Dynamic Pricing in The Presence of Strategic Consumer with Product and Intertemporal Substitution

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

This study develops a dynamic pricing model with a quality substitutable product, taking into account strategic and myopic consumers. In each of the two periods, the firm can choose between offering a high quality product, a low quality product or both and the corresponding price for the product. Strategic consumers compare current utility with future utility in order to decide the time of purchase and the quality of the product in an attempt to maximize their utilities. Myopic consumers consider only current utility in purchasing of the products. We generate scenarios, prove whether a scenario is feasible and which scenario produces the best profit for the firm. Our result suggests that the firm obtains the best profit when it provides only high quality products in each of the two periods. In other words, the firm does not have to offer quality substitution as intertemporal substitution suffices to maximize the expected profit.

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1 Introduction

Dynamic pricing decisions bear significant importance for firms. In recent years it has been implemented in a wide variety of industries (such as retailing, manufacturing, and E-business) while exploring new and innovation pricing strategies. The emergence of new software tools might reflect the importance of dynamic pricing. For example, "Early users of the new software-Gymboree, J. C. Penney, KB Toys, and ShopKo, among others-are already reporting promising results, with gains in gross margins in the range of 5% to 15%. Retailers are also seeing significant increases in process efficiency. Planners at one chain, for instance, experienced a 20% gain in productivity." (S.C. Friend and P.H.Walker., 2001) Welcome to the New World of Merchandizing. Harvard business Review, 79, November 2001.) The same result (gains in revenue and increased productivity) has been shown by Gallego and Van Ryzin (1997). They find that "price-based rationing" is a more profitable way to limit sales than "quantity-based rationing", since firms reduce the sales and increase revenue at the same time.

Commonly, to account for dynamic pricing the interaction between retailers and consumers is modeled as two-period setting where consumers visit the store in both periods and retailers have an opportunity to update the price. A strategy of decreasing the product price over time enables the seller to take advantage of differences in both consumers' valuations of the product and patience level in order to extract consumers' surplus.

While consumers differ in their patience level, generally academic literature divides the consumers into two broad groups: strategic and myopic. The former consider the future; the latter are impulse buyers. The consideration of both types of consumers is essential to the firm's pricing decision, so we discuss more about these two types of customers. Consumers behave strategically, especially when they purchase durable goods or more expensive products. Therefore, models that account for on strategic customers is more realistic. Models based on strategic consumer behaviour are being used more often recently in the operation literature. Bensanko and Winstion (1990) already reveal the fact that "underestimating the rationality of consumers can have dramatic negative effects on a seller's profit." In contrast, consumers can not always behave strategically. In modern society, consumers all lead very hectic daily lives and they are reluctant to take time: such as spending time to visit a store again or searching for the low price. Many researchers have managed to study this complex and heterogeneous consumer behaviour and reveal that "consumers tend to purchase products spontaneously" (Hughes and Fill, 2007). Therefore, much of the literature has assumed myopic consumer behaviour. However, in this study we have chosen to consider the model dealing with strategic as well as myopic consumers.

Besides the pricing decision, other marketing and operational decisions have to be considered. Product quality is one such consideration. The consideration of pricing and product quality is not only useful, but essential to the firm. The integration of pricing and quality/production decisions is still in its early stage in many firms, but it has the potential to radically improve supply chain efficiencies as well as revenue. Although consumers differ in the evaluation of quality, they generally agree which product is of better quality. Their choice preference depends on quality of the product, their evaluation of the quality, location along the Mussa-Rosen line and the price of the product. Needless to say, it is better for a firm to satisfy these consumer desires. For instance, many firms in the computer software industry offer different prices of a single product (professional and student version) of the same program. When consumers pay the higher price, they are able to download the full package of the new software (high quality). The lower priced student version has limitations (low quality).

Consequently, we are motivated in this dissertation to consider both strategic and myopic consumers with pricing and product decisions over two periods. Expressed simply, we introduce a product that has a value from usage both in the first and the second period, but due to discounting the second period value is lower. While much of literature has assumed a single product, we allow the firm to choose between two different quality levels of the product so that it can sell both in the first period and the second period.

We consider a software developing company with many retailers to distribute the product, say Microsoft. The software can easily be sold in different quality levels. The users purchase the installation kits and based on the price paid, different quality of the product components are installed. They behave as strategic consumers since they purchase durable goods and can be patient.

We follow the well-known Mussa and Rosen's model (1978), while incorporating intertemporal effects. In their model, two populations of opposite preferences exist. While the goods are similar and the goods are sold at the same price, the two populations rank them exactly in the reverse order. Especially, Mussa and Rosen's model derives demand functions based on consumer's utility for differentiated goods, and in their utility function, consumers' valuations of quality vary. We extend their setting by looking at a two period version. Furthermore, we restrict the customer's valuation of the quality to two in our utility function and a firm decides to set four prices through the two periods.

In our setting, consumers obtain utility from using the product in the first period and additional discounted utility from using it in the second period. The firm chooses the product selection that becomes available to consumers, which means that the firm can choose between two configurations: high quality and low quality, and can make one or both (or none) available in each of the two periods.

The firm sets the prices for both the high quality and low quality product price in each of the periods. We restrict the high quality product price to be higher than the low quality product. For both, high or low quality product, the first period price is higher than the second period price. However, no assumption is made regarding the relationship between the first period low quality price and the the second period high quality price. A strategic consumer always maximizes the utility in the selection of a product. In other words, when two qualities of the product are being offered, the strategic consumer chooses the time and product that yield the most utility, given the consumer's location along the Mussa-Rosen model. On the other hand, myopic consumers purchase a product in the period they visit as long as their utilities is positive. If a myopic customer's utility is negative, she will return in the second period and repeats the process. For each period, myopic customers decide which quality of the product to purchase by evaluating their utilities.

First, we enumerate all possible scenarios and derive the corresponding prices for each of these scenarios. Then, we eliminate all non-feasible scenarios and generate conditions for feasibility for the remaining one. Lastly, we compare the remaining scenarios to find the scenario that produces the best profit for the firm and characterize the results.

To summarize our results, a firm is better off using intertemporal substitution rather than product substitution. In other words, the provision of two different qualities in one period is not beneficial to a firm. Specifically, the firm generates the best profit with the model in which it provides only high quality product and sets only one price in each of two periods. This result is in line with that of Bara and Carr (2009). In their model, when a consumer purchases the product, the firm offers a future upgrade price for a slightly higher cost. This would be interpreted as offering two qualities of the product in the first period. However, their result show that offering an upgrade prices (i.e., offering two qualities of the product) is not always optimal.

This paper is organized as follows. Section 2 reviews related literature. Later, we provide some general idea in our two-period model and elaborate our model in Section 3. In Section 4, we test the feasibility. In Section 5, we test optimality and find the best scenario of the model. Section 6 concludes.

2 Related Literature Review

In earlier studies, intertemporal substitution, delaying purchase to a future date, was neglected. Models assumed that a consumer either purchases on the first visit or the sale was lost forever. However, recently revenue management literature has begun to pay more attention to intertemporal substitution and new models allow more opportunity for anticipation of future demand. Strategic consumers refer to those who practice intertemporal substitution and myopic consumers refer to those who make a purchase decision at the time of their arrival.

Consumers behave more strategically when they face firms' dynamic pricing mechanisms, (changing prices over periods). The economic literature has considered the dynamic pricing mechanism in the presence of the strategic consumer. Modeling the interaction between strategic consumers and retailers can be found in the famous study of Coas(1972), in which a monopolist sells a durable good to a large group of consumers with different valuations. He shows how the seller sets the price in a way that results in $perfect\ segmentation$: initially charge a high price to customers with high valuation, and later sequentially reduce the prices to customers with low valuation. However, if the high valuation consumers anticipate the future price decrease, they wait for a lower price. This leads the seller to offer the product at marginal cost. Coas suggests ways to avoid this result for the firm. One is that the seller makes a contract with the buyers, not to sell more than a given quantity of the product, referred to as capacity rationing. The other is that the seller makes a commitment that if the future price is lower, then the seller will buy back the purchased good.

Bensanko and Winstion (1990) extends management science literature on intertemporal pricing to include the assumption that consumers are intertemporal utility maximizers. The study characterizes a subgame perfect Nash equilibrium involving a strategic seller and consumers. The study finds that the demand of strategic consumers is more price elastic than that of myopic. In addition, the numerical analysis shows that "underestimating the rationality of consumers can have dramatic negative effect on a seller's profit."

One of the earliest studies, optimal dynamic pricing under strategic consumers with capacity limitations, by Aviv and Pazgal (2008), analyzes a model with a single price reduction at a fixed point in time T. A fixed *premium* price is charged prior to a fixed time T and a discount price p is charged after time T. Consumers' arrival is determined by a Poisson process and their valuations of the product vary across the population. Those consumers arriving before time T wait to purchase the product if it is beneficial from them. However, consumers arriving after time T do not have an incentive to wait. The seller commits a fixed price path and under this assumption the seller has to choose the discount price p. The study considers two discounting strategies: contingent and fixed discounts. In the former case, the magnitude of the discount depends on the remaining inventory. The latter indicates that the discounting factor is announced at the beginning. The authors also test the seller's discount price in both contingent and fixed discount cases. Their results can be grouped. First, pre-commitment is of benefit to a firm when consumers behave strategically. Second, the presence of strategic consumers affect how the firm's inventory influences the depth of discount. Last, ignorance of the strategic consumer induces significant losses to the firm.

Levin et al. (2007) study monopoly and duopoly settings with a fixed number of consumers whose valuations are random along the horizon. They introduce a dynamic model that includes an internal price guarantee instrument. They consider that the initial price guarantee provides a consumer with compensation if the price of the product decreases below the *strike price*. Consumers can choose whether to accept or reject the guarantee. When they buy, they pay a fee. A price guarantee encourages an early purchase. For the firm, an increase in the number of early purchases reduces the uncertainty of late purchases. It also improves consumer satisfaction by capacity planning. In addition, the fee of the guarantee from the consumer exceeds potential average loss. The collected fee provides additional revenue.

Su (2007) studies the impact of strategic consumer behavior in which he allows pricing to increase or decrease over periods. Consumers are modeled in two dimensions: customers having high or low valuations and *high patient* or *impatient*. In this way, it can be studied by four segments of consumers: strategic-high, strategic-low, patient-high and patient-low. The result shows that increasing prices are optimal when high valuation consumers are more strategic but decreasing prices are optimal when high valuation consumers are more myopic. It also emphasizes that strategic behavior implies that when prices are high initially, demand is not lost; rather it cumulates in sales if prices are lowered eventually. Furthermore, scarcity causes strategic consumers to compete and therefore leads to purchases at higher prices.

3 The Model

In this section, we now formally describe our two-period dynamic pricing model. The complete notation is provided at the end of this section. Let $p_{\theta t}$ be the price of the product of quality θ in period t, where $\theta \in \{H, L\}$, denoting high and low quality respectively, $t \in \{1, 2\}$.

Over two periods, a firm may sell two different qualities of the same product: high and low quality. The firm predetermines the pricing path over the two periods but announces the prices sequentially. We restrict the price such that (i) the high quality product price to be higher than the low quality product and (ii) in each case, (high or low quality), the first period price is higher than the second period price (i.e., $p_{H1} > p_{L1}$, $p_{H2} > p_{L2}$, $p_{H1} > p_{H2}$ and $p_{L1} > p_{L2}$). In addition, all decision variables and constants are positive through out this dissertation. We assume that the firm is able to offer enough units to meet all realized demand. That is, we abstract away from inventory considerations.

Consumers is uniformly distributed along the linear line $[0, 1]$ and each customer's location is denoted by i. Consumers have heterogeneous utilities, denoted by $u(\cdot)$ and a unit demand for a product. In each period, consumers arrive continuously. We distinguish two types of consumers: myopic and strategic.

Myopic consumers visit the firm in the first period and purchase the product with the quality that offers the highest positive utility. When the utility is less than zero or less (i.e., $u(\cdot) \leq 0$), the consumers may exit and return in the second period, at which time, she makes the same purchasing decision.

By rational expectations, strategic consumers correctly predict the the firm's choice of quality for the second period and the corresponding prices. Thus, strategic consumers visit the firm in the first period, and choose the time and product that yield the most utility, given the consumer's location, i. Thus, a strategic consumer may delay her purchase to the second period although positive utility could be obtained in the first period as she is interested in maximizing her utility over the entire remaining time.

Demand Side

We describe consumers' derivation of utility. When consumers arrive to the store, they observe the firm's quality choices (i.e., whether a high quality product, α_H, is offered, a low quality product,α_L, is offered or both) and the prices (p_{H1}, $p_{L1} p_{H2}$, andp_{L2}). Each consumer has a utility $u_{\theta t}$ from purchasing a product of quality θ in period t. Consumers who purchase in the first period, obtain both utilities from the first and an additional discounted utility from the second period. Consumers who purchase in the second period, obtain only the discounted utility in the second period. Especially, a consumer located at i obtains a utility of $i\alpha_\theta$ from a product of quality θ if he owns the product in the first period and obtains a utility of $i\delta\alpha_\theta$ from a product if he owns the product in the second period as well. This $u_{\theta t}(\alpha_\theta, \delta, p_{\theta t}\,; i)$:

$$u_{\theta t} = \begin{cases} i\;\alpha_\theta\;(1+\delta)\; - \; p_{\theta 1} \\ i\;\alpha_\theta\;(\delta)\; - \; p_{\theta 2}, \end{cases}$$

where $\theta \in \; \{H, L\}$.

Figure 1 illustrates one of scenarios. The following explanation is based on Figure 1.

The first period

Consider the first period. Strategic consumers are intertemporal utility-maximizers, since they select the time and product that yield the highest utility. For example, suppose that the utility from purchasing high quality product in the first period is u_{H1} and the utility from purchasing high quality product in the second period is u_{H2} (see upper right in Figure 1). Strategic consumers purchase the high quality product in the first period, since for these consumers $u_{H1} \geq u_{H2}$, while some other strategic consumers will purchase this product in the second period as for these consumers $u_{H2} \geq u_{H1}$.

A strategic consumer chooses the product quality and time which provide the maximum

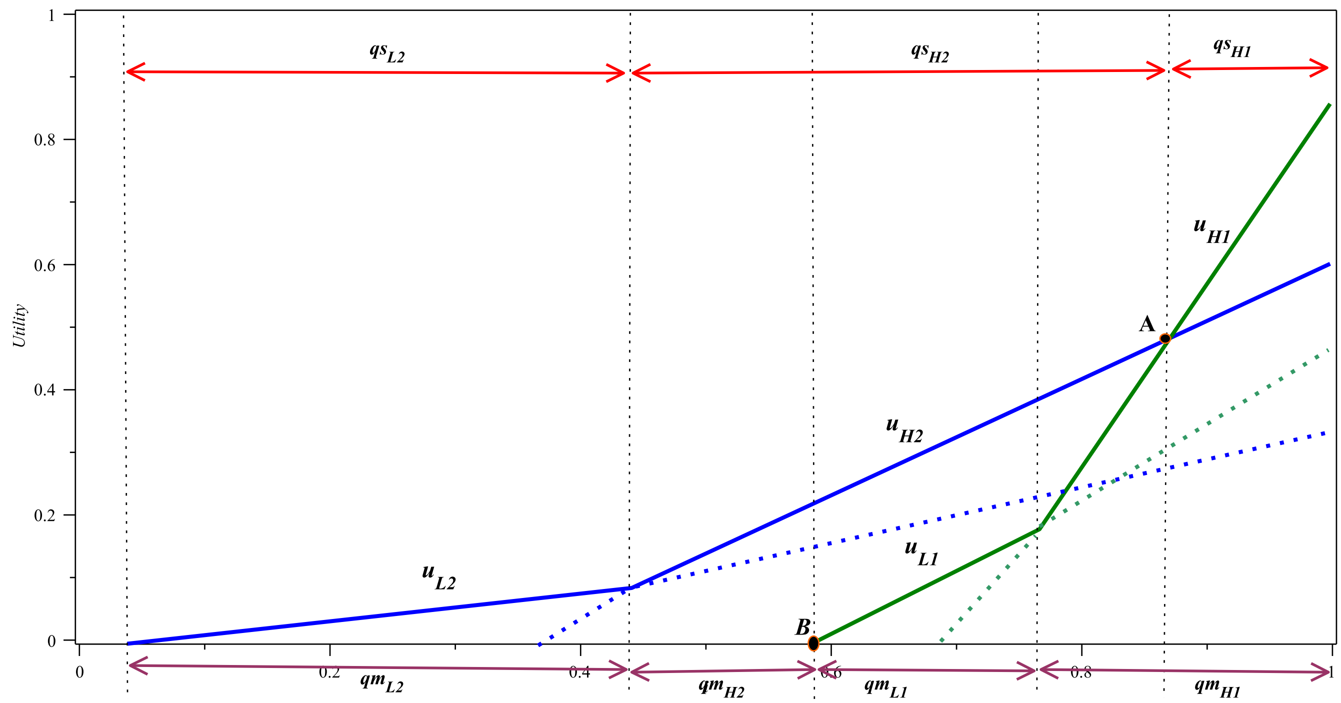


Figure 1: An Example of Consumers' Utilities and Corresponding Demands

utility, $max\{u_{H1}, u_{L1}, u_{H2}, u_{L2}\}$. Strategic consumers' demand for the product θ in period t is expressed as $q^s_{\theta t}$.

Consider the myopic consumers. Myopic consumers purchase the product once they visit a store if the utility is positive. In the first period, the consumers face the decision of which quality of the product to purchase. Some of the myopic consumers choose to purchase the high quality products, since for these consumers $u_{H1} \geq u_{L1}$ while some others will choose the low quality product if in their case $u_{L1} \geq u_{H1}$. Let θ_1 and θ_2 to be the quality choices in period 1 and 2, respectively. In general, given $u_{\theta_1 1}$ and $u_{\theta_2 1}$, a myopic consumer in the first period purchases α_{θ_1} in the period 1 if and only if $u_{\theta_1 1} \geq u_{\theta_2 1}$ where $\theta_1, \theta_2 \in \{H, L\}, \theta_1 \neq \theta_2$. The number of units of quality, θ_t , purchased by myopic consumers in the period t is expressed as $q^m_{\theta t}$, where $t \in \{1, 2\}$.

The Second period

In this period, strategic and myopic consumers behave in the same way. Given $u_{\theta_1 2}$ and $u_{\theta_2 2}$, a myopic consumer purchases α_{θ_1} in period 2 if and only if $u_{\theta_1 2} \geq u_{\theta_2 2}$, where $\theta_1, \theta_2 \in$

$\{H, L\}, \theta_1 \neq \theta_2$.

The Firm's Pricing Decision

We now study the firm's optimization problem. We have seen how the consumers' demand to be determined, so we omit the demand consideration. See the strategic consumers' demand depicted by the two-sided arrow in the upper part and the myopic consumers' demand depicted by the two-sided arrow in the lower part in Figure 1. Note that a strategic consumer at point A has the same utility if she purchases a high quality product in each of the periods. Therefore, she might buy now or later. A myopic consumer at point B has the same utility if she purchases or not in the first period.

Now, the firm is able to anticipate the quantity demand by both the strategic and myopic consumers. Hence, The expected revenue from myopic consumer; demand in both periods is

$$Rev_m = \sum_{t \in \{1,2\}} \sum_{\theta \in \{H,L\}} q_{\theta t}^m \cdot p_{\theta t}.$$

The expected revenue from the strategic consumer; demand in both periods is

$$Rev_s = \sum_{t \in \{1,2\}} \sum_{\theta \in \{H,L\}} q_{\theta t}^s \cdot p_{\theta t}.$$

Hence, the total revenue expected to the firm in both periods is

$$\Pi = \sum_{t \in \{1,2\}} \sum_{\theta \in \{H,L\}} \left(\beta \cdot Rev_s + (1-\beta) \cdot Rev_m\right).$$

Finally, the firm is able to set the prices (i.e., p_{H1}, p_{H2}, p_{L1} and p_{L2}) over two periods by differentiating with respect to the price in each case.

Notation

We summarize the notation used in this dissertation. α_θ refers the quality product θ with

$0 \leq \alpha_L \leq \alpha_H \leq 1$, where $\theta \in \{H, L\}$

β is a share of strategic consumers with $0 \leq \beta \leq 1$

δ is the discounting factor of strategic consumers with $0 \leq \delta \leq 1$

i is the consumer location with $0 \leq i \leq 1$.

$u_{\theta t}$ is the consumer utility from product quality θ in the period t, where $\theta \in \{H, L\}, t \in \{1, 2\}$.

i_t^{**} is the intersection point of two utility functions, high and low quality in period t.

$i_{\theta t}^{*}$ is the i-intercept of the θ quality product utility function in period t.

$i_{\theta 1 \theta 2}^{*}$ is the intercept point of the two utility functions in 1st period and 2nd period.

$U_{i_t^{**}}$ is the consumer utility at i=i_t^{**}.

$U_{i_{\theta t}^{*}}$ is the consumer utility at i=$i_{\theta t}^{*}$.

$U_{i_{\theta_1 \theta_2}^{*}}$ is the consumer utility at i=$i_{\theta_1 \theta_2}^{*}$.

4 Model Analysis

In this section, we study the different scenarios that may occur based on the retailer's choice of qualities to offer in each of the periods and corresponding prices. We distinguish between three main broad groups of scenarios:

Case 1–in which the second period utility function dominates the first period utility function.

Case 2–in which the first period utility function dominates the second period utility function.

Case 3–in which the both period utility functions meet in the range.

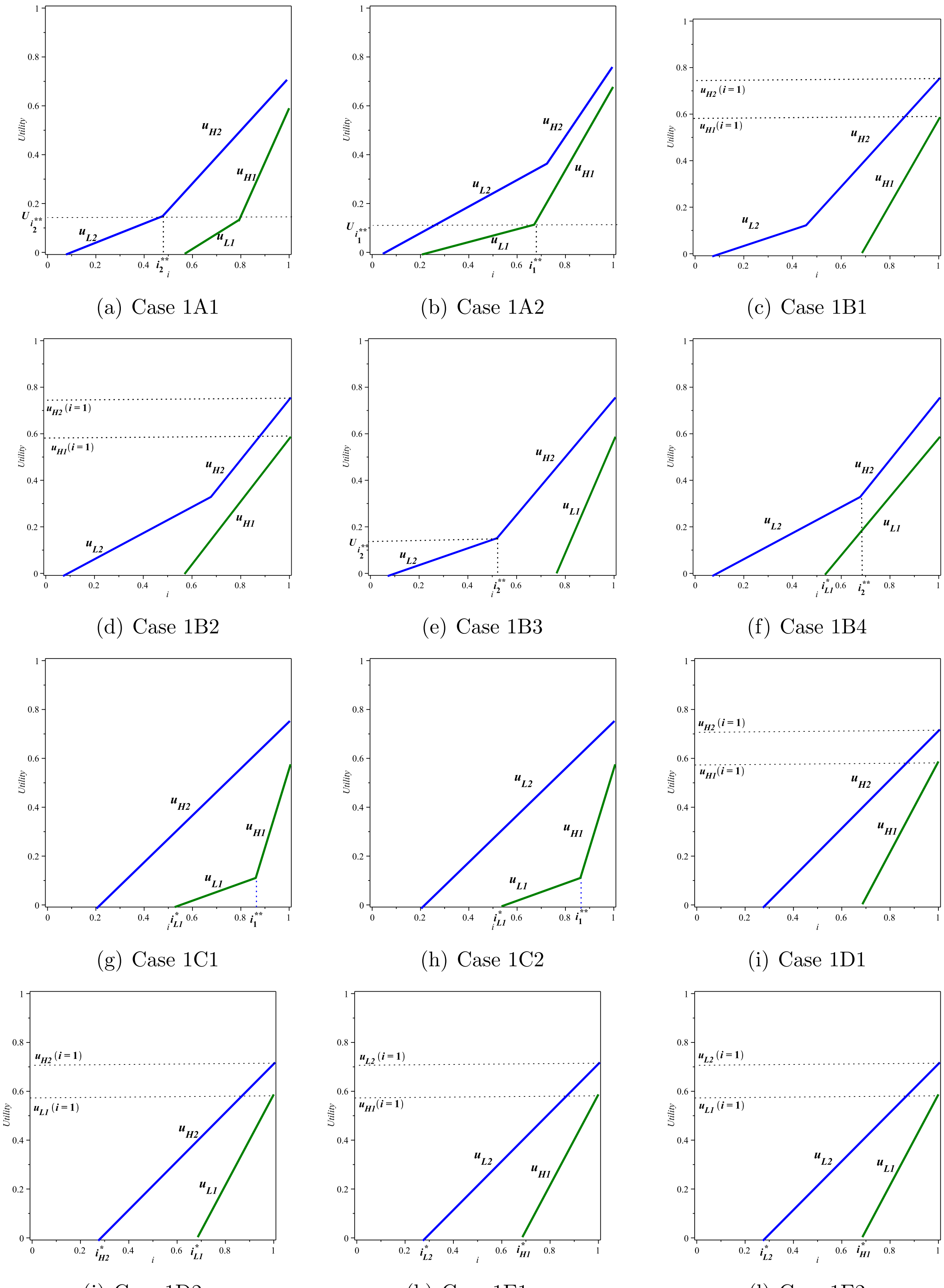
4.1 Scenario Analysis

The retailer can choose to offer only one quality or both qualities of the product in each period. Therefore, we identify the following cases: Case 1, Case 2 and Case 3. Next, we analyze each of cases separately, and consider the different scenarios each case entails. The different scenario described in Figure 2 are based on the potential relationships between the available products in period 1 and period 2, and the points where the utility has intersect. We solve and obtain the prices by assuming each case is feasible.

Case 1

In Case 1, we have several scenarios (see Figure 2). In all scenarios, the utility functions in the second period dominate those in the first period. This implies that all strategic consumers wait for the second period so none of them purchases in the first period. Intuitively, when this happens $max\{u_{H2}, u_{L2}\} > max\{u_{H1}, u_{L1}\}$. We find that only Case 1D2 is feasible. **Case 1A**

In this scenario, the retailer offers both high and low quality of the product in both periods. If the scenario is feasible then (i_2^{**},$U_{i_2^{**}}$) must exist in the first quadrant. In other words, $i_2^{**} > 0$ and $U_{i_2^{**}} > 0$.



(a) Case 1A1 (b) Case 1A2 (c) Case 1B1

(d) Case 1B2 (e) Case 1B3 (f) Case 1B4

(g) Case 1C1 (h) Case 1C2 (i) Case 1D1

(j) Case 1D2 (k) Case 1E1 (l) Case 1E2

Figure 2: Representative Instances for Case 1

Lemma 1 *Case 1A is infeasible.*

Proof. By contradiction.

Assume Case 1A1 is feasible. Then $U_{i_2^{**}} > 0$.

But, $U_{i_2^{**}}$

$= \frac{(p_{H2}-p_{L2})\alpha_H}{\alpha_H-\alpha_L} - p_{H2}$, substituting the optimal prices in this case

$=\left(\frac{\alpha_H\alpha_L\delta(\beta+\beta\delta+1+\delta)}{4\alpha_L+4\alpha_L\delta-\alpha_H\delta+\beta\delta\alpha_H} - \frac{\alpha_L^2\delta(\beta+\beta\delta+1+\delta)}{4\alpha_L+4\alpha_L\delta-\alpha_H\delta+\beta\delta\alpha_H}\right)\alpha_H(\alpha_H-\alpha_L)^{-1} - \frac{\alpha_H\alpha_L\delta(\beta+\beta\delta+1+\delta)}{4\alpha_L+4\alpha_L\delta-\alpha_H\delta+\beta\delta\alpha_H}$

$= \frac{(\alpha_H\alpha_L\delta(\beta+\beta\delta+1+\delta)-\alpha_L^2\delta(\beta+\beta\delta+1+\delta))\alpha_H(\alpha_H-\alpha_L)^{-1}-\alpha_H\alpha_L\delta(\beta+\beta\delta+1+\delta)}{4\alpha_L+4\alpha_L\delta-\alpha_H\delta+\beta\delta\alpha_H}$

$= \frac{\alpha_L\delta(\beta+\beta\delta+1+\delta)((\alpha_H-\alpha_L)\alpha_H(\alpha_H-\alpha_L)^{-1}-\alpha_H)}{4\alpha_L+4\alpha_L\delta-\alpha_H\delta+\beta\delta\alpha_H} = 0$, which violates the assumption that $U_{i_2^{**}}$ should be strictly positive. Hence, Case 1A1 is infeasible. ■

Case 1A2

In this scenario, the retailer offers both high and low quality of the product in both periods. If the scenario is feasible, then $(i_1^{**}, U_{i_1^{**}})$ must exist in the first quadrant. In other words, $i_1^{**} > 0$ and $U_{i_1^{**}} > 0$.

Lemma 2 *Case 1A2 is infeasible.*

Proof. By contradiction.

Assume Case 1A2 is feasible. Then $U_{i_1^{**}} > 0$.

But, in optimality $U_{i_1^{**}} = -\frac{1}{2}\frac{\alpha_L\delta(\delta+1+\beta\delta+\beta)}{4+\beta\,\delta+3\,\delta} < 0$, leading to a contradiction. Thus, Cases 1A2 is infeasible. ■

Case 1B1

In this scenario, the low quality product is not offered in the first period. Hence, whenever $u_{H2} > u_{H1}$, we must have $u_{i^*_{H2}} - u_{i^*_{H1}} \mid_{i=1} > 0$.

Lemma 3 *Case 1B1 is infeasible.*

Proof. By contradiction.

Assume Case 1B1 is feasible. Then $u_{i^*_{H2}} - u_{i^*_{H1}} \mid_{i=1} > 0$.

However, $u_{i^*_{H2}} - u_{i^*_{H1}} \mid_{i=1}$

$$= p_{H1} - p_{H2} - \alpha_H$$
$$= -\frac{\alpha_H\left(2-\delta^2+\beta\delta\right)}{4+3\delta+\beta\delta}$$
$$= -\frac{\alpha_H\left(2-\delta^2+\beta\delta\right)}{4+3\delta+\beta\delta} < 0 \,:\, \text{contradiction.}$$

Hence, the assumption $u_{H2} - u_{H1} \mid_{i=1} > 0$ does not hold and Case 1B1 is infeasible. ■

Case 1B2

In this scenario, the low quality product is not offered in the first period. Hence, whenever $u_{H2} > u_{H1}$, we must have $u_{i^*_{H2}} - u_{i^*_{H1}} \mid_{i=1} > 0$.

Lemma 4 *Case 1B2 is infeasible.*

Proof. By contradiction.

Assume Case 1B2 is feasible. Then $u_{H2} - u_{H1} \mid_{i=1} > 0$.

But, $u_{H2} - u_{H1} \mid_{i=1} = p_{H1} - p_{H2} - \alpha_H$

$$= -\frac{4\alpha_H^2\delta-3\alpha_L\delta^2\alpha_H+\beta\alpha_L\delta^2\alpha_H-4\alpha_L\delta\alpha_H+\alpha_L^2\delta^2-\beta\alpha_L^2\delta^2+2\beta\alpha_L\delta\alpha_H+4\alpha_H^2}{2(4\alpha_H+4\alpha_H\delta-\alpha_L\delta+\beta\alpha_L\delta)}$$
$$= -\frac{\left(4{\alpha_H}^2-4\alpha_L\delta\alpha_H\right)+\left({\alpha_L}^2\delta^2-\beta{\alpha_L}^2\delta^2\right)+\left(4\alpha_H^2\delta-3\alpha_L\delta^2\alpha_H\right)+\left(\beta\alpha_L\delta^2\alpha_H+2\beta\alpha_L\delta\alpha_H\right)}{2(4\alpha_H+3\alpha_H\delta+(\alpha_H-\alpha_L)\delta+\beta\alpha_L\delta)}$$
$$= -\frac{4\alpha_H(\alpha_H-\alpha_L\delta)+(1-\beta){\alpha_L}^2\delta^2+\left(3{\alpha_H}^2\delta-3\alpha_L\delta^2\alpha_H\right)+{\alpha_H}^2\delta+\beta\alpha_L\delta^2\alpha_H+2\beta\alpha_L\delta\alpha_H}{2(4\alpha_H+3\alpha_H\delta+(\alpha_H-\alpha_L)\delta+\beta\alpha_L\delta)}$$
$$= -\frac{4\alpha_H(\alpha_H-\alpha_L\delta)+(1-\beta){\alpha_L}^2\delta^2+3\alpha_H\delta(\alpha_H-\alpha_L)+{\alpha_H}^2\delta+\beta\alpha_L\delta^2\alpha_H+2\beta\alpha_L\delta\alpha_H}{2(4\alpha_H+3\alpha_H\delta+(\alpha_H-\alpha_L)\delta+\beta\alpha_L\delta)} < 0 \,:\, \text{contradiction.}$$

This violates the assumption that u_{H2} dominates and , hence, Case 1B2 is infeasible. ■

Case 1B3

In this scenario, the high quality product is not offered in the first period. Hence, whenever $u_{H2} > u_{L1}$, we must have $u_{i_2^{**}} > 0$.

Lemma 5 *Case 1B3 is infeasible.*

Proof. By contradiction.

Let Case 1B3 is feasible. Then $U_{i_2^{**}} > 0$.

But, $U_{i_2^{**}} = \frac{-\alpha_H p_{L2}+p_{H2}\alpha_L}{\alpha_H-\alpha_L} = \frac{-\alpha_H\left(\frac{\alpha_L^2\delta(\beta+\beta\delta+1+\delta)}{-\alpha_H\delta+\beta\delta\alpha_H+4\alpha_L+4\alpha_L\delta}\right)+\left(\frac{\alpha_H\alpha_L\delta(\beta+\beta\delta+1+\delta)}{-\alpha_H\delta+\beta\delta\alpha_H+4\alpha_L+4\alpha_L\delta}\right)\alpha_L}{\alpha_H-\alpha_L} = 0$, contradiction.

Thus, we conclude that Case 1B3 is infeasible. ■

Case 1B4

In this scenario, the high quality product is not offered in the first period. Hence, whenever $u_{H2} > u_{L1}$, we must have $i_{L1}^* < i_2^{**}$.

Lemma 6 *Case 1B4 is infeasible.*

Proof. By a contradiction.

Assume $i_{L1}^* < i_2^{**}$. Then $i_{L1}^* - i_2^{**} < 0$.

However, $i_{L1}^* - i_2^{**}$

$= \frac{p_{L1}}{\alpha L(1+\delta)} - \frac{p_{H2}-pL2}{\delta(\alpha_H-\alpha_L)}$

$= \frac{\frac{\left(2+\beta\delta^2+4\delta+\beta\delta+2\delta^2\right)\alpha L}{4+3\delta+\beta\delta}}{\alpha L(1+\delta)} - \frac{\left(\frac{1}{2}\frac{\delta\left(-2\alpha_L+2\beta\alpha_L+4\alpha_H+3\alpha_H\delta+\beta\delta\alpha_H-\alpha_L\delta+\beta\alpha_L\delta\right)}{4+3\delta+\beta\delta}\right)-\left(\frac{\alpha_L\delta(\beta\delta+1+\beta+\delta)}{4+3\delta+\beta\delta}\right)}{\delta(\alpha_H-\alpha_L)}$

$= \frac{2+\beta\delta^2+4\delta+\beta\delta+2\delta^2}{(4+3\delta+\beta\delta)(1+\delta)} - \frac{1}{2}$

$= \frac{1}{2}\frac{\delta(\beta+1)}{4+3\delta+\beta\delta} > 0$, which violates the assumption that $i_{L1}^* - i_2^{**} < 0$.

Therefore, Case 1B4 is infeasible. ■

Case 1C1

In this scenario, the low quality product is not offered in the second period. Hence, whenever $u_{H2} > u_{H1}$, we must have $0 < i^*_{L1} < i^{**}_1 < 1$

Lemma 7 *Case 1C1 is infeasible.*

Proof. Showing that CASE 1C is not feasible.

Assumption 1: $i^*_{L1} > 0$

Claim that $i^*_{L1} > 0$

$i^*_{L1} = \frac{\beta\delta+\beta\delta^2+4\alpha_L\delta+2\alpha_L+2\alpha_L\delta^2}{(4\alpha_L+4\alpha_L\delta-\delta+\beta\delta)(1+\delta)} > 0$

$\iff \frac{(\beta\delta+2\alpha_L\delta+2\alpha_L)(1+\delta)}{(4\alpha_L+4\alpha_L\delta-\delta+\beta\delta)(1+\delta)} > 0$

$\iff \frac{(\beta\delta+2\alpha_L\delta+2\alpha_L)}{(4\alpha_L+4\alpha_L\delta-\delta+\beta\delta)} > 0$

$\iff 4\alpha_L + 4\alpha_L\delta - \delta + \beta\delta > 0$

$\iff \alpha_L > \frac{\delta(1-\beta)}{4(1+\delta)}$

Therefore, the assumption 1 holds if and only if $\alpha_L > \frac{\delta(1-\beta)}{4(1+\delta)}$.

Assumption 2: $i^*_{L1} < i^{**}_1$

Claim that $i^*_{L1} < i^{**}_1$

$\iff \frac{p_{L1}\alpha_L}{(1+\delta)} - \frac{p_{H1}-p_{L1}}{(\alpha_H-\alpha_L)(1+\delta)} < 0$

$\iff \frac{1}{2}\frac{(\beta+1)\alpha_H\,\delta}{4\,\alpha_L\,\delta+4\,\alpha_L-\alpha_H\,\delta+\beta\,\alpha_H\,\delta} < 0$

$\iff 4\alpha_L\delta + 4\alpha_L - \delta + \beta\delta < 0$, By substituting $\alpha_H = 1$

$\iff \alpha_L < \frac{\delta(1-\beta)}{4(1+\delta)}$ Thus, the assumption 2 holds iff $\alpha_L < \frac{\delta(1-\beta)}{4(1+\delta)}$.

From, assumption 1 and 2, Case 1C1 is feasible iff $\alpha_L < \frac{\delta(1-\beta)}{4(1+\delta)}$ and $\alpha_L > \frac{\delta(1-\beta)}{4(1+\delta)}$. This implies that there is no value of α_L for which Case 1C1 is feasible. Thus, We conclude that Case 1C1 is infeasible. ■

Case 1C2

In this scenario, the high quality product is not offered in the second period. Hence, whenever $u_{L2} > u_{H1}$, we must have $i^*_{L1} < i^{**}_1$.

Lemma 8 *Case 1C2 is infeasible.*

Proof. By contradiction.

Assume Case 1C2 is feasible. Then $i^*_{L1} < i^{**}_1$.

But, $i^*_{L1} - i^{**}_1 = \frac{1}{2}\frac{(\beta+1)\delta}{3\,\delta+4+\beta\,\delta} > 0$, a contradiction. Hence, Case 1C2 is infeasible. ■

Case 1D1

In this scenario, the retailer offers the high quality product in both periods. Hence, whenever $u_{H2} > u_{H1}$, we must have $u_{i^*_{H2}} - u_{i^*_{H1}} \mid_{i=1} > 0$.

Lemma 9 *Case 1D1 is infeasible.*

Proof. By contradiction.

Assume Case 1D1 is feasible. Then $u_{i^*_{H2}} - u_{i^*_{H1}} \mid_{i=1} > 0$.

But, $u_{i^*_{H2}} - u_{i^*_{H1}} \mid_{i=1}$

$= p_{H1} - p_{H2} - \alpha_H$

$= -\frac{\alpha_H\left(2-\delta^2+\beta\delta\right)}{4+3\delta+\beta\delta}$

$= -\frac{\alpha_H\left(2-\delta^2+\beta\delta\right)}{4+3\delta+\beta\delta} < 0:$ contradiction.

Hence, the assumption $u_{H2} - u_{H1} \mid_{i=1} > 0$ does not hold and Case 1D1 is infeasible. ■

Case 1D2

In this scenario, the retailer offers the low quality product in the first period and the high quality product in the second period. Hence, whenever $u_{H2} > u_{L1}$, we must have $0 < i^*_{H2} < i^*_{L1} < 1, u_{H2} - u_{L1}|_{(i=1)} > 0$.

Lemma 10 *Case 1D2 is feasible if and only if* $\alpha_L > \frac{1}{2}\frac{\alpha_H\delta}{1+\delta}$.

Proof. We test all assumptions.

Assumption 1: $i^*_{H2} > 0$

$i^*_{H2} = \frac{p_{H2}}{\alpha_H\delta} > 0$, which is always true.

Assumption 2: $u_{H2} - u_{L1}|_{(i=1)} > 0$

$u_{H2} - u_{L1}|_{(i=1)} > 0 \iff \alpha_H\delta - p_{H2} > \alpha_L(1+\delta) - p_{L1}$

$p_{H2} < \alpha_H\delta + p_{L1} - \alpha_L(1+\delta) < \alpha_H\delta + \alpha_H(1+\delta) - \alpha_L(1+\delta)$

$= \alpha_H\delta + (\alpha_H - \alpha_L)(1+\delta)$, which is always true since $p_{H2} < \alpha_H\delta$.

Assumption 3: $i^*_{H2} < i^*_{L1}$

Claim that $i^*_{H2} - i^*_{L1} < 0$

$\iff \frac{p_{H2}}{\alpha_H\delta} - \frac{p_{L1}}{\alpha_L(1+\delta)} < 0$

$\iff \frac{\alpha_L(\beta+\beta\delta+1+\delta)}{4\alpha_L\delta+4\alpha_L-\alpha_H\delta+\beta\alpha_H\delta} - \frac{\beta\alpha_H\delta+\beta\alpha_H\delta^2+2\alpha_L\delta^2+2\alpha_L+4\alpha_L\delta}{(4\alpha_L\delta+4\alpha_L-\alpha_H\delta+\beta\alpha_H\delta)(1+\delta)} < 0$

$\iff -\frac{\alpha_L\delta-\beta\alpha_L\delta+\beta\alpha_H\delta-\beta\alpha_L+\alpha_L}{4\alpha_L\delta+4\alpha_L-\alpha_H\delta+\beta\alpha_H\delta} < 0$

$\iff \frac{\alpha_L\delta-\beta\alpha_L\delta+\beta\alpha_H\delta-\beta\alpha_L+\alpha_L}{4\alpha_L\delta+4\alpha_L-\alpha_H\delta+\beta\alpha_H\delta} > 0.$

Now, consider two cases:

(1) The numerator > 0 and the denominator > 0

$\iff \alpha_L > \frac{1}{4}\frac{\alpha_H\delta(1-\beta)}{1+\delta}$ and $\alpha_L > \frac{\beta\alpha_H\delta}{-\delta+\beta\delta+\beta-1}$

Have to check if $\frac{1}{4}\frac{\alpha_H\delta(1-\beta)}{1+\delta} > \frac{\beta\alpha_H\delta}{-\delta+\beta\delta+\beta-1}$

$\frac{1}{4}\frac{\alpha_H\delta(1-\beta)}{1+\delta} - \frac{\beta\alpha_H\delta}{-\delta+\beta\delta+\beta-1} = -\frac{1}{4}\frac{\left(1+2\beta+\beta^2\right)\alpha_H\delta}{-\delta+\beta\delta+\beta-1} = \frac{1}{4}\frac{(\beta+1)^2\alpha H\delta}{(1+\delta)(1-\beta)} > 0$

Thus, $\frac{1}{4}\frac{\alpha_H\delta(1-\beta)}{1+\delta} > \frac{\beta\alpha_H\delta}{-\delta+\beta\delta+\beta-1}$

Hence, $\alpha_L > \frac{1}{4}\frac{\alpha_H\delta(1-\beta)}{1+\delta}$.

(2) The numerator < 0 and the denominator < 0

$\iff \alpha_L < \frac{1}{4}\frac{\alpha_H\delta(1-\beta)}{1+\delta}$ and $\alpha_L < \frac{\beta\alpha_H\delta}{-\delta+\beta\delta+\beta-1}$

Hence, $\alpha_L < \frac{\beta\alpha_H\delta}{-\delta+\beta\delta+\beta-1}$

However, $\alpha_L < \frac{\beta\alpha_H\delta}{-\delta+\beta\delta+\beta-1} = \frac{\beta\alpha_H\delta}{(\delta+1)(\beta-1)} < 0$, which violates the assumption $0 < \alpha_L < 1$

thus, we do not need to consider this case.

Therefore, from (1) and (2) assumption 3 holds $\iff \alpha_L > \frac{1}{4}\frac{\alpha_H\delta(1-\beta)}{1+\delta}$.

Assumption 4: $i^*_{L1} < 1$

Claim that $i^*_{L1} - 1 < 0$

$\iff i^*_{L1} - 1 = \frac{p_{L1}}{\alpha_L(1+\delta)} - 1 = \frac{-2\alpha_L\delta+\alpha_H\delta-2\alpha_L}{4\alpha_L\delta+4\alpha_L-\alpha_H\delta+\beta\alpha_H\delta} < 0$

Now, consider two cases:

(1) The numerator > 0 and the denominator < 0

$\iff \alpha_L < \frac{1}{2}\frac{\alpha_H\delta}{1+\delta}$ and $\alpha_L < \frac{1}{4}\frac{\alpha_H\delta(1-\beta)}{1+\delta}$.

Have to check if $\frac{1}{2}\frac{\alpha_H\delta}{1+\delta} > \frac{1}{4}\frac{\alpha_H\delta(1-\beta)}{1+\delta}$.

$\frac{1}{2}\frac{\alpha_H\delta}{1+\delta} - \frac{1}{4}\frac{\alpha_H\delta(1-\beta)}{1+\delta} = \frac{1}{4}\frac{\alpha_H\delta(\beta+1)}{1+\delta} > 0$

Hence, $\frac{1}{2}\frac{\alpha_H\delta}{1+\delta} > \frac{1}{4}\frac{\alpha_H\delta(1-\beta)}{1+\delta}$.

Hence, $\alpha_L < \frac{1}{4}\frac{\alpha_H\delta(1-\beta)}{1+\delta}$

(2) The numerator < 0 and the denominator > 0

$\iff \alpha_L > \frac{1}{2}\frac{\alpha_H\delta}{1+\delta}$ and $\alpha_L > \frac{1}{4}\frac{\alpha_H\delta(1-\beta)}{1+\delta}$.

Hence, $\alpha_L > \frac{1}{2}\frac{\alpha_H\delta}{1+\delta}$

Therefore, from (1) and (2) assumption 4 holds $\iff \alpha_L < \frac{1}{4}\frac{\alpha_H\delta(1-\beta)}{1+\delta}$ or $\alpha_L > \frac{1}{2}\frac{\alpha_H\delta}{1+\delta}$.

From assumption 1,2,3, and 4, Case 1D2 is feasible if and only if $\alpha_L > \frac{1}{2}\frac{\alpha_H\delta}{1+\delta}$. ■

Case 1E1

In this scenario, the retailer offers the high quality product in the first period and the low quality product in the second period. Hence, whenever $u_{L2} > u_{LH}$, we must have $i^*_{L2} > 0, u_{L2} - u_{H1}|_{(i=1)} > 0$, $i^*_{L2} < i^*_{H1}$, and $i^*_{H1} < 1$.

Lemma 11 *Case 1E1 is infeasible.*

Proof. By contradiction.

Assumption 1: $i^*_{L2} > 0$

$i^*_{L2} = \frac{p_{L2}}{\alpha_L\delta} > 0$, which is always true.

Assumption 2: $u_{L2} - u_{H1}|_{(i=1)} > 0$

$\iff (\alpha_L\delta - p_{L2}) - (\alpha_H(1+\delta) - p_{H1}) > 0$

$\iff \alpha_L\delta - \frac{\alpha_L\alpha_H\delta(\beta+\beta\delta+1+\delta)}{4\alpha_H\delta+4\alpha_H-\alpha_L\delta+\beta\alpha_L\delta} - \left(\alpha_H(1+\delta) - \frac{\alpha_H\left(\beta\alpha_L\delta+\beta\alpha_L\delta^2+2\alpha_H\delta^2+2\alpha_H+4\alpha_H\delta\right)}{4\alpha_H\delta+4\alpha_H-\alpha_L\delta+\beta\alpha_L\delta}\right) > 0$

$\iff \frac{4\alpha_L\delta^2\alpha_H+4\alpha_L\alpha_H\delta-\alpha_L{}^2\delta^2+\alpha_L{}^2\delta^2\beta-\beta\alpha_H\alpha_L\delta-\beta\alpha_H\alpha_L\delta^2-4\alpha_H{}^2\delta-2\alpha_H{}^2\delta^2-2\alpha_H{}^2}{4\alpha_H\delta+4\alpha_H-\alpha_L\delta+\beta\alpha_L\delta} > 0$

Note first, by substituting $\alpha_H = 1$, the numerator becomes:

$4\alpha_L\delta^2 + 4\alpha_L\delta - \alpha_L{}^2\delta^2 + \alpha_L{}^2\delta^2\beta - \beta\alpha_L\delta - \beta\alpha_L\delta^2 - 4\delta - 2\delta^2 - 2$

$= (2\alpha_L\delta^2 - 2\delta^2) + (2\alpha_L\delta^2 - 2) + (-4\delta + 4\alpha_L\delta) - \alpha_L{}^2\delta^2 + (\alpha_L{}^2\delta^2\beta - \beta\alpha_L\delta) - \beta\alpha_L\delta^2$

$= -2\,(1-\alpha_L) - 2(1-\alpha_L\delta^2) - 4\delta(1-\alpha_L) - \alpha_L{}^2\delta^2 - (1-\alpha_L\delta)\beta\alpha_L\delta - \beta\alpha_L\delta^2 < 0$

Now, we need the condition: the denominator < 0;

$\iff 4\alpha_H\delta + 4\alpha_H - \alpha_L\delta + \beta\alpha_L\delta < 0$

$\iff \alpha_L > \frac{4\alpha_H(1+\delta)}{\delta(1-\beta)}$

Hence, the assumption 2 holds iff $\alpha_L > \frac{4\alpha_H(1+\delta)}{\delta(1-\beta)}$.

Assumption 3: $i^*_{L2} < i^*_{H1}$

$\iff \frac{p_{L2}}{\alpha_L\delta} - \frac{p_{H1}}{\alpha_H(1+\delta)} < 0$

$\iff \frac{\alpha_H(\beta+\beta\delta+1+\delta)}{4\alpha_H\delta+4\alpha_H-\alpha_L\delta+\beta\alpha_L\delta} - \frac{\beta\alpha_L\delta+\beta\alpha_L\delta^2+2\alpha_H\delta^2+2\alpha_H+4\alpha_H\delta}{(4\alpha_H\delta+4\alpha_H-\alpha_L\delta+\beta\alpha_L\delta)(1+\delta)} < 0$

$\iff \frac{-\alpha_H\delta+\alpha_H\beta\delta-\beta\alpha_L\delta+\alpha_H\beta-\alpha_H}{4\alpha_H\delta+4\alpha_H-\alpha_L\delta+\beta\alpha_L\delta} < 0$

Note the numerator: $= -(1-\beta)\alpha_H\delta - \beta\alpha_L\delta - (1-\beta)\alpha_H < 0$

Now, we need the condition: the denominator > 0;

$\iff 4\alpha_H\delta + 4\alpha_H - \alpha_L\delta + \beta\alpha_L\delta > 0$

$\iff \alpha_L < \frac{4\alpha_H(1+\delta)}{\delta(1-\beta)}$

Hence, the assumption 3 holds iff $\alpha_L < \frac{4\alpha_H(1+\delta)}{\delta(1-\beta)}$.

Assumption 4: $i^*_{H1} < 1$

$\iff \frac{p_{H1}}{\alpha_H(1+\delta)} < 1 \iff p_{H1} < \alpha_H(1+\delta)$, which is always true.

From the assumptions 1,2,3 and 4, Case 1E1 is feasible if and only if $\alpha_L > \frac{4\alpha_H(1+\delta)}{\delta(1-\beta)}$ and $\alpha_L < \frac{4\alpha_H(1+\delta)}{\delta(1-\beta)}$. This implies that there is no value of α_L for which Case 1E1 is feasible. Therefore, we conclude that Case 1E1 is infeasible. ■

Case 1E2

In this scenario, the retailer offers the low quality product in both periods. Hence, whenever $u_{L2} > u_{L1}$, we must have $u_{L2} - u_{L1}|_{(i=1)} > 0$.

Lemma 12 *Case 1E2 is infeasible.*

Proof. By a contradiction.

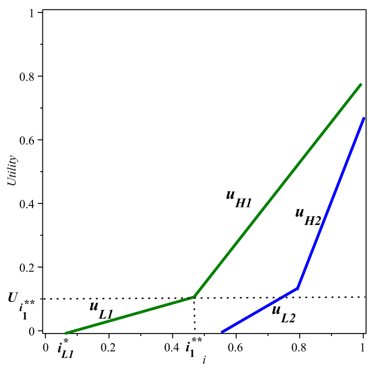
Assume Case 1E2 is feasible. Then $u_{L2} - u_{L1}|_{(i=1)} > 0$. But,

$$u_{L2} - u_{L1}|_{(i=1)} = (\alpha_L\delta - p_{L2}) - (\alpha_L(1+\delta) - p_{L1})$$
$$= \left(\alpha_L\delta - \frac{\alpha_L\delta(\beta+\beta\delta+\delta+1)}{3\delta+4+\beta\delta}\right) - \left(\alpha_L(1+\delta) - \frac{\alpha_L\left(\beta\delta+\beta\delta^2+2\delta^2+2+4\delta\right)}{3\delta+4+\beta\delta}\right)$$
$$= -\frac{\alpha_L\left(-\delta^2+\beta\delta+2\right)}{3\delta+4+\beta\delta} < 0, \text{ since } (-\delta^2+2)+\beta\delta > 0$$

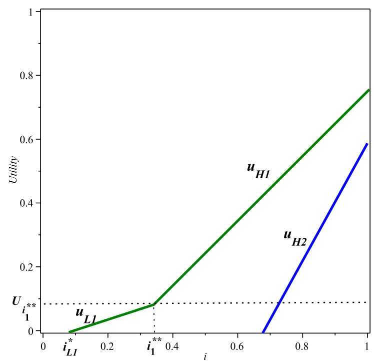
This leads to a contradiction. Therefore, Case 1E2 is infeasible. ■

Case 2

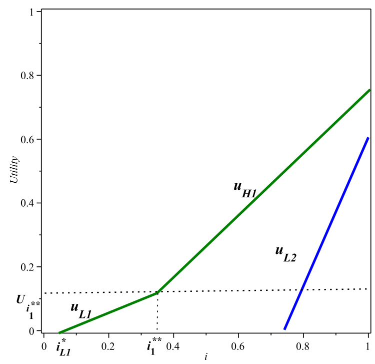
In Case 2 we have several scenarios (see Figure 3). In all scenarios, products in the first period are always preferred over those in the second period. These cases are inferior to other cases. In each case of the possible scenarios in Case 2, we must have $u_{\theta 1} > u_{\theta 2}$, where $\theta \in \{H, L\}$. This implies that strategic consumers purchase the products in the first period. Hence, strategic and myopic consumers behave in the same way, as they all purchase in the first period, if they purchase at all. This leads all consumers purchase in the first period and the firm would not offer the products in the second period. Consequently, this violates our model assumption (two period time horizon) and makes it impossible to obtain the prices in the second period. (i.e., p_{H2} and p_{L2}). Therefore, we cannot prove scenarios. The major proofs are following in two group:



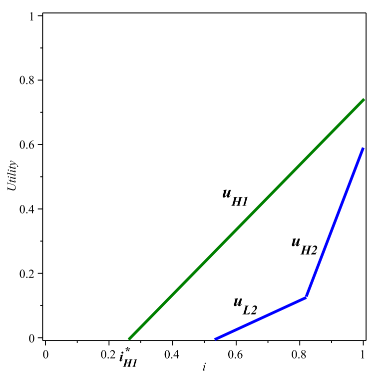
(a) Case 2A



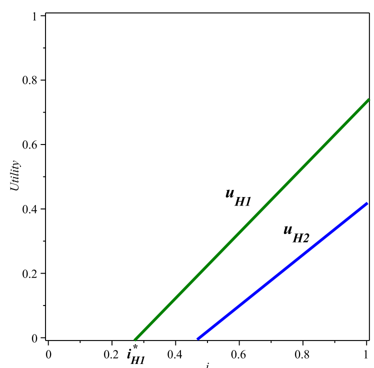
(b) Case 2B1



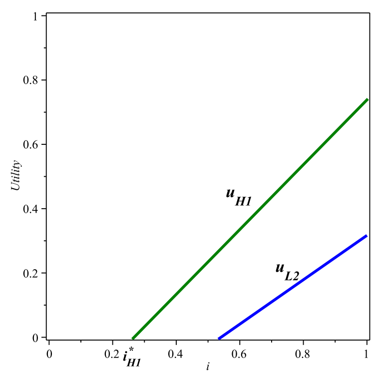
(c) Case 2B2



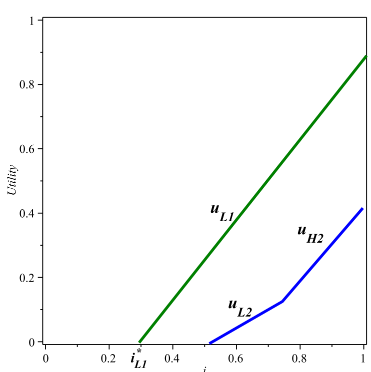
(d) Case 2C1



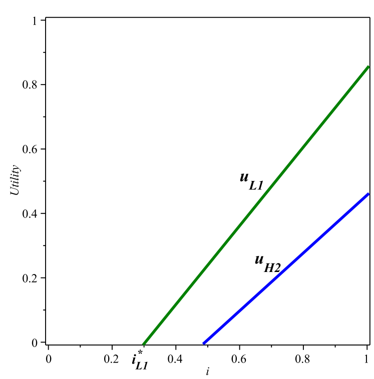
(e) Case 2C2



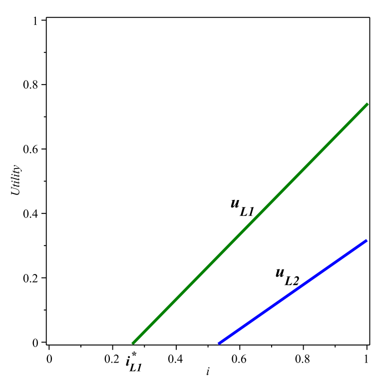
(f) Case 2C3



(g) Case 2D1



(h) Case 2D2



(i) Case 2D3

Figure 3: Representative Instances for Case 2

Case 2A and 2B

Lemma 13 *Case 2A and Case 2B are infeasible.*

Proof. By contradiction.

Assume CASE 2A and CASE 2B are feasible. Then $U_{i_1^{**}} > 0$.

However, $U_{i_1^{**}}$

$= \frac{(p_{H1}-p_{L1})\alpha_H}{\alpha_H-\alpha_L} - p_{H1}$

$= \frac{(p_{H1}-p_{L1})\alpha_H-(\alpha_H-\alpha_L)p_{H1}}{\alpha_H-\alpha_L}$

$= \frac{p_{H1}\alpha_L-p_{L1}\alpha_H}{\alpha_H-\alpha_L}$

substituting optimal prices in these scenarios $p_{H1} = \frac{\alpha_H(1+\delta)}{2}$ and $p_{L1} = \frac{\alpha_L(1+\delta)}{2}$

$= \frac{\frac{\alpha_H(1+\delta)}{2}\alpha_L-\frac{\alpha_L(1+\delta)}{2}\alpha_H}{\alpha_H-\alpha_L}$

$= \frac{\frac{\alpha_H\alpha_L(1+\delta)}{2}-\frac{\alpha_L\alpha_H(1+\delta)}{2}}{\alpha_H-\alpha_L} = 0$, which violates the assumption that $U_{i_1^{**}}$ should be strictly positive.

Therefore, we conclude that Case 2A and Case 2B are infeasible. ■ **Case 2C and 2D**

Lemma 14 *Case 2C and Case 2D are infeasible.*

Proof. By contradiction.

Assume CASE 2C and CASE 2D are feasible. Then $u_{H1} - u_{H2}|_{(i=1)} > 0$.

But, $u_{H1} - u_{H2}|_{(i=1)} = \alpha_H(1+\delta) - p_{H1} - (\alpha_H\delta - p_{H2})$

$= \alpha_H > p_{H1} - p_{H2}$, this can not be shown if whether it is true or not since p_{H2} cane not be determined in this cases. ■

Case 3

In Case 3, we enumerate several scenarios (see Figure 4). In all scenarios, four consumer utility functions meet each other in the range (i.e., $0 < i < 1$). We test feasibility in each case. We find that only Case 3D, Case 3E, Case 3F and Case 3G are feasible.

Case 3A1

In this scenario, u_{H1} and u_{H2} meet in the range. If this is feasible, we must have $U_{i_2^{**}} > 0$.

Lemma 15 *Case 3A1 is infeasible.*

Proof. By contradiction.

Assume Case 3A1 is feasible. Then $U_{i_2^{**}} > 0$.

However, $U_{i_2^{**}}$

$= \frac{-\alpha_H p_{L2} + p_{H2}\alpha_L}{\alpha_H - \alpha_L}$

$= \frac{-\alpha_H (pL2)}{\alpha_H - \alpha_L} + \frac{\alpha_L (pH2)}{\alpha_H - \alpha_L}$

$= -\alpha_H \left(\frac{{\alpha_L}^2 \delta \left(1+\beta+3\beta\delta+2\beta\delta^2+\delta\right)}{4\alpha_L\delta - \alpha_H\delta + 5\beta\alpha_L\delta + 5\beta\alpha_L\delta^2 + \beta\alpha_H\delta - \beta\alpha_H\delta^2 + \beta^2\delta^2\alpha_H - \beta^2\delta^2\alpha_L - \beta^2\delta\alpha_L + 4\alpha_L} \right) (\alpha_H - \alpha_L)^{-1}$

$+\alpha_L \left(\frac{\alpha_H\alpha_L \delta \left(1+\beta+3\beta\delta+2\beta\delta^2+\delta\right)}{4\alpha_L\delta - \alpha_H\delta + 5\beta\alpha_L\delta + 5\beta\alpha_L\delta^2 + \beta\alpha_H\delta - \beta\alpha_H\delta^2 + \beta^2\delta^2\alpha_H - \beta^2\delta^2\alpha_L - \beta^2\delta\alpha_L + 4\alpha_L} \right) (\alpha_H - \alpha_L)^{-1}$

$= -\alpha_H \frac{\alpha_L^2 \delta X}{\alpha_H - \alpha_L} + \alpha_L \frac{\alpha_H \alpha_L \delta X}{\alpha_H - \alpha_L}$

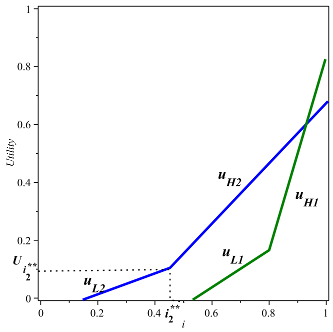
$= \frac{-\alpha_H \alpha_L^2 \delta}{\alpha_H - \alpha_L} X + \frac{\alpha_H \alpha_L^2 \delta}{\alpha_H - \alpha_L} X = 0$:contradiction

,where $X = \frac{1+\beta+3\beta\delta+2\beta\delta^2+\delta}{4\alpha_L\delta - \alpha_H\delta + 5\beta\alpha_L\delta + 5\beta\alpha_L\delta^2 + \beta\alpha_H\delta - \beta\alpha_H\delta^2 + \beta^2\delta^2\alpha_H - \beta^2\delta^2\alpha_L - \beta^2\delta\alpha_L + 4\alpha_L}$.

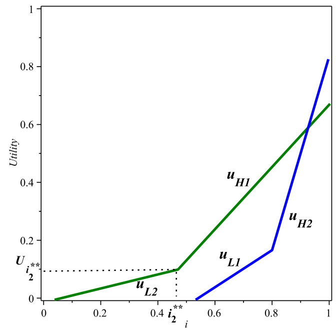
Hence, we conclude that Case 3A1 is infeasible. ■

Case 3A2

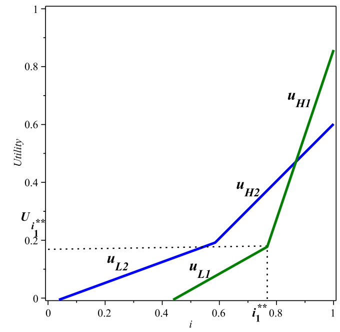
In this scenario, u_{H1} and u_{H2} meet in the range. If Case 3A2 is feasible we must have $U_{i_2^{**}} > 0$.



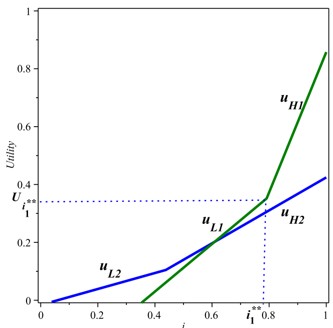
(a) Case 3A1



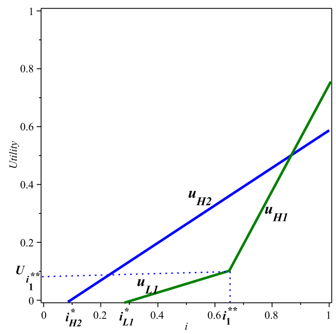
(b) Case 3A2



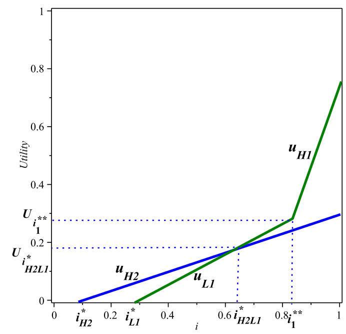
(c) Case 3B1



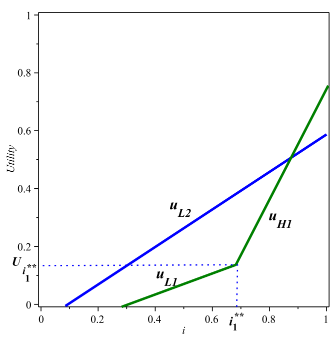
(d) Case 3B2



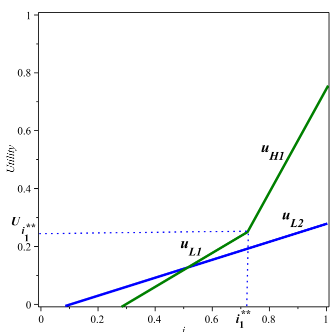
(e) Case 3C1



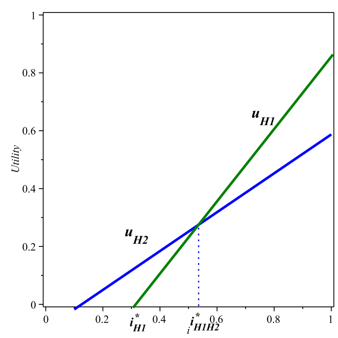
(f) Case 3C2



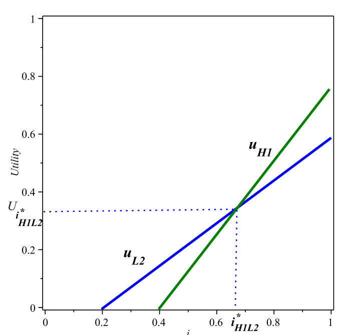
(g) Case 3C3



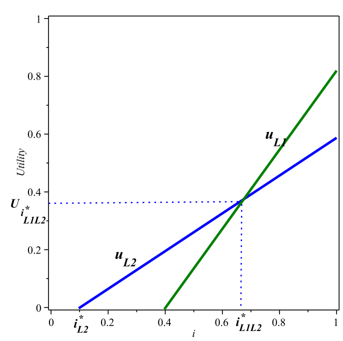
(h) Case 3C4



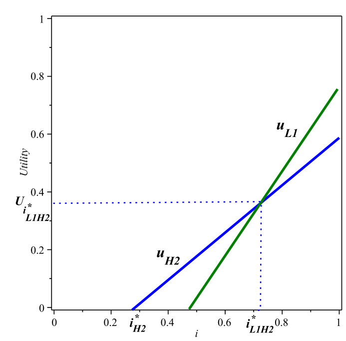
(i) Case 3D



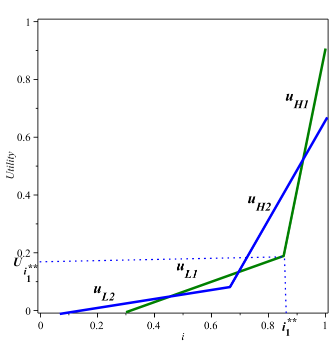
(j) Case 3E

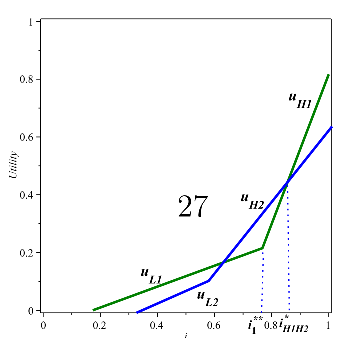


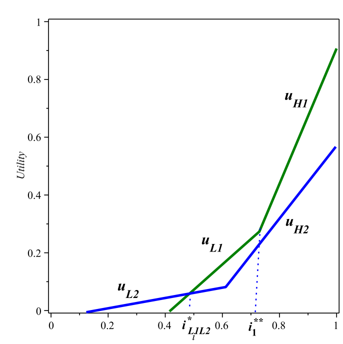
(k) Case 3F



(l) Case 3G







Lemma 16 *Case 3A2 is infeasible.*

Proof. By contradiction.

Assume Case 3A2 is feasible. Then $U_{i_2^{**}} > 0$.

However, $U_{i_2^{**}}$

$= \frac{-\alpha_H p_{L2} + p_{H2}\alpha_L}{\alpha_H - \alpha_L}$

$= \frac{-\alpha_H (pL2)}{\alpha_H - \alpha_L} + \frac{\alpha_L (pH2)}{\alpha_H - \alpha_L}$

$= \frac{\left(\frac{-\alpha_H \alpha_L^2 \delta\left(-\delta^2\alpha_H+\beta\delta^2\alpha_H+\delta^2\alpha_L+\beta\delta^2\alpha_L-\alpha_H\delta+\beta\alpha_L+2\alpha_L\delta+\beta\alpha_H\delta+\alpha_L+2\beta\alpha_L\delta\right)}{\delta^2\alpha_H^2-2\beta\delta^2\alpha_H{}^2+\delta^2\alpha_H{}^2\beta^2+6\beta\delta^2\alpha_L\alpha_H-\delta^2\alpha_H\beta^2\alpha_L-5\delta^2\alpha_L\alpha_H+4\delta^2\alpha_L^2+6\beta\alpha_L\alpha_H\delta-\delta\alpha_H\beta^2\alpha_L-5\alpha_L\alpha_H\delta+8\alpha_L^2\delta+4\alpha_L^2}\right)}{\alpha_H-\alpha_L}$

$+ \frac{\left(\frac{\alpha_L^2 \alpha_H \delta\left(-\delta^2\alpha_H+\beta\delta^2\alpha_H+\delta^2\alpha_L+\beta\delta^2\alpha_L-\alpha_H\delta+\beta\alpha_L+2\alpha_L\delta+\beta\alpha_H\delta+\alpha_L+2\beta\alpha_L\delta\right)}{\delta^2\alpha_H{}^2-2\beta\delta^2\alpha_H{}^2+\delta^2\alpha_H{}^2\beta^2+6\beta\delta^2\alpha_L\alpha_H-\delta^2\alpha_H\beta^2\alpha_L-5\delta^2\alpha_L\alpha_H+4\delta^2\alpha_L^2+6\beta\alpha_L\alpha_H\delta-\delta\alpha_H\beta^2\alpha_L-5\alpha_L\alpha_H\delta+8\alpha_L^2\delta+4\alpha_L^2}\right)}{\alpha_H-\alpha_L}$

$= \frac{-\alpha_H\alpha_L^2 X}{\alpha_H-\alpha_L} + \frac{\alpha_L^2\alpha_H X}{\alpha_H-\alpha_L} = 0 :$ contradiction,

where $X = \frac{\delta\left(-\delta^2\alpha_H+\beta\delta^2\alpha_H+\delta^2\alpha_L+\beta\delta^2\alpha_L-\alpha_H\delta+\beta\alpha_L+2\alpha_L\delta+\beta\alpha_H\delta+\alpha_L+2\beta\alpha_L\delta\right)}{\delta^2\alpha_H^2-2\beta\delta^2\alpha_H{}^2+\delta^2\alpha_H{}^2\beta^2+6\beta\delta^2\alpha_L\alpha_H-\delta^2\alpha_H\beta^2\alpha_L-5\delta^2\alpha_L\alpha_H+4\delta^2\alpha_L^2+6\beta\alpha_L\alpha_H\delta-\delta\alpha_H\beta^2\alpha_L-5\alpha_L\alpha_H\delta+8\alpha_L^2\delta+4\alpha_L^2}$

Hence, we conclude that Case 3A2 is infeasible. ■

Case 3B1

In this scenario, u_{H1} and u_{H2} meet in the range. If this is feasible, we must have $U_{i_1^{**}} > 0$.

Lemma 17 *Case 3B1 is infeasible.*

Proof. By contradiction.

Assume Case 3B1 is feasible. Then $U_{i_1^{**}} > 0$.

However, $U_{i_1^{**}}$

$= \frac{-\alpha_H p_{L1} + p_{H1}\alpha_L}{\alpha_H - \alpha_L}$

$= \frac{1}{2}\frac{\left(\alpha_H+\alpha_H\delta^2\beta+2\beta\delta\alpha_H+\beta\alpha_H+\alpha_H\delta^2+2\alpha_H\delta-\alpha_L\delta+\beta\alpha_L\delta-\delta^2\alpha_L+\delta^2\alpha_L\beta\right)\alpha_L\delta}{(-\beta\delta\alpha_H-7\alpha_H\delta-4\alpha_H-\alpha_H\delta^2\beta-3\alpha_H\delta^2-3\delta^2\alpha_L\beta+\beta^2\alpha_L\delta-5\beta\alpha_L\delta+3\delta^2\alpha_L+4\alpha_L\delta)}$

$= \frac{1}{2}\frac{(1+\delta)(\alpha_H\delta+\beta\delta\alpha_H-\alpha_L\delta+\beta\alpha_L\delta+\alpha_H+\beta\alpha_H)\alpha_L\delta}{(-\beta\delta\alpha_H-7\alpha_H\delta-4\alpha_H-\alpha_H\delta^2\beta-3\alpha_H\delta^2-3\delta^2\alpha_L\beta+\beta^2\alpha_L\delta-5\beta\alpha_L\delta+3\delta^2\alpha_L+4\alpha_L\delta)}$

$= \frac{1}{2}\frac{(1+\delta)((\alpha_H-\alpha_L)\delta+\beta\delta\alpha_H\delta+\beta\alpha_L\delta+\alpha_H+\beta\alpha_H)\alpha_L\delta}{-\beta\delta\alpha_H-4\alpha_H-\alpha_H\delta^2\beta-3\delta^2\alpha_L\beta+\beta\alpha_L\delta(\beta-5)+3\delta^2(\alpha_L-\alpha_H)+4\alpha_L\delta-7\alpha_H\delta}$

$= -\frac{1}{2}\frac{(1+\delta)((\alpha_H-\alpha_L)\delta+\beta\delta\alpha_H\delta+\beta\alpha_L\delta+\alpha_H+\beta\alpha_H)\alpha_L\delta}{\beta\delta\alpha_H+4\alpha_H+\alpha_H\delta^2\beta+3\delta^2\alpha_L\beta+\beta\alpha_L\delta(5-\beta)+3\delta^2(\alpha_H-\alpha_L)+4\delta(\alpha_H-\alpha_L)+3\alpha_H\delta} < 0$: contradiction,

, since the denominator and the numerator are positive.

Hence, we conclude that Case 3B1 is infeasible. ■

Case 3B2

In this scenario, u_{H2} and u_{L1} meet in the range. If this is feasible, we must have $U_{i_1^{**}} > 0$.

Lemma 18 *Case 3B2 is infeasible.*

Proof. By contradiction.

Assume Case 3B2 is feasible. Then $U_{i_1^{**}} > 0$.

However, $U_{i_1^{**}}$

$= \frac{-\alpha_H p_{L1} + p_{H1}\alpha_L}{\alpha_H - \alpha_L}$

$= \frac{1}{2}\frac{\left(-2\,\beta - 2\,\beta\,\delta + 1 + \delta + \beta^2\delta + \beta^2\right)\alpha_L\,\delta}{-4\,\beta\,\delta^2 - 6\,\beta\,\delta - 3\,\delta + \beta^2\delta - 4}$

$= \frac{1}{2}\frac{(\beta-1)^2(1+\delta)\alpha_L\,\delta}{-4\,\beta\,\delta^2 - (6-\beta)\beta\,\delta - 3\,\delta - 4} < 0$: contradiction.

Hence, we conclude that Case 3B2 is not feasible. ■

CASE 3C1

In this scenario, u_{H1} and u_{H2} meet in the range, but the low quality product is not offered in the second period. If this is feasible, we must have $0 < i^*_{H2} < i^*_{L1}$ and $U_{i_1^{**}} > 0$.

Lemma 19 *Case 3C1 is infeasible.*

Proof. By contradiction.

Assume Case 3C1 is feasible.

Then must satisfy assumption (1) $0 < i^*_{H2} < i^*_{L1}$ and assumption (2) $U_{i_1^{**}} > 0$.

Assumption (1) : $0 < i^*_{H2} < i^*_{L1}$

(i) Claim that $i^*_{H2} > 0$

$i^*_{H2} = \frac{\left(1+\beta+3\beta\delta+\delta+2\beta\delta^2\right)\alpha_L}{\beta^2\delta^2\alpha_H+\beta\alpha_H\delta-\alpha_H\delta-\alpha_H\delta^2\beta+4\alpha_L\delta+5\beta\alpha_L\delta^2+5\beta\alpha_L\delta-\beta^2\alpha_L\delta-\beta^2\alpha_L\delta^2+4\alpha_L} > 0$

$\iff \beta^2\delta^2\alpha_H+\beta\alpha_H\delta-\alpha_H\delta-\alpha_H\delta^2\beta+4\alpha_L\delta+5\beta\alpha_L\delta^2+5\beta\alpha_L\delta-\beta^2\alpha_L\delta-\beta^2\alpha_L\delta^2+4\alpha_L > 0$

$\iff \alpha_H < \frac{\alpha_L\left(-4\delta-5\beta\delta^2-5\beta\delta+\beta^2\delta+\beta^2\delta^2-4\right)}{\delta(\beta^2\delta+\beta-1-\beta\delta)}$

Therefore, $i^*_{H2} > 0$ iff $\alpha_H < \frac{\alpha_L\left(-4\delta-5\beta\delta^2-5\beta\delta+\beta^2\delta+\beta^2\delta^2-4\right)}{\delta(\beta^2\delta+\beta-1-\beta\delta)}$

(ii) Let $i_{H2}^* < i_{L1}^*$.

Then $i_{H2}^* - i_{L1}^* = -\frac{(1+\delta)(-\beta\alpha_L\delta+\beta\alpha_H\delta+\alpha_L-\beta\alpha_L)}{\beta^2\delta^2\alpha_H+\beta\alpha_H\delta-\alpha_H\delta-\alpha_H\delta^2\beta+4\alpha_L\delta+5\beta\alpha_L\delta^2+5\beta\alpha_L\delta-\beta^2\alpha_L\delta-\beta^2\alpha_L\delta^2+4\alpha_L} < 0.$

$\iff -\frac{(1+\delta)(\beta\delta\,(\alpha_H-\alpha_L)+\alpha_L(1-\beta)\,)}{\beta^2\delta^2\alpha_H+\beta\alpha_H\delta-\alpha_H\delta-\alpha_H\delta^2\beta+4\alpha_L\delta+5\beta\alpha_L\delta^2+5\beta\alpha_L\delta-\beta^2\alpha_L\delta-\beta^2\alpha_L\delta^2+4\alpha_L} < 0$

$\iff \beta^2\delta^2\alpha_H+\beta\alpha_H\delta-\alpha_H\delta-\alpha_H\delta^2\beta+4\alpha_L\delta+5\beta\alpha_L\delta^2+5\beta\alpha_L\delta-\beta^2\alpha_L\delta-\beta^2\alpha_L\delta^2+4\alpha_L > 0$

$\iff \alpha_H < \frac{\alpha_L\left(-4\delta-5\beta\delta^2-5\beta\delta+\beta^2\delta+\beta^2\delta^2-4\right)}{\delta(\beta^2\delta+\beta-1-\beta\delta)}$

Therefore, $i_{H2}^* < i_{L1}^*$ iff $\alpha_H < \frac{\alpha_L\left(-4\delta-5\beta\delta^2-5\beta\delta+\beta^2\delta+\beta^2\delta^2-4\right)}{\delta(\beta^2\delta+\beta-1-\beta\delta)}$

From (i) and (ii), assumption (1) is satisfied iff $\alpha_H < \frac{\alpha_L\left(-4\delta-5\beta\delta^2-5\beta\delta+\beta^2\delta+\beta^2\delta^2-4\right)}{\delta(\beta^2\delta+\beta-1-\beta\delta)}$

Assumption (2) : $U_{i_1^{**}} > 0$.

Let $U_{i_1^{**}} > 0$.

Then $U_{i_1^{**}} = \frac{-\alpha_H p_{L1}+p_{H1}\alpha_L}{\alpha_H-\alpha_L} > 0$

$\iff -\frac{1}{2}\frac{\left(1+\beta+3\beta\delta+\delta+2\beta\delta^2\right)\alpha_H\alpha_L\delta}{\beta^2\delta^2\alpha_H+\beta\,\alpha_H\,\delta-\alpha_H\,\delta-\alpha_H\,\delta^2\beta+4\,\alpha_L\,\delta+5\,\beta\,\alpha_L\,\delta^2+5\,\beta\,\alpha_L\,\delta-\beta^2\alpha_L\,\delta-\beta^2\alpha_L\,\delta^2+4\,\alpha_L}$

$\iff \beta^2\delta^2\alpha_H+\beta\alpha_H\delta-\alpha_H\delta-\alpha_H\delta^2\beta+4\alpha_L\delta+5\beta\alpha_L\delta^2+5\beta\alpha_L\delta-\beta^2\alpha_L\delta-\beta^2\alpha_L\delta^2+4\alpha_L < 0$

$\iff \alpha_H > \frac{\alpha_L\left(-4\delta-5\beta\delta^2-5\beta\delta+\beta^2\delta+\beta^2\delta^2-4\right)}{\delta(\beta^2\delta+\beta-1-\beta\delta)}$

Therefore, $U_{i_1^{**}} > 0$ iff $\alpha_H > \frac{\alpha_L\left(-4\delta-5\beta\delta^2-5\beta\delta+\beta^2\delta+\beta^2\delta^2-4\right)}{\delta(\beta^2\delta+\beta-1-\beta\delta)}$

From Assumption (1) and (2),

$\alpha_H < \frac{\alpha_L\left(-4\delta-5\beta\delta^2-5\beta\delta+\beta^2\delta+\beta^2\delta^2-4\right)}{\delta(\beta^2\delta+\beta-1-\beta\delta)}$ and $\alpha_H > \frac{\alpha_L\left(-4\delta-5\beta\delta^2-5\beta\delta+\beta^2\delta+\beta^2\delta^2-4\right)}{\delta(\beta^2\delta+\beta-1-\beta\delta)}$

Hence there is no solution of α_H satisfying the Assumption (1) and (2) simultaneously and we conclude that Case 3C1 is infeasible. ■

Case 3C2

In this scenario, u_{L1} and u_{H2} meet in the range, but the low quality product is not offered in the second period. If this is feasible, we must have $0 < i_{H2}^* < i_{L1}^*$ and $i_{H2L1}^* < i_1^{**}$.

Lemma 20 *Case 3C2 is infeasible.*

Proof.

By contradiction.

We solve and obtain the following prices by assuming Case 3C2 is feasible. Assume Case 3C2 is feasible. Then the following assumptions (1),(2),(3),(4) and (5) are satisfied. Assumption 1: $0 < i^*_{H2}$

$i^*_{H2} = \frac{p_{H2}}{\alpha_H \delta} > 0$, which holds.

Assumption 2: $i^*_{H2} < i^*_{L1}$

If $i^*_{H2} < i^*_{L1} \iff \frac{p_{H2}}{\alpha_H \delta} < \frac{p_{L1}}{\alpha_L(1+\delta)} \iff p_{H2} < \frac{\alpha_H \delta p_{L1}}{\alpha_L(1+\delta)}$ ...①

Assumption 3: $i^*_{L1} < i^*_{H2L1}$

If $i^*_{L1} < i^*_{H2L1} \iff \frac{p_{L1}}{\alpha_L(1+\delta)} < \frac{p_{H2}-p_{L1}}{\alpha_H \delta - \alpha_L - \alpha_L \delta}$

$$\text{So}, i^*_{L1} < i^*_{H2L1} \iff \begin{cases} p_{H2} > \frac{\alpha_H \delta p_{L1}}{\alpha_L(1+\delta)}, & \text{if } \alpha_H \delta - \alpha_L - \alpha_L \delta > 0; \quad ...② \\ p_{H2} < \frac{\alpha_H \delta p_{L1}}{\alpha_L(1+\delta)}, & \text{otherwise.} \quad ...③ \end{cases}$$

Note that, $\alpha_H \delta - \alpha_L - \alpha_L \delta < 0 \iff \frac{\alpha_H}{\alpha_L} < \frac{1+\delta}{\delta} \iff \frac{\alpha_L}{\alpha_H} > \frac{\delta}{1+\delta}$ ...④

Assumption 4: $i^*_{H2L1} < i^{**}_1$

If $i^*_{H2L1} < i^{**}_1 \iff \frac{p_{H2}}{\alpha_H \delta} < \frac{p_{H1}-p_{L1}}{(\alpha_H - \alpha_L)(1+\delta)} \iff p_{H2} < \frac{(p_{H1}-p_{L1})\alpha_H \delta}{(\alpha_H-\alpha_L)(1+\delta)}$

If $\alpha_H \delta - \alpha_L - \alpha_L \delta < 0$.

Then we can simplify this inequality equation by using ④ and becomes

$\iff p_{H2} < \frac{(p_{H1}-p_{L1})\alpha_H \delta}{(\alpha_H-\alpha_L)(1+\delta)} = \frac{(p_{H1}-p_{L1})\alpha_H}{(\alpha_H-\alpha_L)} \frac{\delta}{(1+\delta)} = \frac{(p_{H1}-p_{L1})\alpha_L}{(\alpha_H-\alpha_L)}$

$$\text{So}, i^*_{H2L1} < i^{**}_1 \iff \begin{cases} p_{H2} < \frac{(p_{H1}-p_{L1})\alpha_H \delta}{(\alpha_H-\alpha_L)(1+\delta)}, & \text{if } \alpha_H \delta - \alpha_L - \alpha_L \delta > 0; \quad ...⑤ \\ p_{H2} < \frac{(p_{H1}-p_{L1})\alpha_L}{(\alpha_H-\alpha_L)}, & \text{otherwise.} \quad ...⑥ \end{cases}$$

Assumption 5: $U_{i^{**}_1} > U_{i^*_{H2L1}}$

$U_{i^{**}_1} - U_{i^*_{H2L1}} = \frac{-\alpha_H p_{L1} + p_{H1}\alpha_L}{\alpha_H - \alpha_L} - \frac{(-p_{L1}+p_{H2})\alpha_H \delta}{-\alpha_L - \alpha_L \delta + \alpha_H \delta} + p_{H2}$

$$= \frac{\alpha_L(\alpha_H p_{L1} - p_{H1}\alpha_L - \alpha_L\delta p_{H1} + \alpha_H\delta p_{H1} - \alpha_H\delta p_{H2} - p_{H2}\alpha_H + p_{H2}\alpha_L + p_{H2}\alpha_L\delta)}{(-\alpha_L - \alpha_L\delta + \alpha_H\delta)(\alpha_H - \alpha_L)}$$
$$= \frac{\alpha_L(-\alpha_H\delta - \alpha_H + \alpha_L + \alpha_L\delta)p_{H2}}{(-\alpha_L - \alpha_L\delta + \alpha_H\delta)(\alpha_H - \alpha_L)} + \frac{\alpha_L(\alpha_H p_{L1} - p_{H1}\alpha_L - \alpha_L\delta p_{H1} + \alpha_H\delta p_{H1})}{(-\alpha_L - \alpha_L\delta + \alpha_H\delta)(\alpha_H - \alpha_L)} > 0$$

$$\iff \begin{cases} p_{H2} < \frac{\alpha_H p_{L1} - p_{H1}\alpha_L - \alpha_L\delta p_{H1} + \alpha_H\delta p_{H1}}{-\alpha_H\delta - \alpha_H + \alpha_L + \alpha_L\delta}, & \text{if } \alpha_H\delta - \alpha_L - \alpha_L\delta > 0; \quad ...⑦ \\ p_{H2} > \frac{\alpha_H p_{L1} - p_{H1}\alpha_L - \alpha_L\delta p_{H1} + \alpha_H\delta p_{H1}}{-\alpha_H\delta - \alpha_H + \alpha_L + \alpha_L\delta}, & \text{otherwise.} \quad ...⑧ \end{cases}$$

Simplify ⑧ by using ④ and this inequality equation becomes,

$$p_{H2} > \frac{\alpha_H p_{L1} - p_{H1}\alpha_L - \alpha_L\delta p_{H1} + \alpha_H\delta p_{H1}}{-\alpha_H\delta - \alpha_H + \alpha_L + \alpha_L\delta}$$
$$> \frac{\alpha_H p_{L1} - p_{H1}\alpha_L - \alpha_L\delta p_{H1} + \alpha_H\delta p_{L1}}{-\alpha_H\delta - \alpha_H + \alpha_L + \alpha_L\delta}, \text{since } p_{L1} < p_{H1}$$
$$= \frac{(\alpha_H p_{L1} - \alpha_L p_{H1})(1+\delta)}{(-\alpha_H + \alpha_L)(1+\delta)} = \frac{\alpha_H p_{L1} - \alpha_L p_{H1}}{-\alpha_H + \alpha_L} = \frac{\alpha_L p_{H1} - \alpha_H p_{L1}}{\alpha_H - \alpha_L} > \frac{\alpha_H p_{H1} - \alpha_H p_{L1}}{\alpha_H - \alpha_L} = \frac{(p_{H1} - p_{L1})\alpha_L}{(\alpha_H - \alpha_L)}$$

So, $p_{H2} > \frac{(p_{H1} - p_{L1})\alpha_L}{(\alpha_H - \alpha_L)}$ ...⑨

Have to consider two cases: $\alpha_H\delta - \alpha_L - \alpha_L\delta > 0$ and $\alpha_H\delta - \alpha_L - \alpha_L\delta < 0$ separately.

First consider the case if $\alpha_H\delta - \alpha_L - \alpha_L\delta > 0$.

Then from the assumption 2 (①) and assumption 3 (②), $p_{H2} < \frac{\alpha_H\delta p_{L1}}{\alpha_L(1+\delta)}$ and $p_{H2} > \frac{\alpha_H\delta p_{L1}}{\alpha_L(1+\delta)}$. Hence. it is not possible to make a decision to p_{H2}, so fail to satisfy the four assumptions.

Next consider the case if $\alpha_H\delta - \alpha_L - \alpha_L\delta < 0$.

Then from assumption 4 (⑥) and assumption 5 (⑨), $p_{H2} < \frac{(p_{H1} - p_{L1})\alpha_L}{(\alpha_H - \alpha_L)}$ and $p_{H2} > \frac{(p_{H1} - p_{L1})\alpha_L}{(\alpha_H - \alpha_L)}$. Thus, it is not possible to make a decision to p_{H2}, so fail to satisfy the four assumptions. Consequently, the four assumptions above cannot be satisfied so Case 3C2 is infeasible. ■

Case 3C3

In this scenario, u_{H1} and u_{L2} meet in the range, but the high quality product is not offered in each period. If this is feasible, we must have $U_{i_1^{**}} > 0$.

Lemma 21 *Case 3C3 is infeasible.*

Proof. By contradiction.

Assume Case 3C3 is feasible. Then $U_{i_1^{**}} > 0$.

$U_{i_1^{**}} = \frac{-\alpha_H p_{L1} + p_{H1}\alpha_L}{\alpha_H - \alpha_L}$
$= \frac{1}{2}\frac{\left(\delta^2\alpha_H+\beta\delta^2\alpha_H+\beta\alpha_L\delta^2-\delta^2\alpha_L+2\alpha_H\delta+2\beta\alpha_H\delta+\beta\alpha_L\delta-\alpha_L\delta+\alpha_H+\beta\alpha_H\right)\alpha_L\delta}{(-3\delta^2\alpha_H-\beta\delta^2\alpha_H+3\delta^2\alpha_L-3\beta\alpha_L\delta^2-7\alpha_H\delta-\beta\alpha_H\delta-5\beta\alpha_L\delta+\delta\beta^2\alpha_L+4\alpha_L\delta-4\alpha_H)}$
$= \frac{1}{2}\frac{(1+\delta)(-\alpha_L\delta+\beta\alpha_L\delta+\beta\alpha_H\delta+\alpha_H\delta+\beta\alpha_H+\alpha_H)\alpha_L\delta}{(-3\delta^2\alpha_H-\beta\delta^2\alpha_H+3\delta^2\alpha_L-3\beta\alpha_L\delta^2-7\alpha_H\delta-\beta\alpha_H\delta-5\beta\alpha_L\delta+\delta\beta^2\alpha_L+4\alpha_L\delta-4\alpha_H)}$

For the numerator

$-\alpha_L\delta + \alpha_H + \alpha_H\delta + \beta\alpha_L\delta + \beta\alpha_H + \beta\alpha_H\delta$

$= (\alpha_L\delta + \alpha_H + \alpha_H\delta)\,\beta - \alpha_L\delta + \alpha_H + \alpha_H\delta$

$= (\alpha_L\delta + \alpha_H + \alpha_H\delta)\,\beta + (\alpha_H - \alpha_L)\,\delta + \alpha_H > 0$

For the denominator

$-3\delta^2\alpha_H - \beta\delta^2\alpha_H + 3\delta^2\alpha_L - 3\beta\alpha_L\delta^2 - 7\alpha_H\delta - \beta\alpha_H\delta - 5\beta\alpha_L\delta + \delta\beta^2\alpha_L + 4\alpha_L\delta - 4\alpha_H$

$= (-3\delta^2 - \beta\delta^2 - 7\delta - \beta\delta - 4)\,\alpha_H + \delta\beta^2\alpha_L - 3\beta\alpha_L\delta^2 + 3\delta^2\alpha_L - 5\beta\alpha_L\delta + 4\alpha_L\delta$

$= (-3\delta^2 - \beta\delta^2 - 3\delta - 4\delta - \beta\delta - 4)\,\alpha_H + (-5+\beta)\,\beta\alpha_L\delta + 3\delta^2\alpha_L - 3\beta\alpha_L\delta^2 + 4\alpha_L\delta$

$= (-3\delta^2 - \beta\delta^2 - \beta\delta - 3\delta - 4)\,\alpha_H + 4\,(-\alpha_H + \alpha_L)\,\delta + (-5+\beta)\,\beta\alpha_L\delta + 3\delta^2\alpha_L - 3\beta\alpha_L\delta^2$

$= -\left(\beta\delta^2 + \beta\delta + 3\delta + 4\right)\alpha_H - 4\,(\alpha_H - \alpha_L)\,\delta - (5-\beta)\,\beta\alpha_L\delta - 3\delta^2\,(\alpha_H - \alpha_L) - 3\beta\alpha_L\delta^2 < 0$

So $U_{i_1^{**}} = \frac{1}{2}\frac{(1+\delta)(\alpha_L\delta+\alpha_H+\alpha_H\delta)\beta+(\alpha_H-\alpha_L)\delta+\alpha_H}{-(\beta\delta^2+\beta\delta+3\delta+4)\alpha_H-4(\alpha_H-\alpha_L)\delta-(5-\beta)\beta\alpha_L\delta-3\delta^2(\alpha_H-\alpha_L)-3\beta\alpha_L\delta^2} < 0$: contradiction.

Hence, the assumption $U_{i_1^{**}} > 0$ is false and Case 3C3 is infeasible. ■

Case 3C4

In this scenario, u_{L1} and u_{L2} meet in the range, but the high quality product is not offered in the second period. If this is feasible, we must have $U_{i_1^{**}} > 0$.

Lemma 22 *Case 3C4 is infeasible.*

Proof. By contradiction.

Assume Case 3C4 is feasible. Then $U_{i_1^{**}} > 0$.

However, $U_{i_1^{**}} = \frac{-\alpha_H p_{L1} + p_{H1}\alpha_L}{\alpha_H - \alpha_L}$
$= \frac{1}{2}\frac{\left(-2\beta-2\beta\delta+1+\delta+\beta^2\delta+\beta^2\right)\alpha_L\delta}{-4\beta\delta^2-6\beta\delta-3\delta+\beta^2\delta-4}$
$= \frac{1}{2}\frac{(\beta-1)^2(1+\delta)\,\alpha_L\delta}{-4\beta\delta^2-6\beta\delta-3\delta+\beta^2\delta-4}$
$= \frac{1}{2}\frac{(\beta-1)^2(1+\delta)\,\alpha_L\delta}{-4\beta\delta^2-[\,(6-\beta)\beta+3\,]\,\delta-4} < 0$: contradiction.

Hence, the assumption $U_{i_1^{**}} > 0$ is false and Case 3C4 is not feasible. ■

Case 3D

In this scenario, u_{H1} and u_{H2} meet in the range. The high quality product is offered in each period. Case 3D is feasible if and only if $i^*_{H1} < i^*_{H1H2} < 1$.

Lemma 23 *Case 3D is feasible.*

Proof.

Without Loss of Generality (W.L.O.G), substitute $\alpha_H = 1$ into all equations below:

Assumption 1: $i^*_{H1} < i^*_{H1H2}$

$\iff \frac{p_{H1}}{1+\delta} < p_{H1} - p_{H2}$

$\iff p_{H1} < (p_{H1} - p_{H2})(1+\delta)$

$\iff p_{H1}\delta > p_{H2}(1+\delta)$

$\iff p_{H1} > \frac{p_{H2}(1+\delta)}{\delta}$

$\iff p_{H1} - \frac{p_{H2}(1+\delta)}{\delta} > 0$

Substituting $p_{H1} = -\frac{2\left(\beta\delta^3+2\beta\delta^2+\beta\delta+1+2\delta+\delta^2\right)}{-6\beta\delta-4\beta\delta^2-4-3\delta+\beta^2\delta}$ and $p_{H2} = -\frac{\delta\left(\beta+3\beta\delta+2\beta\delta^2+1+\delta\right)}{-6\beta\delta-4\beta\delta^2-4-3\delta+\beta^2\delta}$

$\iff -\frac{2\left(\beta\delta^3+2\beta\delta^2+\beta\delta+1+2\delta+\delta^2\right)}{-6\beta\delta-4\beta\delta^2-4-3\delta+\beta^2\delta} - \frac{-\delta\left(\beta+3\beta\delta+2\beta\delta^2+1+\delta\right)}{-6\beta\delta-4\beta\delta^2-4-3\delta+\beta^2\delta}\frac{1+\delta}{\delta}$

$\iff -\frac{2\left(\beta\delta^3+2\beta\delta^2+\beta\delta+1+2\delta+\delta^2\right)+\left(\beta+3\beta\delta+2\beta\delta^2+1+\delta\right)(1+\delta)}{-6\beta\delta-4\beta\delta^2-4-3\delta+\beta^2\delta}$

$\iff \frac{(1+\delta)^2(\beta-1)}{-6\beta\delta-4\beta\delta^2-3\delta-(4-\beta^2\delta)} > 0$: True.

Hence, assumption 1 is satisfied.

Assumption 2: $i^*_{H1H2} < 1$

$\iff p_{H1} - p_{H2} < 1$

$\iff p_{H1} - p_{H2} - 1 < 0$

$\iff -\frac{-3\beta\delta^2-5\,\beta\,\delta-2+\delta^2+\beta^2\delta}{-6\beta\delta-4\beta\delta^2-4-3\delta+\beta^2\delta}$, by substituting prices

$\iff \frac{(5-\beta)\beta\,\delta+3\,\beta\,\delta^2+(2-\delta^2)}{-6\beta\delta-4\beta\delta^2-3\delta-(4-\beta^2\delta)} < 0$: True.

Hence, assumption 2 is satisfied.

Assumption 1 and 2 above hold so Case 3D is feasible. ■

Case 3E

In this scenario, u_{H1} and u_{L2} meet in the range. The firm offers the high quality product in the first period and the low quality product in the second period. Case 3E is feasible if and only if the following two assumptions hold: $U_{i^*_{H1L2}} > 0$ and $i^*_{H1L2} < 1$.

Lemma 24 *Case 3E is feasible.*

Proof.

We test the following two assumptions are satisfied.

W.L.O.G., substitute $\alpha_H = 1$ into all equations below:

Assumption 1: $U_{i^*_{H1L2}} > 0$

$U_{i^*_{H1L2}} = \frac{(p_{H1}-p_{L2})(1+\delta)}{1+\delta-\alpha_L\delta} - p_{H1}$

Substituting prices $p_{H1} = \frac{2(1+\delta)^2(\beta\alpha_L\delta+\delta-\alpha_L\delta+1)}{X}$ and $p_{L2} = \frac{(1+\delta)(-\alpha_L\delta+\beta\alpha_L\delta+\delta+\beta\delta+1+\beta)\alpha_L\delta}{X}$
$= -\frac{\left(-\delta^2+\beta\delta^2+2\beta\delta-2\delta-1+\beta\right)\alpha_L\delta}{X} = \frac{(1+\delta)^2(1-\beta)\alpha_L\delta}{X} > 0,$

where

$$\begin{aligned} X &= 4\delta^2 - 5\alpha_L\delta^2 + 2 + 6\beta\alpha_L\delta^2 - 2\beta{\alpha_L}^2\delta^2 + 6\beta\alpha_L\delta - \delta^2\beta^2\alpha_L + 8\delta + \delta^2\beta^2{\alpha_L}^2 \\ &\quad + {\alpha_L}^2\delta^2 + 2 - \delta\beta^2\alpha_L \\ &= 4\left(1-\alpha_L\right)\delta^2 + (2-\alpha_L\delta^2) + 4\beta\alpha_L\delta^2 + 2\left(1-\alpha_L\right)\beta\alpha_L\delta^2 + \left(1-\beta\delta\right)\beta\alpha_L\delta + 5\beta\alpha_L\delta \\ &\quad + \left(8-5\alpha_L\right)\delta + \delta^2\beta^2{\alpha_L}^2 + {\alpha_L}^2\delta^2 + (2-\delta\beta^2\alpha_L) > 0 \end{aligned}$$

Thus, assumption 1 is satisfied.

Assumption 2: $i^*_{H1L2} < 1$

$1 - i^*_{H1L2} = 1 - \frac{-p_{L2}+p_{H1}}{-\alpha_L\delta+\alpha_H+\alpha_H\delta} = \frac{Y}{-X} > 0$, where

$$Y = -2\delta^2 - 5\beta\alpha_L\delta^2 + \delta^2\beta^2\alpha_L + 4\alpha_L\delta^2 - \delta^2\beta^2{\alpha_L}^2 + 2\beta{\alpha_L}^2\delta^2 - {\alpha_L}^2\delta^2 - 4\delta - 5\beta\alpha_L\delta$$

$$+\, \delta\beta^2\alpha_L + 4\alpha_L\delta - 2$$

$$= -4\left(1-\alpha_L\right)\delta - 2(1-\alpha_L\delta^2) - 2\left(1-\alpha_L\right)\delta^2 - 2\left(1-\alpha_L\right)\beta\alpha_L\delta^2 - 3\beta\alpha_L\delta^2 - \left(1-\beta\delta\right)\beta\alpha_L\delta$$

$$-\left(4-\beta\right)\beta\alpha_L\delta - \delta^2\beta^2{\alpha_L}^2 - {\alpha_L}^2\delta^2 < 0$$

Thus, assumption 2 is satisfied.

Assumption 1 and 2 are satisfied hence, Case 3E is feasible. ■

Case 3F

In this scenario, u_{L1} and u_{L2} meet in the range. The firm offers the low quality product in each period. Case 3F is feasible if and only if the following three assumptions hold: $i^*_{L2} > 0$, $U_{i^*_{L1L2}} > 0$ and $i^*_{L1L2} < 1$.

Lemma 25 *Case 3F is feasible.*

Proof.

We test assumptions the following:

Assumption 1: $i^*_{L2} > 0$

$i^*_{L2} = \frac{p_{L2}}{\delta\alpha_L} > 0$,True; hence,the assumption 1 holds.

Assumption 2: $U_{i^*_{L1L2}} > 0$

$U_{i^*_{L1L2}} = \frac{p_{L1}+p_{L2}}{\alpha_L} = \frac{\alpha_L\delta\left(2\,\beta\,\delta-\delta^2+\beta\,\delta^2-2\,\delta-1+\beta\right)}{-6\,\beta\,\delta-4\,\beta\,\delta^2-4-3\,\delta+\beta^2\delta} = \frac{-\alpha_L\delta(1+\delta)^2(1-\beta)}{-6\,\beta\,\delta-4\,\beta\,\delta^2-4-(3-\beta^2)\delta} > 0$

Hence, assumption 2 holds.

Assumption 3: $i^*_{L1L2} < 1$

$i^*_{L1L2} - 1 = \frac{p_{L1}-p_{L2}}{\alpha_L} - 1 = \frac{5\,\beta\,\delta+3\,\beta\,\delta^2+2-\beta^2\delta-\delta^2}{-6\,\beta\,\delta-4\,\beta\,\delta^2-4-3\,\delta+\beta^2\delta} = \frac{(5-\beta)\beta\,\delta+3\,\beta\,\delta^2+(2-\delta^2)}{-6\,\beta\,\delta-4\,\beta\,\delta^2-4-(3-\beta^2)\delta} < 0$

Hence, assumption 3 holds.

All three assumptions are satisfied; therefore, Case 3F is feasible. ■

Case 3G

In this scenario, u_{L1} and u_{H2} meet in the range. The firm offers the low quality product in the first period and the high quality product in the second period. Case 3G is feasible if and only if the following three assumptions hold: $i^*_{H2} > 0$, $U_{i^*_{L1H2}} > 0$ and $i^*_{L1H2} < 1$.

Lemma 26 *Case 3G is feasible only if* $\alpha_L > \frac{\delta}{1+\delta}$.

Proof.

We test that all assumptions are satisfied.

W.L.O.G., set $\alpha_H = 1$.

Assumption 1: $i^*_{H2} > 0$

$i^*_{H2} = \frac{p_{H2}}{\alpha_H \delta} > 0$, True; hence, assumption 1 holds.

Assumption 2: $U_{i^*_{L1H2}} > 0$

Let $U_{i^*_{L1H2}} > 0$.

Then $U_{i^*_{L1H2}} = \frac{(-p_{L1}+p_{H2})\alpha_H\delta}{-\alpha_L-\alpha_L\delta+\alpha_H\delta} - p_{H2} = \frac{-p_{L1}\delta\alpha_H+p_{H2}\alpha_L+\alpha_L p_{H2}\delta}{-\alpha_L-\alpha_L\delta+\alpha_H\delta} = \frac{p_{L1}\delta-p_{H2}\alpha_L-\alpha_L p_{H2}\delta}{\alpha_L+\alpha_L\delta-\delta} > 0$.

Have to consider two cases: (1) and (2) below.

(1) The numerator > 0 and the denominator > 0

$\iff \alpha_L < \frac{p_{L1}\delta}{p_{H2}(1+\delta)}$ and $\alpha_L > \frac{\delta}{1+\delta}$

If $\frac{p_{L1}\delta}{p_{H2}(1+\delta)} > \frac{\delta}{1+\delta} \iff \frac{p_{L1}}{p_{H2}} > 1 \iff p_{L1} > p_{H2}$

$$\iff \begin{cases} \frac{\delta}{1+\delta} < \alpha_L < \frac{p_{L1}\delta}{p_{H2}(1+\delta)}, & \text{if } p_{L1} > p_{H2}; \quad ...① \\ \alpha_L < \frac{p_{L1}\delta}{p_{H2}(1+\delta)} \text{ and } \alpha_L > \frac{\delta}{1+\delta} & \text{otherwise.} \quad ...② \end{cases}$$

Note that, ② can be omitted, since two different vales of α_L at the same time does not make sense.

(2) The numerator < 0 and the denominator < 0

$\iff \alpha_L > \frac{p_{L1}\delta}{p_{H2}(1+\delta)}$ and $\alpha_L < \frac{\delta}{1+\delta}$

$$\iff \begin{cases} \alpha_L < \frac{\delta}{1+\delta} \text{ and } \alpha_L > \frac{p_{L1}\delta}{p_{H2}(1+\delta)}, & \text{if } p_{L1} > p_{H2}\,; \quad ...③ \\ \frac{p_{L1}\delta}{p_{H2}(1+\delta)} < \alpha_L < \frac{\delta}{1+\delta}, & \text{otherwise.} \quad ...④ \end{cases}$$

Note that, ③ can be omitted as the same reason with ②.

Hence, assumption 2 holds if ① OR ④

Assumption 3: $i^*_{L1H2} < 1$

Assume $i^*_{L1H2} - 1 < 0$.

Then $i^*_{L1H2} - 1 = \frac{-p_{L1}+p_{H2}}{-\alpha_L - \alpha_L\,\delta + \alpha_H\,\delta} - 1 = \frac{p_{L1}-p_{H2}-\alpha_L-\alpha_L\delta+\delta}{\alpha_L+\alpha_L\,\delta-\delta} < 0$.........⑤

Let's consider two cases: the denominator and the numerator.

(1) The denominator

We draw four different quantities in this scenario (see Figure 5) by Maple.

It suffices that showing the myopic consumer quantity demand is positive.

Figure 5 shows that there is a threshold α_L above which the myopic consumer quantity demand in second period, denoted by Qty_2^m, is positive.

Thus, have to define $\hat{\alpha_L}$.

If $Qty_2^m = 0$

$\iff i^*_{L1} - i^*_{H2} = 0$

$\iff \frac{p_{L1}}{\alpha_L(1+\delta)} - \frac{p_{H2}}{\alpha_H\delta} = 0$

$\iff \frac{\left(-\alpha_L\delta^2+\delta^2\beta\alpha_L+\delta^2-\beta\delta^2-\beta\delta-2\alpha_L\delta+\delta+2\beta\alpha_L\delta-\alpha_L+\beta\alpha_L\right)\alpha_L}{-\delta^2+2\beta\delta^2-\delta^2\beta^2+\delta^2\beta^2\alpha_L-6\delta^2\beta\alpha_L+5\alpha_L\delta^2-4\alpha_L{}^2\delta^2+\delta\beta^2\alpha_L-6\beta\alpha_L\delta+5\alpha_L\delta-8\alpha_L{}^2\delta-4\alpha_L{}^2} = 0,$

by substituting the optimal prices and $\alpha_H = 1$.

$\iff (-\alpha_L\delta^2 + \delta^2\beta\alpha_L + \delta^2 - \beta\delta^2 - \beta\delta - 2\alpha_L\delta + \delta + 2\beta\alpha_L\delta - \alpha_L + \beta\alpha_L)\,\alpha_L = 0 \iff$ $\hat{\alpha_L} = \frac{\delta}{1+\delta}$. Hence, $Qty_2^m > 0$, when $\alpha_L > \hat{\alpha_L} \equiv \frac{\delta}{1+\delta}$.

We conclude that $Qty_2^m > 0$ if and only if $\alpha_L > \frac{\delta}{1+\delta}$.

This leads $\alpha_L + \alpha_L\,\delta - \delta > 0$, which is the denominator of ⑤.

(2) The numerator

Next, consider the numerator of ⑤.

If $\alpha_L > \frac{\delta}{1+\delta} \iff p_{L1} - p_{H2} > 0$, by ① $\iff \frac{p_{L1}-p_{H2}+\delta}{1+\delta} > \frac{\delta}{1+\delta}$, by adding $\frac{\delta}{1+\delta}$ to both sides.

$\iff \alpha_L > \frac{p_{L1}-p_{H2}+\delta}{1+\delta} \iff p_{L1} - p_{H2} - \alpha_L - \alpha_L\delta + \delta < 0$

Hence, the numerator $= p_{L1} - p_{H2} - \alpha_L - \alpha_L\delta + \delta < 0$ and $i^*_{L1H2} - 1 < 0$.

Therefore, assumption 3 holds if and only if $\alpha_L > \frac{\delta}{1+\delta}$....⑥

From assumption 1, 2, and 3, Case 3G is feasible if and only if ether (① and ⑥) or (④ and ⑥) holds. However, there is no value of α_L satisfying the condition: (④ and ⑥).

This leads the condition $\alpha_L > \frac{\delta}{1+\delta}$.

Hence, we conclude that Case 3G is feasible if and only if $\alpha_L > \frac{\delta}{1+\delta}$. ■

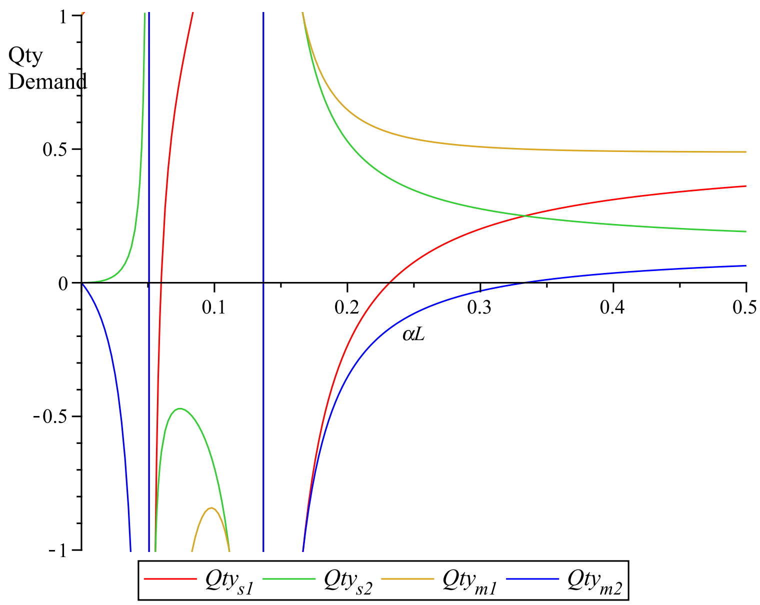


Figure 5: The Quantity Demand in Case 3G

Case 3H1

In this scenario, four consumer utility functions meet each other. If Case 3G is feasible then we must have $U_{i_1^{**}} > 0$.

Lemma 27 *Case 3H1 is infeasible.*

Proof. By contradiction.

Assume Case 3H1 is feasible. Then $U_{i_1^{**}} > 0$.

However, $U_{i_1^{**}} = \frac{1}{2}\frac{\left(-2\beta-2\beta\delta+1+\delta+\beta^2\delta+\beta^2\right)\alpha_L\delta}{-4\beta\delta^2-6\beta\delta-3\delta+\beta^2\delta-4}$

$= \frac{\left(\beta^2-2\beta+1\right)\delta+(\beta^2-2\beta+1)}{2(-4-4\beta\delta^2+(-6\beta-3+\beta^2)\delta)}$

$= \frac{(\delta+1)(\beta-1)^2}{2(-4-4\beta\delta^2-([6-\beta]\beta+3)\delta)} < 0$: contradiction. ■

Case 3H2

In this scenario, three consumer utility functions meet each other. The firm does not offer the low quality product in the second period. If Case 3H2 is feasible then we must have $U_{i_1^{**}} > 0$.

Lemma 28 *Case 3H2 is infeasible.*

Proof. By contradiction.

Assume Case 3H2 is feasible. Then $i^*_{H1H2} > i_1^{**}$.

But, $i^*_{H1H2} - i_1^{**}$

$= \frac{-p_{H2}+p_{H1}}{\alpha_H} - \frac{p_{H1}-p_{L1}}{(\alpha_H-\alpha_L)(1+\delta)}$

$= \frac{-\frac{1}{2}\alpha_H^2\delta-\frac{1}{2}\alpha_H^2\delta^2+\frac{1}{2}\alpha_H\delta\alpha_L+\frac{1}{2}\alpha_L\alpha_H\delta^2+\alpha_H\left(\frac{1}{2}\alpha_H+\frac{1}{2}\alpha_H\delta\right)\delta-\left(\frac{1}{2}\alpha_H+\frac{1}{2}\alpha_H\delta\right)\alpha_L-\alpha_L\delta\left(\frac{1}{2}\alpha_H+\frac{1}{2}\alpha_H\delta\right)+\frac{1}{2}\alpha_H\alpha_L(1+\delta)}{\alpha_H(\alpha_H-\alpha_L)(1+\delta)}$

Canceling the terms, the numerator is equal to zero.

Hence, $i^*_{H1H2} - i_1^{**} = 0$,which violates the assumption $i^*_{H1H2} > i_1^{**}$.

Consequently, Case 3H2 is infeasible. ■

Case 3H3

In this scenario, two qualities of the product are offered in both periods and u_{L1} and u_{L2}. If Case 3H3is feasible then we must have $i_1^{**} > i_{L1L2}^{*}$.

Lemma 29 *Case 3H3 is infeasible.*

Proof. By contradiction.

Assume Case 3H3 is feasible. Then $i_1^{**} > i_{L1L2}^{*}$.

But, $i_1^{**} - i_{L1L2}^{*}$

$= \frac{1}{2}\frac{\delta\left(-2\beta\delta+2\delta+3+\beta^2-4\beta\right)}{(-6\beta\delta-4\beta\delta^2-4-3\delta+\beta^2\delta)}$

$= \frac{1}{2}\frac{\delta\left(-2\beta\delta+2\delta+3+\beta^2-4\beta\right)}{(-6\beta\delta-4\beta\delta^2-4-3\delta+\beta^2\delta)}$

$= \frac{1}{2}\frac{\delta(\beta-1)((\beta-3)-2\delta)}{-4(1+\beta\delta^2)-([6-\beta]\beta+3)\delta} < 0$: contradiction.

Hence, $i_1^{**} - i_{L1L2}^{*} < 0$, which violates the assumption $i_1^{**} - i_{L1L2}^{*} > 0$.

Consequently, Case 3H3 is infeasible. ■

4.2 The Practical Use and Corresponding Scenarios

We discuss each of our five feasible scenarios in terms of practical application.

First, Case 1D2 shows that the high quality utility function in the second period dominates the low quality utility function in the first period. Particularly, in Case 1D2 all strategic consumers delay their purchases to the second period and they only purchase high quality products. Myopic consumers all buy the low quality product in the first period. This is the extreme case, which is hardly found in practice although this scenario is mathematically feasible.

Second, Case 3D is interpreted as offering only the high quality product in both periods and results in the best profit to the firm. This indicates that there is no need to produce two different qualities of the same products. In fact, one might intuitively know this result, but this theoretical study proves it.

Third, Case 3E suggests that the firm provides the high quality products in the first period and the low quality products in the second period. In the introduction section, we show an example in the computer software industry. This scenario works well in that example. Once the firm releases the new product, it sells the professional version of the product at the high price in the first period and sells the student version of the product at the low price in the second period. this scenario turns to the best if $\beta = 1$.

Fourth, we notice that offering the low quality products in both periods, Case 3F, is inferior than offering the high quality products in both periods, Case 3D, since no firm sells low quality products in both periods. Thus, we omit the discussion of Case 3F. Although it might be profitable in the short run, it might hurt the profit in the long run since this scenario does not meet consumer desire for the high quality products.

Last, Case 3G denotes the instance wherein the firm provides the low quality products in the first period and the high quality products in the second period. This strategy also has been used in software industry. For example, Ahnlab, which is the well-known Korean Anti-Virus software producer, offers the first version of a new product (low quality product) at the beginning. It revises errors if any exist or increases performance according to consumer comments and offers the complete product later (high quality product). One study of upgrading pricing related to this scenario can be found in Bala and Carr (2009).

We use Maple Optimization package and LINGO, optimization software, to obtain the optimal maximum possible profit of the each of feasible scenarios (see Table 1). However, many of the optimal solutions in each case explain that in order to capitalize, the firm should not have the strategic consumers ($\beta = 0$) but this is hardly found in practice and these cases are too extreme. Consequently, we choose three feasible scenarios which are considered to be more practical and test by setting different values for the constants, particularly, beta and delta. Unexpectedly, the feasible scenario Case 3G is extremely beneficial (see Figure 6) if the difference between two qualities is very large (say $\alpha_H = 1$ and $\alpha_L = 0.25$).

Scenario	Case 1D2	Case 3D	Case 3E	Case 3F	Case 3G
α_H	1	1	1	NA	1
α_L	0.85	NA	0.0795	1	1
β	0	0	1	0	0
δ	1	1	1	1	1
p_{H1}	NA	1.14	1.14	NA	NA
p_{L1}	0.99	NA	NA	1.14	1.14
p_{H2}	0.29	0.29	NA	NA	NA
p_{L2}	NA	NA	0.29	0.29	0.29
Profit	0.50	0.57	0.50	0.57	0.57

Table 1: Optimal Solution of The Feasible Scenarios

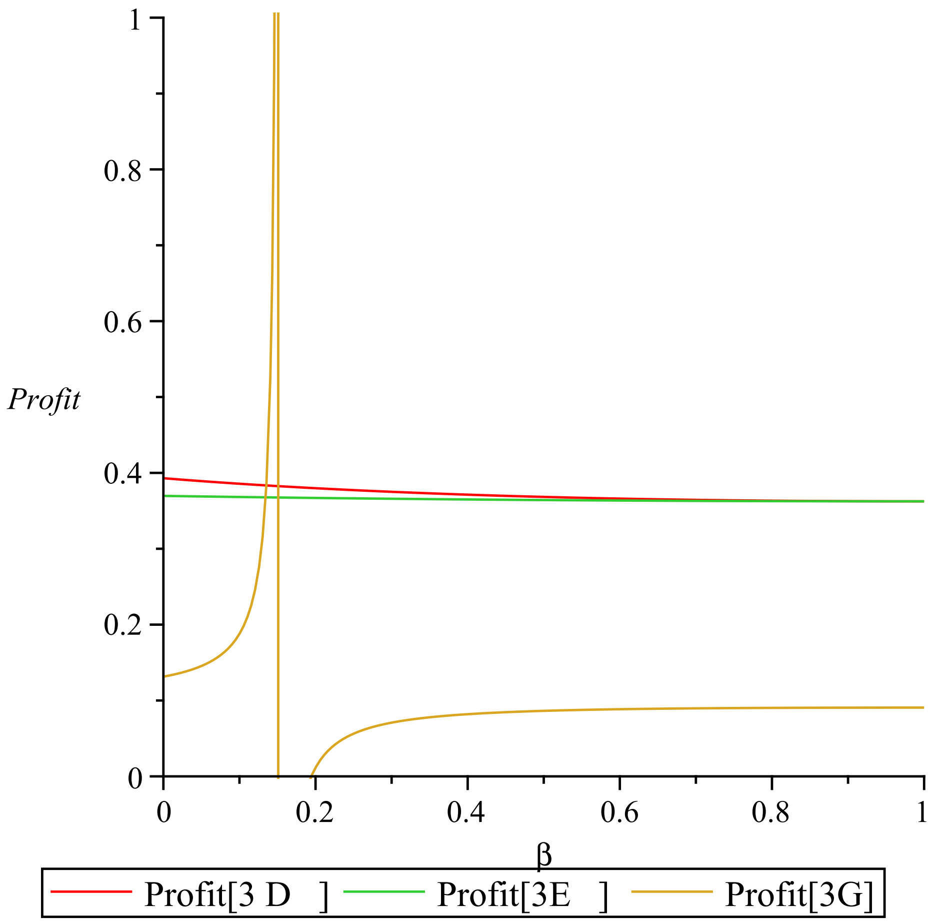


Figure 6: Feasible Solution Analysis (note: Case 3G is feasible only for $\beta > 0.2$)

5 The Optimal Scenario

We have identified the five feasible scenarios: Case 1D2, Case 3D, Case 3E, Case 3F and Case 3G. In this section, we find the scenario which offers the best profit for the firm. We first show that the profit of Case 3D dominates all other feasible scenarios.We then further characterize the best scenario, Case 3D.

Proposition 1 *Case 3D yields the highest profit for the firm with $\alpha_H = 1$, if $\alpha_L > \frac{\delta}{1+\delta}$.*

Proof.

The proof is followed easily from requires the following four Lemmas: *Lemma* 30, 31, 32 and 40.

From *Lemma* 30: $\pi^{3D} \geq \pi^{1D2}$, if $\alpha_L > \frac{\delta}{2(1+\delta)}$.

From *Lemma* 31: $\pi^{3D} \geq \pi^{3E}$ along with our global assumption $0 \leq \alpha_L \leq \alpha_H \leq 1$.

From *Lemma* 32: $\pi^{3D} \geq \pi^{3F}$, when $0 \leq \alpha_L \leq \alpha_H \leq 1$.

From *Lemma* 33: $\pi^{3D} \geq \pi^{3G}$, if$\alpha_L > \frac{\delta}{1+\delta}$.

Note that, $\frac{\delta}{1+\delta} > \frac{\delta}{2(1+\delta)}$. Hence, Case 3D offers the best profit for the firm if $\alpha_L > \frac{\delta}{(1+\delta)}$ with a setting $\alpha_H = 1$. ■

Lemma 30 *The profit of Case 3D dominates the profit of Case 1D2, when $\alpha_L > \frac{\delta}{2(1+\delta)}$ (i.e., $\pi^{3D} \geq \pi^{1D2}$).*

Proof.

Recall that Case 1D2 is feasible only when $\alpha_L > \hat{\alpha_L} \equiv \frac{\alpha_H \delta}{2(1+\delta)}$. Let this threshold $\hat{\alpha_L} \equiv \frac{\alpha_H \delta}{2(1+\delta)}$.

We show that $\pi^{3D} \geq \pi^{1D2}$, when $\alpha_L > \hat{\alpha_L}$.

Let $\triangle\pi = \pi^{3D} - \pi^{1D2}$

$$= -\frac{\left(\beta\delta^3+2\beta\delta^2+\beta\delta+1+2\delta+\delta^2\right)\alpha_H}{-6\beta\delta-4\beta\delta^2-4-3\delta+\beta^2\delta} - \frac{\alpha_L\left(\beta\delta\alpha_H+\beta\alpha_H\delta^2+\alpha_L\delta^2+\alpha_L+2\alpha_L\delta-\beta\alpha_L-2\beta\alpha_L\delta-\beta\alpha_L\delta^2\right)}{4\alpha_L\delta+4\alpha_L-\alpha_H\delta+\beta\delta\alpha_H}$$

$$= \frac{2{\alpha_H}^2\delta^2(\delta+1)^2(\beta-1)(\beta+1)^2}{(4\alpha_L\delta+4\alpha_L-\alpha_H\delta+\beta\delta\alpha_H)^3}.$$

Then $\frac{\partial \triangle\pi}{\partial \alpha_L} = -\frac{(\delta+1)(\alpha_H\delta-2\alpha_L\delta-2\alpha_L)(\beta-1)(\beta\delta\alpha_H+2\alpha_L\delta+2\alpha_L)}{(4\alpha_L\delta+4\alpha_L-\alpha_H\delta+\beta\delta\alpha_H)^2}$.

Now, consider the first order condition.

$$\frac{\partial \triangle\pi}{\partial \alpha_L} = 0 \iff \alpha_L = \frac{\alpha_H\delta}{2(1+\delta)} \equiv \hat{\alpha_L}$$

Next, consider the second order condition.

$$\frac{\partial^2 \triangle\pi}{\partial \alpha_L^2} = \frac{2{\alpha_H}^2\delta^2(\delta+1)^2(\beta-1)(\beta+1)^2}{(4\alpha_L\delta+4\alpha_L-\alpha_H\delta+\beta\delta\alpha_H)^3}$$

$$\frac{\partial^2 \triangle\pi}{\partial \alpha_L^2} \ |_{\alpha_L=\hat{\alpha_L}} = \frac{2(\beta-1)(\delta+1)^2}{(\beta+1)\alpha_H\delta} < 0$$

We have shown that $\frac{\partial \triangle\pi}{\partial \alpha_L} \ |_{\alpha_L=\hat{\alpha_L}} = 0$ and $\frac{\partial^2 \triangle\pi}{\partial \alpha_L^2} \ |_{\alpha_L=\hat{\alpha_L}} < 0$, which means that $\pi^{3D} - \pi^{1D2}$ has the local maximum at $\hat{\alpha_L} = \frac{\alpha_H\delta}{2(1+\delta)}$, and it shows that the profit difference is concave downward near $\hat{\alpha_L}$.

Next, we have to check in which range of α_L, $\pi^{3D} - \pi^{1D2}$ is positive.

W.L.O.G., set $\alpha_H = 1$. First, check the low bound of α_L. Substituting $\alpha_L = 0$, the profit difference becomes $\pi^{3D} - \pi^{1D2}|_{(\alpha_L=0)} = \frac{\beta\delta^3+2\beta\delta^2+\beta\delta+1+2\delta+\delta^2}{6\beta\delta+4\beta\delta^2+3\delta+4-\beta^2\delta} > 0$.

Now, check the upper range of α_L. Substituting $\alpha_L = 1$, the profit difference becomes

$$\begin{aligned}\pi^{3D} - \pi^{1D2}|_{(\alpha_L=1)} &= \frac{\beta\Big(-4-6\delta+3\delta^3-7\beta\delta-13\beta\delta^2+\delta^2\beta^2+\beta^2\delta-7\beta\delta^3+\delta^4-\beta\delta^4\Big)}{(-6\beta\delta-4\beta\delta^2-4-3\delta+\beta^2\delta)(3\delta+4+\beta\delta)} \\ &= \frac{\beta(\delta+1)\Big(\delta^3-\beta\delta^3+2\delta^2-6\beta\delta^2-7\beta\delta+\beta^2\delta-2\delta-4\Big)}{(-6\beta\delta-4\beta\delta^2-4+\beta^2\delta-3\delta)(3\delta+4+\beta\delta)} \\ &= \frac{\beta(\delta+1)\Big((\delta^3-4)-\beta\delta^3+2\delta(\delta-1)-6\beta\delta^2+\beta\delta(-7+\beta)\Big)}{(-6\beta\delta-4\beta\delta^2-4+\beta^2\delta-3\delta)(3\delta+4+\beta\delta)} > 0,\end{aligned}$$

since that the denominator and the numerator are negative. Therefore, we have shown that $\pi^{3D} - \pi^{1D2}$ is concave downward and has the local maximum at $\hat{\alpha_L}$. In addition, $\pi^{3D} - \pi^{1D2}|_{(\alpha_L=0)} > 0$ and $\pi^{3D} - \pi^{1D2}|_{(\alpha_L=1)} > 0$. This means that $\pi^{3D} - \pi^{1D2} > 0$, where $\alpha_L \in [0,1]$. With the feasibility of Case 1D2 ($\alpha_L > \hat{\alpha_L} \equiv \frac{\alpha_H\delta}{2(1+\delta)}$ from *Lemma* 10 and the setting $\alpha_H = 1$, we conclude that the profit of Case 3D dominates the profit of Case 1D2 if $\alpha_L > \frac{\delta}{2(1+\delta)}$. ■

Lemma 31 *The profit of Case 3D dominates the profit of Case 3E (i.e., $\pi^{3D} \geq \pi^{3E}$) .*

Proof.

We prove that $\pi^{3D} \geq \pi^{3E}$.

$\pi^{3D} - \pi^{3E} =$

$$\frac{-\left(\beta\delta^3+2\beta\delta^2+\beta\delta+1+2\delta+\delta^2\right)\alpha_H}{-6\beta\delta-4\beta\delta^2-4-3\delta+\beta^2\delta}$$

$$-\frac{-\left(\beta\alpha_L\delta^3+2\beta\alpha_L\delta^2+\beta\alpha_L\delta+\delta^3\alpha_H-\alpha_L\delta^3-2\delta^2\alpha_L+3\delta^2\alpha_H-\alpha_L\delta+3\alpha_H\delta+\alpha_H\right)\alpha_H{}^2}{-4\alpha_H{}^2\delta^2+5\alpha_L\delta^2\alpha_H-6\beta\alpha_L\delta^2\alpha_H+\delta^2\alpha_H\beta^2\alpha_L+2\beta\alpha_L{}^2\delta^2-\delta^2\beta^2\alpha_L{}^2-\delta^2\alpha_L{}^2-8\alpha_H{}^2\delta+5\alpha_H\alpha_L\delta-6\beta\alpha_H\alpha_L\delta+\delta\alpha_H\beta^2\alpha_L-4\alpha_H{}^2}$$

W.L.O.G., assuming $\alpha_H = 1$ to make the calculation more tractable, $\pi^{3D} - \pi^{3E}$ becomes

$$\frac{-\delta(\delta+1)^2(-1+\alpha_L)(\beta-1)^2(\beta\alpha_L\delta^2+\alpha_L\delta-\delta-1)}{(-6\beta\delta-4\beta\delta^2-4-3\delta+\delta\beta^2)(4\delta^2-\delta^2\beta^2\alpha_L+6\beta\alpha_L\delta^2-5\alpha_L\delta^2+\delta^2\beta^2\alpha_L{}^2-2\beta\alpha_L{}^2\delta^2+\alpha_L{}^2\delta^2+8\delta-\delta\beta^2\alpha_L+6\beta\alpha_L\delta-5\alpha_L\delta+4)}$$

(1) Showing that the numerator < 0

For the last term :

$(\beta\alpha_L\delta^2 + \alpha_L\delta - \delta - 1) = (\beta\alpha_L\delta^2 - \delta) + (\alpha_L\delta - 1) = -\delta(\beta\alpha_L 1 - \delta) - (1 - \alpha_L\delta) < 0$

Consequently, this leads the numerator< 0.

(2) Showing that the denominator < 0

For the first term :

$(-6\beta\delta - 4\beta\delta^2 - 4 - 3\delta + \delta\beta^2) = -6\beta\delta - 4\beta\delta^2 - 3\delta - (4 - \delta\beta^2) < 0$

For the second term :

$(4\delta^2 - \delta^2\beta^2\alpha_L + 6\beta\alpha_L\delta^2 - 5\alpha_L\delta^2 + \delta^2\beta^2\alpha_L{}^2 - 2\beta\alpha_L{}^2\delta^2 + \alpha_L{}^2\delta^2 + 8\delta - \delta\beta^2\alpha_L + 6\beta\alpha_L\delta - 5\alpha_L\delta + 4)$

$= (\delta^2\beta^2 - 2\beta\delta^2 + \delta^2)\alpha_L{}^2 + (-\delta^2\beta^2 + 6\beta\delta^2 - \delta\beta^2 + 6\beta\delta - 5\delta - 5\delta^2)\,\alpha_L + 4\delta^2 + 8\delta + 4$

$= \delta^2\,(\beta - 1)^2\,\alpha_L^2 + (-\delta^2\beta^2 + 6\beta\delta^2 - \delta\beta^2 + 6\beta\delta - 5\delta - 5\delta^2)\,\alpha_L + 4\delta^2 + 8\delta + 4 > 0$

, Since $-\delta^2\beta^2\alpha_L + 6\beta\delta^2\alpha_L - \delta\beta^2\alpha_L + 6\beta\delta\alpha_L - 5\delta\alpha_L - 5\delta^2\alpha_L + 4\delta^2 + 8\delta + 4$

$= (6\beta\delta^2\alpha_L + 6\beta\delta\alpha_L + 4\delta^2 + 8\delta + 4) - (\delta^2\beta^2\alpha_L + \delta\beta^2\alpha_L + 5\delta\alpha_L + 5\delta^2\alpha_L)$

$= (6\beta\delta^2\alpha_L - \delta^2\beta^2\alpha_L) + (6\beta\delta\alpha_L - \delta\beta^2\alpha_L) + (8\delta - 5\delta\alpha_L) + (4\delta^2 - 4\delta^2\alpha_L) + (4 - \delta^2\alpha_L)$

$= \beta\delta^2\alpha_L(6 - \beta) + \beta\delta\alpha_L(6 - \beta) + \delta(8 - 5\alpha_L) + 4\delta^2(1 - \alpha_L) + (4 - \delta^2\alpha_L) > 0$

Hence, the denominator < 0

From (1) and (2), $\pi^{3D} \geq \pi^{3E} = \frac{\text{The numerator}<0}{\text{The denominator}<0} > 0$ (i.e., $\pi^{3D} \geq \pi^{3E}$).

Therefore, we conclude that the profit of Case 3D dominates the profit of Case 3E. ■

Lemma 32 *The profit of Case 3D dominates the profit of Case 3F* ($\pi^{3D} \geq \pi^{3F}$).

Proof.

We prove that $\pi^{3D} \geq \pi^{3F}$. $\pi^{3D} - \pi^{3F} = -\frac{(\delta+1)^2(\beta\delta+1)(\alpha_H-\alpha_L)}{-6\beta\delta-4\beta\delta^2-4-3\delta+\beta^2\delta} = -\frac{(\delta+1)^2(\beta\delta+1)(\alpha_H-\alpha_L)}{-6\beta\delta-4\beta\delta^2-4-3\delta+\beta^2\delta} > 0$.

Hence, the profit of Case 3D dominates the profit of Case 3F. ■

Lemma 33 *The profit of Case 3D dominates the profit of Case 3G if* $\alpha_L > \frac{\delta}{(1+\delta)}$ *(i.e.,* $\pi^{3D} \geq \pi^{3G}$*).*

Proof.

First, test if a threshold exists.

First of all, find this point $\hat{\alpha_L}$.

Profit 3D − Profit 3G

$$= -\frac{\left(\beta\delta^3+2\beta\delta^2+\beta\delta+1+2\delta+\delta^2\right)\alpha_H}{-6\beta\delta-4\beta\delta^2-4-3\delta+\beta^2\delta}$$

$$-\frac{\left(\beta\delta^3\alpha_H+2\beta\alpha_H\delta^2+\beta\delta\alpha_H+\alpha_L\delta^3-\delta^3\alpha_H-2\alpha_H\delta^2+3\alpha_L\delta^2-\alpha_H\delta+3\alpha_L\delta+\alpha_L\right)\alpha_L{}^2}{\alpha_H{}^2\delta^2+\delta^2\alpha_H{}^2\beta^2-2\beta\alpha_H{}^2\delta^2-\delta^2\alpha_H\beta^2\alpha_L-5\alpha_H\delta^2\alpha_L+6\beta\alpha_H\alpha_L\delta^2+4\alpha_L{}^2\delta^2-\delta\alpha_H\beta^2\alpha_L-5\alpha_H\alpha_L\delta+6\beta\alpha_H\alpha_L\delta+8\alpha_L{}^2\delta+4\alpha_L{}^2}$$

$$= \frac{-(\delta+1)^2(-\alpha_L+\alpha_H)W}{(-6\beta\delta-4\beta\delta^2-4-3\delta+\beta^2\delta)X}$$

$$= -\frac{Y}{(-6\beta\delta-4\beta\delta^2-4-3\delta+\beta^2\delta)((4\delta^2+8\delta+4)\alpha_L{}^2+(-\delta^2\alpha_H\beta^2-5\alpha_H\delta^2+6\beta\delta^2\alpha_H-\delta\alpha_H\beta^2-5\alpha_H\delta+6\beta\alpha_H\delta)\alpha_L+\alpha_H{}^2\delta^2+\delta^2\alpha_H{}^2\beta^2-2\beta\alpha_H{}^2\delta^2)}$$

$$
\begin{aligned}
\text{Where}, W =& \beta^3\delta^3{\alpha_H}^2 - 2\beta^2\delta^3{\alpha_H}^2 + \beta\delta^3{\alpha_H}^2 - 4\beta\delta^3\alpha_H\alpha_L + 4\beta^2\delta^3\alpha_H\alpha_L + 4\beta\delta^3{\alpha_L}^2 \\
&+ {\alpha_H}^2\delta^2 + \delta^2{\alpha_H}^2\beta^2 - 2\beta{\alpha_H}^2\delta^2 - \beta^3\delta^2\alpha_H\alpha_L - \beta\alpha_H\alpha_L\delta^2 - 4\alpha_H\delta^2\alpha_L + 6\delta^2\alpha_H\beta^2\alpha_L \\
&- \delta^2\beta^2{\alpha_L}^2 + 3{\alpha_L}^2\delta^2 + 10\beta{\alpha_L}^2\delta^2 - \delta\alpha_H\beta^2\alpha_L + 6\beta\alpha_H\alpha_L\delta - 5\alpha_H\alpha_L\delta + 7{\alpha_L}^2\delta \\
&- \delta{\alpha_L}^2\beta^2 + 6\beta\delta{\alpha_L}^2 + 4{\alpha_L}^2 \\
X =& {\alpha_H}^2\delta^2 + \delta^2{\alpha_H}^2\beta^2 - 2\beta{\alpha_H}^2\delta^2 - \delta^2\alpha_H\beta^2\alpha_L - 5\alpha_H\delta^2\alpha_L + 6\beta\alpha_H\alpha_L\delta^2 + 4{\alpha_L}^2\delta^2 \\
&- \delta\alpha_H\beta^2\alpha_L - 5\alpha_H\alpha_L\delta + 6\beta\alpha_H\alpha_L\delta + 8{\alpha_L}^2\delta + 4{\alpha_L}^2 \\
Y =& \big(-4 - 21\delta^2 + \delta^4\beta^2 + 3\beta^2\delta^3 - 4\beta\delta^5 - 18\delta^4\beta - 30\beta\delta^3 \\
&-15\delta - 3\delta^4 - 13\delta^3 + 3\delta^2\beta^2 + \beta^2\delta - 22\beta\delta^2 - 6\beta\delta\big)\,{\alpha_L}^3 \\
&+ \big(30\beta\delta^3\alpha_H + \beta^3\alpha_H\delta^2 + 7\delta^4\alpha_H + 4\alpha_H - 7\delta^2\alpha_H\beta^2 - 4\delta^5\alpha_H\beta^2 + 35\alpha_H\delta^2 \\
&+8\delta^5\beta\alpha_H - 15\delta^4\alpha_H\beta^2 + 20\alpha_H\delta + 11\beta\delta^2\alpha_H - 18\delta^3\alpha_H\beta^2 + 26\delta^3\alpha_H + 27\delta^4\beta\alpha_H \\
&+2\alpha_H\beta^3\delta^3 + \beta^3\delta^4\alpha_H\big)\,{\alpha_L}^2 \\
&+ \big(6\beta{\alpha_H}^2\delta + 3\beta{\alpha_H}^2\delta^3 - 5\delta^5\beta{\alpha_H}^2 - \beta^3\delta^5{\alpha_H}^2 + 13\beta{\alpha_H}^2\delta^2 + 3\delta^2{\alpha_H}^2\beta^2 - 15{\alpha_H}^2\delta^2 \\
&-9\delta^4\beta{\alpha_H}^2 - 5{\alpha_H}^2\delta + 6\delta^5{\alpha_H}^2\beta^2 - \delta{\alpha_H}^2\beta^2 - \beta^3{\alpha_H}^2\delta^2 - 15{\alpha_H}^2\delta^3 - 3\beta^3\delta^4{\alpha_H}^2 \\
&-3\beta^3\delta^3{\alpha_H}^2 + 15\delta^3{\alpha_H}^2\beta^2 + 17\delta^4{\alpha_H}^2\beta^2 - 5\delta^4{\alpha_H}^2\big)\,\alpha_L \\
&-2\beta^2\delta^5{\alpha_H}^3 - 3\beta{\alpha_H}^3\delta^3 + {\alpha_H}^3\delta^2 + \delta^2{\alpha_H}^3\beta^2 + {\alpha_H}^3\delta^4 + 2{\alpha_H}^3\delta^3 + \beta^3\delta^5{\alpha_H}^3 \\
&+\beta^3{\alpha_H}^3\delta^3 + \beta\delta^5{\alpha_H}^3 + 2\beta^3\delta^4{\alpha_H}^3 - 3\beta^2\delta^4{\alpha_H}^3 - 2\beta{\alpha_H}^3\delta^2
\end{aligned}
$$

Let $\triangle\pi = \pi^{3D} - \pi^{3G}$. Then $\frac{\partial\triangle\pi}{\partial\alpha_L} = -\frac{\alpha_L(1+\delta)^2(2\alpha_L\delta+2\alpha_L-\alpha_H\delta+\beta\alpha_H\delta)A}{B}$,

where

$$
\begin{aligned}
A =& 2\delta^2{\alpha_H}^2\beta^2 - \delta^2\alpha_H\beta^2\alpha_L - \delta\alpha_H\beta^2\alpha_L - 4\beta{\alpha_H}^2\delta^2 + 5\beta\alpha_H\alpha_L\delta^2 \\
&+ 5\beta\alpha_H\alpha_L\delta + 2{\alpha_H}^2\delta^2 - 4\alpha_H\delta^2\alpha_L + 2{\alpha_L}^2\delta^2 - 4\alpha_H\alpha_L\delta + 4{\alpha_L}^2\delta + 2{\alpha_L}^2 \\
B =& \big({\alpha_H}^2\delta^2 + \delta^2{\alpha_H}^2\beta^2 - 2\beta{\alpha_H}^2\delta^2 - \delta^2\alpha_H\beta^2\alpha_L - 5\alpha_H\delta^2\alpha_L + 6\beta\alpha_H\alpha_L\delta^2 \\
&+4{\alpha_L}^2\delta^2 - \delta\alpha_H\beta^2\alpha_L - 5\alpha_H\alpha_L\delta + 6\beta\alpha_H\alpha_L\delta + 8{\alpha_L}^2\delta + 4{\alpha_L}^2\big)^2
\end{aligned}
$$

$\frac{\partial \triangle\pi}{\partial \alpha_L} = 0 \iff \alpha_L = 0, \frac{-\alpha_H\delta(\beta-1)}{2(1+\delta)}, \frac{\left(-1+\frac{\beta}{4}-\frac{1}{4}\sqrt{-8\beta+\beta^2}\right)\delta(\beta-1)\alpha_H}{1+\delta}, \frac{\left(-1+\frac{\beta}{4}+\frac{1}{4}\sqrt{-8\beta+\beta^2}\right)\delta(\beta-1)\alpha_H}{1+\delta}$

Now, we have the threshold $\hat{\alpha_L} \equiv \frac{-\alpha_H\delta(\beta-1)}{2(1+\delta)}$, since two other points are not Real number.

Next, have to check if Profit 3D > Profit 3G holds above $\hat{\alpha_L}$.

$\frac{\partial^2 \triangle\pi}{\partial \alpha_L^2} = \frac{-2{\alpha_H}^2\delta^2(\delta+1)^2(-1+\beta)^2 C}{D}$,

where

$$
\begin{aligned}
C =& \big(3\alpha_L{\alpha_H}^2\delta^3 - 3\delta^3\alpha_H{\alpha_L}^2 + 3\alpha_L{\alpha_H}^2\delta^2 + 3\beta\delta^3{\alpha_H}^3 - 6\alpha_H\delta^2{\alpha_L}^2 + \beta^3\delta^3{\alpha_H}^3 - 3\alpha_H{\alpha_L}^2\delta \\
&-3\beta^2\delta^3{\alpha_H}^3 + \delta^3{\alpha_L}^3 + 3{\alpha_L}^3\delta^2 + 3{\alpha_L}^3\delta - {\alpha_H}^3\delta^3 + {\alpha_L}^3 + 3\delta^2\beta^2{\alpha_L}^3 \\
&+3\delta\beta^2{\alpha_L}^3 + \beta^2{\alpha_L}^3 - 18\beta\delta^2{\alpha_L}^3 - 18\beta\delta{\alpha_L}^3 - 6\beta{\alpha_L}^3 + \delta^3\beta^2{\alpha_L}^3 - 6\delta^3\beta{\alpha_L}^3 \\
&-6\delta^2\alpha_H\beta^2{\alpha_L}^2 + 12\beta\alpha_H{\alpha_L}^2\delta^2 - 3\delta\alpha_H\beta^2{\alpha_L}^2 + 6\beta\alpha_H{\alpha_L}^2\delta - 6\alpha_L\beta{\alpha_H}^2\delta^3 \\
&+3\alpha_L\delta^3{\alpha_H}^2\beta^2 + 3\alpha_L\delta^2{\alpha_H}^2\beta^2 - 6\alpha_L\beta{\alpha_H}^2\delta^2 - 3\delta^3\alpha_H\beta^2{\alpha_L}^2 + 6\delta^3\beta\alpha_H{\alpha_L}^2\big) \\
D =& \big({\alpha_H}^2\delta^2 + \delta^2{\alpha_H}^2\beta^2 - 2\beta{\alpha_H}^2\delta^2 - \delta^2\alpha_H\beta^2\alpha_L - 5\alpha_H\delta^2\alpha_L + 6\beta\alpha_H\alpha_L\delta^2 + 4{\alpha_L}^2\delta^2 \\
&-\delta\alpha_H\beta^2\alpha_L - 5\alpha_H\alpha_L\delta + 6\beta\alpha_H\alpha_L\delta + 8{\alpha_L}^2\delta + 4{\alpha_L}^2\big)^3
\end{aligned}
$$

Substituting $\hat{\alpha_L}$, $\frac{\partial^2 \triangle\pi}{\partial \alpha_L^2} = \frac{2(\delta+1)(\beta\delta+\delta+\beta+1)}{\delta\alpha_H(3\beta+\beta^3-3\beta^2-1)} = \frac{2(\delta+1)^2(\beta+1)}{\delta\alpha_H(-1+\beta)^3} < 0$.

We have shown that $\frac{\partial \triangle\pi}{\partial \alpha_L} |_{\alpha_L=\hat{\alpha_L}} = 0$ and $\frac{\partial^2 \triangle\pi}{\partial \alpha_L^2} |_{\alpha_L=\hat{\alpha_L}} < 0$.

This means that $\pi^{3D} - \pi^{3G}$ has a local maximum at $\hat{\alpha_L} \equiv \frac{\alpha_H\delta(1-\beta)}{2(1+\delta)}$.

In other words, the profit difference is concave downward near $\hat{\alpha_L}$.

Next, we have to check in which range of α_L, $\pi^{3D} - \pi^{3G}$ is positive.

W.L.O.G., set $\alpha_H = 1$. First, check the lower bound of α_L. Substituting $\alpha_L = 0$, the profit difference becomes $\pi^{3D} - \pi^{3G}|_{(\alpha_L=0)}$
$= -\frac{-2\beta\delta^2-2\beta^2\delta^5-3\beta^2\delta^4+\beta^3\delta^3+2\beta^3\delta^4+\beta^3\delta^5+\beta\delta^5-3\beta\delta^3+\delta^2+2\delta^3+\delta^4+\delta^2\beta^2}{(-6\beta\delta-4\beta\delta^2-4-3\delta+\beta^2\delta)(\delta^2+\delta^2\beta^2-2\beta\delta^2)}$
$\frac{-\delta^2\left(-2\beta-2\beta^2\delta^3-3\delta^2\beta^2+\beta^3\delta+2\beta^3\delta^2+\beta^3\delta^3+\beta\delta^3-3\beta\delta+1+2\delta+\delta^2+\beta^2\right)}{(-6\beta\delta-4\beta\delta^2-4-3\delta+\beta^2\delta)\delta^2(\beta-1)^2}$
$= \frac{-\delta^2\left\{\beta\delta^3(\beta-1)^2+(2\beta+1)(\beta-1)^2\delta^2+(\beta+2)(\beta-1)^2\delta+(\beta-1)^2\right\}}{(-6\beta\delta-4\beta\delta^2-4-3\delta+\beta^2\delta)\delta^2(\beta-1)^2} > 0$, since both the denominator and the numerator are negative.

Now, check the upper bound of α_L. Substituting $\alpha_L = 1$, the profit difference becomes $\pi^{3D} - \pi^{3G}|_{(\alpha_L=1)} = \frac{\beta\delta^3+2\beta\delta^2+\beta\delta+1+2\delta+\delta^2}{4\beta\delta^2-\beta^2\delta+3\delta+6\beta\delta+4} - \frac{\beta\delta^3+2\beta\delta^2+\beta\delta+1+2\delta+\delta^2}{4\beta\delta^2-\beta^2\delta+3\delta+6\beta\delta+4} = 0$. Therefore, we have shown that $\pi^{3D} - \pi^{1D2}$ is concave downward and has the local maximum at $\hat{\alpha_L}$. In addition, $\pi^{3D} - \pi^{3G}|_{(\alpha_L=0)} > 0$ and $\pi^{3D} - \pi^{1D2}|_{(\alpha_L=1)} = 0$. This implies that $\pi^{3D} - \pi^{3G} > 0$, where $\alpha_L \in [0, 1)$. With the feasibility of Case 3G ($\alpha_L > \frac{\delta}{1+\delta}$ by *Lemma* 26), we conclude that the profit of Case 3D dominates the profit of Case 3G if $\alpha_L > \frac{\delta}{1+\delta}$. ■

Analysis of Case 3D

In general, the increase in delta means that the consumer utility in the first period is increased. Both types of consumers tend to purchase in the first period at the high price. This leads to higher profits. The profit increases a maximum of 104% when the plausible parameters are set at $\{\alpha_H = 1, \alpha_L = 0.5, \beta = 0.5\}$. Many studies have shown that the increased number of strategic consumers has a negative effect on the firm's profit. In our study the profit loss is 8.3% when the parameters are set at $\{\alpha_H = 1, \alpha_L = 0.5, \delta = 0.5\}$.

We analyze whether product substitution is necessary, but our analysis suggests that offering only one quality of the same product in each period is the optimal strategy for the firm.

6 Conclusion and Remarks

It is well known that the presence of strategic consumers has a negative effect on a firm's profit. In this thesis, we study product and intertemporal substitution in a dynamic pricing model in the presence of strategic consumers. We enumerate scenarios, prove feasibilities and choose the best scenario which provides the most profit for the firm. Our result shows that a firm is better offering only a single quality of the product. Particularly, the highest profit is realized when only the high quality product is offered in each of the two periods. Clearly, fewer strategic consumers mean higher profit. In addition, the increase in the discount factor, δ, allows a strategic consumer to purchase at a high price. Therefore the firm must consider carefully the discount factor particularly in our optimal scenario (Case 3D).

Strictly speaking, we have assumed that both types of consumers are utility maximizers and simply set the consumer linear utility function with quality variation in the form $u(\cdot)$, which may not be realistic. There are studies showing people are not utility maximizers. However, the contribution of this dissertation is to show that product substitution may be beneficial to a firm in some setting.

Although we find the optimal scenario, other feasible scenarios (Case 3E, Case 3G) also can be used if the condition, assumptions for a scenario, is satisfied. Our analysis also has other implications. For examples, firms can choose pricing decisions according to their expectations: how many strategic consumers would visit, how to set quality difference, which quality of product to sell, the effect on their profit if they provide two qualities of the product in the same period (if they have to use this scenario). This study is applicable to a firm's pricing decision depending on the firm's operation environment.

This study may be extended to multiple periods. If a model can be focused on durable goods, then capacity constraint and time limitation can be dealt with it. Furthermore, for durable goods it is much easier to distinguish between strategic and myopic consumers as well as to manage inventory. Any consumer's waiting cost or the cost of learning can be inserted into the utility function although Su (2007) found that when the waiting cost is significantly high, consumers are myopic; otherwise they are strategic. In the future our research plan is to consider a setting in which the firm announces product prices dynamically in each period in a multiple time horizon, and consumers react dynamically to the announced prices.

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