Equiangular Lines and Antipodal Covers

by

Mirhamed Mirjalalieh Shirazi

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Author's declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

It is not hard to see that the number of equiangular lines in a complex space of dimension d is at most d^2. A set of d^2 equiangular lines in a d-dimensional complex space is of significant importance in Quantum Computing as it corresponds to a measurement for which its statistics determine completely the quantum state on which the measurement is carried out. The existence of d^2 equiangular lines in a d-dimensional complex space is only known for a few values of d, although physicists conjecture that they do exist for any value of d.

The main results in this thesis are:
1. Abelian covers of complete graphs that have certain parameters can be used to construct sets of d^2 equiangular lines in d-dimensional space;

2. we exhibit infinitely many parameter sets that satisfy all the known necessary conditions for the existence of such a cover; and

3. we find the decompose of the space into irreducible modules over the Terwilliger algebra of covers of complete graphs.

A few techniques are known for constructing covers of complete graphs, none of which can be used to construct covers that lead to sets of d^2 equiangular lines in d-dimensional complex spaces. The third main result is developed in the hope of assisting such construction.

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Contents

Chapter 1

Introduction

A set of unit vectors in a d-dimensional space (complex or real) is said to be *equiangular* if the angle between any two of them is constant. Here the angle is defined as the absolute value of the usual inner product in the d-dimensional space. Gerzon (Theorem 3.5 in [16]) proved that there cannot be more than $\binom{d+1}{2}$ equiangular lines in d-dimensional real space. With a similar proof, there cannot be more than d^2 equiangular lines in d-dimensional complex space. These are called *absolute bounds* and are presented in Section 2.2.

Finding d^2 equiangular lines in complex space of dimension d is of major interest in Quantum Computing as it corresponds to measurement where its statistics determine completely the quantum state on which the measurement is carried out (see for example D'Ariano et. al. [5] or Busch [4]). Sets of d^2 equiangular lines in dimension d are only known to exist for

$$d \in \{2, \cdots, 15, 19, 24, 35, 48\}.$$

It is proved for $d = 2, 3$ by Delsarte et. al. [7], for $d = 4, 5$ by Zauner [25], for $d = 6$ by Grassl [12], for $d = 7, 19$ by Appleby [1], for $d = 8, ..., 13, 15$ by Grassl [13], and recently for $d = 14, 24, 35, 48$ by Grassl [20]. The problem is open for any other value of d. Renes et. al. [19] gave numerical solutions for any $d \leqslant 45$ and conjectured that a set of d^2 equiangular lines in d-dimensional complex space exists for any value of d. Recently a computer study by Grassl [20] found numerical solutions for any $d \leqslant 67$.

Lemmens and Seidel [16, p. 495] observed that a set of n equiangular lines in dimension d is equivalent to a Hermitian matrix with zero diagonal entries and all off diagonal entries of norm one, such that the multiplicity of

the least eigenvalue is $n - d$. They call any matrix with these properties, a *Seidel matrix*. Lemmens and Seidel [16] were the first to study the problem of finding the maximum number of equiangular lines in a given dimension. We introduce the Seidel matrix and the absolute bounds in Chapter 2. If the absolute bound is tight, then the Seidel matrix of the lines have only two eigenvalues (Godsil and Royle [11, p. 252]). Therefore, to find sets of equiangular lines of maximum size, we need to find $n \times n$ Seidel matrices with $\sqrt{n}$ as the multiplicity of the least eigenvalue. In Chapter 7, we see that from antipodal distance regular graphs of diameter three with certain parameters, we can construct such Seidel matrices.

A graph is called *distance regular* if, given any triple of non-negative integers (i, j, k), any two vertices of the graph u and v at distance i, the number of vertices at distance j from u and k from v only depends on (i, j, k) and not our choice of vertices. A distance regular graph is called *antipodal* if "being at maximum distance" is a transitive relation on the vertices of the graph.

For a graph of diameter d, and any $0 \leqslant k \leqslant d$, define A_k to be a 01 square matrix with its rows and columns indexed by the vertices of the graph, such that, the (i, j)-entry of A_k is 1 if and only if the i-th vertex and the j-th vertex are at distance k. The matrices $A_0, \cdots A_d$ are called the *distance matrices* of the graph. The distance matrices of a distance regular graph form a basis for the algebra generated by the adjacency matrix of the graph, over the field of real numbers. This algebra is called the *Bose-Mesner* algebra of the graph. We see an introduction to distance regular graphs in Chapter 3.

Each antipodal distance regular graph of diameter three is a cover of some complete graph. If a graph Y is an *r-fold cover* of another graph X, then there is a partition of the vertices of Y into sets of size r, which are called *fibres* of the cover. Each fibre corresponds to a vertex in X. Two fibres corresponding to two adjacent vertices in X have a 1-factor between them. Two fibres corresponding to two non-adjacent vertices in X have no edge between them. An introduction to covering graphs is given in Chapter 4.

In Chapter 5, we focus specifically on antipodal distance regular graphs of diameter three. We present the common parameters associated to antipodal distance regular graphs of diameter three, and the necessary conditions

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these parameters must satisfy. Then, in Chapter 6, we present some known techniques to construct antipodal distance regular graphs of diameter three.

In Chapter 7, we see that from antipodal distance regular graphs of diameter three with certain parameters, we can construct d^2 equiangular lines in d-dimensional complex space. We find parameters that not only give us d^2 equiangular lines in d-dimensional complex space, but also satisfy the necessary conditions in Chapter 5.

The antipodal distance regular graphs of diameter three constructed in Chapter 6 do not give us d^2 equiangular lines in d-dimensional complex space. In the hope of assisting us to construct antipodal distance regular graphs of diameter three which do give us d^2 equiangular lines in d-dimensional complex space, we study the *Terwilliger algebra* of antipodal distance regular graphs of diameter three in Chapter 8.

The Bose-Mesner algebra is the most common algebra associated to a distance regular graph. We can get information on the feasibility of the parameters of a distance regular graph by studying its Bose-Mesner algebra. Terwilliger [23] introduced a new algebra which, apart from the adjacency matrix, also includes the diagonal matrices with the first column of distance matrices as the diagonal. We find the decomposition of the space into irreducible modules over the Terwilliger algebra of antipodal distance regular graphs of diameter three in Chapter 8. We will use it to get information on the parameters of the graph. Finally, in Chapter 9, we see some possible directions for future research.

I should emphasize that Chapters 2 to 6 are mainly review. Chapters 7 and 8 contain the new material. Let G be an antipodal distance regular graph of diameter three. Then G is a cover of K_n for some n. Let r be the size of each fibre of the cover, and let c_2 be the number of common neighbours of two vertices at distance two in G. In Chapter 7, we prove that, if for some positive integer t,

$$ n = \left(t^2 - 1\right)^2, \qquad rc_2 = (t-1)^2(t^2 + t - 1), $$

then we can find $(t^2 - 1)^2$ equiangular lines in $(t^2 - 1)$-dimensional complex space.

In Chapter 8, we prove the following theorem using Terwilliger algebra:

1.0.1 Theorem. *Let G be an antipodal distance regular graph of diameter three with parameters n, r, c_2 and eigenvalues $n - 1, -1, \theta, \tau$ with multiplicities $1, n - 1, m_\theta, m_\tau$. If $n > m_\theta - r + 3$, then $\frac{(r-1)\tau - 1}{r}$ is an eigenvalue of the first neighbourhood, and hence r divides $\tau + 1$ if τ is an integer. Analogously, if $n > m_\tau - r + 3$, then $\frac{(r-1)\theta - 1}{r}$ is an eigenvalue of the first neighbourhood, and hence r divides $\theta + 1$ if θ is an integer.*

Chapter 2

Equiangular Lines

Any line in d-dimensional complex space can be represented by a unit vector. A set of unit vectors $\{x_1, \cdots, x_n\}$ in d-dimensional complex space is said to be *equiangular* if the inner product of any two of them has the same absolute value, and by *absolute value* we mean the Euclidean distance from the origin in the complex plane. Here the inner product of x_i and x_j is defined canonically, i.e. $x_i^* x_j$, where x_i^* is the conjugate transpose of x_i. We can define a set of real equiangular lines in the same fashion. None of the results in this chapter is new.

In this chapter, we investigate the problem of finding upper bounds for the number of equiangular lines in a space of given dimension. Gerzon (Theorem 3.5 in [16]) proved that the number of equiangular lines in d-dimensional real space cannot be more than $\binom{d+1}{2}$. Similar proof implies that there cannot be more than d^2 equiangular lines in d-dimensional complex space. These bounds are called the *absolute bounds* and are presented in Section 2.2.

Sets of equiangular lines of maximum size were first studied by Lemmens and Seidel [16], and subsequently by many others. The complex case is of special interest In Quantum Computing, where a set of d^2 equiangular lines in d-dimensional complex space is called a *Symmetric Informationally Complete Positive Operator Valued Measure*, and corresponds to a measurement where its statistics determine completely the quantum state on which the measurement is carried out (see for example D'Ariano et. al. [5] or Busch [4]).

It is proved that a set of d^2 equiangular lines in d-dimensional complex space exists for $d = 2, 3$ by Delsarte et. al. [7], for $d = 4, 5$ by Zauner [25],

for $d = 6$ by Grassl [12], for $d = 7, 19$ by Appleby [1], for $d = 8, ..., 13, 15$ by Grassl [13], and recently for $d = 14, 24, 35, 48$ by Grassl [20]. The problem is open for any other value of d. Renes et. al. [19] gave numerical solutions for any $d \leqslant 45$ and conjectured that a set of d^2 equiangular lines in d-dimensional complex space exists for any value of d. Recently a computer study by Grassl [20] found numerical solutions for any $d \leqslant 67$.

Terminology: In this chapter, whenever we say "a d-dimensional space", we mean the space could be either $\mathbb{R}^d$ or $\mathbb{C}^d$. If we want to specify which one we mean, we will say "d-dimensional real space" or "d-dimensional complex space".

2.1 Seidel Matrices

A square matrix with complex entries is called a *Seidel matrix* [11, p. 250] if

(a) S is Hermitian;

(b) all diagonal entries of S are zero; and

(c) all off-diagonal entries of S lie on the unit circle in the complex plane.

Some sources [14] call Seidel matrices, *signature matrices*.

In this section, we see that a set of n equiangular lines in dimension d is equivalent to a Seidel matrix with the least eigenvalue of multiplicity $n - d$. The results in this section are due to Seidel [16, p. 495]. Although Seidel's arguments were for the real case, the exact same arguments apply to the complex case as well.

Let $\{x_1, \cdots, x_n\}$ be a set of equiangular lines in a d-dimensional space. Then for all i and j, $1 \leqslant i, j \leqslant n$, and for some $\gamma \in \mathbb{R}$ we have

$$|x_i^* x_j|^2 = \begin{cases} \gamma & \text{if} \quad i \neq j, \\ 1 & \text{if} \quad i = j. \end{cases}$$

Note that, by the Cauchy-Schwarz inequality, we get

$$\gamma = |\langle x_i, x_j \rangle|^2 \leqslant |x_i|^2 |x_j|^2 = 1,$$

and if $\gamma = 1$, then x_i is a multiple of x_j. But all x_i's are unit vectors. So if $\gamma = 1$, then all x_i's are equal. This case is not interesting, since if all x_i's are equal, we can make n as large as we want. So we always assume

$$\gamma < 1.$$

We define the *Gram matrix* [11, p. 250] of $\{x_1, \cdots, x_n\}$ as the matrix with $x_i^* x_j$ as its (i, j)-entry. Therefore all diagonal entries of G are 1 and all off-diagonal entries of G have absolute value $\sqrt{\gamma}$. Also G is clearly Hermitian, i.e. $G^* = G$. Therefore we can write G as

$$G = I + \sqrt{\gamma}S,$$

where S is a Seidel matrix. We call S the *Seidel matrix* of $x_1, \cdots, x_n$. We also call the above properties the *Seidel properties*. Now we show that this procedure is reversible. This observation is due to Lemmens and Seidel [16, p. 495].

2.1.1 Theorem. *Any square matrix that satisfies the Seidel properties is the Seidel matrix for some set of equiangular lines. More explicitly, if we have an $n \times n$ matrix satisfying the Seidel properties, and with m as the multiplicity of its least eigenvalue, then it is a Seidel matrix for a set of n equiangular lines in dimension $n - m$.*

Proof. . Let S be an $n \times n$ matrix that satisfies the Seidel properties. Since S is Hermitian, all the eigenvalues of S are real. Let τ be the smallest eigenvalue of S. We show that G defined as

$$G = I - \frac{1}{\tau}S \tag{2.1.1}$$

is the Gram matrix for some set of equiangular lines. To do this, first, we show that G is positive semidefinite.

Since S is Hermitian, so is G. So to prove that G is positive semidefinite, it is enough to show that all its eigenvalues are non-negative. But from Equation 2.1.1, each eigenvalue of G is of the form

$$1 - \frac{1}{\tau}\lambda,$$

where λ is an eigenvalue of S.

Now we show that τ is negative. Since S satisfies the Seidel properties, all of its diagonal entries must be zero, and so its trace must be zero. So the sum of the eigenvalues of S must be zero. Therefore the least eigenvalue, τ, must be negative.

Hence, $1 - \lambda/\tau \geqslant 0$; and equality occurs if and only if λ is equal to τ. Therefore G is positive semidefinite with rank

$$d = n - m,$$

where m is the multiplicity of τ as an eigenvalue of S. Thus by linear algebra, there exist a $d \times n$ matrix U such that

$$G = U^*U.$$

Let $x_1, \cdots, x_n$ be the columns of U. Then the (i,j)-entry of G is $x_i^* x_j$. On the other hand, by Equation 2.1.1, and since S satisfies the Seidel properties, we get all diagonal entries of G are one and all off-diagonal entries of G have absolute value $-1/\tau$. So $\{x_1, \cdots, x_n\}$ is a set of equiangular lines with G as its Gram matrix, and with S as its Seidel matrix. □

2.2 Absolute Bound

In this section we see that the number of equiangular lines in a d-dimensional complex space cannot exceed d^2, whereas in a d-dimensional real space it cannot exceed $\binom{d+1}{2}$. These bounds are called *absolute bounds*; and were proved originally by Gerzon (Theorem 3.5 in Lemmens and Seidel [16]). Here, we follow the proof of Godsil and Royle [11, p. 251], which has the following lemma as the main part of the proof.

2.2.1 Lemma. *Let $\{x_1, \cdots, x_n\}$ be a set of equiangular lines in a d-dimensional space, and, for each i, $1 \leqslant i \leqslant n$, let X_i denote the projection operator, $x_i x_i^*$, onto x_i. Then $\{X_1, \cdots, X_n\}$ is linearly independent over $\mathbb{R}$.*

Proof. Let the real numbers $c_1, \cdots, c_n$ be such that

$$\sum_{i=1}^{n} c_i X_i = 0.$$

Multiply both sides by X_j from left to get

$$\sum_{i=1}^{n} c_i X_j X_i = 0.$$

Now by taking the trace of both sides, we get

$$\sum_{i=1}^{n} c_i \operatorname{tr}(X_j X_i) = 0.$$

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But we have

$$
\begin{aligned}
\mathrm{tr}(X_j X_i) &= \mathrm{tr}(x_j x_j^* x_i x_i^*) \\
&= \mathrm{tr}(x_i^* x_j x_j^* x_i) \\
&= (x_i^* x_j)\,\overline{(x_i^* x_j)} \\
&= |x_i^* x_j|^2 .
\end{aligned}
$$

Now since $\{x_1, \cdots, x_n\}$ is equiangular, for some γ we have

$$
|x_i^* x_j|^2 = \begin{cases} \gamma & \text{if} \quad i \neq j, \\ 1 & \text{if} \quad i = j. \end{cases}
$$

Thus we get

$$
\begin{aligned}
0 &= \sum_{i=1}^{n} c_i \, \mathrm{tr}(X_j X_i) \\
&= \left(\sum_{i=1}^{n} c_i - c_j \right) \gamma + c_j \\
&= (1 - \gamma) c_j + \gamma \sum_{i=1}^{n} c_i.
\end{aligned}
$$

Now, since the left hand side does not depend on j, and since $1 - \gamma$ is nonzero, we get all c_i's are equal. Let c be their common value. Then we get

$$
\sum_{i=1}^{n} c X_i = 0.
$$

If c is non-zero, we get

$$
\sum_{i=1}^{n} X_i = 0.
$$

But this is a contradiction, since for any i we have

$$
\mathrm{tr}\,(X_i) = \mathrm{tr}\,(x_i x_i^*) = x_i^* x_i = 1, \tag{2.2.1}
$$

and therefore,

$$
\mathrm{tr}\left(\sum_{i=1}^{n} X_i \right) = n.
$$

So c must be zero, which means that $\{X_1, \cdots, X_n\}$ is an independent set. $\square$

The following proof is due to Godsil and Royle [11, p. 251]. Although they proved it for the real case, similar argument gives the result in the complex case.

2.2.2 Theorem. *The number of equiangular lines in $\mathbb{C}^d$ is at most d^2 and in $\mathbb{R}^d$ is at most $\binom{d+1}{2}$.*

Proof. Let $\{x_1, \cdots, x_n\}$ be a set of equiangular lines in a d-dimensional space (complex or real). As before, for each i, $1 \leqslant i \leqslant n$, we denote the projection onto x_i by X_i. Then by Lemma 2.2.1, we get $\{X_1, \cdots, X_n\}$ is linearly independent over $\mathbb{R}$. So n cannot be larger than the dimension of the vector space of all $d \times d$ Hermitian matrices over $\mathbb{R}$. To find this dimension, note that each Hermitian matrix is determined by its diagonal entries and half of its off-diagonal entries. In the complex case, we have two degrees of freedom for each of the $(d^2 - d)/2$ off-diagonal entries, whereas we have just one degree of freedom for each of the d diagonal entries, since the diagonal entries of Hermitian matrices are real. Therefore we get the dimension as

$$2\left(\frac{d^2 - d}{2}\right) + d = d^2.$$

But clearly, in the real case, we have just one degree of freedom everywhere. So the dimension in the real case is

$$\frac{d^2 - d}{2} + d = \frac{d^2 + d}{2} = \binom{d+1}{2}.$$

This completes the proof. □

2.3   Relative Bound

The absolute bound gives us an upper bound on the number of equiangular lines in terms of the dimension of the space. In this section, we find an upper bound on the number of lines which also depends on the common angle between the lines. The new bound is called the *relative bound*, and is the same for the real and complex case, unlike the absolute bound. In the end of his section, we also see that if the relative bound is tight, then the Seidel matrix of the lines has only two eigenvalues. The relative bound is stated in Theorem 2.3.2. It follows from Theorem 2.3.1 and Theorem 2.3.2 that the relative bound is tight if the absolute bound is tight. All the results in this section can be found in Godsil and Royle [11] (Sections 11.3 and 11.4).

As in the previous section, we always assume $\{x_1, \cdots, x_n\}$ is a set of equiangular lines in a d-dimensional space (complex or real), and for each i, $1 \leqslant i \leqslant n$, we denote the projection onto x_i by X_i. We also assume γ is the constant such that for any distinct i and j,

$$|x_i^* x_j|^2 = \gamma.$$

2.3.1 Theorem. *If the absolute bound is tight, then*

$$\sum_{i=1}^{n} X_i = \frac{n}{d} I. \tag{2.3.1}$$

Proof. First note that, by Lemma 2.2.1, $\{X_1, \cdots, X_n\}$ is linearly independent. On the other hand, since the absolute bound is tight, by the proof of Theorem 2.2.2, n must be equal to the dimension of the vector space of all $d \times d$ Hermitian matrices over $\mathbb{R}$. So $\{X_1, \cdots, X_n\}$ must be a basis for this vector space. Therefore, we must be able to find real numbers $\{c_1, \cdots, c_n\}$ such that

$$I = \sum_{i=1}^{n} c_i X_i.$$

Following exactly the proof of Lemma 2.2.1, we get all c_i's must be equal. Let c be their common value. So we get

$$I = c \sum_{i=1}^{n} X_i.$$

To find c, we take the trace of both sides. By Equation 2.2.1, we know that the trace of each X_i is one. So we get

$$c = \frac{d}{n}.$$

This completes the proof. □

We present the relative bound in the next theorem, which is proved in Godsil and Royle [11, p. 253]. It follows from the next theorem and Theorem 2.3.1 that the relative bound is in a sense weaker than the absolute bound. To be more explicit, the relative bound is tight if the absolute bound is tight.

2.3.2 Theorem. *If $d\gamma < 1$, then*

$$n \leqslant \frac{d - d\gamma}{1 - d\gamma}.$$

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Furthermore, the bound is tight if and only if

$$\sum_{i=1}^{n} X_i = \frac{n}{d} I.$$

Proof. First note that, for any matrix M, we have M^*M is positive semidefinite, and so its trace is non-negative. For any real number α, let

$$M(\alpha) = \alpha I - \sum_{i=1}^{n} X_i.$$

Then, since $M(\alpha)$ is Hermitian, we get

$$\operatorname{tr}\left(M(\alpha)^*M(\alpha)\right) = \alpha^2 d - 2\alpha \sum_{i=1}^{n} \operatorname{tr}(X_i) + \sum_{i=1}^{n} \operatorname{tr}\left(X_i^2\right) + 2\sum_{i\neq j} \operatorname{tr}(X_i X_j).$$

But as we saw in the proof of Lemma 2.2.1, we have $\operatorname{tr}(X_i)$ is one, and

$$\operatorname{tr}(X_i X_j) = |x_i^* x_j|^2 = \begin{cases} \gamma & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$$

So we get

$$\operatorname{tr}\left(M(\alpha)^*M(\alpha)\right) = \alpha^2 d - 2\alpha n + n + (n^2 - n)\gamma.$$

By setting the derivative of the right hand side to zero, we find that it is minimum when α is $\frac{n}{d}$. Since the above expression must be non-negative for any value of α, we get

$$0 \leqslant \left(\frac{n}{d}\right)^2 d - 2\left(\frac{n}{d}\right) n + n + (n^2 - n)\gamma = \frac{n}{d}\left(-n + d + d\gamma(n-1)\right),$$

which implies

$$n(1 - d\gamma) \leqslant d - d\gamma.$$

Thus, if $d\gamma < 1$, we get

$$n \leqslant \frac{d - d\gamma}{1 - d\gamma}.$$

Also note that equality holds if and only if

$$\operatorname{tr}\left(M\left(\frac{n}{d}\right)^* M\left(\frac{n}{d}\right)\right) = 0,$$

and this happens only if $M(\frac{n}{d})$ is the zero matrix. This completes the proof. $\square$

By Theorem 2.1.1, to find a set of equiangular lines, all we need is a square matrix that satisfies the Seidel properties. The next theorem in Godsil and Royle [11, p. 252] shows that if we want the relative bound to be tight, then the matrix we are looking for must have only two eigenvalues.

2.3.3 Theorem. *Let* $\{x_1, x_2, \cdots x_n\}$ *be a set of equiangular lines in a d-dimensional space, with γ as the square of the angle between the lines. If the relative bound is tight, then the eigenvalues of the Seidel matrix are*

$$-\frac{1}{\sqrt{\gamma}}, \qquad \frac{n-d}{\sqrt{\gamma}d},$$

with multiplicities $n - d$ and d, respectively.

Proof. Let U be a $d \times n$ matrix with $x_1, \cdots, x_n$ as its columns. Then the (i, j)-entry of U^*U is $x_i^*x_j$. So U^*U is the Gram matrix, G, of the lines. On the other hand we have

$$UU^* = x_1x_1^* + \cdots + x_nx_n^* = X_1 + \cdots + X_n.$$

But since the relative bound is tight, by Corollary 2.3.2, we get

$$UU^* = \frac{n}{d}I.$$

Thus, we have

$$G^2 = U^*UU^*U = \frac{n}{d}U^*U = \frac{n}{d}G.$$

Since G is Hermitian, we get that the minimal polynomial of G is

$$x^2 - \frac{n}{d}x.$$

Therefore, the eigenvalues of G are 0 and $\frac{n}{d}$. Since $UU^* = \frac{n}{d}I_d$, then U and I_d have the same rank. Since $U^*U = G$, then G and U have the same rank. So G has rank d, and thus the null space of G has rank $n - d$. Therefore, 0 has multiplicity $n - d$ and $\frac{n}{d}$ has multiplicity d as eigenvalues of G. Now note that by the definition of the Seidel matrix,

$$S = \frac{1}{\sqrt{\gamma}}(G - I),$$

and the result follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

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Chapter 3

Algebra of Distance Regular Graphs

It is always nice to be able to apply Algebra in Graph Theory. It provides us with powerful algebraic techniques in dealing with our graph theoretical problems. One type of graphs that the algebraic techniques can be effectively applied to is *distance regular graphs*, which were defined in Chapter 1.

The *adjacency matrix* of a graph is defined to be a 01-matrix with its rows and columns indexed by the vertices of the graph, where the (i, j)-entry is 1 if and only if the i-th vertex and the j-th vertex are adjacent. In other words, the (i, j)-entry is 1 if and only if the distance between the i-th vertex and the j-th vertex is 1. So we denote the adjacency matrix by A_1, and we define A_k for other values of k similarly, i.e. the (i, j)-entry of A_k is 1 if and only if the distance between the i-th vertex and the j-th vertex is k. If d is the diameter of the graph, the matrices $A_0, A_1, \cdots, A_d$ are called the *distance matrices* of the graph.

In Section 3.3, we see that the combinatorial definition of the distance regular graphs given in Chapter 1, has a purely algebraic interpretation in terms of the *distance matrices* of the graph. More explicitly, we see that a graph of diameter d is distance regular is and only if the algebra generated by its distance matrices has dimension $d + 1$.

The set of distance matrices of a distance regular graph is an important example of an *Association Scheme*. The algebra generated by the matrices of an association scheme is called the *Bose-Mesner Algebra* of the scheme.

3.1 Distance Regular Graphs

The line graph of the Petersen graph is illustrated in Figure 3.1. Choose an ordered triple of non-negative integers, for example $(2, 1, 2)$. Let u and v be two vertices at distance 2 as illustrated in Figure 3.1. Now, we see that there are exactly two vertices at distance 1 from u and at distance 2 from v. Such vertices are denoted by x and y in Figure 3.1. Now, if we choose any two other vertices at distance 2, and we count the number of vertices at distance 1 from u and at distance 2 from v, we see that the answer is again two.
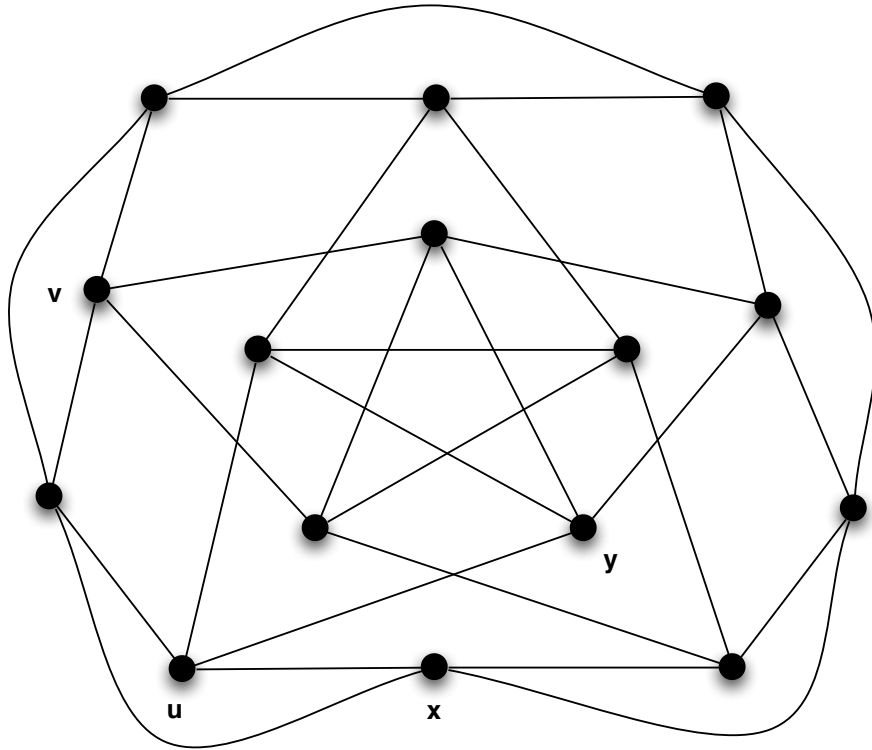

Figure 3.1: line graph of the Petersen graph

By the definition of distance regular graphs in Chapter 1, for any ordered triple of non-negative integers (i, j, k), and any two vertices u and v at distance k from each other, the number of vertices at distance i from u and

j from v only depends on (i, j, k). So if a graph is distance regular, for each triple of non-negative integers (i, j, k), we have a constant associated to it, which we denote by $p_{i,j}(k)$. So in the line graph of the Petersen graph, we have $p_{2,1}(2)$ is equal to 2. Note that we only need to consider the case where i, j, and k are all less than or equal to the diameter of the graph, since otherwise, $p_{i,j}(k)$ is equal to zero. The constants $p_{i,j}(k)$ are called the *intersection numbers* of the distance regular graph.

3.2 Distance Partition

The line graph of the Petersen graph is illustrated again in Figure 3.2. The vertices are partitioned according to their distance from the vertex on the left. This is called the *distance partition* of the graph with respect to that vertex. Given any ordered pair of sets in the partition, for each vertex in the first set, the number of its neighbors in the second set is a constant. Any partition of the vertices of a graph with that property is called an *equitable partition*. Let

$$\mathcal{B} = \{V_0, \cdots, V_d\}$$

be the distance partition of a regular graph G with respect to vertex v of G, where V_i is the set of vertices at distance i from v. So

$$V_0 = \{v\}.$$

If $\mathcal{B}$ is an equitable partition, then there are $2d$ integers $b_0, \cdots, b_{d-1}$, and, $c_1, \cdots, c_d$ such that, for each vertex of V_i, the number of its neighbors in V_{i-1} and V_{i+1} are c_i and b_i, respectively. Note that, since $\mathcal{B}$ is a distance partition, there are no edges between V_i and V_j if $|i - j| > 1$. The sequence

$$\{b_0, \cdots, b_{d-1}; c_1, \cdots, c_d\}$$

is called the *intersection array* of G with respect to v. Note that, since we assumed G is regular, we get the graph induced on each V_i is regular with valency

$$a_i = k - c_i - b_i,$$

where k is the valency of G. Some authors [2, p. 157] include a_i's in the intersection array. We did not add them since they can be determined from c_i's and b_i's.

Note that if G is distance regular, then b_i is just the intersection number $p_{i+1,1}(i)$ we defined for distance regular graphs in Section 3.1. Similarly, c_i

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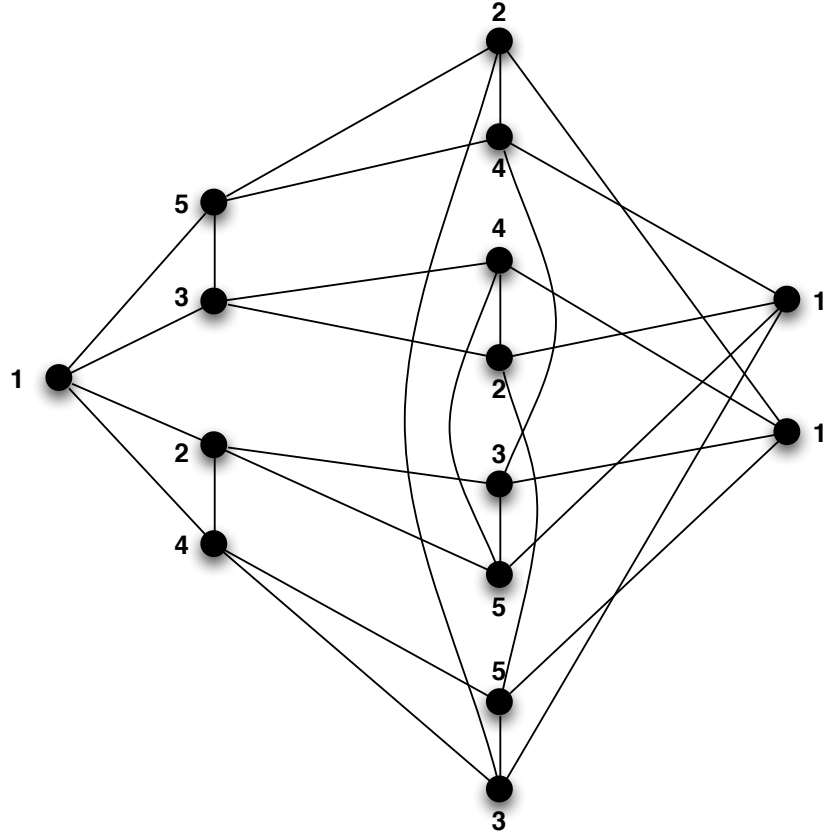
Figure 3.2: Distance partition

is $p_{i-1,1}(i)$ and a_i is $p_{i,1}(i)$. Also since the maximum distance from v is d, we must have $p_{d,d}(0)$ is nonzero, and, $p_{d+i,d+i}(0) = 0$ for any positive integer i. Therefore, the distance partition with respect to any vertex must have $d + 1$ cells.

Above arguments imply that the distance partition of a distance regular graph with respect to any of its vertices is equitable with the same intersection array. Actually the converse holds as well, by Proposition 20.8 in Biggs [2, p. 160]. So we have the following theorem.

3.2.1 Theorem. *A graph is distance regular if and only if the distance partition of the graph with respect to any of its vertices is equitable with the same intersection array.*

Note that we defined the intersection array of a distance regular graph using just the intersection numbers $p_{i+1,1}(i)$ and $p_{i-1,1}(i)$. So for a distance regular graph of diameter d, while we can have up to $(d+1)^3$ intersection numbers, we only have $2d$ of them in the intersection array. By Proposition 20.8 in Biggs [2, p. 160], all intersection numbers can be determined from the intersection array.

3.3 Distance Matrices

The *adjacency matrix* of a graph is defined to be a 01-matrix with its rows and columns indexed by the vertices of the graph, where the (i,j)-entry is 1 if and only if the i-th vertex and the j-th vertex are adjacent. The adjacency matrix of the graph in Figure 3.3 is as follows.

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

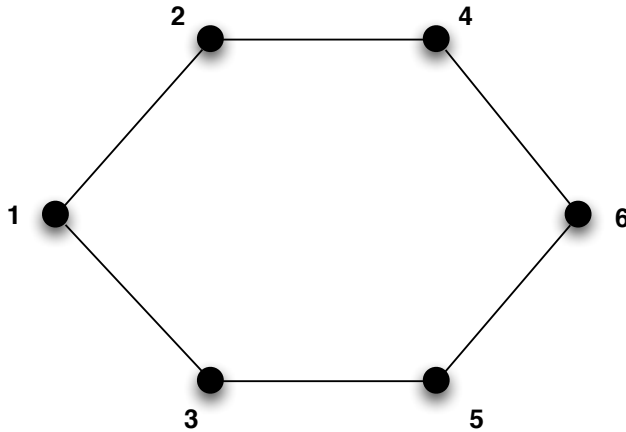Two vertices are adjacent if and only if they are at distance one from

Figure 3.3: cycle of length six

each other. So we represent the adjacency matrix by A_1, and we define

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A_k analogously for other non-negative inetegers. More explicitly, A_k is a 01-matrix with its rows and columns indexed by the vertices of the graph, where the (i, j)-entry is 1 if and only if the distance between the i-th vertex and the j-th vertex is k. So for the graph in Figure 3.3, we have

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since the cycle of length 6 has diameter three, for any k larger than 3, we have A_k is the zero matrix. For a general graph, if k is larger than the diameter of the graph, then A_k is the zero matrix. Also, A_0 is always the identity matrix. If d is the diameter of the graph, the matrices $A_0, A_1, \cdots, A_d$ are called the *distance matrices* of the graph.

Let A and B be two matrices of the same size. Define the *Schur product* $A \circ B$ of A and B to be a matrix with the same size as A and B with its (i, j)-entry obtained by multiplying the (i, j)-entries of A and B.

Let G be a graph of diameter d, with distance matrices $A_0, A_1, \cdots, A_d$. Then, for any (i, j), exactly one of the distance matrices has 1 in its (i, j) position. This implies that, for any distinct i and j,

$$A_i \circ A_j = 0,$$

and

$$A_0 + A_1 + \cdots + A_d = J$$

where J denotes the all ones matrix.

Since the Schur multiplication is distributive over addition, we can prove that $A_0, A_1, \cdots, A_d$ are linearly independent. So the dimension of the algebra generated by $A_0, A_1, \cdots, A_d$ is at least $d+1$. The following is a standard fact on distance regular graphs (see for example Biggs [2, p. 160]).

3.3.1 Theorem. *A graph of diameter d is distance regular if and only if the dimension of the algebra generated by its distance matrices is equal to $d + 1$.*

Proof. Let $A_0, A_1, \cdots, A_d$ denote the distance matrices of the graph. Since the set of distance matrices is a linearly independent set of size $d + 1$, the algebra generated by the distance matrices has dimension $d + 1$ if and only if any matrix in the algebra can be written as a linear combination of the distance matrices. But all matrices in the algebra can be written as a linear combination of the distance matrices if and only if for any i and j, we can write $A_i A_j$ as a linear combination of the distance matrices. In other words, there exist constants $p_{i,j}(t)$ such that

$$A_i A_j = \sum_{t=0}^{d} p_{i,j}(t) A_t.$$

But the (r, s)-entry of the left hand side is the number of vertices at distance i from the r-th vertex and at distance j from the s-th vertex, while if the distance between the r-th vertex and the s-th vertex is k then the (r, s)-entry of the right hand side is $p_{i,j}(k)$, which is a constant independent of r and s. But this is exactly the definition of distance regular graphs. □

3.4 Bose-Mesner Algebra

Let G be a distance regular graph of diameter d, with $A_0, A_1, \cdots, A_d$ as its distance matrices. Then, by Theorem 3.3.1, there exist constants $p_{i,j}(k)$ such that

$$A_i A_j = \sum_{t=0}^{d} p_{i,j}(k) A_k. \tag{3.4.1}$$

Note that by the proof of Theorem 3.3.1, the constants $p_{i,j}(k)$ are just the intersection numbers that we defined in Section 3.1. So we could define the *intersection numbers* of a distance regular graph alternatively, as the constants $p_{i,j}(k)$ that satisfy Equation 3.4.1.

Since all the distance matrices are symmetric, the right hand side of Equation 3.4.1 is always symmetric. So, for any i and j, we have $A_i A_j$ is a symmetric matrix. But if X and Y are two symmetric matrices, then

$$(XY)^T = Y^T X^T = YX.$$

So X and Y commute if and only if XY is symmetric. So the distance matrices of a distance regular graph commute. Consequently, the algebra generated by the distance matrices of a distance regular graph is a commutative algebra.

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Let $\mathcal{A}$ denote the set of distance matrices, $\{A_0, A_1, \cdots, A_d\}$, of a distance regular graph. Then we have

- $A_0 = I$.

- $\sum_{i=0}^{d} A_i = J$.

- $A_i^T \in \mathcal{A}$ for each i.

- $A_i A_j = A_j A_i \in \text{span}(\mathcal{A})$.

Any set of 01-matrices satisfying the above four conditions is called an *Association Scheme*. The algebra generated by the matrices of an association scheme is called the *Bose-Mesner Algebra* of the scheme.

Not every association scheme comes from a distance regular graph (see Zhdan-Pushkin and Ustimenko [26, p. 697]). An association scheme consisting of distance matrices of a distance regular graph is called a *metric association scheme* (see for example Godsil [9, p. 13]). An important property exclusive to metric association schemes is that each A_i can be written as a polynomial of degree i in A_1 (Lemma 20.6 in Biggs [2]). To make our arguments simpler, we only deal with metric association schemes in this chapter, although most of what we say holds for any association scheme.

3.5   Eigenvalues

For real symmetric matrices, the minimal polynomial (monic polynomial of minimum degree that annihilates the matrix), does not have repeated roots. So the number of distinct eigenvalues is equal to the degree of the minimal polynomial, which in turn is equal to the dimension of the algebra generated by the matrix.

On the other hand, in a distance regular graph, since each distance matrix is a polynomial in terms of the adjacency matrix, we get that the algebra generated by the distance matrices is just the algebra generated by the adjacency matrix. So, in a distance regular graph the number of distinct eigenvalues is equal to the dimension of the algebra generated by the distance matrices. Therefore by Theorem 3.3.1, any distance regular graph of diameter d has $d + 1$ distinct eigenvalues.

It is a very useful fact that the distance matrices in a distance regular graph are polynomials in terms of the adjacency matrix of the graph, in

	Eigenvalues				
A_0	1	1	1	$\cdots$	1
A_1	λ_0	λ_1	λ_2	$\cdots$	λ_d
A_2	$p_2(\lambda_0)$	$p_2(\lambda_1)$	$p_2(\lambda_2)$	$\cdots$	$p_2(\lambda_d)$
A_3	$p_3(\lambda_0)$	$p_3(\lambda_1)$	$p_3(\lambda_2)$	$\cdots$	$p_3(\lambda_d)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
A_d	$p_d(\lambda_0)$	$p_d(\lambda_1)$	$p_d(\lambda_2)$	$\cdots$	$p_d(\lambda_d)$

Table 3.1: eigenvalues of the distance matrices

particular in finding the eigenvalues of the distance matrices. Suppose that λ is an eigenvalue of a square matrix A, with corresponding eigenvector x. So we have

$$Ax = \lambda x.$$

Then for any polynomial p, we get

$$p(A)x = p(\lambda)x,$$

and so x is also an eigenvector of $p(A)$, but corresponding to eigenvalue $p(\lambda)$.

Let G be a distance regular graph of diameter d with $A_0, \cdots, A_d$ as its distance matrices. Then A_1 has $d+1$ distinct eigenvalues say $\lambda_0, \cdots \lambda_d$. For each i, $0 \leqslant i \leqslant d$, let p_i be a polynomial of degree i such that

$$A_i = p_i(A_1).$$

Then, for any $0 \leqslant i, j \leqslant d$, any eigenvector of A_1 corresponding to eigenvalue λ_j is also an eigenvector of A_i, but corresponding to eigenvalue $p_i(\lambda_j)$. The table of eigenvalues of the distance matrices is given in Table 3.1. Note that

$$p_0(x) = 1, \qquad p_1(x) = x.$$

When $i \neq 1$, then $p_i(\lambda_0), p_i(\lambda_1), \cdots, p_i(\lambda_d)$ are not necessarily distinct. We have A_i has at most $d+1$ distinct eigenvalues with its eigenspaces generated by (one or more) eigenspace(s) of A_1. For simplicity, we denote $p_i(\lambda_j)$ by $p_i(j)$. Let

$$\delta_{ij} = \begin{cases} 0 & \text{if} \quad i \neq j, \\ 1 & \text{if} \quad i = j, \end{cases}$$

denote the Kronecker delta.

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3.5.1 Lemma. *Let G be an n vertex distance regular graph of diameter d, with $E_0, E_1, \cdots, E_d$ as projections onto its eigenspaces. Then, for any i and j, we have*

(a) $E_i E_j = \delta_{ij} E_i$, and

(b) $A_i = \sum_{j=0}^{d} p_i(j) E_j$.

Proof. Any projection is an idempotent, and so for any i, we have

$$E_i^2 = E_i.$$

On the other hand, since A_1 is symmetric, eigenvectors corresponding to distinct eigenvalues of A_1 are orthogonal, and so for any distinct i and j, we get

$$E_i E_j = 0.$$

Since A_1 is symmetric, its eigenspaces span $\mathbb{R}^n$. Therefore, for any x in $\mathbb{R}^n$, we get

$$E_0 x + E_1 x + \cdots E_d x = x.$$

Hence,

$$E_0 + E_1 + \cdots E_d = I.$$

So for any i, we get

$$A_i E_0 + A_i E_1 + \cdots A_i E_d = A_i.$$

Since, for any j, each column of E_j is an eigenvector of A_i corresponding to eigenvalue $p_i(j)$, we have

$$A_i E_j = p_i(j) E_j,$$

and therefore,

$$p_i(0) E_0 + p_i(1) E_1 + \cdots p_i(d) E_d = A_i.$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Let G be a distance regular graph of diameter d. Let $W_0, W_1, \cdots, W_d$ be the eigenspaces of G corresponding to its eigenvalues. For each i, $0 \leqslant i \leqslant d$, define U_i to be a matrix with its columns forming an orthonormal basis for W_i. Then, if E_i denotes the projection onto W_i, we have

$$E_i = U_i U_i^T.$$

This explicit formulation for the projection matrices can be used to give another proof for Lemma 3.5.1.

3.6 Krein Parameters

Let G be a distance regular graph of diameter d with v vertices. Let $\mathcal{A}$ denote the algebra generated by the distance matrices of G, and let $E_0, E_1, \cdots E_d$ be the projections onto the eigenspaces of G. By Lemma 3.5.1, all of the distance matrices can be written in terms of the projection matrices. But the set of distance matrices has the same size as the set of projection matrices, and therefore it forms a basis for $\mathcal{A}$. So the set of projection matrices must also be a basis for $\mathcal{A}$. So all of the projection matrices can be written as linear combinations of the distance matrices.

On the other hand, since

$$A_i \circ A_j = \delta_{ij} A_i,$$

and since the set of distance matrices forms a basis for $\mathcal{A}$, we get that $\mathcal{A}$ is closed under the Schur product. In particular, we get that $E_i \circ E_j$ is in $\mathcal{A}$, and hence, can be written as a linear combination of projection matrices. Therefore, we can find constants $q_{i,j}(k)$ such that

$$E_i \circ E_j = \frac{1}{v} \sum_{k=0}^{d} q_{i,j}(k) E_k.$$

Recall that v is the number of vertices of the graph. The constants $q_{i,j}(k)$ are called the *Krein parameters* of the Bose-Mesner Algebra. Following Delsarte [6], we have a scaling factor of $1/v$ in our definition. This is more common, although some may follow the definition by Seidel [22] which is without the scaling factor.

The following makes clear the duality between distance matrices and projection matrices. Note that the all ones matrix, which we denote by J, is the identity matrix with respect to the Schur product.

$$
\begin{array}{ll}
A_i \circ A_j = \delta_{ij} A_i, & E_i E_j = \delta_{ij} E_i, \\
A_0 + \cdots + A_d = J, & E_0 + \cdots + E_d = I, \\
A_i A_j = \sum_{k=0}^{d} p_{i,j}(k) A_k, & E_i \circ E_j = \frac{1}{v} \sum_{k=0}^{d} q_{i,j}(k) E_k.
\end{array}
$$

So the Krein parameters for projection matrices are analogues of intersection numbers for distance matrices. The following result in Brouwer, Cohen, and Neumaier [3, p. 132] gives the Krein parameters in terms of the eigenvalues of the distance matrices.

3.6.1 Theorem. *Let G be a distance regular graph with v vertices. Let the eigenvalues of G be $\lambda_0, \lambda_1, \cdots, \lambda_d$ with multiplicities $m_0, m_1, \cdots, m_d$. Then we have*

$$q_{i,j}(k) = \frac{m_i m_j}{v} \sum_{s=0}^{d} \frac{p_s(i)p_s(j)p_s(k)}{p_s(0)^2}. \qquad (3.6.1)$$

Chapter 4

Covering Graphs

Consider the graphs X and Y in the picture below. Above each vertex in X there are three vertices in Y, and above each edge there is a 1-factor. We say that Y is a 3-fold cover of X.
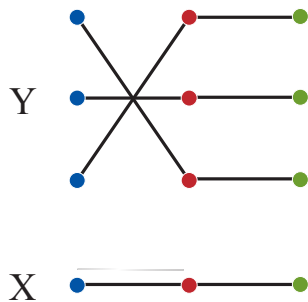
Figure 4.1: Y is a 3-fold cover of X

In general, let X and Y be two graphs with a surjective map ϕ from the vertex set of Y to the vertex set of X, such that the pre-image of each vertex in X is an independent set of size r in Y. Then Y together with ϕ is called an *r-fold cover* of X if, for any two adjacent vertices X, there is a 1-factor between their pre-images in Y, and for any two non-adjacent vertices in X, there is no edge between their pre-images. If ϕ is clear from the context, we just say that Y is an r-fold cover of X. The pre-image of each vertex in X is called a *fibre* of the cover. So fibres partition the vertex set of the cover into sets of size r.

4.0.2 Example. C_6 is a 2-fold cover of C_3 as illustrated in Figure 4.2.

Figure 4.2: C_6 as a 3-fold cover of C_3

Remark. Covering spaces defined in topology are closely related to graph coverings that we deal with here. For a short introduction on covering spaces see Jost [15] (Section 1.3). In fact if a graph Y is a cover of another graph X then, regarding $V(X)$ and $V(Y)$ as discrete topological spaces, we have $V(Y)$ together with the map that sends e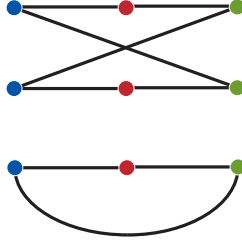ach fibre in Y to the corresponding vertex in X is a covering space of $V(X)$. On occasion, when we introduce a new concept about graph coverings we point out the connection to its analogue in topological covering spaces. Nevertheless, no knowledge of topological covering spaces is needed in reading this thesis as we never take advantage of these connections.

4.1 Preliminaries

Note that if Y is an r-fold cover of X, then, by contracting each fibre in Y, we get X with each edge replaced by r multiple edges. That might justify the name of "cover". But is that property enough to characterize covers? With a little thought you can see the answer is "No". That property just guarantees that there are r edges between any two fibres. However we need the extra condition that the r edges between any two fibres form a 1-factor.

As each 1-factor can be viewed as a permutation, we can describe any cover with a so-called *symmetric arc function*. We define the *arc set* of a graph X to be the set of ordered pairs (u, v) such that uv is an edge in X. A *symmetric arc function of index r* is a function f from the arc set of X to S_r, the symmetric group on r elements, such that $f(u, v) = f(v, u)^{-1}$, for

each $uv \in E(X)$. This condition is to guarantee that the cover defined by f is a simple graph. For instance, in Figure 4.3 the value of symmetric arc
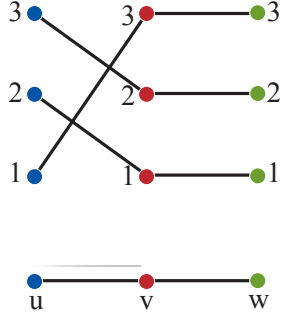

Figure 4.3: representing 1-factors by permutations

function on (u, v) is (132), while the value of the symmetric arc function on (v, u) is (123) which is the inverse of (132).

Observe that in Figure 4.3, if we apply the permutation (132) to the fibre corresponding to u and change the value of the symmetric arc function on (u, v) and (v, u) to the identity permutation, then the covering graph remains unchanged. In general, given a graph, X, and an r-fold cover of it, Y, if we permute the vertices in each fibre, then we can change the symmetric arc function so that the covering graph, Y, remains the same. More explicitly, if for each $u \in X$ we apply permutation τ_u to the fibre corresponding to u, then to get the same covering graph Y all we need to do is to re-define the symmetric arc function on arc (u, v) as

$$\tau_v f(u, v) \tau_u^{-1}.$$

As we saw above, in Figure 4.3, we are able to apply a permutation to u such that the symmetric arc function defining the covering graph is the identity permutation on arc (u, v). In general, by an easy induction, we can see that given any cover Y of another graph X and any spanning forest Z of X we can always apply permutations to each fibre so that the symmetric arc function associated with the cover is the identity on all the arcs of Z. We call a symmetric arc function *normalized*, if it is the identity on every arc of some spanning forest in X.

Now we want to see how we can get information about the structure of the cover from the symmetric arc function defining it. More on this will be

presented in the next section. Let the graph Y be an r-fold cover of another graph X with normalized symmetric arc function f. Define $\langle f \rangle$ to be the subgroup of S_r generated by the image of f. Then it is not hard to see that Y is connected if and only if $\langle f \rangle$ is transitive. We state this as a lemma here, a proof of which can be found in Godsil and Hensel [10] (Lemma 7.1).

4.1.1 Lemma. *Let Y be an r-fold cover of the connected graph X with normalized symmetric arc function f. Then Y is connected if and only if $\langle f \rangle$ is transitive.*

Remark. Let the graph Y be a cover of another graph X. When X or Y is not connected nothing interesting happens, as explained below. If X is not connected, then Y is just a disjoint union of covers of the connected components of X. Now suppose that X is connected but Y is not connected. Let H be a connected component of Y. Then for any two adjacent vertices in X, there must be a 1-factor between the restrictions of their corresponding fibres to H. So H is a cover of X itself. Thus Y is just a disjoint union of covers of X. We always assume that X and Y are both connected, and hence by Lemma 4.1.1, we always assume $\langle f \rangle$ is transitive.

4.2   Regular Covers

In this section we consider the covers that have the "maximum symmetry" that we can hope for, and then we see some interesting properties of these covers that are more relevant here. First we need to say what we mean by "maximum symmetry". For the algebraic prerequisites of this section and the following sections of this chapter the reader is referred to Drozd and Kirichenko [8].

In topology an automorphism of a covering space is defined to be a homeomorphism that fixes each fibre. It turns out that for graph coverings also the group of automorphisms that fix each fibre is more relevant than the whole group of automorphisms. If Y is a cover of X, we denote the group of automorphisms of Y that fix each fibre by $Aut_X(Y)$. In this section we see how this group is related to the structure of the graph.

4.2.1 Lemma. *Let the connected graph Y be an r-fold cover of the graph X. Then the group of automorphisms of Y that fix each fibre, $Aut_X(Y)$, acts semi-regularly on each fibre.*

Proof. Let $g \in Aut_X(Y)$ fix some $u \in V(Y)$. Then g must map the set of neighbours of u to itself. But u has at most one neighbour in each fibre.

So since g fixes each fibre it must fix each neighbour of u. By repeating this argument we conclude that g has to fix all the vertices of Y, since Y is connected. Thus g must be the identity permutation.

Now if u and v are two vertices of Y, and g_1 and g_2 are two members of $Aut_X(Y)$ that both map u to v then $g_1^{-1}g_2$ is an element of g that maps u to itself and so as we proved $g_1^{-1}g_2$ must be the identity of $Aut_X(Y)$. Therefore $g_1 = g_2$. This proves that $Aut_X(Y)$ acts semi-regularly on each fibre. □

Let X, Y and $Aut_X(Y)$ be as defined in the lemma. Let u be a vertex in Y. There are only r possibilities for the image of u under any element of $Aut_X(Y)$. Since $Aut_X(Y)$ acts semi-regularly on each fibre, by the pigeon-hole principle we get

$$|Aut_X(Y)| \leqslant r.$$

If equality holds, then u can be mapped to any other vertex in its fibre by elements of $Aut_X(Y)$, which means that $Aut_X(Y)$ acts regularly on each fibre. This also shows that if $Aut_X(Y)$ acts regularly on one fibre, then it acts regularly on all fibres. So $Aut_X(Y)$ acts regularly on each fibre if and only if it has size r. We say that Y is a *regular cover* if $Aut_X(Y)$ acts regularly on each fibre. Note that regular covering spaces are defined the same way in topology.

The following lemma shows that symmetric arc functions are closely related to the group of automorphisms of the cover that fix each fibre. A proof of the following lemma can be found in Godsil and Hensel [10].

4.2.2 Lemma. *Let the connected graph Y be an r-fold cover of the graph X with normalized symmetric arc function f. Then the group of automorphisms of Y that fix each fibre, $Aut_X(Y)$, is regular if and only if $\langle f \rangle$ is regular. Moreover, if $\langle f \rangle$ is regular then it is isomorphic to $Aut_X(Y)$.*

4.3 Abelian Covers

A regular cover is called an *Abelian cover* if the group of automorphisms of the cover that fix each fibre is an Abelian group. Let Y be an Abelian cover of X with normalized symmetric arc function f. Then by Lemma 4.2.2, $\langle f \rangle$ is Abelian of size r. A *representation* of a finite Abelian group is a group homomorphism from the group to the multiplicative group of complex numbers. A representation is called a *trivial representation* if it maps everything to 1.

Let ϕ be a non-trivial representation of $\langle f \rangle$. Define $A(X)^{\phi(f)}$ to be the matrix with its rows and columns indexed by the vertices of X, and with the (u, v)-entry equal to $\phi(f(u, v))$ if $uv \in E(X)$ and equal to zero otherwise. Note that since $\langle f \rangle$ has size r, and since ϕ is a homomorphism, everything in the image of ϕ must be an r-th root of unity.

In Section 2.1, we saw that a set of equiangular lines is equivalent to a *Seidel matrix*. As we defined there, a Seidel matrix is a Hermitian matrix with off-diagonal entries of norm one, and zero diagonal entries. The next lemma says we can find such matrices from Abelian covers.

4.3.1 Lemma. *Let Y be an Abelian cover of a complete graph, X, with normalized symmetric arc function f, and let ϕ be a non-trivial representation of $\langle f \rangle$. Then $A(X)^{\phi(f)}$ satisfies the Seidel properties.*

Proof. First note that since X has no loops, all diagonal entries of $A(X)^{\phi(f)}$ are zero. Also, each non-zero entry of $A(X)^{\phi(f)}$ is a root of unity and hence has norm one. But all off-diagonal entries of $A(X)^{\phi(f)}$ are non-zero since X is the complete graph. So it just remains to prove that $A(X)^{\phi(f)}$ is Hermitian. For any edge uv of X we always have

$$f(u, v) = f(v, u)^{-1},$$

and since ϕ is a homomorphism, we get

$$\phi(f(u, v)) = \phi(f(v, u))^{-1}.$$

But since everything in the image of ϕ has norm one, we get

$$\phi(f(u, v)) = \overline{\phi(f(v, u))}.$$

So $A(X)^{\phi(f)}$ is Hermitian. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

By Theorem 2.3.3, if we want a set of equiangular lines of maximum size, the Seidel matrix must have exactly two eigenvalues. By Godsil and Hensel [10] (Lemma 8.2.), if X is a complete graph and the cover Y is distance regular, then $A(X)^{\phi(f)}$ has only two eigenvalues. Any antipodal distance regular graph of diameter three is a cover of some complete graph. The next chapter is devoted to such covers.

Chapter 5

Antipodal Distance Regular Covers of Diameter Three

The graph in Figure 5.1 is the line graph of the Petersen graph. As we saw in Section 3.1, the line graph of the Petersen graph is distance regular, and we can check that it has diameter three. We labelled the vertices of the line graph of the Petersen graph in Figure 5.1 so that two vertices are at distance three if and only if they have the same label.

A distance regular graph G of diameter d is called *antipodal* if we can partition the vertices of G so that two distinct vertices are at distance d if and only if they lie in the same cell of the partition. In other words, a distance regular graph G is antipodal if and only if being at distance 0 or d is an equivalence relation on $V(G)$. So the line graph of the Petersen graph is an antipodal distance regular graph of diameter three.

On the other hand, in Figure 5.1 if we consider all vertices of the same label to be a fibre, then we have five fibres, and there is a 1-factor between any two fibres. Therefore the line graph of the Petersen graph is a 3-fold cover of K_5.

In the line graph of the Petersen graph, two vertices are at distance three if and only if they lie in the same fibre. Therefore, two vertices are at distance two if and only if they are non-adjacent, and they lie in different fibres. So the number of common neighbors of two non-adjacent vertices in different fibres is just c_2 in the intersection array of the graph.

We will see in Theorem 5.1.1 that being a cover of a complete graph with the property that the number of common neighbors of two non-adjacent

vertices in different fibres is a constant, is equivalent to being an antipodal distance regular graph of diameter three.
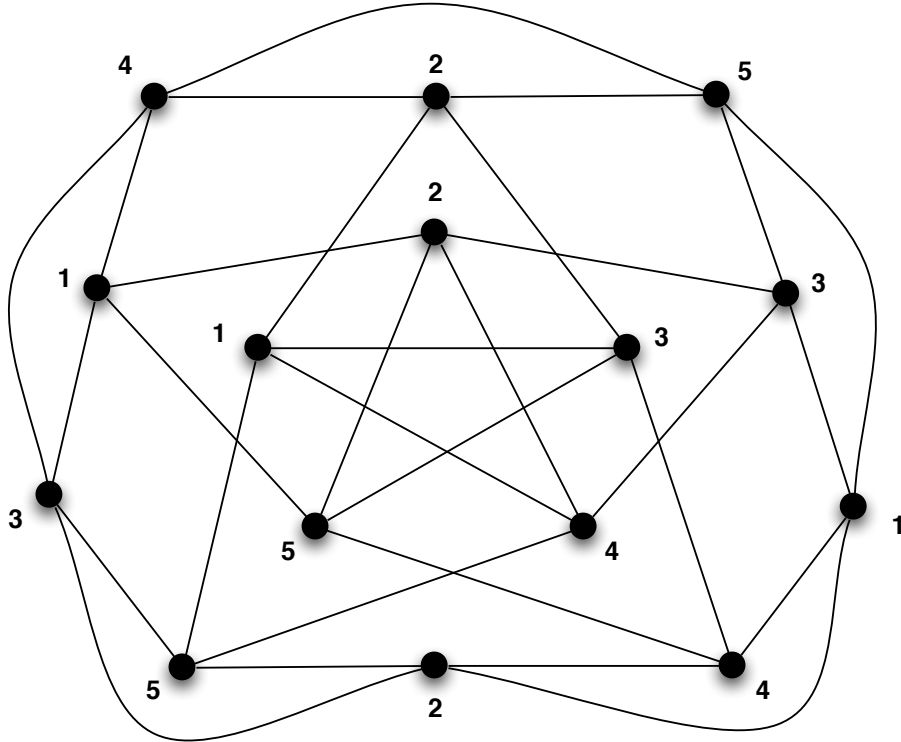
Figure 5.1: A 3-fold cover of K_5

For covers of complete graphs which are antipodal distance regular of diameter three, all the numbers in the intersection array can be expressed in terms of three parameters: the size of the complete graph, the size of each fibre, and the number of common neighbors of two non-adjacent vertices in different fibres. In Section 5.3, we see conditions on the parameters that are necessary for the existence of antipodal distance regular graphs of diameter three. All of the results in this chapter can be found in Godsil and Hensel [10].

5.1 Covers of Complete Graphs

By Theorem 3.2.1, we can prove that a regular graph is distance regular, by proving the numbers in the intersection array are well defined. By the following lemma, if we have a cover of a complete graph where any two non-adjacent vertices in different fibres are at distance two from each other, then to prove that it is distance regular, we just need to show that c_2 is well defined. A short proof of the following lemma is given in Godsil and Hensel [10, p.208]. We give a more detailed and explanatory proof here.

Before proving the lemma, recall that in a distance regular graph, c_i, a_i, and b_i are defined to be the number of neighbours of each vertex in V_i that lie in V_{i-1}, V_i, and V_{i+1} respectively. Also note that, if G is an antipodal distance regular graph of diameter $d \geqslant 3$ with distance partition $\{V_0, \cdots, V_d\}$, then all the vertices in V_d are at distance d from each other. This implies that

$$b_{d-1} = 1,$$

since if $b_{d-1} \geqslant 2$ then two vertices in V_d are at distance two from each other and if $b_{d-1} = 0$ then G is disconnected.

5.1.1 Lemma. *A graph is antipodal distance regular of diameter three if and only if it is a cover of a complete graph with the property that the number of common neighbours of two non-adjacent vertices in different fibres is a non-zero constant.*

Proof. Suppose that G is an antipodal distance regular graph of diameter three. Then in the intersection array of G we must have $b_2 = 1$ as indicated above. Let $\mathcal{B} = \{V_1, \cdots V_n\}$ be the partition of the vertices of G with two distinct vertices being in the same set of the partition if and only if they are at distance three from each other. First we show that G is a cover of K_n with $\mathcal{B}$ being the partition of its vertices into fibres. Note that each V_i is an independent set, since any two vertices in V_i are at distance three from each other.

Now we just need to show there is a 1-factor between V_i and V_j for any distinct i and j. Let $x \in V_i$. If y_1 and y_2 are two neighbours of x in V_j, then there is a path of length two, namely $y_1 x y_2$, between y_1 and y_2. But this contradicts that all the vertices in V_j are at distance three from each other. So x has at most one neighbour in V_j. Thus, we can find a vertex $y \in V_j$ that is not adjacent to x. Therefore, the distance between x and y is two. Also note that $V_j \backslash \{y\}$ is the set of vertices at distance three from y. So

if we consider the distance partition with respect to y then the number of neighbours of x in V_j is equal to b_2 which is one. We proved each vertex in V_i has exactly one neighbour in V_j. Similarly each vertex in V_j has exactly one neighbour in V_i. Therefore there is a 1-factor between V_i and V_j. Now note that any two non-adjacent vertices in different fibres are at distance two from each other, and so they have c_2 common neighbours. This proves one direction of the lemma.

Conversely, suppose that G is a cover of some complete graph K_n with each fibre of size r with the property that there is a constant say α such that any two non-adjacent vertices in different fibres have α common neighbours. To show that G is distance regular, we need to be able to find the intersection array of G for a distance partition of $V(G)$ with respect to some arbitrary vertex v. Let $\{V_0, \cdots, V_d\}$ be the distance partition with respect to v. First note that since G is a cover of K_n, then G is regular of valency $n - 1$. Therefore,

$$b_0 = n - 1,$$

and so $|V_1| = n - 1$. Also clearly,

$$c_1 = 1.$$

On the other hand, by the definition of α, then c_2 is defined and

$$c_2 = \alpha.$$

By the assumption of the theorem, the number of common neighbours of any two non-adjacent vertices in different fibres is non-zero. This implies that any two non-adjacent vertices in different fibres are at distance two from each other. Let u and w be two vertices in some fibre B of G. Since B is an independent set, $d(u, w) \geqslant 2$. Also since there is just a 1-factor between B and any other fibre of G, then u and w cannot have common neighbours. So $d(u, w) \geqslant 3$. Let B' be another fibre of G. Let u' be the unique neighbour of u in B'. Now w and u' are two non-adjacent vertices in different fibres, and so $d(w, u') = 2$. But u and u' are adjacent which implies that $d(u, w) = 3$.

So far, we proved that two vertices in different fibres are at distance one or two from each other and two distinct vertices in the same fibre are at distance three from each other. This implies $|V_2| = (r-1)n$ and $|V_3| = r-1$. Since V_3 is an independent set and G is regular of valency $n - 1$, then c_3 is defined, and

$$c_3 = n - 1.$$

Let $v_1 \in V_2$. Since $\{v\} \cup V_3$ is a fibre of G, then v is adjacent to exactly one vertex in $\{v\} \cup V_3$. But $v_1 \in V_2$ means that v_1 is at distance two from v. Therefore v_1 is adjacent to exactly one vertex of V_3. Thus b_2 is defined, and

$$b_2 = 1.$$

It just remains to prove that b_1 is defined. Let x and y be two adjacent vertices of G. Let A and B be the fibres of G which contain x and y respectively. Note that other than x, each neighbour of y lies outside A and so is adjacent to exactly one of the vertices in A. Thus we get

$$N(y)\backslash\{x\} = \dot{\bigcup_{z \in A}}(N(y) \cap N(z)),$$

where by the dot on $\cup$ we mean this is a disjoint union. Therefore

$$|N(y) \cap N(x)| = n - 2 - \sum_{z \in A\backslash x} |N(y) \cap N(z)| = n - 2 - (r-1)c_2.$$

The last equality holds since for any $z \in A\backslash x$ we have $d(z,x) = 2$ and so $|N(y) \cap N(z)| = c_2$.

So we proved that any vertex $u \in V_1$ has $n - 2 - (r-1)c_2$ neighbours in V_1, or equivalently

$$a_1 = n - 2 - (r-1)c_2,$$

and since u has valency $n - 1$ and is adjacent to v, then the number of neighbours of u in V_2 is

$$n - 1 - 1 - (n - 2 - (r-1)c_2) = (r-1)c_2.$$

Therefore b_1 is defined, and

$$b_1 = (r-1)c_2.$$

Therefore G is distance regular of diameter three. Since we already saw that two vertices are at distant three if and only if they are in the same fibre, G is antipodal as well. □

From now on, we mostly focus on antipodal distance regular covers of diameter three. By Lemma 5.1.1 there is a nice characterization of them. Since we defined being antipodal only for distance regular graphs, we refer to them as diameter three antipodal covers.

5.2 Parameters

Let G be a diameter three antipodal cover of the complete graph K_n with each fibre of size r. Also suppose that any two vertices in different fibres have c_2 common neighbours. By Lemma 5.1.1, G is an antipodal distance regular graph of diameter three. As can be seen in the proof of Lemma 5.1.1 the intersection array of G is

$$\{n - 1, (r - 1)c_2, 1; 1, c_2, n - 1\}.$$

So all the numbers in the intersection array of G can be written in terms of n, r and c_2. All the other usual parameters of a distance regular graph can be written in terms of its intersection array, and hence in the case of diameter three antipodal covers they can be written in terms of n, r and c_2. We call

$$n, r, c_2$$

the *parameters* of G. We define new parameters,

$$\delta = a_1 - c_2 = n - rc_2 - 2, \quad \Delta = \delta^2 + 4(n - 1), \qquad (5.2.1)$$

that appear a lot in our computations. It turns out that δ and Δ are the sum and the difference of two of the eigenvalues of G.

5.2.1 Theorem. *Let G be a diameter three antipodal cover with parameters n, r, c_2. Then the eigenvalues of G are*

$$n - 1, \quad -1, \quad \theta = \frac{\delta + \sqrt{\Delta}}{2}, \quad \tau = \frac{\delta - \sqrt{\Delta}}{2}, \qquad (5.2.2)$$

with multiplicities

$$m_{n-1} = 1, \quad m_{-1} = n - 1, \quad m_\theta = \frac{n(r - 1)\tau}{\tau - \theta}, \quad m_\tau = \frac{n(r - 1)\theta}{\theta - \tau}. \quad (5.2.3)$$

Proof. We use the well-known fact that for any graph with an equitable partition, if we contract all the vertices in the same set of the partition, then the characteristic polynomial of the resulting multigraph divides the characteristic polynomial of the original graph (see Godsil and Royle [11, p. 197]).

Since G is distance regular, the distance partition with respect to any vertex is equitable. If we contract all the vertices in the same set of the

partition, then we get a multigraph on four vertices with adjacency matrix

$$\begin{bmatrix} a_0 & b_0 & 0 & 0 \\ c_1 & a_1 & b_1 & 0 \\ 0 & c_2 & a_2 & b_2 \\ 0 & 0 & c_3 & a_3 \end{bmatrix}$$

The eigenvalues of this matrix are

$$n-1, \quad -1, \quad \theta = \frac{\delta + \sqrt{\Delta}}{2}, \quad \tau = \frac{\delta - \sqrt{\Delta}}{2},$$

So G has at least four distinct eigenvalues. But as we saw in the beginning of Section 3.5, a distance regular graph of diameter d has exactly $d+1$ distinct eigenvalues. Therefore G has exactly four distinct eigenvalues.

Now we need to find the multiplicity of each eigenvalue. For each λ, let m_λ denote the multiplicity of λ as an eigenvalue of G. First note that, since G is a connected $(n-1)$-regular graph, then $n-1$ is a simple eigenvalue, and hence

$$m_{n-1} = 1.$$

Now we get three equations in terms of m_{-1}, m_θ and m_τ. We get these equations by using the following standard facts in linear algebra.

- The eigenvalues of the kth power of any matrix are k^{th} powers of the eigenvalues of the matrix.

- For any $k \geqslant 0$, in the kth power of the adjacency matrix of a graph, the (i,j)-entry counts the number of walks of length k from vertex i to vertex j.

Now, we get our three equations by applying the above facts for $k = 0$, $k = 1$ and $k = 2$. For any vertex in any simple graph, the number of closed walks of length 0 is one, the number of closed walks of length 1 is zero, and the number of closed walks of length 2 is the degree of that vertex. Therefore we get the following three equations. Note that since G is an r-fold cover of K_n, it has rn vertices.

- $1 + m_{-1} + m_\theta + m_\tau = rn$.

- $(1)(n-1) + m_{-1}(-1) + m_\theta\theta + m_\tau\tau = 0$.

- $(1)(n-1)^2 + m_{-1}(-1)^2 + m_\theta\theta^2 + m_\tau\tau^2 = rn(n-1)$.

Solving for m_{-1}, m_θ and m_τ we get

$$m_{-1} = n - 1, \quad m_\theta = \frac{n(r-1)\tau}{\tau - \theta}, \quad m_\tau = \frac{n(r-1)\theta}{\theta - \tau}.$$

and we are done. □

5.3   Necessary Conditions

In the previous section we introduced a set of parameters for antipodal covers of diameter three. The following theorem lists necessary conditions for the existence of antipodal covers of diameter three in terms of those parameters. None of these conditions are new. Note that these are just necessary conditions, which means that, satisfaction of those conditions by a set of parameters does not guarantee the existence of a diameter three antipodal cover with those parameters. The proof of this theorem is given in the following sections of this chapter.

5.3.1 Theorem. *Let G be a diameter three antipodal cover with parameters n, r, c_2, where n, r and c_2 are integers such that $n \geqslant 3$, $r \geqslant 2$ and $c_2 \geqslant 1$. Let θ, τ, m_θ, m_τ and δ be as defined in Equations 5.2.2 and 5.2.3. Then the following are necessary conditions.*

1. *[10, p. 210] $(r-1)c_2 \leqslant n - 2$.*

2. *[10, p. 210] The following numbers must be integers:*

$$\frac{n(r-1)\tau}{\tau - \theta}, \quad \frac{n(r-1)\theta}{\theta - \tau}.$$

3. *[10, Lemma 3.2] If $\delta \neq 0$, then θ and τ are integers.*

4. *[10, Lemma 3.2] If $\delta = 0$, then $\theta = -\tau = \sqrt{n-1}$.*

5. *[10, p. 210] If n is even, then c_2 is even.*

6. *[10, Theorem 3.4] If $c_2 = 1$ then*

   - *$(n - r) \mid (n - 1)$,*
   - *$(n - r)(n - r + 1) \mid rn(n - 1)$,*
   - *$(n - r)^2 \leqslant n - 1$.*

7. *[3, Theorem 4.2.16] $n \leqslant c_2(2r - 1)$.*

8. [21] If $r > 2$, then
$$\theta^3 \geqslant n - 1.$$

9. [18] Let the $q_{i,j}(k)$'s be the Krein parameters defined in Section 3.6, and suppose that

$$\theta \neq 1, \quad \tau \neq -1, \quad \theta^3 \neq n - 1.$$

Then, for $r > 2$, we have

$$rn \leqslant \frac{1}{2}m_\theta(m_\theta + 1), \qquad rn \leqslant \frac{1}{2}m_\tau(m_\tau + 1);$$

and, for $r = 2$, we have

$$n \leqslant \frac{1}{2}m_\theta(m_\theta + 1), \qquad n \leqslant \frac{1}{2}m_\tau(m_\tau + 1).$$

10. [10, p. 230] Let $r > 2$ and $\beta \in \{\theta, \tau\}$ be an integer. If

$$n > m_\beta - r + 3,$$

then
$$\beta + 1 \mid c_2.$$

11. If G is an Abelian cover, then any odd prime that divides r must divide n as well.

We should emphasize that except Condition 11, all the conditions in Theorem 5.3.1 are known. We prove Condition 11 in Section 5.6.

Note that

$$(\theta + 1)(\tau + 1) = \theta\tau + \delta + 1 = (1 - n) + (n - 2 - rc_2) + 1 = -rc_2.$$

Therefore in Condition 10 of Theorem 5.3.1, $\theta + 1 \mid c_2$ is equivalent to $r \mid \tau + 1$, and $\tau + 1 \mid c_2$ is equivalent to $r \mid \theta + 1$.

Now, we see a proof of Conditions 1 and 7 of Theorem 5.3.1. Condition 6 of Theorem 5.3.1 is proved in Godsil and Hensel [10, p. 210]. A proof of Condition 10 of Theorem 5.3.1 can be found in Godsil and Hensel [10, p. 230]. It also follows from a more general result that we prove in Theorem 8.10.6. We present proofs of Conditions 2, 3, 4 and 5 of Theorem 5.3.1 in

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Section 5.4, proofs of Conditions 8 and 9 of Theorem 5.3.1 in Section 5.5, and proof of the last condition of Theorem 5.3.1 in Section 5.6.

For Condition 1 of Theorem 5.3.1, note that all numbers in the intersection array must be non-negative. As we showed in the end of proof of Lemma 5.1.1, we have $a_1 = n - 2 - (r-1)c_2$. Now a_1 being non-negative gives us the first condition in Theorem 5.3.1.

The seventh condition of Theorem 5.3.1 follows from Theorem 4.2.16 in Brouwer, Cohen and Neumaier [3] which states that, for any k-regular distance regular graph we have $k \geqslant 2a_1 + 3 - c_2$. Therefore, in the case of a diameter three antipodal cover, we get

$$n - 1 \geqslant 2(n - (r-1)c_2 - 2) + 3 - c_2,$$

and so

$$n \leqslant c_2(2r - 1).$$

5.4   Integrality Conditions

In this section, we see proofs for Conditions 2, 3, 4, and 5, of Theorem 5.3.1, following their proofs in Godsil and Hensel [10, p. 210]. By Theorem 5.2.1, the numbers in the second condition of Theorem 5.3.1 are just m_θ and m_τ, the multiplicities of the eigenvalues θ and τ, and hence must be integers. Although we got this condition for free, it serves as an important factor in ruling out many parameter sets as infeasible.

Now we prove the third condition of Theorem 5.3.1, following the proof given in Godsil and Hensel [10, p. 210].

5.4.1 Lemma. *If $\delta \neq 0$, then θ and τ are integers.*

Proof. By Theorem 5.2.1,

$$\delta = \theta + \tau, \tag{5.4.1}$$

and so

$$m_\tau - m_\theta = \frac{n(r-1)\delta}{\theta - \tau}.$$

So if $\delta \neq 0$ then $\theta - \tau$ must be a rational number. But again from Theorem 5.2.1,

$$\theta - \tau = \sqrt{\Delta},$$

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and, since

$$\Delta = \delta^2 + 4(n-1)$$

is an integer, we conclude that if $\delta \neq 0$ then $\sqrt{\Delta}$ must be an integer. Also the definition of Δ implies that δ and $\sqrt{\Delta}$ have the same parity, and so

$$\theta = (\delta + \sqrt{\Delta})/2, \quad \tau = (\delta - \sqrt{\Delta})/2$$

are integers. $\qquad\square$

The fourth condition of Theorem 5.3.1 follows from 5.4.1 and

$$\theta\tau = 1 - n. \tag{5.4.2}$$

Now note that if $\delta \neq 0$ and n is even then by 5.4.2, we get θ and τ are both odd, and therefore by 5.4.1, δ is even. Also if $\delta = 0$, then δ is clearly even. So we always have: if n is even, then δ is even. On the other hand, the first neighbourhood in the distance partition is a_1-regular of size $n - 1$, and so if n is even, then a_1 is even. Now the fifth condition of Theorem 5.3.1 follows, since by definition

$$\delta = a_1 - c_2.$$

5.5 Conditions from Krein Parameters

Conditions 8 and 9 of Theorem 5.3.1 come from the theory of association schemes. As we mentioned in Section 3.4, the distance matrices of any distance regular graph form an association scheme. Let the $q_{i,j}(k)$'s be the Krein parameters of a distance regular graph, defined in Section 3.6, and let m_i denote the multiplicity of the i-th eigenvalue of the graph. Then the following bound, proved by Neumaier [18], is called the *absolute bound*.

$$\sum_{q_{i,j}(k)\neq 0} m_k \leqslant \left\{ \begin{array}{ll} m_i m_j & i \neq j \\ \frac{1}{2}m_i(m_i + 1), & i = j. \end{array} \right. \tag{5.5.1}$$

To be able to apply this bound for antipodal distance regular graphs of diameter three, we need to see which Krein parameters are zero.

5.5.1 Lemma. *For an antipodal distance regular graph of diameter three with parameters n, r, c_2 and eigenvalues*

$$p_1(0) = n - 1, \quad p_1(1) = -1, \quad p_1(2) = \theta, \quad p_1(3) = \tau,$$

we have a Krein parameter $q_{i,j}(k)$ is equal to zero if and only if

(a)

$$\{i, j, k\} \in \{\{0,0,1\}, \{0,0,2\}, \{0,0,3\}, \{0,1,2\},$$
$$\{0,1,3\}, \{0,2,3\}, \{1,1,2\}, \{1,1,3\}\}.$$

(b) $\{i, j, k\} = \{1, 2, 2\}$, and $\tau = -1$.

(c) $\{i, j, k\} = \{1, 3, 3\}$, and $\theta = 1$.

(d) $\{i, j, k\} = \{2, 2, 2\}$, and $r = 2$.

(e) $\{i, j, k\} = \{3, 3, 3\}$, and ($r = 2$ or $\theta^3 = n - 1$).

(f) $\{i, j, k\} = \{2, 2, 3\}$, and ($r = 2$ or $\tau = -1$).

(g) $\{i, j, k\} = \{2, 3, 3\}$, and $r = 2$.

Proof. By Equation 3.6.1, we have

$$q_{i,j}(k) = \frac{m_i m_j}{v} \sum_{s=0}^{d} \frac{p_s(i) p_s(j) p_s(k)}{p_s(0)^2}.$$

Let $\{A_0, A_1, A_2, A_3\}$ be the distance matrices of the graph. Note that $p_s(i)$ is just the i-th eigenvalue of A_s. So to use the above equation, we need to find the eigenvalues of the distance matrices. But A_0 is just the identity matrix. The eigenvalues of A_1 are given, and A_3 is an adjacency matrix for n copies of K_r. Now we can find the eigenvalues of A_2, since

$$A_2 = J - A_0 - A_1 - A_3.$$

So we get the eigenvalues of the distance matrices as in Table 5.1. The first column in the table corresponds to the all ones eigenvector, and hence the first column must add up to rn, while in any other column the corresponding eigenvector sums to zero, and hence the column must add up to zero.

Note that $q_{i,j}(k)$ is zero if and only if

$$\sum_{s=0}^{d} \frac{p_s(i) p_s(j) p_s(k)}{p_s(0)^2} \tag{5.5.2}$$

is zero. But 5.5.2 is symmetric in i, j, and k. Therefore, the order of i, j, k does not matter in determining whether $q_{i,j}(k)$ is zero. Now, by computing 5.5.2 for each multi-set $\{i, j, k\}$, we get the result. As an example, we

	Eigenvalues			
A_0	1	1	1	1
A_1	$n-1$	-1	θ	τ
A_2	$(n-1)(r-1)$	$1-r$	$-\theta$	$-\tau$
A_3	$r-1$	$r-1$	-1	-1

Table 5.1: eigenvalues of distance matrices

compute 5.5.2 when $\{i,j,k\} = \{1,2,2\}$:

$$
\begin{aligned}
\sum_{s=0}^{3} \frac{p_s(1)p_s(2)p_s(2)}{p_s(0)^2} &= \frac{1(1)^2}{1^2} + \frac{(-1)\theta^2}{(n-1)^2} + \frac{(1-r)(-\theta)^2}{(n-1)^2(r-1)^2} + \frac{(r-1)(-1)^2}{(r-1)^2} \\
&= 1 - \frac{\theta^2}{(n-1)^2} - \frac{\theta^2}{(n-1)^2(r-1)} + \frac{1}{r-1} \\
&= \left(1 - \frac{\theta^2}{(n-1)^2}\right)\left(1 + \frac{1}{r-1}\right).
\end{aligned}
$$

Since $r \geqslant 2$, and θ is always positive, $q_{1,2,2}$ is zero only if $\theta = n-1$. Since $\theta\tau = 1-n$, then $\theta = n-1$ is equivalent to $\tau = -1$. □

Now we are ready to prove Condition 9 of Theorem 5.3.1. Actually, what we prove is that when we apply Inequality 5.5.1 to antipodal distance regular graphs of diameter three with $\theta \neq 1$, $\tau \neq -1$, and $\theta^3 \neq n-1$, then what we get is exactly Condition 9 of Theorem 5.3.1. To be able to apply Inequality 5.5.1, we need to know which Krein parameters are zero. By Lemma 5.5.1, this depends on whether $r = 2$ or $r > 2$.

First, suppose that $r > 2$. Then by Lemma 5.5.1, we have $q_{i,j,k} = 0$ if and only if

$$
\begin{aligned}
\{i,j,k\} \in \ \ &\{\{0,0,1\},\{0,0,2\},\{0,0,3\},\{0,1,2\},\{0,1,3\}, \\
&\{0,2,3\},\{1,1,2\},\{1,1,3\},\{1,2,2\},\{1,3,3\}\}.
\end{aligned}
$$

Let $C_{i,j}$ denote Inequality 5.5.1 applied for the pair (i,j), when $r > 2$. By Lemma 5.5.1,

- $C_{0,0} : m_0 \leqslant \frac{1}{2}m_0(m_0 + 1)$.

- $C_{0,1} : m_1 \leqslant m_0 m_1$.

- $C_{0,2} : m_2 \leqslant m_0 m_2$.

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- $C_{0,3} : m_3 \leqslant m_0 m_3$.

- $C_{1,1} : m_0 + m_1 \leqslant \frac{1}{2} m_1 (m_1 + 1)$.

- $C_{1,2} : m_2 + m_3 \leqslant m_1 m_2$.

- $C_{1,3} : m_2 + m_3 \leqslant m_1 m_3$.

- $C_{2,2} : m_0 + m_1 + m_2 + m_3 \leqslant \frac{1}{2} m_2 (m_2 + 1)$.

- $C_{2,3} : m_1 + m_2 + m_3 \leqslant m_2 m_3$.

- $C_{3,3} : m_0 + m_1 + m_2 + m_3 \leqslant \frac{1}{2} m_3 (m_3 + 1)$.

Now note that since

$$m_0 = 1, \quad m_1 = n - 1, \quad m_2 = m_\theta, \quad m_3 = m_\tau,$$

$C_{0,0}, C_{0,1}, C_{0,2}, C_{0,3}$ and $C_{1,1}$ are trivially satisfied. Also $C_{2,2}$ and $C_{3,3}$ give Condition 9 of Theorem 5.3.1 for $r > 2$.

Now we show that the other $C_{i,j}$'s give no restriction on the parameters of the cover. $C_{1,2}$ is equivalent to

$$1 + \frac{m_\tau}{m_\theta} \leqslant m_1.$$

But since
$$m_\theta = \frac{-\tau n (r - 1)}{\theta - \tau}, \qquad m_\tau = \frac{\theta n (r - 1)}{\theta - \tau},$$

we have
$$\frac{m_\tau}{m_\theta} = \frac{\theta}{-\tau}.$$

Therefore $C_{1,2}$ is equivalent to

$$\frac{\theta}{-\tau} \leqslant n - 2.$$

On the other hand, $\theta\tau = 1 - n$ implies

$$\frac{\theta}{-\tau} = \frac{\theta^2}{n - 1}.$$

So we get
$$\theta^2 \leqslant (n - 2)(n - 1). \tag{5.5.3}$$

If $\delta = 0$, then $\theta = \sqrt{n-1}$, and so 5.5.3 is satisfied. If $\delta \neq 0$, then θ and τ are integers, and since by our assumption $-\tau \neq 1$, we get $\theta \leqslant n-2$, and hence 5.5.3 is satisfied again.

Similarly, $C_{1,3}$ is equivalent to

$$\tau^2 \leqslant (n-2)(n-1),$$

which is satisfied since $\theta \neq 1$.

The only thing remaining to be checked is $C_{2,3}$. It is equivalent to

$$rn - 1 \leqslant m_\theta m_\tau.$$

First note that $m_\theta \neq 1$, since otherwise, we get

$$1 - \frac{\theta}{\tau} = n(r-1).$$

If $\delta = 0$, then $\theta = -\tau$, and so we get a contradiction. If $\delta \neq 0$, then θ and τ are integers, and so the left hand side in the above equation is at most $1 + (n-2)/2$, which is a contradiction. Similarly, we have $m_\tau \neq 1$. Now note that m_θ and m_τ always add up to $n(r-1)$. So $m_\theta m_\tau$ is minimum when one of them is 2 and the other one is $nr - n - 2$. Thus we get

$$m_\theta m_\tau \geqslant 2(nr - n - 2) = nr - 1 + (nr - 2n - 3) \geqslant nr - 1,$$

since r and n are both at least three.

Now suppose that $r = 2$. Let $D_{i,j}$ denote Inequality 5.5.1 applied for the pair (i, j), when $r = 2$. By Lemma 5.5.1, when $r = 2$, beside from the Krein parameters that are zero when $r > 2$, we also have $q_{i,j,k} = 0$ for

$$\{i, j, k\} \in \{\{2, 2, 2\}, \{3, 3, 3\}, \{2, 2, 3\}, \{2, 3, 3\}\}.$$

This implies that $D_{i,j}$ and $C_{i,j}$ are the same, except when

$$\{i, j\} \in \{\{2, 2\}, \{2, 3\}, \{3, 3\}\}.$$

We have

- $D_{2,2} : m_0 + m_1 \leqslant \frac{1}{2} m_2 (m_2 + 1)$;

- $D_{2,3} : m_1 \leqslant m_2 m_3$; and

- $D_{3,3} : m_0 + m_1 \leqslant \frac{1}{2} m_3 (m_3 + 1)$.

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$D_{2,2}$ and $D_{3,3}$ give Condition 9 of Theorem 5.3.1 for $r = 2$.

Also, $D_{2,3}$ is equivalent to

$$n - 1 \leqslant m_\theta m_\tau.$$

As before neither m_θ nor m_τ can be 1, and, since they add up to n, $m_\theta m_\tau$ is minimum when one of them is 2 and the other one is $n - 2$. So we get

$$m_\theta m_\tau \geqslant 2(n - 2) \geqslant n - 1 + (n - 3) \geqslant n - 1.$$

Therefore, $D_{2,3}$ gives no further restriction on the parameters of the cover.

Condition 8 of Theorem 5.3.1 comes from the *Krein conditions* discovered by Scott [21] that the Krein parameters of any association scheme must be non-negative. When we compute the Krein parameters in Lemma 5.5.1, we see that when $r > 2$, $q_{3,3}(3)$ is non-negative if and only if $\theta^3 \geqslant n - 1$. For any other value of i, j, and k, $q_{i,j}(k)$ is trivially non-negative, and hence we do not get any new restriction on the parameters.

5.6 Quotienting

In this section, we see that from each Abelian antipodal cover of a complete graph, we can derive other Abelian antipodal covers of the same complete graph but with smaller fibre sizes. The following theorem is of central importance when we study the connection between covers and equiangular lines in Chapter 7. We also use this theorem to prove Condition 11 of Theorem 5.3.1 at the end of this section. The following theorem follows from Theorem 6.2 in Godsil and Hensel [10, p. 218].

5.6.1 Theorem. *Let Y be an Abelian antipodal distance regular cover of diameter three with parameters n, r, c_2. Let G be the group of automorphisms of Y fixing its fibres and let H be a subgroup of G of size t. Let π be the partition of the vertices of Y with its cells as the orbits of H on the fibres. Then Y/π is an Abelian antipodal distance regular cover of diameter three with parameters*

$$n, \frac{r}{t}, tc_2.$$

Now to prove Condition 11 of Theorem 5.3.1, we also need the following theorem in Godsil and Hensel [10, p. 228]. A regular cover is called *cyclic* if the group of automorphisms of the cover which fix each fibre is a cyclic group.

5.6.2 Theorem. *Let G be a cyclic r-fold cover of K_n, with $r > 2$. Then r divides n.*

Condition 11 of Theorem 5.3.1 is stated in the following theorem.

5.6.3 Theorem. *Let Y be an Abelian antipodal distance regular cover of diameter three with parameters n, r, c_2. Then any odd prime that divides r must divide n as well.*

Proof. Let G be the group of automorphisms of Y fixing its fibres. Since Y is an Abelian cover, G is Abelian of size r. Let p be a prime that divides r. Then we can find a subgroup H of G such that

$$[G : H] = p.$$

So by Theorem 5.6.1, we can find an Abelian antipodal distance regular cover of K_n with each fibre of size p. Thus, the group of automorphisms of this cover which fix each fibre has size p, and hence is cyclic. Therefore, by Theorem 5.6.2, if $p > 2$ then p divides n. □

Chapter 6

Basic Constructions

In this chapter, some constructions of diameter three antipodal covers will be presented. None of these constructions are new. The notation we use for the parameters of a diameter three antipodal cover are the ones introduced in Section 5.2. We start with the following theorem in Godsil and Hensel [10, Section 4].

6.0.4 Theorem. *For fixed values of r and δ there are only finitely many covers with those parameters unless $\delta \in \{-2, 0, 2\}$.*

In this chapter, we construct an infinite family of covers for fixed values of r and δ when δ is equal to -2, 0 or 2. The constructions make use of symplectic forms.

6.1   Symplectic Forms

Let V be a vector space over the field $\mathbb{F}$. A bilinear function

$$B : V \times V \to \mathbb{F}$$

is called a *symplectic form* on V if for all u in V we have

$$B(u, u) = 0.$$

Note that if B is a symplectic form on V, then, for any u, v in V, we have

$$0 = B(u + v, u + v) = B(u, v) + B(v, u),$$

and so

$$B(u, v) = -B(v, u).$$

A symplectic form, B, is called *non-degenerate* if, for any u in V, $B(u,x) = 0$ for all x in V, implies $u = 0$. Note that if B is a symplectic form over V, then, for each u in V, we can define a linear function, f_u, from V to $\mathbb{F}$ by

$$f_u(v) = B(u,v).$$

Since the image is at most 1-dimensional, f_u has to be either onto, or the zero function. We say B is non-degenerate if and only if f_u is onto for all u in $V\backslash\{0\}$. So if B is non-degenerate, then, for each u in $V\backslash\{0\}$, the null space of f_u is a hyperplane and, for each a in $\mathbb{F}$, the solutions to $f_u(x) = a$ is just a coset of that hyperplane. Therefore if V is d-dimensional, then the number of solutions to

$$f_u(x) = a$$

is $|\mathbb{F}|^{d-1}$ regardless of choices of u and a. We make use of this fact in Section 6.3.

As an example of bilinear forms, let $\mathbb{F}$ be a field. If $V = \mathbb{F}^d$ and A is an arbitrary $d \times (d-2)$ matrix over $\mathbb{F}$, we can define $B(u,v)$ to be the determinant of the matrix obtained by adding u and v to A as its last two columns. Then by properties of the determinant function, B is a symplectic form. However, clearly B is non-degenerate if and only if d is equal to 2.

As another example, let $\mathbb{F}$ be a field, $V = \mathbb{F}^d$ and A be a $d \times d$ anti-symmetric matrix over $\mathbb{F}$, i.e.

$$A^T = -A.$$

For any u and v in V, define

$$B(u,v) = u^T A v.$$

Then clearly B is bilinear. Also note that since $B(u,v) \in \mathbb{F}$,

$$B(u,v) = (B(u,v))^T = (u^T A v)^T = v^T A^T u = -v^T A u = -B(v,u).$$

In particular,

$$B(u,u) = -B(u,u).$$

So if the characteristic of $\mathbb{F}$ is not equal to 2, then $B(u,u) = 0$ and so B is a symplectic form.

If the characteristic of $\mathbb{F}$ is equal to 2 and A is anti-symmetric (which is the same thing as being symmetric when $\mathbb{F}$ has characteristic 2) with zero diagonal we get

$$u^T A u = \sum_{i,j} u_i a_{ij} u_j = \sum_i u_i^2 a_{ii} + \sum_{i<j} 2 u_i a_{ij} u_j = 0.$$

So again we get B is a symplectic form. Note that, regardless of the characteristic of the field, B is non-degenerate if and only if A is invertible.

6.2 Covers with $\delta = 0$

Mathon [17] constructs a diameter three antipodal cover for any feasible parameter set with $\delta = 0$ and $n - 1$ a prime power. The following theorem is an alternative construction due to Brouwer, Cohen and Neumaier [3].

6.2.1 Theorem. *Let $\{n, r, c_2\}$ be a feasible parameter set for an antipodal distance regular cover of diameter three, and let $\delta = n - rc_2 - 2$. Suppose that $\delta = 0$ and $q = n - 1$ is a prime power. Let $\mathbb{F}$ be the field of order q and let $\mathbb{F}^*$ be the multiplicative group of $\mathbb{F}$. So*

$$|\mathbb{F}^*| = q - 1 = n - 2 = \delta + rc_2 = rc_2.$$

Let K be a subgroup of $\mathbb{F}^*$ of order c_2. Let V be a vector space of dimension two over $\mathbb{F}$, and B be a non-degenerate symplectic form on V. Then the graph G defined as follows is a diameter three antipodal cover with parameters $\{n, r, c_2\}$:

$$V(G) = \{Ku \mid u \in V \backslash \{0\}\},$$

$$Ku \sim Kv \iff B(u, v) \in K.$$

Proof. We give the proof in three steps. First, we prove that the adjacency relation on the vertices of G is well-defined. Then we prove that G is a cover of K_n, and finally we prove that the number of common neighbours of two non-adjacent vertices in different fibres is a constant, which by Theorem 5.1.1 guarantees that G is a diameter three antipodal cover.

To prove that the adjacency relation on the vertices of G is well-defined, we need to prove that $B(u, v) \in K$ if and only if $B(v, u) \in K$ for any $u, v \in V$. Since $B(v, u) = -B(v, u)$ it suffices to prove that $-1 \in K$. If q is even then $-1 = 1 \in K$. So suppose that q is odd. Then $n = q + 1$ is even and so by

the fifth feasibility condition in Theorem 5.3.1, then $c_2 = |K|$ is even. Since K is cyclic it has a generator, say α. Then $\alpha^{c_2/2}$ is a non-trivial element of order two in K, so it is -1.

Now we prove that G is a cover of K_n. For this, first we need to introduce the fibres. To make our arguments easier, we represent $V(G)$ as a set of equivalence classes on $V \backslash \{0\}$ with the equivalence relation defined as

$$u \equiv_1 v \Longleftrightarrow v = ku$$

for some $k \in K$. So if $\overline{u}$ denotes the equivalence class containing u then we represent the vertices of $V(G)$ as

$$V(G) = \{\overline{u} \mid u \in V\}.$$

Note that, by this definition of $V(G)$, it immediately follows that

$$|V(G)| = \frac{|V \backslash \{0\}|}{|K|} = \frac{q^2 - 1}{c_2} = \frac{(q+1)rc_2}{c_2} = (q+1)r = nr.$$

Now we claim that the set of equivalence classes of the following equivalence relation on $V(G)$ is a set of fibres of G as a cover of K_n:

$$\overline{u} \equiv_2 \overline{v} \Longleftrightarrow \overline{u} = \alpha \overline{v},$$

for some $\alpha \in \mathbb{F}^*$. Note that by our definition, each fibre has size $[\mathbb{F}^* : K] = r$. We must prove that each vertex in one fibre has exactly one neighbour in any other fibre. Let $\overline{u}$ and $\overline{v}$ belong to two different fibres. The set of neighbours of $\overline{u}$ in the fibre containing $\overline{v}$ is

$$\{\overline{\alpha v} \mid \alpha \in \mathbb{F}^*, B(u, \alpha v) \in K\}.$$

But we have

$$B(u, \alpha v) \in K \Longleftrightarrow \alpha = kB(u,v)^{-1} \Longleftrightarrow \overline{\alpha v} = \overline{B(u,v)^{-1}v},$$

Therefore, the set of neighbours of $\overline{u}$ in the fibre containing $\overline{v}$ is exactly one.

Let $\{d_1, \cdots, d_r\}$ be a complete set of representatives for $\mathbb{F}^*/K$. Now $\{B(d_1 v_1, d_i v_2) \mid 1 \leqslant i \leqslant r\} = \{d_i d_1 B(v_1, v_2) \mid 1 \leqslant i \leqslant r\}$ is a complete set of representatives for $\mathbb{F}^*/K$ since we are just multiplying each element of another set of representatives, namely $\{d_1, \cdots, d_r\}$, by an element of $\mathbb{F}^*$, namely $d_1 B(v_1, v_2)$. Therefore exactly one value in the set belongs to K, and that is exactly what we wanted to prove.

So far, we have shown that G is an r-fold cover of K_n. To prove G is a diameter three antipodal cover with parameters n, r, c_2, the only thing remaining to be proved is that any two non-adjacent vertices in different fibres have exactly c_2 common neighbours. Let $d_i v_1$ and $d_j v_2$ be two non-adjacent vertices in different fibres. We have already proved that each vertex in a fibre has exactly one neighbour in any other fibre. Let $d_{i'} v_2$ be the neighbour of $d_i v_1$ in A_2. Also let $d_{j'} v_1$ be the neighbour of $d_j v_2$ in A_1. So by the definition of adjacency we have

$$B(d_i v_1, d_{i'} v_2) \in K \quad \text{and} \quad B(d_j v_2, d_{j'} v_1) \in K.$$

Now for any $k, k' \in K$ we have

$$\begin{aligned}
B(d_i v_1, k d_{j'} v_1 + k' d_{i'} v_2) &= k d_i d_{j'} B(v_1, v_1) + k' B(d_i v_1, d_{i'} v_2) \\
&= k' B(d_i v_1, d_{i'} v_2) \in K.
\end{aligned}$$

Similarly we get

$$B(d_j v_2, k d_{j'} v_1 + k' d_{i'} v_2) \in K.$$

Therefore all elements of the set

$$S = \{k d_{j'} v_1 + k' d_{i'} v_2 \mid k, k' \in K\}$$

are common neighbours of $d_i v_1$ and $d_j v_2$. However they do not give rise to distinct vertices in G. Two elements of the set give rise to the same vertex if and only if we can get from one to the other by multiplying an element of K. So we can always multiply by k'^{-1} to make the coefficient of v_2 vanish. Now it is easy to see that

$$\{k d_{j'} v_1 + d_{i'} v_2 \mid k \in K\}$$

gives $|K| = c_2$ distinct vertices of G. Therefore $d_i v_1$ and $d_j v_2$ have c_2 common neighbours. To prove that they have exactly c_2 common neighbours we show that any other common neighbour of them must be in S. Let v be a common neighbour of $d_i v_1$ and $d_j v_2$. Then

$$B(d_i v_1, v) \in K \quad \text{and} \quad B(d_j v_2, v) \in K.$$

But since $B(d_i v_1, d_{i'} v_2) \in K$ we can find $k_1 \in K$ such that

$$B(d_i v_1, v) = k_1 B(d_i v_1, d_{i'} v_2).$$

Similarly, we can find $k_2 \in K$ such that

$$B(d_j v_2, v) = k_2 B(d_j v_2, d_{j'} v_1).$$

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By the bilinear property of B we get

$$B(d_i v_1, v) = B(d_i v_1, k_1 d_{i'} v_2),$$

and so
$$B(d_i v_1, v - k_1 d_{i'} v_2) = 0.$$

Similarly,
$$B(d_j v_2, v - k_2 d_{j'} v_1) = 0.$$

Therefore, we get

$$B(d_i v_1, v - k_1 d_{i'} v_2 - k_2 d_{j'} v_1) = B(d_i v_1, v - k_1 d_{i'} v_2) - k_2 d_i d_{j'} B(v_1, v_1) = 0 - 0 = 0.$$

Similarly,
$$B(d_j v_2, v - k_1 d_{i'} v_2 - k_2 d_{j'} v_1) = 0.$$

But V is 2-dimensional, which means that any $u \in V$ can be written as a linear combination of $d_i v_1$ and $d_j v_2$. So we get $B(u, v - k_1 d_{i'} v_2 - k_2 d_{j'} v_1) = 0$ for all $u \in V$. Since B is non-degenerate we get

$$v - k_1 d_{i'} v_2 - k_2 d_{j'} v_1 = 0,$$

so $v \in S$ as desired.

6.3 Covers with $\delta = -2$

Let $\mathbb{F}$ be a field of order q where q is a prime power. Let V be a $2j$-dimensional vector space over $\mathbb{F}$ and B be a non-degenerate symplectic form on V. Define the graph G as follows:

$$V(G) = \{(\alpha, u) \mid \alpha \in \mathbb{F}, u \in V\},$$

with (α, u) adjacent to (β, v) if and only if $u \neq v$ and $B(u, v) = \alpha - \beta$. G is undirected since $B(v, u) = -B(u, v) = \beta - \alpha$. For each $u \in V$ define

$$A_u = \{(\alpha, u) \mid \alpha \in \mathbb{F}\}.$$

First note that, by the definition of adjacency, each A_u is an independent set. We prove that G is a cover of $K_{q^{2j}}$ with the sets A_u's as its fibres. To prove this, we just need to show that if u and v are distinct elements of V and $\alpha \in \mathbb{F}$, then (α, u) has exactly one neighbour in A_v. But this is obvious since (α, u) is adjacent to (β, v) if and only if $\beta = \alpha - B(u, v)$.

Now to prove that G is a diameter three antipodal cover, we just need to prove that the number of common neighbours of two non-adjacent vertices in different fibres is a constant. Let (α, u) and (β, v) be two non-adjacent vertices in different fibres. The set of common neighbours is

$$A = \{(\gamma, w) \in \mathbb{F} \times V \mid w \neq u, B(u, w) = \alpha - \gamma, w \neq v, B(v, w) = \beta - \gamma\}.$$

Now define

$$B = \{w \in V \mid w \neq u, w \neq v, B(u - v, w) = \alpha - \beta\}.$$

Note that there is a canonical one to one correspondence between A and B in the sense that if $(\gamma, w) \in A$ then $w \in B$ and conversely if $w \in B$ then there is a unique $\gamma \in \mathbb{F}$ such that $(\gamma, w) \in A$. Now to find $|B|$ note that neither u nor v can be a solution to $B(u - v, w) = \alpha - \beta$, since if $B(u - v, u) = \alpha - \beta$ or $B(u - v, v) = \alpha - \beta$ then in both cases we get $B(u, v) = \alpha - \beta$ which contradicts that (α, u) and (β, v) are two non-adjacent vertices of G. Therefore $|B|$ is just the number of solutions to $B(u - v, w) = \alpha - \beta$ which, as we mentioned in Section 6.1, is q^{2j-1}. Therefore G is a diameter three antipodal cover with parameters (q^{2j}, q, q^{2j-1}).

6.4 Covers from Strongly Regular Graphs

Let G be a diameter three antipodal cover with parameters n, r, c_2. Then G is distance-regular and any two vertices are at distance at most two from each other unless they lie in the same fibre. Therefore if we join any two vertices in the same fibre by an edge the resulting graph has diameter two. The natural question that arises is whether this new graph, H, is still distance-regular. Distance-regular graphs of diameter two are called *strongly regular graphs* and have been extensively studied (see for example Godsil and Royle [11]).

Note that if H is to be strongly regular, the number of common neighbours of two adjacent vertices must be a constant. If two adjacent vertices lie in different fibres then since there is a 1-factor between any two fibres no new common neighbour is added in H. So as we established in Chapter 5 the number of common neighbours is $a_1 = n - c_2(r - 1) - 2$. On the other hand, if two adjacent vertices lie in the same fibre then the common neighbours are exactly all the other vertices in that fibre, hence there are $r - 2$ of them. Therefore for H to be strongly regular we must have $n - c_2(r-1) - 2 = r - 2$, or equivalently

$$n = c_2(r - 1) + r. \tag{6.4.1}$$

We see shortly that this obvious necessary condition is indeed sufficient as well. More importantly, we see that the converse holds as well, that is, any strongly regular graph with the appropriate parameters and a partition of its vertices into cliques of an appropriate fixed size gives rise to an antipodal cover of diameter three.

To prove that Condition 6.4.1 is sufficient for H to be strongly regular it suffices to show that the number of common neighbours of two non-adjacent vertices, say u and v, in H is a constant. Let C_u and C_v be the fibres containing u and v respectively. Since u has exactly one neighbour in C_v and v has exactly one neighbour in C_u and since the subgraphs of H induced on C_u and C_v are cliques we get exactly two common neighbours for u and v in addition to c_2 common neighbours they already had in G. Therefore u and v have exactly $c_2 + 2$ common neighbours in H. This proves H is a strongly regular graph.

The degree of each vertex in H is $(n - 1) + (r - 1) = n + r - 2$. The intersection array of H as a distance-regular graph is

$$\{n + r - 2, (n + r - 2) - 1 - (r - 2); 1, c_2 + 2\} = \{n + r - 2, n - 1; 1, c_2 + 2\}.$$

For reasons that we explain later we parametrize the covers that satisfy 6.4.1 as follows: $r = s + 1$ and $c_2 = t - 1$ which implies $n = st + 1$. The intersection array then becomes $\{1, t + 1; s(t + 1), st\}$. So we have just proved that any cover with parameters $(n, r, c_2) = (st + 1, s + 1, t - 1)$ gives rise to a strongly regular graph with intersection array $\{1, t + 1; s(t + 1), st\}$ which has the additional structure of having its vertices partitioned into cliques of size $s + 1$. Now we prove the converse, i.e. for any such strongly regular graph by deleting all the edges inside each clique in the partition we get an antipodal cover of diameter three with former cliques as its fibres and with parameters $(n, r, c_2) = (st + 1, s + 1, t - 1)$.

Let H be a strongly regular graph with intersection array $\{1, t + 1; s(t + 1), st\}$ where its vertices partition into cliques of size $s + 1$. From the intersection array it follows $|V(H)| = (s + 1)(st + 1)$ and therefore the number of cliques in the partition is $st + 1$. Let G be the graph obtained from H by deleting all the edges inside each clique of the partition. To prove that G is a diameter three antipodal cover with the former cliques as its fibres, first we need to show there is a 1-factor between any two fibres. Let C_1 and C_2 be two fibres of G and $u \in C_1$. If u has two neighbours in C_2, say v_1 and v_2 then u is a common neighbour of v_1 and v_2. But all the vertices in

$C_2 \backslash \{v_1, v_2\}$ are also common neighbours of v_1 and v_2 in H. Therefore the number of common neighbours of v_1 and v_2 in H is greater than $s - 1$. But from the intersection array of H the number of common neighbours of two adjacent vertices in H is $s(t + 1) - st - 1 = s - 1$. This is a contradiction. Therefore u has at most one neighbour in any fibre of G. On the other hand, since u has valency $s(t + 1)$ in H and it has s neighbours in C_1 it must have exactly one neighbour in any of the st fibres other than C_1. This proves there is a 1-factor between any two fibres.

Now let $u \in C_1$ and $v \in C_2$ be non-adjacent. Since there is a 1-factor between C_1 and C_2 we have u and v have exactly two common neighbours in $C_1 \cup C_2$ as vertices of H. Since in total they have $t + 1$ common neighbours in H they must have $t - 1$ common neighbours in G. Thus G is an antipodal distance regular cover of diameter three, with parameters $(n, r, c_2) = (st + 1, s + 1, t - 1)$.

For $(n, r, c_2) = (st + 1, s + 1, t - 1)$, we get

$$\delta = n - 2 - rc_2 = st + 1 - 2 - (s + 1)(t - 1) = s - t.$$

For any power of 2, say q, if $s = q + 1$ and $t = q - 1$, then there exists a strongly regular graph with intersection array $\{1, t + 1; s(t + 1), st\}$ where its vertices partition into cliques of size $s + 1$ (see Godsil and Hensel [10]). Therefore, there exist an infinite family of antipodal covers with

$$\delta = s - t = q + 1 - (q - 1) = 2.$$

Chapter 7

Equiangular Lines from Covers

As we mentioned in Section 2.1, each set of equiangular lines corresponds to a Seidel matrix. By Lemma 4.3.1, for each Abelian cover Y of a complete graph K_n with symmetric arc function f, and any non-trivial representation of the the group generated by the image of f, $A(X)^{\phi(f)}$ is a Seidel matrix.

Now the question is whether we can find covers which give us sets of equiangular lines of maximum size, in the sense that the absolute bound is tight. In this chapter, we find the parameters of an antipodal distance regular graph of diameter three which give us sets of equiangular lines of maximum size.

If the absolute bound is tight, so is the relative bound. So by Theorem 2.3.2, if $A(X)^{\phi(f)}$ is a Seidel matrix for a set of equiangular lines of maximum size, then it must have only two eigenvalues. The following theorem in Godsil and Hensel [10, p. 225] (Lemma 8.2) guarantees this.

7.0.1 Theorem. *Let G be an Abelian cover of K_n which is antipodal of diameter three with eigenvalues -1, $n-1$, θ and τ. Let f be a normalized symmetric arc function defining G and ϕ be a non-trivial representation of $\langle f \rangle$. Then $A(K_n)^{\phi(f)}$ has only two eigenvalues, namely, θ and τ, with multiplicities*

$$\overline{m}_\theta = \frac{n\tau}{\tau - \theta}, \qquad \overline{m}_\tau = \frac{n\theta}{\theta - \tau}.$$

7.0.2 Remark. *If Y is an r-fold cover of K_n, then all we can say about the off-diagonal entries of $A(X)^{\phi(f)}$ is that they are r-th roots of unity. So in general, we cannot guarantee that they are real unless r is equal to two.*

7.1 New Feasibility Condition for Abelian Covers

As we saw in the beginning of this chapter, antipodal distance regular covers of diameter three which are Abelian covers can be used to construct equiangular lines. So the absolute bound for equiangular lines gives us a feasibility condition on the parameters of the cover. This has been made explicit in the next theorem.

7.1.1 Theorem. *Let G be an Abelian cover of K_n which is antipodal of diameter three with eigenvalues -1, $n - 1$, θ and τ. Then we have*

$$\tau \leqslant -\sqrt{\sqrt{n} + 1}.$$

Furthermore, if equality occurs then there is a set of equiangular lines of size $\overline{m}_\theta^2$ in dimension $\overline{m}_\theta$.

Proof. Let f be a normalized symmetric arc function defining G and ϕ be a non-trivial representation of $\langle f \rangle$. By Lemma 4.3.1 and Theorem 2.1.1, $A(K_n)^{\phi(f)}$ is a Seidel matrix for a set of n equiangular lines in dimension $n - m$, where m is the multiplicity of the least eigenvalue of $A(K_n)^{\phi(f)}$.

By Theorem 7.0.1, $A(K_n)^{\phi(f)}$ has only two eigenvalues, θ and τ. Therefore $n - m$ is just the multiplicity of the larger eigenvalue of $A(K_n)^{\phi(f)}$, i.e. θ. So we get n equiangular lines in dimension $\overline{m}_\theta$. Now we can apply the absolute bound. Note that since in general, the lines are complex, we have to use the absolute bound for complex equiangular lines. So we must have

$$n \leqslant \overline{m}_\theta^2,$$

or equivalently

$$\overline{m}_\theta \geqslant \sqrt{n}.$$

Now since $\theta\tau$ is equal to $1 - n$, we get

$$\overline{m}_\theta = \frac{n\tau}{\tau - \theta} = \frac{n}{1 - \frac{\theta}{\tau}} = \frac{n}{1 + \frac{n-1}{\tau^2}}.$$

Since $\overline{m}_\theta$ is strictly increasing as a function of $|\tau|$, and is equal to $\sqrt{n}$ when $|\tau|$ is equal to

$$\sqrt{\sqrt{n} + 1},$$

we must have

$$|\tau| \geqslant \sqrt{\sqrt{n} + 1},$$

and if equality happens then we get a set of equiangular lines of size $\overline{m}_\theta^2$ in dimension $\overline{m}_\theta$. □

Note that by the proof of Theorem 7.1.1, $\overline{m}_\theta$ is monotonic in $|\tau|$ and to get an equiangular set of lines with size of order the square of the dimension we need $|\tau|$ to be of order $n^{1/4}$.

7.2 Parameters

Let G be an Abelian antipodal cover of diameter three, with parameters n, r, c_2, and eigenvalues -1, $n-1$, θ and τ. Theorem 7.1.1 gives the value of τ in terms of n that leads to a set of complex equiangular lines of maximum size. Since n, r, c_2 are all integers, we get the following theorem. In the following theorem, rc_2 can be determined in terms of t, but not r or c_2 separately.

7.2.1 Theorem. *For an Abelian distance regular cover of K_n with parameters n, r, c_2, and eigenvalues -1, $n-1$, θ and τ, we obtain a set of complex equiangular lines of maximum size if and only if*

$$n = (t^2 - 1)^2, \qquad \tau = -t;$$

where t can be any positive integer larger than one. If the above holds the other parameters of the cover are

$$\theta = t(t^2 - 2), \quad \delta = (t^2 - 3)t$$
$$\overline{m}_\theta = t^2 - 1, \quad \overline{m}_\tau = (t^2 - 2)(t^2 - 1)$$

Proof. First note that

$$\delta = \theta + \tau = \frac{1 - n}{\tau} + \tau.$$

Since

$$\tau = -\sqrt{\sqrt{n} + 1}$$

we get

$$\delta = (\sqrt{n} - 2)(\sqrt{\sqrt{n} + 1}).$$

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But we also have

$$\delta = n - 2 - rc_2.$$

So δ is an integer. On the other hand, since we have a set of complex equiangular lines of maximum size, we must have $\overline{m}_\theta = \sqrt{n}$. So $\sqrt{n}$ must be an integer. This together with the fact that

$$\delta = (\sqrt{n} - 2)(\sqrt{\sqrt{n} + 1})$$

is an integer, implies the $\sqrt{\sqrt{n} + 1}$ is an integer. Set

$$t = \sqrt{\sqrt{n} + 1}.$$

So we get

$$n = (t^2 - 1)^2, \qquad \tau = -t.$$

The rest of the parameters can be computed according to their definitions. □

7.3 Feasible Parameters

Now the question is: for what values of t, do the parameters defined in Theorem 7.2.1, satisfy all the feasibility conditions given in Theorem 5.3.1.

7.3.1 Theorem. *Let G be an Abelian cover with parameters n, r, c_2, such that for some positive integer t, we have*

$$n = (t^2 - 1)^2, \qquad rc_2 = (t - 1)^2(t^2 + t - 1).$$

Then

$$r \mid t - 1.$$

Proof. By Theorem 7.2.1, θ is an integer. By Condition 10 in Theorem 5.3.1, if n is larger than $m_\theta - r + 3$, then $\theta + 1$ divides c_2. Now by substituting $(t^2 - 1)^2$ for n and

$$\overline{m}_\theta(r - 1) = (t^2 - 1)(r - 1)$$

for m_θ, and some straightforward computation, $n > m_\theta - r + 3$ reduces to

$$r < t^2 + 1 - \frac{1}{t^2 - 2};$$

and since r is an integer, we get

$$r \leqslant t^2.$$

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On the other hand, we have

$$\theta + 1 = t(t^2 - 2) + 1 = t^3 - 2t + 1 = (t - 1)(t^2 + t - 1).$$

Thus, Condition 10 in Theorem 5.3.1 becomes: if $r \leqslant t^2$, then $(t-1)(t^2+t-1)$ divides c_2. But from

$$rc_2 = (t - 1)^2(t^2 + t - 1),$$

we get: if $r \leqslant t^2$, then $r \mid t - 1$.

Now we prove that r must always be less than or equal to t^2, and hence, we are done. We show that r always divides $(t - 1)^2$, which implies $r \leqslant t^2$. Since

$$rc_2 = (t - 1)^2(t^2 + t - 1),$$

to prove that r divides $(t - 1)^2$, it suffices to prove that r and $t^2 + t - 1$ are co-prime. Suppose on the contrary that p is a prime that divides both r and $t^2 + t - 1$. Since $t^2 + t - 1$ is always odd, p must be odd. So by Theorem 5.6.3, since p divides r, then p divides n. Thus we have

$$p \mid t^2 - 1, \qquad p \mid t^2 + t - 1,$$

which implies $p \mid \{(t^2 + t - 1) - (t^2 - 1)\}$. Therefore, $p \mid t$, so $p \mid t^2$ and consequently $p \mid 1$, a contradiction. $\qquad \square$

Now we show that apart from some lower bounds on t, r and c_2, the condition $r \mid t - 1$ is all we need to guarantee that all feasibility conditions stated in Theorem 5.3.1 are satisfied.

7.3.2 Theorem. *A parameter set (n, r, c_2) satisfies all the conditions in Theorem 5.3.1, and a cover with the corresponding parameters gives a set of complex equiangular lines of maximum size if and only if there is a positive integer t such that*

$$n = \left(t^2 - 1\right)^2, \qquad rc_2 = (t - 1)^2(t^2 + t - 1),$$

where $t \geqslant 3$, $r \geqslant 4$, $c_2 \geqslant 2$, and

$$r \mid t - 1.$$

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Proof. We need to check that when

$$n = \left(t^2 - 1\right)^2, \qquad rc_2 = (t-1)^2(t^2 + t - 1),$$

the conditions in Theorem 5.3.1 give no restrictions on the parameters of the cover, except the ones given in the theorem. The first feasibility condition in Theorem 5.3.1, states that $(r-1)c_2 \leqslant n-2$ which is equivalent to $\delta + c_2 \geqslant 0$. Since $\delta = (t^2 - 3)t$ and c_2 are always positive this condition always holds. Also Conditions 2, 3 and 4 are trivially always satisfied and so we get no restriction on t. Condition 5 states that if n is even then c_2 is even. So if t is odd then c_2 is even. But since

$$rc_2 = (t-1)^2(t^2 + t - 1),$$

and

$$r \mid t - 1,$$

we get

$$t - 1 \mid c_2,$$

and hence if t is odd, then c_2 is even.

In Condition 6 we have if $c_2 = 1$ then $(n-r)^2 \leqslant n - 1$. Now we prove that $(n-r)^2 > n - 1$ and so $c_2 \neq 1$. We have

$$
\begin{aligned}
(n-r)^2 - (n-1) &= (n - rc_2)^2 - (n-1) \\
&\geqslant \left((t^2 - 1)^2 - (t-1)^2(t^2 + t - 1)\right)^2 - \left((t^2 - 1)^2 - 1\right) \\
&\geqslant (t-1)^4(t+2)^2 - (t^2 - 2)t^2
\end{aligned}
$$

which is greater than zero if $t = 2$. If $t \geqslant 3$ we get

$$
\begin{aligned}
(n-r)^2 - (n-1) &\geqslant (t-1)^4(t+2)^2 - (t^2 - 2)t^2 \\
&\geqslant t^2 \left((t-1)^4 - (t^2 - 2)\right) \\
&= t^2 \left((t^2 - 3t + 1)(t^2 - t + 1) + 2\right),
\end{aligned}
$$

which is greater than zero since $(t^2 - 3t + 1)$ and $(t^2 - t + 1)$ are both positive for any $t \geqslant 3$. So we always get $(n-r)^2 > n - 1$. Thus by condition 6 we must have $c_2 \neq 1$ and therefore $c_2 \geqslant 2$. Note that $c_2 \geqslant 2$ implies that $t \geqslant 3$, since for $t = 2$ we get $rc_2 = (t-1)^2(t^2 + t - 1) = 5$ and so $c_2 = 5$ and $r = 1$ which is a contradiction.

Now we prove that Condition 7 is always satisfied and hence it gives no restrictions on the parameters. Condition 7 states that $n \leqslant c_2(2r - 1)$ or equivalently $2rc_2 - n \geqslant c_2$. But we have

$$2rc_2 - n = 2(t - 1)^2(t^2 + t - 1) - (t^2 - 1)^2 = (t - 1)^2(t^2 - 3).$$

Now since $rc_2 = (t - 1)^2(t^2 + t - 1)$, we have $(t - 1)^2(t^2 - 3) \geqslant c_2$ if and only if

$$r \geqslant \frac{t^2 + t - 1}{t^2 - 3}.$$

But

$$\frac{t^2 + t - 1}{t^2 - 3} = 1 + \frac{t + 2}{t^2 - 3}$$

is less than 2 when $t = 3$; and, is decreasing for any $t \geqslant 3$. Therefore, for any $t \geqslant 3$,

$$\frac{t^2 + t - 1}{t^2 - 3} < 2.$$

But the size of each fibre, r, is at least 2, and so for any $t \geqslant 3$,

$$r \geqslant \frac{t^2 + t - 1}{t^2 - 3},$$

as desired.

Condition 8 states that $\theta^3 \geqslant n - 1$. Substituting the parameters in terms of t we get

$$(t^2 - 2)^3 t^3 \geqslant t^2(t^2 - 2),$$

which is equivalent to

$$(t^2 - 2)^2 t \geqslant 1,$$

and this is clearly satisfied.

Now using Condition 9, we prove that $r \geqslant 4$. First note that $r \neq 2$, since otherwise, by Remark 7.0.2, we have n real equiangular lines in dimension $\sqrt{n}$ which contradicts the absolute bound for real equiangular lines. Noting that $m_\theta = \overline{m}_\theta(r - 1)$ and $\overline{m}_\theta = \sqrt{n}$, by Feasibility Condition 9 in Theorem 5.3.1, we must have

$$\begin{aligned} rn &\leqslant \frac{1}{2}m_\theta(m_\theta + 1) \\ &= \frac{1}{2}\sqrt{n}(r - 1)(\sqrt{n}(r - 1) + 1). \end{aligned}$$

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Simplify this to get

$$rn \leqslant \frac{1}{2}(r-1)^2 n + \frac{1}{2}(r-1)\sqrt{n}.$$

Note that if $r \geqslant 4$ then $r < \frac{1}{2}(r-1)^2$ and so the condition is satisfied. However, if $r = 3$, the condition is reduced to $n \leqslant \sqrt{n}$ which is never satisfied. Therefore $r \geqslant 4$. It also follows that the other condition in Feasibility Condition 9, i.e.

$$rn \leqslant \frac{1}{2}m_\tau (m_\tau + 1)$$

is satisfied for any $r \geqslant 4$, since m_τ is larger than m_θ.

Condition 10 says that if n is larger than $m_\theta - r + 3$, then $\theta + 1$ divides c_2, but this has already been considered in Theorem 7.3.1. Condition 10 also says that if n is larger than $m_\tau - r + 3$, then $\tau + 1$ divides c_2. First note that

$$m_\tau = \overline{m}_\tau (r-1).$$

Also since

$$\overline{m}_\theta = \sqrt{n},$$

and

$$\overline{m}_\theta + \overline{m}_\tau = n,$$

we get

$$\overline{m}_\tau = n - \sqrt{n}.$$

Now since $r \geqslant 4$ we have

$$
\begin{aligned}
m_\tau - r + 3 - n &= \overline{m}_\tau (r-1) - r + 3 - n \\
&= (n - \sqrt{n})(r-1) - r + 3 - n \\
&= r(n - \sqrt{n} - 1) - (2n - \sqrt{n} - 3) \\
&\geqslant 4(n - \sqrt{n} - 1) - (2n - \sqrt{n} - 3) \\
&= 2n - 3\sqrt{n} - 1,
\end{aligned}
$$

which is greater than zero for any $n \geqslant 4$. But as we showed above $t \geqslant 3$ and since $n = (t^2 - 1)^2$ we get $n \geqslant 64$. Therefore, we always have

$$n < m_\tau - r + 3,$$

and so we do not get any more restriction from condition 10. $\qquad \square$

$\sqrt{n}$	n	r	c_2	θ	τ	m_θ	m_τ
35	1225	5	205	204	-6	140	4760
63	3969	7	497	496	-8	378	23436
120	14400	5	2620	1309	-11	480	57120
143	20449	11	1705	1704	-12	1430	203060
195	38025	13	2717	2716	-14	2340	453960
224	50176	7	6692	3345	-15	1344	299712
255	65025	5	12195	4064	-16	1020	259080
288	82944	17	5797	5796	-18	4608	1322496
399	159201	19	7961	7960	-20	7182	2858436

Table 7.1: feasible parameters for $t \leqslant 20$

Unfortunately, the following result eliminates the possibility of covers with small fibre sizes.

7.3.3 Theorem. *Let G be an Abelian cover with parameters n, r, c_2, such that for some positive integer t, we have*

$$n = (t^2 - 1)^2, \qquad rc_2 = (t - 1)^2(t^2 + t - 1).$$

Then r is not divisible by 2 or 3.

Proof. Let p be a prime that divides r. Let H be a subgroup of index p of the group of automorphisms of G fixing its fibres. Let π be the partition of the vertices of G with its cells as the orbits of H on the fibres. Then by Theorem 5.6.1, G/π is an Abelian antipodal distance regular cover of diameter three with parameters

$$n, p, \frac{r}{p}c_2.$$

Now note that θ, τ, $\overline{m}_\theta$ and $\overline{m}_\tau$ are all defined in terms of rc_2 and n. But rc_2 and n remain unchanged in the new cover. So the feasibility conditions we found in Theorem 7.3.2 apply here. In particular, we must have the fibre size, p, is at least 4. □

A list of feasible parameters for $t \leqslant 20$ is given in Table 7.1.

7.4 Real Case

In this section, we see that, for Abelian antipodal covers of complete graphs, if the fibre size is even, then we can find sets of real equiangular lines, and

hence, we can apply the absolute bound given in Theorem 2.2.2 for real equiangular lines, which gives us a stronger feasibility condition than the one given in Theorem 7.1.1.

7.4.1 Theorem. *Let G be an Abelian cover of K_n that is antipodal of diameter three with parameters n, r, c_2 and eigenvalues -1, $n - 1$, θ and τ. Suppose that r is even. Then we have*

$$\tau \leqslant -\sqrt{\frac{\sqrt{1 + 8n} + 3}{2}}.$$

Furthermore, if equality occurs, then there is a set of real equiangular lines of size $\binom{\overline{m}_\theta + 1}{2}$ in dimension $\overline{m}_\theta$.

Proof. Let H be a subgroup of index 2 of the group of automorphisms of G fixing its fibres. Let π be the partition of the vertices of G with its cells as the orbits of H on the fibres. Then by Theorem 5.6.1, G/π is an Abelian antipodal distance regular cover of diameter three with parameters

$$n, 2, \frac{r}{2}c_2.$$

Now note that θ, τ, $\overline{m}_\theta$ and $\overline{m}_\tau$ are all defined in terms of rc_2 and n. But rc_2 and n remain unchanged in the new cover. So as we saw in the proof of Theorem 7.1.1, we get n equiangular lines in dimension $\overline{m}_\theta$. Since we are working with 2-fold covers, by Remark 7.0.2, we get real equiangular lines. So by applying the absolute bound for real equiangular lines we get

$$n \leqslant \binom{\overline{m}_\theta + 1}{2},$$

or equivalently,

$$\overline{m}_\theta \geqslant \frac{-1 + \sqrt{1 + 8n}}{2}.$$

Now since $\theta\tau$ is equal to $1 - n$, we get

$$\overline{m}_\theta = \frac{n\tau}{\tau - \theta} = \frac{n}{1 - \frac{\theta}{\tau}} = \frac{n}{1 + \frac{n-1}{\tau^2}}.$$

Since $\overline{m}_\theta$ is strictly increasing as a function of $|\tau|$, and is equal to $\frac{-1+\sqrt{1+8n}}{2}$ when $|\tau|$ is equal to

$$\sqrt{\frac{\sqrt{1 + 8n} + 3}{2}},$$

we must have

$$|\tau| \geqslant \sqrt{\frac{\sqrt{1 + 8n} + 3}{2}},$$

and if equality happens then we get a set of real equiangular lines of size $\binom{\overline{m}_\theta + 1}{2}$ in dimension $\overline{m}_\theta$. □

Now similar to Theorem 7.2.1, we get the following theorem.

7.4.2 Theorem. *For an Abelian distance regular cover of K_n with parameters n, r, c_2, and eigenvalues -1, $n - 1$, θ and τ, we obtain a set of real equiangular lines of maximum size if and only if r is even, and*

$$n = \frac{(t^2 - 2)(t^2 - 1)}{2}, \qquad \tau = -t;$$

where t can be any positive integer larger than one. If the above holds the other parameters of the cover are

$$\theta = \frac{t(t^2 - 3)}{2}, \qquad \delta = \frac{t(t^2 - 5)}{2}$$
$$\overline{m}_\theta = t^2 - 2, \quad \overline{m}_\tau = \frac{(t^2 - 2)(t^2 - 3)}{2}$$

Considering all the feasibility conditions in Theorem 5.3.1, we get the following analogue of Theorem 7.3.2.

7.4.3 Theorem. *Let n, r, c_2 be the parameters of an Abelian distance regular cover of K_n. Suppose that r is even, and for some positive integer t,*

$$n = \frac{(t^2 - 2)(t^2 - 1)}{2}, \qquad rc_2 = \frac{1}{2}(t - 1)^3(t + 2).$$

Then, $\{n, r, c_2\}$ satisfies all the feasibility conditions of Theorem 5.3.1 if and only if

(a) $c_2 \geqslant 2$ and $t \geqslant 3$;

(b) if t is odd then c_2 is even;

(c) if $r \leqslant \frac{t^2 + 1}{2}$ then $r \mid t - 1$; and

(d) any odd prime that divides r must divide $t - 1$ as well.

Chapter 8

Terwilliger Algebra of Covers of Complete Graphs

We defined the Bose-Mesner algebra of an association scheme to be the algebra generated by its distance matrices. Fix a vertex u of a distance regular graph of diameter d. For each i, $0 \leqslant i \leqslant d$, we call the vertices at distance i from u, the i-th neighbourhood of u. Let F_i be a diagonal matrix with the characteristic vector of the i-th neighbourhood of u as its diagonal. The *Terwilliger algebra* of the distance regular graph with respect to u is defined to be the algebra generated by the distance matrices and the diagonal matrices $F_0, F_1, \cdots, F_d$. Since F_i's have the characteristic vectors of the neighbourhoods of u as their diagonals, the Terwilliger algebra would be useful to get information on the neighbourhoods. We can have different Terwilliger algebras based on the choice of the vertex of origin. However, in this chapter, our vertex of origin is arbitrary. So all the materials in this chapter apply to any Terwilliger algebra with respect to any vertex.

Note that, any matrix in the Bose-Mesner algebra of a distance regular graph is a polynomial in terms of the adjacency matrix of the graph. So a basis of eigenvectors for the adjacency matrix is also a basis of eigenvectors for the Bose-Mesner algebra. Therefore, if v is the number of vertices of the graph, we can write $\mathbb{R}^v$ as a direct sum of one dimensional spaces invariant under the Bose-Mesner Algebra. A subspace invariant under the algebra is called a *module* over that algebra. A module is called *irreducible* if it does not contain a proper non-zero submodule. Let $\mathbb{T}$ be a Terwilliger algebra of a distance regular graph. By a $\mathbb{T}$-*module*, we mean a module over $\mathbb{T}$.

A useful way to study an algebra is to find the decomposition of the space

into irreducible modules over the algebra. For the Bose-Mesner algebra, this decomposition is just a decomposition into one dimensional subspaces. For a Terwilliger algebra, although this decomposition is still possible, it is more complex. In general, the modules in the decomposition are not one dimensional.

In this chapter, we find the decomposition of $\mathbb{R}^{rn}$ into irreducible $\mathbb{T}$-modules, where $\mathbb{T}$ is a Terwilliger algebra of an antipodal cover of diameter three with parameters n, r, c_2. Such a decomposition has been found for distance regular graphs of diameter two, that is, strongly regular graphs, by Tomiyama and Yamazaki [24].

We can also use a Terwilliger algebra to get restrictions on the parameters of a distance regular graph. Feasibility Condition 10 in Theorem 5.3.1 follows from Theorem 8.10.6 that we prove in this chapter using a Terwilliger algebra.

8.1 Distance Partition

Suppose X is a distance regular cover of a compete graph with parameters n, r, c_2. Let $\{V_0, V_1, V_2, V_3\}$ be the distance partition of X with respect to some vertex, illustrated in Figure 8.1. We usually refer to V_i as the i-th neighbourhood. The rows and columns of the distance matrices of X are indexed by the vertices of X, so the ordering of the vertices affects the distance matrices. In this section, we order the vertices of X so that the distance matrices have a simple form. It will be very useful in the remaining sections of this chapter, since distance matrices are involved extensively in our computations.

We have

$$|V_0| = 1, \quad |V_1| = n - 1, \quad |V_2| = (n-1)(r-1), \quad |V_3| = r - 1.$$

We can check that

$$|V_0| + |V_1| + |V_2| + |V_3| = rn = |V(X)|$$

We represent the second neighbourhood by a rectangle consisting of $n - 1$ rows and $r - 1$ columns. We place the vertices in the second neighbourhood
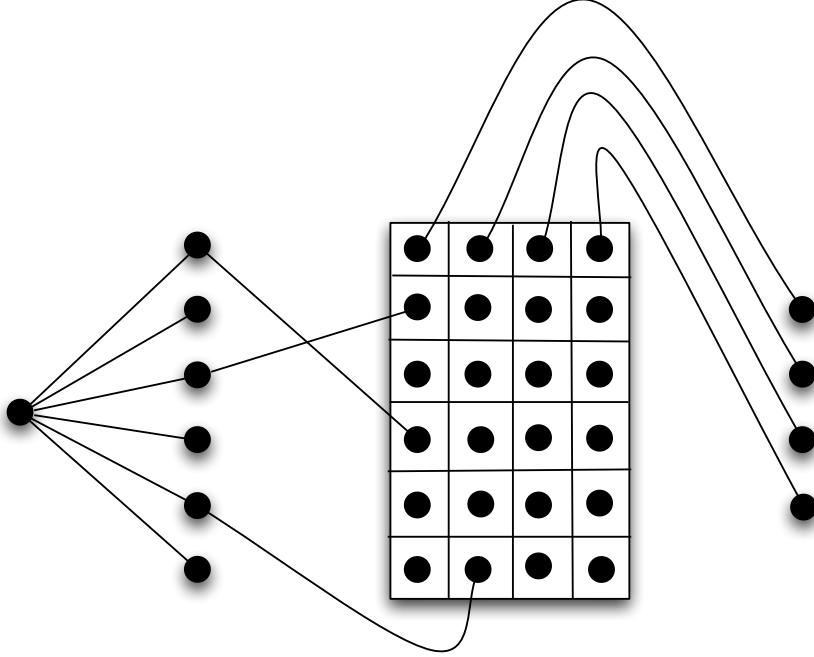
Figure 8.1: distance partition of an antipodal cover

such that for any i, $1 \leqslant i \leqslant n-1$, the i-th vertex in the first neighbourhood together with the i-th row of the second neighbourhood form a fibre of the cover. Then, we move the vertices in each row such that for any j, $1 \leqslant j \leqslant r-1$, the j-th column of V_2 is the neighbourhood of the j-th vertex of V_3.

Let π be a partition of a finite set S. The *characteristic matrix* of π is defined to be the matrix whose columns are the characteristic vectors of the cells of π. So it is a $|S| \times |\pi|$ matrix with $0, 1$ entries whose columns sum to the all ones vector. The column space of the characteristic matrix is the space of vectors defined on S which are constant on the cells of π. Also, the null space of the transpose of the characteristic matrix is the space of vectors defined on S which sum to zero on the cells of π. The space of the vectors defined on S can be written as the direct sum of the vectors constant on each cell of π and the vectors summing to zero on the cells of π.

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Remark. The reason we represent V_2 by a rectangle is that the two partitions associated with it, the row partition and the column partition, come up naturally when we study the $\mathbb{T}$-modules of $\mathbb{R}^{rn}$. Define R and C be the characteristic matrices of the row partition and the column partition, respectively.

From the way we arranged the vertices of X, we get the adjacency matrix of X given below. Here B_1 and B_2 are the adjacency matrices of the graphs induced on the first and second neighbourhood, respectively. Also N is the matrix recording the adjacency between the second and first neighbourhood.

$$A = \begin{bmatrix} 0 & \mathbb{1}_{n-1}^T & 0 & 0 \\ \mathbb{1}_{n-1} & B_1 & N^T & 0 \\ 0 & N & B_2 & C \\ 0 & 0 & C^T & 0 \end{bmatrix}. \tag{8.1.1}$$

Also we get

$$A_3 = \begin{bmatrix} 0 & 0 & 0 & \mathbb{1}_{r-1}^T \\ 0 & 0 & R^T & 0 \\ 0 & R & RR^T - I & 0 \\ I_{r-1} & 0 & 0 & J - I \end{bmatrix}. \tag{8.1.2}$$

Since

$$A_2 = J - I - A - A_3,$$

we can find the block decomposition of A_2 as well.

We always partition any vector $w \in \mathbb{R}^{rn}$ according to the distance partition. In other words, we represent w as

$$w = \begin{bmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \end{bmatrix}, \qquad w_i \in V_i.$$

for all i, $0 \leqslant i \leqslant 3$. When we say w sums to zero on V_i, we mean that the sum of entries of w_i is zero, i.e. $\mathbb{1}^T w_i = 0$.

8.2 Summary of the Decomposition

Since a Terwilliger Algebra is semi-simple, we can decompose $\mathbb{R}^{rn}$ as the direct sum of $\mathbb{T}$-modules. In this section we see what the $\mathbb{T}$-modules in that

dimension	multiplicity	parametrized by
1	$(n-2)(r-3)+m$	eigenvectors of B_2 summing to zero on the cells of both the row and the column partitions
2	$r-2$	eigenvectors of B_2 which sum to zero and are constant on the cells of the column partition
2	m	eigenvectors of B_2 which sum to zero and are constant on the cells of the row partition
3	$n-2-m$	eigenvectors of B_1 which sum to zero and the corresponding eigenvalues are not μ_θ or μ_τ
4	1	the all ones vector

Table 8.1: $\mathbb{T}$-module decomposition of $\mathbb{R}^{rn}$

decomposition look like, what their dimensions are, and how many times a $\mathbb{T}$-module of a certain dimension appears in that decomposition. We also analyze each $\mathbb{T}$-module and we will see how they are related to each other. We will leave the proofs of the results in this section for the remaining sections of this chapter.

In computing the $\mathbb{T}$-module decomposition of $\mathbb{R}^{rn}$, two numbers turn out to be special. To refer to them more easily in the future, we give them names here.

$$\mu_\theta = \frac{(r-1)\theta - 1}{r}, \qquad \mu_\tau = \frac{(r-1)\tau - 1}{r} \qquad (8.2.1)$$

Table 8.1 gives a summary of the $\mathbb{T}$-module decomposition of $\mathbb{R}^{rn}$. For now, assume m is just a parameter. We will see shortly what m explicitly is.

We refer to each $\mathbb{T}$-module by its dimension. The only problem that arises is for $\mathbb{T}$-modules of dimension two, since there are two types of them in Table 8.1. To distinguish them, we call $\mathbb{T}$-modules of dimension two either *row type* or *column type* according to whether they are parametrized by vectors constant on the cells of the row partition or the column partition.

The following theorem makes clear the connection between row type $\mathbb{T}$-modules of dimension two and $\mathbb{T}$-modules of dimension three. The first part

of the following theorem is proved in Theorem 8.10.5, and, the second part, in Corollary 8.7.2.

8.2.1 Theorem. *For any non-trivial eigenvalue μ of B_1, we have*

$$\mu_\theta \leqslant \mu \leqslant \mu_\tau.$$

Moreover, if u is a non-zero vector which sums to zero, and, is defined on the first neighbourhood, then u is an eigenvector of B_1 with the corresponding eigenvalue μ_θ or μ_τ if and only if Ru is an eigenvector of B_2.

Note that, by Theorem 8.2.1, each non-trivial eigenvector u of B_1 gives rise to an irreducible $\mathbb{T}$-module in the decomposition of $\mathbb{R}^{rn}$. If the corresponding eigenvalue is one of μ_θ and μ_τ, then u gives rise to a row type $\mathbb{T}$-module of dimension two; but if the corresponding eigenvalue is anything else, then u gives rise to a $\mathbb{T}$-module of dimension three. Since B_1 is of order $n-1$ we get the multiplicity of $\mathbb{T}$-modules of dimension three as $n-2-m$, as presented in Table 8.1. We also get

$$0 \leqslant m \leqslant n-2.$$

In Section 8.10, we will discuss the cases where the above lower bound or upper bound on m is tight.

The following theorem, proved in Theorem 8.9.3, completely characterizes the column type $\mathbb{T}$-modules of dimension two. Recall that by Theorem 5.2.1, δ which was defined in Equation 5.2.1, is the sum of two eigenvalues, θ and τ, of the cover.

8.2.2 Theorem. *Let v be a vector defined on the second neighbourhood. If v sums to zero and is constant on the cells of the column partition, then v is an eigenvector of B_2 corresponding to the eigenvalue δ.*

Note that since the second neighbourhood has $r-1$ columns, it automatically follows from Theorem 8.2.2 that the multiplicity of the column type $\mathbb{T}$-modules of dimension two is $r-2$, as presented in Table 8.1.

By Table 8.1, we see that any $\mathbb{T}$-module of dimension one or two is parametrized by an eigenvector of B_2. By Theorem 8.2.2 the corresponding eigenvalue in the case of column type $\mathbb{T}$-modules of dimension two is δ. We will prove that the corresponding eigenvalues in the case of $\mathbb{T}$-modules of dimension one are θ and τ. The following theorem, proved in Section 8.7, tells us what the corresponding eigenvalues are in the case of row type $\mathbb{T}$-modules of dimension two.

8.2.3 Theorem. *For any non-trivial eigenvector of B_2 that is constant on the cells of the row partition, the corresponding eigenvalue must be one the following:*

$$\frac{\theta + 1}{r} - 1, \qquad \frac{\tau + 1}{r} - 1.$$

8.3 Local Properties

To compute the $\mathbb{T}$-modules of A, we need to know how the matrices in the block decompositions of A_i's, given in 8.1.1 and 8.1.2, are related to each other. Since these blocks record the adjacency in subgraphs of X, we call the relations between them *local properties*. Some local properties are given in Lemma 8.3.1, and Theorem 8.3.2. Note that, for $i = 1, 2$, B_i is just the adjacency matrix of the i-th neighbourhood. Also N is the matrix that records the adjacency between the second and the first neighbourhood.

Our main tool in finding these relations is the following special case of Equation 3.4.1 that holds for any distance regular graph.

$$AA_i = a_i A_i + b_{i-1} A_{i-1} + c_{i+1} A_{i+1}. \tag{8.3.1}$$

For each i, by considering the corresponding blocks in two sides of 8.3.1, we get several equalities involving block parts of A. Two of them which are going to be frequently used in this chapter are stated here as a lemma.

8.3.1 Lemma. *We have*

$$R^T N = J - I - B_1,$$

$$B_2 R = J - N - R.$$

Proof. Rewriting Equation 8.3.1 for $i = 3$ we get

$$AA_3 = a_3 A_3 + b_2 A_2 = A_2 = J - I - A - A_3.$$

By comparing the (2,2)-entry of both sides we get

$$N^T R = J - I - B_1$$

By comparing the (3,2)-entry of both sides we get

$$B_2 R = J - N - R.$$

This completes the proof. $\qquad \square$

The following corollary is of central importance in finding the relation between the $\mathbb{T}$-modules of dimension three and row type $\mathbb{T}$-modules of dimension two which were introduced in Section 8.2.

8.3.2 Corollary. *Let u be a non-zero vector defined on the first neighbourhood. Suppose that, u sums to zero. Then the following are equivalent:*

(a) $\{Nu, Ru\}$ is a linearly dependent set;

(b) for some real number μ, $B_1u = \mu u$, and $Nu + \frac{1+\mu}{r-1}Ru = 0$; and

(c) Ru is an eigenvector of B_2.

Proof. First to prove that (a) implies (b), note that since u is non-zero and R has full column rank, then, Ru is non-zero. So $\{Nu, Ru\}$ is a linearly dependent set if and only if Nu is a scalar multiple of Ru. Thus for some scalar α we have

$$Nu = \alpha Ru.$$

Multiplying both sides by R^T from the left we get

$$R^T Nu = \alpha R^T Ru.$$

Since columns of R are mutually orthogonal and each sum to $r - 1$, we get

$$R^T R = (r - 1)I,$$

and so

$$R^T Nu = \alpha(r - 1)u.$$

But by Lemma 8.3.1, we have

$$R^T N = J - I - B_1.$$

Applying both sides of the above equality to u we get

$$R^T Nu = -u - B_1 u.$$

Therefore

$$\alpha(r - 1)u = -u - B_1 u,$$

so u is an eigenvector of B_1. Let

$$B_1 u = \mu u.$$

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Then we get

$$\alpha = -\frac{1 + \mu}{r - 1}.$$

Thus

$$Nu = -\frac{1 + \mu}{r - 1} Ru.$$

Now we prove that (b) implies (c). From Lemma 8.3.1 we have

$$B_2 Ru = -Nu - Ru.$$

Since Nu is a scalar multiple of Ru it immediately follows that Ru is an eigenvector of B_2.

Finally to prove that (c) implies (a), we use Lemma 8.3.1 again to get

$$B_2 Ru = -Nu - Ru.$$

Since $B_2 Ru$ is a scalar multiple of Ru it immediately follows that Nu is a scalar multiple of Ru. $\qquad\square$

8.4 Eigenspaces of the Cover

Since each $\mathbb{T}$-module can be written as a direct sum of one-dimensional spaces generated by eigenvectors of A, it would be useful to get more information about the eigenspaces of A. By Theorem 5.2.1, the eigenvalues of A are $n-1$, -1, θ and τ. $n-1$ is the trivial eigenvalue with its eigenspace generated by the all ones vector. The following lemma specifies the eigenspace corresponding to -1.

8.4.1 Lemma. *The eigenspace of A corresponding to the eigenvalue -1 is exactly the space of vectors which sum to zero, and are constant on every fibre of the cover.*

Proof. Let

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \end{bmatrix},$$

be a vector defined on the first neighbourhood. Then Ru is defined on the second neighbourhood. Since R is the characteristic vector of the row partition, then for each i, Ru has the constant value of u_i on the i-th row of

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the second neighbourhood. On the other hand, as we mentioned in Section 8.1, for any i, $1 \leqslant i \leqslant n - 1$, the i-th vertex in the first neighbourhood together with the i-th row of the second neighbourhood form a fibre of the cover. This implies that, the space of vectors which are constant on every fibre of the cover is the following set:

$$\left\{ \begin{bmatrix} a \\ u \\ Ru \\ a\mathbb{1} \end{bmatrix} : a \in \mathbb{R}, u \in \mathbb{R}^{n-1} \right\}.$$

Thus, the space of vectors which sum to zero, and are constant on every fibre of the cover is the following set:

$$S = \left\{ \begin{bmatrix} a \\ u \\ Ru \\ a\mathbb{1} \end{bmatrix} : a \in \mathbb{R}, u \in \mathbb{R}^{n-1}, \mathbb{1}^T u + a = 0 \right\}.$$

By Theorem 5.2.1, the multiplicity of -1, as an eigenvalue of A, is $n - 1$. It is not hard to see that S is a vector space of dimension $n - 1$ as well. Therefore, all we need to prove is that, any vector in S is an eigenvector of A corresponding to the eigenvalue -1. Using Lemma 8.3.1, we get

$$A \begin{bmatrix} a \\ u \\ Ru \\ a\mathbb{1} \end{bmatrix} = \begin{bmatrix} 0 & \mathbb{1}_{n-1}^T & 0 & 0 \\ \mathbb{1}_{n-1} & B_1 & N^T & 0 \\ 0 & N & B_2 & C \\ 0 & 0 & C^T & 0 \end{bmatrix} \begin{bmatrix} a \\ u \\ Ru \\ a\mathbb{1} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbb{1}^T u \\ a\mathbb{1} + B_1 u + N^T Ru \\ Nu + B_2 Ru + aC\mathbb{1} \\ C^T Ru \end{bmatrix}$$

$$= \begin{bmatrix} \mathbb{1}^T u \\ a\mathbb{1} + Ju - u \\ Ju - Ru + a\mathbb{1} \\ Ju \end{bmatrix}.$$

Now since

$$\mathbb{1}^T u = -a,$$

we get

$$Ju = \mathbb{1}\mathbb{1}^T u = -a\mathbb{1}.$$

Therefore
$$
\begin{bmatrix} \mathbb{1}^T u \\ a\mathbb{1} + Ju - u \\ Ju - Ru + a\mathbb{1} \\ Ju \end{bmatrix} = - \begin{bmatrix} a \\ u \\ Ru \\ a\mathbb{1} \end{bmatrix},
$$

and we are done. $\square$

Since A is a real symmetric matrix, the eigenspaces corresponding to distinct eigenvalues are orthogonal, and the subspace generated by the union of all eigenspaces is $\mathbb{R}^{rn}$ itself. So by Lemma 8.4.1, the union of the eigenspaces corresponding to the eigenvalues θ and τ is the space of all vectors which sum to zero on every fibre.

8.5 T-Modules of Dimension One

Starting in this section, we compute the T-module decomposition of $\mathbb{R}^{rn}$. Let W be an irreducible T-module. We consider all the possibilities for W according to its support, or equivalently, for which values of i, $0 \leqslant i \leqslant 3$, $F_i W$ is zero. For different cases, we get T-modules of different dimensions. For some cases we do not get any T-module at all. This section and the following two sections cover all the possible cases. In this section we just consider the cases where the support of W is contained in just one neighbourhood.

It is not hard to check that the algebra generated by the all ones vector is an irreducible T-module. It is called the *trivial module*. It has dimension 4, and
$$
\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \mathbb{1} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \mathbb{1} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ \mathbb{1} \end{bmatrix} \right\}
$$
is a basis.

We assume W is not the trivial module and so it lies in the orthogonal complement to the trivial module. In particular,
$$
F_0 W = 0,
$$
which means the first entry of any vector in W is zero.

We will use the basis $\{I, A, J, A_3\}$ for the algebra generated by A. It is a basis since

$$A_2 = J - (I + A + A_3).$$

Now we start going through the different cases.

Case 1: If the support of W is contained in the third neighbourhood, then if $x \in W$, for some w defined on the third neighbourhood we have

$$x = \begin{bmatrix} 0 \\ 0 \\ 0 \\ w \end{bmatrix}.$$

Since W is a $\mathbb{T}$-module, we get

$$Ax = \begin{bmatrix} 0 \\ 0 \\ Cw \\ 0 \end{bmatrix} \in W.$$

So we must have

$$Cw = 0.$$

But since C is the characteristic matrix of the column partition, it has full column rank. Hence, we must have

$$w = 0,$$

and so

$$x = 0.$$

Therefore

$$W = 0,$$

and so there is no irreducible $\mathbb{T}$-module in this case.

Case 2: If the support of W is contained in the first neighbourhood, then if $x \in W$, for some u defined on the first neighbourhood we have

$$x = \begin{bmatrix} 0 \\ u \\ 0 \\ 0 \end{bmatrix}.$$

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Since W is a $\mathbb{T}$-module, we get

$$A_3 x = \begin{bmatrix} 0 \\ 0 \\ Ru \\ 0 \end{bmatrix} \in W.$$

So we must have

$$Ru = 0.$$

But since R is the characteristic matrix of the row partition, it has full column rank. Hence, we must have

$$u = 0,$$

and so

$$x = 0.$$

Therefore

$$W = 0,$$

and so there is no irreducible $\mathbb{T}$-module in this case.

Case 3: If the support of W is contained in the second neighbourhood, then if $x \in W$, for some v defined on the second neighbourhood we have

$$x = \begin{bmatrix} 0 \\ 0 \\ v \\ 0 \end{bmatrix}.$$

Since W is a $\mathbb{T}$-module, we get

$$Ax = \begin{bmatrix} 0 \\ N^T v \\ B_2 v \\ C^T v \end{bmatrix} \in W.$$

So we must have

$$N^T v = 0, \qquad C^T v = 0.$$

Then we get

$$Ax = \begin{bmatrix} 0 \\ 0 \\ B_2 v \\ 0 \end{bmatrix} \in W,$$

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and so for any non-negative integer k, we get

$$A^k x = \begin{bmatrix} 0 \\ 0 \\ B_2^k v \\ 0 \end{bmatrix} \in W.$$

Since B_2 is symmetric and since the span of the vectors

$$\left\{ B_2^k v \mid k = 0, 1, 2, ... \right\}$$

is B_2-invariant, the span must contain an eigenvector of B_2.

So we can choose a non-zero vector v, defined on the second neighbour-hood, such that

$$x = \begin{bmatrix} 0 \\ 0 \\ v \\ 0 \end{bmatrix} \in W, \qquad B_2 v = \lambda v, \qquad N^T v = 0, \qquad C^T v = 0.$$

Then

$$Ax = \lambda x,$$

and so x spans a one-dimensional $\mathbb{T}$-module.

Also note that, since

$$A_3 x = \begin{bmatrix} 0 \\ R^T v \\ \left(RR^T - I \right) v \\ 0 \end{bmatrix} \in W,$$

we must have

$$R^T v = 0.$$

Now we show that the condition

$$N^T v = 0$$

follows from the conditions

$$B_2 v = \lambda v, \qquad R^T v = 0.$$

By Lemma 8.3.1, we have

$$R^T B_2 = J - N^T - R^T.$$

Applying both sides of the above equality to v we get

$$\lambda R^T v = -N^T v - R^T v,$$

which, because $R^T v = 0$, implies

$$N^T v = 0.$$

To sum up, W is a $\mathbb{T}$-module in Case 3 if and only if it is spanned by a vector whose restriction on the second neighbourhood is an eigenvector of B_2 which satisfies

$$B_2 v = \lambda v, \qquad C^T v = 0, \qquad R^T v = 0.$$

8.6 Column Type $\mathbb{T}$-Modules of Dimension Two

In the previous section, we explored all the cases where the support of a $\mathbb{T}$-module is contained in just one neighbourhood. We start exploring the cases where the support is contained in the union of two neighbourhoods in this section. We will consider only one case in this section, which will give us a $\mathbb{T}$-module of dimension two.

Case 4: Suppose that the support of W is contained in the union of the second and the third neighbourhood. Then if $x \in W$, for some v and w defined on the second and third neighbourhood, respectively, we have

$$x = \begin{bmatrix} 0 \\ 0 \\ v \\ w \end{bmatrix}.$$

Since W is a $\mathbb{T}$-module, we get

$$AF_2 x = A \begin{bmatrix} 0 \\ 0 \\ v \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ N^T v \\ B_2 v \\ C^T v \end{bmatrix} \in W.$$

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So we must have
$$N^T v = 0.$$

On the other hand, we have
$$F_2 A F_2 x = \begin{bmatrix} 0 \\ 0 \\ B_2 v \\ 0 \end{bmatrix} \in W,$$

and for any non-negative integer k, we get
$$(F_2 A F_2)^k x = \begin{bmatrix} 0 \\ 0 \\ B_2^k v \\ 0 \end{bmatrix} \in W.$$

Since B_2 is symmetric and since the span of the vectors
$$\left\{ B_2^k v \mid k = 0, 1, 2, \ldots \right\}$$

is B_2-invariant, the span must contain an eigenvector of B_2.

Thus we can choose a non-zero vector v defined on the second neighbourhood such that
$$x = \begin{bmatrix} 0 \\ 0 \\ v \\ 0 \end{bmatrix} \in W, \qquad B_2 v = \lambda v, \qquad N^T v = 0.$$

Then we have
$$A x = \begin{bmatrix} 0 \\ 0 \\ \lambda v \\ C^T v \end{bmatrix},$$

and
$$A_3 x = \begin{bmatrix} 0 \\ R^T v \\ \left(R R^T - I \right) v \\ 0 \end{bmatrix} \in W.$$

So we must have
$$R^T v = 0,$$

which implies

$$\left(RR^T - I \right) v = -v,$$

and therefore we get

$$A_3 x = -x.$$

Thus

$$\begin{bmatrix} 0 \\ 0 \\ v \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ C^T v \end{bmatrix}$$

span a T-invariant subspace. This $\mathbb{T}$-module is 2-dimensional since by our assumption W is non-zero on the second and the third neighbourhood.

This $\mathbb{T}$-module is irreducible, since otherwise it contains a 1-dimensional T-invariant subspace W' generated by

$$\begin{bmatrix} 0 \\ 0 \\ \alpha v \\ \beta C^T v \end{bmatrix},$$

for some real numbers α and β. If α is non-zero, then as we showed above W' must contain

$$\begin{bmatrix} 0 \\ 0 \\ v \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ C^T v \end{bmatrix},$$

and so W' is equal to W. If α is zero, then the support of W' is only on the third neighbourhood, but we have seen already in Case 1 that this is not possible.

To sum up, W is an irreducible $\mathbb{T}$-module of Case 4 if and only if it is generated by

$$\begin{bmatrix} 0 \\ 0 \\ v \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 \\ 0 \\ 0 \\ C^T v \end{bmatrix},$$

where v satisfies

$$B_2 v = \lambda v, \qquad R^T v = 0, \qquad C^T v \neq 0.$$

As we saw at the end of Case 3, the condition

$$N^T v = 0,$$

follows from the above conditions.

Now we prove that in Case 4,

$$\lambda = \delta,$$

and

$$CC^T v = (n-1)v.$$

First note that, since A is symmetric and W is invariant under A, it must contain an eigenvector for A. Let γ be the corresponding eigenvalue. Thus there exist real numbers α and β, not simultaneously zero, such that

$$\gamma \begin{bmatrix} 0 \\ 0 \\ \alpha v \\ \beta C^T v \end{bmatrix} = A \begin{bmatrix} 0 \\ 0 \\ \alpha v \\ \beta C^T v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \lambda \alpha v + \beta CC^T v \\ \alpha C^T v \end{bmatrix}.$$

By Theorem 5.2.1,

$$\gamma \in \{n-1, -1, \theta, \tau\},$$

and so γ is non-zero. If α is zero, then $\beta CC^T v$ is zero. Also if α is zero, then β is non-zero, and so $CC^T v$ is zero, which implies $C^T v$ is zero, which contradicts our hypothesis. Therefore α is non-zero, and so we can assume α is equal to one. Hence,

$$\begin{cases} (\gamma - \lambda)v = \beta CC^T v, \\ \gamma \beta C^T v = C^T v, \end{cases}$$

which implies

$$\beta = \frac{1}{\gamma},$$

and

$$CC^T v = \gamma(\gamma - \lambda)v.$$

Thus v is an eigenvector of CC^T. It is not hard to see that

$$CC^T = J_{n-1} \otimes I_{r-1},$$

where $\otimes$ denotes the Kronecker product. Thus, the only eigenvalues of CC^T are 0 and $n-1$. Since $C^T v$ is non-zero, we get

$$CC^T v = (n-1)v.$$

Therefore

$$\gamma(\gamma - \lambda) = n - 1. \qquad (8.6.1)$$

On the other hand, since γ is an eigenvalue of A, then by Theorem 5.2.1,

$$\gamma \in \{-1, n-1, \theta, \tau\}.$$

If γ is equal to -1 or $n-1$, then by Equation 8.6.1, λ is equal to $n-2$. So $n-2$ must be an eigenvalue of B_2. But B_2 is an a_2-regular graph where

$$a_2 = n - 1 - b_2 - c_2 = n - 2 - c_2.$$

This is a contradiction, since no eigenvalue of B_2 can exceed a_2. Therefore γ is either θ or τ. But when γ is equal to θ or τ, Equation 8.6.1 implies

$$\lambda = \theta + \tau = \delta.$$

8.7   T-Modules of Dimension Two and Three

In this section, we continue exploring the cases where the support of W is on the union of two neighbourhoods. In the previous section, we considered Case 4, where the support of a T-module is contained in the union of the second and the third neighbourhood. In this section, we consider the case where the support of a T-module is contained in the union of the first and the third neighbourhood (Case 5), and the case where the support of a T-module is contained in the union of the first and the second neighbourhood (Case 6). We will show that Case 5 never happens. However, we will find T-modules of dimension two and three in Case 6. At the end of this section, we also consider the case where the support of a T-module is not zero on any neighbourhood (Case 7), and we will prove in this case that no T-module can be irreducible.

Case 5: Suppose that the support of W is contained in the union of the first and the third neighbourhood. If $x \in W$, then, for some u defined on the first neighbourhood and some w defined on the third neighbourhood we have

$$x = \begin{bmatrix} 0 \\ u \\ 0 \\ w \end{bmatrix}.$$

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Since W is a $\mathbb{T}$-module, we get

$$A_3 F_1 x = A_3 \begin{bmatrix} 0 \\ u \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ Ru \\ 0 \end{bmatrix} \in W.$$

Thus we must have

$$Ru = 0,$$

which, as in Case 2, implies

$$W = 0.$$

Therefore, there is no irreducible $\mathbb{T}$-module in this case.

Case 6: Now suppose the support of W is on the union of the first and the second neighbourhood. This is the most complex case. We will work on it until the end of Theorem 8.7.1. If $x \in W$, then, for some u defined on the first neighbourhood and some v defined on the second neighbourhood we have

$$x = \begin{bmatrix} 0 \\ u \\ v \\ 0 \end{bmatrix}.$$

Since W is a $\mathbb{T}$-module, we get

$$AF_1 x = A \begin{bmatrix} 0 \\ u \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbb{1}^T u \\ B_1 u \\ Nu \\ 0 \end{bmatrix} \in W.$$

Thus we must have

$$\mathbb{1}^T u = 0.$$

On the other hand, we have

$$F_1 A F_1 x = \begin{bmatrix} 0 \\ B_1 u \\ 0 \\ 0 \end{bmatrix} \in W,$$

and for any non-negative integer k, we get

$$(F_1 A F_1)^k x = \begin{bmatrix} 0 \\ B_1^k u \\ 0 \\ 0 \end{bmatrix} \in W.$$

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Since B_1 is symmetric and since the span of the vectors

$$\left\{ B_1^k u \mid k = 0, 1, 2, \ldots \right\}$$

is B_1-invariant, it must contain an eigenvector of B_1.

So we can choose a non-zero vector u defined on the first neighbourhood such that

$$x = \begin{bmatrix} 0 \\ u \\ 0 \\ 0 \end{bmatrix} \in W, \qquad B_1 u = \mu u, \qquad \mathbb{1}^T u = 0.$$

We must also have

$$Ax = \begin{bmatrix} \mathbb{1}^T u \\ B_1 u \\ Nu \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \lambda u \\ Nu \\ 0 \end{bmatrix} \in W, \qquad A_3 x = \begin{bmatrix} 0 \\ 0 \\ Ru \\ 0 \end{bmatrix} \in W.$$

Also note that

$$\mathbb{1}^T u = 0$$

implies

$$Jx = 0.$$

Thus

$$\begin{bmatrix} 0 \\ u \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ Nu \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ Ru \\ 0 \end{bmatrix}$$

span a T-invariant subspace of $\mathbb{R}^{rn}$. Hence W is the subspace of $\mathbb{R}^{rn}$ spanned by the above vectors. Since u is non-zero and since R has full column rank, we have Ru is non-zero, and so the dimension of W is at least two. Now the dimension of W is either two or three depending on whether $\{Nu, Ru\}$ is linearly dependent or linearly independent. By Corollary 8.3.2, there is a one-to-one correspondence between $\mathbb{T}$-modules of dimension two and the set of eigenvectors of B_2 in $\mathrm{col}(R)$.

8.7.1 Theorem. *Let u be a non-trivial eigenvector of B_1 with corresponding eigenvalue μ. Let W be the irreducible $\mathbb{T}$-module in the decomposition of $\mathbb{R}^{rn}$, which is generated by*

$$
\begin{bmatrix} 0 \\ u \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ Nu \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ Ru \\ 0 \end{bmatrix}.
$$

Then W has dimension two if and only if μ is equal to μ_θ or μ_τ.

Proof. Note that μ is equal to μ_θ or μ_τ if and only if

$$
\mu\left(1 + \frac{1}{r-1}\right) + \frac{1}{r-1} \in \{\theta, \tau\}.
$$

First suppose that W has dimension two. Then by Corollary 8.3.2 we have

$$
Nu = -\frac{1+\mu}{r-1}Ru. \tag{8.7.1}
$$

We show that

$$
\begin{bmatrix} 0 \\ -(r-1)u \\ Ru \\ 0 \end{bmatrix}
$$

is an eigenvector of A corresponding to the eigenvalue

$$
\mu\left(1 + \frac{1}{r-1}\right) + \frac{1}{r-1}.
$$

Then since, by Theorem 5.2.1, the only eigenvalues of A are $n-1$, -1, θ, and τ, we get

$$
\left(1 + \frac{1}{r-1}\right) + \frac{1}{r-1} \in \{n-1, -1, \theta, \tau\}.
$$

But

$$
\begin{bmatrix} 0 \\ -(r-1)u \\ Ru \\ 0 \end{bmatrix}
$$

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is not a multiple of the all ones vector; and by Lemma 8.4.1 is not an eigenvector of A corresponding to the eigenvalue -1 either. So

$$\left(1 + \frac{1}{r-1}\right) + \frac{1}{r-1} \in \{\theta, \tau\},$$

and we are done.

We have

$$A \begin{bmatrix} 0 \\ -(r-1)u \\ Ru \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -(r-1)B_1u + N^T Ru \\ -(r-1)Nu + B_2 Ru \\ C^T Ru \end{bmatrix}.$$

By Lemma 8.3.1, we have

$$N^T Ru = Ju - u - B_1 u = -(1 + \mu)u,$$

and

$$B_2 Ru = Ju - Nu - Ru = -Nu - Ru.$$

So by 8.7.1 we get

$$-(r-1)Nu + B_2 Ru = -rNu - Ru = \left(\mu + \frac{1 + \mu}{r-1}\right) Ru.$$

Also since W is zero on the third neighbourhood, $C^T Ru = 0$. Therefore

$$A \begin{bmatrix} 0 \\ -(r-1)u \\ Ru \\ 0 \end{bmatrix} = \left(\mu\left(1 + \frac{1}{r-1}\right) + \frac{1}{r-1}\right) \begin{bmatrix} 0 \\ -(r-1)u \\ Ru \\ 0 \end{bmatrix},$$

and we are done.

To prove the converse, suppose that

$$\mu\left(1 + \frac{1}{r-1}\right) + \frac{1}{r-1} \in \{\theta, \tau\}. \qquad (8.7.2)$$

Define

$$v = Nu + \frac{1 + \mu}{r-1} Ru.$$

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We prove that if v is non-zero, then

$$
x = \begin{bmatrix} 0 \\ 0 \\ v \\ 0 \end{bmatrix}
$$

is an eigenvector of A, and hence it spans a 1-dimensional T-submodule of W. So we can decompose W as a sum of this 1-dimensional submodule and other submodule(s). So W must contain an irreducible submodule of dimension at most two which is not zero on V_1, but is zero on V_3. But in the previous cases, we saw that such a module does not exist except in Case 6 where we need v to be zero. So v must be zero, and hence W is a 2-dimensional $\mathbb{T}$-module.

Suppose 8.7.2 is satisfied. By Case 3, to prove x spans a 1-dimensional $\mathbb{T}$-module, we just need to show that v is an eigenvector of B_2 which sums to zero on the cells of the row partition as well as the cells of the column partition. First note that

$$
A \begin{bmatrix} 0 \\ 0 \\ Nu \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ N^T Nu \\ B_2 Nu \\ C^T Nu \end{bmatrix} \in W,
$$

and so we must have

$$
C^T Nu = 0.
$$

On the other hand, note that the (i,j)-entry of $C^T R$ is just the size of the intersection of the i-th column and the j-th row of the second neighbourhood. So we get

$$
C^T R = J,
$$

and so

$$
C^T Ru = Ju = \mathbb{1}\mathbb{1}^T u = 0.
$$

Therefore

$$
C^T v = C^T Nu + \frac{1+\mu}{r-1} C^T Ru = 0.
$$

So v sums to zero on the cells of the column partition.

On the other hand, we have

$$
R^T v = R^T Nu + \frac{1+\mu}{r-1} R^T Ru.
$$

But by Lemma 8.3.1 we have

$$R^T N u = (J - I - B_1)u = -(1 + \mu)u.$$

Thus

$$R^T v = -(1 + \mu)u + \frac{1 + \mu}{r - 1}(r - 1)u = 0,$$

and so v sums to zero on the cells of the row partition.

Now the only thing remaining to be checked is that v is an eigenvector of B_2. To compute $B_2 v$ we need to compute $B_2 N u$ and $B_2 R u$. By Lemma 8.3.1 we have

$$B_2 R u = (J - N - R)u = -N u - R u.$$

To compute $B_2 N u$, we find an expression for $B_2 N$ first. Note that by setting i to be 1 in 8.3.1 we get

$$
\begin{aligned}
A^2 &= a_1 A + b_0 A_0 + c_2 A_2 & (8.7.3) \\
&= a_1 A + (n - 1)I + c_2 (J - I - A - A_3) & (8.7.4) \\
&= \delta A + (n - 1)I + c_2 (J - I - A_3) & (8.7.5)
\end{aligned}
$$

By comparing the (3,2)-entry of both sides we get

$$N B_1 + B_2 N = \delta N + c_2 J - c_2 R.$$

Therefore,

$$B_2 N u = (\delta - \mu)N u - c_2 R u.$$

Now we have

$$
\begin{aligned}
B_2 v &= B_2 N u + \frac{1 + \mu}{r - 1} B_2 R u \\
&= (\delta - \mu)N u - c_2 R u - \frac{1 + \mu}{r - 1}(N u + R u) \\
&= -\left(\frac{1 + \mu}{r - 1} + \mu - \delta\right) N u - \left(\frac{1 + \mu}{r - 1} + c_2\right) R u.
\end{aligned}
$$

By a straightforward computation, we can see that 8.7.2 is equivalent to

$$c_2 = \frac{1 + \mu}{r - 1}\left(\frac{1 + \mu}{r - 1} + \mu - \delta - 1\right),$$

and so we get

$$
\begin{aligned}
B_2 v &= -\left(\frac{1+\mu}{r-1} + \mu - \delta\right) Nu - \frac{1+\mu}{r-1}\left(\frac{1+\mu}{r-1} + \mu - \delta\right) Ru \\
&= -\left(\frac{1+\mu}{r-1} + \mu - \delta\right) v.
\end{aligned}
$$

Thus v is an eigenvector of B_2 and we are done. $\qquad\square$

8.7.2 Corollary. *Let u be a non-zero vector which sums to zero, and, is defined on the first neighbourhood. Then, u is an eigenvector of B_1 with the corresponding eigenvalue μ_θ or μ_τ if and only if Ru is an eigenvector of B_2.*

Proof. Let u be a non-zero vector which sums to zero, and, is defined on the first neighbourhood. By Corollary 8.3.2, Ru is an eigenvector of B_2 if and only if u is an eigenvector of B_1 such that $\{Ru, Nu\}$ is a linearly dependent set. On the other hand, by Theorem 8.7.1, u is an eigenvector of B_1 such that $\{Ru, Nu\}$ is a linearly dependent set if and only if u is an eigenvector of B_1 with corresponding eigenvalue μ_θ or μ_τ. $\qquad\square$

Proof of Theorem 8.2.3. Let v be a non-trivial eigenvector of B_2 that is constant on the cells of the row partition. Then, $v = Ru$ for some non-zero vector u defined on the first neighbourhood. Since v sums to zero, and $R^T R = (r-1)I$, we get that u sums to zero. By Corollary 8.7.2, $B_1 u = \mu u$, where μ is either μ_θ or μ_τ. On the other hand, by Corollary 8.3.2,

$$
Nu = -\frac{1+\mu}{r-1} Ru.
$$

Now by Lemma 8.3.1,

$$
B_2 Ru = Ju - Nu - Ru = \frac{1+\mu}{r-1} Ru - Ru = \left(\frac{1+\mu}{r-1} - 1\right) Ru.
$$

So Ru is an eigenvector of B_2 corresponding to eigenvalue

$$
\frac{1+\mu}{r-1} - 1.
$$

But

$$
\frac{1+\mu_\theta}{r-1} - 1 = \frac{\theta+1}{r} - 1,
$$

and,

$$
\frac{1+\mu_\tau}{r-1} - 1 = \frac{\tau+1}{r} - 1.
$$

This completes the proof. $\qquad\square$

Case 7: Now suppose that the support of W is non-zero on every neighbourhood. Then there exists a non-zero vector $u \in \mathbb{R}^{|V_1|}$ such that

$$\begin{bmatrix} 0 \\ u \\ 0 \\ 0 \end{bmatrix} \in W.$$

Then following the same arguments in the beginning of Case 6, we see that W must contain a submodule of Case 6. So W cannot be irreducible. Therefore no irreducible $\mathbb{T}$-module exists in this case.

8.8 Thinness

Let W be a $\mathbb{T}$-module. We say W is *thin* if for each i, we have

$$\dim(F_i W) \leqslant 1.$$

For each i, let E_i be the projection onto the i-th eigenspace of G. We say that W is *dual thin* if

$$\dim(E_i W) \leqslant 1.$$

In this section, we investigate the thinness and dual thinness of all the $\mathbb{T}$-modules of $\mathbb{R}^{rn}$, that we found in the previous three sections.

8.8.1 Theorem. *In the decomposition of $\mathbb{R}^{rn}$ into irreducible $\mathbb{T}$-modules, the $\mathbb{T}$-modules of dimension three are dual thin, but not thin. All the other $\mathbb{T}$-modules are both thin and dual thin.*

Proof. First we investigate the thinness of the $\mathbb{T}$-modules. If W is any $\mathbb{T}$-module other than the $\mathbb{T}$-module of dimension three, then clearly for any i, $1 \leqslant i \leqslant 3$, we have

$$\dim\left(F_i W \right) \leqslant 1,$$

and so W is thin. If W is a $\mathbb{T}$-module of dimension three, then by Corollary 8.3.2 we have Nu and Ru are not multiples of each other, and so we get

$$\dim(F_2 W) = 2.$$

Thus no $\mathbb{T}$-module of dimension three is thin.

Now we prove that all the irreducible $\mathbb{T}$-modules in the decomposition of $\mathbb{R}^{rn}$ are dual thin. We already know that the trivial module is dual thin. Also any $\mathbb{T}$-module of dimension one is clearly dual thin. Let W be any

$\mathbb{T}$-module of dimension two. Suppose that W is not dual thin. Then, since W is the direct sum of $E_0 W$, $E_1 W$, $E_2 W$, and $E_3 W$, we must have for some i,

$$\dim (E_i W) = 2,$$

and for any j not equal to i,

$$\dim (E_j W) = 0.$$

Thus

$$W = E_i W.$$

Therefore W is a subspace of some eigenspace of A. On the other hand, as we saw in the sections corresponding to row type and column type $\mathbb{T}$-modules of dimension two, in both cases we can choose a non-zero $w \in W$ with its support only a subset of V_2. Then w spans a 1-dimensional submodule of W which contradicts the irreducibility of W.

Now let W be a $\mathbb{T}$-module of dimension three. Then as we saw in Section 8.7, W is generated by

$$\begin{bmatrix} 0 \\ u \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ Nu \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ Ru \\ 0 \end{bmatrix}.$$

On the other hand,

$$A = (n - 1)E_0 + (-1)E_1 + \theta E_2 + \tau E_3.$$

By Lemma 8.4.1, the eigenspace of A corresponding to the eigenvalue -1 is the set of vectors which sum to zero and are constant on every fibre of the cover. So here $E_1 W$ is the subspace generated by

$$\begin{bmatrix} 0 \\ u \\ Ru \\ 0 \end{bmatrix}.$$

Thus

$$\dim (E_1 W) = 1.$$

Also since each vector in W is orthogonal to the all ones vector, we have

$$\dim (E_0 W) = 0.$$

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So if W is not dual thin, we must have one of E_2W and E_3W is zero, and the other one is 2-dimensional. Suppose that

$$\dim(E_2W) = 2, \qquad E_3W = 0.$$

The case where $\dim(E_3W) = 2$ and $E_2W = 0$ is similar. Define

$$v = Nu + \frac{1+\mu}{r-1}Ru.$$

Define $x \in \mathbb{R}^{rn}$ to be v on V_2 and zero everywhere else. We prove that x is orthogonal to the eigenspace of A corresponding to the eigenvalue -1, which we found in Lemma 8.4.1. A typical vector, y, in the eigensapce of A corresponding to -1 must have the following form, for some u' which sums to zero, and, is defined on the first neighbourhood:

$$\begin{bmatrix} 0 \\ u' \\ Ru' \\ 0 \end{bmatrix}.$$

Then we get

$$y^T x = u'^T R^T N u + \frac{1+\mu}{r-1} u'^T R^T R u.$$

But by Lemma 8.3.1 we have

$$R^T N u = (J - I - B_1)u = -(1+\mu)u.$$

Also since the columns of R are the characteristic vectors of the row partition and since each row has size $r-1$ we get

$$R^T R = (r-1)I.$$

Therefore we get

$$y^T x = u'^T \left(R^T N u + \frac{1+\mu}{r-1} R^T R u \right) = 0.$$

Thus x is orthogonal to the eigensapce of A corresponding to the eigenvalue -1, or equivalently,

$$E_1 x = 0.$$

So $E_i x$ is zero for any i other than two, which implies that x is an eigenvector of A corresponding to the eigenvalue θ. Thus x spans a 1-dimensional submodule of W which contradicts the irreducibility of W. $\qquad \square$

8.9   Multiplicities

In the previous sections of this chapter, we found all the $\mathbb{T}$-modules that appear in the decomposition of $\mathbb{R}^{rn}$ into irreducible $\mathbb{T}$-modules. In this section, we find the number of times that a $\mathbb{T}$-module of a certain type appears in that decomposition, which we call the *multiplicity* of that $\mathbb{T}$-module.

Before finding the multiplicities, we go through the decomposition of each $\mathbb{T}$-module into 1-dimensional subspaces spanned by eigenvectors of A. In each decomposition, we cannot have two eigenvectors belonging to the same eigenspace of A, since each $\mathbb{T}$-module is dual thin. Furthermore, the 1-dimensional eigenspace of A spanned by the all ones vector which corresponds to the largest eigenvalue of A is contained in the trivial module. For eigenvalue λ of A, let W_λ denote the eigenspace of A corresponding to λ.

The trivial module. Since it has dimension four, it must have one eigenvector in each eigenspace of A.

Modules of dimension three. They contain one eigenvector in W_{-1}, one in W_θ and one in W_τ.

Row type modules of dimension two. A $\mathbb{T}$-module corresponding to the eigenvector u of the first neighbourhood can be written as the direct sum of 1-dimensional subspaces spanned by

$$
w_1 = \begin{bmatrix} 0 \\ u \\ Ru \\ 0 \end{bmatrix}, \qquad w_2 = \begin{bmatrix} 0 \\ -(r-1)u \\ Ru \\ 0 \end{bmatrix},
$$

where $w_1 \in W_{-1}$, and

$$
Aw_2 = \left(\mu \left(1 + \frac{1}{r-1} \right) + \frac{1}{r-1} \right) w_2,
$$

where

$$
\mu \left(1 + \frac{1}{r-1} \right) + \frac{1}{r-1} \in \{\theta, \tau\}.
$$

Column type modules of dimension two. They can be written as the direct sum of 1-dimensional subspaces spanned by

$$w_1 = \begin{bmatrix} 0 \\ 0 \\ v \\ \frac{1}{\theta}C^T v \end{bmatrix}, \qquad w_2 = \begin{bmatrix} 0 \\ 0 \\ v \\ \frac{1}{\tau}C^T v \end{bmatrix},$$

where $w_1 \in W_\theta$, and $w_2 \in W_\tau$.

Modules of dimension one. They are 1-dimensional subspaces spanned by eigenvectors of A corresponding to the eigenvalue θ or τ.

Now we find the multiplicity of each $\mathbb{T}$-module. The following is a standard lemma (see Godsil and Royle [11, p. 197]) that we are going to use here.

8.9.1 Lemma. *Let G be a graph with adjacency matrix A. Let P be a characteristic matrix of a partition π of $V(X)$. Then π is equitable if and only if A and PP^T commute.*

8.9.2 Lemma. B_2 and CC^T commute.

Proof. First, we see a combinatorial proof, and then an algebraic one.

Combinatorial proof. By Lemma 8.9.1 we just need to show that the column partition is equitable. Let i and j be two columns of V_2. First assume that i and j are not equal. Note that by our arrangement of vertices in Section 8.1, the i-th and j-th columns of V_2 are just the neighbourhoods of the i-th and j-th vertices of V_3. Since any vertex in the i-th column of V_2 is not in the same fibre as the j-th vertex of V_3, their distance is two. So any vertex in the i-th columns of V_2 has c_2 neighbours in the j-th column of V_2.

Now assume that i and j are equal. Then the subgraph induced on the i-th column of V_2 must be a_1-regular, since the i-th column of V_2 is just the neighbourhood of the i-th vertex of V_3.

Algebraic proof. As we saw in 8.7.5, we have

$$A^2 = \delta A + (n-1)I + c_2 \left(J - I - A_3 \right).$$

Comparing the $(4,3)$ block of both sides, we get

$$C^T B_2 = \delta C^T + c_2 J.$$

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Multiplying both sides by C from the left we get

$$CC^T B_2 = \delta CC^T + c_2 J.$$

Taking the transpose of both sides we get

$$B_2 CC^T = \delta CC^T + c_2 J.$$

Thus B_2 and CC^T commute. □

For i equal to 1, 3 and 4, let m_i denote the multiplicity of $\mathbb{T}$-modules of dimension i in the decomposition of $\mathbb{R}^{rn}$ into irreducible $\mathbb{T}$-modules. Also let m_2, m_2' denote the multiplicities of column type $\mathbb{T}$-modules of dimension two and row type $\mathbb{T}$-modules of dimension two. Then since a $\mathbb{T}$-module of dimension four is just the trivial module, we have

$$m_4 = 1.$$

Now we prove that $m_2 = r - 2$.

8.9.3 Theorem. *Any vector in* $\mathrm{col}(C) \cap \ker(\mathbb{1}^T)$ *is an eigenvector of B_2 corresponding to the eigenvalue δ. Therefore $m_2 = r - 2$.*

Proof. By Lemma 8.9.2, we can partition the eigenvectors of B_2 as the ones in $\ker(C^T)$ and the ones in $\mathrm{col}(C)$. Note that the all ones vector is an eigenvector of B_2 which lies in $\mathrm{col}(C)$. Thus there is a basis of size $r - 2$ for $\mathrm{col}(C) \cap \ker(\mathbb{1}^T)$ consisting of eigenvectors of B_2. Now as shown in Section 8.6 when we studied column type $\mathbb{T}$-modules of dimension two, any eigenvector of B_2 in $\mathrm{col}(C) \cap \ker(\mathbb{1}^T)$ must correspond to the eigenvalue δ of B_2. Therefore any vector in $\mathrm{col}(C) \cap \ker(\mathbb{1}^T)$ must be an eigenvector of B_2 corresponding to the eigenvalue δ. □

As we saw in Section 8.7, any non-trivial eigenvector of B_1 gives rise to a row type $\mathbb{T}$-module of dimension two or a $\mathbb{T}$-module of dimension three. So we must have

$$m_3 + m_2' = n - 2. \tag{8.9.1}$$

Also since the dimensions have to match, we get

$$4m_4 + 3m_3 + 2m_2 + 2m_2' + m_1 = rn,$$

and so

$$4 + 3m_3 + 2(r - 2) + 2(n - 2 - m_3) + m_1 = rn.$$

m_1	$(n-2)(r-3)+m$
m_2	$r-2$
m_2'	m
m_3	$n-2-m$
m_4	1

Table 8.2: multiplicity of $\mathbb{T}$-modules

Thus

$$m_3 + m_1 = (r-2)(n-2). \tag{8.9.2}$$

By Theorem 8.7.1, an eigenvector of B_1 gives rise to a row type $\mathbb{T}$-module of dimension two if and only if the corresponding eigenvalue is equal to μ_θ or μ_τ. So m_2' is just the sum of the multiplicities of μ_θ and μ_τ as eigenvalues of B_1, which we denoted by m in Section 8.2. Now by Equations 8.9.1, and 8.9.2, we get the multiplicities of each $\mathbb{T}$-module as below, which were also presented in Table 8.1 where we gave a summary of the decomposition in Section 8.2.

8.10   Applications

In the previous section we saw that, m, the sum of the multiplicities of μ_θ and μ_τ as eigenvalues of B_1 determines the multiplicities in the $\mathbb{T}$-algebra decomposition. We always have

$$0 \leqslant m \leqslant n-2.$$

In this section, we see that both the lower bound and upper bound are achieved. We also investigate what happens when the bounds are tight. In particular, in Corollary 8.10.2, we will see that, $m = n-2$ if and only if $r = 2$.

8.10.1 Lemma. *We have $m = n-2$ or equivalently $m_3 = 0$ if and only if B_2 and RR^T commute.*

Proof. First suppose that B_2 and RR^T commute. Since both matrices are symmetric, we can simultaneously diagonalize them. In particular, we can partition the eigenvectors of B_2 as the ones in $\ker(RR^T)$ and the ones in $\operatorname{col}(RR^T)$, or equivalently, we can partition the eigenvectors of B_2 as the ones in $\ker(R^T)$ and the ones in $\operatorname{col}(R)$. So we must be able to find an orthonormal basis for $\operatorname{col}(R)$ consisting of eigenvectors of B_2. One of the

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elements in the basis is the all ones vector which corresponds to the trivial eigenvalue of B_2. Since $\mathrm{col}(R)$ has dimension $n - 1$, we must have a basis of size $n - 2$ for $\mathrm{col}(R) \cap \ker(\mathbb{1}^T)$ which consists of eigenvectors of B_2. But by Corollary 8.3.2, any element of that basis gives rise to a row type $\mathbb{T}$-module of dimension two. Therefore $m = n - 2$.

For the converse, suppose that $m = n - 2$. We prove that we can partition the eigenvectors of B_2 as the ones in $\ker(R^T)$ and the ones in $\mathrm{col}(R)$. This would be enough to conclude that we can simultaneously diagonalize B_2 and RR^T. This is true since for any $u \in \mathbb{R}^{n-1}$ we have

$$RR^T Ru = R((n - 1)I)u = (n - 1)Ru,$$

and so any vector in $\mathrm{col}(R)$ is an eigenvector for RR^T corresponding to the eigenvalue $n - 1$.

Since $m = n - 2$, by Corollary 8.3.2, we can find a basis of size $n - 2$ for $\mathrm{col}(R) \cap \ker(\mathbb{1}^T)$ which consists of eigenvectors of B_2. Adding the all ones vector to that set, we can find a basis of size $n - 1$ for $\mathrm{col}(R)$ consisting of eigenvectors of B_2.

Now to show that we can find a basis for $\ker(R^T)$ consisting of eigenvectors of B_2, we will prove that we can find a basis for each of

$$\ker(R^T) \cap \ker(C^T), \qquad \ker(R^T) \cap \mathrm{col}(C),$$

consisting of eigenvectors of B_2. First note that by Theorem 8.9.3, we can always find a basis for $\ker(R^T) \cap \mathrm{col}(C)$ consisting of eigenvectors of B_2. On the other hand,

$$m = n - 2$$

implies

$$m_1 = (r - 2)(n - 2).$$

But as we saw in Section 8.5, each $\mathbb{T}$-module of dimension one arises from an eigenvector of B_2 in $\ker(R^T) \cap \ker(C^T)$. So we can find $(r - 2)(n - 2)$ linearly independent eigenvectors for B_2 which all lie in $\ker(R^T) \cap \ker(C^T)$. Since $\ker(R^T) \cap \ker(C^T)$ has dimension $(r - 2)(n - 2)$ we are done. $\qquad \square$

8.10.2 Corollary. *We have $m = n - 2$ if and only if $r = 2$.*

Proof. By Lemma 8.10.1, $m = n - 2$ if and only if B_2 and RR^T commute. But by Lemma 8.9.1, B_2 and RR^T commute if and only if the row partition

of the second neighbourhood is an equitable partition for the graph induced on it. On the other hand, note that there is a perfect matching between row i and row j of the second neighbourhood if and only if vertex i and vertex j of the first neighbourhood are adjacent. Also if there is no edge between vertex i and vertex j of the first neighbourhood, then there is a matching between row i and row j which is just one edge short of being a perfect matching. Since each row has size $r-1$, for any $r \geqslant 3$, the row partition is an equitable partition if and only if any two vertices in the first neighbourhood are adjacent. But then G is disconnected, which is a contradiction. If $r = 2$, then the row partition is trivially an equitable partition. $\qquad\square$

8.10.3 Theorem. *When $r = 2$, the first neighbourhood and the second neighbourhood are isomorphic strongly regular graphs with eigenvalues μ_θ, μ_τ and the valency, a_1.*

Proof. If $r = 2$, then R is just the identity matrix, and therefore by Lemma 8.3.1 we get

$$B_2 = B_1.$$

As we saw in Section 8.7, any non-trivial eigenvector of B_1 gives rise to a $\mathbb{T}$-module of dimension three or a row type $\mathbb{T}$-module of dimension two. But by Corollary 8.10.2, we get $m = n - 2$, and so any non-trivial eigenvector of B_1 gives rise to a row type $\mathbb{T}$-module of dimension two. But as we saw in Section 8.7, this means that any non-trivial eigenvalue of B_1 is equal to μ_θ or μ_τ. This completes the proof. $\qquad\square$

In

$$0 \leqslant m \leqslant n - 2,$$

we completely characterized the case where the upper bound is tight. Now, we try to investigate the case where the lower bound is tight. In Theorem 8.10.7, we prove that, $m = 0$ for triangle-free covers.

8.10.4 Lemma. *If m and δ are both non-zero, then either*

$$r \mid \theta + 1,$$

or

$$r \mid \tau + 1,$$

Proof. First note that m is equal to the sum of the multiplicities of μ_θ and μ_τ as eigenvalues of B_1. If m is non-zero, then either μ_θ or μ_τ must be an eigenvalue of B_1. Also by Theorem 5.3.1, if δ is non-zero, then θ and τ are integers. Since

$$\mu_\theta = \theta - \frac{\theta + 1}{r}, \qquad \mu_\tau = \tau - \frac{\tau + 1}{r},$$

the result follows. $\qquad\square$

It is not hard to see that if μ_θ and μ_τ are both eigenvalues of B_1, then

$$r \mid n, \qquad r \mid c_2.$$

8.10.5 Theorem. *Each non-trivial eigenvalue mu of B_1 satisfies*

$$\mu_\tau \leqslant \mu \leqslant \mu_\theta.$$

Proof. We have

$$(N + \alpha R)^T (N + \alpha R) = N^T N + \alpha R^T N + \alpha N^T R + (r-1)\alpha^2 I.$$

Simplifying this we get

$$\begin{aligned}
(N + \alpha R)^T (N + \alpha R) &= (r-1)\alpha^2 I + 2\alpha(J - I - B_1) \\
&\quad + (-B_1^2 + \delta B_1 + (c_2 - 1)J + (n - 1 - c_2)I).
\end{aligned}$$

Since the left hand side is positive semi-definite for any value of α, all its eigenvalues must be non-negative. If μ is a non-trivial eigenvalue of B_1, then the following is an eigenvalue of the right hand side, and hence must be non-negative for any α:

$$(r-1)\alpha^2 + 2\alpha(-1 - \mu) + (-\mu^2 + \delta\mu + n - 1 - c_2).$$

Thus the discriminant of the above quadratic equation must be non-positive. But by computing the discriminant, we see that it turns out to be

$$(\mu - \mu_\theta)(\mu - \mu_\tau).$$

Hence

$$\mu_\tau \leqslant \mu \leqslant \mu_\theta,$$

and we are done. $\qquad\square$

Feasibility Condition 10 for antipodal covers that we stated in Theorem 5.3.1 is due to Godsil and Hensel [10]; but it also follows from the following theorem.

8.10.6 Theorem. *Let G be an antipodal distance regular graph of diameter three with parameters n, r, c_2 and eigenvalues $n-1, -1, \theta, \tau$ with multiplicities $1, n-1, m_\theta, m_\tau$. If $n > m_\theta - r + 3$, then μ_τ is an eigenvalue of the first neighbourhood, and hence r divides $\tau + 1$ if τ is an integer. Analogously, if $n > m_\tau - r + 3$, then μ_θ is an eigenvalue of the first neighbourhood, and hence r divides $\theta + 1$ if θ is an integer.*

Proof. Let W_λ denote the eigenspace of G corresponding to eigenvalue λ. As we saw in Section 8.9, each row type $\mathbb{T}$-module of dimension two contains one eigenvector in W_{-1} and one in W_θ or W_τ, depending on whether it comes from an eigenvector of the first neighbourhood corresponding to μ_θ or μ_τ. Suppose that μ_τ is not an eigenvalue of the first neighbourhood. Then each row type $\mathbb{T}$-module of dimension two contains one eigenvector in W_{-1} and one in W_θ.

Also from Section 8.9, we saw that column type $\mathbb{T}$-modules of dimension two, $\mathbb{T}$-modules of dimension three, as well as the trivial module each contain exactly one eigenvector in W_θ. Therefore,

$$m_\theta \geqslant m_2' + m_2 + m_3 + m_4 = m + (r - 2) + (n - 2 - m) + 1 = n + r - 3.$$

Similarly, if μ_θ is not an eigenvalue of the first neighbourhood, then

$$m_\tau \geqslant n + r - 3.$$

This completes the proof. □

We end this chapter by the following theorem on triangle-free covers.

8.10.7 Theorem. *Let $\mathbb{T}$ be a Terwilliger Algebra of a triangle-free antipodal cover of diameter three with parameters n, r, c_2, where r is at least three. Then, no row type $\mathbb{T}$-module of dimension two appears in the decomposition of $\mathbb{R}^{rn}$ into irreducible $\mathbb{T}$-modules.*

Proof. As we saw in Section 8.7, the multiplicity of a row type $\mathbb{T}$-module of dimension two is equal to the sum of the multiplicities of μ_θ and μ_τ as eigenvalues of B_1. So we just need to show that μ_θ and μ_τ cannot be eigenvalues of B_1. Suppose on the contrary that one of them is an eigenvalue of B_1. But since the cover is triangle free, we have

$$B_1 = 0.$$

So one of μ_θ and μ_τ must be zero, which implies either θ or τ is equal to $\frac{1}{r-1}$. Thus $\frac{1}{r-1}$ must be an integer, which implies that r is equal to two. □

Chapter 9

Future Work

The possible directions for future work are presented in this chapter.

1. In Theorem 7.1.1, we found a new feasibility condition for Abelian covers which are antipodal distance regular of diameter three. One could ask if this bound holds for non-Abelian covers as well. I could not find any counterexample.

2. By Theorem 7.3.2, Abelian covers with certain parameters give a set of equiangular lines of maximum size. We can investigate if we can construct Abelian covers with these parameters.

3. Another area of research would be to see whether any existing set of equiangular lines of maximum size comes from an Abelian cover. A Seidel matrix for a set of equiangular lines which comes from an abeian cover is more specific than a Seidel matrix for a general set of equiangular lines. To be more explicit, a Seidel matrix for a set of equiangular lines which comes from an Abelian cover with parameters n, r, c_2 have r-th roots of unity as its off-diagonal entries, while the off-diagonal entries of a Seidel matrix only need to be of norm one. By Table 7.1, the smallest case where we get a set of equiangular lines of maximum size is when $(n, r, c_2) = (1225, 5, 205)$, where we get 35^2 equiangular lines in a 35-dimensional complex space. Recently, Grassl [13] found a set of 35^2 equiangular lines in dimension 35. It would be interesting to see whether Grassl's lines come from an Abelian cover with parameters $1225, 5, 205$.

4. In Theorem 8.10.5, we found a lower bound and an upper bound on the non-trivial eigenvalues of the first neighbourhood of antipodal dis-

tance regular graphs of diameter three. What other information on the neighbourhoods does follow from a Terwilliger algebra?

5. In Theorem 8.10.6, we proved one of the known feasibility conditions for antipodal distance regular covers of diameter three, using a Terwilliger algebra. Is it possible to derive new feasibility conditions for antipodal distance regular covers of diameter three, using a Terwilliger algebra?

6. Another possible area for future research is to work out a Terwilliger algebra of other interesting graphs, such as covers of complete bipartite graphs, or incidence graphs of symmetric designs.

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