ are Hilbert spaces and (A, B, C) is a triple of operators such that*

[CW1] *A is the generator of a C_0-semigroup $\mathcal{T}$ on $\mathcal{H}$.*

[CW2] *B is an admissible control operator for $\mathcal{T}$.*

[CW3] *C is an admissible observation operator for $\mathcal{T}$.*

then the input/output map of the system is bounded if and only if there exists a real number δ such that the system transfer function, $G(s)$, satisfies

$$\sup_{\text{Re}\, s > \delta} \| G(s) \|_{\mathcal{L}(\mathcal{U}, \mathcal{Y})} < \infty.$$

The function $G(s)$ is said to be **proper** *if the above inequality holds.* ■

For the remainder of this section, we study the meaning of transfer functions. Consider a finite dimensional state-space realization

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \qquad x(0) = x_0, \\ y(t) &= Cx(t), \end{aligned}$$

where A, B and C are constant matrices of appropriate dimensions. (For simplicity we assume the system is strictly proper) The input/output map is well defined and given by

$$y(t) = C \int_0^t \exp(A(t-\sigma)) Bu(\sigma) \, d\sigma. \tag{3.1}$$

Let $f(t) = C\exp(At)B$ then the output is simply the convolution of $f(t)$ and $u(t)$. For $s \in \rho(A)$, $f(t)$ is Laplace transformable. Taking the Laplace transform on both sides of equation (3.1) gives

$$\hat{y}(s) = F(s)\hat{u}(s), \tag{3.2}$$

The function $F(s)$ is the *system transfer function*.

For bounded linear operators B and C, the input-output map is described by

$$y(t) = C\int_0^t \mathcal{T}(t-s)Bu(s)\,ds. \tag{3.3}$$

The output is the convolution of $C\mathcal{T}(t)B$, with the input and the system transfer function is simply the Laplace transform of $C\mathcal{T}(t)B$.

When B and C are unbounded operators, the input-output map stated above is no longer well-defined. From equation (2.34), we know that the output equation can be described by

$$y(t) = C(\mu - A)^{-1}(\mu x(t) - \dot{x}(t)) + G_\mu u(t), \tag{3.4}$$

where $G_\mu \in \mathcal{L}(\mathcal{U},\mathcal{Y})$. If (A,B,C) satisfy **[S0]-[S2]**, it was shown in [Curtain, 1988a] that for $s,\mu \in \rho(A)$, $(sI - A)^{-1}B \in \mathcal{L}(\mathcal{U},\mathcal{H})$ and $C(\mu I - A)^{-1} \in \mathcal{L}(\mathcal{H},\mathcal{Y})$. The operator $(sI - A)^{-1}B$ describes the input/state map in frequency domain and the operator $C(\mu I - A)^{-1}$ describes the state/output map in frequency domain. Hence we may take the Laplace transform from both sides of equation (3.4) to obtain

$$\hat{y}(s) = C(\mu I - A)^{-1}(\mu - s)(sI - A)^{-1}B\hat{u}(s) + G_\mu\hat{u}(s).$$

The system transfer function is defined to be

$$F(s) = C(\mu I - A)^{-1}(\mu - s)(sI - A)^{-1}B + G_\mu.$$

From the above equation we see that if (A, B, C) satisfy the assumptions of Theorem 3.0.10 then $F(s) = G_s$. It was also shown in [Curtain, 1988a] that there exist real constants m_1, m_2 and c such that for all Re $s \geq c$ we have

$$\begin{aligned} \left\|(sI - A)^{-1}B\right\|_{\mathcal{L}(\mathcal{U},\mathcal{H})} &\leq \frac{m_1}{\sqrt{\text{Re } s}}, \\ \left\|C(sI - A)^{-1}\right\|_{\mathcal{L}(\mathcal{H},\mathcal{Y})} &\leq \frac{m_2}{\sqrt{\text{Re } s}}. \end{aligned}$$

Hence there exists a real constant M such that $\|F(s)\|_{\mathcal{L}(\mathcal{U},\mathcal{Y})} \leq M|s|$ for all Re $s > c$, $F(s)$.

Theorem 3.0.11 : ([Zemanian, 1972, Theorem 6.5-1]) *A necessary and sufficient condition for a function $F(s) \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ to be the Laplace transform of a distribution f with support of $f \subset [0, \infty)$ are that there exists some half plane* Re $s \geq c$ *on which F is a $\mathcal{Y}$-valued analytic function and there be a polynomial P for which*

$$\|F(s)\|_{\mathcal{L}(\mathcal{U},\mathcal{Y})} \leq P(|s|) \qquad \text{Re } s \geq c \tag{3.5}$$

where $P(|s|)$ is some polynomial in $|s|$. ■

Since $F(s)$ is the Laplace transform of a distribution $f(t)$, hence $y(t)$ is simply the convolution of this distribution and the input.

In the above argument, we assumed that (A, B, C) satisfy assumptions **[S0]-[S2]**. This results in a transfer function that is well-defined on some half-plane.

The remaining question is then how can we describe the input/output map when we don't know that (A, B, C) satisfy **[S0]-[S2]**. As we shall see later, this is of particular importance since our methodology centers on the boundary control system formulation and not the state-space realization.

Let $L^2_{loc}([0,\infty);\mathcal{U})$ denote the space of locally square-integrable functions on $\mathcal{U}$. For any real σ, define the space $L^2_\sigma([0,\infty);\mathcal{U})$ to be $u \in L^2_{loc}([0,\infty);\mathcal{U})$ such that

$$||u||^2_\sigma = \int_0^\infty \exp(-2\sigma t)||u(t)||^2_{\mathcal{U}}\, dt < \infty.$$

Suppose an input/output map $\mathcal{G}$ is linear, shift-invariant and causal and the output is given by $y = \mathcal{G}u$. If $\mathcal{G}$ is a continuous mapping on $C_0^\infty(\Re^n)$ into $C^\infty(\Re^n)$ then by [Yosida, 1971, Theorem 2] there exists a unique distribution f such that $\mathcal{G}u = f * u$. That is, the input-output map is the convolution of a distribution with its input. If this distribution is Laplace transformable then we define the transfer function of the system to be the Laplace transform of this distribution.

In the next section, we discuss previous results for showing boundedness of the input/output map. In Section 3.2 we give a representation for the system transfer function purely in terms of the boundary control formulation (Theorem 3.2.2). This result justifies taking the formal Laplace transform of the system of differential equations. This method has a few advantages. First, our representation does not require the computation of (A, B, C). Second, this method is particularly useful for boundary control systems in more than one spatial dimension where the system transfer function is hard to obtain. It also has consequences for controller design since the transfer function rather than the state-space representation can be used. Our approach transforms the question of boundedness of the input/output map of a boundary control system to boundedness (in some sense defined later) of the solution to a related elliptic problem (Theorem 3.2.6). In Sections 3.5 - 3.8 we show boundedness of the input/output map for a large class of problems with either Dirichlet, Neumann or Robin boundary control.

3.1 Previous Results

One of the main techniques used in establishing well-posedness conditions uses spectral expansion of the underlying semigroup. This technique is applicable to showing boundedness of input/state, state/output and input/output maps. For example, in [Curtain, 1988b] it was shown that the state/output map of an Euler-Bernoulli beam with velocity sensing of the transverse beam vibrations is bounded. In the same paper, it was also shown that an undamped wave equation with Neumann boundary control and pointwise velocity observation is well-posed. In [Curtain and Weiss, 1989], it was shown that the one dimensional heat equation with Dirichlet boundary control and point observation is well-posed under a suitable choice of state space. In [Morris, 1992], well-posedness of an accelerometer control system was shown. In [Avalos *et al.*, 2000; Avalos *et al.*, 1999] the boundedness of the input/output map of a structural acoustics control system is studied with several different types of observation. Pritchard and Salamon [Pritchard and Salamon, 1987] showed that if the spectral expansion of the control operator B and observation operator C satisfy certain assumptions, then the system

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y = Cx(t)$$

satisfies well-posedness assumptions **[S1]** and **[S2]**. In the same paper, they obtain similar results for the system

$$\ddot{z}(t) = Az(t) + Bu(t), \qquad y = Cz(t).$$

The spectral expansion method requires the availability of the eigenvalues of the system (or at least estimates of them) and also that the corresponding eigenvectors must form a Riesz basis. For multi-dimensional problems it is difficult to calculate the eigenfunctions and eigenvalues of the underlying semigroup. Hence this methodology doesn't seem promising for a large class of problems.

Another method for justifying boundedness of the input/output map is through Theorem 3.0.10. That is, the input/output map is bounded if and only if the system transfer function is proper. In fact, most existing results on boundedness of the input/output map, e.g. [Curtain and Weiss, 1989; Morris, 1992], for boundary control systems do so by showing that the system transfer function is well-defined and bounded in some right-half of the complex-plane. The difficulty is that the transfer function has only been rigorously obtained for a few one-dimensional systems with an explicit Riesz basis. Moreover, the three linear operators (A, B, C) that govern the state-space realization are difficult to obtain.

In [Banks and Morris, 1994], boundedness of the input/output map was shown for a class of structural control systems with point measurement of acceleration by showing that the system transfer function is proper. However, unlike the examples given above, justification of properness for the transfer function was not computed directly. Instead, they show that the infinitesimal generator A generates a uniformly bounded analytic semigroup on some appropriate space hence implying properness of the system transfer function. This result is generalized in Section 3.3.

For completeness we mention that several other authors have used different techniques to study boundedness of the state/output map and input/state map. For more details see for example [Lasiecka and Triggiani, 1991], [Lasiecka and Triggiani, 1999],[Grabowski, 1990], [Grabowski, 1995], [Grabowski and Callier, 1996b] and [Grabowski and Callier, 1996a].

Thus our objective is to derive conditions that guarantee boundedness of the input/output map for a general class of boundary control systems without computing a state space realization

3.2 Boundedness of Input/Output Map

In this section, we present a general technique to obtain the transfer function of a boundary control system and present some preliminary results on properness. We first give a formal result explicitly defining the transfer function in terms of an elliptic problem associated with the boundary control system. Recall that a boundary control system (without the output equation) is described through the double (Δ, Γ) where $\Delta \in \mathcal{L}(\mathcal{Z}, \mathcal{H})$, $\Gamma \in \mathcal{L}(\mathcal{Z}, \mathcal{U})$. The corresponding abstract elliptic problem is defined below:

Definition 3.2.1 : The *abstract elliptic problem* $(\Delta, \Gamma)_e$ corresponding to the boundary control system (Δ, Γ), is

$$\left.\begin{aligned} \Delta z &= sz, \quad s \in \mathbb{C} \\ \Gamma z &= u, \quad u \in \mathcal{U} \end{aligned}\right\} \tag{3.6}$$

where $z(s) \in \mathcal{Z}$ and input $u \in \mathcal{U}$. We denote the solution by $z(s)$.

Let μ indicate the growth bound of the semigroup associated with Δ. The elliptic problem (3.6) has a unique solution $z(s)$ for all u and Re $s > \mu$. In fact, the system transfer function may be described through the solutions to the abstract elliptic problem (3.6).

Theorem 3.2.2: *Let (Δ, Γ, K) define a boundary control system. Suppose Γ is onto and that $\ker \Gamma$ is dense in $\mathcal{H}$. Define $\mathcal{W}$, A and $D(A)$ as in Theorem 2.3.3. Then there exists a $\mu \in \Re$ such that the transfer function, $G(s)$, of the boundary control system (Δ, Γ, K) is given by*

$$G(s)u = Kz(s) \quad \forall s \in \mathbb{C}, \text{ with Re } s > \mu, \tag{3.7}$$

where $z(s)$ is the solution to the abstract elliptic problem (3.6) *with input u.*

Proof: Let μ denote the growth bound of the C_0-semigroup generated by A. By Lemma 2.1.14a, for all $s \in \mathbb{C}$ with Re $s > \mu$, $s \in \rho(A)$. Let ι denote the canonical injection from $\mathcal{W}$ to $\mathcal{Z}$ and define the linear operator $C \in \mathcal{L}(\mathcal{W}, \mathcal{Y})$ by $C = K\iota$. For any given $u \in \mathcal{U}$, choose z so that $\Gamma z = u$. Then $G(s) \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ and by Remark 2.7 in [Salamon, 1987] we have Theorem 2.3.4 it is defined by

$$G(s)\Gamma z = Kz - C(sI - A)^{-1}(sz - \Delta z). \tag{3.8}$$

Now for any $u \in \mathcal{U}$ and any $s \in \mathbb{C}$, with Re $s > \mu$, let z solve the associated elliptic problem. That is, $\Delta z(s) = sz(s)$ and $\Gamma z(s) = u$. From equation (3.8) we have

$$\begin{aligned} G(s)u &= Kz - C(sI - A)^{-1}(sz - \Delta z) \\ &= Kz(s). \end{aligned}$$

This is precisely (3.7). ∎

That is, the solution to (3.7) gives a representation of the transfer function of a boundary control system. The representation of $G(s)$ obtained above is not surprising as the abstract elliptic problem (3.6) is simply the "formal Laplace transform" (with respect to t) of the boundary control system. We say that it is the "formal Laplace transform" since $\Gamma \in \mathcal{L}(\mathcal{Z}, \mathcal{H})$ and $\Delta \in \mathcal{L}(\mathcal{Z}, \mathcal{H})$. Thus it is unclear whether the Laplace transform of Γz is simply $\Gamma \hat{z}$ and $\Delta z = \Delta \hat{z}$. Theorem 3.2.2 is a justification of such a process.

As an example, we compute the transfer function for a 1-D system using equation (3.7).

Example 3.2.3: [1-D HEAT EQUATION WITH NEUMANN BOUNDARY CONTROL CONTINUED] Let the output be temperature measurement at a point x_1, $0 \le x_1 \le 1$. We have shown in Chapter 2 that the input/state map and state/output map is bounded. For ease of reference, we recopy the system equations

$$\left.\begin{aligned} \frac{\partial z}{\partial t} &= \frac{\partial^2 z}{\partial x^2}, & x \in [0,1] \\ z(x,0) &= 0, & x \in [0,1] \\ \frac{\partial z}{\partial x}(0,t) &= 0, & t > 0 \\ \frac{\partial z}{\partial x}(1,t) &= u(t), & t > 0 \\ y(t) &= z(x_1,t). & \end{aligned}\right\} \tag{3.9}$$

The elliptic problem corresponding to (3.9) is

$$\left.\begin{aligned} \frac{d^2 z}{dx^2} &= sz, \\ z'(0) &= 0, \\ z'(1) &= \hat{u}, \end{aligned}\right\} \tag{3.10}$$

with output equation $y = Kz = z(x_1)$. The solution to the abstract elliptic problem is

$$z(x,s) = \frac{\hat{u}\cosh(\sqrt{s}\,x)}{\sqrt{s}\sinh\sqrt{s}}.$$

Using the definition of growth bound, we see that $\mu = 0$. By Theorem 3.2.2 we have for all $s \in \mathbb{C}$ with Re $s > 0$, the transfer function of the system is given by

$$\begin{aligned} G(s)\hat{u} &= K\left(\frac{u\cosh(\sqrt{s}\,x)}{\sqrt{s}\sinh\sqrt{s}}\right) \\ &= \frac{\hat{u}\cosh(\sqrt{s}\,x_1)}{\sqrt{s}\sinh\sqrt{s}}. \end{aligned}$$

This is exactly the transfer function one would obtain by formally taking the Laplace transform of (3.9). Moreover the transfer function is clearly proper hence the input/output map is bounded ∎

The following example shows that if the boundary condition is not chosen correctly, it leads to an improper system transfer function. Hence examining the nature of the input/output map is useful in determining whether the mathematical model of the system is physically reasonable.

Example 3.2.4: [EULER BERNOULLI BEAM WITH KELVIN-VOIGT DAMPING] Consider the Euler-Bernoulli beam with Kelvin-Voigt damping. The beam is assumed to be fixed at $x = 0$ and free at $x = 1$. Then a mathematical model for the motion of the transverse displacement is

$$\left.\begin{array}{rcll} \frac{\partial^2 z}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left[EI\frac{\partial^2 z}{\partial x^2} + c_d I\frac{\partial^3 z}{\partial x^2 t}\right] & = & 0, & x \in (0,1) \\ z(0,t) & = & 0, & t \geq 0 \\ \frac{\partial z}{\partial x}(0,t) & = & 0, & t \geq 0 \\ \frac{\partial^2 z}{\partial x^2}(1,t) & = & 0, & t \geq 0 \\ \frac{\partial^3 z}{\partial x^3}(1,t) & = & u(t), & t \geq 0 \\ y(t) & = & \frac{\partial z}{\partial t}(1,t), & \end{array}\right\} \quad (3.11)$$

where EI and $c_d I$ are physical constants. We shall compute the system transfer function via Theorem 3.2.2. The elliptic problem associated with (3.11) is

$$\left.\begin{array}{rcl} (EI + sc_d I)\frac{\partial^4 z}{\partial x^4} & = & -s^2 z, \\ z(0) & = & 0, \\ z'(0) & = & 0, \\ z''(1) & = & 0, \\ z'''(1) & = & u, \end{array}\right\} \quad (3.12)$$

with output equation $y = Kz(1) = sz(1)$. The solution to the abstract elliptic problem is

$$z(x) = A\cosh(mx) + B\sinh(mx) - A\cos(mx) - B\sin(mx).$$

where

$$
\begin{aligned}
m &= \left(\frac{-s^2}{EI + sc_d I}\right)^{\frac{1}{4}} \\
A &= \frac{-u(\sinh(m) + \sin(m))}{2m^3(1 + \cosh(m)\cos(m))}, \\
B &= \frac{-A\Big(\cosh(m) + \cos(m)\Big)}{\sinh(m) + \sin(m)}.
\end{aligned}
$$

Thus the system transfer function is given by

$$
G(s)u = sz(1) = \frac{s\Big(\sinh(m)\cos(m) - \cosh(m)\sin(m)\Big)u}{m^3\Big(1 + \cosh(m)\cos(m)\Big)}.
$$

One can show that

$$
\begin{aligned}
\lim_{|s|\to\infty} \sinh(m)\cos(m) - \cosh(m)\sin(m) &= c_1 \exp(2|m|) \\
\lim_{|s|\to\infty} 1 + \cosh(m)\cos(m) &= c_2 \exp(2|m|)
\end{aligned}
$$

for some constants c_1 and c_2. So

$$
\lim_{|s|\to\infty} \frac{(\sinh(m)\cos(m) - \cosh(m)\sin(m))}{(1 + \cosh(m)\cos(m))} = \frac{c_1}{c_2} > 0.
$$

Thus $G(s)$ is improper since $\lim_{|s|\to\infty} \frac{s}{m^3} = \infty$.

The boundary conditions imposed on the beam are supposed to reflect the conditions that the moment force is zero at $x = 1$ and the shear force is equal to $u(t)$ at $x = 1$. That is we want,

$$
M = 0, \qquad M_x = 0,
$$

where M denotes the moment (See [Banks *et al.*, 1995] for details).

The original set of boundary conditions are incorrect since the moment M is equal to

$\frac{\partial z^2}{\partial x^2}$ only when there is no damping in the system. The correct boundary conditions are

$$EI\frac{\partial^2 z}{\partial x^2} + c_d I\frac{\partial^3 z}{\partial x^2 \partial t}(1,t) = 0 \qquad t \geq 0,$$
$$EI\frac{\partial^3 z}{\partial x^3} + c_d I\frac{\partial^4 z}{\partial x^3 \partial t}(1,t) = u(t). \qquad t \geq 0$$

With these boundary conditions, the resulting transfer function is

$$G(s)u = sz(1) = \frac{s\Big(\sinh(m)\cos(m) - \cosh(m)\sin(m)\Big)u}{m^3\Big(EI + sc_d I\Big)\Big(1 + \cosh(m)\cos(m)\Big)}.$$

Now $G(s)$ is proper since $\lim_{|s|\to\infty} \frac{s}{m^3(EI + sc_d I)} \to 0.$ ■

For a given observation operator K, the properness of the transfer function depends entirely on the behavior of the solution to $(\Delta, \Gamma)_e$ as s varies. More importantly by Theorem 3.0.10, the boundedness of the input/output map of a boundary control system is dependent entirely on the properness of the system transfer function. The following theorem provides a sufficient condition for the properness of the transfer function of a boundary control system. Consequently, it provides a means of establishing boundedness of the input/output map of a boundary control system.

Definition 3.2.5: Let $(\mathcal{V}, \|\cdot\|_{\mathcal{V}})$ be a normed linear space with $\mathcal{V} \subset \mathcal{H}$. We say that the solution, $z(s)$, to the abstract elliptic problem (3.6) is *uniformly bounded with respect to the $\mathcal{V}$ norm* if there exist constants $\mu_1 \in \Re$ and $M \in \Re^+$ such that

$$\|z(s)\|_{\mathcal{V}} \leq M\|u\|_{\mathcal{U}} \tag{3.13}$$

for all $u \in \mathcal{U}$ and for all $s \in \mathbb{C}$ with Re $s > \mu_1$.

This result is now immediate.

Theorem 3.2.6: [SUFFICIENT CONDITION FOR PROPERNESS OF SYSTEM TRANSFER FUNC-

TION] *Suppose* (Δ, Γ, K) *defines a boundary control system. Let* $\mathcal{V}$ *be a normed linear space satisfying* $\mathcal{Z} \subset \mathcal{V} \subset \mathcal{H}$. *If the solution to* $(\Delta, \Gamma)_e$ *is uniformly bounded with respect to the* $\mathcal{V}$ *norm, then for all observation operators* $K \in \mathcal{L}(\mathcal{V}, \mathcal{Y})$, *the transfer function,* $G(s)$, *associated with the boundary control system* (Δ, Γ, K) *is proper.*

Proof: By assumption there exist constants μ_1 and M such that inequality (3.13) holds. Let A be as defined in Theorem 3.2.2 with growth bound ω_0. Choose $\mu = \max\{\mu_1, \omega_0\}$ and the result follows. ■

Thus, boundedness of the input/output map of a boundary control system can be proven by showing uniform boundedness of the solution, $z(s)$, to a family of elliptic problems. We mention a few advantages of our approach. First, boundedness of the input/output map can be justified without constructing $z(s)$ or the transfer function. Second, Theorem 3.2.6 states that uniform boundedness of the solution to the elliptic problem $(\Delta, \Gamma)_e$ in the $\mathcal{V}$ norm implies boundedness of the input/output map for the class of boundary control systems $\{(\Delta, \Gamma, K) \mid K \in \mathcal{L}(\mathcal{V}, \mathcal{Y})\}$. Third, there exist a large number of results on solutions to elliptic partial differential equations, although not on uniform boundedness of solutions. Finally, we avoid the computation of the linear operators (A, B, C) required in the state space realization formulation.

Example 3.2.7: [1-D HEAT EQUATION WITH NEUMANN BOUNDARY CONTROL CONTINUED] The solution to the corresponding elliptic problem is

$$z(x, s) = \frac{u \cosh(\sqrt{s}\, x)}{\sqrt{s}\, \sinh \sqrt{s}}.$$

Let $\mathcal{V} = H^1(0, 1), \mathcal{U} = \Re$ and $\mu_1 = 1$. Then for all $s \in \mathbb{C}$ with Re $s > 1$ we have

$$\begin{aligned} \|z\|^2_{L^2(0,1)} &\leq \frac{|u|^2 \cosh 2}{16 \sinh 2} + \frac{|u|^2}{8 \sinh^2 2}, \\ \left\|\frac{dz}{dx}\right\|^2_{L^2(0,1)} &\leq \frac{|u|^2 \cosh 2}{2 \sinh 2} + \frac{|u|^2}{2 \sinh^2 2}. \end{aligned}$$

Hence $\|z\|_{H^1(0,1)} \leq \sqrt{\frac{2\cosh 2}{\sinh 2}}|u|$. Thus by Theorem 3.2.6, the input/output map is bounded for all $\mathcal{K} \in \mathcal{L}(\mathcal{H}^1(0, 1), \Re)$. In particular, this holds for $Kz = z(x_1, t)$. ■

We now provide some conditions for uniform boundedness of the solution to $(\Delta,\Gamma)_e$ with respect to $\mathcal{V}$ by rewriting $(\Delta,\Gamma)_e$ as two subproblems.

Proposition 3.2.8 : *Let (Δ,Γ) define a boundary control system and $\mathcal{V}$ be as defined in Theorem 3.2.6. For some real μ and $Re\ s > \mu$, define the problems $(\Delta,\Gamma)_{e1}$ and $(\Delta,\Gamma)_{e2}$ by:*

$$(\Delta,\Gamma)_{e1} := \begin{cases} \Delta f &= 0, \\ \Gamma f &= u. \end{cases} \tag{3.14}$$

$$(\Delta,\Gamma)_{e2} := \begin{cases} \Delta w &= sw + sf, \qquad s \in \mathbb{C} \\ \Gamma w &= 0. \end{cases} \tag{3.15}$$

The solution to $(\Delta,\Gamma)_e$ is uniformly bounded with respect to the $\mathcal{V}$ norm if the following two conditions hold:

[C1] *There exists $f \in \mathcal{V}$ such that f solves $(\Delta,\Gamma)_{e1}$ and*

$$\|f\|_{\mathcal{V}} \leq C_1 \|u\|_{\mathcal{U}}, \tag{3.16}$$

for some positive constant C_1.

[C2] *Let $f \in \mathcal{V}$ denote the solution to $(\Delta,\Gamma)_{e1}$. There exists $w \in \mathcal{V}$ such that w solves $(\Delta,\Gamma)_{e2}$ and*

$$\|w\|_{\mathcal{V}} \leq C_2 \|f\|_{\mathcal{V}} \tag{3.17}$$

for some positive constant C_2, independent of s.

Proof: The result is immediate by noting that $w+f$ solves the original elliptic problem $(\Delta,\Gamma)_e$. ■

3.3 Sesquilinear Forms

In this section we consider problems where the operator A is defined via a coercive sesquilinear form.

Definition 3.3.1 : A sesquilinear form $a : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ is said to be *coercive* if it satisfies the following two conditions: there exist constants $c_1, c_2 > 0$ and $k \geq 0$ such that

[Q1] $|\, a(\phi, \psi)\, | \leq c_1 ||\phi||_{\mathcal{V}} ||\psi||_{\mathcal{V}}, \qquad \forall \phi, \psi \in \mathcal{V},$

[Q2] $\text{Re } a(\phi, \phi) + k||\phi||^2_{\mathcal{H}} \geq c_2 ||\phi||^2_{\mathcal{V}}, \qquad \forall \phi \in \mathcal{V}.$

In particular, we show that given a boundary control system (Δ, Γ), if the solution to the subproblem $(\Delta, \Gamma)_{e_1}$ satisfies **[C1]** of Proposition 3.2.8 in $\mathcal{V}$ norm, then the solution to the abstract elliptic problem corresponding to (Δ, Γ) is uniformly bounded with respect to the $\mathcal{V}$ norm.

Each $a(\cdot, \cdot)$ on $\mathcal{V}$ defines a unique $\mathcal{A} \in \mathcal{L}(\mathcal{V}, \mathcal{V}^*)$ given by

$$(\mathcal{A}\phi)(\cdot) \triangleq -a(\phi, \cdot). \tag{3.18}$$

From Equation (3.18), we define A in $\mathcal{H}$, also known in the literature as the realization of $\mathcal{A}$ in $\mathcal{H}$, by

$$\begin{aligned} D(A) &= \{\phi \in \mathcal{V} \mid \mathcal{A}\phi \in \mathcal{H}\}, \\ A\phi &\triangleq \mathcal{A}\phi \qquad \text{for} \quad \phi \in D(A). \end{aligned} \tag{3.19}$$

It is well-known that A generates an analytic C_0-semigroup in $\mathcal{H}$. Here we establish a bound for A in the $\mathcal{V}$ norm.

Definition 3.3.2 : Let $\mathcal{V}$ be a reflexive Banach space and $\mathcal{H}$ be a Hilbert space. Suppose that $\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}^*$. Then $(\mathcal{V}, \mathcal{H}, \mathcal{V}^*)$ is called a *Gelfand triple*.

Lemma 3.3.3 : *Suppose $(\mathcal{V}, \mathcal{H}, \mathcal{V}^*)$ defines a Gelfand triple. Let $a(\phi, \psi)$ be a sesquilinear form defined on $\mathcal{V} \times \mathcal{V} \to \mathbb{C}$ satisfying* **[Q1]** *and* **[Q2]**. *Define the linear operators $\mathcal{A}$, A as in equations*

(3.18) *and* (3.19). *Then there exists a positive constant* C *such that for all* $\lambda \in \mathbb{C}$ *with* $Re\ \lambda \geq k$ *we have*

$$\| (\lambda I - \mathcal{A})^{-1} g \|_{\mathcal{V}} \leq \frac{C}{|\lambda|} \|g\|_{\mathcal{V}}, \qquad \forall g \in \mathcal{V}. \tag{3.20}$$

Proof: Let $\iota_{\mathcal{V}}$ and $\iota_{\mathcal{H}}$ denote the embeddings from $\mathcal{V} \to \mathcal{H}$ and from $\mathcal{H} \to \mathcal{V}^*$. Denote the inner product on $\mathcal{H}$ by $\langle \cdot , \cdot \rangle_{\mathcal{H}}$. By the definition of $\iota_{\mathcal{H}}$ we have for all $g \in \mathcal{V}$ and $h \in \mathcal{H}$,

$$(\iota_{\mathcal{H}} h)(g) = \langle \iota_{\mathcal{V}} g, h \rangle_{\mathcal{H}}. \tag{3.21}$$

Assume without loss of generality that $\|\phi\|_{\mathcal{V}^*} \leq \|\phi\|_{\mathcal{H}} \leq \|\phi\|_{\mathcal{V}}$. It is well known that the sesquilinear form associated with $\mathcal{A}^*$ is $-a^*(v, \phi) = -\overline{a(\phi, v)}$ and it also satisfies **[H1]** and **[H2]**. By Theorem 8.5 in [Tanabe, 1997] it follows that for any λ with Re $\bar{\lambda}$ = Re $\lambda \geq k$

$$\|(\bar{\lambda} I - \mathcal{A}^*)^{-1} v\|_{\mathcal{V}^*} \leq \frac{1 + c_1}{|\lambda|} \|v\|_{\mathcal{V}^*}, \qquad \forall v \in \mathcal{V}^*$$

The remainder of the proof is similar to [Banks and Morris, 1994, Thm 3.1]. Let $\iota_{\mathcal{N}}$ denote the natural embedding of $\mathcal{V}$ in $\mathcal{V}^{**}$. Note that $\mathcal{A} : \mathcal{V} \to \mathcal{V}^*$ and $\mathcal{A}^* : \mathcal{V}^{**} \to \mathcal{V}^*$. Hence, for all $v \in \mathcal{V}^*$ and $g \in \mathcal{V}$,

$$\left((\bar{\lambda} I - \mathcal{A}^*)^{-1} v\right)(g) = \left(\iota_{\mathcal{H}} \iota_{\mathcal{V}} \iota_{\mathcal{N}}^{-1} (\bar{\lambda} I - \mathcal{A}^*)^{-1} v\right)(g).$$

For any $v \in \mathcal{V}^*$ and $g \in \mathcal{V}$, set $\phi = \iota_{\mathcal{H}} \iota_{\mathcal{V}} g \in \mathcal{V}^*$. Then from the definition of dual operators we have

$$\begin{aligned}
v\left((\lambda I - \mathcal{A})^{-1} g\right) &= v\left((\lambda I - \mathcal{A})^{-1} \phi\right) \\
&= \left((\bar{\lambda} I - \mathcal{A}^*)^{-1}(v)\right)(\phi) \\
&= \phi\left(\iota_{\mathcal{N}}^{-1} (\bar{\lambda} I - \mathcal{A}^*)^{-1} v\right) \\
&= (\iota_{\mathcal{H}} \iota_{\mathcal{V}} g)\left(\iota_{\mathcal{N}}^{-1} (\bar{\lambda} I - \mathcal{A}^*)^{-1} v\right) \\
&= \langle \iota_{\mathcal{V}} g, \iota_{\mathcal{V}} \iota_{\mathcal{N}}^{-1} (\bar{\lambda} I - \mathcal{A}^*)^{-1} v \rangle_{\mathcal{H}} \\
&= \left(\iota_{\mathcal{H}} \iota_{\mathcal{V}} \iota_{\mathcal{N}}^{-1} (\bar{\lambda} - \mathcal{A}^*)^{-1} v\right)(g)
\end{aligned}$$

$$\begin{aligned} &\le \frac{1+c_1}{|\lambda|}\|v\|_{\mathcal{V}^*}\|g\|_{\mathcal{V}} \\ &= \frac{1+c_1}{|\lambda|}\|v\|_{\mathcal{V}^*}\|g\|_{\mathcal{V}}. \end{aligned}$$

Since v is arbitrary we have the desired result. ■

The following result is an immediate consequence of Theorem 3.2.6 and Lemma 3.3.3.

Theorem 3.3.4: *Let $(\mathcal{V}, \mathcal{H}, \mathcal{V}^*)$ define a Gelfand triple. Suppose (Δ, Γ, K) defines a boundary control system where Δ is derived via a coercive sesquilinear form $a : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ and $K \in \mathcal{L}(\mathcal{V}, \mathcal{Y})$. Assume the solution to $(\Delta, \Gamma)_{e_1}$ satisfies inequality (3.16). Then the system transfer function is proper.*

Proof: Write $(\Delta, \Gamma)_e$ as $(\Delta, \Gamma)_{e_1}$ and $(\Delta, \Gamma)_{e_2}$ as in Proposition 3.2.8. The result follows by setting $g = sf$ in Lemma 3.3.3. ■

We can use this result to show that the input/output map of a class of second order systems is bounded.

Example 3.3.5: Consider a second order system in a Hilbert state space $\mathcal{H}$ of the form

$$\left.\begin{aligned} \frac{\partial^2 v}{\partial t^2} + A_D\frac{\partial v}{\partial t} + A_S v(t) &= 0 \qquad x \in \Omega,\quad t > 0, \\ \Gamma_1 v &= u \qquad x \in \partial\Omega \quad t > 0, \end{aligned}\right\} \tag{3.22}$$

where A_D and A_S are assumed to have been derived from the coercive sesquilinear forms $a_D : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ and $a_S : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$. Let $z = \left[v, \frac{\partial v}{\partial t}\right]'$, $\tilde{u} = [u, 0]'$, $\Gamma = [\Gamma_1, 0]'$ and $\Delta = \begin{pmatrix} 0 & I \\ -A_S & -A_D \end{pmatrix}$. We can rewrite (3.22) as a first order system

$$\begin{aligned} \dot{z} &= \Delta z, \qquad x \in \Omega, \quad t > 0 \\ \Gamma v &= \tilde{u}. \qquad x \in \partial\Omega \quad t > 0 \end{aligned}$$

We assume that there exists a constant C_1 such that the solution to $(\Delta, \Gamma)_{e_1}$ satisfies assumption [C1] of Proposition 3.2.8. In [Banks and Morris, 1994] it was shown that Δ is also derived from a coercive sesquilinear form $a : (\mathcal{V} \times \mathcal{V}) \times (\mathcal{V} \times \mathcal{V}) \to \mathbb{C}$ hence by Lemma 3.3.3 there exists a constant C_2 such that the solution to $(\mathcal{A}, \Gamma)_{e_2}$ satisfies assumption **[C2]** of Proposition 3.2.8. Hence the solution to (3.22) is uniformly bounded with respect to the $\mathcal{V}$ norm. So the input/output map is bounded for all observation operators $K \in \mathcal{L}(\mathcal{V}, \mathcal{Y})$.

Many mathematical models of physical problems fit into this framework. One example is the Euler-Bernoulli beam with Kelvin-Voigt damping. Here the operators $A_S\phi = EI\nabla^4\phi$ and $A_D\phi = d_dI\nabla^4\phi$. ∎

3.4 Uniformly Elliptic Boundary Value Problems

In the remaining sections, we shall look at boundedness of solutions to uniformly elliptic boundary value problems. We concentrate on linear second order differential operators. We begin with some background theory and then show that under certain standard assumptions, solutions to uniformly elliptic boundary value problems of order 2 with either Dirichlet, Neumann or Robin boundary control are uniformly bounded. Finally in Section 3.8 we generalize the results for Neumann and Robin boundary control problems to higher order uniformly elliptic operators.

3.4.1 Uniformly Elliptic Operators

In many mathematical models of physical problems the resulting operator $\mathcal{A}$ is uniformly elliptic thus it is important to analyze the boundedness of the input/output map to uniformly elliptic boundary value problems. There exist a large number of results on solutions to uniformly elliptic boundary value problems. Of importance to us are the *a priori* estimates to the solution. Unfortunately, the existing results do not explicitly give conditions on uniform boundedness to the abstract elliptic problem, thus our focus lies on obtaining such results.

The existing estimates theorem generally makes various smoothness assumptions on the coefficients of the differential operator, the coefficients of the boundary operator and the domain of interest. We do the same here:

Let Ω be an open set in $\Re^n$ and let $D' = (D_1, D_2, \ldots, D_n)$ denote the differential operator. A linear second order differential operator in Ω is defined by

$$L(x, D) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x)D_{ij} + \sum_{i=1}^{n} c_i(x)D_i + d(x). \tag{3.23}$$

where the coefficients $a_{ij}(x), c_i(x), d(x)$ are real coefficients. We assume that the coefficients are are sufficiently smooth and that the operator L is uniformly elliptic in Ω. More precisely,

[H1a] (Smoothness Condition 1) The coefficients $a_{ij}(x)$ are bounded and uniformly continuous in $\bar{\Omega}$ and the remaining coefficients are bounded and measurable in Ω.

[H1b] (Uniform Ellipticity)Define the principal part of L by

$$L^0(x, D) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x)D_{ij} = D'A(x)D.$$

where $A(x)$ is an $n \times n$ positive definite matrix with components $a_{ij}(x)$. Then L is *uniformly elliptic in* Ω if there exists a positive constant c_L such that for all $x \in \Omega, \xi \in \Re^n$,

$$L^0(x, \xi) \geq c_L|\xi|^2.$$

Since our analysis is based on the boundary control system formulation, we shall no longer refer to the state-space realization. We shall now use the symbol B to define our boundary operator. This is consistent with convention in the elliptic partial differential equation literature.

The boundary operator B is defined by

$$B(x, D) = b_0(x) + \sum_{i=1}^{n} b_{1i}(x)D_i = b_0(x) + B_1'(x)D. \tag{3.24}$$

where $B_1'(x) = (b_{11}(x), \ldots, b_{1n}(x))$ and $D' = (D_1, \ldots, D_n)$. So $B_1'(x) = 0$ for Dirichlet boundary control and $b_0(x) = 0$ for Neumann boundary control. Let $\partial\Omega$ indicate the boundary of Ω. We impose

[H2] (Smoothness Condition 2) The coefficients of B are real. Also, $b_0(x) \in C^2(\partial\Omega)$ and $b_{1i}(x) \in C^1(\partial\Omega)$, for $i = 1, \ldots, n$.

Estimates of the solution to a uniformly elliptic boundary value problem depend on regularity properties of the region Ω. We use a regularity property known as C^m regularity, a concept first introduced by [Browder, 1961].

Definition 3.4.1: Let Ω be an open set in $\Re^n$ with boundary $\partial\Omega$. Then Ω is said to be *uniformly regular of class* C^m if there exists a family of open sets $\{O_i\}$ of $\Re^n$ and of homeomorphisms $\{\Phi_i\}$ of O_i onto the unit ball $\{y : |y| < 1\}$ in $\Re^n$, an integer N and a constant M such that the following conditions are satisfied:

[UR1] Let $O_i' = \Phi_i^{-1}(\{y \in \Re^n \ : \ |y| < 1/2\})$. Then $\bigcup_{i=1}^{\infty} O_i'$ contains the $1/N$ neighborhood of $\partial\Omega$.

[UR2] For each i,

$$\begin{aligned} \Phi_i(O_i \cap \Omega) &= \{y \ : \ |y| < 1, y_1 > 0\}, \\ \Phi_i(O_i \cap \partial\Omega) &= \{y \ : \ |y| < 1, y_1 = 0\}. \end{aligned}$$

[UR3] Any $(N + 1)$ distinct sets of $\{O_i\}$ have an empty intersection.

[UR4] Let $\Psi_i = \Phi_i^{-1}$. Then Ψ_i, Φ_i are mappings of class C^m. Let Φ_{ik}, Ψ_{ik} be the kth components of Φ_i, Ψ_i respectively. Then

$$|D^\alpha \Phi_{ik}(x)| \le M, \quad |D^\alpha \Psi_{ik}(y)| \le M, \quad |\Phi_{i1}(x)| \le M \mathrm{dist}(x, \partial\Omega)$$

for $|\alpha| =\le m$, $x \in O_i$, $|y| < 1, k = 1, \cdots, n$, and $i = 1, 2, \cdots$ where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $|\alpha| = \sum_i^n \alpha_i$.

In general, it is non-trivial to show that a region is uniformly regular of class C^m. For our work, we are only concerned with bounded sets Ω in $\Re^n$ and cylinders of the form $\Omega \times \Re$ in $\Re^{n+1}$. It was stated without details in [Treves, 1975, p.237] that for bounded sets with sufficiently smooth boundary, there exist mappings $\{\Phi_i\}$ such that **[UR2]** holds. We give a more complete discussion of these points below. If Ω is bounded then there is a finite open cover for the boundary. If the boundary is sufficiently smooth then it is possible to choose a covering such that **[UR1]** and **[UR2]** hold. Conditions **[UR3]** and **[UR4]** hold trivially since the covering is finite. If Ω is bounded with sufficiently smooth boundary, then $\Omega \times \Re$ is also uniformly regular.

As an example, we show that the unit disk in $\Re^2$ is uniformly regular.

Example 3.4.2: Let $U := \{(x_1, x_2) \ : \ x_1^2 + x_2^2 < 1\}$. (See Figure 3.1.)

Consider the point on the boundary of the disk, $(-1, 0)$ and define

$$\Phi_1(x_1, x_2) = (x_1 + \sqrt{1 - x_2^2}, x_2).$$

The inverse mapping is

$$\Phi_1^{-1}(y_1, y_2) = (y_1 - \sqrt{1 - y_2^2}, y_2).$$

Let D be the disc of radius 1. Define $O_1 = \Phi_1^{-1}(D)$. This region is shown in Figure 3.1. We now show that **[UR2]** is satisfied for $i = 1$.

$$\begin{aligned}\Phi_1(O_1 \cap U) &= \Big\{(y_1, y_2)\Big\} \\ &= \left\{\left(x_1 + \sqrt{1 - x_2^2}, x_2\right) \ : \ x_1^2 + x_2^2 < 1 \text{ and } (x_1, x_2) \in O_1\right\}.\end{aligned}$$

Since $(x_1, x_2) \in O_1$, $x_1 < 0$. Thus, $x_1^2 + x_2^2 < 1$ implies that $\sqrt{1 - x_2^2} > |x_1|$. Hence $y_1 > 0$. By a similar argument, for points $(y_1, y_2) \in \Phi_1(O_1 \cap \partial U)$, $y_1 = 0$.

All the remaining O_i's are simply a shift and rotation of O_1. They are constructed so that they overlap as shown in Figure 3.1. Let $R_i(x_1, x_2)$ denote the mapping that shifts and rotates O_i to O_1. Define $\Phi_i(x_1, x_2) = \Phi_1(x_1, x_2)R_i(x_1, x_2)$. We have constructed a finite set of regions 0_i and maps Φ_i so that **[UR1]** and **[UR2]** hold. Since the maps are C^2 and there are only a finite number of them, **[UR3]** and **[UR4]** hold. ■

Theorem 3.4.3: *Let $\Omega \in \Re^n$ be uniformly regular of class C^m. Then $Q = \Omega \times \Re$ is uniformly regular of class C^m.*

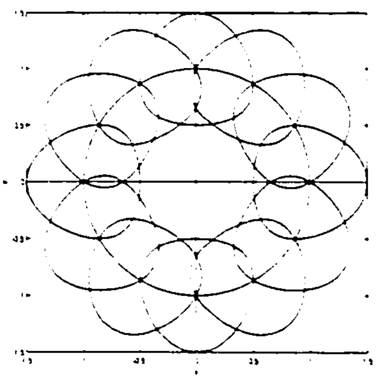


Figure 3.1: Open sets around boundary for unit circle in $\Re^2$

Proof: Let the sets $O_i, i = 1..n$ and maps $o_i, i = 1..n$ be such that **[UR1]**-**[UR4]** are satisfied for Ω. Let the sets O_i' be as in **[UR1]**. Choose $h > 0$ so that the set in $\Re^{n+1}$

$$\{(o_i(x), t), x \in o_i', t \in [-h, h]\}$$

lies inside the disc of radius $\frac{1}{2}$.

For each integer k, define the set $Q_k = \Omega \times [\,-h + kh, h + kh\,]$. Then $Q = \bigcup_{k=-\infty}^{\infty} Q_k$. Note that $(x, t) \in \partial Q_k$ is also a boundary point of Q if and only if $x \in \partial\Omega$. We shall call such boundary points of Q_k *true boundary points*. We see that $\partial Q = \bigcup_{k=-\infty}^{\infty}$ *true boundary points of* Q_k

Consider first $k = 0$ so we have $Q_0 = \Omega \times [-h, h]$ and define $\Phi_{0,i}(x, t) = (o_i(x), t),\ i = 1, \ldots, n$. Then $\Phi_{0,i}^{-1}(y, \tau) = (o_i^{-1}(y), \tau)$. Define $O_{0,i}(x, t) = \Phi_{0,i}^{-1}(D)$ where D is the unit ball in $\Re^{n+1}$. By construction of o_i, and definition of h condition **[UR1]** is satisfied. Condition **[UR2]** is satisfied for the true boundary points of $Q_{0,i}$.

For non-zero k, Q_k is simply a translation of Q_0 along the t-axis. We define the sets $O_{k,i}$ and maps $\Phi_{k,i}$ as for Q_0 and obtain that **[UR1]** is satisfied, and **[UR2]** is satisfied for the true boundary points of Q_k.

Thus, the entire set of sets $O_{k,i}$ and maps $\Phi_{k,i}$ satisfy **[UR1]**,**[UR2]** for Q. Condition **[UR3]** is satisfied for some N since only a finite number of Q_i intersect with any Q_k. Condition **[UR4]** follows since the full set of maps was constructed by shifting a finite set. Hence $Q = \Omega \times \Re$ is uniformly regular of class C^2. ■

In addition to **[H1a]**, **[H1b]** and **[H2]**, we assume throughout, unless stated otherwise, that Ω, L and B also satisfy the following:

[H3] Ω is uniformly regular of class C^2.

[H4] (Root Condition) Let $L^0(x, D)$ denote the principal part of $L(x, D)$. For every pair of linearly independent real vectors ξ and η, the polynomial $L^0(x, \xi + \tau\eta)$ in τ has an equal number of roots with positive and negative imaginary parts.

[H5] (Complementing Condition) Let $B^0(x, D)$ denote the principal part of $B(x, D)$. Let x be an arbitrary point on $\partial\Omega$ and n be the outward normal unit vector to $\partial\Omega$ at x. For each tangential vector $\xi \neq 0$ to $\partial\Omega$ at x, let $\hat{\tau}$ be the root of the polynomial $L^0(x, \xi + \tau n)$ with positive imaginary part. Then $\hat{\tau}$ is not a root of $B^0(x, \xi + \tau n)$.

If $n \geq 3$, then the Root Condition is satisfied for all uniformly elliptic operators [Tanabe, 1997, p130]. If the coefficients of L are real, then the Root Condition is also satisfied when $n = 2$. (see Appendix A for detail)

3.5 Uniformly Elliptic Operators With Dirichlet Boundary Control

Let L be a second order differential operator as defined in Equation (3.23) with $d(x) \leq 0$, $B(x, D) = b_0(x)$ and $\Omega \subset \Re^n$, $n = 1, 2, 3$. We shall show that if Ω, L, B satisfy hypotheses **[H1]**-**[H5]** and Ω satisfies an additional assumption , then the solution to the abstract elliptic problem

$$\left.\begin{aligned} Lz &= sz, && \text{in } \Omega \\ Bz &= u, && \text{on } \partial\Omega \end{aligned}\right\} \qquad (3.25)$$

is uniformly bounded with respect to the $\sup_{x\in\Omega} |\cdot|$ norm. This implies boundedness of the input/output map for the corresponding boundary control system.

The following definition is also due to Browder [Browder, 1961].

Definition 3.5.1: Let Ω be an open set in $\Re^n$. Then Ω is said to be *locally regular of class C^{2m}* if for each point x on the boundary $\partial\Omega$, there exists a neighborhood O_x and a homeomorphism Φ_x of O_x onto the unit ball $\{y : |y| < 1\}$ in $\Re^n$ such that

$$\Phi_x(O_x \cap \Omega) = \{y \ : \ |y| < 1, y_1 > 0\}, \qquad \Phi_x(O_x \cap \partial\Omega) = \{y \ : \ |y| < 1, y_1 = 0\},$$

and so that each component of both Φ_x and Φ_x^{-1} is $2m$ times continuously differentiable.

In addition to uniformly regularity of class C^2, we further assume that

[H6] Ω is locally regular of class C^4.

The following result in [Tanabe, 1997] shows that the solution to the subproblem $(L, B)_{e_2}$ satisfies assumption **[C2]**.

Theorem 3.5.2: *Let $\Omega \subset \Re^n$ be bounded and $F \in C(\bar{\Omega})$. For $z \in H^2(\Omega)$ solving*

$$\begin{aligned} Lz &= sz + F, && \text{in } \Omega \\ z &= 0, && \text{on } \partial\Omega \end{aligned}$$

we have

$$\sup_{x\in\Omega} |z(x)| \leq \frac{C}{|s|} \sup_{x\in\Omega} |F(x)|. \tag{3.26}$$

■

To prove boundedness of the input/output map we also require the Maximum Principle and existence of a solution to $Lf = 0$ with a Dirichlet boundary condition.

Theorem 3.5.3: (**e.g.** [Gilbarg and Trudinger, 1977, Theorem 8.1])
Let $\Omega \subset \Re^n$ be bounded and $f \in H^1(\Omega)$ satisfy $Lf \geq 0$ (≤ 0) in Ω. Then

$$\sup_{x\in\Omega} f(x) \leq \sup_{x\in\partial\Omega} \max\{f(x), 0\}. \qquad \left(\inf_{x\in\Omega} f(x) \geq \inf_{x\in\partial\Omega} \min\{f(x), 0\}\right) \blacksquare$$

The following theorem is a combination of Theorems 8.6, 8.8 and 8.12 in [Gilbarg and Trudinger, 1977].

Theorem 3.5.4: *Let $\Omega \subset \Re^n$ be bounded and L and Ω satisfy assumptions* [**H1**]-[**H6**] *and $u \in H^2(\Omega)$, then there exists a unique $f \in H^2(\Omega)$ that solves*

$$\begin{aligned} Lf &= 0, && \text{in } \Omega \\ f &= u, && \text{on } \partial\Omega \end{aligned}$$

Proof: Theorem 8.6 says that given $u \in H^1(\Omega)$, the partial differential equation

$$\begin{aligned} Lf &= 0, && \text{in } \Omega \\ f &= u, && \text{on } \partial\Omega \end{aligned} \tag{3.27}$$

is uniquely solvable with $f \in H^1(\Omega)$ provided that L is uniformly elliptic in Ω and the coefficients of L is bounded. Theorems 8.8 and 8.12 state that if the coefficients $a_{ij}(x)$ are uniformly Lipschitz and if Ω is of class C^2 then the solution f to equation (3.27) is in $H^2(\Omega)$. Thus these assumptions are weaker than those assumed in [**H1**]-[**H6**], and the result holds. ■

We can now state our main theorem for this section.

Theorem 3.5.5: *Let $\Omega \subset \Re^n$ be bounded and suppose $\{\Omega, L, B\}$ be as defined above and satisfy assumptions* [**H1**]-[**H6**]. *Then the input/output map to the boundary control system* (3.25) *is bounded for all observation operators $K \in \mathcal{L}(C(\Omega), \mathcal{Y})$.*

Proof: Write (L, B) as $(L, B)_{e1}$ and $(L, B)_{e2}$ as in Proposition 3.2.8. Then the solution to the abstract elliptic problem (L, B) is uniformly bounded in the $\sup_{x\in\Omega} |\cdot|$ norm if there exist constants

C_1 and C_2 such that inequalities (3.16) and (3.17) holds.

By Theorem 3.5.4, the subproblem $(L, B)_{e1}$ is uniquely solvable. So there exists f such that $Lf = 0$ $(L(-f) = 0)$ in Ω and $f = u$ on $\partial\Omega$. Hence by Theorem 3.5.3

$$\sup_{x\in\Omega} |f(x)| \leq \sup_{x\in\partial\Omega} |f(x)| = \sup_{x\in\partial\Omega} |u(x)|.$$

Thus inequality (3.16) holds with $C_1 = 1$ and $\mathcal{V} = C(\Omega), \mathcal{U} = C(\partial\Omega)$. The existence of C_2 is evident from inequality (3.26). Therefore by Theorem 3.2.6 the system transfer function associated with (L, B, K) is proper for all observation operators $K \in \mathcal{L}(C(\Omega), \mathcal{Y})$. That is, the input/output map of the boundary control system (L, B, K) is bounded. ■

Let K denote a point observation operator. For a general bounded domain $\Omega \subset \Re^n$, $z \in H^2(\Omega)$ does not necessarily imply that $z \in C(\Omega)$ hence Kz is not well-defined. However, for regions $\Omega \subset \Re^n$ where $n \leq 3$, we may use the Sobolev Imbedding Theorem to show that $z \in C(\Omega)$.

Theorem 3.5.6: (e.g. [Taylor, 1996, Corollary 1.4]) *If $s > n/2 + k$, then $H^s(\Re^n) \subset C^k(\Re^n)$.* ■

For $s = 2, k = 0$ and $n = 1, 2$ or 3, $s > n/2$. Hence $z \in H^2(\Omega)$ implies $z \in C(\Omega)$ thus point evaluation is valid. So the input/output map to the boundary control system (3.25) with point observation is bounded.

3.6 A priori Estimates Theorems

We now list a series of estimate theorems that are needed to prove our main results in the next two sections. These two theorems from Tanabe (for the special case $m = p = 2$) are vital for showing uniform boundedness of solutions to Neumann/Robin boundary control problems.

Theorem 3.6.1: ([Tanabe, 1997, Theorem 4.10]) *Let Ω be uniformly regular of class C^2 and $L(x, D)$, $B(x, D)$ be as defined in Equations (3.23) and (3.24). Suppose that $L(x, D)$, and $B(x, D)$ satisfy assumptions* **[H2]-[H5]**. *Then there exists a positive constant m_1 such that for* *all $z \in H^2(\Omega)$ the following inequality holds:*

$$\|z\|_{H^2(\Omega)} \leq m_1 \left[\|Lz\|_{L^2(\Omega)} + [Bz]_{1/2,\partial\Omega} + \|z\|_{L^2(\Omega)}\right]. \tag{3.28}$$

■

For $u \in H^1(\Omega)$, the norm $[\cdot]_{1/2,\partial\Omega}$ is defined by

$$[u(x)]_{1/2,\partial\Omega} = \inf\{\|z\|_{H^1(\Omega)} : z \in H^1(\Omega), z = u \text{ on } \partial\Omega\}. \tag{3.29}$$

Theorem 3.6.2: ([Tanabe, 1997, Lemma 5.7])) *Let L, B and Ω be as defined above and satisfy assumptions* **[H1]-[H5]**. *Let $\theta \in [-\pi, \pi)$ be fixed but arbitrary and t be a new real variable. Set*

$$\begin{aligned} Q &= \Omega \times \Re, \\ \mathcal{L}_\theta(x, D) = \mathcal{L}_\theta(x, D_x, D_t) &= L(x, D_x) + \exp(i\theta) D_t^2, \end{aligned}$$

and $\mathcal{B}(x, D_x)$ to be the extension of $B(x, D_x)$ to $\partial Q = \partial\Omega \times \Re$.

If $\mathcal{L}_\theta, \mathcal{B}, Q$ also satisfy **[H1]-[H5]** *then there exists a constant M_θ such that for any $z \in H^2(\Omega)$, $u \in H^{1-m_j}(\Omega)$*[1] *satisfying $Bz = u$ on $\partial\Omega$ and any s satisfying $\arg s = \theta$, $|s| > M_\theta$ the following inequality holds:*

$$|s|^{1/2}\|z\|_{H^1(\Omega)} + \|z\|_{H^2(\Omega)} \leq M_\theta \left[\|(L - s)z\|_{L^2(\Omega)} + |s|^{1-m_j/2}\|u\|_{L^2(\Omega)} + \|u\|_{H^{2-m_j}(\Omega)}\right]. \tag{3.30}$$

■

[1] m_j=0 if B is a Dirichlet b.c. and $m_j = 1$ if B is a Neumann or Robin b.c.

3.7 Uniformly Elliptic Operators With Neumann/Robin Boundary Control

Let L and B be as defined in Equation (3.23) and (3.24) with $B_1'(x) \neq 0$. Hence B represents a Neumann boundary control when $b_0(x) = 0$ and a Robin boundary control otherwise. We shall show that if Ω is bounded and $\{\Omega, L, B\}$ satisfy hypotheses **[H1]**-**[H5]**, then the solution to the abstract elliptic problem

$$\left.\begin{array}{rcll} Lz &=& sz & \text{in } \Omega, \\ Bz &=& u & \text{on } \partial\Omega, \end{array}\right\} \tag{3.31}$$

is uniformly bounded with respect to the $H^1(\Omega)$ norm. This implies boundedness of the input/output map for the corresponding boundary control system. For the remaining of this chapter we shall assume that Ω is a bounded subset of $\Re^n$.

For any $\theta \in [-\pi, \pi)$, define $Q, \mathcal{L}_\theta$ and $\mathcal{B}$ by

$$\left.\begin{array}{rrcl} & Q &:=& \Omega \times \Re, \\ & \mathcal{L}_\theta(x, D) = \mathcal{L}_\theta(x, D_x, D_t) &:=& L(x, D_x) + \exp(i\theta) D_t^2, \\ \text{and} & \mathcal{B}(x, D_x) &:=& \text{the extension of } B(x, D_x) \text{ to } \partial Q = \partial\Omega \times \Re. \end{array}\right\} \tag{3.32}$$

From Theorem 3.6.2 we know that given $\theta \in [-\pi, \pi)$, if $\{L, B, \Omega\}$ and $\{\mathcal{L}_\theta, \mathcal{B}, Q\}$ both satisfy **[H1]**-**[H5]**, then there exists a constant M_θ such that the following *a priori* estimate holds for any $z \in H^2(\Omega)$, $u \in H^1(\Omega)$ satisfying $Bz = u$ on $\partial\Omega$ and any s satisfying $\arg s = \theta$, $|s| > M_\theta$:

$$|s|^{1/2} \|z\|_{H^1(\Omega)} + \|z\|_{H^2(\Omega)} \leq M_\theta \left[\|(L - s)z\|_{L^2(\Omega)} + |s|^{1/2} \|u\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)} \right].$$

In particular, if z solves $Lz = sz$ then the above inequality implies

$$\|z\|_{H^1(\Omega)} \leq M_\theta \left(\|u\|_{L^2(\Omega)} + \frac{1}{|s|^{1/2}} \|u\|_{H^1(\Omega)} \right).$$

If in addition $|s| > 1$ then

$$\|z\|_{H^1(\Omega)} \leq 2M_\theta \|u\|_{H^1(\Omega)} .$$

Now by Theorem 3.0.10, the input/output map is bounded if and only if $G(s)$ is bounded on some open right half plane. Thus we seek to prove that for $\theta \in [-\pi/2, \pi/2]$, there exists M_θ such that the *a priori* estimate holds and moreover that this constant can be chosen independent of θ. This will then imply that the solution to (3.31) is uniformly bounded with respect to the H^1 norm and thus the input/output map is bounded for any observation operator $K \in \mathcal{L}(\mathcal{H}^1(\Omega), \mathcal{Y})$.

First we show that Q is uniformly regular of class C^2 and for each $\theta \in [-\pi/2, \pi/2]$, $\mathcal{L}_\theta, \mathcal{B}, Q$ satisfy assumptions **[H1]**,**[H2]**,**[H4]** and **[H5]**. This ensures the existence of $\mathcal{M}_\theta$.

Lemma 3.7.1: *Let $L(x, D_x)$, $B(x, D_x)$ and Ω satisfy assumptions* **[H1]-[H5]**. *For any $\theta \in [-\pi/2, \pi/2]$, define $\mathcal{L}_\theta, \mathcal{B}$ and Q be as in Equation (3.32). Then Q is uniformly regular of class C^2 and $\{\mathcal{L}_\theta, \mathcal{B}\}$ satisfy assumptions* **[H1]**,**[H2]**,**[H4]** *and* **[H5]** *in Q.*

Proof: Since Ω satisfies **[H3]**, Q is uniformly regular.

Next we show that $\mathcal{L}_\theta$ is uniformly elliptic. That is, there exists a positive constant c_1 such that for all $(\xi, \eta) \in \Re^n \times \Re$ and $x \in \Omega$ the following inequality holds:

$$|\mathcal{L}^0_\theta(x, \xi, \eta)| \geq c_1 \left(|\xi|^2 + \eta^2\right) .$$

By assumption, there exists a positive constant c_L such that for all $x \in \Omega, \xi \in \Re^n$,

$$|L^0(x, \xi)| \geq c_L |\xi|^2.$$

Since the matrix A associated with L^0 is positive definite, this means $L^0(x, \xi) \geq 0$ for all $x \in \Omega$ and $\xi \in \Re^n$. Let $c = \min\{c_L^2, 1\}$. Then for any $(x, t) \in \Omega \times \Re$, $(\xi, \eta) \in \Re^n \times \Re$, and $\theta \in [-\pi/2, \pi/2]$ we have

$$\begin{aligned}
\left|\mathcal{L}^0_\theta((x,t),(\xi,\eta))\right|^2 &= |L^0(x,\xi) + \exp(i\theta)\eta^2|^2 \\
&= |L^0(x,\xi)|^2 + 2\cos(\theta) L^0(x,\xi)\eta^2 + \eta^4
\end{aligned}$$

$$\begin{aligned}
&\geq c_L^2|\xi|^4 + \eta^4 \\
&\geq c\left(|\xi|^4 + \eta^4\right) \\
&\geq \frac{c}{2}\left(|\xi|^4 + 2|\xi|^2\eta^2 + \eta^4\right) \\
&= \frac{c}{2}\left(|\xi|^2 + \eta^2\right)^2
\end{aligned}$$

This implies the inequality

$$|\mathcal{L}_\theta^0(x,\xi,\eta)| \geq \sqrt{\frac{c}{2}}\left(|\xi|^2 + \eta^2\right),$$

which proves that $\mathcal{L}$ is uniformly elliptic in Q.

Clearly **[H2]** holds. Also since $n \geq 2$, $n+1 \geq 3$, the Root Condition holds. It remains to show that **[H5]** is satisfied.

Let (x,t) be an arbitrary point on ∂Q, n_1 be the unit outward normal vector to $\partial\Omega$ at x and ξ_1 be any non-zero tangential vector to $\partial\Omega$ at x. The outward normal unit vector to ∂Q at (x,t) is then $n = (n_1', 0)$ and any non-zero tangential vector has the form $\xi = (\xi_1', 0)$. Let $\hat{\tau}$ be a root of $B^0(x,\xi+\tau n)$. Then $\hat{\tau}$ is a root of $B^0(x,\xi_1+\tau n_1)$ which by assumption is not a root of $L^0(x,\xi_1+\tau n_1)$. This implies that

$$\mathcal{L}(x,\xi+\hat{\tau}n) = L(x,\xi_1+\hat{\tau}n_1) + \exp(i\theta)(\xi_2+\hat{\tau}n_2)^2 = L(x,\xi_1+\hat{\tau}n_1) \neq 0.$$

Hence $\hat{\tau}$ is not a root of $\mathcal{L}(x,\xi+\hat{\tau}n)$. So $\{\mathcal{L}_\theta, \mathcal{B}\}$ satisfies **[H5]**. ∎

For each $\theta \in [-\pi/2, \pi/2]$, $\mathcal{L}_\theta, \mathcal{B}, Q$ satisfy **[H1]**,**[H2]**,**[H4]** and**[H5]** thus the hypotheses of Theorem 3.6.2 have been verified. It remains to show that M_θ may be chosen independent of θ in this range. The following lemma is needed to prove this claim.

Lemma 3.7.2 *Let $\mathcal{L}_\theta(x,D)$ be as defined above. Then $\mathcal{L}_\theta$ is continuous in θ. That is, for any $\epsilon > 0$, there exists $\delta > 0$ such that whenever $|\theta_1 - \theta_2| < \delta$, $\theta_1, \theta_2 \in [-\pi/2, \pi/2]$ we have*

$$||\mathcal{L}_{\theta_1}v - \mathcal{L}_{\theta_2}v||_{L^2(Q)} < \epsilon||v||_{H^2(Q)}, \qquad \forall v \in H^2(Q).$$

Proof: For any $0 < \epsilon < \sqrt{2}$, choose $\delta = \arccos\left(1 - \frac{\epsilon^2}{2}\right)$, where arccos denotes the principal branch, then if $|\theta_1 - \theta_2| < \delta$ and $\theta_1, \theta_2 \in [-\pi/2, \pi/2]$ we have

$$\begin{aligned} ||\mathcal{L}_{\theta_1} v - \mathcal{L}_{\theta_2} v||_{L^2(Q)} &\leq |\exp(i\theta_1) - \exp(i\theta_2)|||v||_{H^2(Q)} \\ &= \sqrt{(2 - 2\cos(\theta_1 - \theta_2))}||v||_{H^2(Q)} \\ &= \sqrt{(2 - 2\cos(|\theta_1 - \theta_2|))}||v||_{H^2(Q)}. \end{aligned}$$

Since $\epsilon < \sqrt{2}$, $\delta < \pi/2$ hence the function $f(x) = 2 - 2\cos(x)$ is non-negative and monotone increasing on the interval $[0, \delta]$. Thus

$$\begin{aligned} ||\mathcal{L}_{\theta_1} v - \mathcal{L}_{\theta_2} v||_{L^2(Q)} &< \sqrt{(2 - 2\cos(\delta))}||v||_{H^2(Q)} \\ &= \epsilon||v||_{H^2(Q)}. \end{aligned}$$

For any $\epsilon \geq \sqrt{2}$, choose $\delta = \pi/2$, then if $|\theta_1 - \theta_2| < \pi/2$ and $\theta_1, \theta_2 \in [-\pi/2, \pi/2]$ we have

$$\begin{aligned} ||\mathcal{L}_{\theta_1} v - \mathcal{L}_{\theta_2} v||_{L^2(Q)} &\leq \sqrt{(2 - 2\cos(|\theta_1 - \theta_2|))}||v||_{H^2(Q)} \\ &< \sqrt{2}||v||_{H^2(Q)} \\ &< \epsilon||v||_{H^2(Q)}. \end{aligned}$$

■

Due to Theorem 3.6.1, for each $\theta \in [-\pi/2, \pi/2]$, there exists a constant m_θ such that for any $v \in H^2(Q)$,

$$||v||_{H^2(Q)} \leq m_\theta \left(||\mathcal{L}_\theta v||_{L^2(Q)} + [\mathcal{B}v]_{1/2,\partial Q} + ||v||_{L^2(Q)}\right). \tag{3.33}$$

For each θ, define $m(\theta) = \inf\{m_\theta : \text{inequality (3.33) holds.}\}$. The infimum exists since clearly 1 is a lower bound for m_θ. The next theorem proves that $m(\theta)$ is bounded above.

Theorem 3.7.3 *Let $m(\theta)$ be as defined above. Then $\{m(\theta): -\pi/2 \leq \theta \leq \pi/2\}$ is bounded above. Hence there exists a positive constant $\bar{m}$ such that the following inequality holds for all*

$\theta \in [-\pi/2, \pi/2]$:

$$\|v\|_{H^2(Q)} \leq \bar{m} \left(\|\mathcal{L}_\theta v\|_{L^2(Q)} + [\mathcal{B}v]_{1/2,\partial Q} + \|v\|_{L^2(Q)} \right). \tag{3.34}$$

Proof: Suppose not. Then for each n, there exists $\theta_n \in [-\pi/2, \pi/2]$ such that $m(\theta_n) > n$. The sequence $\{\theta_n\}$ is bounded thus it contains a convergent subsequence $\{\theta_{k_n}\}$ which converges to $\bar{\theta} \in [-\pi/2, \pi/2]$.

Theorem 3.6.1 ensures that $m(\bar{\theta})$ is positive and finite, thus there exists some n such that $m(\bar{\theta}) < n$. Let $\epsilon = \frac{1}{m(\bar{\theta})} - \frac{1}{n} > 0$. By Lemma 3.7.2, there exists $N > n$ such that for all $k_n > N$, (k_n are the indices of the convergent subsequence)

$$\|\mathcal{L}_{\bar{\theta}} v - \mathcal{L}_{\theta_{k_n}} v\|_{L^2(Q)} < \epsilon \|v\|_{H^2(Q)}, \qquad \forall v \in H^2(Q).$$

Pick a k_n such that $m(\theta_{k_n}) - 1 > n$. By definition $m(\theta_{k_n})$ is the smallest constant such that for all $v \in H^2(Q)$, inequality (3.33) holds. Thus there exists some $v_0 \in H^2(Q)$ such that

$$\|v_0\|_{H^2(Q)} > (m_{\theta_{k_n}} - 1) \left(\|\mathcal{L}_{\theta_{k_n}} v_0\|_{L^2(Q)} + [\mathcal{B}v_0]_{1/2,\partial Q} + \|v_0\|_{L^2(Q)} \right).$$

But then

$$\begin{aligned}
\epsilon \|v_0\|_{H^2(Q)} &= \left(\frac{1}{m(\bar{\theta})} - \frac{1}{n} \right) \|v_0\|_{H^2(Q)} \\
&< \left(\frac{1}{m(\bar{\theta})} - \frac{1}{m(\theta_{k_n}) - 1} \right) \|v_0\|_{H^2(Q)} \\
&< \left(\|\mathcal{L}_{\bar{\theta}} v_0\|_{L^2(Q)} + [\mathcal{B}v_0]_{1/2,\partial Q} + \|v_0\|_{L^2(Q)} \right) \\
&\quad - \left(\|\mathcal{L}_{\theta_{k_n}} v_0\|_{L^2(Q)} + [\mathcal{B}v_0]_{1/2,\partial Q} + \|v_0\|_{L^2(Q)} \right) \\
&\leq \|\mathcal{L}_{\bar{\theta}} v_0 - \mathcal{L}_{\theta_{k_n}} v_0\|_{L^2(Q)} \\
&< \epsilon \|v_0\|_{H^2(Q)},
\end{aligned}$$

a contradiction. Thus $m(\theta)$ is bounded above. Let $\bar{m} = \sup\{m(\theta), -\pi/2 \leq \theta \leq \pi/2\}$. Then for any $\theta \in [-\pi/2, \pi/2]$ and $v \in H^2(Q)$, inequality (3.34) holds. ∎

We now state a modification of Theorem 3.6.2.

Theorem 3.7.4: *Let L, B and Ω be as defined above and satisfy assumptions* **[H1]-[H5]**.

Then there exists a positive constant R such that for any $z \in H^2(\Omega)$, $u \in H^1(\Omega)$ satisfying $Bz = u$ on $\partial\Omega$ and any complex number s on the open right half plane $\mathbb{C}_{R^2} := \{s : \text{Re } s > R^2\}$ the following inequality holds:

$$|s|^{1/2}\|z\|_{H^1(\Omega)} + \|z\|_{H^2(\Omega)} \leq m\left[\|(L-s)z\|_{L^2(\Omega)} + |s|^{1/2}\|u\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)}\right]. \quad (3.35)$$

where m is a positive constant dependent only on L and Ω.

Proof: The proof is along the lines given in Tanabe except that we show the constant is independent of θ.

Let ζ be a function in $C^\infty(-\infty, \infty)$ such that $\zeta(t) = 0$ for $|t| > 1$, $\zeta(t) = 1$ for $|t| < 1/2$. Let m_1 be a constant chosen such that $\|\zeta\|_{H^2(\mathbb{R})} \leq m_1$. Let $\bar{m} = \max\{m(\theta), -\pi/2 \leq \theta \leq \pi/2\}$ and $m_2 = \max\{\bar{m}, m_1\}$. Define

$$R := \text{largest root of the quadratic } r^2 - 6m_2^2 r - 6m_2^2.$$

We note that R is necessarily positive and real. In fact $R = \dfrac{6m_2^2 + m_2\sqrt{36m_2^2 + 24}}{2}$. Moreover since $m(\theta)$ is bounded below by 1, $\bar{m}$ and hence m_2 is always greater than 1. Thus $R > 6$.

For any $z \in H^2(\Omega)$ and any $s \in \mathbb{C}_{R^2}$ set $\theta = \arg s$, $r = |s|^{1/2}$ and $v(x,t) = \zeta(t)\exp(irt)z(x)$. Clearly $v \in H^2(Q)$ hence equation (3.34) implies

$$\begin{aligned} \|v\|_{H^2(Q)} &\leq \bar{m}\left(\|\mathcal{L}_\theta v\|_{L^2(Q)} + [\mathcal{B}v]_{1/2,\partial Q} + \|v\|_{L^2(Q)}\right) \\ &\leq m_2\left(\|\mathcal{L}_\theta v\|_{L^2(Q)} + [\mathcal{B}v]_{1/2,\partial Q} + \|v\|_{L^2(Q)}\right). \end{aligned} \quad (3.36)$$

Now a lower bound for $\|v\|_{H^2(Q)}$, an upper bound for $[\mathcal{B}v]_{1/2,\partial Q}$ and an upper bound for $\|\mathcal{L}_\theta v\|_{L^2(Q)}$ need to be computed. The final inequality is then obtained via simple algebra.

First we compute a lower bound for $\|v\|_{H^2(Q)}$. By definition of $\|\cdot\|_{H^2(Q)}$ we have

$$\begin{aligned}
\|v\|^2_{H^2(Q)} &= \sum_{|\alpha|+k\le 2} \int_{-\infty}^{\infty}\int_\Omega \left|D_x^\alpha D_t^k v(x,t)\right|^2 dxdt \\
&\ge \sum_{|\alpha|+k\le 2} \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_\Omega \left|D_x^\alpha D_t^k exp(irt)z(x)\right|^2 dxdt \\
&= \sum_{k=0}^{2} (r)^{2k} \sum_{|\alpha|+k\le 2} \int_\Omega \left|D_x^\alpha z(x)\right|^2 dx \\
&= \sum_{k=0}^{2} (r)^{2k} \|z\|^2_{H^{2-k}(\Omega)} \\
&\ge (r)^{2k} \|z\|^2_{H^{2-k}(\Omega)}.
\end{aligned}$$

for any $k = 0, 1, 2$. Hence

$$\|v\|_{H^2(Q)} \ge (r)^k \|z\|_{H^{2-k}(\Omega)}.$$

for any $k = 0, 1, 2$.

Thus

$$3\|v\|_{H^2(Q)} \ge \sum_{k=0}^{2} (r)^k \|z\|_{H^{2-k}(\Omega)}. \tag{3.37}$$

Next we compute an upper bound for $[\mathcal{B}v]_{1/2,\partial Q}$. By definition of $[\cdot]_{1/2,\partial\Omega}$ we have for $Bz \in H^2(\Omega)$ such that $z = u$ on $\partial\Omega$.

$$\begin{aligned}
[\mathcal{B}v]^2_{1/2,\partial Q} &= [\zeta(t)\exp(irt)Bz(x)]^2_{1/2,\partial Q} \\
&= [\zeta(t)\exp(irt)u]^2_{1/2,\partial Q} \\
&\le \|\zeta(t)\exp(irt)u\|^2_{H^1(Q)} \\
&= \sum_{|\alpha|+k\le 1} \int_{-\infty}^{\infty}\int_\Omega \left|D_x^\alpha D_t^k \zeta(t)\exp(irt)u\right|^2 dxdt \\
&= \int_{-\infty}^{\infty}\int_\Omega |\zeta(t)\exp(irt)u|^2\, dxdt + \int_{-\infty}^{\infty}\int_\Omega |\zeta(t)\exp(irt)Du|^2\, dxdt \\
&\quad + \int_{-\infty}^{\infty}\int_\Omega |\zeta'(t)\exp(irt)u + ir\zeta(t)\exp(irt)u|^2\, dxdt \\
&\le m_1^2\|u\|^2_{L^2(\Omega)} + m_1^2\|Du\|^2_{L^2(\Omega)} + m_1^2\|u\|^2_{L^2(\Omega)} + 2rm_1^2\|u\|^2_{L^2(\Omega)} + r^2m_1^2\|u\|^2_{L^2(\Omega)}
\end{aligned}$$

$$= 2m_1^2 \|u\|_{L^2(\Omega)}^2 + m_1^2 \|Du\|_{L^2(\Omega)}^2 + (2r + r^2)m_1^2 \|u\|_{L^2(\Omega)}^2 .$$

Since $r = |s|^{1/2} > R > 6$, $2r < r^2$. Hence

$$\begin{aligned}
[\mathcal{B}v]_{1/2,\partial Q}^2 &\leq 2m_1^2 \left(\|u\|_{H^1(\Omega)}^2 + r^2 \|u\|_{L^2(\Omega)}^2 \right) \\
&\leq 2m_1^2 \left(r \|u\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)} \right)^2 \\
&\leq 2m_2^2 \left(r \|u\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)} \right)^2 .
\end{aligned}$$

Thus

$$[\mathcal{B}v]_{1/2,\partial Q} \leq \sqrt{2} m_2 \left(r \|u\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)} \right) . \tag{3.38}$$

This is the upper bound on $[\mathcal{B}v]_{1/2,\partial Q}$. Now we calculate an upper bound on $\mathcal{L}_\theta v$.

Substituting the expression for $v(x,t)$ into $\mathcal{L}_\theta v$, we find

$$\mathcal{L}_\theta v = \zeta(t) \exp(irt)(L - r^2 \exp(i\theta))z + 2ir \exp(i\theta)\zeta'(t) \exp(irt)z + \exp(i\theta)\zeta''(t) \exp(irt)z.$$

Therefore

$$\begin{aligned}
\|\mathcal{L}_\theta v\|_{L^2(Q)} &\leq \left\| \zeta(t) \exp(irt)(L - r^2 \exp(i\theta))z \right\|_{L^2(Q)} + 2 \|r \exp(i\theta)\zeta'(t) \exp(irt)z\|_{L^2(Q)} \\
&\quad + \|\exp(i\theta)\zeta''(t) \exp(irt)z\|_{L^2(Q)} \\
&\leq m_1 \left(\left\| (L - r^2 \exp(i\theta))z \right\|_{L^2(\Omega)} + 2r \|z\|_{L^2(\Omega)} + \|z\|_{L^2(\Omega)} \right) \\
&\leq m_2 \left(\left\| (L - r^2 \exp(i\theta))z \right\|_{L^2(\Omega)} + 2r \|z\|_{L^2(\Omega)} + \|z\|_{L^2(\Omega)} \right) .
\end{aligned} \tag{3.39}$$

Also,

$$\|v\|_{L^2(Q)} \leq m_2 \|z\|_{L^2(\Omega)} . \tag{3.40}$$

Substituting inequality (3.37) into (3.36) we obtain,

$$r^2 \|z\|_{L^2(\Omega)} + r \|z\|_{H^1(\Omega)} + \|z\|_{H^2(\Omega)} \leq 3m_2 \left(\|\mathcal{L}_\theta v\|_{L^2(Q)} + [\mathcal{B}v]_{1/2,\partial Q} + \|v\|_{L^2(Q)} \right) . \tag{3.41}$$

Next, substitute inequalities (3.38), (3.39) and (3.40) into inequality (3.41) obtain

$$\begin{aligned} r^2 \|z\|_{L^2(\Omega)} &+ r\|z\|_{H^1(\Omega)} + \|z\|_{H^2(\Omega)} \\ &\leq 3m_2^{\frac{3}{2}} \Big(\|(L - r^2\exp(i\theta))z\|_{L^2(\Omega)} + 2r\|z\|_{L^2(\Omega)} + \|z\|_{L^2(\Omega)} \\ &+ \sqrt{2}r\|u\|_{L^2(\Omega)} + \sqrt{2}\|u\|_{H^1(\Omega)} + \|z\|_{L^2(\Omega)} \Big). \end{aligned} \tag{3.42}$$

After rearrangement we obtain

$$\begin{aligned} (r^2 &- 6m_2^{\frac{3}{2}}r - 6m_2^{\frac{3}{2}})\|z\|_{L^2(\Omega)} + r\|z\|_{H^1(\Omega)} + \|z\|_{H^2(\Omega)} \\ &\leq 3\sqrt{2}m_2^{\frac{3}{2}} \Big(\|(L - r^2\exp(i\theta))z\|_{L^2(\Omega)} + +r\|u\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)} \Big). \end{aligned} \tag{3.43}$$

By definition of R we have $r^2 - 6m_2^{\frac{3}{2}}r - 6m_2^{\frac{3}{2}} \geq 0$. Hence Equation (3.43) implies

$$r\|z\|_{H^1(\Omega)} + \|z\|_{H^2(\Omega)} \leq 3\sqrt{2}m_2^{\frac{3}{2}} \Big(\|(L - r^2\exp(i\theta))z\|_{L^2(\Omega)} + r\|u\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)} \Big). \tag{3.44}$$

Substituting back $s = r^2\exp(i\theta)$ above and defining $m = 3\sqrt{2}m_2^{\frac{3}{2}}$, we have the desired result. ■

The boundedness of the input/output map for

$$\left.\begin{aligned} \frac{\partial z}{\partial t} &= Lz & x \in \Omega,\quad t > 0, \\ Bz &= u & x \in \partial\Omega,\quad t > 0, \\ y &= Kz. & \end{aligned}\right\} \tag{3.45}$$

is now immediate.

Corollary 3.7.5 : *Let L, Ω and B be defined as above and satisfy* **[H1]-[H5]**. *Then the input/output map to the boundary control system* (3.45) *is bounded for all observation operators* $K \in \mathcal{L}(H^1(\Omega), \mathcal{Y})$.

Proof: By Theorem 3.7.4, the solution to the abstract elliptic problem (L, B) is uniformly bounded with respect to the $H^1(\Omega)$ norm. Hence by Theorem 3.2.6, the system transfer function associated with (L, B, K) is proper for all observation operators $K \in \mathcal{L}(H^1(\Omega), \mathcal{Y})$. Thus by Theorem 3.0.10

the input/output map to the boundary control system (L, B, K) is bounded. ■

REMARK 3.7.6 We note that if B is Dirichlet boundary control then $m_j = 0$. Using the same technique as Theorem 3.7.4 we can show that there exists a positive constant R such that for any $z \in H^2(\Omega)$, $u \in H^2(\Omega)$ satisfying $Bz = u$ on $\partial\Omega$ and any complex number s on the open right half plane $\mathbb{C}_{R^2} := \{s : \text{Re } s > R^2\}$ the following inequality holds:

$$|s|^{1/2} \|z\|_{H^1(\Omega)} + \|z\|_{H^2(\Omega)} \leq m \left[\|(L - s)z\|_{L^2(\Omega)} + |s| \, \|u\|_{L^2(\Omega)} + \|u\|_{H^2(\Omega)} \right].$$

where m is a positive constant dependent only on L and Ω. Unfortunately this only implies that the solution to $Lz = sz$ in Ω and $Bz = u$ on $\partial\Omega$ satisfies

$$\|z\|_{H^1(\Omega)} \leq m|s|^{1/2} \|u\|_{H^2(\Omega)}.$$

So we cannot conclude that the solution is uniformly bounded in the H^1 norm.

3.8 Higher Order Uniformly Elliptic Operators with Neumann/Robin Boundary Control

The results in Section 3.7 can be generalized to higher order uniformly elliptic operators with appropriate adjustments to assumptions **[H1]-[H5]**.

Let Ω be a bounded open set in $\Re^n$ and L be a linear differential operator of order m, m even, in Ω defined by:

$$L(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha, \tag{3.46}$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $D^\alpha = (D_1^{\alpha_1}, D_2^{\alpha_2}, \ldots, D_n^{\alpha_n})$. Here, m takes the form $4k$ or $4k + 2$

where k a natural number. Define the $m/2$ boundary operator by

$$B_j(x, D) = \sum_{|\beta| \leq j} b_{j\beta}(x) D^{\beta}, \qquad j = 1, \ldots, m/2 \tag{3.47}$$

[HH1a] (Smoothness Condition 1) The coefficients a_m are bounded and uniformly continuous in $\bar{\Omega}$ and the remaining coefficients are bounded and measurable in Ω.

[HH1b] (Uniform Ellipticity) Let L^0 denote the principal part of L. There exists a positive constant c_L such that for all $x \in \Omega, \xi \in \Re^n$,

$$L^0(x, \xi) \geq c_L |\xi|^m.$$

[HH2] (Smoothness Condition 2) Let $B_j(x, D)$ be given by Equation (3.47) with real coefficients. For $|\beta| \leq j$, $j = 1, \ldots, m/2$, $b_{j\beta}(x) \in C^{m-j}(\partial\Omega)$ and all its derivatives of order up to $m - j$ are all bounded and uniformly continuous on $\partial\Omega$.

[HH3] Ω is uniformly regular of class C^m.

[HH4] (Root Condition) For every pair of linearly independent real vectors ξ and η, the polynomial, $L^0(x, \xi + \tau\eta)$, in τ has equal number of roots with positive and negative imaginary parts.

[HH5] (Complementing Condition) For each $j = 1, \ldots, m/2$, let $B_j^0(x, D)$ denote the principal part of $B_j(x, D)$. Let x be an arbitrary point on $\partial\Omega$ and n be the outward normal unit vector to $\partial\Omega$ at x. For each tangential vector $\xi \neq 0$ to $\partial\Omega$ at x, let $\tau_1(x, \xi), \ldots, \tau_{m/2}(x, \xi)$ be the roots of the polynomial $L^0(x, \xi + \tau n)$ with positive imaginary part. Then a linear combination of $\{B_j^0(x, \xi + \tau n)\}_{j=1}^{m/2}$ is divisible by $\Pi_{j=1}^{m/2}(\tau - \tau_j(x, \xi))$ if and only if all the coefficients vanish.

REMARK 3.8.1 Since L satisfies the Root condition, the order of m is necessarily even.

To prove our result we rely on a more general case of Theorems 3.6.1 and 3.6.2.

Theorem 3.8.2: ([Tanabe, 1997, Theorem 4.10], $p = 2$) *Let Ω be uniformly regular of class C^m and $L(x, D)$, $B_j(x, D)$ be as defined in Equations (3.46) and (3.47). Suppose that $L(x, D)$, and*

$B_j(x, D)$ satisfy assumptions **[HH1]-[HH5]**. *Then there exists a positive constant m_1 such that for all $z \in H^m(\Omega)$ the following inequality holds:*

$$\|z\|_{H^m(\Omega)} \leq m\left[\|Lz\|_{L^2(\Omega)} + \sum_{j=1}^{m/2}[B_j z]_{m-j-1/2,\partial\Omega} + \|z\|_{L^2(\Omega)}\right]. \tag{3.48}$$

■

For $u \in H^q(\Omega)$, the norm $[\cdot]_{q-1/2,\partial\Omega}$ is defined by

$$[u(x)]_{q-1/2,\partial\Omega} = \inf\{\|z\|_{H^q(\Omega)} : z \in H^q(\Omega), z = u \text{ on } \partial\Omega\}. \tag{3.49}$$

Theorem 3.8.3: ([Tanabe, 1997, Lemma 5.7]), $p = 2$) *Let L, B_j and Ω be as defined above and satisfy assumptions* **[HH1]-[HH5]**. *Let $\theta \in [-\pi, \pi)$ be fixed but arbitrary and t be a new real variable. Define $\mathcal{L}_\theta$, $\mathcal{B}_j$ and Q by*

$$\left.\begin{aligned} Q &= \Omega \times \Re, \\ \mathcal{L}_\theta(x, D) = \mathcal{L}_\theta(x, D_x, D_t) &= L(x, D_x) - (-1)^{m/2}\exp(i\theta)D_t^m, \\ \mathcal{B}_j(x, D_x) &= \text{extension of } B_j(x, D_x) \text{ to } \partial Q = \partial\Omega \times \Re. \end{aligned}\right\} \tag{3.50}$$

Suppose $\mathcal{L}_\theta$, $\{\mathcal{B}_j\}_{j=1}^{m/2}$, Q also satisfy **[HH1]-[HH5]**. *Then there exists a constant M_θ such that for any $z \in H^m(\Omega)$, $u_j \in H^{m-j}(\Omega), j = 1, \ldots, m/2$, satisfying $B_j z = u_j$ on $\partial\Omega$ and any s satisfying $\arg s = \theta$, $|s| > M_\theta$ the following inequality holds:*

$$\sum_{j=0}^{m}|s|^{\left(1-\frac{j}{m}\right)}\|z\|_{H^j(\Omega)} \leq M_\theta\left[\|(L-s)z\|_{L^2(\Omega)} + \sum_{j=1}^{m/2}|s|^{\left(1-\frac{j}{m}\right)}\|u_j\|_{L^2(\Omega)} + \sum_{j=1}^{m/2}\|u_j\|_{H^{m-j}(\Omega)}\right]. \tag{3.51}$$

■

Lemma 3.8.4: *Let $L(x, D_x), B_j(x, D_x)$ and Ω satisfy assumptions* **[HH1]-[HH5]** *and $\mathcal{L}, \mathcal{B}_j$*

and Q be as defined in Equation (3.50). Then Q is uniformly regular of class C^m. Moreover

(i) *If $m = 4k$, then for each $\theta \in [-\pi, -\pi/2] \cup [\pi/2, \pi]$, $\{\mathcal{L}_\theta, \{\mathcal{B}_j\}_{j=1}^{2k}\}$ satisfy assumptions* **[HH1]**,**[HH2]**,**[HH4]** *and* **[HH5]** *in Q.*

(ii) *If $m = 4k+2$, then for each $\theta \in [-\pi/2, \pi/2]$, $\{\mathcal{L}_\theta, \{\mathcal{B}_j\}_{j=1}^{2k+1}\}$ satisfy assumptions* **[HH1]**,**[HH2]**,**[HH4]** *and* **[HH5]** *in Q.*

Proof: Since Ω satisfies **[HH3]**, Q is uniformly regular. Next we show that $\mathcal{L}_\theta$ is uniformly elliptic. That is, there exists a positive constant c_1 such that for all $(\xi, \eta) \in \Re^n \times \Re$ and $x \in \Omega$ the following inequality holds:

$$|\mathcal{L}_\theta^0(x, \xi, \eta)| \geq c_1 \left(|\xi|^2 + \eta^2\right)^{m/2}.$$

First we note that for any two real numbers $a, b > 0$ and natural number m,

$$\begin{aligned}(a^2 + b^2)^m &= \sum_{n=0}^{m} \binom{m}{n} a^{2m-2n} b^{2n} \\ &\leq \max\{a, b\}^{2m} \sum_{n=0}^{m} \binom{m}{n}.\end{aligned}$$

Now by assumption, there exists a positive constant c_L such that for all $x \in \Omega$, $\xi \in \Re^n$,

$$|L^0(x, \xi)| \geq c_L |\xi|^m.$$

Let $(x, t) \in \Omega \times \Re$, $(\xi, \eta) \in \Re^n \times \Re$ be fixed but arbitrary and set $c = \min\{c_L^2, 1\}$.

If $m = 4k$, then for $\theta \in [-\pi, -\pi/2] \cup [\pi/2, \pi]$ we have

$$\begin{aligned}\left|\mathcal{L}_\theta^0((x, t), (\xi, \eta))\right|^2 &= |L^0(x, \xi) - \exp(i\theta)\eta^{4k}|^2 \\ &= |L^0(x, \xi)|^2 - 2\cos(\theta) L^0(x, \xi)\eta^{4k} + \eta^{8k} \\ &\geq c_L^2 |\xi|^{8k} + \eta^{8k} \qquad \text{(since } \cos(\theta) \leq 0 \text{)} \\ &\geq c\left(|\xi|^{8k+4} + \eta^{8k+4}\right)\end{aligned}$$

$$\geq \frac{c}{\sum_{n=0}^{4k}\binom{4k}{n}} \sum_{n=0}^{4k}\binom{4k}{n} \max\{\ |\xi|, \eta\ \}^{8k}$$
$$\geq \frac{c}{\sum_{n=0}^{4k}\binom{4k}{n}} \left(|\xi|^2+\eta^2\right)^{4k}.$$

This implies the inequality

$$|\mathcal{L}^0_\theta(x,\xi,\eta)| \geq \sqrt{\frac{c}{\sum_{n=0}^{4k}\binom{4k}{n}}} \left(|\xi|^2+\eta^2\right)^{2k}$$

which proves that for $m = 4k$ and $\theta \in [-\pi, -\pi/2] \cup [\pi/2, \pi]$, $\mathcal{L}_\theta$ is uniformly elliptic in Q.

Similarly, if $m = 4k+2$ then for any and $\theta \in [-\pi/2, \pi/2]$ we have

$$\begin{aligned}
\left|\mathcal{L}^0_\theta((x,t),(\xi,\eta))\right|^2 &= |L^0(x,\xi) + \exp(i\theta)\eta^{4k+2}|^2 \\
&= |L^0(x,\xi)|^2 + 2\cos(\theta) L^0(x,\xi)\eta^{4k+2} + \eta^{8k+4} \\
&\geq c_L^2|\xi|^{8k+4} + \eta^{8k+4} \\
&\geq c\left(|\xi|^{8k+4} + \eta^{8k+4}\right) \\
&\geq \frac{c}{\sum_{n=0}^{4k+2}\binom{4k+2}{n}} \sum_{n=0}^{4k+2}\binom{4k+2}{n} \max\{\ |\xi|, \eta\ \}^{8k+4} \\
&\geq \frac{c}{\sum_{n=0}^{4k+2}\binom{4k+2}{n}} \left(|\xi|^2+\eta^2\right)^{4k+2}.
\end{aligned}$$

This implies the inequality

$$|\mathcal{L}^0_\theta(x,\xi,\eta)| \geq \sqrt{\frac{c}{\sum_{n=0}^{4k+2}\binom{4k+2}{n}}} \left(|\xi|^2+\eta^2\right)^{2k+1}$$

which proves that for $m = 4k+2$ and $\theta \in [-\pi/2, \pi/2]$, $\mathcal{L}_\theta$ is uniformly elliptic in Q. Assumption

[HH2] clearly holds as does **[HH4]** for $n \geq 2$. It remains to show that **[HH5]** is satisfied.

Let (x, t) be an arbitrary point on ∂Q, n_1 be the unit outward normal vector to $\partial\Omega$ at x and ξ_1 be any non-zero tangential vector to $\partial\Omega$ at x. The outward normal unit vector to ∂Q at (x, t) is then $n = (n_1', 0)$ and any non-zero tangential vector has the form $\xi = (\xi_1', 0)$. By definition $\mathcal{B}_j = B_j$, moreover the roots of $\mathcal{L}(x, \xi + \tau n)$ are exactly the roots of $L(x, \xi_1 + \tau n_1)$. Since $\{L, \{B_j\}_{j=1}^{m/2}\}$ satisfy **[HH5]** so must $\{\mathcal{L}, \{\mathcal{B}_j\}_{j=1}^{m/2}\}$. ■

For each $\theta \in [-\pi/2, \pi/2]$, $\mathcal{L}_\theta$, $\{\mathcal{B}_j\}_{j=1}^{2k+1}$, Q satisfy **[HH1]-[HH5]** and for each $\theta \in [-\pi, -\pi/2] \cup [\pi/2, \pi]$, $\mathcal{L}_\theta$, $\{\mathcal{B}_j\}_{j=1}^{2k}$, Q satisfy **[HH1]-[HH5]**. Thus the hypotheses of Theorem 3.8.3 have been verified. One can generalize the results in Lemma 3.7.2 and Theorem 3.7.3 with obvious modifications. Thus M_θ may be chosen independent of θ in this range. We state the generalizations without proof below.

Lemma 3.8.5 *Let $\mathcal{L}_\theta(x, D)$ be as defined above. If $m = 4k$, then for any $\epsilon > 0$ there exists $\delta > 0$ such that whenever $|\theta_1 - \theta_2| < \delta$, $\theta_1, \theta_2 \in [-\pi, -\pi/2] \cup [\pi/2, \pi]$ we have,*

$$||\mathcal{L}_{\theta_1} v - \mathcal{L}_{\theta_2} v||_{L^2(Q)} < \epsilon ||v||_{H^{4k}(Q)} \qquad \forall v \in H^{4k}(Q).$$

If $m = 4k+2$, then for any $\epsilon > 0$ there exists $\delta > 0$ such that whenever $|\theta_1 - \theta_2| < \delta$, $\theta_1, \theta_2 \in [-\pi/2, \pi/2]$ we have

$$||\mathcal{L}_{\theta_1} v - \mathcal{L}_{\theta_2} v||_{L^2(Q)} < \epsilon ||v||_{H^{4k+2}(Q)} \qquad \forall v \in H^{4k+2}(Q). \qquad ■$$

Due to Theorem 3.8.2 (inequality (3.48)), for each $\theta \in [-\pi, -\pi/2] \cup [\pi/2, \pi]$, there exists a constant m_θ such that for any $v \in H^{4k}(Q)$,

$$||v||_{H^{4k}(Q)} \leq m_\theta \left[||\mathcal{L}_\theta v||_{L^2(Q)} + \sum_{j=1}^{2k} [\mathcal{B}_j v]_{4k-j-1/2, \partial Q} + ||z||_{L^2(Q)} \right]. \qquad (3.52)$$

Theorem 3.8.6: *Let $\mathcal{L}_\theta(x, D)$ be as defined above with $m = 4k$. For each θ, define $m(\theta) = $* inf$\{m_\theta$: inequality *(3.52)* holds.$\}$. *Then $\{m(\theta) : \theta \in [-\pi, -\pi/2] \cup [\pi/2, \pi]\}$ is bounded above. Hence there exists a positive constant $\bar{m}$, independent of θ such that the following inequality* *holds:*

$$||v||_{H^{4k}(Q)} \leq \bar{m}\left[||\mathcal{L}_\theta v||_{L^2(Q)} + \sum_{j=1}^{2k} [\mathcal{B}_j v]_{4k-j-1/2,\partial Q} + ||v||_{L^2(Q)}\right]. \quad \blacksquare \tag{3.53}$$

Similarly, for each $\theta \in [-\pi/2, \pi/2]$, there exists a constant m_θ such that for any $v \in H^{4k+2}(Q)$,

$$||v||_{H^{4k+2}(Q)} \leq m_\theta\left[||\mathcal{L}_\theta v||_{L^2(Q)} + \sum_{j=1}^{2k+1} [\mathcal{B}_j v]_{4k+2-j-1/2,\partial Q} + ||z||_{L^2(Q)}\right]. \tag{3.54}$$

Theorem 3.8.7 *Let $\mathcal{L}_\theta(x, D)$ be as defined above with $m = 4k + 2$. For each θ, define $m(\theta) =$* inf$\{m_\theta :$ inequality *(3.54)* holds.$\}$. *Then $\{m(\theta): -\pi/2 \leq \theta \leq \pi/2\}$ is bounded above. Hence there exists a positive constant $\bar{m}$, independent of θ such that the following inequality holds:*

$$||v||_{H^{4k+2}(Q)} \leq \bar{m}\left[||\mathcal{L}_\theta v||_{L^2(Q)} + \sum_{j=1}^{2k+1} [\mathcal{B}_j v]_{4k+2-j-1/2,\partial Q} + ||v||_{L^2(Q)}\right]. \tag{3.55}$$

■

We now state the generalization of Theorem 3.7.4, one for $m = 4k$ and another for $m = 4k + 2$.

Theorem 3.8.8: *Let L, B_j and Ω be as defined above with $m = 4k$, k a natural number, and satisfy assumptions* **[HH1]-[HH5]**. *Then there exists a positive constant R such that for any $z \in H^{4k}(\Omega)$, $u_j \in H^{4k-j}(\Omega)$, $j = 1, \ldots, 2k$, satisfying $B_j z = u_j$ on $\partial\Omega$ and any complex number s on the open right half plane $\mathbb{C}_{R^{4k}} := \{s : \text{Re } s > R^{4k}\}$ the following inequality holds:*

$$\sum_{j=1}^{4k} |s|^{\left(1-\frac{j}{4k}\right)} ||z||_{H^j(\Omega)} \leq m\left[||(L+s)z||_{L^2(\Omega)} + \sum_{j=1}^{2k} |s|^{\left(1-\frac{j}{4k}\right)} ||u_j||_{L^2(\Omega)} + \sum_{j=1}^{2k} |s|^{\left(1-\frac{j+1}{4k}\right)} ||u_j||_{H^{4k-j}(\Omega)}\right] \tag{3.56}$$

where m is a positive constant dependent only on L and Ω.

Proof: Let ζ be a function in $C^\infty(-\infty, \infty)$ such that $\zeta(t) = 0$ for $|t| > 1$, $\zeta(t) = 1$ for $|t| < 1/2$. Let m_1 be a constant chosen such that $||\zeta||_{H^2(\Re)} \leq m_1$. Let $\bar{m} = \max\{m(\theta), \theta \in [-\pi, -\pi/2] \cup [\pi/2, \pi]\}$

and $m_2 = \max\{\bar{m}, m_1\}$. Define the polynomial $P(r)$ of degree $4k$ by

$$P(r) := r^{4k} - (4k+1)m_2^2 \sum_{l=0}^{4k-1} \binom{4k}{l} r^l - (4k+1)m_2^2.$$

Choose R to be a positive real number such that $P(r) \geq 0$ for all $r > R$. Since $m(\theta)$ is bounded below by 1, m and hence m_2 is always greater than 1. Observe that since $\sum_{l=0}^{4k-1}\binom{4k}{l} \geq 1$ and $m_2^2 > 1$

$$\begin{aligned} P(1) &= 1 - (4k+1)m_2^2 \sum_{l=0}^{4k-1} \binom{4k}{l} - (4k+1)m_2^2 \\ &< -8k - 1 \\ &< 0, \end{aligned}$$

thus $R > 1$. For any $z \in H^{4k}(\Omega)$ and any $s \in \mathbb{C}_{R^{4k}}$ set $\gamma = \arg s - \text{sign}(\arg s)\pi$,[2] $r = |s|^{1/(4k)}$ and $v(x,t) = \zeta(t)\exp(irt)z(x)$. Clearly $v \in H^{4k}(Q)$ and $\gamma \in [-\pi, -\pi/2] \cup [\pi/2, \pi]$, hence equation (3.53) implies

$$\begin{aligned} \|v\|_{H^{4k}(Q)} &\leq \bar{m}\left[\|\mathcal{L}_\gamma v\|_{L^2(Q)} + \sum_{j=1}^{2k} [\mathcal{B}_j v]_{4k-j-1/2,\partial Q} + \|v\|_{L^2(Q)}\right] \\ &\leq m_2 \left[\|\mathcal{L}_\gamma v\|_{L^2(Q)} + \sum_{j=1}^{2k} [\mathcal{B}_j v]_{4k-j-1/2,\partial Q} + \|v\|_{L^2(Q)}\right]. \end{aligned} \qquad (3.57)$$

Now a lower bound for $\|v\|_{H^{4k}(Q)}$, an upper bound for $[\mathcal{B}_j v]_{4k-j-1/2,\partial Q}$ and an upper bound for $\|\mathcal{L}_\gamma v\|_{L^2(Q)}$ needs to be computed. The final inequality is then obtained via simple algebra.

By definition of $\|\cdot\|_{H^{4k}(Q)}$ we have

$$\begin{aligned} \|v\|^2_{H^{4k}(Q)} &= \sum_{|\alpha|+\beta\leq 4k} \int_{-\infty}^{\infty}\int_\Omega \left|D_x^\alpha D_t^\beta v(x,t)\right|^2 dx dt \\ &\geq \sum_{|\alpha|+\beta\leq 4k} \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_\Omega \left|D_x^\alpha D_t^\beta exp(irt)z(x)\right|^2 dx dt \end{aligned}$$

[2] sign(x) denotes the sign of x

$$
\begin{aligned}
&= \sum_{\beta=0}^{4k}(r)^{2\beta}\sum_{|\alpha|+\beta\le 4k}\int_\Omega \left|D_x^\alpha z(x)\right|^2 dx \\
&= \sum_{\beta=0}^{4k}(r)^{2\beta}\,\|z\|^2_{H^{4k-\beta}(\Omega)} \\
&\ge (r)^{2\beta}\,\|z\|^2_{H^{4k-\beta}(\Omega)}\,.
\end{aligned}
$$

for any $\beta = 0, 1, \ldots, 4k$. Hence

$$
\|v\|_{H^{4k}(Q)} \ge (r)^{\beta}\,\|z\|_{H^{4k-\beta}(\Omega)}\,.
$$

for any $\beta = 0, 1, \ldots, 4k$. Thus

$$
(4k+1)\,\|v\|_{H^{4k}(Q)} \ge \sum_{\beta=0}^{4k}(r)^{\beta}\,\|z\|_{H^{4k-\beta}(\Omega)}\,. \tag{3.58}
$$

By definition of $[\cdot]_{m-j-1/2,\partial\Omega}$ we have

$$
\begin{aligned}
[B_j v]^2_{4k-j-1/2,\partial Q} &= [\zeta(t)\exp(irt)B_j z(x)]^2_{4k-j-1/2,\partial Q} \\
&= [\zeta(t)\exp(irt)u_j]^2_{4k-j-1/2,\partial Q} \\
&\le \|\zeta(t)\exp(irt)u_j\|^2_{H^{4k-j}(Q)} \\
&= \sum_{|\alpha|+\beta\le 4k-j}\int_{-\infty}^{\infty}\int_\Omega \left|D_x^\alpha D_t^\beta \zeta(t)\exp(irt)u_j\right|^2 dxdt \\
&\le \sum_{\beta=0}^{4k-j}\sum_{|\alpha|\le 4k-j-\beta}\int_{-\infty}^{\infty}\int_\Omega \left|D_x^\alpha u_j\right|^2\left|D_t^\beta\zeta(t)\exp(irt)\right|^2 dxdt \\
&= \sum_{\beta=0}^{4k-j}\sum_{|\alpha|\le 4k-j-\beta}\int_{-\infty}^{\infty}\int_\Omega \left|D_x^\alpha u_j\right|^2\left|\sum_{l=0}^{\beta}\binom{\beta}{l}\zeta^{(\beta-l)}(t)(ir)^l\exp(irt)\right|^2 dxdt \\
&\le \sum_{\beta=0}^{4k-j}\sum_{|\alpha|\le 4k-j-\beta}\int_\Omega \left|D_x^\alpha u_j\right|^2\int_{-\infty}^{\infty}\sum_{l=0}^{\beta} r^{2l}\binom{\beta}{l}^2\sum_{l=0}^{\beta}\left|\zeta^{(\beta-l)}(t)\right|^2 dtdx \\
&\le \sum_{\beta=0}^{4k-j} m_1^2(\beta+1)\sum_{l=0}^{\beta} r^{2l}\binom{\beta}{l}^2\sum_{|\alpha|\le 4k-j-\beta}\int_\Omega \left|D_x^\alpha u_j\right|^2 dx
\end{aligned}
$$

$$
\begin{aligned}
&< (4k+1-j)m_1^2 \sum_{l=0}^{4k+1-j} r^{2l} \left[\sum_{\beta=l}^{4k-j} \binom{\beta}{l}^2 \|u_j\|^2_{H^{4k-j-\beta}(\Omega)} \right] \\
&\leq (4k+1-j)m_1^2 \sum_{l=0}^{4k-j} r^{2l} \|u_j\|^2_{H^{4k-j-l}(\Omega)} \sum_{\beta=l}^{4k-j} \binom{\beta}{l}^2 \\
&\leq (4k+1-j)m_1^2 \hat{\beta}^2 \sum_{l=0}^{4k-j} r^{2l} \|u_j\|^2_{H^{4k-j-l}(\Omega)} \\
&\leq (4k+1-j)m_1^2 \hat{\beta}^2 \left(\sum_{l=0}^{4k-j} r^{l} \|u_j\|_{H^{4k-j-l}(\Omega)} \right)^2
\end{aligned}
$$

where

$$\hat{\beta}^2 = \sum_{\beta=0}^{4k-1} \binom{\beta}{l}^2.$$

Thus

$$
\begin{aligned}
[\mathcal{B}_j v]_{4k-j-1/2,\partial Q} &\leq m_1 \hat{\beta} \sqrt{4k+1-j} \sum_{l=0}^{4k-j} r^l \|u_j\|_{H^{4k-j-l}(\Omega)} \\
&\leq m_2 \hat{\beta} \sqrt{4k+1-j} \left(r^{4k-j} \|u_j\|_{L^2(\Omega)} + (4k-j) r^{4k-j-1} \|u_j\|_{H^{4k-j}(\Omega)} \right)
\end{aligned}
\tag{3.59}
$$

This is the upper bound on $[\mathcal{B}_j v]_{4k-j-1/2,\partial Q}$ for each $j = 1, \ldots, 2k$. Now we calculate an upper bound on $\mathcal{L}_\theta v$.

Let $\theta = \arg s$, then substituting the expression of $v(x,t)$ and θ into $\mathcal{L}_\gamma v$, we find

$$
\begin{aligned}
\mathcal{L}_\gamma v &= \zeta(t)\exp(irt)Lz - \exp(i\gamma)zD_t^{4k}(\zeta(t)\exp(irt)) \\
&= \zeta(t)\exp(irt)Lz + \exp(i\theta)zD_t^{4k}(\zeta(t)\exp(irt)) \\
&= \zeta(t)\exp(irt)Lz + \exp(i\theta)z\sum_{l=0}^{4k}\binom{4k}{l}\zeta^{(4k-l)}(t)(ir)^l\exp(irt) \\
&= \zeta(t)\exp(irt)(L + r^{4k}\exp(i\theta))z + \exp(i\theta)z\sum_{l=0}^{4k-1}\binom{4k}{l}\zeta^{(4k-l)}(t)(ir)^l\exp(irt).
\end{aligned}
$$

Therefore

$$\begin{aligned}
\|\mathcal{L}_\gamma v\|_{L^2(Q)} &\leq \left\|\zeta(t)\exp(irt)(L + r^{4k}\exp(i\theta))z\right\|_{L^2(Q)} \\
&\quad + \left\|\exp(i\theta)z\sum_{l=0}^{4k-1}\binom{4k}{l}\zeta^{(4k-l)}(t)(ir)^l\exp(irt)\right\|_{L^2(Q)} \\
&\leq m_1\left\|(L + r^{4k}\exp(i\theta))z\right\|_{L^2(\Omega)} + \left\{m_1\sum_{l=0}^{4k-1}\binom{4k}{l}r^l\right\}\|z\|_{L^2(\Omega)} \\
&\leq m_2\left\{\left\|(L + r^{4k}\exp(i\theta))z\right\|_{L^2(\Omega)} + \left[\sum_{l=0}^{4k-1}\binom{4k}{l}r^l\right]\|z\|_{L^2(\Omega)}\right\}.
\end{aligned} \tag{3.60}$$

Also,

$$\|v\|_{L^2(Q)} \leq m_2\|z\|_{L^2(\Omega)}. \tag{3.61}$$

Substituting inequality (3.58) into (3.57) we obtain,

$$\sum_{\beta=0}^{4k}(r)^\beta\|z\|_{H^{4k-\beta}(\Omega)} \leq (4k+1)m_2\left[\|Lv\|_{L^2(Q)} + \sum_{j=1}^{2k}[B_j v]_{4k-j-1/2,\partial Q} + \|v\|_{L^2(Q)}\right]. \tag{3.62}$$

Substitution of inequalities (3.59), (3.60) and (3.61) into inequality (3.62) gives

$$\begin{aligned}
\sum_{\beta=0}^{4k}(r)^\beta\|z\|_{H^{4k-\beta}(\Omega)} &\leq (4k+1)m_2^2\left\{\left\|(L + r^{4k}\exp(i\theta))z\right\|_{L^2(\Omega)}\right. \\
&\quad + \left[\sum_{l=0}^{4k-1}\binom{4k}{l}r^l\right]\|z\|_{L^2(\Omega)} \\
&\quad + \sum_{j=1}^{2k}\hat{\beta}\sqrt{4k+1-j}\Big(r^{4k-j}\|u_j\|_{L^2(\Omega)} \\
&\quad \left. + (4k-j)r^{4k-j-1}\|u_j\|_{H^{4k-j}(\Omega)}\Big) + \|z\|_{L^2(\Omega)}\right\}.
\end{aligned} \tag{3.63}$$

After rearrangement we obtain

$$\left[r^{4k} - (4k+1)m_2^2 \sum_{l=0}^{4k-1} \binom{4k}{l} r^l - (4k+1)m_2^2\right] \|z\|_{L^2(\Omega)} + \sum_{\jmath=0}^{4k-1} (r)^{\jmath} \|z\|_{H^{4k-\jmath}(\Omega)}$$
$$\leq (4k+1)m_2^2 \Bigg[\|(L - r^{4k}\exp(i\theta))z\|_{L^2(\Omega)}$$
$$+ \sum_{j=1}^{2k} \jmath\sqrt{4k+1-j}\left(r^{4k-j}\|u_j\|_{L^2(\Omega)} + (4k-j)r^{4k-j-1}\|u_j\|_{H^{4k-j}(\Omega)}\right)\Bigg]. \quad (3.64)$$

By definition of R, we have

$$r^{4k} - (4k+1)m_2^2 \sum_{l=0}^{4k-1} \binom{4k}{l} r^l - (4k+1)m_2^2 \geq 0.$$

Also, we can re-index $\sum_{\jmath=0}^{4k-1}(r)^{\jmath}\|z\|_{H^{4k-\jmath}(\Omega)}$ as $\sum_{j=1}^{4k}(r)^{4k-j}\|z\|_{H^j(\Omega)}$. Hence Equation (3.64) implies

$$\sum_{j=1}^{4k}(r)^{4k-j}\|z\|_{H^j(\Omega)} \leq (4k+1)m_2^2\Bigg[\|(L + r^{4k}\exp(i\theta))z\|_{L^2(\Omega)}$$
$$+ \sum_{j=1}^{2k} \jmath\sqrt{4k+1-j}\Big(r^{4k-j}\|u_j\|_{L^2(\Omega)} +$$
$$(4k-j)r^{4k-j-1}\|u_j\|_{H^{4k-j}(\Omega)}\Big)\Bigg]$$
$$\leq (4k+1)^{5/2}m_2^2\jmath\Bigg[\|(L + r^{4k+1}\exp(i\theta))z\|_{L^2(\Omega)}$$
$$+ \sum_{j=1}^{2k} r^{4k-j}\|u_j\|_{L^2(\Omega)} + r^{4k-j-1}\|u_j\|_{H^{4k-j}(\Omega)}\Bigg].$$

Substituting back $s = r^{4k}\exp(i\theta)$ into the above inequality and defining $m = (4k+1)^{5/2}m_2^2\jmath$ we have the desired result. ∎

Thus if $z \in H^{4k}(\Omega)$ satisfies $Lz = -s$ and $B_j z = u_j$ on $\partial\Omega$, we see from equation (3.56) that

$$\sum_{j=1}^{4k} |s|^{\left(1-\frac{j}{4k}\right)} \|z\|_{H^j(\Omega)} \leq m \left[+ \sum_{j=1}^{2k} |s|^{\left(1-\frac{j}{4k}\right)} \|u_j\|_{L^2(\Omega)} + \sum_{j=1}^{2k} |s|^{\left(1-\frac{j+1}{4k}\right)} \|u_j\|_{H^{4k-j}(\Omega)} \right].$$

Hence Theorem 3.8.8 implies that for sufficiently smooth u_j's, say $u_j \in H^{4k}(\Omega)$ for all $j = 1, \ldots, 2k$, then the input/output map of the boundary control system

$$\begin{aligned} \frac{\partial z}{\partial t} &= -Lz, & & x \in \Omega, \quad t > 0 \\ B_j z &= u_j, & \forall j = 1, \ldots, 2k, \quad & x \in \partial\Omega_j, \quad t > 0 \\ y &= Kz, & & \end{aligned}$$

is bounded for all observation operators $K \in \mathcal{L}(H^1(\Omega), \mathcal{Y})$.

Theorem 3.8.9: *Let L, B_j and Ω be as defined above with $m = 4k+2$, k a natural number, and satisfy assumptions* **[HH1]-[HH5]**. *Then there exists a positive constant R such that for any $z \in H^{4k+2}(\Omega)$, $u_j \in H^{4k+2-j}(\Omega)$, $j = 1, \ldots, 2k+1$, satisfying $B_j z = u_j$ on $\partial\Omega$ and any complex number s on the open right half plane $\mathbb{C}_{R^{4k+2}} := \{s : \mathrm{Re}\, s > R^{4k+2}\}$ the following inequality holds:*

$$\begin{aligned} \sum_{j=1}^{4k+2} |s|^{\left(1-\frac{j}{4k+2}\right)} \|z\|_{H^j(\Omega)} \leq \; & m \Bigg[\|(L-s)z\|_{L^2(\Omega)} + \sum_{j=1}^{2k+1} |s|^{\left(1-\frac{j}{4k+2}\right)} \|u_j\|_{L^2(\Omega)} \\ & + \sum_{j=1}^{2k+1} |s|^{\left(1-\frac{j+1}{4k+2}\right)} \|u_j\|_{H^{4k+2-j}(\Omega)} \Bigg] \end{aligned} \tag{3.65}$$

where m is a positive constant dependent only on L and Ω.

Proof: The proof follows the same structure as that of Theorem 3.8.8, thus we shall only highlight the important steps. Let ζ be a function in $C^\infty(-\infty,\infty)$ such that $\zeta(t) = 0$ for $|t| > 1$, $\zeta(t) = 1$ for $|t| < 1/2$. Let m_1 be a constant chosen such that $\|\zeta\|_{H^2(\mathbb{R})} \leq m_1$. Let $\bar{m} = \max\{m(\theta), -\pi/2 \leq \theta \leq \pi/2\}$ and $m_2 = \max\{\bar{m}, m_1\}$. Thus m_2 is always greater than 1.

Define the polynomial $P(r)$ of degree $4k+2$ by

$$P(r) := r^{4k+2} - (4k+3)m_2^2 \sum_{l=0}^{4k+1} \binom{4k+2}{l} r^l - (4k+3)m_2^2.$$

Choose R to be a positive real number such that $P(r) \geq 0$ for all $r > R$. Observe that

$$P(1) < -8k - 5 < 0,$$

thus $R > 1$. For any $z \in H^{4k+2}(\Omega)$ and any $s \in \mathbb{C}_{R^{4k+2}}$ set $\theta = \arg s$, $r = |s|^{1/(4k+2)}$ and $v(x,t) = \zeta(t)\exp(irt)z(x)$. Clearly $v \in H^{4k+2}(Q)$ hence equation (3.55) implies

$$\|v\|_{H^{4k+2}(Q)} \leq m_2 \left[\|\mathcal{L}_\theta v\|_{L^2(Q)} + \sum_{j=1}^{2k+1} [\mathcal{B}_j v]_{4k+2-j-1/2,\partial Q} + \|v\|_{L^2(Q)} \right]. \tag{3.66}$$

Lower bound for $\|v\|_{H^{4k+2}(Q)}$, and upper bounds for $[\mathcal{B}_j v]_{4k+2-j-1/2,\partial Q}$ and $\|\mathcal{L}_\theta v\|_{L^2(Q)}$ can be obtained using the same technique as in Theorem (3.8.8). The resulting bounds are given below:

$$(4k+3)\|v\|_{H^{4k+2}(Q)} \geq \sum_{\jmath=0}^{4k+2} (r)^{\jmath} \|z\|_{H^{4k+2-\jmath}(\Omega)}. \tag{3.67}$$

For each $j = 1, \ldots, 2k+1$ we have

$$[\mathcal{B}_j v]_{4k+2-j-1/2,\partial Q} \leq m_2 \beta \sqrt{4k+3-j} \left(r^{4k+2-j} \|u_j\|_{L^2(\Omega)} + (4k+2-j) r^{4k+2-j-1} \|u_j\|_{H^{4k+2-j}(\Omega)} \right). \tag{3.68}$$

where

$$\beta^2 = \sum_{\jmath=0}^{4k+1} \binom{\jmath}{l}^2.$$

Also,

$$\|\mathcal{L}_\theta v\|_{L^2(Q)} \leq m_2 \left\{ \left\| (L - r^{4k+2}\exp(i\theta))z \right\|_{L^2(\Omega)} + \left[\sum_{l=0}^{4k+1} \binom{4k+2}{l} r^l \right] \|z\|_{L^2(\Omega)} \right\}. \tag{3.69}$$

and

$$\|v\|_{L^2(Q)} \leq m_2 \|z\|_{L^2(\Omega)} . \tag{3.70}$$

Substituting inequality (3.67) into (3.66) we obtain,

$$\sum_{\jmath=0}^{4k+2} (r)^{\jmath} \|z\|_{H^{4k+2-\jmath}(\Omega)} \leq (4k+3)m_2 \left[\|Lv\|_{L^2(Q)} + \sum_{j=1}^{2k+1} [B_j v]_{4k+2-j-1/2,\partial Q} + \|v\|_{L^2(Q)} \right] . \tag{3.71}$$

Next, substitute inequalities (3.68), (3.69) and (3.70) into inequality (3.71) gives

$$\begin{aligned} \sum_{\jmath=0}^{4k+2} (r)^{\jmath} \|z\|_{H^{4k+2-\jmath}(\Omega)} \leq \; & (4k+3)m_2^2 \Bigg\{ \left\|(L - r^{4k+2}\exp(i\theta))z\right\|_{L^2(\Omega)} \\ & + \left[\sum_{l=0}^{4k+1} \binom{4k+2}{l} r^l \right] \|z\|_{L^2(\Omega)} \\ & + \sum_{j=1}^{2k+1} \hat{\jmath}\sqrt{4k+3-j} \Big(r^{4k+2-j} \|u_j\|_{L^2(\Omega)} \\ & + (4k+2-j) r^{4k+2-j-1} \|u_j\|_{H^{4k+2-j}(\Omega)} \Big) + \|z\|_{L^2(\Omega)} . \Bigg\} \end{aligned} \tag{3.72}$$

After rearrangement we obtain

$$\begin{aligned} \Bigg[r^{4k+2} \; - \; & (4k+3)m_2^2 \sum_{l=0}^{4k+1} \binom{4k+2}{l} r^l - (4k+3)m_2^2 \Bigg] \|z\|_{L^2(\Omega)} + \sum_{\jmath=0}^{4k+1} (r)^{\jmath} \|z\|_{H^{4k+2-\jmath}(\Omega)} \\ \leq \; & (4k+3)m_2^2 \Bigg[\left\|(L - r^{4k+2}\exp(i\theta))z\right\|_{L^2(\Omega)} \\ & + \sum_{j=1}^{2k+1} \hat{\jmath}\sqrt{4k+3-j} \Big(r^{4k+2-j} \|u_j\|_{L^2(\Omega)} + (4k+2-j) r^{4k+2-j-1} \|u_j\|_{H^{4k+2-j}(\Omega)} \Big) \Bigg] . \end{aligned} \tag{3.73}$$

By definition of R, we have

$$r^{4k+2} - (4k+3)m_2^2 \sum_{l=0}^{4k+1} \binom{4k+2}{l} r^l - (4k+3)m_2^2 \geq 0.$$

Also, we can re-index $\sum_{\jmath=0}^{4k+1}(r)^{\jmath} \|z\|_{H^{4k+2-\jmath}(\Omega)}$ as $\sum_{j=1}^{4k+2}(r)^{4k+2-j} \|z\|_{H^j(\Omega)}$. Hence Equation (3.73) implies

$$\begin{aligned}
\sum_{j=1}^{4k+2}(r)^{4k+2-j} \|z\|_{H^j(\Omega)} \leq\ & (4k+3)m_2^2 \Bigg[\|(L - r^{4k+2}\exp(i\theta))z\|_{L^2(\Omega)} \\
& + \sum_{j=1}^{2k+1} \jmath\sqrt{4k+3-j}\Big(r^{4k+2-j}\|u_j\|_{L^2(\Omega)} + \\
& (4k+2-j)r^{4k+2-j-1}\|u_j\|_{H^{4k+2-j}(\Omega)}\Big)\Bigg] \\
\leq\ & (4k+3)^{5/2}m_2^2\jmath\Big[\|(L - r^{4k+2}\exp(i\theta))z\|_{L^2(\Omega)} \\
& + \sum_{j=1}^{2k+1} r^{4k+2-j}\|u_j\|_{L^2(\Omega)} + r^{4k+2-j-1}\|u_j\|_{H^{4k+2-j}(\Omega)}\Big]
\end{aligned}$$

Substituting back $s = r^{4k+2}\exp(i\theta)$ above and defining $m = (4k+3)^{5/2}m_2^2\jmath$ we have the desired result. ■

Hence if $u_j \in H^{4k+2}(\Omega)$ for all $j = 1, \ldots, 2k+1$, then the input/output map of the boundary control system

$$\begin{aligned}
\frac{\partial z}{\partial t} &= Lz, & & x \in \Omega, \quad t > 0 \\
B_j z &= u_j, & \forall j = 1, \ldots, 2k+1, \quad & x \in \partial\Omega_j, \quad t > 0 \\
y &= Kz,
\end{aligned}$$

is bounded for all observation operators $K \in \mathcal{L}(H^1(\Omega), \mathcal{Y})$.

Chapter 4

Controller Design

To approximate multi-dimensional systems numerically, one makes use of finite-dimensional state-space approximations. To obtain good accuracy, often a high order state-space approximation is necessary. These approximations require a lot of computer memory and controller design is difficult. The availability of the transfer function can be advantageous for multi-dimensional systems since typically the number of inputs and outputs is relatively low. Knowing the poles and zeros of the system allows us to determine the system response, thus it is important that they be approximated correctly. Although traditional approximation schemes generally yield good approximations of the poles of the transfer function, this is not the case for the approximations of the zeros of the system. Thus direct approximation of the transfer function can yield an approximation that is more representative of the true dynamics, particularly the zero dynamics.

In this chapter we investigate the prospect of using pole/zero estimates to approximate the transfer function and use these approximates to compute a finite-dimensional controller. We compare our results with those obtained using finite element approximations.

4.1 Transfer Function Approximations

Given the transfer function of the open loop system, the closed loop transfer function of the feedback control system in Figure 4.1 is $(1 + KG(s))^{-1}$, where K is assumed to be a constant. Suppose that $KG(\infty) \neq -1$,(that is $1 + KG(s)$ is not strictly proper) and that $G(s)$ is stable. Then the poles of the closed loop system are the the zeros $1 + KG(s)$. For small values of K, the roots of $1 + KG(s)$ approach the poles of $G(s)$. While for large values of K, the roots of $1 + KG(s)$ approach the zeros of $G(s)$. For a given constant K, if any of the poles are on the right half plane then the closed loop system will be unstable, a scenario that a designer would not want. The root locus is a plot of the closed loop poles with constant proportional gain K, varying from 0 to ∞, and unity feedback. Hence, we can obtain the range of allowable feedback gain K for which the resulting closed loop system will be stable. Thus any finite-dimensional approximation for the open loop transfer function should have a root locus plot that resembles that of the exact closed loop system. We first examine transfer function approximation by revisiting Example 3.2.3 with

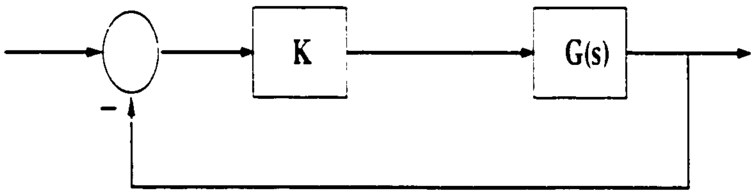
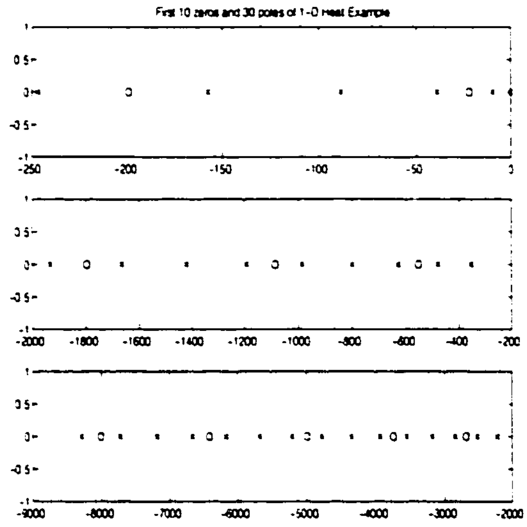
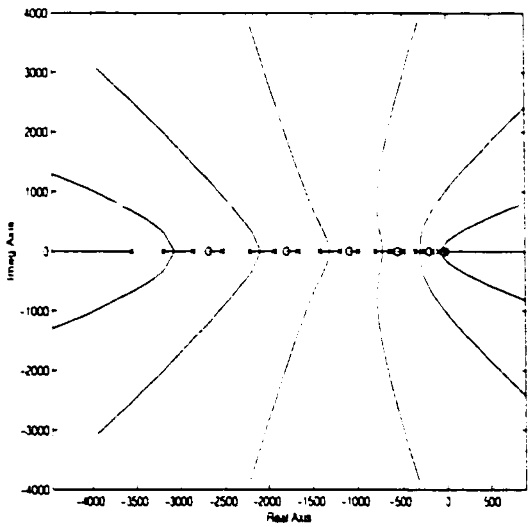


Figure 4.1: Block Diagram of A Feedback Control System

observation at $x_1 = \frac{1}{3}$. The transfer function is $G(s) = \frac{\cosh(\frac{\sqrt{s}}{3})}{\sqrt{s}\sinh(\sqrt{s})}$. The exact transmission zeros and poles are $-\frac{9}{4}(2n+1)^2\pi^2$ and $-n^2\pi^2$ respectively where $n = 0, 1, 2, \ldots$. By plotting the exact zeros and poles we see that there are 3 poles between any two successive transmission zeros (see Figure 4.2a). Figure 4.2b and 4.3 shows the root locus plot using the first 20 poles with 6 zeros and 30 poles with 10 transmission zeros of $G(s)$. Both plots give a similar pattern. In particular, if we restrict the axes of Figure 4.3 to be those of Figure 4.2b, we see that both plots have the same number of branches (four) going into the right half plane. Thus we suspect that the root locus plot of the exact transfer function $G(s)$ also possess the same pattern as well.



(a) Pole/Zero



(b) Root Locus(6 zeros, 20 poles of $G(s)$)

Figure 4.2: Pole/Zero and Root Locus Plot for $G(s) = \frac{\cosh(\frac{\sqrt{s}}{3})}{\sqrt{s}\sinh(\sqrt{s})}$

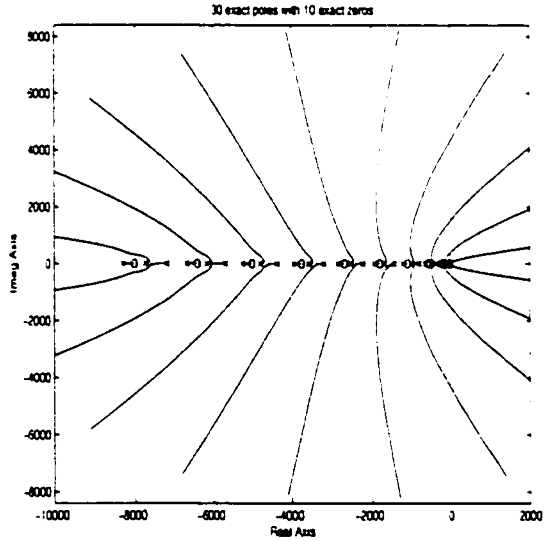
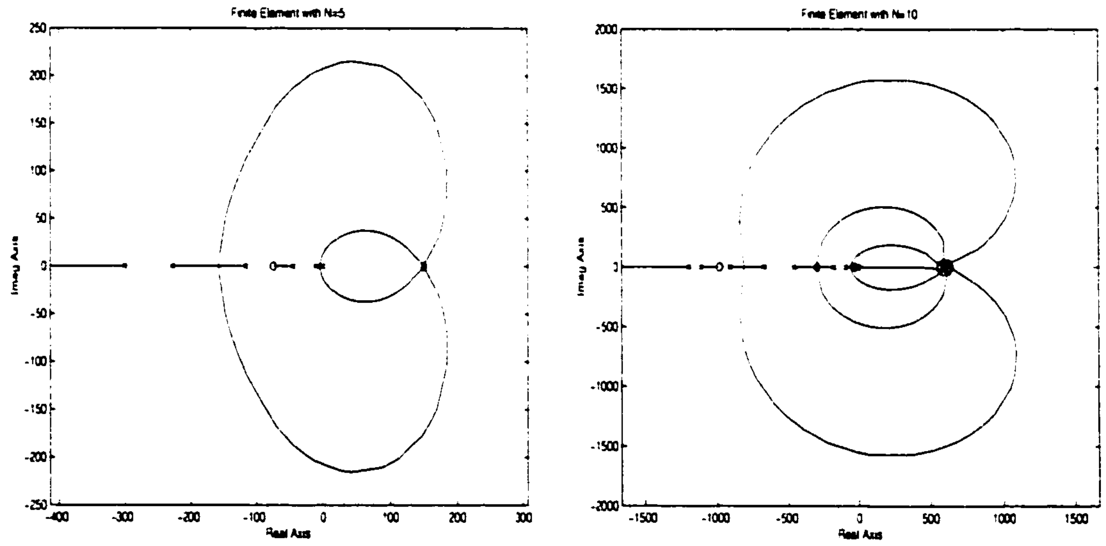


Figure 4.3: Pole/Zero and Root Locus Plot for $G(s) = \frac{\cosh(\frac{\sqrt{s}}{3})}{\sqrt{s}\sinh(\sqrt{s})}$ with 10 zeros, 30 poles

To use these approximating systems for finite-dimensional controller design, it is necessary to ensure that the resulting controller will yield closed loop convergence since the controller needs to stabilize the infinite-dimensional system. This is attained by ensuring that the approximation converges in the graph topology. For details on the graph topology, we refer the reader to Appendix A.1.

We seek to find approximations of $G(s)$ whose root locus plot qualitatively resembles that of Figure 4.3. First we consider finite element methods with linear splines (see Appendix A for detail). We know that this approximation scheme converges in the graph topology [Morris, 1994] and so any controller design scheme will yield closed loop convergence. Figures 4.4 and 4.5 show the root locus plot with approximation orders 5, 10 and 30 respectively. Although 5 elements is sufficiently high for simulation of a one-dimensional heat equation, the corresponding root locus plots for $n = 5, 10$ are poor. It isn't until an order of 30 that the plot appears qualitatively similar to the root locus plots in Figures 4.2 and 4.3. Next we consider approximating $G(s)$ using direct



(a) N=5 (b) N=10

Figure 4.4: Root Locus Plot for Heat Example Using Finite Element Approximation

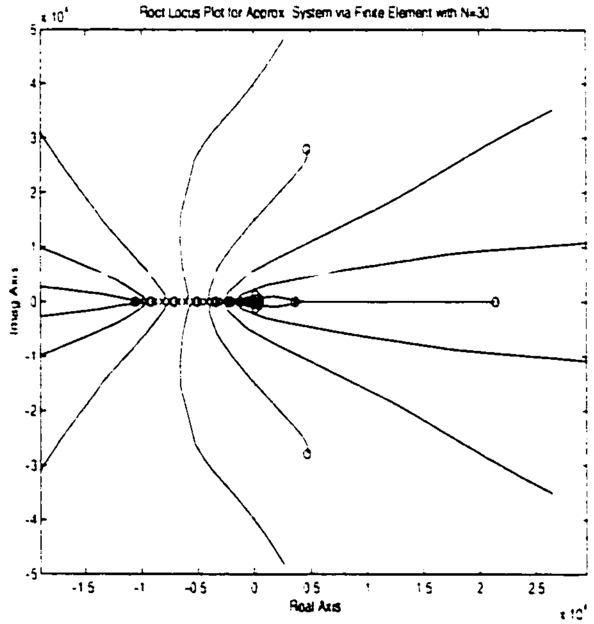


Figure 4.5: Root Locus Plot for Heat Example Using Finite Element Approximation with N=30

pole/zero estimates. Consider the following system where the roles of input $u(t)$ and output $y(t)$ have been reversed and $y(t)$ is set to zero.

$$\left.\begin{array}{rcll} \frac{\partial z}{\partial t} & = & \frac{\partial^2 z}{\partial x^2}, & x \in \left[0, \frac{1}{3}\right] \\ z(x, 0) & = & 0, & x \in \left[0, \frac{1}{3}\right] \\ z\left(\frac{1}{3}, t\right) & = & 0, & t > 0 \\ \frac{\partial z}{\partial x}(0, t) & = & 0, & t > 0 \end{array}\right\} \tag{4.1}$$

The transmission zeros are the poles of the zero dynamics system (4.1). Hence we can approximate the poles and zeros directly by applying the finite element method to systems (3.9) and (4.1) respectively.

The number of zeros and the number of poles one should use in approximating $G(s)$ are not obvious. In other words, what should the $(\#ofpoles)$ be with respect to $(\#ofzeros)$? Unfortunately at present, there is no known methodology to obtain this answer. We therefore resort to trial and error.

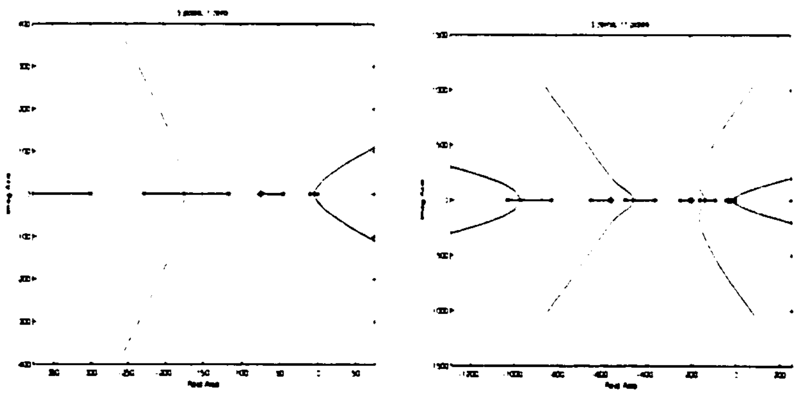
By observing the placement pattern of the exact zeros and poles. (Figure 4.2a) one speculates that good root locus plots may be obtained if we choose the relationships between number of poles and zeros to be one of the following:

1. (Number of poles) = 3× (Number of zeros)

2. (Number of poles) = 3 × (Number of zeros) + 1

3. (Number of poles) = 3 × (Number of zeros)+2

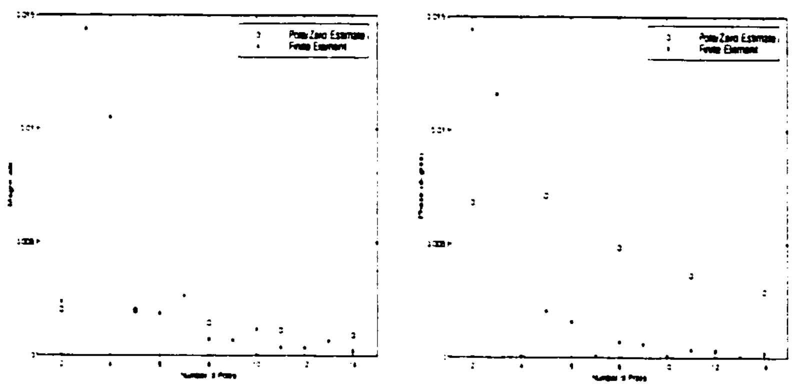
The gain is chosen to be $\frac{\prod_{3m+2} -p_j}{\prod_{i,j\neq 0}^{m} -z_j}$ where p_j and z_j denotes poles and zeros and m is the number of zeros in order to match the residue at $s = 0$. The root locus plots corresponding to each of them were plotted and numerical results indicate that the best choice is relation 3. The results are shown in Figure 4.6. Thus if the relative degree of the approximation is chosen correctly, a much lower order of approximation is needed to yield a root locus plot that is qualitatively more comparable to that given in Figure 4.3. Since the finite element approximations converge in the



(a) 1 zero, 5 poles (b) 3 zeros, 11 poles

Figure 4.6: Root Locus Plot for Heat Example Using Pole/Zero Estimates : # of poles = 3(# of zeros) + 2

graph topology, we can use a sufficiently high order of approximation to represent the 'exact' transfer function. For our example, we picked 30 elements. As a final comparison, we plot the error between the approximated transfer function and $G(s)$. The result is shown in Figure 4.7. We



(a) Magnitude Error (b) Phase Error

Figure 4.7: Error plot between Approximated and Exact Transfer Function ($0.1 \le \omega \le 100$) for the Heat Example

see that the pole/zero approximations give better approximations than low order finite element approximations. However, the finite element approximations perform better if $N \ge 5$. Thus this example did not give us any conclusive results. Since both approximation scheme converges quickly, it was difficult to examine which methodology, if any, is superior. To further investigate our hypothesis, we study another example, a one-dimensional acoustic duct.

Consider one-dimensional plane waves in a duct where a controlled pressure P_c is applied at a point x_a. If one neglects nonlinear terms, then one can model the particle displacement in state-space form as follows: (for complete discussion, see [Grad, 1997] and [Morris, 1998])

$$\frac{\partial^2 z}{\partial t^2} = c^2\frac{\partial^2 z}{\partial x^2} + \delta(x - x_a)\frac{P_c(t)}{\rho} \tag{4.2a}$$

$$z(x,0) = 0, \tag{4.2b}$$

$$\frac{\partial z}{\partial t}(x,0) = 0, \tag{4.2c}$$

$$\frac{\partial z}{\partial x}(0,t) = 0, \tag{4.2d}$$

$$\frac{\partial z}{\partial x}(L,t) = -\frac{K}{c}\frac{\partial z}{\partial t}(L,t) \tag{4.2e}$$

$$y(t) = \frac{\partial z}{\partial x}(x_s, t). \tag{4.2f}$$

Values in Table 4.2 were used for our simulations. The resulting transfer function is

$z(x,t)$	particle displacement (m)
x	position of the particle along the duct (m)
t	time (s)
L	length of duct (m)
c	wave speed (m/s)
ρ	density of the medium in the duct (kg/m^3)
x_s	position where the pressure is sensed or measured and fed back through the controller $(0 \leq x_s \leq L)$
x_a	position where the pressure generated by the controller is applied to the duct $(0 \leq x_a \leq L)$[1]
$P_c(t)$	control pressure applied at $x = x_a$

Table 4.1: One-dimensional duct: variable definitions

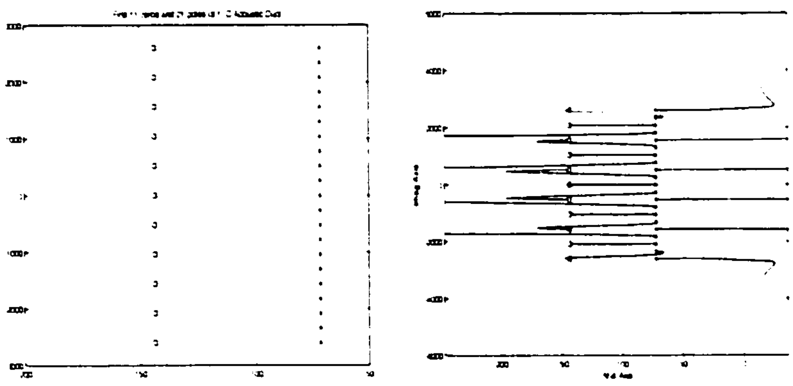
c	$331 m/s$, the speed of sound in air
ρ	$1.29\ kg/m^3$
K	0.7, one end of duct is partially reflective/absorptive
L	$4m$
x_s	$2m$
x_a	$0m$

Table 4.2: Values of c, ρ, K, L, x_s, x_a used in simulation

$$G(s) = \frac{\exp(\frac{-2s}{331})\left(\exp\left(\frac{-4s}{331}\right) - \alpha\right)}{2\left(\exp\left(\frac{-8s}{331}\right) - \alpha\right)}, \tag{4.3}$$

where $\alpha = (1 + K)/(1 - K)$. It has transmission zeros at $\frac{331}{8}\left(\ln\frac{1}{\alpha} \pm 2n\pi i\right)$ and poles at $\frac{331}{4}\left(\ln\frac{1}{\alpha} \pm 2n\pi i\right)$, $n = 0, 1, 2, \ldots$. So any infinite strip of the form $\{(x, y) \mid x \in \Re \text{ and } -4n\pi \leq y \leq 4n\pi, n \in \mathbb{N}\}$ contains $2n + 1$ zeros and $4n + 1$ poles. (see Figure 4.8a)

We first consider finite element methods with linear splines (see Appendix A for detail). The exact root locus plot using the first 41 poles and 21 zeros is shown in Figure 4.8b. The root locus



(a) Pole/Zero (b) Root Locus

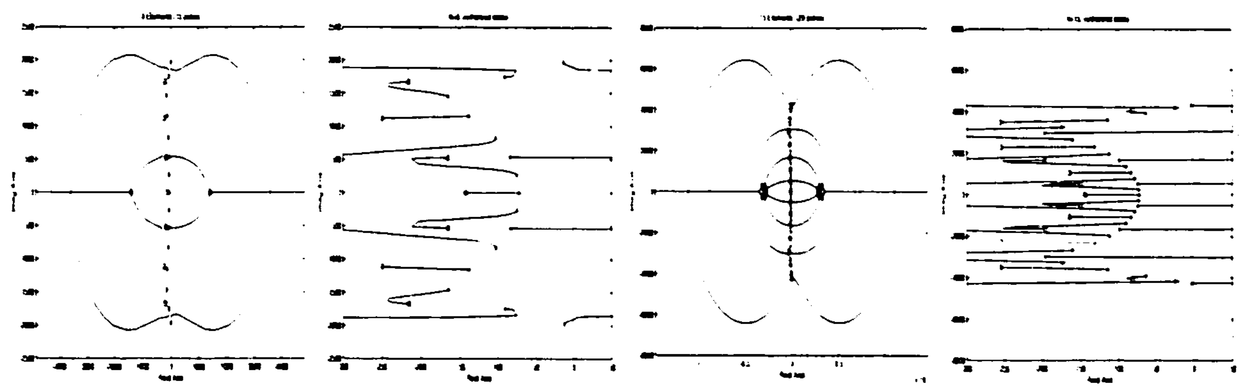
Figure 4.8: Pole/Zero and Root Locus Plot using 41 poles and 21 zeros of $G(s) = \dfrac{\exp(\frac{-2s}{331})\,(\exp(\frac{-4s}{331}) - \alpha)}{2\,(\exp(\frac{-8s}{331}) - \alpha)}$.

plot with $N = 8$ and $N = 15$ are given in Figure 4.9. As discussed in [Grad, 1997], the finite element method gives a number of zeros and poles that do not lie on the line $x = \frac{331}{8} \ln \frac{1}{\alpha}$ and $x = \frac{331}{4} \ln \frac{1}{\alpha}$ respectively. If we view the root locus plot only on an interval around the exact zeros and poles location then it does resemble that of Figure 4.8. For direct approximations we approximate the poles and zeros in the same manner as before. Again we need to choose a suitable relationship between the number of poles and zeros.

Through trial and error, we arrive at the relationship

$$(\text{Number of Poles}) = 4 \times (\text{Number of Zeros}) - 5.$$

The gain is chosen to be $\dfrac{\prod_{4m-5} -p_j}{\prod_m -z_j}$ where p_j and z_j denotes poles and zeros and m is the number of zeros. The corresponding root locus plot is given in Figure 4.10. The error plot is shown in Figure 4.11. It is evident that the pole/zero estimates gives much better results for low orders.



(a) N=8 (b) N=8 (enhanced) (c) N=15 (d) N=15 (enhanced)

Figure 4.9: Root Locus Plot for Duct Example Using Finite Elements Method

We shall compare the different approximation methods in controller design in the next section.

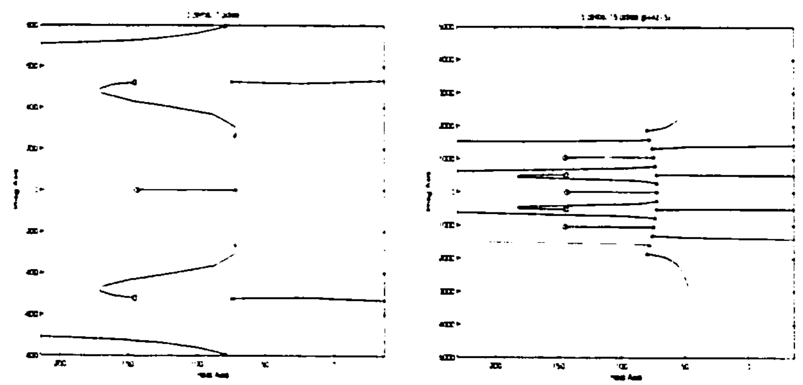
4.2 Finite Dimensional Controller Design

One of the most well-known controller design methods is to use linear state feedback together with a state estimator. One example is the Linear Quadratic Regulator method (LQR). Estimators are very common in practical situations as the full state $x(t)$ is not always available for measurement.

The discussion in this section assumes that the system can be written as a finite-dimensional system in state-space form. Consider a linear time-invariant (LTI) system given by

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t). \end{aligned}\right\} \tag{4.4}$$

where A, B and C are constant matrices of appropriate dimensions. From Figure 4.12 we see that in essence, the estimator takes the plant input $u(t)$ and output $y(t)$ as inputs and determines an estimate $\hat{x}(t)$ for $x(t)$. The error between $\hat{x}(t)$ and $x(t)$ should shrink as time progresses. Thus we should incorporate some error information as input to the estimator to ensure that a proper



(a) 3 zeros, 7 poles (b) 5 zeros, 15 poles

Figure 4.10: Root Locus Plot for Duct Example Using Pole/Zero Estimates

correction can be made to $\hat{x}(t)$. A logical choice is the output $y(t)$. But there are some underlying assumptions one must have on the system in order for $y(t)$ to be "useful". If the matrix C in (4.4) is the zero matrix, then measuring $y(t)$ will give us no information! Suppose the estimator is of the form

$$\frac{d\hat{x}}{dt} = A\hat{x}(t) + Bu(t) + FC\Big(x(t) - \hat{x}(t)\Big) \tag{4.5}$$

where F is some constant matrix and can be thought of as possibly a magnification of the error $C\Big(\hat{x}(t) - x(t)\Big)$. We see that if $\hat{x}(t) = x(t)$ then equation (4.5) is just the first equation of System (4.4). Subtracting Equation (4.5) from the first equation of System (4.4) gives,

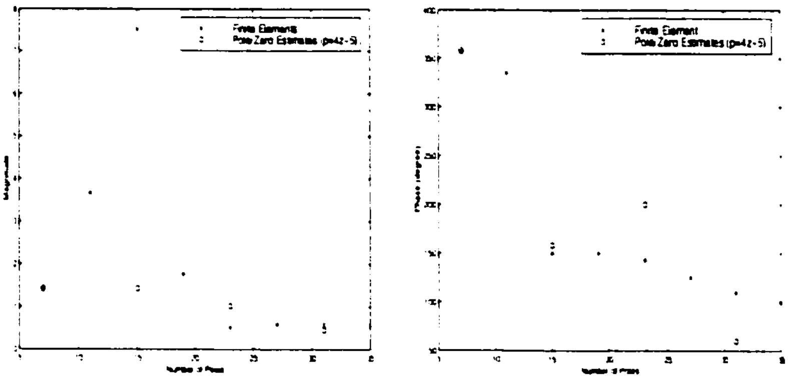
$$\frac{d}{dt}e(t) = (A - FC)e(t),$$

where the error $e(t) = x(t) - \hat{x}(t)$.

So $\hat{x}(t)$ tends to $x(t)$ provided the eigenvalues of $(A - FC)$ have negative real parts.

Definition 4.2.1: A matrix A is *Hurwitz* if all eigenvalues of A have negative real parts.

Definition 4.2.2: System 4.4 is *detectable* if there exists F such that $A - FC$ is Hurwitz.



(a) Magnitude Error (b) Phase Error

Figure 4.11: Error plot between Approximated and Exact Transfer Function ($0 \leq \omega \leq 2500$)

If the system is detectable then its estimation can be accomplished by suitable choice of F. In practice, F is often chosen so that the system behaves in some "best possible" manner.

We now turn to the discussion of linear state feedback. The argument is similar to that above. Let r denote the reference input and K be some constant gain vector. Suppose for the moment that all the states are available for measurement. (We will discuss what happens when we use both state estimator and state feedback on a system later.) Then from Figure 4.12, we have $u(t) = r(t) - Kx(t)$. So the realization of the feedback system is

$$\begin{aligned} \dot{x}(t) &= (A - BK)x(t) + Br(t) \\ y(t) &= Cx(t). \end{aligned} \tag{4.6}$$

Definition 4.2.3: System 4.4 is *stabilizable* if there exists K such that $A - BK$ is Hurwitz.

The goals of controller design are to obtain a stable closed loop system and also to improve the performance in some sense of the overall system. As in the estimator case, if we can choose K such that the eigenvalues of $A - BK$ have negative real parts then the closed loop system will be stable. This is possible if the system is stabilizable. This choice of K, like the choice of F,

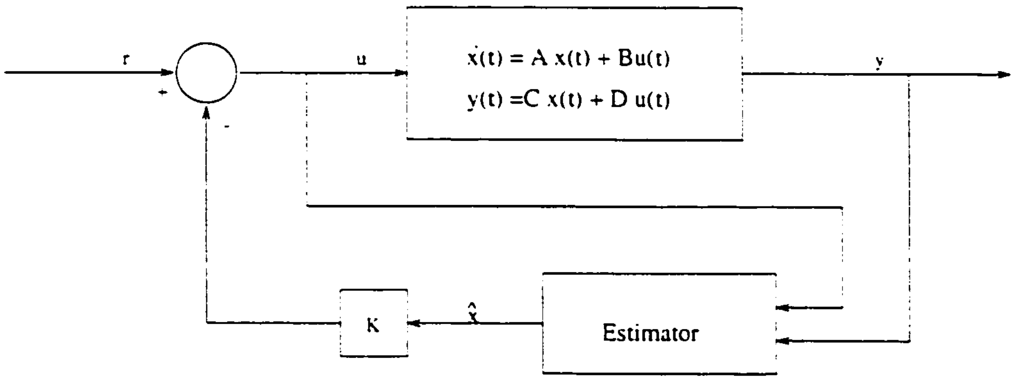


Figure 4.12: Block diagram of estimator with constant state feedback of a LTI system

is non-unique. A common method is to choose K and F so that they solve a linear quadratic regulator problem. This is the topic of discussion in the next section.

The discussions on state estimator and state feedback were done independently. One might ask what happens when we use the estimator to find $\hat{x}$ and then use it to obtain the state feedback $u = r - K\hat{x}$. It turns out that design of the state feedback and state estimator can be done separately. To see this, assume we are designing a state feedback and a state estimator for the LTI system given in equation (4.4), then

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B(r(t) - K\hat{x}(t))\\ \frac{d\hat{x}}{dt} &= (A - FC)\hat{x}(t) + FCx(t) + B(r(t) - K\hat{x}(T))\\ u(t) &= r(t) - K\hat{x}(t).\end{aligned}$$

Hence

$$\begin{pmatrix}\dot{x}(t)\\ \frac{d\hat{x}}{dt}\end{pmatrix} = \underbrace{\begin{pmatrix}A & -BK\\ FC & (A - FC) - BK\end{pmatrix}}_{\bar{A}}\begin{pmatrix}x(t)\\ \hat{x}(t)\end{pmatrix} + \begin{pmatrix}B\\ B\end{pmatrix}r(t)$$

The eigenvalues for $\bar{A}$ are given by the solutions to

$$E(\lambda) = \det \begin{pmatrix} \lambda I - A & BK \\ -FC & \lambda I - (A - FC) + BK \end{pmatrix} = 0.$$

Now let

$$P = \begin{pmatrix} I & 0 \\ -I & I \end{pmatrix}, \text{ so } P^{-1} = \begin{pmatrix} I & 0 \\ I & I \end{pmatrix}.$$

Then since $\det(P) = 1$

$$E(\lambda) = \det\Big(P^{-1}(\lambda I - \bar{A})P\Big) = \det \begin{pmatrix} \lambda I - (A - BK) & BK \\ 0 & \lambda I - (A - FC) \end{pmatrix}$$

The eigenvalues of $\bar{A}$ are just the eigenvalues of state feedback and state estimator! This is known as a *Separation Principle.*

4.3 Linear Quadratic Regulator

The linear quadratic regulator (LQR) problem can be stated as follows. Consider a linear time-invariant system,

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t). \end{aligned}\right\} \tag{4.7}$$

Let Q and R be symmetric positive semi-definite and positive definite matrices. Define the cost function as

$$\mathcal{J}(u) = \int_0^\infty x^T(t)Qx(t) + u^T(t)Ru(t)\,dt. \tag{4.8}$$

The objective is to minimize $\mathcal{J}$ subject to

$$\dot{x}(t) = Ax(t) + Bu(t).$$

The role of Q and R can be thought of as a weight on the state variables $x(t)$ and the control input $u(t)$. The weight on $x(t)$, that is the matrix Q, can be best thought of as weighting the importance of each state $x_i(t)$. If each state is equally important, then a possible choice of Q is the identity matrix.

The LQR problem as given above cannot always be solved. To see this consider the following example given in Anderson and Moore [Anderson and Moore, 1990].

Example 4.3.1 : Consider the system

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{A} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B} u, \quad \begin{pmatrix} x_1(0) \\ x_1(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

with performance indices $R = 1$ and $Q = I_2$. With the given initial conditions, the cost is

$$\mathcal{J}(u) = \int_0^\infty u^2(t) + \exp(2t)\, dt.$$

Clearly $\mathcal{J}(u) \geq \int_0^\infty \exp(2t)\, dt = \infty$. So for any choice of $u(t)$, $\mathcal{J}$ is always infinite. To see what the problem is here, note that for any choice of $K = [K_1\ K_2]$ we have

$$A - BK = \begin{pmatrix} 1 & 0 \\ -K_1 & 1 - K_2 \end{pmatrix}.$$

Hence $\lambda = 1$ is always an eigenvalue of $A - BK$. That is the pair (A, B) is not stabilizable. ∎

We are now ready to state the theorem regarding the optimal solution to the LQR problem.

Theorem 4.3.2 : (e.g. [Anderson and Moore, 1990]) *Given a stabilizable system*

$$\dot{x}(t) = Ax(t) + Bu(t),$$

a positive semi-definite matrix Q and positive definite matrix R, the optimal solution to

$$J(u) = \int_0^\infty x^T(t)Qx(t) + u^T(t)Ru(t)\,dt$$

subject to

$$\dot{x}(t) = Ax(t) + Bu(t),$$

is given by $u^*(t) = -K^*x(t)$ *where*

$$K^* = R^{-1}B^TP,$$

and P denotes the unique positive semi-definite solution to the Algebraic Riccati Equation

$$PA + A^TP + Q - PBR^{-1}B^TP = 0.$$

■

This gives us the solution to our state feedback problem. What about the state estimator? How can we find a suitable F? Since the dual of system (4.7) is

$$\left.\begin{aligned} \dot{x}(t) &= -A^Tx(t) + C^Tu(t) \\ y(t) &= B^Tx(t). \end{aligned}\right\} \qquad (4.9)$$

Thus the pair (A, C) is detectable if and only if the pair (A^T, C^T) is stabilizable. Hence again we can use Theorem 4.3.2 to obtain the solution to a state estimator problem.

There exist a number of techniques to solve the Riccati equation. For a brief discussion, see Grad [Grad, 1993]. The Matlab program CARE was used to solve our algebraic Riccati equations. High accuracy of the algebraic Riccati equation solution is essential to ensure exactness in the design of the finite dimensional controller. We use Newton's Method and Newton's Method with exact line search to improve the solution obtained by CARE. Pseudo codes for the two methods were given in [Benner and Byers, 1998].

4.4 1-D Heat Equation With Neumann Boundary Control

In this section, we use the LQR method to find a controller for system (4.10) using the approximating system resulting from the finite elements or pole/zeros estimate. For details on the computation of the approximating system see Appendix A.

$$\begin{aligned}
\frac{\partial z}{\partial t} &= \frac{\partial^2 z}{\partial x^2}, && (4.10a)\\
z(x,0) &= 0, && (4.10b)\\
\frac{\partial z}{\partial x}(0,t) &= 0, && (4.10c)\\
\frac{\partial z}{\partial x}(L,t) &= u(t), && (4.10d)\\
y(t) &= z(\tfrac{1}{3},t). && (4.10e)
\end{aligned}$$

Let the choice of performance index be denoted by Q_N^F and R_N^F. They must be chosen so that the weighting on the state and the control doesn't vary from one approximating system to the next. Since each state w_k in this finite element method corresponds to the value of the approximation at position $x = k/N$, we can determine Q_N^F as follows: Let

$$\Psi_N(x) = [v_0(x)\ v_1(x)\ \ldots\ v_{N-1}(x)\ v_N(x)].$$

Note that

$$\begin{aligned}
z_N^T(x,t)z_N(x,t) &= V_N'(t)\Psi_N'(x)\Psi_N(x)V_N(t)\\
&= W_N'(t)NM_N^{-1}\Psi_N'(x)\Psi_N(x)NM_N^{-1}W_N(t).
\end{aligned}$$

Integrating both sides with respect to x from 0 to 1 gives

$$\int_0^1 z_N^T(x,t)z_N(x,t)\,dx = W_N'(t)NM_N^{-1}W_N(t).$$

A suitable choice of Q_N^F is therefore NM_N^{-1}. R is set to be 1.

We also need to determine an appropriate choice of Q_N^F from computing the state estimator. Note that the constraint of the LQR problem for the state estimator is

$$\dot{z} = A^T z + C^T u.$$

From equation (4.2), we have

$$\frac{1}{N} M_N \dot{V}_N(t) = N S_N V_N(t) + b_N u(t).$$

Hence

$$\begin{aligned} \dot{V}_N(t) &= N^2 M_N^{-1} S_N V_N(t) + N M_N^{-1} b_N u(t) \\ &= A^T V_N(t) + N M_N^{-1} b_N u(t). \end{aligned}$$

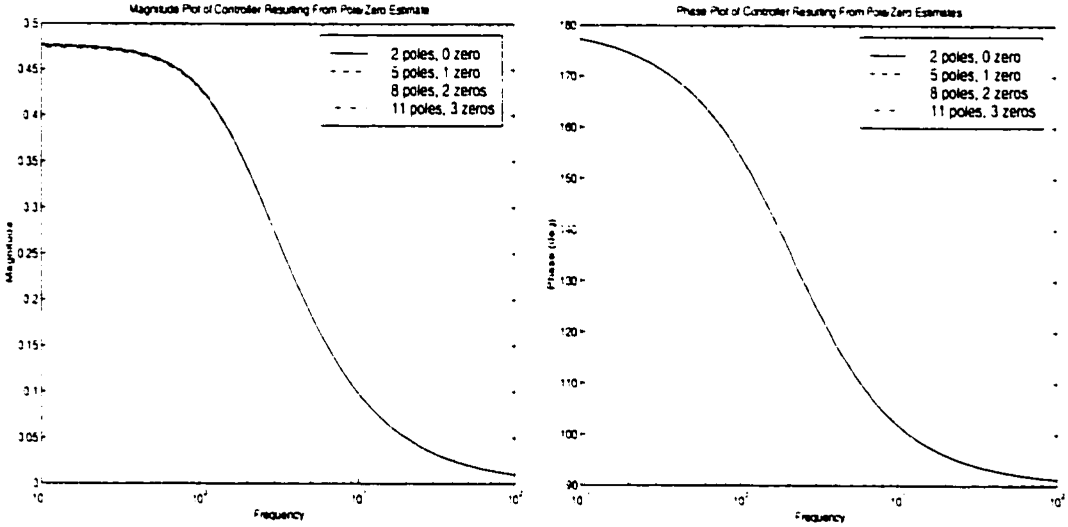
So

$$\begin{aligned} \int_0^1 z_N^T(x,t) z_N(x,t)\, dx &= \int_0^1 V_N'(t) \Psi_N'(x) \Psi_N(x) V_N(t) \\ &= V_N'(t) \frac{1}{N} M_N V_N(t). \end{aligned}$$

Therefore we must choose $Q_N^F = \frac{1}{N} M_N$. The choice of R is again 1.

Figures 4.13–4.16 show the magnitude/phase plot[2] of the resulting controller and its resulting closed loop sensitivity. Not surprisingly (since the approximate transfer functions from both methods converge quickly) both controllers converge very rapidly, with the direct approximation converging slightly faster, so we cannot conclusively decide which method is preferable.

[2] We note that linear scale was used in the magnitude plot



(a) magnitude (b) phase

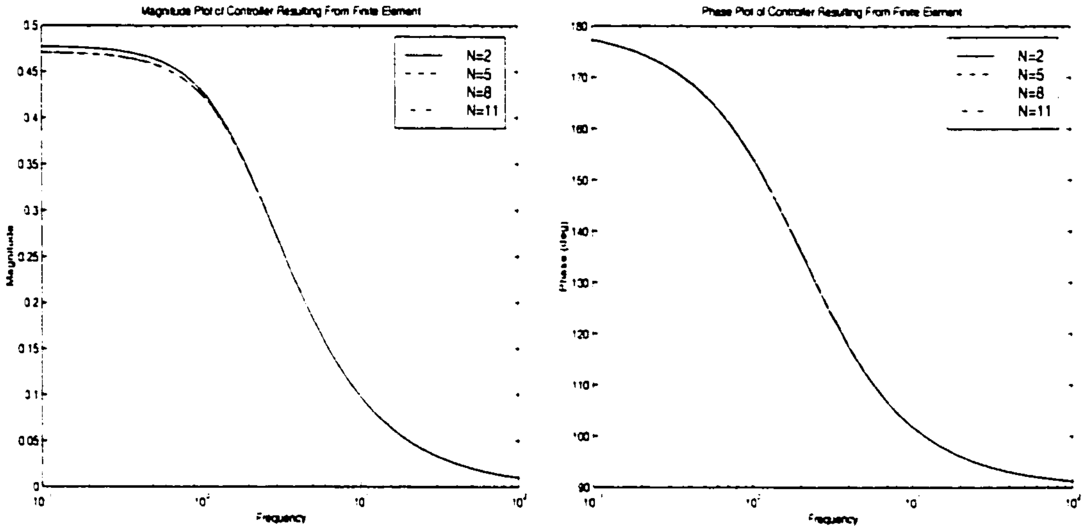
Figure 4.13: Magnitude and Phase Plot of LQR Controller for Heat Example Using Pole/Zero Estimates

4.5 1-D Acoustic Duct

From Section 4.1 we see that the pole/zero estimates transfer function converges faster those the finite elements transfer function of the same order. We suspect that the controllers resulting from pole/zero estimates will converge faster than those obtained from finite elements approximation. For details on computation of the approximating system see Appendix A.

Again we use the LQR method to find a controller for system (4.2). We choose $Q_N^F = C_N^{F'} C_N^F$ for the feedback estimator and $Q_N^F = B_N^F B_N^{F'}$ for the state estimator. The choice of $R = 1$.

We use the controller obtained from using the finite element approximations with 46 elements as a benchmark comparison. The results are shown in Figures 4.17, 4.18 and 4.19. One can clearly see that the controller obtained using the pole/zero estimates converges much faster than that obtained from using the finite elements one. In particular, note the huge difference in the phase convergence and that although the approximating schemes gives slight difference in G_n (Figure 4.11), it results in a significant difference in controllers.



(a) magnitude (b) phase

Figure 4.14: Magnitude and Phase Plot of LQR Controller for Heat Example Using Finite Element

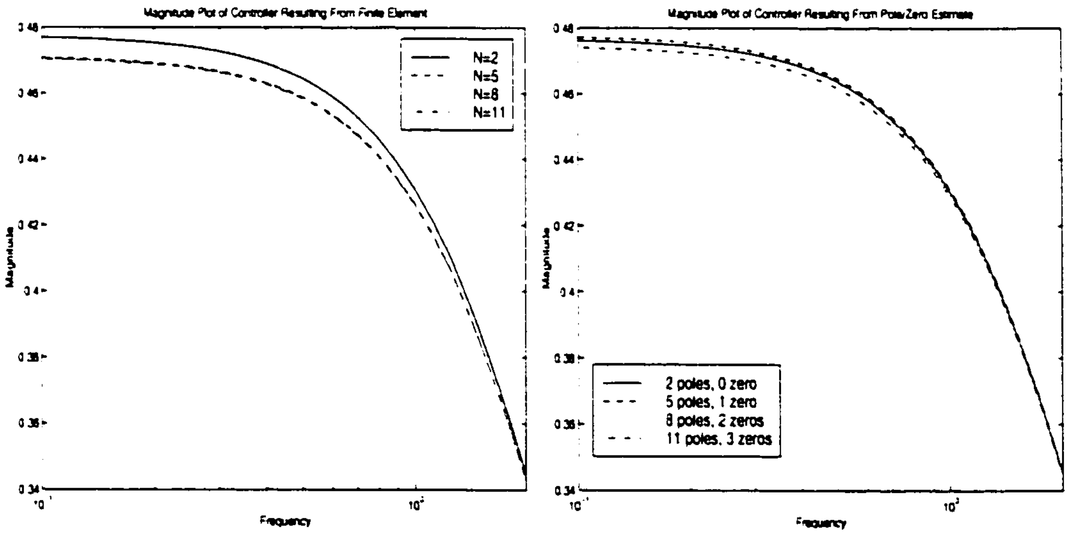
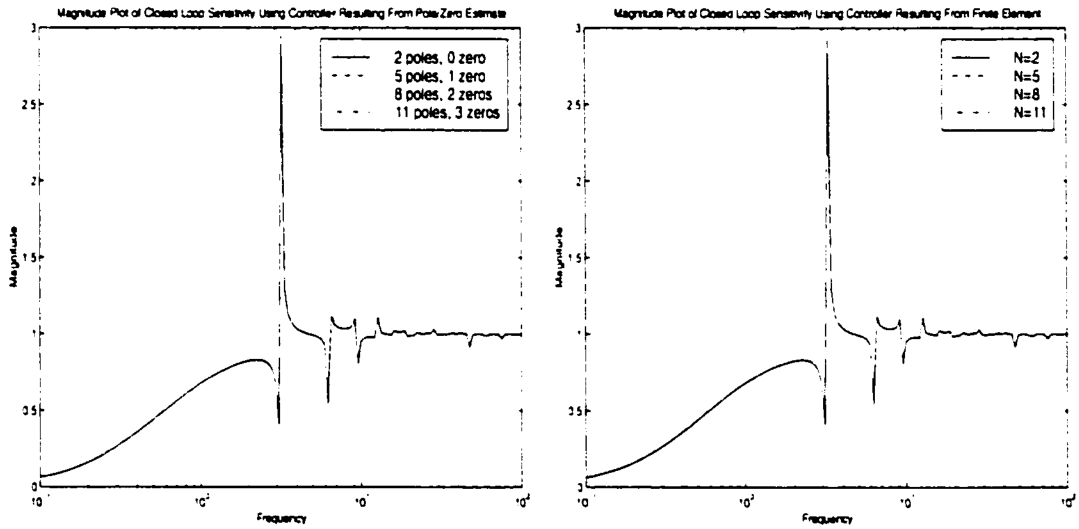


Figure 4.15: Magnitude Plot Comparison for Heat Example Controllers with Enhanced Scaling



(a) Using Pole/Zero Estimates Controller

(b) Using Finite Elements Controller

Figure 4.16: Magnitude Plot of Closed Loop Sensitivity $(1 + G(s)C_n)^{-1}$ for Heat Example. $G(s)$ is the exact transfer function

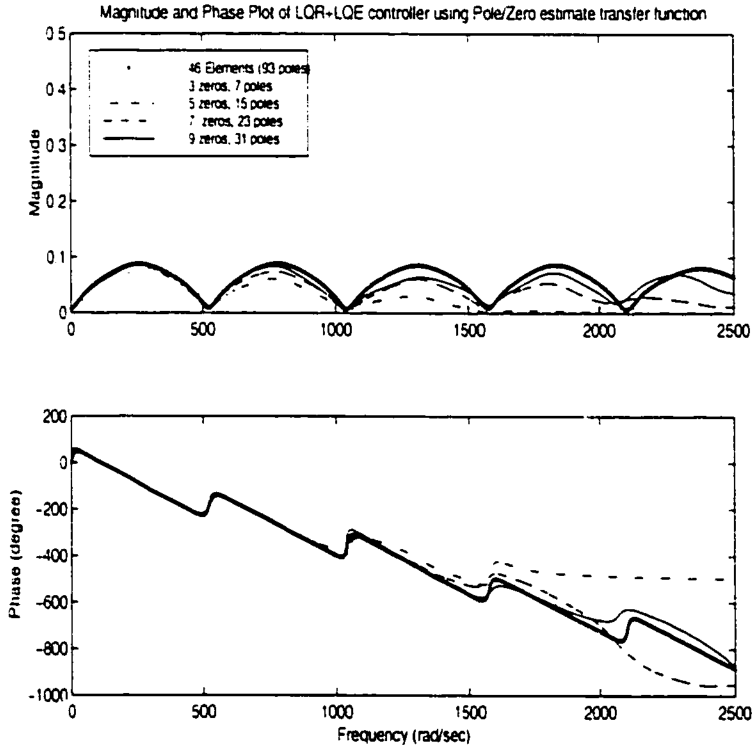


Figure 4.17: Magnitude and Phase Plot of LQR+LQE Controller for Duct Example Using Pole/Zero Estimates

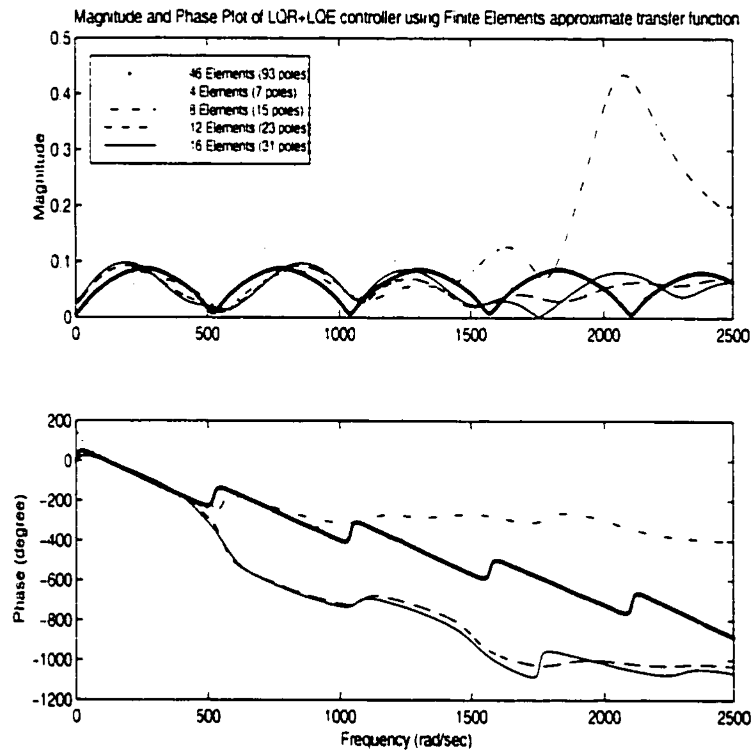
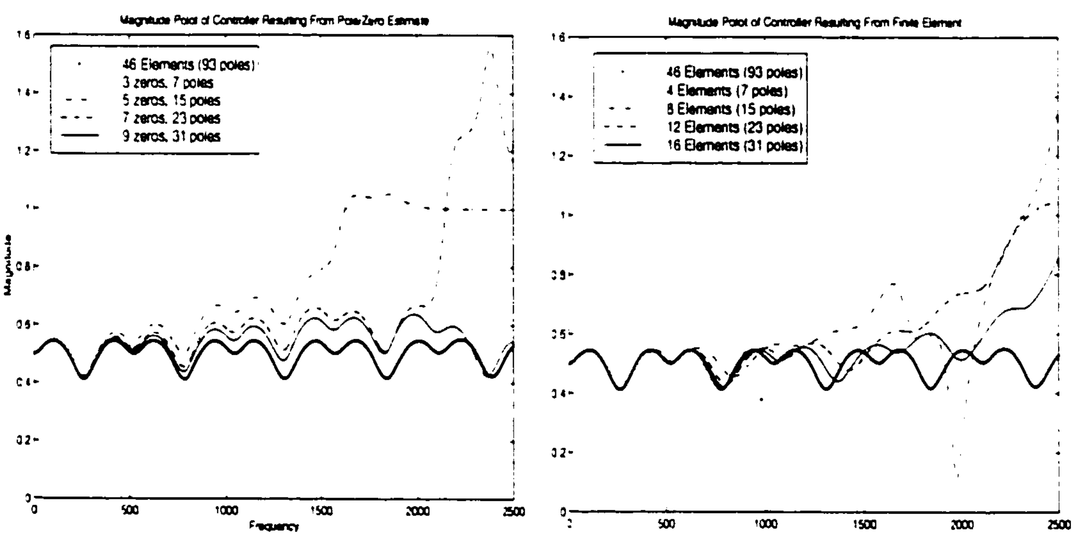


Figure 4.18: Magnitude and Phase Plot of LQR+LQE Controller for Duct Example Using Finite Elements



(a) Using Pole/Zero Estimates Controller

(b) Using Finite Elements Controller

Figure 4.19: Magnitude Plot of Closed Loop Sensitivity $(1 + G(s)C_n)^{-1}$ for Duct Example. $G(s)$ is the exact transfer function

Chapter 5

Conclusions and Future Research

Continuity of the input/output map for boundary control systems has traditionally been justified via either the definition or through properness of the system transfer function. These techniques require explicit expressions for either the input/output map or the system transfer function. In this thesis, we propose an alternate method. In particular, we derived a sufficient condition for properness of the system transfer function. This in turn implies boundedness of the input/output map. Our method transformed properness of the system transfer function to boundedness of solutions to a related elliptic problem. Our technique has the advantage that no explicit expression for the system transfer function is required. Moreover, we avoid the computation of a state space realization.

Although there exist a large number of *a priori* estimates to solutions of uniformly elliptic problems, none addresses the question of uniform boundedness of its solutions. Thus a good portion of this thesis was to establish such bounds. As a consequence, we were able to show boundedness of the input/output map for a class of second order uniformly elliptic operators with Dirichlet, Neumann or Robin boundary control in a general bounded spatial region with varying coefficients. The result was generalized to higher order operators with Neumann or Robin boundary control. Immediate future research includes extending our result on Dirichlet boundary control to higher order operators. Also, we would like to generalize our results to more general operators and also systems that are second order in time. Also of interest are necessary conditions for boundedness of the input/output map.

Although traditional approximation methods generally give good system poles approximations, the approximations for the system zeros are generally quite poor. Thus in Chapter 4 we examine the convergence of transfer function approximation through direct approximations. With this method, the poles of the system are approximated using finite element while the zeros of the system are approximated by the poles of the corresponding zero dynamic system using finite elements. This produces better approximations to the zeros.

Next we investigated the usefulness of direct approximation of the transfer function for controller design. The one-dimensional acoustic duct example indicated that direct approximation of the transfer function can give much faster convergence of the controller sequence than traditional approximation methods. This result is very encouraging although much more research still lies ahead.

Since the poles and zeros are computed independently, we need to determine the correct relationship between number of zeros and number of poles of the exact system. In both of our examples, the relationship between the zeros and poles was obtained through study of the exact zeros and poles of the system and a certain degree of trial and error. How to determine an appropriate relationship between the number of poles and number of zeros is an open question.

Appendix A

Appendix

A.1 Graph Topology

In this section we introduce the concept of the graph topology. It gives a way of measuring closeness of two transfer function. For a complete discussion see [Vidyasagar. 1985]. We begin with a few notations and definition:

1. The notation H_∞ denotes the set of functions f that are analytic in the open right half plane $Re(s) > 0$ with norm

$$\| f \|_\infty = \sup_{\omega \in \mathcal{R}} \lim_{x \downarrow 0} \bar{\sigma}(f(x + j\omega)).$$

   The set of matrices whose elements are in H_∞ is denoted by $M(H_\infty)$. The norm of a function in $M(H_\infty)$ is the induced matrix norm

$$\| F \|_\infty = \sup_{\omega \in \mathcal{R}} \lim_{x \downarrow 0} \bar{\sigma}(F(x + j\omega)).$$

   where $\bar{\sigma}$ denotes the largest singular value of the matrix F.

2. The set of proper rational functions in H_∞ with real coefficients is denoted by S. The set

of matrices whose elements are in S is denoted by $M(S)$.

Definition A.1.1: [COPRIME FACTORIZATION] Given $G(s) \in \mathcal{M}(H_\infty)$, an ordered pair (N, D), $N, D, \in M(H_\infty)$ is a *right coprime factorization (rcf)* of G if

$$\begin{array}{ll} (i) & \det D \neq 0, \\ (ii) & XN + YD = I, \text{ for some } X, Y \in M(H_\infty), \\ (iii) & G = ND^{-1}. \end{array}$$

Similarly, an ordered pair (N, D), $N, D, \in M(H_\infty)$ is a *left coprime factorization (lcf)* of G if

$$\begin{array}{ll} (i) & \det D \neq 0, \\ (ii) & NX + DY = I, \text{ for some } X, Y \in M(H_\infty), \\ (iii) & G = D^{-1}N. \end{array}$$

We shall denote the set of $G(s)$ that have a right and left coprime factorization over $M(H_\infty)$ by $\mathcal{R}(H_\infty)$. A coprime factorization is not unique; however it is unique up to multiplication by a certain matrix called a unimodular matrix. (see e.g. [Vidyasagar, 1985])

Definition A.1.2: A square matrix $U \in M(S)$ is called a *unimodular matrix* if its determinant is nonzero and independent of s and $U^{-1} \in M(S)$. The set of all unimodular matrices whose elements are in S is denoted by $U(S)$.

Closeness of two transfer function can be studied through closeness of their respective coprime factorizations. We define *a basic neighborhood* of $G(s)$ to be one that consists of all transfer functions whose rcf's are "close" to a rcf of $G(s)$. More precisely, let $G(s)$ has an rcf (N, D) and let (X, Y) be such that $XN + YD = I$. Set $\mu = \mu(N, D) = 1/\|[X \;\; Y]\|_\infty$, then whenever $\left\| \begin{bmatrix} N_1 - N \\ D_1 - D \end{bmatrix} \right\|_\infty < \mu$, we have

$$\| XN_1 + YD_1 - I \|_\infty \quad = \quad \| X(N_1 - N) + Y(D_1 - D) \|_\infty$$

$$
\begin{aligned}
&= \left\| [X \;\; Y] \begin{bmatrix} N_1 - N \\ D_1 - D \end{bmatrix} \right\|_\infty \\
&\leq \| [X \;\; Y] \|_\infty \left\| \begin{bmatrix} N_1 - N \\ D_1 - D \end{bmatrix} \right\|_\infty \\
&< 1.
\end{aligned}
$$

This shows that $XN_1 + YD_1$ has an inverse in $M(H_\infty)$ so we can write

$$XN_1 + YD_1 = u \qquad\qquad u \in U(S)$$

then

$$(u^{-1}X)N_1 + (u^{-1}Y)D_1 = I,$$

so the pair (N_1, D_1) is also a right coprime. We can define neighborhoods of G as follows:

Definition A.1.3: [Basic Neighborhood] *Let $0 < \epsilon < \mu(N, D)$ be fixed. A basic neighborhood of G is defined as the set $N(P, N, D, \epsilon)$ given by*

$$N(G, N, D, \epsilon) = \left\{ G_1 = N_1 D_1^{-1} : \left\| \begin{bmatrix} N_1 - N \\ D_1 - D \end{bmatrix} \right\|_\infty < \epsilon. \right\}$$

By varying ϵ over all possible positive values less than μ and varying (N, D) over all rcf of G, where G ranges over all elements of $R(H_\infty)$, we obtain a collection of basic neighborhoods. This collection of neighborhoods together with the set $\mathcal{R}(H_\infty)$ generates a topology. We refer to it as the *graph topology.*

Using the concept of basic neighborhoods we can define what is meant by a convergent sequence in the graph topology.

Definition A.1.4: Suppose G_i is a sequence in $\mathcal{R}(H_\infty)$ and that $G \in \mathcal{R}(H_\infty)$. Then G_i *converges to G in the graph topology* if and only if every basic neighborhood of G contains all but a finite number of terms of the sequence G_i.

Theorem A.1.5: ([Vidyasagar, 1985]) *Let G_n be a sequence in $\mathcal{R}(H_\infty)$ and use $\mathcal{S}(G)$ to denote* *the set of controllers that stabilize G.*

(1) *Suppose G_n converges to $G \in \mathcal{R}(H_\infty)$ and $C \in \mathcal{S}(G)$. Then there exists an N such that $C \in \mathcal{S}(G_n)$ for all $n \geq N$. Moreover, the closed loop transfer matrix $H(G_n, C)$ converges to $H(G, C)$ in $M(H_\infty)$.*

(2) *Conversely, suppose there exists a $C \in \mathcal{S}(G)$ and an N such that $C \in \mathcal{S}(G_n)$ for all $n \geq N$, and the closed loop transfer matrix matrix $H(G_n, C)$ converges to $H(G, C)$ in $M(H_\infty)$. Then G_n converges to G in the graph topology.* ■

A.2 Root Condition

Here we show that all second order uniformly elliptic operators L with real coefficients satisfy the Root Condition.

First note that

$$\begin{aligned} L^0(x, D) &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} D_i D_j \\ &= D' A(x) D, \end{aligned}$$

where $D' = (D_1, D_2, \cdots, D_n)$ and $A(x)$ is the $n \times n$ matrix with components $a_{ij}(x)$. Since L is uniformly elliptic and the coefficients a_{ij}'s are real, the matrix A is positive definite. Let ξ and η be two linearly independent real vectors, then the roots of the polynomial $L^0(x, \xi + \tau\eta) = \tau^2\eta' A\eta + \tau(\xi' A\eta + \eta' A\xi) + \xi' A\xi$, in τ has an equal number of roots with positive and negative imaginary parts if $4(\eta' A\xi\eta' A\xi - \eta' A\eta\xi' A\xi) < 0$.

Let λ_i and v_i denotes the eigenvalues and normalized eigenvectors of A. Write $\eta = \sum c_i v_i$ and

$\xi = \sum d_i v_i$. Then

$$\begin{aligned}
\eta' A\xi\eta' A\xi - \quad & \eta' A\eta\xi' A\xi \\
&= (\eta' A\xi)^2 - \eta' A\eta\xi' A\xi \\
&= \sum c_i v_i' A \sum d_i v_i - \sum c_i^2 v_i' A v_i \sum d_i^2 v_i' A v_i \\
&= (\sum c_i d_i \lambda_i v_i' v_i)^2 - \sum c_i^2 \lambda_i v_i' v_i \sum d_i^2 \lambda_i v_i' v_i \\
&= (\sum c_i d_i \lambda_i)^2 - \sum c_i^2 \lambda_i \sum d_i^2 \lambda_i \\
&= \sum_{i=1}^n \sum_{j=i+1}^n -(c_i^2 d_j^2 + c_j^2 d_i^2 - 2c_i d_i c_j d_j)\lambda_i \lambda_j \\
&= \sum_{i=1}^n \sum_{j=i+1}^n -(c_i d_j - c_j d_i)^2 \lambda_i \lambda_j .
\end{aligned}$$

Since λ_i and λ_j are eigenvalues of A, they are positive. So $\eta' A\xi\eta' A\xi - \eta' A\eta\xi' A\xi < 0$. That is, the root of $L^0(x, D)$ occurs in conjugate pairs.

A.3 1-D Heat Equation with Neumann Boundary Control

In this section we derive the state space approximation of Example 3.2.3 with observation at $x_1 = \frac{1}{3}$. Let

$$z_N(x, t) = \sum_{k=0}^{N} v_k(t) v_k(x).$$

and choose the basis functions $v_k(x)$ $(k = 0, 1, \ldots, N)$ to be the linear splines defined by

$$v_k(x) = \begin{cases} \dfrac{x - x_{k-1}}{\Delta_x} & x_{k-1} \leq x \leq x_k \\ \dfrac{x_{k+1} - x}{\Delta_x} & x_k \leq x \leq x_{k+1} \end{cases} \tag{A.1}$$

where $\Delta_x = 1/N$ and $x_k = k\Delta_x$. Observe that

(i) $v_k(0) = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{otherwise} \end{cases}$

and

(ii) $v_k(1) = \begin{cases} 1 & \text{if } k = N \\ 0 & \text{otherwise.} \end{cases}$

It is easy to verify that

(iii) $\int_0^1 v_i(x)v_j(x)\,dx = \begin{cases} \frac{2\Delta_x}{3} & \text{if } i=j,\ i\neq 0,N \\ \frac{\Delta_x}{3} & \text{if } i=j,\ i=0,N \\ \frac{\Delta_x}{6} & \text{if } \mid i-j\mid = 1 \\ 0 & \text{otherwise} \end{cases}$

and

(iv) $\int_0^1 v_i'(x)v_j'(x)\,dx = \begin{cases} \frac{2}{\Delta_x} & \text{if } i=j,\ i\neq 0,N \\ \frac{1}{\Delta_x} & \text{if } i=j,\ i=0,N \\ \frac{-1}{\Delta_x} & \text{if } \mid i-j\mid = 1 \\ 0 & \text{otherwise.} \end{cases}$

The exact solution $z(x,t)\in L_2(0,1)$ is such that for all $v\in H^1(0,1)$ we have

$$\begin{aligned}\left(\frac{\partial z}{\partial t}, v\right) &= \left(\frac{\partial^2 z}{\partial^2 x}, v\right)\\ &= \frac{\partial z}{\partial x}v\bigg|_0^1 - \left(\frac{\partial z}{\partial x}, \frac{\partial v}{\partial x}\right)\\ &= uv(1) - \left(\frac{\partial z}{\partial x}, \frac{\partial v}{\partial x}\right).\end{aligned}$$

For all j, the approximation $z_N(x,t)$ satisfies

$$\begin{aligned}\left(\frac{\partial z_N}{\partial t}, v_j\right) &= \left(\frac{\partial^2 z_N}{\partial^2 x}, v_j\right)\\ &= \frac{\partial z_N}{\partial x}v_j\bigg|_0^1 - \left(\frac{\partial z_N}{\partial x}, \frac{\partial v_j}{\partial x}\right)\\ &= uv_j(1) - \left(\frac{\partial z_N}{\partial x}, \frac{\partial v_j}{\partial x}\right).\end{aligned}$$

Substituting $z_N(x,t)$ by its sum series representation we obtain

$$\sum_{k=0}^{N} v_k'(t)\int_0^1 v_k(x)v_j(x)\,dx = uv_j(1) - \sum_{k=0}^{N} v_k(t)\int_0^1 v_k'(x)v_j'(x)\,dx.$$

So,

$$\frac{\Delta_x}{3}v_0'(t) + \frac{\Delta_x}{6}v_1'(t) \quad = \quad -\frac{1}{\Delta_x}v_0(t) + \frac{1}{\Delta_x}v_1(t)$$

$$\frac{\Delta_x}{6}v'_{N-1}(t) + \frac{\Delta_x}{3}v'_N(t) = u + \frac{1}{\Delta_x}v_{N-1}(t) - \frac{1}{\Delta_x}v_N(t).$$

and for all $j = 1, \dots, N-1$ we have

$$\frac{\Delta_x}{6}v'_{j-1}(t) + \frac{2\Delta_x}{3}v'_j(t) + \frac{\Delta_x}{6}v'_{j+1}(t) = \frac{1}{\Delta_x}v_{j-1}(t) - \frac{2}{\Delta_x}v_j(t) + \frac{1}{\Delta_x}v_{j+1}(t).$$

Let $V_N(t) = [\, v_0(t), \dots, v_N(t)\,]^T$. Then we can represent the above in matrix form

$$\begin{aligned}
\frac{1}{N}M_N \dot{V}_N(t) &= NS_N V_N(t) + b_N u(t) && \text{(A.2)}\\
y(t) &= z_N(\tfrac{1}{3}, t)\\
&= \sum_{k=0}^{N} v_k(t)v_k(1/3)\\
&= \begin{cases} v_{\frac{N}{3}}(t)v_{\frac{N}{3}}(\frac{1}{3}) & \text{if } \mathrm{mod}(N,3)=0\\ v_{\frac{N-1}{3}}(t)v_{\frac{N-1}{3}}(\frac{1}{3}) + v_{\frac{N+2}{3}}(t)v_{\frac{N+2}{3}}(\frac{1}{3}) & \text{if } \mathrm{mod}(N,3)=1\\ v_{\frac{N-2}{3}}(t)v_{\frac{N-2}{3}}(\frac{1}{3}) + v_{\frac{N+1}{3}}(t)v_{\frac{N+1}{3}}(\frac{1}{3}) & \text{if } \mathrm{mod}(N,3)=1 \end{cases}\\
&= \begin{cases} v_{\frac{N}{3}}(t) & \text{if } \mathrm{mod}(N,3)=0\\ \frac{2}{3}v_{\frac{N-1}{3}}(t) + \frac{1}{3}v_{\frac{N+2}{3}}(t) & \text{if } \mathrm{mod}(N,3)=1\\ \frac{1}{3}v_{\frac{N-2}{3}}(t) + \frac{2}{3}v_{\frac{N+1}{3}}(t) & \text{if } \mathrm{mod}(N,3)=1 \end{cases} && \text{(A.3)}
\end{aligned}$$

where M_N, S_N are $(N+1)\times(N+1)$ matrices, b_N is a $(N+1)$ column vector and c_N is a $(N+1)$ row vector given by

$$M_N = \begin{pmatrix} 1/3 & 1/6 & 0 & \cdots & \cdots & 0\\ 1/6 & 2/3 & 1/6 & 0 & \cdots & \vdots\\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & \ddots & 0\\ \vdots & \ddots & \ddots & 1/6 & 2/3 & 1/6\\ 0 & \cdots & \cdots & 0 & 1/6 & 1/3 \end{pmatrix}, S_N = \begin{pmatrix} -1 & 1 & 0 & \cdots & \cdots & 0\\ 1 & -2 & 1 & 0 & \cdots & \vdots\\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & \ddots & 0\\ \vdots & \ddots & \ddots & 1 & -2 & 1\\ 0 & \cdots & \cdots & 0 & 1 & -1 \end{pmatrix}. \qquad \text{(A.4)}$$

$$
\begin{aligned}
b_N &= (\ 0 \cdots\cdots 1\ )^T, \\
c_N &= \begin{cases} (\ \underbrace{0\cdots 0}_{\frac{N}{3}}\ 1\ 0\cdots 0\ ) & \text{if } \mathrm{mod}(N,3)=0 \\ (\ \underbrace{0\cdots 0}_{\frac{N-1}{3}}\ \frac{2}{3}\ \frac{1}{3}\ 0\cdots 0\ ) & \text{if } \mathrm{mod}(N,3)=1 \\ (\ \underbrace{0\cdots 0}_{\frac{N-2}{3}}\ \frac{1}{3}\ \frac{2}{3}\ 0\cdots 0\ ) & \text{if } \mathrm{mod}(N,3)=2 \end{cases}
\end{aligned}
$$

Let $W_N(t) = \frac{1}{N} M_N V_N(t)$, and define $A_N^F = N^2 S_N M_N^{-1}$, $B_N^F = b_N$ and $C_N^F = N c_N M_N^{-1}$. Then Equations (A.2), (A.3) become

$$
\left.\begin{aligned}
\dot{W}_N(t) &= A_N^F W_N(t) + B_N^F u(t) \\
y(t) &= C_N^F W_N(t),
\end{aligned}\right\} \tag{A.5}
$$

This enables us to compute the finite element approximations of order N. It also allows us to compute the poles for the direct approximations.

We know that the zeros of system (4.10) are the poles of the zero dynamics system. That is, (4.10a) together with initial condition (4.10b), boundary conditions (4.10c) and $z(\frac{1}{3}, t) = 0$. To obtain the zeros for the direct approximations, we need to obtain its corresponding state space approximations. Let $v_k(x)$ be the linear splines defined in (A.1) but with $\Delta x = \frac{1}{3N}$. Then the exact solution $z(x,t) \in L_2(0,1)$ is such that for all $v \in H^1(0,1)$ we have

$$
\begin{aligned}
\left(\frac{\partial z}{\partial t}, v\right) &= \left(\frac{\partial^2 z}{\partial^2 x}, v\right) \\
&= \frac{\partial z}{\partial x} v\Big|_{x=1/3} - \left(\frac{\partial z}{\partial x}, \frac{\partial v}{\partial x}\right).
\end{aligned}
$$

So if $v(1/3) = 0$ then the above becomes

$$
\left(\frac{\partial z}{\partial t}, v\right) = -\left(\frac{\partial z}{\partial x}, \frac{\partial v}{\partial x}\right).
$$

For all $k = 0, \ldots, N-1$, $v_k(1/3) = 0$, hence we obtain

$$\frac{1}{3N}\tilde{M}_N \dot{\tilde{V}}_N(t) = 3N\tilde{S}_N\tilde{V}_N(t),$$

where $\tilde{M}_N$ and $\tilde{S}_N$ denote the $N \times N$ submatrices of M_N and S_N with the $(N+1)th$ row and column removed and $\tilde{V}_N(t) = [\ v_0(t), \ldots, v_{N-1}(t)\ ]^T$. This allows us to compute zeros for the direct approximations by computing the eigenvalue of the matrix $9N^2\tilde{S}_N\tilde{M}_N^{-1}$.

A.4 1-D Acoustic Duct

Let $v_k(x)$ be as defined in Equation (A.1) (with $\Delta x = L/N$) and set

$$z_N(x,t) = \sum_{k=0}^{N} v_k(t)v_k(x).$$

The exact solution $z(x,t) \in L_2(0,L)$ is such that for all $v \in H^1(0,L)$ we have

$$\begin{aligned}\left(\frac{\partial^2 z}{\partial t^2}, v\right) &= c^2\left(\frac{\partial^2 z}{\partial^2 x}, v\right) + \left(\delta(x - x_a)\frac{P_c(t)}{\rho}, v\right)\\ &= c^2\frac{\partial z}{\partial x}v\Big|_0^L - c^2\left(\frac{\partial z}{\partial x}, \frac{\partial v}{\partial x}\right) + \frac{P_c(t)v(x_a)}{\rho}\\ &= -Kc\frac{\partial z}{\partial t}(L,t)v(L) - c^2\left(\frac{\partial z}{\partial x}, \frac{\partial v}{\partial x}\right) + \frac{P_c(t)v(x_a)}{\rho} + \frac{P_d(t)v(x_d)}{\rho}.\end{aligned}$$

Thus for all j, the approximation $z_N(x,t)$ must satisfy

$$\left(\frac{\partial^2 z_N}{\partial t^2}, v\right) = -\frac{K}{c}\frac{\partial z_N}{\partial t}(L,t)v(L) - \left(\frac{\partial z_N}{\partial x}, \frac{\partial v}{\partial x}\right) + \frac{P_c(t)v(x_a)}{\rho}.$$

Substituting $z_N(x,t)$ by its sum series representation we get

$$\sum_{k=0}^{N} v_k''(t)\int_0^L v_k(x)v_j(x)\,dx = -Kc\sum_{k=0}^{N} v_k'(t)v_k(L)v_j(L) - c^2\sum_{k=0}^{N} v_k(t)\int_0^L v_k'(x)v_j'(x)\,dx + \frac{P_c(t)v_j(x_a)}{\rho}.$$

So,

$$\begin{aligned}
\frac{\Delta_x}{3} v_0''(t) + \frac{\Delta_x}{6} v_1''(t) &= -\frac{c^2}{\Delta_x} v_0(t) + \frac{c^2}{\Delta_x} v_1(t) + \frac{P_c(t) v_0(x_a)}{\rho} \\
\frac{\Delta_x}{6} v_{N-1}''(t) + \frac{\Delta_x}{3} v_N''(t) &= -Kcv_N' + \frac{c^2}{\Delta_x} v_{N-1}(t) - \frac{c^2}{\Delta_x} v_N(t)) + \frac{P_c(t) v_N(x_a)}{\rho}.
\end{aligned}$$

and for all $j = 1, \ldots, N-1$ we have

$$\frac{\Delta_x}{6} v_{j-1}''(t) + \frac{2\Delta_x}{3} v_j''(t) + \frac{\Delta_x}{6} v_{j+1}''(t) = \frac{c^2}{\Delta_x} v_{j-1}(t) - \frac{2c^2}{\Delta_x} v_j(t) + \frac{c^2}{\Delta_x} v_{j+1}(t) + \frac{P_c(t) v_j(x_a)}{\rho}.$$

Let $V_N(t) = [\ v_0(t), \ldots, v_N(t)\ ]^T$. Then we can represent the above in matrix form

$$\frac{L}{N} M_N \ddot{V}_N(t) = K_N \dot{V}_N(t) + \frac{c^2 N}{L} S_N V_N(t) + b_N P_c(t). \tag{A.6}$$

where M_N, S_N are as defined in Equation (A.4) and

$$K_N = \begin{pmatrix} 0_{N\times N} & 0 \\ 0 & -Kc \end{pmatrix}, \quad [b_N]_j = \frac{v_j(x_a)}{\rho}.$$

Set $u(t) = P_c(t)$ and define

$$W_N(t) = \begin{pmatrix} V_N(t) \\ \dot{V}_N(t) \end{pmatrix}, \quad A_N^F = \begin{pmatrix} 0_{(N+1)\times(N+1)} & I_{(N+1)\times(N+1)} \\ \frac{c^2N^2}{L^2} M_N^{-1} S_N M_N & \frac{N}{L} M_N^{-1} K_N \end{pmatrix}, \quad B_N^F = \begin{pmatrix} 0_{(N+1)\times 1} \\ M_N^{-1} b_N \end{pmatrix}.$$

Then Equation (A.6) can be written in the standard form

$$\dot{W}_N(t) = A_N^F W_N(t) + B_N^F u(t). \tag{A.7}$$

Define $[c_N]_j = v_j'(x_s)$ and $C_N^F = (c_N\ 0_{1\times(N+1)}\ )$. Then the output equation is given by

$$y(t) == C_N^F W_N(t). \tag{A.8}$$

Appendix B

Program Listings

B.1 Source Code For LQR design of 1-D Heat Equation with Neumann Boundary Control

FINITE DIMENSIONAL APPROXIMATIONS

```
function [NZZ,NP,RZ,RP,sys_N,M_N]=Heat_felinear(N,estzeros,estpoles,rz,rp,obpt)
% HEAT_FELINEAR : Finite element approximations using linear splines for
% 1-D heat equation with Neumann boundary control at x=1 and point
% observation at either 1/3, 1/2 or 1.
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% LIST OF REQUIRED INPUTS:                                           %
%%        N --> order of approximations                              %
%% estzeros --> # of estimated zeros to keep from the approximations %
%% estpoles --> # of estimated poles to keep from the approximations %
%%     obpt --> Point observation pos, either 1/3, 1/2 or 1          %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% LIST OF RETURN OUTPUTS:                                        %
%%    NP --> corrsponding 'estpoles' no of poles, col. vector     %
%%   NZZ --> corr 'estzeros' no of poles from zero dyn,col. vector %
%%    RP --> Real value of the first estpoles of system           %
%%    RZ --> Real value of the first estzeros of system           %
%% sys_N --> state space realization of order N in sys form       %
%%  M_N  --> resulting stiff matrix of order N                    %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%


% Written by Ada Cheng on Apr. 28th, 2000.
  format long
  ni=nargin;
  if ni~=6
    disp('Incorrect number of inputs.  Please type help Heat_felinear.');
  end
  %-------------------------------------- STEP 1 ----------------------------
  %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
  % Set up the real zeros and real poles %
  %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
  RZ=[];
  RP=[];
  for loop=0:rz-1
    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
    % Measurements taken at x=1/3 %
    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
    if (obpt==1/3 )
      RZ=[RZ; -9/4*(2*loop+1)^2*pi^2];
```

```
    end
    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
    % Measurements taken at x=1/2 %
    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
    if (obpt==1/2)
      RZ=[RZ;-(2*loop+1)^2*pi^2];
    end
    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
    % Measurements taken at x=1   %
    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
    if (obpt==1)
      RZ=[RZ;-1/4*(2*loop+1)^2*pi^2];
    end
  end
  for loop=0:rp-1
    RP=[RP;-loop^2*pi^2];
  end
  RZ=sort(RZ);                          % sort poles and zeros,
  RP=sort(RP);                          % since all real, doesn't
                                        % need esort
%--------------------------------------- STEP 2 -----------------------------
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Matrix Setup For Heat Equation With Neumann B.C.         %
% System is in the form 1/N*M_N \dot z = N*S_N z + b_N u,  %
%                                   y = c_N z              %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Initialize all matrices as the zero matrix
M_N=zeros(N+1,N+1); S_N=M_N; b_N=zeros(N+1,1); c_N=b_N';
```

```
M_N(1,1)=1/3; M_N(1,2)=1/6;
S_N(1,1)=-1; S_N(1,2)=1;
for j=2:N
  M_N(j,j-1)=1/6; M_N(j,j)=2/3; M_N(j,j+1)=1/6;
  S_N(j,j-1)=1; S_N(j,j)=-2; S_N(j,j+1)=1;
end
M_N(N+1,N)=1/6; M_N(N+1,N+1)=1/3;
S_N(N+1,N)=1; S_N(N+1,N+1)=-1;
b_N(N+1,1)=1;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% observation at x=1/3 x \in [0,1] %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
if (obpt==1/3)
  if mod(N,3)==0
    c_N(1,N/3+1)=1;
  end
  if mod(N,3)==1
    c_N(1,(N-1)/3+1)=2/3;
    c_N(1,(N+2)/3+1)=1/3;
  end
  if mod(N,3)==2
    c_N(1,(N+1)/3+1)=2/3;
    c_N(1,(N-2)/3+1)=1/3;
  end
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% observation at x=1/2 x \in [0,1] %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

```
if (obpt==1/2)
  if mod(N,2)==0
    c_N(1,N/2+1)=1;
  end
  if mod(N,2)==1
    c_N(1,(N+1)/2+1)=1/2;
    c_N(1,(N-1)/2+1)=1/2;
  end
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% observation at x=1   x \in [0,1] %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
if (obpt==1)
  c_N(1,N+1)=1;
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Transform System To Standard form:     %
% set w= M_N z                           %
% System becomes \dot w = A_N w + B_N u, %
%                    y = C_N w           %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
invM_N=inv(M_N);
A_N=N^2*S_N*invM_N;
B_N=b_N;
C_N=N*c_N*invM_N;
D_N=0;
sys_N=ss(A_N,B_N,C_N,D_N);            % ss. output data
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

```
% Find poles of approx system %
% p - poles                   %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
p=eig(A_N); p=sort(p);
%------------------------------------- STEP 3 -----------------------------
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Poles of zero dynamics system --> zeros of real system %
% zz - poles from zero dynamics system                    %
% boundary conditions are z'(0)=0 and z(1/3)=0            %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%The (N+1)th row is always invalid, therefore it is removed
M_zdyn=M_N(1:N,1:N);
S_zdyn=S_N(1:N,1:N);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% observation at x=1/3 x \in [0,1] %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
if (obpt==1/3)
  % \Delta x = 1/(3*N) instead of 1/N
  A_zdyn=(3*N)^2*S_zdyn*inv(M_zdyn);
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% observation at x=1/2 x \in [0,1] %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
if ( obpt==1/2 )
  % \Delta x = 1/(2*N) instead of 1/N
  A_zdyn=(2*N)^2*S_zdyn*inv(M_zdyn);
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

```
% observation at x=1   x \in [0,1] %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
if (obpt==1)
  % \Delta x is 1/N
  A_zdyn=N^2*S_zdyn*inv(M_zdyn);
end
zz=eig(A_zdyn); zz=sort(zz);           % poles of zero dyn. system
%------------------------------------- STEP 4 ----------------------------
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Record required number of system poles,zeros dynamics poles %
% and put them into desired output format.                    %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
if estzeros==0                        % this corresponds to N=1
  NZZ=[];
else
  % all zeros are negative. zz is sorted in ascending order so the
  % first 'estzeros' zeros is from length(zz)-estzeros+1:length(zz)
  NZZ=zz(length(zz)-estzeros+1:length(zz),1);
end
NP=p(length(p)-estpoles+1:length(p),1);
return
```

B.2 Source Code For LQR design of 1-D Acoustic Duct

<u>FINITE DIMENSIONAL APPROXIMATIONS</u>

```
function [message,sys_N,M_N,zz,p]=Duct_approx(N,felement,polezero,x_s,x_a,c,rho,K,L)
% Give state space model of Finite Elements approx, relevant outputs
% are (sys_N,M_N).  The resulting sys_N has dimension 2N+1.
```

```
% Give zeros/poles of direct approx, relevant outputs are (zz,p).
% The length of zz is N and the length of p is 2N+1.
%
% Written by Ada Cheng May 12th, 2000.
  format long
  ni=nargin
  if ni==2 | ni==3
  c=331;
  rho=1.29;
  K=0.7;
  L=4;
  x_s=2;
  x_a=0;
  end
  if ni==2
    polezero='no';
  end
  if floor(N*x_s/L) == N*x_s/L & x_s ~=0 & x_s ~=L
    s=sprintf('x_s corresponds to element, cannot compute approximations');
    message='abort'; sys_N=[]; M_N=[]; NZZ=[]; NP=[];
    disp(s);
    return
  end
  Deltax=L/N;
  %-------------------------------------- STEP 1 -----------------------------
  %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
  % Matrix Setup For Acoustic Duct                                %
  % System is in the form                                         %
```

```
% 1/N M_N \ddot z = K_N \dot z + c^2N/L S_N z + b_N_1 u, %
%                y = c_N z                                 %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Initialize all matrices as the zero matrix
M_N=zeros(N+1,N+1); S_N=M_N;
b_N_1=zeros(N+1,1); c_N=zeros(1,N+1);
M_N(1,1)=1/3; M_N(1,2)=1/6;
S_N(1,1)=-1; S_N(1,2)=1;
for j=2:N
  M_N(j,j-1)=1/6; M_N(j,j)=2/3; M_N(j,j+1)=1/6;
  S_N(j,j-1)=1; S_N(j,j)=-2; S_N(j,j+1)=1;
end
M_N(N+1,N)=1/6; M_N(N+1,N+1)=1/3;
S_N(N+1,N)=1; S_N(N+1,N+1)=-1;
K_N(N+1,N+1)=-K*c;
% Set up b_N_1
j=floor(N*x_a/L);
if j==N*x_a/L
  b_N_1(j+1,1) = 1;
else
  b_N_1(j+1,1) = j+1 -x_a/Deltax;
  b_N_1(j+2,1) = x_a/Deltax - j;
end
b_N_1=b_N_1/rho;
% Set up c_N
j=floor(N*x_s/L);                      %x_s doesn't correspond to element
c_N(1,j+1)=-1/Deltax;
c_N(1,j+2)=1/Deltax;
```

```
c_N=-rho*c^2*c_N;
%------------------------------------ STEP 2 ----------------------------
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Transform System To Standard form:      %
% set w= [v \dot v]'                %
% System becomes \dot w = A_N w + B_N u, %
%                      y = C_N w          %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
M=L/N*M_N; D=-K_N; G=-c^2*N/L*S_N;        %Janet's Setup
A_N=[zeros(N+1,N+1), eye(N+1,N+1); -inv(M)*G, -inv(M)*D];
B_N=[zeros(N+1,1) ; inv(M)*b_N_1];
C_N=[c_N, zeros(1,N+1)];
D_N=0;
tempp=eig(A_N);
if strcmp(felement,'yes')==1
  [A_N,B_N,C_N,D_N]=minreal(A_N,B_N,C_N,D_N);
  sys_N=ss(A_N,B_N,C_N,D_N);
else
  sys_N=[];
end
%p=eig(A_N);
p=tempp(find(abs(real(tempp))>10^(-8)));% p is sorted in ascending
p=esort(p);                            % order by real part.
p=p(length(p):-1:1);
if strcmp(polezero,'yes')==1
  %------------------------------------ STEP 3 ---------------------------
  %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
  % Poles of zero dynamics system --> zeros of real system %
```

```
    % zz - poles from zero dynamics system                     %
    % boundary conditions are z'(0)=0 and z'(x_s)=0            %
    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
    M=x_s/N*M_N; G=-c^2*N/x_s*S_N;      % Delta x = x_s/N
    A_zdyn=[zeros(N+1,N+1), eye(N+1,N+1); -inv(M)*G, -inv(M)*D];
    zz=eig(A_zdyn);
    zz=myminreal(zz,tempp);
  else
    zz=[];p=[];
  end
  message='continue';
  return
```

ZERO/POLE TO STATE SPACE MODEL

```
function sys=Duct_zp2ss(z,p,k)
% [a,b,c,d]=myzp2ss(z,p,k,choice) computes a state space form for
% SISO system with poles at p,zeros at z ( number of zeros must be
% less than number of poles ) and gain k.  The inputs p
% and z are assumed to be column vectors.
%
% Written by Ada Cheng April 26th, 2000.
  if length(z) >= length(p)
    disp(['The number of zeros are greater than or equal to the number' ...
        ' of poles, cannot use this program to find state space form.'])
    return
  end
  [r,c]=size(p);
  if r > 1 & c > 1
    disp('Error: Input poles is not a vector')
```

```
  return
end
[r,c]=size(z);
if r > 1 & c > 1
  disp('Error: Input zeros is not a vector')
  return
end
if imag(k) ~=0
  disp(['Gain is not real, cannot use this method to find state' ...
        ' space form']);
  return
end
len_z=length(z);
len_p=length(p);
if mod(len_z,2)==1 & mod(len_p,2)==0
  s=sprintf('Error! No. of zeros is odd, but no. of poles is even.');
  disp(s)
  return
end
if mod(len_z,2)==0 & mod(len_p,2)==1
  s=sprintf('Error! No. of zeros is even, but no. of poles is odd.');
  disp(s)
  return
end
if mod(len_z,2)==0                    % all paired up
  no_group_zero=len_z/2;
  single='n';
else                                  % one group of 1
```

```
    no_group_zero=1+(len_z-1)/2;        % remaining paired up
    single='y';
  end
  kbar=abs(k)^(1/no_group_zero);
  if len_z==1
    no_poles_per_group=len_p-1;
  else
    no_poles_per_group=floor((len_p-1)/(no_group_zero));
    if mod(no_poles_per_group,2)~=0
      no_poles_per_group=no_poles_per_group-1;
    end
  end
  no_group_poles_remain=( (len_p-1) - no_poles_per_group*no_group_zero)/2;
  sys=ss([],[],[],[]);
  j=1;
  zj=j;
  pj=j;
  test_z=[];
  test_p=[];
  while zj <=length(z)
    if zj==j
      tz=z(zj,1);
      tp=p(pj,1);
      zj=zj+1;
      pj=pj+1;
      tp=[tp;p(pj:pj+no_poles_per_group-1,1)];
      pj=pj+no_poles_per_group;
      if no_group_poles_remain > 0
```

```
      tp=[tp;p(pj:pj+1,1)];
      pj=pj+2;
      no_group_poles_remain=no_group_poles_remain-1;
    end
    test_z=[test_z;tz];
    test_p=[test_p;tp];
    sys1=zpk(tz,tp,sign(k)*kbar);
  else
    tz=z(zj:zj+1,1);
    zj=zj+2;
    tp=p(pj:pj+no_poles_per_group-1,1);
    pj=pj+no_poles_per_group;
    if no_group_poles_remain > 0
      tp=[tp;p(pj:pj+1,1)];
      pj=pj+2;
      no_group_poles_remain=no_group_poles_remain-1;
    end
    test_z=[test_z;tz];
    test_p=[test_p;tp];
    sys1=zpk(tz,tp,kbar);
  end
  if isempty(find(real(tp) >0)) ==1   %do balreal
    sys1=balreal(sys1);
  end
  tsys=sys;
  sys=Wave_series(sys,sys1);
    [a,b,c,d]=ssdata(sys);
    [Z,P]=ss2zp(a,b,c,d);
```

```
    if length(Z)~=length(test_z) | length(P)~=length(test_p)
      disp('Incorrect length \n')
      keyboard
    else
      zeros_err=(esort(test_z)-esort(Z))./esort(test_z)*100;
      poles_err=(esort(test_p)-esort(P))./esort(test_p)*100;
      if ( max(abs(zeros_err)) > 1 )
        disp('Zeros inaccuracy')
        max(abs(zeros_err))
        keyboard
      end
      if ( max(abs(poles_err)) > 1)
        disp('Poles inaccuracy')
        max(abs(poles_err))
      end
    end
end
[a,b,c,d]=ssdata(sys);
[Z,P]=ss2zp(a,b,c,d);
%Check to make sure ss form correct
if (length(Z) ~= length(z)) | (length(P) ~= length(p))
  disp('ERROR!!!Length of Zeros or Poles not correct from ss realization!!!')
  keyboard
else
  zeros_err=(esort(z)-esort(Z))./esort(z)*100;
  poles_err=(esort(p)-esort(P))./esort(p)*100;
  if ( max(abs(zeros_err)) > 1 )
    disp('Zeros inaccuracy')
```

```
      max(abs(zeros_err))
      keyboard
    end
    if ( max(abs(poles_err)) > 1)
      disp('Poles inaccuracy')
      max(abs(poles_err))
    end
  end
  return
```

B.3 Misc Source Code

Algebraic Riccati Equation Solver [1]

```
function X=myare(A,B,C,Q,R,tol)
% X=myare(A,B,C,Q,R) improves the accuracy (via Newton's Method or
% Exact Line Search) of the solution to the continuous-time algebraic
% Riccati equation
%                          -1
%       A'X + X'A - XBR  B'X +Q =0
% obtained my CARE.  The default value of tol is 10^(-12)
% List of additional non-standard matlab file required: myfmin
%
% Written by Ada Cheng May 24th, 2000.
  format long
  omaxj=50; maxj=omaxj; tol=10^(-12);
  [X,garb,garb,RR_care]=care(A,B,Q,R);
  for exact=0:1                    %run NM (exact=0) and ELS (exact=1)
```

[1] The algorithm was given by [Benner and Byers, 1998]

```
ricX=X*A+A'*X-X*B*inv(R)*B'*X+Q;
RR_care=norm(ricX,'fro')/max(1,norm(X,'fro'));
RR=RR_care;
t=1;
j=0;
while RR > tol & j <max(omaxj,maxj) & t~=0
  K=inv(R)*B'*X;
  Abar=A-B*K;
  Cbar=ricX;
  N=lyap(Abar',Cbar);
  V=N*B*inv(R)*B'*N;
  alpha=trace(ricX^2);
  beta=trace(ricX*V);
  gamma=trace(V^2);
  if exact==1
    if abs(gamma) < eps
      t=1;
    else
      t=myfmin(alpha,beta,gamma);
    end
  else
    t=1;
  end
  X=X+t*N;
  j=j+1;
  ricX=X*A+A'*X-X*B*inv(R)*B'*X+Q;
  RR=norm(ricX,'fro')/max(1,norm(X,'fro'));
  if j==maxj                              %option to increase maxj
```

```
      s=sprintf('\n Original Error is %0.5g\n',RR_care); disp(s);
      s=sprintf('Error after %d iterations with Exact=%d is %0.5g',j,exact,RR);
      disp(s);
      if exact==1 & RR_NM < tol
        s=sprintf('\nStopping ELS Iterations. Use NM result');disp(s);
      else
        keyboard
      end
    end
  end
  if exact==0                          % Save NM RR,j and X
    RR_NM=RR;
    j_NM=j;
    X_NM=X;
  else                                 % Save ELS RR,j and X
    RR_exact=RR;
    j_exact=j;
    X_exact=X;
  end
end                                    % end of for loop
s=sprintf(['\n----------------------------------------------------------' ...
           '-------------']);disp(s);
s=sprintf([' CARE  Newton Method  No. of iter   Exact Line Search   ' ...
           'No. of iter']);disp(s);
s=sprintf(['-----------------------------------------------------------' ...
           '-----------']);disp(s);
s=sprintf(['%0.5g   %0.5g          %d          %0.5g          ' ...
           '%d',RR_care,RR_NM,j_NM,RR_exact,j_exact]);disp(s);
```

```
  s=sprintf(['-----------------------------------------------------------' ...
             '-----------']);disp(s);
  if RR_NM < RR_exact
    X=X_NM;
    RR=RR_NM;
  else
    X=X_exact;
    RR=RR_exact;
  end
  if RR > RR_care
    X=origX;
    RR=RR_care;
  end
  return
function t=myfmin(alpha,beta,gamma)
% minimize the function alpha*(1-t)^2-2*beta*(1-t)^t^2+gamma*t^4
% over the interval [0,2]
%
% Written by Ada Cheng May 24th, 2000.
  format long
  soln=roots([4*gamma 6*beta 2*alpha-4*beta -2*alpha]);
  rsoln=soln(find(imag(soln)==0));
  rsoln=rsoln(find(real(rsoln)>=0 & real(rsoln)<=2));
  test_pt=[0;2;rsoln];
  func_eval=f_eval(alpha,beta,gamma,test_pt);
  [garb,index]=min(func_eval);
  t=test_pt(index);
  return
```

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