Lower Order Terms of Moments of L-Functions

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Given a positive integer k, Conrey, Farmer, Keating, Rubinstein and Snaith conjectured a formula for the asymptotics of the k-th moments of the central values of quadratic Dirichlet L-functions. The conjectured formula for the moments is expressed as sum of a $k(k+1)/2$ degree polynomial in $\log |d|$. In the sum, d varies over the set of fundamental discriminants. This polynomial, called the moment polynomial, is given as a k-fold residue. In Part I of this thesis, we derive explicit formulae for first k lower order terms of the moment polynomial.

In Part II, we present a formula bounding the average of $S(t, f)$, the remainder term in the formula for the number of zeros of an L-function, $L(s, f)$, where f is a newform of weight k and level N. This is Turing's method applied to cuspforms. We carry out the improvements to Turing's original method including using techniques of Booker and Trudgian. These improvements have application to the numerical verification of the Riemann Hypothesis

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Dedicated to Youn Su

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Chapter 1

Results

1.1 Moments of $L(\frac{1}{2}, \chi_d)$

Let k be a positive integer. Let $S_0(X)$ be the set of fundamental discriminants d, with $|d| < X$. Let

$$S^+(X) = \{d \in S_0(X) | d > 0\}, \tag{1.1}$$

and

$$S^-(X) = \{d \in S_0(X) | d < 0\}. \tag{1.2}$$

be the sets of positive and negative fundamental discriminants with $|d| < X$. Let $\chi_d(n) = \left(\frac{d}{n}\right)$, be the Kronecker symbol, and $L(s, \chi_d)$ be the corresponding L-function. These L-functions are described in Section 2.1. Conrey, Farmer, Keating, Rubinstein and Snaith [CFK$^+$05] conjectured an asymptotic expansion for the moments of $L(\frac{1}{2}, \chi_d)$ as a sum of a polynomial over fundamental discriminants, namely:

$$\sum_{d \in S^{\pm}(X)} L(\tfrac{1}{2}, \chi_d)^k = \sum_{d \in S^{\pm}(X)} Q_k^{\pm}(\log |d|)(1 + O(|d|^{-\epsilon})). \tag{1.3}$$

The polynomial $Q_k^+(x)$ or $Q_k^-(x)$ is used, depending upon whether the sum is over $S^+(x)$ or $S^-(x)$. Both are polynomials of degree $\frac{k(k+1)}{2}$, given implicitly as k-fold residues in (2.67).

In Part I, we develop an explicit formula, as a function of k, for the coefficients of the polynomials $Q_k^{\pm}(x)$. This formula can be used for analyzing the coefficients; for examples, they can be used to see how the coefficients behave asymptotically as k increases. We can also use these formulae in a computer program to numerically compute the coefficients.

To simplify the notation below, let $Q_k(x)$ stand for either of $Q_k^{\pm}(x)$.

Theorem 1.1.1. *Write*

$$Q_k(x) = c_0(k)x^{\frac{k(k+1)}{2}} + c_1(k)x^{\frac{k(k+1)}{2}-1} + \cdots + c_{\frac{k(k+1)}{2}}(k). \tag{1.4}$$

Then the leading coefficient $c_0(k)$ of $Q_k(x)$ is

$$c_0(k) = \frac{a_k}{2^k}\left(\prod_{j=0}^{k-1}\frac{(2j)!}{(k+j)!}\right), \tag{1.5}$$

where

$$a_k = \prod_p \left[\left(1-\frac{1}{p}\right)^{\frac{k(k+1)}{2}}\left(\frac{1}{2}\left(\left(1-\frac{1}{\sqrt{p}}\right)^k + \left(1+\frac{1}{\sqrt{p}}\right)^k\right)+\frac{1}{p}\right)\left(1+\frac{1}{p}\right)^{-1}\right], \tag{1.6}$$

and the coefficients $c_1(k), \ldots, c_k(k)$ of $Q_k(x)$ are given by

$$c_r(k) = c_0(k)\sum_{|\lambda|=r} b_\lambda(k)N_\lambda(k). \tag{1.7}$$

The quantities $N_\lambda(k)$ are polynomials in k of degree at most $2r^2$. The polynomials $N_\lambda(k)$ are determined by finding determinants of certain matrices whose entries are binomial coefficients, and doing polynomial interpolation (Section 4.2). The values b_λ, defined in (3.7)*, are the Taylor coefficients of a multivariate holomorphic function.*

The coefficients $c_r(k)$ and b_λ are different for $Q_k^+(x)$ and $Q_k^-(x)$.

1.2 Upper and lower Bounds for $\int_{t_1}^{t_2} S(t,f)\,dt$

In order to numerically check the Riemann hypothesis for an L-function in a certain portion of the critical strip, we must find all the zeros of the L-function in that region and verify that they indeed lie on the critical line. Once found, the data can also be used to corroborate or refute predictions about the statistics of zeros of the L-function.

Finding zeros of an L-function on the critical line up to a height t is done in two steps. The first step is to find zeros on the critical line. The next step is to verify that all the zeros present in the interval of concern have been found. Part II of this thesis concerns this second step for the zeros of L-functions associated to newforms.

Let $L(s,f)$ be an L-function associated to a newform f of weight k and level N (A standard reference for the theory of modular forms is [DS05]). Let Ω be the subset of complex plane obtained by removing the horizontal lines $\{x + it_\rho \mid x \in (-\infty, 1]\}$ for every t_ρ which is an ordinate of a zero ρ of $L(s,f)$: i.e.

$$\Omega = \mathbb{C} - \bigcup_{\rho} \{x + it_\rho \mid x \in (-\infty, 1]\}. \tag{1.8}$$

Since $L(s,f)$ does not have any zero in Ω, and Ω is contractible, $\log L(s,f)$ is an analytic function defined in this domain. We choose the branch of log such that

$$\lim_{\sigma\to\infty} \log L(\sigma + it, f) = 0. \tag{1.9}$$

Then $S(t,f)$ is defined as

$$S(t,f) = \frac{1}{\pi} \Im \log L(\tfrac{1}{2} + it). \tag{1.10}$$

Note that $S(t,f)$ is not defined for any t which is an ordinate of a zero of $L(s,f)$.

For t, which is not an ordinate of a zero of $L(s,f)$, it is easy to see that $S(t,f)$ is the change in $\frac{1}{\pi}\arg L(\sigma + it, f)$, as σ varies along the straight line from $\infty + it$ to $\frac{1}{2} + it$. The change in arg is measured by continuous variation on this line, starting with the value 0 at infinity.

Let $\vartheta(T, f) = \arg\Gamma(\frac{1}{2} + it + \frac{k-1}{2}) - t\log\frac{\sqrt{N}}{2\pi}$. Let $N(T, f)$ be the number of zeros of $L(s, f)$ in the critical strip up to height T. Using the argument principle, see for example [Dav00, chapter 15], Lemma 1.2.1 gives a relationship between $N(T, f)$ and $S(T, f)$.

Lemma 1.2.1 (See also Lemma 5.2.1)**.** *Let f be a newform of weight k and level N. Let $L(s, f)$ be the L-function associated to this newform. Then*

$$\begin{aligned} N(T, f) &= \frac{1}{\pi}\vartheta(T, f) + S(T, f) \\ &= \frac{k-1}{4} + \frac{T}{\pi}\log\left(\frac{T\sqrt{N}}{2\pi}\right) - \frac{T}{\pi} + S(T, f) + O(\frac{1}{T}). \end{aligned}$$

When we find the zeros of the L-function $L(s, f)$ using a computer up to a height t in the critical strip, we immediately obtain $N(t, f)$. Using Lemma 1.2.1 we also obtain $S(t, f)$. We would like to detect the error when we miss any zero or find a spurious zero. This is done by finding $\int_{t_1}^{t_2} S(t, f)\,dt$ using the computationally determined $S(t, f)$, and checking whether this lies within the theoretical bound for $\int_{t_1}^{t_2} S(t, f)\,dt$.

In Part II of this thesis, we give the theoretical bounds for the integral $\int_{t_1}^{t_2} S(t, f)\,dt$.

Lemma 1.2.2 is first used to reduce the problem of finding bounds on $\int_{t_1}^{t_2} S(t, f)\,dt$ to finding bounds for $\int_{\frac{1}{2}}^{\infty} \log|L(\sigma + it, f)|\,d\sigma$.

Lemma 1.2.2. *[TH86, 9.9] If t_1, t_2 are not the ordinates of zeros of $L(s, f)$, then writing $s = \sigma + it$, the following equality holds:*

$$\int_{t_1}^{t_2} S(t, f)\,dt = \frac{1}{\pi}\int_{\frac{1}{2}}^{\infty} \log|L(\sigma + it_2, f)|\,d\sigma - \frac{1}{\pi}\int_{\frac{1}{2}}^{\infty} \log|L(\sigma + it_1, f)|\,d\sigma. \tag{1.11}$$

If we want to find an upper and a lower bound for $\int_{t_1}^{t_2} S(t, f)\,dt$, Lemma 1.2.2 shows that it is enough to find an upper bound and a lower bound for $\int_{\frac{1}{2}}^{\infty} \log|L(\sigma + it, f)|\,d\sigma$.

In Part II, we prove Theorems 1.2.3 and 1.2.4; these give an upper and lower bound for $\int_{\frac{1}{2}}^{\infty} \log|L(\sigma+it,f)|\,d\sigma$.

Theorem 1.2.3. *Let $L(s,f)$ be an L-function associated to a newform f of weight k and level N, and $t > 2+\frac{k}{2}$. Then for $s=\sigma+it$,*

$$\int_{\frac{1}{2}}^{\infty} \log|L(s,f)|\,d\sigma \le a + b\log t + \frac{c}{t}, \tag{1.12}$$

where,

$$a = 0.6 \times \log(\sqrt{N}) + 2.70746797960673, \tag{1.13}$$

$$b = 0.18, \tag{1.14}$$

$$\text{and} \qquad c = 0.09 \times k + 1.02. \tag{1.15}$$

Theorem 1.2.4. *With the same assumptions as in Theorem 1.2.3,*

$$-\int_{\frac{1}{2}}^{\infty} \log|L(s,f)|\,d\sigma \le a + b\log t. \tag{1.16}$$

Here

$$b = 0.36 \times (\log 4 - 1) = 0.13906597\ldots, \tag{1.17}$$

and

$$a = -J(0.6) + 0.36\log 4\left[\log\frac{\sqrt{N}}{2\pi} + 18.88207258 - \sum_{q|N}\frac{\log q}{q^{1.1}-1}\right] - 0.36\log\frac{\sqrt{N}}{2\pi} + 2\epsilon_{t,k}, \tag{1.18}$$

where

$$J(0.6) = -3.69607634894834 + \sum_{q|N}\frac{1}{\log q}\left[\mathrm{Li}_2\left(\frac{-1}{q^{1.7}}\right) - 2\mathrm{Li}_2\left(\frac{-1}{q^{1.1}}\right)\right], \tag{1.19}$$

and

$$\epsilon_{t,k} = \frac{2}{\left(\frac{t}{\frac{k}{2}+2}\right)^2 - 1}. \tag{1.20}$$

The function $\mathrm{Li}_m(x)$ is defined in (7.5).

Part I

Lower order Terms of Moments of $L(\frac{1}{2}, \chi_d)$

Chapter 2

L-functions, Random Matrix Theory, and other Background Information

In this chapter we introduce the basic definitions, the classical results, and more recent results and conjectures upon which this thesis is built.

2.1 Dirichlet L-functions

This section presents a quick overview of the classical definitions and results in the theory of Dirichlet L-functions. More details can be found in [IR90, Dav00, Rad73]. A Dirichlet character is a map $\chi : \mathbb{Z} \to \mathbb{C}$ induced by a homomorphism $\psi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$;

$$\chi(n) = \begin{cases} \psi(n) & \text{if } \gcd(n, N) = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

The Dirichlet character χ is said to be *primitive* if the homomorphism ψ does not factor through $(\mathbb{Z}/M\mathbb{Z})^\times$ for some M dividing N. The *conductor* of a character is the smallest such M through which it factors.

Definition 2.1.1. Let χ be a Dirichlet character and $s \in \mathbb{C}$. For $\Re s > 1$, the series

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \tag{2.2}$$

defines the associated Dirichlet L-function.

A priori, this function is not defined on the whole complex plane. For non-trivial characters, it is defined for $\Re s > 0$. Theorem 2.1.2 indicates that there is an analytic continuation of Dirichlet L-functions to the whole complex plane.

Theorem 2.1.2 ([Dav00, Chapter 9])**.** *Let χ be a primitive character modulo N. Let*

$$\Lambda(s) = \pi^{-\frac{s+q}{2}} N^{\frac{s+a}{2}} \Gamma(\tfrac{s+a}{2}) L(s,\chi), \tag{2.3}$$

where $a = 1$ if $\chi(-1) = -1$ and 0 otherwise. Then

$$\Lambda(s) = \frac{g(\chi)}{i^a N^{\frac{1}{2}}} \overline{\Lambda(1-\bar{s})}. \tag{2.4}$$

Here $g(\chi)$ is the Gauss sum $\sum_{n=1}^{N} \chi(n) e^{\frac{2\pi i n}{N}}$. The absolute value of $g(\chi)$ is $\sqrt{N}$.

Remark. Equation (2.4) is called the functional equation of the Dirichlet L-function $L(s,\chi)$.

The L-functions of concern in this Part of the thesis are those associated with primitive quadratic characters. It is known [Dav00, Chap. 5] that all such quadratic characters are of the form

$$\chi_d(n) = \left(\frac{n}{d}\right). \tag{2.5}$$

Here $\left(\frac{a}{b}\right)$ is the Kronecker symbol [IR90, p.202], and d is a fundamental discriminant, i.e. for some square free D,

$$d = \begin{cases} D & \text{if } D \equiv 1 \mod 4, \\ 4D & \text{if } D \equiv 2,3 \mod 4. \end{cases} \tag{2.6}$$

The fundamental discriminants are products of distinct prime discriminants d', where

$$d' = \begin{cases} (-1)^{\frac{p-1}{2}} p & \text{for } p \text{ odd prime,} \\ -4, \pm 8 & \text{for } p = 2. \end{cases} \tag{2.7}$$

The conductor of the character χ_d is $|d|$. If $d < 0$, the character is odd and if $d > 0$ then the character is even, that is $\chi_d(-1) = \operatorname{sgn} d$.

There is another version of the functional equation (2.4). Let $\varepsilon_d = \frac{g(\chi_d)}{i^a \sqrt{d}}$, $\gamma_d(s) = \left(\frac{\pi}{d}\right)^{-\frac{s}{2}} \Gamma(\frac{s+a}{2})$, and $X_d(s) = \frac{\gamma_d(1-s)}{\gamma_d(s)}$. Then (2.4) can be rewritten as $L(s, \chi_d) = \varepsilon_d X_d(s) L(1-s, \chi_d)$. For every primitive quadratic characters χ_d, the corresponding ε_d is always 1 [Ayo63, p. 372]. Define

$$Z(s, \chi_d) = X_d^{-\frac{1}{2}}(s) L(s, \chi_d). \tag{2.8}$$

Then $Z(s, \chi_d) = Z(1-s, \chi_d)$.

Note that $X_d(\frac{1}{2}) = \frac{\gamma_d(1-\frac{1}{2})}{\gamma_d(\frac{1}{2})} = 1$. Hence

$$L(\tfrac{1}{2}, \chi_d) = Z(\tfrac{1}{2}, \chi_d). \tag{2.9}$$

Therefore finding the moments $\sum_{d \in S(X)} L(\frac{1}{2}, \chi_d)^k$ is equivalent to finding the moments $\sum_{d \in S(X)} Z(\frac{1}{2}, \chi_d)^k$.

2.2 Random matrix theory models

In this section, we shall assume the generalized Riemann hypothesis.

2.2.1 Riemann zeta function

The Riemann zeta function is the most studied of the L-functions. In their 1999 Bulletin of the AMS paper, Katz and Sarnak [KS99b] presented theoretical evidence along with numerics of Rubinstein [Rub98] for a possible relationship between spacing distributions of zeros of an L-function and the spacing distributions of eigenvalues of members from classical groups. The definition and properties of the classical groups can be found in [Wey97].

Let γ_j be the imaginary part of the j^{th} zero of the zeta function sorted by distance from the real axis. Then $\#\{\gamma_j | 0 \leq \gamma_j \leq T\} \sim \frac{T \log T}{2\pi}$. On average, the consecutive zeros become closer to each other as we move up the critical line. Let $\hat{\gamma}_j = \gamma_j \log \gamma_j$. Then the average distance between the consecutive $\hat{\gamma}_j$ is 1. The distances between the $\hat{\gamma}_j$ are called the normalized spacings between zeros.

Let ϕ be a Schwarz class test function (defined in [Rud91, p.149]), such that

$$\hat{\phi}(\xi) = \int_{-\infty}^{\infty} \phi(x) e^{-2\pi i x \xi} dx \tag{2.10}$$

has support in $(-1, 1)$. Montgomery [Mon73] showed that for such ϕ,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{1 \leq j \neq k \leq N} \phi(\hat{\gamma}_j - \hat{\gamma}_k) = \int_{-\infty}^{\infty} \phi(x) \left(1 - \left(\frac{\sin \pi x}{\pi x}\right)^2\right) dx. \tag{2.11}$$

He conjectured that (2.11) holds for all Schwartz class test functions. If we take ϕ to be the characteristic function of a small interval (α, β), then assuming the conjecture we expect

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\substack{1 \leq j \neq k \leq N \\ \alpha < \hat{\gamma}_j - \hat{\gamma}_k < \beta}} 1 = \int_{\alpha}^{\beta} \left(1 - \left(\frac{\sin \pi x}{\pi x}\right)^2\right) dx. \tag{2.12}$$

If we take (α, β) to be a small interval, then the conjecture says

$$\frac{1}{\beta - \alpha} \lim_{N\to\infty} \left(\frac{1}{N} \sum_{\substack{1 \leq j \neq k \leq N \\ \alpha < \hat{\gamma}_j - \hat{\gamma}_k < \beta}} 1 \right) \sim 1 - \left(\frac{\sin \pi \alpha}{\pi \alpha}\right)^2. \tag{2.13}$$

We can see from Figure 2.1 that the experimental data confirms the conjectural prediction very accurately. In Figure 2.1, the left hand side of (2.13) is plotted for several small intervals.

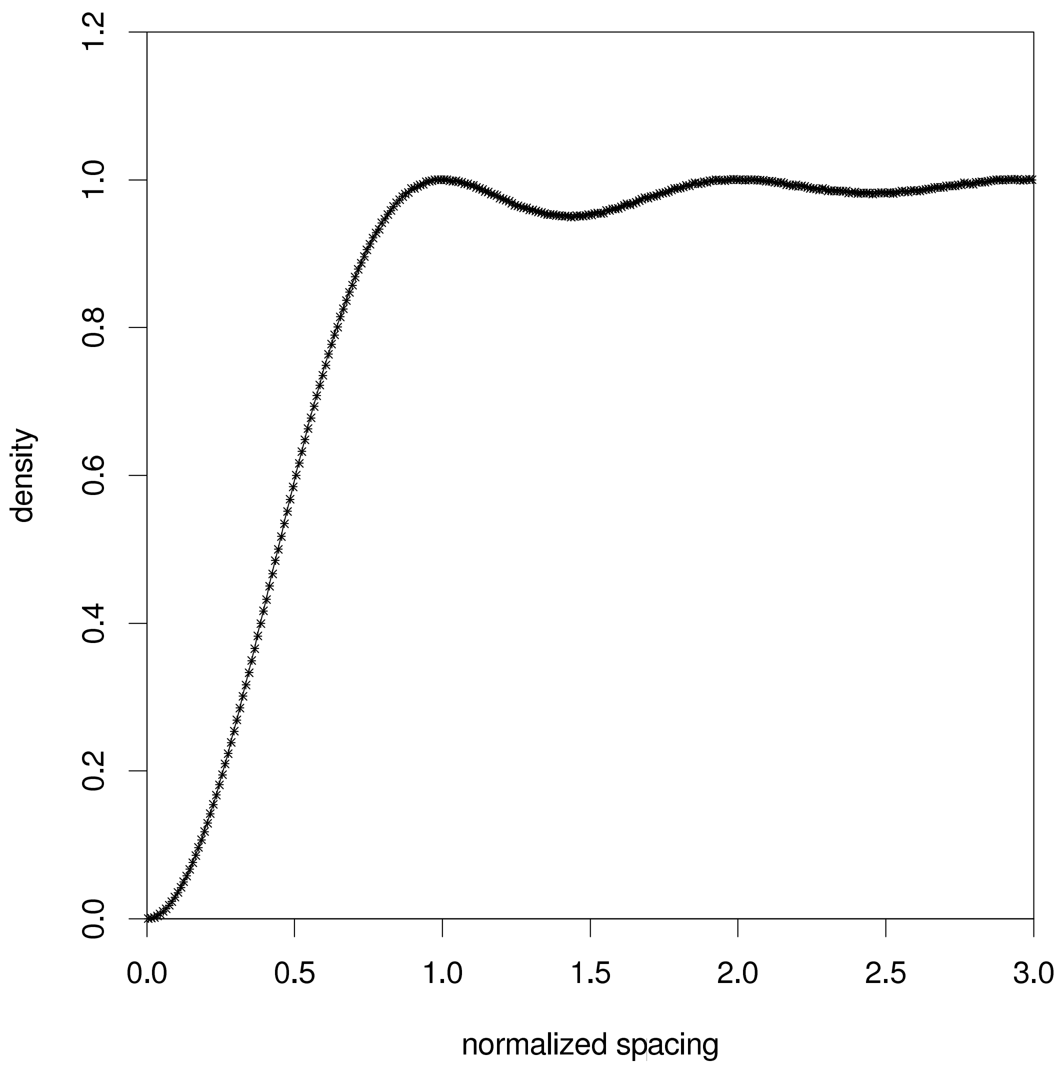

Figure 2.1: Pair correlation for zeros of Zeta function based on 10^8 zeros of the Zeta function near 10^{20} versus $1 - \left(\frac{\sin \pi x}{\pi x}\right)^2$.[Odl]

The function $\left(1 - \left(\frac{\sin \pi x}{\pi x}\right)^2\right)$ in (2.11) is called the pair correlation function of $\zeta(s)$. It also happens to be the same as the pair correlation of the normalized eigenangles of matrices in the unitary group $U(N)$, as explained below.

Let A be an $N \times N$ unitary matrix, and $e^{i\theta_1}, \ldots, e^{i\theta_N}$ be the eigenvalues of A. Here we assume that $0 \le \theta_1 \le \cdots \le \theta_N < 2\pi$ are the eigenangles. We normalize the eigenangles so that the expected difference is 1; so $\hat{\theta} := \frac{N}{2\pi}\theta$ is the normalized eigenangle. For a Schwartz function f, the limit

$$\lim_{N\to\infty} \frac{1}{N} \int_{U(N)} \sum_{i \neq j} f(\hat{\theta}_i - \hat{\theta}_j)\, dA \tag{2.14}$$

exists, and is equal to

$$\int_{-\infty}^{\infty} f(v) \left(1 - \left(\frac{\sin \pi v}{\pi v}\right)^2\right) dv. \tag{2.15}$$

The measure dA is the Haar measure such that the measure of $U(N)$ is 1.

2.2.2 Zeta function of smooth curves

Let C be a smooth projective curve over $\mathbb{F}_q$ of genus g. Let N_n be the number of fixed points of the endomorphism of C which takes each ordinate to its q^n power. The zeta function of the smooth curve C is defined as

$$\zeta(T, C) = \exp\left(\sum_{n=0}^{\infty} \frac{N_n T^n}{n}\right). \tag{2.16}$$

It is known [Del80] that the zeta function (2.16) can be written as

$$\zeta(T, C) = \frac{P(T, C)}{(1-T)(1-qT)}, \tag{2.17}$$

where numerator $P(T, C)$ in the right hand side of (2.17) is a polynomial of degree g. The Riemann hypothesis for curves says that the zeros of $P(T)$ lie on $|T| = \frac{1}{\sqrt{q}}$. It was proved by Deligne [Del80]. The zeros ρ_j of $P(T, C)$ are of the form

$$\rho_j = \frac{1}{\sqrt{q}} e^{i\theta_j}, \quad 0 \le \theta_1 \le \theta_2 \le \cdots \le \theta_{2g} < 2\pi. \tag{2.18}$$

They looked at the distribution of zeros for some families of curves, and showed that they are distributed like the eigenangles of the matrices in classical compact groups. For each of the classical compact groups, they found a family of algebraic curves, such that the statistics of the zeros of the zeta function of these curves and the statistics of eigenangles of the matrices in the corresponding group are the same. A more precise statement can be found in [KS99a, Chapters 1 and 10] and [KS99b].

2.2.3 Other families of L-functions

Motivated by their own work on curves, Katz and Sarnak [KS99b] made predictions for the zeros of L-functions associated to automorphic forms. To each automorphic form f in a family $\mathcal{F}$, let $L(s, f)$ be its L-function, and c_f its conductor. They made the assumption that the set $\mathcal{F}_X = \{f : |c_f| \leq X\}$ is finite for all positive real X. They studied various statistics of $\mathcal{F}_X$ as $X \to \infty$.

Assuming the generalized Riemann hypothesis, let the non trivial zeros of $L(s, f)$ be

$$\frac{1}{2} + i\gamma_f^{(j)}. \tag{2.19}$$

Order them as

$$\cdots \leq \gamma_f^{(-1)} \leq 0 \leq \gamma_f^{(1)} \leq \gamma_f^{(2)} \leq \ldots \tag{2.20}$$

Define the j^{th} normalized zero of $L(s, f)$ to be

$$\frac{\gamma_f^{(j)} \log c_f}{2\pi}. \tag{2.21}$$

The 1-level density for an L-function measures how many normalized zeros lie in a prescribed interval (assuming the generalized Riemann hypothesis, GRH) on the critical line. More generally, one can measure the density by summing the normalized zeros against a Schwartz test function ϕ. The 1-level density of zeros for a given L-function $L(s,f)$, with respect to the weight function ϕ is defined to be:

$$S(f,\phi)=\sum_j \phi\left(\frac{\gamma_f^{(j)}\log c_f}{2\pi}\right).$$

One can then ask, for a collection of L-functions, and a given ϕ, whether the average 1-level density of zeros has a limiting behaviour, i.e. whether

$$\lim_{X\to\infty}\frac{\sum_{c_f\le X}S(f,\phi)}{\#\mathcal{F}_X} \tag{2.22}$$

exists. This quantity measures how dense the normalized zeros are.

Katz and Sarnak conjectured that for various naturally arising families of L-functions, the limit does exist and, in each case, coincides with the average 1-level density for the normalized eigenangles of large matrices from the various classical compact groups. By average, we mean according to the Haar measure, and large means in the limit as the matrix size tends to infinity.

The limit in each case was predicted by Katz and Sarnak to be given by a formula of the form

$$\int_{-\infty}^{\infty}\phi(x)G(x)dx, \tag{2.23}$$

where $G(x)$ is called the limiting 1-level density function, and depends on the specific family.

Katz and Sarnak [KS99b] were able to check their prediction (2.23) theoretically for several families, but with severe restrictions on the function ϕ. They assumed that the Fourier transform of ϕ has support in a prescribed bounded interval. This interval was different for each of the families that they studied. In their paper, they included numerics from Rubinstein's thesis [Rub98] which supported a connection to the classical compact groups [Rub05]. They were able to exhibit families of L-functions for each of the classical groups; the statistics of the normalized zeros of the family of L-functions are modelled closely by the statistics of the normalized eigenangles of the matrices in the corresponding compact classical group.

The family of L-functions consisting of quadratic Dirichlet L-functions $L(s,\chi_d)$ studied in Part I of this thesis was one of the families examined by Katz and Sarnak. They predicted and found evidence for an underlying unitary symplectic behaviour for this collection of L-functions. In particular the statistics of the eigenvalues of matrices in $USp(2N) := U(2N) \cap Sp(2N)$ model those of zeros of this family of L-functions.

For example, averaging over fundamental discriminants $d \in S(X)$, Katz and Sarnak predict that

$$\lim_{X\to\infty} \frac{1}{|S(X)|} \sum_{d\in S(X)} \sum_{j=}^{\infty} \phi\left(\frac{\gamma_d^{(j)} \log|d|}{2\pi}\right) = \int_0^\infty \phi(x) \left(1 - \frac{\sin(2\pi x)}{2\pi x}\right) dx. \tag{2.24}$$

The function $\left(1 - \frac{\sin(2\pi x)}{2\pi x}\right)$ is called the 1-level density of the family $\{L(s,\chi_d)\}$. This coincides with the 1-level density function for eigenangles of $USp(2N) = U(2N) \cap Sp(2N)$, as $N \to \infty$; that is

$$\lim_{N\to\infty} \frac{1}{N} \int_{USp(2N)} \sum_{j=1}^{N} \phi(\hat{\theta}_j)\ dA = \int_0^\infty \phi(t) \left(1 - \frac{\sin 2\pi t}{2\pi t}\right)\ dt, \tag{2.25}$$

where ϕ is a Schwartz function, and $\hat{\theta}_1, \ldots, \hat{\theta}_{2N}$ are the normalized eigenangles of matrices in $USp(2N)$. The eigenvalues of matrices in $USp(2N)$ occur in conjugate pairs. The eigenangles of the eigenvalues below the real axis are excluded from the sum in the left hand side of (2.25). Assuming the GRH for $L(\frac{1}{2},\chi_d)$, Özlük and Snyder [OS93] proved (2.24) for ϕ with support in $(-2,2)$.

We show two plots, courtesy of Rubinstein [Rub05], supporting the unitary symplectic behaviour of $L(s,\chi_d)$. Figure 2.2 is a plot of the zeros up to height 30 for all $|d| < 20000$. We see the density of zeros fluctuating. One can also see secondary terms in the

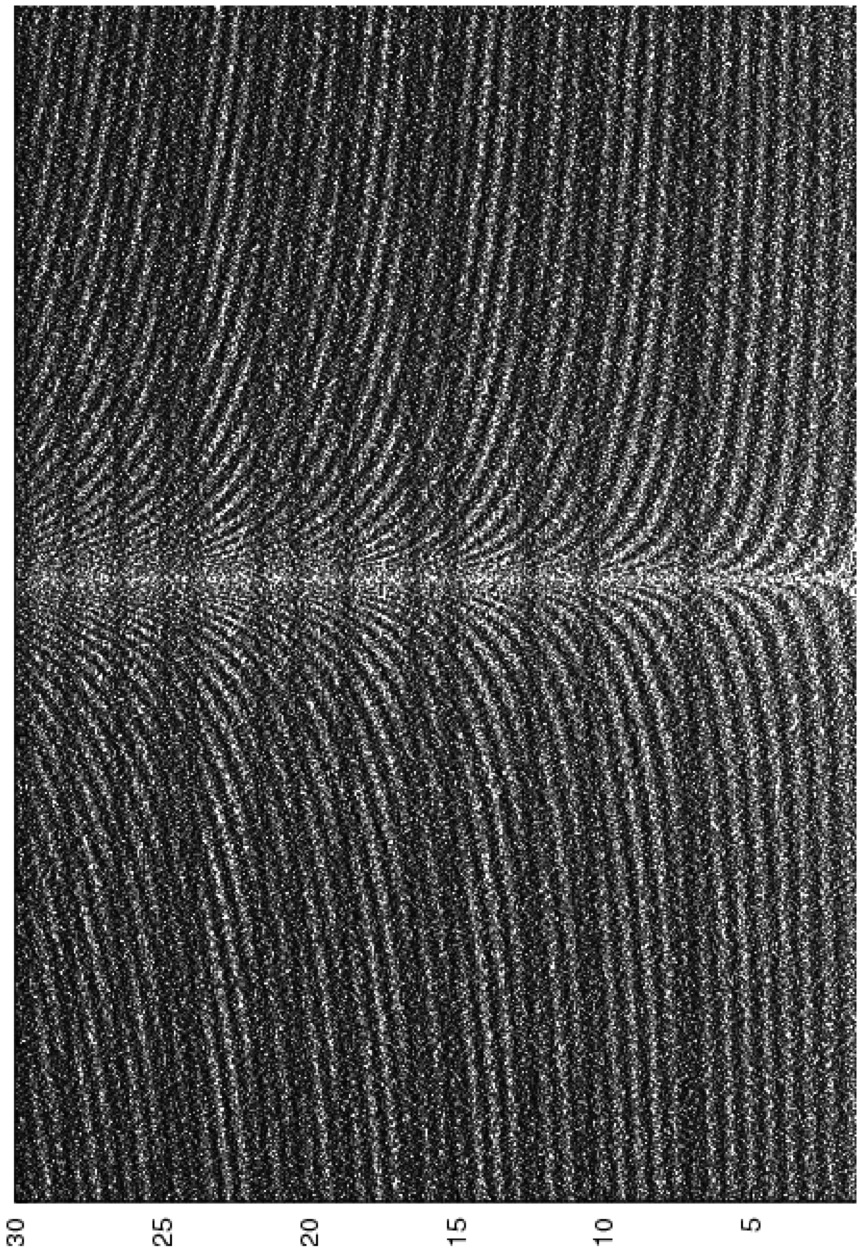

Figure 2.2: Plot of zeros of $L(s, \chi_d)$ vs. d. One sees that the density of zeros fluctuates as the imaginary part increases. The bands are from the lower order terms. *Courtesy of Michael Rubinstein [Rub05].*

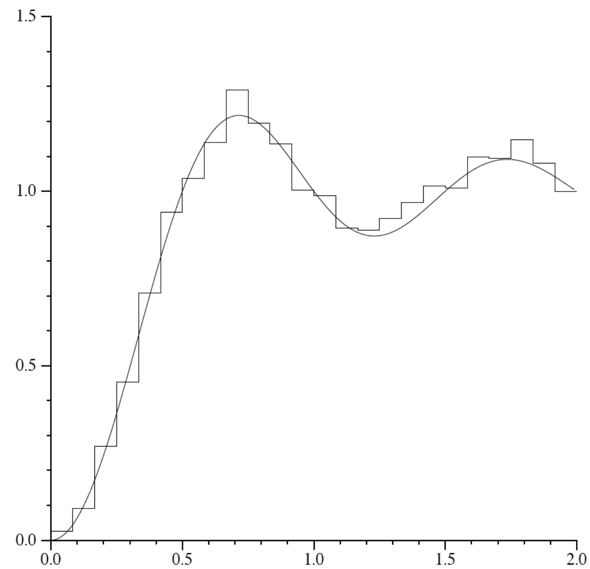

Figure 2.3: 1-level density of zeros of $L(s,\chi_d)$ for 7,000 values of $|d| \approx 10^{12}$. Compared against the random matrix theory prediction, $1 - \sin(2\pi x)/(2\pi x)$. *Courtesy of Michael Rubinstein [Rub98].*

density appearing, for example, at around height 7. These secondary terms were studied in the thesis of Duc Khiem Huynh [Huy09] and capture number theoretic information not captured in the density at the level of the main term. Figure 2.3 depicts the one level density plotted against the prediction for $\phi(x)$ the characteristic function of intervals of length 1/10.

2.3 From densities to moments

In the previous section we described some of the similarities in the statistics of the eigenangles of matrices from the classical compact groups to the zeros of L-functions. These were amongst the first connections made between random matrix theory and families of L-functions. Keating and Snaith investigated these connections further, first for the Riemann

Zeta function [KS00a] and then for other families of L-functions [KS00b]. They looked at moments in random matrix theory and used their results to provide conjectures for the moments of L-functions.

2.3.1 Moments of the zeta function

In [KS00b], Keating and Snaith calculate moments of

$$Z(U,\theta) := \det(e^{i\theta}I - U). \tag{2.26}$$

Let $U \in U(N)$ and $Z(U,\theta)$, defined in (2.26) be its characteristic polynomial. Define

$$M_{U(N)}(2k) := \int_{U(N)} |Z(U,\theta)|^{2k} \, dU. \tag{2.27}$$

For $k > -\frac{1}{2}$, this happens to be independent of θ, but dependent on N. Weyl's integration formula for integrating class functions over $U(N)$ [Wey97] gives

$$M_{U(N)}(2k) = \frac{1}{N!(2\pi)^N} \int_{[0,2\pi]^N} \prod_{1\le j<l\le N} |\exp(i\theta_l) - \exp(i\theta_j)|^2 \times |Z(U,\theta)|^{2k} d\theta_1 \ldots d\theta_N. \tag{2.28}$$

Keating and Snaith applied Selberg's integral to show that (2.28) equals

$$M_{U(N)}(2k) = \prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(2k+j)}{\Gamma(k+j)^2}. \tag{2.29}$$

If $2k$ is a positive even integer then (2.29) simplifies to

$$\begin{aligned} M_{U(N)}(2k) &= \prod_{j=0}^{k-1} \frac{j!}{(j+k)!} \prod_{i=0}^{k-1} (N+i+j+1) \\ &\sim \prod_{j=0}^{k-1} \frac{j!}{(j+k)!} N^{k^2}, \quad \text{as } N \to \infty. \end{aligned} \tag{2.30}$$

Conrey and Ghosh [CG98] conjectured the following form for the moments of $\zeta(\frac{1}{2}+it)$. We would like to emphasize that a_k in this section is not the same as in (1.6).

Conjecture 2.3.1 (Conrey and Ghosh).

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^{2k} dt \sim T \frac{a_k g_k}{k^2!} \log(T)^{k^2} \tag{2.31}$$

where $g_k \in \mathbb{Z}$ *and*

$$a_k = \prod_p \left(1-p^{-1}\right)^{k^2} \sum_{n=0}^{\infty} \binom{n+k-1}{n}^2 p^{-n}. \tag{2.32}$$

The inner sum is ${}_2F_1(k,k;1;1/p)$*, where* ${}_2F_1$ *is the Gauss hypergeometric function [AAR99, Chapter 2].*

Comparing the above conjecture with formula (2.30) for the moments of $Z(U,\theta)$, Keating and Snaith [KS00b] conjectured that

$$\frac{1}{T}\int_0^T |\zeta(\tfrac{1}{2}+it)|^{2k} dt \sim a_k \prod_{j=0}^{k-1} \frac{j!}{(j+k)!} \log(T)^{k^2}; \tag{2.33}$$

i.e.

$$g_k = k^2! \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}. \tag{2.34}$$

This produces $g_1 = 1$, $g_2 = 2$, $g_3 = 42$, and $g_4 = 24024$. The asymptotic equality (2.33) is a theorem of Hardy for $k = 1$, and a theorem of Ingham for $k = 2$ [TH86, chapter 7]. Conrey and Ghosh [CG98] used number theoretic heuristics to conjecture (2.33) for $k = 3$. Conrey and Gonek used similar heuristics to conjecture (2.33) for $k = 4$.

Keating and Snaith's heuristic justification for replacing N in (2.30) by $\log T$ in (2.33) was based on an ad hoc comparison of the mean density of zeros: $\frac{\log T}{2\pi}$ for $\zeta(s)$ in a unit interval at height T versus $\frac{N}{2\pi}$ for unitary eigenangles in $U(N)$.

After the work of Keating and Snaith, two new approaches to the moment conjecture for the Riemann zeta function provided the same predictions as (2.33). The work of Conrey, Farmer, Keating, Rubinstein and Snaith [CFK$^+$05] uses number theoretic heuristics to predict the full asymptotics for the moments of the zeta function. Their work was guided by corresponding results in random matrix theory. The work of Gonek, Hughes, and Keating [GHK07] employs both number theoretic and random matrix theory statistics to explain how a_k and g_k arise.

2.3.2 Moments of $L(\frac{1}{2}, \chi_d)$

Following their success in obtaining a plausible conjecture for the moments of $\zeta(s)$, Keating and Snaith [KS00a] predicted that

$$\frac{1}{|S_0(X)|}\sum_{d\in S_0(X)} L(\tfrac{1}{2}, \chi_d) \sim g_k \frac{a_k}{\Gamma(1+\frac{1}{2}k(k+1))}(\log X^{\frac{1}{2}})^{\frac{1}{2}k(k+1)}, \tag{2.35}$$

where a_k is defined in (1.6), and

$$g_k = \Gamma\left(1+\frac{1}{2}k(k+1)\right)\left(\prod_{j=1}^{k}(2j-1)!!\right)^{-1}. \tag{2.36}$$

Keating and Snaith proved that the k^{th} moment of the characteristic polynomial $Z(U,0)$,

$$M_{USp}(N,k) := \int_{USp(2N)} |Z(U,0)|^k \, dU, \tag{2.37}$$

satisfies

$$M_{USp}(N,k) \sim \left(\prod_{j=1}^{k}(2j-1)!!\right)^{-1} N^{\frac{1}{2}k(k+1)}. \tag{2.38}$$

To conjecture (2.35), they used a heuristic justification similar to the one in Section 2.3.1.

Subsequently Conrey, Farmer, Keating, Rubinstein, and Snaith [CFK+05] gave a more precise prediction for the moments of $L(\frac{1}{2}, \chi_d)$. Their prediction and the heuristics that lead to it are described in Section 2.4.

2.4 Recipe for conjecturing the full asymptotic of moments of $L(\frac{1}{2}, \chi_d)$

This section provides an exposition of the procedure carried out in [CFK+05, section 4] for conjecturing the full asymptotics for the moments $\sum_{d\in S(X)} L(\frac{1}{2}, \chi_d)^k$, where $S(X)$ is either $S^+(X)$ or $S^-(X)$ as defined in (1.1) and (1.2). Here $Z(s, \chi_d)$ is defined in (2.8). We prefer to work with $Z(s, \chi_d)$, rather than $L(s, \chi_d)$, because its functional equation is symmetric, namely:

$$Z(s, \chi_d) = Z(1 - s, \chi_d). \tag{2.39}$$

The following are the heuristic steps in the recipe for obtaining the moment conjecture as applied to our moment problem.

1. Start with the product of shifted L-functions

$$\sum_{d\in S(X)} Z\left(\frac{1}{2} + \alpha_1, \chi_d\right) \dots Z\left(\frac{1}{2} + \alpha_k, \chi_d\right), \tag{2.40}$$

   where all α_j are distinct, and $\Re\alpha_j > 0$.

2. Then the approximate functional equation for $Z(s, \chi_d)$ takes the form,

$$Z(s, \chi_d) = X_s(s)^{-\frac{1}{2}} \sum_{m<\sqrt{d}} \frac{\chi_d(n)}{m^s} + X_d(1 - s)^{\frac{1}{2}} \sum_{n<\sqrt{d}} \frac{\chi_d(n)}{n^{1-s}} + \text{ remainder}. \tag{2.41}$$

3. Drop the remainder in (2.41), and substitute this approximation for $Z(s, \chi_d)$ into (2.40). Multiply this out to obtain 2^k products. Each term is of the form

$$\text{product of } X_d(\tfrac{1}{2} \pm \alpha_j) \times \sum_{n_1\dots n_k} \text{summand}, \tag{2.42}$$

   where summand is a product of some $n_i^{-\frac{1}{2}-\alpha_i}$ and some $n_j^{-\frac{1}{2}+\alpha_j}$.

4. Replace each summand by its expected value as d ranges over the fundamental discriminants, and complete the sums by extending them to infinity. While most of these sums diverge, one can use analytic continuation to give meaning to them.

5. Let the resulting sum over all 2^k terms be $M_d(\alpha_1, \dots, \alpha_k)$. The conjecture is that

$$\sum_{d\in S(X)} \left(\prod_{i=1}^{k} Z(\frac{1}{2}+\alpha_i, \chi_d)\right) \sim \sum_{d\in S(X)} M_d(\alpha_1, \dots, \alpha_k). \tag{2.43}$$

In the rest of this section, we shall follow the recipe to obtain a formula for $M_d(\alpha_1, \dots, \alpha_k)$.

The third step of the recipe says that we should substitute (2.41) for each factor into (2.40). This gives, as an approximation to (2.40) of

$$\sum_{d\in S(X)} \prod_{j=1}^{k} \left(X_d(\tfrac{1}{2}+\alpha_j)^{-\frac{1}{2}} \sum \frac{\chi_d(n)}{n^{\frac{1}{2}+\alpha_j}} + X_d(\tfrac{1}{2}-\alpha_j)^{-\frac{1}{2}} \sum \frac{\chi_d(n)}{n^{\frac{1}{2}-\alpha_j}} \right). \tag{2.44}$$

As we shall see later, the shifted moments allow us to work with what would have been a divergent sum. Let $T = \{1, -1\}^k$. Multiplying out the right hand side of (2.44), we obtain

$$\sum_{d\in S(X)} \sum_{\varepsilon\in T} \prod_{j=1}^{k} \left(X_d^{-\frac{1}{2}}(\tfrac{1}{2}+\varepsilon_j\alpha_j) \sum \frac{\chi_d(n)}{n^{\frac{1}{2}+\varepsilon_j\alpha_j}} \right). \tag{2.45}$$

Exchanging the order of the inner most summation with the product in (2.45), we can write expression (2.45) as

$$\sum_{d\in S(X)} \sum_{\varepsilon\in T} \left(\prod_{j=1}^{k} X_d^{-\frac{1}{2}}(\tfrac{1}{2}+\varepsilon_j\alpha_j) \right) \sum_{n_1,\dots,n_k} \frac{\chi_d(n_1 \dots n_k)}{n_1^{\frac{1}{2}+\varepsilon_1\alpha_1} \dots n_k^{\frac{1}{2}+\varepsilon_k\alpha_k}}. \tag{2.46}$$

In the inner most sum of (2.46), group the terms satisfying $n_1 \dots n_k = m$ for each $m \geq 1$ to get

$$\sum_{d\in S(X)} \sum_{\varepsilon\in T} \left(\prod_{j=1}^{k} X_d^{-\frac{1}{2}}(\tfrac{1}{2}+\varepsilon_j\alpha_j) \right) \left(\sum_{m=1}^{\infty} \sum_{n_1 \dots n_k = m} \frac{\chi_d(m)}{n_1^{\frac{1}{2}+\varepsilon_1\alpha_1} \dots n_k^{\frac{1}{2}+\varepsilon_k\alpha_k}} \right). \tag{2.47}$$

The heuristics say that we should replace $\chi_d(m)$ in (2.47) by its average value. This average value is obtained in Lemma 2.4.1.

Lemma 2.4.1 ([Jut81]). *Let* $a_m = \prod_{p|m} (1 + p^{-1})^{-1}$. *Then*

$$\lim_{X\to\infty} \frac{1}{|S(X)|} \sum_{d<X} \chi_d(m) = \begin{cases} a_m & \text{if } m \text{ is a square,} \\ 0 & \text{otherwise.} \end{cases} \tag{2.48}$$

Following the recipe, we replace (2.47) by

$$\sum_d \sum_{\varepsilon\in T} \left(\prod_{j=1}^k X_d^{-\frac{1}{2}}(\tfrac{1}{2} + \varepsilon_j \alpha_j) \right) \left(\sum_{m=1}^{\infty} \sum_{n_1 \dots n_k = m^2} \frac{a_{m^2}}{n_1^{\frac{1}{2}+\varepsilon_1\alpha_1} \dots n_k^{\frac{1}{2}+\varepsilon_k\alpha_k}} \right). \tag{2.49}$$

We wish to give meaning to the sum over m, as this sum actually diverges unless all $\varepsilon_i = 1$. Recall that we are assuming $\Re\alpha_j > 0$.

Let

$$R(\alpha_1, \dots, \alpha_k) = \sum_{m=1}^{\infty} \sum_{n_1 \dots n_k = m^2} \frac{a_{m^2}}{n_1^{\frac{1}{2}+\alpha_1} \dots n_k^{\frac{1}{2}+\alpha_k}}. \tag{2.50}$$

Then (2.49) can be written as

$$\sum_d \sum_{\epsilon\in T} \left(\prod_{j=1}^k X_d^{-\frac{1}{2}}(\tfrac{1}{2} + \varepsilon_j \alpha_j) \right) R(\varepsilon_1\alpha_1, \dots, \varepsilon_k\alpha_k). \tag{2.51}$$

The sum (2.50) defining $R(\alpha_1, \dots, \alpha_k)$ converges only when each $\Re\alpha_i > 0$. Therefore to make sense of (2.51), we analytically extended the function defined by $R(\alpha_1, \dots, \alpha_k)$ in (2.50) as follows. For now, we assume all $\Re\alpha_j$ are greater than 0. The function $R(\alpha_1, \dots, \alpha_k)$ has an Euler product given by

$$R(\alpha_1, \dots, \alpha_k) = \prod_p R_p(\alpha_1, \dots, \alpha_k), \tag{2.52}$$

where

$$R_p(\alpha_1, \dots, \alpha_k) = 1 + \left(1 + \frac{1}{p}\right)^{-1} \sum_{j=1}^{\infty} \sum_{e_1+\dots+e_k=2j} \prod_{i=1}^k \frac{1}{p^{e_i(\frac{1}{2}+\alpha_i)}}. \tag{2.53}$$

For all $\Re\alpha_j > 0$, the right hand side of (2.53) is equal to

$$1 + (1+p^{-1})^{-1} \left[\frac{1}{2} \left(\prod_{j=1}^k \left(1 + \frac{1}{p^{\frac{1}{2}+\alpha_j}}\right)^{-1} + \prod_{j=1}^k \left(1 - \frac{1}{p^{\frac{1}{2}+\alpha_j}}\right)^{-1} \right) - 1 \right]. \tag{2.54}$$

If we multiply (2.52) by

$$\prod_p \prod_{1\le i\le j\le k}\left(1-\frac{1}{p^{1+\alpha_i+\alpha_j}}\right), \tag{2.55}$$

the resulting expression converges in a neighbourhood of $(\alpha_1,\dots,\alpha_k)=(0,\dots,0)$. The product (2.55) is the reciprocal of the Euler product of a product of zeta functions; more precisely

$$\left[\prod_{1\le i\le j\le k}\zeta(1+\alpha_i+\alpha_j)\right]\times\left[\prod_p \prod_{1\le i\le j\le k}\left(1-\frac{1}{p^{1+\alpha_i+\alpha_j}}\right)\right]=1. \tag{2.56}$$

Therefore,

$$R(\alpha_1,\dots,\alpha_k)=\prod_{1\le i\le j\le k}\zeta(1+\alpha_i+\alpha_j)\prod_p\left(R_p(\alpha_1,\dots,\alpha_k)\prod_{1\le i\le j\le k}\left(1-\frac{1}{p^{1+\alpha_i+\alpha_j}}\right)\right). \tag{2.57}$$

is a meromorphic function defined in a neighbourhood of $(\alpha_1,\dots,\alpha_k)=(0,\dots,0)$.

Let

$$X(s,a)=\pi^{s-\frac12}\frac{\Gamma\left(\frac{1-s+a}{2}\right)}{\Gamma\left(\frac{a+s}{2}\right)}. \tag{2.58}$$

Note that $X_d(s)=X(s,a)|d|^{\frac12-s}$, where $a=1$ for $d<0$ and 0 otherwise. We have thus arrived at the shifted moment conjecture in [CFK+05], namely,

$$\begin{aligned}\sum_{d\in S(X)} & Z(\tfrac12+\alpha_1,\chi_d)\dots Z(\tfrac12+\alpha_k,\chi_d)\\ &\sim\sum_{\varepsilon\in T}\prod_{j=1}^k X(\tfrac12+\varepsilon_j\alpha_j,a)^{\frac12}\sum_{d\in S(X)}R(\varepsilon_1\alpha_1,\dots,\varepsilon_k\alpha_k)|d|^{\frac12\sum_{j=1}^k\varepsilon\alpha_j}\\ &=\sum_{d\in S(X)}\left(\sum_{\varepsilon\in T}\prod_{j=1}^k X(\tfrac12+\varepsilon_j\alpha_j,a)^{\frac12}R(\varepsilon_1\alpha_1,\dots,\varepsilon_k\alpha_k)|d|^{\frac12\sum_{j=1}^k\varepsilon\alpha_j}\right).\end{aligned} \tag{2.59}$$

Let $M_d(\alpha_1,\dots,\alpha_k)$ be denote the quantity in parenthesis in (2.59). Finally we invoke Lemma 2.4.2 [CFK+05] to obtain an expression for $M_d(\alpha_1,\dots,\alpha_k)$. Let $\Delta(z_1,\dots,z_k)$ be the usual Vandermonde determinant,

$$\Delta(z_1,\ldots,z_k)=\det\begin{pmatrix}1 & z_1 & z_1^2 & .. & z_1^{k-1}\\ 1 & z_2 & z_2^2 & .. & z_2^{k-1}\\ \vdots & & & & \vdots\\ 1 & z_k & z_k^2 & .. & z_k^{k-1}\end{pmatrix}=\prod_{1\le i<j\le k}(z_j-z_i). \tag{2.60}$$

Lemma 2.4.2 ([CFK⁺05, Lemma 2.5.4])**.** *Suppose F is a symmetric function of k variables, regular near $(0,\ldots,0)$ and $f(s)$ has a simple pole at 0, and let*

$$K(a_1,\ldots,a_k)=F(a_1,\ldots,a_k)\prod_{1\le i\le j\le k}f(a_i+a_j). \tag{2.61}$$

Then

$$\sum_{\varepsilon_j=\pm1}K(\varepsilon_1\alpha_1,\ldots,\varepsilon_k\alpha_k)=\frac{(-1)^{\binom{k}{2}}2^k}{(2\pi i)^k k!}\oint\cdots\oint K(z_1,\ldots,z_k)\frac{\Delta(z_1^2,\ldots,z_k^2)^2\prod_{j=1}^k z_j}{\prod_{j=1}^k\prod_{j=1}^k(z_i-\alpha_i)(z_j-\alpha_j)}. \tag{2.62}$$

Using Lemma 2.4.2 in conjunction with (2.59), gives the second form of the conjecture of Conrey, Farmer, Keating, Rubinstein, and Snaith.

Conjecture 2.4.3. *[CFK⁺05] The following holds*

$$\sum_{d\in S(X)}Z(\tfrac12+\alpha_1,\chi_d)\ldots Z(\tfrac12+\alpha_k,\chi_d)$$
$$\sim\sum_{d\in S(X)}\frac{(-1)^{\binom{k}{2}}2^k}{(2\pi i)^k k!}\oint\cdots\oint K(z_1,\ldots,z_k)\frac{\Delta(z_1^2,\ldots,z_k^2)^2\prod_{j=1}^k z_j}{\prod_{j=1}^k\prod_{j=1}^k(z_i-\alpha_i)(z_j-\alpha_j)}dz_1\ldots dz_k, \tag{2.63}$$

where

$$K(z_1,\ldots,z_k)=\prod_{j=1}^k X(\tfrac12+z_j,a)\prod_{1\le i\le j\le k}\zeta(1+z_i+z_j)$$
$$\prod_p\left(R_p(z_1,\ldots,z_k)\prod_{1\le i\le j\le k}\left(1-\frac{1}{p^{1+z_i+z_j}}\right)\right)\exp\left(\frac12\log|d|\sum_{j=1}^k z_j\right). \tag{2.64}$$

Specializing to $\alpha_j = 0$ gives

Conjecture 2.4.4. *[CFK+05] Let $S(X)$ be either $S^+(X)$ or $S^-(X)$, the set all positive or all negative fundamental discriminants d such that $|d| \leq X$, and $X_d(s) = |d|^{\frac{1}{2}-s}X(s,a)$, where $X(s,a)$ is defined in (2.58). That is, $X_d(s)$ is the factor in the functional equation $L(s,\chi_d) = \epsilon_d X_d(s) L(1-s,\chi_d)$. Let A_k be the Euler product, absolutely convergent in $\{(z_1,\ldots,z_k):\ |\Re z_j| < \frac{1}{2}\}$, defined by*

$$A_k(z_1,\ldots,z_k) = \prod_p \prod_{1\leq i\leq j\leq k}\left(1-\frac{1}{p^{1+z_i+z_j}}\right) \times \left(\frac{1}{2}\left(\prod_{j=1}^k\left(1-\frac{1}{p^{\frac{1}{2}+z_j}}\right)^{-1} + \prod_{j=1}^k\left(1+\frac{1}{p^{\frac{1}{2}+z_j}}\right)^{-1}\right)+\frac{1}{p}\right)\left(1+\frac{1}{p}\right)^{-1}. \quad (2.65)$$

Let

$$G(z_1,\ldots,z_k) = A_k(z_1,\ldots,z_k)\prod_{j=1}^k X(\tfrac{1}{2}+z_j,a)^{\frac{-1}{2}}\prod_{1\leq i\leq j\leq k}\zeta(1+z_i+z_j). \quad (2.66)$$

Let $Q_k(x)$ be the polynomial given by the k-fold residue,

$$Q_k(x) = \frac{(-1)^{k(k-1)/2}2^k}{k!}\frac{1}{(2\pi i)^k}\oint\cdots\oint\frac{G(z_1,\ldots,z_k)\Delta(z_1^2,\ldots,z_k^2)^2}{\prod_{j=1}^k z_j^{2k-1}}e^{\frac{x}{2}\sum_{j=1}^k z_j}\,dz_1\ldots dz_k. \quad (2.67)$$

Then summing over fundamental discriminants d we have

$$\sum_{d\in S(X)} L(\tfrac{1}{2},\chi_d)^k = \sum_{d\in S(X)} Q_k(\log|d|)(1+O(|d|^{-\epsilon})). \quad (2.68)$$

The quantity $Q_k(x)$ in (2.67) is a polynomial of degree $\frac{k(k+1)}{2}$. To evaluate (2.68), we use the Taylor series of the exponential function to write $e^{\frac{x}{2}\sum_{j=1}^k z_j} = \sum_{u=0}^\infty\left(\frac{x}{2}\sum_{j=1}^k z_j\right)^u/u!$. We then integrate term by term. Only the first $\frac{k(k+1)}{2}+1$ integrals will be non zero, proving that $Q_k(x)$ is of degree $\frac{k(k+1)}{2}$.

The residue picks up the coefficient of $\prod_{j=1}^k z_j^{2k-2}$ of the numerator, which has degree $k(2k-2)$. Now, the Vandermonde squared has degree $2k(k-1)$ in $z_1,\ldots,z_k$ because

$$\Delta(z_1^2,\ldots,z_k^2) = \prod_{1\leq i<j\leq k}(z_j^2 - z_i^2). \quad (2.69)$$

However the product of zetas has poles which cancel $\frac{k(k+1)}{2}$ of the factors of the Vandermonde determinants. This is because

$$\zeta(1+z_i+z_j) = \frac{1}{z_i+z_j} + \gamma + \gamma_1(z_i+z_j)^2 + \dots \tag{2.70}$$

where $\gamma, \gamma_1, \dots$ are the generalized Euler constants. and each $\frac{1}{z_i+z_j}$ cancels a factor of the Vandermonde squared (see (3.5)).

Therefore, in the multivariate Taylor expansion of $\exp(\frac{x}{2}\sum_{j=1}^{k} z_j)$, we need only take terms up to degree $\frac{k(k+1)}{2}$. Hence the highest power of x appearing is $\frac{k(k+1)}{2}$ and $Q_k(x)$ is a polynomial of degree $\frac{k(k+1)}{2}$ in x.

Alderson and Rubinstein [AR] carried out extensive computations and checked how the numerical values of the moments compare with the values predicted by the above conjecture. Figure 2.4 shows the ratio of the predicted moments and computed moments for $k = 1, \dots, 4$, and Figure 2.5 shows the same for $k = 5, \dots, 8$.

2.5 Moments from random matrix theory

While the heuristic derivation of the asymptotic formula for the moments of $L(\frac{1}{2}, \chi_d)$ relies on number theoretic tools, Conrey et al. [CFK+05] were guided by a similar formula in random matrix theory.

For A, an $N \times N$ matrix in the unitary group, let A^* be the conjugate transpose of A. From the definition of the unitary group we know that this is the inverse of A. We define

$$\Lambda_A(s) = \sum_{n=0}^{N} a_n s^n := (-1)^N \det A^* \, s^N \, \det(I - As^{-1}). \tag{2.71}$$

The following is true:

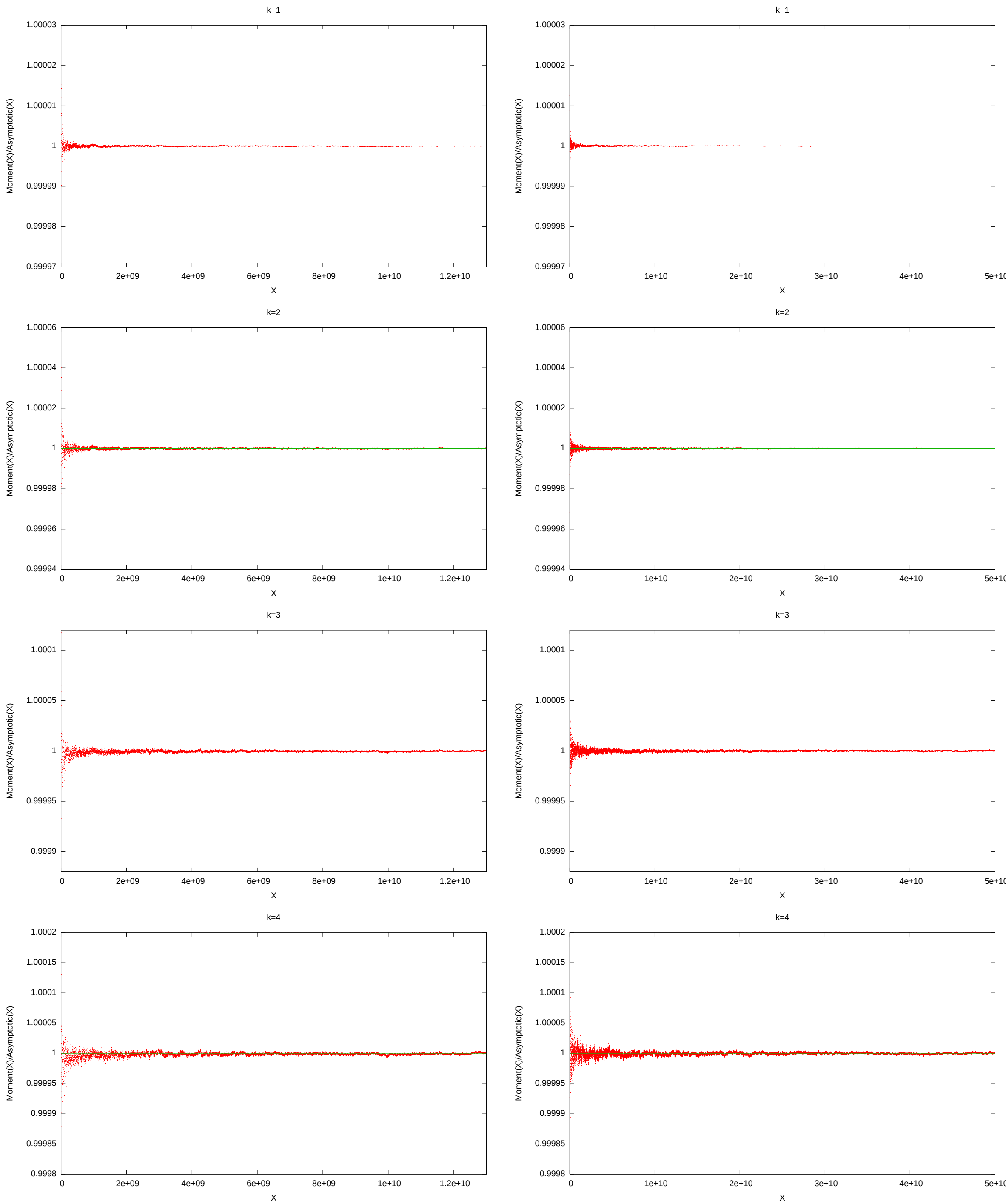

Figure 2.4: Ratio of predicted moment and numerically determined moment $\sum_{d\in S^{\pm}(X)} L(\frac{1}{2},\chi_d)^k$ for $k = 5,\ldots,8$ where $d > 0$ in the left column, and $d < 0$ in the right column. *Courtesy: Matt Alderson and Michael Rubinstein [AR]*

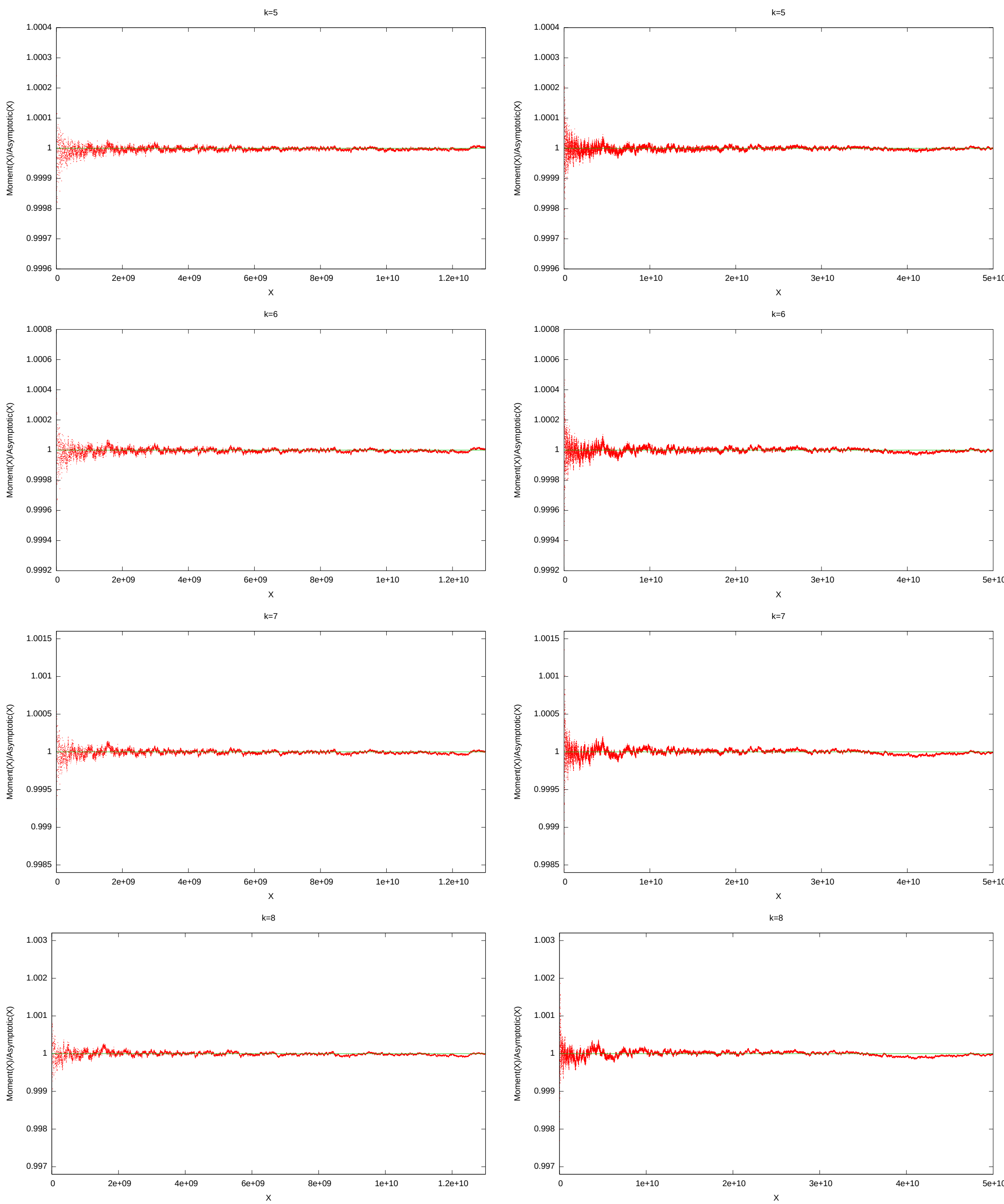

Figure 2.5: Ratio of predicted moment and numerically determined moment $\sum_{d\in S^{\pm}(X)} L(\frac{1}{2},\chi_d)^k$ for $k=5,\ldots,8$ where $d>0$ in the left column, and $d<0$ in the right column. *Courtesy: Matt Alderson and Michael Rubinstein [AR]*

- $\Lambda_A(s)$ satisfies the functional equation,

$$\Lambda_A(s) = (-1)^N \det A^* \, s^N \Lambda_{A^*}(\frac{1}{s}). \tag{2.72}$$

- All the eigenvalues lie on the unit circle $|s| = 1$. The unit circle is the random matrix theory analogue of the critical line.

- There is an approximate functional equation for $\Lambda_A(s)$,

$$\Lambda_A(s) = \sum_{m=0}^{\frac{N-1}{2}} a_m s^m + (-1)^N \det A^* s^N \sum_{n=0}^{\frac{N-1}{2}} \bar{a}_n s^{-n}. \tag{2.73}$$

- The critical value for $\Lambda_A(s)$ is $s = 1$; it corresponds to $s = \frac{1}{2}$ in the case of L-functions.

- Let $\mathcal{Z}_A(s) = ((-1)^N / \det A^*)^{-\frac{1}{2}} s^{-\frac{N}{2}} \Lambda_A(s)$. Then

$$\mathcal{Z}_A(s) = \overline{\mathcal{Z}}_A\left(\frac{1}{s}\right). \tag{2.74}$$

In this thesis we study the symplectic family of L-functions $L(s, \chi_d)$ where d ranges within positive fundamental discriminants, or within negative fundamental discriminants. Let $G(N)$ be a closed subgroup of $U(N)$. For $\alpha = (\alpha_1, \ldots, \alpha_k)$, define

$$J_k(G(N), \alpha) = \int_{G(N)} \mathcal{Z}_A(e^{-\alpha_1}) \ldots \mathcal{Z}_A(e^{-\alpha_k}) \, dA. \tag{2.75}$$

These are the shifted moment of $\mathcal{Z}$. If we set $\alpha_i = 0$ for all i, then we obtain the usual moments of $\mathcal{Z}_A(1)$. Conrey et al. [CFK+05] prove the following formula for the moments of $\mathcal{Z}_A(1)$ averaged over $USp(2N)$.

Theorem 2.5.1 ([CFK+05, Theorem 1.5.4])**.** *Let $USp(2N) = U(2N) \cap Sp(2N)$, and*

$$G(z_1, \ldots, z_k) = \prod_{1 \le i \le j \le k} (1 - e^{-z_i - z_j})^{-1}. \tag{2.76}$$

Then

$$J_k(USp(N),\alpha) = \frac{(-1)^{k(k-1)/2}2^k}{k!}\frac{1}{(2\pi i)^k} \\ \times \oint \dots \oint \frac{G(z_1,\dots,z_k)\Delta(z_1^2,\dots,z_k^2)^2 \prod_{j=1}^k z_j}{\prod_{i=1}^k \prod_{j=1}^k (z_i-\alpha_i)(z_j+\alpha_j)} e^{N\sum_{j=1}^k z_j} dz_1 \dots dz_k. \quad (2.77)$$

The contours of the above integral are small circles around 0 *enclosing* $(\alpha_1,\dots,\alpha_k)$.

Specializing to $(\alpha_1,\dots,\alpha_k) = (0,\dots,0)$*, the equation* (2.77) *is expressed in [KS00b] using the Selberg integral in a closed form as a polynomial in* N *of degree* $\frac{k(k+1)}{2}$*, namely,*

$$J_k(USp(2N),0) = \left(2^{k(k+1)/2}\prod_{j=1}^k\left(\frac{j!}{(2j)!}\right)\right)\prod_{1\le i\le j\le k}\left(N+\frac{i+j}{2}\right). \quad (2.78)$$

Note that there are certain similarities between the random matrix moments (2.77) and the moments $\sum_{d\in S(X)} L(\frac{1}{2},\chi_d)^k$ in (2.67). The function $(1-e^{-z_i-z_j})^{-1}$ is the analogue of the Riemann zeta function. Note that the same Vandermonde determinants occur in both the formulae. However, there is no simple analogue of (2.78) for the polynomial $Q_k(x)$ given implicitly by (2.67).

The arithmetic factor $A(z_1,\dots,z_k)$ which occurs in (2.67) does not occur in (2.77).

In Chapters 3 and 4, we derive formulae for the coefficients of the moment polynomial $Q_k(x)$ defined in (2.67).

Chapter 3

Leading Coefficient of the Moment Polynomial

In Section 3.1 we state the formulae for the first $k+1$ coefficients of the moment polynomial $Q_k(x)$. In order to compute the coefficients of the moment polynomial, we reformulate the problem in Section 3.2. In Section 3.3 we calculate the leading coefficient of the moment polynomial, $Q_k(x)$, and show that it agrees with the leading coefficients as conjectured by an alternate method (via the Selberg integral) by Keating and Snaith [KS00a]. The reformulation in Section 3.2 will also be used to calculate the lower order terms in Chapter 4.

3.1 The main theorem

The goal of Chapters 3 and 4 is to find the coefficients of the polynomial $Q_k(x)$, defined in (2.67). We would like to investigate the lower order terms which appear in $Q_k(x)$.

Theorem 3.1.1. *Let $Q_k(x) = Q^{\pm}(x)$ be as in* (2.67). *Let*

$$Q_k(x) = c_0(k)x^{\frac{k(k+1)}{2}} + c_1(k)x^{\frac{k(k-1)}{2}-1} + \cdots + c_{\frac{k(k+1)}{2}}(k). \tag{3.1}$$

The leading coefficient,

$$c_0(k) = \frac{a_k}{2^k}\left(\prod_{j=0}^{k-1}\frac{(2j)!}{(k+j)!}\right), \tag{3.2}$$

and for $r = 1, \ldots, k$, we have

$$c_r(k) = c_0(k)\sum_{|\lambda|=r} b_\lambda(k)N_\lambda(k), \tag{3.3}$$

where $N_\lambda(k)$ is a polynomial in k of degree at most $2r^2$, $a_k = A_k(0, \ldots, 0)$ is defined in (2.65), *and the b_λs are the Taylor coefficients of a holomorphic function, defined precisely in* (3.7) *and* (3.8). *The b_λs and $N_\lambda(k)$'s, and hence $c_r(k)$ for $r = 1, \ldots, k$, are different for the sum over $S^+(X)$ and the sum over $S^-(X)$, the subsets of positive and negative fundamental discriminants respectively.*

Theorem 3.1.1 will be proved in Chapters 3 and 4. We shall prove the equality (3.2) in Proposition 3.3.1, and the equality (3.3) in Section 4.2.

3.2 Reformulating the problem

We begin by rewriting the integrand on the right hand side of (2.67) as a ratio of a holomorphic function and a monomial. The function $G(z_1, \ldots, z_k)$ in (2.66) has a pole in each z_j at $(0, \ldots, 0)$ coming from the product of the zeta functions. These poles are eliminated by a portion of the Vandermonde determinants. Note that

$$\Delta(z_1^2, \ldots, z_k^2)^2 = \left(\prod_{1\le i\le j\le k}(z_i + z_j)\right)\frac{\Delta(z_1, \ldots, z_k)\Delta(z_1^2, \ldots, z_k^2)}{2^k\prod_{j=1}^k z_j}. \tag{3.4}$$

Specifically each factor (z_i+z_j) occurring in the Vandermonde determinants cancels a pole coming from $\zeta(1+z_i+z_j)$. The equality (3.4) can be seen by the expanding one of the factors on the left hand side of (3.4),

$$\begin{aligned}\Delta(z_1^2,\dots,z_k^2) &= \prod_{i>j}(z_i^2-z_j^2)\\ &= \Delta(z_1,\dots,z_k)\prod_{i>j}(z_i+z_j)\\ &= \Delta(z_1,\dots,z_k)\frac{\prod_{i\geq j}(z_i+z_j)}{2^k\prod_{j=1}^k z_j}. \end{aligned} \tag{3.5}$$

Substituting (3.4) into (2.67), we obtain that $Q_k(x)$ equals

$$\frac{(-1)^{k(k-1)/2}2^k}{k!}\frac{1}{(2\pi i)^k}\oint\cdots\oint A_k(z_1,\dots,z_k)$$
$$\prod_{j=1}^k X(\tfrac{1}{2}+z_j,a)^{-\frac{1}{2}}\prod_{1\leq i\leq j\leq k}(z_i+z_j)\zeta(1+z_i+z_j)$$
$$\frac{\Delta(z_1,\dots,z_k)\Delta(z_1^2,\dots,z_k^2)}{2^k\prod_{j=1}^k z_j^{2k-1}\prod_{j=1}^k z_j}\exp\left(\frac{x}{2}\sum_{j=1}^k z_j\right)dz_1\dots dz_k. \tag{3.6}$$

Now the integrand is written as a ratio of a function which is holomorphic in a neighbourhood of $(0,\dots,0)$ and a monomial.

Our next step is to find the Taylor expansion of the arithmetic factors in (3.6). We exploit the fact that the function is symmetric in its variables, and group together the terms with the same exponents in the multivariate Taylor series. We explain this more precisely below.

Let $a_k = A_k(0,\dots,0)$, and $\tau=(\tau_1,\dots,\tau_k)\in\mathbb{Z}_+^k$. Define $|\tau|=\sum\tau_i$, and $z^\tau = z_1^{\tau_1}\dots z_k^{\tau_k}$. Let m_λ be the monomial symmetric function [Mac95, chapter 1] corresponding to a partition λ. A monomial symmetric function of a partition $\lambda=\lambda_1\geq\lambda_2\geq\dots$ is a polynomial in $z_1,\dots,z_k$ consisting of all monomials of the form $z_{i_1}^{\lambda_1}z_{i_2}^{\lambda_2}\dots$. Let

$$\sum_{i=0}^{\infty}\left(\sum_{|\tau|=i}b_\tau z^\tau\right)=\sum_{i=0}^{\infty}\sum_{|\lambda|=i}b_\lambda m_\lambda(z) \tag{3.7}$$

be the power series expansion of

$$\frac{1}{a_k}A_k(z_1,\ldots,z_k)\prod_{j=1}^{k}X(\tfrac{1}{2}+z_j,a)^{-\frac{1}{2}}\prod_{1\le i\le j\le k}(z_i+z_j)\zeta(1+z_i+z_j). \tag{3.8}$$

We divide the expression by a_k to ensure that the constant term in the power series is 1. We shall calculate the Taylor series of (3.8) by calculating the Taylor series of its logarithm. This calculation is simpler if the constant term is 1, i.e. $b_0 = 1$ in (3.7). To calculate (2.67), we calculate the sum of integrals,

$$\frac{(-1)^{k(k-1)/2}2^k}{k!}\frac{a_k}{(2\pi i)^k}\sum_{i=0}^{\infty}\sum_{|\lambda|=i}b_\lambda\oint\cdots\oint m_\lambda(z_1,\ldots,z_k)$$
$$\frac{\Delta(z_1,\ldots,z_k)\Delta(z_1^2,\ldots,z_k^2)}{2^k\prod_{j=1}^k z_j^{2k}}\exp\left(\frac{x}{2}\sum_{j=1}^{k}z_j\right)dz_1\ldots dz_k. \tag{3.9}$$

Only finitely many integrals in the sum (3.9) are nonzero. Each of the integrals in (3.9) pick up the coefficient of $z_1^{2k-1}\ldots z_k^{2k-1}$ in the Taylor expansion of the numerator of the corresponding integrand. If $\deg m_\lambda(z_1,\ldots,z_k)+\deg\Delta(z_1,\ldots,z_k)+\deg\Delta(z_1^2,\ldots,z_k^2) > \deg(z_1^{2k-1}\ldots z_k^{2k-1})$, that is $|\lambda| > \frac{k(k+1)}{2}$, then in the Taylor expansion of the numerator of (3.9) the coefficient of $z_1^{2k-1}\ldots z_k^{2k-1}$ will be 0.

The above discussion can also be used to see that the degree of the polynomial $Q_k(x)$ is $\frac{k(k+1)}{2}$. Given a λ in the sum (3.9), the coefficient of the monomial $z_1^{2k-1}\ldots z_k^{2k-1}$ in the Taylor expansion of the numerator of the integrand is a constant times $x^{\frac{k(k+1)}{2}-|\lambda|}$.

3.3 The leading term

In this section, we shall calculate the leading coefficient of $Q_k(x)$, i.e. the coefficient $c_0(k)$ of $x^{\frac{k(k+1)}{2}}$. The calculation will also provide insight into how to calculate the lower order terms of $Q_k(x)$.

Proposition 3.3.1. *The leading coefficient $c_0(k)$ of $Q_k(x)$ in* (3.1) *is*

$$\frac{a_k}{2^k}\left(\prod_{j=0}^{k-1}\frac{(2j)!}{(k+j)!}\right). \tag{3.10}$$

The leading term in (3.1) corresponds to $i = 0$ in the sum (3.9). In this case there is only one integral within the inner summation sign,

$$\frac{(-1)^{\frac{k(k-1)}{2}}2^k}{k!(2\pi i)^k}a_k\oint\cdots\oint\frac{\Delta(z_1,\ldots,z_k)\Delta(z_1^2,\ldots,z_k^2)}{2^k\prod_{j=1}^k z_j^{2k}}\exp(\tfrac{x}{2}\sum_{j=1}^k z_j)dz_1\ldots dz_k. \tag{3.11}$$

To find the above integral, we first use substitution to eliminate x from the integrand. Then we introduce new variables $x_1,\ldots,x_k$ to calculate a more general integral. Making the problem more general in fact allows us to simplify the integral.

Substituting $\frac{2}{x}z_j$ for z_j in (3.11) we obtain

$$\frac{(-1)^{\frac{k(k-1)}{2}}}{k!(2\pi i)^k}a_k\oint\cdots\oint\frac{(\frac{2}{x})^{\frac{k(k-1)}{2}}(\frac{2}{x})^{k(k-1)}\Delta(z_1,\ldots,z_k)\Delta(z_1^2,\ldots,z_k^2)}{(\frac{2}{x})^{2k^2-k}\prod_{j=1}^k z_j^{2k}}$$
$$\exp(\sum_{j=1}^k z_j)\,dz_1\cdots dz_k. \tag{3.12}$$

We pull the x outside the integral, and obtain that the integral (3.11) is

$$\frac{(-1)^{\frac{k(k-1)}{2}}}{k!(2\pi i)^k}a_k\left(\frac{x}{2}\right)^{\frac{k(k+1)}{2}}\oint\cdots\oint\frac{\Delta(z_1,\ldots,z_k)\Delta(z_1^2,\ldots,z_k^2)}{\prod_{j=1}^k z_j^{2k}}\exp(\sum_{j=1}^k z_j)\,dz_1\cdots dz_k. \tag{3.13}$$

As mentioned earlier, we introduce new variables $x_1,\ldots,x_k$ and work with a more general integral. Therefore, consider

$$\frac{(-1)^{\frac{k(k-1)}{2}}}{k!(2\pi i)^k}a_k\left(\frac{x}{2}\right)^{\frac{k(k+1)}{2}}\oint\cdots\oint\frac{\Delta(z_1,\ldots,z_k)\Delta(z_1^2,\ldots,z_k^2)}{\prod_{j=1}^k z_j^{2k}}\exp(\sum_{j=1}^k x_jz_j)\,dz_1\cdots dz_k. \tag{3.14}$$

The expression (3.13) is equal to expression (3.14) evaluated at $(x_1,\ldots,x_k) = (1,\ldots,1)$.

The integral (3.14) is not easy to evaluate. We can see that if we somehow eliminated the polynomial coming from the Vandermonde determinants, then we can write the rest of the integral as a product of integrals in one variable. We shall introduce a partial differential operator which will help us move the Vandermonde determinants from inside the integral to the outside. Note that for a polynomial $P(x_1, \dots, x_k)$ in k variables,

$$P\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_1}\right) \exp\left(\sum_{j=1}^{k} x_j z_j\right) = P(z_1, \dots, z_k) \exp\left(\sum_{j=1}^{k} x_j z_j\right). \tag{3.15}$$

Let

$$q(z_1, \dots, z_k) = \Delta(z_1, \dots, z_k)\Delta(z_1^2, \dots, z_k^2). \tag{3.16}$$

Then (3.14) equals

$$\frac{(-1)^{\frac{k(k-1)}{2}}}{k!(2\pi i)^k} a_k \left(\frac{x}{2}\right)^{\frac{k(k+1)}{2}} \oint \cdots \oint q\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k}\right) \frac{\exp(\sum_{j=1}^{k} x_j z_j)}{\prod_{j=1}^{k} z_j^{2k}}\, dz_1 \cdots dz_k. \tag{3.17}$$

Pulling the differential operator outside the integral (Leibniz's rule) we conclude that (3.17) equals

$$\frac{(-1)^{\frac{k(k-1)}{2}}}{k!(2\pi i)^k} a_k \left(\frac{x}{2}\right)^{\frac{k(k+1)}{2}} q\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k}\right) \oint \cdots \oint \frac{\exp(\sum_{j=1}^{k} x_j z_j)}{\prod_{j=1}^{k} z_j^{2k}}\, dz_1 \cdots dz_k. \tag{3.18}$$

The integrand in (3.18) can be written as a product of integrals in one variable,

$$\frac{(-1)^{\frac{k(k-1)}{2}} a_k}{k!(2\pi i)^k} \left(\frac{x}{2}\right)^{\frac{k(k+1)}{2}} q\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k}\right) \prod_{j=1}^{k} \oint \frac{\exp(x_j z_j)}{z_j^{2k}} dz_j. \tag{3.19}$$

Each integral in the product of integrals appearing in (3.19) is easy to evaluate. By expanding $\exp(x_j z_j) = \sum_{n=0}^{\infty} \frac{(x_j z_j)^n}{n!}$; the residue is $\frac{x^{2k-1}}{(2k-1)!}$, the coefficient of z_j^{2k-1}. We obtain that (3.19) equals

$$\frac{(-1)^{\frac{k(k-1)}{2}} a_k}{k!} \left(\frac{x}{2}\right)^{\frac{k(k+1)}{2}} q\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k}\right) \prod_{j=1}^{k} \frac{x_j^{2k-1}}{(2k-1)!}. \tag{3.20}$$

For the leading term (3.12), an expression equal to (3.11), we have turned our residue computation into the question of determining the result of applying $q(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k})$ to $\prod_{j=1}^{k} \frac{x^{2k-1}}{(2k-1)!}$, and finding the value of the resulting polynomial at $(1, \dots, 1)$. This calculation is done in Lemma 3.3.5. The proof of Lemma 3.3.5 uses Lemmas 3.3.2, 3.3.3, and 3.3.4.

Lemma 3.3.2, a variant of [CFK$^+$08, Lemma 2.1], gives a formula for applying the differential operator $\Delta\left(\frac{\partial^2}{\partial x_1^2},\dots,\frac{\partial^2}{\partial x_k^2}\right)$ to a product of functions.

Lemma 3.3.2.

$$\Delta\left(\frac{\partial^2}{\partial x_1^2},\dots,\frac{\partial^2}{\partial x_k^2}\right)\prod_{i=1}^{k} f_i(x_i) = \left|f_i^{(2j-2)}(x_i)\right|_{k\times k}. \tag{3.21}$$

Lemma 3.3.3 gives a formula for applying a product of differentials to a determinant of functions.

Lemma 3.3.3. *Let $f_1(x),\dots,f_k(x)$ be smooth functions of one variable. Then*

$$\frac{\partial^{n_1}}{\partial x_1^{n_1}}\cdots\frac{\partial^{n_k}}{\partial x_k^{n_k}}\begin{vmatrix} f_1(x_1) & \cdots\cdots & f_k(x_1)\\ \vdots & \ddots & \vdots\\ \vdots & \ddots & \vdots\\ f_1(x_k) & \cdots\cdots & f_k(x_k)\end{vmatrix} = \begin{vmatrix} f_1^{(n_1)}(x_1) & \cdots\cdots & f_k^{(n_1)}(x_1)\\ \vdots & \ddots & \vdots\\ \vdots & \ddots & \vdots\\ f_1^{(n_k)}(x_k) & \cdots\cdots & f_k^{(n_k)}(x_k)\end{vmatrix}. \tag{3.22}$$

Proof. It is easy to see if we first look at a simple case, say $\frac{\partial}{\partial x_1}$ applied to the determinant on the left hand side of (3.22). □

Lemma 3.3.4. *The following equality holds:*

$$\det\left(\binom{k}{2j-i-1}_{1\le i,j\le k}\right) = 2^{\binom{k}{2}}. \tag{3.23}$$

Proof. (I. P. Goulden) By reversing the rows and columns, we obtain

$$\det\left(\binom{k}{2j-i-1}\right) = \det\left(\binom{k}{k-2j+i}\right) \tag{3.24}$$

which equals

$$\det\left(\binom{k}{\lambda_j - j + i}\right), \tag{3.25}$$

where $\lambda_j = k - j$.

Let $e_n(x_1, \ldots, x_k)$ be the elementary symmetric polynomials. It is the symmetric polynomial consisting of all monomials $x_{i_1} \cdots x_{i_n}$ with $i_1, \ldots, i_n$ distinct. It is clear from the definition that $e_n(1, \ldots, 1) = \binom{k}{n}$. Expression (3.25) equals

$$\det \left(e_{\lambda_j - j + i}(1, \ldots, 1) \right)_{ij}. \tag{3.26}$$

The Jacobi-Trudi identity [Mac95, (3.5) p.41] is

$$\det \left(e_{\lambda_j - j + i}(x_1, \ldots, x_k) \right) = s_{\lambda'}(x_1, \ldots, x_k). \tag{3.27}$$

Here λ is a partition, and λ' is the conjugate partition [Mac95, p.2]. The polynomial s_λ is the Schur symmetric polynomial [Mac95]. The Schur symmetric polynomial of a partition λ is defined as

$$s_\lambda(x_1, \ldots, x_k) = \frac{\det \left(x_i^{\lambda_j + k - j} \right)_{1 \le i,j \le k}}{\det \left(x_i^{k-j} \right)_{1 \le i,j \le k}}. \tag{3.28}$$

If $\lambda = (k-1, k-2, \ldots, 1, 0)$, then $\lambda = \lambda'$. Now we can see that the determinant (3.26) is $s_{(k-1,k-2,\ldots,0)}(1, \ldots, 1)$. This equals $2^{\binom{k}{2}}$, since

$$s_{(k-1,\ldots,0)}(x_1, \ldots, x_k) = \frac{\Delta(x_1^2, \ldots, x_k^2)}{\Delta(x_1, \ldots, x_k)} = \prod_{i<j} (x_i + x_j). \tag{3.29}$$

□

Lemma 3.3.5. *Let* $q(z_1, \ldots, z_k) = \Delta(z_1, \ldots, z_k)\Delta(z_1^2, \ldots, z_k^2)$, *then*

$$q\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k} \right) \prod_{i=1}^{k} \frac{x_j^{2k-1}}{(2k-1)!} \tag{3.30}$$

evaluated at $(x_1, \ldots, x_k) = (1, \ldots, 1)$ *is*

$$(-1)^{\frac{k(k-1)}{2}} \times k! \left(\prod_{j=0}^{k-1} \frac{(2j)!}{(k+j)!} \right) 2^{\frac{k(k-1)}{2}}. \tag{3.31}$$

Proof. To prove the Lemma, we relate the value of (3.30) evaluated at $(x_1, \ldots, x_k) = (1, \ldots, 1)$ to a determinant of a matrix whose entries are binomial coefficients. We then use an identity for binomial coefficients to rewrite the determinant as a product of two determinants, and evaluate each of them separately.

Let $f(x) = \frac{x^{2k-1}}{(2k-1)!}$. Applying Lemma 3.3.2, we can deduce that

$$\Delta\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k}\right) \Delta\left(\frac{\partial^2}{\partial^2 x_1^2}, \ldots, \frac{\partial^2}{\partial x_k^2}\right) \prod_{j=1}^{k} f(x_j) \tag{3.32}$$

equals

$$\Delta\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k}\right) \left|f^{(2(j-1))}(x_i)\right|_{k\times k}. \tag{3.33}$$

Expanding the Vandermonde determinant of partial differential operators, we obtain

$$\sum_{\mu\in S_k} \operatorname{sgn}(\mu) \frac{\partial^{\mu_1 - 1}}{\partial x_1^{\mu_1-1}} \cdots \frac{\partial^{\mu_k-1}}{\partial x_k^{\mu_k-1}} \begin{vmatrix} f(x_1) & f^{(2)}(x_1) & \cdots & f^{(2(k-1))}(x_1) \\ f(x_2) & f^{(2)}(x_2) & \cdots & f^{(2(k-1))}(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ f(x_k) & f^{(2)}(x_k) & \cdots & f^{(2(k-1))}(x_k) \end{vmatrix}, \tag{3.34}$$

where $\mu_1, \ldots, \mu_k$ is the image of the permutation μ of $1, \ldots, k$. Applying Lemma 3.3.3, we can see that (3.34) equals

$$\sum_{\mu\in S_k} \operatorname{sgn}(\mu) \begin{vmatrix} f^{(\mu_1-1)}(x_1) & f^{(\mu_1+1)}(x_1) & \cdots & f^{(\mu_1-1+2(k-1))}(x_1) \\ f^{(\mu_2-1)}(x_2) & f^{(\mu_2+1)}(x_2) & \cdots & f^{(\mu_2-1+2(k-1))}(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ f^{(\mu_k-1)}(x_k) & f^{(\mu_k+1)}(x_k) & \cdots & f^{(\mu_k-1+2(k-1))}(x_k) \end{vmatrix}. \tag{3.35}$$

Let $f(x) = \frac{x^{2k-1}}{(2k-1)!}$. Expression (3.35) evaluated at $(x_1, \ldots, x_k) = (1, \ldots, 1)$ is

$$\sum_{\mu\in S_n} \operatorname{sgn}(\mu) \begin{vmatrix} \frac{1}{(2k-\mu_1)!} & \frac{1}{(2k-\mu_1-2)!} & \cdots & \frac{1}{(-\mu_1+2)!} \\ \frac{1}{(2k-\mu_2)!} & \frac{1}{(2k-\mu_2-2)!} & \cdots & \frac{1}{(-\mu_2+2)!} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{(2k-\mu_k)!} & \frac{1}{(2k-\mu_k-2)!} & \cdots & \frac{1}{(-\mu_k+2)!} \end{vmatrix}. \tag{3.36}$$

Rearranging the rows to cancel the effect of μ and evaluating at $(x_1, \ldots, x_k) = (1, \ldots, 1)$, we get (3.36) equals

$$k! \begin{vmatrix} \frac{1}{(2k-1)!} & \frac{1}{(2k-3)!} & \cdots & \frac{1}{1!} \\ \frac{1}{(2k-2)!} & \frac{1}{(2k-4)!} & \cdots & \frac{1}{0!} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{k!} & \frac{1}{(k-1)!} & \cdots & 0 \end{vmatrix}. \tag{3.37}$$

We can convert the determinant (3.37) into a determinant of matrices whose entries are binomial coefficients. Multiply the j^{th} column by $\frac{1}{(2(j-1))!}$ and the i^{th} row by $(2k-i)!$ to see that (3.37) equals

$$k! \frac{0!2! \cdots (2k-2)!}{(2k-1)!(2k-2)! \cdots k!} \begin{vmatrix} \binom{2k-1}{0} & \binom{2k-1}{2} & \cdots & \binom{2k-1}{2k-2} \\ \binom{2k-2}{0} & \binom{2k-2}{2} & \cdots & \binom{2k-2}{2k-2} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{k}{0} & \binom{k}{2} & \cdots & \binom{k}{2k-2} \end{vmatrix}. \tag{3.38}$$

The matrix in (3.38) is

$$\left(\binom{2k-i}{2(j-1)} \right)_{k \times k}. \tag{3.39}$$

We shall factor the matrix (3.39) into a product of two matrices. The determinant of the first will be a power of -1, and the determinant of the second will be a power of 2. A version of Chu-Vandermonde identity [Ask75, p. 69] says that for positive integers a and b such that $0 \leq b \leq a-1$,

$$\binom{a+b}{n} = \sum_{l=1}^{a} \binom{b}{l-1} \binom{a}{n-l+1}. \tag{3.40}$$

This can be seen by equating the coefficients of $(1+x)^{a+b}$ and $(1+x)^a(1+x)^b$. Applying the identity (3.40) to $\binom{2k-i}{2(j-1)}$ with $a = k$ and $b = k-i$, we obtain

$$\binom{2k-i}{2(j-1)} = \sum_{l=1}^{k} \binom{k-i}{l-1} \binom{k}{2(j-1)-l+1}. \tag{3.41}$$

The identity (3.41) allows us to decompose (3.39) as a product of two matrices,

$$\begin{pmatrix} \binom{k-1}{0} & \binom{k-1}{1} & \cdots\cdots\cdots & & \binom{k-1}{k-1} \\ \binom{k-2}{0} & \binom{k-2}{1} & \cdots & \binom{k-2}{k-2} & 0 \\ & & \iddots & & \\ \binom{1}{0} & \binom{1}{1} & & & \vdots \\ \binom{0}{0} & 0 & & \cdots & 0 \end{pmatrix} \begin{pmatrix} \binom{k}{0} & \binom{k}{2} & \binom{k}{4} & \cdots & \binom{k}{2k-2} \\ 0 & \binom{k}{1} & \binom{k}{3} & \cdots\cdots\cdots & \\ 0 & \binom{k}{0} & \cdots\cdots\cdots & & \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \binom{k}{-k+1} & \binom{k}{-k+3} & \cdots\cdots & & \binom{k}{k-1} \end{pmatrix}$$

$$= \left(\binom{k-i}{l-1}\right)_{(i,l)} \left(\binom{k}{2(j-1)-l+1}\right)_{(l,j)}. \quad (3.42)$$

The first factor of (3.42) is an lower triangular matrix with its rows reversed. Its determinant $(-1)^{k(k-1)/2}$. By Lemma 3.3.4 the second factor has determinant $2^{\binom{k}{2}}$. □

Applying Lemma 3.3.5 to (3.20), we find that the leading coefficient is:

$$\begin{aligned} &\frac{a_k}{k!}\left(\frac{x}{2}\right)^{\frac{k(k+1)}{2}}\left(k!\frac{0!2!\cdots(2k-2)!}{(2k-1)!\cdots k!}\right)2^{\frac{k(k-1)}{2}} \\ =&a_k\left(\frac{x}{2}\right)^{\frac{k(k+1)}{2}}\left(\prod_{j=0}^{k-1}\frac{(2j)!}{(k+j)!}\right)2^{\frac{k(k-1)}{2}}. \end{aligned} \quad (3.43)$$

Hence the coefficient of the leading term is

$$\frac{a_k}{2^k}\left(\prod_{j=0}^{k-1}\frac{(2j)!}{(k+j)!}\right). \quad (3.44)$$

This proves Proposition 3.3.1.

In this chapter, the primary concern was calculating the leading term of $Q_k(x)$. In Chapter 4, we shall investigate the other lower order terms.

Chapter 4

Lower Order Terms of the Moment Polynomial

In Chapter 3, we calculated the leading coefficient of $Q_k(x)$. We calculate the second highest order coefficient of the moment polynomial $Q_k(x)$ in Section 4.1. In Section 4.2, we prove (3.3) of Theorem 3.1.1. We initially obtain a formula given in terms of certain $k \times k$ determinants of binomial coefficients. A large part of Section 4.2 is devoted to showing that each such determinant can be expressed as a power of 2, depending on k, times a polynomial in k. In Section 4.3, we state a conjecture concerning these polynomials and present experimental evidence to support it.

4.1 Second term

The calculation of the second highest order term can be reduced to that of the leading term, and it is therefore worthwhile to treat this case separately.

The second highest order term in (3.9) has only one integral within the inner sum; there is only one partition of 1. The corresponding monomial symmetric function is $\sum_{i=1}^{k} z_i$. The integral we will compute is

$$\frac{(-1)^{\frac{k(k-1)}{2}} a_k 2^k}{k!(2\pi i)^k} \oint \cdots \oint b_{(1)} \sum_{i=1}^{k} z_i \frac{\Delta(z_1, \ldots, z_k)\Delta(z_1^2, \ldots, z_k^2)}{2^k \prod_{j=1}^{k} z_j^{2k}} \exp\left(\frac{x}{2}\sum_{j=1}^{k} z_j\right) dz_1 \ldots dz_k. \tag{4.1}$$

We use the subscript $_{(1)}$ on b to emphasize that this is the coefficient of the monomial symmetric function corresponding to the partition (1) of 1. We can see that $1 + b_{(1)} \sum_{i=1}^{k} z_i$ are the first two terms of the multivariate Taylor series for (3.8) as well as for $\exp(b_{(1)} \sum_i z_i)$. The Taylor series of $\exp(b_{(1)} \sum_{j=1}^{k} z_j)$ and (3.8) have identical constant and linear terms, hence the two leading terms of the polynomial given implicitly by (4.2) are identical to those of $Q_k(x)$. We shall consider the integral

$$\frac{(-1)^{k(k-1)/2}}{k!} \frac{a_k}{(2\pi i)^k} \oint \cdots \oint \exp(b_{(1)} \sum_{j=1}^{k} z_j) \frac{\Delta(z_1, \ldots, z_k)\Delta(z_1^2, \ldots, z_k^2)}{\prod_{j=1}^{k} z_j^{2k}} \exp(\frac{x}{2} \sum_{j=1}^{k} z_j) dz_1 \ldots dz_k. \tag{4.2}$$

The integral (4.2) is also a polynomial of degree $\frac{k(k+1)}{2}$ in x. This will be used to reduce the problem of calculating the second highest order term to a problem similar to the calculation of the leading order term done in Section 3.3, by absorbing the $\exp(b_{(1)} \sum_{j=1}^{k} z_j)$ into $\exp(\frac{x}{2} \sum_{j=1}^{k} z_j)$.

In the last paragraph we saw that the two leading terms of

$$\frac{(-1)^{k(k-1)/2}}{k!} \frac{a_k}{(2\pi i)^k} \oint \cdots \oint \frac{\Delta(z_1, \ldots, z_k)\Delta(z_1^2, \ldots, z_k^2)}{\prod_{j=1}^{k} z_j^{2k}} e^{(\frac{x}{2}+b_1)\sum_{j=1}^{k} z_j} dz_1 \ldots dz_k \tag{4.3}$$

are the same as the two leading terms of (3.9), hence (3.6) i.e. $Q_k(x)$, as a polynomial in x. Substituting $u_i = (x/2 + b_1)z_i$, we see that (4.3) is

$$\frac{(-1)^{k(k-1)/2}}{k!} a_k \left(\frac{x}{2} + b_{(1)}\right)^{\frac{k(k+1)}{2}} \frac{1}{(2\pi i)^k} \oint \cdots \oint \frac{\Delta(u_1, \ldots, u_k)\Delta(u_1^2, \ldots, u_k^2)}{\prod_{j=1}^{k} u_j^{2k}} du_1 \ldots du_k. \tag{4.4}$$

The integral in (4.4) has already been evaluated since it is identical to the integral in (3.13), which we calculated in Section 3.3. This is

$$a_k \left(\frac{x}{2} + b_{(1)}\right)^{\frac{k(k+1)}{2}} \frac{0!2!\cdots(2k-2)!}{(2k-1)!(2k-2)!\cdots k!} 2^{k(k-1)/2}. \tag{4.5}$$

Here $b_{(1)}$ is the coefficient of the linear monomials in the Taylor expansion of

$$\frac{1}{a_k} A_k(z_1, \dots, z_k) \prod_{j=1}^{k} X(\tfrac{1}{2} + z_j, a)^{-\frac{1}{2}} \prod_{1 \le i \le j \le k} ((z_i + z_j)\zeta(1 + z_i + z_j)) \, . \tag{4.6}$$

The linear part of the Taylor expansion of (4.6) is also that of

$$\log A_k(z_1, \dots, z_k) - \log a_k - \tfrac{1}{2} \sum_{j=1}^{k} \log X(\tfrac{1}{2} + z_j, a) + \sum_{1 \le i \le j \le k} \log ((z_i + z_j)\zeta(1 + z_i + z_j)) \, , \tag{4.7}$$

as we see by comparing the Taylor expansion of (4.6), $1 + b_{(1)} \sum_j z_k + \dots$, with that of $\log(1+w) = w - w^2/2 + \dots$. Since (4.6) is symmetric in z_j, all the linear monomials have the same coefficient.

Using maple, it is not hard to verify the following:

- The Taylor expansion of $\log(s\zeta(1+s))$ is

$$\gamma_0 s + \text{ higher order terms.} \tag{4.8}$$

  Here γ_0 is Euler's constant.

- The linear part in the Taylor expansion of A_k, defined in (2.65), is

$$\sum_p \left(\frac{(k+1)\log p}{p-1} + \frac{-\frac{1}{2}\frac{(1-1/\sqrt{p})^{-k}}{\sqrt{p}-1} + \frac{1}{2}\frac{(1+1/\sqrt{p})^{-k}}{\sqrt{p}+1}}{\frac{1}{2}(1-1/\sqrt{p})^{-k} + \frac{1}{2}(1+1/\sqrt{p})^{-k} + \frac{1}{p}} \log p \right) \sum_{j=1}^{k} z_n. \tag{4.9}$$

- The Taylor expansion of $-\frac{1}{2}\log \prod_{j=1}^{k} X(\frac{1}{2} + z_j, a)$ is

$$-\frac{1}{2}\left(\log \pi - \psi\left(\frac{1/2 + a}{2}\right)\right) \sum_{j=1}^{k} z_j + \text{ higher order terms.} \tag{4.10}$$

  The function $\psi(x)$ is the digamma function; $\psi(x) = \frac{d}{dx} \log \Gamma(x)$.

The above list has everything to compute $b_{(1)}$;

$$b_{(1)} = -\frac{1}{2}\left(\log\pi - \psi\left(\frac{1/2+a}{2}\right)\right) + (k+1)\gamma_0$$
$$+ \sum_p \left(\frac{(k+1)\log p}{p-1} + \frac{-\frac{1}{2}\frac{(1-1/\sqrt{p})^{-k}}{\sqrt{p}-1} + \frac{1}{2}\frac{(1+1/\sqrt{p})^{-k}}{\sqrt{p}+1}}{\frac{1}{2}(1-1/\sqrt{p})^{-k} + \frac{1}{2}(1+1/\sqrt{p})^{-k} + \frac{1}{p}} \log p \right). \quad (4.11)$$

Therefore the second highest order term is

$$a_k \frac{0!2!\cdots(2k-2)!}{(2k-1)!(2k-2)!\cdots k!} 2^{\frac{k(k-1)}{2}} b_{(1)} \frac{k(k+1)}{2} \left(\frac{x}{2}\right)^{\frac{k(k+1)}{2}-1}$$
$$= \frac{a_k}{2^k}\left(\prod_{j=0}^{k-1} \frac{j!}{(k+j)!}\right) k(k+1) b_{(1)} x^{\frac{k(k+1)}{2}-1}. \quad (4.12)$$

4.2 Further lower order terms

In this section we calculate a general integral occurring in the sum of integrals (3.9). Let λ be a partition. We shall calculate

$$\frac{(-1)^{k(k-1)/2}2^k}{k!} \frac{a_k}{(2\pi i)^k} b_\lambda \oint \cdots \oint m_\lambda(z_1,\ldots,z_k)$$
$$\frac{\Delta(z_1,\ldots,z_k)\Delta(z_1^2,\ldots,z_k^2)}{2^k \prod_{j=1}^k z_j^{2k}} \exp\left(\frac{x}{2}\sum_{j=1}^k z_j\right) dz_1 \ldots dz_k. \quad (4.13)$$

We first sketch the steps involved in calculating the leading coefficient of $Q_k(x)$. Recall that when we calculated the leading coefficient in Section 3.3, we had to calculate the integral

$$\oint \ldots \oint \frac{q_0(z_1,\ldots,z_k)}{\prod_{j=1}^k z_j^{2k}} \exp\left(\frac{x}{2}\sum_{j=1}^k z_j\right) dz_1 \ldots dz_k, \quad (4.14)$$

where

$$q_0(z_1,\ldots,z_k) = \Delta(z_1,\ldots,z_k)\Delta(z_1^2,\ldots,z_k^2). \quad (4.15)$$

We made a substitution to pull x out of the integral. Then we introduced variables $x_1, \ldots, x_k$ to calculate the more general integral

$$\oint \ldots \oint q_0(z_1, \ldots, z_k) \exp\left(\sum_{j=1}^{k} x_j z_j\right) dz_1 \ldots dz_k. \tag{4.16}$$

The integral (4.16) was then calculated by introducing a differential operator

$$\oint \ldots \oint q_0\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k}\right) \exp\left(\sum_{j=1}^{k} x_j z_j\right) dz_1 \ldots dz_k. \tag{4.17}$$

The integral (4.17) now becomes a product of integrals

$$q_0\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k}\right) \prod_{j=1}^{k} \oint \exp\left(x_j z_j\right) dz_j. \tag{4.18}$$

This allowed us to separate integrals, and resulted when $x_1 = \ldots = x_k = 1$ in Lemma 3.3.5 for the leading coefficient. The steps sketched in this paragraph are explained in more detail between (3.11) and (3.20).

We now describe how to modify this approach with the addition of the monomial $m_\lambda(z_1, \ldots, z_k)$. Let

$$q(z_1, \ldots, z_k) = m_\lambda(z_1, \ldots, z_k)\Delta(z_1, \ldots, z_k)\Delta(z_1^2, \ldots, z_k^2). \tag{4.19}$$

Following the same steps as that of evaluation of the leading term, the expression (4.13) becomes

$$\frac{(-1)^{\frac{k(k-1)}{2}} a_k b_\lambda}{k!} \left(\frac{x}{2}\right)^{\frac{k(k+1)}{2} - |\lambda|} \left(q\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k}\right) \prod_{j=1}^{k} \frac{x_j^{2k-1}}{(2k-1)!}\right)_{\text{evaluated at } x_j = 1}. \tag{4.20}$$

This section is devoted to calculating (4.20).

Let $f(x) = x^{2k-1}/(2k-1)!$. Let $|\lambda| = \sum_i \lambda_i = l$, and $l(\lambda) = L$, that is L is the number of non zero elements of the partition λ. Let

$$\lambda = l^{m_l} \cdots 1^{m_1}. \tag{4.21}$$

The integer m_j is the number of j's in the partition. We assume $\lambda_j = 0$ for $j > L$. There are $\binom{k}{L}\binom{L}{m_1,m_2,\ldots}$ monomials in $m_\lambda(x_1,\ldots,x_k)$ [Sta99, 7.8]. Here $\binom{L}{m_1,m_2,\ldots}$ is the multinomial coefficient. Since we are working with symmetric functions, it is enough to compute (4.13), i.e. (4.20), for one monomial of $m_\lambda\left(\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_k}\right)$. Therefore,

$$q\left(\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_k}\right)\prod_{j=1}^{k} f(x_j)\Bigg|_{\text{evaluated at } x_j=1} \tag{4.22}$$

equals

$$\binom{k}{L}\binom{L}{m_1,m_2,\ldots}\frac{\partial^l}{\partial x_1^{\lambda_1}\ldots\partial x_L^{\lambda_L}}\Delta\left(\frac{\partial}{\partial x_1}\cdots\frac{\partial}{\partial x_k}\right)\Delta\left(\frac{\partial^2}{\partial x_1^2}\cdots\frac{\partial^2}{\partial x_k^2}\right)\prod_{j=1}^{k} f(x_j) \tag{4.23}$$

evaluated at $(x_1,\ldots,x_k) = (1,\ldots,1)$. We already have the expression for the effect of Vandermonde determinant operators in (3.35). Therefore by Lemma 3.3.3, the expression (4.23) equals

$$\binom{k}{L}\binom{L}{m_1,m_2,\ldots}\frac{\partial^l}{\partial x_1^{\lambda_1}\ldots\partial x_L^{\lambda_L}}\sum_{\mu\in S_k}\operatorname{sgn}(\mu)\det\left(f^{(\mu_i-1)+2(j-1))}(x_i)\right). \tag{4.24}$$

The expression (4.24) is equal to

$$\binom{k}{L}\binom{L}{m_1,m_2,\ldots}\sum_{\mu\in S_k}\operatorname{sgn}(\mu)\det\left(f^{(\mu_i-1+2(j-1)+\lambda_i)}(1)\right). \tag{4.25}$$

In each summand of (4.25), rearrange the rows so as to reverse the effect of μ. We get

$$\binom{k}{L}\binom{L}{m_1,m_2,\ldots}\sum_{\nu\in S_k}\det\left(f^{(i-1+2(j-1)+\lambda_{\nu_i})}(1)\right). \tag{4.26}$$

Here ν is μ^{-1}. The expression (4.26) is

$$\binom{k}{L}\binom{L}{m_1,m_2,\ldots}\sum_{\nu\in S_k}\det\left(\frac{1}{(2k-1-(i-1)-2(j-1)-\lambda_{\nu_i})!}\right)_{ij}. \tag{4.27}$$

In the sum (4.27), each determinant inside the sum is of the form

$$\det\left(\frac{1}{(2k-1-(i-1)-2(j-1)-d_i)!}\right)_{ij}, \tag{4.28}$$

and $\sum d_i = l$. In Proposition 4.2.1, we determine a necessary condition for the determinant (4.28) to be non zero. This condition will imply that a large portion of terms in (4.27) are zero.

Proposition 4.2.1. *Consider the determinant*

$$\det\left(\frac{1}{(2k-1-(i-1)-2(j-1)-d_i)!}\right)_{ij}. \tag{4.29}$$

Assume that $\sum_i d_i = l$*, and* $l < k$*. The determinant* (4.29) *is zero if any of* $d_1, \ldots, d_{k-l}$ *is non zero.*

Proof. Let u be a number between 1 and k such that d_u is non zero. The u^{th} row in the matrix is

$$\left(\frac{1}{(2k-1-(u-1)-2(j-1)-d_u)!}\right)_{1\le j\le k}. \tag{4.30}$$

Now look at the row which is d_u rows below the row u in the matrix (4.29). Let this be row v where $v = u + d_u$. Row v,

$$\left(\frac{1}{(2k-1-(v-1)-2(j-1)-d_v)!}\right)_{1\le j\le k}, \tag{4.31}$$

is identical to row u if d_v is zero. We have a necessary condition for the matrix to have a non zero determinant; for every u such that $d_u \neq 0$, either d_{u+d_u} is also non zero or $u + d_u > k$. We look at this cascading process, and see that if we start at a row above the row $k-l$, that is if $d_u \neq 0$ for some $u \le k-l$, then we cannot go down beyond row k since all d_i add to l. Hence we will have two identical rows. We can then conclude that we obtain non zero determinants in (4.29) only when $d_u = 0$ for $1 \le u \le k-l$. □

The number of possible non zero terms in the sum of determinants (4.27) is therefore $l! \times (k-l)!$. Let $(a_1, \ldots, a_l)$ be a permutation $(\lambda_{\sigma_1}, \ldots, \lambda_{\sigma_l})$ of $\lambda_1, \ldots, \lambda_l$ for $\sigma \in S_l$. We shall assume $a_i = 0$ for $i > l$. Then the expression (4.27) can be written as

$$\binom{k}{L}\binom{L}{m_1, m_2, \ldots}(k-l)! \sum_{(a_1,\ldots,a_l)} \left| \begin{array}{cccc} \frac{1}{(2k-1)!} & \frac{1}{(2k-3)!} & \cdots & \frac{1}{1!} \\ \frac{1}{(2k-2)!} & \frac{1}{(2k-4)!} & \cdots & \frac{1}{0!} \\ \vdots & \vdots & & \vdots \\ \frac{1}{(k+l)!} & \frac{1}{(k-l-2)!} & & \vdots \\ \hdashline \frac{1}{(k+l-1-a_l)!} & \frac{1}{(k+l-3-a_l)!} & & \vdots \\ \vdots & \vdots & & \vdots \\ \frac{1}{(k-a_1)!} & \frac{1}{(k-a_1-2)!} & \cdots & 0 \end{array} \right| . \tag{4.32}$$

In the expression (4.32), there are $k-l$ rows above the horizontal dashed line and l rows below the dotted line. Now consider one specific term in the sum (4.32). As in the calculation of the leading coefficient, multiply its i^{th} row by $(2k-i)!$ and its j^{th} column by $\frac{1}{(2(j-1))!}$. This enables us to write the determinant in a term of (4.32) as a product of a known quantity and a determinant of binomial coefficients,

$$\frac{\prod_{j=1}^{k}(2(j-1))!}{\prod_{i=1}^{k}(2k-i)!} \times \left| \begin{array}{cccc} \binom{2k-1}{0} & \binom{2k-1}{2} & \cdots\cdots & \binom{2k-1}{2k-2} \\ \binom{2k-2}{0} & \binom{2k-2}{2} & \cdots\cdots & \binom{2k-2}{2k-2} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{k+l}{0} & \binom{k+l}{2} & \ddots & \binom{k+l}{2k-2} \\ \binom{k+l-1-a_l}{0} & \binom{k+l-1-a_l}{2} & \cdots\cdots & \binom{k+l-1-a_l}{2k-2} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{k-a_1}{0} & \binom{k-a_1}{2} & \cdots\cdots & \binom{k-a_1}{2k-2} \end{array} \right|$$

$$\times (k+l-1)_{a_l}(k+l-2)_{a_{l-1}} \cdots (k)_{a_1}. \tag{4.33}$$

Here $(x)_n$ is the falling factorial $x(x-1)\ldots(x-n+1)$. The last factor, the product of falling factorials, is a polynomial of degree l in k. The expression (4.33) is the analogue of (3.38) for the leading coefficient. Here the difference is the presence of the product of falling factorials and $a_1, \dots, a_l$ in the determinant. These are accounted for by the fact that the $(2k-i)!$ is not entirely cancelled by the numerator of the binomial coefficients in the last l rows.

We calculate the determinant occurring as a factor in (4.33) in Proposition 4.2.2. We show that the determinant is $2^{\binom{k}{2}}$ times a polynomial in k of degree at most $2l^2 - l$. Hence (4.33) equals

$$\frac{\prod_{j=1}^{k}(2(j-1))!}{\prod_{i=1}^{k}(2k-i)!} \times 2^{\binom{k}{2}} \times \{\text{polynomial in } k \text{ of degree at most } 2l^2\}. \tag{4.34}$$

Recall that (4.34) is the value of a term inside the summation in (4.32), and (4.32) is the value of the factor involving partial differential operators in (4.20). Thus (4.20) is

$$\frac{(-1)^{\frac{k(k-1)}{2}} a_k b_\lambda}{k!} \left(\frac{x}{2}\right)^{\frac{k(k+1)}{2} - |\lambda|} \binom{k}{m_0, \dots, m_l} (k-l)! \left(\prod_{i=0}^{k-1} \frac{(2i)!}{(k+i)!}\right) \\ \times 2^{\binom{k}{2}} \times \{\text{polynomial in } k \text{ of degree at most } 2l^2\}. \tag{4.35}$$

Recall that $c_0(k) = \frac{a_k}{2^k} \left(\prod_{j=0}^{k-1} \frac{(2j)!}{(k+j)!}\right)$. Observing that $c_0(k)$ occurs as a factor in (4.35), we see that (4.35) equals

$$c_0(k)\ \frac{b_\lambda 2^{|\lambda|}}{\prod_{j=1}^{l} m_j!} \times \{\text{polynomial in } k \text{ of degree } 2l^2\}. \tag{4.36}$$

This proves Theorem 3.1.1 provided we prove Proposition 4.2.2.

Proposition 4.2.2. *Let $\varepsilon = (\varepsilon_1, \dots, \varepsilon_k)$ be a partition padded with 0s if needed. Let the weight of ε be r; that is $|\varepsilon| = r$. We assume that r is less than or equal to k. Let $m = r + l(\varepsilon)$ where $l(\varepsilon)$ is the number of non zero ε_i. Let M be a $k \times k$ matrix,*

$$M = \begin{pmatrix} \binom{2k-1-\varepsilon_k}{0} & \binom{2k-1-\varepsilon_k}{2} & \cdot\cdot & \binom{2k-1-\varepsilon_k}{2k-2} \\ \binom{2k-2-\varepsilon_{k-1}}{0} & \binom{2k-2-\varepsilon_{k-1}}{2} & \cdot\cdot & \binom{2k-2-\varepsilon_{k-1}}{2k-2} \\ \vdots & \vdots & & \vdots \\ \binom{2k-k-\varepsilon_1}{0} & \binom{2k-k-\varepsilon_1}{2} & \cdot\cdot & \binom{2k-k-\varepsilon_1}{2k-2} \end{pmatrix}. \tag{4.37}$$

Then

$$\det M = 2^{\binom{k}{2}} \times \{ \textit{ polynomial in } k \textit{ of degree at most } r(m-1)\}. \tag{4.38}$$

Remark. At the end of the chapter, we conjecture that the degree is in fact at most $2r$.

Proof of Proposition 4.2.2. To calculate the determinant of (4.37), we use the same underlying idea as in the calculation of the determinant of the matrix (3.39), which arose when we computed the leading coefficient. In this case, recall we used the Chu-Vandermonde identity (3.40) to split the matrix (3.39) as a product of two matrices: the determinant of the first being a power of -1, and the second being a power of 2. The identity (3.40) cannot be applied to M as we did when computing the leading coefficient. If we try to do the same with M, b will go out of the range in which the identity is valid. To circumvent this, we extend the $k \times k$ matrix M to a $(k+r) \times (k+r)$ matrix of the following type:

$$\widetilde{M} = \left(\begin{array}{cccc:c} \binom{2k-1-\varepsilon_k}{0} & \binom{2k-1-\varepsilon_k}{2} & & \binom{2k-1-\varepsilon_k}{2k-2} & \\ \binom{2k-2-\varepsilon_{k-1}}{0} & \binom{2k-2-\varepsilon_{k-1}}{2} & & \binom{2k-2-\varepsilon_{k-1}}{2k-2} & * \\ \vdots & \vdots & & & \\ \binom{2k-k-\varepsilon_1}{0} & \binom{2k-k-\varepsilon_1}{2} & & \binom{2k-k-\varepsilon_1}{2k-2} & \\ \hdashline & & & & \begin{matrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{matrix} \\ & & 0 & & \end{array}\right). \tag{4.39}$$

There are k rows above the dashed horizontal line, and k columns before the dashed vertical line. Notice that the top left corner of $\widetilde{M}$ is M, and we can put any quantity in the positions represented by $*$ in $\widetilde{M}$. In particular,

$$\det M = \det \widetilde{M}. \tag{4.40}$$

With $a = k + r - i - \varepsilon_{k-i+1}$ and $b = k - r$ in (3.40),we obtain

$$\binom{2k - i - \varepsilon_{k-i+1}}{2j-2} = \sum_{l=1}^{k+r} \binom{k + r - i - \varepsilon_{k-i+1}}{l-1} \binom{k-r}{2j - 2 - (l-1)}, \tag{4.41}$$

We had to extend the matrix to ensure that $a = k+r-i-\varepsilon_{k-i+1}$ and $b = k-r$ are always positive. We shall use (4.41) to write $\widetilde{M}$ as a product of matrices A and B, which are shown in (4.42) and (4.44). The first k rows of A and the first k columns of B are determined by the identity (4.41). The first factor on the right hand side of (4.41) for $1 \leq l \leq k+r$ are the elements of the i^{th} row of A. The second factor of (4.41) for $1 \leq l \leq k+r$ are the elements of j^{th} column of B. The rest of the entries of A and B will be chosen to ensure that the product AB is of the form as given in (4.39). There is no unique choice to achieve this. Our choice for A and B is given in (4.42) and (4.44).

Let

$$A = \left(\begin{array}{cccccc}
\binom{k+r-1-\varepsilon_k}{0} & \binom{k+r-1-\varepsilon_k}{1} & \multicolumn{3}{c}{\cdots\cdots\cdots\cdots\cdots\cdots} & \binom{k+r-1-\varepsilon_k}{k+r-1} \\
\binom{k+r-2-\varepsilon_{k-1}}{0} & \binom{k+r-2-\varepsilon_{k-1}}{1} & \multicolumn{3}{c}{\cdots\cdots\cdots\cdots\cdots\cdots} & \binom{k+r-2-\varepsilon_{k-1}}{k+r-1} \\
\vdots & & & & & \vdots \\
\binom{k+r-k-\varepsilon_1}{0} & \binom{k+r-k-\varepsilon_1}{1} & \multicolumn{3}{c}{\cdots\cdots\cdots\cdots\cdots\cdots} & \binom{k+r-k-\varepsilon_1}{k+r-1} \\
\hdashline
 & & & & & \\
0 & \binom{k-r}{0} & -\binom{k-r}{1} & \multicolumn{2}{c}{\cdots\cdots\cdots\cdots} & 0 \\
0 & 0 & 0 & \binom{k-r}{0} & -\binom{k-r}{1} & 0 \\
\vdots & & & & & \vdots \\
0 & 0 & \multicolumn{3}{c}{\cdots\cdots\cdots\cdots\cdots\cdots} & (-1)^{k-r}\binom{k-r}{k-r}
\end{array} \right). \tag{4.42}$$

There are k rows above the dashed line and r rows below the dashed line. We can write the above matrix concisely as

$$A = \left(\begin{array}{c}
\binom{k - i - \varepsilon_{k-i+1}}{j-1}_{\substack{1 \leq i \leq k \\ 1 \leq j \leq k+r}} \\
\hdashline
(-1)^j \binom{k-r}{j - 2(i-k)}_{\substack{k+1 \leq i \leq k+r \\ 1 \leq j \leq k+r}}
\end{array} \right). \tag{4.43}$$

Let B be a $(k+r)\times(k+r)$ matrix,

$$B = \left(\begin{array}{cccc:ccc}
\binom{k-r}{0} & \binom{k-r}{2} & \cdots\cdots & \binom{k-r}{2k-2} & 0 & 0 & 0 \\
\binom{k-r}{-1} & \binom{k-r}{1} & \binom{k-r}{3} & \binom{k-r}{2k-3} & 1 & 0 & 0 \\
\vdots & & & \vdots & 0 & 0 & 0 \\
 & & & & 0 & 1 & 0 \\
\vdots & & & \vdots & \vdots & & \vdots \\
0 & 0 & \cdots\cdots & \binom{k-r}{k-r-1} & 0 & 0 & 0
\end{array} \right). \tag{4.44}$$

Let C be the submatrix consisting of the r right most columns of B. In each of the columns of C, there is exactly one 1 and the rest are 0. The 1s are in the first r even rows, that is in the $(2i, i)$ position of C for $1 \le i \le r$. We can summarize the matrix as

$$B = \left(\begin{array}{c:c} \binom{k-r}{2j-2-(i-1)}_{\substack{1\le i\le k+r \\ 1\le j\le k}} & C \end{array} \right). \tag{4.45}$$

By (4.41), we have

$$\widetilde{M} = AB. \tag{4.46}$$

To calculate the determinant of A we shall perform a series of row operations on A, which do not change the determinant. After this we shall multiply the resulting matrix on the right by another matrix of determinant 1 to obtain a matrix of the form:

$$\left(\begin{array}{c|c} 0 & \begin{matrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{matrix} \\ \hline U & V \end{array} \right). \tag{4.47}$$

The upper right submatrix of (4.47) has 1s on its back-diagonal, and 0s elsewhere. The matrix (4.47) has the same determinant as A. We will then show that the determinant of U is a power of 2 times a polynomial in k.

Notice that most of the rows in (4.42) are of the form $\left(\binom{k+r-i}{j-1}\right)_{1\leq j\leq k+r}$, since all but $l(\varepsilon)$ rows in the top portion of (4.42) have $\varepsilon_{k-i+1} = 0$. These rows are consecutive rows of Pascal's triangle, and we can simplify them without changing the determinant by subtracting row_{i+1} from row_i in the first $k - l(\varepsilon) - 1$ rows, and repeating it. On each repetition we use one fewer row.

More precisely, we express the sequence of row operations mentioned above as a product of matrices applied to A. In the next paragraph, $D_{a,b}$ is introduced as a shorthand for this sequence of row operations.

For $a \geq 2$ and $b \leq a - 1$, let $D_{a,b}$ be the $a \times a$ matrix with diagonal entries 1 and the first b entries above the diagonal entries -1.

$$D_{a,b} = \begin{pmatrix} 1 & -1 & & & & & \\ & 1 & -1 & & & & \\ & & \ddots & & & & \\ & & & 1 & -1 & & \\ & & & & 1 & 0 & \\ & & & & & \ddots & 0 \\ & & & & & & 1 \end{pmatrix}. \tag{4.48}$$

If S is another $a \times a$ matrix, then $D_{a,b}S$ is an operation on S in which row_i becomes $row_i - row_{i+1}$ for $1 \leq i \leq b$. This operation leaves any row beyond the b^{th} row unchanged.

Recall $m = r + l(\varepsilon)$ and let $K = k + r$, then multiplying A on the left by

$$\prod_{i=1}^{K-m} D_{K,i}, \tag{4.49}$$

that is performing a sequence of row operations on A as explained in the last paragraph, we see that $\left(\prod_{i=1}^{K-m} D_{K,i}\right) A$ equals

$$\left(\begin{array}{cccccccc}
 & & \binom{m}{0} & \binom{m}{1} & \cdots\cdots\cdots & \binom{m}{m} & & \\
 & \binom{m}{0} & \binom{m}{1} & \cdots\cdots\cdots & \binom{m}{m} & & & \\
 & & & \vdots & & & & \\
 & \cdots\cdots\cdots\cdots\cdots & & & & & & \\
\binom{m}{0} & \binom{m}{1} & \cdots\cdots\cdots & \binom{m}{m} & & & & \\
\hdashline
\binom{r+l(\varepsilon)-\varepsilon_{l(\varepsilon)}}{0} & \cdots & & \binom{r+l(\varepsilon)-\varepsilon_{l(\varepsilon)}}{2j-2} & & \cdots & & \\
 & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & & & & & & \\
\binom{r-\varepsilon_1}{0} & \cdots & & \binom{r-\varepsilon_1}{2j-2} & \cdots\cdots\cdots & & & \\
\hdashline
0 & \binom{k-r}{0} & -\binom{k-r}{1} & \cdots\cdots\cdots\cdots\cdots\cdots & & 0 & & \\
0 & 0 & 0 & \binom{k-r}{0} & -\binom{k-r}{1} & \cdots\cdots\cdots & 0 & \\
 & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & & & & & &
\end{array}\right). \tag{4.50}$$

There are $k - l(\varepsilon)$ rows in the top segment. There are $l(\varepsilon)$ rows in the middle segment. There are r rows in the bottom segment.

The top $k - l(\varepsilon)$ rows of (4.50) are identical, though each is shifted by one column. They contain the coefficients of $(1+x)^m$. We shall use the fact the $(1+x)^m \times (1+x)^{-m} = 1$. We shall multiply (4.50) on the right by a lower triangular matrix with entries which are coefficients of $(1+x)^{-m}$. This operation will not change the determinant but result in only one non zero entry in the top $k - l(\varepsilon)$ rows. Let

$$H^m = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & \binom{-m}{i-j} & & \ddots & \\ & & & & 1 \end{pmatrix}. \tag{4.51}$$

Note that H^m is the m^{th} power of $H = H^1$. If we multiply the matrix (4.50) by H^m on the right, we obtain the matrix

$$\left(\prod_{i=1}^{K-m} D_{K,i}\right) A\, H^m = \left(\begin{array}{c|ccc} & 0 & & 1 \\ 0 & & \iddots & \\ & 1 & & 0 \\ \hline & & & \\ U & & V & \end{array}\right), \tag{4.52}$$

where U is a $m \times m$ matrix whose top $l(\varepsilon)$ rows are independent of k and each entry in the bottom k rows, as we shall see shortly, is a product of 2^{k-r} and a polynomial in k of degree $m - 1$.

The last r rows of A, defined in (4.42), have entries which are binomial coefficients in the expansion of $(x - 1)^{k-r}$. Up to sign, the entries of the bottom r rows of A are the coefficients of decreasing powers of x in $(x-1)^{k-r}$. Note that all coefficients of the binomial expansion occur in all of the bottom r rows of A, hence of (4.50). Because the $D_{k,i}$'s in (4.52) leave the bottom r rows untouched, only the entries of H^m affect the bottom r rows of A.

The ij entry in the bottom r rows of (4.52), i.e. with $k+1 \le i \le k+r$, and $1 \le j \le r$, is equal to

$$\sum_{l=j}^{k+r} (-1)^l \binom{k-r}{l-2(i-k)} \binom{-m}{l-j} = (-1)^{l+j} \sum_{l=0}^{k+r-j} \binom{k-r}{l+j-2(i-k)} \binom{-m}{l}. \tag{4.53}$$

Now

$$\binom{-m}{l} = (-1)^l \binom{m-1+l}{l}. \tag{4.54}$$

Let $a = j - 2(i-k)$. Therefore $-2r+1 \le a \le r-2$, and (4.53) equals

$$(-1)^j \sum_{l=0}^{\infty} \binom{k-r}{a+l} \binom{m-1+l}{l}. \tag{4.55}$$

Notice that both sums in (4.53) and (4.55) terminate when $l = k-r-a$ (and that $k-r-a < k+r-j$ since $j \le r < 2r$), since $\binom{k-r}{a+l}$ is 0 when $l > k-r-a'$.

Also notice that, when $a \le 0$, the sum in (4.55) can be written as

$$(-1)^j \sum_{l=0}^{\infty} \binom{k-r}{l} \binom{m-1+l-a}{l-a}. \tag{4.56}$$

We therefore distinguish the two cases $a \ge 0$ and $a \le 0$.

Assume $a \ge 0$. Let $T = k-r$, $(x)_n = x(x-1)\dots(x-n+1)$ and $[x]_n = x(x+1)\dots(x+n-1)$, then (4.55) is

$$\begin{aligned}\sum_{l=0}^{\infty} \frac{(T)_{a+l}}{(a+l)!} \frac{(m+l-1)+l}{l!} &= \frac{(T)_a}{a!} \sum_{l=0}^{\infty} \frac{(T-a)_l}{(a+l)_l} \times \frac{(m+l-1)_l}{l!} \\ &= \binom{T}{a} \sum_{l=0}^{\infty} \frac{[a-T]_l}{[a+1]_l} \times \frac{[m]_l}{l!} (-1)^l,\end{aligned} \tag{4.57}$$

which is equal to

$$\binom{T}{a} {}_2F_1(m, a-T; a+1; -1). \tag{4.58}$$

For $a \le 0$, we get that (4.56) equals

$$\binom{m+|a|-1}{|a|} {}_2F_1(m+|a|, -T; |a|+1; -1). \tag{4.59}$$

We know that ${}_2F_1(\alpha, \beta; \alpha; z) = \frac{1}{(1-z)^\beta}$. We use the identity [AS64, 15.2.26],

$$\gamma(z-1)\, {}_2F_1(\alpha, \beta+1; \gamma; z) = ((\gamma-\alpha)z-\beta)\, {}_2F_1(\alpha, \beta+1; \gamma+1; z) + (\beta-\alpha)\, {}_2F_1(\alpha, \beta; \gamma+1; z). \tag{4.60}$$

By repeated application of (4.60) to (4.58) to increment $a+1$ until it equals m (note that $a \leq r-1$, and $m = r + l(\varepsilon)$, hence $a+1 < m$), and of (4.60) to (4.59) to increment $|a|+1$ until it equals $|a|+m$, we see that (4.58) is 2^T times a polynomial of degree $m-1$ and (4.59) is 2^T times a polynomial of degree $m-1$.

In the submatrix U of (4.52) there are r rows with these entries. These are the entries ij with $k+1 \leq i \leq k+r$, $1 \leq j \leq r$. Now recall that $T = k-r$, $-2r+1 \leq a = j - 2(i-k) \leq r-2$. So we can conclude that the determinant of A is a polynomial in k of degree $r(m-1)$ times $2^{rT} = 2^{r(k-r)}$;

$$\det A = \{\text{Polynomial in } k \text{ of degree } r(m-1)\} \times 2^{r(k-r)}. \tag{4.61}$$

The determinant of B (4.44) can be simplified by expanding along the sparse sub matrix in the right r columns. This gives the determinant of the $k \times k$ matrix,

$$\left(\begin{array}{ccc|cccc} 1 & & * & & & & \\ & \ddots & & & * & & \\ & & 1 & & & & \\ \hline & & & \binom{k-r}{0} & \binom{k-r}{2} & \cdots\cdot & \binom{k-r}{2k-2r-2} \\ & 0 & & \binom{k-r}{-1} & \binom{k-r}{1} & & \binom{k-r}{2k-2r-3} \\ & & & \vdots & & \ddots & \vdots \\ & & & \binom{k-r}{-k+r+1} & \binom{k-r}{-k+r+2} & \cdots\cdot & \binom{k-r}{k-r-1} \end{array}\right). \tag{4.62}$$

We have already computed this determinant in Lemma 3.3.4; the determinant is $2^{\binom{k-r}{2}}$.

Combining the information in the last two paragraphs, we conclude that (4.37) is

$$\{\text{Polynomial in } k \text{ of degree } r(m-1)\} \times 2^{\binom{k}{2}}, \tag{4.63}$$

where m is always less than or equal to $2r$ since it is less than or equal to r plus the length of a partition of r, $m = r + l(\varepsilon)$.

□

4.3 A conjecture

We conclude this chapter with a conjecture we mentioned in a remark following Proposition 4.2.2.

Conjecture 4.3.1. *Given a partition λ of weight w, then for $k \geq w$ the determinant of the matrix*

$$\begin{pmatrix} \binom{k-\lambda_1}{0} & \binom{k-\lambda_1}{2} & \cdot\cdot & \binom{k-\lambda_1}{2k-2} \\ \binom{k+1-\lambda_2}{0} & \binom{k+1-\lambda_2}{2} & \cdot\cdot & \binom{k+1-\lambda_2}{2k-2} \\ \vdots & & & \vdots \\ \binom{2k-1-\lambda_k}{0} & \binom{2k-1-\lambda_k}{k+1} & \cdot\cdot & \binom{2k-1-\lambda_k}{2k-2} \end{pmatrix} \tag{4.64}$$

is

$$2^{\binom{k}{2}-w} \times p(k). \tag{4.65}$$

Here $p(k)$ is an integer valued polynomial of degree w; that is, it is integer linear combination of binomial coefficients, $\sum_i \alpha_i \binom{k}{r_i}$, for some integers α_i and non negative integers r_i.

Table 4.1 gives a list of polynomial $p(k)$ for some partitions. It is known that any integer valued polynomial can be written as an integer linear combinations of the set of binomial coefficients,

$$\left\{ \binom{k}{n} \mid 0 \leq n < \infty \right\}. \tag{4.66}$$

Partition	Polynomial	Coefficients
(1)	$k+1$	$[1,1]$
(2)	$\frac{1}{2}k^2+\frac{3}{2}k+1$	$[1,2,1]$
(1, 1)	$\frac{1}{2}k^2+\frac{1}{2}k-1$	$[-1,1,1]$
(3)	$k^3+k^2+\frac{11}{6}k+1$	$[1,3,3,1]$
(2, 1)	$\frac{1}{3}k^3+k^2-\frac{1}{3}k-2$	$[-2,1,4,2]$
(1, 1, 1)	$\frac{1}{6}k^3-\frac{7}{6}k+1$	$[-1,1,1,1]$
(4)	$\frac{1}{24}k^4+\frac{5}{12}k^3+\frac{35}{24}k^2+\frac{25}{12}k+1$	$[1,4,6,4,1]$
(3, 1)	$\frac{1}{8}k^4+\frac{3}{4}k^3+\frac{7}{8}k^2-\frac{7}{4}k-3$	$[-3,0,8,9,1]$
(2, 2)	$\frac{1}{12}k^4+\frac{1}{3}k^3-\frac{1}{12}k^2-\frac{4}{3}k-1$	$[-1,-1,3,5,2]$
(2, 1, 1)	$\frac{1}{8}k^4+\frac{1}{4}k^3-\frac{9}{8}k^2-\frac{5}{4}k+3$	$[3,-2,1,6,3]$
(1, 1, 1, 1)	$\frac{1}{24}k^4-\frac{1}{12}k^3-\frac{13}{24}k^2+\frac{19}{12}k-1$	$[-1,1,-1,1,1]$

Table 4.1: Polynomials $p(k)$ for partitions up to partition of 4.

The polynomials in the middle column of of Table 4.1 can be written as integer linear combinations of (4.66). The third column gives this representation. For example, the polynomial corresponding to the partition $(2,1)$ is $\frac{1}{3}k^3+k^2-\frac{1}{3}k-2$. The third column of Table 4.1 says that

$$\frac{1}{3}k^3+k^2-\frac{1}{3}k-2=-2\binom{k}{0}+\binom{k}{1}+4\binom{k}{2}+2\binom{k}{3}. \tag{4.67}$$

We have numerically verified the conjecture for all 271 partitions of weight up to 12. We present more evidence in the Appendix for weights up to 8.

Part II

Upper and Lower bounds for $\int_{t_1}^{t_2} S(t)\,dt$

Chapter 5

Introduction to Turing's method

Numerically finding zeros of an L-function in an interval on the critical line involves two steps. The first step is to search for a list of zeros of the L-function in the critical strip. This is done by looking for changes in sign of the Hardy Z-function on the critical line. Once we have a list of zeros, the second step is to verify that we have found all of them. In this thesis we are concerned with this second step for modular form L-functions. All of the current methods for verification are generalizations of Turing's method for the Riemann zeta function [Tur53]. In Section 5.1, we discuss the history of Turing's method. In Section 5.2, we introduce newforms and their L-functions; and in Chapters 6 and 7, we extend Turing's method to these L-functions.

5.1 History of Turing's method

In his work on computing the zeros of the Riemann Zeta function in an interval on the critical line, Turing devised a way to prove, given a set of zeros in an interval, that all the zeros of the Riemann zeta function in this interval have been found. We need some background information to describe this method. Let $N(T)$ be the number of zeros of $\zeta(\sigma+it)$ for $0<\sigma<1$, $0<t<T$. For any t which is not an ordinate of a zero of $\zeta(s)$, define

$$S(t)=\frac{1}{\pi}\arg\zeta(\tfrac{1}{2}+it). \tag{5.1}$$

The arg in (5.1) is measured by continuous variation along the line from $\infty+it$ to $\frac{1}{2}+it$ starting with the value 0. Define

$$\vartheta(t)=\arg\Gamma(\tfrac{1}{2}+it)-\frac{t}{2}\log\pi+1. \tag{5.2}$$

Then the argument principle combined with the functional equation gives (see [Dav00, p.98])

$$N(T)=\frac{1}{\pi}\vartheta(t)+S(t). \tag{5.3}$$

Using Stirling's approximation of $\log\Gamma(z)$ we can deduce that

$$N(T)=\frac{T}{2\pi}\log\frac{T}{2\pi}-\frac{T}{2\pi}+\frac{7}{8}+S(T)+O(T^{-1}). \tag{5.4}$$

Turing used a result of Littlewood [Lit24, Theorem7] in order to verify that all the zeros have been found. Littlewood showed that $\int_0^T S(t)\,dt=O(\log T)$. Turing proved an explicit version of Littlewood's result. Lehman [Leh70] later corrected a few mistakes in Turing's work. It is his version which we have reproduced here.

Theorem 5.1.1 ([Leh70])**.** *If $t_2>t_1>168\pi$, then*

$$\left|\int_{t_1}^{t_2} S(t)\,dt\right|\le 1.91+0.114\log\frac{t_2}{2\pi} \tag{5.5}$$

When we are computing zeros of $\zeta(s)$, we can easily obtain $N(T)$ by counting the number of zeros we have found up to a height T. This gives a way of computing $S(T)$,

$$S(T)=N(T)-\frac{1}{\pi}\vartheta(t).$$

If we miss a zero say at $\frac{1}{2}+it_0$ while computing zeros between t_1 and t_2, i.e. $N(t)$ is wrong after t_0, then, for $t > t_0$, the computed value of $S(t)$ will be off the actual value by at least a non zero integer. If we use this erroneous $S(t)$ to numerically compute $\int_{t_1}^{x} S(t)\,dt$, it will deviate from the real integral by a non zero integer multiple of $x-t_0$ for $x > t_0$. The bound for $\int_{t_1}^{x} S(t)\,dt$ given by (5.5) is a linear function of $\log x$. The error grows exponentially faster as a function of x than the bound (5.5) on $\int_{t_1}^{x} S(t)\,dt$. We can show that we have found all the zeros in the interval (t_1, t_2), if we verify that the inequality on $\int_{t_1}^{x} S(t)\,dt$ given by (5.5) up to $x = t_2 + 1.91 + 0.114 \log \frac{t_2}{2\pi}$. Note that to verify that we have found all the zeros up to t_2, we have to compute $S(t)$ up to $x = t_2 + 1.91 + 0.114 \log \frac{t_2}{2\pi}$. To compute $S(t)$, we have to find zeros up to $t_2 + 1.91 + 0.114 \log \frac{t_2}{2\pi}$.

When computing zeros of the Hardy Z-function, one looks for sign changes in the Z-function by advancing along the critical line in small increments. Skipping over a sign change misses two zeros. Hence an even number of consecutive zeros are missed. That makes it twice as easy to detect violations of (5.5).

The proof of Theorem 5.1.1 uses the following lemma by Titchmarsh. In fact, the proof of Turing's method for every L-function uses this lemma.

Lemma 5.1.2 ([TH86, Theorem 9.9])**.** *Let* $S(t) = \frac{1}{\pi} \arg L(\frac{1}{2} + it)$. *Then*

$$\int_{t_1}^{t_2} S(t)\,dt = \int_{\frac{1}{2}}^{\infty} \log|L(\sigma + it_1)|d\sigma - \int_{\frac{1}{2}}^{\infty} \log|L(\sigma + it_2)|d\sigma$$

If we can find upper and lower bounds for $\int_{\frac{1}{2}}^{\infty} \log|L(\sigma+it)|\,d\sigma$, this lemma immediately gives us an upper and lower bound for $\int_{t_1}^{t_2} S(t)\,dt$. Theorem 5.1.1 is proved by finding these bounds.

The zeros of the Riemann zeta function have been numerically studied for over 100 years. For Dirichlet L-functions, Rumely [Rum93] verified the generalized Riemann hypothesis numerically up to a certain height on the critical line. He verified the Riemann hypothesis up to $t = 10000$ for Dirichlet characters with conductors up to 13. He also verified the generalized Riemann hypothesis for a few other Dirichlet L-functions up to lower heights. Tollis [Tol97] verified the generalized Riemann hypothesis for Dedekind zeta functions ζ_K of number fields K. Their Turing type results are stated below.

Theorem 5.1.3 ([Rum93, Theorem 3])**.** *Let $L(s,\chi)$ be a Dirichlet L-function, and χ be a primitive Dirichlet character modulo Q. Then for $t > 50$,*

$$-3.4507 - 0.24 \log\left(\frac{Qt}{2\pi}\right) \leq \int_{\frac{1}{2}}^{\infty} \log|L(\sigma+it,\chi)|\, d\sigma \leq 2.3288 + 0.15 \log\left(\frac{Qt}{2\pi}\right).$$

Theorem 5.1.4 ([Tol97])**.** *Let K be a number field of degree N and discriminant D_K. If $t > 40$, then*

$$\begin{aligned} - 3.4489N - 0.24 \log\left[|D_K|\left(\frac{t}{2\pi}\right)^N\right] &\leq \int_{\frac{1}{2}}^{\infty} \log|\zeta_K(s)|\ ds \\ &\leq 0.8252 + 2.329N + 0.1407 \log\left[|D_K|\left(\frac{t}{2\pi}\right)^N\right]. \end{aligned}$$

Trudgian [Tru09] improved the bounds for the Riemann zeta function, Dirichlet L-function, and Dedekind zeta function.

Theorem 5.1.5 ([Tru09])**.**

- *Let $S(t) = \frac{1}{\pi}\arg\zeta(1/2+it)$ then for $t_2 > t_1 > t_0 > 168\pi$*

$$\left|\int_{t_1}^{t_2} S(t)\, dt\right| \leq 2.066 + 0.0585 \log t_2$$

- *Let $S_\chi(t) = \frac{1}{\pi}\arg L_\chi(\frac{1}{2}+it)$, where χ is a Dirichlet character modulo Q, and L_χ is the Dirichlet L-function then*

$$\left|\int_{t_1}^{t_2} S_\chi(t)\, dt\right| \leq 1.9744 + 0.0833 \log\frac{Qt_2}{2\pi}$$

- *Let* $S_K(t) = \frac{1}{\pi}\arg\zeta_K(\frac{1}{2}+it)$, *where* K *is a number field of degree* N, *then*

$$\left|\int_{t_1}^{t_2} S_K(t)\,dt\right| \le a + bN + g\log\left(|D_K|\left(\frac{t_2}{2\pi}\right)^N\right)$$

  D_K *is the discriminant. Here* a, b, g *are to be found by optimization. The optimized* $a, b,$ *and* g *depend on the number field and the region on the critical line where the computations are being carried out.*

5.2 Definitions and notations

Define

$$\Gamma_0(N) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \;:\quad c \equiv 0 \mod N\right\}. \tag{5.6}$$

Let f be a newform [AL70, p.145] (see also [DS05, p.187]) of weight k and level N; that is, $f \in S_k(\Gamma_0(N))$ [DS05], then f can be written as

$$f(z) = \sum_{n\ge 1} a_n e^{2\pi i n z}. \tag{5.7}$$

Recall that f is also an eigenfunction of the Atkin-Lehner involution ω_N, where

$$\omega_N = \begin{pmatrix} & -1 \\ N & \end{pmatrix}. \tag{5.8}$$

We further assume that $a_1 = 1$. Define an L-function associated to the modular form,

$$\tilde{L}(s,f) = \sum_{n\ge 1}\frac{a_n}{n^s}. \tag{5.9}$$

This Dirichlet series converges absolutely in the half plane $\Re s > \frac{k+1}{2}$. Deligne [Del80] showed that the coefficients above satisfy $a_p \le 2p^{\frac{k-1}{2}}$. It is known [Li75, Theorem 3] that this L-function has analytic continuation to $\mathbb{C}$. It has an Euler product representation,

$$\tilde{L}(s,f) = \prod_{p\mid N}(1-a_p p^{-s})^{-1}\prod_{p\nmid N}\left(1-a_p p^{-s}+p^{k-1}p^{-2s}\right)^{-1}. \tag{5.10}$$

Let $F(s,f) = N^{s/2}(2\pi)^{-s}\Gamma(s)\tilde{L}(s,f)$. Then $F(s,f)$ satisfies the functional equation

$$F(s,f) = \varepsilon F(k-s,f), \tag{5.11}$$

where ε is a root of unity. More precisely ε is $(-1)^{\frac{k}{2}}$ times the eigenvalue of the Atkin-Lehner involution. In this thesis, we will only use the fact that it is a root of unity. More generally if given a character ψ modulo N and f a newform in $S_k(\Gamma_0(N),\psi)$ [Li75], then

$$F(s,f) = \varepsilon \bar{F}(k-s,f) \tag{5.12}$$

where $\bar{G}(s) = \overline{G(\bar{s})}$. Let $\Lambda(s,f) = F(s+\frac{k-1}{2},f)$. Then Λ satisfies

$$\Lambda(s,f) = \varepsilon\bar{\Lambda}(1-s,f) \tag{5.13}$$

Define $L(s,f) = \tilde{L}(s+\frac{k-1}{2},f)$. This L-function has $\Re s = 1/2$ as its critical line, and the Dirichlet series converges absolutely for $\Re s > 1$. This L-function satisfies the product formula

$$L(s,f) = \prod_{p|N}(1-A_p p^{-s})^{-1}\prod_{p\nmid N}(1-A_p p^{-s}+p^{-2s}), \text{ where } A_p = \frac{a_p}{p^{\frac{k-1}{2}}}. \tag{5.14}$$

Let $S(t,f)$ be as defined in page 3. We can write $S(t,f)$ as

$$S(t,f) = -\frac{1}{\pi}\Im\left(\int_{\frac{1}{2}}^{\infty}\frac{L'(\sigma+it,f)}{L(\sigma+it,f)}\,d\sigma\right) \qquad \text{for } s=\sigma+it. \tag{5.15}$$

For $t_2 > t_1 > 0$ we shall find the upper and lower bounds for

$$\int_{t_1}^{t_2} S(t,f)\,dt. \tag{5.16}$$

Recall that the verification of Riemann hypothesis for an L-function requires a relationship between $N(T,f)$, the number of zeros up to height T, $\vartheta(T,f) = \arg\Gamma(\frac{1}{2}+it+\frac{k-1}{2}) - t\log\frac{\sqrt{N}}{2\pi}$ and $S(T,f) = \frac{1}{\pi}\arg L(\frac{1}{2}+iT,f)$. The following lemma gives the needed modular form analogue of (5.4).

Lemma 5.2.1. *Let f be a newform of weight k and level N. Let $L(s,f)$ be the L-function associated to this newform. Then*

$$\begin{aligned} N(T,f) &= \frac{1}{\pi}\vartheta(T,f) + S(T,f) \\ &= \frac{k-1}{4} + \frac{T}{\pi}\log\left(\frac{T\sqrt{N}}{2\pi}\right) - \frac{T}{\pi} + S(T,f) + O(\frac{1}{T}). \end{aligned}$$

Reference for proof. The proof of this statement is exactly the same as that of (5.4), which can be found in [Dav00, Chapter 15]. □

By Lemma 5.1.2, to find an upper and lower bound for $\int_{t_1}^{t_2} S(t,f)dt$, it is enough to find the upper and lower bounds for $\int_{\frac{1}{2}}^{\infty} \log|L(\sigma+it,f)|\,d\sigma$. In the next two chapters we shall find an upper bound and a lower bound for this integral.

5.3 Results

Let $s=\sigma+it$. Proposition 5.3.1 gives an upper bound for $\int_{\frac{1}{2}}^{\infty}\log|L(s,f)|\,d\sigma$. Proposition 5.3.2 gives a lower bound for $\int_{\frac{1}{2}}^{\infty}\log|L(s,f)|\,d\sigma$.

Proposition 5.3.1. *Let η be a real number between 0 and 0.5. Then*

$$\begin{aligned} \int_{1/2}^{1+\eta} \log|L(s,f)|d\sigma &\leq (\eta+1/2)\left[\log\left(\frac{\sqrt{N}}{2\pi}\right) + 2\log(\zeta(1+\eta))\right] \\ &+ \frac{1}{t}\left[\frac{(1+k)}{2}(1+\eta)(\eta+\frac{1}{2}) + (1+\eta-\frac{k+1}{2})(\frac{(1+\eta)^2-\frac{1}{4}}{2}) + \left(\frac{(\eta+1)^3-\frac{1}{8}}{3}\right)\right] \\ &+ \frac{(\eta+\frac{1}{2})^2}{2}\log t + 2I(1+\eta). \end{aligned} \tag{5.17}$$

Proposition 5.3.2. *Let $0<d<0.5$. Let $J(d)$ be as in (7.10), and $\epsilon_{t,k}$ be as in (7.23). Then for $t>\frac{k}{2}+2$,*

$$-\int_{\frac{1}{2}}^{\infty} \log|L(s,f)|\,ds \leq a + b\log t, \tag{5.18}$$

where

$$a = -J(d) + d^2 \log 4 \left[\log \frac{\sqrt{N}}{2\pi} - 2\frac{\zeta'(d+\frac{1}{2})}{\zeta(d+\frac{1}{2})} - \sum_{q|N} \frac{\log q}{q^{d+\frac{1}{2}} - 1} \right] - d^2 \log \frac{\sqrt{N}}{2\pi} + 3\epsilon_{t,k}, \quad (5.19)$$

and

$$b = d^2(\log 4 - 1). \quad (5.20)$$

Proposition 5.3.1 is proved in Chapter 6, and Proposition 5.3.2 is proved in Chapter 7.

5.4 Discrete version of Turing's inequality

In this section, we explain the application of Turing's inequality to verify that all zeros up to a given height of an L-function have been found, and in which only computations on the critical line are needed. In order to make the exposition easy we assume that $L(\frac{1}{2}, f) \neq 0$. We shall use the inequality developed in earlier chapters and Turing's method as explained in Edward's book [Edw01, p.173]. We assume that we have a sequence of numbers h_m satisfying certain conditions which will be clarified below.

Let g_m be the Gram points of $L(s,f)$, i.e values of t such that $\vartheta(t,f) = m\pi$. At these points $S(t,f)$ has integral values. Let h_m be real numbers such that $(-1)^m Z(g_m+h_m) \geq 0)$ and $g_m + h_m$ is an increasing sequence. One should think of this as small adjustments to the Gram points needed to identify sign changes in $Z(t,f)$. When we have a Gram point g_m such that $N(g_m,f) = m$ (i.e. $S(g_m,f) = 0$), we always choose h_m to be 0.

Fix an m such that h_m is 0. Recall that $N(T,f) = \frac{1}{\pi}\vartheta(T,f) + S(T,f)$. Let $C(t)$ be an increasing step function which is 0 at $g_m + h_m$ and i at $g_{m+i} + h_{m+i}$ for $i > 0$; i.e. $C(t) = i$ for $g_{m+i} + h_{m+i} \leq t < g_{m+i+1} + h_{m+i+1}$. The Hardy Z-function $Z(t,f)$ changes sign between these points. In other words, $Z(t,f)$ has a zero between these points. $N(t,f)$ must increase by one between these points, hence

$$N(t,f) \geq N(g_m,f) + C(t). \tag{5.21}$$

Let

$$C_1(t) = \left\lfloor \frac{1}{\pi}\vartheta(t,f) \right\rfloor. \tag{5.22}$$

Clearly $C_1(t)+1 > \frac{1}{\pi}\vartheta(t,f) \geq C_1(t)$. Note that $\frac{1}{\pi}\vartheta(g_m,f) = m$. Substituting $S(g_m,f)+m$ for $N(g_m,f)$, and $S(t,f) + \frac{1}{\pi}\vartheta(t,f)$ for $N(t,f)$ in (5.21), we obtain

$$S(g_m,f) \leq S(t,f) + \frac{1}{\pi}\vartheta(t,f) - m - C(t) \tag{5.23}$$

$$\leq S(t,f) + C_1(t) + 1 - m - C(t). \tag{5.24}$$

Integrating this from g_m to g_{m+k} we obtain

$$S(g_m,f)(g_{m+k} - g_m) \leq \int_{g_m}^{g_{m+k}} S(t,f) + 1\,dt + \sum_{i=1}^{k-1} h_{m+i}, \tag{5.25}$$

giving the upper bound

$$S(g_m,f) \leq 1 + \frac{\int_{g_m}^{g_{m+k}} S(t,f)\,dt + \sum_{i=1}^{k-1} h_{m+i}}{g_{m+k} - g_m}. \tag{5.26}$$

If the last sum does not increase very fast, we get an upper bound on $S(g_m,f)$ which will be violated if any zero is missed.

Similarly we can find a lower bound for $S(g_m,f)$. Recall that we have fixed g_m such that $S(g_m,f) = 0$. Analogous to $C(t)$, define $R(t)$ for $g_{m-k} + h_{m-k} \leq t \leq g_m$ as a decreasing step function which is 0 at g_m:

$$R(t) = s, \quad \text{for} \;\; g_{m-s-1} + h_{m-s-1} < t \leq g_{m-s} + h_{m-s}, \tag{5.27}$$

where s is an integer greater than or equal to 0. It is easy to see that

$$N(t,f) \leq N(g_m,f) - R(t), \tag{5.28}$$

for t between g_{m-k} and g_m. Similar to $C_1(t)$, we define $R_1(t)$ between g_{m-k} and g_m. It is a decreasing step function which takes the value 0 at g_m; more precisely

$$R_1(t) = u, \qquad \text{for } \; g_{m-u-1} < t \le g_{m-u}, \tag{5.29}$$

where u is an integer greater than or equal to 0. From the definition of $R(t)$, it is easy to verify that

$$\left\lfloor \frac{1}{\pi} \vartheta(t, f) \right\rfloor = m - R_1(t) - 1. \tag{5.30}$$

Using the fact that $N(t, f) = \frac{1}{\pi}\vartheta(t, f) + S(t, f)$, and substituting (5.30) in (5.28), we see

$$S(t, f) + m - R_1(t) - 1 \le S(g_m, f) + m - R(t). \tag{5.31}$$

Hence

$$S(t, f) + R(t) - R_1(t) - 1 \le S(g_m, f). \tag{5.32}$$

Integrating (5.32) between g_{m-k} and g_m, we obtain

$$\int_{g_{m-k}}^{g_m} S(t, f) - 1 \; dt - \sum_{j=1}^{k-1} h_{m-j} \le (g_m - g_{m-k}) S(g_m). \tag{5.33}$$

Hence

$$S(g_m) \ge -1 + \frac{\int_{g_{m-k}}^{g_m} S(t, f) - \sum_{j=1}^{k-1} h_{m-j}}{g_m - g_{m-k}}. \tag{5.34}$$

Proposition 5.4.1. *Equations* (5.26) *and* (5.34) *give the upper and lower bounds for* $S(t)$ *at the Gram points* g_m. *The bounds for the integral in the numerator are given by Proposition 5.3.1 and Proposition 5.3.2.*

The discrete version of Turing's inequality can be used to find Gram points g_m such that $S(g_m, f) = 0$. We choose a Gram point such that $S(g_m, f)$ is an even integer, i.e. $(-1)^m Z(g_m, f) \ge 0$. We find a few zeros on either side of this Gram point, which is then used to find the h_js. If we can show, using (5.26) and (5.34), that $-2 < S(g_m, f) < 2$, then we immediately obtain the required Gram point. Otherwise start with another Gram point where $S(t, f)$ is an even integer.

The ability to find Gram points such that $S(g_m, f) = 0$ can be used to find $N(T, f)$ for any T. We know that $N(g_m, f) = m$ when g_m is a Gram point such that $S(g_m, f) = 0$. To find $N(T, f)$, we find two Gram points g_m and g_n such that $g_m \leq T \leq g_n$, and $S(g_m, f) = S(g_n, f) = 0$. We just have to find all $n-m$ zeros between g_m and g_n to determine $N(T, f)$.

Chapter 6

Upper Bound for $\int_{\frac{1}{2}}^{\infty} \log |L(\sigma + it, f)| \, d\sigma$

The upper bound for the integral $\int_{\frac{1}{2}}^{\infty} \log |L(\sigma + it, f)| \, d\sigma$ is found by dividing the integral into two parts, an integral from $\frac{1}{2} + it$ to $1 + c + it$, and an integral from $1 + c + it$ to $\infty + it$, for some positive real c,

$$\int_{\frac{1}{2}}^{\infty} \log |L(\sigma + it, f)| \, d\sigma = \int_{\frac{1}{2}}^{1+c} \log |L(\sigma + it, f)| \, d\sigma + \int_{1+c}^{\infty} \log |L(\sigma + it, f)| \, d\sigma. \quad (6.1)$$

The number c is left as an unknown which is to be calculated during computation. It will be determined by where on the critical line one is performing the computation, so that we have a tight bound.

We find an upper bound for each of the integrals on the right hand side of (6.1). For the first integral, we will find an upper bound for $|L(s, f)|$ in a vertical strip containing $\frac{1}{2} \leq \Re s \leq 1 + c$. For the second integral, we find an upper bound in terms of a similar integral for the Riemann zeta function.

In Section 6.1, we find an upper bound for $|L(s, f)|$ in a vertical strip containing the strip $\frac{1}{2} \leq \Re s \leq 1 + c$. In Section 6.2, we use the bound found in Section 6.1 to find an upper bound for $\int_{\frac{1}{2}}^{\infty} \log |L(\sigma + it, f)| \, d\sigma$.

6.1 Bound for $L(s, f)$ in critical strip

To find an upper bound for $|L(s, f)|$ in a vertical strip containing the critical strip, we will use a theorem of Rademacher, a variant of the Phrägmen-Lindelöf theorem. Theorem 6.1.1 states that for analytic functions satisfying certain growth conditions, we can find an upper bound for its absolute value inside a vertical strip if we know its behaviour on the left and right edges of the vertical strip.

Theorem 6.1.1 ([Rad59, Theorem 1])**.** *Let $f(s)$ be a regular analytic function in the strip $S(a, b) = \{s|\, a \leq \Re s \leq b\}$ and satisfy for certain positive constants e and C*

$$|f(s)| < Ce^{|t|^e}. \tag{6.2}$$

Suppose moreover that

$$\begin{cases} |f(a+it)| \leq A|Q+a+it|^{\alpha} \\ |f(b+it)| \leq B|Q+b+it|^{\beta} \end{cases} \tag{6.3}$$

with

$$Q + a > 0, \tag{6.4}$$

$$\alpha \geq \beta. \tag{6.5}$$

Then inside the strip $S(a, b)$

$$|f(s)| \leq \left(A|Q+s|^{\alpha}\right)^{\frac{b-\sigma}{b-a}} \left(B|Q+s|^{\beta}\right)^{\frac{\sigma-a}{b-a}}. \tag{6.6}$$

Lemma 6.1.2 is an analogue of Lemma 1 of [Rad59], which gives an upper bound for $\left|\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}\right|$ in the strip $|\Re s| \leq \frac{1}{2}$. In Lemma 6.1.2, we find a similar bound for the function $\Gamma(1 - s + \frac{k-1}{2})/\Gamma(s + \frac{k-1}{2})$.

Lemma 6.1.2. *For $k > 1$, $-\frac{1}{2} \leq \sigma \leq \frac{1}{2}$ we have*

$$\left|\frac{\Gamma\left(1-s+\frac{k-1}{2}\right)}{\Gamma\left(s+\frac{k-1}{2}\right)}\right| \leq \left|\frac{k+1}{2} + s\right|^{1-2\sigma}. \tag{6.7}$$

Proof. We use Theorem 6.1.1 with $a = -\frac{1}{2}$, $b = \frac{3}{2}$, and

$$f(s) = \frac{\Gamma\left(1 - s + \frac{k-1}{2}\right)}{\Gamma\left(s + \frac{k-1}{2}\right)}. \tag{6.8}$$

On the line $s = -\frac{1}{2} + it$,

$$\begin{aligned}\left|\frac{\Gamma(1 - s + \frac{k-1}{2})}{\Gamma(s + \frac{k-1}{2})}\right| &= \left|\frac{\Gamma\left(\frac{3}{2} - it + \frac{k-1}{2}\right)}{\Gamma\left(-\frac{1}{2} + it + \frac{k-1}{2}\right)}\right| \\ &= \left|\frac{(\frac{k}{2} - it)(\frac{k}{2} - 1 - it)\Gamma(\frac{k}{2} - 1 - it)}{\Gamma(\frac{k}{2} - 1 + it)}\right| \\ &= \left|\left(\frac{k}{2} - it\right)\left(\frac{k}{2} - 1 - it\right)\right| \\ &\leq \left|\frac{k}{2} + it\right|^2 .\end{aligned} \tag{6.9}$$

The second equality in (6.9) uses the identity, $s\Gamma(s) = \Gamma(s+1)$. On the line $s = 1/2 + it$, we can check that

$$\left|\frac{\Gamma(1 - s + \frac{k-1}{2})}{\Gamma(s + \frac{k-1}{2})}\right| = \left|\frac{\Gamma\left(\frac{1}{2} - it + \frac{k-1}{2}\right)}{\Gamma\left(\frac{1}{2} + it + \frac{k-1}{2}\right)}\right| = 1. \tag{6.10}$$

Applying Theorem 6.1.1, with $A = B = 1$, $Q = \frac{k}{2} + \frac{1}{2}$, $\alpha = 2$, and $\beta = 0$, we get the result. □

In Lemma 6.1.3 we shall find a bound for $L(s, f)$ in the critical strip by finding a bound for the left and right edges of a slightly wider strip, and use the log convexity bound of Theorem 6.1.1.

Lemma 6.1.3. *If $0 < \eta < \frac{1}{2}$, $s = \sigma + it$ and $-\eta < \sigma < 1 + \eta$, then we have*

$$|L(s, f)| \leq \left(\frac{\sqrt{N}}{2\pi}\left|\frac{k+1}{2} + s\right|\right)^{1+\eta-\sigma} \zeta(1+\eta)^2. \tag{6.11}$$

Proof. To find an upper bound for $|L(s, f)|$ in the vertical strip $-\eta \leq \Re s \leq 1 + \eta$, we shall find an upper bound for the left and right edges of the vertical strip, and use Theorem 6.1.1.

For $0 < \eta$, let $s = 1 + \eta + it$ be a point the right edge of the vertical strip, then we have

$$\begin{aligned} |L(1+\eta+it, f)| &= \left| \prod_{p \nmid N} \left(1 - A_p p^{-s} + p^{-2s}\right)^{-1} \prod_{q \mid N} \left(1 - A_q q^{-s}\right)^{-1} \right| \\ &\le \prod_{p \nmid N} \left(1 - p^{-1-\eta}\right)^{-2} \prod_{q \mid N} \left(1 - q^{-1-\eta}\right)^{-1} \\ &= \zeta(1+\eta)^2 \prod_{q \mid N} \left(1 - q^{-1-\eta}\right) < \zeta(1+\eta)^2. \end{aligned} \tag{6.12}$$

The first inequality in (6.12) makes use of Deligne's bound $|A_p| < 2$ for $p \nmid N$ and $|A_q| \le 1$ if $q \mid N$. We have found an upper bound for $L(s, f)$ on the right edge of the vertical strip $-\eta < \sigma < 1 + \eta$.

For the left edge, we use the functional equation satisfied by $L(s, f)$,

$$\begin{aligned} & N^{\frac{s+\frac{k-1}{2}}{2}} (2\pi)^{-(s+\frac{k-1}{2})} \Gamma\left(s + \frac{k-1}{2}\right) L(s, f) \\ &\qquad = \varepsilon N^{\frac{(1-s)+\frac{k-1}{2}}{2}} (2\pi)^{-(1-s+\frac{k-1}{2})} \Gamma\left(1 - s + \frac{k-1}{2}\right) L(1-s, f). \end{aligned} \tag{6.13}$$

Using (6.13), we get

$$|L(s, f)| = \left| N^{\frac{1}{2}-s} (2\pi)^{2s-1} \frac{\Gamma(1 - s + \frac{k-1}{2})}{\Gamma(s + \frac{k-1}{2})} L(1-s, f) \right|. \tag{6.14}$$

For $s = -\eta + it$, using (6.12) and Lemma 6.1.2, we get

$$|L(-\eta+it, f)| \le N^{\frac{1}{2}+\eta} (2\pi)^{-2\eta-1} \left| \frac{k+1}{2} + (-\eta + it) \right|^{1+2\eta} \zeta(1+\eta)^2. \tag{6.15}$$

An application of Theorem 6.1.1 gives us: for $-\eta < \sigma < 1 + \eta$,

$$|L(s, f)| \le \left(\frac{\sqrt{N}}{2\pi} \left| \frac{k+1}{2} + s \right| \right)^{1+\eta-\sigma} \zeta(1+\eta)^2. \tag{6.16}$$

□

6.2 Upper bound for $\int_{\frac{1}{2}}^{\infty} \log |L(\sigma + it)| d\sigma$

In this section we use the bound for $|L(s, f)|$ found in the Section 6.1 to obtain a bound for $\int_{\frac{1}{2}}^{\infty} \log |L(\sigma + it, f)| d\sigma$.

In the following theorem, we state the bound in terms of several parameters. Introduction of these parameters gives us a more complicated looking expression, but the advantage is that we can optimize the bound depending upon the region in which we are computing. The bound has the form $a + b \log t$. Ideally we would like both a and b to be small, but there is always a trade off. If we try to decrease one, the other increases. When t is large, we would like b to be small, and when t is small we would like a to be small. Trudgian [Tru09] introduced similar parameters to improve on earlier results for $\zeta(s)$, Dirichlet L-functions and Dedekind zeta functions

Theorem 6.2.1. *Let k be an even integer greater than 1. Let c and η be real numbers satisfying $0 < c \leq \eta \leq \frac{1}{2}$. Then for $t > \frac{k}{2} + 2$ (See Figure 6.1) the following inequality holds*

$$\int_{\frac{1}{2}}^{\infty} \log |L(\sigma + it, f)| d\sigma \leq (c + \frac{1}{2}) \left[\log \left(\frac{\sqrt{N}}{2\pi} \right) + 2 \log(\zeta(1+\eta)) \right] + \frac{1}{t} \left[\frac{k+1}{2}(1+\eta)(c + \frac{1}{2}) + (1 + \eta - \frac{k+1}{2}) \left(\frac{(1+c)^2 - \frac{1}{4}}{2} \right) \right] + \log t \left[\frac{(\eta + \frac{1}{2})^2 - (\eta - c)^2}{2} \right] + 2I(1+c), \quad (6.17)$$

where

$$I(\alpha) = \int_{\alpha}^{\infty} \log \zeta(\sigma) d\sigma. \quad (6.18)$$

Proof. Outside the critical strip, $L(s, f)$ tends quickly to 1 as $\Re s$ increases. We therefore split the integral on the left hand side of (6.17). We separate the part inside the critical

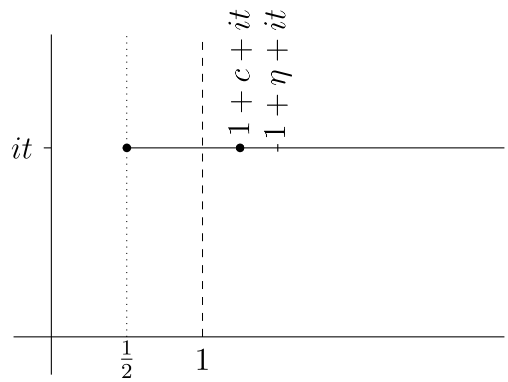

Figure 6.1: Path of integration

strip from the part outside the critical strip,

$$\int_{\frac{1}{2}}^{\infty} \log|L(\sigma+it,f)|d\sigma = \int_{\frac{1}{2}}^{1+c} \log|L(\sigma+it,f)|d\sigma + \int_{1+c}^{\infty} \log|L(\sigma+it,f)|d\sigma. \tag{6.19}$$

In the second integral on the right hand side of (6.19), we are integrating in the region outside the critical strip. This is the simpler part. By (6.12), the second integral is bounded above by $2I(1+c)$. The first integral will be bounded above using the bound for $L(s,f)$ given by (6.11). In the interval $\frac{1}{2} < \sigma < 1+c$

$$\log|L(s,f)| \leq \log\left(\frac{\sqrt{N}}{2\pi}\right) + (1+\eta-\sigma)\log\left|\frac{k+1}{2}+\sigma+it\right| + 2\log\zeta(1+\eta). \tag{6.20}$$

The first and last terms do not depend on σ. We can simplify the expression slightly by using the inequality $\log(1+x) \leq x$ for $x \geq 0$. The logarithm in the middle term of the right hand side of (6.20) can be simplified a little more;

$$(1+\eta-\sigma)\log\left|\frac{k+1}{2}+\sigma+it\right| \leq (1+\eta-\sigma)\left[\log t + \log\left(1+\frac{\frac{k+1}{2}+\sigma}{t}\right)\right]. \tag{6.21}$$

In the inequality (6.21), $\log\left(1+\frac{\frac{k+1}{2}+\sigma}{t}\right) \leq \frac{\frac{k+1}{2}+\sigma}{t}$. Integrating (6.20) and using the inequality (6.21), we obtain

$$\int_{1/2}^{1+c} \log|L(\sigma+it,f)|d\sigma \leq (c+1/2)\left[\log\left(\frac{\sqrt{N}}{2\pi}\right)+2\log(\zeta(1+\eta))\right] + \log t\left[\frac{(\eta+\frac{1}{2})^2}{2}-\frac{(\eta-c)^2}{2}\right] + \frac{1}{t}\left[\frac{(1+k)}{2}(1+\eta)(c+\frac{1}{2})+(1+\eta-\frac{k+1}{2})(\frac{(1+c)^2-\frac{1}{4}}{2})+\left(\frac{(c+1)^3-\frac{1}{8}}{3}\right)\right]. \quad (6.22)$$

Hence the upper bound of $\int_{\frac{1}{2}}^{\infty} \log|L(\sigma+it,f)|\,d\sigma$ is the sum of the right hand side of (6.22) and $2I(1+c)$,

$$(c+1/2)\left[\log\left(\frac{\sqrt{N}}{2\pi}\right)+2\log(\zeta(1+\eta))\right]+\log t\left[\frac{(\eta+\frac{1}{2})^2}{2}-\frac{(\eta-c)^2}{2}\right] + \frac{1}{t}\left[\frac{(1+k)}{2}(1+\eta)(c+\frac{1}{2})+(1+\eta-\frac{k+1}{2})(\frac{(1+c)^2-\frac{1}{4}}{2})+\left(\frac{(c+1)^3-\frac{1}{8}}{3}\right)\right] + 2I(1+c). \quad (6.23)$$

□

In the following corollary, we let $\eta = c$ and obtain a slightly simpler expression.

Corollary 6.2.2. *If we take $c = \eta$ in the above then we have the following inequality*

$$\int_{1/2}^{1+\eta} \log|L(\sigma+it,f)|d\sigma \leq (\eta+1/2)\left[\log\left(\frac{\sqrt{N}}{2\pi}\right)+2\log(\zeta(1+\eta))\right] + \frac{1}{t}\left[\frac{(1+k)}{2}(1+\eta)(\eta+\frac{1}{2})+(1+\eta-\frac{k+1}{2})(\frac{(1+\eta)^2-\frac{1}{4}}{2})+\left(\frac{(\eta+1)^3-\frac{1}{8}}{3}\right)\right] + \frac{(\eta+\frac{1}{2})^2}{2}\log t+2I(1+\eta). \quad (6.24)$$

Chapter 7

Lower Bound for $\int_{\frac{1}{2}}^{\infty} \log|L(\sigma+it,f)|\, d\sigma$

In this chapter we derive a lower bound for $\int_{\frac{1}{2}}^{\infty} \log|L(\sigma+it,f)|\, ds$ of the form $a+b\log t$, and give formulae for a and b in terms of a parameter d that we will introduce. In Chapter 8, we make a specific choice for d. The purpose of this chapter is to find a and b. To calculate the lower bound for

$$\int_{\frac{1}{2}}^{\infty} \log|L(\sigma+it,f)|d\sigma, \tag{7.1}$$

we rewrite the integral as

$$\int_{\frac{1}{2}}^{\infty} \log|L(\sigma+it,f)|\, d\sigma = \int_{\frac{1}{2}}^{\frac{1}{2}+d} \log\left|\frac{L(\sigma+it,f)}{L(\sigma+d+it,f)}\right| d\sigma \\ + \int_{\frac{1}{2}+d}^{\infty} \log|L(\sigma+it,f)|\, d\sigma + \int_{\frac{1}{2}+d}^{\frac{1}{2}+2d} \log|L(\sigma+it,f)|\, d\sigma. \tag{7.2}$$

We shall assume that $d > \frac{1}{2}$. As in [Tru09], d will be determined later. A lower bound for the second and the third integral on the right hand side can be found using Lemma 7.1.2. A lower bound for the sum of the second and the third integrals is given by $J(d)$ defined in (7.10). The lower bound for the first integral on the right hand side of (7.2), the only integral whose path crosses the critical strip, is found in Section 7.2.

7.1 Bound for the integral over a path lying outside the critical strip

Let f be a newform of weight k and level N. We know that $L(s,f)$ has an Euler product valid for $\Re s > 1$, namely,

$$L(s,f) = \prod_{q|N} \left(1 - \frac{\alpha_q}{q^s}\right)^{-1} \prod_{p\nmid N} \left(1 - \frac{\alpha_{p,1}}{p^s}\right)^{-1} \left(1 - \frac{\alpha_{p,2}}{p^s}\right)^{-1}. \tag{7.3}$$

Here α_q and $\alpha_{p,i}$ have absolute value less than or equal to 1. From this we can easily conclude:

Lemma 7.1.1. *For $\sigma > 1$,*

$$|L(\sigma + it, f)| \geq \frac{\zeta(2\sigma)^2}{\zeta(\sigma)^2} \prod_{q|N} \left(1 + \frac{1}{q^\sigma}\right) \tag{7.4}$$

Proof. Starting with (7.3), we have

$$|L(\sigma + it, f)| \geq \prod_{q|N} \left(1 + \frac{1}{q^\sigma}\right)^{-1} \prod_{p\nmid N} \left(1 + \frac{1}{q^\sigma}\right)^{-2} = \frac{\zeta(2\sigma)^2}{\zeta(\sigma)^2} \prod_{q|N} \left(1 + \frac{1}{q^\sigma}\right).$$

□

In Lemma 7.1.2, for $a < b$, we find a lower bound for $\int_a^b \log|L(\sigma+it,f)|\,d\sigma$. Note that the path of integration is completely outside the critical strip. For $|z| < 1$, the function $\mathrm{Li}_m(z)$ is defined to be

$$\mathrm{Li}_m(z) = \sum_{n=1} \frac{z^n}{n^m}. \tag{7.5}$$

Lemma 7.1.2. *For $1 < \sigma_1 < \sigma_2$,*

$$\int_{\sigma_1}^{\sigma_2} \log|L(\sigma+it,f)|\,d\sigma \geq 2\int_{\sigma_1}^{\sigma_2} \log\zeta(2\sigma) - \log\zeta(\sigma)\,d\sigma \\ + \sum_{q|N} \frac{1}{\log q} \left[\mathrm{Li}_2\left(-\frac{1}{q^{\sigma_2}}\right) - \mathrm{Li}_2\left(-\frac{1}{q^{\sigma_1}}\right)\right]. \tag{7.6}$$

Proof. The inequality (7.4) gives us

$$\log|L(\sigma+it,f)| \geq 2\log\zeta(2\sigma) - 2\log\zeta(\sigma) + \sum_{q|N}\log\left(1+\frac{1}{q^\sigma}\right). \tag{7.7}$$

Recall that when $|z| \leq 1$, $\log(1+z) = \sum_{n\geq 1}\frac{(-1)^{n-1}z^n}{n}$. For $\sigma > 0$ and $q > 1$,

$$\log\left(1+\frac{1}{q^\sigma}\right) = \sum_{n\geq 1}\frac{(-1)^{n-1}}{q^{n\sigma}}. \tag{7.8}$$

Using equation (7.8), and integrating term by term we obtain

$$\int_{\sigma_1}^{\sigma_2}\log\left(1+\frac{1}{q^\sigma}\right)d\sigma = \left[\frac{1}{\log q}\sum_{n\geq 1}\frac{(-1)^n}{n^2 q^{nd}}\right]_{\sigma_1}^{\sigma_2} = \left[\frac{1}{\log q}\mathrm{Li}_2\left(\frac{-1}{q^\sigma}\right)\right]_{\sigma_1}^{\sigma_2}. \tag{7.9}$$

This proves the lemma. □

A straight forward application of Lemma 7.1.2 to each of the second and third integrals of (7.2) tells us that their sum is greater than or equal to $J(d)$, where

$$\begin{aligned} J(d) = &\int_{2d+1}^{\infty}\log\zeta(\sigma)d\sigma - 2\int_{d+\frac{1}{2}}^{\infty}\log\zeta(\sigma)d\sigma - \sum_{q|N}\frac{1}{\log q}\mathrm{Li}_2\left(\frac{-1}{q^{d+\frac{1}{2}}}\right)\\ &+\int_{2d+1}^{4d+1}\log\zeta(\sigma)d\sigma - 2\int_{d+\frac{1}{2}}^{2d+\frac{1}{2}}2\log\zeta(\sigma)d\sigma + \sum_{q|N}\frac{1}{\log q}\mathrm{Li}_2\left(\frac{-1}{q^{2d+\frac{1}{2}}}\right)\\ &-\sum_{q|N}\frac{1}{\log q}\mathrm{Li}_2\left(\frac{-1}{q^{d+\frac{1}{2}}}\right). \end{aligned} \tag{7.10}$$

In the $I(\alpha)$ notation of (6.18), the right hand side of (7.10) can be written as

$$\begin{aligned} &I(2d+1) - 4I(d+\frac{1}{2}) + I(2d+1) - I(4d+1) + I(2d+\frac{1}{2})\\ &-\sum_{q|N}\frac{1}{\log q}\mathrm{Li}_2\left(\frac{-1}{q^{d+\frac{1}{2}}}\right) + \sum_{q|N}\frac{1}{\log q}\mathrm{Li}_2\left(\frac{-1}{q^{2d+\frac{1}{2}}}\right) - \sum_{q|N}\frac{1}{\log q}\mathrm{Li}_2\left(\frac{-1}{q^{d+\frac{1}{2}}}\right). \end{aligned} \tag{7.11}$$

7.2 Bound for the integral over a path crossing the critical strip

We now come to the problem of finding a lower bound for the integral whose path crosses into the critical strip, i.e the first integral in (7.2). For the first integral in (7.2), we use the Weierstrass product formula for $\Lambda(s, f)$,

$$\Lambda(s,f) = e^{As+B} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}}. \tag{7.12}$$

Equation (7.12) and the definition of $\Lambda(s, f)$ in Section 5.2 gives

$$L(s,f) = \left(\frac{2\pi}{\sqrt{N}}\right)^{s+\frac{k-1}{2}} \frac{1}{\Gamma\left(s+\frac{k-1}{2}\right)} \; e^{As+B} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}}. \tag{7.13}$$

Let $s = \sigma + it$, then

$$\begin{aligned}\int_{\frac{1}{2}}^{\frac{1}{2}+d} \log\left|\frac{L(s,f)}{L(s+d,f)}\right| d\sigma = &- \int_{\frac{1}{2}}^{\frac{1}{2}+d} \log\left|\frac{\Gamma(s+\frac{k-1}{2})}{\Gamma(s+d+\frac{k-1}{2})}\right| d\sigma \\ &+ \int_{\frac{1}{2}}^{\frac{1}{2}+d} \sum_{\rho} \log\left|\frac{s-\rho}{s+d-\rho}\right| d\sigma \\ &+ \int_{\frac{1}{2}}^{\frac{1}{2}+d} d \log \frac{\sqrt{N}}{2\pi}\, d\sigma \\ =& I_1 + I_2 + d^2 \log \frac{\sqrt{N}}{2\pi}. \end{aligned} \tag{7.14}$$

We will find lower bounds for I_1 and I_2. To obtain a lower bound for I_1 of (7.14), we will use Lemma 7.2.1. To obtain a lower bound for I_2, we will use Lemma 7.2.2.

Lemma 7.2.1 ([Leh70, Lemma 8, p.308]). *If $\Re z > 0$, then*

$$\frac{\Gamma'(z)}{\Gamma(z)} = \log z - \frac{1}{2z} + \Theta\left(\frac{2}{\pi^2 |(\Im z)^2 - (\Re z)^2|}\right) \tag{7.15}$$

where $f(x) = \Theta(g(x))$ means that $|f(x)| \leq g(x)$.

For $s = \sigma + it$, the fundamental theorem of calculus gives

$$-\int_{\frac{1}{2}}^{\frac{1}{2}+d} \log\left|\frac{\Gamma(s+\frac{k-1}{2})}{\Gamma(s+d+\frac{k-1}{2})}\right| d\sigma = -\int_{\frac{1}{2}}^{\frac{1}{2}+d}\int_{o}^{d} \Re\frac{\Gamma'(s+\frac{k-1}{2}+\xi)}{\Gamma(s+\frac{k-1}{2}+\xi)} d\xi d\sigma. \tag{7.16}$$

Similar to the integrals involved in the upper bound, we limit d to be in $(\frac{1}{2}, 1]$. The mean value theorem of differential calculus tells us that the integral on the right hand side of (7.16) is

$$-d^2\frac{\Gamma'(\sigma+it)}{\Gamma(\sigma+it)}, \tag{7.17}$$

for some σ such that $\frac{k}{2} < \sigma < \frac{k}{2} + 2d$. Since we are assuming that $\frac{1}{2} < d \leq 1$, σ satisfies $\frac{k}{2} < \sigma < \frac{k}{2} + 2$. We shall estimate the value of (7.17) for $t > \frac{k}{2} + 2$. This is one of the reasons we introduced this condition in the statements of Theorems 1.2.3 and 1.2.4. Using Lemma 7.2.1, we see that

$$\Re\frac{\Gamma'(\sigma+it)}{\Gamma(\sigma+it)} = \Re\log(\sigma+it) - \Re\frac{1}{2(\sigma+it)} + \Theta\left(\frac{2}{\pi^2|t^2-\sigma^2|}\right). \tag{7.18}$$

For $t > \sigma$, the right hand side of (7.18) is

$$\log t + \Theta\left(\frac{\sigma^2}{2t^2}\right) - \frac{\sigma}{2(\sigma^2+t^2)} + \Theta\left(\frac{2}{\pi^2|t^2-\sigma^2|}\right). \tag{7.19}$$

Let $C = \frac{t}{\frac{k}{2}+2}$, then (7.19) equals

$$\log t + \Theta\left(\frac{1}{2C^2} + \frac{1}{k(C^2+1)} + \frac{8}{\pi^2k^2(C^2-1)}\right), \tag{7.20}$$

which simplifies to

$$\log t + \Theta\left(\frac{1}{2C^2} + \frac{1}{k(C^2+1)} + \frac{8}{\pi^2k^2(C^2-1)}\right). \tag{7.21}$$

We know that the weight of the newform, k, is at least 2, and $\frac{1}{C^2+1} < \frac{1}{C^2} < \frac{1}{C^2-1}$. We can simplify (7.21) further and obtain

$$\Re\frac{\Gamma'(\sigma+it)}{\Gamma(\sigma+it)} = \log t + \Theta\left(\frac{2}{C^2-1}\right). \tag{7.22}$$

Using (7.22) in (7.16), we see that (7.16) is greater than

$$-d^2\log t - d^2\epsilon_{t,k}, \quad \text{where} \quad \epsilon_{t,k} = \frac{2}{\left(\frac{t}{\frac{k}{2}+2}\right)^2 - 1}, \tag{7.23}$$

giving a lower bound for I_1 defined in (7.14).

To obtain a lower bound for I_2 we use a lemma by Booker [Boo06, Lemma 4.4]. The following form is from [Tru09, Lemma 2.10].

Lemma 7.2.2 ([Tru09, Lemma 2.10])**.** *Let w be a complex number such that $\Re w \le \frac{1}{2}$. Then for $\frac{1}{2} < d \le 1$,*

$$\int_0^d \log\left|\frac{(x+d+w)(x+d-\bar{w})}{(x+w)(x-\bar{w})}\right| dx \le d^2(\log 4)\Re\left(\frac{1}{d+w} + \frac{1}{d-\bar{w}}\right). \tag{7.24}$$

Recall I_2 defined in (7.14):

$$I_2 = \int_{\frac{1}{2}}^{\frac{1}{2}+d} \sum_\rho \log\left|\frac{s-\rho}{s+d-\rho}\right| d\sigma = \int_0^d \sum_\rho \log\left|\frac{\sigma+\frac{1}{2}+it-\rho}{\sigma+d+\frac{1}{2}+it-\rho}\right| d\sigma. \tag{7.25}$$

Using Lemma 7.2.2, we can find a lower bound for $\int_0^d \log\left|\frac{\sigma+\frac{1}{2}+it-\rho}{\sigma+d+\frac{1}{2}+it-\rho}\right| d\sigma$ if the root is on the critical line, and the sum of integrals corresponding to ρ and $1-\bar{\rho}$ if ρ is off the critical line. If $\rho = \frac{1}{2} + ir$ is on the critical line,

$$\int_0^d \log\left|\frac{\sigma+\frac{1}{2}+it-\rho}{\sigma+d+\frac{1}{2}+it-\rho}\right| d\sigma = \int_0^d \log\left|\frac{\sigma+i(t-r)}{\sigma+d+i(t-r)}\right| d\sigma. \tag{7.26}$$

Using $w = i(t-r)$ in (7.24), we obtain a lower bound for the left hand side of (7.26),

$$\begin{aligned}\int_0^d \log\left|\frac{\sigma+\frac{1}{2}+it-\rho}{\sigma+d+\frac{1}{2}+it-\rho}\right| d\sigma &= \int_0^d \log\left|\frac{\sigma+i(t-r)}{\sigma+d+i(t-r)}\right| d\sigma \\ &\ge -d^2\log 4\ \Re\frac{1}{d+i(t-r)} = -d^2\log 4\ \Re\frac{1}{d+\frac{1}{2}+it-\rho}. \end{aligned}\tag{7.27}$$

If $\rho = u + iv$ is off the critical line, then $1 - \bar{\rho}$ is also a zero. Using $w = u - \frac{1}{2} + i(t - v)$ in (7.24), we get

$$\int_0^d \log \left| \frac{\sigma + \frac{1}{2} + it - \rho}{\sigma + d + \frac{1}{2} + it - \rho} \right| d\sigma + \int_0^d \log \left| \frac{\sigma + \frac{1}{2} + it - (1 - \bar{\rho})}{\sigma + d + \frac{1}{2} + it - (1 - \bar{\rho})} \right| d\sigma$$
$$\geq -d^2 \log 4 \, \Re \left(\frac{1}{d + \frac{1}{2} + it - \rho} + \frac{1}{d + \frac{1}{2} + it - (1 - \bar{\rho})} \right). \quad (7.28)$$

From inequalities (7.27) and (7.28), we conclude

$$\sum_\rho \int_0^d \log \left| \frac{\sigma + \frac{1}{2} - \rho}{\sigma + \frac{1}{2} + d - \rho} \right| d\sigma \geq -d^2 \log 4 \sum_\rho \Re \frac{1}{d + \frac{1}{2} + it - \rho}. \quad (7.29)$$

Hence

$$- I_2 \leq d^2 \log 4 \sum_\rho \Re \frac{1}{d + \frac{1}{2} + it - \rho}. \quad (7.30)$$

To obtain a lower bound for I_2, it is enough to find and upper bound for the right hand side of (7.30). We calculate an upper bound of (7.30) in the rest of this section.

Taking the real part of the logarithmic derivative of the equation

$$\Lambda(s, f) = \left(\frac{\sqrt{N}}{2\pi} \right)^{s + \frac{k-1}{2}} \Gamma \left(s + \frac{k-1}{2} \right) L(s, f) = e^{A + Bs} \prod_\rho \left(1 - \frac{s}{\rho} \right) e^{\frac{s}{\rho}}, \quad (7.31)$$

we obtain

$$\Re \sum_\rho \frac{1}{s - \rho} = \log \frac{\sqrt{N}}{2\pi} + \Re \frac{\Gamma'(s + \frac{k-1}{2})}{\Gamma(s + \frac{k-1}{2})} + \Re \frac{L'(s, f)}{L(s, f)}. \quad (7.32)$$

Note that the left hand side of (7.32) evaluated at $d + \frac{1}{2} + it$ is present in the right hand side of (7.30). We can find an upper bound for $-I_2$ if we find an upper bound for each of the terms in the right hand side of (7.32). The first term is just a constant. An upper bound for the second term is found using Lemma 7.2.1. For the last term, we use the Euler product (7.3) for $L(s, f)$. If we take the real part of the logarithmic derivative of the Euler product, we obtain

$$\Re \frac{L'(s, f)}{L(s, f)} \leq -2 \frac{\zeta'(\sigma)}{\zeta(\sigma)} - \sum_{q | N} \frac{\log q}{q^\sigma - 1}. \quad (7.33)$$

Equation (7.23) provides an upper bound on $\frac{\Gamma'(\sigma+it)}{\Gamma(\sigma+it)}$ appearing in the right hand side of (7.32), and (7.33) provides an upper bound on $\frac{L'(\sigma+it,f)}{L(\sigma+it,f)}$. Using these upper bounds in (7.30), we obtain

$$-I_2 \leq d^2 \log 4 \left[\log \frac{\sqrt{N}}{2\pi} + \log t + \epsilon_{t,k} - 2\frac{\zeta'(d+\frac{1}{2})}{\zeta(d+\frac{1}{2})} - \sum_{q|N} \frac{\log q}{q^{d+\frac{1}{2}}-1} \right]. \tag{7.34}$$

7.3 The lower bound

Lemma 7.3.1. *Let $J(d)$ be as in* (7.10)*, and $\epsilon_{t,k}$ be as in* (7.23)*. Then for $t > \frac{k}{2}+2$,*

$$-\int_{\frac{1}{2}}^{\infty} \log|L(\sigma+it,f)|\, d\sigma \leq a + b\log t, \tag{7.35}$$

where

$$a = -J(d) + d^2 \log 4 \left[\log \frac{\sqrt{N}}{2\pi} - 2\frac{\zeta'(d+\frac{1}{2})}{\zeta(d+\frac{1}{2})} - \sum_{q|N} \frac{\log q}{q^{d+\frac{1}{2}}-1} \right] - d^2 \log \frac{\sqrt{N}}{2\pi} + 3\epsilon_{t,k}, \tag{7.36}$$

and

$$b = d^2(\log 4 - 1). \tag{7.37}$$

In Chapter 8, we make a specific choice for d.

Chapter 8

Numerical Values

8.1 Upper bound

The right hand side of (6.24) is of the form

$$a + b \log t + \frac{c}{t}. \tag{8.1}$$

To compute numerical values of a, b, and c, we have to find an upper bound and a lower bound for $I(\alpha) = \int_\alpha^\infty \log \zeta(\sigma) d\sigma$ for $\alpha > 1$. Note that for $\sigma > 1$, $\log \zeta(\sigma)$ is a convex function, so using the trapezoidal rule for integrating gives an upper bound. For $\alpha < \beta < \infty$,

$$\int_\alpha^\infty \log \zeta(\sigma) d\sigma = \int_\alpha^\beta \log \zeta(\sigma) d\sigma + \int_\beta^\infty \log \zeta(\sigma) d\sigma. \tag{8.2}$$

We use the trapezoidal rule to compute an upper bound for the first integral on the right hand side of (4.8). For an upper bound on the second integral, we use the inequality derived from the Euler product:

$$\int_{\beta}^{\infty} \log \zeta(\sigma) d\sigma = \sum_{p} \sum_{n} \int_{\beta}^{\infty} \frac{1}{np^{n\sigma}} d\sigma \tag{8.3}$$

$$= \sum_{n \geq 1} \sum_{p} \frac{1}{n^2 p^{n\beta} \log p} \tag{8.4}$$

$$\leq \sum_{n \geq 2} \frac{1}{n^{\beta}} = \zeta(\beta) - 1. \tag{8.5}$$

If we choose $\beta = 20$, then $\int_{20}^{\infty} \log \zeta(\sigma) < 10^{-6}$. If we choose $\eta = 0.1$ in Corollary 6.2.2, then we find an upper bound for the integral $\int_{\frac{1}{2}}^{\infty} \log |L(s, f)| d\sigma$ for $t > \frac{k}{2} + 2$ of

$$a + b \log t + \frac{c}{t}, \tag{8.6}$$

where

$$a = 0.6 \times \log(\sqrt{N}) + 2.70746797960673 \tag{8.7}$$

$$b = 0.18 \tag{8.8}$$

$$c = 0.09 \times k + 1.02. \tag{8.9}$$

8.2 Lower bound

For the lower bound we use $d = .6$ in (7.35) – (7.37), and get the following values for a and b in the lower bound:

$$a = J(0.6) + 0.36 \log 4 \left[\log \frac{\sqrt{N}}{2\pi} + 2 \times 9.441036290 - \sum_{q|N} \frac{\log q}{q^{1.1} - 1} \right] - 0.36 \log \frac{\sqrt{N}}{2\pi} + 2\epsilon_{t,k}, \tag{8.10}$$

$$b = 0.36 \times (\log 4 - 1), \tag{8.11}$$

where

$$J(0.6) = -3.69607634894834 + \sum_{q|N} \frac{1}{\log q} \left[\mathrm{Li}_2\left(\frac{-1}{q^{1.7}}\right) - 2\mathrm{Li}_2\left(\frac{-1}{q^{1.1}}\right) \right], \tag{8.12}$$

$$\frac{\zeta'(1.1)}{\zeta(1.1)} = -9.441036390, \tag{8.13}$$

and $\varepsilon_{t,k}$ is given in (7.23).

8.3 Example

Let E be the elliptic curve of conductor 11 given by the equation $y^2 - y = x^3 - x^2$. From Wiles' work we know that the L-function of this elliptic curve is the same as that of a modular form of weight $k = 2$ and level $N = 11$. Then

$$\int_{\frac{1}{2}}^{\infty} \log |L(\sigma + it)| d\sigma \leq 3.42683658183951 + 0.18 \log t, \tag{8.14}$$

and

$$-\int_{\frac{1}{2}}^{\infty} \log |L(\sigma + it)| d\sigma \leq 12.888062 + 0.139066 \log t + 2\epsilon_{t,2}. \tag{8.15}$$

Appendix

In Section 4.3, we provided evidence that the polynomial in $p(k)$ in (4.65) is of degree $w = |\lambda|$. In this Appendix, we provide more evidence for the conjecture.

In the following tables there are two columns. The first column is a partition, and the second column is a representation of the polynomial $p(k)$ for that partition. The polynomial $p(k)$ is an integer valued polynomial. It can be uniquely written as an integer linear combination of polynomials from the set

$$\left\{ \binom{k}{n} \;:\; n \in \mathbb{Z},\ \text{and } n \geq 0 \right\}. \tag{A-1}$$

The right column of the tables give the coefficients when $p(k)$ is written as an integer linear combination of the basis (A-1). For example, $p(k)$ for partition $(3, 1, 1)$ is

$$6\binom{k}{0} - 2\binom{k}{1} - \binom{k}{2} + 15\binom{k}{3} + 18\binom{k}{4} + 6\binom{k}{5}. \tag{A-2}$$

Table 1 tells us that the coefficients are $[6, -2, -1, 15, 18, 6]$. This is used to create the polynomial (A-2).

Table 1: Partitions of 5

Partition	Polynomial
(5)	[1, 5, 10, 10, 5, 1]
(4, 1)	[-4, -2, 12, 23, 16, 4]
(3, 2)	[-2, -4, 5, 19, 17, 5]
(3, 1, 1)	[6, -2, -1, 15, 18, 6]
(2, 2, 1)	[2, 0, -2, 8, 13, 5]
(2, 1, 1, 1)	[-4, 3, -2, 1, 8, 4]
(1, 1, 1, 1, 1)	[1, -1, 1, -1, 1, 1]

Table 2: Partitions of 6

Partition	Polynomial
(6)	[1, 6, 15, 20, 15, 6, 1]
(5, 1)	[-5, -5, 15, 45, 49, 25, 5]
(4, 2)	[-3, -9, 3, 42, 63, 39, 9]
(4, 1, 1)	[10, 0, -5, 26, 56, 40, 10]
(3, 3)	[-1, -4, -1, 16, 29, 20, 5]
(3, 2, 1)	[5, 3, -8, 18, 64, 56, 16]
(3, 1, 1, 1)	[-10, 5, -1, -2, 24, 30, 10]
(2, 2, 2)	[-1, 1, -1, 1, 13, 15, 5]
(2, 2, 1, 1)	[-3, 1, 1, -3, 15, 24, 9]
(2, 1, 1, 1, 1)	[5, -4, 3, -2, 1, 10, 5]
(1, 1, 1, 1, 1, 1)	[-1, 1, -1, 1, -1, 1, 1]

Table 3: Partitions of 7

Partition	Polynomial
(7)	[1, 7, 21, 35, 35, 21, 7, 1]
(6, 1)	[-6, -9, 16, 75, 114, 89, 36, 6]
(5, 2)	[-4, -16, -6, 70, 160, 156, 74, 14]
(5, 1, 1)	[15, 5, -10, 36, 127, 145, 75, 15]
(4, 3)	[-2, -11, -12, 36, 114, 129, 68, 14]
(4, 2, 1)	[9, 11, -16, 20, 173, 255, 155, 35]
(4, 1, 1, 1)	[-20, 5, 4, -8, 48, 110, 80, 20]
(3, 3, 1)	[3, 5, -7, 3, 77, 131, 87, 21]
(3, 2, 2)	[-3, 3, -1, -3, 51, 109, 81, 21]
(3, 2, 1, 1)	[-9, -1, 8, -12, 43, 145, 125, 35]
(3, 1, 1, 1, 1)	[15, -9, 4, 0, -3, 35, 45, 15]
(2, 2, 2, 1)	[2, -2, 2, -2, 6, 41, 44, 14]
(2, 2, 1, 1, 1)	[4, -2, 0, 2, -4, 24, 38, 14]
(2, 1, 1, 1, 1, 1)	[-6, 5, -4, 3, -2, 1, 12, 6]
(1, 1, 1, 1, 1, 1, 1)	[1, -1, 1, -1, 1, -1, 1, 1]

Table 4: Partitions of 8

Partition	Polynomial
(8)	[1, 8, 28, 56, 70, 56, 28, 8, 1]
(7, 1)	[-7, -14, 14, 112, 224, 238, 146, 49, 7]
(6, 2)	[-5, -25, -25, 95, 325, 445, 325, 125, 20]
(6, 1, 1)	[21, 14, -14, 42, 238, 386, 309, 126, 21]
(5, 3)	[-3, -21, -39, 43, 275, 457, 379, 161, 28]
(5, 2, 1)	[14, 26, -21, 3, 344, 764, 736, 344, 64]
(5, 1, 1, 1)	[-35, 0, 14, -14, 76, 285, 335, 175, 35]
(4, 4)	[-1, -8, -18, 9, 105, 194, 172, 77, 14]
(4, 3, 1)	[7, 18, -13, -22, 227, 630, 685, 350, 70]
(4, 2, 2)	[-6, 6, 4, -17, 110, 396, 484, 266, 56]
(4, 2, 1, 1)	[-19, -11, 23, -21, 69, 465, 675, 405, 90]
(4, 1, 1, 1, 1)	[35, -14, 0, 8, -11, 80, 190, 140, 35]
(3, 3, 2)	[-3, 2, 3, -12, 53, 238, 321, 189, 42]
(3, 3, 1, 1)	[-6, -5, 10, -9, 22, 226, 368, 238, 56]
(3, 2, 2, 1)	[7, -7, 5, -1, 7, 205, 395, 280, 70]
(3, 2, 1, 1, 1)	[14, -2, -7, 13, -16, 84, 272, 232, 64]
(3, 1, 1, 1, 1, 1)	[-21, 14, -8, 3, 1, -4, 48, 63, 21]
(2, 2, 2, 2)	[-1, 1, -1, 1, -1, 19, 56, 49, 14]
(2, 2, 2, 1, 1)	[-3, 3, -3, 3, -3, 17, 89, 91, 28]
(2, 2, 1, 1, 1, 1)	[-5, 3, -1, -1, 3, -5, 35, 55, 20]
(2, 1, 1, 1, 1, 1, 1)	[7, -6, 5, -4, 3, -2, 1, 14, 7]
(1, 1, 1, 1, 1, 1, 1, 1)	[-1, 1, -1, 1, -1, 1, -1, 1, 1]

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