Motion Planning and Observer Synthesis for a Two-Span Web Roller Machine

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

A mathematical model for a Two-Span Web Roller machine is defined in order to facilitate motion planning, motion tracking and state observer design for tracking web tension and web velocity. Differential Flatness is utilized to create reference trajectories that are tracked with a high convergence rate. Flatness also allows for nominal input torque generation without integration. Constraints on the inputs are satisfied through the motion planning phase. A partial state feedback linearization is performed and an exponential tracking dynamic feedback controller is defined. An exponential Kalman-related tension observer is also defined with semi-optimal gain formulation. The observer takes advantage of the bilinearity of the dynamics up to additive output nonlinearity. The closed-loop system is simulated in MatLab with comparisons to reference trajectories previously employed in literature. The importance of proper motion planning is demonstrated by producing excellent performance compared with existing tracking and tension observing methods.

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Chapter 1

Introduction

A two-span web roller is a simple mechanical device designed to facilitate operations on, or alterations to a *web*, which is a thin, elastic material that is wound onto a roller (See Figure 1). This technology has wide applications in industry. These include printing on paper, film and textile, alteration of fabric, and making parts for various electronic devices from metal sheets. Careful control of the web tension is an important aspect of quality assurance. Large variations in tension can greatly affect product quality. And careful control of web velocity is important as the operating velocities are increased in order to increase productivity. This is particularly important during the start-up phase. The control objective is to guide the system from rest to a desired operating state, or more generally, from one state to another quickly and smoothly. This is normally solved by designing a powerful tracking controller to force the system to its operating state through high levels of convergence. In this thesis, the approach is to properly define the reference trajectories through the use of motion planning. The desired motions or trajectories are developed by considering a model with simplified dynamics that are differentially flat. This flat system is shown to be an ideal platform to generate criteria to define these trajectories. A procedure for tracking these trajectories is then developed. Tension observing is also an important issue. Apparatus for measuring web tension is expensive and not readily adaptable for changing conditions. A tension observer is designed to estimate the tension with sufficient accuracy so as to preserve the tracking procedure.

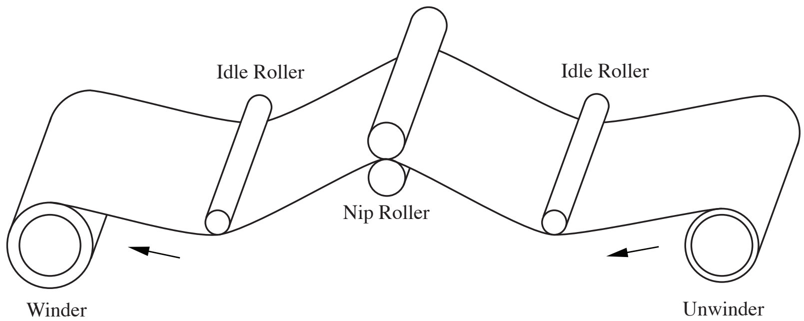

Figure 1.1: Two span web roller with tension observers.

Web roller technology has been extensively analyzed in literature. Various control and observer methods have been developed. The aim of this thesis is to design a simple method to achieve the tracking objective with comparable or improved performance of existing methods. The model for the web roller developed in this thesis does not deal with the dynamics of the torque actuation nor the tension and velocity measurement. The model follows the works in [3], [22] and [21] which are reviewed in [19] and are standard for tension and velocity control.

Controller designs for the web roller have been investigated. An automatically tuning PID controller for the two-span web roller using a genetic algorithm was employed in [4]. Other control methods involving gain tuning for the web roller include a *Sliding-Mode* control [1], *Inverse Linear Quadratic* (ILQ) optimal control for a Hot-Strip Mill [11], and an H_∞ robust control strategy [15], [14]. The objective of this thesis is to design a tracking procedure that, as a result of proper motion planning, does not require significant gain tuning or optimization as those currently employed.

Tension observers are also a popular topic as the continued use of mechanical tension tensors has become unnecessarily costly [7]. An approximate error linearization method, a simple tension estimator assuming a rigid web, and a sliding-mode observer are defined in [19], [18]. A simple observer with observer gain construction is presented in [1]. Low-pass filter methods for tension observers is presented in [17]. A nonlinear observer design is needed that does not make inappropriate assumptions on the system variables. By taking advantage of the form of the web roller dynamics, a simple observer with tunable gains will be defined that can accurately track the trajectories defined using motion planning and thus provide a high convergence rate in the presence of disturbances.

This thesis is organized as follows. Part I contains a review of the relevant theory for motion planning, tracking and observance. In Part II, these concepts are applied to the two-span web roller with simulated results in MatLab. Chapter 2 reviews differential flatness presented in [16] as it applies to motion planning and tracking. Chapter 3 reviews some concepts and examples of nonlinear observers presented in [5]. Chapter 4 reviews common modeling practices for web roller machines. A model is developed for use with the motion planning and observer models discussed in Part I. In Chapter 5 differential flatness is applied to this mathematical model to define suitable reference trajectories. A review of the tracking controller developed in [1] is presented in Chapter 6 with application to the flatness-based motion planning procedure with simulation results for various reference trajectories. An observer is developed for the web roller in Chapter 7 and simulations are conducted for the closed loop system with the controller and observer.

Part I

Theory

Chapter 2

Flatness

This chapter contains a review of the concept of differential flatness as presented in [16]. A flat system is one whose integral curves (curves that satisfy the differential equations) can be mapped in a one-to-one way to ordinary curves (which need not satisfy any differential equation) in a suitable space, whose dimension is possibly different than that of the original system state space. With regard to control theory, this presents an excellent platform for trajectory planning and tracking for dynamical systems. Exploiting this concept of equivalent systems will provide insight into the nature of the evolution of the dynamical system and hence provide a simple and effective approach to the motion tracking problem.

2.1 Differentially Flat Systems

Consider a general nonlinear autonomous control system

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) = f_u(x(t)) \\ y(t) &= h(x(t)) \end{aligned} \tag{2.1}$$

with state variables $x(t) \in X$, an n-dimensional manifold, control inputs $u(t) \in U \subseteq \mathbb{R}^m$, outputs $y(t) \in \mathbb{R}^p$, vector field:

$$f : U \times X \to TX, \; (f_u : X \to TX)$$

and output map $h : X \to \mathbb{R}^p$. Denote $\chi_u(t, x_0)$ as the time-dependent solution to (2.1) with input $u(t)$ and initial condition $x(t{=}0) = x_0$. The dynamics are *differentially flat* if all state and control variables can be expressed in terms of a *flat output* and a finite number of its derivatives. This flat output is determined from the system variables and a finite number of the input derivatives. This is formalized in the following definition.

Definition 2.1.1 (Differentially Flat System)**.** *A system of the form* (2.1) *is Differentially Flat (or simply Flat) if and only if there exists a flat output* $\mu \in \mathbb{R}^m$*, multi-integers* $s = (s_1, \ldots, s_m)$ *and* $r = (r_1, \ldots, r_m)$ *with* $\sum_{i=1}^{m}(r_i + 1) \geq n$*, an m-dimensional map* $\psi : X \times (\mathbb{R}^m)^{s+1} \to \mathbb{R}^m$ *and an (n+m)-dimensional map* $(\varphi_0, \varphi_1) : \mathbb{R}^{(m+2)r} \to \mathbb{R}^n \times \mathbb{R}^m$*, such that*

$$\mu = (\mu_1, \ldots, \mu_m) = \psi(x, u, \dot{u}, \ldots, u^{(s)}) \tag{2.2}$$

implies that

$$x = \varphi_0(\mu, \dot{\mu}, \ldots, \mu^{(r)}) \tag{2.3}$$

and

$$u = \varphi_1(\mu, \dot{\mu}, \ldots, \mu^{(r+1)}) \tag{2.4}$$

satisfying

$$\frac{d\varphi_0}{dt} = f(\varphi_0, \varphi_1)$$

where

$$u^{(s)} = (u_1^{(s_1)}, \ldots, u_m^{(s_m)}), \quad \mu^{(r)} = (\mu_1^{(r_1)}, \ldots, \mu_m^{(r_m)})$$

For simplicity let $\bar{\mu} = (\mu, \dot{\mu}, \ldots, \mu^{(q)})$ for some $q \in \mathbb{R}$ (μ and a finite number of its derivatives). Having the ability to express all system variables as a function of a flat output and a finite number of its successive derivatives proves very useful in motion planning and tracking. Another advantage the flat system provides is that the system dynamics can be put into a useful form.

Theorem 2.1.2 (Theorem 6.2 from [16])**.** *Every flat system is endogenous dynamic feedback linearizable and the closed-loop system is diffeomorphic to the linear controllable system in canonical form*

$$\begin{aligned} &\mu_1^{(r_1+1)} = \nu_1 \\ &\vdots \\ &\mu_m^{(r_m+1)} = \nu_m. \end{aligned} \tag{2.5}$$

Remark A more formal definition of differential flatness is that a flat system is *Lie-Bäcklund equivalent* to a *trivial system*: a system without differential constraints. Hence the map (φ_0, φ_1) is referred to as the flat system's *Lie-Bäcklund isomorphism* and the linear system (2.5) of Theorem 2.1.2 as its *trivial system.* A consequence of this equivalence is that flat outputs can be defined which have no differentiability constraints, but can be mapped injectively to the integral curves of the flat system.

2.2 Flatness and Motion Planning

Planning the motion of a reference trajectory is an important part of trajectory tracking. The goal is to find a trajectory that meets design requirements and that the system will not have trouble tracking. The concept of Flatness is a valuable tool in motion planning for differential systems. Using the Lie-Bäcklund isomorphism, trajectories can be designed *a priori* for all state variables and constraints on these trajectories can be easily generated from constraints on the system variables which simplifies the planning procedure. This section will introduce the basics of the theory that will be employed to perform motion planning for the web roller.

Consider the system (2.1), with endpoint constraints

$$x(t_i) = x_i, \quad x(t_f) = x_f$$

and

$$u(t_i) = u_i, \quad u(t_f) = u_f.$$

The problem of motion planning is to find trajectories that satisfy these constraints and the dynamic equations (2.1). Normally this requires a method of approximation or integration along the vector field $f(x, u)$, but for dynamically flat systems it can be done quite simply. The endpoint constraints can be transformed into constraints on the flat output and its first *r+1* time derivatives by equation (2.2).

$$\mu_1(t_i), \ldots, \mu_1^{(r_1+1)}(t_i), \ldots, \mu_m(t_i), \ldots, \mu_m^{(r_m+1)}(t_i) \tag{2.6}$$

and

$$\mu_1(t_f), \ldots, \mu_1^{(r_1+1)}(t_f), \ldots, \mu_m(t_f), \ldots, \mu_m^{(r_m+1)}(t_f) \tag{2.7}$$

These represent $2r + 3$ conditions on the reference trajectories for the flat outputs. Polynomials of degree $2r + 3$ are chosen because they can easily be constructed to fulfill the requirements. Let $T = t_f - t_i$, the duration, and $\sigma(t) = (t - t_i)/T$, the normalized time, then the trajectories have the form

$$\mu_j(t) = \sum_{k=0}^{2r_j+3} a_{j,k}\sigma^k(t), \quad j = 1, \ldots, m.$$

The coefficients $a_{j,k}$ are determined by evaluating the derivatives of the polynomials at the endpoints and equating them with the values given by the endpoint constraints. The derivatives are

$$\mu_j^{(k)}(t) = \frac{1}{T^k} \sum_{l=k}^{2r_j+3} \frac{l!}{(l-k)!} a_{j,l}\sigma^{l-k}(t), \quad j = 1, \ldots, m$$

and so at the initial time, t_i, $\sigma = 0$

$$\mu_j^{(k)}(t_i) = \frac{k!}{T^k} a_{j,k}, \quad k = 0, \ldots, r+1,\ j = 1, \ldots, m \tag{2.8}$$

and at the final time, t_f, $\sigma = 1$

$$\mu_j^{(k)}(t_f) = \frac{1}{T^k} \sum_{l=k}^{2r_j+3} \frac{l!}{(l-k)!} a_{j,l}, \quad k = 0, \ldots, r+1,\ j = 1, \ldots, m. \tag{2.9}$$

By equating these values with their constraints (2.6) and (2.7), the required trajectories can be generated.

2.2.1 Rest-to-Rest Trajectories

For rest-to-rest trajectories, the initial and final positions are equilibrium points of the system, i.e. $\dot{x}(t_i) = \dot{u}(t_i) = \dot{x}(t_f) = \dot{u}(t_f) = 0$. It can be shown that the corresponding points $\mu(t_i)$ and $\mu(t_f)$ are equilibrium points for the trivial system (2.5) (Levine Theorem 5.2 [16]) and that therefore the first r derivatives of the flat output must vanish

$$x(t_i) = \varphi_0(\mu(t_i), 0, \ldots, 0), \quad x(t_f) = \varphi_0(\mu(t_f), 0, \ldots, 0)$$

and

$$u(t_i) = \varphi_1(\mu(t_i), 0, \ldots, 0), \quad u(t_f) = \varphi_1(\mu(t_f), 0, \ldots, 0)$$

By setting equations (2.8) and (2.9) equal to zero for $k > 0$ and solving for the polynomial coefficients $a_{j,k}$, the resulting trajectories are

$$\mu_j(t) = \mu_j(t_i) + (\mu_j(t_f) - \mu_j(t_i))(\sigma(t))^{r_j+2} \left(\sum_{k=0}^{r_j+1} \alpha_{j,k} (\sigma(t))^k \right), \quad j = 1, \ldots, m \tag{2.10}$$

with $\alpha_{j,0}, \ldots, \alpha_{j,r+1}$ given by

$$\begin{bmatrix} 1 & 1 & \ldots & 1 \\ r_j+2 & r_j+3 & \ldots & 2r_j+3 \\ (r_j+1)(r_j+2) & (r_j+2)(r_j+3) & \ldots & (2r_j+2)(2r_j+3) \\ \vdots & & & \vdots \\ (r_j+2)! & \dfrac{(r_j+3)!}{2} & \ldots & \dfrac{(2r_j+3)!}{(r_j+2)!} \end{bmatrix} \begin{bmatrix} \alpha_{j,0} \\ \vdots \\ \alpha_{j,r_j+1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{2.11}$$

This $(r+1)$th-order polynomial satisfies the given endpoint constraints and the equilibrium constraints of the system equations. The order of the polynomials may be increased and the additional coefficients found by making the higher order derivatives vanish at the endpoints. The result would provide a more *smooth* transition between the rest positions and would therefore have a greater likelihood of being tracked effectively. To demonstrate this consider the piecewise defined rest-to-rest trajectory

$$\mu(t) = \begin{cases} \mu(t_i), & t < t_i \\ \mu(t_i) + (\mu(t_f) - \mu(t_i))(\sigma(t))^{r+2}\left(\sum_{k=0}^{r+1} \alpha_{j,k}(\sigma(t))^k\right), & t_i \leq t \leq t_f \\ \mu(t_f), & t > t_f \end{cases}$$

with prescribed endpoint positions $\mu(t_i)$ and $\mu(t_f)$, and $\alpha_{j,k}$ as above. This is $(r+1)$-times continuous differentiable and therefore the corresponding nominal state and control trajectories $(x, u) = (\varphi_0(\mu, \dot{\mu}, \dots, \mu^{(r)}), \varphi_1(\mu, \dot{\mu}, \dots, \mu^{(r+1)}))$ will not contract any discontinuities from their inputs.

2.2.2 Path Constraints

Often in practice additional constraints on the state and control variables beyond the fixed initial and final location may need to be satisfied by their reference trajectories. For instance, the system variables or their derivatives may need to remain bounded. As with the previous section, the Lie-Bäcklund isomorphism allows these constraints to be interpreted as additional constraints on the flat outputs. Consider path constraints of the form

$$\|g(x, \dot{x}, \dots, u, \dot{u}, \dots)\| \leq C_g$$

for a continuously differentiable function g. Using the Lie-Bäcklund isomorphism, this can be expressed as a path constraint on the flat output and its derivatives

$$\left\|G(\mu, \dot{\mu}, \dots, \mu^{(q)})\right\| \leq C_g$$

where $\mu^{(q)} = (\mu_1^{(q_1)}, \dots, \mu_j^{(q_j)})$. With the polynomial form developed in Section 2.2 this can be satisfied by choosing an appropriate duration, T. The maximum value of the derivatives of the flat output is inversely proportional to the duration. Using the normalized time derivatives

$$\mu^{(k)}(t) = \frac{1}{T^k}\frac{d^k\mu}{d\sigma^k}(\sigma(t)), \quad \forall k \geq 1 \tag{2.12}$$

the path constraint becomes

$$\left\|G\left(\mu, \frac{1}{T}\frac{d\mu}{d\sigma}, \dots, \frac{1}{T^q}\frac{d^q\mu}{d\sigma^q}\right)\right\| \leq C_g$$

and the supremum of the derivatives can be expressed as

$$\max_{t\in[t_i,t_f]} \left\|\mu^{(k)}(t)\right\| = \frac{1}{T^k} \max_{\sigma\in[0,1]} \left\|\frac{d^k\mu}{d\sigma^k}(\sigma)\right\|, \quad \forall k \geq 1. \tag{2.13}$$

The supremum of the normalized derivative on the right-hand-side of (2.13) can be calculated explicitly. An upper bound on the norm of the constraint function can now be established in the form

$$\left\|G\left(\mu, \frac{1}{T}\frac{d\mu}{d\sigma}, \ldots, \frac{1}{T^q}\frac{d^q\mu}{d\sigma^q}\right)\right\| \leq \sum_{k=0}^{q} C_k \left(\frac{1}{T^k}\right) \leq C_g, \quad C_k \in \mathbb{R}_+ \tag{2.14}$$

The coefficients C_k are found using norm inequalities $\|a \pm b\| \leq \|a\| + \|b\|$ and $\|a \cdot b\| \leq \|a\|\,\|b\|$, and, if necessary, the *Mean Value Theorem* (MVT)

$$\left\|G\left(\mu, \frac{1}{T}\frac{d\mu}{d\sigma}, \ldots, \frac{1}{T^q}\frac{d^q\mu}{d\sigma^q}\right)\right\| \leq \|G(\mu, 0, \ldots, 0)\| + \sum_{j=1}^{m}\sum_{k=1}^{q} \left\|\frac{\partial G}{\partial \mu_j^{(k)}}\right\|_\infty \frac{1}{T^k} \left\|\frac{d^k\mu_j}{d\sigma^k}\right\|$$

where the value of

$$\left\|\frac{\partial G}{\partial \mu_j^{(k)}}\right\|_\infty = \max_{\sigma\in[0,1]} \left\|\frac{\partial G}{\partial \mu_j^{(k)}}(\bar{\mu}(\sigma))\right\|$$

can be found by recursive application of MVT, if necessary, until the constraint G is bounded above by a q-th order polynomial in $1/T$. A minimum value for the duration T^* is given by the positive real valued root of

$$\sum_{k=0}^{q} C_k z^k - C_g.$$

with $z = 1/T$. For high values of the integer q, finding the roots may not be practical and so an alternative approach is proposed. By appropriate choice of upper bounds on the flat output derivatives

$$\|\dot{\mu}_j(t)\| \leq C_{j,1}, \ldots, \left\|\mu_j^{(q)}(t)\right\| \leq C_{j,q}, \quad j = 1, \ldots, m$$

such that the inequality

$$\begin{aligned}
\left\|G(\mu, \dot{\mu}, \ldots, \mu^{(q)})\right\| &= \left\|G\left(\mu, \frac{1}{T}\frac{d\mu}{d\sigma}, \ldots, \frac{1}{T^q}\frac{d^q\mu}{d\sigma^q}\right)\right\| \\
&\leq \|G(\mu, 0, \ldots, 0)\| + \sum_{j=1}^{m}\sum_{k=1}^{q} \left\|\frac{\partial G}{\partial \mu_j^{(k)}}\right\| \frac{1}{T^k} \left\|\frac{d^k\mu_j}{d\sigma^k}\right\| \\
&\leq \|G(\mu, 0, \ldots, 0)\| + \sum_{j=1}^{m}\sum_{k=1}^{q} C_{j,k} \left\|\frac{\partial G}{\partial \mu_j^{(k)}}\right\| \\
&\leq C_g
\end{aligned}$$

is satisfied. This is done by choosing a duration that satisfies

$$T \geq \left\{ \frac{1}{C_{j,1}} \max_{\sigma\in[0,1]} \left\| \frac{d\mu_j}{d\sigma} \right\|, \ldots, \left(\frac{1}{C_{j,q}} \max_{\sigma\in[0,1]} \left\| \frac{d^q\mu_j}{d\sigma^q} \right\| \right)^{\frac{1}{q}} \right\}_{j=1,\ldots,m}$$

This is essentially a *guess and check* method for finding the minimum duration. Choosing an upper bound $C_{j,k}$ is equivalent to choosing a lower bound on the duration by

$$T^k \geq \left\| \frac{d^k\mu_j}{d\sigma^k} \right\| \frac{1}{C_{j,k}}$$

Additional path constraints can be addressed independently resulting in multiple lower bounds on the duration. By choosing a duration greater than all of these, the polynomial reference trajectories will necessarily satisfy all of the path constraints.

2.3 Flatness and Tracking

The problem of developing a reference trajectory for the state and control variables, x and u, respectively, for system (2.1) was shown to be equivalent to finding a reference trajectory for the flat output. In the absence of disturbances, measurement and actuation error, and system instabilities, the *open loop* control (2.4) given by the Lie-Bäcklund isomorphism would produce the desired result. These disturbances do however occur in practise, therefore a *closed loop* control scheme is devised so that deviations from the nominal reference can be corrected in real time. Consider the error between the reference, μ^*, developed using motion planning, and the actual flat output μ:

$$e_i = \mu_i - \mu_i^*, \quad i = 1, \ldots m.$$

The trajectory is tracked if this error converges to zero. This is achieved by stabilizing the error dynamics

$$\begin{aligned} e_1^{(r_1+1)} &= \mu_1^{(r_1+1)} - (\mu_1^*)^{(r_1+1)} + w_1 \\ &\vdots \\ e_m^{(r_m+1)} &= \mu_m^{(r_m+1)} - (\mu_m^*)^{(r_m+1)} + w_m \end{aligned}$$

where $w_1, \ldots, w_m$ are unmeasured disturbance terms. By Theorem 2.1.2 there exists a dynamic feedback so that the system dynamics can be expressed as $(\mu_i)^{(r_i+1)} = \nu_i$. Denote the derivatives of the flat output by $(\mu_i^*)^{(r_i+1)} = \nu_i^*$. The error dynamics become

$$\begin{aligned} e_1^{(r_1+1)} &= \nu_1 - \nu_1^* + w_1 \\ &\vdots \\ e_m^{(r_m+1)} &= \nu_m - \nu_m^* + w_m \end{aligned} \tag{2.15}$$

A feedback rule for the input ν is needed such that these error dynamics are stable at the origin. These is achieved by applying the dynamic feedback

$$\nu_i = \nu_i^* - \sum_{j=0}^{r_i} k_{i,j} e_i^{(j)}, \quad i = 1, \ldots, m$$

The error dynamics become

$$e_i^{(r_i+1)} = -\sum_{j=0}^{r_i} k_{i,j} e_i^{(j)} + w_i, \quad i = 1, \ldots, m.$$

If the disturbance terms dissipate over time so that $w_i \to 0$ as $t \to \infty$ $i = 1, \ldots, m$, and the gains $k_{i,j}$ are chosen so that the polynomials

$$P_i(s) = s_i^{(r_i+1)} + \sum_{j=0}^{r_i} k_{i,j} s_i^{(j)} \tag{2.16}$$

are *Hurwitz* (which means the roots of $P_i(t)$, and therefore the poles of the error dynamics, are in the left-half of the complex plane) then the flat output will converge exponentially to its reference. And by the differentiability of the Lie-Bäcklund isomorphism, the original state and control variables must converge to their references exponentially.

Chapter 3

Nonlinear Observers

A closed loop control scheme like the tracking procedure defined in Section 2.3 requires knowledge of the state variables during operation. Physical, economic and other constraints may restrict the availability of state measurement. State observers are used to approximate the unmeasured states and hence preserve the possibility of feedback. If possible, an observer will force the error between the approximate and actual state variables to zero. The possibility of a convergent observer design is characterized by the observability properties of the system. Solutions to this problem are well-defined for linear systems. Nonlinear systems can be linearized or the nonlinearities can be ignored or assumed to be bounded, but this usually provides only a locally convergent solution. Convergent observers have been developed in recent years for different classes of nonlinear systems. This chapter presents a review of observability and observer design for special classes of nonlinear systems as presented in [5].

3.1 Observability

The basis of observability is that for systems without full-state knowledge, the outputs of the system under different initial conditions remain distinguishable for varying inputs:

Definition 3.1.1 (Observability). *A system of the form* (2.1) *is observable on* $[0, t]$ *for an input* $u \in U$ *if for every pair* $x_0, x_0' \in X$ *satisfying* $x_0 \neq x_0'$, $\exists s \in [0, t]$ *such that*

$$h(\chi_u(s, x_0)) \neq h(\chi_u(s, x_0'))$$

or equivalently

$$\int_0^t \|h(\chi_u(\tau, x_0)) - h(\chi_u(\tau, x_0'))\| \, d\tau > 0.$$

For nonlinear systems, observability is often dependent on the choice of inputs. The inputs of Definition 3.1.1 with which the system is made observable are called *Universal Inputs*. If every input for a system is universal, then the system is *Uniformly Observable*. For some observer designs, this notion of observability is not always enough to guarantee that all trajectories can be observed accurately in the presence of disturbances. A stronger observability condition is achieved through the restriction to inputs that are *Regularly Persistent*:

Definition 3.1.2 (Regularly Persistent Input)**.** *An input u is regularly persistent for system* (2.1) *if $\exists t_i, T$ s.t. $\forall x_{t-T} \neq x'_{t-T}$, $\forall t > t_i$*

$$\int_{t-T}^{t} \left\| h(\chi_u(\tau, x_{t-T})) - h(\chi_u(\tau, x'_{t-T})) \right\| d\tau > \beta(\left\| x_{t-T} - x'_{t-T} \right\|)$$

for some class K function β (monotonic positive definite function):

$$\begin{aligned} &\beta(0) = 0 \\ &\beta(x) > 0 \quad \forall x > 0 \\ &\beta \text{ is non-decreasing.} \end{aligned}$$

Regular persistency is essential to observer design for bilinear and state-affine systems [2], [10]. For state-affine systems, $\dot{x}(t) = A(u(t))x(t) + B(u(t))$, regularly persistent inputs satisfy

$$\exists t_0, T, \alpha : \quad \Gamma(t - T, t) \geq \alpha I > 0, \quad \forall t \geq t_0 \tag{3.1}$$

where the *Observability Grammian*, $\Gamma(t_1, t_2)$, is given by

$$\Gamma(t_1, t_2) = \int_{t_1}^{t_2} \Phi_u^T(s, t_1) C^T C \Phi_u(s, t_1) ds$$

with the *Transition Matrix*, $\Phi_u(s, t)$, satisfying

$$\frac{\partial \Phi_u(s, t)}{\partial s} = A(u(t))\Phi_u(s, t), \quad \Phi_u(t, t) = I$$

Once the observability properties of a system have been identified, an observer can be defined.

3.2 Observer Form

The function of an observer is to generate a variable, $\hat{x}$, that well-approximates the actual state using the knowledge of the state dynamics. Through feedback, dynamics for the approximate state are chosen so that $\forall s \geq t_i$ if $\hat{x}(s) = x(s)$ then $\hat{x}(t) = x(t)$ $\forall t > s$ in the absence of disturbances. As well, deviations of the approximation from the actual state should be corrected by monitoring the output error, $h(\hat{x}(t)) - y(t)$, so that the approximation converges to the actual state. Under observability restrictions, such as those discussed in the previous section, these dynamics take the following observer form:

$$\begin{aligned}\dot{\hat{x}}(t) &= f(u(t), \hat{x}(t)) - k(t, \hat{y}(t) - y(t)) \\ \hat{y}(t) &= h(\hat{x}(t))\end{aligned}$$

with the observer gain function $k : \mathbb{R}^{p+1} \to \mathbb{R}^n$ chosen such that observer error, $\varepsilon(t) = \hat{x}(t) - x(t)$, converges to zero, meaning $\|\hat{x}(t) - x(t)\| \to 0$ as $t \to \infty$. Convergence is typically achieved by stabilizing the dynamics of the observer error at its origin, $\varepsilon = 0$. Methods have been derived for choosing an effective gain for different forms of the system equations. It is usually sufficient to choose gains of the form $k(t, \hat{y}(t) - y(t)) = K(t)(\hat{y}(t) - y(t))$.

3.2.1 Kalman Filter

The observer gains can be derived from the state dynamics to produce optimal convergence qualities. One example is the *Kalman Observer* for LTV systems [12]. The following is a refined version of this observer presented in [5]. Consider an LTV system of the form

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t)\end{aligned}$$

For $A(t)$ and $C(t)$ bounded the following Kalman-related observer is proposed [12], [2], [10], [6], [8]

$$\dot{\hat{x}}(t) = A(t)\hat{x}(t) + B(t)u(t) - K(t)(C\hat{x}(t) - y(t)).$$

The observer gain, $K(t)$, is given by

$$K(t) = P(t)C^T W^{-1}$$

where the $n \times n$ matrix $P(t)$ satisfies the Riccati equation

$$\begin{aligned}\dot{P}(t) &= A(t)P(t) + P(t)A^T(t) - P(t)C^T W^{-1} C P(t) + Q + \delta P(t) \\ P(0) &= P^T(0) > 0\end{aligned}$$

with symmetric positive definite gain

$$W = W^T > 0$$

and

$$\delta > 2\,\|A(t)\| \quad \text{or} \quad Q = Q^T > 0.$$

To clarify the notation for the positive definiteness of the constant gains in the Riccati equation:

Definition 3.2.1 (Definiteness)**.** *A matrix $M \in \mathbb{R}^{j \times j}$ is Positive Definite (resp. Semi-definite) iff it satisfies:*

1. $x^T M x > 0$ *(resp.* $x^T M x \geq 0$*)* $\forall x \in \mathbb{R}^j$
2. $x^T M x = 0 \Rightarrow x = 0$ *(for definiteness only)*

and the property is denoted by $M > 0$ (resp. $M \geq 0$). For Negative Definite and Semi-definite, the inequalities are reversed.

An alternative form for the observer is given

$$K(t) = S^{-1}(t) C^T W^{-1}$$

where

$$\begin{aligned} \dot{S}(t) &= -A^T(t)S(t) - S(t)A(t) + C^T W^{-1} C - S(t)QS(t) - \delta S(t) \\ S(0) &= S^T(0) > 0 \end{aligned}$$

For the case of $\delta = 0$, this resembles the classic Kalman filter in which the gain is optimal in the sense of minimizing the cost

$$\int_0^t [(C(s)z(s) - y(s))^T W^{-1} (C(s)z(s) - y(s)) + \nu^T(s) Q^{-1} \nu(s)] ds \\ + (z_0 - \hat{x}_0)^T P_0^{-1} (z_0 - \hat{x}_0)$$

subject to

$$\begin{aligned} \dot{z}(t) &= A(t)z(t) + \nu(t) \\ y(t) &= C(t)z(t). \end{aligned}$$

There exist numerous adaptations of the Kalman observer for nonlinear systems. This result can readily be extended to state-affine systems by way of substituting $A(u(t))$ into the Riccati equation. Additionally this can be extended to apply to dynamics that are affine in their unmeasured states up to additive output nonlinearity [9] [6]:

$$\begin{aligned} \dot{x}(t) &= A(u(t), y(t))x(t) + B(u(t), y(t)) \\ y(t) &= Cx(t). \end{aligned} \tag{3.2}$$

These dynamics admit an observer of the form

$$\dot{\hat{x}}(t) = A(u(t), y(t))\hat{x}(t) + B(u(t), y(t)) - K(t)(C\hat{x}(t) - y(t)).$$

With the gain, $K(t)$, given by

$$\begin{aligned}
\dot{P}(t) &= A(u(t), y(t))P(t) + P(t)A^T(u(t), y(t)) - P(t)C^T W^{-1} C P(t) + Q + \delta P(t) \\
P(0) &= P^T(0) > 0 \\
K(t) &= P(t)C^T W^{-1}
\end{aligned}$$

with $W = W^T > 0$ and $\delta > 2\,\|A(u(t), y(t))\|$ or $Q = Q^T > 0$. The error between the approximate and actual state variables will converge exponentially to zero if the extended inputs $v(t) = (u(t), C\chi_u(t, x_0))$ are *regularly persistant*. The observability property can be shown by the observability grammian method (3.1) with the extended input. Proofs for the convergence of the Kalman-related filter can be found in [6], [10] for the case of $\delta > \|A(t)\|$, and [8] for $V = V^T > 0$.

Part II

Two-Span Web Roller

Chapter 4

Mathematical Modelling

The function of web roller technology is to facilitate action along a thin flexible material called a web that can be wound around cylindrical shafts called rollers. Web material is transported from an *unwinder* roller onto a *winder* roller after passing through two *nip* rollers. This is performed by applying torque independently to the unwinder, winder and one of the nip rollers. As a result, the tension on either side of the nip and the velocity of the web through the nip may be controlled independently. For many applications, these three values must be carefully regulated. This is particularly important when using web material that is highly flexible and when employing high web velocities to provide a high production rate. As well, deviations in web tension can result in poor product quality. The purpose of this project is to design the trajectories for the two-span web roller and the controller and observer to track them, the mathematical model must be both accurate and functional. Traditionally linear approximations are used since procedures for stabilizing, controlling and observing linear systems are well-defined. However, using a linear model for the web roller or applying linear approximations when developing these procedures does not capture its dynamic behaviour and results in poor performance. Much advancement has been made in the field of nonlinear control theory so that less drastic assumptions can be made while still allowing for useful performance methods. In this chapter a mathematical model will be developed with suitable properties to accommodate the motion planning, tracking and observer procedures to be addressed in later chapters. Early works in developing dynamical models for tension controllers were done in [3], [22] and [21]. These and other sources for elastic web tension models are reviewed in [19] upon which the following section is accordingly based.

4.1 Dynamical System Model

A basic two-span web roller or tension controller consists of a web, winding and unwinding rollers, a nip, and two idle rollers to facilitate tension measurement. In the model shown in Figure 4.1, the web material is fed from right to left by motors which drive (independently) the unwinder and winder rollers which release and gather the web respectively and a motor which drives one of the nip rollers.

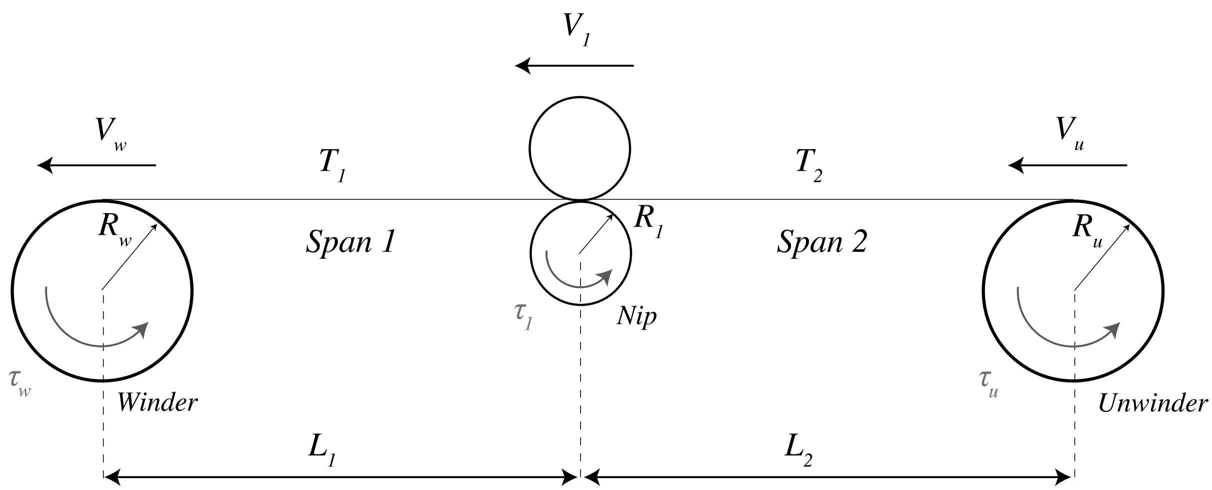

Figure 4.1: Free-body diagram of the web roller.

The length of web between the winder and the nip, which is assumed to have a constant length L_1 and uniform tension T_1, is denoted *Span 1*. Likewise, the constant length L_2 between the nip and unwinder is denoted *Span 2* and has uniform tension T_2. The winder and unwinder rollers have variable radii r_w and r_u, respectively, with initial conditions: $r_w(0) = R_w$ and $r_u(0) = R_u$. The angular velocity of the winder, unwinder and nip are denoted ω_w, ω_u and ω_1, respectively.

Assumption 4.1.1. *Friction between the web and the rollers is sufficient so that slip or separation does not occur.*

The velocity of the web approaches the peripheral velocity of a roller at the point of contact. Therefore the velocity of the web entering Span 2 is $V_u = r_u\omega_u$, exiting Span 2 and entering Span 1 it is $V_1 = R_1\omega_1$, and exiting Span 1 it is $V_w = r_w\omega_w$.

Assumption 4.1.2. *The motors on the three rollers are able to track torque application to the roller shafts.*

The dynamics of the torque actuation are ignored such that the torques can be viewed as inputs to the system belonging to a suitable bounded set.

$$u = (\tau_1, \tau_w, \tau_u) \tag{4.1}$$

Assumption 4.1.3. *The angular velocity and radius of each roller are able to be measured and load cells are able to measure web tension. This is done in real time without affecting the dynamics of the web roller.*

A measurable position of the system consists of the radii and angular velocity of the winder and unwinder rollers, the angular velocity of the nip roller and the tension of the web within the two spans. The case of unmeasured tension is addressed in Chapter 7.

$$x = (T_1, T_2, \omega_1, \omega_w, \omega_u, r_w, r_u) \tag{4.2}$$

Assumption 4.1.4. *Average winder and unwinder radii vary linearly with the angular position of the roller.*

If one revolution of a roller corresponds to an increase/decrease in radius by the thickness, H, of the web, then

$$\begin{aligned} r_w(t) &= R_w + \frac{H}{2\pi}\theta_w(t) \\ r_u(t) &= R_u - \frac{H}{2\pi}\theta_u(t) \end{aligned} \tag{4.3}$$

where $R_w, R_u \in \mathbb{R}$ are the initial radii with θ_w and θ_u the angular positions of the winding and unwinding rollers in radians satisfying $\theta_w(t_0) = 0$ and $\theta_u(t_0) = 0$. Therefore

$$\begin{aligned} \dot{r}_w &= \frac{H}{2\pi}\omega_w \\ \dot{r}_u &= -\frac{H}{2\pi}\omega_u \end{aligned}$$

for angular velocities ω_w and ω_u in radians-per-second.

Definition 4.1.5 (Newton's Second Law for Rotation Systems)**.** *The rate of change in angular momentum of a rotating body equals the sum of torques applied to the body.*

For each roller, the rate of change in angular momentum equals the angular acceleration multiplied by the angular moment of inertia for the roller: $J(r)\dot{\omega}$. The moment of the roller as the radius increases or decreases changes according to

$$J(r) = J(r_0) + \pi\varrho W(r^4 - r_0^4)$$

where ϱ is the web density and W the width. The torques applied to each roller are the control torque input, torque applied by the web tension, and frictional torque at the axis of rotation.

Assumption 4.1.6. *Frictional torque on the rollers depends linearly on angular velocity.*

The torque applied by friction in the opposite direction of rotation for each roller is therefore

$$
\begin{aligned}
F_1 &= B_1 \omega_1 \\
F_w &= B_w \omega_w \\
F_u &= B_u \omega_u
\end{aligned}
$$

for coefficients of viscous friction B_w, B_u and B_1 determined experimentally. Newton's Law for the three rollers is therefore

$$
\begin{aligned}
J_w(r_w)\dot{\omega}_w &= -r_w T_1 - B_w \omega_w + \tau_w \\
J_1 \dot{\omega}_1 &= R_1(T_1 - T_2) - B_1 \omega_1 + \tau_1 \\
J_u(r_u)\dot{\omega}_u &= r_u T_2 - B_u \omega_u + \tau_u
\end{aligned}
$$

Assumption 4.1.7. *Web thickness H and width W (and therefore Cross-sectional area, A) are constant. Strain occurs only along the direction of web transport.*

The linear density of the web under tension T_1 (resp. T_2) is ρ_1 (resp. ρ_2) corresponding to strain ϵ_1 (resp. ϵ_2). If the un-stretched linear density of the web is ρ_0, then the density of the web experiencing a strain of ϵ_k is

$$\rho_k = \frac{\rho_0}{1 + \epsilon_k}, \quad k = 1, 2.$$

Assumption 4.1.8. *Strain of the web material is small enough for a first order approximation $1/(1+\epsilon) \approx (1-\epsilon)$ (usually less than 0.01 [21]).*

This gives

$$\rho_k = \rho_0(1 - \epsilon_k), \quad k = 1, 2. \tag{4.4}$$

Assumption 4.1.9. *The linear density of the web entering Span 2 at the unwinder is equal to the un-stretched density of the web.*

In practice, there is normally a wound-on tension associated with the unwinder, however, varying this has been shown to have little effect on observer estimates [19].

Definition 4.1.10 (Conservation of Mass)**.** *The change of mass in an open system must equal the difference between the mass entering the system and the mass exiting the system.*

For the web roller this means the rate of change in the mass of web between adjacent rollers must equal the mass flow rate entering the span minus the exiting flow rate. Mass flow rate is given by multiplying the linear velocity of the web by its linear density. The flow rate exiting from Span 1 (onto the winder) is $V_w\rho_1$, the flow rate entering Span 1 (through the nip) is $V_1\rho_2$. The mass of web in Span 1, at an instant dt, is $L_1\rho_1$.

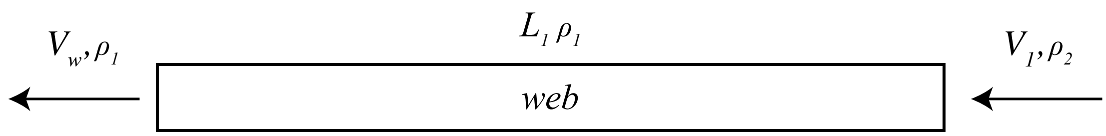

Figure 4.2: Mass flow through Span 1.

The flow rate entering Span 2 (from the unwinder) is $V_u\rho_0$ and the exiting flow rate to the nip is $V_1\rho_2$. The mass of web in Span 2 is $L_2\rho_2$.

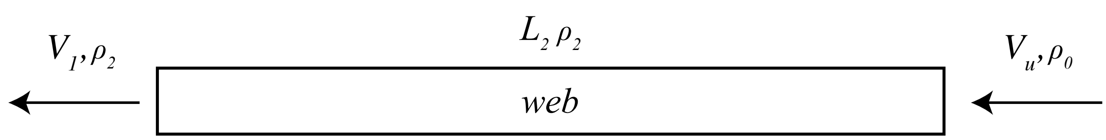

Figure 4.3: Mass flow through Span 2.

Assuming constant span lengths, Conservation of Mass for each span gives the dynamics

$$\begin{aligned} L_1\dot{\rho}_1 &= V_1\rho_2 - V_w\rho_1 \\ L_2\dot{\rho}_2 &= V_u\rho_0 - V_1\rho_2. \end{aligned}$$

Substituting expression (4.4) for the linear densities and canceling out the un-stretched density ρ_0 gives

$$\begin{aligned} -L_1\dot{\epsilon}_1 &= V_1(1-\epsilon_2) - V_w(1-\epsilon_1) \\ -L_2\dot{\epsilon}_2 &= V_u - V_1(1-\epsilon_2). \end{aligned}$$

Assumption 4.1.11. *The web satisfies Hooke's Law: tension is proportional to strain so that $T_k = EA\epsilon_k$ with E, Young's modulus of the material, and A, the cross-sectional area.*

Applying Hooke's law, these dynamics can be expressed in term of tension.

$$-L_1\dot{T}_1 = V_1(EA - T_2) - V_w(EA - T_1)$$
$$-L_2\dot{T}_2 = V_uEA - V_1(EA - T_2).$$

The seventh order dynamical system representation for the web roller is therefore

$$\dot{T}_1 = -\frac{r_w}{L_1}\omega_w T_1 + \frac{R_1}{L_1}\omega_1 T_2 + \frac{EA}{L_1}(r_w\omega_w - R_1\omega_1) \tag{4.5a}$$
$$\dot{T}_2 = -\frac{r_u}{L_2}\omega_u T_2 + \frac{EA}{L_2}(R_1\omega_1 - r_u\omega_u) \tag{4.5b}$$
$$\dot{\omega}_1 = \frac{R_1}{J_1}(T_1 - T_2) - \frac{B_1}{J_1}\omega_1 + \frac{\tau_1}{J_1} \tag{4.5c}$$
$$\dot{\omega}_w = -\frac{r_w}{J_w(r_w)}T_1 - \frac{B_w}{J_w(r_w)}\omega_w + \frac{\tau_w}{J_w(r_w)} \tag{4.5d}$$
$$\dot{\omega}_u = \frac{r_u}{J_u(r_u)}T_2 - \frac{B_u}{J_u(r_u)}\omega_u + \frac{\tau_u}{J_u(r_u)} \tag{4.5e}$$
$$\dot{r}_w = \frac{H}{2\pi}\omega_w \tag{4.5f}$$
$$\dot{r}_u = -\frac{H}{2\pi}\omega_u \tag{4.5g}$$

where

$$J_w(r_w) = J_{w0} + \frac{\pi}{2}\varrho W(r_w^4 - R_w^4)$$
$$J_u(r_u) = J_{u0} + \frac{\pi}{2}\varrho W(r_u^4 - R_u^4)$$

with initial conditions $r_w(0) = R_w$ and $r_u(0) = R_u$.

4.2 Simplified Flat Model

A simplified version of the dynamics (4.5) are used to facilitate the motion planning and tracking procedure. Consider a model with fixed roller radii, with (4.3) given by

$$r_w(t) = R_w$$
$$r_u(t) = R_u, \quad \forall t \geq 0$$

and, by extension, fixed inertia $J_w = J_{w0}$ and $J_u = J_{u0}$. In the start-up scenario, the objective is to guide the states to a desired operating point quickly and efficiently. Therefore the average radii of the winder and unwinder should not change significantly. However, to maintain accuracy for operation at the steady state, the varying radii model must be adopted for controller design. The simplified model will, however, be useful for developing reference trajectories. The system dynamics, now of fifth order, are

$$\dot{T}_1 = -\frac{R_w}{L_1}\omega_w T_1 + \frac{R_1}{L_1}\omega_1 T_2 + \frac{EA}{L_1}(R_w\omega_w - R_1\omega_1) \tag{4.6a}$$

$$\dot{T}_2 = -\frac{R_u}{L_2}\omega_u T_2 + \frac{EA}{L_2}(R_1\omega_1 - R_u\omega_u) \tag{4.6b}$$

$$\dot{\omega}_1 = \frac{R_1}{J_1}(T_1 - T_2) - \frac{B_1}{J_1}\omega_1 + \frac{\tau_1}{J_1} \tag{4.6c}$$

$$\dot{\omega}_w = -\frac{R_w}{J_w}T_1 - \frac{B_w}{J_w}\omega_w + \frac{\tau_w}{J_w} \tag{4.6d}$$

$$\dot{\omega}_u = \frac{R_u}{J_u}T_2 - \frac{B_u}{J_u}\omega_u + \frac{\tau_u}{J_u}. \tag{4.6e}$$

These dynamics have a particular structure which is valuable for motion planning and tracking: they are differentially flat. Differentially flat systems were introduced in Chapter 2. Under Definition 2.1.1, flatness is shown by defining a suitable flat output. In general this is not a simple task, however, for the simplified model, there exists a favorable result.

Proposition 4.2.1. *The simplified system* (4.6) *is differentially flat.*

Proof. To prove flatness, it is necessary and sufficient to define any flat outputs $\mu \in \mathbb{R}^3$ which satisfy the conditions of Definition 2.1.1. This is equivalent to defining the maps $\psi(x, u, \dot{u}, \ldots, u^{(s)})$, $\varphi_0(\mu, \dot{\mu}, \ldots, \mu^{(r)})$ and $\varphi_1(\mu, \dot{\mu}, \ldots, \mu^{(r+1)})$. For simplicity of the motion planning procedure, the trajectories that are to be tracked are chosen as the flat outputs:

$$\mu_1 = T_1 \tag{4.7}$$

$$\mu_2 = T_2 \tag{4.8}$$

$$\mu_3 = \omega_1. \tag{4.9}$$

This is a well defined flat output under (2.2). It remains to represent all other system variables in function of these flat outputs and a finite number of its successive derivatives. Solving (4.6a) and (4.6b) for ω_w and ω_u respectively yields:

$$\omega_w = \frac{L_1\dot{\mu}_1 + R_1\mu_3(EA - \mu_2)}{R_w(EA - \mu_1)}. \tag{4.10}$$

and

$$\omega_u = \frac{EAR_1\mu_3 - L_2\dot{\mu}_2}{R_u(EA + \mu_2)} \tag{4.11}$$

Solving (4.6c) for τ_1 and substituting the expressions (4.10) and (4.11) into (4.6d) and (4.6e), respectively, and solving for τ_w and τ_u gives:

$$\tau_1 = J_1\dot{\mu}_3 + R_1(\mu_2 - \mu_1) + B_1\mu_3 \tag{4.12}$$

$$\begin{aligned}\tau_u = {} & J_u \frac{(EAR_1\dot{\mu}_3 - L_2\ddot{\mu}_2)}{R_u(EA+\mu_2)} + R_u\mu_2 \\ & + \frac{EAR_1\mu_3 - L_2\dot{\mu}_2}{R_u(EA+\mu_2)}\left(B_u - \frac{J_u\dot{\mu}_2}{EA+\mu_2}\right)\end{aligned} \tag{4.13}$$

$$\begin{aligned}\tau_w = {} & J_w \frac{(L_1\ddot{\mu}_1 + R_1\dot{\mu}_3(EA-\mu_2) - R_1\mu_3\dot{\mu}_2)}{R_w(EA-\mu_1)} + R_w\mu_1 \\ & + \frac{L_1\dot{\mu}_1 + R_1\mu_3(EA-\mu_2)}{R_w(EA-\mu_1)}\left(\frac{J_w\dot{\mu}_1}{EA-\mu_1} + B_w\right).\end{aligned} \tag{4.14}$$

□

The Lie-Bäcklund isomorphism for this flat system, by which the state and control variables are retrieved from the flat output, are of the form

$$\begin{aligned} x &= \varphi_0(\mu_1, \mu_2, \mu_3, \dot{\mu}_1, \dot{\mu}_2) \\ u &= \varphi_1(\mu_1, \mu_2, \mu_3, \dot{\mu}_1, \dot{\mu}_2, \dot{\mu}_3, \ddot{\mu}_1, \ddot{\mu}_2).\end{aligned}$$

This is a diffeomorphism provided $R_u(EA+T_2) \neq 0$ and $R_w(EA-T_1) \neq 0$ which is satisfied if $T_1 \neq EA$. The multi-integer r of Definition 2.1.1 is therefore given by

$$r = \{r_1, r_2, r_3\} = \{1, 1, 0\}. \tag{4.15}$$

This flat model will be used to plan and track the motion of the web roller. Flat outputs are not unique, however, the chosen outputs are the state variables that need to be tracked and are therefore an ideal choice.

Chapter 5

Motion Planning

Typically, to improve the convergence of state variables toward a fixed path, controller design and feedback gains are modified. However, proper motion planning can be used to define trajectories that will be tracked more reliably and with less energy investment. For the web roller the control objective is to steer the system from rest to the operating position $T_1 = T_1^*$, $T_2 = T_2^*$ and $\omega_1 = \omega_1^*$. In Section 2.2 the concept of developing a *well-defined* nominal flat output for a flat system was introduced. Trajectories were formally defined for a rest-to-rest path, to switch the system smoothly from one state to another. Criteria were developed to satisfy constraints on the state and input variables. In Section 4.2 the fixed-radii model was shown to be differentially flat and the states to be tracked, T_1, T_2 and ω_1, are flat outputs. By investigating the flat system, suitable polynomial reference trajectories can be defined and the Lie-Bäcklund isomorphism will provide nominal inputs: a point of reference for the torques that will be required to track them.

5.1 Flatness-Based Trajectory Generation

The states T_1, T_2 and ω_1 are given fixed initial and final positions. Each are assumed to be at rest (constant) at zero prior to time t_0 and to remain at rest at the operating position. By considering a simplified flat system model, this constitutes a full-state rest-to-rest trajectory planning problem: guiding the roller between equilibrium positions. With the varying radii model (2.1), the winder and unwinder radii (and therefore angular velocities) must continue to increase and decrease, respectively, to maintain the operating position. In the start-up scenario, in order to reach a desired state from rest, a simple set of trajectories to track are the step inputs

$$T_1(t) = \begin{cases} 0, & t_i \leq t < t_f \\ T_1^*, & t \geq t_f \end{cases} \tag{5.1a}$$

$$T_2(t) = \begin{cases} 0, & t_i \leq t < t_f \\ T_2^*, & t \geq t_f \end{cases} \tag{5.1b}$$

$$\omega_1(t) = \begin{cases} 0, & t_i \leq t < t_f \\ \omega_1^*, & t \geq t_f \end{cases} \tag{5.1c}$$

This is a common choice of reference for rest-to-rest motions, however, without tuning feedback gains, tracking these trajectories usually results in large over-shoot, rising time and settling time. The step input does not constitute a *natural* motion of the flat system as defined in Chapter 2. In attempt to solve this, a continuous piecewise-linear objective function is considered

$$T_1(t) = \begin{cases} 0, & t_i \leq t < t_0 \\ T_1^* \dfrac{t - t_0}{t_f - t_0}, & t_0 \leq t < t_f \\ T_1^*, & t \geq t_f \end{cases} \tag{5.2a}$$

$$T_2(t) = \begin{cases} 0, & t_i \leq t < t_0 \\ T_2^* \dfrac{t - t_0}{t_f - t_0}, & t_0 \leq t < t_f \\ T_2^*, & t \geq t_f \end{cases} \tag{5.2b}$$

$$\omega_1(t) = \begin{cases} 0, & t_i \leq t < t_0 \\ \omega_1^* \dfrac{t - t_0}{t_f - t_0}, & t_0 \leq t < t_f \\ \omega_1^*, & t \geq t_f \end{cases} \tag{5.2c}$$

and there is a notable improvement in the controller's tracking response [4]. This suggests a correlation between the differentiability of the reference trajectories and the accuracy of the system response to tracking them. More specifically, the references should match some differentiability property of the system dynamics. For differentially flat system, this notion is revealed by the trivial system. By Theorem 2.1.2 there exists an endogenous dynamic feedback to transform the flat dynamics (4.6) into the form

$$\begin{aligned} \ddot{\mu}_1 &= \nu_1 \\ \ddot{\mu}_2 &= \nu_2 \\ \dot{\mu}_3 &= \nu_3. \end{aligned}$$

The initial and final positions are equilibrium positions of the trivial system as well. This places endpoint constraints on the derivatives of the flat output

$$\mu(0) = \dot{\mu}(0) = 0, \quad \ddot{\mu}_1(0) = \ddot{\mu}_2(0) = 0$$

and at the terminal point

$$\mu(t_f) = (T_1^*, T_2^*, \omega_1^*), \quad \dot{\mu}(t_f) = 0, \quad \ddot{\mu}_1(t_f) = \ddot{\mu}_2(t_f) = 0.$$

The polynomial trajectories developed in Section 2.2.1 for the rest-to-rest case can be employed. Equation (2.10) for $j = 1$, 2 and 3 gives

$$\mu_j(t) = \mu_j(0) + (\mu_j(t_f) - \mu_j(0))(\sigma(t))^{r_j+2} \left(\sum_{k=0}^{r_j+1} \alpha_{j,k}(\sigma(t))^k \right) \tag{5.3}$$

For the nip angular velocity, represented by the flat output μ_3, the multi-integer gives $r_3 = 0$. In this case equation (2.11) reads

$$\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

which gives

$$\alpha = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The flatness-based reference for μ_3 by (5.3) is therefore

$$\mu_3^*(t) = \omega_1^* \left(\frac{t}{t_f} \right)^2 \left(3 - 2 \left(\frac{t}{t_f} \right) \right).$$

This cubic polynomial is illustrated in Figure 5.1

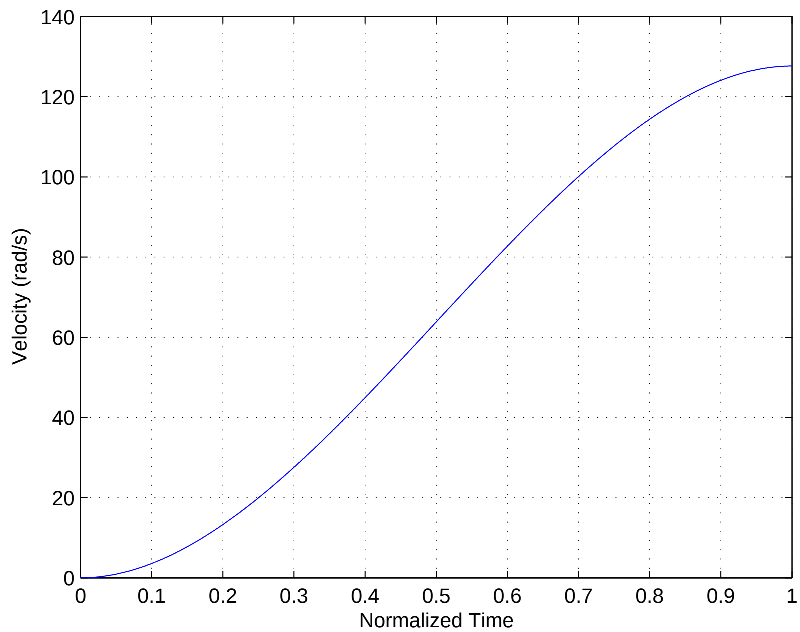

Figure 5.1: Flatness based reference (5.4c) for the nip angular velocity.

For the web tensions, represented by the outputs μ_1 and μ_2, with $r_1 = r_2 = 1$, yields

$$\begin{pmatrix} 1 & 1 & 1 \\ 3 & 4 & 5 \\ 6 & 12 & 20 \end{pmatrix} \alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

giving

$$\alpha = \begin{pmatrix} 10 & -4 & 1/2 \\ -15 & 7 & -1 \\ 6 & -3 & 1/2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and the remaining reference trajectories are therefore

$$\mu_1^*(t) = T_1^* \left(\frac{t}{t_f}\right)^3 \left(10 - 15\left(\frac{t}{t_f}\right) + 6\left(\frac{t}{t_f}\right)^2\right)$$
$$\mu_2^*(t) = T_2^* \left(\frac{t}{t_f}\right)^3 \left(10 - 15\left(\frac{t}{t_f}\right) + 6\left(\frac{t}{t_f}\right)^2\right)$$

This fifth order polynomials are illustrated in Figure 5.2.

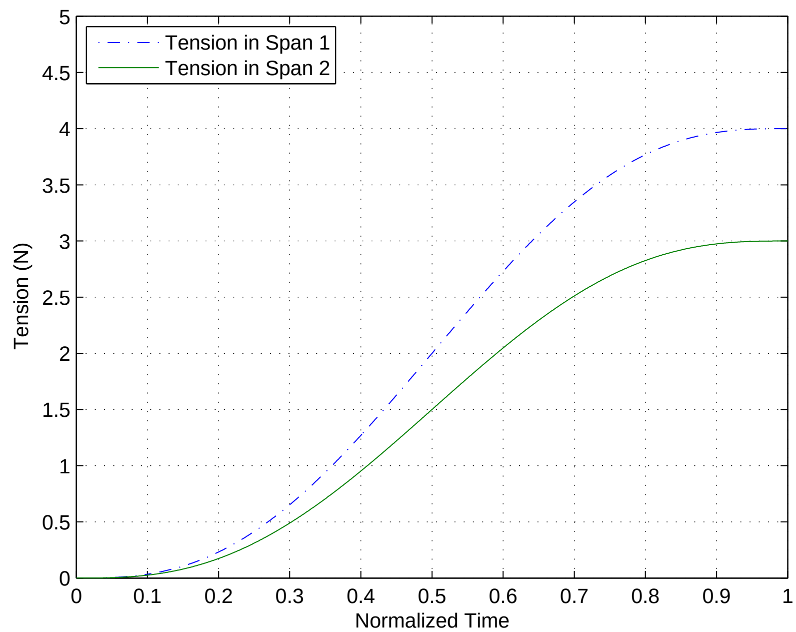

Figure 5.2: Flatness based references (5.4a) and (5.4b) for the web tensions.

The nominal velocity of the nip roller is continuously differentiable and the nominal tensions are twice continuously differentiable. These trajectories guide the state variables from an equilibrium position (of the flat dynamics) at $t = t_0 = 0$, to an equilibrium position at $t = t_f$. The duration time $T = t_f$ is tunable and the following section demonstrates the relationship between this parameter and the control variables.

$$\mu_1^*(t) = \begin{cases} T_1^* \left(\frac{t}{t_f}\right)^3 \left(10 - 15\left(\frac{t}{t_f}\right) + 6\left(\frac{t}{t_f}\right)^2\right), & 0 \le t \le t_f \\ T_1^*, & t > t_f \end{cases} \tag{5.4a}$$

$$\mu_2^*(t) = \begin{cases} T_2^* \left(\frac{t}{t_f}\right)^3 \left(10 - 15\left(\frac{t}{t_f}\right) + 6\left(\frac{t}{t_f}\right)^2\right), & 0 \le t \le t_f \\ T_2^*, & t > t_f \end{cases}. \tag{5.4b}$$

$$\mu_3^*(t) = \begin{cases} \omega_1^* \left(\frac{t}{t_f}\right)^2 \left(3 - 2\left(\frac{t}{t_f}\right)\right), & 0 \le t \le t_f \\ \omega_1^*, & t > t_f. \end{cases}. \tag{5.4c}$$

5.2 Constraints

One of the advantages that the flatness-based references provide is they can be tracked more easily, meaning that, in most cases, even for low values of the feedback gains there will be less tracking error than that resulting from a non flatness based reference for which careful tuning may be required to guarantee adequate convergence. Even in cases where the gains can be increased to provide a comparable convergence with that of the flatness based reference, this may have unpredictable and possibly adverse effects on the state and control variables. For instance, forcing a higher convergence rate through feedback may result in an unacceptable increase on the control input. For a flat system this is well-accomodated by the Lie-Bäcklund isomorphism whereby constraints on the state or control variables or their derivatives can be transformed directly into constraints on the reference trajectories during the motion planning stage. Physical (mechanical and electrical) limitations of the roller would impose a maximum on the derivative of the torque. This can be satisfied by imposing a minimum for the rising time T. The nominal torque inputs given by the Lie-Bäcklund isomorphism are shown in Figure

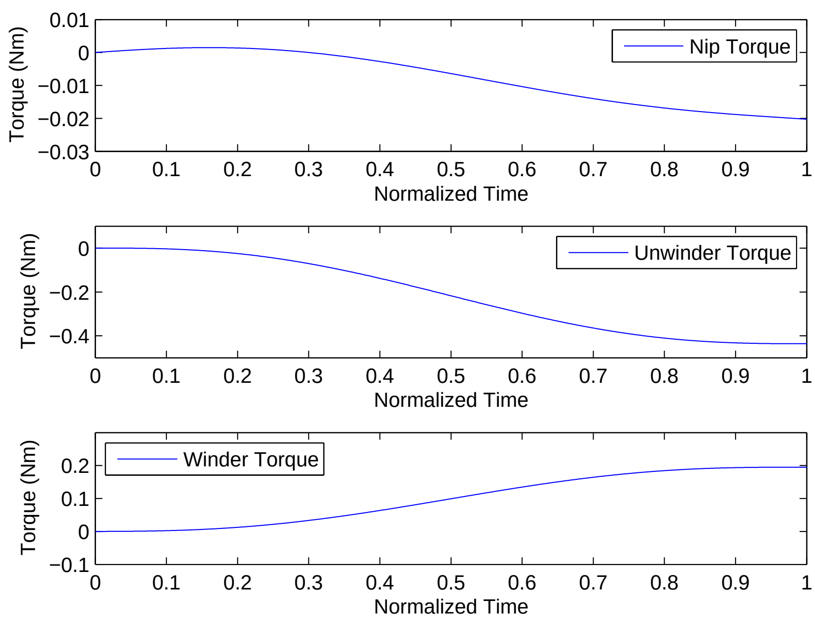

Figure 5.3: Nominal torque inputs (4.12), (4.13) and (4.14).

The flat outputs are adjusted so that the nominal input satisfies the upper bound constraint

$$\|\dot{u}^*\| \leq \|\dot{u}_{max}\| = (\|\dot{\tau}_{1,max}\|, \|\dot{\tau}_{2,max}\|, \|\dot{\tau}_{3,max}\|). \tag{5.5}$$

Given the Lie-Bäcklund isomorphism, the rate of change of the input torque can be expressed in terms of the flat output

$$\begin{aligned}
\dot{\tau}_1 &= J_1\ddot{\mu}_3 + R_1(\dot{\mu}_2 - \dot{\mu}_1) + B_1\dot{\mu}_3 \\
\dot{\tau}_u &= J_u\frac{EAR_1\ddot{\mu}_3 - L_2\mu_2^{(3)}}{R_u(EA+\mu_2)} + \frac{EAR_1\dot{\mu}_3 - L_2\ddot{\mu}_2}{R_u(EA+\mu_2)}\left(B_u - \frac{2J_u\dot{\mu}_2}{EA+\mu_2}\right) \\
&\quad + \frac{EAR_1\mu_3 - L_2\dot{\mu}_2}{R_u(EA+\mu_2)^2}\left(\frac{2J_u(\dot{\mu}_2)^2}{EA+\mu_2} - J_u\ddot{\mu}_2 - B_u\dot{\mu}_2\right) + R_u\dot{\mu}_2 \\
\dot{\tau}_w &= J_w\frac{L_1\mu_1^{(3)} + R_1(\ddot{\mu}_3(EA-\mu_2) - 2\dot{\mu}_3\dot{\mu}_2 - \mu_3\ddot{\mu}_2)}{R_w(EA-\mu_1)} + R_w\dot{\mu}_1 \\
&\quad + \frac{L_1\ddot{\mu}_1 + R_1(\dot{\mu}_3(EA-\mu_2) - \mu_3\dot{\mu}_2)}{R_w(EA-\mu_1)}\left(B_w + \frac{2J_w\dot{\mu}_1}{EA-\mu_1}\right) \\
&\quad + \frac{L_1\dot{\mu}_1 + R_1\mu_3(EA-\mu_2)}{R_w(EA-\mu_1)^2}\left(\frac{2J_w(\dot{\mu}_1)^2}{EA-\mu_1} - J_w\ddot{\mu}_1 - B_w\dot{\mu}_1\right).
\end{aligned}$$

Using the normalized flat output derivative relationship (2.12), this can be expressed in terms of the normalized flat output

$$\begin{aligned}
\dot{\tau}_1 &= \frac{J_1}{T^2}\frac{d^2\mu_3}{d\sigma^2} + \frac{R_1}{T}\left(\frac{d\mu_2}{d\sigma} - \frac{d\mu_1}{d\sigma}\right) + \frac{B_1}{T}\frac{d\mu_3}{d\sigma} \\
\dot{\tau}_u &= \frac{J_u\Lambda_u}{R_u}\left(\frac{EAR_1}{T^2}\frac{d^2\mu_3}{d\sigma^2} - \frac{L_2}{T^3}\frac{d^3\mu_2}{d\sigma^3}\right) + \frac{R_u}{T}\frac{d\mu_2}{d\sigma} \\
&\quad + \frac{\Lambda_u}{R_u}\left(\frac{EAR_1}{T}\frac{d\mu_3}{d\sigma} - \frac{L_2}{T^2}\frac{d^2\mu_2}{d\sigma^2}\right)\left(B_u - \frac{2J_u\Lambda_u}{T}\frac{d\mu_2}{d\sigma}\right) \\
&\quad + \frac{\Lambda_u^2}{R_u}\left(EAR_1\mu_3 - \frac{L_2}{T}\frac{d\mu_2}{d\sigma}\right)\left(\frac{2J_u\Lambda_u}{T^2}\left(\frac{d\mu_2}{d\sigma}\right)^2 - \frac{J_u}{T^2}\frac{d^2\mu_2}{d\sigma^2} - \frac{B_u}{T}\frac{d\mu_2}{d\sigma}\right) \\
\dot{\tau}_w &= \frac{J_w\Lambda_{w,1}}{R_w}\left(\frac{L_1}{T^3}\frac{d^3\mu_1}{d\sigma^3} + R_1\left(\frac{\Lambda_{w,2}}{T^2}\frac{d^2\mu_3}{d\sigma^2} - \frac{2}{T^2}\frac{d\mu_3}{d\sigma}\frac{d\mu_2}{d\sigma} - \frac{\mu_3}{T^2}\frac{d^2\mu_2}{d\sigma^2}\right)\right) + \frac{R_w}{T}\frac{d\mu_1}{d\sigma} \\
&\quad + \frac{\Lambda_{w,1}}{R_w}\left(\frac{L_1}{T^2}\frac{d^2\mu_1}{d\sigma^2} + R_1\left(\frac{\Lambda_{w,2}}{T}\frac{d\mu_3}{d\sigma} - \frac{\mu_3}{T}\frac{d\mu_2}{d\sigma}\right)\right)\left(B_w + \frac{2J_w\Lambda_{w,1}}{T}\frac{d\mu_1}{d\sigma}\right) \\
&\quad + \frac{\Lambda_{w,1}^2}{R_w}\left(\frac{L_1}{T}\frac{d\mu_1}{d\sigma} + R_1\Lambda_{w,2}\mu_3\right)\left(\frac{2J_w\Lambda_{w,1}}{T^2}\left(\frac{d\mu_1}{d\sigma}\right)^2 - \frac{J_w}{T^2}\frac{d^2\mu_1}{d\sigma^2} - \frac{B_w}{T}\frac{d\mu_1}{d\sigma}\right)
\end{aligned}$$

where $\Lambda_u = 1/(EA + \mu_2(\sigma))$, $\Lambda_{w,1} = 1/(EA - \mu_1(\sigma))$ and $\Lambda_{w,2} = EA - \mu_2(\sigma)$. By applying norm inequalities, a measurable upper bound can be established for the nominal torque derivatives. The results are polynomial functions in $1/T$

$$\|\dot{\tau}_1\| \leq a_1 \frac{1}{T^2} + b_1 \frac{1}{T}$$
$$\|\dot{\tau}_u\| \leq a_u \frac{1}{T^3} + b_u \frac{1}{T^2} + c_u \frac{1}{T}$$
$$\|\dot{\tau}_w\| \leq a_w \frac{1}{T^3} + b_w \frac{1}{T^2} + c_w \frac{1}{T}.$$

with coefficients

$$a_1 = J_1 \left\| \frac{d^2\mu_3}{d\sigma^2} \right\|$$
$$b_1 = R_1 \left(\left\| \frac{d\mu_2}{d\sigma} \right\| + \left\| \frac{d\mu_1}{d\sigma} \right\| \right) + B_1 \left\| \frac{d\mu_3}{d\sigma} \right\|$$

for the nip,

$$\begin{aligned}
a_u &= \frac{J_u L_2}{R_u} \left(\|\Lambda_u\| \left\| \frac{d^3\mu_2}{d\sigma^3} \right\| + 3\|\Lambda_u\|^2 \left\| \frac{d^2\mu_2}{d\sigma^2} \right\| \left\| \frac{d\mu_2}{d\sigma} \right\| + 2\|\Lambda_u\|^3 \left\| \frac{d\mu_2}{d\sigma} \right\|^3 \right) \\
b_u &= \frac{J_u EAR_1}{R_u} \left(\|\Lambda_u\| \left\| \frac{d^2\mu_3}{d\sigma^2} \right\| + 2\|\Lambda_u\|^2 \left\| \frac{d\mu_3}{d\sigma} \right\| \left\| \frac{d\mu_2}{d\sigma} \right\| \right) \\
&\quad + \frac{J_u EAR_1}{R_u} \|\mu_3\| \left(\|\Lambda_u\|^2 \left\| \frac{d^2\mu_2}{d\sigma^2} \right\| + 2\|\Lambda_u\|^3 \left\| \frac{d\mu_2}{d\sigma} \right\|^2 \right) \\
&\quad + \frac{B_u L_2}{R_u} \left(\|\Lambda_u\| \left\| \frac{d^2\mu_2}{d\sigma^2} \right\| + \|\Lambda_u\|^2 \left\| \frac{d\mu_2}{d\sigma} \right\|^2 \right) \\
c_u &= R_u \left\| \frac{d\mu_2}{d\sigma} \right\| + \frac{B_u EAR_1}{R_u} \left(\|\Lambda_u\| \left\| \frac{d\mu_3}{d\sigma} \right\| + \|\Lambda_u\|^2 \|\mu_3\| \left\| \frac{d\mu_2}{d\sigma} \right\| \right)
\end{aligned}$$

for the unwinder and

$$
\begin{aligned}
a_w &= \frac{J_w L_1}{R_w}\left(\|\Lambda_{w,1}\|\left\|\frac{d^3\mu_1}{d\sigma^3}\right\| + 3\|\Lambda_{w,1}\|^2\left\|\frac{d^2\mu_1}{d\sigma^2}\right\|\left\|\frac{d\mu_1}{d\sigma}\right\| + 2\|\Lambda_{w,1}\|^3\left\|\frac{d\mu_1}{d\sigma}\right\|^3\right) \\
b_w &= \frac{J_w R_1\|\Lambda_{w,1}\|}{R_w}\left(\|\Lambda_{w,2}\|\left\|\frac{d^2\mu_3}{d\sigma^2}\right\| + 2\left\|\frac{d\mu_3}{d\sigma}\right\|\left\|\frac{d\mu_2}{d\sigma}\right\| + \|\mu_3\|\left\|\frac{d^2\mu_2}{d\sigma^2}\right\|\right) \\
&\quad + \frac{J_w R_1\|\Lambda_{w,2}\|}{R_w}\|\mu_3\|\left(\|\Lambda_{w,1}\|^2\left\|\frac{d^2\mu_1}{d\sigma^2}\right\| + 2\|\Lambda_{w,1}\|^3\left\|\frac{d\mu_1}{d\sigma}\right\|^2\right) \\
&\quad + \frac{2J_w R_1\|\Lambda_{w,1}\|^2}{R_w}\left\|\frac{d\mu_1}{d\sigma}\right\|\left(\|\Lambda_{w,2}\|\left\|\frac{d\mu_3}{d\sigma}\right\| + \|\mu_3\|\left\|\frac{d\mu_2}{d\sigma}\right\|\right) \\
&\quad + \frac{B_w L_1}{R_w}\left(\|\Lambda_{w,1}\|\left\|\frac{d^2\mu_1}{d\sigma^2}\right\| + \|\Lambda_{w,1}\|^2\left\|\frac{d\mu_1}{d\sigma}\right\|^2\right) \\
c_w &= R_w\left\|\frac{d\mu_1}{d\sigma}\right\| + \frac{B_w R_1\|\Lambda_{w,1}\|}{R_w}\left(\|\Lambda_{w,2}\|\left\|\frac{d\mu_3}{d\sigma}\right\| + \|\mu_3\|\left\|\frac{d\mu_2}{d\sigma}\right\|\right) \\
&\quad + \frac{B_w R_1\|\Lambda_{w,1}\|^2\|\Lambda_{w,2}\|}{R_w}\|\mu_3\|\left\|\frac{d\mu_1}{d\sigma}\right\|
\end{aligned}
$$

for the winder. Therefore upper bounds are computed for the first three derivatives of μ_1^* and μ_2^* and the first two derivatives of μ_3^*. The nominal flat output polynomial (5.4a) for μ_1^* has derivatives

$$
\begin{aligned}
\frac{d\mu_1^*}{d\sigma} &= T_1^*(30\sigma^2 - 60\sigma^3 + 30\sigma^4), \\
\frac{d^2\mu_1^*}{d\sigma^2} &= T_1^*(60\sigma - 180\sigma^2 + 120\sigma^3), \\
\frac{d^3\mu_1^*}{d\sigma^3} &= T_1^*(60 - 360\sigma + 360\sigma^2), \quad 0 \le \sigma \le 1.
\end{aligned}
$$

The upper bounds are found either at the endpoints or by solving for the critical points to give

$$
\begin{aligned}
\max_{\sigma\in[0,1]} \|\mu_1^*(\sigma)\| &= T_1^* \quad \text{at} \quad \sigma = 1 \\
\max_{\sigma\in[0,1]} \left\|\frac{d\mu_1^*}{d\sigma}\right\| &= \frac{15T_1^*}{8} \quad \text{at} \quad \sigma = 1/2 \\
\max_{\sigma\in[0,1]} \left\|\frac{d^2\mu_1^*}{d\sigma^2}\right\| &= \frac{10T_1^*}{\sqrt{3}} \quad \text{at} \quad \sigma = 1/2 + \sqrt{3}/6 \\
\max_{\sigma\in[0,1]} \left\|\frac{d^3\mu_1^*}{d\sigma^3}\right\| &= 60T_1^* \quad \text{at} \quad \sigma = 0
\end{aligned}
$$

Likewise, the upper bounds for the output μ_2^* are T_2^*, $15T_2^*/8$, $10T_2^*/\sqrt{3}$ and $60T_2^*$, respectively. Repeating this procedure for third flat output μ_3^* yields

$$\frac{d\mu_3^*}{d\sigma} = \omega_1^*(6\sigma - 6\sigma^2),$$
$$\frac{d^2\mu_3^*}{d\sigma^2} = \omega_1^*(6 - 12\sigma), \quad 0 \le \sigma \le 1$$

with the supremums

$$\max_{\sigma\in[0,1]} \left\| \frac{d\mu_3^*}{d\sigma} \right\| = \frac{3\omega_1^*}{2} \quad \text{at} \quad \sigma = 1/2$$
$$\max_{\sigma\in[0,1]} \left\| \frac{d^2\mu_3^*}{d\sigma^2} \right\| = 6\omega_1^* \quad \text{at} \quad \sigma = 0.$$

The positivity property $\mu_2 \ge 0$ gives the upper bound $\|\Lambda_u\| \le 1/EA$. By Assumption 4.1.11 (Hooke's Law) and given that the strain in the web is typically small ($\epsilon < 0.01$) the web tension can be assumed to satisfy $T < EA$. This yields the upper bounds $\|\Lambda_{w,1}\| \le 1/(EA - T_1^*)$ and $\|\Lambda_{w,2}\| \le EA$. The coefficients for the nip torque therefore must satisfy

$$a_1 \le 6J_1\omega_1^*$$
$$b_1 \le \frac{15}{8}R_1(T_1^* + T_2^*) + \frac{3}{2}B_1\omega_1^*,$$

and so the inequality constraint $\|\dot{\tau}_1^*\| \le \|\dot{\tau}_{1,max}\|$ becomes

$$6J_1\omega_1^*\frac{1}{T^2} + \left(\frac{R_1}{2}(T_1^* + T_2^*) + B_1\omega_1^*\right)\frac{1}{T} - \|\dot{\tau}_{1,max}\| \le 0$$

This is satisfied if the duration is greater than or equal to the nominal duration T_n^* which is the positive solution to the quadratic equation

$$a\left(\frac{1}{T^*}\right)^2 + b\left(\frac{1}{T^*}\right) + c = 0$$

where $a, b \ge 0$ and $c < 0$. This has the solution

$$\frac{1}{T^*} = \frac{\sqrt{b^2 - 4ac} - b}{2a}.$$

The nominal duration is therefore

$$T_n^* = \frac{12J_1\omega_1^*}{\sqrt{\left(\frac{15R_1}{8}(T_1^* + T_2^*) + \frac{3B_1}{2}\omega_1^*\right)^2 + 24J_1\omega_1^*\|\dot{\tau}_{1,max}\|} - \left(\frac{15R_1}{8}(T_1^* + T_2^*) + \frac{3B_1}{2}\omega_1^*\right)}.$$

For the winder and unwinder torque derivatives, the upper bounds on the coefficients become

$$
\begin{aligned}
a_u &\le \frac{J_u L_2}{R_u}\left(60\frac{T_2^*}{EA} + \frac{225}{4\sqrt{3}}\left(\frac{T_2^*}{EA}\right)^2 + \frac{3375}{256}\left(\frac{T_2^*}{EA}\right)^3\right) \\
b_u &\le \frac{J_u EA R_1 \omega_1^*}{R_u}\left(\frac{6}{EA} + \frac{45T_2^*}{8(EA)^2} + \frac{10T_2^*}{\sqrt{3}(EA)^2} + \frac{225(T_2^*)^2}{32(EA)^3}\right) \\
&\quad + \frac{B_u L_2}{R_u}\left(\frac{10T_2^*}{\sqrt{3}EA} + \frac{225(T_2^*)^2}{64(EA)^2}\right) \\
c_u &\le \frac{15R_u T_2^*}{8} + \frac{B_u EA R_1}{R_u}\left(\frac{3\omega_1^*}{2EA} + \frac{15T_2^*\omega_1^*}{8(EA)^2}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
a_w &\le \frac{J_w L_1}{R_w}\left(60\frac{T_1^*}{EA - T_1^*} + \frac{225}{4\sqrt{3}}\left(\frac{T_1^*}{EA - T_1^*}\right)^2 + \frac{3375}{256}\left(\frac{T_1^*}{EA - T_1^*}\right)^3\right) \\
b_w &\le \frac{J_w R_1 \omega_1^*}{R_w(EA - T_1^*)}\left(6EA + \frac{45T_2^*}{8} + \frac{10T_2^*}{\sqrt{3}}\right) \\
&\quad + \frac{J_w EA R_1 \omega_1^*}{R_w}\left(\frac{10T_1^*}{\sqrt{3}(EA - T_1^*)^2} + \frac{225(T_1^*)^2}{32(EA - T_1^*)^3}\right) \\
&\quad + \frac{15J_w R_1 \omega_1^* T_1^*}{4R_w(EA - T_1^*)^2}\left(\frac{3EA}{2} + \frac{15T_1^*}{8}\right) \\
&\quad + \frac{B_w L_1}{R_w}\left(\frac{10T_1^*}{\sqrt{3}(EA - T_1^*)} + \frac{225(T_1^*)^2}{64(EA - T_1^*)^2}\right) \\
c_w &\le \frac{15R_w T_1^*}{8} + \frac{B_w R_1 \omega_1^*}{R_w(EA - T_1^*)}\left(\frac{3EA}{2} + \frac{15T_2^*}{8} + \frac{15EAT_1^*}{8(EA - T_1^*)}\right)
\end{aligned}
$$

As with the nip torque, the winder and unwinder constraints are satisfied if the duration is greater than or equal to a nominal duration that is the positive solution to a cubic equation of the form

$$
a\left(\frac{1}{T^*}\right)^3 + b\left(\frac{1}{T^*}\right)^2 + c\left(\frac{1}{T^*}\right) + d = 0
$$

with $a, b, c \ge 0$ and $d < 0$. This has the solution

$$
\frac{1}{T^*} = \frac{1}{3a}\left(\frac{p}{\sqrt[3]{2}} - \frac{\sqrt[3]{2}}{p}r - b\right) \tag{5.6}
$$

where p and r are given by the expressions

$$
p = \sqrt[3]{q + \sqrt{q^2 + 4r^3}}, \quad q = 9abc - 2b^3 - 27da^3, \quad r = 3ac - b^2.
$$

The coefficients a, b and c for the unwinder (resp. winder) are the upper bounds on a_u, b_u and c_u (resp. a_w, b_w and c_w) and the coefficient d is $-\|\dot{\tau}_{u,max}\|$ (resp. $-\|\dot{\tau}_{w,max}\|$). Solving (5.6) for both cases provides two more lower bounds for the duration T_u^* and T_w^*. All three path constraints on the torque derivatives are satisfied if the duration is chosen so that

$$T \geq \max\{T_n^*, T_u^*, T_w^*\}.$$

For any duration greater than this nominal value, the path constraint will be satisfied, assuming a tracking procedure is employed that can adequately track the reference outputs in the presence of disturbances. This procedure essentially places upper bounds on the derivatives of the outputs. Frequently a tracking control law with poor convergence qualities will result in a 'delayed reaction' of the system output resulting in a longer rise time than prescribed. In this case the upper bound would not be violated.

Chapter 6

Controller Design

The control inputs for the Two Span Web Roller are the torques on each of the winders. The control objective is to design the inputs which will achieve the desired system output: the nominal reference trajectories developed in Chapter 5. In simulation, this can be achieved by applying the inputs given by the Lie-Bäcklund isomorphism. This open-loop scheme is rarely applied in practice because modeling error and disturbances will cause the actual trajectories to deviate from the references. A feedback or closed-loop control law is needed that can measure deviations from system outputs and correct them automatically. For this process it is assumed that all state variables are measurable. The alternative case will be dealt with in Section 7.

In this chapter, the feedback linearization scheme defined in Section 2.3 is applied to the simplified flat model. This method is then adapted for the general model by a partial feedback linearization scheme presented in [1]. Results for the tracking procedure applied to the step (5.1), piecewise linear (5.2), and flatness-based reference trajectories are then simulated.

6.1 Tracking with the Simplified Model

The simplified flat dynamics for the web roller were used to develop reference trajectories for the flat outputs. If the system is assumed to be flat, then by Section 2.3 there exists an exponential tracking procedure. This is achieved by first linearizing the system dynamics and then stabilizing the linear dynamics of the error between the flat output and its reference. By Theorem 2.1.2 there exists an endogenous dynamic feedback to transform the flat dynamics (4.6) into the form

$$
\begin{aligned}
\ddot{\mu}_1 &= \nu_1 \\
\ddot{\mu}_2 &= \nu_2 \\
\dot{\mu}_3 &= \nu_3
\end{aligned}
$$

The states which need to be tracked are chosen as the flat outputs by (4.7) so the dynamics in terms of the state variable x are

$$
\begin{aligned}
\ddot{T}_1 &= \nu_1 \\
\ddot{T}_2 &= \nu_2 \\
\dot{\omega}_1 &= \nu_3.
\end{aligned}
\tag{6.1}
$$

It remains to find the feedback in function of the state variables and the intermediate input ν (i.e. $u(x, \nu)$). The left-hand-side of these dynamics can be expressed in terms of x and u by differentiating the original dynamical equations.

$$
\begin{aligned}
\ddot{T}_1 &= -\frac{R_w}{L_1}(\dot{\omega}_w T_1 + \omega_w \dot{T}_1) + \frac{R_1}{L_1}(\dot{\omega}_1 T_2 + \omega_1 \dot{T}_2) + \frac{EA}{L_1}(R_w \dot{\omega}_w - R_1 \dot{\omega}_1) \\
\ddot{T}_2 &= -\frac{R_u}{L_2}(\dot{\omega}_u T_2 + \omega_u \dot{T}_2) + \frac{EA}{L_2}(R_1 \dot{\omega}_1 - R_u \dot{\omega}_u) \\
\dot{\omega}_1 &= \frac{R_1}{J_1}(T_1 - T_2) - \frac{B_1}{J_1}\omega_1 + \frac{\tau_1}{J_1}
\end{aligned}
$$

and substituting the expressions for the derivatives to obtain the form

$$
\begin{pmatrix} \ddot{T}_1 \\ \ddot{T}_2 \\ \dot{\omega}_1 \end{pmatrix} = \mathcal{A}(x) + \mathcal{B}(x)u \tag{6.2}
$$

with the vector $\mathcal{A}(x) \in \mathbb{R}^3$ given component-wise

$$\begin{aligned}
\mathcal{A}_1(x) = & -\frac{R_w(EA - T_1)}{L_1}\left(\frac{R_w T_1 + B_w \omega_w}{J_w} + \frac{R_w \omega_w^2}{L_1}\right) \\
& + \frac{R_1(T_2 - EA)}{L_1}\left(\frac{R_1(T_1 - T_2) - B_1\omega_1}{J_1} - \frac{R_w \omega_w \omega_1}{L_1}\right) \\
& + \frac{R_1\omega_1}{L_1 L_2}(EAR_1\omega_1 - (T_2 + EA)R_u\omega_u) \\
\mathcal{A}_2(x) = & \frac{R_u(T_2 + EA)}{L_2}\left(\frac{-R_u T_2 + B_u\omega_u}{J_u} + \frac{R_u\omega_u^2}{L_2}\right) \\
& + \frac{EAR_1}{L_2}\left(\frac{R_1(T_1 - T_2) - B_1\omega_1}{J_1} - \frac{R_u\omega_u\omega_1}{L_2}\right) \\
\mathcal{A}_3(x) = & \frac{R_1}{J_1}(T_1 - T_2) - \frac{B_1}{J_1}\omega_1
\end{aligned}$$

and the matrix $\mathcal{B}(x)$

$$\mathcal{B}(x) = \begin{bmatrix} \frac{R_1}{L_1 J_1}(T_2 - EA) & \frac{R_w}{L_1 J_w}(EA - T_1) & 0 \\ EAR_1/(L_2 J_1) & 0 & -\frac{R_u}{L_2 J_u}(T_2 + EA) \\ 1/J_1 & 0 & 0 \end{bmatrix}.$$

The dynamic feedback which transforms the flat dynamics into the trivial dynamics is found by equating 6.1 with 6.2 which gives

$$u = \mathcal{B}^{-1}(x)(\nu - \mathcal{A}(x)).$$

This feedback is defined only if the matrix $\mathcal{B}$ is invertible. The inverse matrix

$$\mathcal{B}^{-1}(x) = \begin{bmatrix} 0 & 0 & J_1 \\ \frac{L_1 J_w}{R_w(EA - T_1)} & 0 & \frac{R_1 J_w(EA - T_2)}{R_w(EA - T_1)} \\ 0 & \frac{-L_2 J_u}{R_u(T_2 + EA)} & \frac{EAR_1 J_u}{R_u(T_2 + EA)} \end{bmatrix}$$

exists if $T_1 \neq EA$. The system remains open-loop because of the free control input ν. The purpose of this feedback is to linearize the system dynamics with respect to the flat output and hence linearize the dynamics for the error, $e = \mu - \mu^*$, between this output and its nominal trajectory developed in Chapter 5. The error dynamics with the substitution of the trivial system are

$$\begin{aligned}
\ddot{e}_1 &= \nu_1 - \ddot{\mu}_1^* \\
\ddot{e}_2 &= \nu_2 - \ddot{\mu}_2^* \\
\dot{e}_3 &= \nu_3 - \dot{\mu}_3^*.
\end{aligned}$$

This can be made exponentially stable at the origin with the feedback control

$$\begin{aligned}
\nu_1 &= \ddot{\mu}_1^* - k_{1,1}(\dot{x}_1 - \dot{\mu}_1^*) - k_{1,0}(x_1 - \mu_1^*) \\
\nu_2 &= \ddot{\mu}_2^* - k_{2,1}(\dot{x}_2 - \dot{\mu}_2^*) - k_{2,0}(x_2 - \mu_2^*) \\
\nu_3 &= \dot{\mu}_3^* - k_{3,0}(x_3 - \mu_3^*)
\end{aligned}$$

if the real valued gains $k_{i,j}$ are chosen such that the polynomials

$$\begin{aligned}
P_1(s) &= s^2 + k_{1,1}s + k_{1,0} \\
P_2(s) &= s^2 + k_{2,1}s + k_{2,0} \\
P_3(s) &= s + k_{3,0}
\end{aligned} \tag{6.3}$$

are Hurwitz. This is a simple and effective tracking procedure, however, the varying radii and moments of inertia cannot be ignored if the controller is to be adaptable to different materials and operating conditions. Therefore this procedure will be reconstructed in the framework of the general model.

6.2 Tracking with the General System Model

Over the chosen duration time, T, the radii of the unwinder and winder rollers may not significantly increase and so the constant radii model and therefore the controller defined in the previous section may prove useful. However, as one of the goals of this paper is to utilize a non-restrictive model which will widen the application of the controller and observer design, the controller will be adapted for the system model with varying radii. Multiple tracking methods have been designed for the web roller [4], [1], [11], [15], [14]. This section will outline the partial feedback linearization scheme developed for the general system in [1] with an emphasis on the consequences of the flatness-based approach to reference trajectory generation. Without a flat output characterization, full state feedback linearization is no longer guaranteed. However, this is not necessary as it suffices to linearize the states T_1, T_2 and ω_1. Once this is achieved, the error stabilization method can be employed just as in the flat case. In order to linearize the second derivative of the tensions, a change of variables $z = \phi(x)$ is performed

$$z = \begin{pmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \\ x_3 \\ x_6 \\ x_7 \end{pmatrix} = \begin{pmatrix} T_1 \\ T_2 \\ (-r_w \omega_w T_1 + R_1 \omega_1 T_2 + EA(r_w \omega_w - R_1 \omega_1))/L_1 \\ (-r_u \omega_u T_2 + EA(R_1 \omega_1 - r_u \omega_u))/L_2 \\ \omega_1 \\ r_w \\ r_u \end{pmatrix}$$

The inverse transform $x = \phi^{-1}(z)$ is linear in all state variables except the winder and unwinder angular velocities

$$\begin{aligned} \omega_w(z) &= \frac{R_1 z_5 (EA - z_2) + L_1 z_3}{z_6 (EA - z_1)} \\ \omega_u(z) &= \frac{EAR_1 z_5 - L_2 z_4}{z_7 (EA + z_2)}. \end{aligned}$$

The condition for diffeomorphism is the same as that for the Lie-Bäcklund isomorphism and for the existence of the endogenous dynamic feedback for the flat system controller: $T_1 \neq EA$. Therefore, the dynamics (4.5) can be expressed in terms of the transform state

$$\begin{aligned} \dot{z}_1 &= z_3 \\ \dot{z}_2 &= z_4 \\ \dot{z}_3 &= \alpha_1(z) + \beta_1^T(z) u \\ \dot{z}_4 &= \alpha_2(z) + \beta_2^T(z) u \\ \dot{z}_5 &= \alpha_3(z) + \beta_3^T(z) u \\ \dot{z}_6 &= \frac{H}{2\pi} \omega_w(z) \\ \dot{z}_7 &= \frac{-H}{2\pi} \omega_u(z) \end{aligned}$$

with

$$
\begin{aligned}
\alpha_1(z) =& \frac{EA - z_1}{L_1}\left(\frac{H(\omega_u(z))^2}{2\pi} - \frac{z_6}{J_w(z_6)}(z_6 z_1 + B_w\omega_w(z))\right) + \frac{R_1 z_5 z_4}{L_1} \\
&+ \frac{R_1(z_2 - EA)}{L_1 J_1}(R_1(z_1 - z_2) - B_1 z_5) - \frac{z_6 z_3 \omega_w(z)}{L_1} \\
\alpha_2(z) =& \frac{z_2 + EA}{L_2}\left(\frac{H(\omega_u(z))^2}{2\pi} - \frac{z_7}{J_u(z_7)}(z_7 z_2 - B_u\omega_u(z))\right) \\
&+ \frac{EAR_1}{L_2 J_1}(R_1(z_1 - z_2) - B_1 z_5) - \frac{z_7 z_4 \omega_u(z)}{L_2} \\
\alpha_3(z) =& (R_1(z_1 - z_2) - B_1 z_5)/J_1
\end{aligned}
$$

and with $\beta(z) \in \mathbb{R}^{3\times 3}$ so that $\beta^T(z) = [\beta_1(z), \beta_2(z), \beta_3(z)]$ with

$$
\beta(z) = \begin{bmatrix} \frac{R_1}{L_1 J_1}(z_2 - EA) & \frac{z_6}{L_1 J_w(z_6)}(EA - z_1) & 0 \\ EAR_1/(L_2 J_1) & 0 & -\frac{z_7}{L_2 J_u(z_7)}(z_2 + EA) \\ 1/J_1 & 0 & 0 \end{bmatrix}
$$

and the moments of inertia

$$
\begin{aligned}
J_w(z_6) &= J_{w0} + \frac{\pi}{2}\varrho W(z_6^4 - R_w^4) \\
J_u(z_7) &= J_{u0} + \frac{\pi}{2}\varrho W(z_7^4 - R_u^4).
\end{aligned}
$$

The states to be tracked are now z_1, z_2 and z_5 for which the reference trajectories are μ_1^*, μ_2^* and μ_3^*. By choosing the feedback control

$$
u = \beta^{-1}(z)(\nu - \alpha(z)) \tag{6.4}
$$

the desired partially linearized dynamics are obtained

$$
\begin{aligned}
\dot{z}_3 &= \nu_1 \\
\dot{z}_4 &= \nu_2 \\
\dot{z}_5 &= \nu_3.
\end{aligned}
$$

As with the flat case, the inverse matrix

$$
\beta^{-1}(x) = \begin{bmatrix} 0 & 0 & J_1 \\ \frac{L_1 J_w(z_6)}{z_6(EA - z_1)} & 0 & \frac{R_1 J_w(z_6)(EA - z_2)}{z_6(EA - z_1)} \\ 0 & \frac{-L_2 J_u(z_7)}{z_7(z_2 + EA)} & \frac{EAR_1 J_u(z_7)}{z_7(z_2 + EA)} \end{bmatrix}
$$

exists when $T_1 \neq EA$. This first order system is equivalent to the trivial flat system: the linearized dynamics (6.1), ignoring the dynamics of z_6 and z_7, hence the term *partial* linearization. The control inputs ν can therefore be chosen as in the case of the flat system

$$
\begin{aligned}
\nu_1 &= \ddot{\mu}_1^* - k_{1,1}(\dot{x}_1 - \dot{\mu}_1^*) - k_{1,0}(x_1 - \mu_1^*) \\
\nu_2 &= \ddot{\mu}_2^* - k_{2,1}(\dot{x}_2 - \dot{\mu}_2^*) - k_{2,0}(x_2 - \mu_2^*) \\
\nu_3 &= \dot{\mu}_3^* - k_{3,0}(x_3 - \mu_3^*)
\end{aligned}
$$

so that the error dynamics become exponentially stable at the origin for appropriately chosen gains. This controller resembles a *Proportional-Derivative* (PD) controller. A more general form of this type of controller for error stabilization is the *Proportional-Integral-Derivative* (PID) controller, in which an integrator for the error is added to the feedback which can improve the convergence quality of the controller. Refer to [20] for information on PID control. The error dynamics with the integrator become

$$
\begin{aligned}
\nu_1 &= \ddot{\mu}_1^* - k_{D,1}(\dot{x}_1 - \dot{\mu}_1^*) - k_{P,1}(x_1 - \mu_1^*) - k_{I,1}\int_0^t (x_1(s) - \mu_1^*(s))ds + w_1(t) \\
\nu_2 &= \ddot{\mu}_2^* - k_{D,2}(\dot{x}_2 - \dot{\mu}_2^*) - k_{P,1}(x_2 - \mu_2^*) - k_{I,2}\int_0^t (x_2(s) - \mu_2^*(s))ds + w_2(t) \\
\nu_3 &= \dot{\mu}_3^* - k_{P,3}(x_3 - \mu_3^*) - k_{I,3}\int_0^t (x_3(s) - \mu_3^*(s))ds + w_3(t)
\end{aligned}
\tag{6.5}
$$

where the noise terms $w_j(t)$ have been added to represent disturbances and uncertainty. For the tracker to converge, the characteristic polynomials (6.3) must be Hurwitz. With the integral term, the capability of convergence is improved.

Remark Although it will be referred to as such, the controller design presented is not strictly a PID controller. The control design closely resembles a PID controller applied to the error dynamics after partial-state feedback linearization.

Various *gain-scheduling* or gain tuning methods have been derived for designing optimal values for PID gains. The flatness-based reference trajectories, however, are tracked reliably by the controller (6.5) without such gain selection methods or software tools. The procedure was simulated to track the step inputs (5.1), piecewise linear inputs (5.2) and the polynomial inputs designed with flatness (5.4). The following simulation parameters were provided by Dr. Choi [4]:

Table 6.1: Simulation Parameters

Parameter	Value	Units
Nip Roller Radius, R_1	0.02535	m
Initial Winder Radius, R_w	0.0483	m
Initial Unwinder Radius, R_u	0.1455	m
Nip Moment of Inertia, J_1	1.95×10^{-5}	kg-m^2
Initial Winder Moment of Inertia, J_{w0}	6.83×10^{-4}	kg-m^2
Initial Unwinder Moment of Inertia, J_{u0}	5.91×10^{-2}	kg-m^2
Length of Span 1, L_1	1.490	m
Length of Span 2, L_2	1.335	m
Coefficient of Viscous Friction on Nip, B_1	2.533×10^{-5}	kg-m^2/s
Coefficient of Viscous Friction on Winder, B_w	2.533×10^{-5}	kg-m^2/s
Coefficient of Viscous Friction on Unwinder, B_u	2.533×10^{-5}	kg-m^2/s
Thickness, H	1×10^{-4}	m
Width, W	0.12	m
Cross-Sectional Area, A	1.2×10^{-5}	m^2
Young's Modulus, E	2.7×10^{9}	N/m^2
Density, ϱ	700	kg/m^3

The initial choice for the PID gains is: $k_{P,1} = k_{P,2} = 30$, $k_{P,3} = 1$, $k_{D,1} = k_{D,2} = 3$, $k_{I,1} = k_{I,2} = 6$, $k_{I,3} = 0$. These were derived by modifying the gains suggested by the *Ziegler-Nichols* method [23]. The simulations were run in MatLab using the *ode45* differential equation solver, with a random number generator simulating noise in the error dynamics. The results of the simulation for the three references are shown in Figures 6.1, 6.2 and 6.3.

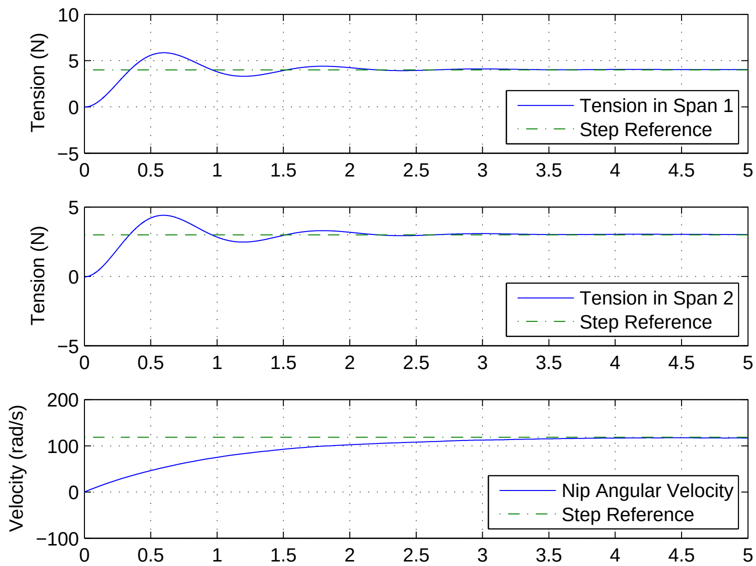

Figure 6.1: System response to tracking step input (5.1) without gain tuning.

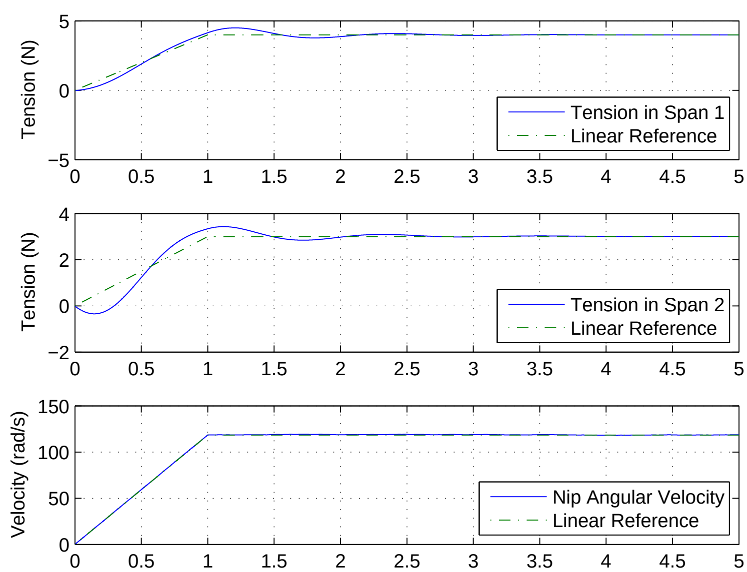

Figure 6.2: System response to tracking piecewise linear input (5.2) without gain tuning.

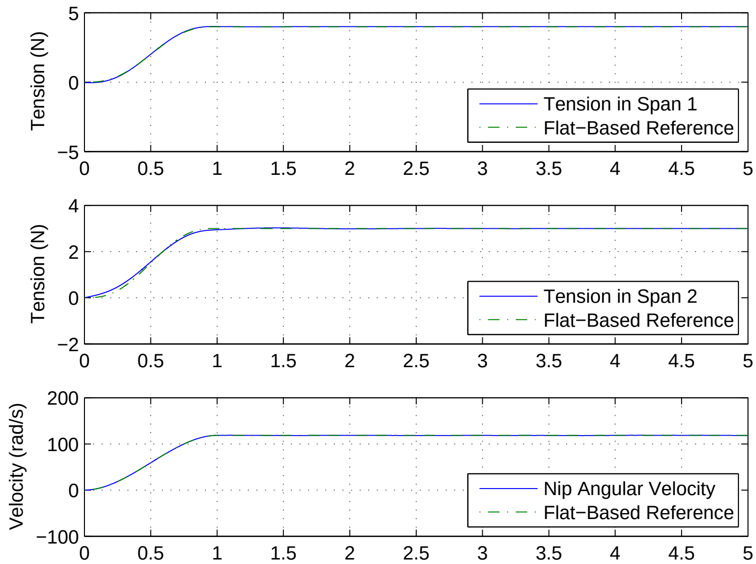

Figure 6.3: System response to tracking flatness-based input (5.4) without gain tuning.

As expected, the polynomial trajectories are tracked much better than the other trajectories and the desired final state, T_1^*, T_2^*, ω_1^*, is reached more quickly. It is uncertain whether the PID controller can be tuned to track the step or piece-wise linear reference trajectories to result in a comparable rise time and settling time without saturating or exceeding the input constraints and/or acquiring a heightened energy requirement. However, by adjusting the feedback gains, the response to the step input can be made to match the rise time of the flatness-based trajectories. This is shown in Figure 6.4. The response with these adjusted gains for the other references is also shown in Figures 6.5 and 6.6. This simulation was run with the PID gains: $k_{P,1} = k_{P,2} = 48$, $k_{P,3} = 5$, $k_{D,1} = k_{D,2} = 15$, $k_{I,1} = k_{I,2} = 6$, $k_{I,3} = 0$.

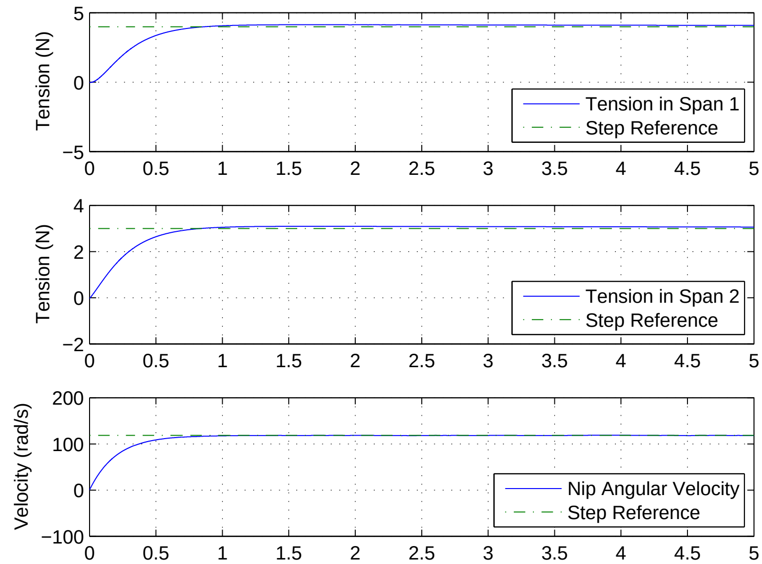

Figure 6.4: Step input response with adjusted feedback gains.

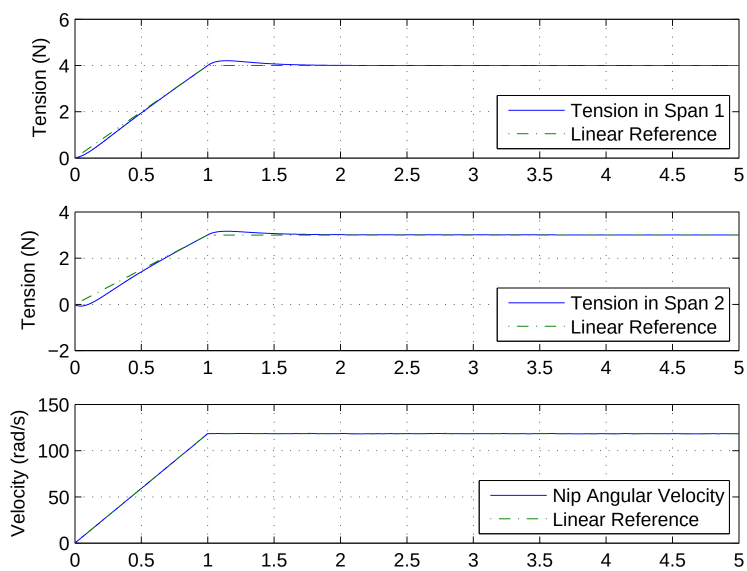

Figure 6.5: Linear input response with adjusted feedback gains.

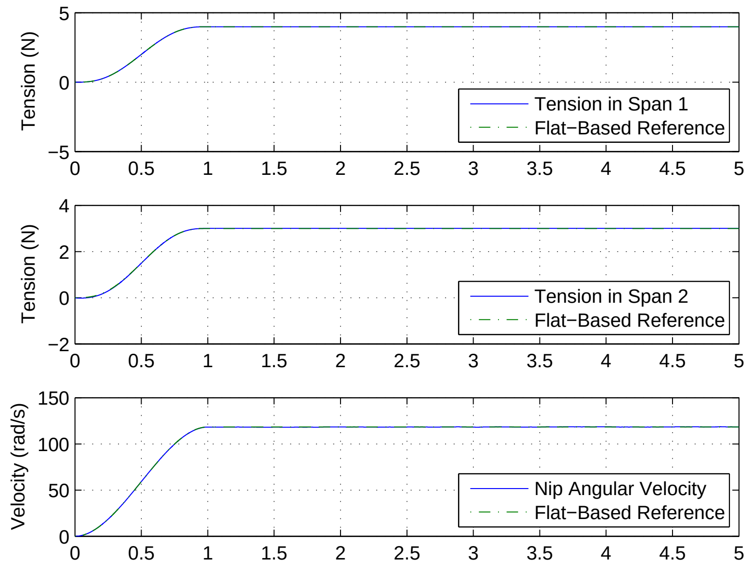

Figure 6.6: Flat-based input response with adjusted feedback gains.

The corresponding torque inputs for each of these tracking procedures are shown in Figures 6.7, 6.8 and 6.9, respectively. The reference torques (dashed lines) are computed with the Lie-Bäcklund isomorphism with the respective references as inputs to (4.12), (4.14), and (4.13).

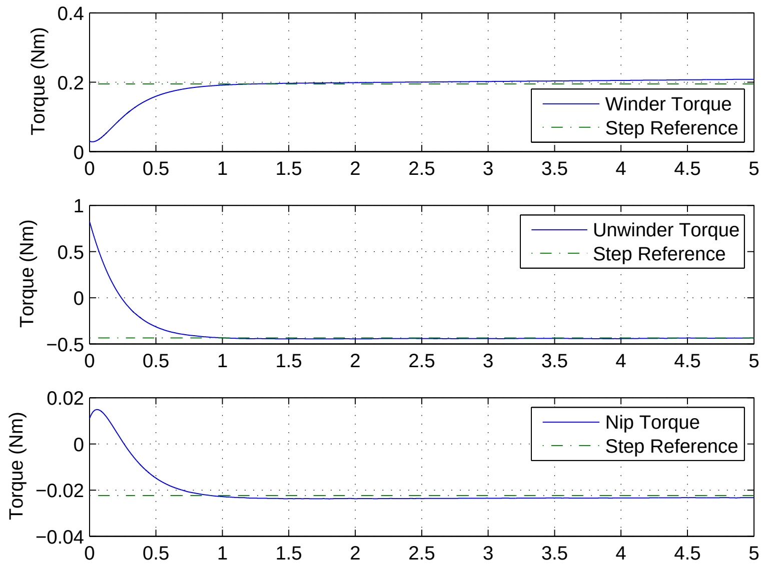

Figure 6.7: Torque response to step input tracking.

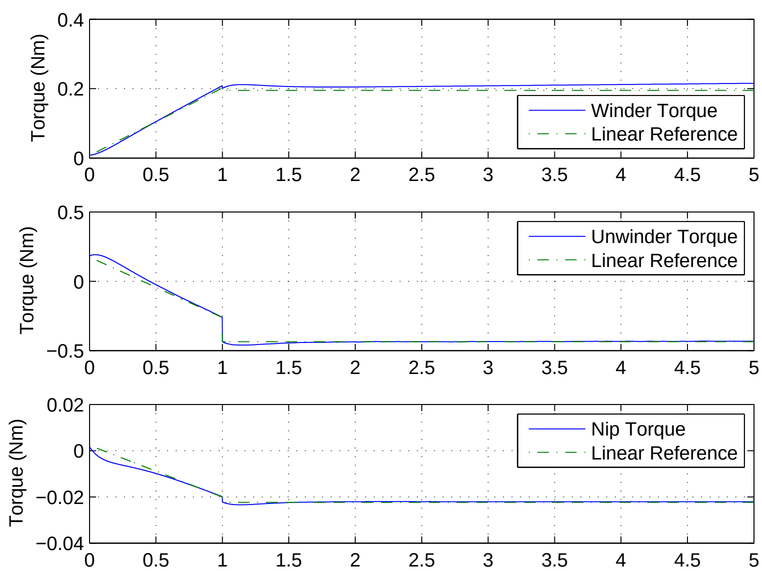

Figure 6.8: Torque response to linear input tracking.

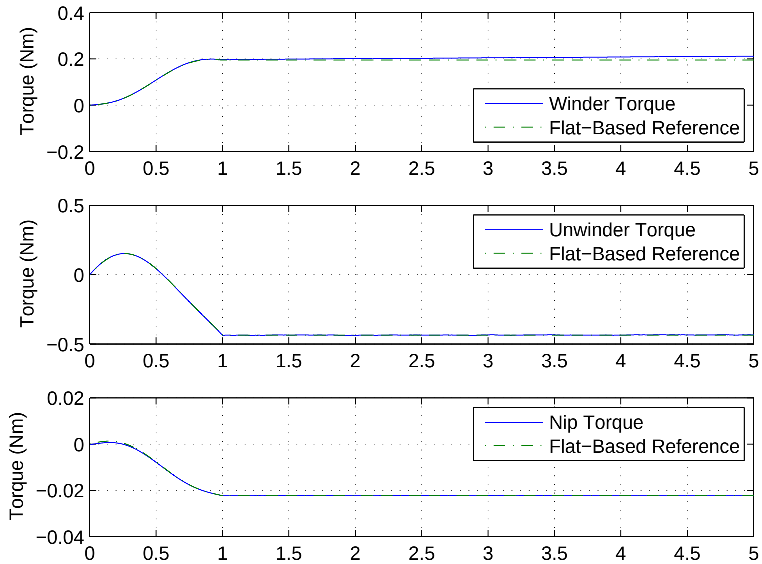

Figure 6.9: Torque response to flat-based input tracking.

The input response for the linear (ramp) trajectory contains a near discontinuity at the end of the duration $t = t_f$. This would require a large energy investment and would most likely violate the input constraints design in Section 5.2. The input response for the step input reaches the operating position before that of the flatness-based tracking procedure although it isn't clear whether the input constraints are violated. The benefit of the flatness-based design is that the nominal trajectories represent a calculated limit that the system can and will track with high enough gains. To accentuate this point a simulation was run with high PID gains: $k_{P,1} = k_{P,2} = 3000$, $k_{P,3} = 10$, $k_{D,1} = k_{D,2} = 150$, $k_{I,1} = k_{I,2} = 600$, $k_{I,3} = 0$.

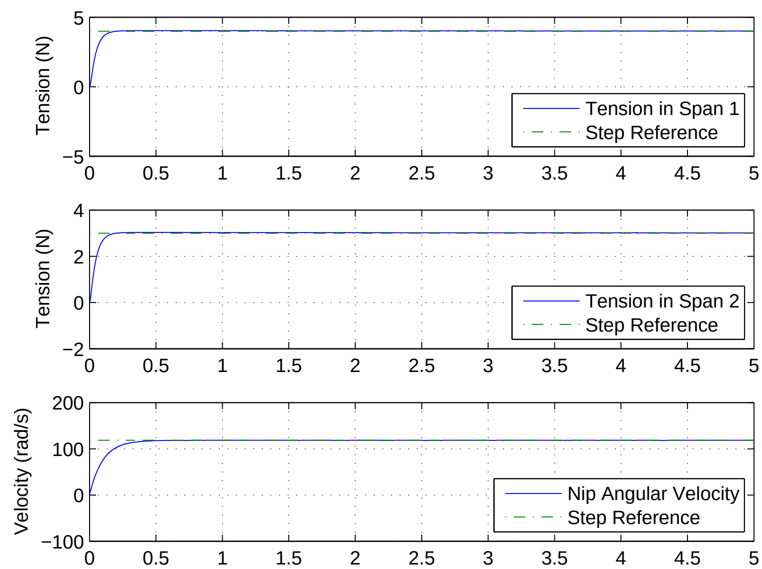

Figure 6.10: Step input response with high feedback gains.

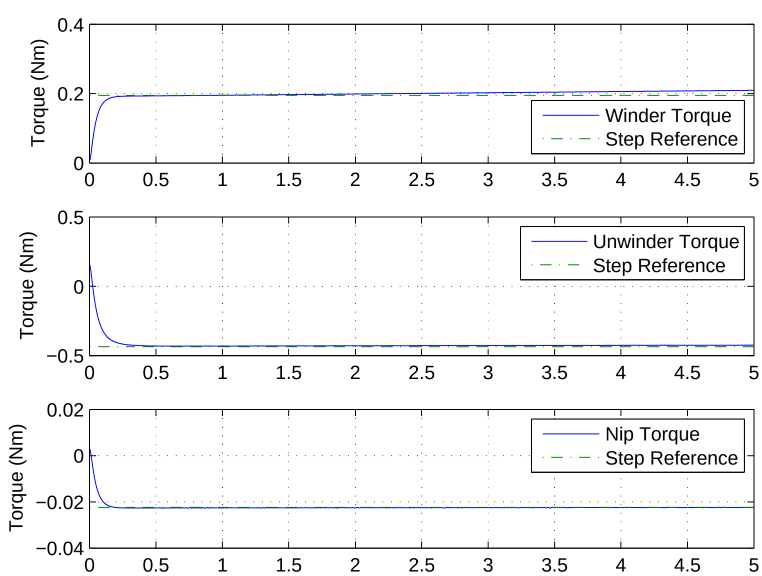

Figure 6.11: Torque response to step input tracking with high gains.

The torque inputs in this scenario most likely violate the constraints and would require a large energy investment. The jump discontinuity observed in Figure 6.5 for the torque response to the linear reference worsens with these high gains and the flat-based response is closer to the reference, as expected.

The partial feedback linearization scheme with a PID controller have proven to be effective at tracking the the flatness-based reference trajectories. The motion planning procedure produced references that were tracked without any significant gain tuning as the continuous nominal torques were easily obtained by the tracker.

Chapter 7

Observer Design

The controller design implemented to track the web tension and nip velocity assumes full state knowledge. Current values of the roller radii, velocity and web tension are used to construct the necessary torque inputs. While the angular velocity and radius of a roller may be measured with sufficient accuracy and with minor cost, tension in a material is often difficult to measure efficiently. Load cells and Dancer rollers are two types of mechanical apparatuses than can be attached to a web roller. Both can provide accurate measurements, however the cost of installation and maintenance, which includes adjustments to accommodate different operating conditions, can be significant [7]. State observers are economic, easy to install and can easily be adjusted for changes in operating conditions, provided the assumptions made in developing the system model in Chapter 4 remain valid. A state observer approximates the unmeasured states which are transmitted to the controller in a manner that maintains stability and accurate tracking capability of the closed-loop system. To this end, a nonlinear observer is designed to approximate the web tensions. First, this problem's feasibility is verified by investigating the observability properties of the system (4.5). Tension observers have been investigated for the web roller [19], [18], [1], [17], [11]. Typically, however, only constant observer gains are offered with weak notions of error stability corresponding to pole placement theory for autonomous system. A variable gain observer is needed with a strong sense of stability.

7.1 Observer Form

For the roller dynamics (4.5), with the observability restriction, the vector field f is both control-affine and affine in the unmeasured states, and the functional h is linear. The dynamics can be written in the form:

$$
\begin{aligned}
\dot{x}(t) &= A(y(t))x(t) + B(y(t))u(t) \\
y(t) &= Cx(t)
\end{aligned}
\tag{7.1}
$$

with

$$
A(y(t)) = \begin{bmatrix}
\frac{-r_w\omega_w}{L_1} & \frac{\omega_1 R_1}{L_1} & -\frac{EAR_1}{L_1} & \frac{r_w EA}{L_1} & 0 & 0 & 0 \\
0 & -\frac{r_u\omega_u}{L_2} & \frac{EAR_1}{L_2} & 0 & -\frac{r_u EA}{L_2} & 0 & 0 \\
\frac{R_1}{J_1} & -\frac{R_1}{J_1} & -\frac{B_1}{J_1} & 0 & 0 & 0 & 0 \\
-\frac{r_w}{J_w(r_w)} & 0 & 0 & -\frac{B_w}{J_w(r_w)} & 0 & 0 & 0 \\
0 & r_u/J_u(r_u) & 0 & 0 & -\frac{B_u}{J_u(r_u)} & 0 & 0 \\
0 & 0 & 0 & \frac{h}{2\pi} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{h}{2\pi} & 0 & 0
\end{bmatrix}
$$

$$
B(y(t)) = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
1/J_1 & 0 & 0 \\
0 & 1/J_w(r_w) & 0 \\
0 & 0 & 1/J_u(r_u) \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad
C = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

Alternatively, the drift matrix $A(y(t))$ may be defined as

$$A(y(t)) = \begin{bmatrix} \dfrac{-r_w \omega_w}{L_1} & \dfrac{\omega_1 R_1}{L_1} & -\dfrac{EAR_1}{L_1} & 0 & 0 & \dfrac{\omega_w EA}{L_1} & 0 \\ 0 & -\dfrac{r_u \omega_u}{L_2} & \dfrac{EAR_1}{L_2} & 0 & 0 & 0 & -\dfrac{\omega_u EA}{L_2} \\ \dfrac{R_1}{J_1} & -\dfrac{R_1}{J_1} & -\dfrac{B_1}{J_1} & 0 & 0 & 0 & 0 \\ -\dfrac{r_w}{J_w(r_w)} & 0 & 0 & -\dfrac{B_w}{J_w(r_w)} & 0 & 0 & 0 \\ 0 & r_u/J_u(r_u) & 0 & 0 & -\dfrac{B_u}{J_u(r_u)} & 0 & 0 \\ 0 & 0 & 0 & \dfrac{h}{2\pi} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\dfrac{h}{2\pi} & 0 & 0 \end{bmatrix}$$

while maintaining the the structure (7.1). Under this construction, an observer of the following form can be implemented

$$\dot{\hat{x}}(t) = A(y(t))\hat{x}(t) + B(y(t))u(t) - K(t)(C\hat{x}(t) - y(t)).$$

The error dynamics for this observer are

$$\dot{\varepsilon}(t) = (A(y(t)) - K(t)C)\varepsilon(t)$$

where $\varepsilon(t) = \hat{x}(t) - x(t)$. Appropriate choice of the observer gains $K(t)$ would be those that result in the stability of the equilibrium $\varepsilon = 0$. This is difficult to determine since the output $y(t) = Cx(t)$ and the error are not independent and therefore the dynamics are nonlinear and time-varying. Linearization of error dynamics is often used, however this typically only gives local convergence. Constant gain will normally only be effective if the system satisfies certain observability restrictions like universal observability. The focus of this chapter is to define restrictions on the observer gain such the unmeasured states can be approximated accurately and so that the controller developed in Chapter 6 can still effectively track the flatness-based trajectories.

7.2 Kalman-Related Observer

Linearization methods and High Gain observers that ignore nonlinearities or force them to zero often don't have good convergence qualities for tracking non-periodic reference trajectories. An observer is needed that will accurately approximate the web tension as they are tracked along the flatness-based trajectories. However, to approach this problem, an open-loop scheme is considered, meaning that the controller will be ignored and the input torques are assumed free.

In Section 3.2.1 an exponentially convergent Kalman-related observer for nonlinear systems that are state-affine up to additive output nonlinearity was presented. The web dynamics fall into a subset of the class of nonlinear systems (3.2) in that they are also control-affine. The web roller therefore admit the following observer.

$$\dot{\hat{x}}(t) = A(y(t))\hat{x}(t) + B(y(t))u(t) - K(t)(C\hat{x}(t) - y(t)).$$

The gain, $K(t) \in \mathbb{R}^{n\times n}$, is given by

$$K(t) = P(t)C^T W^{-1}$$

where the matrix $P(t) \in \mathbb{R}^{n\times n}$ satisfies the Riccati equation

$$\begin{aligned} &\dot{P}(t) = P(t)A^T(y(t)) + A(y(t))P(t) - P(t)C^T W^{-1} CP(t) + Q + \delta P(t) \\ &P(0) = P^T(0) > 0 \end{aligned}$$

with

$$W = W^T > 0$$

and

$$\delta > 2\,\|A(y(t))\| \quad \text{or} \quad Q = Q^T > 0.$$

Alternatively, the gain may be given by

$$K(t) = S^{-1}(t)C^T W^{-1}$$

where the matrix $S(t) \in \mathbb{R}^{n\times n}$ satisfies the Riccati equation

$$\begin{aligned} &\dot{S}(t) = -A^T(y(t))S(t) - S(t)A(y(t)) + C^T W^{-1} C - S(t)QS(t) - \delta S(t) \\ &S(0) = S^T(0) > 0 \end{aligned}$$

As mentioned, the error dynamics are non-autonomous and therefore the stability method of pole placement, which was employed to stabilize the tracker error dynamics and is also commonly used for tension observers, will not guarantee good convergence. For non-autonomous systems stability is harder to prove.

Theorem 7.2.1 (Lyapunov Stability [13])**.** *Consider the general nonautonomous system*

$$\dot{x}(t) = f(t, x(t))$$

with equilibrium $x = 0$. If there exists a continuously differentiable Lyapunov function $V(t,x)$, and positive integers c_1, c_2 and c_3 such that

$$c_1 \|x\|^2 \le V(t,x) \le c_2 \|x\|^2$$

and

$$\dot{V}(t,x) \le -c_3 \|x\|^2$$

then the equilibrium is globally exponentially stable.

With the Lyapunov tool, convergence of the Kalman-related observer can be shown.

Proposition 7.2.2. *The Kalman Observer is an exponential observer for system* (2.1) *if the outputs are regularly persistent.*

Sketch of Proof *See [6], [10] and [8] for the detailed elements of the proof.*

Choosing a Lyapunov candidate

$$V(t,\varepsilon(t)) = \langle S(t)\varepsilon(t), \varepsilon(t)\rangle$$

the symmetric positive definite property

$$c_1 \|x\|^2 \le V(t,x) \le c_2 \|x\|^2$$

is derived from the regular persistency of the outputs. For the transition matrix

$$\frac{\partial \Phi(s,t)}{\partial s} = A(y(t))\Phi(s,t), \quad \Phi(t,t) = I$$

the solution of the Riccati equation yields

$$S(t) = e^{-\delta t}\Phi^T(0,t)S(0)\Phi(0,t) + \int_0^t e^{-\delta(t-s)}\Phi^T(s,t)C^T C\Phi(s,t)ds.$$

Applying regular persistency gives the lower bound on the Lyapunov function and the upper bound can be established using norm inequalities. Secondly, by taking the time derivative

$$\begin{aligned}
\dot{V}(t,\varepsilon(t)) =& \langle \dot{S}(t)\varepsilon(t), \varepsilon(t)\rangle + \langle S(t)\dot{\varepsilon}(t), \varepsilon(t)\rangle + \langle S(t)\varepsilon(t), \dot{\varepsilon}(t)\rangle \\
=& \langle(-A^T(y(t))S(t) - S(t)A(y(t)) + C^T W^{-1} C \\
& \quad - S(t)QS(t) - \delta S(t))\varepsilon(t), \varepsilon(t)\rangle \\
& + \langle S(t)(A(y(t)) - S^{-1}(t)C^T W^{-1} C)\varepsilon(t), \varepsilon(t)\rangle \\
& + \langle S(t)\varepsilon(t), (A(y(t)) - S^{-1}(t)C^T W^{-1} C)\varepsilon(t)\rangle \\
=& - \langle S(t)QS(t)\varepsilon(t), \varepsilon(t)\rangle - \langle \delta S(t)\varepsilon(t), \varepsilon(t)\rangle - \langle C^T W^{-1} C\varepsilon(t), \varepsilon(t)\rangle
\end{aligned}$$

the property

$$\dot{V}(t,x) \leq -c_3 \|x\|^2$$

is given by the symmetry and positive definiteness of Q*,* W *and* $S(t)$*. The exponential convergence of the observer is tunable by the constant* δ *or the minimum eigenvalues of the gain matrices.*

Using the Lie-Bäcklund isomorphism with the flat-based references, a base-line value for the gain δ is established.

$$\delta > 2\|A(C\varphi_0(\mu,\dot{\mu}), R_w, R_u)\|$$

The remainder the gains $P(0)$, Q and W are chosen as diagonal matrices with positive entries, with heavier weighting on the T_1 and T_2 terms. Without changing the PID controller gains, the closed loop system maintains accurate tracking capabilities for the flat-based reference as observed in Figure 7.2. The gains used are $k_{P,1} = k_{P,2} = 48$, $k_{P,3} = 5$, $k_{D,1} = k_{D,2} = 15$, $k_{I,1} = k_{I,2} = 6$, $k_{I,3} = 0$.

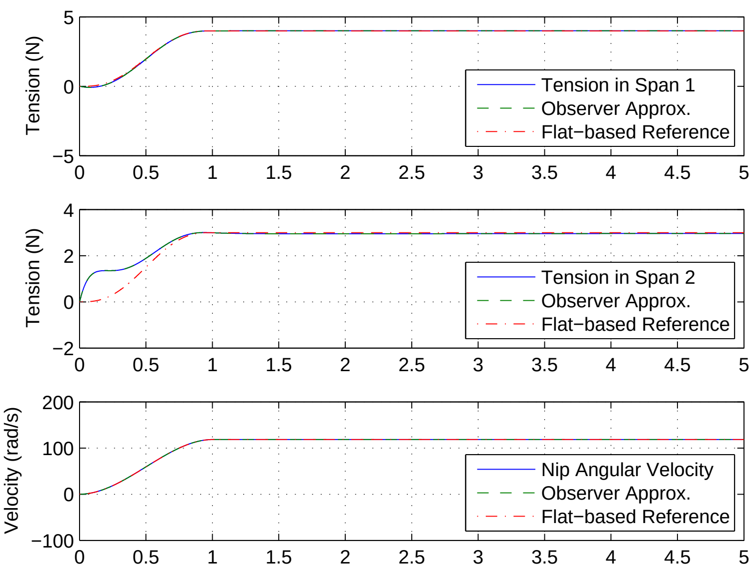

Figure 7.1: Flat-based input response to observer without altering PID gains.

The convergence of the observer error in Figures 7.2 is much faster than that of the tracker. This is resolved by increasing the PID gains as shown in Figure 7.2 to the high gain selection $k_{P,1} = k_{P,2} = 3000$, $k_{P,3} = 10$, $k_{D,1} = k_{D,2} = 150$, $k_{I,1} = k_{I,2} = 600$, $k_{I,3} = 0$.

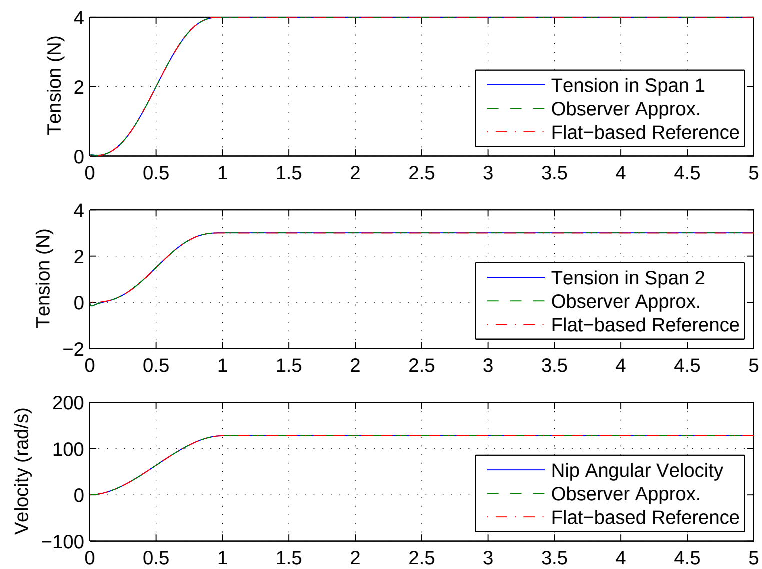

Figure 7.2: Flat-based input response to observer with high PID gains.

Similar to a high gain approach, little tuning is needed beyond providing gains that are high enough to ensure the necessary convergence rate for any application. For the step input, as with the full-state observance case, careful tuning of the PID gains is needed to produce the result in Figure 7.3

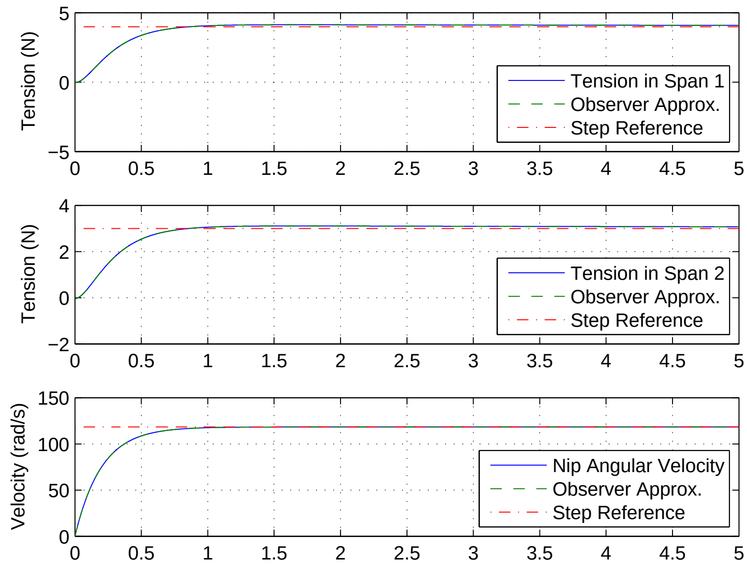

Figure 7.3: Step input response with tuned PID and observer gains.

The Kalman observer connected with the controller exhibits a high convergence rate toward the actual state variable, although it seams to weaken the convergence of the actual state toward the reference. This is solved by increasing the controller gains. Therefore with sufficiently high observer and tracker gains, the closed-loop system should be effective in regulating the web tension and speed for the two-span web roller with varying applications.

Chapter 8

Conclusions

A dynamical model for the two-span web roller was derived. Under a fixed radii assumption for the winder and unwinder, the system dynamics were shown to be flat. Reference trajectories for the tensions of the two spans of the web and the angular velocity of the nip were developed using flatness to guide the system to a prescribed operating position. The rise time of these polynomial trajectories was related explicitly to the maximum derivative of the nominal torque input by the Lie-Bäcklund isomorphism so that input constraints could be satisfied. Trajectories were successfully tracked by the non-flat system with varying radii using a partial feedback linearization scheme and a constant gain PID-related controller which produced exponential convergence of the web tension and nip velocity to their reference trajectories. The flatness-based trajectories were tracked more reliably than the step and linear inputs without gain tuning. Using the bilinear form of the dynamics, up to additive output nonlinearity, a Kalman-related observer was designed to estimate the web tension. This produced exponential convergence of the estimated state variables to their actual values. The time-dependent observer gains, like the controller gains were given sufficiently high values, without requiring gain scheduling methods, and the resulting closed-loop controller-observer system performed well with fast convergence toward the reference trajectory. The steady-state values for the web tension and velocity were reached reliably within the prescribed rise-time.

Without an appropriate platform for comparison, it is difficult to assess the performance of the control design developed in this thesis compared with other previously tested designed. However, both the tracker and observer designs can easily be implemented, and if the dynamical model employed is an accurate representation of the web roller system, then the proposed design should produce a high convergence rate and should not exhibit the characteristic oscillatory behaviour of conventional controller and observer methods. Motion planning has been shown to improve over current designs. Simple gain selection for the feedback constructions facilitates accurate tracking along a path which, because of its level of differentiability, results in less over shoot, and settling time, and input constraints in the system are not saturated.

8.1 Future Work

Exponential stability of the controller was proven and references were given for the proof of exponential stability of the observer. It would be reasonable to hypothesize that the stability of the closed-loop system is exponential, but it has yet to be proven.

Polynomial trajectories were chosen for the flatness-based references because they could easily forced to follow the endpoint and input constraints. Other types of trajectories could be investigated which may be tracked more effectively, or to possibly reduce the maximum rise time needed to satisfy the input constraints.

The Lie-Bäcklund isomorphism could be used for optimization of the reference trajectories. Increasing the dimension of the flat outputs would provide extra degrees of freedom. An optimization procedure could then be carried out with respect to the extra coefficients in the polynomials by expressing the cost function in terms of the flat output via the isomorphism.

Applying gain tuning methods, such as those referenced in the literature, for selecting optimal PID controller gains may further improve the convergence quality and further reduce the amount of manual tuning.

References

[1] N.R. Abjadi, J. Soltani, and J. Askari. Nonlinear sliding-mode control of a multi-motor web-winding system without tension sensor. *IET Control Theory and Applications*, 3(4):419–427, 2009. 2, 3, 43, 46, 59

[2] G. Bornard, N. Couenne, and F. Celle. Regularly persistent observer for bilinear systems. In *New Trends in Nonlinear Control Theory*, volume 122, pages 130–140, Berlin, 1988. Springer-Verlag. 16, 17

[3] G. Brandenburg. New mathematical models for web tension and register error. In *Proceedings of the third International IFAC Conference on Instrumentation and Automation in the Paper, Rubber and Plastics Industries*, pages 411–438, Brussels, Belgium, 1976. 2, 23

[4] K. H. Choi, T. T. Thanh, B. S. Yang, and D. S. Kim. A control system design of automatically tuning controller for roll to roll web system using the modified genetic algorithm. In *Proceedings of the 7th Asian Control Conference*, pages 1691–1696, Hong Kong, China, August 2009. 2, 32, 46, 49

[5] G. Besançon, editor. *Nonlinear Observers and Applications*. Lecture Notes in Control and Information Sciences. Springer-Verlag, Berlin, 2007. 3, 15, 17

[6] G. Besançon, G. Bornard, and H. Hammouri. Observer synthesis for a class of nonlinear control systems. *European Journal of Control*, 3(1):176–193, 1996. 17, 18, 19, 63

[7] N. A. Ebler, R. Arnason, G. Michaelis, and N. D'Sa. Tension control: dancer rolls or load cells. In *Proceedings of Pulp and Paper Industry Technical Conference*, pages 61–72, Portland, OR, 1992. 2, 59

[8] J.P. Gauthier and A.K. Kupka. *Deterministic Observation Theory and Applications*. Cambridge University Press, 1997. 17, 19, 63

[9] H. Hammouri and F. Celle. Some results about nonlinear systems equivalence for the observer synthesis. In *New Trends in Systems Theory*, volume 7 of *Progress in Systems and Control Theory*, pages 332–339. Birkhäuser, 1991. 18

[10] H. Hammouri and J. De Leon Morales. Observer synthesis for state-affinne systems. In *Proceedings of the 29th IEEE Conf. on Decision and Control*, Honolulu, HI, December 1990. 16, 17, 19, 63

[11] I. C. Hwang and S. B. Kim. Feedback linearization control of the looper system in hot strip mills. *KSME International Journal*, 17(11):1608–1615, 2003. 2, 46, 59

[12] R. E. Kalman and R. S. Bucy. New results in linear filtering and prediction theory. *ASME Transactions, Journal of Basic Engineering*, 83:95–108, 1961. 17

[13] H. K. Khalil. *Nonlinear Systems.* Prentice Hall, Upper Saddle River, NJ, 2^{nd} edition, 1996. 62

[14] D. Knittel, E. Laroche, D. Gigan, and H. Koç. Tension control for winding systems with two-degrees-of-freedom H_∞ controllers. *IEEE Transactions on Industry Applications*, 9(1):113 – 120, 2003. 2, 46

[15] H. Koç, D. Knittel, M. de Mathelin, and G. Abba. Modeling and robust control of winding systems for elastic webs. *IEEE Transactions on Control Systems Technology*, 10(2):197–208, 2002. 2, 46

[16] J. Lévine. *Analysis and Control of Nonlinear Systems: A Flatness-based Approach.* Mathematical Engineering. Springer-Verlag, Berlin, 2009. 3, 7, 8, 10

[17] K. C. Lin, M. C. Tsai, and K. Y. Chen. Web tension control of a start-up process using observer techniques with friction and inertia compensation. In *27th Annual Conference of the IEEE Industrial Electronics Society*, volume 1, pages 529–534, Piscataway, NJ, 2001. 2, 59

[18] A. F. Lynch. *Nonlinear tension observers for web machines.* PhD thesis, University of Toronto, Canada, 1999. 2, 59

[19] A. F. Lynch, S. A. Bortoff, and K. Röbenack. Nonlinear observer synthesis and error linearization. *Automatica*, 40(9):1517–1524, 2004. 2, 23, 26, 59

[20] K. A. Morris. *Introduction to Feedback Control.* Harcourt/Academic Press, San Diego, CA, 2001. 49

[21] J. J. Shelton. Dynamics of web tension control with velocity or torque control. In *Proceedings of the 1986 American control conference*, pages 1423–1427, Seattle, WA, 1986. 2, 23, 26

[22] D. P. D. Whitworth and M. C. Harrison. Tension variations in pliable material in production machinery. *Applied Mathematical Modelling*, 7(3):189–196, 1983. 2, 23

[23] J. B. Ziegler and N. B. Nichols. Optimum settings for automatic controllers. *ASME Transactions*, 64:759–768, 1942. 50