A 4d Lorentzian Spin Foam Model With Timelike Surfaces

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

We construct a 4d Lorentzian spin foam model capable of describing both spacelike and timelike surfaces. To do so we use a coherent state approach inspired by the Riemannian FK model. Using the coherent state method we reproduce the results of the EPRL model for Euclidean tetrahedra and extend the model to include Lorentzian tetrahedra. The coherent states of spacelike/timelike triangles are found to correspond to elements of the discrete/continuous series of $SU(1,1)$. It is found that the area spectrum of both spacelike and timelike surfaces is quantized. A path integral for the quantum theory is defined as a product of vertex amplitudes. The states corresponding to timelike triangles are constructed in a basis diagonalised with respect to a noncompact generator. A derivation of the matrix elements of the generators of $SL(2,\mathbb{C})$ in this basis is provided.

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Dedication

To my grandparents:

Walter and Verna Bish

Bill and Stella Hnybida

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Chapter 1

Introduction

The General Theory of Relativity describes the dynamical relationship between matter and geometry. Spacetime is the field which dictates the kinematics of matter while the presence of matter causes the geometry of spacetime to curve. As best as can be observed, the spacetime field of our universe has four dimensions: three spatial and one temporal. It is therefore reasonable to represent spacetime as a smooth four dimensional manifold; the geometry is completely described by a metric while the dynamics is dictated by Einstein's equations. Since spacetime is a dynamical field there is no static background for which one can make reference to and thus only relational quantities have physical meaning. This is referred to as background independence and is a cornerstone of the classical theory.

How should one go about quantizing such a theory? Why should one want to quantize it in the first place? The answer to the latter is that the theory of General Relativity predicts singularities in the geometry of spacetime. Singularities are points in the manifold at which the curvature or other physical quantities diverge. For example such singularities exist at the center of black holes in which all the matter of a body is crushed to a single point. Another example of a singularity is the big bang from which it is believed that the universe itself expanded. Due to general theoretical arguments such as the singularity theorems [1] as well as a large amount of indirect observational evidence it is widely believed that black holes do exist in our universe and that the big bang did indeed happen. Therefore as the paradigm of General Relativity is pushed to its limits a complete physical description of these phenomena is needed more than ever.

Since these singularities are highly localized, it is believed that it is our negligence of Quantum Mechanics that is causing these divergences. As of yet, a satisfactory theory of quantum gravity has yet to be discovered. Some have even gone so far as to claim that General Relativity and Quantum Mechanics are fundamentally incompatible. One observation that might lead to such a claim is that Schrödinger's equation is formulated in terms of an external time whereas General Relativity is background independent. This is by no means a dead end, however, since there are other formulations of Quantum Mechanics which are generally covariant such as Feynman's path integral formulation.

Spin foam models attempt to define a quantum theory of gravity in this way as a sum over geometries. A naive attempt to construct such a theory might be to define the following path integral

$$Z = \int e^{iS[g_{\mu\nu}]} d\mu[g_{\mu\nu}], \tag{1.1}$$

where S is the Einstein-Hilbert action. In this way one might be able to calculate the transition amplitude between two 3d metrics. The main problem with this formulation is that a consistent definition of the measure $d\mu[g_{\mu\nu}]$ has not been found (except for the case of a perturbative expansion on a flat background [2]). This is due to the fact that if spacetime is truly continuous then there should be no restriction on the complexity of its geometry or topology. In this way one could imagine holes, handles, and ripples in spacetime ad infinitum. One solution to this problem is to treat spacetime as a discrete structure. In this way the number of possible geometries becomes much more manageable mathematically.

The assumption that space and time are quantized is motivated by a canonical approach to quantum gravity known as Loop Quantum Gravity [3]. Loop Quantum Gravity is a theory of quantized geometry based on holonomies and hence loops. These loop states form an algebra analogous to a Yang-Mills gauge theory and remarkably they provide a basis for a separable Hilbert space of a Quantum Field theory. Moreover, this Hilbert space is also spanned by a basis of states known as spin networks which admit the compelling physical interpretation as quantized spatial geometries.

These spin networks are simply graphs with links labeled by spins j_l and nodes labeled by intertwiners. Spin network states are eigenstates of the $SU(2)$ gauge invariant area operator $\mathbf{A}$ having the discrete spectrum [4]

$$\mathbf{A} = 8\pi\gamma\hbar G \sum_i \sqrt{j_i(j_i+1)}, \tag{1.2}$$

where γ is a free dimensionless parameter of the theory called the Immirzi parameter. Each eigenvalue of $\mathbf{A}$ is interpreted as the area of a surface which intersects Γ across n of its links labeled by the spins j_i for $i = 1, ..., n$. Notice that since the fundamental representation of $SU(2)$ is given by $j = 1/2$ this formula implies that there exists a minimal quantum of area.

The transition amplitude $\langle s'|s\rangle$ from one spin network s to another spin network s' can then be calculated as a sum over paths. The paths in this case are referred to as spin foams and consist of sequences of spin networks which are bounded by s and s'. A spin foam can be represented as a 2-complex which consists of vertices, edges, and faces where now the faces are labeled by spins and the edges are labeled by intertwiners.

The interpretation just described of a spin foam as a spin network history is derived from the canonical theory. One can also arrive at a similar interpretation of spin foams from a covariant starting point, that is, directly from a path integral. In the canonical approach a spin network is an arbitrary embedded graph which can have any valence or even knots. In the covariant approach however, for simplicity one restricts to the case of triangulations which are taken to be 4-valent. In this thesis we will work from the covariant direction by first constructing a classical theory of gravity on a discretized space and then quantizing the theory by specifying a path integral.

We construct a path integral of General Relativity indirectly by first quantizing a simpler theory known as BF theory and then imposing constraints on the quantum BF theory so that it agrees with General Relativity in the classical limit.[1] The main difference between most spin foam models is simply the way in which these simplicity constraints are imposed on the quantum BF theory.

One of the first models for imposing the simplicity constraints was proposed by Barrett and Crane [5]. The simplicity constraints were imposed as strong operator equations in the quantum theory and resulted in a simple and elegant theory. However, it was shown that in the semiclassical limit the graviton propagator lacked the correct degrees of freedom to describe gravity [6]. This is because the simplicity constraints form a set of second class constraints and it is known that imposing second class constraints strongly can result in the loss of physical degrees of freedom [7].

New spin foam models were later proposed in which the simplicity constraints were imposed weakly after quantization. First, models of 4d Euclidean gravity were proposed by Engle, Pereira, Rovelli, and Livine (EPRL) [8] as well as by Freidel and Krasnov (FK) [9]. The EPRL model imposes the simplicity constraints directly on the quantum operators using what is called the Master constraint while the FK model employs a semiclassical method using coherent states to impose the simplicity constraints on expectation values of the constraint equations. Although the two models were constructed by very different approaches, they were nevertheless shown to be closely related [10]. A Lorentzian version of the EPRL model was also defined in [8], however a Lorentzian version of the FK model was not.

A serious weakness of the EPRL model in Lorentzian signature is that it is only defined

[1] The name BF theory comes from the form of the Lagrangian of the theory which is composed of the wedge product of a two form B and a curvature two form F.

for spacelike surfaces. That is, the EPRL model is restricted to Lorentzian triangulations consisting of only spacelike triangles. Thus, in particular, one can only consider spacelike boundary hypersurfaces consisting of Euclidean tetrahedra. This is troublesome because, as argued by Oeckl [11], if one treats spacetime quantum mechanically then any quantum measurement will require the specification of not only the initial and final states on spacelike boundary hypersurfaces, but also the specification of the state on the timelike boundary surface connecting them.[2] A final motivation for the generalisation to timelike surfaces is the coupling of other fields in the spin foam model. For example, if one were to couple an electromagnetic field to a model with only spacelike surfaces then such a field would only have a description in terms of magnetic fields. Thus the generalisation of spin foam models to include timelike surfaces is imperative.

The reason the EPRL model is restricted to only spacelike surfaces is due to ambiguities in the Master Constraint method of imposing the simplicity constraints in Lorentzian signature. We will overcome this weakness by using a different method of imposing the simplicity constraints. We begin by defining a Lorentzian spin foam model in the spirit of the FK model by constructing a new set of coherent states for the the gauge group $SL(2,\mathbb{C})$. We then find that the coherent state method does not share the ambiguities of the Master Constraint method in Lorentzian signature and can thus be used to quantize timelike surfaces. Hence, we extend the EPRL model to general triangulations and general boundary states. This work has resulted in the following publications [12] and [13].

The thesis is organised as follows. In Chapter 2 a formulation of classical General Relativity is given in terms of the Hilbert-Palatini action. In Chapter 3 we give a derivation of the BF theory path integral and the EPRL method of imposing the simplicity constraints. The original work of this thesis is contained in Chapters 4 and 5. In Chapter 4 the EPRL model is rederived using the coherent state method and the EPRL model is then extended to the case of timelike surfaces. In Chapter 5 we propose a spin foam model which is applicable for both spacelike and timelike surfaces.

The construction of the coherent states for $SL(2,\mathbb{C})$ required considerable mathematical machinery, some of which was not contained in the literature. For instance, in order to derive the coherent states corresponding to timelike surfaces the action of the generators of $SL(2,\mathbb{C})$ decomposed into unitary irreducible representations (irreps) of $SU(1,1)$ and diagonalized with respect to a noncompact generator was needed and had to be derived. This had not been done before to our knowledge and is part of further work in the process of publication [14]. This original work and other information about $SL(2,\mathbb{C})$ is contained in the Appendices.

The Appendices are organised as follows. Appendix A contains standard information about the unitary irreducible representations of $SL(2,\mathbb{C})$ as can be found in [15]. In

[2]While it can be argued that a connected boundary hypersurface can also be constructed using only spacelike hypersurfaces, such a construction is impractical in general.

Appendix B we derive the matrix elements of $SL(2,\mathbb{C})$ in the continuous series of $SU(1,1)$ in a basis diagonalized with respect to the generator K^1. In Appendix C we list the matrix elements of generators of $SL(2,\mathbb{C})$ in the various bases used in the thesis. Finally, in Appendix D we give the explicit parameterization of various quotient spaces which are used to relate the coherent states in the spin foam model to the normal vectors of classical triangles.

Chapter 2

Classical General Relativity

The mathematical formalism of General Relativity is that of differential geometry which in its standard formulation is described by a metric and a connection. In the absence of torsion the connection is determined by the metric. We will assume the knowledge of this standard formulation of General Relativity in terms of a Lorentzian metric and the corresponding Levi-Civita connection and we will use the following notations and conventions.

2.1 Notations and Conventions

Let $\mathcal{M}$ be a smooth oriented Lorentzian manifold equipped with a metric g of the signature convention $(+,-,-,-)$. We will denote 4d Minkowski space by $\mathbb{M}^4$ which has the metric $\eta^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$ and 3d Minkowski space by $\mathbb{M}^3$ which has the metric $\eta^{\mu\nu} = \mathrm{diag}(1,-1,-1)$. Here spacetime indices will be denoted by Greek indices with the range $\mu = 0,1,2,3$. The metric is used to raise and lower indices and thus gives an isomorphism between vectors in the tangent space and one forms in the cotangent space.

Differential forms are antisymmetric tensor products of one forms. The wedge product of a p form and a q form is given by the antisymmetrized tensor product

$$(V \wedge U)_{\mu_1 \cdots \mu_{p+q}} = V_{[\mu_1 \cdots \mu_p} U_{\mu_{p+1} \cdots \mu_{p+q}]}, \tag{2.1}$$

where we have denoted the antisymmetrization of indices by square brackets. The antisymmetrization of indices is given by

$$A^{[ab]} \equiv A^{ab} - A^{ba}, \tag{2.2}$$

$$A^{[abc]} \equiv A^{[ab]c} + A^{[ca]b} + A^{[bc]a}, \tag{2.3}$$

and so on. The Hodge dual of a p form $V_{\mu_1\cdots\mu_p}$ is defined by

$$\star V = \frac{1}{p!}\epsilon^{\nu_1\cdots\nu_p}{}_{\mu_1\cdots\mu_{n-p}} V_{\nu_1\cdots\nu_p}, \tag{2.4}$$

where $\star V$ is an $n-p$ form. Here $\epsilon^{\alpha\beta\gamma\delta}$ is the totally antisymmetric symbol such that $\epsilon^{0123}=1$. Furthermore, we have that

$$\star\star V = (-1)^{p(n-p)-1} V. \tag{2.5}$$

Integration of a scalar function ϕ over $\mathcal{M}$ is given by

$$\int_{\mathcal{M}} \phi(x) = \int_{\mathbb{R}^4} d^4x\,\sqrt{|g|}\,\phi(x), \tag{2.6}$$

where $\sqrt{|g|}d^4x = \epsilon_{\mu_0\cdots\mu_3} dx^{\mu_1}\otimes\cdots\otimes dx^{\mu_3}$ is the volume form on $\mathcal{M}$ and $g=\det g_{\mu\nu}$. The integration over $\mathbb{R}^4$ is taken to be the Lebesgue measure on each of the coordinate charts of an oriented atlas. We choose units in which $\hbar = G = c = 1$ and for simplicity we will ignore dimensionless constants in the classical actions.

2.2 Tetrad Formalism

Let g be a metric on a manifold $\mathcal{M}$. On a chart of $\mathcal{M}$ with coordinates x^μ the metric can be given in the coordinate basis by

$$g = g^{\mu\nu}\,\partial_\mu\otimes\partial_\nu = g_{\mu\nu}\,dx^\mu\otimes dx^\nu. \tag{2.7}$$

Unless the metric is flat, the coordinate basis is not orthonormal. We can construct an orthonormal basis for a general nondegenerate metric by defining a set of basis vectors e^μ_I for $I=0,1,2,3$ called tetrads which satisfy

$$g^{\mu\nu} = e^\mu_I e^\nu_J \eta_{IJ}, \tag{2.8}$$

and

$$e^I_\mu e^\mu_J = \delta^I_J. \tag{2.9}$$

The one forms e^I_μ compose the inverted tetrad matrix and are referred to as the cotetrad. We note that the definition of a tetrad (or cotetrad) is equivalent to the definition of a metric. The capital Latin indices are referred to as internal indices and are raised/lowered by the η_{IJ} while the usual Greek letter spacetime indices are raised/lowered by $g_{\mu\nu}$. In terms of the metric the volume form is given by $e\, d^4x$ where

$$e = \det(e_{\alpha I}) = \frac{1}{4!}\epsilon^{\alpha\beta\gamma\delta}\epsilon^{IJKL}e_{\alpha I}e_{\beta J}e_{\gamma K}e_{\delta L}. \tag{2.10}$$

We can define a metric compatible covariant derivation by requiring that it annihilate e^I_μ as in

$$\nabla_\alpha e_{\beta I} \equiv \partial_\alpha e_{\beta I} - \Gamma^\gamma_{\alpha\beta}e_{\gamma I} + \omega_{\alpha I}{}^J e_{\beta J} = 0, \tag{2.11}$$

where $\Gamma^\gamma_{\alpha\beta}$ and $\omega_{\alpha I}{}^J$ are referred to as the affine and spin connections respectively.

2.3 Einstein-Hilbert Action

In this section we review the derivation of Einstein's equations from variations of the Einstein-Hilbert action using the tetrad notation [16]. The Einstein-Hilbert Lagrangian consists of simply the Ricci scalar which can be expressed as a function of the affine connection or the spin connection. If one assumes vanishing torsion then both the spin connection and the affine connection can be written in terms of the tetrad which is thus taken to be the only independent variable. The torsion tensor is given by the antisymmetric part of the affine connection therefore if we take

$$\Gamma^\gamma_{[\alpha\beta]} = 0, \tag{2.12}$$

then from Eq. (2.11) one obtains the unique solution

$$\Gamma^\epsilon_{\alpha\beta} = \frac{1}{2}g^{\epsilon\gamma}\left(\partial_\alpha g_{\epsilon\beta} + \partial_\beta g_{\epsilon\alpha} - \partial_\epsilon g_{\alpha\beta}\right). \tag{2.13}$$

We have written $\Gamma^\epsilon_{\alpha\beta}$ in terms of the metric for familiarity but this can expressed in terms of the tetrad by a simple substitution. Putting Eq. (2.13) into Eq. (2.11) one can also solve for the spin connection in terms of the tetrad as

$$\omega_\alpha^{IK} = \frac{1}{2} e^{\epsilon[I} \left(\partial_{[\alpha} e_{\epsilon]}^{K]} + e^{K]\beta} e_\alpha^L \partial_\beta e_{\epsilon L} \right). \tag{2.14}$$

We define the Riemann tensor in terms of the covariant derivative as follows

$$\nabla_{[\alpha} \nabla_{\beta]} v_\epsilon = R_{\alpha\beta\epsilon}{}^\mu v_\mu, \tag{2.15}$$

$$\nabla_{[\alpha} \nabla_{\beta]} v_I = R_{\alpha\beta I}{}^J v_J, \tag{2.16}$$

for all vectors $v_\epsilon = e_\epsilon^I v_I$. Geometrically the commutator of covariant derivatives represents the parallel transport of a vector around an infinitesimal rectangle and the Riemann tensor is the corresponding transformation tensor. Note that from these definitions it follows that

$$R_{\alpha\beta I}{}^J = R_{\alpha\beta\epsilon}{}^\mu e_I^\epsilon e_\mu^J. \tag{2.17}$$

Using Eqs. (2.15) or (2.16) the Riemann tensor can be given in terms of the affine or spin connection respectively by

$$R_{\alpha\beta}{}^\epsilon{}_\mu = \partial_{[\alpha} \Gamma^\epsilon_{\beta]\mu} + \Gamma^\epsilon_{[\alpha\rho} \Gamma^\rho_{\beta]\mu}, \tag{2.18}$$

$$R_{\alpha\beta}{}^{IJ} = \partial_{[\alpha} \omega_{\beta]}^{IJ} + \omega_{[\alpha}{}^{IK} \omega_{\beta]K}{}^J. \tag{2.19}$$

Finally we can use the tetrads to construct the Ricci scalar by contracting the first and third indices of the Riemann tensor with the second and fourth indices respectively. Thus the Einstein-Hilbert Lagrangian in the tetrad formalism is given by

$$\mathcal{L}_{EH} = e\, e_I^\alpha e_J^\beta R_{\nu\beta}{}^{IJ}(e_K^\gamma), \tag{2.20}$$

Note that $e\, d^4x$ is the volume form in the tetrad formalism. Varying the action with respect to the tetrad one gets the following equation of motion

$$e_K^\alpha e_I^\gamma e_J^\beta R_{\gamma\beta}{}^{KJ} - \frac{1}{2} e_I^\alpha e_K^\gamma e_J^\beta R_{\gamma\beta}{}^{KJ} = 0, \tag{2.21}$$

which when multiplied by e_μ^I gives the familiar Einstein equations for pure gravity

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0. \tag{2.22}$$

2.4 Hilbert Palatini Action

The difference between the Hilbert Palatini action and the Einstein Hilbert action is that torsion is not assumed to vanish a priori. The spin connection is therefore not determined uniquely by the tetrad and is instead taken to be an independent variable. Moreover, we can formulate everything in terms of the spin connection using the language of differential forms. Indeed, one can define the spin connection by an exterior covariant derivation by

$$Dv^I \equiv dv^I + \omega^{IJ} \wedge v_J, \tag{2.23}$$

for all vectors v^I. Defining the curvature tensor in analogy with Eq. (2.16)

$$D^2 v_I = F_I{}^J \wedge v_J, \tag{2.24}$$

we obtain Eq. (2.19) in differential forms[1]

$$F^{IJ} = d\omega^{IJ} + \omega^{IK} \wedge \omega_K^J. \tag{2.25}$$

The Hilbert Palatini Lagrangian has the same form as the Einstein Hilbert Lagrangian except the curvature scalar is now a function of the spin connection independently. It is thus given by[2]

$$\mathcal{L}_{HP} = e e_I^\alpha e_J^\beta F_{\alpha\beta}{}^{IJ}(\omega_\gamma^{KL}). \tag{2.26}$$

Varying with respect to the tetrad and the spin connection gives the following two equations of motion respectively

$$e_K^\alpha e_I^\gamma e_J^\beta F_{\gamma\beta}{}^{KJ} - \frac{1}{2} e_K^\gamma e_J^\beta e_I^\alpha F_{\gamma\beta}{}^{KJ} = 0, \tag{2.27}$$

$$D_\alpha(\epsilon^{\alpha\beta\gamma\delta} \epsilon_{IJKL} e_\gamma^K e_\delta^L) = 0. \tag{2.28}$$

The first equation of motion is precisely Eq. (2.21) with R replaced by F. The second equation of motion can be written as

[1] We use the letter F to denote the curvature tensor instead of R since it is not equivalent to the Riemann tensor in the presence of torsion.

[2] One can also include a cosmological constant to the action but we have omitted it since it will not be implemented into the spin foam models.

$$D(e^K \wedge e^L) = 0, \tag{2.29}$$

and can be shown to be equivalent to the zero torsion condition as follows. First expand in a basis of three forms

$$D(e^I \wedge e^J) = De^I \wedge e^J - e^I \wedge De^J, \tag{2.30}$$
$$= T^I_{KL} e^K \wedge e^L \wedge e^J - T^J_{KL} e^I \wedge e^K \wedge e^L, \tag{2.31}$$
$$= \left(T^I_{KL}\delta^J_M + T^J_{KM}\delta^I_L\right) e^K \wedge e^L \wedge e^M, \tag{2.32}$$

for some coefficients T^I_{JK}. Then setting $J = M$ we have

$$2T^I_{[LK]} + T^J_{[KJ]}\delta^I_L - T^J_{[LJ]}\delta^I_K = 0. \tag{2.33}$$

Finally, setting $I = L$ we obtain $T^I_{[IJ]} = 0$ which when put back into Eq. (2.33) implies $T^I_{[LK]} = 0$ and thus

$$De^I = T^I_{LK} e^L \wedge e^K = 0, \tag{2.34}$$

which is the zero torsion condition. Together the two equations of motion are equivalent to Einstein's equations.

We can generalise the Lagrangian in Eq. (2.26) by adding what is called a Holst term without changing the classical equations of motion. To see this more clearly we will first write the Lagrangian in the language of forms. Using the identity

$$e e^{[\alpha}_I e^{\beta]}_J = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}\epsilon_{IJKL} e^K_\gamma e^L_\delta, \tag{2.35}$$

we can write Eq. (2.26) as

$$\mathcal{L}_{HP} = \epsilon_{IJKL} e^K \wedge e^L \wedge F^{IJ}(\omega). \tag{2.36}$$

Further, we will use the Hodge $\star$ notation where

$$\star(e^I \wedge e^J) = \frac{1}{2}\epsilon^{IJ}{}_{KL} e^K \wedge e^L, \tag{2.37}$$

in which case the Hilbert-Palatini Lagrangian takes the form[3]

$$\mathcal{L}_{HP} = \star(e_I \wedge e_J) \wedge F^{IJ}. \tag{2.38}$$

We will now add the Holst term which has the form $e \wedge e \wedge F$ to this Lagrangian to get

$$\mathcal{L}_{Holst} = \star(e_I \wedge e_J) \wedge F^{IJ} - \frac{1}{\gamma}(e_I \wedge e_J) \wedge F^{IJ}, \tag{2.39}$$

where $\gamma > 0$ is a free parameter referred to as the Immirzi parameter. Adding the Holst term does not change the classical equations of motion of the Hilbert-Palatini action. This is because varying with respect to ω^{IJ} still produces the equation of motion (2.28) while varying the Holst term with respect to e^I gives

$$F^I_J \wedge e^J = 0, \tag{2.40}$$

which is the first Bianchi identity when combined with Eq. (2.28), i.e. zero torsion. Therefore, the action in Eq. (2.39) produces the same equations of motion as the Hilbert Palatini action, namely the Einstein equations.

The action in Eq. (2.39) will be our starting point for constructing a spin foam model of quantum gravity. We will find that the inclusion of the Holst term, and hence the Immirzi parameter, will be essential to this construction. Interestingly, this action is also the starting point of Loop Quantum gravity for which the Holst term is also required [17].

[3] We will omit the brackets for $\star$ when it is clear from the context. In this case there is no ambiguity since the Lagrangian must be a 4-form.

Chapter 3

Spin Foam Models

It is not known how to quantize the Hilbert Palatini action directly. However the quantization of a simpler model called BF theory is much more well understood. We first review the quantization of BF theory and then we review the EPRL method of imposing the simplicity constraints on the quantum BF theory. The resulting spin foam model should then agree with the Hilbert Palatini action in Eq. (2.39) in the classical limit.

3.1 BF theory

The classical BF action is defined in analogy to Eq. (2.39) as

$$S = \int_{\mathcal{M}} \left[B_{IJ} \wedge F^{IJ} + \frac{1}{\gamma} (\star B)_{IJ} \wedge F^{IJ} \right] \tag{3.1}$$

where F is the curvature and B is an arbitrary 2-form. For simplicity we consider an oriented manifold $\mathcal{M}$ without boundary but we note that the generalisation to manifolds with boundary is straightforward [18]. If B is constrained to be[1]

$$B^{IJ} = \star(E^I \wedge E^J), \tag{3.2}$$

then one recovers the Hilbert-Palatini action with a Holst term. [2] Equation (3.2) is referred to as the simplicity constraint because the two form B is constrained to be a

[1]We will use a capital E to denote the tetrad since lower case e will be used later to denote edges of a triangulation.

[2]Notice that we could equivalently constrain $B = E \wedge E$ and recover Eq. (2.39) with Immirzi parameter $-1/\gamma$. Thus there are two equivalent sectors of BF theory which are both equivalent to Eq. (2.39) but have different values for the Immirzi parameter.

simple bivector. The reason for considering this constrained BF theory rather than the Hilbert-Palatini action directly is that the two form B can be integrated out in a path integral providing a tremendous simplification.

We will approximate this continuous theory by a discrete lattice theory. Any Lorentzian manifold can be approximated sufficiently well by a piecewise flat manifold, i.e. flat 4-simplicies glued together in such a way as to approximate the continuous geometry. Simplicies are the building blocks of polyhedra, i.e. points, lines, triangles, etc and a simplicial complex is a set of simplicies glued together in an appropriate manner[3]. Finally, a triangulation of a manifold is a simplicial complex for which the union of all the simplicies in the complex is homeomorphic to the manifold, i.e. one which respects the topology. Note that the orientation of $\mathcal{M}$ defines an orientation of the simplicies in a triangulation.

Let Δ be such a triangulation of $\mathcal{M}$ consisting of 4-simplicies σ, tetrahedra t, triangles, edges and verticies. The dual complex Δ^* is composed of verticies v, edges e, and faces f where each 4-simplex $\sigma \in \Delta$ is dual to a vertex $v \in \Delta^*$, each tetrahedron $t \in \Delta$ is dual to an edge $e \in \Delta^*$, and each triangle in Δ is dual to a face $f \in \Delta^*$. Note that the 2-simplicies in the dual complex are in general not triangles but polygons hence we refer to them as faces. A summary of these labels is contained in table 3.1.

n-simplex	Δ label	Δ^* label	name
0		v	vertex
1		e	edge
2		f	face
3	t		tetrahedron
4	σ		4-simplex

Table 3.1: Labels of n-simplicies in a 4d triangulation. We will not require names for the 0,1,2-simplicies in Δ therefore we have left these fields blank. On the other hand, the 3,4-simplicies are, in general, not defined in Δ^*.

The curvature of the simplicial geometry is concentrated on the faces in the following sense. The metric inside each 4-simplex is flat and although it may be possible to construct a flat coordinate system covering adjacent 4-simplicies, in general it is impossible to construct a flat coordinate system covering all of the 4-simplicies surrounding a face. This formulation of curvature on a discretized space is referred to as Regge calculus [20].

With this in mind, we wish to formulate the holonomy around a face in Δ^*. Let x^μ be a chart covering the tetrahedron t of a 4-simplex σ and let $E^I_\mu(t)$ be the associated orthonormal basis given by

[3] For more details see [19].

$$E^I(t) = E^I_\mu(t) dx^\mu. \tag{3.3}$$

Since the metric is flat inside σ we can construct a common coordinate system y^μ covering all five tetrahedra inside σ. Let $E(v)$ be the orthonormal basis corresponding to y^μ given by

$$E^I(v) = E^I_\mu(v) dy^\mu. \tag{3.4}$$

It follows that the cotetrad fields $E^I_\mu(t)$ for the five tetrahedra in σ must be related to $E^I_\mu(v)$ by matrices[4] $(g_{ve})^I{}_J \in SO(3,1)$ defined by

$$(g_{ve})^I{}_J E(v)^J_\mu = E(t)^I_\mu. \tag{3.5}$$

Note that $e \in \Delta^*$ is the edge dual to the tetrahedron t and one can view the variables g_{ve} as a discretized connection on Δ^*. Further, we denote the gauge transformation from one tetrahedron t to an adjacent tetrahedron t' (or equivalently the parallel transport in Δ^* from edges e to e') by

$$G_{ee'} \equiv g_{ev} g_{ve'}, \tag{3.6}$$

where $g_{ev} \equiv (g_{ve})^{-1}$. The holonomy around a face $f \in \Delta^*$ with n verticies is therefore given by

$$G_f \equiv G_{e_1e_2} G_{e_2e_3} \cdots G_{e_ne_1} \equiv \prod_{v \subset f} g_{ev} g_{ve'}. \tag{3.7}$$

The second equality is condensed notation as a product over the verticies in f of the connection variables g_{ev} and $g_{ve'}$ where it is understood that e' follows e around the oriented face f. See Fig. 3.1.

We can relate the discrete variables G_f and B_f to the continuous variables ω^{IJ}_α and e^I_α as follows. The holonomy variables are related to the continuous connection $\omega(t)^{IJ}$ defined on t by

[4]Here we take the gauge group to be $SO(3,1)$ which is the homogeneous Lorentz group. In the quantum theory we will use $SL(2,\mathbb{C})$ which is the double cover of $SO(3,1)$.

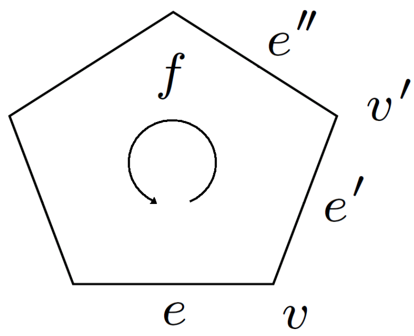

Figure 3.1: An arbitrary face $f \in \Delta^*$ is composed of verticies v, v',... and edges e, e',... The connection is defined on half edges $g_{ev} \in SL(2,\mathbb{C})$ and the holonomy around the face is given by the product $G_f = g_{ev} g_{ve'} g_{e'v'} \cdots$.

$$G_f^{IJ} = \mathcal{P} \exp \oint_f \omega^{IJ}(t), \tag{3.8}$$

which is the path ordered exponential around the face f. Also, the discrete analog of Eq. (3.2) can be defined by the following surface integral

$$B_f^{IJ} = \int_f \star \left(E(t)^I \wedge E(t)^J \right). \tag{3.9}$$

The classical discrete analog of the action in Eq. (3.1) is given by

$$S = \sum_{f \in \Delta^*} \mathrm{tr} \left[B_f G_f + \frac{1}{\gamma} \star B_f G_f \right]. \tag{3.10}$$

To see this recall that in the continuum limit

$$G_f \equiv \mathbb{1} + F_f, \tag{3.11}$$

where F_f is the curvature around the face f and thus

$$
\begin{aligned}
S &= \sum_{f\in\Delta^*} \mathrm{tr}\left[B_f G_f + \frac{1}{\gamma}\star B_f G_f\right], \\
&= \sum_{f\in\Delta^*} \mathrm{tr}\left[B_f \cdot \mathbb{1} + \frac{1}{\gamma}\star B_f \cdot \mathbb{1}\right] + \mathrm{tr}\left[B_f F_f + \frac{1}{\gamma}\star B_f F_f\right], \\
&= \sum_{f\in\Delta^*} \mathrm{tr}\left[B_f F_f + \frac{1}{\gamma}\star B_f F_f\right], \\
&\approx \int_{\mathcal{M}} \mathrm{tr}\left[B\wedge F + \frac{1}{\gamma}\star B\wedge F\right]. \qquad (3.12)
\end{aligned}
$$

where we used the fact that the trace of B_f (and hence $\star B_f$) vanishes since it is antisymmetric.

So far we have constructed a classical theory of gravity on a discretized space given by the action in Eq. (3.10) and subject to the simplicity constraints in Eq. (3.2). The quantization of this theory consists of promoting the classical variables B_f and G_f to operators and then specifying the dynamics by amplitudes from a path integral. The variables B_f are promoted to right invariant vector fields on the gauge group $SL(2,\mathbb{C})$ which are isomorphic to elements of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ while the holonomies G_f are promoted to operators on the Hilbert space of unitary irreducible representations of $SL(2,\mathbb{C})$ [18].

$SL(2,\mathbb{C})$ is the group of all unimodular, two by two, complex matrices. Its Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ is spanned by the six generators J^i, K^i for $i = 1,2,3$ which satisfy the following commutation relations

$$[J^i, J^j] = i\epsilon^{ij}{}_k J^k, \qquad (3.13)$$

$$[J^i, K^j] = i\epsilon^{ij}{}_k K^k, \qquad (3.14)$$

$$[K^i, K^j] = -i\epsilon^{ij}{}_k J^k. \qquad (3.15)$$

In the fundamental representation these generators are given by the Pauli matrices as

$$J^i = \frac{\sigma_i}{2}, \qquad K^i = \frac{i\sigma_i}{2}. \qquad (3.16)$$

The generators of the Lorentz group also satisfy these commutation relations and so $SL(2,\mathbb{C})$ is locally isomorphic to the Lorentz group. In fact, $SL(2,\mathbb{C})$ is the double cover of the Lorentz group, which is what we will use in the quantum theory. To construct unitary operators in the quantum theory we will need to work with the unitary irreducible representations (irreps) of $SL(2,\mathbb{C})$. The irreps of $SL(2,\mathbb{C})$ are labeled by a positive integer n and a real number ρ. For a derivation of these irreps of $SL(2,\mathbb{C})$ see Appendix A.2.

The classical action in Eq. (3.10) is then used to formulate the following path integral

$$Z = \int \prod_{f\in\Delta^*} dB_f \prod_{ev\in\Delta^*} dg_{ev} e^{iS}, \tag{3.17}$$

where dB_f is taken to be the Lebesgue measure and dg_{ev} is a measure of $SL(2,\mathbb{C})$ for the connection variables g_{ev} on the half edges which compose the holonomies G_f. The integration of the B_f's can be done explicitly giving [21]

$$Z = \int_{SL(2,\mathbb{C})} \prod_{ev\in\Delta^*} dg_{ev} \prod_{f\in\Delta^*} \delta(G_f). \tag{3.18}$$

where $\delta(g)$ is the delta distribution on $SL(2,\mathbb{C})$. As such, $\delta(g)$ admits a character decomposition of the form [22]

$$\delta(g) = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} d\rho (n^2+\rho^2) \mathrm{tr}\left[D^{(\rho,n)}(g)\right], \tag{3.19}$$

where $D^{(\rho,n)}(g)$ is the operator corresponding to g in the (ρ,n) representation of $SL(2,\mathbb{C})$. Therefore the partition function can be written as

$$Z = \sum_{n_f=0}^{\infty} \int_{-\infty}^{\infty} d\rho_f \int_{SL(2,\mathbb{C})} \prod_{ev\in\Delta^*} dg_{ev} \prod_{f\in\Delta^*} (n_f^2+\rho_f^2) \mathrm{tr}\left[D^{(\rho,n)}(G_f)\right], \tag{3.20}$$

$$= \sum_{n_f=0}^{\infty} \int_{-\infty}^{\infty} d\rho_f \prod_{f\in\Delta^*} (n_f^2+\rho_f^2) A_f((\rho_f,n_f);G_f), \tag{3.21}$$

where we used Eq. (3.7) to define the face amplitude

$$A_f((\rho_f,n_f);G_f) = \int\limits_{SL(2,\mathbb{C})} \prod_{v\subset f} dg_{ev}dg_{ve'} \,\mathrm{tr}\left[D^{(\rho_f,n_f)}(g_{ev}g_{ve'})\right]. \tag{3.22}$$

Notice that the sum over n_f and the integral over ρ_f have been moved to the left. This is because we first colour the faces of Δ^* with representation labels (ρ_f, n_f) and then compute the product of all the face amplitudes A_f for each face $f \in \Delta^*$. In the end we sum over all possible colourings (ρ_f, n_f).

Expressing the partition function as a product of face amplitudes is most natural but one can organize it in other ways. One could instead consider each of the faces containing a vertex v and then express the partition function as a product of vertex amplitudes A_v. The use of vertex amplitudes is more conventional and this is how we will express the final partition function once we specify how the trace in Eq. (3.22) is to be calculated.

In order for the partition function in Eq. (3.21) to describe gravity the simplicity constraints need to be enforced so that the discretized BF action in Eq. (3.10) agrees with the Hilbert Palatini action (2.39) in the classical limit. We will find that the simplicity constraints will manifest themselves in the quantum theory as constraints on the representations (ρ, n).

3.2 Simplicity Constraints

The simplicity constraints can be formulated in a concise way by the following argument [9]. The constraint that $\star B$ is a simple bivector is equivalent to the requirement that there exists a 4-vector U which is orthogonal to it. This is because if $\star B$ is not simple then it must be the sum of two bivectors which are composed of four 4-vectors which span $\mathbb{M}^4$. Hence $\star B$ cannot be orthogonal to any other 4-vector which proves the contrapositive. Conversely, if B is simple then it is a bivector which spans a two dimensional subspace and thus there must exist a 4-vector U which is orthogonal to it.

Hence, the simplicity constraints in full generality are equivalent to the existence of a a unit norm 4-vector U such that

$$U \cdot (\star B) = 0. \tag{3.23}$$

Moreover this form of the simplicity constraints is advantageous because it selects the sector of BF theory for which the Immirzi parameter is equal to γ (as opposed to $-1/\gamma$) and $\star B$ will be a simple bivector of the form

$$\star B = E_1 \wedge E_2. \tag{3.24}$$

If we further enforce the closure constraint[5]

$$\sum_{f \in t} B_f(t) = 0, \tag{3.25}$$

the bivectors $B_f(t)$ form a closed tetrahedron in a 3d subspace. Furthermore, the bivectors $\star B_f$ also form a closed tetrahedron and U can be taken to be the normal vector to this tetrahedron (in 4 dimensions). See Fig. 3.2.

Since U is orthogonal to $\star B_f$ for each $f \in t$ this implies that B is parallel to U so

$$B = AU \wedge N, \tag{3.26}$$

where N is a unit 4-vector such that $U \cdot N = 0$. The constant A corresponds to the area of parallelogram spanned by E_1 and E_2 as

$$A = \sqrt{|E_1^2 E_2^2 - (E_1 \cdot E_2)^2|}. \tag{3.27}$$

Furthermore, since $N \cdot (\star B) = 0$ we can interpret N as the unit normal vector to the triangle $E_1 \wedge E_2$.

Since B is parallel to U it must also be orthogonal to $\star B$, therefore we also get the weaker constraint

$$\star B \cdot B = 0. \tag{3.28}$$

In accord with the discretized action in Eq. (3.10) we add a Holst term by defining

$$J \equiv B + \frac{1}{\gamma} \star B. \tag{3.29}$$

[5]Note that the closure constraint is imposed dynamically as an equation of motion of the discretized action Eq. in (3.10). For a derivation see [18].

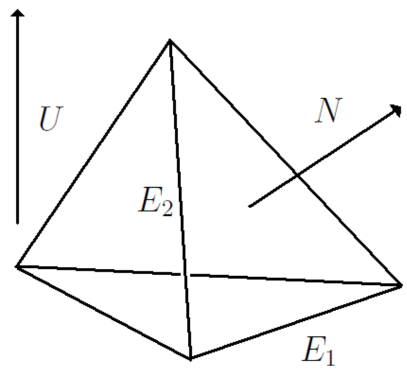

Figure 3.2: Classical variables for a tetrahedron $t \in \Delta$. Each bivector $E_1 \wedge E_2$ represents a triangle having a unit normal N. The unit normal to the tetrahedron is U.

We have labeled this bivector with the letter J as it will be closely related to the generators of $SL(2,\mathbb{C})$ in the quantum theory. Inverting this equation for B in terms of J gives

$$B = \frac{\gamma^2}{1+\gamma^2}\left(J - \frac{1}{\gamma} \star J\right). \tag{3.30}$$

Thus in terms of J Eq. (3.23) becomes

$$U \cdot \left(\star J + \frac{1}{\gamma} J\right) = 0. \tag{3.31}$$

Also, in terms of J Eq. (3.28) becomes

$$\begin{aligned} \star B \cdot B &= \frac{\gamma^4}{(1+\gamma^2)^2}\left(\star J + \frac{1}{\gamma}J\right) \cdot \left(J - \frac{1}{\gamma} \star J\right), \\ &= \frac{\gamma^4}{(1+\gamma^2)^2}\left(\left(1 - \frac{1}{\gamma^2}\right) \star J \cdot J + \frac{2}{\gamma} J \cdot J\right) \end{aligned} \tag{3.32}$$

which implies

$$\left(1 - \frac{1}{\gamma^2}\right) \star J \cdot J + \frac{2}{\gamma} J \cdot J = 0. \tag{3.33}$$

Recall that Eq. (3.33) is implied by Eq. (3.31). The reason for writing the two constraints separately is that the EPRL model imposed them in different ways as will be demonstrated in the next section.

3.3 The EPRL Model

If one imposes the simplicity constraints directly into the quantum theory, that is as strong operator equations, then one arrives at the Barret Crane model [5]. This model has been shown to lack the correct dynamics for the free graviton propagator in the semi-classical limit [23]. This is due to the fact that the simplicity constraints are imposed too strongly, i.e. on individual 4-simplicies, and neglects the fact that neighboring 4-simplicies share tetrahedra and are thus not independent. The EPRL model remedies this oversight by imposing Eq. (3.31) in a weaker manner by using what is called the Master Constraint. On the contrary Eq. (3.33) will be shown to be of first class in the quantum theory and can thus be imposed strongly.

3.3.1 Simplicity Constraints

In the quantum theory the classical bivectors J^{IJ} are promoted to elements of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$. Therefore $\star J \cdot J$ and $J \cdot J$ become the two Casimirs of $\mathfrak{sl}(2,\mathbb{C})$ which are denoted by

$$C_1 = J \cdot J \qquad \text{and} \qquad C_2 = \star J \cdot J. \tag{3.34}$$

Defining the usual rotation and boost operators by

$$J^i \equiv \frac{1}{2}\epsilon^{0i}{}_{jk}J^{jk} \qquad \text{and} \qquad K^i \equiv J^{0i}, \tag{3.35}$$

the Casimirs can be expressed as

$$C_1 = 2\delta_{ij}(J^iJ^j - K^iK^j) \qquad \text{and} \qquad C_2 = -4\delta_{ij}J^iK^j. \tag{3.36}$$

This implies that the constraint given in Eq. (3.33) commutes with all other constraints since the Casimirs are by definition invariant operators. Hence Eq. (3.33) can be imposed directly in the quantum theory as a strong operator equation

$$\left(1-\frac{1}{\gamma^2}\right)C_2+\frac{2}{\gamma}C_1=0. \tag{3.37}$$

Equation (3.31) however should not be imposed strongly. To impose Eq. (3.31) we will assume that U is timelike and gauge fix it to $U=(1,0,0,0)$. The result for a general timelike vector U will then hold by gauge invariance[6]. As we will see, the EPRL method of solving the constraints is only applicable to this choice of gauge in which U is timelike. Proceeding, Eq. (3.31) with this gauge choice becomes

$$\star B^{0i}=0. \tag{3.38}$$

Using Eq. (3.30) this reads

$$\star B^{0i}=\frac{\gamma^2}{1+\gamma^2}\left(\star J^{0i}+\frac{1}{\gamma}J^{0i}\right)=0, \tag{3.39}$$

which in terms of J^i and K^i becomes

$$J^i+\frac{1}{\gamma}K^i=0. \tag{3.40}$$

The imposition of Eq. (3.40) is done so weakly by the so called Master constraint. The Master Constraint is simply the sum of the squares of the constraints in Eq. (3.40) and is given by

$$\sum_i\left(J^i+\frac{1}{\gamma}K^i\right)^2=0. \tag{3.41}$$

An important point to note is that the Master constraint is equivalent to the original set of constraints (classically) because the squares of the constraints in Eq. (3.40) are positive definite which forces each constraint to vanish separately. The reason that each constraint is positive definite is because the isotropy subgroup for U is $SU(2)$ when U is timelike and

[6] An overall transformation of all the triangles in a tetrahedron can be absorbed into the connection variables which are integrated over in the path integral

therefore $\vec{J}$ and $\vec{K}$ always have positive norm since they transform as Euclidean 3-vectors. [7]

For spacelike U this is no longer the case. The isotropy subgroup for spacelike U is $SU(1,1)$ and so the analogous constraints involve vectors which transform as 3d Minkowski vectors which do not have positive norm. Hence for spacelike U it is unclear how to interpret the Master constraint since the simplicity constraints in Eq. (3.40) are no longer required to vanish individually.

Proceeding, Eq. (3.41) can be written in terms of the Casimirs of $SL(2,\mathbb{C})$ using Eq. (3.36) as

$$\begin{aligned}\sum_i \left(J^i + \frac{1}{\gamma}K^i\right)^2 &= J^i J_i + \frac{2}{\gamma}J^i K_i + \frac{1}{\gamma^2}K^i K_i,\\ &= \vec{J}^2 + \frac{2}{\gamma}\left(-\frac{C_2}{4}\right) + \frac{1}{\gamma^2}\left(J^2 - \frac{C_1}{2}\right),\\ &= \vec{J}^2\left(1+\frac{1}{\gamma^2}\right) - \frac{C_1}{2\gamma^2} - \frac{C_2}{2\gamma}. \end{aligned} \tag{3.42}$$

Using the relation for the Casimirs in Eq. (3.37) we have

$$\begin{aligned}\sum_i \left(J^i + \frac{1}{\gamma}K^i\right)^2 &= \vec{J}^2\left(1+\frac{1}{\gamma^2}\right) - \frac{1}{4\gamma}\left(1-\frac{1}{\gamma^2}\right)C_2 - \frac{C_2}{2\gamma},\\ &= \left(1+\frac{1}{\gamma^2}\right)\left(\vec{J}^2 - \frac{C_2}{4\gamma}\right), \end{aligned} \tag{3.43}$$

which implies

$$C_2 = 4\gamma \vec{J}^2. \tag{3.44}$$

Now we have Eqs. (3.37) and (3.44) as constraints on the Casimirs of $SL(2,\mathbb{C})$ and hence the representation labels (n,ρ). Substituting the eigenvalues of the Casimirs as given in Appendix A.2 as

$$C_1 = \frac{1}{2}(n^2 - \rho^2 - 4), \tag{3.45}$$

$$C_2 = n\rho, \tag{3.46}$$

[7] An isotropy subgroup (or little group) is the subgroup for which a vector is invariant. In this case the vector $U = (1,0,0,0)$ is invariant under all 3d spatial rotations so the isotropy subgroup of $SL(2,\mathbb{C})$ is $SU(2)$.

the two constraint equations become

$$n\rho\left(\gamma - \frac{1}{\gamma}\right) = \rho^2 - n^2, \tag{3.47}$$

and

$$n\rho = 4\gamma j(j+1). \tag{3.48}$$

Equation (3.47) has the two solutions $\rho = \gamma n$ or $\rho = -n/\gamma$ which reflect the two sectors of BF theory mentioned earlier. Substituting $\rho = \gamma n$ into Eq. (3.48) gives

$$n^2 = 4j(j+1), \tag{3.49}$$

while substituting $\rho = -n/\gamma$ gives

$$-n^2 = \gamma^2 j(j+1). \tag{3.50}$$

Eq. (3.50) has no solution while Eq. (3.49) cannot be solved in general for integer n and half-integer j. However, notice that in Eq. (3.49) we have approximately $j = n/2$ which holds for large j which is the classical limit.

Moreover, this value of j is not completely arbitrary, in fact it happens to be the lowest weight representation in the decomposition of $SL(2,\mathbb{C})$ into irreps of $SU(2)$ (see Appendix A.3) and is thus the only "special" value of j. Therefore the EPRL model proposes the constraints $\rho = \gamma n$ on the representation of $SL(2,\mathbb{C})$ and $j = n/2$ on the representation of the subgroup $SU(2)$.

3.3.2 Area Spectrum

The area spectrum of the theory can be calculated as follows. From Eq. (3.26)

$$B^{0i} = A\left(U^0 N^i - U^i N^0\right), \tag{3.51}$$

$$= AN^i, \tag{3.52}$$

while Eq. (3.30) gives

$$B^{0i} = \frac{\gamma^2}{1+\gamma^2}\left(J^{0i} - \frac{1}{\gamma} \star J^{0i}\right), \tag{3.53}$$

$$= \frac{\gamma^2}{1+\gamma^2}\left(K^i - \frac{1}{\gamma} J^i\right), \tag{3.54}$$

$$= \frac{\gamma^2}{1+\gamma^2}\left(-\gamma J^i - \frac{1}{\gamma} J^i\right), \tag{3.55}$$

$$= \frac{-\gamma^2}{1+\gamma^2}\left(\frac{\gamma^2+1}{\gamma}\right) J^i, \tag{3.56}$$

$$= -\gamma J^i. \tag{3.57}$$

Therefore

$$AN^i = -\gamma J^i, \tag{3.58}$$

and $N = (0, \vec{N})$ where $\vec{N}$ is a point on S^2. The interpretation of $\vec{N}$ is the normal vector of the triangle $E^i \wedge E^j$ embedded in a Euclidean subspace. Squaring both sides of Eq. (3.58) we have

$$A^2 = \gamma^2 \vec{J}^2. \tag{3.59}$$

Taking expectation values and the square root

$$A = \gamma\sqrt{j(j+1)}, \tag{3.60}$$

which is exactly the area spectrum of Loop Quantum Gravity quoted in Eq. (1.2).

Chapter 4

The Coherent State Formulation

Instead of imposing the simplicity constraints directly onto the operators J^{IJ} as is done in the EPRL model one can instead impose the simplicity constraints on the expectation values of these operators using semiclassical states. This was the approach of the Freidel-Krasnov (FK) model for 4d Euclidean spin foam models [9]. The semiclassical states were taken to be coherent states which were first introduced by Livine and Speziale to rewrite the BF theory amplitudes as in Eq. (3.22) into a form with a more intuitive geometrical interpretation as a superposition of semiclassical wavepackets [24]. The FK model took the coherent state approach one step further and imposed the condition that these coherent states should also satisfy the simplicity constraints as expectation values.[1] However, an analogous Lorentzian spin foam model was not fully constructed using coherent states of $SL(2, \mathbb{C})$.

In this chapter we derive a new set of coherent states for the Lorentzian theory and we will impose the simplicity constraints on expectation values in the spirit of the FK model. For timelike U we arrive at the same constraints on representation labels as the EPRL model. We then use the coherent state method to derive constraints for spacelike U which has not been done before.

4.1 Coherent state derivation of the EPRL constraints

As in the EPRL model we assume that U is timelike and gauge fix $U = (1, 0, 0, 0)$. Recall that the isotropy subgroup for $U = (1, 0, 0, 0)$ is $SU(2)$. Since triangles in a tetrahedron with $U = (1, 0, 0, 0)$ are in a 3d Euclidean subspace it follows that $SU(2)$ transformations

[1]The FK model was shown to be similar but distinct from the EPRL model with Euclidean signature. [10]

do not take these triangles outside of this subspace. Similarly, we want to choose quantum states which are closed under $SU(2)$ transformations. As shown in Appendix A.3 one can decompose each irrep of $SL(2,\mathbb{C})$ labeled by (ρ, n) into a direct sum of irreps of $SU(2)$. Thus we have the following isomorphism

$$\mathcal{H}_{(\rho,n)} \simeq \bigoplus_{j=n/2}^{\infty} \mathcal{D}_j \,, \qquad , \tag{4.1}$$

where $\mathcal{D}_j$ is the spin j unitary irreducible representation of $SU(2)$ spanned by the states $|jm\rangle$ for $m = -j, ..., j$. We will digress here for a moment to review some facts about $SU(2)$.

The subgroup $SU(2)$ is generated by the subset of generators J^1, J^2, and J^3 which satisfy the commutation relations in Eq. (A.1) and possess the Casimir operator

$$\vec{J}^2 = \delta_{ij} J^i J^j. \tag{4.2}$$

for $i = 1, 2, 3$. The action of these generators on the standard irreps is given by

$$\vec{J}^2 |j\,m\rangle = j(j+1)|j\,m\rangle, \tag{4.3}$$

$$J^3 |j\,m\rangle = m|j\,m\rangle, \tag{4.4}$$

$$\langle j\,m'|j\,m\rangle = \delta_{m'\,m}. \tag{4.5}$$

We will denote the corresponding states in $\mathcal{H}_{(\rho,n)}$ by $|\Psi_{j\,m}\rangle$ and take these states to be our quantum states. Since these states are isomorphic to $SU(2)$ irreps they are also eigenvectors of the $SU(2)$ operators $\vec{J}^2$ and J^3 as in[2]

$$\vec{J}^2 |\Psi_{j\,m}\rangle = j(j+1)|\Psi_{j\,m}\rangle, \tag{4.6}$$

$$J^3 |\Psi_{j\,m}\rangle = m|\Psi_{j\,m}\rangle, \tag{4.7}$$

$$\langle \Psi_{j'\,m'}|\Psi_{j\,m}\rangle = \delta_{j'\,j}\delta_{m'\,m}. \tag{4.8}$$

In what follows we will also require the matrix elements of the generator K^3 which is given by the following action on the states

[2] The matrix elements of the generators outside of the subgroup $SU(2)$ (the boost operators) are calculated in [25]. For a compilation of these matrix elements see Appendix C.

$$
\begin{aligned}
K^3|\Psi_{j\,m}\rangle = &(\rho/2 + i(j+1))C_{j+1}|\Psi_{j+1\,m}\rangle \\
&- mA_j|\Psi_{j\,m}\rangle \\
&+ (\rho/2 - ij)C_j|\Psi_{j-1\,m}\rangle.
\end{aligned} \tag{4.9}
$$

where

$$A_j = \frac{\rho n}{4j(j+1)}, \tag{4.10}$$

$$C_j = \frac{\sqrt{n^2/4 - j^2}\sqrt{m^2 - j^2}}{j\sqrt{2j-1}\sqrt{2j+1}}. \tag{4.11}$$

From these quantum states we would now like to construct a set of coherent states which are to be maximally classical states. By this we mean that the uncertainties in the configuration variables, which in this case are the vectors $\vec{J}$ and $\vec{K}$, should be minimal.

Thus, we will construct our coherent states by minimizing the variances of the vectors $\vec{J}$ and $\vec{K}$. To be precise, we will require our coherent states to satisfy the following set of criteria[3]

$$\frac{\Delta J}{|\vec{J}|} = O\left(\frac{1}{\sqrt{|\vec{J}|}}\right), \tag{4.12}$$

$$\langle\vec{J}\rangle + \frac{1}{\gamma}\langle\vec{K}\rangle = O(1), \tag{4.13}$$

$$\frac{\Delta K}{|\vec{K}|} = O\left(\frac{1}{\sqrt{|\vec{K}|}}\right). \tag{4.14}$$

The first and third conditions assert that the coherent states should be peaked around the classical values of $\vec{J}$ and $\vec{K}$. The second condition implies that the expectation values of the states should satisfy the simplicity constraints as was pioneered in the FK model. Note that in general the variances will be nonzero but we can at least minimize them to the orders given in the criteria above.

[3]For a detailed analysis of coherent states on $SU(2)$ and $SU(1,1)$ see [26].

Coherent states on $SU(2)$ were studied in detail by Perelomov [26] and are given by states with minimal variance in $\vec{J}$. This variance is calculated as

$$\begin{aligned}(\Delta J)^2 &= \langle \vec{J}^2 \rangle^2 - \langle \vec{J} \rangle^2, \\ &= \langle jm|\vec{J}^2|jm\rangle^2 - \langle jm|J^3|jm\rangle^2, \\ &= j(j+1) - m^2, \end{aligned} \tag{4.15}$$

therefore one should take $m = \pm j$ in which case

$$\frac{\Delta J}{|\vec{J}|} = \frac{\sqrt{j}}{j} = \frac{1}{\sqrt{j}}, \tag{4.16}$$

which agrees with Eq. (4.12). Note that we only needed to consider the 3-component of $\vec{J}$ since J^1 and J^2 are linear combinations of raising and lowering operators and thus have vanishing expectation value. Also, since the states $|j\,j\rangle$ and $|j\,-j\rangle$ are related by a gauge transformation we can take our coherent states to be given by $m = j$ without any loss of generality.

Moreover, we can use this gauge invariance to define a general set of coherent states by acting on the state $|j\,j\rangle$ by the group action as

$$|j\,g\rangle \equiv D^j(g)|j\,j\rangle, \tag{4.17}$$

where $g \in SU(2)$. This has the effect of rotating the J^3 angular momentum axis to an arbitrary direction. Here the state $|j\,j\rangle$ is a maximally classical state corresponding to a triangle in the classical theory. The classical analog of Eq. (4.17) is thus the rotation of this triangle in a 3d Euclidean subspace. This analogy between the classical and the quantum theory can be demonstrated more explicitly in the following way.

Each element $g \in SU(2)$ can be factored as $g_0 h$ where $h \in U(1)$ and g_0 is a representative of the coset $SU(2)/U(1)$. In Appendix D.1 we show that

$$\langle j\,j|g_0^\dagger \vec{J} g_0|j\,j\rangle = j\vec{N}, \tag{4.18}$$

where $\vec{N} \in S^2$. Referring to Eq. (3.58) we see that we can interpret $\vec{N}$ as the quantum analog of the normal vector to the classical triangle. The reference state $|j\,j\rangle$ corresponds to $\vec{N} = (0,0,1)$ which is the direction of the J^3 angular momentum axis as intuited above.

Furthermore, Eq. (4.18) establishes the isomorphism $SU(2)/U(1) \cong S^2$. Thus each generalised coherent state $|j\, g\rangle$ is determined (up to a phase) by a vector $\vec{N} \in S^2$. In a completeness relation these phases will cancel and it will be sufficient to consider the states

$$|j\, \vec{N}\rangle \equiv |j\, g(\vec{N})\rangle. \tag{4.19}$$

For simplicity, we will continue to use the reference state $|j\, j\rangle$ in calculations since the results obtained will hold in general by gauge invariance. We will refer to the corresponding generalised coherent states in $\mathcal{H}_{(\rho,n)}$ by $|\Psi_{j\, g}\rangle$.

Now imposing the simplicity constraint on expectation values as in Eq. (4.13) we get

$$\langle\Psi_{j\, j}|\vec{J}|\Psi_{j\, j}\rangle = -\frac{1}{\gamma}\langle\Psi_{j\, j}|\vec{K}|\Psi_{j\, j}\rangle, \tag{4.20}$$

$$j = -\frac{1}{\gamma}(-jA_j), \tag{4.21}$$

which implies

$$\gamma = A_j = \frac{\rho n}{4j(j+1)}. \tag{4.22}$$

Finally, the variance of $\vec{K}$ is given by

$$\begin{aligned}(\Delta K)^2 &= \langle\vec{K}^2\rangle^2 - \langle\vec{K}\rangle^2,\\ &= \langle\Psi_{j\, j}|\vec{J}^2 - \frac{1}{4}(n^2-\rho^2-4)|\Psi_{j\, j}\rangle^2 - \langle\Psi_{j\, j}|K^3|\Psi_{j\, j}\rangle^2,\\ &= j(j+1) - \frac{1}{4}(n^2-\rho^2-4) - (jA_j)^2,\end{aligned} \tag{4.23}$$

and using Eq. (4.22)

$$\begin{aligned}(\Delta K)^2 &= \frac{\rho n}{4\gamma} - \frac{1}{4}(n^2-\rho^2-4) - j^2\gamma^2,\\ &= \frac{\rho n}{4\gamma} - \frac{1}{4}(n^2-\rho^2) - \gamma^2 j(j+1) + \gamma^2 j + 1,\\ &= \frac{\rho n}{4\gamma} - \frac{1}{4}(n^2-\rho^2) - \frac{\gamma\rho n}{4} + \gamma^2 j + 1,\\ &= \frac{1}{4}\Big(\rho - \gamma n\Big)\Big(\rho + \frac{n}{\gamma}\Big) + \gamma^2 j + 1.\end{aligned} \tag{4.24}$$

Therefore choosing either $\rho = \gamma n$ or $\rho = -n/\gamma$ makes the first term vanish and thus

$$\frac{\Delta K}{|\vec{K}|} = \frac{\sqrt{\gamma^2 j + 1}}{\gamma j} = O\left(\frac{1}{\sqrt{|\vec{K}|}}\right), \tag{4.25}$$

which satisfies Eq. (4.14). Note that Eq. (4.22) is precisely Eq. (3.48) from the EPRL model and therefore $\rho = -n/\gamma$ gives a contradiction while $\rho = \gamma n$ implies that we should use the lowest weight representation $j = n/2$. Thus we have arrived at precisely the same constraints as the EPRL model.

The fact that such different methods of imposing the simplicity constraints led to the same result is fascinating and lends credibility to the two quantization procedures. Notice that both the EPRL model and the coherent state method used Eq. (3.31) to arrive at the same equations Eqs. (3.44) and (4.22) respectively but by different means. The EPRL model used the Master constraint while the coherent state method imposed the simplicity constraints on expectation values.

Likewise, it is even more interesting that the EPRL model and the coherent state method both arrived at the same constraint $\rho = \gamma n$. The EPRL model used Eq. (3.33) as a strong operator equation in the quantum theory while the coherent state method minimized the variance in $\vec{K}$. In [13] it is shown how $(\Delta K)^2$ is related to Eq. (3.33) but it is still remarkable that the two lines of reasoning converge.

4.2 Extension of the EPRL model

In Section 3.3 we saw that the master constraint provides a sensible means for imposing the simplicity constraints assuming U is timelike. However, for spacelike U the validity of the master constraint becomes unclear and a different technique is needed. Furthermore, we showed that for timelike U the EPRL model agrees with the coherent state method. Thus the coherent state method should also provide a sensible means of imposing the simplicity constraints for the case of spacelike U.

In this chapter we will construct a set of coherent states corresponding to triangles in a 3d Minkowskian tetrahedron, i.e. one with spacelike U. In this case the triangles can be either spacelike or timelike. In doing so we will construct a new spin foam model which will generalise the EPRL model to include timelike surfaces.

4.2.1 Classical Variables

First we investigate the classical simplicity constraints and areas as was done in Section 3.3 except now for $U = (0, 0, 0, 1)$. In this case Eq. (3.33) becomes

$$\star B^{3i} = 0, \tag{4.26}$$

for $i = 0, 1, 2$. Using Eq. (3.39)

$$\star B^{3i} = \frac{\gamma^2}{1+\gamma^2}\left(\star J^{3i} + \frac{1}{\gamma}J^{3i}\right) = 0, \tag{4.27}$$

and again using J^i and K^i defined in Eq. (3.35) we have

$$\star B^{30} = \frac{\gamma^2}{1+\gamma^2}\left(\star J^{30} + \frac{1}{\gamma}J^{30}\right) = \frac{\gamma^2}{1+\gamma^2}\left(-J^3 - \frac{1}{\gamma}K^3\right), \tag{4.28}$$

$$\star B^{31} = \frac{\gamma^2}{1+\gamma^2}\left(\star J^{31} + \frac{1}{\gamma}J^{31}\right) = \frac{\gamma^2}{1+\gamma^2}\left(K^2 - \frac{1}{\gamma}J^2\right), \tag{4.29}$$

$$\star B^{32} = \frac{\gamma^2}{1+\gamma^2}\left(\star J^{32} + \frac{1}{\gamma}J^{32}\right) = \frac{\gamma^2}{1+\gamma^2}\left(K^1 - \frac{1}{\gamma}J^1\right), \tag{4.30}$$

therefore

$$K^1 - \frac{1}{\gamma}J^1 = 0, \tag{4.31}$$

$$K^2 - \frac{1}{\gamma}J^2 = 0, \tag{4.32}$$

$$J^3 + \frac{1}{\gamma}K^3 = 0. \tag{4.33}$$

Defining

$$\begin{pmatrix} F^0 \\ F^1 \\ F^2 \end{pmatrix} = \begin{pmatrix} J^3 \\ K^1 \\ K^2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} G^0 \\ G^1 \\ G^2 \end{pmatrix} = \begin{pmatrix} K^3 \\ -J^1 \\ -J^2 \end{pmatrix} \tag{4.34}$$

we can write the constraints in Eqs. (4.31), (4.32), and (4.33) in an form analogous to Eq. (3.40) as

$$F^i + \frac{1}{\gamma} G^i = 0. \tag{4.35}$$

for $i = 0, 1, 2$. Moreover $\vec{F}$ and $\vec{G}$ transform as Minkowski 3-vectors under $SU(1,1)$ transformations just like $\vec{J}$ and $\vec{K}$ transform as Euclidean 3-vectors under $SU(2)$ transformations. This can be seen by the following commutation relations

$$[F^i, F^j] = i\epsilon^{ij}{}_k F^k, \tag{4.36}$$

$$[F^i, G^j] = i\epsilon^{ij}{}_k G^k, \tag{4.37}$$

$$[G^i, G^j] = -i\epsilon^{ij}{}_k F^k. \tag{4.38}$$

We can compute the area of the classical bivectors B as follows. From Eq. (3.26)

$$B^{3i} = A\left(U^3 N^i - U^i N^3\right), \tag{4.39}$$

$$= A N^i \tag{4.40}$$

and from Eq. (3.30)

$$B^{30} = \frac{\gamma^2}{1+\gamma^2}\left(J^{30} + \frac{1}{\gamma} \star J^{30}\right), \tag{4.41}$$

$$= \frac{\gamma^2}{1+\gamma^2}\left(-K^3 + \frac{1}{\gamma} J^3\right). \tag{4.42}$$

Using $K^3 = -\gamma J^3$

$$B^{30} = \frac{\gamma^2}{1+\gamma^2}\left(\gamma J^3 + \frac{1}{\gamma} J^3\right), \tag{4.43}$$

$$= \frac{\gamma^2}{1+\gamma^2}\left(\frac{\gamma^2+1}{\gamma}\right) J^3, \tag{4.44}$$

$$= \gamma J^3. \tag{4.45}$$

Therefore

$$AN^0 = \gamma J^3, \tag{4.46}$$

and similarly

$$AN^1 = \gamma K^2 \qquad \text{and} \qquad AN^2 = -\gamma K^1. \tag{4.47}$$

Therefore

$$A \begin{pmatrix} N^0 \\ N^1 \\ N^2 \end{pmatrix} = \gamma \begin{pmatrix} F^0 \\ F^2 \\ -F^1 \end{pmatrix}, \tag{4.48}$$

and so $N = (\vec{N}, 0)$ where $\vec{N} = (N^0, N^1, N^2)$. Hence taking the square

$$A^2 N^2 = \gamma^2 \left((F^0)^2 - (F^1)^2 - (F^2)^2\right) = \gamma^2 Q. \tag{4.49}$$

The vector N is now a Minkowski 3-vector and can thus have norm ± 1. When N is timelike it corresponds to a point on the two-sheeted timelike hyperboloid (Fig. 4.1)

$$\mathbb{H}_+ \cup \mathbb{H}_-, \qquad \mathbb{H}_\pm = \{\vec{N} \,|\, \vec{N}^2 = 1,\ N^0 \gtrless 0\}. \tag{4.50}$$

When N is spacelike it corresponds to a point on the one-sheeted spacelike hyperboloid (Fig. 4.2)

$$\mathbb{H}_{\text{sp}} = \{\vec{N} \,|\, \vec{N}^2 = -1\}. \tag{4.51}$$

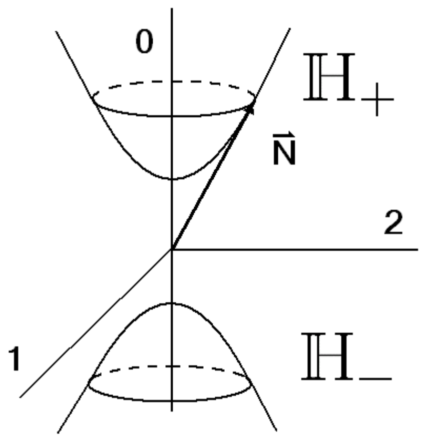

Figure 4.1: Points on the two-sheeted timelike hyperboloid $\mathbb{H}_\pm$ correspond to future/past timelike normal vectors N of spacelike triangles respectively. Moreover $\mathbb{H}_+ \cup \mathbb{H}_-$ is isomorphic to the quotient $SU(1,1)/U(1)$ where $U(1)$ is the one parameter subgroup generated by J^3.

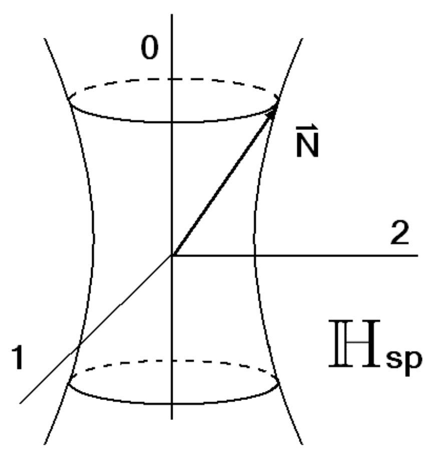

Figure 4.2: Points on the one-sheeted spacelike hyperboloid $\mathbb{H}_{\rm sp}$ correspond to spacelike normal vectors N of timelike triangles respectively. Moreover $\mathbb{H}_{\rm sp}$ is isomorphic to the quotient $SU(1,1)/(G_1 \otimes \mathbb{Z}_2)$ where G_1 is the one parameter subgroup generated by K^1.

Since A and γ are positive this suggests that the sign of Q will determine whether N is timelike or spacelike[4].

When $\vec{F}$ and $\vec{G}$ are promoted to operators in the quantum theory Q will correspond to the Casimir on $SU(1,1)$. Likewise, the sign of this Casimir acting on the quantum states will also indicate whether the state is timelike or spacelike (except for a few exceptions).

4.2.2 Quantum States

As we saw in Section 4.1 the irreps of $SL(2,\mathbb{C})$ are decomposable into a direct sum of irreps of the compact subgroup $SU(2)$. In an analogous way the irreps of $SL(2,\mathbb{C})$ can also be decomposed into a direct sum of irreps of the noncompact subgroup $SU(1,1)$. However, since $SU(1,1)$ is noncompact the decomposition is more complicated than in the $SU(2)$ case as it admits a discrete series of irreps $\mathcal{D}_j^{\pm}$ labeled by half integers j and a continuous series $\mathcal{C}_s^{\epsilon}$ labeled by positive real numbers s. The decomposition is described in Appendix A.4 and is given by the following direct sum

$$\mathcal{H}_{(\rho,n)} = \left(\bigoplus_{-j\geq 1}^{n/2} \mathcal{D}_j^{+} \oplus \int_0^{\infty} ds\, \mathcal{C}_s^{\epsilon} \right) \oplus \left(\bigoplus_{-j\geq 1}^{n/2} \mathcal{D}_j^{-} \oplus \int_0^{\infty} ds\, \mathcal{C}_s^{\epsilon} \right). \tag{4.52}$$

The appearance of a discrete series and a continuous series is unfortunately complicated but completely necessary in the following sense. As mentioned above for U spacelike the triangles of a tetrahedron can be either spacelike or timelike. As we will see, the states of the discrete series will represent spacelike triangles while states in the continuous series will represent timelike triangles.

Both the discrete series and the continuous series can be constructed as eigenstates of J^3 and Q. For the discrete series $\mathcal{D}_j^{\tau}$

$$Q|j\ m\rangle = j(j+1)|j\ m\rangle, \qquad j = -1, -\frac{3}{2}, ..., \tag{4.53}$$

$$J^3|j\ m\rangle = m|j\ m\rangle, \qquad \tau m = j, j+1, j+2, ... \tag{4.54}$$

$$\langle j\ m'|j\ m\rangle = \delta_{m\,m'}, \tag{4.55}$$

where $\tau = \pm 1$ corresponds to the positive/negative series. For the continuous series $\mathcal{C}_s^{\epsilon}$

[4] Note that the definition of Q is consistent with the convention $(+,-,-,-)$ for the spacetime metric and $(+,-,-)$ for the 3d Minkowski metric.

$$Q|j\,m\rangle = j(j+1)|j\,m\rangle, \qquad j = -\frac{1}{2} + is, \quad 0 \le s < \infty, \tag{4.56}$$

$$J^3|j\,m\rangle = m|j\,m\rangle, \qquad m = \pm\epsilon, \pm(\epsilon+1), \pm(\epsilon+2), ... \tag{4.57}$$

$$\langle j\,m'|j\,m\rangle = \delta_{m\,m'}, \tag{4.58}$$

where $\epsilon = 0, 1/2$ depending on whether $n/2$ is integer or half integer.

Notice that for the discrete series $Q = j(j+1) > 0$ (for $j < -1$) while for the continuous series $Q = -\frac{1}{4} - s^2 < 0$. Referring to Eq. (4.49) this suggests that N should be timelike for the discrete series and spacelike for the continuous series.

However, for the continuous series we cannot describe spacelike normal vectors using eigenstates of J^3 since from Eq. (4.48)

$$\langle \vec{N} \rangle \propto \begin{pmatrix} \langle F^0 \rangle \\ \langle F^2 \rangle \\ -\langle F^1 \rangle \end{pmatrix} = \begin{pmatrix} m \\ 0 \\ 0 \end{pmatrix}, \tag{4.59}$$

which is inconsistent as a spacelike vector. Therefore we should instead choose a basis of $SU(1,1)$ which diagonalizes one of the spacelike generators, either K^1 or K^2. We will use eigenstates of K^1 as derived in [27]

$$K^1|j\,\lambda\,\sigma\rangle = \lambda|j\,\lambda\,\sigma\rangle, \tag{4.60}$$

where $-\infty < \lambda < \infty$ is a continuous eigenvalue since K^1 generates a noncompact subgroup. The spectrum of K^1 in the continuous series is twofold degenerate and is labeled by an additional label $\sigma = 0, 1$ (See Appendix B).

Finally, we will define our coherent states by the following conditions in analogy with Eqs. (4.12), (4.13), and (4.14) by

$$\frac{\Delta F}{|\vec{F}|} = O\left(\frac{1}{\sqrt{|\vec{F}|}}\right), \tag{4.61}$$

$$\langle \vec{F} \rangle + \frac{1}{\gamma}\langle \vec{G} \rangle = O(1), \tag{4.62}$$

$$\frac{\Delta G}{|\vec{G}|} = O\left(\frac{1}{\sqrt{|\vec{G}|}}\right). \tag{4.63}$$

Notice that since the vectors $\vec{F}$ and $\vec{G}$ are Minkowski 3-vectors one should define the uncertainties as Lorentz invariant quantities. Therefore we define

$$(\Delta F)^2 = \langle F^i F_i \rangle - \langle F^i \rangle \langle F_i \rangle, \tag{4.64}$$

$$(\Delta G)^2 = \langle G^i G_i \rangle - \langle G^i \rangle \langle G_i \rangle. \tag{4.65}$$

where now $\Delta F \equiv \sqrt{|(\Delta F)^2|}$ and $|\vec{F}| \equiv \sqrt{|\langle F^i \rangle \langle F_i \rangle|}$ and similarly for G.

4.3 Describing Spacelike Surfaces: Coherent States for the Discrete Series

The standard basis of irreps in $\mathcal{D}_j^{\pm}$ can be constructed as eigenvectors of J^3 and Q as described in the previous section. The corresponding states in $\mathcal{H}_{(\rho,n)}$ which we denote by $|\Psi^{\tau}_{j\,m}\rangle$ will also be eigenvectors of J^3 and Q as in

$$Q|\Psi^{\tau}_{j\,m}\rangle = j(j+1)|\Psi^{\tau}_{j\,m}\rangle, \tag{4.66}$$

$$J^3|\Psi^{\tau}_{j\,m}\rangle = m|\Psi^{\tau}_{j\,m}\rangle, \tag{4.67}$$

$$\langle \Psi^{\tau}_{j'\,m'}|\Psi^{\tau}_{j\,m}\rangle = \delta_{j'\,j}\delta_{m'\,m}. \tag{4.68}$$

Using these matrix elements the variance in F is computed as

$$\begin{aligned}(\Delta F)^2 &= \langle F^i F_i \rangle^2 - \langle F^i \rangle \langle F_i \rangle, \\ &= \langle j\,m|\vec{J}^2|j\,m\rangle^2 - \langle j\,m|J^3|j\,m\rangle^2, \\ &= j(j+1) - m^2. \end{aligned} \tag{4.69}$$

Hence taking $m = \pm j$ the states $|\Psi^{\pm}_{j\,\pm j}\rangle$ satisfy

$$\frac{\Delta F}{|\vec{F}|} = \frac{\sqrt{|j|}}{|j|} = \frac{1}{\sqrt{|j|}}, \tag{4.70}$$

which agrees with Eq. (4.61). Note that unlike the $SU(2)$ case we cannot relate the states $|j\,j\rangle$ and $|j\ -j\rangle$ by a gauge transformation. This is because $|j\,j\rangle$ is a state in the positive discrete series and negative discrete series which are disconnected spaces (as will become clear shortly).

We can again construct a general set of coherent states by acting on the reference states $|j\ \pm j\rangle$ with $SU(1,1)$ transformations as

$$|j\,g\rangle_+ \equiv D^j(g)|j\,j\rangle, \tag{4.71}$$

$$|j\,g\rangle_- \equiv D^j(g)|j\ -j\rangle, \tag{4.72}$$

for $g \in SU(1,1)$. As in the $SU(2)$ case this has the effect of rotating the J^3 angular momentum axis to an arbitrary direction, however in this case the axis is "rotated" in a 3d Minkowski subspace. To make this more precise we will consider the expectation value of $\vec{F}$ to relate the state $|j\,g\rangle$ to a normal vector as was done in Section 4.1.

Each element $g \in SU(1,1)$ can be factored as $g_0 h$ where $h \in U(1)$ and g_0 is a representative of the coset $SU(1,1)/U(1)$. In Appendix D.2 we show that

$$\langle j\ \pm j|g_0^\dagger \vec{F} g_0|j\ \pm j\rangle = \pm j\vec{N}, \tag{4.73}$$

where $\vec{N} \in \mathbb{H}_\pm$ which is the one-sheeted timelike hyperboloid shown in Fig. 4.1. Referring to Eq. (4.48) we see that we can once again interpret $\vec{N}$ as the quantum analog of the normal vector to the triangle. The reference states $|j\ \pm j\rangle$ corresponds to $\vec{N} = (\pm 1, 0, 0)$ which is the direction of the J^3 angular momentum axis for the positive and negative discrete series.

Furthermore, Eq. (4.73) establishes the isomorphism $SU(1,1)/U(1) \cong \mathbb{H}_+ \cup \mathbb{H}_-$. Therefore each generalised coherent state $|j\,g\rangle$ is determined (up to a phase) by a vector $\vec{N} \in S^2$. In a completeness relation these phases will cancel and it will be sufficient to consider the states

$$|j\,\vec{N}\rangle \equiv |j\,g(\vec{N})\rangle_\pm. \tag{4.74}$$

This also shows why the positive and negative discrete series are disconnected.

We label the corresponding generalised coherent states in $\mathcal{H}_{(\rho,n)}$ by $|\Psi^\tau_{jg}\rangle$. Again for simplicity we will continue to use the reference states $|\Psi^\pm_{j\,\pm j}\rangle$ in calculations since the general result will follow by gauge invariance.

In what follows we will also need the matrix element of K^3 which is given by [28]

$$K^3|\Psi^\tau_{j\,m}\rangle = \tau(\rho/2 + i(j+1))C_{j+1}|\Psi^\tau_{j+1\,m}\rangle$$
$$- mA_j|\Psi^\tau_{j\,m}\rangle$$
$$+ \tau(\rho/2 + i(j+1))C_j|\Psi^\tau_{j-1\,m}\rangle. \tag{4.75}$$

where A_j and C_j are given in Eqs. (4.10) and (4.11). The simplicity constraint in Eq. (4.62) is thus

$$\langle\Psi^\pm_{j\,\pm j}|\vec{F}|\Psi^\pm_{j\,\pm j}\rangle = -\frac{1}{\gamma}\langle\Psi^\pm_{j\,\pm j}|\vec{G}|\Psi^\pm_{j\,\pm j}\rangle, \tag{4.76}$$
$$\pm j = -\frac{1}{\gamma}(\mp jA_j), \tag{4.77}$$

which implies

$$\gamma = A_j = \frac{\rho n}{4j(j+1)}. \tag{4.78}$$

Again we only needed to consider the 0 components of $\vec{F}$ and $\vec{G}$ since the other components are linear combinations of raising/lowering operators and thus have vanishing expectation values. Finally we compute the variance in G using the reference states $|\Psi^+_{j\,j}\rangle$ and we note that one obtains the same result using the state $|\Psi^-_{j\,-j}\rangle$. First note that we can write

$$\vec{G}^2 = Q - \frac{1}{2}C_1, \tag{4.79}$$

where C_1 is the $SL(2,\mathbb{C})$ Casimir given in Eq. (A.9). Therefore

$$(\Delta G)^2 = \langle G^iG_i\rangle^2 - \langle G^i\rangle\langle G_i\rangle,$$
$$= \langle\Psi^+_{j\,j}|Q - \frac{1}{4}(n^2 - \rho^2 - 4)|\Psi^+_{j\,j}\rangle^2 - \langle\Psi^+_{j\,j}|K^3|\Psi^+_{j\,j}\rangle^2,$$
$$= j(j+1) - \frac{1}{4}(n^2 - \rho^2 - 4) - (jA_j)^2. \tag{4.80}$$

Using Eq. (4.78)

$$
\begin{aligned}
(\Delta G)^2 &= \frac{\rho n}{4\gamma} - \frac{1}{4}(n^2 - \rho^2 - 4) - j^2\gamma^2, \\
&= \frac{\rho n}{4\gamma} - \frac{1}{4}(n^2 - \rho^2) - \gamma^2 j(j+1) + \gamma^2 j + 1, \\
&= \frac{\rho n}{4\gamma} - \frac{1}{4}(n^2 - \rho^2) - \frac{\gamma \rho n}{4} + \gamma^2 j + 1, \\
&= \frac{1}{4}\Big(\rho - \gamma n\Big)\Big(\rho + \frac{n}{\gamma}\Big) + \gamma^2 j + 1.
\end{aligned} \tag{4.81}
$$

Choosing either $\rho = \gamma n$ or $\rho = -n/\gamma$ makes the first term vanish and thus

$$
\frac{\Delta G}{|\vec{G}|} = \frac{\sqrt{|\gamma^2 j + 1|}}{|\gamma j|} = O\left(\frac{1}{\sqrt{|\vec{G}|}}\right), \tag{4.82}
$$

which satisfies Eq. (4.63). Once again, as was found for the $SU(2)$ coherent states Eq. (4.78) is precisely Eq. (3.48) from the EPRL model and therefore $\rho = -n/\gamma$ gives a contradiction while $\rho = \gamma n$ implies we should use the lowest weight irreps $j = -n/2$. Note that states in the discrete series are not equivalent to states in the $SU(2)$ basis but they belong to the same total Hilbert space. In fact states in the $SU(2)$ and $SU(1,1)$ discrete series even have an inner product which is given in Eq. (A.6). We will next investigate the quantization of timelike triangles for which we will find a qualitatively different set of constraints.

4.4 Describing Timelike Surfaces: Coherent States for the Continuous Series

Now we will consider coherent states for the continuous series $\mathcal{C}_s^\epsilon$. As mentioned previously, states of the continuous series represent triangles having spacelike normal vectors. As such we should not use a basis of eigenvectors of J^3 which is the temporal component of the 3d Minkowski vector $\vec{F}$. Rather, we choose a basis of $SU(1,1)$ consisting of eigenvectors of the spatial component K^1 as in

$$
K^1|j\,\lambda\,\sigma\rangle = \lambda|j\,\lambda\,\sigma\rangle, \tag{4.83}
$$

$$
Q|j\,\lambda\,\sigma\rangle = j(j+1)|j\,\lambda\,\sigma\rangle, \tag{4.84}
$$

$$
\langle j\,\lambda'\,\sigma'|j\,\lambda\,\sigma\rangle = \delta_{\sigma'\sigma}\delta(\lambda' - \lambda), \tag{4.85}
$$

where $-\infty < \lambda < \infty$ and $\sigma = 0, 1$. The construction of these states is given in [29].

We label the corresponding basis states of $\mathcal{H}_{(\rho,n)}$ by $|\Psi_{j\,\lambda\,\sigma}\rangle$ for which the inner product of two such states is given by

$$\langle \Psi_{j'\,\lambda'\,\sigma'} | \Psi_{j\,\lambda\,\sigma} \rangle = \frac{\delta(s'-s)}{\mu_\epsilon(s)} \delta(\lambda'-\lambda)\delta_{\sigma'\sigma}, \tag{4.86}$$

where

$$\mu_\epsilon(s) = \begin{cases} -i\tanh(\pi s), & \epsilon = 0, \\ -i\coth(\pi s), & \epsilon = \frac{1}{2}. \end{cases} \tag{4.87}$$

Note that $\epsilon = 0, 1/2$ if $n/2$ is either an integer or half-integer respectively. To compute expectation values we must therefore introduce a regularization scheme. Hence we define the smearing function

$$f_\delta(x) = \begin{cases} 1/\sqrt{\delta}, & |x| \leq \delta/2, \\ 0, & |x| \geq \delta/2, \end{cases} \tag{4.88}$$

and a set of smeared states

$$|\Psi_{j\,\lambda\,\sigma}(\delta)\rangle \equiv \int_0^\infty ds' \int_{-\infty}^\infty d\lambda' \sqrt{\mu_\epsilon(s')} f_\delta(s'-s) f_\delta(\lambda'-\lambda) |\Psi_{j'\lambda'\sigma}\rangle, \tag{4.89}$$

for which the inner product is normalized and we can then take $\delta \to 0$.[5] For example the expectation value of K^1 is calculated as

[5]This is merely a formalism which might seem more complicated than it is. The expectation values after taking $\delta \to 0$ are always what one would expect naively.

$$
\begin{aligned}
\langle \Psi_{j\,\lambda\,\sigma}(\delta)|K^1|\Psi_{j\,\lambda\,\sigma}(\delta)\rangle = & \int_0^\infty ds'' ds' \int_{-\infty}^\infty d\lambda'' d\lambda' \sqrt{\mu_\epsilon(s'')}\sqrt{\mu_\epsilon(s')} f_\delta(s''-s) f_\delta(s'-s) \\
& \times f_\delta(\lambda''-\lambda) f_\delta(\lambda'-\lambda) \langle \Psi_{j''\lambda''\sigma}|K^1|\Psi_{j'\lambda'\sigma}\rangle, \qquad (4.90) \\
= & \int_0^\infty ds'' ds' \int_{-\infty}^\infty d\lambda'' d\lambda' \sqrt{\mu_\epsilon(s'')}\sqrt{\mu_\epsilon(s')} f_\delta(s''-s) f_\delta(s'-s) \\
& \times f_\delta(\lambda''-\lambda) f_\delta(\lambda'-\lambda) \frac{\lambda}{\mu_\epsilon(s')} \delta(s''-s')\delta(\lambda''-\lambda'), \qquad (4.91) \\
= & \lambda \int_0^\infty ds' \int_{-\infty}^\infty d\lambda' f_\delta^2(s'-s) f_\delta^2(\lambda'-\lambda), \qquad (4.92) \\
= & \lambda \int_{s-\delta/2}^{s+\delta/2} \int_{\lambda-\delta/2}^{\lambda+\delta/2} \frac{1}{\delta^2}, \qquad (4.93) \\
= & \lambda. \qquad (4.94)
\end{aligned}
$$

With these formalities taken care of we compute the variance of $\vec{F}$ as

$$
\begin{aligned}
(\Delta F)^2 &= \langle F^i F_i\rangle^2 - \langle F^i\rangle\langle F_i\rangle, \\
&= \langle \Psi_{j\,\lambda\,\sigma}(\delta)|Q|\Psi_{j\,\lambda\,\sigma}(\delta)\rangle^2 + \langle \Psi_{j\,\lambda\,\sigma}(\delta)|K^1|\Psi_{j\lambda\sigma}(\delta)\rangle^2, \\
&\overset{\delta\to 0}{=} j(j+1) + \lambda^2, \\
&= -\frac{1}{4} - s^2 + \lambda^2. \qquad (4.95)
\end{aligned}
$$

We can minimize ΔF by taking $\lambda = \sqrt{s^2 + \frac{1}{4}}$ which is rather cumbersome. We instead choose the simpler value $\lambda = s$ in which case

$$
\frac{\Delta F}{|\vec{F}|} = \frac{\sqrt{1/4}}{s} = O\left(\frac{1}{|\vec{F}|}\right), \qquad (4.96)
$$

which satisfies Eq. (4.61). We note that if one chooses either value for λ one will arrive at the same end result. Once again we can generalise these coherent states over $SU(1,1)$ by the group action as

$$|\Psi^{\tau}_{j\,g\,\delta}\rangle \equiv D^{(\rho,n)}(g)|\Psi^{\tau}_{j\,s\,1}\rangle, \tag{4.97}$$

where $g \in SU(1,1)$[6]. Once again this has the effect of rotating the K^1 axis to an arbitrary direction and we can relate the generalised coherent states to a set of classical normal vectors.

In this case, instead of factoring $SU(1,1)$ by $U(1)$ which is the one parameter subgroup generated by J^3 we will factor by the one parameter subgroup generated by K^1 which we will call G_1. In Appendix D.2 it is shown explicitly how each element $g \in SU(1,1)$ can be factored as $g_0 h$ where $h \in G_1$ and g_0 is a representative of the coset $SU(1,1)/(G_1 \otimes \mathbb{Z}_2)$[7]. It is then shown that

$$\langle j\,s\,1|g_0^{\dagger}\vec{F}g_0|j\,s\,1\rangle = \pm s\vec{N}, \tag{4.98}$$

where $\vec{N} \in \mathbb{H}_{\text{sp}}$ which is the one-sheeted spacelike hyperboloid shown in Fig. 4.2. Referring to Eq. (4.48) we see that we can interpret $\vec{N}$ as the quantum analog of the normal vector to the triangle. The reference states $|j\,s\,1\rangle$ corresponds to $\vec{N} = (0,1,0)$ which is the direction of the K^1 axis.

Furthermore, Eq. (4.98) establishes the isomorphism $SU(1,1)/(G_1 \otimes \mathbb{Z}_2) \cong \mathbb{H}_{\text{sp}}$. Therefore each generalised coherent state $\Psi^{\tau}_{jg\delta}$ is determined (up to a phase) by a vector $\vec{N} \in S^2$. In a completeness relation the phases will cancel and it will be sufficient to consider the states

$$|\Psi^{\tau}_{j\,\vec{N}\,\delta}\rangle \equiv |\Psi^{\tau}_{j\,g(\vec{N})\,\delta}\rangle_{\pm}. \tag{4.99}$$

We will use the reference state $|\Psi^{\tau}_{j\,s\,1}\rangle$ in calculations since the general result will follow by gauge invariance

To calculate the simplicity constraint and ΔG in Eqs. (4.62) and (4.63) we need the matrix elements of the generator J^1. These matrix elements were not found anywhere in the literature and were needed to be calculated. A derivation is contained in Appendix B. The result is

[6]We have chosen $\sigma = 1$ as a convention since both values of σ cover the space $\mathbb{H}_{\text{sp}}$.

[7]The factor of $\mathbb{Z}_2$ is related to a redundancy in the parameterization.

$$
\begin{aligned}
J^1|\Psi_{j\lambda\sigma}\rangle = &-\frac{1}{2}(\rho/2+i(j+1))((j+1)^2+\lambda^2)\hat{C}_{j+1}|\Psi_{j+1\,\lambda\sigma'}\rangle \\
&+\lambda A_j|\Psi_{j\lambda\sigma}\rangle \\
&-2(\rho/2-ij)\hat{C}_j|\Psi_{j-1\,\lambda\sigma'}\rangle,
\end{aligned} \tag{4.100}
$$

where A_j and $\hat{C}_j$ are given by

$$
A_j = \frac{n\rho}{4j(j+1)}, \tag{4.101}
$$

$$
\hat{C}_j = \frac{1}{j}\frac{\left(\frac{n^2}{4}-j^2\right)^{1/2}}{\sqrt{2j+1}\sqrt{2j-1}}. \tag{4.102}
$$

Therefore the simplicity constraint in Eq. (4.62) gives

$$
\begin{aligned}
0 &= \langle\Psi_{js\sigma}(\delta)|\vec{F}|\Psi_{js\sigma}(\delta)\rangle - \frac{1}{\gamma}\langle\Psi_{js\sigma}(\delta)|\vec{G}|\Psi_{js\sigma}(\delta)\rangle, \\
&\overset{\delta\to 0}{=} s - \frac{1}{\gamma}(sA_j),
\end{aligned} \tag{4.103}
$$

or

$$
\gamma = A_j = \frac{\rho n}{4j(j+1)}. \tag{4.104}
$$

We again note that only the F^1 component has a nonvanishing expectation value since F^0 and F^2 are linear combinations of raising/lowering operators. Finally, the variance in G is given by

$$
\begin{aligned}
(\Delta G)^2 &= \langle G^i G_i\rangle^2 - \langle G^i\rangle\langle G_i\rangle, \\
&= \langle\Psi_{js\sigma}(\delta)|Q - \frac{1}{4}(n^2-\rho^2-4)|\Psi_{js\sigma}(\delta)\rangle^2 + \langle\Psi_{js\sigma}(\delta)|K^1|\Psi_{js\sigma}(\delta)\rangle^2, \\
&\overset{\delta\to 0}{=} j(j+1) - \frac{1}{4}(n^2-\rho^2-4) + (sA_j)^2,
\end{aligned} \tag{4.105}
$$

where we again used Eq. (4.79) to write G^2 in terms of Q and C_1. Using Eq. (4.104)

$$\begin{aligned}(\Delta G)^2 &= \frac{\rho n}{4\gamma} - \frac{1}{4}(n^2 - \rho^2 - 4) + s^2\gamma^2, \\ &= \frac{\rho n}{4\gamma} - \frac{1}{4}(n^2 - \rho^2) + \gamma^2\left(s^2 + \frac{1}{4}\right) - \frac{\gamma^2}{4} + 1, \\ &= \frac{\rho n}{4\gamma} - \frac{1}{4}(n^2 - \rho^2) + \frac{\gamma\rho n}{4} - \frac{\gamma^2}{4} + 1, \\ &= \frac{1}{4}\Big(\rho - \gamma n\Big)\left(\rho + \frac{n}{\gamma}\right) - \frac{\gamma^2}{4} + 1. \end{aligned} \tag{4.106}$$

Once again choosing either $\rho = \gamma n$ or $\rho = -n/\gamma$ makes the first term vanish and thus

$$\frac{\Delta G}{|\vec{G}|} = \frac{\sqrt{|-\gamma^2/4 + 1|}}{\gamma s} = O\left(\frac{1}{|\vec{G}|}\right), \tag{4.107}$$

which satisfies Eq. (4.63). Substituting $\rho = \gamma n$ into Eq. (4.104) we get

$$n^2 = 4j(j+1) = 4\left(-\frac{1}{4} - s^2\right), \tag{4.108}$$

which is a contradiction since n is a positive integer and s is a positive real number. Next substituting $n = -\gamma\rho$ into Eq. (4.104) we get

$$\gamma = -\frac{\rho^2\gamma}{4j(s^2 + 1/4)}, \tag{4.109}$$

or

$$\frac{\rho}{2} = -\sqrt{s^2 + 1/4}. \tag{4.110}$$

Therefore we have arrived at the constraints $n = -\gamma\rho$ and $\rho/2 = -\sqrt{s^2 + 1/4}$ which are qualitatively different from the constraints for spacelike triangles. Note that this also implies $\rho < -1$ and thus $n > \gamma$. Finally the area spectrum is given by the expectation value of Eq. (4.49) as

$$A = \gamma\sqrt{-\langle Q\rangle} = \gamma\sqrt{s^2 + 1/4} = -\gamma\frac{\rho}{2} = \frac{n}{2}. \tag{4.111}$$

from which we conclude that the area spectrum of timelike surfaces is quantized.

Chapter 5

A Spin Foam Model for General Lorentzian 4-Geometries

We will construct a spin foam model of quantum gravity by imposing the simplicity constraints on the BF theory partition function in Eq. (3.22). A summary of the constraints found in Sections 4.1 and 4.2 is given in Table 5.1.

To impose the simplicity constraints we will restrict the representations (n, ρ) of $SL(2, \mathbb{C})$ and the representations j of the little groups to those summarized in Table 5.1. Further, we will restrict the possible quantum states to those of the physical Hilbert space. States in the physical Hilbert space are those which represent simple bivectors. If a state is simple then it will be closed under the appropriate little groups which depends on whether the normal vectors U, N are either spacelike or timelike.

Since the coherent states we constructed in Sections 4.1 and 4.2 satisfy the simplicity constraints as expectation values we will take these to be our physical states. Moreover, these coherent states form an overcomplete basis. We will construct completeness relations as integrals over these coherent states and use these completeness relations as projectors onto the physical Hilbert space. These projectors will be functions of U and ζ where $\zeta = \pm 1$ will be used to specify whether N is either timelike or spacelike respectively.

5.1 Projector onto the Physical Hilbert Space

The completeness relation in an irrep $\mathcal{D}_j$ of $SU(2)$ can be written as an integral over coherent states as [26]

Classical data	$U=(1,0,0,0)$ N spacelike $\star B$ spacelike	$U=(0,0,0,1)$ N timelike $\star B$ spacelike	$U=(0,0,0,1)$ N spacelike $\star B$ timelike
little group	SU(2)	SU(1,1)	SU(1,1)
reference coherent states	$\lvert j\, j\rangle \in \mathcal{D}_j$	$\lvert j\, \pm j\rangle \in \mathcal{D}_j^{\pm}$	$\lvert j\, s\, 1\rangle \in \mathcal{C}_s^{\epsilon}$
quotient space	$\vec{N} \in S^2$	$\vec{N} \in \mathbb{H}_{\pm}$	$\vec{N} \in \mathbb{H}_{\mathrm{sp}}$
constraints on (ρ, n)	$\rho = \gamma n$	$\rho = \gamma n$	$n = -\gamma \rho$
constraints on j	$j = n/2$	$j = -n/2$	$s^2 + 1/4 = \rho^2/4$
area spectrum	$\gamma\sqrt{j(j+1)}$	$\gamma\sqrt{j(j+1)}$	$\gamma\sqrt{s^2+1/4}$

Table 5.1: The constraints on representation labels obtained in Sections 4.1 and 4.2. U is the normal to a tetrahedron which is either timelike or spacelike and gauge fixed to be $U = (1,0,0,0)$ or $U = (0,0,0,1)$ respectively. For U timelike the normal vectors N of triangles in the tetrahedron must be spacelike and correspond to irreps of $SU(2)$. For U spacelike N can be either timelike or spacelike and correspond to irreps of the discrete or continuous series of $SU(1,1)$ respectively.

$$\mathbb{1}_j = \sum_{m=-j}^{j} |j\,m\rangle\langle j\,m| = (2j+1) \int\limits_{SU(2)} dg |j\,g\rangle\langle j\,g|. \tag{5.1}$$

Therefore the projector onto the subspace of $\mathcal{H}_{(\rho,n)}$ isomorphic to $\mathcal{D}_j$ is given by the following integral over coherent states

$$P_j = (2j+1) \int\limits_{SU(2)} dg |\Psi_{j\,g}\rangle\langle\Psi_{j\,g}|, \tag{5.2}$$

where the generalised coherent states in the $SU(2)$ basis are given by

$$|\Psi_{jg}\rangle \equiv D^{(\rho,n)}(g)|\Psi_{jj}\rangle, \qquad g \in SU(2). \tag{5.3}$$

Similarly, the completeness relations for the irrep D_j^τ of the discrete series can also be written as an integral over coherent states as [26]

$$\mathbb{1}_j^\pm = \sum_{\pm m=-j}^{\infty} |j\,m\rangle\langle j\,m| = (2j+1) \int\limits_{SU(1,1)} dg |j\,g\rangle_\pm\langle j\,g|_\pm, \tag{5.4}$$

and thus the projector onto the subspace $D_j^\pm$ of $\mathcal{H}_{(\rho,n)}$ is given by

$$P_j^\tau = (2j+1) \int\limits_{SU(1,1)} dg |\Psi^\tau_{j\,g}\rangle\langle\Psi^\tau_{j\,g}|, \tag{5.5}$$

where the generalised coherent states in the $SU(1,1)$ discrete series are given by

$$|\Psi^\pm_{jg}\rangle \equiv D^{(\rho,n)}(g)|\Psi^\pm_{j\,\pm j}\rangle, \qquad g \in SU(1,1). \tag{5.6}$$

The completeness relation for the continuous series is more subtle due to the smearing of the states. The derivation of the projector for the continuous series is given in [13] by

$$P_j^\tau(\delta) = (2j+1) \int\limits_{SU(1,1)} dg |\Psi^\tau_{jg\delta}\rangle\langle\Psi^\tau_{jg\delta}|, \tag{5.7}$$

where $|\Psi^\tau_{jg\delta}\rangle$ is defined in Eq. (4.97). Writing the little group subspaces as

$$H(\zeta, U) \equiv \begin{cases} SU(2), & \text{if} \quad \zeta = 1, \quad U = (1,0,0,0), \\ SU(1,1), & \text{if} \quad \zeta = \pm 1, \quad U = (0,0,0,1), \\ 0, & \text{if} \quad \zeta = -1, \quad U = (1,0,0,0), \end{cases} \tag{5.8}$$

we can combine Eqs. (5.2), (5.5), and (5.7) into

$$P_j(\zeta, U, \delta) = (2j+1) \sum_{\tau=\pm} \int\limits_{H(\zeta,U)} dh |\Psi^\tau_{jh\delta}\rangle\langle\Psi^\tau_{jh\delta}|, \tag{5.9}$$

where $h \in H(\zeta, U)$ and there is only one value of τ for $SU(2)$. Now setting j to the values given in Table 5.1

$$j = \begin{cases} n/2, & \text{if} \quad \zeta = 1, \quad U = (1,0,0,0), \\ -n/2, & \text{if} \quad \zeta = 1, \quad U = (0,0,0,1), \\ -\frac{1}{2} + \frac{i}{2}\sqrt{n^2/\gamma^2 - 1}, & \text{if} \quad \zeta = -1, \quad U = (0,0,0,1), \end{cases} \tag{5.10}$$

the projector in Eq. (5.9) projects onto the physical Hilbert space for all cases of ζ and U. Explicitly this is given by

$$P^{(\rho,n)}_{\text{phys}}(\zeta, U, \delta) = d_n(\zeta, U) \sum_{\tau} \int\limits_{H(\zeta,U)} dh |\Psi^\tau_{jh\delta}\rangle\langle\Psi^\tau_{jh\delta}|, \tag{5.11}$$

where

$$d_n(\zeta, U) \equiv \begin{cases} n+1, & \text{if} \quad \zeta = 1, \quad U = (1,0,0,0), \\ \theta(n-2)(1-n), & \text{if} \quad \zeta = 1, \quad U = (0,0,0,1), \\ \theta(n-\gamma) i \sqrt{n^2/\gamma^2 - 1}, & \text{if} \quad \zeta = -1, \quad U = (0,0,0,1), \\ 0, & \text{if} \quad \zeta = -1, \quad U = (1,0,0,0), \end{cases} \tag{5.12}$$

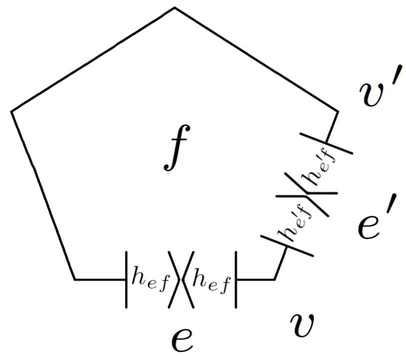

Figure 5.1: A pictorial representation of the insertion of the projector in Eq. (5.11) into the BF theory face amplitudes in Eq. (3.22) which computes the trace over the physical Hilbert space. A projector is inserted on each edge of the face which sandwiches pairs of connection variables g_{ev} and $g_{ve'}$ (attached to half edges) between coherent states $\langle\Psi^{\tau_{ef}}_{j_{ef}h_{ef}\delta}|$ and $|\Psi^{\tau_{e'f}}_{j_{e'f}h_{e'f}\delta}\rangle$. The result is the face amplitude given in Eq. (5.14).

where $\theta(x)$ is the Heaviside function which vanishes for $x < 0$ and is unity for $x > 0$. This then takes into account the constraint that $n \geq 2$ for the discrete series and $n > \gamma$ for the continuous series.

One can equivalently express these projectors as integrals over the spaces of normal vectors S^2, $\mathbb{H}_\pm$, and $\mathbb{H}_{\text{sp}}$, which is done in [13]. However because of subtleties in the smearing of states in the continuous series the notation for this is more cumbersome. Therefore we choose to express the projectors of coherent states as integrals over the little groups $H(\zeta, U)$ as in Eq. (5.11).

5.2 Partition function

The strategy for imposing the simplicity constraints on the BF theory partition function in Eq. (3.21) is to restrict the $SL(2,\mathbb{C})$ representation labels (ρ, n) and to compute the trace in the BF theory face amplitudes in Eq. (3.22) using the coherent states we have constructed. To do this we will insert the projector in Eq. (5.11) at each edge of a given face, i.e. between the representation matrices of the two connection variables in Eq. (3.22). This is shown pictorially in Fig. 5.1 for two of the edges of a face but in actuality a projector is inserted at all the edges around the face completing a loop.

In order to use the projector in Eq. (5.11) one needs geometrical data a priori, namely a value for U_e for each edge[1] and a value $\zeta_f = \pm 1$ for each face in order to choose the correct little group. Hence assuming this data is specified, the projectors are inserted at each edge and the face amplitude in Eq. (3.22) becomes

$$d_{n_f}(\zeta_f, U_e) \int\limits_{SL(2,\mathbb{C})} \prod_{v\subset f} dg_{ev} dg_{ve'} \int\limits_{H(U_e,\zeta_f)} \prod_{e\subset f} dh_{ef} \sum_{\tau_{ef}} \langle \Psi^{\tau_{ef}}_{j_{ef} h_{ef} \delta} | D^{(\rho_f, n_f)}(g_{ev} g_{ve'}) | \Psi^{\tau_{e'f}}_{j_{e'f} h_{e'f} \delta} \rangle. \tag{5.13}$$

We must compute the product of all the face amplitudes before doing the sums over the geometrical data. Therefore, we pull out the integral over $H(U_e, \zeta_f)$, the sum over τ_{ef} and the factor $d_{n_f}(\zeta_f, U_e)$ into the partition function. Moreover, we express label the amplitudes in terms of vertices by defining

$$A_v((\rho_f, n_f); h_{ef}, \tau_{ef}, \delta) = \int\limits_{SL(2,\mathbb{C})} \prod_{v\subset f} dg_{ev} dg_{ve'} \langle \Psi^{\tau_{ef}}_{j_{ef} h_{ef} \delta} | D^{(\rho_f, n_f)}(g_{ev} g_{ve'}) | \Psi^{\tau_{e'f}}_{j_{e'f} h_{e'f} \delta} \rangle, \tag{5.14}$$

and the final partition function of the theory is given by

$$Z = \sum_{U_e, \zeta_f, \tau_{ef}} \sum_{n_f} d_{n_f}(\zeta_f, U_e) \int\limits_{H(U_e,\zeta_f)} dh_{ef} \prod_{f\in\Delta^*} (1+\gamma^{2\zeta_f})\, n_f^2 \lim_{\delta\to 0} \prod_{v\in\Delta^*} A_v((\zeta_f \gamma^{\zeta_f}, n_f); h_{ef}, \tau_{ef}, \delta). \tag{5.15}$$

The partition function can be understood by the following recipe. We begin by assigning the following geometrical data to Δ^* in the following order:

1. A timelike/spacelike normal vector $U_e = (1, 0, 0, 0)$ or $U_e = (0, 0, 0, 1)$ for each edge e.
2. A spacelike/timelike normal vector for each face f given by $\zeta_f = \pm 1$ respectively.
3. An integer n_f to each face f to specify an $SL(2, \mathbb{C})$ representation $(n_f, \zeta_f n_f^{\zeta_f})$.
4. A coherent state $|\Psi^{\tau_{ef}}_{j_{ef} h_{ef} \delta}\rangle$ to each edge e and adjacent face f.

Once this data is specified one can compute the vertex amplitudes in Eq. (5.14) by integrating over the connection variables. One then multiplies all the vertex amplitudes together with the other factors in the partition function. Finally one sums over all possible values for U_e, ζ_f, τ_{ef}, all representations n_f, and integrates over all the coherent states parameterized by h_{ef}.

[1] Recall that an edge $e \in \Delta^*$ is dual to a tetrahedron $t \in \Delta$, for which U_e is the normal vector.

Chapter 6

Conclusion

We began with a review of classical General Relativity in the tetrad formalism. We first derived the equations of motion for the Einstein-Hilbert action in which torsion was assumed to vanish. We then reviewed the Hilbert-Palatini formulation of General Relativity in which torsion was not assumed to vanish in the action but was obtained instead as an equation of motion. Finally, it was shown that the Holst term could be added to the action without changing the classical equations of motion while introducing a free parameter known as the Immirzi parameter. The Hilbert-Palatini action with a Holst term was our starting point for the quantization of General Relativity.

We then reviewed the quantization of a similar but simpler theory known as BF theory. The classical BF action was seen to be equivalent to the Hilbert Palatini action with a Holst term provided that the B two forms were constrained to be simple bivectors. The BF action was then formulated on a discretized space and expressed in terms of holonomies. The discretized BF action was quantized and a path integral was defined. The path integral was then reduced to a product over face amplitudes and a sum over representations of $SL(2,\mathbb{C})$. All that was left was to impose the simplicity constraints on the BF theory path integral.

A key step before imposing the simplicity constraints was the gauge fixing of the normal vectors of the tetrahedra. Before gauge fixing one must first choose whether a normal vector is timelike or spacelike. If one assumes that a normal vector is timelike and gauge fixes it to $(1,0,0,0)$ then all the triangles in the tetrahedron will be forced to be spacelike. The EPRL model consisted of tetrahedra of solely this type. Alternatively, the normal vector could be assumed spacelike and gauge fixed to $(0,0,0,1)$ in which case the triangles in the tetrahedron could be either spacelike or timelike. It was our objective to impose the simplicity constraints for this alternative possibility and thus extend the EPRL model to timelike surfaces.

Since the Master Constraint method of the EPRL model was unsuitable for imposing the simplicity constraints for the $(0, 0, 0, 1)$ case an alternative method was needed. The alternative method we used was a coherent state method similar to the one used in the Euclidean FK model. Since the FK model was only defined for Euclidean signature we first had to formulate a Lorentzian version of the model. To do this we needed to construct a new set of coherent states for $SL(2, \mathbb{C})$. These coherent states were constructed such that the uncertainties in the corresponding bivectors were minimal and such that they satisfied the classical simplicity constraints as expectation values. These constraints translated into constraints on the $SL(2, \mathbb{C})$ representation labels as well as the little group representation labels where the little group was determined by the normal vector of the tetrahedron.

We then proceeded to use the coherent state method for the same case considered in the EPRL model in which the normal vectors of the tetrahedra were chosen to be timelike and gauge fixed to $(1, 0, 0, 0)$. In this case the tetrahedra were Euclidean and the coherent states were constructed as states in the unitary irreducible representations of the little group $SU(2)$. What was found by using the coherent state method for this case was precisely the constraints of the EPRL model. This provided support for the EPRL results since it followed from an independent derivation. Moreover, it also confirmed the validity of the coherent state method as a means of imposing the simplicity constraints for the case of Lorentzian tetrahedra.

The same procedures used to reproduce the EPRL results were then applied to the case of Lorentzian tetrahedra. In this case the coherent states were constructed as states in the unitary irreducible representations of the little group $SU(1, 1)$. States in the discrete/continuous series of $SU(1, 1)$ were found to correspond to classical spacelike/timelike triangles respectively. The area spectrum of the timelike states was shown to be discrete and thus the discreteness of area was extended to timelike surfaces.

In deriving the coherent states in the continuous series it was necessary to use a basis of $SU(1, 1)$ which was diagonalised with respect to one of the noncompact generators. The matrix elements of the generators of $SL(2, \mathbb{C})$ were required but had not been calculated before in this basis. A derivation of these matrix elements was given in Appendix B.

A path integral was then constructed as a sum over geometries with spacelike and timelike surfaces. This was done by restricting the representation labels in the quantum BF theory and computing the trace in the face amplitudes using states in the physical Hilbert space, i.e. those satisfying the simplicity constraints. The final partition function was then given by a sum over all possible choices of geometrical data (normal vectors of tetrahedra and triangles), a sum over representations of $SL(2, \mathbb{C})$ (restricted by the simplicity constraints), an integral over the coherent states, and a product of vertex amplitudes.

We have thus extended the EPRL model to timelike surfaces. This new model allows one to consider both spacelike and timelike boundary states such as for a finite region.

The inclusion of timelike surfaces could also be necessary to connect with a Hamiltonian version of the theory. For example, in the case of causal dynamical triangulations we know that the inclusion of timelike and null edges is required [30].

Further, we have lifted the restriction of the EPRL model to triangulations with only spacelike triangles. The freedom to use these more general triangulations might be necessary to avoid artifacts or distortions that could arise when the triangulations are restricted.

One possible extension of this model would be to include null surfaces. States corresponding to null surfaces belong to what is called the complementary series of $SL(2, \mathbb{C})$ which have zero measure in the Plancherel decomposition [31]. Thus these states are not included in the completeness relations which is why we we did not require them in the path integral. The inclusion of null surfaces would then permit fully general triangulations.

Finally, we note that since all quantization procedures are merely rules of thumb the validity of this theory is to be determined by the investigation of its behaviour. Such investigations include the asymptotics in the large area limit, the derivation of the graviton propagator, as well as the descriptions of intertwiners in terms of tetrahedra which have all been done for previous models [32] [23] [33].

Appendix A

$SL(2, \mathbb{C})$ in a Basis of J^3 Eigenvectors

In this appendix we review the construction of the unitary irreducible representations of $SL(2, \mathbb{C})$. We also review the decomposition of these irreps into direct sums of $SU(2)$ and $SU(1,1)$ irreps. The irreps of $SU(2)$ and $SU(1,1)$ are decomposed into the standard bases as eigenvectors of J^3. These basis states are given in terms of the Wigner functions defined in A.5. Finally, in A.6 we review the construction of an inner product between states in the $SU(2)$ basis and states in the $SU(1,1)$ discrete series.

A.1 Lie Algebra of $SL(2, \mathbb{C})$

The Lie algebra of $SL(2, \mathbb{C})$ is spanned by the six generators J^i, K^i for $i = 1, 2, 3$ which satisfy the following commutation relations

$$[J^i, J^j] = i\epsilon^{ij}{}_k J^k, \tag{A.1}$$

$$[J^i, K^j] = i\epsilon^{ij}{}_k K^k, \tag{A.2}$$

$$[K^i, K^j] = -i\epsilon^{ij}{}_k J^k. \tag{A.3}$$

In the fundamental representation these generators are given by the Pauli matrices as

$$J^i = \frac{\sigma_i}{2}, \qquad K^i = \frac{i\sigma_i}{2}, \tag{A.4}$$

and correspond to the one-parameter subgroups

$$a_1(\psi) = \begin{pmatrix} \cos(\psi/2) & i\sin(\psi/2) \\ i\sin(\psi/2) & \cos(\psi/2) \end{pmatrix}, \qquad b_1(u) = \begin{pmatrix} \cosh(u/2) & -\sinh(u/2) \\ -\sinh(u/2) & \cosh(u/2) \end{pmatrix}, \tag{A.5}$$

$$a_2(\theta) = \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}, \qquad b_2(t) = \begin{pmatrix} \cosh(t/2) & i\sinh(t/2) \\ -i\sinh(t/2) & \cosh(t/2) \end{pmatrix}, \tag{A.6}$$

$$a_3(\varphi) = \begin{pmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{pmatrix}, \qquad b_3(\alpha) = \begin{pmatrix} e^{-\alpha/2} & 0 \\ 0 & e^{\alpha/2} \end{pmatrix}, \tag{A.7}$$

where

$$a_i(\tau) \equiv e^{iJ^i\tau}, \qquad b_i(\tau) \equiv e^{iK^i\tau}. \tag{A.8}$$

Moreover, there exist two Casimir operators which are given by

$$C_1 = 2\delta_{ij}(J^iJ^j - K^iK^j), \qquad C_2 = -4\delta_{ij}J^iK^j. \tag{A.9}$$

The subgroup $SU(2)$ is generated by the subset of generators J^1, J^2, and J^3 which satisfy the commutation relations in Eq. (A.1) and produce the Casimir operator

$$\vec{J}^2 = \delta_{ij}J^iJ^j. \tag{A.10}$$

The subgroup $SU(1,1)$ is generated by the subset of generators K^1, K^2, and J^3 which satisfy the commutation relations

$$[J^3, K^1] = iK^2, \qquad [J^3, K^2] = -iK^1, \qquad [K^1, K^2] = -iJ^3, \tag{A.11}$$

and produce the Casimir operator

$$Q = (J^3)^2 - (K^1)^2 - (K^2)^2. \tag{A.12}$$

A.2 Representation of $SL(2,\mathbb{C})$

In this Appendix we construct the unitary irreducible representations of $SL(2,\mathbb{C})$ as can be found in [15]. Let F be a function of two complex variables z_1 and z_2. We define the operator $D(g)$ for $g \in SL(2,\mathbb{C})$ on the space of such functions by

$$D(g)F(z_1,z_2) = F(az_1 + cz_2, bz_1 + dz_2), \tag{A.13}$$

where

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{A.14}$$

and $ac - bd = 1$. This can be viewed as a matrix multiplication in the following sense

$$\begin{pmatrix} z_1 & z_2 \\ \cdots & \cdots \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} az_1 + cz_2 & bz_1 + dz_2 \\ \cdots & \cdots \end{pmatrix}. \tag{A.15}$$

This provides a representation of $SL(2,\mathbb{C})$ since

$$\begin{aligned} D(g_1)D(g_2)F(z_1,z_2) &= D(g_1)\Big[D(g_2)F\Big](z_1,z_2), \\ &= D(g_2)F(a_1z_1 + c_1z_2, b_1z_1 + d_1z_2), \\ &= F\Big(a_2(a_1z_1 + c_1z_2) + c_2(b_1z_1 + d_1z_2), \\ &\qquad b_2(a_1z_1 + c_1z_2) + d_2(b_1z_1 + d_1z_2)\Big), \\ &= F\Big((a_1a_2 + b_1c_2)z_1 + (c_1a_2 + d_1c_2)z_2, \\ &\qquad (a_1b_2 + b_1d_2)z_1 + (c_1b_2 + d_1d_2)z_2\Big), \\ &= D(g_1g_2)F(z_1,z_2), \end{aligned} \tag{A.16}$$

where

$$g_1g_2 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1c_2 & c_1b_2 + d_1d_2 \end{pmatrix}. \tag{A.17}$$

To find the irreducible representations we will consider only those functions which are homogeneous of degree λ_1 and λ_2 respectively. By this we mean that for any complex number $\alpha \neq 0$

$$F(\alpha z_1, \alpha z_2) = \alpha^{\lambda_1}\overline{\alpha}^{\lambda_2}F(z_1, z_2). \tag{A.18}$$

Note that $\lambda_1 - \lambda_2$ much be an integer in order for such a function to be well defined[1]. Next, we define a scalar product on the space of these homogeneous functions by

$$\langle F_1|F_2\rangle = \int \overline{F_1(z_1, z_2)}F_2(z_1, z_2)d\mu(z_1, z_2), \tag{A.19}$$

where the measure $d\mu(z_1, z_2)$ is defined to be invariant under the transformations in Eq. (A.13). This establishes a Hilbert space consisting of all such square integrable functions. Moreover, the operators $D(g)$ on this Hilbert space are unitary since

$$\begin{aligned}\langle D(g)F_1|D(g)F_2\rangle &= \int \overline{F_1(az_1 + cz_2, bz_1 + dz_2)}F_2(az_1 + cz_2, bz_1 + dz_2)d\mu(z_1, z_2),\\ &= \int \overline{F_1(z_1', z_2')}F_2(z_1', z_2')d\mu(z_1', z_2'),\\ &= \langle F_1|F_2\rangle.\end{aligned} \tag{A.20}$$

where we used the invariance of the measure. Furthermore, we get the condition

$$\begin{aligned}\|F\|^2 &= \langle F|F\rangle,\\ &= \int \overline{F(z_1, z_2)}F(z_1, z_2)d\mu(z_1, z_2),\\ &= \int \overline{F(\alpha z_1, \alpha z_2)}F(\alpha z_1, \alpha z_2)d\mu(\alpha z_1, \alpha z_2),\\ &= \int \overline{\alpha}^{\overline{\lambda_1}}\alpha^{\overline{\lambda_2}}\overline{F(\alpha z_1, \alpha z_2)}\alpha^{\lambda_1}\overline{\alpha}^{\lambda_2}F(\alpha z_1, \alpha z_2)\alpha^2\overline{\alpha}^2 d\mu(z_1, z_2),\\ &= \alpha^{\lambda_1+\overline{\lambda_2}+2}\overline{\alpha}^{\overline{\lambda_1}+\lambda_2+2}\|F\|^2,\\ &= \left|\alpha^{\lambda_1+\overline{\lambda_2}+2}\right|^2\|F\|^2,\end{aligned} \tag{A.21}$$

[1]To see this take $\alpha = e^{2\pi i}$.

which implies $\lambda_1 + \overline{\lambda_2} + 2 = 0$. As we noted above $\lambda_1 - \lambda_2$ must be an integer therefore we have the solution

$$\lambda_1 = \frac{n}{2} + \frac{i\rho}{2} - 1, \tag{A.22}$$

$$\lambda_2 = -\frac{n}{2} + \frac{i\rho}{2} - 1, \tag{A.23}$$

where n is an integer and ρ is real. The representation just described is referred to as the Principle series in which each pair (n, ρ) labels a unitary irreducible representation (irrep) of $SL(2, \mathbb{C})$. It can be shown that two irreps (n, ρ) and (n', ρ') are equivalent if $(n, \rho) = (-n', -\rho')$ therefore we will restrict n to be positive.

The Hilbert space corresponding to the representation (n, ρ) with the inner product given in Eq. (A.19) will be denoted by $\mathcal{H}_{(\rho,n)}$. Operators corresponding to $SL(2, \mathbb{C})$ transformations will be denoted $D^{(\rho,n)}(g)$. The $SL(2, \mathbb{C})$ Casimirs in these representations take the values

$$C_1 = \frac{1}{2}(n^2 - \rho^2 - 4), \tag{A.24}$$

$$C_2 = n\rho. \tag{A.25}$$

A.3 Reduction of $SL(2, \mathbb{C})$ into Irreps of $SU(2)$

We will now review the decomposition of the irreps of $SL(2, \mathbb{C})$ into a direct sum of irreps of $SU(2)$ as given in [15]. To do this, notice that by homogeneity the above representation is completely determined by its action on the unit sphere since

$$F(z_1, z_2) = (|z_1|^2 + |z_2|^2)^{i\rho/2-1} F(u_1, u_2), \tag{A.26}$$

where

$$u_1 = \frac{z_1}{\sqrt{|z_1|^2 + |z_2|^2}}, \tag{A.27}$$

$$u_2 = \frac{z_2}{\sqrt{|z_1|^2 + |z_2|^2}}, \tag{A.28}$$

and thus $|u_1|^2+|u_2|^2 = 1$. Furthermore, the action of $SL(2,\mathbb{C})$ elements can be represented entirely on this subspace since

$$\begin{aligned} D^{(\rho,n)}(g)F(u_1,u_2) &= F(au_1+cu_2, bu_1+du_2), \\ &= \left(|au_1+cu_2|^2+|bu_1+du_2|^2\right)^{i\rho/2-1} F(u_1', u_2'), \end{aligned} \tag{A.29}$$

where

$$u_1' = \frac{au_1+cu_2}{(|au_1+cu_2|^2+|bu_1+du_2|^2)^{1/2}}, \tag{A.30}$$

$$u_2' = \frac{bu_1+du_2}{(|au_1+cu_2|^2+|bu_1+du_2|^2)^{1/2}}. \tag{A.31}$$

and thus $|u_1'|^2+|u_2'|^2 = 1$. Finally, we associate the normalized spinor (u_1, u_2) with an element of $SU(2)$ by

$$u = \begin{pmatrix} u_1 & u_2 \\ -\overline{u}_2 & \overline{u}_1 \end{pmatrix}, \tag{A.32}$$

so that we can consider the function space to be defined over $SU(2)$ and we will therefore write

$$f(u) \equiv F(u_1, u_2). \tag{A.33}$$

Due to homogeneity, however, this identification is only unique up to a phase such that

$$\begin{aligned} f(\gamma u) &= F(e^{i\omega}u_1, e^{i\omega}u_2), \\ &= e^{i(\lambda_1-\lambda_2)\omega}F(u_1,u_2), \\ &= e^{in\omega}f(u), \end{aligned} \tag{A.34}$$

where

$$\gamma = \begin{pmatrix} e^{i\omega} & 0 \\ 0 & e^{-i\omega} \end{pmatrix}. \tag{A.35}$$

Therefore, we will have to enforce the covariance condition in Eq. (A.34) when viewing the function space to be over $SU(2)$.

We can recast the inner product in Eq. (A.19) as in integral over $SU(2)$

$$\langle f|f'\rangle = \int_{SU(2)} \overline{f(u)} f'(u) d\mu(u), \tag{A.36}$$

where $\mu(u)$ is the Haar measure on $SU(2)$. An explicit parametrization of the measure is given by

$$d\mu(u) = (2\pi)^{-2} dr d\theta_1 d\theta_2, \tag{A.37}$$

where

$$(u_1, u_2) = (\sqrt{r}e^{i\theta_1}, \sqrt{1-r}e^{i\theta_2}), \tag{A.38}$$

for $0 \le r \le 1$ and $0 \le \theta_1, \theta_2 \le 2\pi$.

We may now invoke the Plancherel theorem which asserts that the functions

$$(2j+1)^{1/2} D^j_{m_1\, m_2}(u) \qquad \text{for} \qquad j = 0, \frac{1}{2}, 1, \frac{3}{2}, ... \tag{A.39}$$

for $-j \le m_1, m_2 \le j$ are a complete and orthonormal basis for the Hilbert space $\mathcal{H}_{(\rho,n)}$ of functions over the group $SU(2)$. The functions $D^j_{m_1\, m_2}(u)$ are the matrix elements of $SU(2)$ which can be parameterized as

$$\begin{aligned} D^j_{m_1\, m_2}(u) &= \langle j\, m_1 | D^j(u) | j\, m_2\rangle, \\ &= \langle j\, m_1 | e^{i\theta_1 J^3} e^{i\theta J^2} e^{i\theta_2 J^3} | j\, m_2\rangle, \\ &= e^{im_1\theta_1} d^j_{m_1\, m_2}(\theta) e^{im_2\theta_2}, \end{aligned} \tag{A.40}$$

where

$$d^j_{m_1\, m_2}(\theta) \equiv \langle j m_1 | e^{i\theta J^2} | j\, m_2\rangle. \tag{A.41}$$

The explicit form of the functions $d^j_{m_1\, m_2}(\theta)$ are given in Appendix A.5. Enforcing the covariance condition in Eq. (A.34) we first notice that

$$\begin{aligned}
\gamma u &= \begin{pmatrix} e^{i\omega} & 0 \\ 0 & e^{-i\omega} \end{pmatrix} \begin{pmatrix} e^{i\theta_1/2} & 0 \\ 0 & e^{-i\theta_1/2} \end{pmatrix} e^{i\eta J^2} e^{i\theta_2 J^3}, \\
&= \begin{pmatrix} e^{i(\theta_1+2\omega)/2} & 0 \\ 0 & e^{-i(\theta_1+2\omega)/2} \end{pmatrix} e^{i\eta J^2} e^{i\theta_2 J^3}, \\
&= e^{i(\theta_1+2\omega)J^3} e^{i\eta J^2} e^{i\theta_2 J^3},
\end{aligned} \tag{A.42}$$

therefore

$$\begin{aligned}
D^j_{m_1\, m_2}(\gamma u) &= e^{im_1(\theta_1+2\omega)} d^j_{m_1\, m_2}(\eta) e^{im_2\theta_2}, \\
&= e^{2im_1\omega} D^j_{m_1\, m_2}(u).
\end{aligned} \tag{A.43}$$

Now consider the inner product

$$\begin{aligned}
\langle D^j_{m_1\, m_2} | f \rangle &= \int_{SU(2)} \overline{D^j_{m_1\, m_2}(u)} f(u) d\mu(u), \\
&= \int_{SU(2)} \overline{D^j_{m_1\, m_2}(\gamma u)} f(\gamma u) d\mu(u), \\
&= e^{-i(n-2m_1)\omega} \int_{SU(2)} \overline{D^j_{m_1\, m_2}(u)} f(u) d\mu(u), \\
&= e^{-i(n-2m_1)\omega} \langle D^j_{m_1\, m_2} | f \rangle,
\end{aligned} \tag{A.44}$$

where we used the invariance of the measure. This implies that $m_1 = n/2$ and therefore the subset of functions

$$|\Psi_{j\, m}\rangle \equiv (2j+1)^{1/2} D^j_{n/2\, m}(u) \qquad \text{for} \qquad j = \frac{n}{2}, \frac{n}{2}+1, \frac{n}{2}+2, ..., \tag{A.45}$$

and for $-j \leq m \leq j$ also form a complete and orthonormal basis for $\mathcal{H}_{(\rho,n)}$. Thus we may expand any function $f(u)$ as

$$f(u) = \sum_{j=n/2}^{\infty} \sum_{m=-j}^{j} (2j+1) \langle D^j_{n/2\, m} | f \rangle D^j_{n/2\, m}(u), \tag{A.46}$$

with the inner product defined in Eq. (A.36). This basis for the representation space of $SL(2,\mathbb{C})$ using irreps of $SU(2)$ is referred to as the Canonical basis. In terms of the standard Hilbert spaces $\mathcal{D}_j$ of $SU(2)$ we thus have the isomorphism

$$\mathcal{H}_{(\rho,n)} \simeq \bigoplus_{j=n/2}^{\infty} \mathcal{D}_j \,, \tag{A.47}$$

and the completeness relation

$$\mathbb{1}_{(\rho,n)} = \sum_{j=n/2}^{\infty} \sum_{m=-j}^{j} |\Psi_{j\,m}\rangle\langle\Psi_{j\,m}| \,. \tag{A.48}$$

A.4 Reduction of $SL(2,\mathbb{C})$ into Irreps of $SU(1,1)$

Similarly, the irreps of $SL(2,\mathbb{C})$ can be decomposed into a direct sum of irreps of $SU(1,1)$ [15]. Indeed, the same representation of $SL(2,\mathbb{C})$ can be completely specified by its action on the unit hyperboloid

$$|v_1|^2 - |v_2|^2 = \tau, \tag{A.49}$$

where $\tau = \pm 1$. By homogeneity we have

$$F(z_1, z_2) = [\tau(|z_1|^2 - |z_2|^2)]^{i\rho/2-1} F(v_1, v_2), \tag{A.50}$$

where

$$v_1 = \frac{z_1}{\sqrt{\tau(|z_1|^2 - |z_2|^2)}}, \qquad v_2 = \frac{z_2}{\sqrt{\tau(|z_1|^2 - |z_2|^2)}}, \tag{A.51}$$

and

$$\tau = +1 \qquad \text{for} \qquad |z_1|^2 \geq |z_2|^2, \tag{A.52}$$

$$\tau = -1 \qquad \text{for} \qquad |z_1|^2 \leq |z_2|^2. \tag{A.53}$$

Again the action of $SL(2,\mathbb{C})$ elements can be represented entirely on this hyperboloid since

$$D^{(\rho,n)}(g)F(v_1,v_2) = F(av_1+cv_2, bv_1+dv_2),$$
$$= \left(\tau|av_1+cv_2|^2 - \tau|bv_1+dv_2|^2\right)^{i\rho/2-1} F(v_1', v_2'), \tag{A.54}$$

where

$$v_1' = \frac{av_1+cv_2}{(\tau|av_1+cv_2|^2 - \tau|bv_1+dv_2|^2)^{1/2}}, \tag{A.55}$$
$$v_2' = \frac{bv_1+dv_2}{(\tau|av_1+cv_2|^2 - \tau|bv_1+dv_2|^2)^{1/2}}, \tag{A.56}$$

and thus $|v_1'|^2 - |v_2'|^2 = \tau$.

We would now like to express F as a function of the group $SU(1,1)$ so that we can use the Plancherel decomposition theorem. Given a point (v_1, v_2) on the τ branch of the unit hyperboloid we can parameterize an element of $SU(1,1)$ in the following way[2]

$$v = \begin{pmatrix} v_1 & v_2 \\ \overline{v}_2 & \overline{v}_1 \end{pmatrix}, \qquad \tau = +1, \tag{A.57}$$
$$v = \begin{pmatrix} \overline{v}_2 & \overline{v}_1 \\ v_1 & v_2 \end{pmatrix}, \qquad \tau = -1. \tag{A.58}$$

In view of Eqs. (A.52) and (A.53) the two values of τ correspond to two disjoint subspaces of $\mathbb{C}^2$. Thus, in order to express F over $SU(1,1)$ and still represent the entire Hilbert space of $SL(2,\mathbb{C})$ we need to use both branches of the hyperboloid. In other words, each function F corresponds to a pair of functions f_τ over $SU(1,1)$ given by

$$f_\tau(v) = [\tau(|z_1|^2 - |z_2|^2)]^{-i\rho/2+1} F(z_1, z_2). \tag{A.59}$$

As in the $SU(2)$ case we have the following covariance condition

$$f_\tau(\gamma v) = e^{i\tau n\omega} f_\tau(v), \tag{A.60}$$

[2]Note that these two paramterizations are related via multiplication on the left by the Pauli matrix σ_1.

where γ is given in Eq. (A.35). Notice the factor of τ in the exponent of the phase factor which is a result of the relation $\gamma\sigma_1 = \sigma_1\gamma^{-1}$. Furthermore, since the functions f_τ are defined on disjoint subspaces of $\mathbb{C}^2$, the inner product in Eq. (A.19) is given by

$$\langle F|F'\rangle = \sum_\tau \int_{SU(1,1)} \overline{f_\tau(v)} f'_\tau(v) d\mu(v), \tag{A.61}$$

where $d\mu(v)$ is an invariant measure on $SU(1,1)$. An explicit parametrization of this measure is given by

$$d\mu(v) = (2\pi)^{-2} dw d\theta_1 d\theta_2, \tag{A.62}$$

where

$$(v_1, v_2) = (\sqrt{w}e^{i\theta_1}, \sqrt{w-1}e^{i\theta_2}), \qquad \tau = +1, \tag{A.63}$$

$$(v_1, v_2) = (\sqrt{w-1}e^{i\theta_1}, \sqrt{w}e^{i\theta_2}), \qquad \tau = -1, \tag{A.64}$$

for $1 \le w < \infty$ and $0 \le \theta_1, \theta_2 \le 2\pi$. Here the measure factor is not unique since $SU(1,1)$ is not compact so we just choose it to match the inner product over $SU(2)$.

We will again use the Plancherel theorem to decompose the functions $f_\tau(v)$ into matrix elements of $SU(1,1)$. However, since $SU(1,1)$ is not compact the decomposition will involve an integral over a continuous set of functions as well as a sum over a discrete set. Indeed, the set of functions

$$(2j+1)^{1/2} D^j_{m_1\, m_2} \quad \text{for} \quad j = -1, -\frac{3}{2}, ..., \qquad \text{discrete series,} \tag{A.65}$$

$$(2j+1)^{1/2} D^j_{m_1\, m_2} \quad \text{for} \quad j = -\frac{1}{2} + is, \quad 0 < s < \infty, \quad \text{continuous series,} \tag{A.66}$$

form a complete and orthonormal basis of $\mathcal{H}_{(\rho,n)}$. The functions $D^j_{m_1\, m_2}$ are the matrix elements of $SU(1,1)$ which can be parameterized as

$$\begin{aligned} D^j_{m_1\, m_2}(v) &= \langle j\, m_1 | D^j(v) | j\, m_2 \rangle, \\ &= \langle j\, m_1 | e^{i\theta_1 J^3} e^{i\eta K^2} e^{i\theta_1 J^3} | j\, m_2 \rangle, \\ &= e^{i m_1 \theta_1} d^j_{m_1\, m_2}(\eta) e^{i m_2 \theta_2}, \end{aligned} \tag{A.67}$$

where

$$d^j_{m_1\,m_2}(\eta) \equiv \langle j m_1 | e^{i\eta K^2} | j\, m_2 \rangle. \tag{A.68}$$

The explicit form of the functions $d^j_{m_1\,m_2}(\eta)$ is give in Appendix A.5. Enforcing the covariance condition in Eq. (A.60) we first notice that

$$\begin{aligned}
\gamma v &= \begin{pmatrix} e^{i\omega} & 0 \\ 0 & e^{-i\omega} \end{pmatrix} \begin{pmatrix} e^{i\theta_1/2} & 0 \\ 0 & e^{-i\theta_1/2} \end{pmatrix} e^{i\eta K^2} e^{i\theta_2 J^3}, \\
&= \begin{pmatrix} e^{i(\theta_1+2\omega)/2} & 0 \\ 0 & e^{-i(\theta_1+2\omega)/2} \end{pmatrix} e^{i\eta K^2} e^{i\theta_2 J^3}, \\
&= e^{i(\theta_1+2\omega)J^3} e^{i\eta K^2} e^{i\theta_2 J^3}
\end{aligned} \tag{A.69}$$

therefore

$$\begin{aligned}
D^j_{m_1\,m_2}(\gamma v) &= e^{i m_1(\theta_1+2\omega)} d^j_{m_1\,m_2}(\eta) e^{i m_2 \theta_2}, \\
&= e^{2 i m_1 \omega} D^j_{m_1\,m_2}(v),
\end{aligned} \tag{A.70}$$

Now consider the inner product

$$\begin{aligned}
\langle D^j_{m_1\,m_2} | f_\tau \rangle &= \int_{SU(1,1)} \overline{D^j_{m_1\,m_2}(v)} f_\tau(v) d\mu(v), \\
&= \int_{SU(1,1)} \overline{D^j_{m_1\,m_2}(\gamma v)} f_\tau(\gamma v) d\mu(v), \\
&= e^{-i(\tau n - 2m_1)\omega} \int_{SU(1,1)} \overline{D^j_{m_1\,m_2}(v)} f_\tau(v) d\mu(v), \\
&= e^{-i(\tau n - 2m_1)\omega} \langle D^j_{m_1\,m_2} | f_\tau \rangle,
\end{aligned} \tag{A.71}$$

where we used the invariance of the measure. This implies that $m_1 = \tau n/2$ and therefore the subset of functions

$$|\Psi^+_{j\,m}\rangle \equiv (2j+1)^{1/2} \begin{pmatrix} D^j_{n/2\,m}(v) \\ 0 \end{pmatrix}, \qquad |\Psi^-_{j\,m}\rangle \equiv (2j+1)^{1/2} \begin{pmatrix} 0 \\ D^j_{-n/2\,m}(v) \end{pmatrix}, \tag{A.72}$$

$$|\Psi^+_{s\,m}\rangle \equiv (2j+1)^{1/2} \begin{pmatrix} D^{-\frac{1}{2}+is}_{\tau n/2\,m}(v) \\ 0 \end{pmatrix}, \qquad |\Psi^-_{s\,m}\rangle \equiv (2j+1)^{1/2} \begin{pmatrix} 0 \\ D^{-\frac{1}{2}+is}_{\tau n/2\,m}(v) \end{pmatrix}, \tag{A.73}$$

also form a complete and orthonormal basis of $\mathcal{H}_{(\rho,n)}$. For the discrete series $\mathcal{D}_j^\tau$

$$j=-1,-2,... \qquad \text{or} \qquad j=-\frac{3}{2},-\frac{5}{2},..., \qquad \text{with} \qquad \tau m = j, j+1, j+2, ..., \tag{A.74}$$

such that $j-n/2$ is an integer. Note that $j=-1/2$ is excluded from the Plancherel decomposition because it is not normalizable [28]. For the continuous series $\mathcal{C}_s^\epsilon$

$$0 \le s < \infty, \qquad \text{with} \qquad \pm\, m = \epsilon, \epsilon+1, \epsilon+2, ... \tag{A.75}$$

where $\epsilon = 0, \frac{1}{2}$ such that $\epsilon - n/2$ is an integer.

Furthermore, using the inner product in Eq. (A.61) we have the following orthogonality relations

$$\langle \Psi^\tau_{j\,m} | \Psi^{\tau'}_{j'\,m'} \rangle = \delta_{j'j}\delta_{m'm}\delta_{\tau'\tau}, \tag{A.76}$$

$$\langle \Psi^\tau_{s\,m} | \Psi^{\tau'}_{s'\,m'} \rangle = \frac{\delta(s'-s)}{\mu_\epsilon(s)}\delta_{m'm}\delta_{\tau'\tau}, \tag{A.77}$$

$$\langle \Psi^\tau_{j\,m} | \Psi^{\tau'}_{s'\,m'} \rangle = 0. \tag{A.78}$$

In terms of the Hilbert spaces of the discrete series $\mathcal{D}_j^\tau$ and the continuous series $\mathcal{C}_s^\epsilon$ of $SU(1,1)$ we have the isomorphism

$$\mathcal{H}_{(\rho,n)} = \left(\bigoplus_{-j>1/2}^{n/2} \mathcal{D}_j^+ \oplus \int_0^\infty ds\, \mathcal{C}_s^\epsilon \right) \oplus \left(\bigoplus_{-j>1/2}^{n/2} \mathcal{D}_j^- \oplus \int_0^\infty ds\, \mathcal{C}_s^\epsilon \right), \tag{A.79}$$

and the completeness relation

$$\begin{aligned} \mathbb{1}_{(\rho,n)} = & \sum_{-j>1/2}^{\infty} \sum_{m=j}^{\infty} |\Psi^+_{j\,m}\rangle\langle\Psi^+_{j\,m}| + \int_0^\infty ds\, \mu_\epsilon(s) \sum_{\pm m=\epsilon}^{\infty} |\Psi^+_{s\,m}\rangle\langle\Psi^+_{s\,m}| \\ & + \sum_{-j>1/2}^{\infty} \sum_{-m=j}^{\infty} |\Psi^-_{j\,m}\rangle\langle\Psi^-_{j\,m}| + \int_0^\infty ds\, \mu_\epsilon(s) \sum_{\pm m=\epsilon}^{\infty} |\Psi^-_{s\,m}\rangle\langle\Psi^-_{s\,m}|, \end{aligned} \tag{A.80}$$

where

$$\mu_\epsilon(s) = \begin{cases} -i\tanh(\pi s) & \epsilon = 0, \\ -i\coth(\pi s) & \epsilon = \frac{1}{2}. \end{cases} \tag{A.81}$$

A.5 Representation matrices of $SU(2)$ and $SU(1,1)$

The $SU(2)$ matrix elements can be parameterized by

$$\begin{aligned} D^j_{m_1\,m_2}(u) &= \langle j\; m_1|D^j(u)|j\; m_2\rangle, \\ &= \langle j\; m_1|e^{i\theta_1 J^3}e^{i\theta J^2}e^{i\theta_2 J^3}|j\; m_2\rangle, \\ &= e^{im_1\theta_1}d^j_{m_1\,m_2}(\theta)e^{im_2\theta_2}, \end{aligned} \tag{A.82}$$

where

$$d^j_{m_1\,m_2}(\theta) \equiv \langle jm_1|e^{i\theta J^2}|j\; m_2\rangle. \tag{A.83}$$

Similarly, the matrix elements of $SU(1,1)$ can be parameterized as

$$\begin{aligned} D^j_{m_1\,m_2}(v) &= \langle j\; m_1|D^j(v)|j\; m_2\rangle, \\ &= \langle j\; m_1|e^{i\theta_1 J^3}e^{itK^2}e^{i\theta_1 J^3}|j\; m_2\rangle, \\ &= e^{im_1\theta_1}b^j_{m_1\,m_2}(z(t))e^{im_2\theta_2}, \end{aligned} \tag{A.84}$$

where

$$b^j_{m_1\,m_2}(t) \equiv \langle jm_1|e^{itK^2}|j\; m_2\rangle = \sqrt{(-1)^{m_2-m_1}}d^j_{m_1 m_2}(it). \tag{A.85}$$

Explicitly we have

$$d^j_{m_1\,m_2}(\theta) = \frac{1}{(m_1-m_2)!}N^j_{m_1 m_2}F^j_{m_1 m_2}(z(\theta)), \tag{A.86}$$

where

$$F^j_{m_1 m_2}(z) = (1-z)^{(m_1+m_2)/2}\,(z)^{(m_1-m_2)/2}\,{}_2F_1\left(-j+m_1, j+m_1+1; m_1-m_2+1; z\right), \tag{A.87}$$

$$N^j_{m_1 m_2} = \left[\prod_{l=0}^{m_1-m_2-1}(j+m_1-l)(j-m-l)\right]^{\frac{1}{2}} = \left[\frac{(j+m_1)!(j-m_2)!}{(j+m_2)!(j-m_1)!}\right]^{\frac{1}{2}}, \tag{A.88}$$

$$z(\theta) = \frac{1}{2}(1-\cos\theta), \tag{A.89}$$

for $m_1 - m_2 \geq 0$ and $m_1 + m_2 \geq 0$. Note that in the parametrization of $SU(2)$ in Eq. (A.38) we have $z = r$ and for $SU(1,1)$ in Eqs. (A.63) and (A.64) we have $z = w$. For the other three possibilities for m_1 and m_2 we use [3]

$$\begin{array}{ll} d^j_{m_1\,m_2}(z) = (-1)^{m_1-m_2} d^j_{m_2\,m_1}(z), & m_1+m_2 \geq 0, \quad m_1 - m_2 \leq 0, \\ d^j_{m_1\,m_2}(z) = (-1)^{m_1-m_2} d^j_{-m_1\,-m_2}(z), & m_1+m_2 \leq 0, \quad m_1 - m_2 \leq 0, \\ d^j_{m_1\,m_2}(z) = d^j_{-m_2\,-m_1}(z), & m_1+m_2 \leq 0, \quad m_1 - m_2 \geq 0. \end{array} \tag{A.90}$$

Notice that $F^j_{m_1 m_2}(z(\theta)) = F^{-j-1}_{m_1 m_2}(z(\theta))$ since the hypergeometric function is symmetric in the first two arguments. Moreover from Eq. (A.88) we see that $(N^j_{m_1 m_2})^2 = (N^{-j-1}_{m_1 m_2})^2$ and for large $|j|$ $N^j_{m_1 m_2} \sim j^{m_1 - m_2}$ therefore

$$N^j_{m_1 m_2} = (-1)^{m_1 - m_2} N^{-j-1}_{m_1 m_2}, \tag{A.91}$$

Hence we have the relation

$$d^j_{m_1\,m_2}(z) = (-1)^{m_1-m_2} d^{-j-1}_{m_1\,m_2}(z). \tag{A.92}$$

This relation will be useful when comparing elements of the $SU(2)$ basis for which $j > 0$ to elements in the $SU(1,1)$ discrete series for which $j < 0$.

A.6 Inner product between the $SU(2)$ and $SU(1,1)$ bases

Elements of the $SU(2)$ basis and the $SU(1,1)$ discrete series belong to the same Hilbert space $\mathcal{H}_{(\rho,n)}$ which was constructed in Appendix A.2. Therefore it is meaningful to compute the inner product between such vectors. A discussion of this is contained in [15]. To do so we will relate elements of $SU(2)$ to elements of $SU(1,1)$ by the virtue that they both represent the same homogeneous pair of variables (z_1, z_2). We will then parameterize an element of $SU(2)$ using the parametrization of an element of $SU(1,1)$ and once both vectors share the same parametrization we will compute their inner product using Eq. (A.61).

Let (u_1, u_2) and (v_1, v_2) correspond to elements $u \in SU(2)$ and $v_\tau \in SU(1,1)$ respectively. Using the parameterizations in Eqs. (A.38), (A.63), and (A.64) these matrices are given by

[3] See [34].

$$u = \begin{pmatrix} \sqrt{r}e^{i\theta_1} & \sqrt{1-r}e^{i\theta_2} \\ -\sqrt{1-r}e^{-i\theta_2} & \sqrt{r}e^{-i\theta_1} \end{pmatrix}, \qquad 0 \leq r \leq 1, \tag{A.93}$$

$$v_+ = \begin{pmatrix} \sqrt{w}e^{i\theta_1} & \sqrt{w-1}e^{i\theta_2} \\ \sqrt{w-1}e^{-i\theta_2} & \sqrt{w}e^{-i\theta_1} \end{pmatrix}, \qquad 1 \leq w \leq \infty, \tag{A.94}$$

$$v_- = \begin{pmatrix} \sqrt{w-1}e^{i\theta_1} & \sqrt{w}e^{i\theta_2} \\ \sqrt{w}e^{-i\theta_2} & \sqrt{w-1}e^{-i\theta_1} \end{pmatrix}, \qquad 1 \leq w \leq \infty, \tag{A.95}$$

$$\tag{A.96}$$

and $0 \leq \theta_1, \theta_2 \leq 2\pi$.

Since these pairs correspond to the homogeneous variables (z_1, z_2) the following ratios must be equal

$$\frac{u_1}{u_2} = \frac{v_1}{v_2}. \tag{A.97}$$

Therefore from Eq. (A.97) we have the relations

$$\tau = +1: \qquad \frac{r}{1-r} = \frac{w}{w-1} \qquad \Longrightarrow \qquad r = \frac{w}{2w-1} \quad \text{and} \quad \frac{1}{2} \leq r \leq 1, \tag{A.98}$$

$$\tau = -1: \qquad \frac{r}{1-r} = \frac{w-1}{w} \qquad \Longrightarrow \qquad r = \frac{w-1}{2w-1} \quad \text{and} \quad 0 \leq r \leq \frac{1}{2}. \tag{A.99}$$

Therefore we define the $SU(2)$ elements

$$u_+(v) = \left(\sqrt{\frac{w}{2w-1}}e^{i\theta_1}, \sqrt{\frac{w-1}{2w-1}}e^{i\theta_2} \right), \qquad 1 \leq w \leq \infty, \tag{A.100}$$

$$u_-(v) = \left(\sqrt{\frac{w-1}{2w-1}}e^{i\theta_1}, \sqrt{\frac{w}{2w-1}}e^{i\theta_2} \right), \qquad 1 \leq w \leq \infty, \tag{A.101}$$

and the inner products between basis vectors of the $SU(2)$ basis and the $SU(1,1)$ discrete series can then be computed by

$$\langle \Psi_{j'\,m'} | \Psi^{\tau}_{j\,m} \rangle = \sqrt{2j'+1}\sqrt{2j+1} \int_{SU(1,1)} \overline{D^{j'}_{n/2\,m'}(u_\tau(v))} D^{j}_{\tau n/2\,m}(v) d\mu(v). \tag{A.102}$$

where the representation matrices for $SU(2)$ and $SU(1,1)$ are given in Appendix A.5.

Appendix B

The Continuous Series in a Basis of K^1 Eigenvectors

We now wish to construct a basis of eigenvectors of the generator K^1 for the continuous series. Since this generator corresponds to a noncompact subgroup, the eigenvalues λ will be continuous and these states will thus be non-normalizable. The basis vectors are therefore postulated to satisfy

$$Q|j\,\lambda\,\sigma\rangle = j(j+1)|j\,\lambda\,\sigma\rangle \qquad j=-\frac{1}{2}+is, \qquad 0<s<\infty, \tag{B.1}$$

$$K^1|j\,\lambda\,\sigma\rangle = \lambda|j\,\lambda\,\sigma\rangle, \qquad -\infty<\lambda<\infty, \tag{B.2}$$

$$\langle\, j\lambda\,\sigma|j\,\lambda'\,\sigma'\rangle = \delta_{\sigma\,\sigma'}\delta(\lambda-\lambda'), \tag{B.3}$$

where $\sigma = 0, 1$ resolves a two-fold degeneracy for states diagonalized with respect to K^1. This degeneracy arises because there exists an outer automorphism[1]

$$(J^3, K^1, K^2) \to (-J^3, K^1, -K^2), \tag{B.4}$$

which is realised by conjugation by an operator P where

$$P|j\,m\rangle = e^{i\pi m}|j\ -m\rangle, \tag{B.5}$$

$$P|j\,\lambda\,\sigma\rangle = (-1)^\sigma|j\,\lambda\,\sigma\rangle. \tag{B.6}$$

[1]Refer to the commutation relations in Eq. (A.11).

Since $[K^1, P] = 0$ both operators can be diagonalized simultaneously.

We can then express the matrix elements of $SU(1,1)$ using a combination of this basis $|j\,\lambda\,\sigma\rangle$ and the standard basis $|j\,m\rangle$ by

$$\begin{aligned} D^j_{m\lambda\sigma}(v) &= \langle j\,m|D^j(v)|j\,\lambda\,\sigma\rangle, \\ &= \langle j\,m|e^{-i\varphi J^3}e^{-itK^2}e^{iuK^1}|j\,\lambda\,\sigma\rangle, \\ &= e^{-im\varphi}d^j_{m\lambda\sigma}(t)e^{i\lambda u}, \end{aligned} \tag{B.7}$$

where

$$d^j_{m\lambda\sigma}(t) \equiv \langle j\,m|e^{-itK^2}|j\,\lambda\,\sigma\rangle. \tag{B.8}$$

The functions $d^j_{m\lambda\sigma}(t)$ are given in [27] by

$$\begin{aligned} d^j_{m\lambda\sigma}(t) = \left[\frac{\Gamma(m-j)}{\Gamma(m+j+1)}\right]^{1/2} &\frac{2^{j-1}\Gamma(-j+i\lambda)}{i^\sigma \sin\frac{\pi}{2}(-j+\sigma-i\lambda)} \\ &\times \left\{\frac{2^{-m}F^j_{m\lambda}(t)}{\Gamma(-m-j)\Gamma(m+1+i\lambda)} - \frac{(-1)^\sigma 2^m F^j_{-m\lambda}(-t)}{\Gamma(m-j)\Gamma(-m+1+i\lambda)}\right\}, \end{aligned} \tag{B.9}$$

where

$$\begin{aligned} F^j_{m\lambda}(t) = \left(\cosh\frac{t}{2} - i\sinh\frac{t}{2}\right)^{m-i\lambda} &\left(\cosh\frac{t}{2} + i\sinh\frac{t}{2}\right)^{m+i\lambda} \\ &\times {}_2F_1\Big(m-j, m+j+1; m+1+i\lambda; (1+i\sinh t)/2\Big). \end{aligned} \tag{B.10}$$

B.1 Differential Operators

We wish to find the differential operators corresponding to the parametrization of $SU(1,1)$ used to define the matrix elements in Eq. (B.7), i.e.

$$v = e^{-i\varphi J^3}e^{-itK^2}e^{iuK^1}. \tag{B.11}$$

By applying the derivative operators ∂_φ, ∂_t, and ∂_u to v we can solve for the infinitesimal generators K^1, K^2, and K^3 as differential operators acting on the right. To do this we use the commutation relations in Eq. (A.11) and the Hadamard lemma

$$e^A B e^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + ... \tag{B.12}$$

from which we can derive

$$\partial_u v = v(iK^1), \tag{B.13}$$

$$\partial_t v = v\left(-iK^2 \cosh u + iJ^3 \sinh u\right), \tag{B.14}$$

$$\partial_\varphi v = v\left(-iJ^3 \cosh u \cosh t + iK^2 \sinh u \cosh t + iK^1 \sinh t\right), \tag{B.15}$$

Solving for the differential operators we obtain

$$K^1 = -i\partial_u, \tag{B.16}$$

$$K^2 = i\,\mathrm{sech}t \sinh u \partial_\varphi + i\cosh u \partial_t - i\tanh t \sinh u \partial_u, \tag{B.17}$$

$$J^3 = i\,\mathrm{sech}t \cosh u \partial_\varphi + i\sinh u \partial_t - i\tanh t \cosh u \partial_u. \tag{B.18}$$

The generators of $SL(2,\mathbb{C})$ outside of $SU(1,1)$, namely J^1, J^2, and K^3 can be determined from the following definitions

$$J^i f_\tau(v) = -i\frac{\partial}{\partial\psi} D^{(\rho,n)}(a_i(\psi)) f_\tau(v)\Big|_{\tau=0}, \qquad K^i f_\tau(v) = -i\frac{\partial}{\partial\psi} D^{(\rho,n)}(b_i(\psi)) f_\tau(v)\Big|_{\tau=0}, \tag{B.19}$$

where $a_i(\psi)$, $b_i(\psi)$ are the one parameter subgroup generated by J^i, K^i and are given in Appendix A.1. For the generator J^1 we find

$$J^1 f_\tau(v) = (\frac{\rho}{2} + i)\sinh t f_\tau(v) - i\frac{\partial}{\partial\psi} f_\tau(v')\Big|_{\psi=0}. \tag{B.20}$$

where we've used

$$
\begin{aligned}
v &= e^{-i\varphi J^3} e^{-itK^2} e^{iuK^1}, \\
&= \begin{pmatrix} e^{i\varphi/2}\cosh\frac{t}{2}\cosh\frac{u}{2} + ie^{i\varphi/2}\sinh\frac{t}{2}\sinh\frac{u}{2} & -e^{i\varphi/2}\cosh\frac{t}{2}\sinh\frac{u}{2} - ie^{i\varphi/2}\sinh\frac{t}{2}\cosh\frac{u}{2} \\ -e^{-i\varphi/2}\cosh\frac{t}{2}\sinh\frac{u}{2} + ie^{-i\varphi/2}\sinh\frac{t}{2}\cosh\frac{u}{2} & e^{-i\varphi/2}\cosh\frac{t}{2}\cosh\frac{u}{2} - ie^{-i\varphi/2}\sinh\frac{t}{2}\sinh\frac{u}{2} \end{pmatrix} \\
&\equiv \begin{pmatrix} v_1 & v_2 \\ \overline{v}_2 & \overline{v}_1 \end{pmatrix}. \qquad \text{(B.21)}
\end{aligned}
$$

for $\tau = +1$ and $\sigma_1 v$ for $\tau = -1$. In the fundamental representation we have

$$
\begin{aligned}
J^1 &= (\frac{\rho}{2} + i)\sinh t - iv^{-1}\frac{\partial}{\partial\psi}v'\Big|_{\psi=0}, && \text{(B.22)} \\
&= (\frac{\rho}{2} + i)\sinh t + \frac{1}{2}\begin{pmatrix} -\sinh u\cosh t & \cosh t\cosh u \\ -\cosh t\cosh u & \sinh u\cosh t \end{pmatrix}, && \text{(B.23)} \\
&= \left(\frac{\rho}{2} + i\right)\sinh t + \cosh t\cosh u K^2 - \cosh t\sinh u J^3. && \text{(B.24)}
\end{aligned}
$$

Substituting in the expressions for K^2 and J^3 from Appendix B.1

$$
\begin{aligned}
J^1 = &\left(\frac{\rho}{2} + i\right)\sinh t + i\cosh t\cosh u\,(\,\mathrm{sech} t\sinh u\partial_\varphi + \cosh u\partial_t - \tanh t\sinh u\partial_u) \\
&- i\cosh t\sinh u\,(\,\mathrm{sech} t\cosh u\partial_\varphi + \sinh u\partial_t - \tanh t\cosh u\partial_u)\,, \\
= &\left(\frac{\rho}{2} + i\right)\sinh t + i\cosh t\partial_t. \qquad \text{(B.25)}
\end{aligned}
$$

The remaining differential operators can be found from commutation relations to be

$$
J^2 = \left(\frac{\rho}{2} + i\right)\cosh t\sinh u - i\tanh t\cosh u\partial_\varphi + i\sinh t\sinh u\partial_t + i\,\mathrm{sech} t\cosh u\partial_u, \quad \text{(B.26)}
$$

$$
K^3 = -\left(\frac{\rho}{2} + i\right)\cosh t\cosh u - i\tanh t\sinh u\partial_\varphi - i\sinh t\cosh u\partial_t - i\,\mathrm{sech} t\sinh u\partial_u. \quad \text{(B.27)}
$$

B.2 Action of J^1 on Eigenstates of K^1

We will now consider the action of J^1 on the matrix elements given in Eq. (B.7). Replacing j_0 with $-i\lambda$ in Eq. (A9) of [35] we have

$$
\begin{aligned}
&- i \sinh t F\Big(m-j, m+j+1; m+1+i\lambda; (1+i\sinh t)/2\Big) \\
&= \frac{(j+m+1)(j+i\lambda+1)}{(j+1)(2j+1)} F\Big(m-(j+1)), m+(j+1)+1; m+1+i\lambda; (1+i\sinh t)/2\Big) \\
&\quad + \frac{(j-m)(j-i\lambda)}{j(2j+1)} F\Big(m-(j-1), m+(j-1)+1; m+1+i\lambda; (1+i\sinh t)/2\Big) \\
&\quad - \frac{im\lambda}{j(j+1)} F\Big(m-j, m+j+1; m+1+i\lambda; (1+i\sinh t)/2\Big). \qquad \text{(B.28)}
\end{aligned}
$$

from which the action on the functions $F^j_{m\lambda}(t)$ is given by

$$
\sinh t F^j_{m\lambda}(t) = iC^{j+1}_{m\lambda} F^{j+1}_{m\lambda}(t) + C^j_{m\lambda} F^j_{m\lambda}(t) + iC^{j-1}_{m\lambda} F^{j-1}_{m\lambda}(t), \qquad \text{(B.29)}
$$

where

$$
C^{j+1}_{m\lambda} = \frac{(j+m+1)(j+i\lambda+1)}{(j+1)(2j+1)}, \qquad \text{(B.30)}
$$

$$
C^{j}_{m\lambda} = \frac{m\lambda}{j(j+1)}, \qquad \text{(B.31)}
$$

$$
C^{j-1}_{m\lambda} = \frac{(j-m)(j-i\lambda)}{j(2j+1)}. \qquad \text{(B.32)}
$$

Now let us consider the action of second term in the differential operator of J^1. First define

$$
G_{m\lambda}(t) = \left(\cosh\frac{t}{2} - i\sinh\frac{t}{2}\right)^{m-i\lambda} \left(\cosh\frac{t}{2} + i\sinh\frac{t}{2}\right)^{m+i\lambda}, \qquad \text{(B.33)}
$$

so that

$$
F^j_{m\lambda}(t) = G_{m\lambda}(t) F(a,b;c;z), \qquad \text{(B.34)}
$$

where

$$a = m - j, \tag{B.35}$$

$$b = m + j + 1, \tag{B.36}$$

$$c = m + 1 + i\lambda, \tag{B.37}$$

$$z = \frac{1 + i \sinh t}{2}. \tag{B.38}$$

One can show

$$\cosh t \frac{\partial}{\partial t} G_{m\lambda}(t) = (-\lambda + m \sinh t) G_{m\lambda}(t), \tag{B.39}$$

and so

$$\cosh t \frac{\partial}{\partial t} F^{j}_{m\lambda}(t) = G_{m\lambda}(t) \Big(-\lambda + im(1-2z) + 2iz(1-z)\frac{\partial}{\partial z}\Big) F(a,b;c;z), \tag{B.40}$$

where we used $-i \sinh t = (1-2z)$ and $\cosh^2 t = 4z(1-z)$. From Bateman 2.8 (27) we have

$$z(1-z)\frac{\partial}{\partial z} F(a,b;c;z) = (m + i\lambda) F(a-1, b-1, c-1, z) - \left[i\lambda + m(1-2z)\right] F(a,b;c;z). \tag{B.41}$$

and following a lengthy derivation [14]

$$\begin{aligned} &F(a-1, b-1; c-1; z) \\ &= -\frac{(j+i\lambda+1)(-m+i\lambda+1)(j+m+1)}{2(m+i\lambda)(j-i\lambda)(j+1)} F(a-1, b+1; c; z) \\ &\quad - \frac{(-j+m-1)(-j+m)}{2(m+i\lambda)j} F(a+1, b-1; c; z) \\ &\quad + \frac{-j+m-1}{2(m+i\lambda)} \left[\frac{(-m+i\lambda+1)(j+i\lambda+1)}{(j-i\lambda)(j+1)} - \frac{j+i\lambda+1}{j-i\lambda} + \frac{j+m}{j} \right] F(a,b;c;z) \\ &\quad - \frac{(j+i\lambda+1)(-j+m-1)}{2(m+i\lambda)(j-i\lambda)} (1-2z) F(a,b;c;z), \end{aligned} \tag{B.42}$$

which together give

$$\cosh t \frac{\partial}{\partial t} F^j_{m\lambda}(t) = ijC^{j+1}_{m\lambda} F^{j+1}_{m\lambda}(t) - C^j_{m\lambda} F^j_{m\lambda}(t) - i(j+1) C^{j-1}_{m\lambda} F^{j-1}_{m\lambda}(t), \tag{B.43}$$

where

$$C^{j+1}_{m\lambda} = \frac{(j+m+1)(j+i\lambda+1)}{(j+1)(2j+1)}, \tag{B.44}$$

$$C^{j}_{m\lambda} = \frac{m\lambda}{j(j+1)}, \tag{B.45}$$

$$C^{j-1}_{m\lambda} = \frac{(j-m)(j-i\lambda)}{j(2j+1)}. \tag{B.46}$$

Combining both terms in Eqs. (B.29) and (B.43) the action of the full differential operator is given by

$$J^1 F^j_{m\lambda}(t) = \left(\frac{i\rho}{2} - j - 1\right) C^{j+1}_{m\lambda} F^{j+1}_{m\lambda}(t) + \frac{\rho}{2} C^j_{m\lambda} F^j_{m\lambda}(t) + \left(\frac{i\rho}{2} + j\right) C^{j-1}_{m\lambda} F^{j-1}_{m\lambda}(t). \tag{B.47}$$

Write

$$d^j_{m\lambda\sigma} = S^j_m \left(T^j_{m\lambda\sigma} F^j_{m\lambda}(t) - (-1)^\sigma T^j_{-m\lambda\sigma} F^j_{-m\lambda}(-t)\right), \tag{B.48}$$

where

$$S^j_m = \left[\frac{\Gamma(m-j)}{\Gamma(m+j+1)}\right]^{1/2}, \tag{B.49}$$

and

$$T^j_{m\lambda\sigma} = \frac{2^{j-m-1}\Gamma(-j+i\lambda)}{i^\sigma \sin\frac{\pi}{2}(-j+\sigma-i\lambda)\Gamma(-m-j)\Gamma(m+1+i\lambda)}. \tag{B.50}$$

Then from Eq. (B.47) we have

$$
\begin{aligned}
J^1 T^j_{m\lambda\sigma} F^j_{m\lambda}(t) = &\left(\frac{i\rho}{2} - j - 1\right) \frac{(j+i\lambda+1)(j-i\lambda+1)}{2(j+1)(2j+1)\cot\frac{\pi}{2}(-(j+1)+\sigma-i\lambda)} T^{j+1}_{m\lambda\sigma} F^{j+1}_{m\lambda}(t) \\
&+ \frac{m\rho\lambda}{2j(j+1)} T^j_{m\lambda\sigma} F^j_{m\lambda}(t) \\
&+ \left(\frac{i\rho}{2} + j\right) \frac{2(m-j)(m+j)}{j(2j+1)\cot\frac{\pi}{2}(-(j-1)+\sigma-i\lambda)} T^{j-1}_{m\lambda\sigma} F^{j-1}_{m\lambda}(t). \qquad \text{(B.51)}
\end{aligned}
$$

To obtain the action of J^1 on the second term in Eq. (B.48) we take $m \to -m$ and $t \to -t$. However, under this transformation $J^1 \to -J^1$. This minus sign is canceled in the j term due to the factor of m but the $j+1$ and $j-1$ terms are invariant. To compensate for these minus signs notice that $T^j_{m\lambda\sigma} \to i\cot\frac{\pi}{2}(-j+\sigma-i\lambda)T^j_{m\lambda\sigma'}$. Thus changing $\sigma \to \sigma'$ in the $j+1$ and $j-1$ terms in Eq. (B.51) gives

$$
\begin{aligned}
J^1 T^j_{m\lambda\sigma} F^j_{m\lambda}(t) = &\, i\left(\frac{i\rho}{2} - j - 1\right) \frac{(j+i\lambda+1)(j-i\lambda+1)}{2(j+1)(2j+1)} T^{j+1}_{m\lambda\sigma'} F^{j+1}_{m\lambda}(t) \\
&+ \frac{m\rho\lambda}{2j(j+1)} T^j_{m\lambda\sigma} F^j_{m\lambda}(t) \\
&+ i\left(\frac{i\rho}{2} + j\right) \frac{2(m-j)(m+j)}{j(2j+1)} T^{j-1}_{m\lambda\sigma'} F^{j-1}_{m\lambda}(t). \qquad \text{(B.52)}
\end{aligned}
$$

Taking $m \to -m$ and $t \to -t$ makes $J^1 \to -J^1$ therefore

$$
\begin{aligned}
J^1 T^j_{-m\lambda\sigma} F^j_{-m\lambda}(-t) = &-i\left(\frac{i\rho}{2} - j - 1\right) \frac{(j+i\lambda+1)(j-i\lambda+1)}{2(j+1)(2j+1)} T^{j+1}_{-m\lambda\sigma'} F^{j+1}_{-m\lambda}(-t) \\
&+ \frac{m\rho\lambda}{2j(j+1)} T^j_{-m\lambda\sigma} F^j_{-m\lambda}(-t) \\
&- i\left(\frac{i\rho}{2} + j\right) \frac{2(m-j)(m+j)}{j(2j+1)} T^{j-1}_{-m\lambda\sigma'} F^{j-1}_{-m\lambda}(-t). \qquad \text{(B.53)}
\end{aligned}
$$

Finally

$$
J^1 d^j_{m\lambda\sigma}(t) = S^j_m J^1 \Big(T^j_{m\lambda\sigma} F^j_{m\lambda}(t) - (-1)^\sigma T^j_{-m\lambda\sigma} F^j_{-m\lambda}(-t) \Big), \qquad \text{(B.54)}
$$

and the substitution $(-1)^\sigma = -(-1)^{\sigma'}$ gives the correct form of the matrix element for the $j+1$ and $j-1$ terms. Therefore

$$J^1 d^j_{m\lambda\sigma}(t) = \frac{i(j+i\lambda+1)(j-i\lambda+1)\left[(m-j-1)(m+j+1)\right]^{1/2}}{2(j+1)(2j+1)}\left(\frac{i\rho}{2}-j-1\right) d^{j+1}_{m\lambda\sigma'}$$
$$+ \frac{m\rho\lambda}{2j(j+1)} d^j_{m\lambda\sigma}$$
$$+ \frac{2i\left[(m-j)(m+j)\right]^{1/2}}{j(2j+1)}\left(\frac{i\rho}{2}+j\right) d^{j-1}_{m\lambda\sigma'}. \tag{B.55}$$

Now define the states

$$|\Psi^+_{j\,\lambda\,\sigma}\rangle = \sqrt{2j+1}\begin{pmatrix} D^j_{\frac{n}{2}\lambda\sigma} \\ 0 \end{pmatrix} \qquad |\Psi^-_{j\,\lambda\,\sigma}\rangle = \sqrt{2j+1}\begin{pmatrix} 0 \\ D^j_{\frac{-n}{2}\lambda\sigma} \end{pmatrix} \tag{B.56}$$

in which case

$$J^1|\Psi^\tau_{j\,\lambda\,\sigma}\rangle = -\frac{1}{2}(\rho/2+i(j+1))((j+1)^2+\lambda^2)\widetilde{C}_{j+1}|\Psi^\tau_{j+1\,\lambda\,\sigma'}\rangle$$
$$+ \lambda A_j|\Psi^\tau_{j\,\lambda\,\sigma}\rangle$$
$$- 2(\rho/2-ij)\widetilde{C}_j|\Psi^\tau_{j-1\,\lambda\,\sigma'}\rangle, \tag{B.57}$$

where

$$\widetilde{C}_j = \frac{\sqrt{n^2/4-j^2}}{j\sqrt{2j-1}\sqrt{2j+1}}, \tag{B.58}$$
$$A_j = \frac{n\rho}{4j(j+1)}. \tag{B.59}$$

Appendix C

Matrix Elements of the Generators of $SL(2,\mathbb{C})$

Here we list the matrix elements of the generators of $SL(2,\mathbb{C})$ on states in the various bases of $SL(2,\mathbb{C})$ discussed in the Appendix. We provide the matrix elements for J^3, K^3 or K^1 and J^1 where the remaining matrix elements can be expressed as ladder operators and calculated by the commutation relations. For more details see [14].

For $SU(2)$ in a J^3 basis

$$
\begin{aligned}
J^3|\Psi_{j\,m}\rangle &= m|\Psi_{j\,m}\rangle, && \text{(C.1)}\\
K^3|\Psi_{j\,m}\rangle &= (\rho/2 + i(j+1))C_{j+1}|\Psi_{j+1\,m}\rangle \\
&\quad - mA_j|\Psi_{j\,m}\rangle \\
&\quad + (\rho/2 - ij)C_j|\Psi_{j-1\,m}\rangle, && \text{(C.2)}
\end{aligned}
$$

where

$$
\begin{aligned}
A_j &= \frac{\rho n}{4j(j+1)}, && \text{(C.3)}\\
C_j &= \frac{\sqrt{n^2/4 - j^2}\sqrt{m^2 - j^2}}{j\sqrt{2j-1}\sqrt{2j+1}}. && \text{(C.4)}
\end{aligned}
$$

For the $SU(1,1)$ discrete series in a J^3 basis

$$J^3|\Psi^\tau_{j\,m}\rangle = m|\Psi^\tau_{j\,m}\rangle, \tag{C.5}$$

$$\begin{aligned} K^3|\Psi^\tau_{j\,m}\rangle &= \tau(\rho/2 + i(j+1))C_{j+1}|\Psi^\tau_{j+1\,m}\rangle \\ &\quad - mA_j|\Psi^\tau_{j\,m}\rangle \\ &\quad + \tau(\rho/2 - ij)C_j|\Psi^\tau_{j-1\,m}\rangle, \end{aligned} \tag{C.6}$$

For the $SU(1,1)$ continuous series in a K^1 basis

$$K^1|\Psi^\tau_{j\,\lambda\,\sigma}\rangle = \lambda|\Psi^\tau_{j\,\lambda\,\sigma}\rangle, \tag{C.7}$$

$$\begin{aligned} J^1|\Psi^\tau_{j\,\lambda\,\sigma}\rangle &= -\frac{1}{2}(\rho/2 + i(j+1))((j+1)^2 + \lambda^2)\widetilde{C}_{j+1}|\Psi^\tau_{j+1\,\lambda\,\sigma'}\rangle \\ &\quad + \lambda A_j|\Psi^\tau_{j\,\lambda\,\sigma}\rangle \\ &\quad - 2(\rho/2 - ij)\widetilde{C}_j|\Psi^\tau_{j-1\,\lambda\,\sigma'}\rangle, \end{aligned} \tag{C.8}$$

where

$$\widetilde{C}_j = \frac{\sqrt{n^2/4 - j^2}}{j\sqrt{2j-1}\sqrt{2j+1}}. \tag{C.9}$$

Appendix D

Parametrization of Quotient Spaces

D.1 Isomorphism between $SU(2)/U(1)$ and S^2

In this Appendix we derive the explicit isomorphism between elements in the quotient $SU(2)/U(1)$ and the sphere S^2. This isomorphism is used to relate coherent states in the $SU(2)$ basis to Euclidean normal vectors of classical triangles.

An element $u \in SU(2)$ is parameterized by

$$u = e^{iJ^3\varphi} e^{iJ^2\theta} e^{iJ^3\psi}, \tag{D.1}$$

where

$$-\pi < \varphi \leq \pi \qquad -2\pi < \theta \leq 2\pi \qquad -\pi \leq \psi < \pi. \tag{D.2}$$

We can represent an element of the quotient space $SU(2)/U(1)$ by

$$g = e^{iJ^3\varphi} e^{iJ^2\theta}. \tag{D.3}$$

Consider the states $g|j\,m\rangle$ and the expectation values

$$\langle j\,m|g^\dagger J^i g|j\,m\rangle. \tag{D.4}$$

Using the Hadamard lemma in Eq. (B.12) we have the following relations

$$g^\dagger J^1 g = J^1 \cos\varphi + J^2 \sin\alpha - \cos\varphi \sin\theta J^3, \tag{D.5}$$

$$g^\dagger J^2 g = -\sin\varphi \cos\theta J^1 + J^2 \cos\varphi + \sin\varphi \sin\theta J^3, \tag{D.6}$$

$$g^\dagger J^3 g = J^3 \cos\theta + J^1 \sin\theta. \tag{D.7}$$

Therefore

$$\langle j\, m | g^\dagger J^1 g | j\, m\rangle = -m \cos\varphi \sin\theta, \tag{D.8}$$

$$\langle j\, m | g^\dagger J^2 g | j\, m\rangle = m \sin\varphi \sin\theta, \tag{D.9}$$

$$\langle j\, m | g^\dagger J^3 g | j\, m\rangle = m \cos\theta. \tag{D.10}$$

Hence

$$\langle j\, m | g^\dagger J^i g | j\, m\rangle = m N^i, \tag{D.11}$$

where

$$\vec{N} = (-\cos\varphi \sin\theta, \sin\varphi \sin\theta, \cos\theta), \tag{D.12}$$

and $(N^1)^2 + (N^2)^2 + (N^3)^2 = 1$.

D.2 Isomorphism between $SU(1,1)/U(1)$ and $\mathbb{H}_+ \cup \mathbb{H}_-$

In this Appendix we derive the explicit isomorphism between elements in the quotient $SU(1,1)/U(1)$ and the two-sheeted hyperboloid $\mathbb{H}_+ \cup \mathbb{H}_-$. This isomorphism is used to relate coherent states in the $SU(1,1)$ discrete series to timelike normal vectors of classical triangles.

For the parameterization of eigenstates of J^3, an element $v \in SU(1,1)$ is parameterized by

$$v = e^{i\varphi J^3} e^{itK^2} e^{i\psi J^3}. \tag{D.13}$$

where

$$-\pi < \varphi \leq \pi \qquad -2\pi < \psi \leq 2\pi \qquad 0 \leq t < \infty. \tag{D.14}$$

We can represent an element of the quotient space $SU(1,1)/U(1)$ by

$$g = e^{i\varphi J^3} e^{itK^2}. \tag{D.15}$$

Consider the states $g|j\,m\rangle$ and the expectation values

$$\langle j\,m|g^\dagger F^i g|j\,m\rangle. \tag{D.16}$$

Using the Hadamard lemma in Eq. (B.12) we have the following relations

$$g^\dagger J_3 g = J^3 \cosh t + K^1 \sinh t, \tag{D.17}$$
$$g^\dagger K_1 g = J^3 \cos\varphi \sinh t + K^1 \cos\varphi \cosh t + K^2 \sin\varphi, \tag{D.18}$$
$$g^\dagger K_2 g = -J^3 \sin\varphi \sinh t - K^1 \sin\varphi \cosh t + K^2 \cos\varphi, \tag{D.19}$$

Therefore

$$\langle j\,m|g^\dagger J_3 g|j\,m\rangle = m \cosh t, \tag{D.20}$$
$$\langle j\,m|g^\dagger K_1 g|j\,m\rangle = m \cos\varphi \sinh t, \tag{D.21}$$
$$\langle j\,m|g^\dagger K_2 g|j\,m\rangle = -m \sin\varphi \sinh t. \tag{D.22}$$

Hence

$$\langle j\,m|g^\dagger F^i g|j\,m\rangle = mN^i, \tag{D.23}$$

where

$$\vec{N} = \pm(\cosh t, \cos\varphi \sinh t, -\sin\varphi \sinh t), \tag{D.24}$$

and $(N^0)^2 - (N^1)^2 - (N^2)^2 = 1$.

D.3 Isomorphism between $SU(1,1)/(G_1 \otimes \mathbb{Z}_2)$ and $\mathbb{H}_{\text{sp}}$

In this Appendix we derive the explicit isomorphism between elements in the quotient $SU(1,1)/(G_1 \otimes \mathbb{Z}_2)$ and the one-sheeted hyperboloid $\mathbb{H}_{\text{sp}}$. This isomorphism is used to relate coherent states in the $SU(1,1)$ continuous series to spacelike normal vectors of classical triangles.

For the parameterization of eigenstates of K^1 an element $v \in SU(1,1)$ is parameterized by

$$v = e^{-i\varphi J^3} e^{-itK^2} e^{iuK^1}. \tag{D.25}$$

where

$$-2\pi < \varphi \leq 2\pi \qquad 0 \leq t < \infty \qquad 0 \leq u < \infty. \tag{D.26}$$

We can represent an element of the quotient space $SU(1,1)/(G_1 \otimes \mathbb{Z}_2)$ by

$$g = e^{-i\varphi J^3} e^{-itK^2}, \tag{D.27}$$

where the division by $\mathbb{Z}_2$ corresponds to the restriction of $-\pi < \varphi \leq \pi$. For the continuous series we consider the states $g|j\,\lambda\,\sigma\rangle$ and the expectation values

$$\langle j\,\lambda\,\sigma| g^\dagger F^i g |j\,\lambda\,\sigma\rangle. \tag{D.28}$$

Using the Hadamard lemma in Eq. (B.12) we have the following relations

$$g^\dagger J^3 g = -K^1 \sinh t + J^3 \cosh t, \tag{D.29}$$

$$g^\dagger K^1 g = K^1 \cos\varphi \cosh t - K^2 \sin\varphi + J^3 \cos\varphi \sinh t, \tag{D.30}$$

$$g^\dagger K^2 g = K^1 \sin\varphi \cosh t + K^2 \cos\varphi + J^3 \sin\varphi \sinh t, \tag{D.31}$$

and so the expectation values are

$$\langle j\,\lambda\,\sigma|g^\dagger J^3 g|j\,\lambda\,\sigma\rangle = -\lambda \sinh t, \tag{D.32}$$

$$\langle j\,\lambda\,\sigma|g^\dagger K^1 g|j\,\lambda\,\sigma\rangle = \lambda \cos\varphi \cosh t, \tag{D.33}$$

$$\langle j\,\lambda\,\sigma|g^\dagger K^2 g|j\,\lambda\,\sigma\rangle = \lambda \sin\varphi \cosh t, \tag{D.34}$$

$$\tag{D.35}$$

Hence

$$\langle j\,\lambda\,\sigma|g^\dagger F^i g|j\,\lambda\,\sigma\rangle = \lambda N^i, \tag{D.36}$$

where

$$\vec{N} = (-\sinh t, \cos\varphi \cosh t, \sin\varphi \cosh t), \tag{D.37}$$

and $(N^0)^2 - (N^1)^2 - (N^2)^2 = -1$.

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