Stochastic Renewal Process Models for Maintenance Cost Analysis

by

Tianjin Cheng

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

The maintenance cost for an engineering system is an uncertain quantity due to uncertainties associated with occurrence of failure and the time taken to restore the system. The problem of probabilistic analysis of maintenance cost can be modeled as a stochastic renewal-reward process, which is a complex problem. Assuming that the time horizon of the maintenance policy approaches infinity, simple asymptotic formulas have been derived for the failure rate and the cost per unit time. These asymptotic formulas are widely utilized in the reliability literature for the optimization of a maintenance policy. However, in the finite life of highly reliable systems, such as safety systems used in a nuclear plant, the applicability of asymptotic approximations is questionable. Thus, the development of methods for accurate evaluation of expected maintenance cost, failure rate, and availability of engineering systems is the subject matter of this thesis.

In this thesis, an accurate derivation of any m^{th} order statistical moment of maintenance cost is presented. The proposed formulation can be used to derive results for a specific maintenance policy. The cost of condition-based maintenance (CBM) of a system is analyzed in detail, in which the system degradation is modeled as a stochastic gamma process. The CBM model is generalized by considering the random repair time and delay in degradation initiation. Since the expected cost is not informative enough to estimate the financial risk measures, such as Value-at-Risk, the probability distribution of the maintenance cost is derived. This derivation is based on an interesting idea that the characteristic function of the cost can be computed from a renewal-type integral equation, and its Fourier transform leads to the probability distribution. A sequential inspection and replacement strategy is presented for the asset management of a large population of components. The finite-time analyses presented in this thesis can be combined to compute the reliability and

availability at the system level.

Practical case studies involving the maintenance of the heat transport piping system in a nuclear plant and a breakwater are presented. A general conclusion is that finite time cost analysis should be used for a realistic evaluation and optimization of maintenance policies for critical infrastructure systems.

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Dedication

Dedicated to my parents, whose emotional support made this work possible.

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Chapter 1

Introduction

1.1 Background

The basic premise of this thesis is a system, structure or component (SSC) in which the occurrence of failure is uncertain. The unexpected occurrence of failure can have adverse consequences, i.e., risk, to the plant, machinery, and people. The uncertain nature of failure can be attributed to many factors, such as random fluctuations in operating environment (temperature, stress etc.) and loss of system capacity by various processes of degradation (corrosion, wear, fatigue etc.), and many other reasons.

A nuclear reactor is a critical system in which the failure of major equipment can be risky for plant personnel and surrounding environment. In the Canadian nuclear reactor design (CANDU), the reactor core consists of a large number (380–480) of pressure vessels, referred to as fuel channels (Figure 1.2). The fuel channel has two concentric cylinders. The inner tube is called pressure tube which stores the nuclear fuel required for fission reaction. The outer tube is called the calandria tube, which is filled with a gas. The heavy water is the primary coolant, which picks up the heat generated from the fission reaction. The heated heavy water is transferred to steam generators using the feeder pipes (Figure 1.3). A typical steam generator has thousands of thin-walled tubes (2500 – 4000) in which the primary coolant flows, and transfers the heat energy to surrounding light water (i.e., secondary coolant) to produce steam. Steam is finally taken to turbines that drive the electrical generator for producing power.

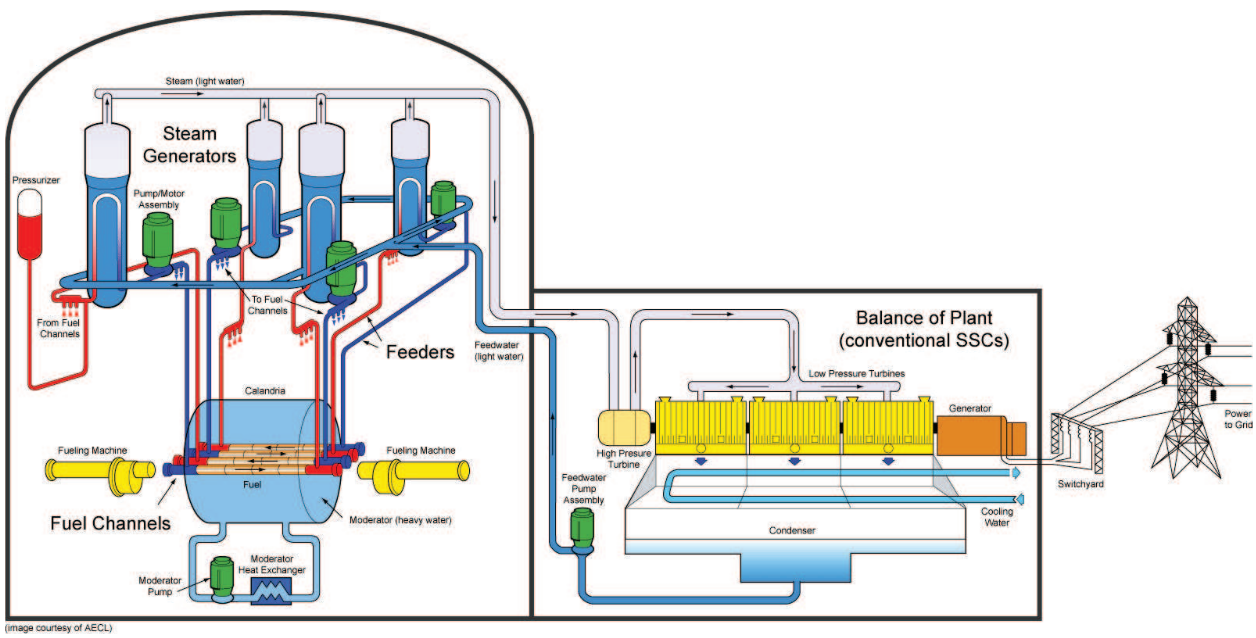
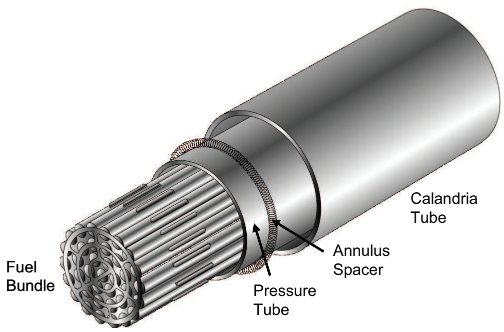
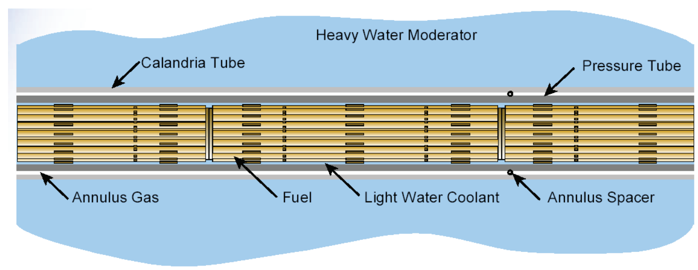


Figure 1.1: A schematic of major systems/components in a CANDU reactor

Because of intensely high temperature, pressure and radiation field, nuclear reactor components can experience various degradation mechanisms. Pressure tubes, feeders, and steam generator (SG, Figure 1.4) tubing are highly critical components in a reactor. The creep deformation of pressure tube diameter can reduce the efficiency of cooling. Feeder pipes experience flow-accelerated corrosion (FAC) and SG tubing is susceptible to corrosion and fretting wear. These degradation mechanisms are fairly uncertain.



(a) A schematic of fuel channel in CANDU reactor



(b) Cross-section of fuel channel

Figure 1.2: A fuel channel

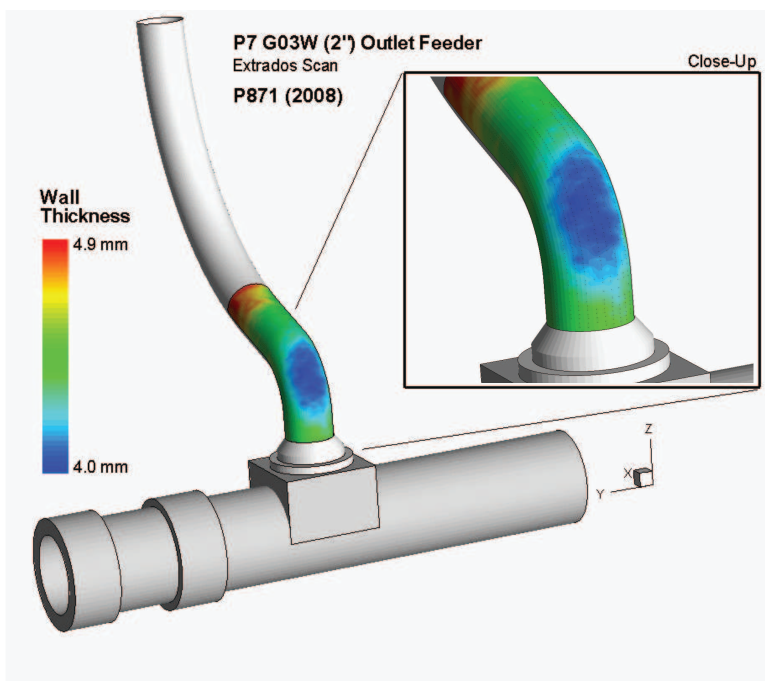


Figure 1.3: A feeder pipe showing the wall thickness loss due to FAC

The equipment reliability is maintained through inspection and maintenance of various components and systems in a systematic manner. In a nuclear plant, maintenance outage is commenced at a regular interval of 1–3 years in which all the major components are inspected and repaired/replaced as per the need. Typical policies are age-based and block

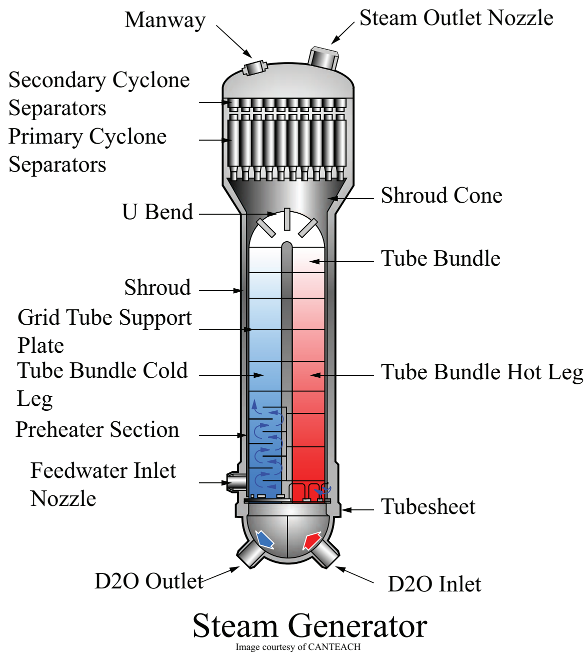


Figure 1.4: Cross-section of SG

replacement, condition-based maintenance, and numerous combinations of other types.

A key responsibility of the plant owner is to ensure high reliability of a system by implementing an optimal maintenance program. "Optimal" means: (1) the system failure rate is below an acceptable regulatory limit, (2) availability exceeding a specified limit, and (3) minimum cost of maintenance over a defined time horizon.

If a system experiences uncertain degradation, the time of occurrence of failure becomes a random variable, referred to as "time to failure". The time to repair of the system can also be modelled as a random variable to account for uncertainties associated with deployment of maintenance staff, detection of failure, and availability of spare parts. When the system is undergoing repair, the revenue (or productivity) may be lost due to loss of functionality. In this context, it is important to investigate the following problems:

(1) In a defined operating life of a system, what could be the cost resulting from failures?

(2) If an inspection and maintenance program is implemented, what should be the inspection interval and criteria for maintenance that would minimize the total maintenance cost?

(3) What are the benefits of a chosen maintenance program in terms of reduction in failure rate and increase in availability?

(4) What should be the maintenance budget for a fixed planning horizon?

The central problem is that the maintenance cost (including costs of inspection, repairs, and failures) in a time interval is not predictable in a deterministic sense, since it is also an uncertain quantity. The general goal of this thesis is to provide more accurate methods for analyzing the maintenance cost, failure rate, and availability of engineering systems with uncertain lifetime.

1.2 Motivation

If the time between failures is a random variable with some known probability distribution and the system is renewed after each failure to as-good-as-new condition, then this process of renewal over a time interval $(0, t]$ can be modelled as a stochastic renewal process. The total cost, $C(t)$, is the sum of costs incurred in $N(t)$ renewals. Since $N(t)$ is a random variable, $C(t)$ is also referred to as a random sum with certain probability distribution.

The derivation of the expected number of renewals can be formulated in terms of a renewal integral equation, and a similar approach can be taken to derive the expected cost. Since solutions of integral equations are somewhat involved, asymptotic limits (as time approaches infinity) have been derived for $N(t)$ and $C(t)$. For example, the asymptotic limit of $N(t)$ is the reciprocal of mean time between failure, and the asymptotic limit for $C(t)$ is the ratio of expected cost in one renewal cycle to the mean time between failure. Because these asymptotic limits are very simple to compute, they have been widely used in reliability-based maintenance modeling and optimization literature [6, 40, 67, 53].

A typical rule of thumb is that asymptotic solution is applicable when the time horizon is greater than three times the mean time between failure. If this condition is not fulfilled, the asymptotic solution will not serve as an adequate approximation. In many mechanical and electrical systems where components are relatively inexpensive and the impact of failure is small, the mean time between failure tends to be much smaller than the planning horizon. In such cases the validity of asymptotic approach is acceptable. However, for critical systems, such as those in a nuclear plant, the high reliability requirement dictates that the mean time between failure should be of the order of the plant operating lifetime. In such cases, the application of asymptotic formulas becomes questionable.

Thus, development of methods for accurate evaluation of expected maintenance cost, failure rate, and availability of highly reliable systems is the motivation for research presented in this thesis. Initially the focus was on the derivation of expected cost, but later it was realized that the standard deviation of cost is also necessary to quantify uncertainty. Also, higher order moments are required to model the distribution tails.

1.3 Research Objectives

(1) Investigate probabilistic approaches for the estimation of maintenance cost associated with condition-based maintenance models by relaxing the asymptotic approximations used in the literature;

(2) Derive statistical moments of maintenance cost (e.g., mean, variance and other higher order moments) for a fixed planning horizon, also referred to as finite-time solutions.

(3) Derive probability distribution of maintenance cost for the evaluation of financial risk measures, such as Value-at-Risk (VaR) and statistical prediction intervals.

(4) Conduct case studies using real-life data to illustrate the applications of analytical/computational methods developed in this thesis.

1.4 Organization of the Thesis

Chapter 2 provides an overview of the theory of stochastic renewal process that is relevant to the research scope of this thesis. Key terminology, definitions and theorems are presented to set the stage for subsequent chapters.

Chapter 3 presents a general derivation of any m^{th} order statistical moment of maintenance cost in a finite time horizon. The moment of cost is derived as a renewal-type integral equation. The proposed formulation can be used to derive results for a specific maintenance policy, so long as it can be modelled as a stochastic renewal-reward process. This general approach would allow the finite time cost analysis of a variety of maintenance policies. Subsequent chapters will use the results presented in this chapter.

Chapter 4 analyzes the cost of condition-based maintenance of a system in which degradation is modelled as a stochastic gamma process. Although the gamma process is widely used in the literature, the finite time mean and variance of cost are derived for the first time in this work. This chapter presents a case study involving CBM of the piping system in a nuclear plant. The CBM model analyzed in Chapter 4 assumes that time required for repair is negligible, and degradation is initiated as soon as the system is put in service. These two assumptions are relaxed in Chapter 5 by considering the repair (or down) time and delay in degradation initiation as random variables. The finite time cost analysis, with and without discounting, presented in this chapter is not yet seen in the existing literature. The evaluation of expected cost is reasonable for finding an optimal maintenance policy among a set of possible alternatives. However, this approach is not informative enough to enable the estimation of financial risk measures, such as percentiles of the cost, also known as Value-at-Risk (VaR). To address this issue, Chapter 6 presents a derivation of the probability distribution of the maintenance cost. The proposed approach is based on formulating a renewal equation for the characteristic function of cost in finite time. Subsequently, the Fourier transform of the characteristic function leads to the probability distribution of the cost.

In Chapter 7, a sequential inspection and replacement model is presented for the asset management of a large population of components in a large infrastructure system. In this approach, the population is divided into δ blocks (or sub-populations) and one block per year is inspected such that it takes δ years to inspect the entire population. Note that all the failed components found through inspection are replaced with new components. The model is based on the concept of delayed renewal process and it is used to predict the expected number of replacements and substandard components in any given year.

Chapter 8 presents the reliability analysis of systems with repairable components. Each component has a random life time and repair time described by general (non-exponential) probability distributions. The time-dependent unavailability and failure rate are derived for each individual component of the system by solving a set of renewal equations. Then, system unavailability and failure rate are computed based on the component level information. This chapter illustrates that models presented in the previous chapters can be used to analyze reliability at the system level.

Conclusions of the thesis are presented in Chapter 9.

Chapter 2

Introduction to Renewal Theory

2.1 Introduction

Renewal process theory had its origin in the studies of population analysis and strategies for replacement of technical components [38]. Later, it was developed as a general topic in the field of stochastic processes [23, 17]. The renewal process became an important part of the reliability theory [6, 55].

This chapter summarizes main aspects of the renewal theory that are relevant to research presented in this thesis. It should be noted that a complete overview of stochastic renewal process is not intended here.

Key terminology related to ordinary and the delayed renewal processes is introduced. Formulas for evaluating the expected number of failures (or renewal function) and the expected maintenance cost are summarized. Illustrative examples are also presented.

2.2 Lifetime Distribution

Let X be the lifetime of a component (system). X (> 0) is a random variable. The cumulative distribution function (CDF) and the survival function (SF) of X are defined as,

$$F_X(x) = \mathsf{P}\{X \le x\}, \quad \overline{F}_X(x) = \mathsf{P}\{X > x\} = 1 - F_X(x). \tag{2.1}$$

Here, $\mathsf{P}\{*\}$ denotes the probability of an event inside the $\{\}$.

If X is continuous, the probability density function (PDF) and the expected value are defined as

$$f_X(x) = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \mathsf{P}\{x < X \le x + \Delta x\} = \frac{\mathrm{d}F_X(x)}{\mathrm{d}x} = -\frac{\mathrm{d}\overline{F}_X(x)}{\mathrm{d}x} \tag{2.2}$$

$$\mathsf{E}[X] = \int_0^\infty x f_X(x) \mathrm{d}x. \tag{2.3}$$

The hazard rate of X is defined by [17]

$$\lambda_X(x) = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \mathsf{P}\{x < X \le x + \Delta x | X > x\}. \tag{2.4}$$

Here the notation $\mathsf{P}\{A|B\}$ represents the probability of event A conditional on event B. $\lambda_X(x)\mathrm{d}x$ is the probability that a component will fail in the interval $(x, x + \mathrm{d}x]$ given that it has survived for a period of x. Since

$$\mathsf{P}\{x < X \le x + \Delta x | X > x\} = \frac{\mathsf{P}\{x < X \le x + \Delta x\}}{\mathsf{P}\{X > x\}} = \frac{F_X(x + \Delta x) - F_X(x)}{\overline{F}_X(x)}, \tag{2.5}$$

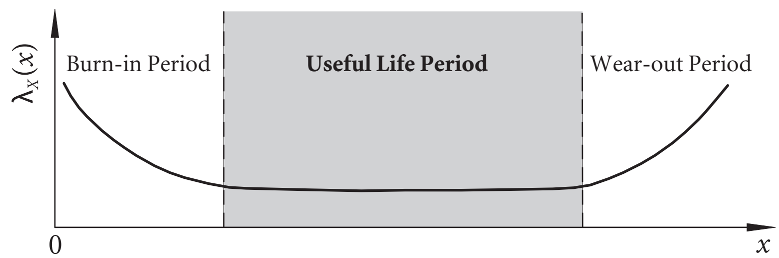


Figure 2.1: A typical hazard rate

the hazard rate can be written as

$$\lambda_X(x) = \frac{f_X(x)}{\overline{F}_X(x)} = -\frac{1}{\overline{F}_X(x)}\frac{\mathrm{d}\overline{F}_X(x)}{\mathrm{d}x}. \tag{2.6}$$

Then we have

$$\overline{F}_X(x) = e^{-\int_0^x \lambda_X(\tau)\mathrm{d}\tau}. \tag{2.7}$$

A typical hazard rate is shown in Figure 2.1, which is usually called a bathtub curve. The hazard rate is often high in the initial phase, known as "infant mortality". This can be explained by the fact that there may be undiscovered defects in a component, which contribute to early failures. When the component has survived the infant mortality period, the hazard rate often stabilizes at a level where it remains constant for a certain period of time. With time, it starts to increase as the component begins to wear out. From the shape of the bathtub curve, the lifetime of a unit may be divided into three typical intervals: the burn-in period, the useful life period, and the wear-out period.

If X is discrete and takes value of x_k, where $k = 1, 2, \cdots$, and $x_k = k\Delta x$, the probability mass function (PMF) of X is defined as

$$f_X(x_k) = \mathsf{P}\{X = x_k\} = F_X(x_k) - F_X(x_{k-1}) = \overline{F}_X(x_{k-1}) - \overline{F}_X(x_k). \tag{2.8}$$

The expected value of X is then given by

$$\mathsf{E}[X] = \sum_{k=1}^{\infty} x_k f_X(x_k). \tag{2.9}$$

The hazard rate in the discrete sense is defined as [39, 3]

$$\lambda_X(x_k) = \mathsf{P}\{X = x_k | X > x_{k-1}\} = \frac{f_X(x_k)}{\overline{F}_X(x_{k-1})} \tag{2.10}$$

Substituting Eq. (2.8) into the above equation gives

$$\lambda_X(x_k) = 1 - \frac{\overline{F}_X(x_k)}{\overline{F}_X(x_{k-1})}. \tag{2.11}$$

Then we have

$$\overline{F}_X(x_k) = \prod_{i=1}^{k} [1 - \lambda_X(x_i)]. \tag{2.12}$$

The Weibull distribution is a typical lifetime distribution used in reliability theory [40]. For continuous time, the Weibull distribution has the following CDF and hazard rate, respectively,

$$F_X(x) = 1 - e^{-(x/\beta)^\alpha}, \quad \lambda_X(x) = \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1}, \tag{2.13}$$

where $x \geq 0$, $\alpha > 0$, and $\beta > 0$. For discrete time, there are multiple types of the Weibull distribution, one of which has the following hazard rate [59, 34, 73]

$$\lambda_X(x) = \begin{cases} (x/\beta)^{\alpha-1} \quad (x = 1, 2, \cdots, \beta) & \text{if } \alpha > 1, \\ \theta x^{\alpha-1} \quad (x = 1, 2, \cdots) & \text{if } 0 < \alpha \leq 1, \end{cases} \tag{2.14}$$

where β is an integer and $0 < \theta < 1$. The above definition preserves the power function form of the hazard rate. Use Eq. (2.12) to compute the SF, and then the CDF and the PMF can be calculated. These four quantities are shown in Figure 2.2.

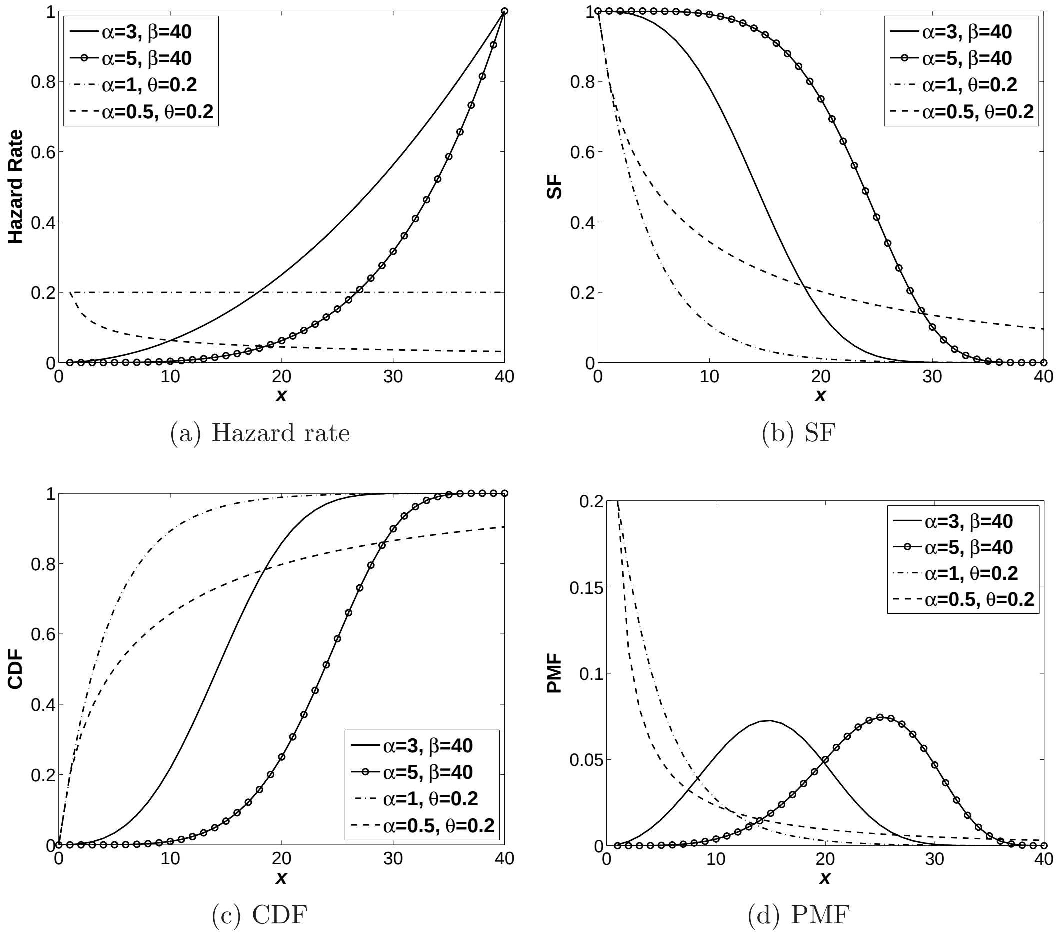
2.3 Ordinary Renewal Process

The following example is used to illustrate the ordinary renewal process. Suppose that we have a population of identical components. The lifetime of any component, denoted by X, is a discrete random variable with probability mass function (PMF)

$$f_X(x) = \mathsf{P}\{X = x\}, \quad x = 0, \Delta t, 2\Delta t, \cdots, \tag{2.15}$$

and $f_X(0) = 0$. We start with a new component at time zero. The component survives a period of X_1. Then it is replaced immediately by a new one. The time for replacement is assumed to be negligible. The second component survives a period of X_2, and fails at time $(X_1 + X_2)$. Then it is also replaced immediately by a new one, and so on and so forth (see Figure 2.3).

Let X_n, $n = 1, 2, \cdots$, be the length of the n^{th} survival period and S_n is the time of the



(a) Hazard rate (b) SF

(c) CDF (d) PMF

Figure 2.2: Discrete Weibull distribution

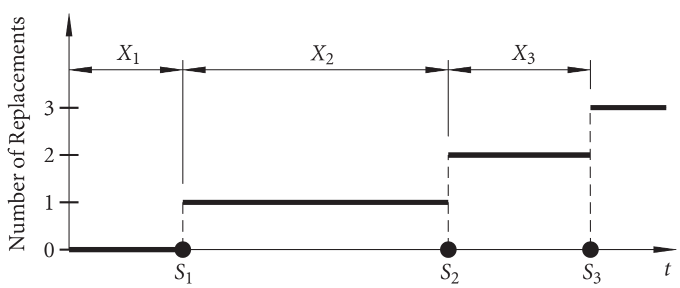


Figure 2.3: Renewal process

n^{th} replacement, i.e.

$$S_n = \sum_{j=1}^{n} X_j, \quad n = 1, 2, \cdots . \tag{2.16}$$

Let $S_0 = 0$. S_0, S_1, $\cdots$, are called renewal times and X_1, X_2, $\cdots$, are called renewal intervals.

A counting process $N(t)$ is defined as

$$N(t) = \max\{n; S_n \leq t\}, \quad t = 0, \Delta t, 2\Delta t, \cdots . \tag{2.17}$$

$N(t)$ is the number of replacements up to time t. $N(t)$ is called the *ordinary renewal process* (ORP) with renewal distribution $f_X(x)$.

Let $M(t) = \mathsf{E}\left[N(t)\right]$. $M(t)$ is called the *renewal function*. In the following we are going to derive $M(t)$. Obviously, we have $M(0) = 0$. For $t > 0$, use the law of total expectation by conditioning on X_1

$$M(t) = \sum_{0<x\leq t} \mathsf{E}\left[N(t)|X_1 = x\right] f_X(x) = \sum_{0<x\leq t} \mathsf{E}\left[1 + N(x,t)|X_1 = x\right] f_X(x). \tag{2.18}$$

In the above equation, the term $N(t)$ is split into $1 + N(x,t)$, where $N(x,t)$ is the number of replacements in the interval $(x, t]$. Note that since X_1, X_2, $\cdots$, are *idd* random variables, given $X_1 = x$, $N(x,t)$ can be considered as an ORP with length of $(t - x)$ with x as the new origin. Thus, $N(x,t)$ is stochastically the same as $N(t - x)$. Hence

$$\mathsf{E}\left[N(x,t)|X_1 = x\right] = \mathsf{E}\left[N(t - x)\right] = M(t - x). \tag{2.19}$$

Substituting Eq. (2.19) into (2.18) gives

$$M(t) = \sum_{0<x\leq t} \left[1 + M(t - x)\right] f_X(t) = \sum_{k=1}^{t/\Delta t} M(t - k\Delta t) f_X(k\Delta t) + F_X(t), \tag{2.20}$$

where $F_X(t) = \sum_{x \le t} f_X(x)$ is the cumulative distribution function (CDF) of X. Equation (2.20) is a discrete renewal equation. The values of $M(\Delta t)$, $M(2\Delta t)$, $\cdots$, can be obtained recursively from this equation with the initial condition $M(0) = 0$.

Define the convolution of two discrete functions $f_1(t)$ and $f_2(t)$ as

$$(f_1 * f_2)(t) = \sum_{0 \le x \le t} f_1(t-x) f_2(x) = \sum_{k=0}^{t/\Delta t} f_1(t - k\Delta t) f_2(k\Delta t).$$

Noting that $f_X(0) = 0$, Eq. (2.20) can be written in a more compact form as

$$M(t) = (M * f_X)(t) + F_X(t). \tag{2.21}$$

Let $I(t)$ be the indicator of a replacement at time t, 1 for yes and 0 for no. Define the *renewal density* as $m(t) = \mathsf{E}\left[I(t)\right]/\Delta t$. Note that $\mathsf{E}\left[I(t)\right]$ is equal to the probability of replacement at time t. Then $m(t)$ is the probability density of replacement or renewal at t. Since $I(t) = N(t) - N(t - \Delta t)$, we have

$$m(t) = \frac{1}{\Delta t}\big[M(t) - M(t - \Delta t)\big]. \tag{2.22}$$

Letting $m(0) = 0$, Eq. (2.21) and (2.22) yield

$$m(t) = (m * f_X)(t) + \frac{1}{\Delta t} f_X(t). \tag{2.23}$$

The above equation is also a renewal equation and $m(t)$ can be determined recursively with the initial condition $m(0) = 0$.

If there exists an integer $\delta > 1$ such that a replacement can only occur at times δ,

$2\delta, \cdots$, i.e. $f_X(x) = 0$ if x is not divisible by δ, then the replacement is called periodic. The greatest δ with this property is called the period of replacement. For non-periodic replacements, we have the following renewal theorem [23]

Theorem 2.1. *(Erdö-Feller-Pollard) If a replacement is not periodic, then the asymptotic renewal density*

$$\lim_{t\to\infty} m(t) = \frac{1}{\mu_X}, \tag{2.24}$$

where μ_X is the expected length of a renewal interval, i.e. $\mu_X = \sum_{x>0} x f_X(x)$.

For periodic replacements, Eq. (2.24) should be changed into $\lim_{k\to\infty} m(k\delta) = \delta/(\mu_X \Delta t)$, where δ is the period of replacement.

Using L'Hôpital's rule [58], equation (2.22) and (2.24) yield

$$\lim_{t\to\infty} \frac{M(t)}{t} = \lim_{t\to\infty} m(t) = \frac{1}{\mu_X}. \tag{2.25}$$

The above equation implies that $M(t) = t/\mu_X + o(t)$, where $o(t)$ is of lower order than t. Hence for a large t, we can use t/μ_X to approximate $M(t)$. A better approximation of $M(t)$ is presented by Feller as follows [22]

$$M(t) = \frac{1}{\mu_X} t + \frac{\mu_X^2 + \sigma_X^2 + \mu_X \Delta t}{2\mu_X^2} - 1 + o(1), \tag{2.26}$$

where σ_X is the standard deviation of the renewal interval, i.e. $\sigma_X^2 = \sum_{x>0} (x - \mu_X)^2 f_X(x)$.

Example 2.1. *Suppose that the time unit is $\Delta t = 1$ and the renewal interval X of a renewal process is a discrete Weibull distributed random variable. The hazard rate is shown by Eq.*

(2.14) *with parameters* $\alpha = 3$ *and* $\beta = 30$*. The PMF of* X *is shown in Figure 2.4. Then the mean and the standard deviation of* X *are* $\mu_X = 12.1$ *and* $\sigma_X = 4.2$*, respectively.*

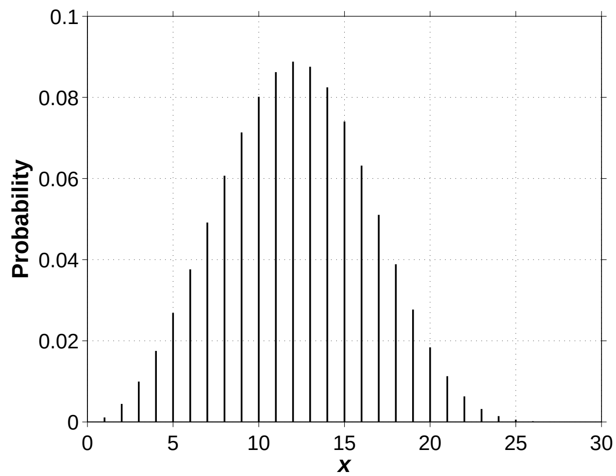
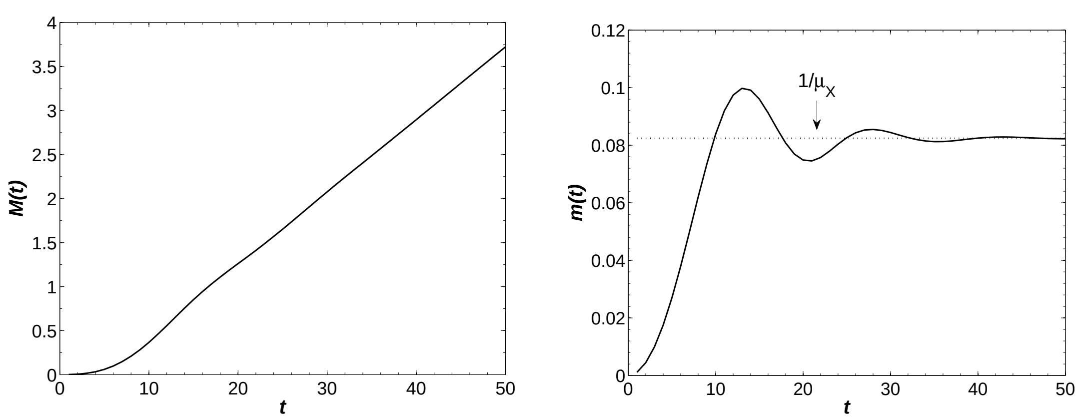


Figure 2.4: PMF of X

The renewal function and the renewal density are show in Figure 2.5. As shown in Figure 2.5b, the renewal density oscillates and then converges to $1/\mu_X$*.*



(a) Renewal function

(b) Renewal density

Figure 2.5: Renewal function & rate of an ordinary renewal process

So far we have only considered discrete time. Letting $\Delta t \to 0$, we will obtain the results for continuous time. The renewal equation (2.21) will still hold for the renewal function $M(t)$ except that now $f_X(x)$ represents the PDF instead of the PMF of X, and the sign $*$ represents continuous convolution instead of discrete convolution, i.e. $(M * f_X)(t) = \int_0^t M(t-x) f_X(x) \mathrm{d}x$. The renewal density will become $m(t) = \mathrm{d}M(t)/\mathrm{d}t$. Equation (2.25) will also still hold [17, 40], while Eq. (2.23) and (2.26) should be modified as

$$m(t) = (m * f_X)(t) + f_X(t), \tag{2.27}$$

$$M(t) = \frac{1}{\mu_X} t + \frac{\mu_X^2 + \sigma_X^2}{2\mu_X^2} - 1 + o(1), \quad \text{as } t \to \infty. \tag{2.28}$$

Equation (2.28) can be found in Chapter 8 of [63].

The method of Laplace transform can be used to solve for $M(t)$ for continuous time. The Laplace transform of a function $g(t)$ is defined as

$$\mathcal{L}\{g\}(s) = \int_0^\infty g(t) e^{-st} \mathrm{d}t.$$

Taking the Laplace transforms of the both sides in Eq. (2.21) follows

$$\mathcal{L}\{M\}(s) = \mathcal{L}\{M\}(s)\mathcal{L}\{f_X\}(s) + \mathcal{L}\{F_X\}(s) \quad \Rightarrow \quad \mathcal{L}\{M\}(s) = \frac{\mathcal{L}\{F_X\}(s)}{1 - \mathcal{L}\{f_X\}(s)}. \tag{2.29}$$

Since $F_X(t) = \int_0^t f_X(x)dx$, we have $\mathcal{L}\{F_X\}(s) = \mathcal{L}\{f_X\}(s)/s$. Then Eq. (2.29) gives

$$M(t) = \mathcal{L}^{-1}\left\{\frac{\mathcal{L}\{f_X\}(s)}{s\left[1 - \mathcal{L}\{f_X\}(s)\right]}\right\}(t), \tag{2.30}$$

where $\mathcal{L}^{-1}$ means the inverse Laplace transform.

Example 2.2. *Suppose that* X *is an exponentially distributed random variable with PDF* $f_X(x) = \lambda e^{-\lambda x}$, $x > 0$. *Then the Laplace transform of* $f_X(x)$ *is* $\mathcal{L}\{f_X\}(s) = \lambda/(\lambda + s)$. *Substituting* $\mathcal{L}\{f_X\}(s)$ *into Eq.* (2.29) *gives* $\mathcal{L}\{M\}(s) = \lambda/s^2$. *Then the renewal function is obtained as* $M(t) = \lambda t$. *Hence the renewal density is* $m(t) = \lambda$, *which is a constant. Note that here* $N(t)$ *is actually a Poisson process with rate parameter* λ, *from which we can also draw the conclusion that* $m(t) = \lambda$.

In general, $\mathcal{L}\{M\}(s)$ is so complicated that the analytical solution of $M(t)$ can not be obtained. In most practical cases, numerical solution of inverse Laplace transform is required.

2.4 Delayed Renewal Process

In an ORP, the probability distribution of inter-arrival times, X_1, X_2, $\cdots$, are *iid*, since we start with a new component at time 0. If we start with an aged component at the beginning, X_1 will have a different probability distribution from those of X_2, X_3, $\cdots$. Suppose that the initial age is a, $a = 0, \Delta t, 2\Delta t, \cdots$. Let $N(t|a)$ be the number of renewal up to t. $N(t|a)$ is called the *delayed renewal process*. Let $f_X(x|a)$ be the PMF of X_1 and $f_X(x)$ be that of the other renewal intervals. Obviously, $N(t)$ and $f_X(x)$ in the previous section can be considered as the special cases of $N(t|a)$ and $f_X(t|a)$ when $a = 0$, respectively.

Since the first component has been aged for a period of a, the PMF of the first renewal interval is equal to

$$f_X(x|a) = \mathsf{P}\{X = x + a | X > a\} = \frac{f_X(x + a)}{\overline{F}_X(a)}. \tag{2.31}$$

In the above equation, X is the lifetime of the first component and $\overline{F}_X$ is the SF of f_X.

Let $M(t|a)$ be the renewal function and $m(t|a)$ the renewal density in the delayed renewal process. In the following we are going to derive these two values. Similar to Eq. (2.18), using the law of total expectation by conditioning on X_1, we have

$$M(t|a) = \sum_{0<x\leq t} \mathsf{E}\left[1 + N(x,t)|X_1 = x\right] f_X(x|a). \tag{2.32}$$

Note that we always start with a new component except in the first renewal interval. Hence given $X_1 = x$, $N(x,t)$ can be considered as an ORP with length $(t-x)$. Then

$$\mathsf{E}\left[N(x,t)|X_1 = x\right] = M(t-x). \tag{2.33}$$

Here $M(t-x)$ is the renewal function of an ORP and can be computed from Eq. (2.21). Substituting Eq. (2.33) into (2.32) gives

$$M(t|a) = \sum_{0<x\leq t} M(t-x) f_X(x|a) + F_X(t|a). \tag{2.34}$$

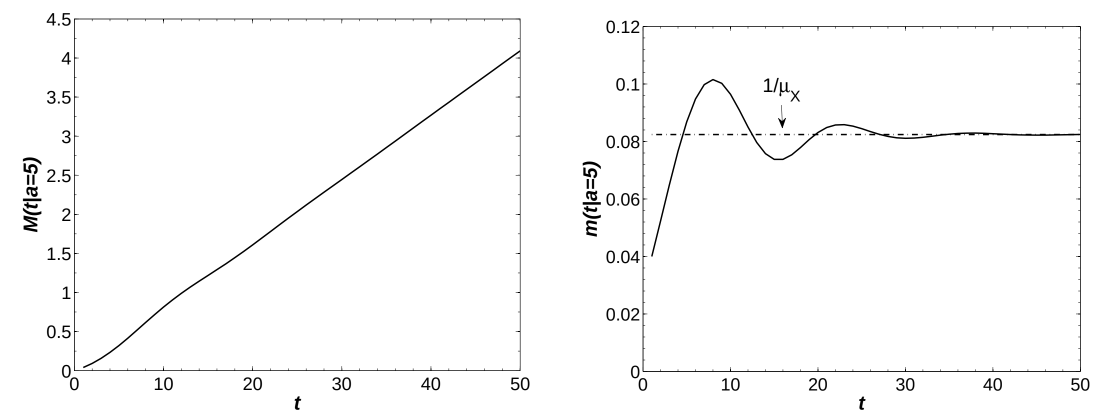
where $F_X(t|a) = \sum_{x\leq t} f_X(x|a)$ is the CDF of X_1.

The renewal density $m(t|a)$ can be similarly obtained as

$$m(t|a) = \sum_{0<x\leq t} m(t-x) f_X(x|a) + \frac{1}{\Delta t} f_X(t|a), \tag{2.35}$$

where $m(t-x)$ is the renewal density of an ORP and can be computed from Eq. (2.23). The asymptotic value of $m(t|a)$ is the same as that of $m(t)$, which is equal to $1/\mu_X$, regardless of the initial age a [23].

Example 2.3. *Take the same parameters as in Example 2.1 except that the initial age at time 0 is* $a = 5$*. The renewal function and the renewal density are shown in Figure 2.6. Comparing Figure 2.5a and 2.6a, we can see that* $M(t|a)$ *is larger than* $M(t)$*, which is because we start with an aged component in the delayed renewal process, leading to more replacements. As shown in Figure 2.6b, the renewal density still oscillates about the value of* $1/\mu_X$ *and asymptotically tends to it.*



(a) Renewal function (b) Renewal density

Figure 2.6: Renewal function & rate of a delayed renewal process

2.5 Summary

This chapter provides an overview of the theory of stochastic renewal process that is relevant to research scope of this thesis. Key terminology, definitions, and theorems are presented to set the context for subsequent chapters.

The renewal process, $N(t)$, is defined as the number of renewals up to time t with inter-renewal times, X_1, X_2, $\cdots$, being independent and identically distributed (*iid*) random variables. The expected number of renewals in a time interval $(0, t]$ is referred to as the renewal function, which can be derived from a renewal equation. In case of the delayed renewal process, X_1 has a different probability distribution than other inter-renewal times. The renewal density, i.e., probability of renewal per unit time, has a asymptotic value that is equal to $1/\mu_X$, where μ_X is the expected length of a renewal interval. This result is called the classical renewal theorem, and it has been fundamental to expected maintenance cost analysis in an asymptotic sense.

Chapter 3

Basic Concepts of Maintenance Cost Analysis

3.1 Introduction

3.1.1 Motivation

A wide variety of maintenance policies can be analytically treated as stochastic renewal-reward processes, so long as the system is renewed after each maintenance work. An accurate evaluation of mean, variance and other higher order moments of the reward (or cost) is an analytically challenging task. For this reason, simple asymptotic expected cost analysis is commonly used in the literature.

Accurate evaluation of expected cost was studied only in a few papers [13, 14, 41] for simple cases, such as age-based replacement policy. Derivation of higher order moments maintenance cost has not been presented.

This chapter presents a general derivation of any m^{th} order statistical moment of maintenance cost in a finite time horizon. The moment of cost is derived as a renewal-type integral equation. This approach also allows the computation of unavailability and failure rate of the system under a given maintenance policy. This chapter presents fundamental formulation that will be frequently utilized in applications presented in the subsequent chapters.

In this thesis, only discrete time is considered unless explicitly stated. The time unit is Δt.

3.1.2 Organization

This chapter is organized as follows. Section 3.2 introduces basic concepts underlying the theory of the renewal-reward process. Section 3.3 presents a general derivation of statistical moments of maintenance cost in a finite time horizon. The asymptotic formulation is discussed in Section 3.4. The unavailability and failure rate are analyzed in Section 3.5. An illustrative example is given in Section 3.6.

3.2 Renewal-Reward Process

3.2.1 Ordinary & General Renewal-Reward Process

Consider an ordinary renewal process $N(t)$. A cost C_n is incurred at the end of each renewal interval T_n, $n = 1, 2, \cdots$ (see Figure 3.1). Here C_n may be a fixed value or a random variable. Assume that pairs $\{T_n, C_n\}$ are *iid* random vectors.

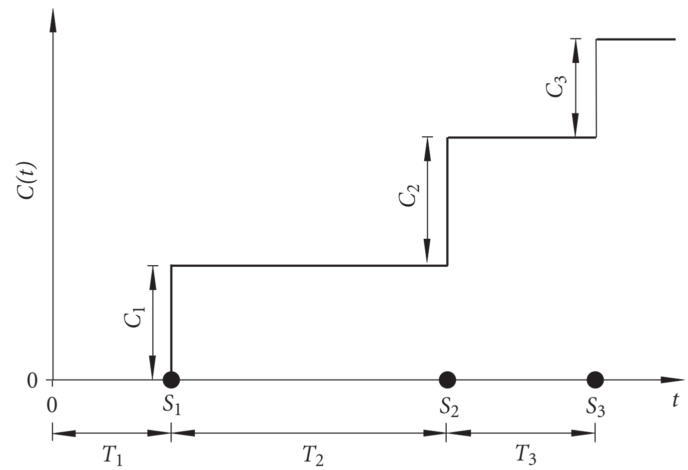


Figure 3.1: Renewal Reward Process

Denote the cumulative cost in the time interval $(t_1, t_2]$ by $C(t_1, t_2)$ and write $C(0, t)$ as $C(t)$ for simplicity. Up to time t, there will be $N(t)$ complete renewal intervals and an incomplete renewal interval $(S_{N(t)}, t]$, where $S_n = \sum_{j=1}^{n} T_n$ is the n^{th} renewal time. There is no cost in $(S_{N(t)}, t]$. Hence

$$C(t) = \sum_{n=1}^{N(t)} C_n. \tag{3.1}$$

$C(t)$ is called the *renewal-reward process* (RRP). The ordinary renewal process can be considered as a special case of the RPP when $C_k \equiv 1$.

In the above model, cost is assumed to be only incurred at the end of each renewal interval. However, in many maintenance policies, as shown in the following, cost may be incurred during renewal intervals. To differentiate these two cases, $C(t)$ in the former case is called the *ordinary renewal-reward process* in this thesis, while that in the latter case is called the *general renewal-reward process* or just simply called the renewal-reward process.

For a general RRP, Eq. (3.1) should be modified as

$$C(t) = \sum_{n=1}^{N(t)} C_n + C\big(S_{N(t)}, t\big), \tag{3.2}$$

since now the cost incurred in $(S_{N(t)}, t]$ may not be equal to 0.

3.2.2 Example: Age Based Replacement

In this maintenance policy, a component is replaced either when it fails (called corrective maintenance, CM) or at an age of t_p (called preventive maintenance, PM), t_p being a predetermined constant, whichever occurs first. This policy is called the *age based replacement policy*, which has been widely discussed in the literature.

In general, CM cost is much larger than PM cost due to loss resulting from component failure. Let L be the lifetime of the component and X be the time to replacement by CM or PM, then

$$X = \min\{L, t_p\} = \begin{cases} L, & \text{if } 0 < L < t_p, \\ t_p, & \text{if } L \geq t_p. \end{cases} \tag{3.3}$$

If time spent on replacement is negligible, i.e. the component is renewed instantly, the associated maintenance cost $C(t)$ is an ordinary RRP. The length of a renewal interval is then given by

$$T = X, \tag{3.4}$$

and the cost incurred in a complete renewal interval is equal to

$$C = \begin{cases} c_{\mathrm{F}}, & \text{if } 0 < L < t_p, \\ c_{\mathrm{P}}, & \text{if } L \geq t_p, \end{cases} \tag{3.5}$$

where c_{F} is the CM cost and c_{P} is the PM cost.

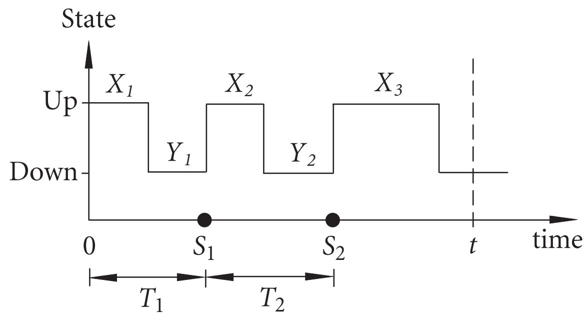


Figure 3.2: The component alternates between the up and the down states.

If time spent on replacement is non-negligible, the component will be in the down state during replacement, resulting in a down time cost due to component unavailability. The down time cost is proportional to the length of down time. As shown in Figure 3.2, where X's are the times to replacement and Y's are the subsequent down times, the component alternates between the up and the down states. The component is renewed only when the replacement is finished. Hence the length of a renewal interval is given by

$$T = X + Y. \tag{3.6}$$

The cost incurred in one complete renewal interval is equal to

$$C = c_{\mathrm{D}}Y + \begin{cases} c_{\mathrm{F}}, & \text{if } 0 < L < t_p, \\ c_{\mathrm{P}}, & \text{if } L \geq t_p, \end{cases} \tag{3.7}$$

where c_{D} is the unit down time cost.

Note that if time spent on replacement is non-negligible the maintenance cost $C(t)$ is a general RRP instead of an ordinary RRP since c_{F}, c_{P}, and c_{D} are all incurred during the renewal interval. The cost incurred in an incomplete renewal interval may not be zero. For example, in Figure 3.2 with $N(t) = 2$, suppose that at the time of $(S_2 + X_3)$, the component fails. Then the cost incurred in the incomplete interval $(S_2, t]$ is equal to

$$C(S_2, t) = c_{\mathrm{F}} + c_{\mathrm{D}}(t - S_2 - X_3).$$

3.2.3 Renewal Argument

The renewal argument discussed in Section 2.3 also applies to the RRP. If τ is a renewal point, then the cost $C(\tau, t)$, $t > \tau$, can be considered as an RRP over an interval $(t - \tau)$, and $C(\tau, t)$ is independent of $C(0, \tau)$. In summary we have the following theorem (renewal argument for the RRP)

Theorem 3.1. *Given that τ is a renewal point of an RRP $C(t)$, $t > \tau$, we have*

(1) *$C(\tau, t)$ is stochastically the same as $C(0, t - \tau)$ or $C(t - \tau)$; and*

(2) *$C(\tau, t)$ is independent of $C(0, \tau)$ or $C(\tau)$.*

For example, as shown in Figure 3.2, after the first renewal point S_1, the component is still subjected to the same age based replacement policy except that the time horizon is reduced to $t - S_1$.

3.3 Moments of Maintenance Cost

3.3.1 General Approach

Let $U_m(t)$ be the m^{th} moment of $C(t)$, defined as

$$U_m(t) = \mathsf{E}\left[C^m(t)\right].$$

with an initial condition that $U_m(0) = 0$. In this section we derive a general expression for $U_m(t)$ as

$$U_m(t) = (U_m * f_T)(t) + G_m(t), \tag{3.8}$$

where f_T is the PMF of the length of the renewal interval T, and $G_m(t)$ is a function associated with the expected cost in one renewal interval and is determined by the maintenance policy.

Equation (3.8) is referred to as a generalized renewal equation [27]. It can be used for a general maintenance policy that can be treated as an RRP. A specific maintenance policy only influences the values of $f_T(t)$ and $G_m(t)$. Once $f_T(t)$ and $G_m(t)$ are given, $U_m(t)$ can be recursively calculated as follows

$$\begin{aligned}
U_m(\Delta t) &= G_m(\Delta t), \\
U_m(2\Delta t) &= U_m(\Delta t) f_T(\Delta t) + G_m(2\Delta t), \\
U_m(3\Delta t) &= U_m(2\Delta t) f_T(\Delta t) + U_m(\Delta t) f_T(2\Delta t) + G_m(3\Delta t), \\
&\vdots \\
U_m(t) &= U_m(t-\Delta t) f_T(\Delta t) + U_m(t-2\Delta t) f_T(2\Delta t) + \cdots + U_m(\Delta t) f_T(t-\Delta t) + G_m(t).
\end{aligned}$$

3.3.2 First Moment

Conditioning on the first renewal time T_1 (see Figure 3.2) and using the law of total expectation, the expected cost, $U_1(t)$, is written as

$$U_1(t) = \sum_{0<\tau\leq t} \mathsf{E}\left[C(t)|T_1 = \tau\right] f_T(\tau) + \mathsf{E}\left[C(t)|T_1 > t\right] \overline{F}_T(t), \tag{3.9}$$

where $\overline{F}_T(t) = \mathsf{P}\{T_1 > t\}$ is the SF of T. In the above equation, $U_1(t)$ is partitioned into two parts associated with events $T_1 \leq t$ and $T_1 > t$. When $T_1 = \tau < t$, split $C(t)$ into two terms: (1) the cost in the first renewal interval (C_1), and (2) the cost in the remaining time horizon, $C(\tau, t)$, such that

$$\mathsf{E}\left[C(t)|T_1 = \tau\right] = \mathsf{E}\left[C_1|T_1 = \tau\right] + \mathsf{E}\left[C(\tau,t)|T_1 = \tau\right] = \mathsf{E}\left[C_1|T_1 = \tau\right] + U(t-\tau). \tag{3.10}$$

In the above equation we used the renewal argument (Theorem 3.1) that $\mathsf{E}\left[C(\tau,t)|T_1 = \tau\right] = U_1(t-\tau)$. This is because given the first renewal point $T_1 = \tau$, $C(\tau, t)$ is stochastically the same as $C(0, \tau)$ or $C(\tau)$.

Substituting Eq. (3.10) into Eq.(3.9) gives

$$U_1(t) = (U_1 * f_T)(t) + G_1(t), \tag{3.11}$$

where

$$G_1(t) = \sum_{0<\tau\le t} \mathsf{E}\left[C_1|T_1=\tau\right] f_T(\tau) + \mathsf{E}\left[C(t)|T_1>t\right] \overline{F}_T(t). \tag{3.12}$$

3.3.3 Second Moment

Similar to Eq.(3.9), the second moment or mean-square of the cost can be written as

$$U_2(t) = \sum_{0<\tau\le t} \mathsf{E}\left[C^2(t)|T_1=\tau\right] f_T(\tau) + \mathsf{E}\left[C^2(t)|T_1>t\right] \overline{F}_T(t), \tag{3.13}$$

When $T_1 = \tau < t$, split $C(t)$ into $C_1 + C(\tau, t)$, which allows to write the first expectation term in the right hand side of Eq.(3.13) as

$$\mathsf{E}\left[C^2(t)|T_1=\tau\right] = \mathsf{E}\left[C_1^2|T_1=\tau\right] + 2\mathsf{E}\left[C_1 C(\tau,t)|T_1=\tau\right] + \mathsf{E}\left[C^2(\tau,t)|T_1=\tau\right]. \tag{3.14}$$

Based on the renewal argument, the last two terms in Eq.(3.14) can be simplified as

$$\begin{aligned} \mathsf{E}\left[C_1 C(\tau,t)|T_1=\tau\right] &= \mathsf{E}\left[C_1|T_1=\tau\right] \mathsf{E}\left[C(\tau,t)|T_1=\tau\right] \\ &= \mathsf{E}\left[C_1|T_1=\tau\right] U_1(t-\tau), \qquad (3.15) \\ \mathsf{E}\left[C^2(\tau,t)|T_1=\tau\right] &= \mathsf{E}\left[C^2(t-\tau)\right] = U_2(t-\tau). \qquad (3.16) \end{aligned}$$

Substituting Eq. (3.14), (3.15) and (3.16) into (3.13), the following renewal equation is obtained

$$U_2(t) = (U_2 * f_T)(t) + G_2(t), \tag{3.17}$$

where $t \geq 1$ and

$$G_2(t) = \sum_{0<\tau\leq t} \mathsf{E}\left[C_1^2 | T_1 = \tau\right] f_T(\tau) + 2 \sum_{0<\tau\leq t} \mathsf{E}\left[C_1 | T_1 = \tau\right] f_T(\tau) U_1(t-\tau)$$
$$+ \mathsf{E}\left[C_1^2(t) | T_1 > t\right] \overline{F}_T(t). \tag{3.18}$$

To compute $U_2(t)$, $U_1(t)$ should be first derived from Eq.(3.11). Finally, the variance ($V(t)$) and the standard deviation ($\sigma(t)$) of cost can be obtained as,

$$V(t) = U_2(t) - U_1^2(t) \quad \text{and} \quad \sigma(t) = \sqrt{V(t)}. \tag{3.19}$$

3.3.4 Higher-Order Moments

For simplicity, define

$$h_m(\tau) = \mathsf{E}\left[C_1^m | T_1 = \tau\right] f_T(\tau) \quad \text{and} \quad \overline{H}_m(t) = \mathsf{E}\left[C^m(t) | T_1 > t\right] \overline{F}_T(t). \tag{3.20}$$

Then $h_m(\tau)$ is the partition of $\mathsf{E}\left[C_1^m\right]$ over the set $\{T_1 = \tau\}$ and $\overline{H}_m(t)$ is that of $\mathsf{E}\left[C^m(t)\right]$ over the set $\{T_1 > t\}$. We have

$$\sum_{\tau>0} h_m(\tau) = \sum_{\tau>0} \mathsf{E}\left[C_1^m | T_1 = \tau\right] f_T(\tau) = \mathsf{E}\left[C_1^m\right] \quad \text{and} \quad \lim_{t\to\infty} \overline{H}_m(t) = 0. \tag{3.21}$$

Using the above definition, $G_1(t)$ in Eq. (3.12) and $G_2(t)$ in Eq. (3.18) can be simplified as

$$G_1(t) = \sum_{0<\tau\leq t} h_1(\tau) + \overline{H}_1(t). \tag{3.22}$$

$$G_2(t) = \sum_{0<\tau\leq t} h_2(\tau) + 2(h_1 * U_1)(t) + \overline{H}_2(t). \tag{3.23}$$

Similar to the derivation of $U_2(t)$, the renewal equation (3.8) can be obtained for the m^{th} moment of $C(t)$, where

$$G_m(t) = \sum_{0<\tau\leq t} h_m(\tau) + \sum_{j=1}^{m-1} \begin{pmatrix} m \\ j \end{pmatrix} (h_j * U_{m-j})(t) + \overline{H}_m(t), \tag{3.24}$$

where $\begin{pmatrix} m \\ j \end{pmatrix} = \dfrac{m!}{j!(m-j)!}$ is the binomial coefficient.

Equation (3.24) shows that computation of the m^{th} moment requires all the moments of order less than m.

Letting $\Delta t \to \infty$, we will obtain the results for continuous time. Then the term $\sum_{0<\tau\leq t} h_m(\tau)$ in Eq. (3.24) should be replaced by $\int_0^t h_m(\tau)\mathrm{d}\tau$ and Eq. (3.8) will still hold.

3.3.5 Computational Procedure

To compute an m^{th} moment of maintenance cost, $U_m(t)$, we need to take the following procedure

(1) For specific maintenance policies, evaluate the renewal distribution $f_T(\tau)$ and the terms $h_j(\tau) = \mathsf{E}\left[C_1^j | T_1 = \tau\right] f_T(\tau)$ and $\overline{H}_j(\tau) = \mathsf{E}\left[C^j(\tau) | T_1 > \tau\right] \overline{F}_T(\tau)$, where $0 < \tau \leq t$ and $1 \leq j \leq m$;

(2) Use Eq. (3.24) to obtain $G_j(t)$ and substitute $G_j(t)$ and $f_T(\tau)$ into Eq. (3.8) to compute $U_j(t)$ recursively.

(2.1) substitute $h_1(\tau)$ and $\overline{H}_1(\tau)$ into Eq. (3.24) to obtain $G_1(\tau)$ and then substitute $G_1(\tau)$ and $f_T(\tau)$ into Eq. (3.8) to obtain $U_1(\tau)$ as

$$\begin{aligned}
U_1(\Delta t) &= G_1(\Delta t), \\
U_1(2\Delta t) &= U_1(\Delta t) f_T(\Delta t) + G_1(2\Delta t), \\
U_1(3\Delta t) &= U_1(2\Delta t) f_T(\Delta t) + U_1(\Delta t) f_T(2\Delta t) + G_1(3\Delta t), \\
&\vdots \\
U_1(t) &= U_1(t - \Delta t) f_T(\Delta t) + U_1(t - 2\Delta t) f_T(2\Delta t) + \cdots + U_1(\Delta t) f_T(t - \Delta t) \\
&\quad + G_1(t)
\end{aligned}$$

(2.2) Substitute $U_1(\tau)$, $h_1(\tau)$, $h_2(\tau)$ and $\overline{H}_2(\tau)$ into Eq. (3.24) to obtain $G_2(t)$ and then substitute $G_2(\tau)$ and $f_T(\tau)$ into Eq. (3.8) to obtain $U_2(\tau)$;

$\vdots$

(2.m) Substitute $U_1(\tau)$ – $U_{m-1}(\tau)$, $h_1(\tau)$ – $h_m(\tau)$, and $\overline{H}_m(\tau)$ into Eq. (3.24) to obtain $G_m(\tau)$ and then substitute $G_m(\tau)$ and $f_T(\tau)$ into Eq. (3.8) to obtain $U_m(\tau)$.

3.4 Asymptotic Formula for Expected Maintenance Cost

In this section, we are going to use Eq. (3.11) to obtain the asymptotic formula of the first moment of cost, $U_1(t)$.

For renewal equations, we have following theorem [21]

Theorem 3.2. *For a given function $g(t)$ which is bounded on finite intervals and a PDF $f(t)$ which has a finite first moment μ, let $z(t)$ be defined by the renewal equation*

$$z(t) = (z * f)(t) + g(t), \quad t > 0$$

Then

$$\lim_{t\to\infty} z(t) = \frac{1}{\mu}\int_0^\infty g(t)dt \tag{3.25}$$

In the above theorem, if all the functions are discrete, then the integral sign in Eq. (3.25) should be changed to the summation sign, i.e. Eq. (3.25) should be modified as

$$\lim_{t\to\infty} z(t) = \frac{1}{\mu}\sum_{t>0} g(t)\Delta t. \tag{3.26}$$

Let $u_1(t) = \left[U_1(t) - U_1(t-\Delta t)\right]/\Delta t$ and $g_1(t) = \left[G_1(t) - G_1(t-\Delta t)\right]/\Delta t$. Differentiating

Eq. (3.11) gives

$$u_1(t) = (u_1 * f_T)(t) + g_1(t).$$

The above equation is still a renewal equation. Using Theorem 3.2, the asymptotic value of $u_1(t)$ is obtained as

$$\lim_{t\to\infty} u_1(t) = \frac{1}{\mu_T}\sum_{t>0} g_1(t)\Delta t = \frac{1}{\mu_T}\lim_{t\to\infty} G_1(t), \tag{3.27}$$

where μ_T is the expected value of the renewal interval T. Equation (3.22) gives that

$$\lim_{t\to\infty} G_1(t) = \sum_{\tau>0} h_1(\tau) + \lim_{t\to\infty} \overline{H}_1(t).$$

Note that

$$\sum_{\tau>0} h_1(\tau) = \sum_{\tau>0} \mathsf{E}\left[C_1 | T_1 = \tau\right] f_T(\tau) = \mathsf{E}\left[C_1\right] \quad \text{and} \quad \lim_{t\to\infty} \overline{H}_1(t) = 0.$$

Denote $\mathsf{E}\left[C_1\right]$ by μ_C. Equation (3.27) becomes

$$\lim_{t\to\infty} u_1(t) = \frac{\mu_C}{\mu_T}. \tag{3.28}$$

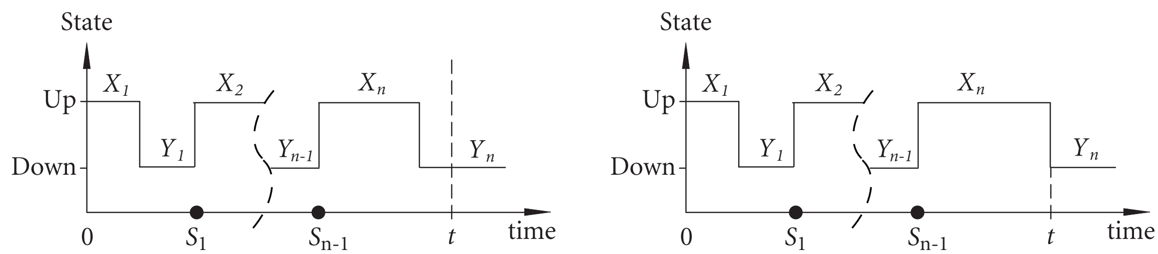
Hence we have the following asymptotic formula of $U_1(t)$

$$U_1(t) = \frac{\mu_C}{\mu_T} t + o(t). \tag{3.29}$$

The above formula has been widely used as an objective function for optimizing maintenance cost in the literature [40]. In this thesis we will show that this is not a precise approximation of the expected maintenance cost.

3.5 Unavailability and Failure Rate

Unavailability is the probability that a component is in the down state, and failure rate is the expected number of failures per unit time. Denote the unavailability at time t by $u_D(t)$ and the failure rate by $u_F(t)$. Then $u_D(t)$ is the probability of the event shown by Figure 3.3a and $u_F(t)\Delta t$ is that shown by Figure 3.3b.



(a) Down at t (b) Failure at t

Figure 3.3: Unavailability and failure rate

In the following we derive $u_D(t)$ and $u_F(t)$ by using Eq. (3.8) for an age based replacement model with finite replacement time (see Section 3.2.2).

Let $I_D(t)$ be the indicator of component state at time t, 1 for down and 0 for up. Then

$$u_D(t) = \mathsf{P}\{I_D(t) = 1\} = \mathsf{E}[I_D(t)]$$

The event of $\{I_D(t) = 1\}$ implies that the component keeps in the down state in the interval of $(t - \Delta t, t]$. Hence $\sum_{0<\tau\leq t} I_D(\tau)\Delta t$ is equal to the total down time up to t.

Note that if the unit costs in the age based replacement model are taken as $c_{\mathrm{D}} = 1$ and $c_{\mathrm{F}} = c_{\mathrm{P}} = 0$ (see Eq. (3.7)), then $C(t)$ will be equal to the total down time up to t. Denote such $C(t)$ by $N_D(t)$ and the associated expected value $U_1(t)$ by $U_D(t)$. Then we have

$$\mathsf{E}\left[\sum_{0<\tau\leq t} I_D(\tau)\Delta t\right] = U_D(t).$$

Therefore unavailability can be obtained as

$$u_D(t) = \frac{\Delta U_D(t)}{\Delta t}, \tag{3.30}$$

where $\Delta U_D(t) = U_D(t) - U_D(t - \Delta t)$. Then we can use Eq. (3.8) to obtain $U_D(t)$ first and then use the above equation to compute $u_D(t)$.

Failure rate $u_F(t)$ can be obtained similarly. Let $I_F(t)$ be the indicator of component failure at time t, 1 for yes and 0 for no. Then

$$u_F(t) = \frac{1}{\Delta t}\mathsf{P}\left\{I_F(t) = 1\right\} = \frac{1}{\Delta t}\mathsf{E}\left[I_F(t)\right],$$

and $\sum_{0<\tau\leq t} I_F(\tau)$ is equal to the number of failures up to t. Taking the unit costs in Eq. (3.7) as $c_{\mathrm{F}} = 1$ and $c_{\mathrm{P}} = c_{\mathrm{D}} = 0$, then $C(t)$ will be equal to the number of failures up to t. Denote $U_1(t)$ by $U_F(t)$, such that

$$u_F(t) = \frac{\Delta U_F(t)}{\Delta t}, \tag{3.31}$$

where $\Delta U_F(t) = U_F(t) - U_F(t - \Delta t)$.

For continuous time, i.e., $\Delta t \to 0$, Eq. (3.30) and (3.31) will become

$$u_D(t) = \frac{\mathrm{d}U_D(t)}{\mathrm{d}t}, \quad u_F(t) = \frac{\mathrm{d}U_F(t)}{\mathrm{d}t}. \tag{3.32}$$

3.6 Example

A numerical example is presented to illustrate the age based replacement policy with finite replacement time as described in Section 3.2.2.

This example is purely illustrative. The units of various quantities in this section are not of any practical relevance.

3.6.1 Input Data

Suppose that time is discretized as 0, 1, 2, $\cdots$. Component lifetime L is a discrete Weibull distributed random variable. The hazard rate is shown by Eq. (2.14) with parameters $\alpha = 4$ and $\beta = 40$. Then the PMF $f_L(l)$, the CDF $F_L(l)$, and the SF $\overline{F}_L(l)$, $l = 1, 2, \cdots, \beta$, can be computed. The mean and the standard deviation of L are $\mu_L = 20$ and $\sigma_L = 5.5$, respectively.

The down time, Y, is a geometrically distributed random variable with PMF

$$f_Y(y) = \phi(1-\phi)^{y-1}, \quad y = 1, 2, \cdots \tag{3.33}$$

where the parameter ϕ is the probability that replacement will be finished at time y. We take $\phi = 0.5$ so that the mean down time is $\mu_Y = 1/\phi = 2$. It is assumed that L and Y are independent of each other.

Unit costs are taken as $c_{\mathrm{F}} = 5$, $c_{\mathrm{P}} = 1$ and $c_{\mathrm{D}} = 0.2$. The length of time horizon is $t = 30$.

3.6.2 Computation

For any age of replacement t_p, the procedure of Section 3.3.5 to obtain the expected maintenance cost $U_1(t)$ requires computation of the following quantities.

(1) $f_T(\tau)$

Note that the time to replacement $X = \min(L, t_p)$. Then $X = 1, 2, \cdots, t_p - 1$, and the PMF of X is obtained as

$$f_X(x) = \begin{cases} f_L(x), & \text{if } x < t_p, \\ \overline{F}_L(t_p - 1), & \text{if } x = t_p. \end{cases}$$

Since X is independent of Y and the length of renewal interval is $T = X + Y$, the PMF of T is equal to

$$f_T(\tau) = (f_X * f_Y)(\tau).$$

(2) $h_1(\tau)$

Using the law of total expectation by conditioning on X_1 and Y_1, $h_1(\tau)$ is obtained as

$$
\begin{aligned}
h_1(\tau) &= \mathsf{E}\left[C_1 | T_1 = \tau\right] f_T(\tau) \\
&= \sum_{x=1}^{\min(\tau, t_p)} \mathsf{E}\left[C_1 | X_1 = x, Y_1 = \tau - x\right] f_X(x) f_Y(\tau - x).
\end{aligned} \tag{3.34}
$$

In the above equation, the event $\{T_1 = \tau\}$ is partitioned into mutually exclusive subevents as $\bigcup_{x=1}^{\min(\tau,t_p)}\{X_1 = x, Y_1 = \tau - x\}$. Given $\{X_1 = x, Y_1 = \tau - x\}$, the maintenance cost of the first renewal interval is equal to

$$
C_1 = \begin{cases} C_{\mathrm{CM}}(x, \tau) = c_{\mathrm{F}} + c_{\mathrm{D}}(\tau - x), & \text{if } x < tp, \\ C_{\mathrm{PM}}(t_p, \tau) = c_{\mathrm{P}} + c_{\mathrm{D}}(\tau - t_p), & \text{if } x = t_p, \end{cases}
$$

Then Eq. (3.34) gives

$$
h_1(\tau) = \begin{cases} \sum_{x=1}^{\tau} C_{\mathrm{CM}}(x, \tau) f_X(x) f_Y(\tau - x), & \text{if } \tau < t_p, \\ \sum_{x=1}^{t_p - 1} C_{\mathrm{CM}}(x, \tau) f_X(x) f_Y(\tau - x) + C_{\mathrm{PM}}(t_p, \tau) f_X(t_p) f_Y(\tau - t_p), & \text{if } \tau \geq t_p. \end{cases}
$$

(3) $\overline{H}_1(t)$

Using the law of total expectation by conditioning on X_1 and Y_1, $\overline{H}_1(t)$ is obtained as

$$
\begin{aligned}
\overline{H}_1(t) &= \mathsf{E}\left[C(t) | T_1 > t\right] \overline{F}_T(t) \\
&= \sum_{x=1}^{\min(t, t_p)} \mathsf{E}\left[C(t) | X_1 = x, Y_1 > t - x\right] f_X(x) \overline{F}_Y(t - x).
\end{aligned} \tag{3.35}
$$

Given $\{X_1 = x, Y_1 > t - x\}$, the maintenance cost up to t is equal to

$$C(t) = \begin{cases} \overline{C}_{\text{CM}}(x,t) = c_{\text{F}} + c_{\text{D}}(t-x), & \text{if } x < tp, \\ \overline{C}_{\text{PM}}(t_p,t) = c_{\text{P}} + c_{\text{D}}(t-t_p), & \text{if } x = t_p, \end{cases}$$

Then Eq. (3.35) gives

$$\overline{H}_1(t) = \begin{cases} \sum_{x=1}^{t} \overline{C}_{\text{CM}}(x,t) f_X(x) \overline{F}_Y(t-x), & \text{if } t < t_p, \\ \sum_{x=1}^{t_p-1} \overline{C}_{\text{CM}}(x,t) f_X(x) \overline{F}_Y(t-x) + \overline{C}_{\text{PM}}(t_p,t) f_X(t_p) \overline{F}_Y(t-t_p), & \text{if } t \geq t_p. \end{cases}$$

Then use Eq. (3.12) to obtain $G_1(t)$ and substitute $G_1(t)$ and $f_T(\tau)$ into Eq. (3.11) to calculate $U_1(t)$ recursively.

3.6.3 Numerical Results

Figure 3.4 shows the expected maintenance cost in a time horizon of $t = 30$ versus the replacement age. The finite time formula shows that the optimal replacement age is $t_p = 15$, for which the minimum cost is 2.8. However, the asymptotic formula shows that the optimal replacement age is $t_p = 13$, for which the minimal cost is 3.6. The asymptotic cost over-predicts the expected maintenance cost by almost 30%. The finite time formulation provides a mroe accurate estimate of the expected cost.

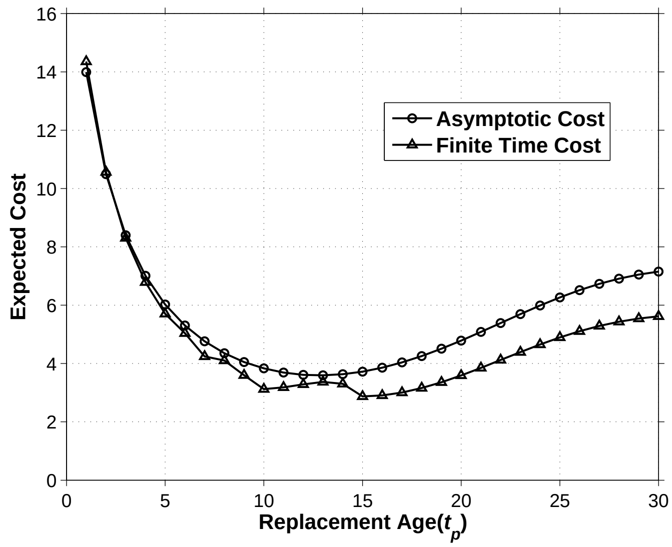


Figure 3.4: Expected cost vs. replacement age

3.7 Summary

This chapter presents a general derivation of any m^{th} order statistical moment maintenance cost in a finite time horizon. The moment of cost is derived as a renewal-type integral equation. The proposed formulation can be used to derive results for a specific maintenance policy, so long as it can be modelled as a stochastic renewal-reward process. This general approach allows the finite time cost analysis of a variety of maintenance policies. Subsequent chapters will use the results presented in this chapter.

Chapter 4

Condition Based Maintenance

4.1 Introduction

4.1.1 Motivation

Critical engineering systems and structures, such as dams, dikes, breakwater, and other protection systems are adversely affected by degradation caused by over-stress and aging related mechanisms such as erosion, corrosion, and fatigue.

To ensure safety and availability of these systems, the condition based maintenance (CBM) policy is often used. Under this policy, the condition of the system is examined through inspections planned at a fixed interval. If the degradation is found to exceed a threshold, the system is preventively replaced prior to onset of a catastrophic failure.

A CBM policy is more appropriate than age-based replacement policy, if the replacement of the system is prohibitively costly, such as in a nuclear plant. The reason is that an age-based policy requires replacement of the system irrespective of its condition, whereas decision-making in a CBM policy is based on the existing condition of the system.

Several variations of CBM models have been discussed in the industrial and maintenance engineering literature, depending on whether or not the inspection schedule is periodic, inspection tools are perfect, failure is detected immediately, or repair duration is negligible. Abdel-Hameed [1] and Park [50] presented models of periodic CBM of components subjected to gamma process degradation. The model of non-periodic CBM was presented by Grall [25] and that of imperfect inspection by Kallen [33]. Castanier [10] studied such a maintenance policy in which both the future maintenance (replacement or imperfect repair) and the inspection schedule depend on the magnitude of degradation. A detailed review of stochastic maintenance models is presented in a recent monograph [40].

The optimization of CBM is based on minimization of maintenance cost with respect to the inspection interval and preventive maintenance criteria while complying with the regulatory limits of reliability and availability.

As discussed before, maintenance cost optimization is based on the asymptotic cost, since its evaluation is quite easy. However, finite-time cost analysis is required for practical engineering systems with relatively finite operating life and financing horizon [44, 11, 47, 12]. The finite time cost analysis of the CBM policy has not been reported in the literature. A recent survey shows that the finite time cost model has been limited to age and block replacements, and minimal repair policy [41]. Christer [13] and Christer & Jack [14] derived the expected finite time cost for an age-based replacement policy in form of a recursive equation. The examples given in these studies showed that the traditional asymptotic solution for optimal age can lead to significant error in comparison to finite time cost. Later Jack [29, 30] applied this approach to analyze a policy in which a component is minimally repaired after failure and replaced after failure. In a recent paper, Jiang [32] presented an optimal solution of age replacement problem for a finite time horizon.

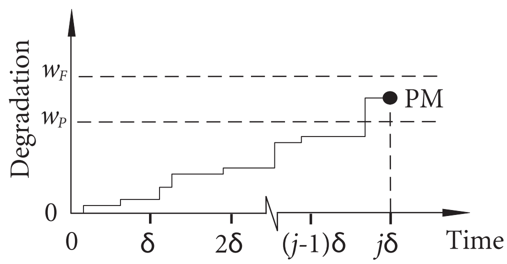
The objective of this chapter is to derive finite-time mean and variance of the maintenance cost of a CBM program. This formulation serves as a foundation to subsequent optimization of the maintenance cost.

4.1.2 Organization

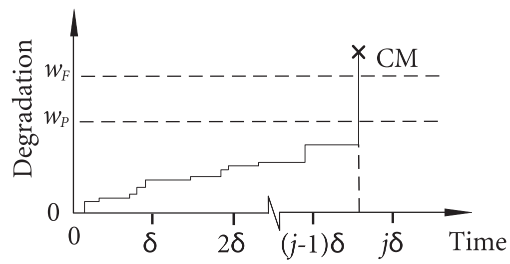
Section 4.2 provides the details of the CBM model discussed in this chapter. Section 4.3.2 introduces the stochastic gamma process model of uncertain degradation process. The mean and variance of the cost, and failure rate are derived in Section 4.4. The method of simulation is given in 4.5. The asymptotic cost is discussed in Section 4.6. Section 4.7 presents a practical example of corrosion in the heat transport piping system of a nuclear power plant.

4.2 Maintenance Model

Figure 4.1 is an illustration of the CBM policy studied in this chapter. Let $W(t)$ be degradation of a component at time t. $W(t)$ is a non-decreasing process, and component failure will occur when $W(t)$ exceeds a critical thresh hold w_F. To avoid component failure, the component is inspected periodically at times δ, 2δ, $\cdots$, and the value of $W(t)$ is measured. If $W(t)$ exceeds a preventive threshold w_P ($< w_F$), the component will be renewed (replaced or repaired into an as-good-as-new condition), called preventive maintenance (PM, Figure 4.1a). If $W(t)$ exceeds w_F between two consecutive inspection times, the component will fail and will be renewed right after failure, referred to as corrective maintenance (CM, Figure 4.1b), which is much more costly than PM.



(a) PM



(b) CM

Figure 4.1: Illustration of CBM

In this chapter, it is assumed that: (1) a component's degradation starts as it is put into service; (2) time for maintenance, whether PM or CM, is negligible; and (3) component failure is self-announcing, i.e., no inspection is needed to detect component failure and then CM will be performed right after failure. After component renewal, the inspection schedule will restart from then on. For example, if the time of renewal is t, then inspection times following that will be $t+\delta$, $t+2\delta$, $\cdots$. In a finite time horizon, there will be multiple renewal intervals (see Figure 4.2).

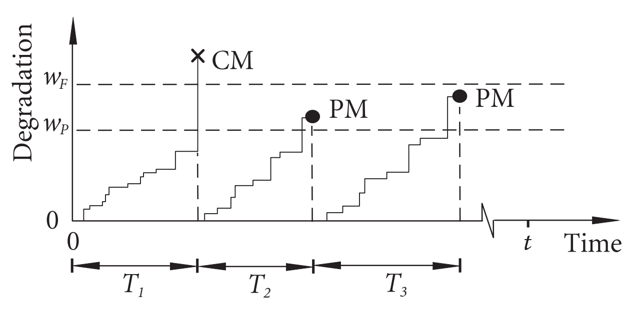


Figure 4.2: Sample path of CBM

In the above model, the total maintenance cost consists of inspection cost, PM cost, and CM cost. The unit costs of these items are denoted as: inspection cost – c_I, CM cost

– c_F, and PM cost – c_P. Then the cost incurred in a PM renewal interval of length τ, τ being a multiple of δ, is equal to

$$C_{\text{PM}}(\tau) = (\tau/\delta)c_I + c_P, \tag{4.1}$$

and that incurred in a CM renewal interval of length τ, τ being of any value, is equal to

$$C_{\text{CM}}(\tau) = \lfloor \tau/\delta \rfloor c_I + c_F, \tag{4.2}$$

where $\lfloor\rfloor$ denotes the floor function. Since the inspection schedule is also renewed after component renewal, the new component is still subject to periodic inspection and replacement, if any, by taking the last renewal point as the new time origin. Hence $\{T_n, C_n\}$, T_n being the length of the nth renewal interval and C_n the associated cost, are *iid* random vectors. Then the total cost up to time t, $C(t)$, is a renewal-reward process.

The above CBM model is the same as that in [50], where it is used to maintain break linings subjected to stochastic wear.

4.3 Stochastic Degradation Process

4.3.1 Background

The theory of stochastic processes has served as a fundamental basis for modeling an uncertain, dynamic process of degradation, and for estimating the maintenance cost by incorporating uncertainties associated with the occurrence of failure and maintenance events over a stipulated service life of the system. Although stochastic maintenance models are common in operation research and queuing theory literature [40], van Noortwijk and his co-workers [66, 69, 67, 68] presented several formal applications of the renewal theory to maintenance cost analysis of structures. Van der Weide et al. [64] used the compound renewal process to model system degradation. Optimization of inspection and repair for the Wiener and gamma processes of degradations was discussed in [42]. Other recent examples of stochastic models for structural maintenance are presented by Rackwitz et al. [54], Streicher et al. [60], Rackwitz & Joanni [53].

4.3.2 Gamma Process

The gamma process is an example of a stochastic cumulative process with a simple mathematical structure that provides an effective tool to model the evolution of damage. The basic mathematical framework of the gamma process model was developed in early 1970's, and then is introduced by van Noortwijk to civil engineering community [69, 67, 68]. Gamma process has been applied to model various types of degradation processes, such as creep in concrete [15], recession of coastal cliffs [26], deterioration of coating on steel structures [43], structural degradation [24] and wall thinning corrosion of pipes in nuclear power plants [75]. A comprehensive review of the gamma process model and its applications was recently published [65].

A gamma process is defined as follows. Denote a gamma distributed random variable with shape parameter ξ and scale parameter β by Gamma(ξ, β). Let $f^{\mathrm{G}}(w; \xi, \beta)$, $F^{\mathrm{G}}(w; \xi, \beta)$ and $\overline{F}^{\mathrm{G}}(w; \xi, \beta)$ be the PDF, the CDF, and the SF of Gamma(ξ, β). Then

$$f^{\mathrm{G}}(w; \xi, \beta) = \frac{w^{\xi-1} e^{-w/\beta}}{\beta^{\xi} \Gamma(\xi)}, \quad F^{\mathrm{G}}(w; \xi, \beta) = \frac{\Gamma(\xi, w/\beta)}{\Gamma(\xi)}, \quad \overline{F}^{\mathrm{G}}(w; \xi, \beta) = 1 - \overline{F}^{\mathrm{G}}(w; \xi, \beta), \tag{4.3}$$

where $\Gamma(a) = \int_0^\infty x^{a-1}e^{-x}\mathrm{d}x$ is the complete gamma function, and $\Gamma(a,b) = \int_0^b x^{a-1}e^{-x}\mathrm{d}x$ is the lower incomplete gamma function.

Let $t_k = k\Delta t$, $k = 0, 1, 2, \cdots$. A stochastic process $W(t)$ is called a gamma process with shape function $\xi(t)$ and scale parameter β if

(1) $W(0) = 0$;

(2) the sequence of one-step differences $W(t_{k-1}, t_k) = W(t_k) - W(t_{k-1})$ is a sequence of independent random variables; and

(3) $W(t_{k-1}, t_k) \sim \text{Gamma}(\xi(t_{k-1}, t_k), \beta)$, where $\xi(t_{k-1}, t_k) = \xi(t_k) - \xi(t_{k-1})$.

Generally, $\xi(t)$ can be taken in a form of a general power law as

$$\xi(t) = \alpha t^\theta. \tag{4.4}$$

When the shape function $\xi(t)$ is nonlinear in time, $W(t)$ is referred to as a non-stationary gamma process. In a special case of $\theta = 1$, $W(t)$ is called a stationary gamma process. Given degradation measurement data collected at various time intervals, α, β and θ can be estimated using the methods of maximum likelihood or the method of moments [7, 43, 65].

As mentioned before, the component fails as soon as the gamma degradation passes a failure level w_F. Hence, the lifetime of the component, denoted by L, is equal to

$$L = \min\{t : W(t) > w_F\}.$$

The PMF of L is then given by

$$
\begin{aligned}
f_L^{\mathrm{G}}(t_k) &= \mathsf{P}\left\{L = t_k\right\} \\
&= \mathsf{P}\left\{W(t_{k-1}) \leq w_F, W(t_k) > w_F\right\} \\
&= \mathsf{P}\left\{W(t_{k-1}) \leq w_F, W(t_{k-1}) + W(t_{k-1}, t_k) > w_F\right\}.
\end{aligned}
$$

Note that the random variables $W(t_{k-1})$ and $W(t_{k-1}, t_k)$ are independent of each other, $W(t_{k-1}) \sim \mathrm{Gamma}\big(\xi(t_{k-1}), \beta\big)$ and $W(t_{k-1}, t_k) \sim \mathrm{Gamma}\big(\xi(t_{k-1}, t_k), \beta\big)$. Then the above probability can be evaluated by conditioning on the $W(t_{k-1})$ as

$$
\begin{aligned}
f_L^{\mathrm{G}}(t_k) &= \int_0^{w_F} f^{\mathrm{G}}\big(w; \xi(t_{k-1}), \beta\big)\mathsf{P}\left\{W(t_{k-1}, t_k) > w_F - w\right\} \mathrm{d}w \\
&= \int_0^{w_F} f^{\mathrm{G}}\big(w; \xi(t_{k-1}), \beta\big)\overline{F}^{\mathrm{G}}\big(w_F - w; \xi(t_{k-1}, t_k), \beta\big)\mathrm{d}w.. \qquad (4.5)
\end{aligned}
$$

For a stationary gamma process with shape function $\xi(t) = \alpha$ and scale parameter β, the average degradation per unit time is equal to [65]

$$
\frac{1}{t}\mathsf{E}\left[W(t)\right] = \alpha\beta. \qquad (4.6)
$$

4.3.3 Simulation of Gamma Process

A variety of simulation methods have been presented in the literature, such as gamma-increment sampling [4], gamma-bridge sampling [19, 56], and compound Poisson simulation [74, 51]. A simple and efficient gamma-increment sampling method is chosen. Samples of independent increment, $\Delta W(t_{k-1}, t_k)$, are simulated from a gamma distribution, $\mathrm{Gamma}(\xi(t_{k-1}, t_k), \beta)$. Then the sample of $W(t_i)$ is obtained as $W(t_i) = \sum_{k=1}^{i} \Delta W(t_{k-1}, t_k)$.

4.4 Maintenance Cost Analysis

Since the total cost $C(t)$ in the CBM model as described in Section 4.2 is a renewal-reward process, we can directly use results of Chapter 3 to obtain the moments of $C(t)$. To use Eq. (3.8) to compute the expected value and the variance of $C(t)$, the renewal distribution $f_T(\tau)$ and the term $G_m(t)$, $m = 1, 2$, should be first derived, as shown in Section 4.4.1 and 4.4.2, respectively.

4.4.1 Renewal Interval Distribution

To derive the renewal distribution $f_T(\tau)$, the following two events are defined at any j^{th} inspection time, $t_j^I = j\delta$, for $j \geq 1$:

$$A_j = \left\{W(t_{j-1}^I) \leq w_P, w_P < W(t_j^I) \leq w_F\right\},$$
$$B_j = \left\{W(t_{j-1}^I \leq w_P, W(t_j^I) > w_F\right\}.$$

Only one of the two events can terminate the renewal interval.

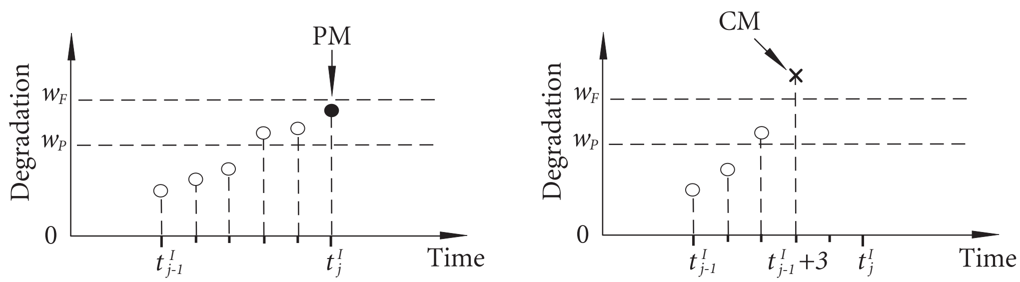
- If A_j occurs, then $T = t_j^I$;
- If B_j occurs, then T takes a value in the set $\left\{t_{j-1}^I + \Delta t, t_{j-1}^I + 2\Delta t, \cdots, t_{j-1}^I + \delta = t_j^I\right\}$.

  For $r = \Delta t, 2\Delta t, \cdots, \delta$, let

  $$B_{j,r} = \left\{W(t_{j-1}^I) \leq w_P, W(t_{j-1}^I + r - \Delta t) \leq w_F, W(t_{j-1}^I + r) > w_F\right\}.$$

  For a fixed j, the sets $\{B_{j,r}\}$ are a partition of B_j. If $B_{j,r}$ occurs, then $T = t_{j-1}^I + r$.

Events A_j and $B_{j,r}$ are illustrated in Figure 4.3 for a specific case of $\Delta t = 1$, $\delta = 5$ and $r = 3$.



(a) PM event A_j (b) CM event $B_{j,r}$

Figure 4.3: Illustration of events A_j and $B_{j,r}$

To derive $f_T(\tau)$, probabilities of the events A_j and $B_{j,r}$ are computed using the same argument as for the evaluation of (4.5), which leads to

$$\mathsf{P}\{A_j\} = \int_0^{w_P} f^{\mathrm{G}}\big(w_1; \xi(t^I_{j-1}), \beta\big) \left[\int_{w_P - w_1}^{w_F - w_1} f^{\mathrm{G}}\big(w_2; \xi(t^I_{j-1}, t^I_j), \beta\big) \mathrm{d}w_2\right] \mathrm{d}w_1, \tag{4.7}$$

and

$$\mathsf{P}\{B_{j,r}\} = \int_0^{w_P} f^{\mathrm{G}}\big(w; \xi(t^I_{j-1}), \beta\big) Q(w_F - w; \xi_1, \xi_2, \beta) \mathrm{d}w, \tag{4.8}$$

where $\xi_1 = \xi(t^I_{j-1}, t^I_{j-1} + r - \Delta t)$, $\xi_2 = \xi(t^I_{j-1} + r - \Delta t, t^I_{j-1} + r)$, and the function

$$Q(w; \xi_1, \xi_2, \beta) = \int_0^w f^{\mathrm{G}}(w'; \xi_1, \beta) \overline{F}^{\mathrm{G}}(w - w'; \xi_2, \beta) \mathrm{d}w'.$$

Then the probability of a renewal interval ending by PM at an inspection time $\tau = t^I_j$ is given by

$$f_{\mathrm{PM}}(\tau) = \mathsf{P}\{A_j\}. \tag{4.9}$$

Similarly, the probability of CM at any time $\tau = t^I_{j-1} + r$, $0 < r \leq \delta$, is given by

$$f_{\mathrm{CM}}(\tau) = \mathsf{P}\left\{B_{j,r}\right\}. \tag{4.10}$$

Then the PMF of T is given by

$$f_T(\tau) = f_{\mathrm{CM}}(\tau) + \begin{cases} f_{\mathrm{PM}}(\tau), & \text{if } \mathrm{mod}(\tau, \delta) = 0, \\ 0, & \text{otherwise.} \end{cases} \tag{4.11}$$

4.4.2 Computation of $G_m(t)$

To compute $G_m(t)$ by using Eq. (3.24), we need to obtain the terms $h_m(\tau)$ and $\overline{H}_m(\tau)$ first.

To derive $h_m(\tau)$, note that the cost incurred in a renewal interval associated with event A_j is equal to $C_{\mathrm{PM}}(\tau)$, where $\tau = j\delta$, and that associated with $B_{j,r}$ is equal to $C_{\mathrm{CM}}(\tau)$, where $\tau = (j-1)\delta + r$. $C_{\mathrm{PM}}(\tau)$ and $C_{\mathrm{CM}}(\tau)$ are given by Eq. (4.1) and (4.2), respectively. Then using the law of total expectation by conditioning on the type of maintenance, $h_m(\tau)$ can be obtained as

$$\begin{aligned} h_m(\tau) &= \mathsf{E}\left[C_1^m | T_1 = \tau\right] f_T(\tau) \\ &= C^m_{\mathrm{CM}}(\tau) f_{\mathrm{CM}}(\tau) + \begin{cases} C^m_{\mathrm{PM}}(\tau) f_{\mathrm{PM}}(\tau), & \text{if } \mathrm{mod}(\tau, \delta) = 0, \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{4.12}$$

To derive $\overline{H}_m(\tau)$, note that if $T_1 > t$, no maintenance will be taken before t. Then only inspection cost is incurred. The number of inspections up to t is equal to $\lfloor t/\delta \rfloor$. Hence

$\overline{H}_m(\tau)$ can be obtained as

$$\overline{H}_m(\tau) = \mathsf{E}\left[C(t)^m | T_1 > t\right] \overline{F}_T(t) = \lfloor t/\delta \rfloor \, c_I \overline{F}_T(t). \tag{4.13}$$

In the above equation, $\overline{F}_T(t)$ is the SF of T and can be obtained from $f_T(\tau)$.

Substitute $h_m(\tau)$ and $\overline{H}_m(\tau)$ into Eq. (3.24) to derive $G_m(t)$ and then the first and the second moment of $C(t)$ can be obtained from Eq. (3.8).

4.4.3 Failure Rate

Note that if we take $c_I = c_P = 0$ and $c_F = 1$, $C(t)$ will become the number of failures up to t. Then failure rate can be obtained from Eq. (3.31).

4.5 Simulation

The Monte Carlo simulation method is also used to evaluate the maintenance cost and verify the results of renewal equation method. Sample paths of gamma degradation process are simulated using the method described in Section 4.3.3. The algorithm for evaluation of maintenance cost is shown in Figure 4.4. Note that ΔW is the degradation increment, W is the total degradation, and C is the maintenance cost.

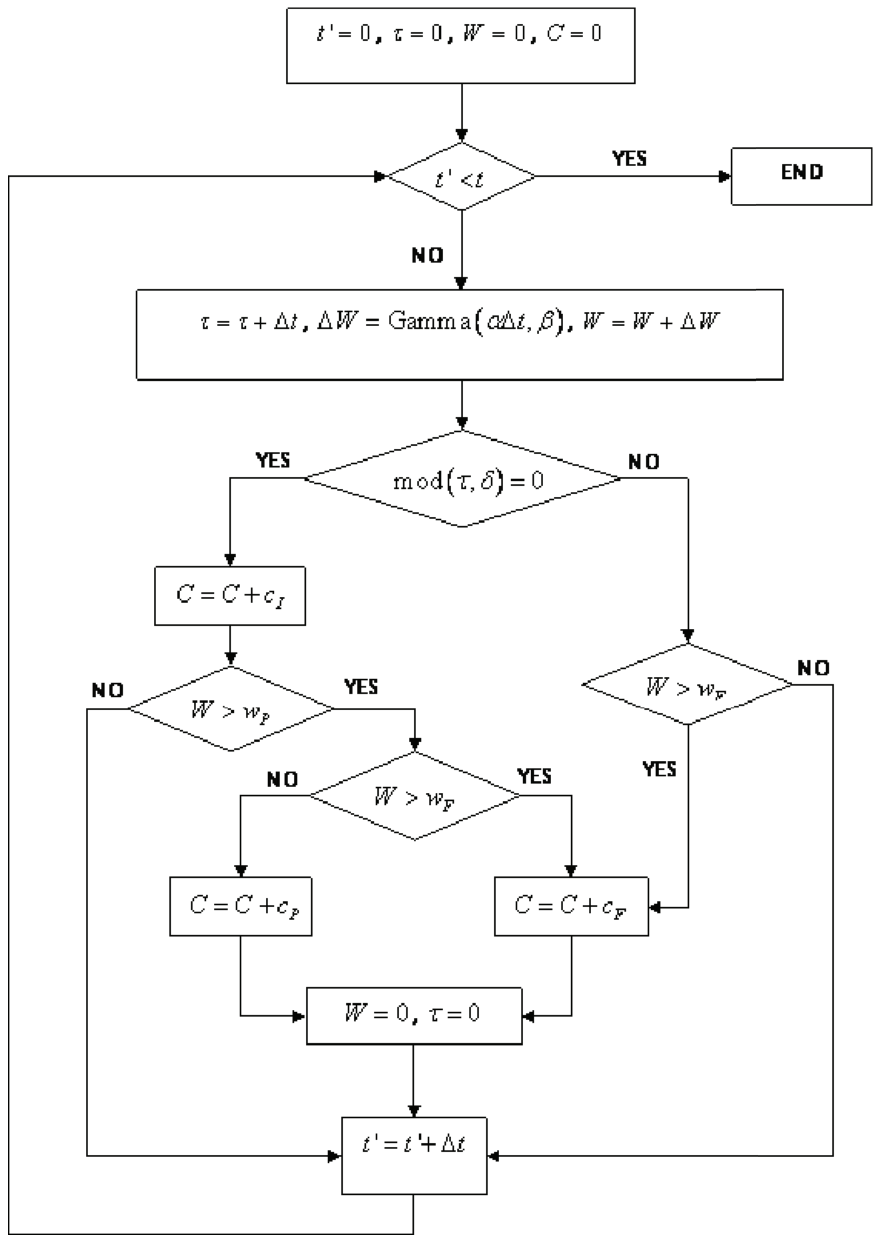


Figure 4.4: Algorithm of simulation

4.6 Asymptotic Cost

The expected length and the expected cost of a renewal interval can be obtained from $f_T(\tau)$ as

$$\mu_T = \sum_{\tau>0} \tau f_{\text{CM}}(\tau) + \sum_{j=1}^{\infty} j\delta f_{\text{PM}}(j\delta), \tag{4.14}$$

$$\mu_C = \sum_{\tau>0} C_{\text{CM}}(\tau) f_{\text{CM}}(\tau) + \sum_{j=1}^{\infty} C_{\text{PM}}(j\delta) f_{\text{PM}}(j\delta). \tag{4.15}$$

Then the asymptotic cost can be obtained from Eq. (3.29).

4.7 Example

Flow accelerated corrosion (FAC) degradation is common in the heat transport piping system (PHTS) of nuclear power plants. The uncertain corrosion process can be modelled as a stochastic gamma process [75]. The following information is gathered from past inspections and design documents [49]. The initial wall thickness of the pipe is 6.50 mm. The minimum required wall thickness is 2.41 mm. The degradation threshold of failure is thus w_F=3.09 mm.

Assume that FAC is a stationary gamma process, i.e. the parameter θ in Eq. (4.4) is equal to 1 and then the shape function is $\xi(t) = \alpha t$. Using wall thickness measurements collected from several inspections, the parameters of the gamma process were estimated as $\alpha = 1.13$/year and $\beta = 0.0882$ mm. The lifetime of the pipe is the time when wall thickness exceeds the threshold of 3.09 mm. Using this criterion, the PMF of the lifetime is computed from equation (4.5) and plotted in Figure 4.5. The mean and the standard deviation of lifetime are 32 and 5.25 years, respectively.

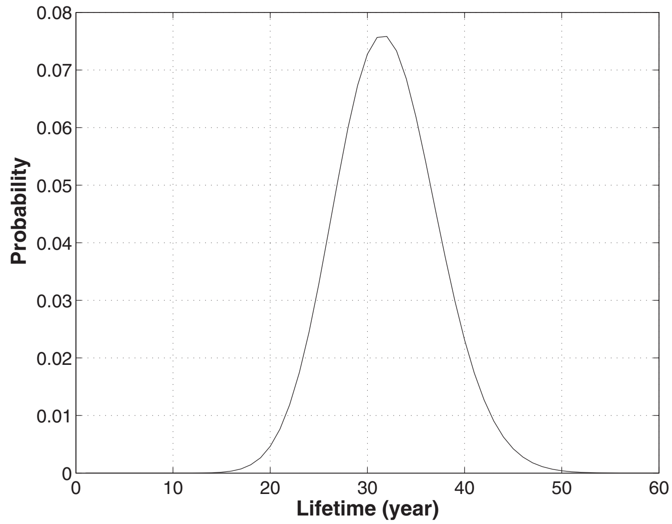


Figure 4.5: Lifetime distribution of pipes affected by corrosion

Cost data are specified in the unit of million\$ as $c_I = 0.01$, $c_P = 1$, and $c_F = 5$. The preventive replacement level is chosen as $w_P = 2.0$ mm based on a regulatory requirement. The maintenance cost is evaluated for a $t = 30$ years time horizon. Use Eq. (4.12) and (4.13) to calculate $h_m(\tau)$ and $\overline{H}_m(t)$. Then the first and the second moments of cost, $U_1(t)$ and $U_2(t)$, can be obtained by substituting $h_m(\tau)$ and $\overline{H}_m(t)$ into Eq. (3.8).

The variation of $U_1(t)$ with the inspection interval is plotted in Figure 4.6. The finite time model results in an optimal inspection interval of 21 years and corresponding cost of 0.82 million\$. The asymptotic formula results in an optimal inspection interval of 6 years and the associated cost is 1.31 million\$, which is about 60% higher than that calculated from the finite time formula. Given that the Canadian reactor design consists of 380 to 480 pipe sections, this cost differential for the entire reactor would be quite large.

Expected cost versus inspection interval results are also evaluated using simulation

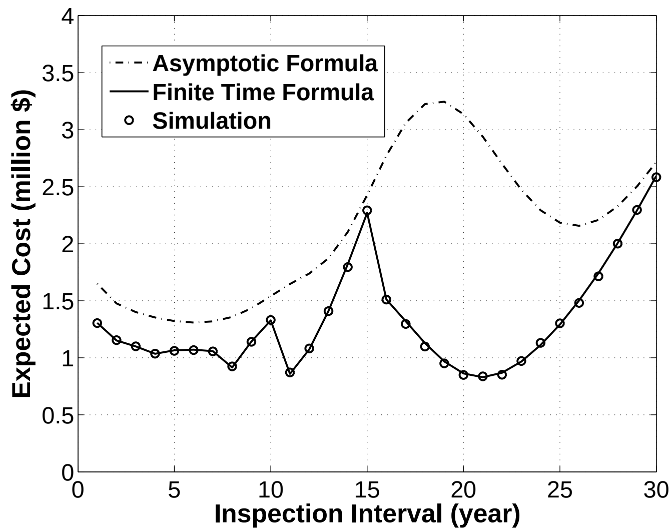


Figure 4.6: Expected cost vs. inspection interval

method. In each simulation run 10^4 samples are used. Figure 4.6 shows that results obtained from the finite time formula are quite close to those obtained from simulation. However, the computational time of simulation method is much larger. Using MATLAB 2010a version, the computational time of the simulation method is 58 seconds, but the finite time formula takes only 1 second.

The standard deviation of cost is equal to $\sigma(t) = \sqrt{U_2(t) - U_1(t)}$. $\sigma(t)$ provides valuable information about potential uncertainty associated with the estimated cost. Figure 4.7 plots the finite time expected cost (U_1) and one standard deviation upper bound ($U_1 + \sigma$) against the inspection interval. This Figure shows that an increase in inspection interval is accompanied with increase in the standard deviation of cost. It makes sense, since the temporal uncertainty associated with gamma process degradation increases as the inspection interval becomes long.

The expected cost curve has 3 competing optimal points (P2, P3 and P4), though the

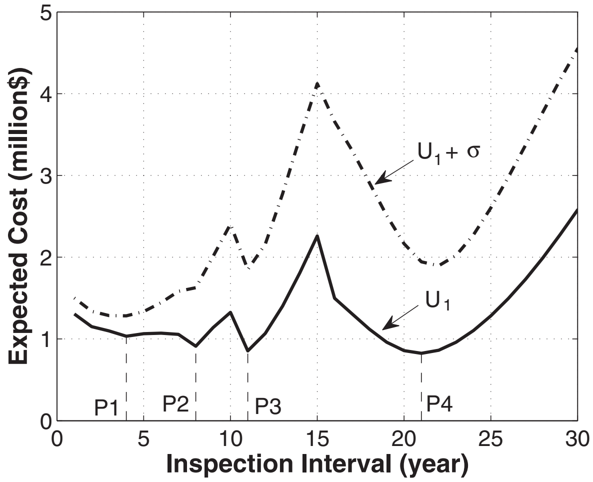


Figure 4.7: Mean and standard deviation of maintenance cost vs. inspection interval

standard deviation associated with them is remarkably different. At the most optimal inspection interval of $\delta = 21$ years (point P4 in Figure 4.7), the standard deviation is equal to $\sigma_D = 1.12$ million\$. There is another competing solution, $\delta = 11$ years (point P3), with slightly higher expected cost of 0.85 million\$, and lower standard deviation of 0.99 million\$. The next optima is point P2 with $\delta = 8$ years, $U_1 = 0.91$ million\$ and $\sigma = 0.72$ million\$. As the inspection interval is reduced, the expected cost increases, but associated standard deviation declines. Therefore, based on the risk tolerance of a decision maker, an appropriate combination of the mean and the standard deviation of cost can be used to determine an optimal policy.

Based on a minimum upper bound, $U_1 + \sigma$, criteria, the most optimal inspection interval is 4 years and associated upper bound cost is 1.28 million\$ (Point P1). It should be remarked that the analysis favors a shorter inspection interval, because the cost of inspection is much smaller (0.01 million\$) than that associated with PM or CM. The results of the finite time cost analysis emphasize the fact that the consideration of variance of the cost is of utmost importance in the optimization of a maintenance program. These results can be used to evaluate the benefit-cost analysis of various maintenance strategies with various combinations of inspection interval and the PM threshold, w_P.

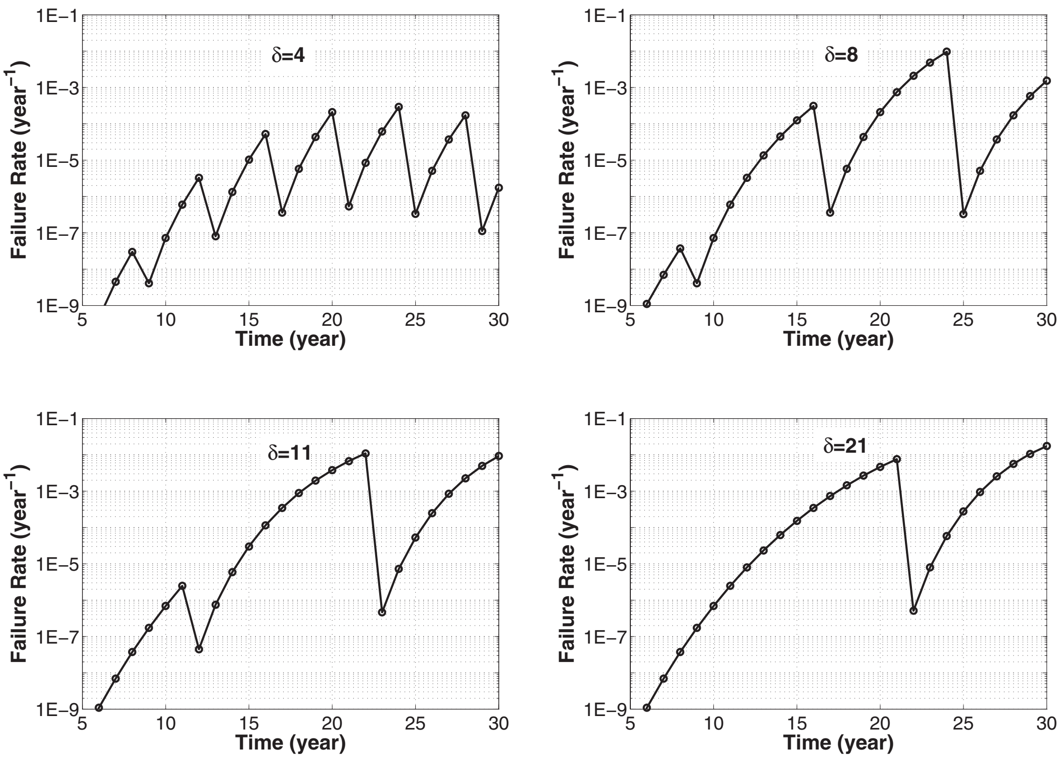


Figure 4.8: Time-dependent failure rate for various inspection intervals (δ years)

The variation of the failure rate over a 30 years time horizon is shown in Figure 4.8 for four competing solutions of optimal inspection intervals. To interpret these results, take a case of the first subfigure for $\delta = 4$, which shows that the failure rate in year 15, for example, is approximately 10^{-5}. This failure rate is a result of inspecting the pipe at 4 year interval and replacing it if its wall thickness loss exceeded w_P (=2.00 mm). The failure rate drops right after the inspection in years, e.g., year 9, 13, 17, $\cdots$, reflecting the benefit of the PM policy. All other subfigures can be interpreted in a similar manner. In summary, Figure 4.8 shows the benefit of the maintenance program in terms of reduction in the failure rate. The smaller the inspection interval, the lower is the failure rate. For a given regulatory limit on the maximum failure rate, Figure 4.8 can help identify a valid optimal solution. If the objective is to keep the failure rate below 10^{-3} per year, inspection intervals of δ=8, 11 and 21 years are not appropriate, since the maximum failure rate in these cases exceeds the limit 10^{-3} per year.

4.8 Summary

This chapter analyzed the cost of CBM of a system in which degradation is modelled as a stochastic gamma process. Although the gamma process is widely used in the literature, the finite time mean and variance of cost are derived for the first time in this work.

This chapter presents a case study involving CBM of the piping system in a nuclear plant. The study illustrates that the asymptotic formula over-predicts the maintenance cost as compared to that obtained from the proposed finite time model. Given that a plant contains a large fleet of piping components, the over-prediction by the asymptotic formula can be substantial, which can adversely affect the maintenance budget at the plant level.

This chapter also emphasizes the fact that the consideration of variance of the cost is of utmost importance in maintenance optimization. In a set of competing optimum solutions based on expected cost, the variance of cost would determine a more robust (less uncertain) solution. From a practical point of view, the utility of an optimum expected cost solution without knowing the associated variance is quite limited. The failure rate is another important quantity of the optimal inspection policy, especially if there is a need to satisfy a regulatory limit on failure rate during the entire service life of the system.

It is concluded that the finite time model should be used for a realistic evaluation and optimization of CBM for safety critical infrastructure systems. The optimal inspection and maintenance should be based on a prudent consideration of an upper bound cost and failure rate. In this context, the asymptotic solutions have limited utility for a decision maker.

The results of this chapter have been published in [44, 11, 47].

Chapter 5

Condition Based Maintenance-An Advanced Model

5.1 Introduction

5.1.1 Motivation

The CBM model analyzed in the previous chapter assumed that time required for repair is negligible, and degradation initiated as soon as the system was put in service. These two assumptions are relaxed in this chapter by considering the repair (or down) time and delay in degradation initiation as random variables. This chapter presents the derivation of the expected maintenance cost in a finite time horizon. In addition, computation of net present value of maintenance cost (or discounted cost) is also formulated.

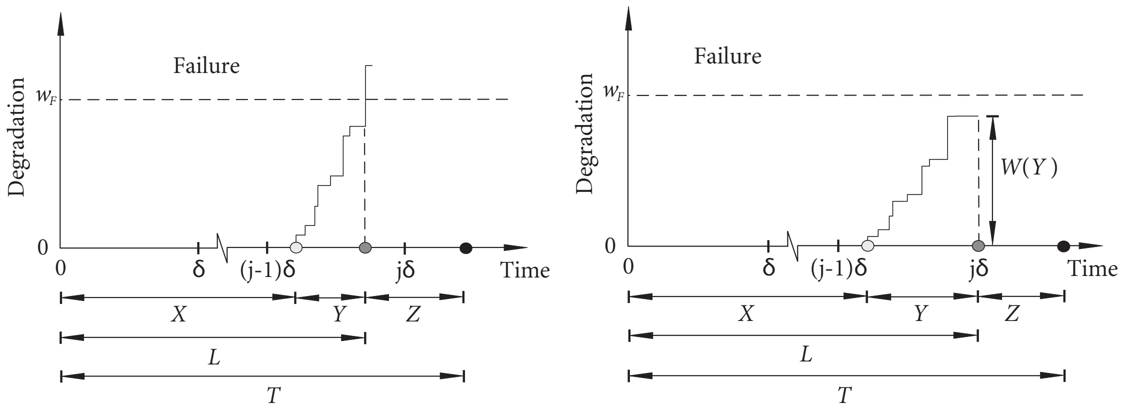
5.1.2 Organization

Section 5.2 describes CBM model, and Section 5.3 presents the computation of the expected cost. The asymptotic cost, unavailability, and failure rate are derived in Section 5.4. The formulation of discounted maintenance cost is given in Section 5.5. A practical case study related to the maintenance of a hydraulic structure is presented in Section 5.6.

5.2 Maintenance Model

This chapter considers a component degradation process that initiates and grows in a stochastic manner over time (see Figure 5.1). Degradation of a component does not accumulate until it is initiated. When degradation exceeds a threshold value w_F, the component fails and a CM is required to restore its condition (see Figure 5.1a). To reduce the risk of failure, it is planned to carry out periodic inspections at interval of δ and a PM is taken to repair any detected degradation, i.e. PM threshold $w_P = 0$ (see Figure 5.1b). The time for maintenance, whether CM or PM, is assumed to be non-negligible. Of course, inspections are not taken any more during maintenance interval. Note that maintenance intervals, PM and CM, could be different from each other. Generally, CM interval is longer than PM interval since CM is unpredicted maintenance and it takes more time to get ready. After maintenance, both the component and the inspection schedule are renewed, i.e. the new component will be inspected at times $\tau + \delta$, $\tau + 2\delta$, $\cdots$, τ being the last renewal time.

As shown in Figure 5.1, the degradation free interval is X, the degradation growth interval is Y, and the maintenance interval is Z. Then the operation interval of this



(a) CM (b) PM

Figure 5.1: Proposed stochastic model of degradation and maintenance (w_F = failure threshold, ○ = initiation of degradation, ● = start of maintenance, ● = end of maintenance)

component and the length of a renewal interval are, respectively,

$$L = X + Y, \tag{5.1}$$

$$T = L + Z = X + Y + Z. \tag{5.2}$$

Denote the degradation growth process by $W(y)$, where y is the time elapsing from degradation initiation. It is assume that (1) X and Z are both random variables; (2) $W(y)$ is a stationary gamma process with shape parameter α and scale parameter β. Due to the uncertainty of $W(y)$, Y is also a random variable; and (3) X is independent of $W(y)$ and Z. Denote the PMF of X by $f_X(x)$. Note that the PMF of Z depends on the type of maintenance (CM or PM). Denote the PMF of CM interval and that of PM by $f_{Z,\text{CM}}(z)$ and $f_{Z,\text{PM}}(z)$, respectively.

The total maintenance cost consists of inspection cost, CM cost, PM cost, and down time cost. The down time cost is due to the unavailability of the component when it is in maintenance and is proportional to the length of the maintenance interval. Different from the previous chapter, the PM cost is composed of a fixed cost and a variable cost. The variable PM cost is proportional to the amount of degradation (see $W(Y)$ in Figure 5.1b). The following notations are used to denote the unit costs of various items: inspection cost–c_I, CM cost–c_F, fixed PM cost–c_P, variable PM cost per unit degradation –c_V , and down time cost per unit time–c_D. Then the cost incurred in a CM renewal interval as shown in Figure 5.1a and that incurred in a PM renewal interval as shown in Figure 5.1b are equal to, respectively,

$$C_{\mathrm{CM}} = \lfloor L/\delta \rfloor c_I + c_F + c_D Z, \tag{5.3}$$

$$C_{\mathrm{PM}} = (L/\delta) c_I + [c_P + c_V W(Y)] + c_D Z. \tag{5.4}$$

Since the inspection schedule is also renewed after component renewal, the new component is still subject to periodic inspection and replacement, if any, by taking the last renewal point as the new time origin. Hence $\{T_n, C_n\}$, T_n being the length of the nth renewal interval and C_n the associated cost, are *iid* random vectors. Then the total cost up to time t, $C(t)$, is a renewal-reward process.

5.3 Maintenance Cost Analysis

In this section, the expected maintenance cost is derived. Since the total cost $C(t)$ in the CBM model as described in Section 5.2 is a renewal-reward process, we can directly use Eq. (3.8) in Chapter 3 to obtain the expected value of $C(t)$. In Chapter 3, we use a subscript below U to denote the order of moments. However, in this chapter, only the first order of cost is considered. Hence we will omit the subscript of U in this chapter, i.e. use $U(t)$ to denote $\mathsf{E}[C(t)]$ for simplicity. Similarly, we use the notations of $h(\tau)$, $\overline{H}(t)$ and $G(t)$ instead of $h_1(\tau)$, $\overline{H}_1(t)$ and $G_1(t)$ in this chapter.

To use Eq. (3.11), the renewal distribution $f_T(\tau)$ and the term $G(t)$ should be first derived, as shown in Section 5.3.1 and 5.3.2, respectively.

5.3.1 Renewal Interval Distribution

As mentioned in Section 5.2 (see Eq. (5.1) and (5.2)), a complete renewal cycle T consists of two subintervals: (1) operation interval L; and (2) the maintenance interval Z. The operation interval L also consists of subintervals: (1.1) degradation free interval X; and (1.2) degradation growth interval Y. In the following, we are going to obtain the probability distribution of L first and then to obtain that of T.

As shown by Figure 5.1a, in a CM renewal interval, the event $\{L = l\}$, denoted by $\mathcal{A}_{\mathrm{CM}}(l)$, implies that component degradation is initiated during $\big[t_{\mathrm{I}}(l,\delta), l\big)$ and then exceeds w_F at time l, i.e.

$$\mathcal{A}_{\mathrm{CM}}(l) = \bigcup_{t_{\mathrm{I}}(l,\delta)\le x<l} \{X = x, W(l-x-\Delta t) \le w_F, W(l-x) > w_F\}, \tag{5.5}$$

where

$$t_{\mathrm{I}}(l,\delta) = \begin{cases} \lfloor l/\delta \rfloor \delta, & \text{if } \mathrm{mod}(l,\delta) \neq 0, \\ l-\delta, & \text{if } \mathrm{mod}(l,\delta) = 0, \end{cases} \tag{5.6}$$

is the inspection time up to but not including l. Let $f_{L,\mathrm{CM}}(l)$ be probability of such an event. Then

$$f_{L,\mathrm{CM}}(l) = \sum_{t_{\mathrm{I}}(l,\delta)\le x<l} \mathsf{P}\{X = x, W(l-x-\Delta t) \le w_F, W(l-x) > w_F\} \tag{5.7}$$

As mentioned in Section 5.2, degradation free interval X is independent of degradation growth $W(y)$. Then the probability in the right hand side of the above equation is equal to $f_X(x)\mathrm{P}\{W(l-x-\Delta t) \le w_F, W(l-x) > w_F\}$, where $f_X(x)$ is the PMF of X. The probability that a gamma process exceeds a critical value of w_F at time t_k has been derived as function $f_L^{\mathrm{G}}(t_k)$ in Eq. (4.5) in Chapter 4. Using this function, the probability of $\{W(l-x-\Delta t) \le w_F, W(l-x) > w_F\}$ can be obtained as $f_L^{\mathrm{G}}(l-x)$. Then Eq. (5.7) gives

$$f_{L,\mathrm{CM}}(l) = \sum_{t_{\mathrm{I}}(l,\delta)\le x<l} f_X(x) f_L^{\mathrm{G}}(l-x). \tag{5.8}$$

As shown by Figure 5.1b, in a PM renewal interval, the event $\{L = l\}$, denoted by $\mathcal{A}_{\mathrm{PM}}(l)$ and l being a multiple of δ, implies that component degradation is initiated during $\big[l-\delta, l\big)$ and does not exceed w_F at time l, i.e.

$$\mathcal{A}_{\mathrm{PM}}(l) = \bigcup_{l-\delta\le x<l} \{X = x, W(l-x) \le w_F\}. \tag{5.9}$$

Let $f_{L,\mathrm{PM}}(l)$ be probability of such a event. As mentioned in Section 4.3.2, a gamma distribution has gamma distributed increments. Then $W(l-x) \sim \mathrm{Gamma}(\alpha(l-x), \beta)$, where α and β are the shape and the scale parameters of the gamma degradation process, respectively. Hence the probability of $\{W(l-x) \le w_F\}$ is equal to $F^{\mathrm{G}}(w_F; \alpha(l-x), \beta)$, where F^{G} is the CDF of the gamma distribution as shown in Eq. (4.3). Then using Eq.

(5.9) and noting that X is independent of $W(y)$, $f_{L,\mathrm{PM}}(l)$ can be obtained as

$$f_{L,\mathrm{PM}}(l) = \sum_{l-\delta \le x < l} f_X(x) F^{\mathrm{G}}(w_F; \alpha(l-x), \beta). \tag{5.10}$$

Finally, since $\mathcal{A}_{\mathrm{CM}}(l)$ and $\mathcal{A}_{\mathrm{PM}}(l)$ are mutually exclusive events and $T = L + Z$, the PMF of T can be obtained as

$$f_T(\tau) = \left(f_{L,\mathrm{CM}} * f_{Z,\mathrm{CM}}\right)(\tau) + \left(f_{L,\mathrm{PM}} * f_{Z,\mathrm{PM}}\right)(\tau), \tag{5.11}$$

where $f_{Z,\mathrm{CM}}(z)$ and $f_{Z,\mathrm{PM}}(z)$ are the PMF of Z for CM and PM, respectively.

5.3.2 Computation of $G(t)$

To compute $G(t)$ by using Eq. (3.24), we need to obtain the following two terms first:

$$h(\tau) = \mathsf{E}\left[C_1 | T_1 = \tau\right] f_T(\tau) \quad \text{and} \quad \overline{H}(t) = \mathsf{E}\left[C(t) | T_1 > t\right] \overline{F}_T(t).$$

(1) $h(\tau)$

Conditioning on L, Z, and the maintenance type, partition the event $\{T = \tau\}$ into mutually exclusive subevents as

$$\{T = \tau\} = \bigcup_{0 < l \le \tau} \left(\{\mathcal{A}_{\mathrm{CM}}(l), Z = \tau - l\} \bigcup \{\mathcal{A}_{\mathrm{PM}}(l), Z = \tau - l\} \right). \tag{5.12}$$

Events $\mathcal{A}_{\mathrm{PM}}(l)$ and $\mathcal{A}_{\mathrm{PM}}(l)$ are given in Eq. (5.5) and (5.9). Then based on the above partition, $h(\tau)$ can be obtained by using the law of total expectation as

$$h(\tau) = \sum_{0<l\le\tau} \Big[C_{\mathrm{CM}}(l,\tau) f_{L,\mathrm{CM}}(l) f_{Z,\mathrm{CM}}(\tau - l) + C_{\mathrm{PM}}(l,\tau) f_{L,\mathrm{PM}}(l) f_{Z,\mathrm{PM}}(\tau - l) \Big], \tag{5.13}$$

where $C_{\mathrm{CM}}(l,\tau)$ and $C_{\mathrm{PM}}(l,\tau)$ are the expected costs incurred in a renewal interval of $\{L = l, T = \tau\}$ with maintenance type CM and PM, respectively. Using Eq. (5.3) and (5.4), $C_{\mathrm{CM}}(l,\tau)$ and $C_{\mathrm{PM}}(l,\tau)$ can be obtained as

$$C_{\mathrm{CM}}(l,\tau) = \lfloor l/\delta \rfloor \, c_I + c_F + c_D(\tau - l), \tag{5.14}$$

$$C_{\mathrm{PM}}(l,\tau) = (l/\delta) c_I + [c_P + c_V V(l)] + c_D(\tau - l), \tag{5.15}$$

where $V(l) = \mathsf{E}\left[W(Y) | \mathcal{A}_{\mathrm{PM}}(l)\right]$ is the expected value of degradation at time l given that a PM is taken at l. Then $V(l)$ is given by

$$V(l) = \int_0^{w_F} w \cdot d\left[F_{W(Y)}\big(w | \mathcal{A}_{\mathrm{PM}}(l)\big) \right], \tag{5.16}$$

where $F_{W(Y)}\big(w|\mathcal{A}_{\mathrm{PM}}(l)\big)$ is the conditional CDF of $W(Y)$ given $\mathcal{A}_{\mathrm{PM}}(l)$. The integral interval in Eq. (5.16) is taken as $0 < w \le w_F$ due to the fact that degradation at the time of PM should be less than or equal to w_F. In Appendix B, $F_{W(Y)}\big(w|\mathcal{A}_{\mathrm{PM}}(l)\big)$ is derived as

$$F_{W(Y)}\big(w|\mathcal{A}_{\mathrm{PM}}(l)\big) = \frac{1}{f_{L,\mathrm{PM}}(l)} \sum_{l-\delta\le x<l} f_X(x) F^{\mathrm{G}}(w_F; \alpha(l-x), \beta). \tag{5.17}$$

Then the value of $h(\tau)$ in Eq. (5.13) can be obtained. Note that the derivation of

$V(l)$ in this thesis is logically simpler and more straight forward than that given by van Noortwijk & van Gelder [69].

(2) $\overline{H}(t)$

Partition the event $\{T_1 > t\}$ into two mutually exclusive subevents as $\{L_1 \le t < T_1\} \bigcup \{L_1 > t\}$. Then $\overline{H}(t)$ is given by

$$\overline{H}(t) = \underbrace{\mathsf{E}\left[C(t)|L_1 \le t < T_1\right]\mathsf{P}\left\{L_1 \le t < T_1\right\}}_{\overline{H}^{(1)}(t)} + \underbrace{\mathsf{E}\left[C(t)|L_1 > t\right]\mathsf{P}\left\{L_1 > t\right\}}_{\overline{H}^{(2)}(t)}. \quad (5.18)$$

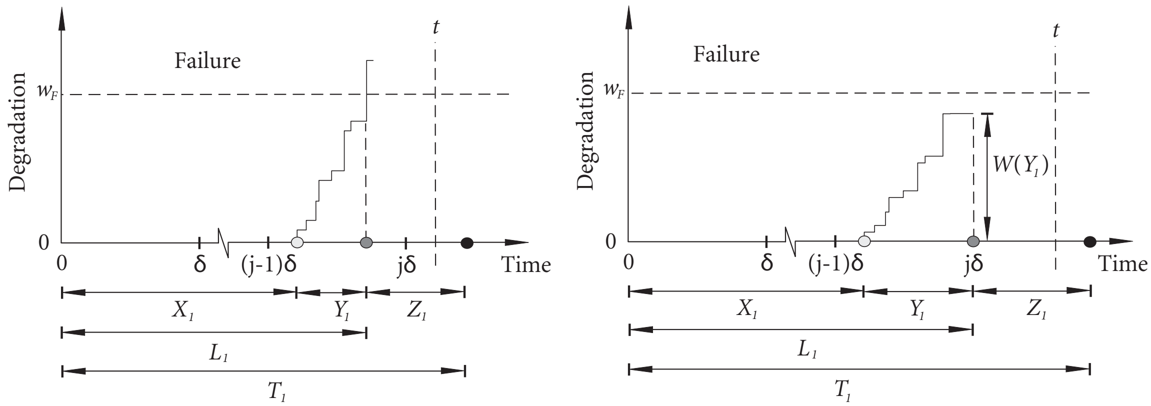
Using the law of total expectation by conditioning on L_1 and the maintenance type, $\overline{H}^{(1)}(t)$ is obtained as

$$\begin{aligned}\overline{H}^{(1)}(t) = \sum_{0<l\le t} \Big\{ & \mathsf{E}\left[C(t)|\mathcal{A}_{\mathrm{CM}}(l), Z_1 > t-l\right]\mathsf{P}\left\{\mathcal{A}_{\mathrm{CM}}(l), Z_1 > t-l\right\} \\ & + \mathsf{E}\left[C(t)|\mathcal{A}_{\mathrm{PM}}(l), Z_1 > t-l\right]\mathsf{P}\left\{\mathcal{A}_{\mathrm{PM}}(l), Z_1 > t-l\right\}\Big\} \\ = \sum_{0<l\le t} & \left[\overline{C}_{\mathrm{CM}}(l,t) f_{L,\mathrm{CM}}(l) \overline{F}_{Z,\mathrm{CM}}(t-l) + \overline{C}_{\mathrm{PM}}(l,t) f_{L,\mathrm{PM}}(l) \overline{F}_{Z,\mathrm{PM}}(t-l)\right]\end{aligned} \quad (5.19)$$

where $\overline{F}_{Z,\mathrm{CM}}(z)$ and $\overline{F}_{Z,\mathrm{PM}}(z)$ are the SF of $f_{Z,\mathrm{CM}}(z)$ and $f_{Z,\mathrm{PM}}(z)$, respectively, $\overline{C}_{\mathrm{CM}}(l,t)$ is the cost incurred in the interval of $(0,t]$ if $\{L_1 = l, T_1 > t\}$ and the maintenance in the first renewal interval is a CM, and $\overline{C}_{\mathrm{CM}}(l,t)$ is that if the maintenance in the first renewal interval is a PM (see Figure 5.2). As shown by Figure 5.2, $\overline{C}_{\mathrm{CM}}(l,t)$ and $\overline{C}_{\mathrm{PM}}(l,t)$ can be obtained as

$$\overline{C}_{\mathrm{CM}}(l,t) = \lfloor l/\delta \rfloor c_I + c_F + c_D(t-l), \quad (5.20)$$

$$\overline{C}_{\mathrm{PM}}(l,t) = (l/\delta)c_I + [c_P + c_V V(l)] + c_D(t-l), \tag{5.21}$$



(a) CM (b) PM

Figure 5.2: $L_1 \leq t < T_1$

In the case of $\{L_1 > t\}$ (see Figure 5.3), only inspection cost is incurred up to t, which is equal to

$$C_I(t) = c_I \lfloor t/\delta \rfloor . \tag{5.22}$$

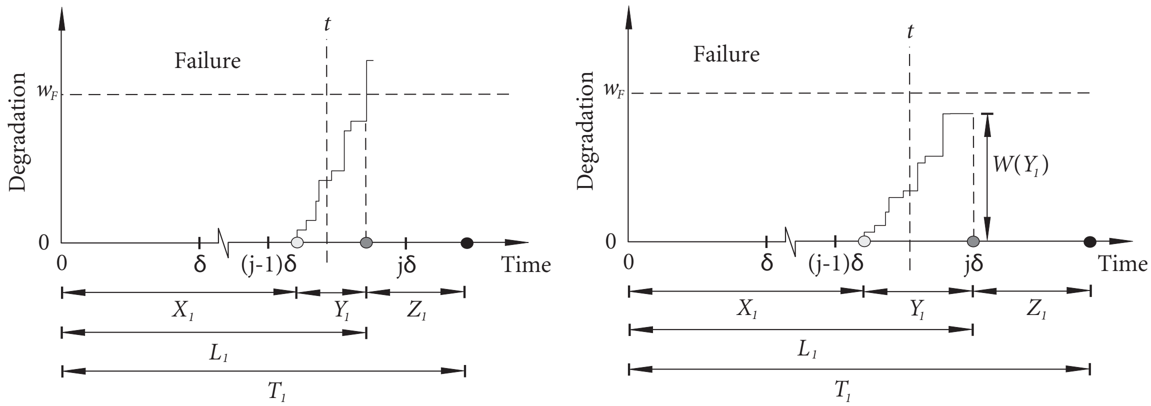
Hence $\overline{H}^{(2)}(t)$ is given by

$$\overline{H}^{(2)}(t) = C_I(t)\overline{F}_L(t), \tag{5.23}$$

where $\overline{F}_L(t)$ is the SF of L and is equal to

$$\overline{F}_L(t) = 1 - \sum_{0<l\leq t} \left[f_{L,\mathrm{CM}}(l) + f_{L,\mathrm{PM}}(l)\right].$$

Then $\overline{H}(t)$ can be obtained by substituting Eq. (5.19) and (5.23) into (5.18).



(a) CM

(b) PM

Figure 5.3: $L_1 \leq t < T_1$

5.4 Asymptotic Cost, Unavailability & Failure Rate

The expected length and the expected cost of a renewal interval can be obtained as

$$\mu_T = \sum_{\tau>0} \tau f_T(\tau), \quad \mu_C = \sum_{\tau>0} h(\tau), \tag{5.24}$$

where $f_T(\tau)$ and $h(\tau)$ are given in Eq. (5.11) and (5.13), respectively. Then the asymptotic cost can be obtained from Eq. (3.29).

Note that if we take $c_D = 1$ and other unit costs as 0, $C(t)$ will become the length of down time up to t. Then unavailability can be obtained from Eq. (3.30). Similarly, if we take $c_F = 1$ and other unit costs as 0, $C(t)$ will become the number of failures up to t. Then failure rate can be obtained from Eq. (3.31).

5.5 Discounted Cost

In this section, cost discounting is considered. The discounted cost over a time horizon is determined by summing the discounted values of the costs over that time horizon. In the following, we suppose that $\Delta t = 1$. Let r be the discount rate per unit time and b the discounted factor. Then

$$\rho = \frac{1}{1+b}.$$

The net present value (NPV) or the discounted value of a cost of c incurred at time t is given by

$$\mathrm{NPV} = c\rho^t.$$

In this section, we use the superscript d to denote the discounted value. Then for a renewal-reward process, the expected discounted cost up to t, denoted by $U^{\mathrm{d}}(t)$, can be written as the following renewal equation (see Appendix C)

$$U^{\mathrm{d}}(t) = \sum_{\tau=1}^{t} \rho^\tau f_T(\tau) U^{\mathrm{d}}(t-\tau) + G^{\mathrm{d}}(t), \tag{5.25}$$

where

$$G^{\mathrm{d}}(t) = \sum_{\tau=1}^{t} h^{\mathrm{d}}(\tau) + \overline{H}^{\mathrm{d}}(t),$$

$$h^{\mathrm{d}}(\tau) = \mathsf{E}\left[C_1^{\mathrm{d}} | T_1 = \tau\right] f_T(\tau).$$

$$\overline{H}^{\mathrm{d}}(t) = \mathsf{E}\left[C^{\mathrm{d}}(t) | T_1 > t\right] \overline{F}_T(t).$$

Equation (3.11) can be considered as a special case of Eq. (5.25) when $\rho = 1$.

The discounted terms $h^{\mathrm{d}}(\tau)$ and $\overline{H}^{\mathrm{d}}(t)$ in the above equation can be still obtained from Eq. (5.13) and (5.18) except that $C_{\mathrm{CM}}(l,\tau)$, $C_{\mathrm{PM}}(l,\tau)$, $\overline{C}_{\mathrm{CM}}(l,\tau)$, $\overline{C}_{\mathrm{PM}}(l,\tau)$ and $C_I(t)$ in these two equations should be replaced with the following discounted terms

$$\begin{aligned}
C^{\mathrm{d}}_{\mathrm{CM}}(l,\tau) &= \sum_{j=1}^{\lfloor l/\delta \rfloor} \rho^{j\delta} c_I + \rho^l c_F + \sum_{i=l+1}^{\tau} \rho^i c_D \\
&= \frac{\rho^\delta \left(1-\rho^{\lfloor l/\delta \rfloor \delta}\right)}{1-\rho^\delta} c_I + \rho^l c_F + \frac{\rho^{l+1}\left(1-\rho^{\tau-l}\right)}{1-\rho} c_D, \qquad (5.26)\\
C^{\mathrm{d}}_{\mathrm{PM}}(l,\tau) &= \sum_{j=1}^{l/\delta} \rho^{j\delta} c_I + \rho^l \left[c_F + V(l)c_V\right] + \sum_{i=l+1}^{\tau} \rho^i c_D \\
&= \frac{\rho^\delta \left(1-\rho^{l}\right)}{1-\rho^\delta} c_I + \rho^l \left[c_P + V(l)c_V\right] + \frac{\rho^{l+1}\left(1-\rho^{\tau-l}\right)}{1-\rho} c_D, \qquad (5.27)\\
\overline{C}^{\mathrm{d}}_{\mathrm{CM}}(l,t) &= \sum_{j=1}^{\lfloor l/\delta \rfloor} \rho^{j\delta} c_I + \rho^l c_F + \sum_{i=l+1}^{t} \rho^i c_D \\
&= \frac{\rho^\delta \left(1-\rho^{\lfloor l/\delta \rfloor \delta}\right)}{1-\rho^\delta} c_I + \rho^l c_F + \frac{\rho^{l+1}\left(1-\rho^{t-l}\right)}{1-\rho} c_D, \qquad (5.28)\\
\overline{C}^{\mathrm{d}}_{\mathrm{PM}}(l,t) &= \sum_{j=1}^{l/\delta} \rho^{j\delta} c_I + \rho^l \left[c_P + V(l)c_V\right] + \sum_{i=l+1}^{t} \rho^i c_D \\
&= \frac{\rho^\delta \left(1-\rho^{l}\right)}{1-\rho^\delta} c_I + \rho^l \left[c_P + V(l)c_V\right] + \frac{\rho^{l+1}\left(1-\rho^{t-l}\right)}{1-\rho} c_D, \qquad (5.29)\\
C^{\mathrm{d}}_I(t) &= \sum_{j=1}^{\lfloor t/\delta \rfloor} \rho^{j\delta} c_I = \frac{\rho^\delta (1-\rho^{\lfloor t/\delta \rfloor \delta})}{1-\rho^\delta} c_I. \qquad (5.30)
\end{aligned}$$

The equivalent average cost rate $u^{\mathrm{d}}(t)$ is often used as an optimization criterion, which is defined as

$$U^{\mathrm{d}}(t) = u^{\mathrm{d}}(t) \sum_{i=1}^{t} \rho^i = \frac{\rho(1-\rho^t)}{1-\rho} u^{\mathrm{d}}(t) \quad \Longrightarrow \quad u^{\mathrm{d}}(t) = \frac{1-\rho}{\rho(1-\rho^t)} U^{\mathrm{d}}(t). \qquad (5.31)$$

In an infinite time horizon, the expected discounted cost can be derived as [69]

$$U^{\mathrm{d}}(\infty) = \frac{\sum_{\tau=1}^{\infty}\sum_{l=1}^{\tau}\left[C_{\mathrm{CM}}^{\mathrm{d}}(l,\tau) f_{L,\mathrm{CM}}(l) f_{Z,\mathrm{CM}}(\tau-l) + C_{\mathrm{PM}}^{\mathrm{d}}(l,\tau) f_{L,\mathrm{PM}}(l) f_{Z,\mathrm{PM}}(\tau-l)\right]}{1-\sum_{\tau=1}^{\infty}\rho^{\tau} f_T(\tau)}. \quad (5.32)$$

Here $C_{\mathrm{CM}}^{\mathrm{d}}(l,\tau)$ and $C_{\mathrm{PM}}^{\mathrm{d}}(l,\tau)$ appropriately include the discounting factor as shown in equations (5.26) and (5.27). Then the asymptotic equivalent average cost can be obtained from Eq. (5.31) as

$$u^{\mathrm{d}}(\infty) = \left(\rho^{-1}-1\right) U^{\mathrm{d}}(\infty). \quad (5.33)$$

5.6 Example

5.6.1 Problem

This section considers an example of maintenance of berm breakwater structure. This problem was first analyzed by van Noortwijk & van Gelder [69]. In the following, time unit is taken as $\Delta t = 1$ year.

Berm breakwaters are used to prevent coastal lines of defence from being affected by severe hydraulic loads from the sea. The main components of a berm breakwater are the core and the armour layer (see Figure 5.4). The armour layer is subjected to longshore transport of stones, resulting from wave attack. Failure of a berm breakwater is caused by the longshore transport of an excessive number of rocks, w_F, in the armour layer after the breach of protection barrier. Therefore, this structure has to be inspected and preventively maintained upon rock displacement is detected at a regular interval.

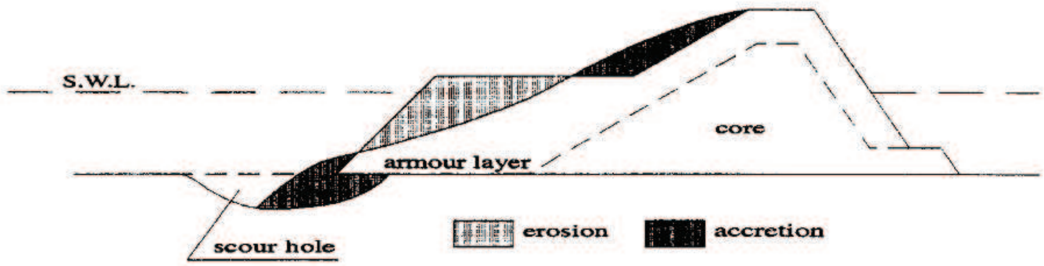


Figure 5.4: The cross-section of a berm breakwater[69]

The process of rock displacement consists two consecutive steps: (1) initiation of an armour breach; and (2) longshore rock transport. These two steps were both considered as stochastic processes in [69]. The PMF of the degradation free interval, X, is assume to be a geometrically distributed random variable with PMF as

$$f_X(x) = \phi(1-\phi)^{x-1},$$

where $x = 1, 2, \cdots$, and ϕ is the probability of armour breach occurrence per unit time.

Assumed that the times for CM and PM are both random variables and are also geometrically distributed with PMF

$$f_{Z,\mathrm{CM}}(z) = \lambda_{\mathrm{CM}}(1-\lambda_{\mathrm{CM}})^{z-1}, \quad f_{Z,\mathrm{PM}}(z) = \lambda_{\mathrm{PM}}(1-\lambda_{\mathrm{PM}})^{z-1},$$

where λ_{CM} and λ_{PM} are the probabilities of completing CM and PM per unit time, respectively. The input data given in [69] are also used here (see Table 5.1). Note that for the stationary gamma process with shape parameter α and scale parameter β, the expected annual growth of degradation is $\theta = \alpha\beta$ (see [65]). Hence in this example, the scale parameter is equal to $\beta = \theta/\alpha = 80$ stones.

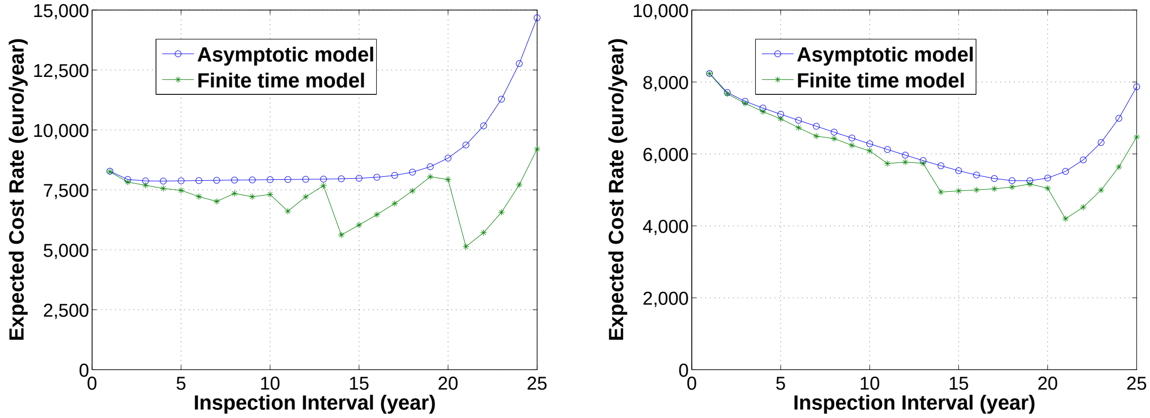
Table 5.1: Data for numerical example

Parameter	Description	Value	Dimension
Δt	time unit	1	year
α	shape parameter of gamma process	1	year^{-1}
θ	average rate of longshore rock transport	80	stone/year
ϕ	probability of armour breach occurrence per unit time	0.4	year^{-1}
λ_{CM}	probability of completing CM per unit time	0.5	year^{-1}
λ_{PM}	probability of completing PM per unit time	0.9	year^{-1}
b	discount rate per year	5%	year^{-1}
ρ	discount factor per year	0.9524	year^{-1}
c_I	unit inspection cost	$1,000$	euro
c_D	unit down time cost	$10,000$	euro/year
c_P	fixed PM cost	$10,000$	euro
c_V	variable PM cost	100	euro/stone
c_F	CM cost	$2,500,000$	euro
w_F	failure threshold	2500	stone
t	time horizon	40	year

Van Noortwijk & van Gelder [69] optimized the inspection interval by minimizing the asymptotic cost rate, and they ignored the time required for maintenance. These two limitations of the past analysis are relaxed in the present analysis. For this reason, few additional input data are added in Table 1, such as the down time cost and probability distribution of CM and PM intervals.

5.6.2 Results

The variation of the expected cost rate with inspection interval is shown in Figure 5.5. Here the expected cost rate is defined as $u(t) = U(t)/t$. Based on the asymptotic cost analysis without discounting (see Figure 5.5a), an optimal inspection interval is found as 4 years with a cost rate of $u = 7,866$ euros/year, while the finite time model leads to a longer



(a) Without discounting

(b) With discounting

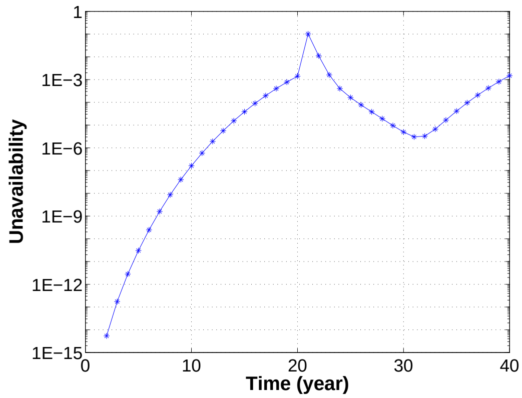
Figure 5.5: Expected cost rate vs. inspection interval

optimal interval of 21 years at a lower cost rate of $u = 5,132$ euros/year. The asymptotic model over-predicts the optimal cost by 53%.

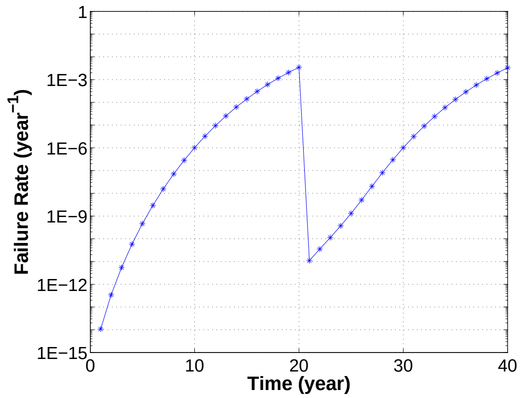
In the case of discounting, the expected equivalent cost rate is computed using an annual discount rate of 5%. Figure 5.5b shows that the asymptotic model leads to an optimal interval of 19 years at an equivalent cost rate of $u^{\mathrm{d}} = 5,254$ euros/year, whereas the finite time analysis leads to an optimal interval of 21 year as before, and an equivalent cost rate of $u^{\mathrm{d}} = 4,196$ euros/year. Here, the asymptotic model over-predicts the optimal cost rate by 25.2%.

These results show that the use of a refined approach based on finite time formulation is necessary to obtain realistic estimates of maintenance cost associated with a maintenance program.

Figure 5.6 shows the time-dependent unavailability and failure rate of the breakwater corresponding to an inspection interval of 20 years, which is close to the optimal interval of 21 years. Initially the unavailability is almost zero and it gradually increases up to the time of PM. There is a jump at time 20 years, because the time required for PM would



(a) Unavailability



(b) Failure Rate

Figure 5.6: Time-dependent unavailability and failure rate for an inspection interval $k = 20$ years

render the structure unavailable with a high probability. The failure rate curve shows a familiar pattern. It increases up to the time of PM and then drop significantly as a effect of renewal through PM. This is because a PM is performed at time 20 years. Hence at time 21 years unavailability will increase while the failure rate will decrease.

5.7 Summary

This chapter presents maintenance cost analysis of a more complex CBM policy, which includes the repair time and delay in degradation initiation as random variables. The finite time cost analysis, with and without discounting, presented in this chapter is not yet seen in the existing literature.

A case study related to the maintenance of breakwaters is analyzed. This problem was originally analyzed by van Noortwijk & van Gelder [69] using the asymptotic cost rate criterion without considering the repair time distribution. A re-analysis of this example illustrates that the asymptotic formula can be a rather crude approximation of the actual expected cost. In addition, the present work evaluates the unavailability failure rate of the structure. It is concluded that the finite time cost formula should be used for a realistic evaluation and optimization of the maintenance policy for critical infrastructure systems.

The results of this chapter have published in [45].

Chapter 6

Probability Distribution of Maintenance Cost

6.1 Introduction

6.1.1 Motivation

The financial risk assessment of a maintenance program deals with different issues, such as, how much capital is required to implement a maintenance policy, and what is the residual risk after the implementation of a maintenance policy. The evaluation of expected cost is reasonable for finding an optimal maintenance policy among a set of alternatives in a relative sense. However, this approach is not informative enough to enable the estimation of financial risk measures, such as percentiles of the cost, also known as Value-at-Risk (VaR). To address these questions, it is clear that complete probability distribution of maintenance cost is required, which would allow accurate prediction of cost and assess the financial risk.

6.1.2 Research Objective and Approach

The objective of this chapter is to derive probability distribution of cost of condition-based maintenance of a system affected by stochastic degradation.

The proposed solution is based on the fact that the characteristic function of a continuous/discrete random variable is the inverse Fourier transform of its probability density/mass function. Therefore, a renewal equation is firstly formulated to evaluate the characteristic function. Then, the Fourier transform of the characteristic function is computed, which leads to complete probability distribution of cost in a finite time setting. Once the cost distribution is derived, financial risk measures, such as VaR, can be easily calculated. The proposed method is applicable to a general stochastic renewal-reward process.

6.1.3 Organization

Section 6.2 presents the terminology and the basic assumptions on cost distribution analysis. Section 6.3 formulates a renewal equation for the characteristic function, and its Fourier transform leads to the probability distribution of maintenance cost. A computation procedure on how to obtain the probability distribution of maintenance cost is given in Section 6.4. Illustrative numerical examples are presented in Section 6.5.

6.2 Terminology and Assumptions

In this chapter, time is discrete and is equal to 0, 1, 2, $\cdots$, i.e. $\Delta t = 1$. Let $C(t)$ be the total cost up to time t for a specific maintenance model. It is assumed that $C(t)$ is a renewal-reward process. Furthermore for any t, $C(t)$ is a discrete random variable with PMF

$$f_C(c, t) = \mathsf{P}\{C(t) = c\}. \tag{6.1}$$

The cost is discretized in a unit of ρ as 0, $\rho, 2\rho, \cdots, n_c\rho$, where $n_c\rho$ is an upper limit of the cost, such that $\mathsf{P}\{C(t) > n_c\rho\} \approx 0$.

Modeling the cost as as a discrete variable is justified on a practical ground. In financial planning, the cost estimates are typically rounded off to hundreds or thousands of dollars, or any other suitable number depending the cost involved. Therefore, treating the cost as a precise continuous variable is unwarranted. For the CBM model in Chapter 4, the unit cost ρ can be taken as the greatest common factor of c_I, c_F, and c_P. Since all the unit costs are a multiple of ρ, $C(t)$ will also be a multiple of ρ. For example, if $c_I = 2$, $c_P = 10$ and $c_F = 20$, then $\rho = 2$.

The upper bound of $C(t)$, $n_c\rho$, can be estimated based on an upper bound number of renewal cycles. The expected number of renewal intervals up to time t is approximately t/μ_T, where μ_T is the expected length of one renewal interval. The probability that the actual number would exceed $3t/\mu_T$ renewals is expected to be negligible. Therefore, $3C_{\text{sup}}t/\mu_T$ is a conservative upper bound of $C(t)$, where C_{sup} is the upper bound of the cost in one renewal interval. Thus n_c can be estimated as

$$n_c \approx \left\lceil \frac{3C_{\text{sup}}t}{\mu_T \rho} \right\rceil, \tag{6.2}$$

where $\lceil * \rceil$ is an integer ceiling function. For the CBM model in Chapter 4, since $c_I \ll c_P$ c_F in general, then C_{sup} can be taken as c_F. For other maintenance models, C_{sup} can be conservatively taken as $3\mu_C$, μ_C being the expected cost incurred in one renewal interval.

The characteristic function (CF) of a random variable X is defined as

$$\phi_X(\omega) = \mathsf{E}\left[e^{i\omega X}\right], \tag{6.3}$$

where $i = \sqrt{-1}$ is the imaginary number and ω is an argument in the circular frequency domain. It is recognized that $\phi_X(\omega)$ is the inverse Fourier transform (FT) of the probability density (or mass) function of X [9]. Therefore, the Fourier Transform of $\phi_X(\omega)$ would recover the PDF/PMF of X.

In the context of financial risk analysis, a p^{th} percentile of the distribution of cost is also referred to as Value-at-Risk, VaR(c, p), defined as

$$\text{VaR}_p(C(t)) = \inf\{x : \mathsf{P}\{C(t) \leq x\} \geq p\} \tag{6.4}$$

For example, 95^{th} percentile means 5% probability that the maintenance cost would exceed this value.

6.3 Characteristic Function of Cost

Let $\phi(\omega, t)$ be the characteristic function of $C(t)$, i.e.

$$\phi(\omega, t) = \mathsf{E}\left[e^{i\omega C(t)}\right]. \tag{6.5}$$

Because the cost is a discrete variable, ω will also be a discrete quantity: 0, $\Delta\omega$, $2\Delta\omega$, $\cdots$, $n_c\Delta\omega$. The unit frequency is defined as [61]

$$\Delta\omega = \frac{2\pi}{[n_c + 1]\rho}. \tag{6.6}$$

In general, a frequency is denoted as $\omega_m = m\Delta\omega$.

Similar to the derivation of renewal equation for expected cost, (3.8), a renewal equation for $\phi(\omega, t)$ is derived based on renewal argument described as follows.

Let T_1 be the length of the first renewal interval. The expectation associated with the CF in Eq. (6.5) can be partitioned into two cases: $T_1 \le t$ and $T_1 > t$. Since T_1 can take any value between 1 and t, the law of total expectation is used to rewrite Eq. (6.5):

$$\phi(\omega, t) = \sum_{\tau=1}^{t} \mathsf{E}\left[e^{i\omega C(t)} | T_1 = \tau\right] f_T(\tau) + \mathsf{E}\left[e^{i\omega C(t)} | T_1 > t\right] \overline{F}_T(t). \tag{6.7}$$

The above equation is valid for any $\omega_0, \omega_1, \cdots, \omega_{n_c}$. But the subscript is avoided in this section for sake of brevity and readability of formulas.

When $T_1 = \tau$ ($\le t$), the cost can be written as a sum: $C(t) = C_1 + C(\tau, t)$, where C_1 is the cost in the first renewal cycle. The second component is equivalent to $C(t - \tau)$ based on the renewal argument discussed in Chapter 3. Therefore, the first expectation in

Eq.(6.7) can be re-written as

$$\begin{aligned}\mathsf{E}\left[e^{i\omega C(t)}|T_1=\tau\right] = \mathsf{E}\left[e^{i\omega C_1}e^{i\omega C(\tau,t)}|T_1=\tau\right] &= \mathsf{E}\left[e^{i\omega C_1}|T_1=\tau\right]\mathsf{E}\left[e^{i\omega C(t-\tau)}\right] \\ &= \mathsf{E}\left[e^{i\omega C_1}|T_1=\tau\right]\phi(\omega,t-\tau). \end{aligned} \tag{6.8}$$

Substituting Eq. (6.8) into (6.7) leads to

$$\phi(\omega,t) = \sum_{\tau=1}^{t} \underbrace{\mathsf{E}\left[e^{i\omega C_1}|T_1=\tau\right] f_T(\tau)}_{f_\phi(\omega,\tau)} \phi(\omega,t-\tau) + \underbrace{\mathsf{E}\left[e^{i\omega C(t)}|T_1>t\right]\overline{F}_T(t)}_{G_\phi(\omega,t)}. \tag{6.9}$$

Thus, the following renewal-type equation is obtained for the CF:

$$\phi(\omega,t) = \sum_{\tau=1}^{t} \phi(\omega,t-\tau)f_\phi(\omega,\tau) + G_\phi(\omega,t) \tag{6.10}$$

Equation (6.10) is similar to Eq. (3.8), except that $f_T(\tau)$ is replaced by $f_\phi(\omega,\tau)$ and $G(t)$ by $G_\phi(\omega,t)$. Here, the initial condition is $\phi(\omega,0)=1$ for all values of ω, since $C(0)=0$.

Equation (6.10) is genetic and it holds for any maintenance models as long as the maintenance cost in these models is a renewal-reward process. The values of $f_\phi(\omega,\tau)$, $G_\phi(\omega,t)$ depends on maintenance models. For the CBM model in Chapter 4, since C_1 can be $C_{\mathrm{PM}}(\tau)$ or $C_{\mathrm{CM}}(\tau)$ depending on the type of renewal (see Eq. (4.1) and (4.2)), $f_\phi(\omega,\tau)$ can be accordingly written as

$$f_\phi(\omega,\tau) = e^{i\omega C_{\mathrm{CM}}(\tau)} f_{\mathrm{CM}}(\tau) + e^{i\omega C_{\mathrm{PM}}(\tau)} f_{\mathrm{PM}}(\tau). \tag{6.11}$$

When $T_1 > t$, only inspection cost, $\lfloor t/\delta \rfloor c_I$, is incurred in $(0,t]$. Thus, $G_\phi(\omega,t)$ can be written as

$$G_{\phi}(\omega, t) = e^{i\omega \lfloor t/\delta \rfloor c_I} \overline{F}_T(t). \tag{6.12}$$

The terms $f_{\mathrm{PM}}(\tau)$, $f_{\mathrm{CM}}(\tau)$, and $\overline{F}_T(t)$ in the above two equations can be obtained from Eq. (4.9) (4.10) and (4.11) in Chapter 4.

Note that $f_{\phi}(\omega, \tau)$, $G_{\phi}(\omega, t)$ and $\phi(\omega, t)$ are all complex variables. Using superscript R and I to denote the real and the imaginary part, respectively, Eq. (6.10) can be written as

$$\begin{cases} \phi^R(\omega, t) = G_{\phi}^R(t) + \sum\limits_{\tau=1}^{t} \left[\phi^R(\omega, t-\tau) f_{\phi}^R(\tau) - \phi^I(\omega, t-\tau) f_{\phi}^I(\tau)\right], \\ \phi^I(\omega, t) = G_{\phi}^I(t) + \sum\limits_{\tau=1}^{t} \left[\phi^R(\omega, t-\tau) f_{\phi}^I(\tau) + \phi^I(\omega, t-\tau) f_{\phi}^R(\tau)\right]. \end{cases} \tag{6.13}$$

Equation (6.13) forms a set of two coupled recursive equations with initial condition $\phi(\omega, 0) = 1$, or $\phi^R(\omega, 0) = 1$ and $\phi^I(\omega, 0) = 0$.

6.4 Computational Procedure

6.4.1 Summary of Variables

The analysis involves the following key variables.

(1) The time horizon for the calculation of maintenance cost is discrete and finite as $0, 1, 2, \cdots, \tau, \cdots, t$.

(2) The renewal cycle length (T) is a discrete random variable, $0, 1, 2, \cdots, \tau, \cdots, \infty$, with PMF $f_T(\tau)$.

(3) The maintenance cost is a discrete variable, $0, \rho, 2\rho, \cdots, n_c\rho$, with PMF $f_C(c,t)$.

(4) The circular frequency, ω is discretized as $0, \Delta\omega, 2\Delta\omega, \cdots, n_c\Delta\omega$, where $\Delta\omega$ is given by Eq. (6.6).

6.4.2 Steps of Computation

The discrete FT of $\phi(\omega, t)$ leads to the PMF of cost as

$$f_C(c,t) = \frac{1}{n_c+1}\sum_{m=0}^{n_c} \phi(\omega_m, t)e^{i\omega_m c}, \tag{6.14}$$

The use of the Fast Fourier transform (FFT) algorithm makes the computation of (6.14) very efficient [61].

First prepare the input data about the specific maintenance model to be studied. Estimate the mean renewal cycle length μ_T and the upper bound of the cost in one renewal cycle C_{sup}. Compute the number of discrete intervals of cost, n_c, using Eq. (6.2). The frequency interval, $\Delta\omega$, is calculated from (6.6).

The computation of $f_C(c,t)$ involves the following steps:

(1) Start with $m = 0$ and compute $\omega_m = m\Delta\omega$.

(2) Compute $\phi(\omega_m, \tau)$ for all the discrete time values $\tau = 1, 2, \cdots, t$, from the renewal equation (6.10) or (6.13)

(2.1) Start with $\tau = 1$ and compute $\phi(\omega_m, \tau)$.

(2.2) Continue computation of $\phi(\omega_m, \tau)$ for $\tau = 2, 3, \cdots t$.

(3) Take the next increment of the frequency ($m = 1$) and repeat the step 2. Continue with computation of $\phi(\omega_m, t)$ until $m = n_c$.

(4) At the end of step (3), a two dimensional array of complex numbers $\phi(\omega_m, \tau)$ of size $(n_c + 1) \times t$ is obtained.

(5) Compute FFT of the last column, $\phi(\omega_m, t)$, resulting in the discrete PMF of the cost, $f_C(c, t)$, where $c = 0, \rho, \cdots, n_c\rho$.

(6) Note that the PMF of cost for any intermediate time $1 \leq \tau \leq t$ can be obtained by taking FFT of the suitable column of CF matrix.

The numerical computation is involved, because a pair of renewal equations has to be solved $(n_c + 1)$ times. However, the use of an efficient FFT algorithm reduces the burden of computation and provides a practical approach to the evaluation of distribution of cost.

6.5 Numerical Results

Numerical examples are presented to illustrate the proposed methodology for deriving the distribution of maintenance cost. The maintenance model is the CBM model as described in Chapter 4.

6.5.1 Input Data

The parameters of gamma process, maintenance thresholds and unit cost data are given in Table 6.1. The time horizon is taken as $t = 60$. Since this example is purely illustrative, the units of the quantities in Table 2 are not of any practical relevance.

Table 6.1: Input data used in numerical example

<table>
<tr><td colspan="2">Variable</td><td>Value</td></tr>
<tr><td rowspan="2">Gamma Degradation Process</td><td>α</td><td>0.4</td></tr>
<tr><td>β</td><td>4</td></tr>
<tr><td>Failure level</td><td>w_F</td><td>20</td></tr>
<tr><td>PM level</td><td>w_P</td><td>15</td></tr>
<tr><td rowspan="3">Unit Cost</td><td>c_I</td><td>0.2</td></tr>
<tr><td>c_P</td><td>1</td></tr>
<tr><td>c_F</td><td>4</td></tr>
<tr><td>Time horizon</td><td>t</td><td>60</td></tr>
</table>

The cost is discretized in the unit of $\rho = 0.2$, which is the unit cost of inspection in the current example. The maintenance cost is considered to have an upper bound, $n_c\rho$, which is estimated based on an upper bound number of CM renewal cycles. The expected number of CM renewal is approximately t/μ_T. The probability that the actual number would exceed $3t/\mu_T$ renewals is expected to be negligible. Since $c_I \ll c_P \ll c_F$, the cost of one CM renewal is about c_F. Therefore, $3c_F t/\mu_T$ is a conservative upper bound of $C(t)$. Thus, n_c can be estimated as

$$n_c \approx \left\lceil \frac{3c_F t}{\mu_T \Delta_c} \right\rceil . \qquad (6.15)$$

6.5.2 Example-1

Expected Cost

For a chosen inspection interval, δ, the expected cost, $U(t)$, was calculated using (3.8) for the time horizon $t = 60$. The inspection interval was varied from 1 to 30 and $U(t)$

was calculated for each case. The variation of $U(t)$ versus δ is plotted in Figure 6.1. The expected cost is minimum for $\delta = 4$, which is an optimal inspection interval.

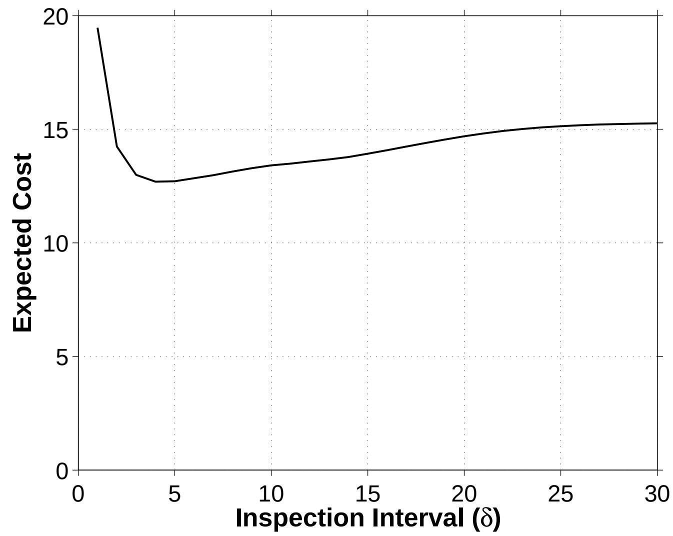


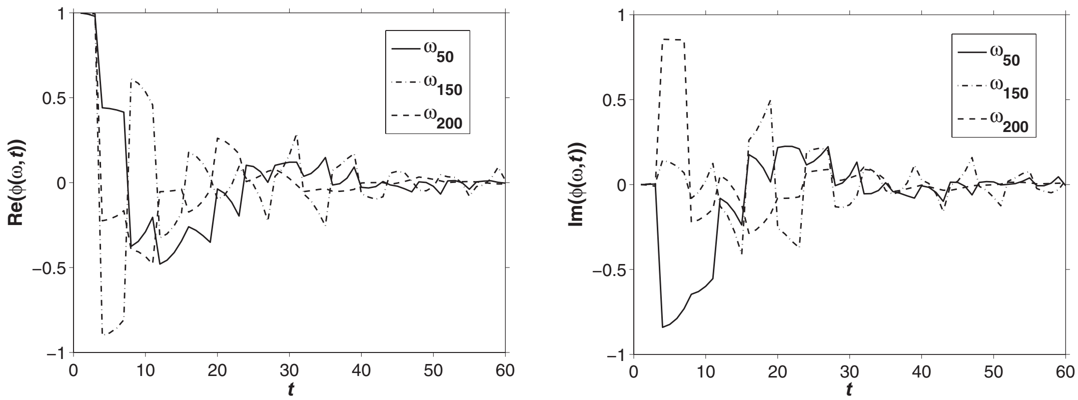
Figure 6.1: Expected cost, $C(t = 60)$, vs. the inspection interval

CF and Distribution of Cost

The PMF of cost is derived corresponding to the optimal inspection interval of $\delta = 4$. In this case, the expected length of a renewal cycle is $\mu_T = 14.2$. Substituting this and $c_F = 4$ in Eq. (6.2), leads to $n_c = 253$. Using Eq. (6.6), the circular frequency, ω, is discretized in steps of $\Delta\omega = 0.12$.

The real and imaginary parts of the characteristic function, $\phi(\omega, t)$ versus t, are plotted in Figure 6.2 for three values of ω.

Using Eq.(6.14), the PMF of $C(t)$ was obtained and plotted in Figure 6.3. The distribution is bounded between 5 and 25 units of cost. The mean and standard deviation of



(a) Real part of CF

(b) Imaginary part of CF

Figure 6.2: CF of cost,$\phi(\omega, t)$, plotted over time

the maintenance cost are 15 and 3.4, respectively. The 95^{th} percentile of the cost is 19.4 units.

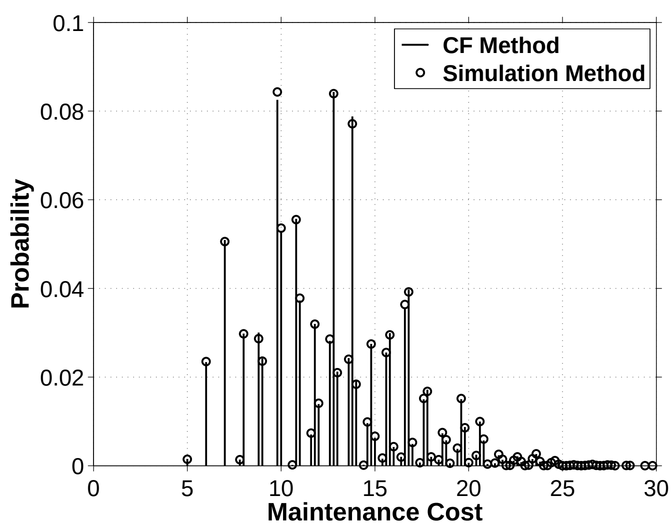


Figure 6.3: PMF of the maintenance cost $C(t)$

PMF of cost was also evaluated using simulation method with 10^5 simulations. The result of simulation is presented in Figure 6.3. PMFs obtained from simulation and CF methods are fairly close, which confirms the validity of proposed CF method. However, the computational time of the simulation method is 72 seconds, which is much larger than that associated with CF method (only 2 seconds). Both methods are implemented in MATLAB 2010a version.

6.5.3 Example-2

In this example, only the scale parameters the gamma process is changed to $\beta = 2$, such that the mean of degradation rate is reduced to $\alpha\beta = 0.8$ per unit of time. This results in an increase in the mean renewal cycle length, and therefore decreases the expected number of renewals in the time horizon. All other parameters are the same as in Table 2.

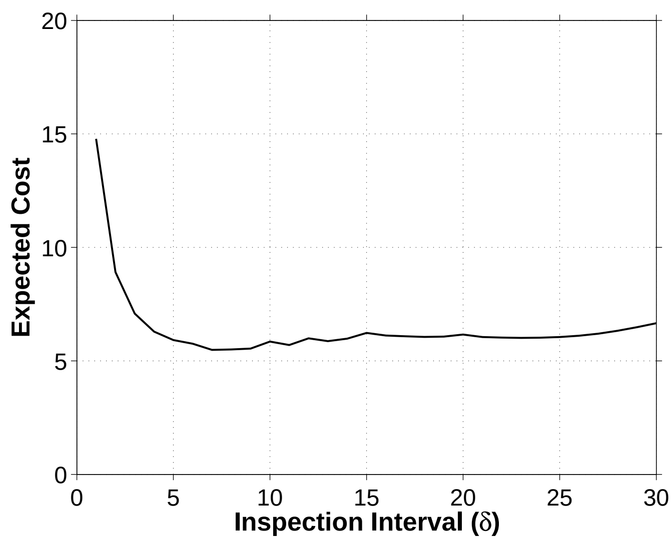


Figure 6.4: Expected cost vs. inspection interval - Example 2

The variation of the expected cost with inspection interval is shown in Figure 6.4, and the optimum interval turns out to be $\delta = 7$. In this case, the expected renewal cycle length is $\mu_T = 23.7$. Substituting this and $c_F = 4$ in Eq. (6.2), leads to $n_c = 152$.

The PMF of cost shown in Figure 6.5 is rather sparse, and bounded by 2 and 12 units

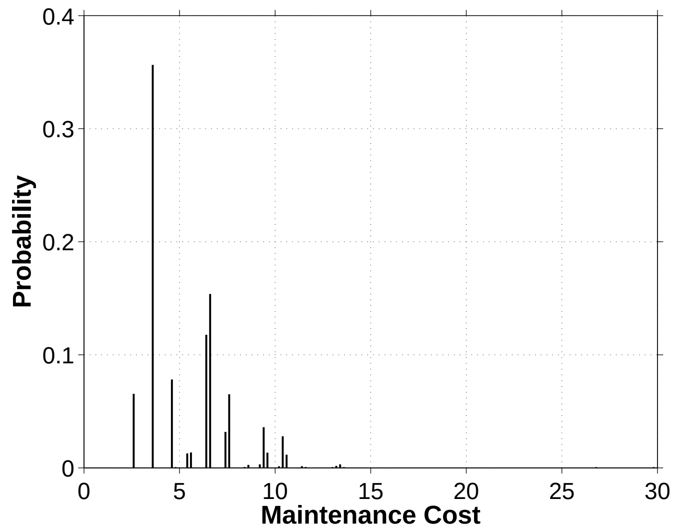


Figure 6.5: PMF of maintenance cost - Example 2

of cost. The mean and standard deviation of the cost are 5.5 and 2.2, respectively. The 95^{th} percentile of the cost is 9 units.

6.6 Summary

In the literature, the optimization of a maintenance program is typically based on the minimization of the asymptotic cost rate. However, expected cost solution (asymptotic or accurate) is not informative enough to enable an accurate prediction of the upper bound of maintenance cost in a fixed time horizon. For the evaluation of measures of financial risk, such as VaR, a complete probability distribution of cost is required. The method presented in this chapter meets this objective.

The main contribution of this chapter is the derivation of the probability distribution of the maintenance cost. It is a general method that can be applied to any maintenance policy that can be treated as a stochastic renewal-reward process.

The proposed approach is based on formulating a renewal equation for the characteristic function of cost in finite time. Subsequently, the Fourier transform of the characteristic function leads to the probability distribution of cost. The CBM policy presented in Chapter 4 is analyzed to illustrate the proposed approach.

It is concluded that the proposed model would serve as a foundation to realistic financial risk assessment and optimization of maintenance of engineering systems.

Chapter 7

Sequential Inspection and Replacement Model

7.1 Introduction

7.1.1 Motivation

This chapter presents a probabilistic approach to manage a large population of components in a large infrastructure system, such as electrical transmission and distribution networks consisting of thousands of poles, cross-arms, switches, and other components.

Latent failure is of primary concern in such systems, which are also referred to as degradation failure, i.e., degradation exceeding an acceptable limit specified by the standard of practice. A degradation failure is detected through inspection only, because the component remains functional despite degradation. Nevertheless, degradation makes a component more vulnerable to failure.

Standards and regulations recommend that components with degradation exceeding a critical limit should be identified and must be preventively removed from the electrical network in order to maintain a high level of reliability. For example, Canadian Standard (CSA C22.3 2001) recommends that a component should be replaced when the total degradation of its capacity exceeds one-third of the installed capacity.

A key challenge in the asset management of such systems is that all the components cannot be inspected in any one given year due to prohibitively large inspection cost and labour requirements associated with a large population. Therefore, the industry typically adopts a sequential inspection and replacement program under which the total population is divided into several blocks or subpopulations and each is inspected in a sequential manner [48]. In each year, only one block is inspected and all failed components in that block are replaced.

Under this asset management policy, the number of blocks to be divided, say δ, is critical to reliability and cost-effectiveness of the electrical network. A small δ will impose a high work load on maintenance stuff and large inspection cost, while a large δ will make a system very vulnerable to failure. Hence we should optimize δ to balance the costs and reliability. This infrastructure renewal problem is in contrast with traditional models of maintenance strategies discussed in the literature [18, 31].

The expected cost for a sequential inspection and replacement policy is derived in this chapter. The basic idea is that components in an individual block are subjected to periodic inspection and replacement, which can be modelled as a renewal-reward process. Then the results of Chapter 3 can be used to obtain the expected cost.

7.1.2 Organization

Section 7.2 gives a detailed description of the inspection and replacement program. The cost analysis of an individual block is presented in Section 7.3. Section 7.4 gives the expected cost of the entire population. The asymptotic formulation is also given in this section. Section 7.5 evaluates expected proportion of failed components and required replacements in any given year. An example that is related to wood poles in electrical network is given in Section 7.6.

7.2 Maintenance Model

7.2.1 Latent Failure and Down Time Cost

In this chapter, the latent failure is considered, which means that failure can be detected only by an inspection. Then there will be a time delay between component failure and its detection, resulting in a down time cost or penalty cost due to the unavailability of the component.

Many examples of the latent failure are given in [62], one of which is the redundant component. Failure of redundant components will not lead to shut-down of a system until the component in service fails. The down time cost of redundant components is referred to the increased risk of system shut-down after failure of redundant components. Another example is the component which will fall into a state with lower performance after failure. The down time cost of this example is referred to the cost of reduced performance.

7.2.2 Sequential Inspection and Replacement Program

Suppose that there is a population of size N of identical components. Component lifetime, denoted by X, is a random variable with PMF $f_X(x)$. Inspection is done to detect and replace failed components. It is assumed that (1) at time 0, all the components are new; (2) components are independent of each other; and (3) the time spent for inspection and replacement is negligible.

Because of lack of resources, it may not be possible to inspect all the components in one year. Hence, the population can be divided into several blocks or subpopulations. Only one block or subpopulation is inspected in a year such that it takes several years to complete the entire population. This program is called the *sequential inspection and replacement program* (SIRP) [48].

Figure 7.1 is an illustration of SIRP. Components are divided into δ blocks. Each block has N/δ components and is sequentially inspected each year. Failed components found during an inspection will be replaced by new ones. For example, in year 1, block 1 is inspected and failed components are replaced, while blocks 2–δ are left uninspected, leading to accumulation of failed components in these blocks. In year 2, block 2 is subject to inspection and replacement, and blocks 1 and 3–δ are unattended. Failed components in blocks 3–δ continue to accumulate. Furthermore, previously inspected block 1 experiences accumulation of failed components as well. This process continues and at the end of year δ, all the components have been inspected. Then in year $(\delta+1)$, block 1 is inspected again, and so on and so for until the end of the planning horizon t, typically 30 to 50 years.

In SIRP, each block is subjected to period inspection, and replacement, if any, at a interval of δ years (see Figure 7.1). For example, block 1 is inspected at time 1, $\delta + 1$,

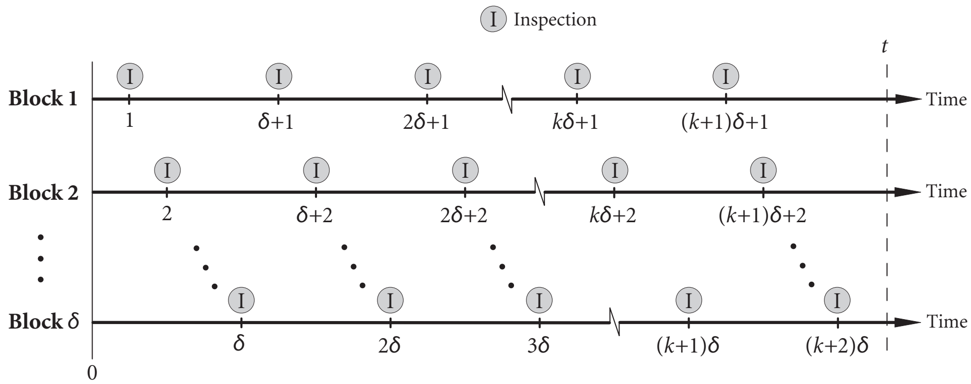


Figure 7.1: Sequential Inspection and Replacement Program

$2\delta + 1, \cdots$, and block δ is inspected at time δ, 2δ, $\cdots$.

Maintenance cost of SIRP includes inspection cost, down time cost, and the replacement cost. Replacement cost is incurred at renewal points. The unit costs of these items are denoted by: inspection cost–c_I, down time cost –c_D, and replacement cost–c_R.

7.3 Maintenance Cost of Block or Sub-population

In this section, the expected maintenance cost of an individual block is derived. Note that since components are independent of each other, we can use a component in each block to represent all the components in that block. For any component, denote the cost in the interval of $(t_1, t_2]$ by $C(t_1, t_2)$ and write $C(0, t)$ as $C(t)$ compactly. Denote $\mathsf{E}[C(t)]$ by $U(t|r)$ for the component in Block r, $r = 1, 2, \cdots, \delta$, and write $U(t|r = \delta)$ as $U(t)$ for simplicity. In this chapter, time unit is taken as $\Delta t = 1$.

7.3.1 Block δ

As shown in Figure 7.1, the component in Block δ is inspected at times δ, 2δ, $\cdots$. Figure 7.2 is an illustration of the state alternation of such components, where Y denotes the length of down time and $T = X + Y$ is the length of a renewal interval. As shown in that figure, in each renewal interval, the component is always inspected at times δ, 2δ, $\cdots$. Hence all the renewal intervals are identically distributed, resulting in that $\{T_n, C_n\}$, $n = 1, 2, \cdot$, are *iid* random vectors. Here C_n is the cost associated with the n^{th} renewal interval. Therefore $C(t)$ is a renewal-reward process. Then $U(t)$ can be obtained from Eq. (3.8) by taking $m = 1$ (note that $U(t)$ is the first moment of $C(t)$) as

$$U(t) = (U * f_T)(t) + G(t), \tag{7.1}$$

where $f_T(t)$ is the PMF of T, and

$$G(t) = \sum_{0<\tau\leq t} \underbrace{\mathsf{E}\left[C_1 | T_1 = \tau\right] f_T(\tau)}_{h(\tau)} + \underbrace{\mathsf{E}\left[C(t) | T_1 > t\right] \overline{F}_T(t)}_{\overline{H}(t)}. \tag{7.2}$$

The asymptotic formula of $U(t)$ is obtained from Eq. (3.29) as

$$U(t) = \frac{\mu_C}{\mu_T} t + o(t), \tag{7.3}$$

where

$$\mu_T = \mathsf{E}\left[T_1\right] = \sum_{\tau>0} f_T(\tau), \quad \mu_C = \mathsf{E}\left[C_1\right] = \sum_{\tau>0} h(\tau). \tag{7.4}$$

The terms $f_T(\tau)$, $h(\tau)$, and $\overline{H}(t)$ in Eq. (7.1) are given as follows.

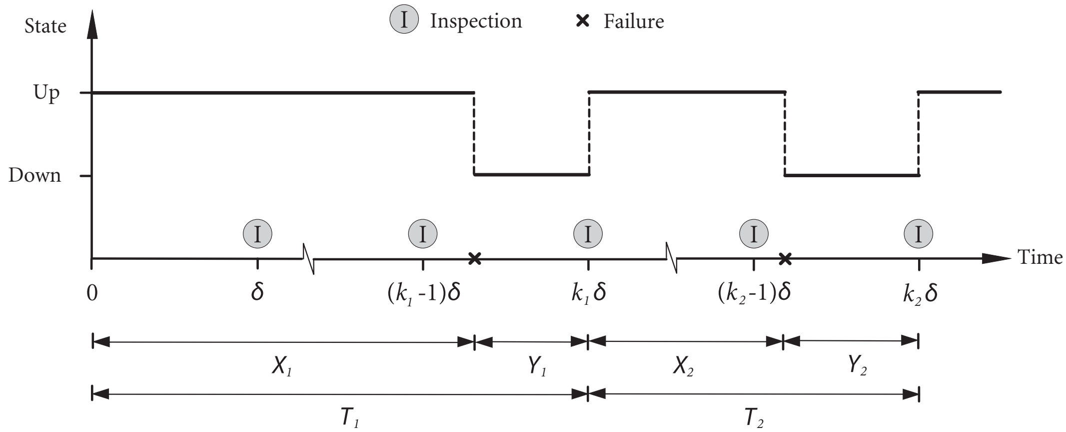


Figure 7.2: State alternation of components in Block δ

(1) $f_T(\tau)$

As shown in Figure 7.2, renewals can only take place at an inspection time. Hence renewal interval T can only take a value of multiple of δ. The event $\{T = k\delta\}$ implies that the component fails in the interval of $((k-1)\delta, k\delta]$, i.e., $(k-1)\delta < X \leq k\delta$. Hence $f_T(\tau)$, $\tau = k\delta$, is obtained as

$$f_T(\tau) = \mathsf{P}\{\tau - \delta < X \leq \tau\} = F_X(\tau) - F_X(\tau - \delta). \tag{7.5}$$

(2) $h(\tau)$

Since $h(\tau) = \mathsf{E}[C_1 | T_1 = \tau] f_T(\tau)$ and $f_T(\tau)$ takes a non-zero value only when τ is a multiple of δ, $h_1(\tau)$ also takes a non-zero value only when τ is a multiple of δ.

To derive $h_1(\tau)$, τ being a multiple of δ, partition the event $\{T_1 = \tau\}$ into mutually exclusive subevents as $\bigcup_{\tau-\delta<x\leq\tau}\{X_1 = x, Y_1 = \tau - x\}$. The event $\{X_1 = x, Y_1 = \tau - x\}$ implies that the component survives for a period of x and then is renewed at the following inspection time (see Figure 7.3). In such a renewal interval, the number

of inspections is equal to $k = \tau/\delta$ and the length of down time is equal to $(\tau - x)$. Therefore the cost incurred in the first renewal interval of $\{X_1 = x, Y_1 = \tau - x\}$ is given by

$$C_1 = c_I(\tau/\delta) + c_D(\tau - x) + c_R.$$

Then using the law of total expectation by conditioning on X_1, $h_1(\tau)$ is obtained as

$$h(\tau) = \sum_{\tau-\delta<x\leq\tau} \Big[c_I(\tau/\delta) + c_D(\tau - x) + c_R\Big] f_X(x). \tag{7.6}$$

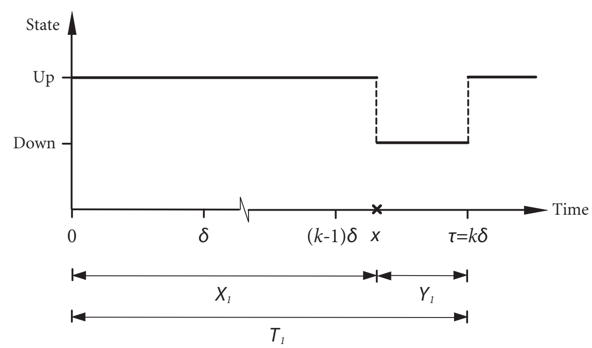


Figure 7.3: Event $\{X_1 = x, Y_1 = \tau - x\}$ for components in Block δ

(3) $\overline{H}(t)$

To derive $\overline{H}(t)$, partition the event $\{T_1 > t\}$ into mutually exclusive subevents as $\left(\bigcup_{\nu(t,\delta)<x\leq t}\{X_1 = x, Y_1 > t - x\}\right)\bigcup\{X_1 > t\}$. Here

$$\nu(t, \delta) = \lfloor t/\delta \rfloor \, \delta \tag{7.7}$$

is the inspection time right before t. The event $\{X_1 = x, Y_1 > t - x\}$ is shown in Figure 7.4, where $(k-1)\delta < t \leq k\delta$. Up to time t, the number of inspections is equal to $\lfloor t/\delta \rfloor$ and the length of down time is equal to $(t-x)$. Note that no replacement cost is incurred up to t since replacement cost is only incurred at renewal points and t is not a renewal point. Hence the cost up to t associated with the event $\{X_1 = x, Y_1 > t-x\}$ is given by

$$C(t) = c_I \lfloor t/\delta \rfloor + c_D(t-x).$$

For the event $\{X_1 > t\}$, only inspection cost is incurred before t. Hence the cost up to t associated with the event $\{X_1 > t\}$ is given by

$$C(t) = c_I \lfloor t/\delta \rfloor .$$

Then using the law of total expectation by conditioning on X_1, $\overline{H}(t)$ is obtained as

$$\overline{H}(t) = \sum_{\nu(t,\delta)<x\leq t} \Big\{ c_I \lfloor t/\delta \rfloor + c_D(t-x) \Big\} f_X(x) + c_I \lfloor t/\delta \rfloor \overline{F}_X(t). \tag{7.8}$$

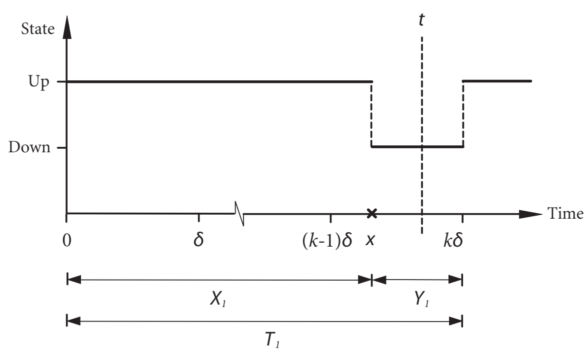


Figure 7.4: Event $\{X_1 = x, Y_1 > t-x\}$ for components in Block δ

7.3.2 Other Blocks

As shown in Figure 7.1, the component in Block r is inspected at times r, $r+\delta$, $r+2\delta$, $\cdots$. Figure 7.5 is an illustration of the state alternation of such components. As shown in that figure, in all the renewal intervals except the first one, the component is still inspected at times δ, 2δ, $\cdots$. However, in the first renewal interval, the component is inspected at times r, $r+\delta$, $r+2\delta$, $\cdots$. Then all the renewal intervals are identically distributed except the first one. Denote the PMF of T_1 by $f_T(\tau|r)$ and the associated CDF and SF by $F_T(\tau|r)$ and $\overline{F}_T(\tau|r)$, respectively. Obviously, $f_T(\tau)$ given in Eq. (7.5) is a special case of $f_T(\tau|r)$ when $r=\delta$.

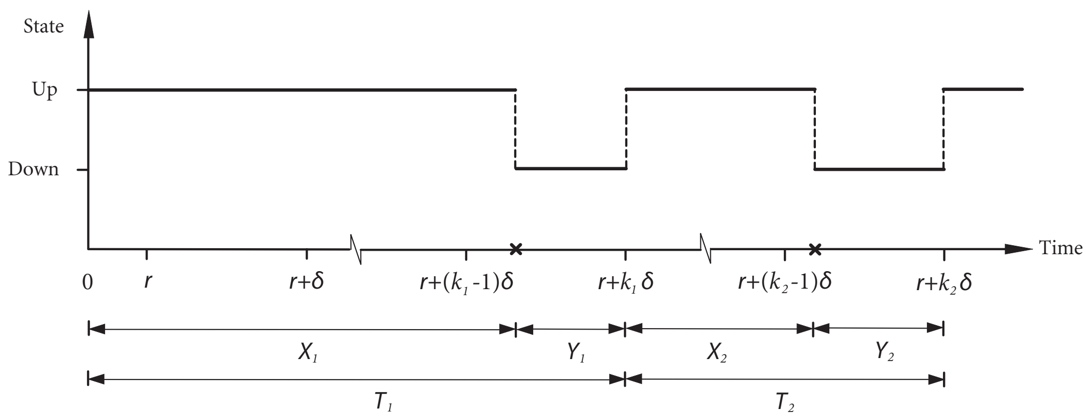


Figure 7.5: State alternation of components in Block r

To derive $U(t|r)$, use the law of total expectation by conditioning on T_1 and then $U(t|r)$ is written as

$$U(t|r) = \sum_{0<\tau\leq t} \mathsf{E}\left[C(t)|T_1=\tau\right] f_T(\tau|r) + \mathsf{E}\left[C(t)|T_1>t\right] \overline{F}_T(t|r). \tag{7.9}$$

Here, $U(t|r)$ is partitioned into two parts associated with events $\{T_1 \leq t\}$ and $\{T_1 > t\}$.

When $T_1 = \tau < t$, split $C(t)$ into $C_1 + C(\tau, t)$, such that

$$\mathsf{E}\left[C(t)|T_1 = \tau\right] = \mathsf{E}\left[C_1|T_1 = \tau\right] + \mathsf{E}\left[C(\tau, t)|T_1 = \tau\right] \tag{7.10}$$

As mentioned above that in all the renewal intervals, except the first one, the component is inspected at times of δ, 2δ, $\cdots$. Then if we take the first renewal point $T_1 = \tau$ as the new time origin, the cost in the remaining interval can be considered as the cost of a component in Block δ. Hence $C(\tau, t)$ will be stochastically the same as the cost of a component in Block δ with time horizon of $(t - \tau)$. Then the term $\mathsf{E}\left[C(\tau, t)|T_1 = \tau\right]$ in Eq. (7.10) can be simplified as

$$\mathsf{E}\left[C(\tau, t)|T_1 = \tau\right] = U(t - \tau). \tag{7.11}$$

Substituting Eq. (7.10) into (7.9), $U(t|r)$ will be obtained as

$$U(t|r) = \sum_{0<\tau\leq t} U(t - \tau) f_T(\tau|r) + G(t|r), \tag{7.12}$$

where

$$G(t|r) = \sum_{0<\tau\leq t} \underbrace{\mathsf{E}\left[C_1|T_1 = \tau\right] f_T(\tau|r)}_{h(\tau|r)} + \underbrace{\mathsf{E}\left[C(t)|T_1 > t\right] \overline{F}_T(t|r)}_{\overline{H}(\tau|r)}. \tag{7.13}$$

The asymptotic formula of Eq. (7.3) still holds for $U(t|r)$ despite of r. The terms $f_T(\tau|r)$, $h(t|r)$, and $\overline{H}(t|r)$ are given as follows.

(1) $f_T(\tau|r)$

Since the inspection times of the first renewal interval are r, $r+\delta$, $r+2\delta$, $\cdots$, T_1 can only take a value of $r + k\delta$, $k = 0, 1, \cdots$. The event $\{T_1 = r + k\delta\}$ implies that the component fails in the interval of $(r+(k-1)\delta, r+k\delta]$, i.e., $r+(k-1)\delta < X \leq r+k\delta$. Hence $f_T(\tau|r)$, $\tau = r + k\delta$, is obtained as

$$f_T(\tau|r) = \mathsf{P}\{\tau - \delta < X \leq \tau\} = F_X(\tau) - F_X(\tau - \delta). \tag{7.14}$$

(2) $h(\tau|r)$

Since $h(\tau|r) = \mathsf{E}[C_1|T_1 = \tau]\, f_T(\tau|r)$ and $f_T(\tau|r)$ takes a non-zero value only when $\tau = r + k\delta$, $h_1(\tau)$ also takes a non-zero value only when $\tau = r + k\delta$.

To derive $h_1(\tau|r)$, τ being equal to $(r+k\delta)$, partition the event $T_1 = \tau$ into mutually exclusive subevents as $\bigcup_{\tau-\delta<x\leq\tau}\{X_1 = x, Y_1 = \tau - x\}$. In a renewal interval of $\{X_1 = x, Y_1 = \tau - x\}$ (see Figure 7.6), the number of inspections is given by $k = (\tau - r)/\delta + 1$ and the length of down time is equal to $(\tau - x)$. Therefore the cost incurred in the first renewal interval of $\{X_1 = x, Y_1 = \tau - x\}$ is given by

$$C_1 = c_I\,[(\tau - r)/\delta + 1] + c_D(\tau - x) + c_R.$$

Then using the law of total expectation by conditioning on X_1, $h_1(\tau|r)$ is obtained as

$$h(\tau|r) = \sum_{\tau-\delta<x\leq\tau} \Big\{c_I\,[(\tau - r)/\delta + 1] + c_D(\tau - x) + c_R\Big\} f_X(x). \tag{7.15}$$

(3) $\overline{H}(t|r)$

To derive $\overline{H}(t|r)$, partition the event $\{T_1 > t\}$ into mutually exclusive subevents as

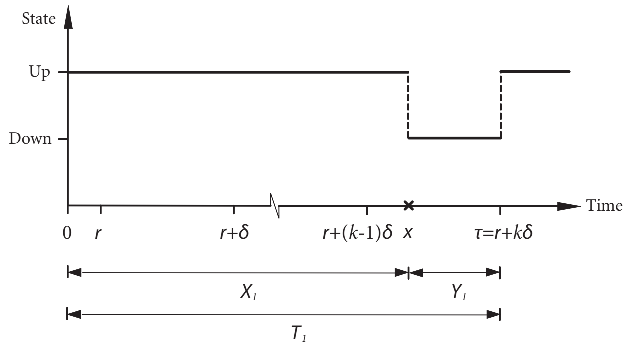


Figure 7.6: Event $\{X_1 = x, Y_1 = \tau - x\}$ for components in Block r

$\left(\bigcup_{\nu(t,\delta|r)<x\leq t}\{X_1 = x, Y_1 > t - x\}\right)\bigcup\{X_1 > t\}$. Here

$$\nu(t, \delta|r) = r + \left\lfloor \frac{t-r}{\delta} \right\rfloor \delta \tag{7.16}$$

is the inspection time right before t. The event $\{X_1 = x, Y_1 > t - x\}$ is shown in Figure 7.7, where $r + (k-1)\delta < t \leq r + k\delta$. Up to time t, the number of inspections is equal to $\lfloor (t-r)/\delta \rfloor + 1$ and the length of down time is equal to $(t - x)$. No replacement cost is incurred up to t since t is not a renewal point. Hence the cost up to t associated with the event $\{X_1 = x, Y_1 > t - x\}$ is given by

$$C(t) = c_I \left(\left\lfloor \frac{t-r}{\delta} \right\rfloor + 1 \right) + c_D(t - x).$$

For the event $\{X_1 > t\}$, only inspection cost is incurred before t. Hence the cost up to t associated with the event $\{X_1 > t\}$ is given by

$$C(t) = c_I \left(\left\lfloor \frac{t-r}{\delta} \right\rfloor + 1 \right).$$

Then using the law of total expectation by conditioning on X_1, $\overline{H}(t|r)$ is obtained as

$$\overline{H}(t|r) = \sum_{\nu(t,\delta|r)<x\leq t} \left\{ c_I \left(\left\lfloor \frac{t-r}{\delta} \right\rfloor + 1 \right) + c_D(t-x) \right\} f_X(x) + c_I \left(\left\lfloor \frac{t-r}{\delta} \right\rfloor + 1 \right) \overline{F}_X(t). \quad (7.17)$$

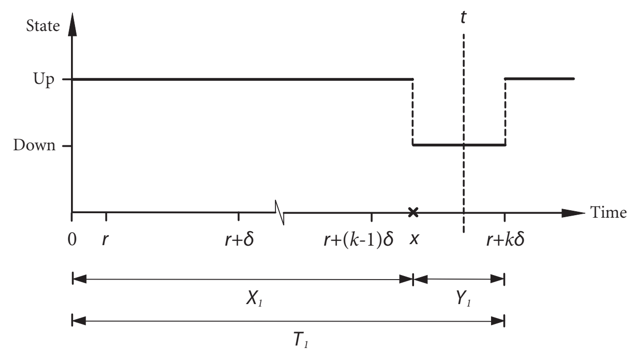


Figure 7.7: Event $\{X_1 = x, Y_1 > t - x\}$ for components in Block r

7.4 Maintenance Cost for Population

Since there are N/δ components in each block and components are independent of each other, the expected cost of Block r will be equal to $U(t|r) \times N/\delta$. Then the expected cost of the entire population is given by

$$U_{\text{total}}(t) = \frac{N}{\delta} \sum_{r=1}^{\delta} U(t|r). \quad (7.18)$$

Define

$$U_{\text{av}}(t) = \frac{1}{N} U_{\text{total}}(t) = \frac{1}{\delta} \sum_{r=1}^{\delta} U(t|r). \quad (7.19)$$

$U_{\text{av}}(t)$ is the average cost per component. Note that the asymptotic formula of Eq. (7.3) holds for any $U(t|r)$, substituting which into Eq. (7.19) gives the asymptotic formula of $U_{\text{av}}(t)$ as

$$U_{\text{av}}(t) = \frac{\mu_C}{\mu_T} t + o(t), \tag{7.20}$$

which is the same as that of $U(t)$.

7.5 Expected Down Components & Replacements

The expected number of down components and replacements are two important terms in the analysis of reliability and maintainability. A large proportion of down components will make a system very vulnerable to failure. The proportion of replacements need to be estimated to raise capital for the maintenance program.

7.5.1 Expected Components in Down State

Take unit costs $c_D = 1$ and $c_I = c_R = 0$, and then $U(t|r)$ becomes the expected length of down time of a component in Block r up to time t. Using Eq. (3.30), the unavailability of the component at time t is obtained as (note that here $\Delta t = 1$)

$$u_D(t|r) = U(t|r) - U(t-1|r). \tag{7.21}$$

Then the expected down components in Block r will be equal to $u_D(t|r) \times N/\delta$. Summing up these values gives the expected down components in the entire population. Then dividing the summation by N gives the expected proportion of down components in the entire population at time t as

$$\begin{aligned} P_D(t) &= \frac{1}{\delta}\sum_{r=1}^{\delta} u_D(t|r) \\ &= \frac{1}{\delta}\sum_{r=1}^{\delta} [U(t|r) - U(t-1|r)] \\ &= U_{\text{av}}(t) - U_{\text{av}}(t-1). \end{aligned} \tag{7.22}$$

Note that the asymptotic value of $u_D(t|r)$ is equal to

$$\lim_{t\to\infty} u_D(t|r) = \frac{\mu_Y}{\mu_T} \tag{7.23}$$

despite of r. Here μ_Y is the value of Y and can be obtained as $\mu_Y = \mu_T - \mu_X$, μ_X being the expected value of X. Then the asymptotic value of $P_D(t)$ is obtained as

$$\lim_{t\to\infty} P_D(t) = \frac{\mu_Y}{\mu_T}, \tag{7.24}$$

which is the same as that of $u_D(t|r)$.

7.5.2 Expected Number of Replacements

Take unit costs $c_R = 1$ and $c_I = c_D = 0$, and then $U(t|r)$ becomes the expected number of replacements of a component in Block r up to time t. Similar to Eq. (3.31), the replacement rate, i.e., the probability density of replacement, at time t is obtained as

$$u_R(t|r) = U(t|r) - U(t-1|r). \tag{7.25}$$

Then the expected replacements in Block r at time t will be equal to $u_D(t|r) \times N/\delta$. Summing up these values gives the expected replacements in the entire population. Then dividing the summation by N gives the expected proportion of replacements in the entire population at time t as

$$P_R(t) = \frac{1}{\delta}\sum_{r=1}^{\delta} u_R(t|r) = U_{\mathrm{av}}(t) - U_{\mathrm{av}}(t-1). \tag{7.26}$$

Note that the asymptotic value of $u_R(t|r)$ is equal to

$$\lim_{t\to\infty} u_R(t|r) = \frac{1}{\mu_T} \tag{7.27}$$

despite of r. Then the asymptotic value of $P_R(t)$ is obtained as

$$\lim_{t\to\infty} P_R(t) = \frac{1}{\mu_T}, \tag{7.28}$$

which is the same as that of $u_R(t|r)$.

7.6 Example

Consider an electrical network consisting of a large population of wood poles. The lifetime of a component (wood pole), X, is a discrete weibull distributed random variable. The hazard rate is shown by Eq. (2.14) with parameters $\alpha = 5$ and $\beta = 35$, such that the mean life time is $\mu_X = 21.5$ and the coefficient of variation (COV) is equal to 0.22. The PMF of X is shown in Figure 7.8.

Suppose that the unit costs are $c_I = 1$, $c_D = 5$, and $c_R = 25$, and the planning horizon is

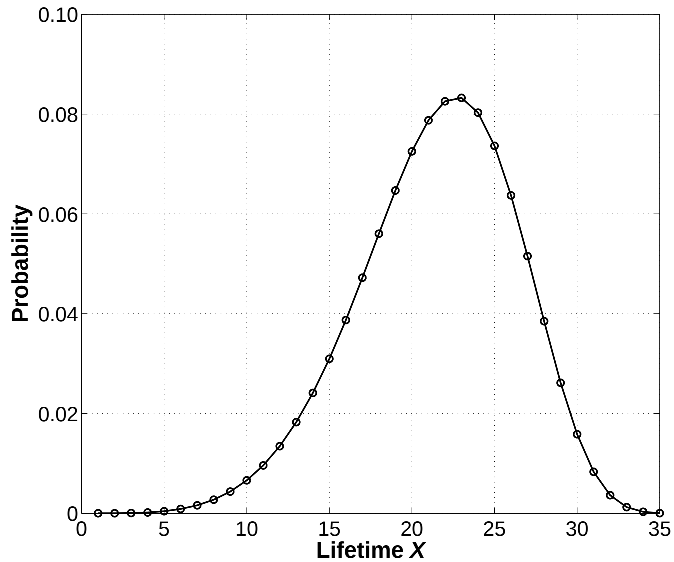


Figure 7.8: PMF of X

$t_m = 40$ years. The average cost per component, $U_{\text{av}}(t_m)$, can be obtained from Eq. (7.19). $U_{\text{av}}(t_m)$ with respect to δ is plotted in Figure 7.9 (the curve of the finite time cost). We can see that the optimal value of δ is $\delta_{\text{opt}} = 5$, for which the average cost $U_{\text{av}}(t_m)$ is minimum $U_{\text{av}}^{\min}(t_m) = 49.2$. If $\delta = 1$, i.e., SIRP is not used and then the entire population will be inspected in one year, then average cost is equal to 73.6, which is larger than $U_{\text{av}}^{\min}(t_m)$ by almost 50%. Hence SIRP has considerable effect on reducing maintenance cost.

The average cost obtained from asymptotic formula (7.20) is also presented in Figure 7.9 (the curve of the asymptotic cost). The difference between the finite time cost and the asymptotic cost is considerable. The asymptotic formula results in an optimal δ of 3 and the minimal average cost of 66.7, which is 35% higher than the finite time cost.

Taking δ as 5, the expected proportions of down components and replacements in the entire population can be obtained from Eq. (7.22) and (7.26), and are plotted in Figure 7.10a and 7.10b, respectively. As a comparison, the cases of $\delta = 2$ and $\delta = 10$ are also

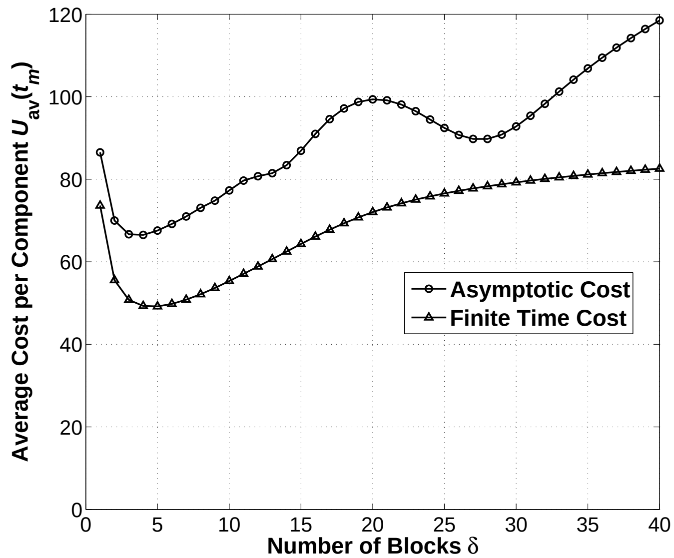
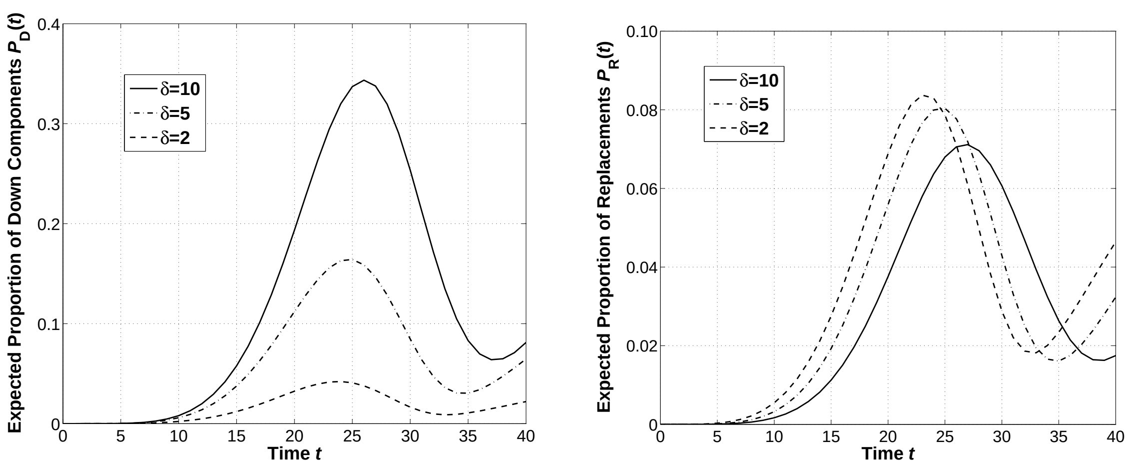


Figure 7.9: Average cost per component $U_{\text{av}}(t_m)$ vs. δ

presented. We can see that P_D increases with δ. The case of $\delta = 2$ has the minimal P_D. This is at the expense of a much more frequent inspections than the other two cases. The values of P_R in the three cases are fairly close. Considering cost and reliability, $\delta = 5$ can be a better choice.



(a) Down components (b) Replacements

Figure 7.10: Expected proportions of down components and replacements in the entire population

7.7 Summary

In case of a large population of components with latent failures, it is not possible to inspect all the components in one year due to limited resources. Hence, the population is divided into several blocks or subpopulations, say δ, and in one year only one block is inspected to replace all failed components. It means that the entire population is inspected over a period of δ years. This policy is called sequential inspection and replacement program (SIRP). The objective is to optimize δ to minimize the expected maintenance cost.

The expected maintenance cost of SIRP in a finite planning horizon is derived by using the theory of renewal-reward process. Using this model, δ can be optimized. An illustrative example related to the asset management of a large population of wood poles used in the electrical network is given.

Chapter 8

System Reliability Analysis

8.1 Introduction

8.1.1 Motivation and Approach

The results and discussions presented in the previous chapters are relevant to reliability and maintenance of a single system. Since a practical engineering system includes many sub-systems and components, it is of interest to estimate maintenance cost at the system level. This topic is explored in this chapter. Reliability analysis of a system with repairable components is a difficult problem to analyze. Various assumptions and approximations are used to simplify the computation of system reliability. A common simplification is to model the failure and repair as an alternate renewal process and assign exponential distributions to the time to failure and time to repair.

Fault-tree analysis is a common method of system reliability analysis used in the nuclear industry [36]. In this approach, all possible pathways of the system failure are investigated as various combinations of component failures. Each pathway is referred to as a cut-set. In the nuclear industry, asymptotic results for failure rate and unavailability are often used to further reduce the computational burden. In summary, the system reliability analysis is also conducted in some asymptotic sense, and the notion of finite-time reliability and cost analysis are not fully explored.

It should be acknowledged that research efforts have been directed towards time-dependent system reliability analysis using methods such as simulation [5, 72], Markov [16, 2] and semi-Markov process [70] models, stochastic Petri nets [35], dynamic fault threes [20] and binary decision diagrams [57]. Our objective is to illustrate a more practical approach of analyzing system reliability once the sub-systems are analyzed using the methods presented in the previous chapters.

The starting point is the reliability block diagram of the system from which cut-sets are derived by usual methods [36]. Assuming the statistical independence of sub-systems [71], system level formula for unavailability and failure rate are derived. Finally, these results are used for maintenance cost estimation.

In passing we note that system reliability analysis of structures has been an active topic of research in civil engineering. However, most of the civil engineering literature deals with systems with non-repairable components only. It means that only probability of first system failure is computed, and therefore, the renewal process models are not utilized. Approximate methods based on First-Order Reliability Method (FORM) have been utilized for time-invariant and time-dependent reliability analysis of structures [28, 37, 52].

8.1.2 Organization

This chapter is organized as follows. Section 8.2 presents the structure of the system and the maintenance model. The derivation of the unavailability and the failure rate for individual components, subsystems and the whole system are given in Sections 8.3, 8.4, and 8.5, respectively. The computation of maintenance cost is given in Section 8.6. Reliability analysis of a system with non-repairable components is presented in Section 8.7. An illustrative example is given in Section 8.8.

8.2 System Model

For the sake of clarity of discussion, we present the model development through an example of a system as shown in Figure 8.1. There are four independent components in this system, denoted as A_i, i =1–4. There are two parallel sub-systems, $\{A_1, A_2\}$ and $\{A_3, A_4\}$, connected in a series. It is assumed that components are independent of each other. This system has been analysed in [46].

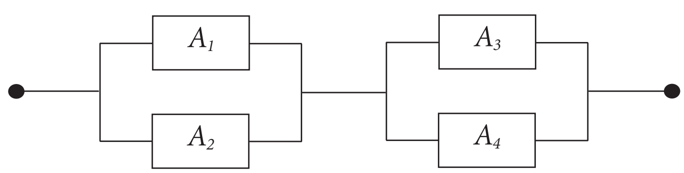


Figure 8.1: Reliability block diagram of the example system

Each component has a random lifetime X. Components are repaired to an as-good-as-new condition upon failure. Time spent on repair is also a random variable Y. In this maintenance model, each component alternates between the up and the down states (see Figure 8.2), generating an alternating renewal process [8]. The length of renewal interval is given by

$$T = X + Y. \tag{8.1}$$

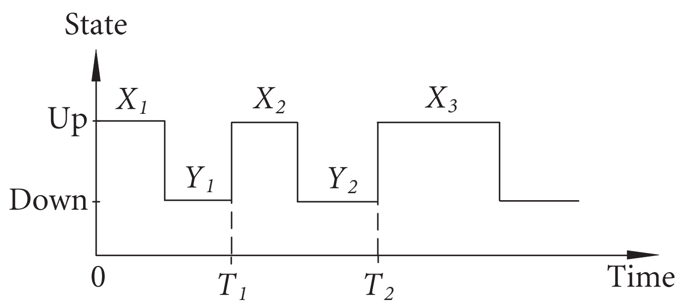


Figure 8.2: State alternation of a component

In this chapter, continuous time is considered. Denote the PDF of X by $f_X(x)$ and that of Y by $f_Y(y)$. Note that the four components in this system are not necessarily the same. Then $f_X(x)$ and $f_Y(y)$ vary with components. It is assumed that X and Y are independent of each other. Then the PDF of T is equal to

$$f_T(\tau) = (f_X * f_Y)(\tau). \tag{8.2}$$

8.3 Component Reliability Analysis

In this section, the unavailability and the failure rate of a specific components are derived.

8.3.1 Unavailability

Denote the length of down time up to time t by $N_D(t)$ and $\mathsf{E}[N_D(t)]$ by $U_D(t)$. As mentioned in Section 3.5, the unavailability of a component is equal to

$$u_D(t) = \frac{\mathrm{d}U_D(t)}{\mathrm{d}t}, \tag{8.3}$$

and $U_D(t)$ can be obtained from Eq. (3.8) as

$$U_D(t) = (U_D * f_T)(t) + \underbrace{\left[\int_0^t h_D(\tau)\mathrm{d}\tau + \overline{H}_D(t)\right]}_{G_D(t)}, \tag{8.4}$$

where

$$h_D(\tau) = \mathsf{E}[N_{D1}|T_1 = \tau]\, f_T(\tau), \quad \overline{H}_D(t) = \mathsf{E}[N_D(t)|T_1 > t]\, \overline{F}_T(t), \tag{8.5}$$

and N_{D1} is the length of down time in the first renewal interval. Using the law of total expectation by conditioning on X_1 (see Figure 8.2), $h_D(\tau)$ is obtained as

$$\begin{aligned} h_D(\tau) &= \int_0^\tau \mathsf{E}[N_{D1}|X_1 = x, Y_1 = \tau - x]\, f_X(x) f_Y(\tau - x)\mathrm{d}x \\ &= \int_0^\tau (\tau - x) f_X(x) f_Y(\tau - x)\mathrm{d}x, \end{aligned} \tag{8.6}$$

and $\overline{H}_D(t)$ is obtained as

$$\begin{aligned} \overline{H}_D(t) &= \int_0^t \mathsf{E}[N_D(t)|X_1 = x, Y_1 > t - x]\, f_X(x)\overline{F}_Y(t - x)\mathrm{d}x \\ &= \int_0^t (t - x) f_X(x)\overline{F}_Y(t - x)\mathrm{d}x. \end{aligned} \tag{8.7}$$

Then $u_D(t)$ can be obtained by substituting Eq. (8.3) into (8.4) as

$$u_D(t) = (u_D * f_T)(t) + g_D(t), \tag{8.8}$$

where (see Appendix D)

$$g_D(t) = \frac{\mathrm{d}G_D(t)}{\mathrm{d}t} = (f_X * \overline{F}_Y)(t). \tag{8.9}$$

The initial condition for $u_D(t)$ is $u_D(0) = 0$.

8.3.2 Failure Rate

Denote the number of failures up to time t by $N_F(t)$ and $\mathsf{E}\left[N_F(t)\right]$ by $U_F(t)$. As mentioned in Section 3.5, the failure rate of a component is equal to

$$u_F(t) = \frac{\mathrm{d}U_F(t)}{\mathrm{d}t}, \tag{8.10}$$

and $U_F(t)$ can be obtained from Eq. (3.8) as

$$U_F(t) = (U_F * f_T)(t) + \underbrace{\left[\int_0^t h_F(\tau)\mathrm{d}\tau + \overline{H}_F(t)\right]}_{G_F(t)}, \tag{8.11}$$

where

$$h_F(\tau) = \mathsf{E}\left[N_{F1} | T_1 = \tau\right] f_T(\tau), \quad \overline{H}_F(t) = \mathsf{E}\left[N_F(t) | T_1 > t\right] \overline{F}_T(t), \tag{8.12}$$

and N_{F1} is the number of failures in the first renewal interval and is equal to 1. Then $h_F(\tau)$ is obtained as

$$h_F(\tau) = f_T(\tau). \tag{8.13}$$

Given $T_1 > t$, $N_F(t)$ is equal to 1 if $X_1 < t$ and 0 otherwise. Then $\overline{H}_F(t)$ is obtained by conditioning on X_1 as

$$\overline{H}_F(t) = \int_0^t f_X(x)\overline{F}_Y(t-x)\mathrm{d}x = (f_X * \overline{F}_Y)(t) \tag{8.14}$$

Then $u_F(t)$ can be obtained by substituting Eq. (8.10) into (8.11) as

$$u_F(t) = (u_F * f_T)(t) + g_F(t), \tag{8.15}$$

where (see Appendix E)

$$g_F(t) = \frac{\mathrm{d}G_F(t)}{\mathrm{d}t} = f_X(t). \tag{8.16}$$

The initial condition for $u_F(t)$ is $u_F(0) = 0$.

8.4 Reliability of a Subsystem

In the following, we will use a superscript $\{i\}$ over u_F or u_D, i =1-4, to imply that this quantity is associated with component A_i. Since subsystem $\{A_1, A_2\}$ is a parallel system, $\{A_1, A_2\}$ is down if and only if both A_1 and A_2 are down. Then the unavailability of

$\{A_1, A_2\}$ is obtained as

$$u_D^{\{1,2\}}(t) = u_D^{\{1\}}(t) u_D^{\{2\}}(t). \tag{8.17}$$

The event that $\{A_1, A_2\}$ fails in the interval $(t, t + \mathrm{d}t]$ implies that (1) one of the two components has already been down at time t and then the other fails in $(t, t + \mathrm{d}t]$; or (2) both of them fail in $(t, t + \mathrm{d}t]$. Hence the probability that $\{A_1, A_2\}$ fails in $(t, t + \mathrm{d}t]$ is equal to

$$u_F^{\{1,2\}}(t)\mathrm{d}t = u_D^{\{1\}}(t) u_F^{\{2\}}(t)\mathrm{d}t + u_D^{\{2\}}(t) u_F^{\{1\}}(t)\mathrm{d}t + u_F^{\{1\}}(t) u_F^{\{2\}}(t)(\mathrm{d}t)^2,$$

where $u_F^{\{1,2\}}(t)$ is the failure rate of the subsystem $\{A_1, A_2\}$. Validly neglecting orders greater than 1 in the above equation gives

$$u_F^{\{1,2\}}(t) = u_D^{\{1\}}(t) u_F^{\{2\}}(t) + u_D^{\{2\}}(t) u_F^{\{1\}}(t). \tag{8.18}$$

The unavailability and the failure rate of the subsystem $\{A_3, A_4\}$, $u_D^{\{3,4\}}(t)$ and $u_F^{\{3,4\}}(t)$, can be obtained similarly.

8.5 Reliability of the System

Note that $\{A_1, A_2\}$ and $\{A_3, A_4\}$ are connected in series. Then the system is down if either one of the two subsystems is down. Hence the unavailability of the system is equal to

$$u_D^{\mathrm{S}}(t) = u_D^{\{1,2\}}(t) + u_D^{\{3,4\}}(t) - u_D^{\{1,2\}}(t) u_D^{\{3,4\}}(t). \tag{8.19}$$

The event that the system fails in the interval $(t, t+\mathrm{d}t]$ implies that at least one of the two subsystems fails in that interval. Hence the probability that the system fails in $(t, t+\mathrm{d}t]$ is equal to

$$u_F^{\mathrm{S}}(t)\mathrm{d}t = \left[1 - u_D^{\{1,2\}}(t)\right] u_F^{\{3,4\}}(t)\mathrm{d}t + \left[1 - u_D^{\{3,4\}}(t)\right] u_F^{\{1,2\}}(t)\mathrm{d}t + u_F^{\{1,2\}}(t)u_F^{\{3,4\}}(t)(\mathrm{d}t)^2,$$

where $u_F^{\mathrm{S}}(t)$ is the failure rate of the system. On the right hand side of the above equation, the first and the second terms are the probability that only one of the two subsystems fails while the other keeps in the up state, and the third term is the probability that both of the two subsystems fail. Validly neglecting orders greater than 1 gives the failure rate of the system as

$$u_F^{\mathrm{S}}(t) = \left[1 - u_D^{\{1,2\}}(t)\right] u_F^{\{3,4\}}(t) + \left[1 - u_D^{\{3,4\}}(t)\right] u_F^{\{1,2\}}(t). \tag{8.20}$$

8.6 Maintenance Cost Analysis

The maintenance cost in a finite time horizon $(0, t]$ consists of the following three parts

$$\begin{aligned} C(t) &= \text{Repair Cost} + \text{System Failure Cost} + \text{System Outage Cost} \\ &= \sum_{i=1}^{4} c_R^{\{i\}} N_F^{\{i\}}(t) + c_F^{\mathrm{S}} N_F^{\mathrm{S}}(t) + c_D^{\mathrm{S}} N_D^{\mathrm{S}}(t), \end{aligned}$$

where $N_F^{\{i\}}(t)$ and $N_F^{\mathrm{S}}(t)$ are the number of failures of component A_i and the system, respectively, $N_D^{\mathrm{S}}(t)$ the length of down time of the system, $c_R^{\{i\}}$ the unit repair cost of A_i, c_F^{S} the unit system failure cost, and c_D the unit system outage cost. Then the expected maintenance cost $U(t) = \mathsf{E}[C(t)]$ is given by

$$U(t) = \sum_{i=1}^{4} c_R^{\{i\}} U_F^{\{i\}}(t) + c_F^{\mathrm{S}} U_F^{\mathrm{S}}(t) + c_D^{\mathrm{S}} U_D^{\mathrm{S}}(t), \tag{8.21}$$

where

$$U_F^{\{i\}} = \mathsf{E}\left[N_F^{\{i\}}(t)\right], \quad U_F^{\mathrm{S}}(t) = \mathsf{E}\left[N_F^{\mathrm{S}}(t)\right], \quad U_D^{\mathrm{S}}(t) = \mathsf{E}\left[N_D^{\mathrm{S}}(t)\right].$$

$U_F^{\{i\}}(t)$ can be derived from Eq. (8.11) for specific components. To derive $U_F^{\mathrm{S}}(t)$ and $U_D^{\mathrm{S}}(t)$, note that we can still obtain Eq. (3.32) for the system by using the same derivation as shown in Section 3.5, i.e.

$$u_D^{\mathrm{S}}(t) = \frac{\mathrm{d}U_D^{\mathrm{S}}(t)}{\mathrm{d}t}, \quad u_F^{\mathrm{S}}(t) = \frac{\mathrm{d}U_F^{\mathrm{S}}(t)}{\mathrm{d}t}.$$

Then $U_F^{\mathrm{S}}(t)$ and $U_D^{\mathrm{S}}(t)$ can be obtained from the failure rate and the unavailability of the system as

$$U_F^{\mathrm{S}}(t) = \int_0^t u_F^{\mathrm{S}}(\tau)\mathrm{d}\tau, \quad U_D^{\mathrm{S}}(t) = \int_0^t u_D^{\mathrm{S}}(\tau)\mathrm{d}\tau. \tag{8.22}$$

8.7 System with Non-repairable Components

In the literature, components are often assumed implicitly to be non-repairable in the analysis of system reliability. In this section, the unavailability, the failure rate, and the expected maintenance cost of systems with non-repairable components are presented. The structure of the system is still the same as that in Section 8.2 except that all the four components are non-repairable.

For non-repairable components, the failure rate is equal to the PDF of the component lifetime, i.e.,

$$u_F(t) = f_X(t). \tag{8.23}$$

A component is down at time t if and only if component lifetime X is less than t. Hence the unavailability of the component is equal to

$$u_D(t) = \mathsf{P}\{L \leq t\} = F_X(t). \tag{8.24}$$

Since the structure of the system is still the same, the reliability of the subsystems and the system can still be obtained by using the methods described in Section 8.4 and 8.5.

Note that in the new model, there is no repair cost. Hence Eq. (8.21) should be modified as

$$U(t) = c_F^{\mathrm{S}} U_F^{\mathrm{S}}(t) + c_D^{\mathrm{S}} U_D^{\mathrm{S}}(t). \tag{8.25}$$

8.8 Example

A numerical example is presented to illustrate the proposed methodology for deriving the system reliability and the expected maintenance cost. This example is purely illustrative. The units of the quantities in this section are not of any practical relevance.

8.8.1 Input Data

The distributions of component lifetime and repair time are given in Table 8.1. Unit system failure cost and system outage cost are taken as $c_F^S = 100$ and $c_D^S = 20$, respectively. Repair cost for any component is equal to $c_R = 20$.

Table 8.1: Distributions of lifetime and repair time of each component

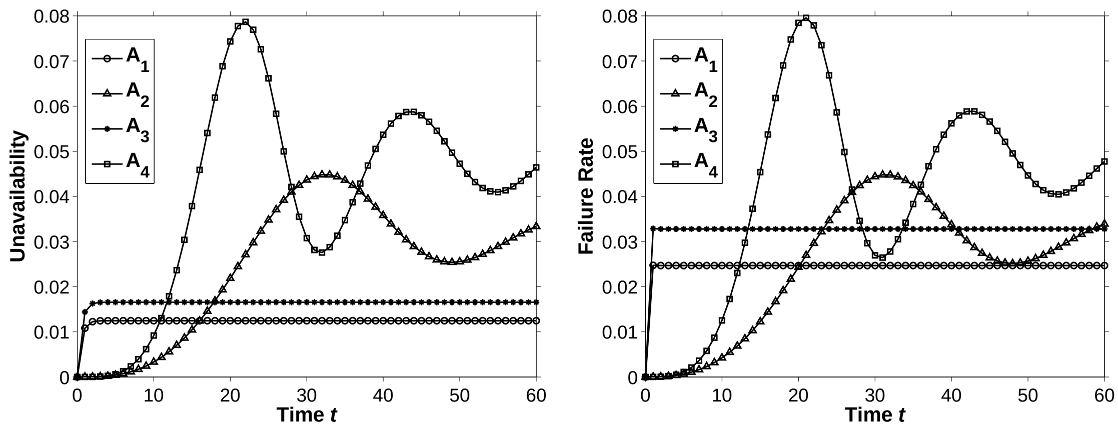
	Lifetime X			Repair Time Y		
	Distribution	Mean	COV	Distribution	Mean	COV
A_1	Exponential	40	1	Exponential	0.5	1
A_2	Weibull	30	0.3	Exponential	1	1
A_3	Exponential	30	1	Exponential	0.5	1
A_4	Weibull	20	0.25	Exponential	1	1

8.8.2 Numerical Results

(1) Reliability of individual components and the system

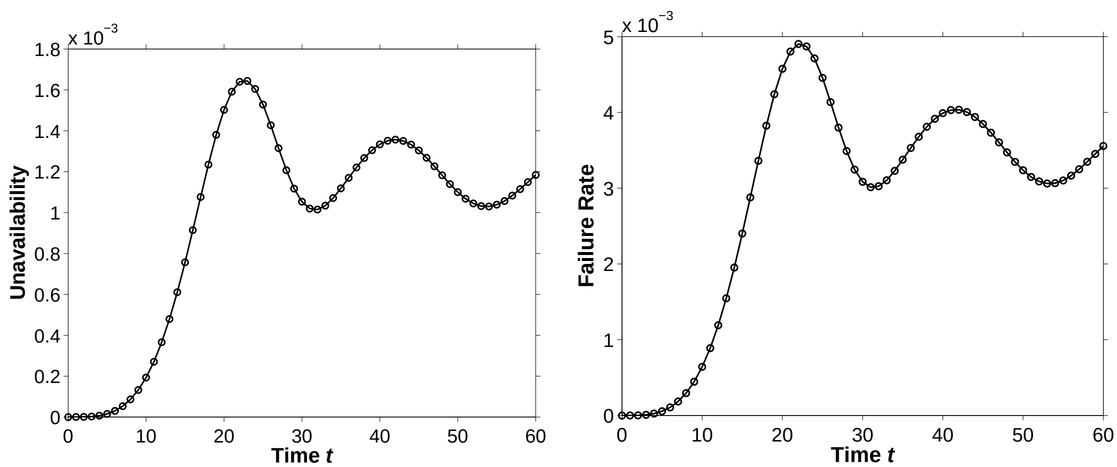
For repairable components, the unavailability and the failure rate can be obtained from Eq. (8.8) and (8.15), respectively. For non-repairable components, these two quantities can be obtained from Eq. (8.24) and (8.23), respectively. Then the unavailability and the failure rate of the subsystems and the system can be obtained from Eq. (8.17), (8.18), (8.19) and (8.20).

For repairable components, the component reliability and the associated system reliability are shown in Figure 8.3 and 8.4, respectively. The reliability of the system with non-repairable components is shown in Figure 8.5. We can see that system reliability is greatly improved by using repairable components. As time t increases, the unavailability of the system with repairable components oscillates about a constant value ($\approx 1.2 \times 10^{-3}$, as shown in Figure 8.4), while that of the system with non-repairable components tends to 1.



(a) Unavailability (b) Failure rate

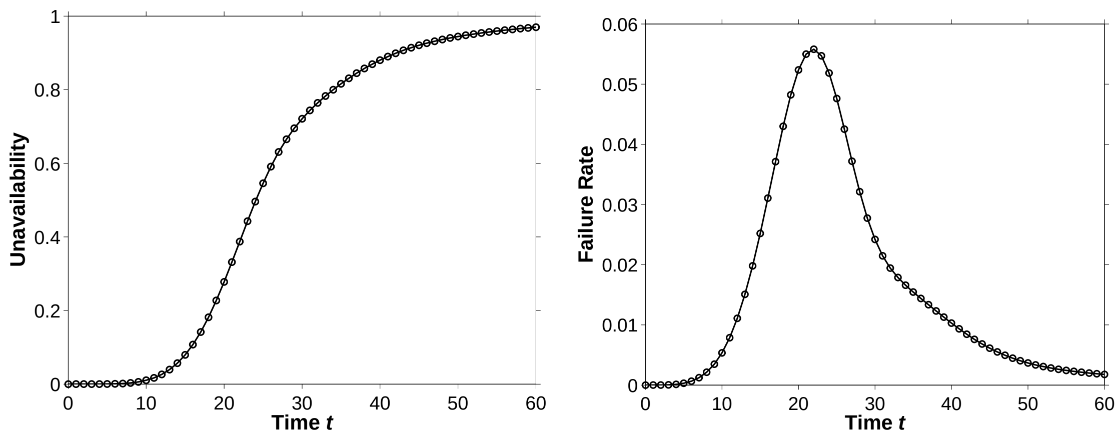
Figure 8.3: Reliability of repairable components



(a) Unavailability (b) Failure rate

Figure 8.4: System reliability with repairable components

(2) Maintenance Cost



(a) Unavailability (b) Failure rate

Figure 8.5: System reliability with non-repairable components

The expected maintenance cost can be obtained from Eq. (8.21) for repairable components or from Eq. (8.25) for non-repairable components and is shown in Figure 8.6. We can see that if $t < 18.5$, the expected maintenance cost with non-repairable components is less than that with repairable components, which is because there is no repair cost for non-repairable components. However, as t increases, the expected maintenance cost with non-repairable components becomes significantly larger than that with repairable components, which is because the system unavailability with non-repairable components is much larger than that with repairable components. Hence if the design time is large, it is better to use repairable components.

8.9 Summary

This chapter presents time-dependent reliability analysis of systems with repairable components. Each component has a random life time and repair time described by general (non-exponential) probability distributions. The time-dependent unavailability and failure

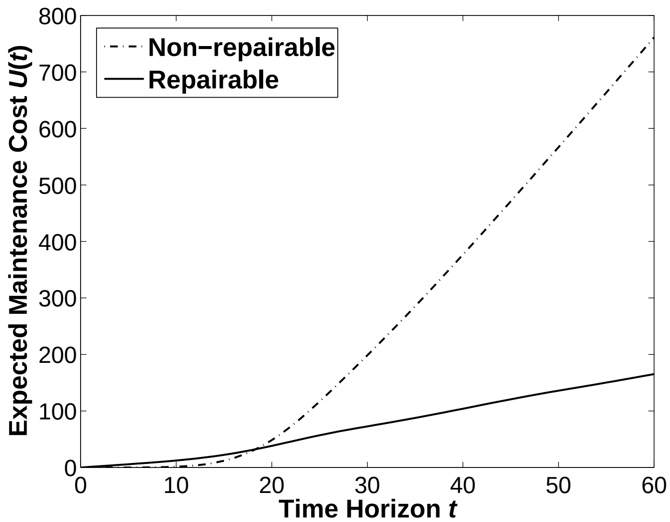


Figure 8.6: Expected maintenance cost vs. time horizon

rate are derived for each individual component of the system by solving a set of renewal equations. Then, system unavailability and failure rate are computed based on the component level information. The analysis method is illustrated using a simple example. The differences between the reliability of the system with repairable components and that with non-repairable components are discussed.

This method can be used to optimize CBM of different components by minimizing maintenance cost of the overall system. The proposed method can be applied to a variety of infrastructure systems, such as bridges, buildings and power systems.

Chapter 9

Conclusion

9.1 Summary of Results

This thesis deals with the probabilistic analysis of maintenance cost for all those failure-repair processes that can be modelled as a stochastic renewal-reward process. A key feature of this process is that time-to-failure and time-to-repair are modelled as random variables. After each repair, preventive or corrective, the system is restored to as-good-as-new condition. The failures can be contributed by an underlying stochastic process describing the degradation in the condition of a system over time. The stochastic gamma process model is used in this thesis. The attention is focussed on condition-based maintenance of system, structures and components that are part of safety critical infrastructure, such as nuclear plants, dams, and dikes. Since existing maintenance cost optimization models rely on asymptotic results for stochastic renewal-reward process, such as the renewal theorem, the cost analysis in a finite-time horizon is a key topic of research interest in this thesis.

A general derivation of any m^{th} order statistical moment of maintenance cost is presented in Chapter 3. The proposed formulation can be used to derive results for a specific maintenance policy, so long as it an be modelled as a stochastic renewal-reward process. In Chapter 4, degradation is modelled as a stochastic gamma process and the cost of condition-based maintenance policy (CBM) is analyzed. Although the gamma process is widely used in the literature, the finite-time mean and variance of cost are derived for the first time in this work. Chapter 5 generalizes the CBM model of Chapter 4 by considering the repair (or down) time and delay in degradation initiation as random variables. The finite time expected cost analysis with discounting is presented in this chapter. It is recognized that the expected cost is not informative enough to enable the estimation of financial risk measures, such as the Value-at-Risk (VaR). To address this issue, Chapter 6 presents a derivation of the probability distribution of the maintenance cost. In Chapter 7, another application of renewal theory is presented to model the sequential inspection and replacement strategy for the asset management of a large population of components, such as electrical networks. Chapter 8 illustrates that the models presented in the previous chapters can be used to analyze reliability at the system level. Here, time-dependent unavailability and failure rate are derived for each individual component of the system by solving a set of renewal equations. Then, system unavailability and failure rate are computed based on the component level information.

Case studies involving the maintenance of heat transport piping system in a nuclear plant and a breakwater are presented in the thesis. A general conclusion is that finite time cost analysis should be used for a realistic evaluation and optimization of maintenance policies for critical infrastructure systems.

9.2 Key Research Contributions

This thesis presents the probabilistic analysis of maintenance cost expected to incur in a finite time horizon for all those maintenance policies that can be modelled as a stochastic renewal-reward process. A particular emphasis is placed on condition-based maintenance policies with a fixed inspection interval and provision for preventive maintenance. The degradation is modelled as a stochastic gamma process. An important aspect of this work is accurate analysis of moments and distribution of cost, instead of relying on asymptotic solutions for optimization of a maintenance policy. The specific contributions are as follows:

(1) Renewal-type integral equations are derived for statistical moments of any m^{th} order moments of the maintenance cost. Specific solutions for the mean and variance of cost are derived for condition-based maintenance policies.

(2) Probability distribution of cost of a condition-based maintenance policy is derived. Here, an interesting idea is that the characteristic function of cost is formulated as a renewal integral equation. The Fourier transform of the characteristic function leads to the probability distribution.

(3) Practical case studies are presented that confirm that the use of asymptotic solutions is not warranted for maintenance cost analysis of highly reliable systems.

9.3 Recommendations for Future Research

Recommendations for future research are as follows:

(1) In the present work inspection is assumed to be perfect for detecting the failure or quantifying the magnitude of degradation. However, inspection instruments in practice are not perfect. The analysis in future should consider the imperfect nature of inspection by incorporating the probability of detection and sizing error.

(2) A fast algorithm should be developed for solving renewal equations, especially in the case of continuous random variables. The order of computation of renewal equations is n^2, n being the number of time steps. If the time step is very small, the computation can be very time consuming.

(3) A higher order approximation of the maintenance cost should be developed, which would be closer to finite time cost estimate.

(4) The system reliability should be analyzed for more complicated maintenance policies adopted at the component level. The approach presented in Chapter 8 can enable this generalization.

APPENDICES

Appendix A

Abbreviations and Notations

CANDU	CANada Deuterium Uranium
CF	Characteristic Fucntion
CBM	Condition Based Maintenance
CDF	Cumulative Distribution Function
COV	Coefficient of Variance
FT	Fourier Transform
FAC	Flow Accelerated Corrosion
PDF	Probability Density Function
ORP	Ordinary Renewal Process
RRP	Renewal-Reward Process
PMF	Probability Mass Function
SIRP	Sequential Inspection and Replacement Program
SF	Survival Function
SG	Steam Generator

VaR	Value-at-Risk

$\mathsf{P}\{\mathcal{E}\}$	Probability of event $\mathcal{E}$
$\mathsf{E}[X]$	Expected value of X
$f_X(x)$	PDF of X
$F_X(x)$	CDF of X
$\overline{F}_X(x)$	Survival function of X
$\text{Gamma}(\xi, \beta)$	Gamma distribution with shape parameter ξ and scale parameter β
$f^{\mathrm{G}}(w; \xi, \beta)$	PDF of $\text{Gamma}(\xi, \beta)$
$F^{\mathrm{G}}(w; \xi, \beta)$	CDF of $\text{Gamma}(\xi, \beta)$
$\overline{F}^{\mathrm{G}}(w; \xi, \beta)$	Survival function of $\text{Gamma}(\xi, \beta)$
$\lfloor\rfloor$	Floor function

Appendix B

Derivation of Eq. (5.17)

Given $\mathcal{A}_{\text{PM}}(l)$, l being a multiple of δ, degradation should have been initiated in the interval $[l-\delta, l)$ (see Figure 5.1b), i.e. $l-\delta \leq X < l$. Then the degradation growth interval is equal to $Y = l - X$. Therefore

$$F_{W(Y)}\big(w|\mathcal{A}_{\text{PM}}(l)\big) = \mathsf{P}\left\{W(l-X) \leq w|\mathcal{A}_{\text{PM}}(l)\right\} = \frac{\mathsf{P}\left\{W(l-X) \leq w, \mathcal{A}_{\text{PM}}(l)\right\}}{\mathsf{P}\left\{\mathcal{A}_{\text{PM}}(l)\right\}}. \quad \text{(B.1)}$$

In the above equation, $\mathsf{P}\left\{\mathcal{A}_{\text{PM}}(l)\right\}$ is equal to $f_{L,\text{PM}}(l)$, which is given in Eq. (5.10). Note that $\mathcal{A}_{\text{PM}}(l)$ means that at time l, degradation does not exceed w_F, i.e. $W(l-X) \leq w_F$. Hence event $\{W(l-X) < w\}$ implies $\mathcal{A}_{\text{PM}}(l)$ since $w \leq w_F$. Then

$$\mathsf{P}\left\{W(l-X) \leq w, \mathcal{A}_{\text{PM}}(l)\right\} = \mathsf{P}\left\{W(l-X) \leq w\right\}.$$

Using the law of total probability by conditioning on X, the above probability is obtained as

$$\mathsf{P}\left\{W(l-X) \leq w\right\} = \sum_{l-\delta \leq x < l} \mathsf{P}\left\{X = x, W(l-x) \leq w\right\}$$

Note that $W(y)$ is a stationary gamma process with shape parameter α and scale parameter β. Then $W(l-x) \sim \text{Gamma}(\alpha(l-x), \beta)$. Hence the probability of $\{W(l-x) \leq w_F\}$ is equal to $F^{\mathrm{G}}(w_F; \alpha(l-x), \beta)$, where F^{G} is the CDF of the gamma distribution as shown in Eq. (4.3). Since X is independent of $W(y)$, the above equation gives

$$\mathsf{P}\left\{W(l-X) \leq w\right\} = \sum_{l-\delta \leq x < l} f_X(x) F^{\mathrm{G}}(w_F; \alpha(l-x), \beta).$$

Substituting the above equation into Eq. (B.1), $F_{W(Y)}\big(w|\mathcal{A}_{\mathrm{PM}}(l)\big)$ is obtained as

$$F_{W(Y)}\big(w|\mathcal{A}_{\mathrm{PM}}(l)\big) = \frac{1}{f_{L,\mathrm{PM}}(l)} \sum_{l-\delta \leq x < l} f_X(x) F^{\mathrm{G}}(w_F; \alpha(l-x), \beta) \tag{B.2}$$

Appendix C

Derivation of Eq. (5.25)

Similar to the derivation of Eq. (3.8), using the law of total expectation by conditioning on the time of first renewal T_1, the expected cost, $U(t)$, is written as

$$U^{\mathrm{d}}(t) = \sum_{0<\tau\leq t} \mathsf{E}\left[C^{\mathrm{d}}(t)|T_1 = \tau\right] f_T(\tau) + \mathsf{E}\left[C^{\mathrm{d}}(t)|T_1 > t\right] \overline{F}_T(t). \tag{C.1}$$

Here, $U^{\mathrm{d}}(t)$ is partitioned into two cases corresponding to $T_1 \leq t$ and $T_1 > t$. When $T_1 = \tau < t$, split $C^{\mathrm{d}}(t)$ into two terms: (1) the cost in the first renewal interval (C_1), and (2) the cost in the remaining time horizon, $C^{\mathrm{d}}(\tau, t)$, such that

$$\mathsf{E}\left[C^{\mathrm{d}}(t)|T_1 = \tau\right] = \mathsf{E}\left[C_1^{\mathrm{d}}|T_1 = \tau\right] + \mathsf{E}\left[C^{\mathrm{d}}(\tau, t)|T_1 = \tau\right] \tag{C.2}$$

Note that given $T_1 = \tau < t$, the non-discounted cost $C(\tau, t)$ is stochastically the same as $C(t-\tau)$ by taking τ as the new time origin. Then taking τ as the present time, the NPV of $C(\tau, t)$ should be stochastically the same as $C^{\mathrm{d}}(t-\tau)$. However, the cost at time τ should be discounted with a factor of ρ^τ if time 0 is taken as the present time. Hence $C^{\mathrm{d}}(\tau, t)$ is stochastically the same as $\rho^{\tau} C(t-\tau)$. Therefore the second term in the right hand side of Eq. (C.2) is given by

$$\mathsf{E}\left[C^{\mathrm{d}}(\tau, t) | T_1 = \tau\right] = \rho^{\tau} U^{\mathrm{d}}(t-\tau). \tag{C.3}$$

Then substituting Eq. (C.2) into (C.1), we can obtain $U^{\mathrm{d}}(t)$ as

$$U^{\mathrm{d}}(t) = \sum_{\tau=1}^{t} \rho^{\tau} f_T(\tau) U^{\mathrm{d}}(t-\tau) + G^{\mathrm{d}}(t), \tag{C.4}$$

where

$$G^{\mathrm{d}}(t) = \sum_{\tau=1}^{t} h^{\mathrm{d}}(\tau) + \overline{H}^{\mathrm{d}}(t), \tag{C.5}$$

$$h^{\mathrm{d}}(\tau) = \mathsf{E}\left[C_1^{\mathrm{d}} | T_1 = \tau\right] f_T(\tau). \tag{C.6}$$

$$\overline{H}^{\mathrm{d}}(t) = \mathsf{E}\left[C^{\mathrm{d}}(t) | T_1 > t\right] \overline{F}_T(t). \tag{C.7}$$

Appendix D

Derivation of Eq. (8.9)

Since

$$G_D(t) = \int_0^t h_D(\tau)\mathrm{d}\tau + \overline{H}_D(t),$$

we have

$$g_D(t) = \frac{\mathrm{d}G_D(t)}{\mathrm{d}t} = h_D(t) + \frac{\mathrm{d}\overline{H}_D(t)}{\mathrm{d}t}.$$

Substituting Eq. (8.6) and (8.7) into the above equation gives

$$g_D(t) = \int_0^t (t-x) f_X(x) f_Y(t-x)\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (t-x) f_X(x)\overline{F}_Y(t-x)\mathrm{d}x. \tag{D.1}$$

The second term on the right hand side of the above equation can be obtained as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (t-x) f_X(x)\overline{F}_Y(t-x)\mathrm{d}x = \int_0^t (t-x) f_X(x)\left[\overline{F}_Y(t-x) - (t-x) f_Y(t-x)\right]\mathrm{d}x$$

$$= (f_X * \overline{F}_Y)(t) - \int_0^t (t-x) f_X(x)(t-x) f_Y(t-x)\mathrm{d}x,$$

substituting which into Eq. (D.1) gives

$$g_D(t) = (f_X * \overline{F}_Y)(t). \tag{D.2}$$

Appendix E

Derivation of Eq. (8.16)

Since

$$G_F(t) = \int_0^t h_F(\tau)\mathrm{d}\tau + \overline{H}_F(t),$$

we have

$$g_F(t) = \frac{\mathrm{d}G_F(t)}{\mathrm{d}t} = h_F(t) + \frac{\mathrm{d}\overline{H}_F(t)}{\mathrm{d}t}.$$

Substituting Eq. (8.13) and (8.14) into the above equation gives

$$g_F(t) = f_T(t) + \frac{\mathrm{d}(f_X * \overline{F}_Y)(t)}{\mathrm{d}t}. \tag{E.1}$$

The second term on the right hand side of the above equation can be obtained as

$$\frac{\mathrm{d}(f_X * \overline{F}_Y)(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t f_X(x)\overline{F}_Y(t-x)\mathrm{d}x$$

$$= f_X(t) - \int_0^t f_X(x) f_Y(t-x) \mathrm{d}x$$
$$= f_X(t) - f_T(t),$$

substituting which into Eq. (E.1) gives

$$g_F(t) = f_X(t). \tag{E.2}$$

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