Mean Curvature Flow in Euclidean spaces, Lagrangian Mean Curvature Flow, and Conormal Bundles

by

Chun Ho Leung

A thesis
presented to the University of Waterloo
in fulfilment of the
thesis requirement for the degree of
Master of Mathematics
in
Pure Mathematics

Waterloo, Ontario, Canada, 2011

AUTHOR'S DECLARATION

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Chun Ho Leung

Abstract

I will present the mean curvature flow in Euclidean spaces and the Lagrangian mean curvature flow. We will first study the mean curvature evolution of submanifolds in Euclidean spaces, with an emphasis on the case of hypersurfaces. Along the way we will demonstrate the basic techniques in the study of geometric flows in general (for example, various maximum principles and the treatment of singularities).

After that we will move on to the study of Lagrangian mean curvature flows. We will make the relevant definitions and prove the fundamental result that the Lagrangian condition is preserved along the mean curvature flow in Kähler-Einstein manifolds, which started the extensive, and still ongoing, research on Lagrangian mean curvature flows. We will also define special Lagrangian submanifolds as calibrated submanifolds in Calabi-Yau manifolds.

Finally, we will study the mean curvature flow of conormal bundles as submanifolds of $\mathbf{C}^n$. Using some tools developed recently, we will show that if a surface has strictly negative curvatures, then away from the zero section, the Lagrangian mean curvature flow starting from a conormal bundle does not develop Type I singularities.

Acknowledgements

First of all I would like to thank my advisor Spiro Karigiannis for his support and encouragement. Without him this work would not have been possible. He is a strong mathematician and a very good teacher. More importantly, he cares about students, and he is always concerned about whether I have got enough sleep. It is a wonderful experience working with him. I would also like to thank my readers, Ruxandra Moraru and Doug Park for their enormous help during the preparation of this thesis.

It has been a fruitful year in University of Waterloo, and I have learned a lot from the faculty members, and fellow graduate students. Waterloo is a nice city, although it is quite too cold in the Winter.

I also want to thank my parents, who have always supported me, especially during this past year, in which I disappointed them a lot. Many thanks go to Ms. Poon Suet Fan, who is a dedicated teacher, and who made me determined to do mathematics. The most important thing I learned from her was that I should always work harder and harder.

A lot of professors have helped me during my study of pure mathematics, and I am much indebted to them: Prof. Ruxandra Moraru, Prof. Thomas Au, Prof. Kathryn Hare, to name a few.

Finally, my heartfelt gratitude goes to my friends, whose friendship and support are very important to me. I would espeically like to thank Eric Kwan, Chris Wong, Raine Chan, Vincent Yip for being there with me during the toughest times of my life.

This thesis is dedicated to my parents, and Ms. Poon Suet Fan.

Contents

1 Basic Settings

In this thesis, all manifolds will be smooth. Let M be an m-dimensional manifold. A submanifold N in M is a subset of M that admits adapted coordinates: for every point $p \in N$ there exists a coordinate neighborhood U of p in M such that in the local coordinates (x_i) corresponding to U, the intersection $N \cap U$ corresponds to $x_{n+1} = ... = x_m = 0$. This n is independent of the point or the choice of chart chosen. In this case, we say N is an embedded n-dimensional submanifold of M and the codimension of N is $m-n$. In the case where the codimension is 1 we call N a hypersurface of M. It can be proved that with the subspace topology, N is an n-dimensional manifold itself and the inclusion map is an embedding. That is, it is a smooth topological embedding that is an immersion (the differential at each point assumes full rank).

Alternatively, we can consider a wider class of submanifolds, the immersed submanifolds. Let N be an abstract manifold, and M as above. If there exists an immersion $i : N \mapsto M$, then we say $i(N)$ is an immersed submanifold of M. In what follows, we will identify N with its image $i(N)$, and the immersion i in this case would become the inclusion. Note that an important difference between an immersed submanifold and an embedded submanifold is that an immersed submanifold with the subspace topology need not be a manifold itself. An example illustrating this point is the figure-eight, which can be thought of as $\mathbf{R}$ immersed in $\mathbf{R}^2$, but with the subspace topology, the subset fails to be a manifold precisely at the centre.

The tangent space at a point p of the submanifold N (embedded or immersed) can be thought of as a subspace of the tangent space at p of the ambient manifold M, via the push-forward of the inclusion. That i_* is injective follows from our very definition of submanifolds, hence this identification is justified. Most of our results will be true for both types of submanifolds, although for simplicity we may just state the proof for the embedded ones.

Now in what follows, we assume that M is equipped with a metric g, which is a smoothly varying inner product on the tangent spaces.

The metric g on M naturally induces a metric g' on N via the pull-back of the inclusion. More specifically, if $U, V \in T_pN$, we define $g'(U,V) = g(i_*(U), i_*(V))$. Another equivalent way to say this is to define g' such that with respect to g' and g, the inclusion is an isometry.

Let $\bar{\nabla}$ be the Levi-Civita connection on M with respect to g, and X, Y be local vector fields on N, and p a point in the intersection of their domains of definition. We can always extend X and Y to local vector fields $\tilde{X}$, $\tilde{Y}$ in M, such that $\tilde{X}$ and $\tilde{Y}$ agree with X and Y respectively on N. Define $\bar{\nabla}^T_X(Y)$ to be the tangential part of $\bar{\nabla}_{\tilde{X}}(\tilde{Y})$ to N, i.e. its projection onto the tangent space of N. It is easy to prove that $\bar{\nabla}^T$ is well-defined, and defines the Levi-Civita connection of N with respect to the induced metric.

The normal part of $\bar{\nabla}_{\tilde{X}}(\tilde{Y})$, namely $\bar{\nabla}^{\perp}_{\tilde{X}}(\tilde{Y}) = \bar{\nabla}_{\tilde{X}}(\tilde{Y}) - (\bar{\nabla}_{\tilde{X}}(\tilde{Y}))^T$, is denoted $B(X,Y)$. This B is called the second fundamental form of N and it is well-defined and symmetric in X, Y (see [2], chapter 6). From the fact that $\bar{\nabla}_X(Y)$

is tensorial in X, symmetry of B also means that $B(X,Y)$ is tensorial in both X and Y.

Note that $B : T_pN \times T_pN \mapsto (T_pN)^\perp$ is symmetric, hence for a normal vector field η, the bilinear map $S_\eta(X,Y) = g(B(X,Y),\eta) = \langle B(X,Y),\eta\rangle$ is also symmetric. So there exists a self-adjoint map $S_\eta : T_pN \mapsto T_pN$ such that $g(S_\eta(X),Y) = S_\eta(X,Y) = g(B(X,Y),\eta)$. It can be easily proved that $S_\eta(X) = -(\bar{\nabla}_X\eta)^T$. Any of these maps may be called the second fundamental form, depending on convention.

Hence, now we have two pieces of information. The first is the intrinsic information on N, here expressed by the tangential component of $\bar{\nabla}$; the second one, given by the second fundamental form, is extrinsic and depends on the way N is embedded in M.

Thus, in addition to the intrinsic curvature of N defined by the metric, B measures the way N sits in M, and this defines other notions of curvature as a measurement of this. In particular, the mean curvature will be of interest to us.

To formulate this, at any point $p \in N$ we take an orthonormal basis $\{e_i\}$ of T_pN. Let η be a normal vector field. The mean curvature H_η with respect to η at p is defined to be $H_\eta = \sum_{i=1}^n S_\eta(e_i,e_i) = \sum_i g(B(e_i,e_i),\eta)$, namely, the metric trace of S_η. Let $\eta_1,\eta_2,...,\eta_{m-n}$ be a local orthonormal basis of normal vector fields. The mean curvature vector $\mathbf{H}$ is then defined to be $\sum_i H_{\eta_i}\eta_i$. Depending on the convention, there may be a factor of $1/n$.

For later use, we need two important equations, the Gauss and Codazzi equations, relating the curvature information of the submanifold N and the ambient manifold M.

The first one compares the difference between the two curvatures. Let $\bar{R}$ be the curvature of M, defined by

$$\bar{R}(X,Y,Z,W) = g(\bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]} Z, W),$$

where $\bar{\nabla}$ is the Levi-Civita connection of the ambient manifold M. Equivalently, we can define

$$\bar{R}(X,Y)Z = \bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]} Z.$$

Similarly, let R be the curvature of N with respect to the induced metric. Then the Gauss equation states that if $X, Y, Z, W \in T_p N$,

$$\bar{R}(X,Y,Z,W) - R(X,Y,Z,W) = \langle B(Y,W), B(X,Z) \rangle - \langle B(X,W), B(Y,Z) \rangle \,. \tag{1.1}$$

Hence the difference between the curvatures is precisely given by the second fundamental form.

The Codazzi equation is about switching indices of the covariant derivative of the second fundamental form. Briefly, it states that switching index corresponds to a curvature term, which is a common phenomenon in geometric analysis. More precisely, for $X, Y, Z \in T_pN$ and η a normal vector field,

$$\left\langle \bar{R}(X,Y)Z, \eta \right\rangle = (\bar{\nabla}_Y B)(X, Z, \eta) - (\bar{\nabla}_X B)(Y, Z, \eta). \tag{1.2}$$

Notationally, we will denote $A_{ij} = B(e_i, e_j)$ at $p \in N$, and $A^{\nu}_{ij} = \langle A_{ij}, \nu \rangle$.

2 Minimal submanifolds and the volume functional

Definition 2.1. Let N be an n-dimensional submanifold of M, as in the last section. Then N is said to be minimal in M if the mean curvature vector $\mathbf{H} = 0$.

A classical interpretation of this condition is as follows: let N_t be a small variation of N, with $t \in (-\epsilon, \epsilon)$. Each N_t is an n-dimensional submanifold of M, varying smoothly with t, and $N_0 = N$. The metric on M induces metrics on each N_t, and it is natural to ask what condition guarantees that N has the smallest or largest volume with respect to all its small variations.
An equivalent mathematical way of formulating this problem is to consider the volume functional. Let $f_t : N \to M$ be a smooth family of immersions. The collection $f = \{f_t\}$ corresponds to the smooth variations of N we considered above.
Let $I(t) = I_f(t) = \int_N d(f_t^* g)$, where g is the metric on M, and $t \in (-\epsilon, \epsilon)$. We use $d(f_t^* g)$ to denote the volume form on N with respect to the metric $f_t^* g$. Differentiating with respect to t,

$$\frac{d}{dt} I(t) = \int_N \frac{d}{dt}(d(f_t^* g)).$$

Denote $g_t = f_t^* g$. Let $p \in M$, and consider a coordinate system (x_i) of N around p. Let e_i be the i-th coordinate vector field, and $g_{t,ij} = g_t(e_i, e_j)$.

Then it is well-known that $dg_t = \sqrt{\det(g_{t,ij})}dx^1 \wedge ... \wedge dx^n$ where dg_t is the volume element with respect to g_t (see [10]). From this, it follows that

$$dg_t = (\sqrt{\det(g_{t,ij})}/\sqrt{\det(g_{0,ij})})dg_0.$$

Hence it suffices to compute $\frac{\partial}{\partial t}(\sqrt{\det(g_{t,ij})}/\sqrt{\det(g_{0,ij})})$.

By considering a normal coordinate system at p, we can assume that at p, $g_{0,ij} = \delta_{ij}$. We have

$$\frac{\partial}{\partial t}(\sqrt{\det(g_{t,ij})}/\sqrt{\det(g_{0,ij})})|_{t=0} = \frac{1}{2}\frac{\partial}{\partial t}(\det(g_{t,ij}))|_{t=0}.$$

On the other hand,

$$\frac{\partial}{\partial t}(\det(g_{t,ij}))|_{t=0} = tr(\frac{\partial}{\partial t}g_{t,ij}|_{t=0}) = \sum_{i=1}^{n}\frac{\partial}{\partial t}g_t(e_i, e_i)|_{t=0}.$$

Now for simplicity we denote $\bar{\nabla}_t = \bar{\nabla}_{\frac{\partial}{\partial t}}$ and $\bar{\nabla}_i = \bar{\nabla}_{e_i}$(recall that $\bar{\nabla}$ is the Levi-Civita connection on the ambient submanifold M). We can interchange the time derivative and space derivative and see that

$$\frac{\partial}{\partial t}g_t(e_i, e_i) = 2g_t(\bar{\nabla}_t e_i, e_i) = 2g_t(\bar{\nabla}_i\frac{\partial}{\partial t}f_t, e_i),$$

which, evaluated at $(p, 0)$ is $2g_0(\bar{\nabla}_i V, e_i)|_p$, where V is the variational field of f_t, evaluated by taking the derivative with respect to t at 0. This V is the direction in which the submanifold N_0 moves under f_t. We can decompose $V = V^T + V^\perp$, where V^T is tangential to N and $V^\perp$ is normal to N. Then

$$\sum_i 2g_0(\bar{\nabla}_i V, e_i) = \sum_i (2g_0(\bar{\nabla}_i V^T, e_i) + 2g_0(\bar{\nabla}_i V^\perp, e_i)) = 2\mathrm{div}_N(V^T) - 2H_{V^\perp},$$

where $H_{V^\perp}$ is the mean curvature with respect to the $V^\perp$ direction and the metric g_0. Integrating on N and dividing by 2, we get

$$\frac{\partial}{\partial t} I(t)|_{t=0} = \int_N \mathrm{div}_N(V^T) - \int_N H_{V^\perp}.$$

Now we assume N is closed (compact without boundary), then, by Stokes' Theorem, this is equal to $-\int_N H_{V^\perp} = -\int_N g_0(\mathbf{H}, V)$.

This leads to the following theorem:

Theorem 2.2. *The closed submanifold N is a critical point of the volume functional if and only if the mean curvature vector vanishes, i.e. N is minimal.*

One direction is immediate. The other direction follows from choosing the variation V to be $\mathbf{H}$.
It also follows that deformation in the direction of mean curvature, i.e. choosing $V = \mathbf{H}$, decreases the volume.
This theorem also shows that the term "minimal" can be misleading; it can be a local maximum, minimum or saddle point of the volume functional of

its variations. There are other conditions to guarantee a submanifold to be volume minimizing in its homology class. One such is the condition of being calibrated, i.e. there is a closed n-form η on M that has value less than or equal to 1 on any orthonormal n-frame in T_pM, but is 1 on T_qN, for any $q \in N$. The proof is just a simple application of Stokes' Theorem. More details of this can be found in [4].

3 The Mean Curvature Flow in Euclidean space

Let N be an n-dimensional submanifold of an m-dimensional Riemannian manifold M, as in the last section.

Due to the technicalities, for now we will only restrict ourselves to studying the case $M = \mathbb{R}^m$.

A motion by mean curvature flow of N is defined to be a smooth family of embeddings $F_t : N \to M$, for $t \in (-\varepsilon, \varepsilon)$ with F_0 the original embedding of N into M (or inclusion), such that at each point $p \in N$,

$$\frac{\partial}{\partial t} F_t(p) = \mathbf{H_t}(p)$$

in the domain of definition of t, where $\mathbf{H_t}(p)$ denotes the mean curvature vector of N_t at $F_t(p)$. We shall drop the subscript t when there is no confusion.

3.1 Short-time Existence and Uniqueness

In $\mathbb{R}^m$, the Levi-Civita connection is the flat one. The mean curvature vector, defined for an orthonormal frame $\{e_i\}$ to be $\sum_{i=1}^n (\bar{\nabla}_{e_i} e_i)^\perp$, is easily seen to be the metric trace of the second fundamental form. Hence for a general coordinate frame (instead of an orthonormal frame), we can write $\mathbf{H} = (g^{ij} \frac{\partial^2 F}{\partial x^i \partial x^j})^\perp$, where g_{ij} is the metric on N.

The equation $\frac{\partial}{\partial t}F_t(p) = \mathbf{H_t}(p)$ can then be written as

$$\frac{\partial}{\partial t}F(p) = (g^{ij}\frac{\partial^2 F}{\partial x^i \partial x^j})^{\perp} = g^{ij}\frac{\partial^2 F}{\partial x^i \partial x^j} - \left\langle g^{ij}\frac{\partial^2 F}{\partial x^i \partial x^j}, \frac{\partial F}{\partial x^k}\right\rangle_{\mathbb{R}^m} g^{kl}\frac{\partial F}{\partial x^l}.$$

In components,

$$\frac{\partial}{\partial t}F^{\alpha}(p) = g^{ij}\frac{\partial^2 F^{\alpha}}{\partial x^i \partial x^j} - g^{ij}g^{kl}\sum_{\beta=1}^{m}\frac{\partial^2 F^{\beta}}{\partial x^i \partial x^j}\frac{\partial F^{\beta}}{\partial x^k}\frac{\partial F^{\alpha}}{\partial x^l}.$$

By computing the symbol, it can be shown that this equation is not strictly parabolic (i.e. the right side is not strictly elliptic). Hence the standard theory of existence and uniqueness theorems of parabolic differential equations does not apply here.

Yet another way of writing the mean curvature flow equation is as follows: $\mathbf{H} = (g^{ij}\frac{\partial^2 F}{\partial x^i \partial x^j})^{\perp} = g^{ij}(\frac{\partial^2 F}{\partial x^i \partial x^j} - (\frac{\partial^2 F}{\partial x^i \partial x^j})^T)$. Notice that the last term is the Levi-Civita connection on N (which we denote by ∇ without a bar), we have $\mathbf{H} = g^{ij}(\frac{\partial^2 F}{\partial x^i \partial x^j} - \nabla_{\frac{\partial F}{\partial x^i}}\frac{\partial F}{\partial x^j})$. Consider now F as a vector-valued function, so component-wise it makes sense to talk about its Laplacian. Recall that for a function h on N (see [7]),

$$\nabla^2_{X,Y}h = \nabla_X \nabla_Y h - \nabla_{\nabla_X Y}h.$$

Hence we have established

$$\mathbf{H}_t = g_t^{ij}\nabla^2_{\frac{\partial F}{\partial x^i},\frac{\partial F}{\partial x^j}}F = \triangle_{g(t)}F,$$

where in the last equation, the metric $g(t) = g_t$ is the induced metric on N by the immersion at time t. The above equation means that each component of $\mathbf{H}$ is the Laplacian on (N, g_t) of the corresponding component of F. The introduction of the subscript t again is to emphasize the fact that this Laplacian depends on the metric $g(t)$ as it evolves - for this reason it is not strictly parabolic as it may seem.

To prove the short-time existence and uniqueness, we will use deTurck's trick, which is to fix a connection through a diffeomorphism.

Theorem 3.1. *If N is compact, then there is a unique short-time solution to the mean curvature flow equation of N.*

Proof. Suppose for some choice of vector field v, the equation

$$\frac{\partial \tilde{F}}{\partial t} = \triangle_{g(t)} \tilde{F} + v^k \nabla_k \tilde{F} \tag{3.1}$$

with initial value F_0 is uniquely solvable for a short time (here we denote $\nabla_k \tilde{F} = \frac{\partial \tilde{F}}{\partial x_k}$ for simplicity). We show that this means that same holds for the mean curvature flow of F_0.

Indeed, consider a family of time-dependent diffeomorphisms $\varphi_t : N \times [0, T) \to N$ of N. Let $F_t(p) = \tilde{F}_t(\varphi_t(p)) := \tilde{F}(\varphi_t(p), t)$ (and similarly, to emphasize time dependence we may write a time-dependent function $A_t(p)$ as $A(p, t)$).

By the chain rule, (3.1) is transformed to

$$
\begin{aligned}
\frac{\partial F_t}{\partial t}(p) &= \frac{\partial}{\partial t}\tilde{F}(\varphi_t(p),t) + \nabla_k \tilde{F}(\varphi_t(p),t)\frac{\partial \varphi_t(p)^k}{\partial t} \\
&= \triangle_{g(t)}\tilde{F}(\varphi_t(p),t) + \nabla_k \tilde{F}(\varphi_t(p),t)(v^k + \frac{\partial \varphi_t(p)^k}{\partial t}) \\
&= \triangle_{g(t)}F(p,t) + \nabla_k \tilde{F}(\varphi_t(p),t)(v^k + \frac{\partial \varphi_t(p)^k}{\partial t}).
\end{aligned}
$$

So to obtain the mean curvature flow equation it suffices to find φ_t such that

$$
\frac{\partial \varphi_t}{\partial t} = -v, \qquad \varphi_0 = \mathrm{id}.
$$

This is a system of ODE, and φ_t exists for compact initial data.
Hence the problem is to choose a suitable vector field v such that the first system is short-time uniquely solvable.
Now choose a fixed connection $\tilde{\nabla}$ on N. Choose the vector field v such that $v^k = g^{ij}(\Gamma^k_{ij} - \tilde{\Gamma}^k_{ij})$, where $\tilde{\Gamma}^k_{ij}$ are the Christoffel symbols for $\tilde{\nabla}$. Then (3.1) becomes

$$
\begin{aligned}
\frac{\partial \tilde{F}}{\partial t} &= (g^{ij}\frac{\partial^2 \tilde{F}}{\partial x^i \partial x^j} - g^{ij}\Gamma^k_{ij}\frac{\partial \tilde{F}}{\partial x^k}) + g^{ij}(\Gamma^k_{ij} - \tilde{\Gamma}^k_{ij})\frac{\partial \tilde{F}}{\partial x^k} \\
&= g^{ij}\frac{\partial^2 \tilde{F}}{\partial x^i \partial x^j} - \tilde{\Gamma}^k_{ij}\frac{\partial F}{\partial x^k}.
\end{aligned}
$$

Then since we have changed Γ^k_{ij} to $\tilde{\Gamma}^k_{ij}$, which is independent of t, this right side expression is strictly elliptic. Hence the theory of parabolic equations applies to show that the equation, hence the mean curvature flow of F_0, has a unique short-time solution (see [12]). $\square$

3.2 Evolution of geometry

In this section, we will study how geometric quantities evolve under the mean curvature flow.

Theorem 3.2. *Under the mean curvature flow, the metric and the volume form evolve as follows:*

$$\frac{\partial}{\partial t} g_{ij} = -2 \langle \mathbf{H}, A_{ij} \rangle \tag{3.2}$$

$$\frac{\partial}{\partial t} \sqrt{\det g} = -|\mathbf{H}|^2 \sqrt{\det g}. \tag{3.3}$$

Proof. As above, $F : M \to \mathbb{R}^m$ is the immersion, and e_i is $\frac{\partial}{\partial u^i} F$.
Interchanging the time and space derivatives, we compute:

$$\begin{aligned}
\frac{\partial g_{ij}}{\partial t} &= \frac{\partial}{\partial t} \left\langle \frac{\partial F}{\partial u^i}, \frac{\partial F}{\partial u^j} \right\rangle \\
&= \left\langle \nabla_t \frac{\partial F}{\partial u^i}, \frac{\partial F}{\partial u^j} \right\rangle + \left\langle \frac{\partial F}{\partial u^i}, \nabla_t \frac{\partial F}{\partial u^j} \right\rangle \\
&= \left\langle \nabla_i \mathbf{H}, \frac{\partial F}{\partial u^j} \right\rangle + \left\langle \frac{\partial F}{\partial u^i}, \nabla_j \mathbf{H} \right\rangle \\
&= - \langle \mathbf{H}, \nabla_i e_j \rangle - \langle \mathbf{H}, \nabla_j e_i \rangle \\
&= - \langle \mathbf{H}, A_{ij} \rangle - \langle \mathbf{H}, A_{ji} \rangle = -2 \langle \mathbf{H}, A_{ij} \rangle
\end{aligned}$$

by symmetry of the second fundamental form.
The second equation is essentially proved in the first variation formula. $\square$

From (3.3), we can see that as long as $|\mathbf{H}|^2$ is bounded, say $|\mathbf{H}|^2 \leq C$

throughout, then $\frac{\partial}{\partial t}\sqrt{\det g} \geq -C\sqrt{\det g}$ as long as the flow exists. Then this means $\sqrt{\det g(t)} \geq \sqrt{\det g_0}e^{-Ct}$ along the flow, hence it stays positive since $\sqrt{\det g_0}$ is positive. This means that under the mean curvature flow, F remains an immersion as long as the flow exists and the second fundamental form is bounded throughout.

There is also an unexpected connection of the Mean Curvature Flow with the Ricci flow. Suppose $\{e_i\}$ is an orthonormal basis of T_pN. Then the Ricci tensor is defined to be (in components) $R_{ij} = \sum_{l=1}^{n} \langle R(e_i, e_l)e_l, e_j\rangle$. From the Gauss equation (1.1) in the first section,

$$R_{ij} = \sum_{l=1}^{n} \langle A_{ij}, A_{ll}\rangle - \sum_{l=1}^{n} \langle A_{lj}, A_{il}\rangle = \langle A_{ij}, \mathbf{H}\rangle - \sum_{l=1}^{n} \langle A_{lj}, A_{il}\rangle .$$

Hence the evolution of the metric becomes $\frac{\partial}{\partial t}g_{ij} = -2R_{ij} - \sum_{l=1}^{n} \langle A_{lj}, A_{il}\rangle$, which is the Ricci flow with a correction term.

The above are only the evolution of intrinsic geometry on the submanifold N. We are also interested in how the extrinsic geometry -the way N is embedded- evolves. For simplicity, we will assume that the codimension is one. We will also point out at the end how we can generalize the formulas to higher codimensions.

Before we start to investigate the evolution of the second fundamental form, we need Simons's identities, which tell us about the Laplacian of the second fundamental form.

Since we are in codimension one, we have only one direction (up to sign) for the normal vector. We shall fix one choice ν, and denote $H = H^\nu := \langle \mathbf{H}, \nu \rangle$. To distinguish from the general situation, we shall denote $h_{ij} = A^\nu_{ij}$, and h to be the second fundamental form tensor $h(X,Y) = \langle B(X,Y), \nu \rangle$.

Lemma 3.3. The following two identities hold:

$$\triangle h_{ij} = \nabla^2_{i,j} H + H h^2_{ij} - |h|^2 h_{ij} \tag{3.4}$$

$$\frac{1}{2} \triangle |h|^2 = \left\langle h_{ij}, \nabla^2_{i,j} H \right\rangle + |\nabla h|^2 + H \operatorname{Tr}(h^3) - |h|^4 \tag{3.5}$$

where $h^2_{ij} = h_{il} g^{lm} h_{mj}$, and $h^3_{uv} = h_{uk} g^{ki} h_{il} g^{lm} h_{mv}$. To avoid confusion, h_{ij} squared will be denoted by $(h_{ij})^2$, while h^2_{ij} will denote tensor components of the tensor h^2, and similarly for h^3. The norm $|h| = (g^{ij} g^{kl} h_{ik} h_{jl})^{1/2}$ is the norm of the tensor h with respect to the metric g.

Proof. At a point p, assume the coordinate system we chose is normal, namely with vanishing Christoffel symbols at p.
Then we have $(\triangle h)_{ij} = \sum_k \nabla_k \nabla_k h_{ij}$ ($\triangle h$ is a tensor whose definition can be found in [11]). From the Codazzi equation (1.2), since the ambient space is flat and we are using a normal coordinate system, $\sum_k \nabla_k \nabla_k h_{ij} = \sum_k \nabla_k \nabla_i h_{kj}$. Switching the first two indices i and k results in curvature terms: $\sum_k \nabla_k \nabla_i h_{kj} = \nabla_i \nabla_k h_{kj} + R_{ikj\alpha} h_{\alpha k} + R_{ikk\beta} h_{\beta j}$ (we omit the summation when it is obvious). Now the Gauss equation (1.1), and again the

Codazzi equation (1.2) give:

$$\begin{aligned}
\Delta h_{ij} &= \nabla_i \nabla_k h_{kj} + (h_{i\alpha} h_{kj} - h_{ij} h_{k\alpha}) h_{\alpha k} + (h_{i\beta} h_{kk} - h_{ik} h_{k\beta}) h_{\beta j} \\
&= \nabla_i \nabla_j h_{kk} + h_{i\alpha} h_{kj} h_{\alpha k} - h_{ij} h_{k\alpha} h_{\alpha k} + h_{i\beta} h_{kk} h_{\beta j} - h_{ik} h_{k\beta} h_{\beta j} \\
&= \nabla_i \nabla_j H + h_{i\alpha} h^2_{j\alpha} - h_{ij} |h|^2 + H h^2_{ij} - h_{ik} h^2_{jk} \\
&= \nabla_i \nabla_j H - |h|^2 h_{ij} + H h^2_{ij}
\end{aligned}$$

as desired. Next, we compute:

$$\begin{aligned}
\frac{1}{2} \triangle |h|^2 &= \tfrac{1}{2} \quad \triangle (\sum_{i,j} (h_{ij})^2) = \frac{1}{2} \nabla_k \nabla_k (h_{ij})^2 = \nabla_k (h_{ij} \nabla_k h_{ij}) \\
&= (\nabla_k h_{ij})^2 + h_{ij} \nabla_k \nabla_k h_{ij} \\
&= |\nabla h|^2 + h_{ij} (\nabla^2_{i,j} H + H h^2_{ij} - |h|^2 h_{ij}) \\
&= |\nabla h|^2 + \left\langle h_{ij}, \nabla^2_{i,j} H \right\rangle + H \operatorname{Tr}(h^3) - |h|^4.
\end{aligned}$$

□

We now derive the evolution of extrinsic geometric quantities.

Theorem 3.4. *(Evolution of extrinsic geometry) The extrinsic geometric quantities evolve under the flow as follows:*

$$\frac{\partial}{\partial t} \nu = -\frac{\partial H}{\partial x_i} \frac{\partial F}{\partial x_j} g^{ij} \tag{3.6}$$

$$\frac{\partial}{\partial t} h_{ij} = \triangle h_{ij} - 2H h^2_{ij} + |h|^2 h_{ij} \tag{3.7}$$

$$\frac{\partial}{\partial t}H = \triangle H + |h|^2 H \tag{3.8}$$

$$\frac{\partial}{\partial t}|h|^2 = \triangle|h|^2 - 2|\nabla h|^2 + 2|h|^4 \tag{3.9}$$

Proof. Denote $u_i = \frac{\partial F}{\partial x_i}$. Since u_i's are a basis, we have:

$$\begin{aligned}
\frac{\partial}{\partial t}\nu &= g^{ij}\left\langle \frac{\partial}{\partial t}\nu, u_i \right\rangle u_j \\
&= -g^{ij}\left\langle \nu, \frac{\partial}{\partial t}u_i \right\rangle u_j = -g^{ij}\left\langle \nu, \nabla_i \mathbf{H} \right\rangle u_j \\
&= -g^{ij}\nabla_i \left\langle \nu, H\nu \right\rangle u_j = -g^{ij}(\nabla_i H)u_j \text{ (since } \left\langle \nabla_i \nu, \nu \right\rangle = 0).
\end{aligned}$$

Similarly, we have:

$$\begin{aligned}
\frac{\partial}{\partial t}h_{ij} &= \frac{\partial}{\partial t}\left\langle \bar{\nabla}_i u_j, \nu \right\rangle \\
&= \left\langle \bar{\nabla}_i \bar{\nabla}_j (H\nu), \nu \right\rangle + \left\langle \bar{\nabla}_i u_j, -g^{uv}(\nabla_u H)u_v \right\rangle \\
&= \left\langle \bar{\nabla}_i((\bar{\nabla}_j H)\nu - Hg^{lk}h_{jl}u_k), \nu \right\rangle - g^{uv}\Gamma^k_{ij}g_{kv}\nabla_u H \\
&= \left\langle (\bar{\nabla}_i \bar{\nabla}_j H)\nu - Hg^{lk}h_{jl}\bar{\nabla}_i u_k, \nu \right\rangle - \Gamma^k_{ij}\nabla_k H \quad (\left\langle \bar{\nabla}_i \nu, \nu \right\rangle = \left\langle u_k, \nu \right\rangle = 0)
\end{aligned}$$

$$\begin{aligned}
&= \bar{\nabla}_i \bar{\nabla}_j H - Hg^{lk}h_{jl}h_{ik} - \Gamma^k_{ij}\nabla_k H \\
&= \nabla^2_{i,j}H - Hg^{lk}h_{jl}h_{ik} \\
&= \triangle h_{ij} - 2Hg^{lk}h_{jl}h_{ik} + |h|^2 h_{ij}
\end{aligned}$$

where we have used Lemma 3.3 in the last line.

To derive the evolution of the mean curvature, we note that by (3.2)

$$\frac{\partial}{\partial t}g^{ij} = -g^{il}\frac{\partial g_{lk}}{\partial t}g^{kj} = 2Hg^{il}h_{lk}g^{kj} = 2Hh^{ij}.$$

Then we see:

$$\begin{aligned}
\frac{\partial}{\partial t}H &= \frac{\partial}{\partial t}(g^{ij}h_{ij}) = (\frac{\partial}{\partial t}g^{ij})h_{ij} + g^{ij}\frac{\partial}{\partial t}h_{ij} \\
&= 2Hh^{ij}h_{ij} + g^{ij}(\triangle h_{ij} - 2Hg^{lk}h_{jl}h_{ik} + |h|^2 h_{ij}) \\
&= g^{ij}\triangle h_{ij} + g^{ij}|h|^2 h_{ij} = \triangle H + |h|^2 H.
\end{aligned}$$

Finally, we compute:

$$\begin{aligned}
\frac{\partial}{\partial t}|h|^2 &= \frac{\partial}{\partial t}(g^{ik}g^{jl}h_{ij}h_{kl}) \\
&= 2Hh^{ik}g^{jl}h_{ij}h_{kl} + g^{ik}(2Hh^{jl})h_{ij}h_{kl} + g^{ik}g^{jl}(\triangle h_{ij} - 2Hh^2_{ij} + |h|^2 h_{ij})h_{kl} \\
&\quad + g^{ik}g^{jl}h_{ij}(\triangle h_{kl} - 2Hh^2_{kl} + |h|^2 h_{kl}) \\
&= 4Hh^{ik}g^{jl}h_{ij}h_{kl} + 2g^{ik}g^{jl}h_{kl}(\triangle h_{ij} - 2Hg^{uv}h_{iu}h_{jv} + |h|^2 h_{ij}) \\
&= 4Hg^{kl}h_{kl} + 2g^{ik}g^{jl}h_{kl}\triangle h_{ij} - 4g^{ik}g^{jl}h_{kl}Hg^{uv}h_{iu}h_{jv} + 2g^{ik}g^{jl}h_{kl}|h|^2 h_{ij} \\
&= 2\langle h, \triangle h\rangle + 2|h|^4.
\end{aligned}$$

Since $2\langle h, \triangle h\rangle = \triangle|h|^2 - 2|\nabla h|^2$, this last equation is equal to

$$\frac{\partial}{\partial t}|h|^2 = \triangle|h|^2 - 2|\nabla h|^2 + 2|h|^4$$

as desired. $\square$

In the case of higher codimensions, many of the quantities above become tensors. For example, the second fundamental form A_{ij}^{ν} will depend on the choice of the unit normal vector field ν. If, in addition, the ambient space is not $\mathbf{R}^n$, then our method above will not work anymore because these are only computations in local coordinates, and the same method may not hold when the ambient space is not flat. A more detailed account, and the parallel computations in higher codimensions, can be found in [18].

3.3 Maximum principles of Elliptic and Parabolic PDEs

One sees from the above evolution equations that many of the geometric quantities evolve by their own Laplacians. This is a general phenomenon that happens with geometric flows. In general, we cannot solve for the quantities directly. Thus there is a need to study the qualitative behaviours of these quantities without knowing the exact solutions.

The following theorem can be found in [20].

Theorem 3.5. *For $t \in [0, T]$, let $g(t)$ be a smooth family of metrics on M. Let $X(t)$ a smooth family of vector fields, and $K(x, t)$ a function on $M \times [0, T]$. Suppose a smooth function u on $M \times [0, T]$ satisfies*

$$\frac{\partial u}{\partial t} \leq \triangle_{g(t)} u + \langle X(t), \nabla u \rangle + K(u, t).$$

Let $\varphi : [0, T] \to \mathbf{R}$ be a function such that $\frac{d}{dt}\varphi = K(\varphi(t), t)$ and $\varphi(0) = \alpha$. If $u(x, 0) \leq \alpha$ for all x, then $u(x, t) \leq \varphi(t)$ for all $t \in [0, T]$.

Proof. Consider φ_ε where $\frac{d\varphi_\varepsilon}{dt} = K(\varphi_\varepsilon(t), t) + \varepsilon$ and $\varphi_\varepsilon = \alpha + \varepsilon$. It is easy to see that such φ_ε exists for all $\varepsilon > 0$ by classical ODE theory, and that $\varphi_\varepsilon \to \varphi$ uniformly as $\varepsilon \to 0$. Then it suffices to prove that $u \leq \varphi_\varepsilon$ for all $\varepsilon > 0$.

Suppose this is false, so for some ε_0, t_0 and $x \in M$ we have $u(x, t_0) > \varphi_\varepsilon(t_0)$. If t_0 is the first such time, then $\frac{\partial}{\partial t}(u - \varphi_\varepsilon) \geq 0$ at (x, t_0). Without loss of generality x is a maximum of u at t_0, so $\triangle u \leq 0$ and $\nabla u = 0$. Hence the original equation becomes $\frac{\partial}{\partial t}u(x, t_0) \leq K(x, t_0) < \frac{\partial \varphi_\varepsilon}{\partial t}$, which is a contradiction. $\square$

Remark 3.6. Clearly the above theorem is still true with $\geq$ replaced by $\leq$, and ε by $-\varepsilon$ everywhere.

In the following, we will denote the second order partial derivatives with respect to x_i and x_j by D^2_{ij}.

Theorem 3.7. *Let $Lu = \sum_{ij} a^{ij} D^2_{ij} u + \langle X(t), \nabla u \rangle + c(x, t)u - \frac{\partial}{\partial t}u$ on $\Omega = \Omega' \times [0, T]$ where Ω' is some bounded domain on which a^{ij} is a positive definite matrix, $X(t)$ is a time-dependent vector field, and $c(x, t)$ is a smooth function on Ω. Assume that $Lu \leq 0$, and assume that u can be extended to $\bar{\Omega}$. Then if u starts off non-negative at $t = 0$ and on $\partial\Omega \times (0, T)$, then it remains so throughout.*

Proof. For now we assume that $Lu < 0$ everywhere.

Let $S = \{(x,t) \in \bar{\Omega} : u(x,t) \leq 0\}$. This set is compact. There is a first $t_0 > 0$ for which there is (x_0, t_0) such that $u(x_0, t_0) \leq 0$. By continuity, since $u(x,t) \geq 0$ for $t < t_0$, we have $u(x_0, t_0) = 0$ and that fixing the time slice t_0, it is easy to see that x_0 is an interior local minimum with respect to the x-component. This means that $\frac{\partial}{\partial t}u \leq 0$ and $\nabla u = 0$ there. So by positive definiteness of a^{ij}, $Lu = \sum_{ij} a^{ij} D^2_{ij} u + \langle X(t), \nabla u \rangle + c(x,t)u - \frac{\partial}{\partial t}u \geq 0$ at that point, contradicting $Lu < 0$.

In the case that $Lu \leq 0$, bound Ω' by $\Omega' \subset \{||x_1|| < d\}$. Consider $u_\varepsilon = u - \varepsilon e^{\alpha x_1}$, where $\alpha > 0$ is to be chosen. It is easy to compute that

$$
\begin{aligned}
Lu_\varepsilon &= Lu - \varepsilon(\alpha^2 a^{11}(x,t) + \alpha X_1(x,t) + c(x,t))e^{\alpha x_1} \\
&\leq -\varepsilon(\alpha^2 a^{11}(x,t) + \alpha X_1(x,t) + c(x,t))e^{\alpha x_1} \\
&\leq -\varepsilon(\alpha^2 a^{11}(x,t) - |\alpha| ||X_1||_\infty - ||c(x,t)||_\infty)e^{\alpha x_1}.
\end{aligned}
$$

Choose $\alpha > 0$ sufficiently large, this is less than 0 (recall $a^{11} > 0$ by positive definiteness), hence we can apply the result in the first case (that $Lu < 0$) to show that $u_\varepsilon \geq 0$ for $\varepsilon > 0$. Letting $\varepsilon \to 0$ we get the result. $\square$

Remark 3.8. The same is true if we replace non-negativity by positivity: if u starts positive, then it remains so throughout.

Corollary 3.9. *If in the above notation, $Lu \leq 0$ and $Lv \geq 0$, then if $u \geq v$ on the boundary, we have that $u \geq v$ throughout.*

Proof. Apply Theorem 3.7 to $u - v$. $\square$

Corollary 3.10. *If the mean curvature is non-negative at time $t = 0$ on a compact hypersurface M, then along the mean curvature flow, the non-negativity is preserved.*

Proof. Apply the above corollary to the evolution equation (3.8) for H:

$$\frac{\partial}{\partial t} H = \triangle H + |h|^2 H,$$

and compare with the zero function. In fact, the positivity would also be preserved, using the strong maximum principle (see [12]). $\square$

Let u be a function defined on some domain Ω in $\mathbf{R}^n$, which is of the form $\cup_{t \in [0,T)} \Omega(t) \times \{t\}$, where each $\Omega(t)$ is some domain not necessarily open in $\mathbf{R}^{n-1}$. For a function $a(X, u, Du)$ of the point $X \in \Omega$, the function u and its derivative Du, the operator $Pu = -\frac{\partial u}{\partial t} + a^{ij}(X, u, Du) D^2_{ij} u + a(X, u, Du)$ is parabolic on $S \subset \Omega \times \mathbf{R} \times \mathbf{R}^n$ if the matrix $a^{ij}(X, z, p)$ is positive definite for all $(X, z, p) \in S$.
The following comparison principle is derived in [12].

Theorem 3.11. *Suppose a^{ij} is independent of z, and there is a constant $k(L)$ (that increases with L) such that $a(X, z, p) - k(L)z$ is an increasing function of z on $\Omega \times [-L, L] \times \mathbf{R}^n$ for $L > 0$. Suppose $u(x, t)$ and $v(x, t)$ are functions such that $Pu \geq Pv$ in the interior of Ω and $u \leq v$ on the boundary* *of Ω, and that P is parabolic with respect to u or v. Then $u \leq v$ throughout Ω.*

Proof. Let $L = \max\{\sup|u|, \sup|v|\}$, and $w = (u-v)e^{\lambda t}$, where λ is a constant to be determined. At a positive interior maximum $X_0 = (x_0, t_0)$ of w, we have

$$(Du - Dv)e^{\lambda t} = 0 \Rightarrow Du = Dv$$

$$D^2_{ij}u - D^2_{ij}v \leq 0$$

$$((u-v)_t + \lambda(u-v))e^{\lambda t} = \frac{\partial}{\partial t}w = 0.$$

Denote $R = (X_0, u(X_0), Du(X_0))$, and $S = (X_0, v(X_0), Dv(X_0))$. At this point, $Du(X_0) = Dv(X_0)$, and by the assumption that a^{ij} is independent of z, $a^{ij}(R) = a^{ij}(S)$. Now we have

$$\begin{aligned}
0 &\leq Pu(X_0) - Pv(X_0) = a^{ij}(R)D_{ij}u - a^{ij}(S)D_{ij}v + [a(R) - a(S)] - (u-v)_t \\
&\leq a^{ij}(R)D^2_{ij}(u-v) + [a(R) - a(S)] + \lambda[u-v] \\
&\leq a(R) - a(S) + \lambda[u-v] \qquad (D^2_{ij}(u-v) \leq 0) \\
&\leq k(L)(u-v) + \lambda(u-v) = (k(L)+\lambda)(u-v).
\end{aligned}$$

Now we could have chosen $\lambda < -k(L)$, which would mean $(k(L)+\lambda)(u-v) < 0$, a contradiction. Hence no interior positive maximum is possible. $\square$

3.4 Some explicit examples

We now discuss some explicit examples of mean curvature flow.

Example 3.12. (Graphs)

Let $u : \mathbf{R}^n \to \mathbf{R}$ be smooth. Then the graph of u is $\{(x, u(x)) : x \in \mathbf{R}^n\}$.

The characterization of a hypersurface M being a graph is that there exists a constant unit vector ω for which $\langle \nu, \omega \rangle > 0$ for a choice of nowhere vanishing normal vector field ν of M. To study the evolution of graphs, we first prove that under the mean curvature flow, a graph remains a graph.

Let ν be a unit normal vector field on M, and $\mathbf{H} = H\nu$. Then from our evolution equation, we have $\frac{\partial}{\partial t}\nu_t = -\frac{\partial H}{\partial x_i}\frac{\partial F}{\partial x_j}g^{ij} = -\nabla H$ by (3.6), where $-\nabla H$ is the coordinate free way of writing this equation ($\nabla H = (\nabla_{e_i} H)e_i$ for an orthonormal tangent frame $\{e_i\}$ of M).

Denote $r_t = \langle \nu_t, w \rangle$, where w is a constant unit vector such that on M, $\langle \nu, w \rangle > 0$. It suffices to prove that $r_t > 0$ for all t. Indeed, under the mean curvature flow,

$$\frac{\partial}{\partial t} r_t = -\langle \nabla H, w \rangle .$$

On the other hand, let e_i be a system of normal coordinates at a point p of the hypersurface, so $\langle e_i, e_j \rangle = \delta_{ij}$ and $\nabla^T_{e_i} e_j = 0$ at p. Then we have

$$\begin{aligned}
\triangle r = e_i(e_i(r_t)) &= e_i(\langle \nabla_{e_i}\nu, w\rangle) = e_i \langle -h_{il}e_l, w\rangle \\
&= -\langle e_i(h_{il})e_l, w\rangle - h_{il}\langle \nabla_{e_i}e_l, w\rangle \\
&= -\langle e_l(h_{ii})e_l, w\rangle - h_{il}\langle h_{il}\nu, w\rangle \text{ (Codazzi equation (1.2))} \\
&= -\langle \nabla H, w\rangle - |h|^2 r.
\end{aligned}$$

This shows that the evolution of r_t satisfies

$$\frac{\partial}{\partial t} r = \triangle r + |h|^2 r.$$

Hence r remains positive along the mean curvature flow by the remark following Theorem 3.7. It is easy to see that the normal vector ν remains a normal vector to the evolving hypersurface:

$$\begin{aligned}
\frac{\partial}{\partial t}\langle \nu, e_j\rangle &= -\langle (\nabla_i H)e_i, e_j\rangle + \langle \nu, \nabla_t e_j\rangle \\
&= -\nabla_j H + \langle \nu, \nabla_j \mathbf{H}\rangle = -\nabla_j H + \langle \nu, \nabla_j(H\nu)\rangle \\
&= -\nabla_j H + \nabla_j H + H\langle \nu, \nabla_j \nu\rangle = 0.
\end{aligned}$$

Hence we have proven:

Proposition 3.13. *Graphs remain graphs along the mean curvature flow.*

Now we will write down the evolution equation for u_t. The graph of u is $\{(x, u(x)) : x \in \mathbf{R}^n\}$. Hence the normal vector field ν is, up to sign,

$\frac{1}{\sqrt{1+|\nabla u|^2}}(-\nabla u, 1)$. Note that the mean curvature in the direction of a normal vector field ν is $-\mathrm{div}_T\nu$, where div_T is the tangential divergence.

Hence the mean curvature is $\mathrm{div}\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}$.

Writing the immersion as

$$F : \mathbf{R}^n \to \mathbf{R}^{n+1} \quad F(p,t) = (x(p,t), u(x(p,t),t)),$$

we want $\frac{\partial F}{\partial t} = H\nu$. So from $\nu = \frac{1}{\sqrt{1+|\nabla u|^2}}(-\nabla u, 1)$, we see that

$$\begin{aligned}
\frac{\partial F}{\partial t} &= (\frac{\partial x}{\partial t}, \left\langle \nabla u, \frac{\partial x}{\partial t} \right\rangle) + (0, \frac{\partial u}{\partial t}) = H\frac{1}{\sqrt{1+|\nabla u|^2}}(-\nabla u, 1) \\
\Rightarrow \frac{\partial x}{\partial t} &= -H\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \\
\Rightarrow \frac{\partial u}{\partial t} &= (1+|\nabla u|^2)H\frac{1}{\sqrt{1+|\nabla u|^2}} = H\sqrt{1+|\nabla u|^2} \\
&= \sqrt{1+|\nabla u|^2}\mathrm{div}\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}.
\end{aligned}$$

This is a quasi-linear parabolic system for u. Indeed, direct expansion gives

$$\frac{\partial u}{\partial t} = \delta_{ij}D^2_{ij}u - \frac{1}{1+|Du|^2}D^2_{ij}uD_iuD_ju = (\delta_{ij} - \frac{D_iuD_ju}{1+|Du|^2})D^2_{ij}u.$$

Let $a^{ij}(X,z,p) = \delta_{ij} - \frac{p_ip_j}{1+|p|^2}$. In the notation of Theorem 3.11, we also have $a = 0$, and from our definition a^{ij} does not depend on z. The equation is parabolic because if $p = 0$, the matrix a^{ij} is the identity, and for $p \neq 0$, for a vector x with $|x| = 1$,

$$x_i a^{ij}(X, z, p) x_j = 1 - \frac{\langle x, p\rangle}{1 + |p|^2} > 0$$

by the Cauchy-Schwarz inequality. Thus Theorem 3.11 applies to this setting. In particular, one consequence will be the following

Theorem 3.14. *(Avoidance principle) Let M_0 and N_0 be hypersurfaces without boundary that do not intersect. Under the mean curvature flow, M_t and N_t remain disjoint.*

Proof. Suppose M_t and N_t intersect at a point p at the first time t_0, then it can be shown that they share the same tangent plane, hence the normal vectors coincide possibly up to a sign.
Using this normal vector, we can form graphical coordinates for M_{t_0} and N_{t_0} near p. Then they can be represented as graphs of two functions u_{t_0} and v_{t_0} respectively. Shortly before t_0, there is $\varepsilon > 0$ such that $u_t - \varepsilon - v_t > 0$. Run the mean curvature flow from there, then both $u - \varepsilon$ and v solve the quasi-linear parabolic system. We have proved that Theorem 3.11 applies, hence at time t_0 we see a contradiction to the maximum principle. □

This is a property of mean curvature flow that happens only in the case of codimension one. In general this does not hold for higher codimensions. For example, one can easily see that two intertwining embedded circles in $\mathbf{R}^3$ would touch each other some time during the flow. Essentially, this avoidance principle works in codimension one because in this case, we can essentially reduce the flow to a scalar PDE.

The above theorem also has the following consequence.

Theorem 3.15. *(Containment principle) If M_0 and N_0 are hypersurfaces of $\mathbf{R}^{n+1}$ and M_0 is contained in the region bounded by N_0, then along the mean curvature flow, as long as the flow exists we have that M_t is contained in N_t for all t.*

Example 3.16. (Sphere) As a sphere and its mean curvature vector $\mathbf{H}$ are rotationally symmetric, the symmetry is preserved under the mean curvature flow. It suffices to calculate how the radius $r(t)$ transforms along the flow. Let $x \in \mathbb{S}^n \subset \mathbf{R}^{n+1}$ be a position vector. Then $\frac{d}{dt}(r(t)x) = \mathbf{H} = Hx$, where H is the mean curvature with respect to the position vector x itself.
To compute the mean curvature, we shall use the spherical coordinates to parametrize $\mathbf{R}^{n+1}$. For notational simplicity, we will only do the case $n = 2$. The general situation is done in exactly the same way.
The parametrization is given by $(r, \theta, \varphi) \mapsto (r\cos\theta\sin\varphi, r\sin\theta\sin\varphi, r\cos\varphi)$. Then the coordinate vectors are given by

$$\begin{aligned} E_1 = \frac{\partial}{\partial r} &= (\cos\theta\sin\varphi, \sin\theta\sin\varphi, \cos\varphi) \\ E_2 = \frac{\partial}{\partial \theta} &= (-r\sin\theta\sin\varphi, r\cos\theta\sin\varphi, 0) \\ E_3 = \frac{\partial}{\partial \varphi} &= (r\cos\theta\cos\varphi, r\sin\theta\cos\varphi, -r\sin\varphi) \end{aligned}$$

It is then clear that these vectors are mutually orthogonal, and $\frac{\partial}{\partial\theta}$, $\frac{\partial}{\partial\varphi}$ have length $r\sin\varphi$ and r respectively. Now we use the formula

$$\Gamma_{ij}^k = \frac{1}{2}\sum_l g^{kl}(\partial_i g_{jl} + \partial_j g_{li} - \partial_l g_{ij}).$$

It is then clear that $\Gamma_{22}^1 = -r\cos^2\varphi$ and $\Gamma_{33}^1 = -r$. So the mean curvature is given by $g^{22}(-r\cos^2\varphi) + g^{33}(-r) = -\frac{2}{r}$, by the standard formula for mean curvature of a surface in $\mathbf{R}^3$ (see for example [2]).

In general, for $\mathbb{S}_r^n \subset \mathbf{R}^{n+1}$, where r is the radius, the mean curvature with respect to the outward normal is $-\frac{n}{r}$.

Going back to our evolution of $r(t)$, plugging the mean curvature back in the equation $\frac{\partial}{\partial t}(r(t)x) = \mathbf{H} = Hx$ we find

$$\frac{dr}{dt} = -\frac{n}{r}.$$

Solving this ODE we have

$$r(t) = \sqrt{r_0^2 - 2nt}.$$

Hence spheres remain spheres along the mean curvature flow, and shrink to a point at time $t = \frac{r_0^2}{2n}$.

In particular, this implies the following:

Proposition 3.17. *Compact hypersurfaces develop singularity in finite time* *under the mean curvature flow.*

Proof. Let M_0 be a compact hypersurface. Choose a large sphere S_0 such that M is strictly contained in the region bounded by it. Let M_t be the solution of the mean curvature flow of M at time t, and similarly for S_t. The containment principle means that M_t is contained in the region bounded by S_t. But S_t shrinks to a point, and develops singularity at a finite time t_0. So either M_t develops a singularity before t_0, at which the flow will not continue, or it shrinks to a point at t_0, where singularity occurs. This completes the proof. $\square$

3.5 A few words on singularities

It can be shown that at the singularity, what will happen is that the norm of the second fundamental form will blow up somewhere. It was shown in [6] that if towards the singularity, the norm of the second fundamental form $|A|^2 = g^{ij}g^{kl}A_{ik}A_{jl}$ remains bounded, then in fact one can uniformly bound all the derivatives of the second fundamental form, and that the metric also converges to a positive definite tensor.

It also follows from the inequality $|F(x,t) - F(x,s)| \leq \int_s^t |\mathbf{H}|$ and the bound on $|\mathbf{H}|$ that the $F(\cdot, t)$ converges to some pointwise limit, and by the above and an application of the Arzela-Ascoli theorem we can conclude that the limiting surface is a smoothly immersed submanifold, with bounded second fundamental form. Hence the flow will continue by local existence.

Therefore, singularities occur precisely when the second fundamental form blows up. Hence it is natural to study the rate at which it grows.

Proposition 3.18. *Let $U(t) = \max_{M_t} |A|^2$, where M_t is a solution to the mean curvature flow with first singularity time T. Then $U(t) \geq \frac{1}{2(T-t)}$.*

Proof. Recall the evolution equation for $|A|^2$:

$$\frac{\partial}{\partial t}|A|^2 = \triangle|A|^2 - 2|\nabla A|^2 + 2|A|^4.$$

At the maximum point, $\triangle|A|^2 \leq 0$. So $\frac{\partial}{\partial t}U \leq 2U(t)^2$.
This means $-\frac{1}{U(t)} \leq 2t + C$, for some constant C. Since the second fundamental form blows up at $t = T$, we have $\frac{1}{U(t)} \to 0$ as $t \to T$, which means $C = -2T$. These combine to give

$$U(t) \geq \frac{1}{2(T-t)}.$$

□

Definition 3.19. The singularity is said to be of type I if there exists a constant C such that the blow-up rate satisfies $U(t) \leq \frac{C}{2(T-t)}$, namely, the slowest possible blow-up rate asymptotically.

A lot of research has been done to understand type I singularities. To study what happens at a type I singularity, we will often do a scaling so that we obtain a modified flow in which the second fundamental form remains bounded. More precisely, let $s = -\frac{1}{2}\log(T-t)$ and $\tilde{F}(p,s) = (2(T-$

$t))^{-1/2}F(p,t)$. The mean curvature flow transforms to

$$\frac{\partial}{\partial s}\tilde{F}(p,s) = \tilde{H}(p,s) + \tilde{F}(p,s).$$

Denote $\alpha = (2(T-t))^{-1/2}$. It can be easily seen with this scaling that $\tilde{g}_{ij} = \alpha^2 g_{ij}$, $\tilde{g}^{ij} = \alpha^{-2} g^{ij}$ and $\tilde{h}_{ij} = \alpha h_{ij}$. Hence

$$|\tilde{A}|^2 = \tilde{g}^{ij}\tilde{g}^{kl}\tilde{h}_{ik}\tilde{h}_{jl} = \alpha^{-2} g^{ij} g^{kl} h_{ik} h_{jl} = (2(T-t))|A|^2.$$

Assuming type I singularity, the second fundamental form of the modified flow remains bounded, so we can see more clearly what happens there before it blows up to a singularity. Also, s goes to ∞ as $t \to T$, so we study the convergence properties as $s \to \infty$.

To give a sense of what happens there, we shall need a well-known formula by Huisken, which is one of the most important tools in studying mean curvature flow.

Denote $\Phi(x_0,T)(x,t) = \frac{1}{(4\pi(T-t))^{n/2}} e^{-\frac{|x-x_0|^2}{4(T-t)}}$. We have the following:

Theorem 3.20. *(Huisken's monotonicity formula) If M_t are closed submanifolds of $\mathbf{R}^n$ satisfying the mean curvature flow for $t < T$, and assuming all quantities are finite, then*

$$\frac{d}{dt}\int_{M_t} \Phi(x_0,T)(x,t)d\mu_t = -\int_{M_t} \Phi(x_0,T)(x,t)|\mathbf{H} + \frac{1}{2(T-t)}(\mathbf{x}-x_0)^{\perp}|^2 d\mu_t$$

where $d\mu_t$ is the volume element of M_t, and $(\mathbf{x}-x_0)^{\perp}$ is the projection of

$\mathbf{x} - x_0$ *onto the normal plane.*

Proof. Denote $\rho = \Phi(x_0, T)$. We shall prove the following:

$$\frac{\partial}{\partial t}(\rho d\mu_t) = (-\triangle\rho - \rho|\mathbf{H} + \frac{1}{2\tau}(\mathbf{x} - x_0)^{\perp}|^2)d\mu_t$$

where $\tau = T - t$. We know that $\frac{\partial}{\partial t}d\mu_t = -|\mathbf{H}|^2 d\mu_t$ by (3.3), and

$$\frac{\partial}{\partial t}\rho = \frac{n\rho}{2\tau} + \rho(-\frac{|\mathbf{x} - x_0|^2}{4\tau^2} - \frac{\langle \mathbf{H}, \mathbf{x} - x_0\rangle}{2\tau}).$$

From $|\mathbf{H} + \frac{1}{2\tau}(\mathbf{x} - x_0)|^2 = |\mathbf{H}|^2 + \frac{1}{\tau}\langle \mathbf{H}, \mathbf{x} - x_0\rangle + \frac{1}{4\tau^2}|\mathbf{x} - x_0|^2$, we get

$$\begin{aligned} \frac{\partial}{\partial t}(\rho d\mu_t) &= (\frac{n}{2\tau} - \frac{|\mathbf{x} - x_0|^2}{4\tau^2} - \frac{\langle \mathbf{H}, \mathbf{x} - x_0\rangle}{2\tau} - |\mathbf{H}|^2)\rho d\mu_t \qquad (3.10) \\ &= (\frac{n}{2\tau} + \frac{1}{2\tau}\langle \mathbf{x} - x_0, \mathbf{H}\rangle - |\mathbf{H} + \frac{1}{2\tau}(\mathbf{x} - x_0)|^2)\rho d\mu_t. \qquad (3.11) \end{aligned}$$

Let $\{e_i\}$ be a basis of vector fields induced from normal coordinates. Now we compute $\mathrm{grad}(\rho) = \rho(-\frac{1}{2\tau}\langle \mathbf{x} - x_0, e_i\rangle)e_i = \frac{-\rho}{2\tau}(\mathbf{x} - x_0)^T$.

For a vector field Y, not necessarily tangential to $M = M_t$, we have the formula $\mathrm{div}_M Y = \langle \nabla_i Y, e_i\rangle = \left\langle \nabla_i Y^T + \nabla_i Y^N, e_i\right\rangle = \mathrm{div}_M Y^T - \langle Y, \mathbf{H}\rangle$. Hence letting $Y = \frac{\rho(\mathbf{x} - x_0)}{2\tau}$, using the product rule, $\mathrm{div}_M Y = \frac{n\rho}{2\tau} + \frac{1}{2\tau}\langle \mathbf{x} - x_0, \mathrm{grad}\rho\rangle = \frac{n\rho}{2\tau} - \frac{\rho}{4\tau^2}|(\mathbf{x} - x_0)^T|^2$, and so we have have

$$\frac{\rho}{2\tau}\langle \mathbf{x} - x_0, \mathbf{H}\rangle = -\frac{n\rho}{2\tau} + \frac{\rho}{4\tau^2}|(\mathbf{x} - x_0)^T|^2 + \mathrm{div}_M Y^T.$$

But $\mathrm{div}_M Y^T = -\mathrm{div}_M \mathrm{grad}(\rho) = -\triangle\rho$. Plugging the above into (3.2), we have

$$\begin{aligned}
\frac{\partial}{\partial t}(\rho d\mu_t) &= (-\triangle\rho - \rho|\mathbf{H} + \frac{\rho}{2\tau}(\mathbf{x} - x_0)|^2 + \frac{\rho}{4\tau^2}|(\mathbf{x} - x_0)^T|^2)d\mu_t \\
&= (-\triangle\rho - \rho|\mathbf{H} + \frac{\rho}{2\tau}(\mathbf{x} - x_0)^\perp|^2)d\mu_t.
\end{aligned}$$

Integrating, using the closedness of M and Stokes' theorem to eliminate the Laplacian we get the desired conclusion. In fact, the above theorem is true for more general submanifolds on which ρ decays fast enough for us to apply Stokes' theorem. □

For the rescaled flow, define $\tilde{\rho}(\mathbf{x}) = e^{-\frac{1}{2}|\mathbf{x}|^2}$. By the same argument as in the above theorem, the formula is tranformed to

$$\frac{\partial}{\partial s}\int_{\tilde{M}_s} \tilde{\rho}\, d\tilde{\mu}_s = -\int_{\tilde{M}_s} \tilde{\rho}|\tilde{\mathbf{H}} + \tilde{\mathbf{x}}^\perp|^2 d\tilde{\mu}_s.$$

At a type I singularity, $|A_t|^2 \leq \frac{C}{2(T-t)}$, so $|\mathbf{H}| \leq \frac{\sqrt{n}C}{(2(T-t))^{1/2}}$. Towards the singularity, the rescaled submanifolds remain bounded:

$$\begin{aligned}
|\tilde{F}(p,t)| &= (2(T-t))^{-1/2}|F(p,t)| \leq (2(T-t))^{-1/2}\int_t^T |\mathbf{H}(p,\tau)|d\tau \\
&\leq \sqrt{n}C(2(T-t))^{-1/2}\int_t^T \frac{1}{2(T-\tau)^{1/2}}d\tau = \text{constant}.
\end{aligned}$$

In fact, it can be proved in this case that the rescaled flow converges to some limiting submanifold. The monotonicity formula, taking the limit as $s \to \infty$, means that at this limiting submanifold we have $\tilde{\mathbf{H}} + \tilde{\mathbf{x}}^\perp = 0$

(this is because at the steady state limit, $\frac{\partial}{\partial s}\int_{\tilde{M}_s}\tilde{\rho}d\tilde{\mu}_s = 0$). Suppose now a submanifold satisfies $\mathbf{H}_0 + \mathbf{x}_0^{\perp} = 0$. The deformation

$$\mathbf{x}(t) = (2(T-t))^{1/2}\mathbf{x}_0$$

satisfies that $(\frac{\partial \mathbf{x}}{\partial t})^{\perp} = \frac{1}{(2(T-t))^{1/2}}\mathbf{H}_0 = \mathbf{H}_t$. This means that up to deformation in the tangential direction, this flow is self-similar (namely, a scaling of the original submanifold). Hence at any type I singularity, the mean curvature flow is asymptotically self similar.

It has been proved in [5] that if a hypersurface of dimension at least 2 is compact with nonnegative mean curvature, then if it also satisfies $\mathbf{H}+\mathbf{x}^{\perp} = 0$, it has to be a sphere.

4 Lagrangian Mean Curvature Flow

The following is a brief introduction to the relevant terminology in symplectic geometry and complex geometry. For a more detailed explanation of the concepts, see [13].

4.1 Definitions

Definition 4.1. Let V be a vector space. A skew-symmetric bilinear map $\alpha : V \times V \to \mathbf{R}$ is called non-degenerate if the induced map $\alpha^* : V \to V^*$ has trivial kernel. If V is equipped with such a structure then we say V is a symplectic vector space.

It can be proved by an analogue of the Gram-Schmidt process that every symplectic vector space with symplectic structure ω admits a basis $\{e_i, f_i\}$ such that $\omega = \sum_i e^i \wedge f^i$. Such basis is called a symplectic basis.
Therefore, not every vector space can have a symplectic structure, only the even dimensional ones can.

Definition 4.2. A manifold M is called symplectic if there is a closed 2-form ω on M such that it is a symplectic form on each of its tangent spaces.

The closedness of ω is to relate the algebraic structure of the symplectic form to the differential geometry of M.
A fundamental theorem in symplectic geometry is that locally there is always a coordinate chart such that the coordinate vector fields induce a symplectic

basis at every point in the domain. Hence essentially, there is no local geometry on a symplectic manifold, because any symplectic form on M can be induced by a coordinate chart. More often we are concerned with the global behaviour, and hence also the topology, of symplectic manifolds when there is only the symplectic structure.

Definition 4.3. Let (M, ω) be symplectic. A half dimensional submanifold N of M is said to be a Lagrangian submanifold of M if at every point of N, the form ω restricts to zero.

Definition 4.4. Let (M, g) be a Riemannian manifold with an almost complex structure J (a smoothly varying endomorphism of tangent spaces of M satisfying $J^2 = -1$). Then we say M is almost Hermitian if $g(JX, JY) = g(X, Y)$ for all X, Y.

It is clear that J is diagonalizable with eigenvalues $\pm i$. At $p \in M$, we denote the i-eigenspace of J (in the complexified tangent space) to be $T_p^{1,0}M$. A tangent vector in $T_p^{1,0}M$ is said to be of type $(1,0)$. Similarly, we define the $(-i)$-eigenspace of J at p to be $T_p^{0,1}M$. Its elements are said to be of type $(0,1)$.

If M is a complex manifold, we can choose J to be the canonical complex structure defined by

$$J\frac{\partial}{\partial x} = \frac{\partial}{\partial y}, \qquad J\frac{\partial}{\partial y} = -\frac{\partial}{\partial x}.$$

A complex almost Hermitian manifold is called a Hermitian manifold.

Now suppose M is Hermitian. Define $\omega(X,Y) = g(JX,Y)$. Suppose ω is closed. Then M is automatically symplectic, with symplectic form ω. In this case, we call M a Kähler manifold, and ω the Kähler form. Another more geometric condition for a Hermitian manifold M to be Kähler is that the almost complex structure is parallel: $\nabla J = 0$. This is also equivalent to the condition that parallel transport preserves the types of complexified tangent vectors.

Definition 4.5. A Kähler manifold is said to be Calabi-Yau if its Riemannian metric is Ricci-flat, namely, the Ricci curvature is zero.

The following theorem is useful for our purpose, but we will not state the proof. A proof can be found in, for example, [8].

Theorem 4.6. *If M^{2n} is Calabi-Yau, then there exists a global non-vanishing holomorphic $(n,0)$-form that is parallel.*

A form of type $(n,0)$ is an n-form such that whenever it takes any vector field of type $(0,1)$ as one of the n input arguments, it will vanish.

It is possible to study Lagrangian submanifolds in the setting of a symplectic manifold. But with additional structures, we can say much more.
For example, suppose M is Kähler-Einstein, namely $\rho = k\omega$ where k is a constant depending only on the metric of M, and ρ is the Ricci form on M, given by $\rho(X,Y) = Ric(JX,Y)$. Define a 1-form on a Lagrangian submanifold N by $\sigma_{\mathbf{H}} = \mathbf{H} \lrcorner \omega$, called the mean curvature 1-form. Using the Codazzi equation

(1.2), it is easy to compute that $d\sigma_H = \rho|_N$. Since N is Lagrangian, by Cartan's magic formula we have $\mathcal{L}_{\mathbf{H}}\omega = d\sigma_{\mathbf{H}} + \mathbf{H}\lrcorner(d\omega) = \rho|_N + 0 = k\omega|_N = 0$. This means that $\mathbf{H}$ is a symplectic vector field (i.e. its corresponding 1-parameter family of diffeomorphisms preserve ω).

4.2 Lagrangian submanifolds in Calabi-Yau Manifolds

Let M^{2n} be Calabi-Yau. Then we can find a global non-vanishing holomorphic parallel $(n, 0)$-form, which we will denote by Ω. Let N^n be a submanifold of M.

Since Ω is parallel, it has constant length. Moreover, it is clear that $\Omega|_N$ is a top form on N. Hence $*_N(\Omega|_N)$ is a function on N, and after normalizing by its length, $*_N(\Omega|_N) = e^{i\theta}$ for some θ. We call N special Lagrangian with phase $e^{i\theta}$ if θ is a constant.

Since we have normalized Ω, it has length 1 everywhere, namely, for E_i orthonormal, $\Omega(E_1, ..., E_n) \leq 1$ with 1 attainable. It is easy to see that Ω is closed because it is a holomorphic $(n, 0)$ form, or we can see it from the fact that it is parallel. That means $\alpha = \mathrm{Re}\,\Omega$ is a calibration (see [4] for discussion on calibrations). We could have chosen $\beta = \mathrm{Im}\,\Omega$, or more generally, an S^1 family of all such choices (given by the real part of Ω rotated by an angle) and get the same conclusion. For this α, we can see the following immediate conclusion.

Proposition 4.7. *In the above notation, N^n of M^{2n} is calibrated with respect* *to α if and only if $e^{i\theta} = 1$, or equivalently, $\theta = 0 \mod 2\pi$.*

To justify the term special "Lagrangian", we have

Proposition 4.8. *If N is special Lagrangian with a phase, then it is automatically Lagrangian.*

Proof. It is easy to see that being special Lagrangian with a phase is equivalent to being calibrated with respect to $\mathrm{Re}(e^{-i\theta}\Omega)$ for some constant θ, and in particular, at any point p, $|\Omega(\Pi)| = 1$, where $\Pi = T_p^{\mathbf{C}}N = q_1 \wedge q_2 \wedge ... \wedge q_n$, for an orthonormal frame $\{q_i\}$ that spans $T_p^{\mathbf{C}}N$. Using the formula $|\det_C A|^2 = \det_R A$, this means $|\Pi \wedge J\Pi| = |\Omega(\Pi)|^2 = 1$ hence by Hadamard's inequality this implies that Π is orthogonal to $J\Pi$. That is, $\omega(u, v) = g(u, Jv) = 0$ for all $u, v \in T_pN$. So N is Lagrangian with respect to ω. $\square$

We also have the following more explicit relationship between θ and the mean curvature vector, whose elegant proof is due to Richard Schoen and can be found in [21].

Theorem 4.9. *For a Lagrangian submanifold N in a Calabi-Yau manifold M, we have*

$$\mathbf{H} = J\nabla\theta$$

on N.

Proof. It is equivalent to proving $\nabla\theta = -J\mathbf{H}$ and hence that for every vector field X on N, we have $X\theta = \langle \nabla\theta, X \rangle = -\langle J\mathbf{H}, X \rangle$. We can choose a local basis e_i of TN induced by normal coordinates such that $\{e_i\}$ is orthonormal at p and since N is Lagrangian, $\{e_i, Je_i\}$ span TM around p. Let f^i be the dual basis of e_i and $g^j = -f^j \circ J$ be dual to Je_j. In this basis, since Ω is holomorphic, $\Omega = e^{i\theta}(e^1 + if^1) \wedge ... \wedge (e^n + if^n)$.

Since Ω is parallel, we have

$$\begin{aligned} e^{-i\theta}\nabla_X\Omega = 0 &= i\nabla_X\theta(f^1 + ig^1) \wedge .... \wedge (f^n + ig^n) \\ &\quad + \sum_k (f^1 + ig^1) \wedge ... \wedge \nabla_X(f^k + ig^k) \wedge ... \wedge (f^n + ig^n). \end{aligned}$$

By the fact that $(f^k + ig^k)(\frac{e_i - iJe_i}{2}) = \delta_{ik}$, we have

$$\begin{aligned} & i(\nabla_X\theta)(f^1 + ig^1) \wedge .... \wedge (f^n + ig^n) \\ &= -\sum_k (f^1 + ig^1) \wedge ... \wedge \nabla_X(f^k + ig^k) \wedge ... \wedge (f^n + ig^n) \\ &= \sum_k (f^1 + ig^1) \wedge ... \wedge (\sum_r b_r(f^r + ig^r)) \wedge ... \wedge (f^n + ig^n), \end{aligned}$$

where $b_r = (\nabla_X(f^k + ig^k))(\frac{e_r - iJe_r}{2})$. Since $(f^r + ig^r) \wedge (f^i + ig^i) = 0$ if $i = r$, we get

$$\begin{aligned} & i(\nabla_X\theta)(f^1 + ig^1) \wedge .... \wedge (f^n + ig^n) \\ &= -\sum_k \nabla_X(f^k + ig^k)(\frac{e_k - iJe_k}{2})(f^1 + ig^1) \wedge ... \wedge (f^n + ig^n) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_k ((f^k + ig^k)(\nabla_X(e_k - iJe_k)))(f^1 + ig^1) \wedge \dots \wedge (f^n + ig^n) \\
&= \frac{1}{2}\sum_k (f^k(\nabla_X(-iJe_k)) + ig^k(\nabla_X e_k))(f^1 + ig^1) \wedge \dots \wedge (f^n + ig^n)
\end{aligned}
$$

(we have chosen $\{e_k\}$ to be induced by normal coordinates, and so $\nabla_X e_k$ is normal and $\nabla_X Je_k$ is tangent, whereas f^k is tangent and g^k is normal). Now this sum is equal to

$$
\begin{aligned}
&i(\nabla_X\theta)(f^1 + ig^1) \wedge \dots . \wedge (f^n + ig^n) \\
&= \frac{1}{2}\sum_k (-if^k(J\nabla_X e_k) + ig^k(\nabla_X e_k))(f^1 + ig^1) \wedge \dots \wedge (f^n + ig^n)
\end{aligned}
$$

due to the fact that $\nabla J = 0$, so that $\nabla_{e_k}(JX) = J\nabla_{e_k}X$. On the other hand, $-\langle J\mathbf{H}, X\rangle = -\sum_i \langle J\nabla_{e_i}e_i, X\rangle = -\sum_i \langle e_i, J\nabla_{e_i}X\rangle$. Now by the formula

$$
\nabla_S T - \nabla_T S = [S, T]
$$

we have that $\nabla_{e_i}X - \nabla_X e_i = [e_i, X]$ which is tangent to N. So $J[e_i, X]$ is normal since N is Lagrangian. Hence

$$
\begin{aligned}
-\langle J\mathbf{H}, X\rangle &= \left\langle \sum_i (\nabla_{e_i}e_i)^\perp, JX \right\rangle = \sum_i \langle \nabla_{e_i}e_i, JX\rangle = -\sum_i \langle e_i, J\nabla_{e_i}X\rangle \\
&= -\sum_i \langle e_i, J\nabla_X e_i + J[e_i, X]\rangle = -\sum_i \langle e_i, J\nabla_X e_i\rangle .
\end{aligned}
$$

So it suffices to prove that

$$\frac{1}{2}\sum_k(-if^k(J\nabla_X e_k)+ig^k(\nabla_X e_k))=-i\sum_i\langle e_i,J\nabla_X e_i\rangle\,.$$

Now $g^k(\nabla_X e_k) = -f_k(J\nabla_X e_k)$ (recall that $g^k = -f^k \circ J$), hence the left hand side is $\sum_k(-if^k(J\nabla_X e_k)) = -i\sum_k \langle e_k, J\nabla_X e_k\rangle$, as desired. □

Corollary 4.10. *If L is a Lagrangian submanifold of a Calabi-Yau manifold M, then L is minimal if and only if L is special Lagrangian with some phase.*

Proof. By Theorem 4.9, $\mathbf{H} = J\nabla\theta$. So $\mathbf{H} = 0$ if and only if θ is constant, since J is invertible. □

Of course, we have seen that a special Lagrangian submanifold has to be area-minimizing hence minimal, because it is calibrated.
We remark that the concept of special Lagrangian submanifolds can be defined on an almost Calabi-Yau manifold, which is a $2n$-dimensional Kähler manifold with a global non-vanishing holomorphic $(n,0)$ form φ. In this case we can conformally deform the Kähler form so that φ has constant length, and this is all we need in the preceding discussion. For details, see [3].

4.3 Lagrangian condition preserved along the Mean Curvature Flow

This section will be based on Smoczyk's paper [19].
Let N_0 be a Lagrangian submanifold in a Kähler Einstein manifold $(M, J, \bar{g}, \omega)$

(i.e. there exists a constant k with $\overline{Ric} = k\bar{g}$, where $\overline{Ric}$ is the Ricci curvature tensor of M), so that ω is zero on N_0. To show that the Lagrangian condition is preserved along the Mean Curvature Flow, we shall prove that ω restricted to N_t is zero as long as the flow exists. To do this we will derive the evolution equation for $|\omega|^2$, and use the maximum principle.

Note that while the general strategy is similar to our techniques in tackling hypersurfaces, the situation is more complicated in that there is more than one normal vector. Hence we will need to derive new evolution equations for quantities like the second fundamental form.

Let $F : L^n \to M^{2n}$ be an immersion, not necessarily Lagrangian, and let $\bar{\nabla}$ and ∇ be the connections on M and L respectively . Define a new tensor (generalizing our second fundamental form in hypersurfaces) by $h(u, v, w) = \left\langle N(u), \bar{\nabla}_v w \right\rangle = -\left\langle \bar{\nabla}_v N(u), w \right\rangle$ where $u, v, w \in T_pL$, $N(u) = (Ju)^{\perp}$.

If L was Lagrangian, then $N(u) = J(u)$ because J maps T_pL to its orthogonal complement in M. We shall assume that, for now, N is an isomorphism along the flow. This assumption will not hurt our arguments to follow, by the following arguments:

First we start with a Lagrangian submanifold L and run the mean curvature flow, and at the beginning $N(u) = J(u)$ since L is Lagrangian, which means that N is an isomorphism at the start. So in any compact neighborhood of $p \in N$, at least for a short time N will remain an isomorphism, by continuity. Suppose now we have proved that L_t remains Lagrangian as long as N is an isomorphism, but along the mean curvature flow there is some time t_0 such that N fails to be an isomorphism on L_{t_0}. Without loss of generality t_0 is the first such time. Then by continuity ω will restrict to zero on L_{t_0} because L_t are Lagrangian for $t < t_0$ (because we have proved the theorem for the case that N is an isomorphism), and this means that L_{t_0} is Lagrangian. So $N = J$ at time t_0, which is an isomorphism.

Let $e_i = \frac{\partial F}{\partial x_i}$, $i = 1, 2, ..., n$ be a basis of coordinate vector fields in TL near some point $p \in L$. Denote $h_{kij} = h(e_k, e_i, e_j)$. One immediate consequence is that $h_{kij} = h_{kji}$. Using this notation, the mean curvature vector is clearly $\mathbf{H} = \eta^{kl} H_k N(e_l)$, where $\eta^{kl} = \langle N(e_k), N(e_l) \rangle$ and $H_k = g^{ij} h_{kij}$, the mean curvature with respect to $N(e_k)$. In this setting, the mean curvature 1-form $\sigma_{\mathbf{H}}$, defined by $\sigma_{\mathbf{H}}(X) = \omega(\mathbf{H}, X)$, can be written as $\sigma_{\mathbf{H}} = -H_i e^i$.

We have the following evolution equations:

Lemma 4.11. Under the mean curvature flow,

$$\frac{\partial}{\partial t}\omega = d\sigma_{\mathbf{H}}$$

Proof. In what follows, we will sum from 1 to $2n$ the indices α, β, γ, and 1 to n the indices i, j, k and so on. Assuming $\{\tilde{e}_\alpha\}$ is a basis of tangent vector fields to the ambient manifold M, we have

$$\begin{aligned}
\frac{\partial}{\partial t}\omega_{ij} &= \frac{\partial}{\partial t}(\omega(\frac{\partial F}{\partial x^i}, \frac{\partial F}{\partial x^j})) \\
&= \frac{\partial}{\partial t}(\omega(\frac{\partial F^\alpha}{\partial x^i}\tilde{e}_\alpha, \frac{\partial F^\beta}{\partial x^j}\tilde{e}_\beta)) = \frac{\partial}{\partial t}(\omega_{\alpha\beta}\frac{\partial F^\alpha}{\partial x^i}\frac{\partial F^\beta}{\partial x^j}) \\
&= (\frac{\partial}{\partial t}\omega_{\alpha\beta})\frac{\partial F^\alpha}{\partial x^i}\frac{\partial F^\beta}{\partial x^j} + \omega_{\alpha\beta}\frac{\partial}{\partial t}(\frac{\partial F^\alpha}{\partial x^i})\frac{\partial F^\beta}{\partial x^j} + \omega_{\alpha\beta}\frac{\partial F^\alpha}{\partial x^i}\frac{\partial}{\partial t}(\frac{\partial F^\beta}{\partial x^j})
\end{aligned}$$

$$= \quad \omega_{\alpha\beta,\gamma}\frac{\partial F^\gamma}{\partial t}\frac{\partial F^\alpha}{\partial x^i}\frac{\partial F^\beta}{\partial x^j} + \omega_{\alpha\beta}\frac{\partial}{\partial x^i}(\frac{\partial F^\alpha}{\partial t})\frac{\partial F^\beta}{\partial x^j} + \omega_{\alpha\beta}\frac{\partial F^\alpha}{\partial x^i}\frac{\partial}{\partial x^j}(\frac{\partial F^\beta}{\partial t}).$$

Since $d\omega = 0$, we have $\omega_{\alpha\beta,\gamma} + \omega_{\beta\gamma,\alpha} + \omega_{\gamma\alpha,\beta} = 0$. Hence

$$\begin{aligned}
\frac{\partial}{\partial t}\omega_{ij} &= -\omega_{\beta\gamma,\alpha}\frac{\partial F^\gamma}{\partial t}\frac{\partial F^\alpha}{\partial x^i}\frac{\partial F^\beta}{\partial x^j} - \omega_{\gamma\alpha,\beta}\frac{\partial F^\gamma}{\partial t}\frac{\partial F^\alpha}{\partial x^i}\frac{\partial F^\beta}{\partial x^j} \\
&\quad + \omega_{\alpha\beta}\frac{\partial}{\partial x^i}(\frac{\partial F^\alpha}{\partial t})\frac{\partial F^\beta}{\partial x^j} + \omega_{\alpha\beta}\frac{\partial F^\alpha}{\partial x^i}\frac{\partial}{\partial x^j}(\frac{\partial F^\beta}{\partial t}) \\
&= \frac{\partial}{\partial x^i}(\omega_{\gamma\beta}\frac{\partial F^\gamma}{dt}\frac{\partial F^\beta}{\partial x^j}) - \omega_{\gamma\beta}\frac{\partial F^\gamma}{\partial t}\frac{\partial^2 F^\beta}{\partial x^i \partial x^j} \\
&\quad - \frac{\partial}{\partial x^j}(\omega_{\gamma\alpha}\frac{\partial F^\gamma}{dt}\frac{\partial F^\alpha}{\partial x^i}) + \omega_{\gamma\alpha}\frac{\partial F^\gamma}{\partial t}\frac{\partial^2 F^\alpha}{\partial x^j \partial x^i} \\
&= \frac{\partial}{\partial x^i}(\omega_{\gamma\beta}\frac{\partial F^\gamma}{dt}\frac{\partial F^\beta}{\partial x^j}) - \frac{\partial}{\partial x^j}(\omega_{\gamma\alpha}\frac{\partial F^\gamma}{dt}\frac{\partial F^\alpha}{\partial x^i}).
\end{aligned}$$

Now $\omega_{\gamma\beta}\frac{\partial F^\gamma}{dt}\frac{\partial F^\beta}{\partial x^j} = \omega(\frac{\partial F}{\partial t}, \frac{\partial F}{\partial x^j}) = \omega(\mathbf{H}, e_j) = \sigma_{\mathbf{H}}(e_j)$. Hence, $\frac{\partial}{\partial t}\omega_{ij} = e_i(\sigma_{\mathbf{H}}(e_j)) - e_j(\sigma_{\mathbf{H}}(e_i)) = (d\sigma_{\mathbf{H}})(e_i, e_j)$, because e_i, e_j have zero bracket (since they are coordinate vector fields). On the other hand,

$$\begin{aligned}
(\frac{\partial}{\partial t}\omega)(e_i, e_j) &= \frac{\partial}{\partial t}(\omega(e_i, e_j)) - \omega(\frac{\partial}{\partial t}e_i, e_j) - \omega(e_i, \frac{\partial}{\partial t}e_j) \\
&= \frac{\partial}{\partial t}(\omega(e_i, e_j)) - \omega(\nabla_i \mathbf{H}, e_j) - \omega(e_i, \nabla_j \mathbf{H}) \\
&= \frac{\partial}{\partial t}(\omega(e_i, e_j)) - g(J\nabla_i \mathbf{H}, e_j) - g(Je_i, \nabla_j \mathbf{H}).
\end{aligned}$$

Now using the fact that J is parallel, we can simplify this to

$$\begin{aligned}
& \frac{\partial}{\partial t}(\omega(e_i, e_j)) + g(J\mathbf{H}, \nabla_i e_j) - g(\nabla_j e_i, J\mathbf{H}) \\
= \ & \frac{\partial}{\partial t}(\omega(e_i, e_j)) + g(J\mathbf{H}, \nabla_i e_j - \nabla_j e_i) \\
= \ & \frac{\partial}{\partial t}(\omega(e_i, e_j)) + g(J\mathbf{H}, [e_j, e_i]) = \frac{\partial}{\partial t}(\omega(e_i, e_j))
\end{aligned}$$

since $\{e_i\}$ is a coordinates basis ($[e_i, e_j] = 0$). Therefore,

$$(\frac{\partial}{\partial t}\omega)(e_i, e_j) = \frac{\partial}{\partial t}(\omega(e_i, e_j)) = (d\sigma_{\mathbf{H}})(e_i, e_j),$$

which proves that $\frac{\partial}{\partial t}\omega = d\sigma_{\mathbf{H}}$. $\square$

Next we have a technical lemma on switching indices:

Lemma 4.12. (index switching)

$$h_{kij} = h_{ikj} - \nabla_j \omega_{ik} \tag{4.1}$$

$$\begin{aligned}
\nabla_l h_{ikj} - \nabla_k h_{ilj} = \bar{R}_{\hat{i}jkl} - \eta^{mn}{\omega_n}^s(h_{mlj}h_{ski} - h_{mkj}h_{sli}) \\
-\eta^{mn}{\omega_i}^s(h_{mkj}h_{nls} - h_{mlj}h_{nks})
\end{aligned} \tag{4.2}$$

$$\begin{aligned}
\nabla_l h_{kij} - \nabla_k h_{lij} = \bar{R}_{\hat{i}jkl} - \nabla_j \nabla_i \omega_{lk} + {\omega_i}^s \bar{R}_{sjkl} + {\omega_k}^s R_{silj} \\
+{\omega_l}^s R_{sijk} - \eta^{mn}{\omega_n}^s(h_{mlj}h_{ski} - h_{mkj}h_{sli})
\end{aligned} \tag{4.3}$$

where an index with a hat means the corresponding component is the image by N, for example $R_{ijk\hat{l}} = \langle R(e_i, e_j)e_k, N(e_l)\rangle$.

The proof mainly uses the Gauss equation (1.1) which is the reason why there are curvature terms. The proof is in the same flavour as all the tensor computations we did before, hence we will omit it, so as to focus on the geometric content. The detailed computations can be found in [19]. The following will be the main part in the proof of our main theorem, that the Lagrangian condition is preserved by the mean curvature flow.

Proposition 4.13. *For each compact interval $[0, T]$, there exists a constant c such that*

$$\frac{\partial}{\partial t}|\omega|^2 \leq \triangle|\omega|^2 + c|\omega|^2.$$

Here ω is restricted to L, and $|\omega|$ is the norm of $\omega|_L$.

Proof. We begin by computing:

$$\frac{\partial}{\partial t}|\omega|^2 = \frac{\partial}{\partial t}(g^{ij}g^{kl}\omega_{ik}\omega_{jl}) = 2\frac{\partial g^{ij}}{\partial t}g^{kl}\omega_{ik}\omega_{jl} + 2g^{ij}g^{kl}\frac{\partial \omega_{ik}}{\partial t}\omega_{jl}.$$

Using the formula $\frac{\partial g^{ij}}{\partial t} = -g^{iu}\frac{\partial g_{uv}}{\partial t}g^{vj}$ and our evolution equations, we get

$$\frac{\partial}{\partial t}|\omega|^2 = -2g^{iu}(-2\eta^{mn}H_m h_{nuv})g^{vj}g^{kl}\omega_{ik}\omega_{jl} + 2g^{ij}g^{kl}(-\nabla_i H_k + \nabla_k H_i)\omega_{jl}$$

where the term $-\nabla_i H_k + \nabla_k H_i$ is from the fact that $\sigma_{\mathbf{H}} = -H_i e^i$. Using our index switching lemma 4.12, since $H_k = g^{mn} h_{kmn}$ and everything is tensorial,

$$\begin{aligned} -\nabla_i H_k + \nabla_k H_i &= g^{mn}(\bar{R}_{\hat{m}nik} - \nabla_n \nabla_m \omega_{ki} + \omega_m{}^s \bar{R}_{snik} + \omega_i{}^s R_{smkn} \\ &\quad +\omega_k{}^s R_{smni} - \eta^{uv}\omega_v{}^s(h_{ukn}h_{sim} - h_{uin}h_{skm})). \end{aligned}$$

Plugging it back, we have

$$\begin{aligned} \frac{\partial}{\partial t}|\omega|^2 &= -2g^{iu}(-2\eta^{mn}H_m h_{nuv})g^{vj}g^{kl}\omega_{ik}\omega_{jl} \\ &\quad +2\omega_{jl}g^{ij}g^{kl}g^{mn}(\bar{R}_{\hat{m}nik} - \nabla_n\nabla_m\omega_{ki} + \omega_m{}^s\bar{R}_{snik} + \omega_i{}^s R_{smkn} \\ &\quad +\omega_k{}^s R_{smni} - \eta^{uv}\omega_v{}^s(h_{ukn}h_{sim} - h_{uin}h_{skm})). \end{aligned}$$

To simplify this, note that $\triangle|\omega|^2 = g^{nm}\nabla_n\nabla_m(g^{ij}g^{kl}\omega_{ik}\omega_{jl}) = g^{nm}g^{ij}g^{kl}\omega_{jl}\nabla_n\nabla_m(\omega_{ik}) + 2|\nabla\omega|^2 = -g^{nm}g^{ij}g^{kl}\omega_{jl}\nabla_n\nabla_m(\omega_{ki}) + 2|\nabla\omega|^2$. Therefore, we have

$$\begin{aligned} \frac{\partial}{\partial t}|\omega|^2 &= 4\eta^{mn}H_m h_{nuv}\omega^{ul}\omega^v{}_l \\ &\quad +2\omega^{ik}\bar{R}_{\hat{p}}{}^p{}_{ik} + \triangle|\omega|^2 - 2|\nabla\omega|^2 + 2\omega^{ik}\omega_m{}^s\bar{R}_s{}^m{}_{ik} + 2\omega^{ik}\omega_i{}^s R_{smk}{}^m \\ &\quad +2\omega^{ik}\omega_k{}^s R_{sp}{}^p{}_i - 2\omega^{ik}\eta^{uv}\omega_v{}^s(h_{uk}{}^m h_{sim} - h_{ui}{}^m h_{skm}) \\ &\leq \triangle|\omega|^2 + 2\omega^{ik}\bar{R}_{\hat{p}}{}^p{}_{ik} + 4\eta^{mn}H_m h_{nuv}\omega^{ul}\omega^v{}_l \\ &\quad +2\omega^{ik}\omega_m{}^s\bar{R}_s{}^m{}_{ik} + 2\omega^{ik}\omega_i{}^s R_{smk}{}^m \\ &\quad +2\omega^{ik}\omega_k{}^s R_{sp}{}^p{}_i - 2\omega^{ik}\eta^{uv}\omega_v{}^s(h_{uk}{}^m h_{sim} - h_{ui}{}^m h_{skm}). \end{aligned}$$

Now we try to bound the absolute value of the terms other than $\triangle|\omega|^2$ by $c|\omega|^2$, for some constant c (which may depend on T in $[0,T]$). Observe that all of these terms, except for $2\omega^{ik}\bar{R}_{\hat{p}}{}^{p}{}_{ik}$, depend quadratically on ω, and they are essentially of the form $2\omega^{sl}\omega^{m}{}_{l}a_{sm}$ for some tensor a_{sm} which depends on η, h, and curvature. We bound them as follows (assume that we are working in normal coordinates which would not affect the bound because both sides are independent of coordinates). By repeated applications of the Cauchy-Schwarz inequality:

$$\begin{aligned} 2\omega^{sl}\omega^{m}{}_{l}a_{sm} &= 2\sum_{s,l,m}\omega_{sl}\omega_{ml}a_{sm} \leq |\omega|^2 + \sum_{s,l}(\sum_{m}\omega_{ml}a_{sm})^2 \\ &\leq |\omega|^2 + n\sum_{s,l,m}(\omega_{ml}a_{sm})^2 \leq (1+n|a|^2)|\omega|^2. \end{aligned}$$

To bound the remaining term $2\omega^{ik}\bar{R}_{\hat{p}}{}^{p}{}_{ik}$, assume that we have chosen a normal coordinates system for the submanifold L such that e_i diagonalizes η_{ij}, and a corresponding coordinate system for M given by $e_i, N(e_i)$. Then clearly $\langle e_i, e_j\rangle = \delta_{ij}$, $\langle e_j, N(e_j)\rangle = 0$ and $\langle N(e_i), N(e_j)\rangle = 0$ for $i \neq j$ because e_i was chosen to diagonalize η. Finally, $\langle N(e_i), N(e_i)\rangle = \langle Je_i - \omega_{il}e_l, Je_i - \omega_{ik}e_k\rangle$ (recall that $N(u) = (J(u))^{\perp}$). Hence $\langle N(e_i), N(e_i)\rangle = 1 - \sum_l \omega_{il}^2 - \sum_k \omega_{ik}^2 + \sum_l \omega_{il}^2 = 1 - a_i$ where

$$a_i = \sum_{l=1}^{n} \omega_{il}^2. \tag{4.4}$$

We have assumed that N is an isomorphism throughout, hence $1 - a_i \neq 0$, and it is bounded by some constant depending on T.

Since the manifold is Kähler-Einstein, there is a constant k such that $k\bar{g}(X,Y) = \overline{Ric}(X,Y)$, where $\overline{Ric}$ is the Ricci tensor. Expressing this equation in the aforementioned basis, and by the fact that

$$\bar{R}(X', Y', Z', W') = \bar{R}(JX', JY', Z', W') = \bar{R}(X', Y', JZ', JW')$$

for vector fields X', Y', Z', W' on M (see for example chapter 6 in [14]), we have

$$\begin{aligned}
k\bar{g}(X,Y) &= \overline{Ric}(X,Y) = \sum_i \bar{R}(X, e_i, e_i, Y) + \sum_i \frac{1}{1-a_i}\bar{R}(X, N(e_i), N(e_i), Y) \\
&= \sum_i \bar{R}(X, e_i, Je_i, JY) + \sum_i \frac{1}{1-a_i}\bar{R}(X, N(e_i), JN(e_i), JY) \\
&= \sum_i \bar{R}(X, e_i, Je_i, JY) + \sum_i \bar{R}(X, N(e_i), JN(e_i), JY) \\
&\quad + \frac{a_i}{1-a_i}\bar{R}(X, N(e_i), JN(e_i), JY).
\end{aligned}$$

Now we use the equation $Je_i = N(e_i) + \omega_{il} e_l$ and get

$$\begin{aligned}
&k\bar{g}(X,Y) \\
= &\sum_i \bar{R}(X, e_i, Je_i, JY) + \sum_i \bar{R}(X, N(e_i), JN(e_i), JY) \\
&+ \frac{a_i}{1-a_i}\bar{R}(X, N(e_i), JN(e_i), JY) \\
= &\sum_{i,l} \bar{R}(X, e_i, N(e_i) + \omega_{il} e_l, JY) + \sum_{i,l} \bar{R}(X, N(e_i), -e_i + \omega_{il} Je_l, JY) \\
&+ \sum_i \frac{a_i}{1-a_i}\bar{R}(X, N(e_i), JN(e_i), JY) \\
= &\sum_{i,l} (\omega_{il}\bar{R}(X, e_i, e_l, JY) + \omega_{il}\bar{R}(X, N(e_i), Je_l, JY))
\end{aligned}$$

$$
\begin{aligned}
&+\sum_i \frac{a_i}{1-a_i}\bar{R}(X, N(e_i), JN(e_i), JY) \\
&+\sum_i (\bar{R}(X, e_i, N(e_i), JY) - \bar{R}(X, N(e_i), e_i, JY)).
\end{aligned}
$$

Now we use the Bianchi identity to get

$$
\bar{R}(X, e_i, N(e_i), JY) - \bar{R}(X, N(e_i), e_i, JY) = -\bar{R}(e_i, N(e_i), X, JY).
$$

Hence,

$$
\begin{aligned}
\sum_i \bar{R}(e_i, N(e_i), X, JY) &= \sum_{i,l} (\omega_{il}\bar{R}(X, e_i, e_l, JY) + \omega_{il}\bar{R}(X, N(e_i), Je_l, JY)) \\
&\quad + \sum_i \frac{a_i}{1-a_i}\bar{R}(X, N(e_i), JN(e_i), JY) - k\bar{g}(X, Y).
\end{aligned}
$$

Choosing $X = e_m$ and $Y = -Je_k$, recall that the index with a hat is the image of that index element by N:

$$
\begin{aligned}
\bar{R}_{i\hat{i}mk} &= k\omega_{km} + \sum_{i,l} (\omega_{il}\bar{R}_{milk} + \omega_{il}\bar{R}(e_m, N(e_i), Je_l, e_k)) \\
&\quad + \sum_i \frac{a_i}{1-a_i}\bar{R}(e_m, N(e_i), JN(e_i), e_k).
\end{aligned}
$$

Multiplying by $2\omega_{km}$, and summing over k, m,

$$
\begin{aligned}
2\omega_{km}\bar{R}_{i\hat{i}mk} &= 2k\omega_{km}^2 + \sum_{i,l,k,m} (2\ \omega_{km}\omega_{il}\bar{R}_{milk} + 2\omega_{km}\omega_{il}\bar{R}(e_m, N(e_i), Je_l, e_k)) \\
&\quad + \sum_{i,k,m} \frac{2a_i}{1-a_i}\omega_{km}\bar{R}(e_m, N(e_i), JN(e_i), e_k).
\end{aligned}
$$

Using the previous argument we can bound every term but the last on the right hand side by some $c_3|\omega|^2$ because they depend quadratically on ω. To bound the last term on the right hand side, we proceed as follows:

$$\begin{aligned}
& 2 \sum_{i,k,m} \frac{a_i}{1-a_i}\omega_{km}\bar{R}(e_m, N(e_i), JN(e_i), e_k) \\
\leq & \sum_i (\frac{a_i}{1-a_i})^2 + \sum_i (\sum_{k,m} \omega_{km}\bar{R}(e_m, N(e_i), JN(e_i), e_k))^2.
\end{aligned}$$

The second term depends quadratically on ω, hence can be bounded by a constant multiple of $|\omega|^2$. Now, recall that $|\frac{1}{1-a_i}|$ is bounded by a constant $C_1 > 0$ depending on T because N is an isomorphism, hence

$$|\frac{a_i}{1-a_i}| = |1 - \frac{1}{1-a_i}| \leq 1 + C_1.$$

On the other hand, $\sum_i a_i = |\omega|^2$ using (4.4), so we have

$$\sum_i (\frac{a_i}{1-a_i})^2 \leq (1+C_1)\sum_i \frac{a_i}{1-a_i} \leq (1+C_1)C_1 \sum_i a_i = (1+C_1)C_1|\omega|^2.$$

This provides a bound for the term $\sum_{i,k,m} \frac{2a_i}{1-a_i}\omega_{km}\bar{R}(e_m, N(e_i), JN(e_i), e_k)$, and since we are working in normal coordinates, we have bounded $2\omega^{ik}\bar{R}_{\hat{p}}{}^{p}{}_{ik}$ by a constant multiple of $|\omega|^2$, which was what we needed to complete the argument. $\square$

Theorem 4.14. *If L_0 is Lagrangian in a Kähler-Einstein manifold M, then the Lagrangian condition is preserved under the mean curvature flow.*

Proof. By the previous proposition, for each compact interval we can find a constant c depending on T such that

$$\frac{\partial}{\partial t}|\omega|^2 \leq \triangle|\omega|^2 + c|\omega|^2.$$

Using Theorem 3.5, we compare $|\omega|^2$ with the zero function and find that $|\omega|^2 \leq 0$ on L_t. Since $|\omega|^2 \geq 0$ also we get the result. $\square$

If M is in addition Calabi-Yau, then we can talk about special Lagrangian submanifolds. Constructing new special Lagrangian submanifolds is an important problem and a lot of research has been done on this. In particular, one approach is to study the behaviour of Lagrangian submanifolds under the mean curvature flow and investigate when it will converge to a special Lagrangian (after rescaling near a singularity). In [21], Thomas and Yau made an important conjecture which says that if, in a Calabi-Yau manifold, a compact embedded Lagrangian submanifold has zero Maslov class (namely, the integral cohomology class of the mean curvature 1-form $\sigma_{\mathbf{H}}$ is zero), then the mean curvature flow has long time existence and converges to a special Lagrangian submanifold in the same Hamiltonion deformation class (it can be proved that in this case, the mean curvature flow preserves the Hamiltonian deformation class).

Schoen and Wolfson in [17] constructed examples that show this is not true if we do not assume the Lagrangian submanifold has zero Maslov class. It is not known whether the examples they constructed have zero Maslov class, and if they did, they will be counterexamples. Incidentally, non-compact examples that develop finite time singularities have also been found (see [15]).

5 Conormal Bundles

In this section, we shall study the mean curvature deformation of a particular class of Lagrangian submanifolds in $\mathbf{C}^n$.

5.1 Bundle Construction of Lagrangian Submanifolds

Let M^p be a submanifold of $\mathbf{R}^n$. Consider the bundle $T^*\mathbf{R}^n$, which can be locally described by

$$(x_1, x_2, ..., x_n, t_1, t_2, ..., t_n) \mapsto (x_1, x_2, ..., x_n, t_1 dx^1 + t_2 dx^2 + ... + t_n dx^n).$$

Clearly this can be identified with $\mathbf{C}^n$, with the complex structure given by $J(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial t_i} = dx^i$. We can define the conormal bundle N^*M of M to be

$$N^*M = \{(x, v) \in T^*\mathbf{R}^n : v(E) = 0 \text{ for any } E \in T_pM\}.$$

Let $x \in M$ and $e_1, ..., e_p$ be a local frame of tangent vectors which are orthonormal at x. Let $e^1, ..., e^p$ be their dual coframe. Similarly, let $\nu_1, ..., \nu_{n-p}$ be a local frame of normal vectors orthonormal at x and let $\nu^1, ..., \nu^{n-p}$ be their dual coframe. They can be chosen to be induced by local coordinates. Notice that $T^*\mathbf{R}^n$ is symplectic, with symplectic structure explained below. Now for a 1-form μ on M, we define the submanifold $X_\mu = N^*M + \mu$ to be locally $\{(x, v + \mu_x) : (x, v) \in N^*M\}$. Namely, it is the conormal bundle translated by μ_x at every x. In what follows, we shall adopt the notations

$\bar{e}_i = (e_i, 0)$ and $\check{e}^i = (0, e^i)$, and $\bar{e}^i$, $\check{e}_i$ are their duals. The same convention of notations apply similarly to ν's.

Proposition 5.1. *In the notation above, X_μ is Lagrangian in $T^*\mathbf{R}^n$ if and only if μ is closed.*

Proof. Let α be the tautological 1-form on $T^*\mathbf{R}^n$, defined by $\alpha|_{(x,\beta)} = \pi^*\beta$, where π is the projection onto the base manifold $\mathbf{R}^n$. The symplectic form ω is given by $d\alpha$, and in a local coordinate system at x described above, it has the representation $\omega = \sum_k \bar{e}^k \wedge \check{e}_k + \sum_l \bar{\nu}^l \wedge \check{\nu}_l$. At a point $(x, \xi + \mu_x)$ the tautological 1-form α is then given by $\pi^*(\xi + \mu) = \pi^*\mu$ when restricted to TM since ξ is in the conormal bundle. Hence $d\alpha = 0$ if and only if $d\pi^*\mu = 0$ if and only if $\pi^* d\mu = 0$ if and only if $d\mu = 0$ on M. $\square$

In fact, it was proved in [1] that in the case of a surface, i.e. $p = 2$, X_μ is special Lagrangian with a phase that depends only on the codimension if and only if M is a minimal surface, and μ is harmonic (see also [9]). Note that $\mathbf{C}^n$ is automatically Calabi-Yau.

5.2 The Mean Curvature

With the natural Euclidean metric on $\mathbf{C}^n$, we can compute the mean curvature vector of X_μ.

Without loss of generality, by parallel transport we can assume that the coordinates we have chosen at the beginning satisfy

$$(\bar{\nabla}_{e_i} e_j)|_x^T = 0 \quad \text{and} \quad (\bar{\nabla}_{e_i} \nu_j)|_x^N = 0$$

at some fixed point x (here $\bar{\nabla}$ is the Levi-Civita connection on $\mathbf{C}^n$). Denote $A_{ij}^k = A_{ij}^{\nu_k} = A_{ji}^k$. Now $\bar{\nabla}_{e_i} e_j$ has no tangential component at x, by our assumption. Hence $\bar{\nabla}_{e_i} e_j = \sum_{k=1}^{n-p} \left\langle \bar{\nabla}_{e_i} e_j, \nu_k \right\rangle \nu_k = \sum_{k=1}^{q} A_{ij}^k \nu_k$, where we denote $q = n - p$. Similarly, $\bar{\nabla}_{e_i} \nu_j = \sum_{k=1}^{p} \left\langle \bar{\nabla}_{e_i} \nu_j, e_k \right\rangle e_k = -\sum_{k=1}^{p} A_{ik}^j e_k$. From these two formulas it follows immediately that $\bar{\nabla}_{e_i} e^j = \sum_{k=1}^{q} A_{ij}^k \nu^k$ and $\bar{\nabla}_{e_i} \nu^j = -\sum_{k=1}^{p} A_{ik}^j e^k$.

Let the embedding of M into $\mathbf{R}^n$ be F. In a local coordinate system, let

$$\Phi : (x^1, x^2, ..., x^p, t_1, t_2, ...t_q) \to (F(x^1, ..., x^p), t_1 \nu^1 + ... + t_q \nu^q + \mu)$$

be a local coordinate system of X_μ. The tangent space to X_μ at x is spanned by the vectors

$$E_i = \Phi_*(\frac{\partial}{\partial x^i}) = (e_i, \nabla_{e_i}(t_1 \nu^1 + ... + t_q \nu^q + \mu))$$

$$F_j = \Phi_*(\frac{\partial}{\partial t_j}) = (0, \nu^j),$$

where $i = 1, 2, ..., p$, and $j = 1, 2, ..., q$. Since we know that, at this particular point x, $\bar{\nabla}_{e_i}\nu^j = -\sum_{k=1}^p A_{ik}^j e^k$, at x we have

$$E_i = \bar{e}_i - \sum_{k=1}^p A_{ik}^\nu \check{e}^k + \nabla_{e_i}\check{\mu},$$

where we denote $\nu = t_i \nu^i$.

The above is just coordinate vectors at the point x. Since we will be taking second derivatives, we have to know also the coordinate vectors around x. In fact, the above argument goes through, except that $\bar{\nabla}_{e_i}\nu^j$ may have conormal components (since we assumed a normal coordinate system at x, this has no conormal components at x). Therefore, we write $\bar{\nabla}_{e_i}\nu^j = -\sum_{k=1}^p A_{ik}^j e^k + B_j \check{\nu}^j$, where $B_j(x) = 0$. The actual expression for B_j is not important for our purpose, as will be obvious in what follows (it will be automatically cancelled out). The tangent space to X_μ is then spanned by the vectors

$$E_i = (e_i, \bar{\nabla}_{e_i}\nu + \bar{\nabla}_{e_i}\mu + \sum_j B_j \nu^j) = \bar{e}_i - \sum_{k=1}^p A_{ik}^\nu \check{e}^k + \sum_j B_j \check{\nu}^j + \bar{\nabla}_{e_i}\check{\mu},$$

where $B_j(x) = 0$ (here $i = 1, ..., p$), and

$$F_j = (0, \nu^j) = \check{\nu}^j,$$

where $j = 1, ..., q$.

The metric at x is then given by

$$h_{ij}(x) = \langle E_i, E_j \rangle = \delta_{ij} + \sum_l A^{\nu}_{il} A^{\nu}_{jl} - \sum_l A^{\nu}_{il} \bar{\nabla}_j \mu_l - \sum_l A^{\nu}_{jl} \bar{\nabla}_i \mu_l,$$

$$\langle E_i, F_j \rangle = 0 \text{ and } \langle F_i, F_j \rangle = \delta_{ij}.$$

The Levi-Civita connection $\bar{\nabla}$ on $\mathbf{C}^n$ is just the flat one. Hence it is clear that $\bar{\nabla}_{F_i} F_j = 0$, and $\bar{\nabla}_{F_i} E_j = 0$. Since $\langle E_i, F_j \rangle = 0$, to compute the mean curvature vector we need only consider $\bar{\nabla}_{E_i} E_j$.

A frame of normal vector fields are given by

$$\gamma_l = \sum_i (A^{\nu}_{il} - \bar{\nabla}_i \mu_l) \bar{e}_i + \check{e}^l, \qquad l = 1, ..., p,$$

$$\gamma_l = \bar{\nu}_j, \qquad l = p+1, ..., n.$$

Denote $u_{ij} = \langle \gamma_i, \gamma_j \rangle$. The mean curvature vector at x is then found to be

$$\begin{aligned} \mathbf{H} &= u^{\alpha\beta} \left\langle h^{ij} \bar{\nabla}_{E_i} E_j, \gamma_\alpha \right\rangle \gamma_\beta \\ &= u^{\alpha\beta} \left\langle h^{ij} \bar{\nabla}_{\bar{e}_i} \bar{e}_j - h^{ij} \sum_{k=1}^{p} \bar{\nabla}_{\bar{e}_i} (A^{\nu}_{jk} \check{e}^k) + h^{ij} \sum_k \bar{\nabla}_{\bar{e}_i} (B_k \nu^k) + h^{ij} \bar{\nabla}_{\bar{e}_i} \bar{\nabla}_{e_j} \check{\mu}, \gamma_\alpha \right\rangle \gamma_\beta. \end{aligned}$$

Using our assumption that $B_k(x) = 0$, and that $\bar{\nabla}_{\bar{e}_i}\check{e}^k$ has no tangential components,

$$\mathbf{H} \;=\; u^{\alpha\beta}\left\langle h^{ij}\bar{\nabla}_{\bar{e}_i}\bar{e}_j - h^{ij}\sum_{k=1}^{p}\bar{\nabla}_{\bar{e}_i}(A^{\nu}_{jk})\check{e}^k + h^{ij}\bar{\nabla}_{\bar{e}_i}\bar{\nabla}_{e_j}\check{\mu}, \gamma_\alpha\right\rangle\gamma_\beta.$$

From here we observe that when $\nu = 0$ and $\mu = 0$, $h_{ij} = u_{ij} = \delta_{ij}$. So in the case of a conormal bundle without twisting, the mean curvature vector at the zero section is the mean curvature vector of the original submanifold M.

Suppose now we have chosen the coordinate vectors such that they diagonalize A^ν at x, with $A^\nu_{ii} = \lambda_i$. In this case,

$$h_{ij}(x) = \delta_{ij}(1+\lambda_i^2) - \lambda_i\bar{\nabla}_i\mu_j - \lambda_j\bar{\nabla}_j\mu_i$$

$$u_{ij}(x) = \delta_{ij}(1+\lambda_i^2) + \sum_l \bar{\nabla}_l\mu_i\bar{\nabla}_l\mu_j - \lambda_j\bar{\nabla}_j\mu_i - \lambda_i\bar{\nabla}_i\mu_j.$$

Further away from the zero section, say changing $\nu \to \lambda\nu$, the product of these metric terms scale at least quadratically. Hence the mean curvature vector is zero asymptotically.

Now run the mean curvature flow on conormal bundles. The key result we want to prove is

Theorem 5.2. *Suppose p is even, and the Gauss-Kronecker curvatures satisfy* $\det A^\nu < 0$ *if* $p = 2 \mod 4$ *and* $\det A^\nu > 0$ *if* $p = 0 \mod 4$*. Then there* *exists a cut function* $\alpha : M \to \mathbf{R}^+$ *such that the submanifold of* X_μ *locally given by* $S_\alpha = \{(x, \nu + \mu) \in X_\mu : |\nu| > \alpha(x)\}$ *does not develop any type I singularity under the mean curvature flow.*

As a very special case, the result holds for surfaces with negative curvatures.

Since the mean curvature vector goes to zero asymptotically, it is natural to think of X_μ as being "almost" minimal, or in this context, "almost" calibrated in some sense. This is a major observation that motivates the theorem. To prove the theorem, we need the concept of almost calibrated submanifolds, which we discuss next.

5.3 Almost Calibrated Submanifolds and Type I Singularities

To see things more clearly, we will go back to the more general case of a Calabi-Yau ambient manifold. Now we restrict our setting to a Lagrangian submanifold L^n in a Calabi Yau manifold $(M^{2n}, J, \bar{g}, \omega)$ which has unit parallel holomorphic $(n, 0)$ form Ω. We have seen that on L, $*_L\Omega = e^{i\theta}$ for some θ which may be a multi-valued function on L. Then L is calibrated with respect to $\operatorname{Re}\Omega$ if $\operatorname{Re}\Omega = 1$ on L. More generally, we have the following

Definition 5.3. A Lagrangian submanifold L is said to be almost calibrated by Ω if there exists a constant $\varepsilon > 0$ such that $\operatorname{Re} *_L\Omega \geq \varepsilon$ on L.

First we derive the evolution equation of $\alpha = *_L\Omega$.

Proposition 5.4.

$$\frac{\partial}{\partial t}\alpha = \triangle\alpha + |\mathbf{H}|^2\alpha \tag{5.1}$$

Proof. Choose a local basis e_i of TL induced by normal coordinates such that $\{e_i\}$ is orthonormal at p and $\{e_i, Je_i\}$ span TM around p, which is possible because L is Lagrangian. Let f^i be the dual basis of e_i and $g^j = -f^j \circ J$ be dual to Je_j. In this basis, $\Omega = e^{i\theta}(e^1 + if^1) \wedge ... \wedge (e^n + if^n)$.

In the proof of the identity $\mathbf{H} = J\nabla\theta$ (Theorem 4.9), we have seen that

$$i\bar{\nabla}_X\theta = \frac{1}{2}\sum_k(-if^k(J\bar{\nabla}_X e_k) + ig^k(\bar{\nabla}_X e_k)) = -i\sum_k \left\langle J\bar{\nabla}_X e_k, e_k\right\rangle.$$

Since $\alpha = e^{i\theta}$, choosing $X = \frac{\partial}{\partial t}$ we have

$$\begin{aligned}\frac{\partial}{\partial t}\alpha &= i\alpha\bar{\nabla}_{\frac{\partial}{\partial t}}\theta = -i\alpha\sum_k\left\langle J\bar{\nabla}_{\frac{\partial}{\partial t}}e_k, e_k\right\rangle\\ &= -i\alpha\sum_k\left\langle J\bar{\nabla}_i\mathbf{H}, e_k\right\rangle.\end{aligned}$$

On the other hand, we have proved that $\mathbf{H} = J\nabla\theta$, so $\nabla\theta = -J\mathbf{H}$, which means $\nabla\alpha = -i\alpha J\mathbf{H}$. Since we are working in normal coordinates,

$$\begin{aligned}\triangle\alpha &= \sum_i\nabla_i\nabla_i\alpha = \sum_i\nabla_i\left\langle -i\alpha J\mathbf{H}, e_i\right\rangle\\ &= -i\sum_i\left\langle(\bar{\nabla}_i\alpha)J\mathbf{H} + \alpha\bar{\nabla}_i(J\mathbf{H}), e_i\right\rangle - i\sum_i\left\langle\alpha J\mathbf{H}, \bar{\nabla}_i e_i\right\rangle\\ &= -i\sum_i\left\langle(\left\langle -i\alpha J\mathbf{H}, e_i\right\rangle)J\mathbf{H}, e_i\right\rangle - i\sum_i\alpha\left\langle J\bar{\nabla}_i\mathbf{H}, e_i\right\rangle - 0\end{aligned}$$

where we have used the fact that J is parallel, and that $J\mathbf{H}$ is in TL while $\bar{\nabla}_i e_i$ has only normal components (because we are working with normal coordinates on L). Plugging in what we got for $\frac{\partial}{\partial t}\alpha$, we have

$$\begin{aligned}\triangle\alpha &= -\sum_i \alpha \langle J\mathbf{H}, e_i\rangle^2 + \frac{\partial}{\partial t}\alpha \\ &= -\sum_i \alpha \langle \mathbf{H}, Je_i\rangle^2 + \frac{\partial}{\partial t}\alpha \\ &= -|\mathbf{H}|^2\alpha + \frac{\partial}{\partial t}\alpha\end{aligned}$$

because Je_i, $i = 1, 2, ..., n$ span NL. This proves the proposition. □

Writing $\mathrm{Re} *_L \Omega = \cos\theta$, we see that $\frac{\partial}{\partial t}\cos\theta = \triangle\cos\theta + |H|^2\cos\theta$. Hence by the parabolic maximum principle (Theorem 3.7), we have

Corollary 5.5. *If L is almost calibrated, then it remains so under the mean curvature flow.*

An important result on singularities of mean curvature flow of almost calibrated submanifolds is as follows.

Theorem 5.6. *If L is almost calibrated, then it does not develop any type I singularity along the mean curvature flow.*

Since many technicalities which are beyond the topics we have discussed will be used in the proof, we will only provide a sketch of the proof. The reader may find a complete proof in [22]. The major technical tools used to understand singularities, including what follows, can also be found in [16].

Denote $\Phi(x_0,T)(x,t) = \frac{1}{(4\pi(T-t))^{n/2}}e^{-\frac{|x-x_0|^2}{4(T-t)}}$. First we need the following variant of Huisken's monotonicity formula (cf. Theorem 3.20):

Lemma 5.7. If f_t are smooth functions on L_t, where L_t are boundaryless solutions to the mean curvature flow, then

$$\frac{d}{dt}\int_{L_t} f_t\Phi(x_0,T) = \int_{L_t}(\frac{\partial f_t}{\partial t} - \triangle f_t)\Phi(x_0,T) - \int_{L_t} f_t|H + \frac{(\mathbf{x}-x_0)^{\perp}}{2(T-t_0)}|^2\Phi(x_0,T)$$

where the integration is with respect to the metric on L_t.

Proof. Denote $\rho = \Phi(x_0,T)$. We have seen in Huisken's monotonicity formula that

$$\frac{\partial}{\partial t}(\rho d\mu_t) = (-\triangle\rho - \rho|\mathbf{H} + \frac{1}{2\tau}(\mathbf{x}-x_0)^{\perp}|^2)d\mu_t$$

where $d\mu_t$ is the volume element of L_t, and $\tau = T-t$. Thus we have

$$\frac{\partial}{\partial t}(f_t\rho d\mu_t) = (\frac{\partial}{\partial t}f_t)\rho d\mu_t + f_t(-\triangle\rho - \rho|\mathbf{H} + \frac{1}{2\tau}(\mathbf{x}-x_0)^{\perp}|^2)d\mu_t.$$

Now we integrate both sides, and use the Green's identities to get

$$\begin{aligned}
\int_{L_t}\frac{\partial}{\partial t}(f_t\rho d\mu_t) &= \int_{L_t}(\frac{\partial}{\partial t}f_t)\rho d\mu_t + \int_{L_t} f_t(-\triangle\rho - \rho|\mathbf{H} + \frac{1}{2\tau}(\mathbf{x}-x_0)^{\perp}|^2)d\mu_t \\
&= \int_{L_t}(\frac{\partial}{\partial t}f_t)\rho d\mu_t - \int_{L_t}(\triangle f_t)\rho d\mu_t + \int_{L_t} f_t(-\rho|\mathbf{H} + \frac{1}{2\tau}(\mathbf{x}-x_0)^{\perp}|^2)d\mu_t \\
&= \int_{L_t}(\frac{\partial}{\partial t}f_t - \triangle f_t)\rho d\mu_t - \int_{L_t}\rho f_t|\mathbf{H} + \frac{1}{2\tau}(\mathbf{x}-x_0)^{\perp}|^2 d\mu_t
\end{aligned}$$

as desired. □

We shall need the following regularity theorem, which is a fundamental tool in the study of regularity of mean curvature flows. The Gaussian density ratio is defined to be $\theta_t(x_0, l) = \int_{M_t} \Phi(x_0, l)(x, 0)$ where M_t is a smooth solution to the mean curvature flow of a k-dimensional submanifold of $\mathbf{R}^N$.

Theorem 5.8. *(White's regularity theorem) There exist constants $\varepsilon(N, k)$ and $C(n, k)$ such that if $\partial M_t \cap B_{2R}$ is empty, and $\theta_t(x, l) \leq 1 + \varepsilon_0$ for all $l \leq R^2$, and $x \in R_{2R}$, and $t \leq R^2$, then the $C^{2,\alpha}$-norm of M_t in B_R is bounded by $Ct^{-1/2}$ for all $t \leq R^2$.*

The motivation and the proof can be found in [23].

Proof. (sketch of proof of Theorem 5.6)

Using the modified monotonicity formula and the evolution equation (5.1) for $\cos\theta$ we obtain

$$\begin{aligned}\frac{\partial}{\partial t} \textstyle\int_{L_t} \Phi(x_0, T)(1 - \cos\theta) d\mu_t = -\textstyle\int \Phi(x_0, T)|\mathbf{H}|^2 \cos\theta d\mu_t \\ -\textstyle\int \Phi(x_0, T)|\mathbf{H} + \tfrac{1}{2(T-t)}(x - x_0)^{\perp}|^2 (1 - \cos\theta) d\mu_t.\end{aligned}$$

Now we do a rescaling to study the type I singularity. Take $\lambda_i \to \infty$, let $s_i = -\lambda_i^2(T - t)$, and rescale the immersions $F_{s_i}^{\lambda_i}(x) = \lambda_i(F_t(x) - y_0)$ where (y_0, T) is the point where the singularity occurs. This is to enlarge the behaviour at the singularity time T by letting $\lambda_i \to \infty$, and by doing this we also have that the second fundamental forms of the rescaled submanifolds are uniformly bounded. It can be proved that the above equations continue to hold after rescaling, with $2(T-t)$ replaced by $-s$.

Hence a type I singularity means there exists a subsequence of rescaled submanifolds with smooth limit satisfying $|\mathbf{H}|^2 = |\mathbf{H} - \frac{1}{2s}(\mathbf{x} - x_0)^\perp|^2 = 0$, and one checks easily that this must be flat. Since the Gaussian density ratio is 1 for a plane, White's theorem can be used to show that (y_0, T) is in fact a smooth point and the flow can be continued.

A more detailed proof can be found in [22], in which Wang proves this result on symplectic surfaces in four dimensional Kähler-Einstein manifolds, but the same argument applies to show the result for almost calibrated submanifolds. □

5.4 Proof of Theorem 5.2

Our aim is to prove that there exists an α for which S_α is almost calibrated, with respect to some $i^{-q}\Omega$.

Let $\Omega = (\bar{e}^1 + i\check{e}_1) \wedge ... \wedge (\bar{e}^p + i\check{e}_p) \wedge (\bar{\nu}^1 + i\check{\nu}_1) \wedge ... \wedge (\bar{\nu}^q + i\check{\nu}_q)$. Once again, by a rotation if necessary, we can assume that the e_k's diagonalize A^ν at x, with $A^\nu(e_k) = \lambda_k e_k$. Then we have

$$(\bar{e}^j + i\check{e}_j)(E_i) = \delta_{ji} - i\lambda_i\delta_{ji} + i\nabla_{e_i}\mu(e_j)$$

$$(\bar{e}^j + i\check{e}_j)(F_i) = 0 = (\bar{\nu}^j + i\check{\nu}_j)(E_i)$$

$$(\bar{\nu}^j + i\check{\nu}_j)(F_i) = i\delta_{ji}.$$

Hence $\Omega(E_1, ..., E_p, F_1, ..., F_q) = i^q \det(\delta_{ji} - i\lambda_i \delta_{ji} + i\nabla_{e_i}\mu(e_j)) = i^q \det(I + i(-D + H))$ where D is the diagonal matrix consisting of the eigenvalues $\lambda_1, ..., \lambda_p$ and H is the matrix $H_{ij} = \nabla_{e_i}\mu(e_j)$.

It suffices to prove that we can choose α so that on S_α, there exists ε such that $\mathrm{Re}(i^{-q}\Omega) \geq \varepsilon \, \mathrm{Im}(i^{-q}\Omega)$. Since p is even, the real part is a polynomial in the λ's with highest degree term being the p-th power $i^p \lambda_1 \lambda_2 ... \lambda_p$. All the terms in the imaginary part have degree less than p.

Suppose that we change $\nu \to \xi\nu$, then the homogenous polynomials of degree d in the λ_i's scale by ξ^d. From this it can be seen that with the assumption on the Gauss-Kronecker curvatures, $i^p \lambda_1 \lambda_2 ... \lambda_p > 0$ always. Hence there exists C_ν such that if $\xi > C_\nu$, then the real part can be made arbitrarily greater than the imaginary part.

Now we vary ν. By compactness of the unit sphere, it is easy to see that there exists a maximum over all the choices of C_ν. We choose $\alpha(x)$ to be this maximum. This completes the proof.

From the proof of the theorem, we see that as the ξ goes to infinity, $\frac{\mathrm{Re}(i^{-q}\Omega)}{||\Omega||}$ can be arbitrarily close to 1, hence it is "close" to being calibrated (the calibration has to be normlized). This confirms with our computation of the mean curvature vector, where it becomes zero asymptotically.

Also, this could be close to the best we can say about this analysis, namely we cannot include also the zero section and expect the entire X_μ to develop no type I singularity. This is because the mean curvature vector at the zero section is exactly the one of the submanifold, hence by local uniqueness, at least in short time the flow of that piece would be the mean curvature flow of the submanifold itself, and this can develop singularities of any type.

Remark 5.9. It can be shown that in the case of a minimal surface M with curvature bounded above by a strictly negative constant, and if $\nabla\mu$ is bounded, then this α can be chosen to be a constant function on M.

References

[1] A. Borisenko, "Ruled special Lagrangian surfaces", in *Minimal surfaces*, 269–285, *Adv. Soviet Math.*, Volume 15 (1993), Amer. Math. Soc., Providence, RI.

[2] M. do Carmo, *Riemannian Geometry*, First edition, Birkhuser, United States (1992).

[3] E. Goldstein, "Calibrations and minimal Lagrangian submanifolds", *M.I.T. Ph.D Thesis*, 2001. Available at: http://hdl.handle.net/1721.1/8639.

[4] R. Harvey and H. B. Lawson, "Calibrated geometries", *Acta Math.* **148** (1982), 47–157.

[5] G. Huisken, "Asymptotic behaviour for singularities of the mean curvature flow", *J. Differential Geom.* **31** (1990), 285–299.

[6] G. Huisken, "Flow by mean curvature of convex surfaces into spheres", *J. Differential Geom.* **20** (1984), 237–266.

[7] J. Jost, *Riemannian Geometry and Geometric Analysis*, Fifth edition, Springer, United States (2008).

[8] D. D. Joyce, *Compact Manifolds with Special Holonomy*, Oxford University Press, United States (2000).

[9] S. Karigiannis, N. Leung, "Some explicit calibrated deformations of calibrated subbundles of Euclidean spaces", *in preparation*.

[10] J. M. Lee, *Riemannian Manifolds*, First edition, Springer, United States (1997).

[11] P. Li, *Lecture Notes on Geometry Analysis*, Lecture Notes Series No. 6, Research Institute of Mathematics, Global Analysis Research Center, Seoul National University, Korea (1993).

[12] G. M. Lieberman, *Second Order Parabolic Differential Equations*, First edition, World Scientific Publishing Company, Singapore (1996).

[13] D. McDuff, D. Salamon, *Introduction to Symplectic Topology*, Second edition, Oxford University Press, United States (1999).

[14] A. Moroianu, *Lectures on Kähler Geometry*, First edition, Cambridge University Press, United States (2007).

[15] A. Neves, "Finite time singularities for Lagrangian mean curvature flow", arXiv: math/1009.1083v2.

[16] A. Neves, "Recent Progress on Singularities of Lagrangian Mean Curvature Flow", arXiv: 1012.2055

[17] R. Schoen, J. Wolfson, *Mean curvature flow and Lagrangian embeddings*, preprint.

[18] K. Smoczyk, "Mean curvature flow in higher codimension - Introduction and survey", arXiv: 1104.3222v2

[19] K. Smoczyk, "A canonical way to deform a Lagrangian submanifold", arXiv: dg-ga/9605005v2.

[20] P. M. Topping, *Lectures on the Ricci flow*, First edition, Cambridge University Press, London (2006).

[21] R. P. Thomas, S. T. Yau, "Special Lagrangians, stable bundles and mean curvature flow", *Comm. Anal. Geom.* **10** (2002), 1075–1113.

[22] M-T. Wang, "Mean curvature flow of surfaces in Einstein four-manifolds", *J. Differential Geom.* **57** (2001), 301–338.

[23] B. White, "A local regularity theorem for mean curvature flow", *Ann. of Math.* **161** (2005), 1487–1519.