Markovian Approaches to Joint-life Mortality with Applications in Risk Management

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

The combined survival status of the insured lives is a critical problem when pricing and reserving insurance products with more than one life. Our preliminary experience examination of bivariate annuity data from a large Canadian insurance company shows that the relative risk of mortality for an individual increases after the loss of his/her spouse, and that the increase is especially dramatic shortly after bereavement. This preliminary result is supported by the empirical studies over the past 50 years, which suggest dependence between a husband and wife.

The dependence between a married couple may be significant in risk management of joint-life policies. This dissertation progressively explores Markovian models in pricing and risk management of joint-life policies, illuminating their advantages in dependent modeling of joint time-until-death (or other exit time) random variables. This dissertation argues that in the dependent modeling of joint-life dependence, Markovian models are flexible, transparent, and easily extended.

Multiple state models have been widely used in historic data analysis, particularly in the modeling of failures that have event-related dependence. This dissertation introduces a common shock factor into a standard Markov joint-life mortality model, and then extends it to a semi-Markov model to capture the decaying effect of the "broken heart" factor. The proposed models transparently and intuitively measure the extent of three types of dependence: the instantaneous dependence, the short-term impact of bereavement, and the long-term association between lifetimes. Some copula-based dependence measures, such as upper tail dependence, can also be derived from Markovian approaches.

Very often, death is not the only mode of decrement. Entry into long-term care and voluntary prepayment, for instance, can affect reverse mortgage terminations.

The semi-Markov joint-life model is extended to incorporate more exit modes, to model joint-life reverse mortgage termination speed. The event-triggered dependence between a husband and wife is modeled. For example, one spouse's death increases the survivor's inclination to move close to kin. We apply the proposed model specifically to develop the valuation formulas for roll-up mortgages in the UK and Home Equity Conversion Mortgages in the US. We test the significance of each termination mode and then use the model to investigate the mortgage insurance premiums levied on Home Equity Conversion Mortgage borrowers.

Finally, this thesis extends the semi-Markov joint-life mortality model to having stochastic transition intensities, for modeling joint-life longevity risk in last-survivor annuities. We propose a natural extension of Gompertz' law to have correlated stochastic dynamics for its two parameters, and incorporate it into the semi-Markov joint-life mortality model. Based on this preliminary joint-life longevity model, we examine the impact of mortality improvement on the cost of a last survivor annuity, and investigate the market prices of longevity risk in last survivor annuities using risk-neutral pricing theory.

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Chapter 1

Introduction

Many insurance products provide benefits that are contingent on the combined survival status of multiple lives. The most common examples are joint-life annuities and insurances sold to married couples. A couple is considered as an entity, and the payoff of the products is contingent on the first death (joint-life status) or the last death (the last survival status) in the entity. Although some products are sold to, for example, business partners, we in this thesis restrict our discussion to benefits payable to married couples. Modeling the combined survival status of the insured lives is a critical problem in pricing and reserving those products. The theory of multiple life contingencies traditionally postulates independence between the remaining lifetimes of joint lives for the sake of simplicity, but there is strong empirical evidence that supports dependence between the time-until-death random variables of a husband and wife.

This introductory chapter serves as background to motivate the study of joint-life mortality risk evaluation and management in the Markovian framework, and is organized as follows. Section 1.1 develops the reasons for dependent modeling of joint time-until-death random variables. Section 1.2 gives a brief review of multiple state models and the Markov and semi-Markov property. Section 1.3 reviews several mortality models that have been proposed in demographic analysis for the modeling of force of mortality. Among these mortality models, the Gompertz law serves as a starting point for the transition intensities of the multiple state model. The first three sections are tied together in Section 1.4, where we raise the research questions that will be answered in this thesis. A detailed exposition of the contents of this thesis is presented in Section 1.5.

1.1 Dependent Joint Lifetimes

Research on the dependence between bivariate lifetimes of coupled lives dates back to the 1960's. See Young at el. (1963), Cox and Ford (1964), and Parkes et al. (1969). Recent research work, such as Frees et al. (1996) and Youn and Shemyakin (2001), focused on more sophisticated modeling of the dependence between couples.

The selection effect of bereavement on the post-bereavement mortality of the surviving spouse was recognized early in the empirical research literature. Young et al. (1963) traced the mortality of widowers up to the end of the fifth year after bereavement. They found the mortality rate during the first six months of bereavement was 40% greater than that for married men of the same age, and the increase gradually fell thereafter to the level of the rate for married men. Parkes et al. (1969) examined the same widowers studied by Young et al. (1963) and showed that the increased mortality was almost confined to the first six months of bereavement, after which mortality falls back to that of married men of the same age. Their research on the selection effect of bereavement underpinned the semi-Markov property specified for the mortality in the widowed status. The instantaneous dependence among the lifetimes of a couple describes the dependence of lives arising from a common exogenous events. Marshall and Olkin (1967) modeled this type of dependence by the “common shock” model.

Overall, the dependence between the future lifetimes of a husband and wife can be categorized as follows:

- the long-term dependence that is due to the common lifestyle shared by a married couple,
- the short term dependence that is caused by the select effect of the "broken heart syndrome",
- the instantaneous dependence as a result of the common shock events exposed to a married couple at the same time.

Joint-life insurance policies and financial products are highly popular and the dependence among multiple lives has been recognized as a consensus. The unrealistic assumption of independence could have a significant financial impact in the industry. The problem of fair pricing for such products requires the construction of a statistical model for the impact of one life's survivorship on another.

1.2 Multiple State Models

Multiple state models, also called multi-state models, or MSMs, are models for a continuous-time stochastic process, which at any time occupies one of a finite number of possible states. The states and transitions allowed between selected pairs of states, specify the features and conditions of a process. Multiple state models have been widely used in demography, biostatistics, and other fields.

Some specific models can be formulated as special examples of multi-state models in general, for example, a competing risk model in which a subject is exposed to many causes of failure. Sverdrup (1965) used a multi-state Markov chain to model disability. Freund (1961) used a homogeneous Markov multi-state model where the hazard of future events is determined by the state at the current time. The "common shock" model also can be easily formulated in the multiple state model framework. Recently, multiple state models have become an widely used tool for the calculation of life contingent functions. Examples can be found in Dickson et al. (2009).

A Markov model assumes that the transition probability depends only on the current time and the state occupied, that is, it is independent of all previous transitions. In a semi-Markov model, the transition probability depends not only on the current time and the state occupied, but also on the time since the previous transition. In other words, the transition rate depends on the duration spent in the current state. Semi-Markov models are an interesting extension of Markov models. A semi-Markov model can be used to study whether the risk fades out, or is increased after an event has happened.

1.3 Models for the Force of Mortality

In multiple state models, the transition intensity, also called force of transition, between states are of interest. In life contingency theory, the distribution of an individual's future lifetime can be represented as a multiple state model with two states, which are 'Alive' and 'Dead'. The concept of force of mortality in life contingencies is identical to the transition intensity in this simple two-state model.

It is a tradition in demographic and survival analysis that mortality models are represented by force of mortality, which are called hazard functions in statistics. This section reviews several formulae that have been proposed for describing the relationship between aging and force of mortality. All those models can serve as a candidate model for the transition intensity of a multiple state model.

In 1825, Benjamin Gompertz developed a simple mathematic relationship between aging and the mortality rates of the elderly. Gompertz (1825) observed that the force of mortality exponentially increases with age, based on empirical data for ages from 30 to 80. This famous law is expressed as

$$\mu(x) = Bc^x,$$

where B is interpreted as the general mortality level and c is the exponential coefficient of mortality growth. Although the Gompertz curve was criticized for overestimating the high age mortalities, it well describes adult mortality. Furthermore, it offers a parsimonious and analytically tractable formula for force of mortality (see, Schoen et al., 2004). Ever since Gompertz' work, mortality models proposed all contain the Gompertz component.

Makeham (1860) added an age-independent component to the Gompertz model, to take into account the force of accidental death that is assumed to be not age related. The force of mortality is modeled as

$$\mu_x = A + Bc^x.$$

As stated in Dickson et al. (2009), the extra constant term improves the model to fit to mortality data at younger ages better than Gompertz' law.

Perks (1932) proposed a four-parameter general logistic model, in a form of

$$\mu_x = \frac{A + Be^{\mu x}}{1 + Ce^{\mu x}}.$$

Beard (1963) proposed a less complex three-parameter logistic mortality, in a form of

$$\mu_x = \frac{Be^{\mu x}}{1 + Ce^{\mu x}}.$$

Logistic models dealt with the problem that the Gompertz law and Makeham's law overestimate mortality at the oldest ages. Thatcher (1999) addressed this problem in a form similar to the Beard model,

$$\mu_x = A + \frac{Be^{\mu x}}{1 + Be^{\mu x}}.$$

He stated that the fully general logistic model (Perks model) was found to be less useful in practice, while the three-parameter logistic model is simpler and more robust. Bongaarts (2005) pointed out that, at lower adult ages the force of mortality estimated by the Makeham's law and the Thatcher model are very similar, because the denominator term in the second term of the Thatcher model is close to 1. However, the two models diverge at the oldest ages due to the fact that the Thatcher model levels off $1 + A$ while the Makeham's force of mortality has no limit.

Kannisto (1992) assumed that the parameter $A = 0$, and gave a simple 2-parameter model

$$\mu_x = \frac{Be^{\mu x}}{1 + Be^{\mu x}},$$

which implies that $\text{logit}(\mu_x) = \ln(B) + \mu \cdot x$.

1.4 Research Questions

The method of modeling the dependence between lifetimes has followed two popular methods: copula methods and multi-state Markov models. Multiple state models are a natural tool in actuarial science. Many actuarial models, such as survival models, disability models, and "death-disease" decrement models, are specific examples of multiple state models in general. Multiple state models are intuitive, flexible, and easily extendable.

For pricing and risk management of joint-life insurance policies and financial products, what is of concern is the correlation between joint lives. A major element of dependence could be called event-related, which means an actual event, such as an accident, or the death of one's spouse, leads to the change in risk. A multi-state model presents itself as an intuitive approach to model such non-static event-related dependencies between a couple, having the advantages of being flexible and transparent. This dissertation makes an attempt to employ multi-state Markovian approaches (we include its semi-Markov case within the general class of Markovian approaches) to joint-life survival analysis for the risk management of joint-life insurance and financial products, addressing three questions:

1. How can we employ a multiple state model to describe the recognized types of dependence between the remaining lifetimes of joint lives?

2. Based on the flexibility and extendability of a Markovian multiple state approaches, how will a Markovian joint-life model be generalized to a more complex product?

3. How can we evaluate joint-life longevity risk in the framework of Markovian multiple state approaches?

These three research questions are not isolated topics. They constitute a progressive study of joint-life Markovian models. Through addressing these three questions, this thesis aims to make its contribution to provide a comprehensive model for pricing and risk management of a wide range of joint life products, including life insurance, annuities, pensions, and reverse mortgage schemes.

1.5 Overview of the Thesis

Each of the next three chapters of this thesis will further develop issues specific to one research question, present the findings, and discuss the future work. This section gives an overview of how this thesis elaborates on three research topics.

In Chapter 2, we build two multi-state Markovian models to measure the extent of three types of dependence between the lifetimes of a married couple, which we will compare with the copula approach. Copulas are a popular method for modeling the dependence among bivariate lifetimes. An attractive advantage of copulas is that they have a parameter-parsimonious model structure. However, the dependence structure in a copula model is not so transparent as a MSM. Moreover, the copula dependence structure remains static over time. As stated in Section 1.2, multiple state models lead themselves most readily to modeling the dependence between the remaining lifetimes of a couple. We set up two models, one Markov and one semi- Markov, and fit the models to a set of bivariate joint-life and last-survivor annuity data from a large Canadian insurance company. We employed two fitted models to examine the impact of dependence on the value of last-survivor annuities, and compare our Markovian models with two copula models considered in previous research on modeling joint-life mortality. Our findings illustrate the advantages of multiple state Markovian approaches in modeling joint-life mortality. This establishes the theoretical basis for our research direction.

In Chapter 3, we apply the semi-Markov joint-life model in the previous chapter to a more complex financial product, the reverse mortgage, also known as equivalently an equity release product. Huge uncertainty about the amount and timing of the future cash flow introduces risks to the pricing and risk management of reverse mortgages. The timing of reverse mortgage repayments is dependent on many exit modes: death, moveout, entering long-term care, refinancing, and other voluntary prepayment. A good reverse mortgage termination model will reduce the uncertainty around the timing of the cash flows from a reverse mortgage, providing reliable information on reverse mortgage terminations. Given its flexibility, the semi-Markov joint-life model is extended to incorporate more decrements. Based on the proposed model, we investigate the prices of the embedded "no negative equity guarantee" in the U.K. equity release schemes and examine the sensitivity of the price to each termination mode assumption. We also assess the fairness of the mortgage insurance premium in the U.S. reverse mortgages for its "non-recourse" guarantee.

In Chapter 4, we extend the semi-Markov joint-life model to examine the market prices of longevity risk in last-survivor annuities. To this end, we propose a semi-Markov joint-life longevity model for the pricing and risk management of last-survivor annuities. An unexpected increase in life expectancy puts enormous pressure on retirement funds and annuity products. Currently, insurance companies usually price annuities based on a life table projected with deterministic mortality reduction factors. Great uncertainty about future mortality improvement speed leads to a controversy over whether the annuity market has sufficiently allowed for mortality improvement in pricing annuity products. In terms of last-survivor products, dependence between joint lives may complicate the situation. Longevity risk may be even greater for couples. We incorporate the stochastic Gompertz law to the semi-Markov joint-life mortality model. The model makes a preliminary attempt at the dependent modeling of joint-life longevity risk. We then use the proposed model to price last survivor annuities in a risk-neutral measure, and compare the market prices of longevity risk in last-survivor annuities and single-life annuities.

Chapter 5 concludes the thesis and discusses future work.

Chapter 2

Markovian Approaches to Joint-life Mortality

2.1 Introduction

Several empirical studies in recent years suggest considerable dependence between the lifetimes of a husband and wife. Denuit et al. (2001) argue that a husband and wife are exposed to some of the same risks, since they share a common lifestyle and may encounter common disasters. Jagger and Sutton (1991) show that there is an increased relative risk of mortality following spousal bereavement. This condition, which they call the broken-heart syndrome, can last for a prolonged period of time. We find evidence of both common shock and broken-heart effects in the data set on which this article is based. All these findings call for appropriate methods to model lifetime dependence, which may have a significant impact on risk management for joint-life insurance policies.

One way to model dependence between lifetimes is to employ copula models. Frees et al. (1996) and Youn and Shemyakin (2001) use a Frank's copula and a Hougaard copula respectively to model joint-life mortality. We refer interested readers to Frees and Valdez (1998), Klugman et al. (2008) and McNeil et al. (2005), who offer comprehensive descriptions of copula models. An attractive advantage of the copula approach is that it allows the correlation structure of the remaining lifetime variables to be estimated separately from their marginal distributions. Nevertheless, choosing a suitable copula may not be straightforward. While we can compare one copula with another, whether either actually fits the dependence structure adequately is often not easy to quantify, and it is rare to have a qualitative or intuitive justification for a specific copula.

Another way to model dependence is to use finite state Markov models. In this approach, possible outcomes are mapped to a number of states. Transitions between states are governed by a matrix of transition intensities. Depending on the properties of transition intensities, models in this approach can be divided into two categories: Markov and semi-Markov. In Markov models, transition intensities depend on the current state only, while in semi-Markov models, transition intensities depend on only the current state and the time elapsed since the last transition (the sojourn time in the current state). Markov and semi-Markov models are highly transparent, as we see clearly from the multiple state model how a change of state, for example, from married to widowed, impacts mortality.

Markov multiple state models have been applied to diverse areas in actuarial science. Sverdrup (1965) and Waters (1984) both considered models where the states represented different health statuses. The first application to joint-life mortality modeling may be by Norberg (1989). Spreeuw and Wang (2008) extend Norberg's work by allowing mortality to vary with the time elapsed since death of spouse. Dickson et al. (2009) explain how finite state Markov models may be used for modelling various insurance benefits, including joint life, critical illness, accidental death and income replacement insurance.

This study, conducted simultaneously with that of Spreeuw and Wang (2008), considers different extensions to Norberg's model. First, we introduce to the model a common shock factor, which is associated with the time of a catastrophe, for example, a plane crash, that affects both lives. Secondly, we extend the original model to a semi-Markov model, which characterizes the broken-heart effect by a smooth parametric function of the time elapsed since bereavement. The parametric function provides more information about how the broken-heart factor diminishes with time than the step function given by Spreeuw and Wang (2008). Furthermore, we offer a comparison between the multiple state and copula approaches, by applying our models to the data set used by Frees et al. (1996) and Youn and Shemyakin (2001).

The rest of this chapter is organized as follows. Section 2 describes the data we use throughout this study. Section 3 illustrates the concept of Markovian modeling with a simple Markov model. Section 4 presents the full semi-Markov model with a parametric function for modeling the broken heart effect. Detailed information about the model structures and the estimation procedure is provided. Section 5 compares annuity prices using the Markov model with the semi-Markov model. Section 6 compares the semi-Markov model with the copulas proposed in Frees et al. (1996) and in Youn and Shemyakin (2001). Section 7 gives concluding remarks.

2.2 The Data

The data used in this chapter were developed in the research reported in Frees et al. (1996), funded by the Society of Actuaries. Youn and Shemyakin (2001) as well as Spreeuw and Wang (2008) also used this data. They comprise 14,947 records of joint and last-survivor annuity contracts over the observation period from December 29, 1988 to December 31, 1993. In Table 2.1 we show a summary of the data after the removal of replicated records and records associated with annuitants of the same sex. It is interesting to note that the death counts for males are almost three times that for females. This observation implies that there are more widowed females than widowed males.

In Table 2.2 we show the breakdown of the data by the following categories: (1) survived to the end of the observation period; (2) died at least 5 days before or after spouse's death; (3) died within 5 days before or after spouse's death. Without cause of death information, we cannot tell if the highly proximate deaths are due to a common disaster, or due to the impact of the broken heart syndrome on the widow(er)'s physiognomy. In our analysis of the data, it seems most likely that couples in group (3) died of the same accident as their spouses. To capture this phenomenon, we introduce a common shock component to the Markovian models. The use of a 5-day cut-off to allocate the deaths to common shock is admittedly somewhat arbitrary. The issues are discussed and analyzed further in Section 4.

2.3 Markov Model

2.3.1 Model Specification

Before we present our extension to Norberg's (1989) work, we illustrate the concept of multiple-state modeling with a simple Markov model.

A stochastic process $\{S_t, t \geq 0\}$ is a Markov process if, for any $u > v > 0$, the conditional probability distribution of S_u, given the whole history of the process up to and including time v, depends on the value of S_v only. In other words, given S_v, the process at time u is independent of the history of the process before time v.

The Markov process for this study can conveniently be represented by the diagram in Figure 2.1. The boxes represent the four possible states that a couple can be in at any time and the arrows indicate the possible transitions between these states. The process has a state space $\{0, 1, 2, 3\}$. We interpret, for example, $S_t = 0$

Age at inception	No. of persons-at-risk	No. of deaths
Females		
60 – 69	6397	177
70 – 79	2376	188
80 – 89	192	49
90+	10	7
Total	8975	421
Males		
60 – 69	4949	373
70 – 79	3684	586
80 – 89	318	130
90+	24	19
Total	8975	1108

Table 2.1: Number of persons-at-risk at the beginning of the observation period and number of deaths during the observation period.

as both husband and wife are alive at time t.

We assume that the force of mortality for an individual depends on his/her marital status, but not on his/her spouse's age. Let us suppose that the current ages of a wife and husband are x and y, respectively. The wife's force of mortality at age $x+t$ is μ^*_{x+t} if she is widowed; the force of mortality from all causes other than common shock, for a wife with husband still living, is denoted by μ_{x+t}. Likewise, the husband's force of mortality at age $y+t$ is μ^*_{y+t} if he is widowed, while μ_{y+t} denotes mortality for a still married man from all causes other than common shock.

	Females	Males
Survived	8554	7867
Died at least 5 days before spouse's death	288	999
Died at least 5 days after spouse's death	81	57
Died within 5 days before/after spouse's death	52	52
Total	8975	8975

Table 2.2: Breakdown of the data by survival status at the end of the observation period.

The common shock factor allows the process to move from state 0 to state 3 directly. Without this transition, simultaneous deaths would not be possible within this model, given the standard Markov model assumptions. We assume that μ^{03}, the intensity of moving from state 0 to state 3 directly, is independent of age (time). This assumption may be relaxed if sufficient information about common shock deaths is available. The use of the common shock transition means that the total force of mortality for a married woman age x + t is $\mu_{x+t}+\mu^{03}$, and similarly for a married man. This is essentially the same common shock model as that described in Bowers et al. (1997), though they do not use the Markov model framework.

In many applications of the model, we require the following transition probabilities:

$$ {}_tp_x^{ij} = \Pr(S_{x+t} = j | S_x = i), \qquad i, j = 0, 1, 2, 3, \qquad x, t \geq 0. $$

The computation of these probabilities requires two technical assumptions, in addition to the Markov assumption. Assumption (1): the probability of two or more transitions in a small interval h is $o(h)$, where $o(.)$ is a function such that $\lim_{h\to 0} o(h)/h = 0$. Assumption (2): ${}_tp_x^{ij}$ is a differentiable function of t. Given these two assumptions, we can compute ${}_tp_x^{ij}$ by the Kolmogorov forward equations,

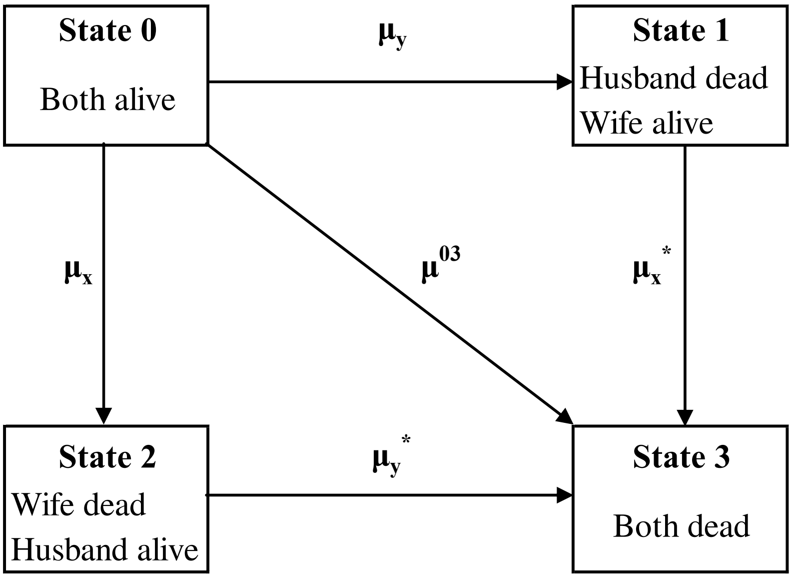

Figure 2.1: Specification of the Markov model.

which can be written in a compact form as

$$\frac{\partial}{\partial t}P(x, x+t) = P(x, x+t)A(x+t), \qquad x, t \geq 0,$$

where $P(x, x+t)$ and $A(x+t)$ are matrices in which the (i, j)th entries are ${}_tp_x^{ij}$ and $\mu^{ij}(x+t)$, respectively. $A(x+t)$ is called the infinitesimal generator matrix, also known as the intensity matrix, in which $\mu^{ij}(x+t) \geq 0$ for every $i \neq j$, and $\mu^{ii}(x+t) = -\sum_{j=0,j\neq i}^{3} \mu^{ij}(x+t)$. Solving this system of partial differential equations, we obtain the following expressions for the transition probabilities:

$$
\begin{aligned}
{}_tp^{00}_{x:y} &= \exp\Big(-\int_0^t \mu_{x+s} + \mu_{y+s} + \mu^{03}\, ds\Big); \\
{}_tp^{11}_{x} &= \exp\Big(-\int_0^t \mu^*_{x+s}\, ds\Big); \\
{}_tp^{22}_{y} &= \exp\Big(-\int_0^t \mu^*_{y+s}\, ds\Big); \\
{}_tp^{01}_{x:y} &= \int_0^t {}_sp^{00}_{x:y}\ \mu_{y+s}\ {}_{t-s}p^{11}_{x+s}\, ds; \\
{}_tp^{02}_{x:y} &= \int_0^t {}_sp^{00}_{x:y}\ \mu_{x+s}\ {}_{t-s}p^{22}_{y+s}\, ds; \\
{}_tp^{13}_{x} &= \int_0^t {}_sp^{11}_{x}\ \mu^*_{x+s}\, ds; \\
{}_tp^{23}_{y} &= \int_0^t {}_sp^{22}_{y}\ \mu^*_{y+s}\, ds.
\end{aligned}
$$

More detail is offered in Dickson et al. (2009).

2.3.2 Parameter Estimation

Let T_x and T_y be the remaining lifetimes of a wife and husband, respectively. The joint density function for T_x and T_y can be expressed as

$$
f_{T_x,T_y}(u,v) = \begin{cases} {}_up^{00}_{x:y}\ {}_{v-u}p^{22}_{y+u}\ \mu_{x+u}\ \mu^*_{y+v}, & \text{if } u < v, \\ {}_vp^{00}_{x:y}\ {}_{u-v}p^{11}_{x+v}\ \mu_{y+v}\ \mu^*_{x+u}, & \text{if } u > v, \\ {}_up^{00}_{x:y}\ \mu^{03}, & \text{if } u = v. \end{cases} \tag{2.3.1}
$$

Technically speaking, $f_{T_x,T_y}(u,v)$ is a probability density function with respect to a probability measure that is a combination of a two dimensional absolutely continuous Lebesque measure and a one dimensional singular measure, with positive mass on the line $T_x = T_y$. Therefore, the probability of moving directly from state 0 to state 3 is non-zero.

Given the joint density function, we can construct the log-likelihood function from which maximum likelihood estimates of transition intensities can be derived. Assuming independence among different couples in the data, the log-likelihood function can be written as a sum of three separate parts, ℓ_1, ℓ_2, and ℓ_3, where

$$
\begin{aligned}
\ell_1 &= \sum_{i=1}^{n}\Big(-\int_0^{v_i} (\mu_{x_i+t} + \mu_{y_i+t} + \mu^{03})\,dt + d_i^1 \ln \mu_{y_i+v_i} + d_i^2 \ln \mu_{x_i+v_i} + d_i^3 \ln \mu^{03}\Big),\\
\ell_2 &= \sum_{j=1}^{m_1}\Big(-\int_0^{u_{1,j}} \mu^*_{x_j+v_j+t}\,dt + h_{1,j} \ln \mu^*_{x_j+v_j+u_{1,j}}\Big),\\
\ell_3 &= \sum_{k=1}^{m_2}\Big(-\int_0^{u_{2,k}} \mu^*_{y_k+v_k+t}\,dt + h_{2,k} \ln \mu^*_{y_k+v_k+u_{2,k}}\Big),
\end{aligned}
$$

where

- n is the total number of couples in the data set,
- m_1 (m_2) is the total number of widows (widowers) in the data set,
- v_i is the time until the ith couple exits state 0, $i = 1, ..., n$,
- $d_i^j = 1$ if the the ith couple moves from state 0 to state j at $t = v_i$, $i = 1, ..., n$, $j = 1, 2, 3$,
- $u_{1,j}$ ($u_{2,k}$) is the time until the jth (kth) widow (widower) exits state 1 or 2, $j = 1, ..., m_1$, $k = 1, ..., m_2$,
- $h_{1,j} = 1$ if the jth widow dies at $t = u_{1,j}$,
- $h_{2,j} = 1$ if the kth widower dies at $t = u_{2,k}$,
- x_i and y_i are the entry ages of the wife and husband of the ith couple, respectively.

By maximizing three parts of log-likelihood function separately, we can get the maximum likelihood estimates of the transition intensities in each state. Note that, for right censored data, d_i^j, $h_{1,j}$, and $h_{2,j}$ will be zero.

To calculate the log-likelihood, there is a need to define 'simultaneous deaths' so that a value of d_i^j can be assigned to each couple. Here we treat the 52 deaths which occurred within 5 days before or after bereavement as simultaneous deaths. A deeper discussion on the cut-off rule is provided in Section 2.4 in which we present the full semi-Markov model.

Although we could assume that the forces of mortality are piecewise constant, in this study we graduate the forces of mortality (except μ^{03}, which is assumed to be invariant with age) using Gompertz' law, $\mu_x = BC^x$, where $B > 0$, and $C > 1$. The parametric graduation is used because it is a more parsimonious approach, it allows us to extrapolate the forces of mortality to extreme ages, it smooths the results, and it enables us to calculate quantities such as annuity values efficiently. Although we use the Gompertz model, we do not claim that it provides the best fit of all possible models. Our point here is to illustrate the combination of the individual mortality model (Gompertz, for simplicity) with the multiple state model which provides the dependency framework.

Assuming that, for both genders, the mortality in state 0 follows Gompertz' law, we can rewrite ℓ_1 as

$$\begin{aligned}\ell_1 = \sum_{i=1}^{n} \Big(-\frac{B_1 C_1^{x_i}(C_1^{v_i} - 1)}{\ln(C_1)} + d_i^2 \ln(B_1 C_1^{x_i+v_i}) - \frac{B_2 C_2^{y_i}(C_2^{v_i} - 1)}{\ln(C_2)} \\ + d_i^1 \ln(B_2 C_2^{y_i+v_i}) - v_i \mu^{03} + d_i^3 \ln(\mu^{03}) \Big),\end{aligned} \tag{2.3.2}$$

where (B_1, C_1) and (B_2, C_2) are the Gompertz parameters for female and male mortality in state 0, respectively. We can rewrite ℓ_2 and ℓ_3 in a similar manner. The maximum likelihood estimate of μ^{03} is 0.1407%, and its standard error is 0.0195% based on the asymptotic variance of the maximum likelihood estimator. Maximum likelihood estimates of other parameters are displayed in Table 2.3. Parameters μ_x^* and μ_y^* have higher standard errors, since the number of individuals who transition to states 1 or 2 is relatively small (see Table 2.2), even in this extensive data set.

In Figure 2.2 we plot the fitted forces of mortality in different states. We observe, for both sexes, an increased force of mortality after bereavement. This observation supports the broken-heart syndrome. We can further deduce from Figure 2.2 that broken-heart effects vary with age, as the mortality curves do not shift in parallel.

	B	Standard error	C	Standard error
Females				
μ_x	9.741×10^{-7}	2.889×10^{-7}	1.1331	0.0047
μ_x^*	2.638×10^{-5}	3.370×10^{-5}	1.1020	0.0181
Males				
μ_y	2.622×10^{-5}	1.038×10^{-5}	1.0989	0.0058
μ_y^*	3.899×10^{-4}	4.057×10^{-4}	1.0725	0.0136

Table 2.3: Estimates of Gompertz parameters in the Markov model.

To examine whether the Gompertz' laws give an adequate fit, we perform a χ-square test. That is

$$\chi^2 = \sum_{j=1}^{k} \frac{(E_j - O_j)^2}{E_j},$$

where E_j is the number of expected observations and O_j is the number of observations in the age interval. The age interval is set up with a conservative condition that each has at least 5 expected observations.

For μ_x, μ_x^*, and μ_y^*, the null hypothesis that the model gives an adequate fit is not rejected at 5% level of significance, but for μ_y, the null hypothesis is marginally rejected (the p-value is 0.042). The fit for μ_y can be improved by using Makeham's law, $\mu_x = A + BC^x$, which increases the p-value for the χ-square to 0.13, indicating an adequate fit. However, for consistency reasons, we use Gompertz' law for all four forces of mortality, even though Makeham's law may better fit μ_y.

In addition, we also fit the other two logistic models of mortality to the data. One is the Beard model (Beard, 1963), $\mu_x = \frac{Be^{\mu x}}{1+Ce^{\mu x}}$, which a three parameter logistic model (Logit-3). The other is the Kannisto model (Kannisto, 1992), $\mu_x = \frac{Be^{\mu x}}{1+Be^{\mu x}}$ (Logit-2). we present in Table 2.4 the estimated log-likelihood value from these four parametric models of mortality.

	Log-likelihood value			
Force of mortality	Gompertz	Logit-2	Logit-3	Makeham
μ_x	-1605.135	-1605.318	-1604.835	-1604.627
μ_x^*	-327.038	-327.035	-327.033	-327.023
μ_y	-4463.311	-4464.085	-4462.905	-4460.447 †
μ_y^*	-179.687	-179.072	-179.016	-179.686

† The Gompertz law is rejected by a likelihood ratio test with p-value= 0.05.

Table 2.4: Estimated log-likelihood value for four parametric models of mortality.

We can see that, for μ_x, μ_x^*, and μ_y^*, the estimated maximum log-likelihood values are very close among the four parametric models. Only in the case that μ_y is fitted by the Makeham's law, is the Gompertz model rejected by a likelihood ratio test at 5% level of significance. Neither a three parameter nor a two parameter logistic model presents higher quality of fitting than the Gompertz law.

As we have stated, the Gompertz model is used mainly for illustration, and these tests are simply to ensure that the model is not radically out of line with the data. Alternative models, for example, non-parametric Kaplan-Meier estimates, can also be used. However, because of very sparse data at high ages, Kaplan-Meier and other non-parametric approaches are of limited usefulness. We reiterate that, our purpose is more to demonstrate the methodology than to give definitive answers. We encourage users to apply their own data sets and determine the models most appropriate to them.

2.4 Semi-Markov Model

2.4.1 Model Specification

The Markov model described above is somewhat rigid, in that the mortality impact of widow(er)hood is assumed to be constant, regardless of the length of time since the spouse's death. While it might be reasonable that mortality of widow(er)s is generally higher than married individuals of the same age, it also seem reasonable to consider that the detrimental impact of bereavement on the surviving spouse's health might be stronger in the months immediately following the spouse's death than it is later on. In fact, the medical and demographic descriptions of the broken heart syndrome generally reference this more short term impact.

An analysis of times to death for widows and widowers can offer us some insight into how the forces of mortality change after bereavement. In Table 2.5 we show the breakdown of post-bereavement death counts by different ranges of W, which denotes the time between bereavement and death (in years). We observe from Table 2.5 that more than half of the deaths occurred during the first year after bereavement. In addition, we see an inverse relationship between death counts and W, suggesting that the deterioration of mortality after bereavement tapers off with time. Therefore, a semi-Markov model, which allows transition intensities to vary with sojourn times, may offer a better representation of post-bereavement mortality.

	Total	$0 < W \leq 1$	$1 < W \leq 2$	$2 < W \leq 3$	$W > 3$
Females	90	51	19	14	6
Males	100	72	16	4	8
Total	190	123	35	18	14

Table 2.5: Breakdown of post-bereavement death counts by different ranges time (W) between bereavement and death.

The need for a semi-Markov model can also be seen from Figure 2.3, which shows, for both sexes, the graduated values of the following three forces of mortality:

- $\mu^*_{x(y)|0}$: the force of mortality during the first year after bereavement;
- $\mu^*_{x(y)|1}$: the force of mortality during the second year after bereavement;
- $\mu^*_{x(y)|2^+}$: the force of mortality beyond the second year after bereavement.[1]

We observe that, at any given age, $\mu^*_{x(y)|0}$ is always highest, followed by $\mu^*_{x(y)|1}$, and then $\mu^*_{x(y)|2^+}$. These curves point to the conclusion that the deterioration of mortality after bereavement tapers off with time, even if age is controlled.

Given the observed pattern of post-bereavement mortality, we use the following parametric functions to model the force of mortality after bereavement:
for widows,

$$\mu^*(x,t) = (1 + a_1 e^{-k_1 t})(\mu_{x+t} + \mu^{03}) = F_1(t)(\mu_{x+t} + \mu^{03});$$

for widowers,

$$\mu^*(y,t) = (1 + a_2 e^{-k_2 t})(\mu_{y+t} + \mu^{03}) = F_2(t)(\mu_{y+t} + \mu^{03}),$$

[1] In calculating these forces of mortality, we treat deaths which occurred within 5 days before or after bereavement as simultaneous deaths. The graduation is based on Gompertz law.

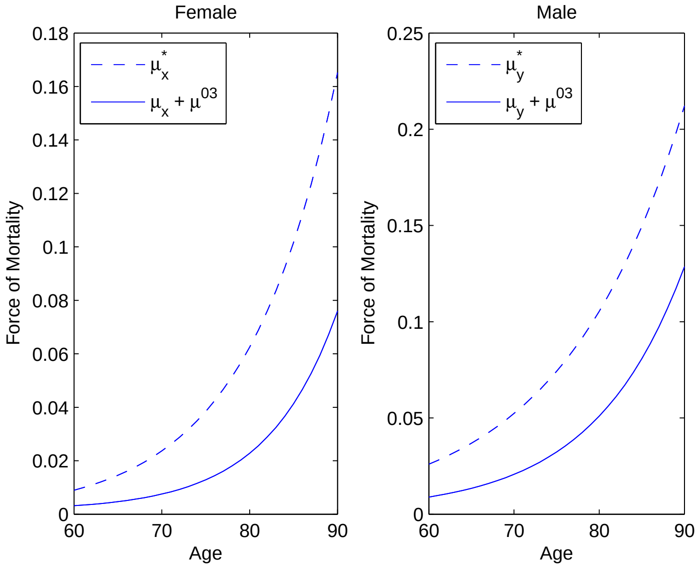

Figure 2.2: Forces of mortality in different states

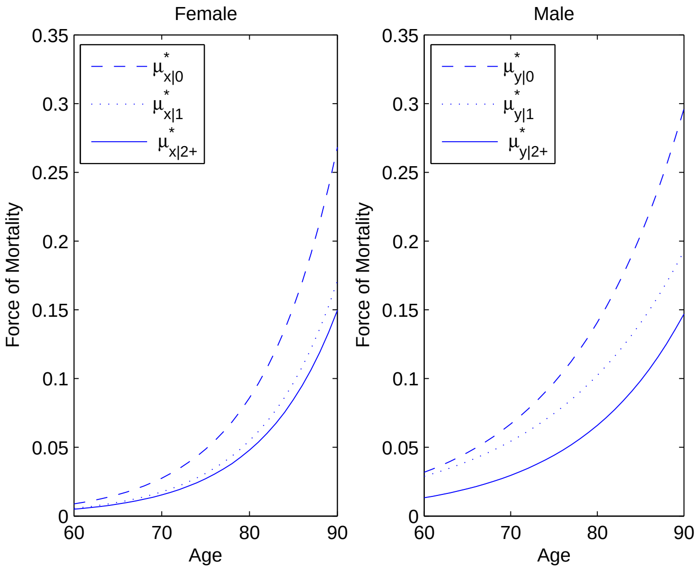

Figure 2.3: Forces of mortality in different years after bereavement

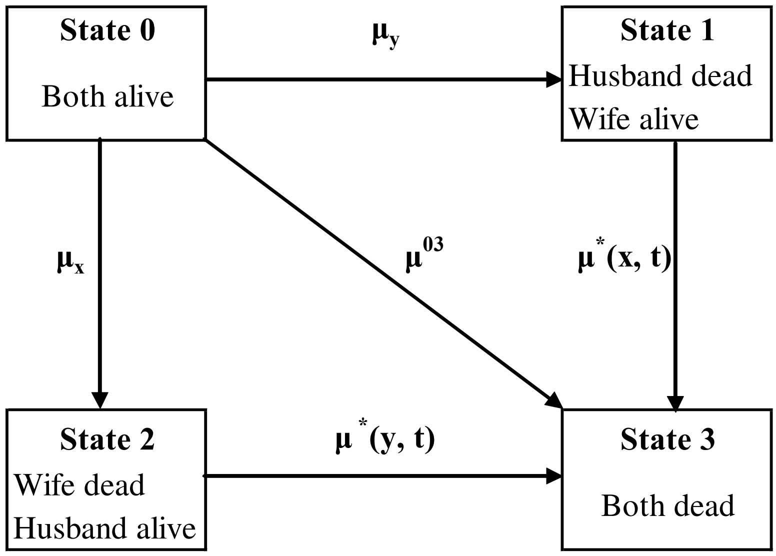

Figure 2.4: Specification of the semi-Markov model

where $a_j > -1$ and $k_j > 0$ for $j = 1, 2$, and t is the time since bereavement. Under this model, the force of mortality after bereavement is proportional to the corresponding force of mortality if bereavement did not occur. Initially, bereavement increases the force of mortality by a percentage of $100a_1\%$ for females and $100a_2\%$ for males. As t increases, the multiplicative factors $F_1(t) = 1 + a_1e^{-k_1t}$ and $F_2(t) = 1 + a_2e^{-k_2t}$ decrease exponentially and finally approach 1, capturing the selection effect of the broken-heart syndrome. Parameters k_1 and k_2 govern the speed at which the selection effect diminishes. The complete specification of the semi-Markov model is shown diagrammatically in Figure 2.4.

2.4.2 Parameter Estimation

Because the semi-Markov extension affects post-bereavement mortality only, there is no change to the meaning and values of μ_x, μ_y, and μ^{03}. Given the estimates of μ_x, μ_y, and μ^{03}, the remaining parameters can be estimated by partial maximum likelihood estimation. The partial likelihood function ℓ_1^p for parameters a_1 and k_1 is given by

$$\ell_1^p = \sum_{j=1}^{m_1} \left(- \int_0^{u_{1,j}} (1 + a_1 e^{-k_1 t})(\hat{B}_1 \hat{C}_1^{x+t} + \mu^{03}) dt \right. \tag{2.4.3}$$
$$\left. + h_{1,j} \ln \left((1 + a_1 e^{-k_1 t})(\hat{B}_1 \hat{C}_1^{x+t} + \mu^{03}) \right) \right),$$

where $\hat{B}_1$ and $\hat{C}_1$ are the maximum likelihood estimate for B_1 and C_1, respectively (see Table 2.3). The partial likelihood function ℓ_2^p for a_2 and k_2 can be obtained by changing the parameters in ℓ_1^p accordingly. By maximizing ℓ_1^p and ℓ_2^p, we can obtain estimates for the semi-Markov parameters.

In the above log-likelihood function ℓ_1^p, B_1, C_1 and μ^{03} have been estimated. This estimation method is called a two-stage estimation. In evaluating the variation in the estimator $\hat{a_1}$, $\hat{k_1}$, the variation in $\hat{B_1}$, $\hat{C_1}$ and $\hat{\mu^{03}}$ from the first step need to be taken into account. A full description of this two-stage estimation procedure as follows.

The full log-likelihood function l is given by

$$\begin{aligned}
\ell \quad &= \ell_1 + \ell_2 + \ell_3 \\
&= \sum_{i=1}^{n} \left(- \int_0^{v_i} (\mu_{x_i+t} + \mu_{y_i+t} + \mu^{03})\, dt + d_i^1 \ln \mu_{y_i+v_i} + d_i^2 \ln \mu_{x_i+v_i} + d_i^3 \ln \mu^{03} \right. \\
&\quad - \int_0^{u_{1,i}} (1 + a_1 e^{-k_1 t})(\mu_{x_i+v_i+t} + \mu^{03}) dt + h_{1,j} \ln \left((1 + a_1 e^{-k_1 t})(\mu_{x_i+v_i+t} + \mu^{03}) \right) \\
&\quad \left. - \int_0^{u_{2,i}} (1 + a_2 e^{-k_2 t})(\mu_{y_i+v_i+t} + \mu^{03}) dt + h_{2,j} \ln \left((1 + a_2 e^{-k_2 t})(\mu_{y_i+v_i+t} + \mu^{03}) \right) \right),
\end{aligned} \tag{2.4.4}$$

where all symbols carry the same meaning as they do in Section 2.3.2. Note that, from equation (2.4.3) and (2.4.4), we know there are $n - m_1$ observations where females did not enter the widowed state, The observed value of $u_{1,j}$ is hence 0, and obviously $h_{1,j}$ is 0, where $j = 1, 2, \ldots n - m_1$.

Define $\alpha_1 = [B_1, C_1, \mu^{03}]'$ and $\beta_1 = [k_1, a_1]'$. The estimating functions for α_1 are given by $\sum_{i=1}^n S_{1,i}(\alpha_1) = \partial\ell/\partial\alpha_1'$. α_1 is obtained as the solution to

$$\sum_{i=1}^n S_{1,i}(\alpha_1) = 0, \tag{2.4.5}$$

Given the estimate of α_1, we define the estimating functions $\sum_{i=1}^n U_{1,i}(\hat{\alpha_1}, \beta_1) = \partial\ell/\partial\beta_1'$. The estimating equation for β_1 are

$$\sum_{i=1}^n U_{1,i}(\hat{\alpha_1}, \beta_1) = 0, \tag{2.4.6}$$

Let $H_{1,i}(\alpha_1, \beta_1) = \begin{pmatrix} S_{1,i}(\alpha_1) \\ U_{1,i}(\alpha_1, \beta_1) \end{pmatrix}$. As the estimating equations $S_{1,i}(\alpha_1)$ are functions of α_1 alone, the information matrix $\partial H_{1,i}(\alpha_1, \beta_1)/\partial(\alpha_1', \beta_1')$ is lower triangular, i.e.,

$$\frac{\partial H_{1,i}(\alpha_1, \beta_1)}{\partial(\alpha_1', \beta_1')} = \begin{pmatrix} \frac{\partial S_{1,i}(\alpha_1)}{\partial\alpha_1'} & 0 \\ \frac{\partial U_{1,i}(\alpha_1,\beta_1)}{\partial\alpha_1'} & \frac{\partial U_{1,i}(\alpha_1,\beta_1)}{\partial\beta_1'} \end{pmatrix}$$

As stated in appendix of Yi and Cook (2002), with probability approaching to 1, there is a unique solution $(\hat{\alpha_1}', \hat{\beta_1}')'$ from the joint estimating equations $\sum_{i=1}^n H_{1,i}(\alpha_1, \beta_1) = 0$, that satisfies

$$n^{1/2} \begin{pmatrix} \hat{\alpha_1} - \alpha_1 \\ \hat{\beta_1} - \beta_1 \end{pmatrix} = -\{\mathsf{E}[\partial H_{1,i}(\alpha_1, \beta_1)/\partial(\alpha_1', \beta_1')]\}^{-1} \cdot n^{-1/2} \sum_{i=1}^n H_{1,i}(\alpha_1, \beta_1) + o_p(1).$$

The solution $(\hat{\alpha_1}', \hat{\beta_1}')'$ from the joint estimating equations is the same as the two-stage solution $\hat{\alpha_1}$, $\hat{\beta_1}$. Further, since $\partial H_{1,i}(\alpha_1, \beta_1)/\partial(\alpha_1', \beta_1')$ is lower triangular, for the estimator β_1 of central interest, we have

$$n^{1/2} \left(\hat{\beta_1} - \beta_1\right) = -\Gamma^{-1} n^{-1/2} \cdot \sum_{i=1}^n Q_{1,i}(\alpha_1, \beta_1) + o_p(1),$$

where $\Gamma = \mathsf{E}[\partial U_{1,i}(\alpha_1, \beta_1)/\partial \beta_1')]$, and

$$Q_{1,i}(\alpha_1, \beta_1) = U_{1,i}(\alpha_1, \beta_1) - \mathsf{E}[\partial U_{1,i}(\alpha_1, \beta_1)/\partial \alpha_1')] \cdot [\partial S_{1,i}(\alpha_1)/\partial \alpha_1']^{-1} \cdot S_{1,i}(\alpha_1).$$

By the central limit theorem, $n^{1/2}\left(\hat{\beta}_1 - \beta_1\right)$ is asymptotically normally distributed with mean 0 and asymptotic variance $\Gamma^{-1}\Sigma[\Gamma^{-1}]'$, where $\Sigma = \mathsf{E}[Q_{1,i}(\alpha_1, \beta_1)Q_{1,i}'(\alpha_1, \beta_1)]$.

The asymptotic distribution for $n^{1/2}\left(\hat{\beta}_2 - \beta_2\right)$, where $\beta_2 = [k_2, a_2]'$, can be obtained similarly. The estimates of a_1, a_2, k_1, and k_2 and their standard errors are shown in Table 2.6.

A limitation of our semi-Markov model is high parameter uncertainty. This is because the semi-Markov parameters are estimated from only a small number of post-bereavement deaths (see Table 2.5). Still, all four parameters are significantly greater than zero,[2] indicating the need for both the parameters for the magnitude of bereavement effect, a_i and the parameters for the speed of decay k_i.

	Central estimate	Standard error
Females		
a_1	3.3786	1.0723
k_1	0.5225	0.3058
Males		
a_2	11.0541	4.6484
k_2	7.9064	3.3335

Table 2.6: Estimates of parameters a_1, a_2, k_1, and k_2 in the semi-Markov model

[2]Assuming normality holds and a 95% level of significance, we say that a parameter θ is significantly greater than zero if $\hat{\theta} - 1.645\sqrt{\mathrm{Var}(\hat{\theta})} > 0$, where $\hat{\theta}$ is the central estimate of θ.

Figure 2.5 displays how the multiplicative factors $F_1(t)$ and $F_2(t)$ vary with time. The upper panel, which focuses on the first year after bereavement, shows that widowers are subject to a much higher broken heart effect shortly after bereavement. However, as the lower panel indicates, the broken heart effect for widows is more persistent than that for widowers.

Since μ_x^* and μ_y^* in the semi-Markov model are functions of both age (x) and time since bereavement (t), conducting a *chi*-square test for each specific force of mortality (as what is done in Section 2.3) would require us to group the deaths not only by x but also by t. In such a grouping, the number of deaths in each group will not be significant enough for us to perform the test with sufficient granularity.

2.4.3 Identifying Common Shock Deaths

In the semi-Markov model, the following two effects are explicitly modeled:

1. the common shock effect: the effect of a catastrophic event that affects both a husband and wife;
2. the broken-heart effect: the effect of spousal death on an individual's mortality.

In building the semi-Markov model, the threshold or cut-off for defining simultaneous deaths is highly important. If the threshold is set too long, some deaths associated with the broken-heart effect will be misclassified as simultaneous deaths, leading to an overestimation of μ^{03}. If the threshold is set too short, some simultaneous deaths will be misclassified, affecting the shape of the multiplicative factors $F_1(t)$ and $F_2(t)$, which are intended to model the broken-heart and not the common shock effect.

The parameter estimates in Table 2.6 are based on cut-off rule of 5 days. As a robustness check, we re-estimate the model parameters using cut-off rules of 0 day (i.e., no common shock), 2 days, and 10 days. The re-estimated multiplicative factors are shown diagrammatically in Figure 2.6. (We show $F_2(t)$ in log scale because the value $F_2(0)$ is extremely large when a 0-day or 2-day cut-off rule is used.)

When a short cut-off rule is used, the resulting multiplicative factors become extremely large for small values of t. For instance, when a 2-day rule is used, the value of $F_2(0)$ is as large as 95, which means mortality immediately after bereavement is 95 times greater than before! Also, when a short cut-off rule is used, the multiplicative factors reduce to 1 swiftly, failing to indicate the broken-heart effect that we should expect given the mortality curves shown in Figure 2.3. These observations indicate that short cut-off rules such as 0-day and 2-day tend to misclassify deaths associated with the common shock effect, distorting the shapes of $F_1(t)$ and $F_2(t)$.

From Figure 2.6 we observe that a reasonable cut-off rule would be at least 5 days. We base the parameters in the semi-Markov model on a cut-off rule of 5 days because it yields a higher log-likelihood value than longer cut-off rules.

Defining simultaneous deaths is to capture common shocks. If a shock happens to a couple and causes pair deaths, we refer this to "simultaneity". Common shock events do happen somewhere around us. Of the 190 pairs of deaths in our data, 24 occurred with one day, 30 occurred with two day, and 52 occurred with five day. We believe some paired deaths were caused by an accident event and hence were "simultaneous". Without causes of death information, it is impossible to distinguish common shock deaths from bereavement caused deaths for sure. Paired deaths within one or two days gap may be due to bereavement, while those with five or even more than five days may be caused by common shock events. We reiterate that our intention is to demonstrate the full flexibility of the model, and consistency with the data set in question. It is not our intention to provide definitive parameter or annuity values under the model. The choice of a 5-day cut-off allows

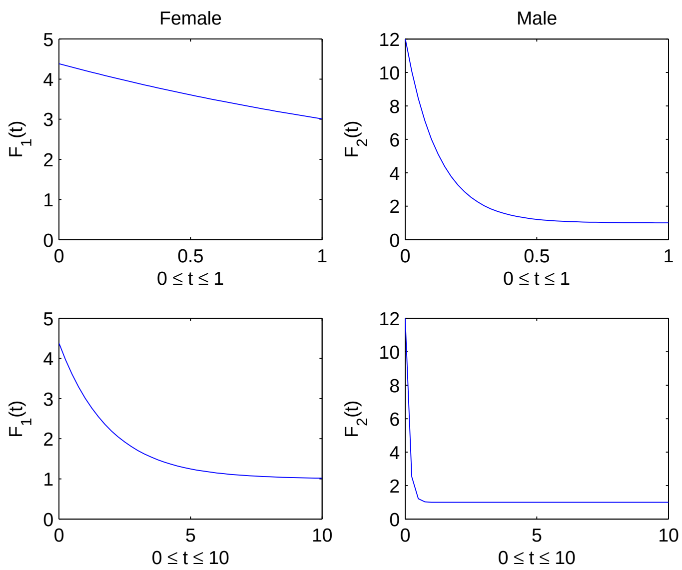

Figure 2.5: Factors $F_1(t)$ and $F_2(t)$ in the semi-Markov model.

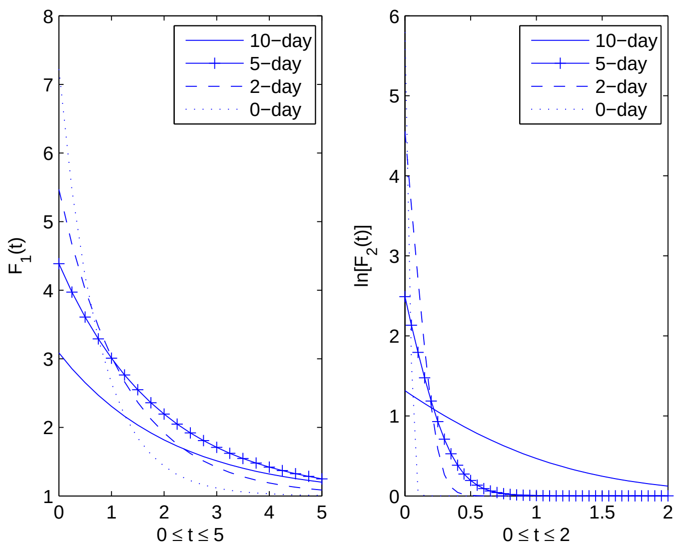

Figure 2.6: Multiplicative factors $F_1(t)$ and $F_2(t)$ (in log scale) on the basis of different cut-off rules for simultaneous deaths.

us to demonstrate the model incorporating, separately, the common shock and broken heart effect.

2.5 Positive Quadratic Dependence

So far, we have studied two parts of the dependence structure implied by the Markovian models. The first part is the common shock factor (μ^{03}), which describes the instantaneous dependence between the lifetimes of a husband and wife in state 0. The second part is the broken heart factor ($F_1(t)$ and $F_2(t)$ in the semi-Markov model), which models the temporary increase in mortality after bereavement. The Markov model differs from the semi-Markov model in that it postulates long-term dependence between the lifetimes of a husband and wife.

To understand the long-term dependence structure implied by the Markov models, we utilize the concept of positive quadrant dependence (PQD), which was first introduced by Lehmann (1966). We let T_x and T_y be the remaining lifetimes of a husband and wife, respectively. The lifetimes T_x and T_y are positively quadrant dependent if

$$\Pr(T_x \leq t, T_y \leq s) \geq \Pr(T_x \leq t)\Pr(T_y \leq s) \qquad \forall\, t, s \geq 0, \tag{2.5.7}$$

or, equivalently,

$$\Pr(T_x > t, T_y > s) \geq \Pr(T_x > t)\Pr(T_y > s) \qquad \forall\, t, s \geq 0. \tag{2.5.8}$$

Expressions (2.5.7) and (2.5.8) state that the probability that both lifetimes are small (or large) is higher under PQD assumption than under the assumption of independence. Another way to interpret PQD can be obtained by rewriting expressions (2.5.7) and (2.5.8) respectively as

$$\Pr(T_x \leq t \mid T_y \leq s) \geq \Pr(T_x \leq t) \qquad \forall\, t, s \geq 0, \tag{2.5.9}$$

and

$$\Pr(T_x > t \mid T_y > s) \geq \Pr(T_x > t) \qquad \forall\, t, s \geq 0. \tag{2.5.10}$$

The inequalities above mean that an individual is expected to live longer if his/her spouse lives for a long time, and to die earlier if his/her spouse dies early. Since these inequalities hold true for all values of t and s, PQD means spousal death (or survival) has long-term effects on the mortality of widows and widowers.

Norberg (1989) proved that the following statements regarding Markov models are true if there is no common shock component:

$$\begin{aligned} \mu_x \equiv \mu_x^* \text{ and } \mu_y \equiv \mu_y^* &\iff T_y \text{ and } T_x \text{ are independent;} \\ \mu_x < \mu_x^* \text{ and } \mu_y < \mu_y^* &\iff T_y \text{ and } T_x \text{ are positive quadrant dependent.} \end{aligned}$$

Without common shock component, two lifetimes are independent if the force of mortality before and after bereavement are equal; while with common shock transition, μ^{03}, in the Markov model, Norberg's conclusion will be changed a little bit.

$$\mu_x + \mu^{03} \ \leq \mu_x^* \text{ and } \mu_y + \mu^{03} \ < \mu_y^* \iff T_y \text{ and } T_x \text{ are positive quadrant dependent.}$$

With $\mu^{03} > 0$, PQD holds even if the force of mortality before and after bereavement are equal. The formal proof of this is given as follows.

If $\Pr(T_x > s|T_y > t)$ is an increasing function of t for each fixed s, then we have

$$\Pr(T_x > s|T_y > t) \geq \Pr(T_x > s|T_y > 0) = \Pr(T_x > s),$$

which immediately implies

$$\Pr(T_x > s, T_y > t) \geq \Pr(T_x > s) \Pr(T_x > t),$$

that is, T_x and T_y are positive quadrant dependent. As a result, it suffices to show that $\Pr(T_x > s|T_y > t)$ is an increasing function of t for each fixed s if $\mu_x^* \geq \mu_x + \mu^{03}$ and $\mu_y^* \geq \mu_y + \mu^{03}$.

When $s \le t$,

$$\begin{aligned}
&\frac{\Pr(T_x > s, T_y > t)}{\Pr(T_y > t)} \\
=\ &\frac{\Pr(T_x > t, T_y > t) + \Pr(s < T_x \le t, T_y > t)}{\Pr(T_x > t, T_y > t) + \Pr(T_x \le t, T_y > t)} \\
=\ &\frac{e^{-\int_0^t \phi(x,y,u)\,du} + \int_s^t e^{-\int_0^\tau \phi(x,y,u)\,du}\ \mu_{x+\tau}\ e^{-\int_\tau^t \mu^*_{y+u}\,du}\,d\tau}{e^{-\int_0^t \phi(x,y,u)\,du} + \int_0^t e^{-\int_0^\tau \phi(x,y,u)\,du}\ \mu_{x+\tau}\ e^{-\int_\tau^t \mu^*_{y+u}\,du}\,d\tau} \\
=\ &1 - \frac{\int_0^s e^{-\int_0^\tau \phi(x,y,u)\,du}\ \mu_{x+\tau}\ e^{-\int_\tau^t \mu^*_{y+u}\,du}\,d\tau}{e^{-\int_0^t \phi(x,y,u)\,du} + \int_0^t e^{-\int_0^\tau \phi(x,y,u)\,du}\ \mu_{x+\tau}\ e^{-\int_\tau^t \mu^*_{y+u}\,du}\,d\tau} \\
=\ &1 - \frac{e^{-\int_0^t \mu^*_{y+u}\,du}\ \int_0^s e^{-\int_0^\tau \gamma_1(x,y,u)\,du}\ \mu_{x+\tau}\,d\tau}{e^{-\int_0^t \phi(x,y,u)\,du} + e^{-\int_0^t \mu^*_{y+u}\,du}\ \int_0^t e^{-\int_0^\tau \gamma_1(x,y,u)\,du}\ \mu_{x+\tau}\,d\tau} \\
=\ &1 - \frac{\int_0^s e^{-\int_0^\tau \gamma_1(x,y,u)\,du}\ \mu_{x+\tau}\,d\tau}{e^{-\int_0^t \gamma(x,y,u)\,du} + \ \int_0^t e^{-\int_0^\tau \gamma_1(x,y,u)\,du}\ \mu_{x+\tau}\,d\tau}.
\end{aligned}$$

where $\phi(x,y,u) = \mu_{x+u} + \mu_{y+u} + \mu^{03}$, and $\gamma_1(x,y,u) = \mu_{x+u} + \mu_{y+u} + \mu^{03} - \mu^*_{y+u}$.

Differentiating both sides with respect to t, we obtain

$$\frac{\partial}{\partial t} \Pr(T_x > s | T_y > t) = C_0(t)(\mu^*_{y+t} - \mu_{y+t} - \mu^{03}),$$

where $C_0(t)$ is a function which is positive for all t. Therefore, for $s \le t$, $\Pr(T_x > s | T_y > t)$ is an increasing function of t for each fixed s if $\mu^*_{y+t} > \mu_{y+t} + \mu^{03}$ for all t.

When $s > t$,

$$\begin{aligned}
&\frac{\Pr(T_x > s, T_y > t)}{\Pr(T_y > t)} \\
=\ &\frac{\Pr(T_x > s, T_y > s) + \Pr(T_x > s, t < T_y \le s)}{\Pr(T_x > t, T_y > t) + \Pr(T_x \le t, T_y > t)} \\
=\ &\frac{e^{-\int_0^s \phi(x,y,u)\,du} + \int_t^s e^{-\int_0^\tau \phi(x,y,u)\,du}\ \mu_{y+\tau} e^{-\int_\tau^s \mu^*_{x+u}\,du}\,d\tau}{e^{-\int_0^t \phi(x,y,u)\,du} + \int_0^t e^{-\int_0^\tau \phi(x,y,u)\,du} \mu_{x+\tau}\ e^{-\int_\tau^t \mu^*_{y+u}\,du}\,d\tau}.
\end{aligned}$$

Differentiating both sides with respect to t, we obtain

$$\begin{aligned}
&\frac{\partial}{\partial t}\Pr(T_x > s|T_y > t) \\
= \;& C_1(t)\Big(-\mu_{y+t}\, e^{\int_t^s \gamma_2(x,y,u)\,du}\Big(1+\int_0^t e^{-\int_\tau^t \gamma_2(x,y,u)\,du}\mu_{x+\tau}\,d\tau\Big) \\
&+\Big(1+\int_t^s e^{-\int_\tau^s \gamma_2(x,y,u)\,du}\mu_{y+\tau}\,d\tau\Big)\Big(\mu_{y+u}+\mu^{03}+\mu^*_{y+u}\int_0^t e^{-\int_\tau^t \gamma_2(x,y,u)\,du}\;\mu_{x+\tau}\,d\tau\Big)\Big),
\end{aligned}$$

where $\gamma_2(x,y,u) = \mu_{x+u} + \mu_{y+u} + \mu^{03} - \mu^*_{x+u}$, and $C_1(t)$ is a function which is positive for all t. The integral $\int_t^s e^{-\int_\tau^s \gamma_2(x,y,u)\,du}\mu_{y+\tau}\,d\tau$ can be evaluated as follows:

$$\begin{aligned}
&\int_t^s e^{-\int_\tau^s \gamma_2(x,y,u)\,du}\mu_{y+\tau}\,d\tau \\
= \;& \int_t^s e^{-\int_\tau^s \gamma_2(x,y,u)\,du}\gamma_2(x,y,\tau)d\tau + \int_t^s e^{-\int_\tau^s \gamma_2(x,y,u)\,du}(\mu^*_{x+\tau}-\mu_{x+\tau}-\mu^{03})d\tau \\
= \;& e^{-\int_t^s \gamma_2(x,y,u)\,du} - 1 + \int_t^s e^{-\int_\tau^s \gamma_2(x,y,u)\,du}(\mu^*_{x+\tau}-\mu_{x+\tau}-\mu^{03})d\tau.
\end{aligned}$$

Substituting the above equation into the expression for $\frac{\partial}{\partial t}\Pr(T_x > s|T_y > t)$ when $s > t$, we obtain

$$\begin{aligned}
&\frac{\partial}{\partial t}\Pr(T_x > s|T_y > t) \\
= \;& \Big(\int_t^s e^{-\int_\tau^s \gamma_2(x,y,u)\,du}(\mu^*_{x+\tau}-\mu_{x+\tau}-\mu^{03})d\tau\Big)\times \\
& \Big(\mu_{y+t}+\mu^{03}+\mu^*_{y+t}\int_0^t e^{-\int_\tau^t \gamma_2(x,y,u)\,du}\mu_{x+\tau}\,d\tau\Big)+(\mu^*_{y+t}-\mu_{y+t})\times \\
& e^{\int_s^t \gamma_2(x,y,u)\,du}\int_0^t e^{-\int_\tau^t \gamma_2(x,y,u)\,du}\mu_{x+\tau}\,d\tau + \mu^{03}e^{\int_s^t \gamma_2(x,y,u)\,du}.
\end{aligned}$$

Since $\mu^*_{y+t} > \mu_{y+t}$ for all y and t, it follows that, for $s > t$, $\Pr(T_x > s|T_y > t)$ is an increasing function of t for each fixed s if $\mu^*_{x+t} > \mu_{x+t} + \mu^{03}$ for all t. In conclusion, positive quadrant dependence holds if $\mu^*_x \geq \mu_x + \mu^{03}$ and $\mu^*_y \geq \mu_y + \mu^{03}$. The proof is still valid with a time variant common shock component μ^{03}_t.

As we see from Figure 2.2, both conditions ($\mu^*_x \geq \mu_x + \mu^{03}$ and $\mu^*_y \geq \mu_y + \mu^{03}$) hold for the estimated Markov model. It would be interesting to see whether PQD holds in the fitted semi-Markov model.

In the semi-Markov model, the multiplicative factors $F_1(t)$ and $F_2(t)$ are strictly greater than 1 for all t. The force of mortality after bereavement is higher than the corresponding force of mortality before bereavement. However, the above formal proof does not apply to this case, since the multiplicative factors are functions of sojourn time in the widowed status. In fact, PQD does not hold, in general, for the semi-Markov model. We see this from Figure 2.7, where the ratio of $\Pr(T_x > t, T_y > s)$ to $\Pr(T_x > t)\Pr(T_y > s)$ for the semi-Markov model is greater than 1 at most points of t and s, but not everywhere.

Examine the ratio of $\Pr(T_x > t, T_y > s)$ to $\Pr(T_x > t)\Pr(T_y > s)$. If PQD exists between random variables T_x and T_y, we will expect the value of ratio is greater than 1 for every pair of t and s. Taking $x = 60$ and $y = 62$ as an example, we plot the values of this ratio in Figure 2.7. We see the Markov model has PQD between T_x and Y_y. However, the ratio from the fitted semi-Markov model is not greater than 1 everywhere.

Take one point as an example. $\frac{\Pr(T_{x=60}>35, T_{y=62}>5)}{\Pr(T_{x=60}>35)\Pr(T_{y=62}>5)} < 1$. From this inequality, it is trivial to derive

$$\Pr(T_{x=60} > 35 | T_{y=62} > 5) < \Pr(T_{x=60} > 35 | 0 < T_{y=62} \leq 5),$$

which means that the probability of a 60-year-old wife living to her 90th birthday is greater if her husband died 30 years before than if her husband died more recently. It means, in some cases, the longer the widowhood period, the greater probability of living longer. The situation may partially be explained by the semi-Markov property which allows for recovery from bereavement. In a sense, the allowance for recovery from bereavement made by the semi-Markov model emphasizes short-term dependence in addition to long-term dependence between the remaining life times of a couple.

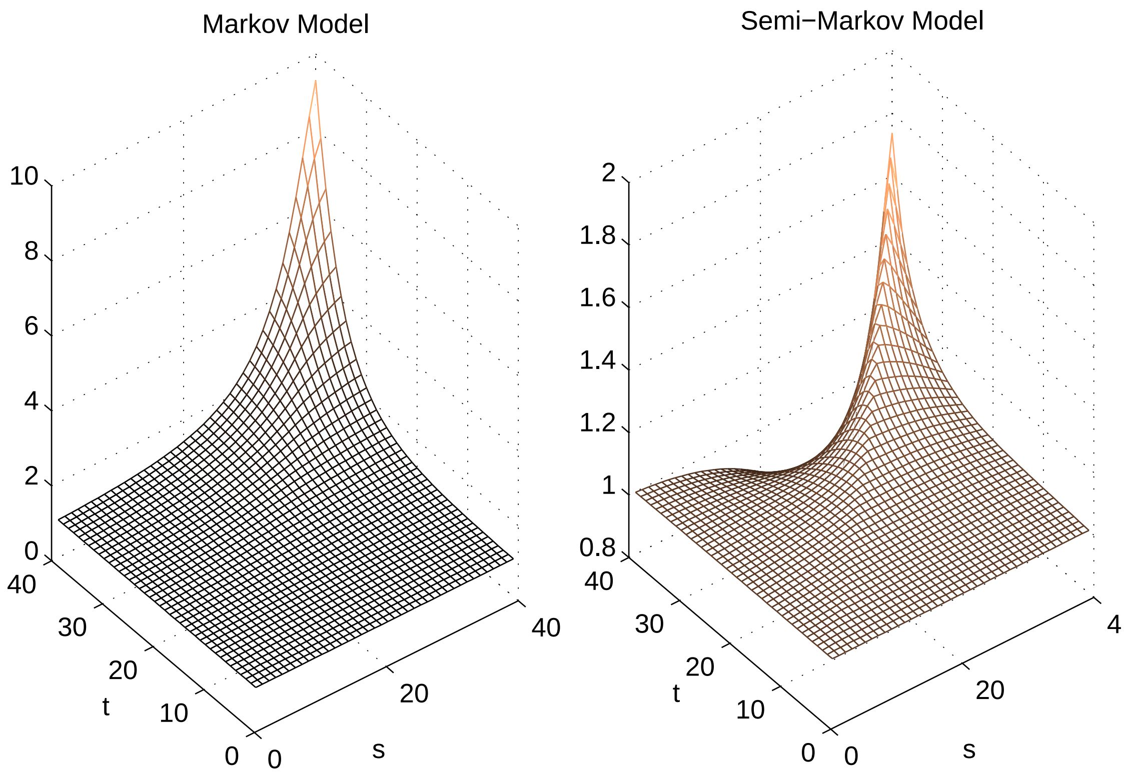

Figure 2.7: Plots of ratios of $\Pr(T_x > t, T_y > s)$ to $\Pr(T_x > t)\Pr(T_y > s)$ for the Markov and semi-Markov model, $x = 60$ and $y = 62$

2.6 Implications for Annuity Values

In Section 2.5, we introduced an important property called positive quadrant dependence, and summarized the dependence structure described by the Markov and semi-Markov models. Note that both Markovian models indicate an increase in mortality after bereavement. However, the implied persistency is different. While the semi-Markov model allows recovery from bereavement, the Markov model assumes that the increase in mortality is permanent. Such a difference has an impact on annuity values, as we now discuss.

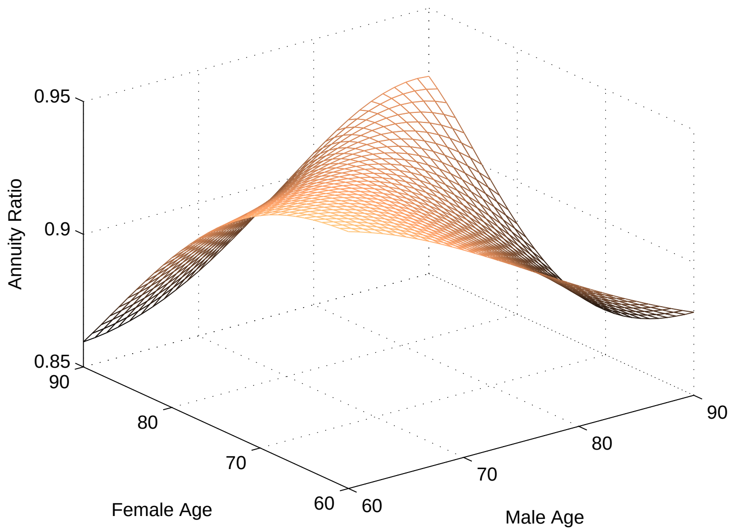

Figure 2.8: Three-dimensional plot of the ratios of Markov-model-based to independent last-survivor annuity values (5% interest is assumed).

First, let us consider the Markov model in Section 2.3. The three-dimensional plot in Figure 2.8 shows the ratios of annuity values using the Markov-model to values assuming independent lifetimes. All ratios in the plot are less than 1, confirming that last-survivor annuities are overpriced if the assumption of independent lifetimes is used. From the three-dimensional plot we also observe that the annuity ratios are lower when the gap between the lives' ages is larger. This observation implies the effect of long-term dependence is more significant when the age gap $|x-y|$ is wider.

The three-dimensional plot in Figure 2.8 is clearly asymmetric. This asymmetry can be explained by the ratios $\mu_x^*/(\mu_x+\mu^{03})$ and $\mu_y^*/(\mu_y+\mu^{03})$. From Figure 2.9 we observe that, the ratios are different from each another, indicating a sex differential in the effect of bereavement on mortality. Such a differential explains why we observe an asymmetry in the plot of annuity ratios.

Next we consider the semi-Markov model from Section 2.4. Figure 2.10 plots

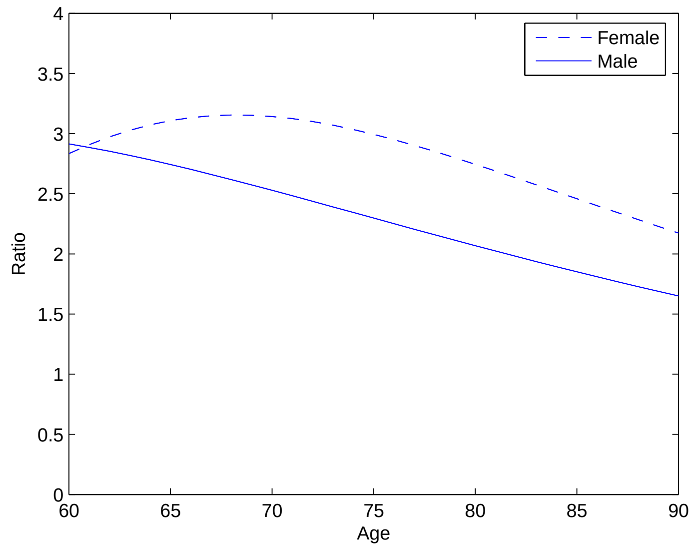

Figure 2.9: Ratios $\mu_x^*/(\mu_x+\mu^{03})$ and $\mu_y^*/(\mu_y+\mu^{03})$ computed from the fitted Markov model.

the ratio of the last-survivor annuity values using the semi-Markov model to those based on the assumption of independence. We observe that these ratios are closer to 1.0; a little lower at most age combinations. We also note that the plot is asymmetric, as we would expect, given the different patterns for males and females of the impact and likelihood of bereavement.

For many annuity contracts, the difference between the ages of a husband and wife is small. We found from our data set that more than 50% of the annuity contracts are sold to couples with an (absolute) age difference of less than 2 years. So the annuity ratios when x and y are close to each other are of particular importance. In Figure 2.11 we plot the annuity ratios for $x = y$, on the basis of both Markov and semi-Markov models. The Markov model results in lower annuity values for younger ages, and higher for more advanced ages, compared with the semi-Markov model. This can be explained by the fact that the semi-Markov model generates higher mortality immediately after bereavement, compared with the Markov model,

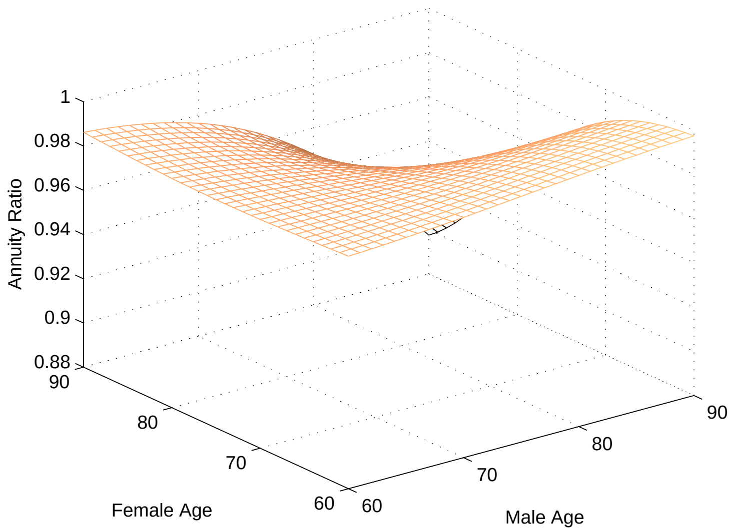

Figure 2.10: Three-dimensional plot of the ratios of semi-Markov-model-based to independent last-survivor annuity values (5% p.y. interest).

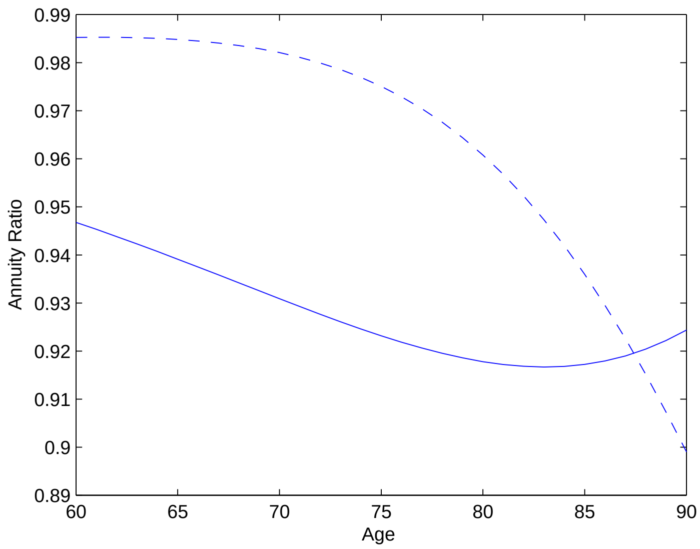

Figure 2.11: Ratios of dependent to independent last-survivor annuity values when $x = y$ (5% interest is assumed).

but later, after a period of recovery, the ultimate rates are lower for widows and widowers under the semi-Markov model. For younger lives, the long term mortality has more impact, so the last survivor may be assumed to live longer, compared with the Markov, so the annuity values are higher. For older lives, the shorter term mortality is more significant as there is no time for long term recovery after bereavement, before extreme old age, and so the last survivor under the semi-Markov model will have heavier overall mortality, resulting in lower annuity values.

2.7 Markovian Models vs. Copulas

Another way to model the dependence between the lifetimes of a husband and wife is to use copulas. A copula $C(u, v)$ is a bivariate distribution function over the unit square with uniform marginals. It allows us to construct bivariate distributions for random variables with known marginal distributions. To illustrate, let us consider two variables X and Y with known marginal distribution functions $F_X(x)$ and $F_Y(y)$, respectively. From $C(u, v)$ and the marginal distribution functions, we can create a bivariate distribution function

$$F_{X,Y}(x, y) = C(F_X(x), F_Y(y)),$$

which introduces a degree of association between the two random variables. A good general introduction to copulas can be found in Frees and Valdez (1998), Klugman et al. (2008) and McNeil et al. (2005).

The use of copulas to model the dependence between two lifetimes has been considered by Frees et al. (1996) and Youn and Shemyakin (2001). Given that their studies are based on the same data as we use in this research, a comparison between their and our findings may offer us some insights into how Markovian approaches are different from copula approaches.

Frees et al. (1996) model the dependence between the lifetimes of a wife and a husband by a Frank's copula, which can be expressed as follows:

$$C(u, v) = \frac{1}{\alpha} \ln\Big(1 + \frac{(e^{\alpha u} - 1)(e^{\alpha v} - 1)}{e^{\alpha} - 1}\Big),$$

where the α is a parameter that controls the degree of association between the two random variables. Values of α less than 1 indicate a positive association, values greater than 1 indicate an inverse interaction, and 1 indicates independence. This copula is symmetric, because $C(u, v) = C(v, u)$.

Using the estimation procedure provided by Frees et al. (1996), we obtain the estimated copula parameter $\hat{\alpha} = -3.64$. The extent of dependence between the two marginal lifetime distributions can be measured by Kendall's τ, which ranges from -1 and 1. We found that the Kendall's τ statistic for this model is 0.36, which indicates a moderate dependence between the two marginal distributions.

On the basis of this copula, we plot in Figure 2.12 the ratios of dependent to independent last-survivor annuity values for a range of ages. There are two significant differences between this plot and the plots based on our Markovian models. First, this plot is roughly symmetric, which means that the marginal distributions of males' and females' lifetimes have approximately the same effect on the ratios. Second, some annuity ratios in this plot are greater than 1. This suggests that under the Frank's copula, the use of the independence assumption may under- or overestimate the value of a last-survivor annuity, depending on the wife's and husband's ages when the annuity is sold.

Youn and Shemyakin (2001) consider a Hougaard copula,[3] which can be written as follows:

$$C(u, v) = \exp(-((-\ln u)^{\theta} + (-\ln v)^{\theta})^{1/\theta}),$$

where $\theta \geq 1$ is a parameter that indicates the extent of dependence. When $\theta = 1$, there is no dependence; and when $\theta \to \infty$, the two underlying random variables

[3]Also known as a Gumbel copula.

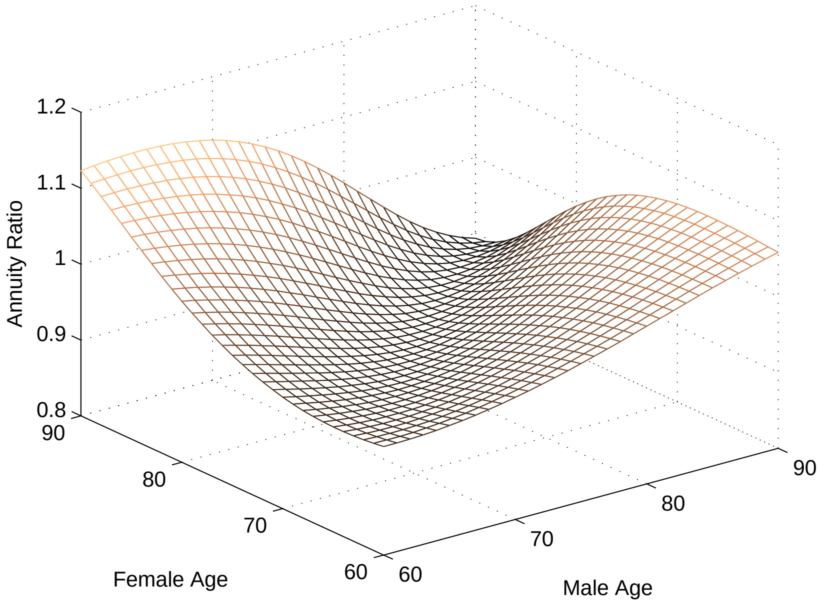

Figure 2.12: Ratios of last-survivor annuity values based on a Frank's copula to last-survivor annuity values based on the independent assumption (5% interest is assumed).

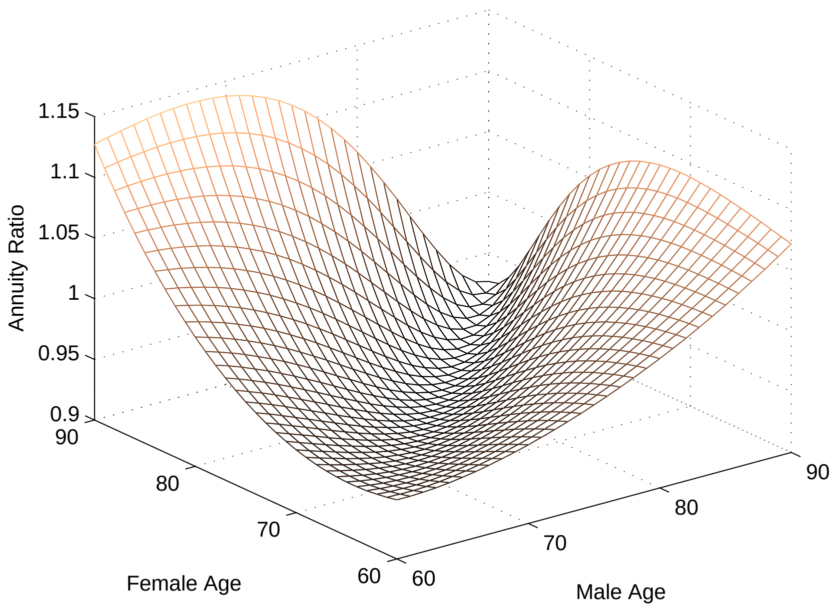

Figure 2.13: Ratios of last-survivor annuity values based on a Hougaard copula to last-survivor annuity values based on the independent assumption (5% interest is assumed).

tend to be perfectly correlated with each other. This copula is also symmetric as $C(u, v) = C(v, u)$. Fitting a Hougaard copula to the annuitants' mortality data, we obtain $\hat{\theta} = 1.57$. The Kendall's τ statistic for this model is 0.36, as for the Frank's copula.

Given the Hougaard copula, we plot in Figure 2.13 the ratios of dependent to independent last-survivor annuity values. This plot shares the same features as that based on the Frank's copula: (1) both plots are roughly symmetric; (2) when the age gap $|x - y|$ is large, annuity ratios in both plots are greater than 1.

In a sense, every joint distribution implicitly contains both a description of individual marginal distribution and a description of their dependence structure. Copulas and Markovian approaches model the dependence between remaining lifetimes of joint lives in different ways. Direct comparison of the dependence structure described by copulas and Markovian models is not easy. but we can gain some insight using the upper tail dependence plots of bivariate remaining lifetimes, T_x and T_y, under each model.

Both the Frank's copula and Hougaard copula are used for the dependence profile of a bivariate age-at-death random vector (X, Y), while we are concerned with the dependence between the future lifetimes of a married couple.

Define the upper tail dependence functions of T_x and T_y as

$$\begin{aligned}\lambda(u) &= \frac{\Pr(T_x > \Psi_1^{\leftarrow}(u), T_y > \Psi_2^{\leftarrow}(u))}{1-u} \\ &= \left(\frac{\Pr(X > x + \Psi_1^{\leftarrow}(u), Y > y + \Psi_2^{\leftarrow}(u))}{\Pr(X > x, Y > y)}\right)/(1-u),\end{aligned}$$

where $\Psi^{\leftarrow}()$ is the inverse of the cdf of $T_{x(y)}$. The function $\lambda(u)$ is the survival probability for (x) given that (y) survives to beyond the $u-$quantile of the marginal distribution (or vice versa). The limiting value $\lambda = \lim_{u \to 1^-} \lambda(u)$ is called the coefficient of the upper tail dependence, provided a limit of $\lambda \in [0, 1]$ exists. If $\lambda = 0$, then T_x and T_y are asymptotically independent in the upper tail; if $\lambda \in (0, 1]$, they are said to have extremal dependence in the upper tail.

The upper tail dependence functions of $T_{x=60}$ and $T_{y=62}$ are graphed in Figure 2.14 for two copulas and two Markovian models. The Hougaard copula has very heavy upper tail dependence compared with all the others. A value for, say, $\lambda(0.999)$ of around 0.5 means that the wife has a 50% chance of surviving to the 99.9th percentile of the lifetime distribution if her husband did so, and only a 0.05% chance of doing so if her husband did not (combining to give a 0.1% chance overall).

Frank's copula and the semi-Markov model do not show upper tail dependence in the limit. Away from the extremes, Frank's copula and the Markov model have similar upper tail dependence, and the semi-Markov model shows the lightest upper tail dependence.

Figure 2.15 gives 5,000 simulated points of $(T_{x=60}, T_{y=62})$ from the Frank's copula, Hougaard copula, Markov model, and semi-Markov model. Two Markovian models exhibit asymmetric dependence structure between T_x and T_y and the simultaneous dependence from the modeled "common shock" effect. Our data set is not large enough for a direct comparison with the upper tail dependence in the data itself.

An advantage of copula approaches is that they involve a smaller number of parameters than Markovian approaches do. To illustrate, let us assume that we use Gompertz law to model the marginal lifetime distributions. Using a Frank's copula or a Hougaard copula, the resulting bivariate distribution will consist of 5 parameters (1 from the copula and 2 from each Gompertz formula). However, if we use the Markov model specified in Section 2.3, then we will be required to estimate 9 parameters (1 from the common shock factor, 4 from the Gompertz formulas for state 0, and 4 from the Gompertz formulas for states 1 and 2), with two further parameters for the semi-Markov version. It is difficult to compare quantitatively the fit of the copula model compared with the Markov model, to assess the benefit from the additional parameters, because the semi-Markov model is fitted using partial likelihood. This means we cannot easily apply likelihood based criteria such as Akaike or Schwartz' Bayes Information.

Although they are more parsimonious than the semi-Markov model, we believe that there are some significant advantages supporting the use of a Markov (or semi-Markov) approach:

1. It is natural in mortality studies to work with the force of mortality rather than the distribution function. The dependence structure in the Markov and semi-Markov models is instantly comprehensible through the impact on the force of mortality of the bereavement event. Using copulas, it is not so transparent. There is no intuitive explanation for why the Hougaard copula fits the data better than Frank's copula, nor is it straightforward to see the

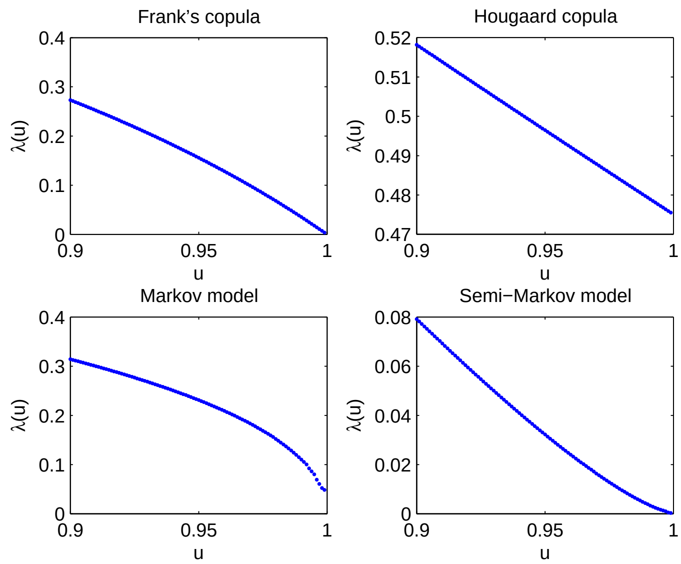

Figure 2.14: Estimates of upper tail dependence functions $\lambda(u)$

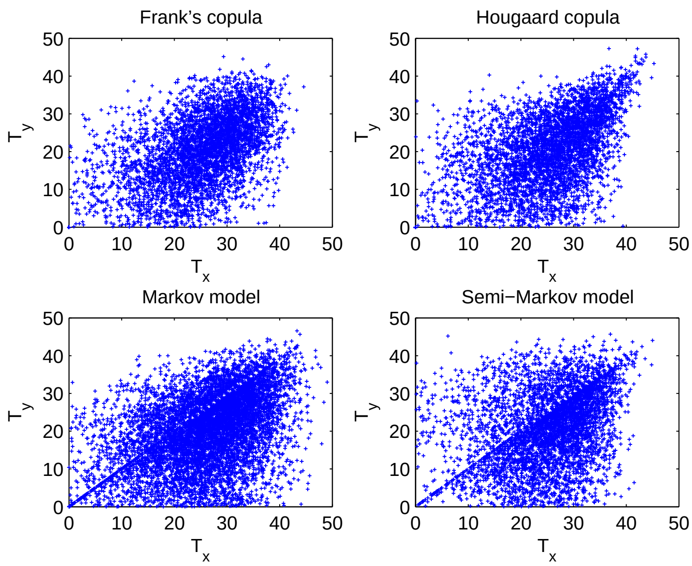

Figure 2.15: Scatterplots of 5,000 simulated points ($T_{x=60}$, $T_{y=62}$) from four fitted models

impact of bereavement on the mortality of the survivor. We cannot tell, from how the copula is formulated, why the plots of annuity ratios (Figures 2.12 and 2.13) are highly symmetric even though the marginal lifetime distributions for males and females are different. In contrast, using Markov models, by considering the transitions between states and the short- and long-term effects of bereavement on mortality, we can easily understand the nature of the dependence between joint lives, and explain intuitively the properties of the surfaces in Figures 2.8 and 2.10.

2. Using a copula results in a dependence structure between lifetimes that is static. By introducing factors $F_1(t)$ and $F_2(t)$ in the semi-Markov model, we allow the impact of the broken-heart syndrome to diminish with time in a way that would be difficult to capture with a copula.

3. According to Sklar's theorem (described in Frees et al. ,1996), there exists a unique copula for any pair of continuous random variables. Therefore, there is a copula behind each Markovian model we presented, although the copula may not have an explicit expression. However, Sklar's theorem does not necessarily mean that the two approaches are equivalent. Suppose that we have single life data that contains information regarding each individual's marital status at the moment of death; we may even know, for widows/widowers the length of their widow(er)hood. Using our Markov approach, this information can be incorporated into the model easily through the log-likelihood function given in equation (2.3.2) or (2.4.3). For copula methods, this information would be insufficient; if we have no information on the age of the spouse, if living, or the age at death, if not, then we cannot use the dependence structure. Only bivariate data can be applied.

   Similarly, we can apply the model to a life knowing only that life's age, sex and marital status (and length of widow(er)hood if relevant). This could improve the pricing and valuation of single life annuities for older married and widowed annuitants.

2.8 Concluding Remarks

That there is dependence between the lifetimes of a husband and wife is intuitive, but the nature of the dependence is not clear from pure empirical observations. Through the Markovian models we fit to the annuitants' mortality data, we have a better understanding of two different aspects of dependence between lifetimes. First, the common shock factor μ^{03} tells us the risk of a catastrophic event that will affect both lives. Second, in the semi-Markov model, factors $F_1(t)$ and $F_2(t)$ measure the impact of spousal death on mortality and the pace that this impact tapers off with time.

We acknowledge the shortcoming that both Markovian models we considered involve a relatively large number of parameters. Given a small volume of data, the parameter estimates tend to have large variances, and a removal or addition of a few data points may affect the maximum likelihood estimates significantly. The lack of data also prohibits us to consider a more sophisticated model specification. With more data, we could possible examine how the common shock factor may vary with the ages of a husband and wife, and perhaps relax the assumption that the force of mortality for an individual is independent of his/her spouse's age.

Very often, death is not the only mode of decrement. Lapses and surrenders, for instance, can affect the pricing for many traditional insurance products. Rather than being static, the intensity rates for lapses and surrenders are known to be dependent on time and policyholders' circumstances (see, e.g., Kim (2005) and Scotchie (2006)). By expanding the state space (i.e., introducing new states), we can easily incorporate multiple modes of decrement into Markovian models. Further, just as we modeled the diminishing impact of the broken heart syndrome, we can explicitly allow the intensity rates for different modes of decrement to vary with, for example, age, gender and sojourn time.

Chapter 3

A Semi-Markov Multiple State Model for Reverse Mortgage Terminations

3.1 Introduction

A reverse mortgage is a loan available to seniors to convert home equity to cash or lifetime income. The elderly borrows money against the value of their home equity and retain full ownership of the home for the whole life of the loan. No repayments of interest or principal are required until the last survivor dies or leaves home (e.g., moves to a long-term care facility) permanently. At that time, the mortgaged home is sold and the proceeds from the sale are used to repay the loan. Usually, a reverse mortgage includes an embedded guarantee which ensures that the borrower will not have to pay back any more than the value of the mortgaged home if it is less than the amount owing on the loan. This guarantee is called the No-Negative-Equity-Guarantee (NNEG) in the UK and the non-recourse provision

(NRP) in the US. We will use the NNEG acronym here to refer to the guarantee whether in the UK or US context.

From the lender's viewpoint, the inclusion of the NNEG is the same as writing a put option on the mortgaged home with the strike price being the outstanding balance of the loan when the loan is terminated. Termination occurs when the property is vacated, or on earlier prepayment. The payoff from the guarantee is determined by the interest rate on the loan, the house price appreciation rate, and the termination date. In this chapter, we specifically focus on the uncertainty associated with the termination date. This piece of uncertainty is critically important, because the longer the loan continues, the more likely that the outstanding balance will exceed the net house value. From an option valuation perspective, the strike price is increasing at the interest rate on the loan, which will be greater than the risk free rate. In this case, the value of the put option is an increasing function of the term. We therefore require a model for reverse mortgage terminations to value the contract, and to better understand the risks and impact of borrower behaviour.

Currently, multivariate regression models are used for reverse mortgage terminations in the US. The model was originally suggested by Chow et al. (2000), and were adapted by the US Department of Housing and Urban Development (2003) and Rodda et al. (2004). These models provide a good fit to the actual termination rates before age 90. However, because they are regression-based, they require substantial economic and non-economic information about the borrowers as input. Another problem of these models is that they assume the probability of loan termination remains level after age 90; this is counter-intuitive, and because there is significant longevity risk inherent in the guarantee, it appears to be a significant weakness. Furthermore, these models do not make explicit allowance for moveouts, health or non-health related.

In this chapter, we utilize and extend the semi-Markov model in Chapter 2, which describes the dependence between the lifetimes of a husband and wife, to model reverse mortgage terminations. The proposed model explicitly incorporates three different modes of termination: death, entrance into a long-term care facility, and voluntary prepayment. In addition, the event-triggered dependency between the lifetimes of a husband and wife is modeled. This feature is of practical importance, because a significant proportion of reverse mortgages are issued to couples (around 40% in the US, according to HUD Report, 2008). The model would also offer a more sophisticated approach to reverse mortgages purchased by widows/widowers.

The rest of this chapter is organized as follows: in Section 2 we provide some background information regarding reverse mortgages in the UK and US, and describe the guarantees embedded in the reverse mortgages sold in these two countries. Section 3 first discusses different modes of reverse mortgage termination; it then describes the semi-Markov multiple state model which we use for modeling these modes of termination. Section 4 applies the model to roll-up mortgages sold in the UK, and examines the importance of each mode of termination to the value of the embedded guarantee. Section 5 applies the proposed model to HECMs sold in the US, and analyzes the adequacy of the guarantee premiums that are currently charged. Finally, Section 6 concludes the work.

3.2 Reverse Mortgages in the UK and US

3.2.1 Contract Design

Reverse mortgages are available in many countries, including the UK, the US, India, Australia, and Japan. In this work, we consider specifically roll-up mortgages in the UK, and Home Equity Conversion Mortgages (HECMs) in the US for illustrative purposes. Below we provide some background information about these two types of reverse mortgage.

In the UK, there are different ways for older home owners to release the equity that has been built up in their home. One way is to use a home reversion, which is not a mortgage but a sale with conditions. Under a home reversion contract, the homeowner sells all or part of his/her property to the provider for an agreed amount, but retains the right to live in the property rent-free until death. Another way is relying on a lifetime mortgage, which permits homeowners to borrow money against the value of their home equity and retain full ownership of the home for the whole life of the loan. Depending on how the loan is taken and repaid, lifetime mortgages are divided into finer classifications. In this work, we consider roll-up mortgages, also called fixed interest lifetime mortgages, which may be regarded as the most straightforward type of lifetime mortgages. Other types of lifetime mortgages include interest-only mortgages and drawdown mortgages. We refer interested readers to the Institute of Actuaries (2005a) for further details.

In a typical roll-up mortgage, the homeowners are advanced a lump sum of money at the outset, and interest on the amount advanced is compounded at a fixed rate. The principal and interest are repaid from the property sale proceeds when the last survivor dies, sells the house, or moves into a long-term care facility permanently. The loan may also be prepaid without a house sale.

Given that the value of the property when the loan is repaid is uncertain, a shortfall in the proceeds from the sale of the home relative to the outstanding mortgage is possible. However, most roll-up mortgages in the UK are sold with the NNEG, which protects the borrower by capping the redemption amount of the mortgage at the lesser of the face amount of the loan and the sale proceeds of the home. Because the interest rate is fixed, borrowers have a financial incentive to repay the loan and refinance when interest rates decline.

In the US, HECMs are the most popular reverse mortgage product, accounting for about 90% of the market share. HECMs are sold to US homeowners who are no younger than 62 years old. In contrast to the roll-up mortgages in the UK, HECMs can be originated with an interest rate adjusted either monthly, annually or fixed. Borrowers choosing a fixed-rate HECM will receive a closed-end loan and will not be able to prepay the loan and draw any additional funds, while borrowers with a variable-rate HECM loan tend to choose payments in the form of a line of credit rather than a single lump sum at the outset of the contract. Usually, variable interest rates are linked to the rate on the one year Constant Maturity Treasury (CMT) bills or the one year London Interbank Offered Rate (LIBOR).

Recently, the US HECMs went through many changes, some of them were driven by the Fannie Mae's policy on purchasing HECM loans. Fannie Mae is the primary investor in the reverse mortgage market. In September 2009, Fannie Mae discontinued purchasing CMT-indexed HECMs, but continued purchasing monthly adjustable-rate LIBOR-indexed HECMs and fixed-rate HECMs. Variable-rate HECM thereafter has shifted to being LIBOR-indexed, and fixed-rate HECM has become more popular. Another move in the HECM program was the introduction, in October 2010, of the HECM Standard and the HECM Saver in replace of the uniform HECM contracts.

All HECMs are insured by the Federal Housing Administration of the US Department of Housing and Urban Development. The purposes of this insurance are twofold. The first is to ensure that borrowers will receive cash advances in a timely manner even if their lender becomes bankrupt. The second is to protect lenders from losses if the price of the mortgaged home falls below the loan balance. In this connection, such insurance may be viewed as an embedded guarantee that is similar to the NNEG in the UK.

Each HECM borrower is required to pay a mortgage insurance premium. The current mortgage insurance premium for the HECM Standard is comprised of a front-end charge of 2% of the maximum claim amount[1] and a monthly charge of

[1]The maximum claim amount is the lesser value of the appraised home equity and the maximum loan limit for the geographical area in which the mortgaged property is located.

1/12 of 1.25% of the outstanding loan balance. The HECM Saver has a front-end charge of 0.01% of the maximum claim amount and the same monthly charge as the HECM Standard, with compensation of lower loan-to-value rates available to borrowers.

3.2.2 The Embedded Guarantee

We let L_t and H_t be the time-t values of the loan and the mortgaged home, respectively. Suppose that the loan is due at time t. If $H_t \geq L_t$, then the lender will obtain the entire value of the loan, L_t, and the balance of the property price passes to the borrowers or their estate. If $H_t < L_t$, then the lender will obtain only H_t through the NNEG. Mathematically, the repayment to the lender is given by

$$\min(L_t, H_t) = L_t - \max(L_t - H_t, 0), \tag{3.2.1}$$

which is the loan value less the payoff from the guarantee. Note that $\max(L_t - H_t, 0)$ is precisely the payoff function for a European put option with a strike price L_t.

The embedded guarantee prevents the borrower from owing more than the value of the mortgaged home when the loan is repaid. The risk that the loan exceeds the home value is borne by the lender (for roll-up mortgages in the UK) or the Federal Housing Administration (for HECMs in the US). This risk is sometimes referred to as the crossover risk.

Let us use a simple roll-up mortgage to illustrate the crossover risk. Assume that the initial value of the mortgaged home is \$300,000 and that the loan to value ratio is 50%. Given the hypothetical trajectory shown in Figure 1 (dashed line), the crossover occurs in about 35 years from now. If the loan is repaid after the crossover, the lender is subject to a loss. If a higher loan-to-value ratio, say 60%, is assumed, the crossover will occur sooner.

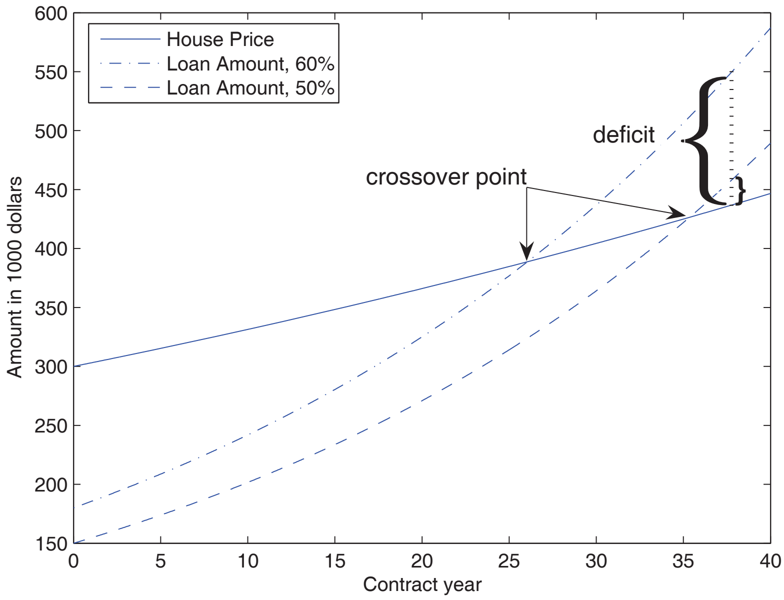

Figure 3.1: Demonstration of the crossover risk.

From Figure 1 we observe that the crossover risk is affected by the loan-to-value ratio and the interest rate at which the loan is accumulated. The risk is also affected by stochastic factors including the appreciation of house prices and the timing of repayment. Some research on the appropriate model for house prices exists (such as Li et al., 2010). The focus of this work, though, is the timing of the repayment date, basing the modeling on the semi-Markov model we describe in the following section.

3.3 A Semi-Markov Multiple State Model

3.3.1 Modes of Termination

A reverse mortgage may terminate for various reasons.

- Death
  Death is a major mode of termination. Its role is particularly important when the homeowner reaches a very high age.

- Entrance to a long-term care (LTC) facility
  Similar to mortality, health-related moveouts plays a predominant role when the homeowner becomes old.

- Moveout for non-health reasons
  A homeowner may move out his/her mortgaged home permanently for a non-health reason, for example, downsizing.

- Refinancing
  A reverse mortgage may be repaid because of a change in the borrower's financial circumstances. In the UK, voluntary prepayments may be associated with refinancing when the market interest rate is lower than the fixed interest rate at which the loan is accumulated. In the US, refinancing may occur if the younger spouse dies, as the maximum loan to property ratio is determined as a function of the age of the younger surviving spouse.

3.3.2 Model Specification

Our model is built upon the semi-Markov multiple state model proposed in Chapter 2 to capture the effect of bereavement. In that model, the force of mortality after bereavement is specified by the following parametric functions:

- widows, age x, t years since bereavement:

$$\tilde{\mu}^f(x,t) = (1 + a^f e^{-k^f t})(\mu^f_{x+t} + \lambda) = F^f(t)(\mu^f_{x+t} + \lambda);$$

- widowers, age y, t years since bereavement:

$$\tilde{\mu}^m(y,t) = (1 + a^m e^{-k^m t})(\mu^m_{y+t} + \lambda) = F^m(t)(\mu^m_{y+t} + \lambda),$$

where λ represents the force of mortality from "common shock" events (events that would cause simultaneous mortality of both lives), and μ^f_{x+t} and μ^m_{y+t} represent the force of mortality for married women and men, respectively, from all causes other than common shock; a^m, a^f, k^m, and k^f are the semi-Markov parameters.

In this chapter, the model is extended to incorporate additional modes of decrement. The complete specification of our proposed model is shown diagrammatically in Figure 3.2. The boxes represent the state process during the lifetime of a reverse mortgage. For example, if the process is still in State 0 at time-t, that means that both husband and wife are alive at t. In States 1 to 4, only one of the joint borrowers is still living at the mortgaged home. In States 5 to 8, the last survivor has either died or permanently left the mortgaged home, and the reverse mortgage is terminated on the transition to any of those states.

The arrows between the states represent the possible transitions, indicating how a reverse mortgage may be terminated. Our model permits transitions from State 0 to 5. This feature captures the simultaneous dependence between joint lifetimes due to common shocks. However, transitions from State 0 to 7 are not permitted, although we can easily relax this assumption if information about long-term care incidence for married couples is available. For convenience we allocate the two non-health related terminations, moveout for non-health reasons and refinancing, to one single state, State 8, labeled as voluntary prepayment.

Note that, State 6 where one spouse is in a LTC facility and one is dead includes two situations. One situation is the borrower has died after his/her spouse moved to a LTC facility; the other situation is that the borrower has moved to a LTC facility after loosing his/her spouse. The model permits transitions from State 3 or 4 to 6 directly, but does not permit transitions from State 3 to 6 via State 1, or from State 4 to 6 via State 2. We assume that lenders will not follow up the borrower's status after he/she moves into a LTC facility. That means we actually do not know the LTC spouse's status on the death of the remaining spouse.

Following Chapter 2, a semi-Markovian approach is used to model the effect of bereavement on mortality. Specifically, μ_x^{15} and μ_y^{25} are obtained by multiplying $F^f(t)$ and $F^m(t)$, respectively, with the corresponding pre-bereavement force of mortality.

Let x and y denote the age of a wife and husband, respectively. The forces of transition from State 0, in which both borrowers are living in the mortgaged home, are denoted by $\mu_{x:y}^{0i}$, for $i = 0, 1, 2, 3, 4, 5, 8$. A slightly different notation is used when there is only one person living in the mortgaged home, as we assume the age of each partner is only relevant while that person is still in the home. For example, we use μ_x^{15} to denote the transition intensity for a widow of age x from State 1 to 5. The age of her husband is not included in the notation as it is irrelevant to the calculations.

Since all transitions are unidirectional, it is straightforward to calculate the transition probabilities by using the Kolmogorov's forward equations. We denote the probability of transition from State 0 to State i at time t by ${}_tp_{x:y}^{0i}$.

We let ${}_{t|}q_{x:y}^{(\tau)}$ be the probability that a reverse mortgage, issued to a wife aged x and a husband aged y at time 0, is in force at time t and will be terminated before time $t+1$. This aggregate one-year termination probability is of particular interest, because the simulation studies in later sections are conducted in annual time steps. We can calculate ${}_{t|}q_{x:y}^{(\tau)}$ by summing the probabilities of transition from State 0 to

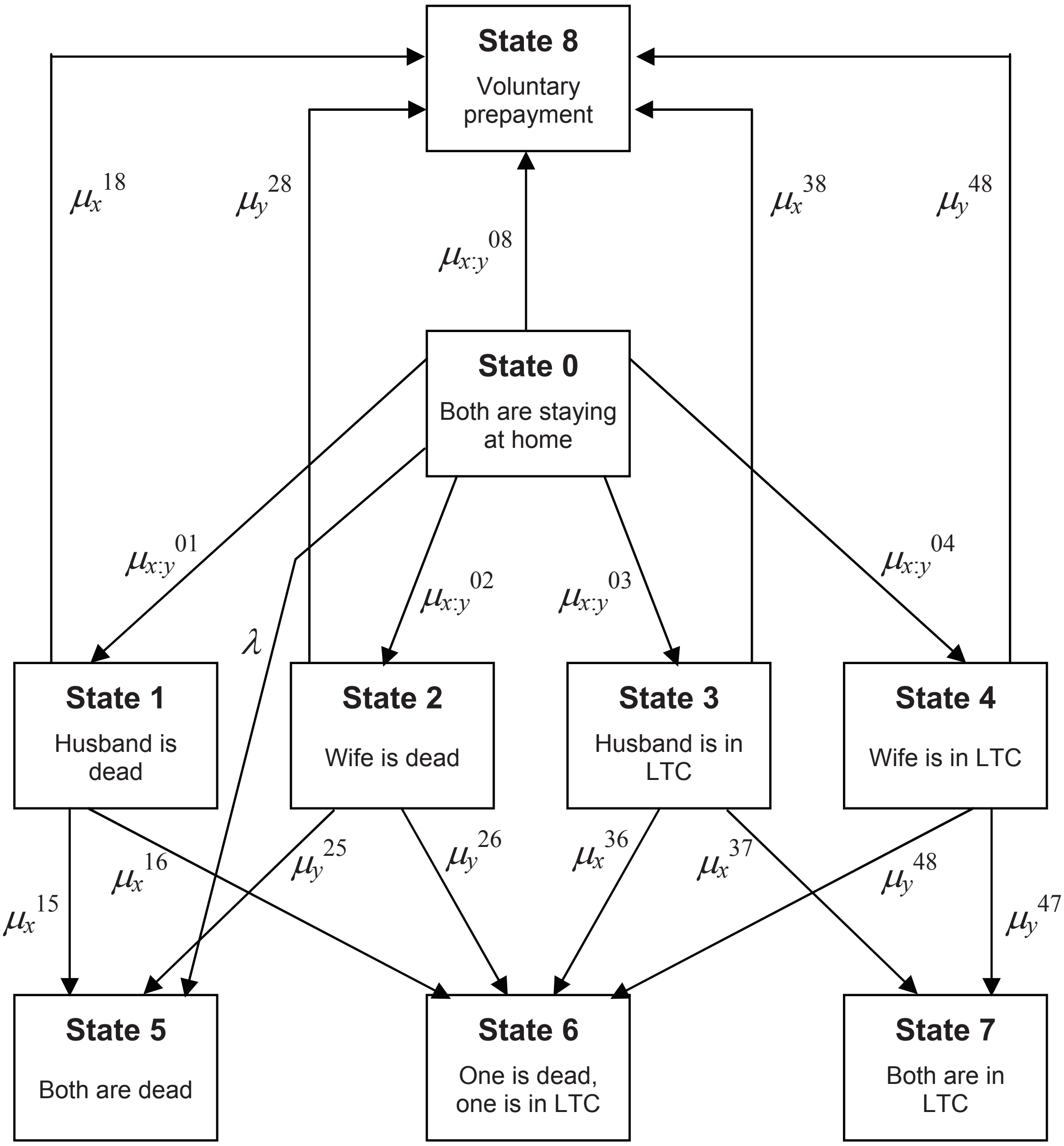

Figure 3.2: A multiple state model for joint-life reverse mortgages.

States 5, 6, 7, and 8; that is,

$$
\begin{aligned}
& {}_{t|}q^{(\tau)}_{x:y} \\
= \; & \int_0^1 {}_tp^{00}_{x:y}\ {}_sp^{00}_{x+t:y+t}\ (\mu^{05}_{x+t+s:y+t+s} + \mu^{08}_{x+t+s:y+t+s})\, ds \\
& + \int_0^1 {}_tp^{01}_{x:y}\ {}_sp^{11}_{x+t}\ (\mu^{15}_{x+t+s} + \mu^{16}_{x+t+s} + \mu^{18}_{x+t+s})\, ds \\
& + \int_0^1 {}_tp^{02}_{x:y}\ {}_sp^{22}_{y+t}\ (\mu^{25}_{y+t+s} + \mu^{26}_{y+t+s} + \mu^{28}_{y+t+s})\, ds \\
& + \int_0^1 {}_tp^{03}_{x:y}\ {}_sp^{33}_{x+t}\ (\mu^{36}_{x+t+s} + \mu^{37}_{x+t+s} + \mu^{38}_{x+t+s})\, ds \\
& + \int_0^1 {}_tp^{04}_{x:y}\ {}_sp^{44}_{y+t}\ (\mu^{46}_{y+t+s} + \mu^{47}_{y+t+s} + \mu^{48}_{y+t+s})\, ds.
\end{aligned}
$$

3.4 Valuing NNEGs in the UK

3.4.1 The Pricing Formula

Let us define the following notation:

- r: the continuously compounded risk-free interest rate;
- g: the continuously compounded rental yield;
- u: the continuously compounded roll-up interest rate on the loan;
- L_t: the value of the reverse mortgage loan at time t, excluding NNEG; $L_t = L_0e^{ut}$;
- H_t: the value of the mortgaged property at time t;
- δ: the average delay in time from the point of home exit until the actual sale of the property.

- c: the cost (as a percentage of the property value) associated with the sale of the property;
- ω: the highest attained age;
- $P(t, S_0, K, r, g, \sigma)$: the time-zero value of a put option on an asset with initial value S_0, volatility σ and dividend yield g; the option matures at time t and has a strike price K.

We use discrete annual time steps and assume that all decrements occur at mid-year, the value of a NNEG written to a wife of age x and a husband of age y can be expressed as

$$\sum_{t=0}^{\omega-\min(x,y)-1} P\left(t+\frac{1}{2}+\delta, H_0(1-c), L_0e^{ut}, r, g, \sigma\right) \ {}_{t|}q^{(\tau)}_{x:y}, \tag{3.4.2}$$

where ${}_{t|}q^{(\tau)}_{x:y}$ is the probability that the loan is terminated between year t to year $t+1$. This probability is calculated on the basis of the semi-Markov multiple state model.

In our illustrations we assume that property prices follow a geometric Brownian motion. The same assumption on house prices is also used by the Institute of Actuaries (2005b). A more realistic econometric model may be used, but we do not explore this aspect of the problem. See Li et al. (2010) for more discussion of house price modeling.

Assuming that the price of the mortgaged property follows a geometric Brownian motion, $P\left(t+\frac{1}{2}+\delta, H_0(1-c), L_0e^{ut}, r, g, \sigma\right)$ can be calculated by the Black-Scholes formula:

$$L_0e^{(u-r)(t+\frac{1}{2}+\delta)}\ N(-d_2)\ -\ H_0(1-c)e^{-g(t+\frac{1}{2}+\delta)}\ N(-d_1), \tag{3.4.3}$$

where $N(\cdot)$ is the cumulative distribution of the standard normal distribution,

Parameter	Assumed value
σ	12%
r	4.75%
g	2%
u	7.5%
δ	0.5 year
c	2.5%
L_0	£30 000

Table 3.1: Assumed values of the parameters in the NNEG pricing formula.

Age of the younger spouse at inception	60	70	80	90
Initial house value	£176 500	£111 000	£81 000	£60 000

Table 3.2: Minimum initial house values. Source: Institute of Actuaries (2005b).

$d_1 = \frac{\ln\left(\frac{H_0(1-c)}{L_0}\right)+(r-u-g+\frac{\sigma^2}{2})\ (t+\frac{1}{2}+\delta)}{\sigma\sqrt{t+\frac{1}{2}+\delta}}$, and $d_2 = d_1 - \sigma\sqrt{t+\frac{1}{2}+\delta}$.

In practice, the roll-up rate u is greater than the risk-free interest rate r. When $u > r$, the value of $P\left(t+\frac{1}{2}+\delta, H_0(1-c), L_0e^{ut}, r, g, \sigma\right)$ will be a strictly increasing function of t. Therefore, decrement assumptions play a critical role in valuing the guarantee.

The assumed parameter values are summarized in Table 3.1. The initial house value H_0 is the minimum assumed starting property value required for a loan of L_0 = £30 000 (see Table 3.2). Note that the value of H_0 is negatively related to the age of the younger spouse at inception – older lives may borrow more, because of the reduced crossover risk.

3.4.2 The Impact of Mortality

Here we examine the impact of mortality assumptions on the value of a NNEG. For now we assume that death is the only mode of decrement.

First we price guarantees written to single lives. Using the joint-life mortality data, we estimate the marginal survival distribution for each gender. Termination probabilities for all durations are calculated accordingly and are substituted into equation (3.4.2) to obtain the guarantee value. The relationship between the value of the guarantee and the age at inception is depicted in Figure 3.3. The guarantee values for females are higher than the corresponding values for males, because female mortality is generally lighter than male mortality.

Next we price guarantees written to joint lives. To examine how the value of a NNEG may be affected by the dependence between two lifetimes, we use two different assumptions. First, we assume independence between the lifetimes of the husband and wife, and use the marginal distributions. Secondly, we use the semi-Markov multiple state model (with States 0, 1, 2, and 5), which is fitted to the same data set. We use simulation (with 100,000 projections) to estimate aggregate termination probabilities, which are then applied to equation (3.4.2) to obtain the guarantee value.

In Figure 3.3 we show the guarantee values simulated with both assumptions. The standard errors in the simulated prices are very small. Take the age combination of $x = 60$ and $y = 62$ as an example, the mean of simulated price is 40.72% with a standard error of 0.0378%. For joint lives, the x-axis in the diagram corresponds to the age of the wife, who is assumed to be two years younger than the husband. From the diagram we observe that the assumption of independence generally leads to an overestimation of NNEG prices. The overestimation is especially significant at high ages. This may be explained by the semi-Markov property, which allows widows and widowers to recover from bereavement. In particular, as younger wid-

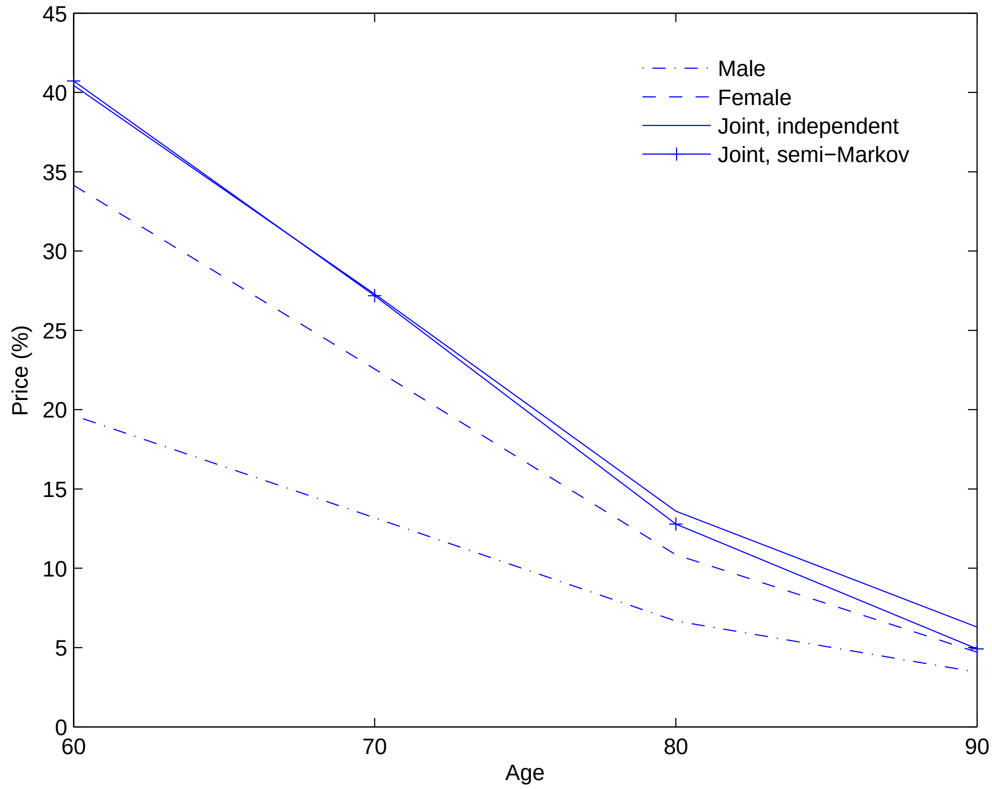

Figure 3.3: Simulated NNEG prices (as a percentage of cash advanced), single lives and joint lives with a 2-year age gap.

owed borrowers have time to recover, the impact of bereavement on the guarantee value is relatively low. The opposite is true for older widowed borrowers.

In assessing the impact of mortality assumption, we do not make allowance for mortality improvements. We focus on the different pricing results from single life models and joint-life models. Most providers in practice usually allow for mortality improvements in pricing the NNEG costs. However, we believe that allowance for mortality improvements will not affect the conclusion of the comparison. Actually, it is not difficult to incorporate mortality improvement assumption into the semi-

Markov model by using a mortality projection model for the force of mortality.

3.4.3 The Impact of Long-Term Care Incidence

We now study the impact of long-term care incidence on the value of a NNEG. The model used here is comprised of States 0 to 7, assuming that mortality and entrance to a long-term care facility are the only two modes of termination.

Generally speaking, people living in long-term care facilities are less healthy than those remaining in their own homes. This means that the introduction of long-term care incidence to the model impacts the at-home mortality. Therefore, besides estimating the additional forces of transition, the forces of transition that are included in the model considered in Section 3.4.2 must also be re-estimated or at least adjusted.

We derive the required forces of transition by a proportional adjustment. Let μ_x^f and μ_y^m be the forces of mortality (from all causes other than common shock) for a wife of age x and a husband of age y, respectively (see Section 3.3.2). We model the forces of transition to a long-term care facility for males and females by $\rho_y^m \mu_y^m$ and $\rho_x^f \mu_x^f$, respectively, where ρ_y^m and ρ_x^f are constants. We model the knock-on impact by assuming males and females at-home mortality are obtained by multiplying their forces of mortality (from all causes other than common shock) with constant proportional factors θ_y^m and θ_x^f, respectively.

We assume that bereavement has an effect on the forces of transition from States 1 and 2. The forces of transition from these states are obtained on the basis of the semi-Markov functions $F^f(t)$ and $F^m(t)$ defined in Section 3.3.2. For instance, the force of transition from State 1 to 5 for a widow of age x, s years after bereavement, is given by $F^f(s)(\theta_x^f \mu_x^f + \lambda)$. The expressions for all forces of transition in the model are provided in Table 3.3.

When both borrowers are alive
$\mu_{x:y}^{01} = \theta_y^m \mu_y^m$
$\mu_{x:y}^{03} = \rho_y^m \mu_y^m$
$\mu_{x:y}^{02} = \theta_x^f \mu_x^f$
$\mu_{x:y}^{04} = \rho_x^f \mu_x^f$
$\mu_{x:y}^{05} = \lambda$
When one of the borrowers is dead (s years after bereavement)
$\mu_x^{15} = F^f(s)(\theta_x^f \mu_x^f + \lambda)$
$\mu_x^{16} = F^f(s)(\rho_x^f \mu_x^f)$
$\mu_y^{25} = F^m(s)(\theta_y^m \mu_y^m + \lambda)$
$\mu_y^{26} = F^m(s)(\rho_y^m \mu_y^m)$
When one of the borrowers is in a long-term care facility
$\mu_x^{36} = \theta_x^f \mu_x^f + \lambda$
$\mu_x^{37} = \rho_x^f \mu_x^f$
$\mu_y^{46} = \theta_y^m \mu_y^m + \lambda$
$\mu_y^{47} = \rho_y^m \mu_y^m$

Table 3.3: Transition intensities for the model in Section 3.4.3.

To estimate the proportional factors for the move from an aggregate model to the at-home/in LTC split model, we make use of the findings in the Equity Release Report of the Institute of Actuaries (2005b). In the report, the following ratios are provided:

- L_y^m: long-term care incidence rate to population mortality rate (males, age y);
- L_x^f: long-term care incidence rate to population mortality rate (females, age

x);

- A_y^m: at-home mortality to population mortality rate (males, age y);
- A_x^f: at-home mortality to population mortality rate (females, age x).

Given these ratios, the proportional factors are calculated by solving the following equations numerically:

$$
\begin{aligned}
L_y^m &= \frac{\int_0^1 e^{-\int_0^s (\theta_y^m+\rho_y^m)\mu_{y+u}^m+(\theta_x^f+\rho_x^f)\mu_{y+u}^f+\lambda du}\ \rho_y^m \mu_{y+s}^m\, ds}{\int_0^1 e^{-\int_0^s \mu_{y+u}^m+\mu_{y+u}^f+\lambda du}\ (\mu_{y+s}^m+\lambda)\, ds}; \\
A_y^m &= \frac{\int_0^1 e^{-\int_0^s (\theta_y^m+\rho_y^m)\mu_{y+u}^m+(\theta_x^f+\rho_x^f)\mu_{y+u}^f+\lambda\, du}\ (\theta_y^m \mu_{y+s}^m+\lambda)\, ds}{\int_0^1 e^{-\int_0^s \mu_{y+u}^m+\mu_{y+u}^f+\lambda du}\ (\mu_{y+s}^m+\lambda)\, ds}; \\
L_x^f &= \frac{\int_0^1 e^{-\int_0^s (\theta_y^m+\rho_y^m)\mu_{x+u}^m+(\theta_x^f+\rho_x^f)\mu_{x+u}^f+\lambda du}\ \rho_x^f \mu_{x+s}^f\, ds}{\int_0^1 e^{-\int_0^s \mu_{x+u}^m+\mu_{x+u}^f+\lambda du}\ (\mu_{x+s}^f+\lambda)\, ds}; \\
A_x^f &= \frac{\int_0^1 e^{-\int_0^s (\theta_y^m+\rho_y^m)\mu_{x+u}^m+(\theta_x^f+\rho_x^f)\mu_{x+u}^f+\lambda\, du}\ (\theta_x^f \mu_{x+s}^f+\lambda)\, ds}{\int_0^1 e^{-\int_0^s \mu_{x+u}^m+\mu_{x+u}^f+\lambda du}\ (\mu_{x+s}^f+\lambda)\, ds}.
\end{aligned}
$$

Note that in equations above, it is assumed that the husband and wife are of the same age. In Table 3.4 we show the estimated proportional factors for ages ≤ 70, 80, 90, and ≥ 100. For ages 71-79, 81-89 and 91-99, the proportional factors are obtained by linear interpolation.

We then use the eight-state model to simulate prices of NNEGs written to joint-borrowers. To examine the impact of long-term care incidence on the NNEG prices, we consider the following four cases:

- Case 1: Central estimate of ρ.
- Case 2: ρ is increased by 30%, other things equal.
- Case 3: ρ is decreased by 30%, other things equal.

Age	ρ^m	θ^m	ρ^f	θ^f
≤ 70	0.05	0.97	0.10	0.95
80	0.07	0.97	0.20	0.90
90	0.15	0.94	0.33	0.85
≥ 100	0.22	0.94	0.46	0.80

Table 3.4: Estimated values of ρ^m, θ^m, ρ^f, and θ^f.

- Case 4: No long-term care incidence.

Table 3.5 shows the NNEG prices under the four assumptions about long-term care incidence. It is assumed in the calculations that the husband is two years older than the wife. From Table 3.5 we observe that, when long-term care incidence is introduced to the model, the resulting guarantee value, expressed as a percentage of cash advanced, will decrease in absolute value by 4.5% and 1% for young-old borrowers and old-old borrowers, respectively. This indicates that entrance to a long-term care facility is a significant mode of termination, and that it requires adequate modeling.

In expressing long-term care incidence rates as a fraction of the corresponding base mortality rates, we have implicitly assumed that the desire to move to a long-term care facility is determined by age-related health conditions only. However, other factors, for example, the quality of long-term care facilities, may also affect one's desire to move. If data permits, such factors may be incorporated into the model by modifying the expressions for the forces of transition accordingly.

3.4.4 The Impact of Voluntary Prepayment

We now consider the full nine-state model, which incorporates three modes of termination including voluntary prepayment. A roll-up mortgage may be prepaid

	NNEG prices (standard errors in brackets)			
Age of wife at inception	60	70	80	90
Case 1	36.19%	23.6%	10.51%	3.85%
	(0.031%)	(0.020%)	(0.013%)	(0.006%)
Case2	34.24%	22.08%	9.58%	3.42%
	(0.032%)	(0.019%)	(0.01%)	(0.005%)
Case 3	38.46%	25.38 %	11.63%	4.39%
	(0.027%)	(0.022%)	(0.013%)	(0.007%)
Case 4	40.72%	27.18%	12.79%	4.92%
	(0.038%)	(0.026%)	(0.014%)	(0.008%)

Table 3.5: Simulated NNEG prices (as a percentage of cash advanced) under different assumptions about long-term care incidence.

voluntarily due to a non-health related moveout or refinancing.

There is little information about non-health related moveouts available in the public domain. In our calculations, we use the assumptions made by the Institute of Actuaries (2005b), which are summarized in Table 3.6. The initial selection effect is modeled by using lower rates for early contract years. We assume further that, after the fifth contract year, the rate of non-health related moveouts is reduced by 0.25% if both lives are still staying in the mortgaged property. This assumes that a borrower may have a greater desire to move out after his/her spouse has died.

The assumptions of prepayment due to non-health related moveouts are set very approximately, as stated in the Institute of Actuaries (2005b). According to the Safe Home Income Plans (SHIP) voluntary code, an equity release mortgage is transportable on moving home provided that the loan-to-value is lower than the maximum allowed. The assumption for prepayment rates may need to take

Contract year	Prepayment rate
1	0.0%
2	0.0%
3	0.15%
4	0.3%
5	0.3%
6+	0.75%

Table 3.6: Allowances for prepayments arising from changes in personal circumstances expressed as a percentage of in force contracts. Source: Institute of Actuaries (2005b).

into account this fact. However, without reliable data, we do not propose further adjustment. Equity release product providers are encouraged to make their own judgement according to firm experience.

As the roll-up interest rate is usually fixed, borrowers may have a financial incentive to refinance when there is a fall in market interest rates. Here we use the remortgaging rates assumed by Hosty et al. (2007), which they claim to be suitable for reverse mortgages sold at a time when interest rates are relatively low but not at the bottom of the market.[2] The assumed remortgaging rates are displayed in Table 3.7.

We incorporate these voluntary prepayment rates into the full nine-state model to price a NNEG. To examine the impact of the assumption, we consider four cases:

[2]More specifically, Hosty et al. (2007) claim that the remortgaging rates in Table 3.7 might be considered best estimates for a provider with robust early repayment charges distributing a flexible product at competitive but not market-leading rates through a broker distribution channel at a time when interest rates are relatively low but not bottom of the market (say headline rates of 6.5% p.a.).

Contract year	Remortgaging rate
1-2	1.0%
3	2.0%
4-5	2.5%
6-8	2.0%
9-10	1.0%
11-20	0.5%
21+	0.25%

Table 3.7: Assumed remortgaging rates. Source: Hosty et al. (2007).

- Case 1: Central rates of voluntary prepayment from Table 3.6 and 3.7.
- Case 2: The rates of voluntary prepayment are increased by 30%, other things equal.
- Case 3: The rates of voluntary prepayment are reduced by 30%, other things equal.
- Case 4: No voluntary prepayment.

The simulated NNEG prices for all four cases are shown in Table 3.8. It is assumed in the calculations that the husband is two years older than the wife. Note that the prices for Case 4 are taken from Section 3.4.3. From Table 3.8 we observe that the NNEG prices drop significantly when voluntary prepayment is taken into account. The effect of voluntary prepayment is even more significant than the effect of long-term care incidence.

The analysis shows that, using reasonable assumptions for the termination model, the impact of health and non-health related move-outs is very significant,

	NNEG prices (standard errors in brackets)			
Age of wife at inception	60	70	80	90
Case 1	24.45%	17.09%	8.16%	3.27%
	(0.031%)	(0.021%)	(0.010%)	(0.005%)
Case2	21.72%	15.51%	7.56%	3.12%
	(0.027%)	(0.020%)	(0.011%)	(0.005%)
Case 3	27.5%	18.83%	8.80%	3.44%
	(0.03%)	(0.022%)	(0.012%)	(0.005%)
Case 4	36.19%	23.6%	10.51%	3.85%
	(0.031%)	(0.020%)	(0.013%)	(0.006%)

Table 3.8: Simulated NNEG prices (as a percentage of cash advanced), under different assumptions about voluntary prepayments

and that the semi-Markov model offers a transparent and flexible approach to the modeling of terminations.

We apply the model to valuing the contracts by allow progressively for each mode, in order to present the significance of each mode of termination. The flexibility and transparency of the semi-Markov termination model guarantee that further extension can be developed in practice, for example, including initial and ongoing expenses for assessing capital requirement.

3.5 Valuing HECM Insurance Premiums in the US

3.5.1 The Pricing Formula

In this section, we use the model and parameters developed above to analyze the premium that US HECM borrowers pay for the NNEG. All HECMs sold in the US are insured by the Federal Housing Administration, with premiums paid by the borrowers. For a loan written to a wife of age x and a husband of age y, the time-0 value of the mortgage insurance can be expressed as follows:

$$\sum_{t=0}^{\omega-\min(x,y)-1} P\left(t+\frac{1}{2}+\delta, H_0(1-c), L_0e^{ut}, r, g, \sigma\right) {}_{t|}q_{x:y}^{(\tau)} \tag{3.5.4}$$

where L_0 is the amount borrowed, including the origination costs and front-end mortgage insurance premium; H_0 is the adjusted property value when the loan is originated. We assume that H_0 is always smaller than the HECM maximum loan limit for the area in which the property is located. Other symbols in the above expression carry the same meaning as they do in Section 3.4.1. It is assumed in our calculations that all loans terminate at mid-year.

On October 4, 2010, the Federal Housing Administration introduced the HECM Saver to give participants a borrowing option with a lower front-end insurance premium. The purpose is to offer a loan with lower origination costs for mortgagors who want to borrow a smaller amount than that available with a HECM Standard. The HECM Standard is to distinguish and separate the HECM Saver from the initial HECM program. In addition, The FHA raised the monthly insurance premium to 1.25% annually of the outstanding loan balance, for both the HECM Standard and the HECM Saver. The rate had been 0.5%.

The HECM Saver is a positive development, but the ongoing insurance premium on all the HECM loans has been raised. Recent decreases in home values during the economic recession lead to an opinion that the FHA has been undercharging for the risk associated with the program. The new borrowing option complies with the suggestion that the front-end mortgage insurance premium of 2% has much room to cut off the high origination cost, for example in Caplin (2002). However, the expected total mortgage insurance premium charged from the borrows increases, regardless of that fact the trend of house prices will be reverse in the wake from the 2007-09 global financial crisis.

Let α be the fair front-end charge, expressed as the a percentage of the home value. The time-0 value of the total mortgage insurance premium can be expressed as

$$\alpha\ H_0 + 0.0125 \times \frac{1}{12} \sum_{k=0}^{\omega-\min(x,y)-1} \left({}_{k|}q_{x:y}^{(\tau)} \sum_{t=1}^{12(k+\frac{1}{2})} L_0 e^{\frac{(u-r)t}{12}} \right). \tag{3.5.5}$$

By actuarial equivalence, we have the following formula for calculating α:

$$\begin{aligned} \alpha \quad = \quad & \frac{1}{H_0} \sum_{k=0}^{\omega-\min(x,y)-1} \left\{ {}_{k|}q_{x:y}^{(\tau)} \left[P\left(k + \frac{1}{2} + \delta, H_0(1-c), L_0 e^{ut}, r, g, \sigma\right) \right.\right. \\ & \left.\left. -0.0125 \times \frac{1}{12} \sum_{t=1}^{12(k+\frac{1}{2})} L_0 e^{\frac{(u-r)t}{12}} \right] \right\}. \end{aligned} \tag{3.5.6}$$

Here we assume again that house prices follow a geometric Brownian motion. In equation (3.5.6) of deriving a fair front-end charge by actuarial equivalence, we ignore the value of the other purpose of the HECM's insurance, that is, protection against the lender's default. Presently, there is no record that a reverse mortgage provider has discontinued its commitment to borrowers, but we acknowledge that some value should be attributed to this part of insurance. However, there is no reliable information for assessing a fair charge for it. We believe that the premium charged for it will be marginal relative to the cost of the non-recourse guarantee. We simply ignore this part of value and treat it as a by-product of the non-recourse guarantee.

Age of the younger spouse at inception	65	70	75	80	85	90
Variable-Rate HECM Standard	0.535	0.572	0.605	0.641	0.684	0.716
Fixed-Rate HECM Standard	0.642	0.671	0.698	0.724	0.754	0.776
Variable-Rate HECM Saver	0.430	0.455	0.478	0.503	0.531	0.552
Fixed-Rate HECM Saver	0.536	0.552	0.566	0.580	0.598	0.610

Table 3.9: Principal limit factors for HECM loans in 90803 Los Angeles.

When H_0 is smaller than the HECM maximum loan limit, the maximum amount that a borrower can borrow is the product of H_0 and the applicable principal limit factor, f, which depends on the expected interest rate and the borrower's age at inception. For example, if the applicable principal limit factor is 0.551 and the value of the mortgaged property is \$100,000 at inception, then the maximum amount that can be borrowed would be \$55,100, including the origination costs paid on the borrower's behalf.

Table 3.9 displays the principal limit factors for different inception ages of the younger spouse and different HECM contracts. These factors are obtained from the online reverse mortgage calculator provided on Wells Fargo Bank's website[3] on 22 April 2011. They are applicable to HECM Standard loans and HECM Saver loans with monthly adjustable one-year- LIBOR-indexed interest rate and fixed-rate respectively. The calculations were based on zip code 90803 Los Angeles, which has the highest number of HECM loans. Our calculations will be based on these principal limit factors.

Borrowers desiring a fixed-rate HECM loan will receive a closed-end loan of 100% of initial principle limit. They will not be able to prepay the loan and draw any additional funds. However, most borrowers choosing variable-rate HECM loans

[3]https://www.benefits-mortgage.com/calculator/entry.do?linkType=mps.

do not use the full principal limit. According to the HECM Actuarial Review Report (2010) provided by the IBM Global Business Services, the line of credit option accounted for 91% of the 2009 book-of-business and 93% of the 2010 book-of-business. Although most borrowers of variable-rate HECM loans use a sizeable amount of their lines of credit at the inception of the contract, they usually do not exhaust the line of credit during the term of the loan.

If the borrower withdraws $100\phi\%$ of the maximum amount that he/she can borrow, then L_0 in equation (3.5.6) can be expressed as $\phi f H_0$. Plugging equation (3.4.3) into equation (3.5.6) and replacing L_0 by $\phi f H_0$, we can show easily that α does not depend on H_0 if we assume house prices follow a geometric Brownian motion.

In our calculations, the following three scenarios are considered: $\phi = 1$, $\phi = 0.9$, and $\phi = 0.8$. The assumed values for other model parameters are described below.

- The house price volatility, σ, is 12%. This is the historical volatility of the Quarterly Purchase-only House Price Index from 1991 to 2008 provided by Office of Federal Housing Enterprise Oversight.

- The continuously compounded risk-free interest rate, r, is 3.4285%. This equivalent to the average annual rate of 3.488% on 10-year US Treasury Bills in April as of 22 April 2011, obtained from the website of the US Department of Treasury.[4]

- The continuously compounded roll-up rate, u, is 4.94%, which is equivalent to an annual interest rate of 5.06% for the Wells Fargos Bank' s fixed-rate HECM loans in April 2011.

As in pricing the UK NNEGs, we assume that $g = 2\%$, $c = 2.5\%$, and $\delta = 0.5$.

[4]http://www.ustreas.gov/offices/domestic-finance/debt-management/interest-rate/yield.shtml.

3.5.2 Decrement Assumptions

When we apply the semi-Markov multiple state model to HECM insurance premiums, the following assumptions are used:

- Mortality and long-term care incidence

  Central assumptions about mortality and long-term care incidence are the same as those used in Section 3.4.

- Refinancing

  In contrast to roll-up mortgages sold in the UK, fixed-rate HECM loans are close-ended, which means borrowers will not be able to refinance the loan. Meanwhile, borrowers of variable-rate HECMs loans have rather low incentive to refinance. The current market practice uses the age of the younger spouse to determine the principal limit of the loan. Therefore, a borrower choosing a variable-rate HECM loan may remortgage when his/her spouse dies, as that may lead to an increase in the principal limit. However, such a refinancing arrangement would not affect the existing mortgage insurance. In this connection, refinancing is not considered when we compute HECM insurance premiums.

- Non-health-related moveouts

  In a US-specific study on mobility, Zhai (2000) argued that mobility is a combined result of increasing health-related and declining non-health-related moveouts, plus a static rate. Zhai went on to derive a U-shaped curve of mobility rates, which says that the rate of mobility (for both health and non-health related reasons) declines from 4.8% at age 60 to 3.2% at age 80, and then rises slowly to about 4.2% at age 105.

  We set our assumptions about non-health related moveouts on the basis of the U-shaped curve provided by Zhai (2000). In particular, since mobility at younger ages is mostly non-health related, we assume that the rate of non-health related moveout is 4% from age 60 to 65. This rate is linearly reduced to 1% for age 90 and above, when mobility is mostly health-related. Following Zhai (2000), we discount the mobility rate by 50% when both borrowers are living in the mortgaged property.

We further model the effect of selection by applying a 80% discount to the mobility rate during the first contract year. The discount is reduced linearly to zero during the tenth contract year.

Having established the decrement assumptions, we can simulate the survival curve for a HECM contract. From the survival curve we can tell the probability that the HECM contract is still in force at a certain time after inception.

In Figure 3.4 we show the survival curves for a HECM contract written to a 62-year-old wife and a 64-year-old husband, when different modes of termination are incorporated into the model. We observe from the diagram that long-term care incidence and voluntary prepayments (non-health related moveouts) would significantly reduce the survival probabilities, thereby shortening the expected duration of the HECM contract.

3.5.3 The Estimated Premiums

In Table 3.10 we display the estimated values of α (the front-end charge as a percentage of house value) in fixed-rate HECMs and various scenarios of variable-rate HECMs. The calculations are based on 100,000 simulations from the multiple state model and the assumption that the husband is two years older than the wife.

We observe that the estimated fair front-end charge for a variable-rate HECM loan will drop significantly if the joint borrowers do not exhaust the available line of credit when the loan is originated. When 80% of the principal limit is advanced,

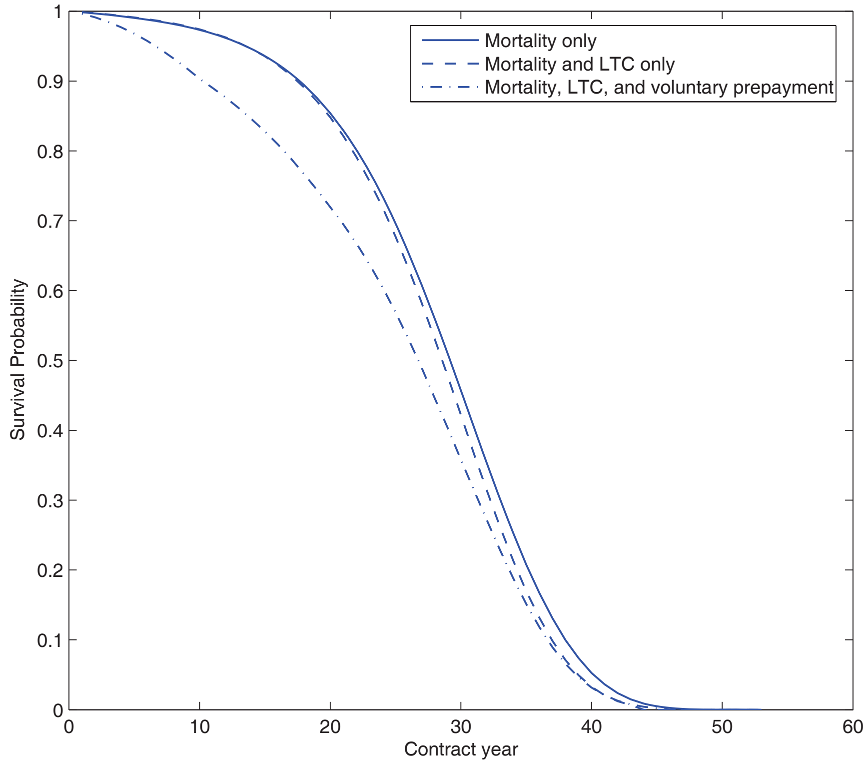

Figure 3.4: Estimated survival curves for a HECM contract written to a 62-year-old wife and a 64-year old husband.

the required front-end charges are negative in almost all cases we consider. This means that the monthly premium itself is more than enough to cover the cost of the embedded guarantee. The average drawdown rate of a variable-rate HECM loan is below 60%, according to, for example, the HECM Actuarial Review Report (2010). This indicates that variable-rate HECM loans are generally sustainable.

However, fixed-rate HECM program is a different story. Borrowers draw 100% of the principal limit because of the close end loan. The principal limit factors, which are determined according to borrowers' age and the expected interest rate, are very high in the current economic situation of low interest rates. According to the model and the assumptions described, the charged front-end premium, 2% in a HECM Standard and 0.01% in a HECM Saver, is insufficient for the risk associated with the non-recourse guarantee. Attention is called for this situation, as 69% of HECMs contracts originated in fiscal year 2010 are fixed-rate loans compared to 11% in 2009 according to the HECM Actuarial Review Report (2010).

The simulation results point to three other important conclusions. First, since the estimated front-end charge decreases with the borrowers' ages, older couples are subsidizing younger couples under the current premium structure. The problem could be ameliorated by using an age-dependent front-end charge as this, according to our results, does not seem to undermine the long-term financial soundness of the HECM insurance fund. Alternatively, lower principal limit factors should be offered to younger borrowers. We expect that this change will be profound, since the US reverse mortgage market has seen a shift to younger elderly homeowners (Bishop and Shan, 2008).

Secondly, The HECM Saver program is a positive development. The lower front-end premium charge makes HECMs more attractive to potential borrowers, and the lower principal limit factors make the HECM Saver much safer to the mortgage insurance fund than the HECM Standard. The monthly premium of 1.25% may be higher than a fair level. However, we note that the GBM assumption for the U.S.

Age of wife at inception	65	70	75	80	85	90
Front-end premium for Variable-rate HECM Standard loans						
$\phi = 1.0$	6.21%	5.52%	4.36%	3.28%	2.57%	1.58%
	(0.006%)	(0.006%)	(0.005%)	(0.004%)	(0.004%)	(0.003%)
$\phi = 0.9$	2.91%	2.23%	1.25%	0.43%	-0.03 %	-0.55 %
	(0.005%)	(0.004%)	(0.004%)	(0.003%)	(0.003%)	(0.002%)
$\phi = 0.8$	0.08%	-0.51%	-1.23%	-1.73%	-1.85%	-1.87%
	(0.004%)	(0.003%)	(0.003%)	(0.002%)	(0.001%)	(0.001%)
Front-end premium for Variable-rate HECM Saver loans						
$\phi = 1.0$	0.18%	-0.62%	-1.45%	-1.99%	-2.17%	-2.12%
	(0.004%)	(0.003%)	(0.002%)	(0.002%)	(0.001%)	(0.001%)
$\phi = 0.9$	-1.72 %	-2.35%	-2.89%	-3.1%	-2.94%	-2.54%
	(0.003 %)	(0.002%)	(0.002%)	(0.001%)	(0.001%)	(0.001%)
$\phi = 0.8$	-3.24%	-3.66%	-3.89%	-3.78%	-3.33%	-2.66%
	(0.002%)	(0.002%)	(0.001%)	(0.001%)	(0.001%)	(0.001%)
Front-end premium for Fixed-rate HECM Standard loans						
$\phi = 1.0$	13.96 %	12.3%	10.21 %	7.94%	6.04%	4.07%
	(0.008%)	(0.008%)	(0.007%)	(0.006%)	(0.006%)	(0.004%)
Front-end premium for Fixed-rate HECM Saver loans						
$\phi = 1.0$	6.27%	4.31%	2.29%	0.55%	-0.57%	-1.28%
	(0.006%)	(0.005%)	(0.004%)	(0.003%)	(0.002%)	(0.001%)

Table 3.10: Simulated front-end premiums (standard errors in brackets) for fixed-rate HECM loans and variable-rate HECM loans with different values of ϕ.

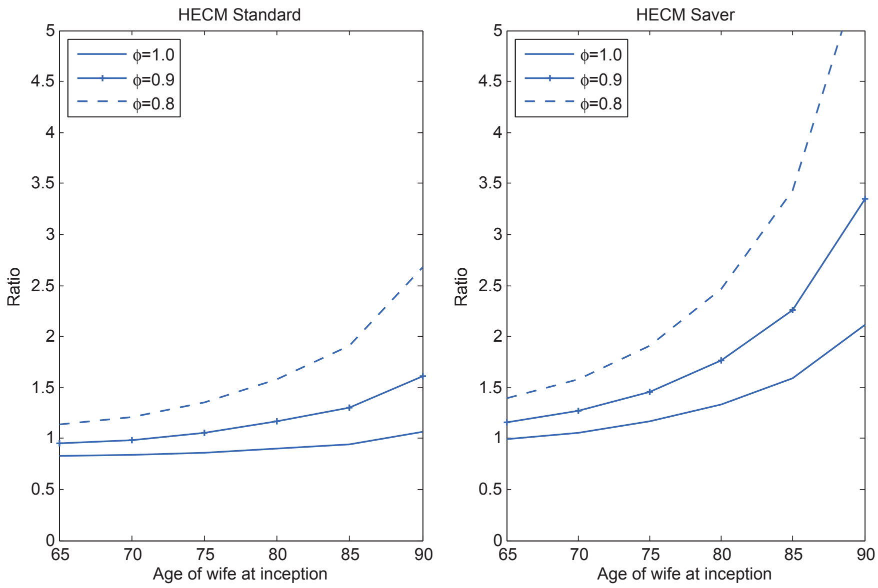

Figure 3.5: Ratios of the present value of total premium charges to the total guarantee value for variable-rate HECMs with different portions of the available line of credit are utilised.

house prices will (probably) lead to an underestimate of the guarantee cost, given that the process is more likely to exhibit fatter tails and auto-correlation.

Thirdly, the estimated front-end charge decreases dramatically when the utilization of the available line of credit is reduced. This implies that, under the current premium structure, borrowers who utilize a smaller portion of the available line of credit are subsidizing those who utilize more. This problem can be understood more easily from Figure 3.5, in which we plot, for three different degrees of utilization, the ratio of HECM's current total premium charge to total time-0 value of the embedded guarantee. We observe from this diagram that the problem of subsidy is particular severe in a HECM Saver loan. A fairer premium structure would use a multiple of the loan utilised rather than the house value as the basis for the front-end charge.

Finally, we calculate the front-end charge for three typical age combinations: (1)

Age of wife at inception	65	70	75	80	85	90
	Scenario 1: husband and wife are of the same age					
premium	14.45%	12.85 %	10.77 %	8.51%	6.58%	4.56%
	(0.009%)	(0.008%)	(0.007%)	(0.006%)	(0.006%)	(0.004%)
	Scenario 2: husband is 1 years older than the wife					
premium	14.20%	12.57%	10.48%	8.22%	6.30%	4.30%
	(0.009%)	(0.008%)	(0.007%)	(0.006%)	(0.006%)	(0.004%)
	Scenario 3: husband is 3 years older than the wife					
premium	13.73%	12.05 %	9.96%	7.69%	5.81%	3.86%
	(0.008%)	(0.007%)	(0.007%)	(0.005%)	(0.006%)	(0.004%)

Table 3.11: Simulated front-end premiums (standard errors in brackets) for fixed-rate HECM Standard loans with different age combinations.

the husband and wife are of the same age; (2) the husband is 1 year older than his wife; (3) the husband is 3 years older than his wife. The calculations are based on 100,000 simulations from the termination model and the fixed-rate HECM Standard loans. The results, which are displayed in Table 3.11, indicate that the front-end charge is quite dependent on the age gap. This effect arises from the dependence among joint-lives, suggesting the advantage of using a joint-life termination model when valuing joint-life reverse mortgages.

3.6 Concluding Remarks

In this chapter, we have proposed a semi-Markov multiple state model for reverse mortgage terminations. The model incorporates not only multiple modes of decrement, but also the statistical dependence between the lifetimes of a husband and wife. This feature is particularly important for valuing joint-life reverse mortgages, which have become increasingly popular in recent years.

Because most data on reverse mortgage terminations are proprietary, some proxy data and assumptions are used in our illustrations. Nevertheless, this does not affect our objectives, to demonstrate how the model can be used to determine a fair price for the NNEG, and to demonstrate the relative importance of different modes of termination. Reverse mortgage providers, who have access to their own decrement data, can easily adapt the multiple state model we propose to suit their own experience.

For the US HECMs, it was found that, in today's interest rate environment, the current principal limit factors for fixed-rate HECMs are extremely high. High loan-to-value ratios make fixed-rate HECMs unsustainable to the risk associated with the non-recourse guarantee. This result may not cause enough concern since fixed-rate HECM loans have only become popular in 2010. However, providers need to review fixed-rate HECMs' principal limit factors.

The HECM Saver contracts charge a negligible front-end mortgage insurance premium. In contrast, the rather low principal limit factors for this new loan option make it more sustainable than the counterpart, the HECM Standard. High front-end charge in the HECM standard is less attractive. On the contrary, it is insufficient for the risk of non-recourse guarantee. The main reason comes from a high loan-to-value ratio, especially for younger borrowers.

What determines the claim from a HECM mortgage insurance is the value of the mortgaged property when the loan is due, usually many years from the time when the loan is written. Hence, in some sense, the heavily front-loaded mortgage insurance premium means that HECMs have front-loaded revenue and back-loaded risk. From a risk management viewpoint, the HECM Saver's premium structure with a lower front-end charge would seem to be more effective for capturing the risk associated with the uncertainty in future house prices.

Readers should keep in mind that our conclusions on HECM mortgage insurance premiums are based on the interest rate and the principal limit factors as of this writing. When interest rates change, the principal limit factors, and hence the estimated mortgage insurance premiums, will change accordingly. It is important to take the change in interest rates, possibly through a stochastic interest rate model, into account when deciding a new premium structure. The use of a model such as ours to determine principal limit factors could improve the product design,

We repeat two caveats around the specific numerical results presented here. The geometric Brownian motion assumption for house prices is probably too thin tailed. In practice, one may consider a house price model that permits autocorrelation and stochastic volatility. For example, Li et al. (2010) fit an ARMA-EGARCH model for house prices in the UK; Chen et al. (2009) use an ARMA-GARCH model for house prices in the US. The use of such models will imply market incompleteness, which adds an extra challenge in the pricing process.

The loan interest rate is another variable that we have not explored extensively. It will, of course, affect how fast a floating rate loan is accumulated. It will also affect the guarantee values through the correlation with house prices, as we have observed painfully through the recent financial crisis. Furthermore, there will be dependence between the termination transition probabilities, especially for the non-health related terminations. For example, it is more likely that a borrower will move and repay his/her reverse mortgage in a booming economy. In a recession, homeowners may be less likely to choose the expensive option of long term care. In future research, it would be interesting to integrate a stochastic interest rate model into the multiple state termination model, possibly through a regime-switching framework.

Chapter 4

Longevity Risk in Last Survivor Annuities

4.1 Introduction

During the past half-century, developed countries have witnessed remarkable mortality improvement leading to the growth of older population and increasing life expectancy (see Khalaf-Allah et al. 2006, Cox et al. 2008, and references therein). A large element of mortality improvement is driven by medical advances. Ongoing support for medical research will continue to lead to further mortality decline (Gallop, 2006). The US National Institute of Health Workshop Report on Aging has estimated that the 65- to 74-year-old age group in the US will be 36 million in 2030 compared with 21.5 million in 2010; the 75- to 84-year-old age group will be 25 million compared with 17 million in 2010; and the 85- to 99-year-old age group will be 5 million, compared with 2.1 million in 2010.

Mortality improvement is anticipated in the foreseen future, while such a process is of great uncertainty in terms of the extent and pace of improvement. Uncertainty in mortality improvement puts enormous pressure on retirement funds and annuity insurance funds. Social security retirement systems and private annuity product providers have come under increasing strain of affording payments for longer than expected periods.

We refer to this higher-than-expected mortality improvement as longevity risk. Given life expectancy for the population as a whole, the idiosyncratic risk that a particular annuitant lives longer than expected could, in principle, be minimized by holding a sufficiently large portfolio of individual policies. Longevity risk is uncertainty about the life expectancy of the population as a whole. It is a systematic risk of the annuity business, having potentially significant impact on the annuity market. Insurance companies and private pension plan providers should incorporate longevity risk in their actuarial calculations. In the language of financial economics there should be a market price for the systematic longevity risk.

Currently, annuity prices in the private annuity market are usually based on the annuity life table projected to the current year and beyond. Deterministic, age specific mortality reduction factors are usually used for mortality projection. US insurance companies generally use the Annuity 2000 Basic Table and Scale G for pricing individual annuities (Doll et al., 2011). UK companies similarly use period mortality tables with some improvement projection methods. The UK Continuous Mortality Investigation Bureau (CMI, for short) used smoothed P-spline estimates of the annual mortality improvement rates in its mortality projection model.

While deterministic modeling can provide best-estimate mortality scenarios, it is inadequate for some applications in the practice. Where longevity and mortality risk constitute a significant risk for insurers, stochastic models can be used in evaluating these risks. Stochastic modeling is able to deliver full probability distributions of the quantities of interest, and allows us to quantify uncertainty and risks adequately, for better risk management.

Dependence between joint-life mortality has not been taken into account in the practice of pricing annuity products either. The forecasting of joint-life longevity may be even more complicated than single lives, because of the dependence between the future lifetime of a husband and wife. The prices for joint-and-last survivor annuities in the current market are quite inconsistent. In the US, prices for last survivor annuities quoted from Immediate Annuity[1] are determined only by the younger age of a husband and wife. In the UK annuity market, there is a similar pricing practice.

Also as a new era of unisex annuity rates is anticipated, because on 1 March 2011 the European Court of Justice ruled that gender may not be used in insurance pricing, the market for last survivor annuities may be growing. It would be harmful to the development of annuity market if prices for joint-life annuities deviate too far from the fair price range. The aim of this chapter is to examine how the annuity market accounts for future improvements in mortality rates and life expectancy when pricing last survivor annuities. For this end, we propose a joint-life longevity risk model, incorporating stochastic mortality dynamics into the semi-Markov joint-life mortality model.

In the current actuarial literature, several stochastic mortality models have been proposed during the past two decades. The Lee-Carter model (Lee and Carter, 1992) and the Cairns-Blake-Dowd (CBD) model (Cairns et al., 2006) are two more popular ones among them. The Lee-Carter method models central mortality rates by two age-specific factors and a time-dependent factor. The model is famous for its parsimonious structure and easy interpretation. Various extensions and methodological improvements have been studied. Readers are referred to Wilmoth (1993), Brouhns et al. (2002), Renshaw and Haberman (2003 and 2006), Li et al. (2010), Delwarde et al. (2007), etc.

The CBD model forecasts the post-60 mortality using two factors that are mea-

[1]It is referred to as the No. 1 web site for income annuities. Immediate annuity rates can be quoted from its web site http://www.immediateannuities.com/.

surable with time. The first factor affects mortality-rate dynamics at all ages, and its downward trend indicates general improvements in mortality over time. The second factor affects mortality-rate dynamics as a coefficient of age. Its increasing trend means mortality improvements have been relatively greater at younger old ages. From another viewpoint, the CBD model might be looked as a dynamic version of the Heligman and Pollard model. Heligman and Pollard (1980) proposed a mortality graduation model for the whole ages in equation of $\frac{q_x}{1-q_x} = A^{(x+B)^C} + De^{-E(\ln x - \ln F)^2} + GH^x$, where q_x is the mortality rate at age x, and $A, B, \cdots, H$ are parameters. The third term stems from the Gompertz exponential, representing senescent mortality.

Applying Gompertz' law in stochastic modeling of mortality rates is not new. McNown and Rogers (1989) applied a univariate time series approach to the Heligman-Pollard model for forecasting the US mortality. Schoen et al. (2004) considered a time factor for a continuously declining mortality, in a form of $\mu(x, t) = Ae^{kx-ct}$, with $A, k > 0$ and $c \geq 0$. Lockwood (2009) fitted univariate time series models to the parameters of a series of Gompertz-Makeham models of order (r, s), or GM(r, s) models, using CMI male assured lives data and the England and Wales population data from age 30 to 90.

The merits of Gompertz' law and its application in mortality forecasting motivate us to propose a stochastic Gompertz model with time dependent parameters for the transition intensities in the semi-Markov joint-life mortality model. The reasons that we choose a preliminary stochastic Gompertz model for mortality forecasting are as follows:

1. the Gompertz' exponential curve per se well describes older age mortality, in a simple, analytically tractable pattern (Schoen et al., 2004, and references therein);

2. the proposed stochastic Gompertz model has parsimonious and readily inter-

pretable parameters;

3. it is convenient to be incorporated into the semi-Markov joint-life mortality model.

Based on the proposed joint-life longevity risk model, we examine the market prices of longevity risk in joint-and-last survivor immediate annuities in the current US and UK annuity market. The remainder of this chapter is organized as follows: Section 2 specifies the semi-Markov joint-life longevity model, where transition intensities are stochastically modeled by Gompertz' law with time-dependent parameters. The model is fitted to the US and UK base annuity life table and historic population data, for the forecasting of annuitants' mortality in the future. Section 3 demonstrates the impact of longevity risk on the prices of last survivor annuities. Section 4 reviews pricing methodologies for longevity/mortality risk and describes the method used in this research. The market prices of longevity risk in last survivor annuities written in the US and UK annuity market are then compared. Section 5 concludes this chapter.

4.2 A Semi-Markov Joint-life Longevity Model

4.2.1 Model Specification

In Chapter 2, we propose a semi-Markov mortality model for the dependent modeling of joint-life mortality, in which the force of mortality after bereavement is modeled as the product of a multiplicative function and the corresponding force of mortality when his/her spouse is still alive. Specifically, the force of mortality for widows, age x, s years after bereavement,

$$\mu^*(x,s) = F^f(s)(\mu_x^f + \lambda); \tag{4.2.1}$$

and for widowers, age y, s years after bereavement,

$$\mu^*(y, s) = F^m(s)(\mu_y^m + \lambda), \tag{4.2.2}$$

where μ_x^f and μ_y^m represent the force of mortality for married women and men, respectively, from all causes other than common shock, λ is the "common shock" parameter. The multiplicative functions are exponentially decreasing, in forms of $F^f(s) = 1 + a^f e^{-k^f s}$ and $F^m(t) = 1 + a^m e^{-k^m s}$, where a^m, a^f, k^m, $k^f > 0$.

We proposed a semi-Markov joint-life longevity model as an extension of the semi-Markov joint-life mortality model. For the joint-life longevity model, we assume that λ is zero, that is, we do not allow for transitions from "common shock" events. The main reason for this assumption is that there is no historic mortality data for "common shocks". We can hardly calibrate a process for the instantaneous transition. If data permits, time-t dependent or independent common shock transition can easily be incorporated, and will not affect the current setting. Figure 4.1 specifies the proposed joint-life longevity model.

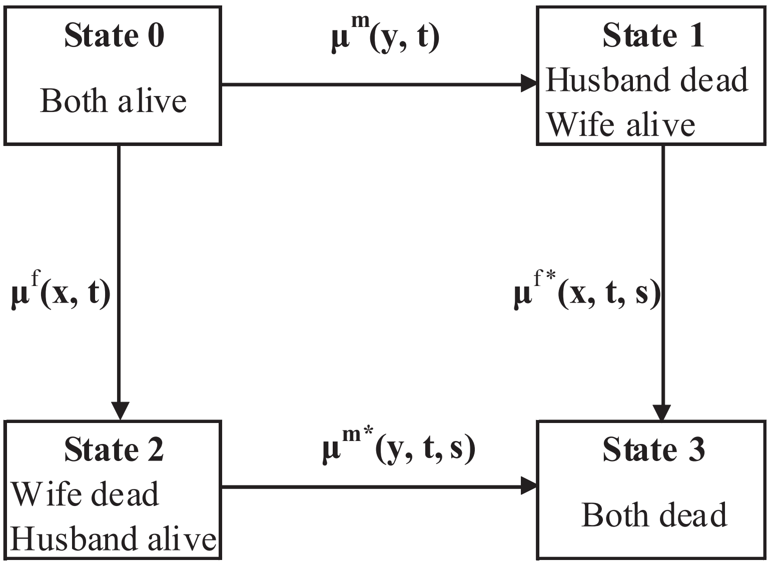

Figure 4.1: Specification of the semi-Markov joint-life longevity model

The transition intensities are stochastically modeled and thereafter time-t dependent. Let $\mu^f(x,t)$ and $\mu^m(y,t)$ denote the force of mortality for a x-age wife and a y-age husband at time t respectively, in the married status; $\mu^{f*}(x,t,s)$ denotes the force of mortality for a x-age widow, at time t, s years after bereavement, while $\mu^{m*}(y,t,s)$ is the force of mortality for a y-age widower and widower, at time t and s years after bereavement.

We assume that the exponentially decaying functions for the bereavement effect identified in Chapter 2 are still valid with the joint-life longevity model. The selection effect of the broken-heart syndrome is well defined by $F^f(s)$ and $F^m(s)$. That is to say, the post-bereavement mortality rates are driven by three factors: age, chronological time, and time since bereavement. Specifically, the proposed semi-Markov joint-life longevity model will have

$$\mu^{f*}(x,t,s) = (1 + a^f e^{-k^f s})\mu^f(x,t); \tag{4.2.3}$$

and,

$$\mu^{m*}(y,t,s) = (1 + a^m e^{-k^m s})\mu^m(y,t). \tag{4.2.4}$$

4.2.2 Stochastic Transition Intensities

We start with Gompertz' law and make a proposal for stochastic Gompertz parameters in the Gompertz formula. The two Gompertz parameters are thereafter time-t dependent.

Gompertz' law models the hazard function of a random lifetime variable for an individual as $\mu_x = Be^{cx}$, for $x > 0$ and $B, c > 0$. Gumbel (1937) and Carriere (1992, 1994) employed informative re-parametrization of the Gompertz formula in a form of

$$\mu_x = \xi \exp\{\xi(x - \gamma)\}, \tag{4.2.5}$$

where γ is the mode parameter and ξ denotes the force of mortality at the modal age, that is, $\mu_\gamma = \xi$. It is trivial to get $\gamma = -\frac{1}{c}\ln(\frac{B}{c})$ and $\xi = c$.

The force of mortality at the modal age coincides with the Gompertz ageing parameter. Carriere (1992) showed that, the aging parameter also coincides with the measure of spread about the mode. The lifetime distribution concentrates about the mode γ when ξ is large, that is, the inverse of the aging parameter represents the spread of the Gompertz distribution.

Allowing the Gompertz parameters to be time-t dependent, we express the forces of mortality $\mu^f(x,t)$ and $\mu^m(y,t)$ for a x-age wife and a y-age husband, at time t, in the married state, mathematically as

$$\mu^f(x,t) = \xi_t^f \exp\{\xi_t^f(x - \gamma_t^f)\}, \tag{4.2.6}$$

and

$$\mu^m(y,t) = \xi_t^m \exp\{\xi_t^m(y - \gamma_t^m)\}, \tag{4.2.7}$$

where ξ_t^f and γ_t^f are the Gompertz mode parameter and aging parameter at time t for females, and ξ_t^m and γ_t^m are for males. Their values determine the time-t mortality profile of females and males in the married status.

By fitting Gompertz' law to historical mortality data, we can get a time series of historic Gompertz parameters. The Human Mortality Database provides historical data of mortality rates, death counts and population size at detailed levels. Maximum likelihood estimation methodology is used to estimate the parameters of the Gompertz distribution. It is assumed that the number of deaths, which is a counting random variable, follows the Poisson distribution (see, e.g., Wilmoth, 1993 and Brouhns et al., 2002).

We fit Gompertz' law to the US historic population period mortality data for age 60 to 109 during years from 1950 to 2007. Figure 4.2 depicts the estimated values of ξ_t and γ_t from year 1950 to 2007. The mortality improvement has occurred with increasing mode parameter γ_t and increasing aging parameter ξ_t. An increasing γ_t causes the left shift of the lifetime distribution. An increasing ξ_t indicates of the concentration about the modal age of the lifetime distribution.

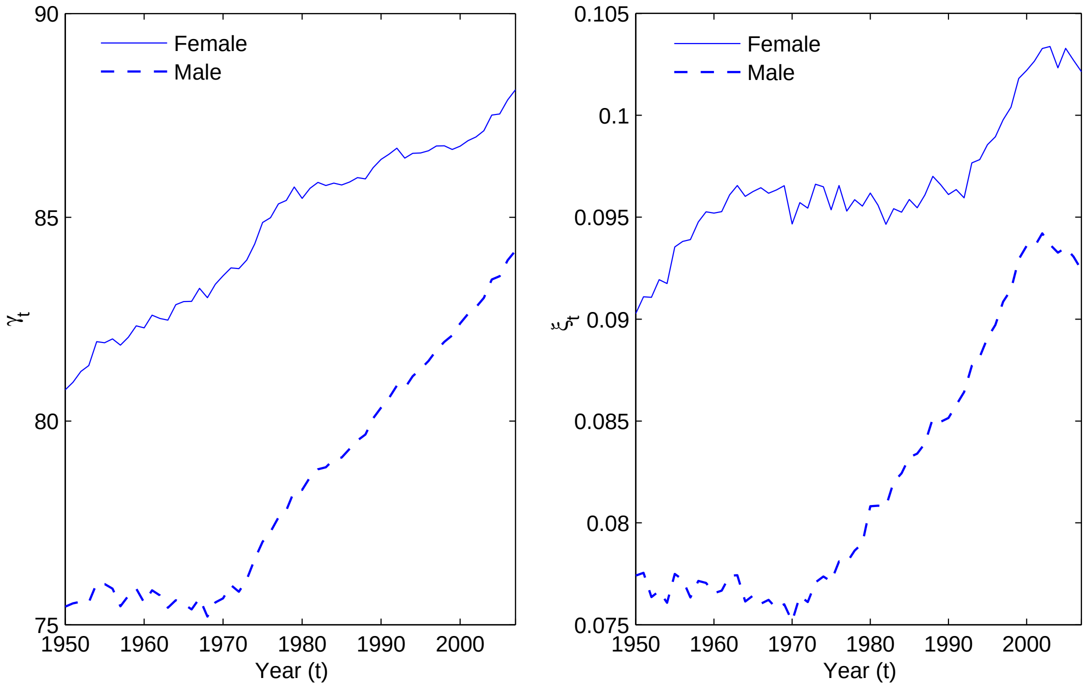

Figure 4.2: Estimated values of ξ_t and γ_t for the US historic mortality data from 1950 to 2007

We also fit Gompertz' law to the England and Wales population data from year 1950 to 2009, a similar period as the US example. Figure 4.3 depicts the estimated values of ξ_t and γ_t. Generally increasing mode parameter γ_t and increasing aging parameter ξ_t indicate a trend of declining mortality; while, the estimated volatilities of the stochastic process for the UK mortality might be greater than the volatilities for the US mortality.

A vector stochastic process is thereafter proposed to model the Gompertz parameters ξ_t and γ_t, modeling their trends, in a correlated form, for the stochastic

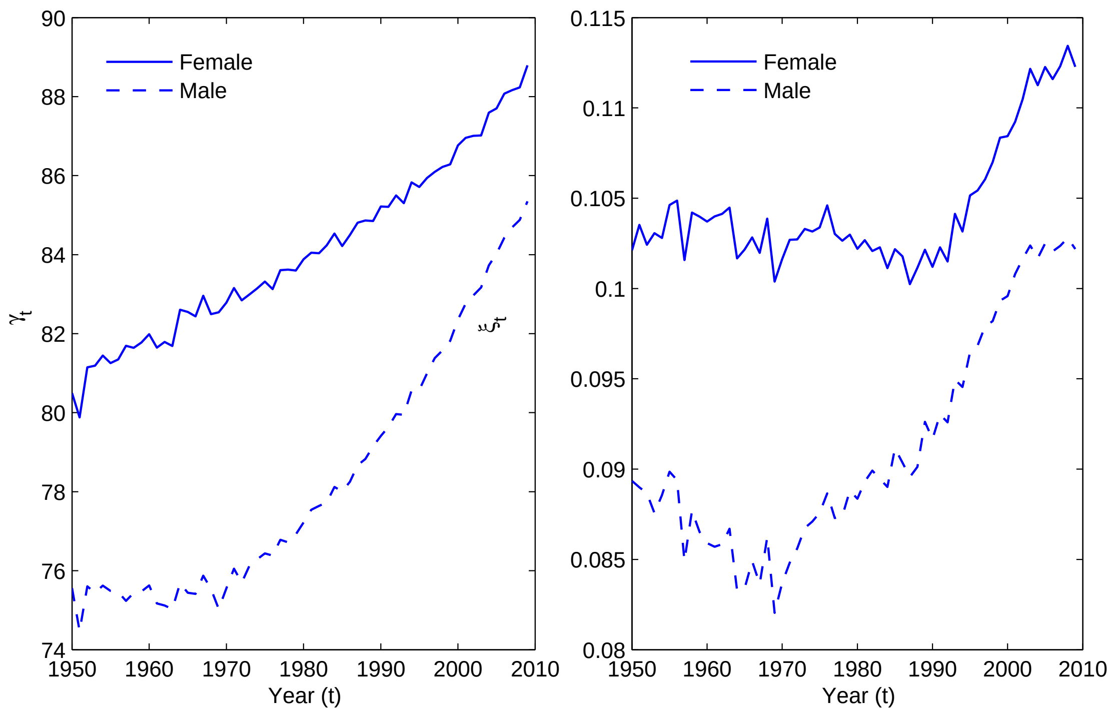

Figure 4.3: Estimated values of ξ_t and γ_t for the England and Wales population mortality data from 1950 to 2009

modeling of $\mu^f(x,t)$ and $\mu^m(y,t)$. Denote $z^f(t)$ and $z^m(t)$ to be the vector stochastic process for the Gompertz parameters for females and males respectively, that is, $z^f(t) = [\xi_t^f, \gamma_t^f]'$ and $z^m(t) = [\xi_t^m, \gamma_t^m]'$. Define the drift vector ν^f and ν^m for females and males respectively by

$$\nu^f = \begin{pmatrix} \nu_1^f \\ \nu_2^f \end{pmatrix} \quad \text{and} \quad \nu^m = \begin{pmatrix} \nu_1^m \\ \nu_2^m \end{pmatrix},$$

the 2×2 - dimensional covariance matrix V^f and V^m for females and males respectively by

$$V^f = \sigma^f \sigma^{f\prime} = \begin{pmatrix} V_{11}^f & V_{12}^f \\ V_{21}^f & V_{22}^f \end{pmatrix} \quad \text{and} \quad V^f = \sigma^m \sigma^{m\prime} = \begin{pmatrix} V_{11}^m & V_{12}^m \\ V_{21}^m & V_{22}^m \end{pmatrix},$$

and the 2-dimensional standard normal random variable by $Z(t)$. Symbol "$\prime$" means the transpose of a matrix.

In discrete time, the vector processes $z^f(t)$ and $z^m(t)$ are modeled as a two-dimensional random walk with drift. Specifically

$$z^f(t+1) = z^f(t) + \nu^f + \sigma^f \ Z(t+1), \tag{4.2.8}$$

and

$$z^m(t+1) = z^m(t) + \nu^m + \sigma^m \ Z(t+1). \tag{4.2.9}$$

The choice of diffusion matrix σ^f and σ^m is not unique but will not make any difference to our analysis, as stated in Cairns et al. (2006). Following Cairns et al. (2006), $\sigma^{f(m)}$ is chosen to be the Cholesky decomposition of $V^{f(m)}$.

Fitting the vector processes $z^f(t)$ and $z^m(t)$ to the estimated historic Gompertz parameters from the US population data for 60 to 109 ages during year 1950 to 2007 (58 observations), we have
for females,

$$\nu^f = \begin{pmatrix} 0.1292 \\ 2.083 \times 10^{-4} \end{pmatrix}, \quad V^f = \sigma^f \sigma^{f\prime} = \begin{pmatrix} 0.0324 & -7.2589 \times 10^{-5} \\ -7.2589 \times 10^{-5} & 5.3877 \times 10^{-7} \end{pmatrix};$$

for males,

$$\nu^m = \begin{pmatrix} 0.1533 \\ 2.632 \times 10^{-4} \end{pmatrix}, \quad V^m = \sigma^m \sigma^{m\prime} = \begin{pmatrix} 0.0475 & 1.6330 \times 10^{-5} \\ 1.6330 \times 10^{-5} & 4.7453 \times 10^{-7} \end{pmatrix}.$$

Similarly, fitting the vector processes $z^f(t)$ and $z^m(t)$ to the estimated historic Gompertz parameters from the England and Wales historic population mortality data for 60 to 109 ages from year 1950 to 2009 (60 observations), we get
for females,

$$\nu^f = \begin{pmatrix} 0.1407 \\ 1.7225 \times 10^{-4} \end{pmatrix}, \quad V^f = \sigma^f \sigma^{f\prime} = \begin{pmatrix} 0.0945 & -2.3514 \times 10^{-4} \\ -2.3514 \times 10^{-4} & 1.587 \times 10^{-6} \end{pmatrix};$$

for males,

$$\nu^m = \begin{pmatrix} 0.1659 \\ 2.1766 \times 10^{-4} \end{pmatrix}, \quad V^m = \sigma^m \sigma^{m\prime} = \begin{pmatrix} 0.1137 & 8.6000 \times 10^{-6} \\ 8.6000 \times 10^{-6} & 2.0164 \times 10^{-6} \end{pmatrix}.$$

The estimated results may be indicative of some information about the trend of mortality during the past half-century. Firstly, there are upward trends in the Gompertz modal parameter and aging parameter, which means that human lifetime distribution is increasingly concentrated around the increasing modal age.

Secondly, the pace of mortality improvement is faster but more volatile for males than for females. However, this feature comes from a short period of data. The conclusion may be reversed in a long run of human mortality evolvement.

Finally, using the historic mortality data as of 1950, the correlation between the two Gompertz parameters is negative for female mortality but positive for male mortality.When we fit the stochastic processes to different period of data, the sign of correlation changes for males. This may be due to the high volatility in male mortality evolvement, or it may indicate that the stochastic process for the Gompertz parameters is not robust to the data.

The proposed stochastic Gompertz model is theoretically feasible, and potentially provides satisfactory forecasts of future mortality rates. However, we acknowledge that the validity and robustness of the model needs to be tested. We would like to leave it to future work, for not been distracted from the main purpose of this study.

Generally, incorporating the stochastic Gompertz' law into the semi-Markov joint-life mortality model is a preliminary step. The multiplicative functions for the select effect of bereavement are derived based on force of mortality modeled by Gompertz' law. The stochastic Gompertz' law is a natural extension.

Other mortality forecasting models, like the Lee-Carter model or other more complicated stochastic parameter models, may also play the role. When more data becomes available in the future, further research work is expected in the area of joint-life mortality forecasting, choosing suitable models and testing the robustness of each method in the joint lives context.

4.2.3 Base Rates of Mortality

Our goal is to examine last survivor annuity prices and the implied market prices of joint-life longevity risk. Base rates of mortality should in principle be related to the mortality experience of annuitants. The currently used annuity life tables in the US and UK annuity market are used as base tables.

A life table gives mortality rates at each age for an individual. It specifies the distribution of the future lifetime random variable T_x for males or females at any age x, regardless of their marital status. It is the aggregate mortality rates for an individual in the status of being married, single, divorced, or widowed with any period of time after bereavement.

From the semi-Markov joint-life model, without mortality projection at this stage, we can derive marginal mortality rates for the married and the widowed. Assume mortality rates for the single or divorced are the same as the aggregate rates. Let h_x be the percentage of population in the married status, and g_x be the proportion in the widowed status; $1 - h_x - g_x$ is for the others. We assume that the aggregate mortality rate is approximately represented by the equation

$$\mu_x^{aggregate} = \frac{h_x}{h_x + g_x}\mu_x^{married} + \frac{g_x}{h_x + g_x}\mu_x^{widowed}.$$

Borrowing information from relevant census study on age-specific marital status, we can approximately estimate the values of the Gompertz parameters for the base mortality rates in the semi-Markov joint-life longevity risk model, by equating the approximately mixed single-life mortality rates to the mortality rates in the referred life table for both males and females. Meanwhile, we can also estimate the values of the Gompertz parameters, by directly fitting Gompertz' law to the mortality rates in the referred life table for both males and females, that is, without the semi-Markov model. The resulting estimates are for individual single-life Gompertz mortality model or independent joint-life mortality rates.

US insurance companies generally use the Annuity 2000 Basic Mortality Table (A2000, for short) as the base mortality for annuity pricing. Using the marital status of the population in 2000 studied by Kreider and Simmons (2003), and the individual mortality rates in A2000, we estimate the parameters for the base rates of mortality in the semi-Markov joint-life longevity model, by fitting the mixed marginal mortality distribution from the semi-Markov joint-life mortality model to the mortality rates in the A2000 life table. The fitting approach is based on a least squares minimization.

Table 4.1 summarizes the estimated parameters for the semi-Markov joint-life mortality model and individual single-life Gompertz mortality model. These parameters are based on the mortality rates at ages beyond 59 in A2000 life table. Single-life Gompertz models specify the aggregate mortality rates for individuals in various marital status. They can be used for independent joint lives or single lives.

The parameters displayed in Table 4.1 indicates several important results. Firstly,

The semi-Markov model		Single-life model	
Parameters	Values	Parameters	Values
γ^f	91.9541	$\gamma^{f,\ Inde}$	90.4699
ξ^f	0.1096	$\xi^{f,\ Inde}$	0.1138
γ^m	89.1318	$\gamma^{m,\ Inde}$	87.1418
ξ^m	0.0870	$\xi^{m,\ Inde}$	0.0972
a^f	3.7804		
k^f	0.3901		
a^m	10.4253		
k^m	0.7754		

Table 4.1: Parameter values for base mortality in the semi-Markov joint-life longevity model and individual single-life mortality model, for the US.

the force of mortality in the married status is generally lower than the marginal, or independent, force of mortality of the same age, which represents the combined rate of mortality of the married and the widowed. Secondly, the effect of bereavement will increase the force of mortality after bereavement by a higher level for males than for females, however males recover from bereavement faster than females. This estimated result is consistent to the result in Chapter 2.

Similarly, using the population marital status information provided by the UK Government Actuary's Department and the UK CMI Series 00 Immediate Annuity Life tables, we can estimate the parameters for the base mortality rates in the semi-Markov joint-life longevity mortality model and single-life Gompertz model applied to the UK annuitants. Table 4.2 summarizes the estimated parameters, which are based on the mortality rates at ages beyond 59 in the CMI Series 00 Immediate Annuity Life tables.

The semi-Markov model		Single-life model	
Parameters	Values	Parameters	Values
γ^f	92.8059	$\gamma^{f,\ Inde}$	90.0127
ξ^f	0.1296	$\xi^{f,\ Inde}$	0.1383
γ^m	88.7920	$\gamma^{m,\ Inde}$	86.6564
ξ^m	0.1044	$\xi^{m,\ Inde}$	0.1202
a^f	8.4790		
k^f	0.3921		
a^m	8.6868		
k^m	0.3603		

Table 4.2: Parameter values for base mortality in the semi-Markov joint-life longevity model and individual single-life mortality model, for the UK.

The estimated parameter values for the UK are slightly different from the values fitted for the US. The difference lies in the parameters for the semi-Markov property, that is, the selection effect of bereavement. From the values for the UK, males and females are subject to a nearly same broken heart effect shortly after bereavement, and they recover from bereavement at a similar speed. Here, we just state the data fitting results. The reasons underlying this difference between the US and UK are beyond the scope of our study.

4.2.4 Joint-life Mortality Projection

We have specified the basic mortality rates, joint-life dependence structure, and a mortality projection model for the semi-Markov joint-life longevity model. We have two ways to proceed for joint-life mortality projection.

One method is to assume the two exponentially decreasing multiplicative functions for the select effect of bereavement keep unchanged with time. We only stochastically model time-t dependent forces of mortality $\mu^f(x,t)$ and $\mu^m(y,t)$ in the married state. The other method is to stochastically project the mortality rates in the married state and the marginal, independent, or aggregate mortality rates at the same time, for females and males respectively. By this way, we stochastically model time-t dependent forces of mortality $\mu^f(x,t)$ and $\mu^m(y,t)$, and the corresponding time-t dependent single-life forces of mortality. The select effect of bereavement, which may evolve with time, is implied by a set of forecasted time-t dependent mortality rates.

The probability that the last survivor of a currently x-age wife and y-age husband at time $t_0 = 2011$ will survive t years from now can be computed as

$$ {}_tp_{\overline{xy}}(t_0) \quad = \quad {}_tp_x^f(t_0) + {}_tp_y^m(t_0) - {}_tp_{x:y}^{00}(t_0), \tag{4.2.10} $$

where

$$ \begin{aligned} {}_tp_x^f(t_0) &= \prod_{j=0}^{t-1} \exp\left\{ -\int_0^1 \left(\mu^{f,\ Inde}(x+j+s, t_0+j)\right) ds \right\}, \\ {}_tp_y^m(t_0) &= \prod_{j=0}^{t-1} \exp\left\{ -\int_0^1 \left(\mu^{m,\ Inde}(y+j+s, t_0+j)\right) ds \right\}, \\ {}_tp_{x:y}^{00}(t_0) &= \prod_{j=0}^{t-1} \exp\left\{ -\int_0^1 \left(\mu^f(x+j+s, t_0+j) + \mu^m(y+j+s, t_0+j)\right) ds \right\} \end{aligned} $$

This approach calls for less computer resources in joint-life mortality projection. Furthermore, it enables comparison between joint-life and single-life mortality projection, because single-life mortality rates are projected at the same time for deriving the projected joint-life mortality rates. In addition, the implicitly defined dependence structure may evolve with time. If we use the first approach, we have to assume that dependence structure is time-invariant.

Choosing this joint-life mortality projection method, we assume that the marginal mortality and the mortality in the married status, for both females and males, have their Gompertz parameters follow the same vector random walk process. It may be argued that mortality in the married status and the marginal (or aggregate) mortality may evolve differently. At the current stage, we have no historic mortality data for married couples to support or test this argument. If better data become available in the future, we can explore this topic further.

Meanwhile, forces of mortality are projected by the vector random walk process that has been fitted using the population mortality database. We acknowledge that this database is not perfect for calibrating the vector stochastic processes for annuitants' mortality improvement. However, without more suitable data, we use the population data to calibrate the process, at least approximately. Furthermore, correlation between the improvements of mortality for males and females has not been examined. We leave it to future work.

4.3 Implication for Last Survivor Annuity Values

Joint and last survivor annuities are typical products associated with joint-life longevity risk in the current annuity market. They provide benefits to the annuitant and his/her spouse until both of them have passed away. These products are not only offered by insurance companies but are also an important benefit to pension plan retirees. If annuitants live longer than expected because of unexpected mortality improvement, the financial soundness of annuity portfolios could be at risk.

In this section, we examine how stochastic joint-life mortality improvement will affect the cost of a last survivor annuity using the model proposed above. The impact of joint-life longevity risk can be measured by the increase in the cost of a last survivor annuity due to the allowance of mortality improvement compared with the corresponding cost when mortality improvement is not allowed.

The quantity of interest, the cost of a last survivor annuity, is a non-linear function of the fitted Gompertz parameters in the current year and the parameters of the vector stochastic Gompertz processes. It is not possible to derive the distribution of the last survivor annuity net premiums and relevant confidence intervals analytically. Monte-Carlo simulation is used here.

4.3.1 Simulation Method

To derive the distribution of the expected annuity value, we simulate the realization of future Gompertz parameters for 5,000 times. From each realization of future Gompertz parameters, we have a surface of mortality for individuals and married couples. We then estimate the present value of annuity payments from each realization of future mortality surface.

Specifically, we simulate the cost of a last survivor annuity taking the following steps:

1. Simulate N trajectories of the Gompertz parameters for individual mortality and joint survival mortality from the year, in which the base mortality is applied to, to the current year and beyond. Each trajectory is simulated based on the Gompertz parameters for the base mortality and the calibrated vector stochastic processes from the historic population mortality data.

2. From each trajectory, compute individual survival probabilities and joint survival probabilities, and estimate the expected present value of annuity payments;

3. From step (2), get an empirical distribution of the cost of a last survivor annuity.

In calibrating the stochastic processes for the Gompertz parameters, there are generally two sources of parameter uncertainty: sampling errors in the historic Gompertz parameters estimated from the Poisson models, and parameter uncertainty in calibrating the stochastic mortality models to the historic Gompertz parameters. A parametric bootstrap simulation technique can be used to allow for parameter uncertainties.

We conduct two simulation methods, with and without allowance for parameter uncertainty. Allowance for parameter uncertainty in the model is more time-consuming but does not make much difference in the simulated annuity values. So, we do not use a parametric bootstrap in the simulation and ignore the trivial impact of parameter uncertainty.

4.3.2 The Results

Allowance for mortality improvement will increase the expected present value of annuity payments. We use a last survivor annuity issued to a 65 year old husband and a 65 year old wife, with annual payments of \$1 (or £1) paid monthly in advance, as an example to illustrate the extent of increase in the annuity value due to mortality improvement. Annuities with more frequent payments in a year are approximately calculated using the Woolhouse's formula, which is discussed in Chapter 5 of Dickson et al. (2009). The interest rate is assumed to be 4.25% , which is an approximate average interest rate on the 20-year US treasure bills and 20-year UK government bonds during April 2011.

Based on the proposed semi-Markov joint-life longevity model, we can simulate a surface of survival probabilities for last survival status. For each scenario, we compute the cost of the annuity with and without allowance for mortality improvement. With no allowance for future mortality improvement, the Gompertz parameters in the future will be the same as the parameters in the current year (t_0=2011).

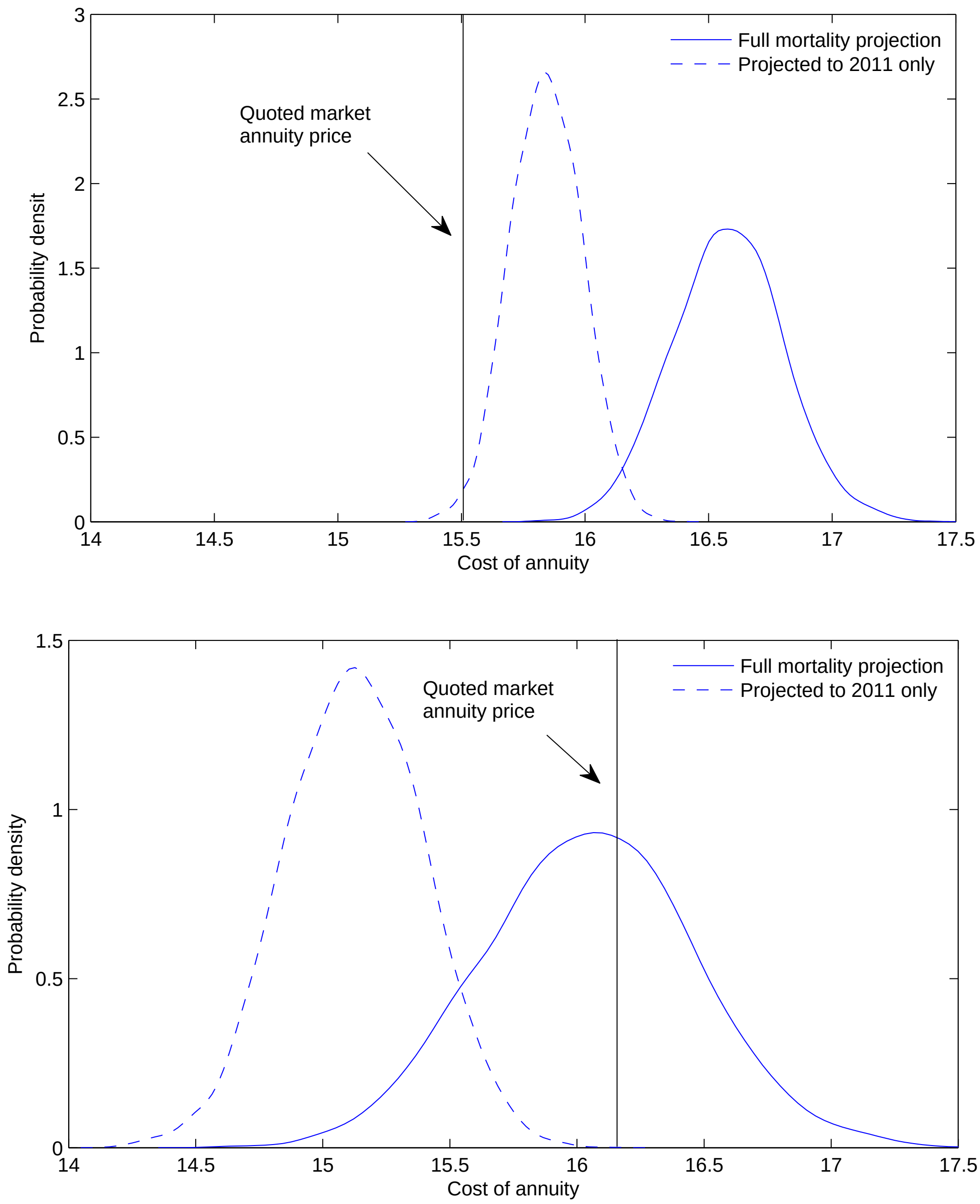

Figure 4.4: Distribution of the cost of last survivor annuity in year 2011 with and without allowance for future mortality improvement, for the US market (Top) and the UK market (Bottom), female age $x = 65$ and male age $y = 65$, interest rate 4.25%.

We depict in Figure 4.4 the simulated cost of the last survivor annuity in the current year 2011 with and without allowance for future mortality improvement, for US market (Top) and the UK market (Bottom) respectively. The simulated empirical distribution is smoothed using a kernel density estimation method. The quoted market price and simulated cost is for an annuity per unit annual benefit paid monthly in advance. No allowance for future mortality improvement means the base rates of mortality have been projected to the current year only.

The information from Figure 4.4 can be summarized as follows. Firstly, allowance for mortality improvement dramatically increases the cost of last survivor annuities. Systematic longevity risk has significant impact on the annuity cost. For pricing annuities, a mortality projection model is critical. In fact, this is a fundamental problem in the annuity market.

Secondly, the modeled joint-life longevity risk from the proposed semi-Markov model is more significant in the US market than in the UK market. From the US model, there is a very short overlap between the distribution of simulated annuity value with and without allowance for mortality improvement.

Thirdly, based on the same interest rate, the annuity value in the UK market may generally be lower than the value in the US market, while the simulated annuity value is more volatile in the UK market. The underlying reason for the annuity volatility is the more volatile mortality improvement calibrated from the historic England and Wales population mortality data.

Finally, the quoted annuity rate in the US market is lower than the simulated annuity rates, which are based on a risk-free interest rate. We would expect that the market annuity rate should be higher than the average of the simulated annuity rate with full mortality projection, if the market sufficiently allows for mortality improvement in their pricing. It appears that, the US annuities are underestimated. The underpricing problem is less in the UK market. In the next section, we further investigate the longevity risk in last survivor annuities.

4.4 The Market Prices of Longevity Risk

In this section we attempt to identify how joint-life longevity risk has been taken into account in the practice of pricing last survivor annuities.The concept of the market price of longevity risk is used to assess how insurers view longevity implicitly in their pricing.

4.4.1 Pricing Method

In the recent literature, several pricing methods have been developed for pricing the longevity/mortality risk. Cairns et al. (2006), Dahl and Møller (2006), and Dahl et al. (2008) use a risk-neutral pricing theory; Wang (1996, 2000, 2001, 2002) has developed a method that uses a one-factor risk distortion operator to drive a risk-distorted measure for universally pricing financial and insurance. Lin and Cox (2005) and Denuit et al. (2007) have applied the Wang transform to pricing mortality risk. Other methods include the utility maximization principle, the principle of equivalent utility, and the Sharpe ratio approach. Chen et al. (2010) investigated connections and differences among the risk-neutral method, the Wang transform and the Sharpe ratio rule. Readers are referred to Chen et al. (2010) and references therein for a review of these methods.

The Wang transform has been widely used in insurance pricing. Cox and Lin (2005) used the Wang transform to distort the best estimated deterministic distribution, ${}_tq_x$, of the remaining lifetime random variable T_x. A market consistent price of mortality risk was deduced for pricing mortality derivatives. However, this distortion method is rather arbitrary, since the risk of interest is the uncertainty in the mortality rates per se rather than individual lifetimes. Chen et al. (2010) stated that the Wang Transform is stable for large probabilities whereas it is highly unstable for small probabilities, and robustness of the Wang transform becomes worse as the maturity becomes longer. In addition, the Wang transform is unable to deliver a risk-adjusted dynamic.

Risk neutral pricing theory is well established. Financial economic theory states that, if the market is arbitrage-free, there exists a risk-neutral measure such that the price of an asset equals the expected discounted payments under the risk-neutral measure. A risk-neutral measure, also called an equivalent martingale measure or Q-measure, is equivalent to a real-world measure which is referred to as P-measure, in the probabilistic sense. In the Q-measure (using risk adjusted probability), the current value of all financial assets is equal to the expected future payoff of the asset discounted at the risk-free rate.

If the market is complete, there exists a unique risk-neutral measure, while, in an incomplete market many risk-neutral risk measures might exist. As pointed out in Cairns et al. (2006), we are far from having a complete market in which all contingent claims can be replicated by self-financed portfolio. There is no liquid market for systematic longevity risk. In a sense, it is difficult to calibrate the risk premium in annuities for systematic longevity risk.

Using the risk-neutral pricing approach, we need to make a further assumption that market players act in an equilibrium setting and this equilibrium selects a market consistent risk-neutral measure. In this research, we follow the method proposed in Cairns et al. (2006) to define such a market-consistent Q-measure. In their method, the risk-adjusted pricing measure $Q(\eta)$ is modeled using an adjustment to the dynamics of the stochastic process of mortality rates. Specifically, under the risk-neutral measure $Q(\eta)$,

$$\begin{aligned} z^{f(m)}(t+1) &= z^{f(m)}(t) + \nu^{f(m)} + \sigma^{f(m)}(\tilde{Z}(t+1) + \eta^{f(m)}) \\ &= z^{f(m)}(t) + \tilde{\nu}^{f(m)} + \sigma^{f(m)}\tilde{Z}(t+1), \end{aligned}$$

where $\tilde{\nu}^{f(m)} = \nu^{f(m)} + \sigma^{f(m)}\eta^{f(m)}$. $\tilde{Z}(t+1)$ is a standard two dimensional normal random variable under Q-measure.

The vector $\eta^{f(m)}$ is the market prices of longevity risk associated with the stochastic processes for Gompertz parameters $\gamma_t^{f(m)}$ and $\xi_t^{f(m)}$. $\eta_1^{f(m)}$ is the market price of longevity risk associated the stochastic process of the Gompertz modal parameter, γ_t, representing left shift in mortality distribution; while $\eta_2^{f(m)}$ is the market price of longevity risk associated the stochastic process of the Gompertz aging parameter, ξ_t, representing dispersion in mortality distribution.

We use the risk-neutral approach to calculate the risk premium for systematic longevity risk based on the idea that the market prices of annuities reflect the uncertainty of longevity risk. Assuming that the longevity and interest risk risks are independent, we can evaluate the market price of joint-life longevity risk using risk-adjusted survival probabilities, which can be simulated from the stochastic mortality processes in the Q-measure.

Let us denote $P(s,\tau)$ to be the price of a zero-coupon bond issued at time s, which pays one dollar at maturity time τ ($\tau \geq s$). Define $\delta(t)$ to be the risk-free interest rate at time t. In the risk neutral measure Q,

$$P(s,\tau) = \mathsf{E}_Q\Big[\exp\Big(-\int_s^\tau \delta(t)dt\Big)|\mathcal{F}_s\Big],$$

where $\{\mathcal{F}_s, s = 0, 1, \dots\}$ is the natural filtration for the process.

Assuming that the longevity and interest rate risks are independent, the cost of an annuity is the present value of contingent payments, discounted by the risk-free interest rate, using the Q-measure. Using risk-neutral survival probabilities, we then derive the price of last survivor immediate annuity issued to a y-year old husband and x-year old wife by the following equation:

$$\ddot{a}_{\overline{xy}}^{\,market}(2011) = 1 + \sum_{\tau\geq 1} P(0,\tau)\mathsf{E}_{Q(\eta^f,\eta^m)}\big[{}_\tau p_{\overline{xy}}|\mathcal{G}_0\big], \tag{4.4.11}$$

where $\ddot{a}_{\overline{xy}}^{\,market}(2011)$ is the market price of a last survivor immediate annuity with \$1 per year paid in advance in year 2011. For annuities with more frequent payments in a year, we approximate $\ddot{a}^{(m)}$, where payment is made $1/m$thly, using the Woolhouse's formula again.

4.4.2 The US Market

We use the prices of last survivor immediate annuities to derive the market price of joint-life longevity risk, since the market of immediate annuities is larger and more transparent than the market of deferred annuities. The risk-free interest rate is assumed to be constant and equal to 4.25%, which is the average interest rate on the US 20-year Treasure bill in April 2010.

The prices of immediate annuities in the US market are quoted from the ImmediateAnnuity.com[2]. The quoted prices are for annuities per \$1 annual benefit paid in advance, in monthly instalment. The ImmediateAnnuity.com claims that the quoted price from its web site is close to the lowest price in the current market. We assume the quoted prices are net of expense.

Theoretically, if the market is consistent, there will be unique market prices of longevity risk. However, the market for annuities is not consistently priced. We actually derive a series of the market prices of joint-life longevity risk using the quoted annuity prices for differen age combinations.

For the convenience of comparison, we assume $\eta_1^f = \eta_2^f$ and $\eta_1^m = \eta_2^m$. That is interpreted as assuming that the market prices of the two elements of risk are same. Our aim is more to demonstrate to what extent the market prices of joint-life longevity risk are reflected in the current market prices of last survivor annuities, than to calculate the exact values of market prices of risk η_1 and η_2, which would require more data and more assumptions.

We quote a series of market prices of last survivor annuities for \$1 paid monthly in advance, without guarantee. From the quoted prices, we know that, in the

[2]Available at: http://www.immediateannuities.com/

US market, the price of a last survivor annuity depends on the age of younger annuitant only. The age of elder annuitant will not change the quoted prices. This phenomenon is not actuarially sound. Annuity payments to a 65-year-old husband and 65-year-old wife are expected to be greater than the payments to a 75-year-old husband and 65-year-old wife. However, the quoted immediate annuity prices for these two couples are same according to the current pricing practice.

Female age	Male age	Market annuity price	η^f	η^m	$\eta^f = \eta^m$
65	65	15.56	-0.1714	-0.8509	-0.4154
66	66	15.23	-0.1238	-0.9116	-0.4096
67	67	14.90	0.0038	-0.9323	-0.3996
68	68	14.81	-10.1144	1.0747	-0.2296
69	69	14.38	0.2181	-1.3702	-0.2682
70	70	14.06	-2.4832	0.7675	-0.2377
71	71	13.65	-0.2085	-0.3234	-0.2520
72	72	13.26	0.3759	-1.5999	-0.2530
73	73	13.16	0.5207	-1.6741	-0.0668
74	74	12.77	0.5484	-1.7395	-0.0532
75	75	12.41			-0.0241

Table 4.3: Market prices of joint-life longevity risk in the US market, η^f and η^m, calibrated from the quoted market prices of immediate last survivor annuities with equal inception ages from 65 to 75.

Table 4.3 displays the quoted prices of last survivor annuities where the age of the female, who is younger, ranges from 65 to 75. According to the current market pricing practice, the quoted annuity price for each age combination applies to all the cases that the female is at the specified age and her spouse is the same age or older. For calibrating the market prices of longevity risk, we assume that the husband and wife are the same age. Each pair of values of η^f and η^m for an age combination is estimated using the quoted annuity values for that age combination and the next one.

The average value of the market price of longevity risk is -1.1434 for females and is -0.7560 for males. The market price of risk for each element of longevity risk, γ_t and ξ_t, is negative for both female and male mortality. It means that the market's view about the left shift in future lifetime distribution is smaller than modeled by the proposed joint-life longevity model; the marked is also less worried about the concentration of future lifetime distribution about the modal age than modeled. This result may indicate an underpricing problem with last survivor annuities in the US market.

The estimated values of η^f and η^m fluctuate dramatically. In addition, it seems that the value of one parameter is reflecting the value of the other. It may be due to that the stochastic processes for female and male mortality are uncorrelated. An increase in the market price of risk for the elements of longevity risk in female mortality rates leading to a decrease in the price for the elements of longevity risk in male mortality rates, and verse visa. A model that allows for correlation between the future mortality improvements in female and male mortality rates may give more reasonable results than the current setting.

From the modeling results, we believe that the US annuity market underprices last survivor annuities. We acknowledge that the constraints $\eta_1^f = \eta_2^f = \eta^f$ and $\eta_1^m = \eta_2^m = \eta^m$ will not exactly reflect the market's view about longevity risk. Dramatic fluctuation in η^f and η^m make it hard to tell a general level of the market's view about longevity risk.

We further assume $\eta^f = \eta^m$ to determine a general extent of the underpricing. Negative market prices of longevity risk at all ages indicate that the market underestimates longevity risk in last survivor annuities. The extent of underpricing is more severe for younger old annuitants. This may be due to cross subsidy or natural hedge between younger annuitants and older annuitants in pricing.

From the estimated prices in the last column in Table 4.3, the average level of the market price of longevity risk is -0.2372. It appears that last survivor annuities are underpriced according to our joint-life longevity model. Antolin (2006)'s argument that the market does not allow adequately for longevity risk is supported here in the case of last survivor annuities.

The market is aggressive in pricing last survivor immediate annuities, perhaps due to very competitive pricing strategy, with low rate of voluntary annuitization. However, unexpected mortality improvements in joint-life mortality could jeopardize the financial solvency of an annuity fund that has not adequately anticipated the possible impact of longevity risk.

4.4.3 The UK Market

The UK annuity market is bigger and more developed than the US market, because of the legal obligation to annuitize substantial proportion of retirement funds. Meanwhile, the UK market was aware of longevity risk earlier than the US market. In addition, it is more liquid because of the availability of longevity risk securitization instruments.

More information can be gleaned by comparing these two markets. The prices of immediate annuities in the UK annuity market are quoted from the Annuity On-line[3], which gives an indication of an averaged annuity price from a number of annuity providers during February 2011. We quote for last survivor annuities for annuitants in good health and non-smoking. We assume again these quoted prices are net of expense.

[3]Available at: http://www.annuities-online.com/

Table 4.4 displays the quoted market prices of unit annuities, and the calibrated market prices of longevity risk. The quoted prices are for annuities per £ 1 annual benefit paid in advance, in monthly instalment, without guarantee, same as the US example above. We use the same age combinations as the US example, and calibrate the market prices of longevity risk assuming that the husband and wife are the same age.

Assuming $\eta_1^f = \eta_2^f$ and $\eta_1^m = \eta_2^m$, the average value of η^f is 0.0647 and of η^m is -0.3437. The market price of risk is positive for the elements of longevity risk in female mortality rates, and negative for the elements in male mortality rates. Underpricing last survivor annuities also appears to the UK annuity market, though to a lesser extent than in the US market. This point is confirmed, as a positive market price of longevity risk is derived if $\eta^f = \eta^m$ is assumed. It could be interpreted that the joint-life longevity risk appears to be more adequately allowed in the UK annuity market than in the US annuity market; in addition, the US market considered the longevity risk of joint lives less consistently than the UK market, because of the wider range of market prices of longevity risk in the US market.

As in the US market, the UK market has a similar pricing practice for last survivor annuities. The male's age will not be a pricing factor unless he is at least two years younger than his wife. The quoted last survivor annuity price for a 65-year old wife and 65-year old husband is applied to all the cases that the wife is aged 65 and the husband is aged 63 or older. That is to say, the last survivor annuity price only depends on the female's age (x) once her spouse's age (y) satisfies $y \geq x - 2$. In that case, the age of the male is ignored in determining the expected annuity payments.

The prices of last survivor annuities in both the US and the UK market do not reflect the difference between single-life mortality and joint-life mortality. They are not based on a joint-life mortality model. Dependence between joint lives,

Female age	Male age	Market annuity price	η^f	η^m	$\eta^f = \eta^m$
65	65	16.17	-0.9635	0.5396	0.0397
66	66	15.77	0.1398	-0.1652	0.0306
67	67	15.40	0.1605	-0.2002	0.0356
68	68	15.01	0.1743	-0.2198	0.0396
69	69	14.62	0.2029	-0.2759	0.0459
70	70	14.20	0.4180	-1.2110	0.0494
71	71	13.78	0.4168	-1.1955	0.0513
72	72	13.34	0.0244	0.0887	0.0494
73	73	12.88	0.4341	-1.2509	0.0484
74	74	12.47	-0.3605	0.4534	0.0695
75	75	11.98			0.0611

Table 4.4: Market prices of joint-life longevity risk in the UK market, η^f and η^m, calibrated from the quoted market prices of immediate last survivor annuities with equal inception ages from 65 to 75.

including the "broken heart" effect, has not been considered. The irrational last survivor annuity pricing structure in the US and UK market implicitly affect the calibrated market price of longevity risk of joint lives, to some extent.

4.4.4 Joint-life vs. Single-life

For further understanding of the results in the previous section, we also examined the market prices of single-life longevity risk in the US and UK annuity market. Recall that, in projecting joint-life mortality rates, the single-life mortality is projected as the marginal mortality distribution. It enable a meaningful comparison

	The US market		The UK market	
Inception age	Males	Females	Males	Females
65	13.17	14.16	14.97	16.09
66	12.86	13.87	14.62	15.68
67	12.53	13.56	14.21	15.31
68	12.21	13.37	13.78	14.92
69	11.92	13.09	13.40	14.52
70	11.63	12.80	13.00	14.10
71	11.28	12.42	12.59	13.68
72	10.93	12.07	12.11	13.23
73	10.71	11.72	11.67	12.77
74	10.39	11.37	11.23	12.35
75	10.11	11.05	10.76	11.86

Table 4.5: Quoted market prices per unit annual benefit paid monthly in advance for single-life immediate annuities in the US and UK market for different inception ages.

between the market prices of longevity risk in joint-life products and in single-life products.

The market prices of single-life longevity risk are calibrated based on the projected marginal single-life mortality, using quotes of single-life immediate annuities in the US and UK market. Table 4.5 lists the quoted market prices of single-life immediate annuities per unit annual benefit in monthly instalments, paid in advance, without guarantee, in the US and UK market. The quoted market prices of single-life annuities in the UK are generally higher than the quoted prices in the US, while the difference narrows with the inception age of annuity.

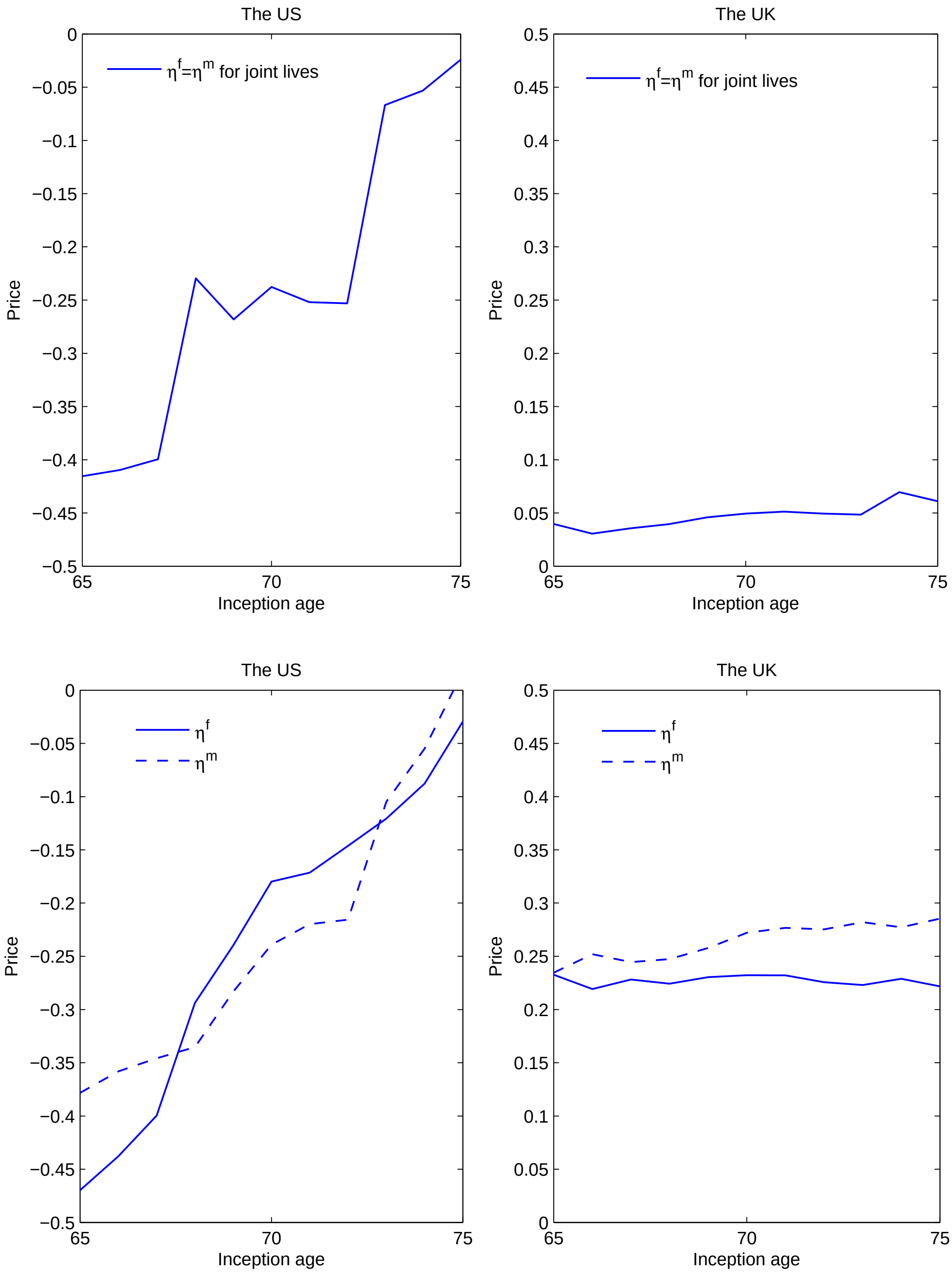

Figure 4.5: The estimated market prices of longevity risk in last survivor annuities (Top) and single-life annuities (Bottom) in the US market (Left) and the UK market (Right) .

Figure 4.5 compares the calibrated market prices of longevity risk in last survivor annuities (Top) and single-life annuities (Bottom) in the US and the UK market. The risk-free interest rate is 4.25%. The market prices of longevity risk of joint lives are taken for the last column in Table 4.3 and 4.4, which are calibrated assuming $\eta^f = \eta^m$.

In the US, the line of market prices of joint-life longevity risk is roughly between the two lines of market prices of single-life longevity risk. However, the market prices of longevity risk, either joint-life or single-life, are generally negative.

In the UK, all the calibrated market prices of longevity risk are positive, although the market prices of longevity risk in last survivor annuities is much lower than the corresponding market price of single-life longevity risk. The market prices of longevity risk are positive in the UK annuity market, but negative in the US market. It indicates that the UK annuity market appears to more adequately allow for longevity risk when pricing immediate annuities than the US annuity market. Furthermore, the market prices of longevity risk in the US are far below zero. The US market does not correctly estimate the future improvements in mortality rates. Underpricing appears to be prevalent in the US annuity market. Mortality assumptions for pricing annuities needs to be reviewed. Further study in fair pricing annuities is required.

4.5 Concluding Remarks

In this chapter, we have mainly focussed on the sustainability and reasonability of the prices of last survivor annuities in the private market. For this end, we propose a semi-Markov joint-life longevity risk model, and investigate the market prices of joint-life longevity risk in the US and UK, using the risk-neutral pricing theory.

The effect of mortality improvement has substantial impact on last survivor annuities. However, market prices of longevity risk in last survivor annuities for two components of mortality processes are quite volatile. Negative market prices of longevity risk calibrated from the prices of last survivor annuities in both the US and the UK market indicate that last survivor annuities may not be well priced currently.

We compare the market prices of joint-life longevity risk against the market prices of single-life longevity risk. The results indicate that the US market systematically underprices joint-life annuities and single-life annuities. The UK annuity market has more conservative allowance for longevity risk when pricing single-life annuities. Unfortunately we do not see consistent pricing of joint-life annuities.

Joint-life pricing structures are irrational in both the US and the UK annuity market. The impact could be destructive for the development of annuity market. Last survivor annuities are likely to become more critical following the European Union ban on gender-specific annuity rates, which will take effect in 2012. Careful attention is called for to avoid underpricing these products. Further study in fare pricing this type of annuities is required.

Deep-deferred annuities, which are also called longevity annuities or an advanced-life delayed annuity, have been introduced to the market recently. They are promoted to provide efficient protection against longevity risk (see, for example, Andrew, 2008; Gong and Webb, 2009). Last survivor deep-deferred annuities could be an important longevity hedge in an increasingly defined contribution world. Negative market prices of longevity risk imply that, the ability of the market to price last survivor deep-deferred annuities adequately is questionable. Pricing these annuities based on the market price of longevity risk calibrated from immediate annuities could be harmful. US and UK Insurers need to re-evaluate their pricing of last survivor annuities and other joint-life products.

The aforementioned remarks are based on the proposed semi-Markov joint-life longevity model, which is built up upon a Gompertz distribution with stochastic parameters for the stochastic modeling of force of mortality. The stochastic Gompertz mortality model is a natural extension of the Gompertz' law. It is most easily incorporated into the semi-Markov joint-life model. However, we acknowledge that mortality forecasting using models with stochastic parameter relies on the accuracy of the underlying parametric model.

Although the Gompertz curve generally fits adult mortality quite well, the exponential relationship between the force of mortality and aging has been criticized as overestimating mortality rates at the advanced ages. The argument of late-life mortality plateau suggests improvement over Gompertz' law at a cost of less parsimonious form. Li et al. (2008) proposed a threshold life table method, integrating the extreme value theory with Gompertz' law, offering a promising approach to the modeling of high age mortality.

Generally speaking, the proposed model is a preliminary step in the modeling and risk management of joint-life longevity risk. Currently, a little research has been done in this area. What we have done here is only a lead-in to future analysis. The correlation in the improvement of mortality for males and females has not been reflected. If there exists some correlation, it may have non-negligible impact on joint-life longevity risk. This should be investigated further, especially if suitable data become available.

Chapter 5

Discussion and Future Research

5.1 Markovian Approaches for Joint-life Mortality

Most insurance companies in practice postulate independence between joint lives. The unrealistic assumption of independence has a potentially significant financial impact on the industry. A model for the impact of one life's survivorship on another is required for products that provide benefits contingent on the combined survival status of multiple lives.

In Chapter 2, we explore Markovian models for the dependent modeling of joint lifetime random variables, as an important alternative to the copula method. A "common shock" factor is introduced into a standard Markov joint-life mortality model, for capturing instantaneous dependence between joint lifetimes. The semi-Markov property is exerted on the force of mortality for the widowed, capturing the decaying effect of the "broken heart" factor

In the proposed semi-Markov joint-life mortality model, a decreasing exponential function has been defined to describe the selection effect of bereavement. However, the parameters are estimated using relatively limited bivariate mortality data from a large Canadian insurance company. Collection of reliable industry-wide bivariate mortality data is highly encouraged. Further research work can thereafter be conducted on the dependent modeling of joint-life mortality in the framework of Markovian approaches.

Another future research topic is to refine the joint-life dependence structure in a Markovian model for joint-life mortality. The relationship between the mortality rates in the married state and in the widowed state could be further explored. The short-term dependence due to the impact of bereavement has been specifically modeled by a multiplicative function that decreases with the time in widowhood, while the long-term dependence due to common life styles shared by a husband and wife may be specified through a component in forces of mortality in the married status.

In addition, we have examined positive quadratic dependence in the Markov and semi-Markov model. Conditions for positive quadratic dependence in the Markov model are derived. Counter examples are used to disprove the existence of positive quadratic dependence in the semi-Markov model. In future work, a potentially challenging topic is to investigate the conditions for positive quadratic dependence in a semi-Markov mortality model.

5.2 Multiple State Model for Reverse Mortgage Terminations

In Chapter 3, the semi-Markov joint-life model is extended to model joint-life reverse mortgage terminations, incorporating other decrements than death. Event-triggered termination, status-dependent termination, and anti-selection effect on termination have been allowed for in the model. A specific semi-Markov multiple state model has been developed for the dependent modeling of joint-life reverse mortgage terminations, incorporating elaborately categorized termination modes. The implication of each termination mode to the value of a No-Negative-Equity-Guarantee in reverse mortgages has been investigated.

The model has a complex but well-organized state structure. With ever-developing computer capacity and improved simulation techniques, the proposed model can be easily applied in practice. As a multiple state model is employed in modeling reverse termination rates, the industry has an incentive to collect detailed termination data. Users can adapt the model according to the available reverse mortgage data. The model could be developed further to account for more realistic termination assumptions.

On the basis of a multiple state model for reverse mortgage terminations, we may consider the following suggestions for further research on reverse mortgages.

- Integrating economic factors into the multiple state termination model

  Economic factors, such as interest rates, and economic cycles, will significantly affect reverse mortgage terminations. For example, a borrower is more likely to move and repay his or her reverse mortgage in light of increases in home values during the economic boom. In a economic recession, homeowners will be less likely to move out. A regime-switching framework offers a potential solution.

- Projecting the financial status of the HECM portion of the Mutual Mortgage Insurance Fund

  The HECM program is now in the Mutual Mortgage Insurance Fund (MMIF). To ensure the financial soundness of MMIF, it is necessary to monitor the viability of the program. The risk of the program to the insurance fund is driven by three main risk factors: termination rates, interest rates, and the appreciation rates of home equity. These three factors are correlated. A model that can dependently model all these factors will lead to more reliable projection of the financial status of the HECM portion of the MMIF than independent assumptions.

5.3 Longevity Risk in Last Survivor Annuities

In Chapter 4, we incorporate a mortality projection method into the semi-Markov joint-life mortality model, to investigate the market prices of longevity risk in last survivor annuities. A preliminary semi-Markov joint-life longevity model has been established, which generalize Gompertz' law to a stochastic process.

The market prices of longevity risk are examined in a risk-neutral measure defined in Cairns et al. (2006). In the research literature on longevity/mortality risk, different methods are proposed for pricing the risk. Each method has its advantages, but there is no agreement on which method is preferable to the other. It is interesting to compare and discuss the market prices of joint-life longevity risk implied by different pricing theories.

Longevity risk has become an increasingly important risk in the annuity market and a hot research topic in the actuarial literature. Recent research on longevity risk mainly focuses on how to recognize, quantify and manage the risk. It is critical to develop a reliable model for the projection of mortality improvements. However, this task has proven to be difficult. We will continue, in future work, to explore solutions for longevity risk management and securitization. In addition to the aforementioned research directions, we may also consider the following topics.

- Incorporating correlation between male and female mortality improvements into the joint-life longevity model

Although male and female mortality rates have experienced different improvement speed (see Anderton et al., 1997), their trends will not diverge too much. In Chapter 4, we do not allow for any correlation between the stochastic dynamics for male and female mortality rates. This presumptive condition leads to some problems when calibrating the market prices of longevity risk in last survivor annuities. A method that allows for co-integration between two stochastic processes or controls the divergence between them can be employed to deal with the situation.

- Integrating the stochastic Gompertz model with the Extreme Value Theory.

  High age mortality and mortality improvement at advanced ages constitute a significant risk in annuity products. A typical product associated with such risk is a deep deferred annuity, where annuity payments commence at the very high ages, say 85 and above. Forecasting of high age mortality is a difficult task because of sparse data at older ages. The stochastic Gompertz model use a mathematical function of age and automatically permits extrapolations, which is not suitable for forecasting mortality at very advanced ages. The model, having been fitted to the mortality data at all ages, may not provide a particularly good fit at the advanced ages.

  Recent research has applied extreme value theory to this problem. The generalized Pareto distribution (GPD) offers a promising approach to the modeling of high age mortality. For example, Han (2003, 2005) applied a transformed generalized Pareto distribution to period life tables. Watts et al. (2006) used generalized extreme value and GPD to investigate advanced age mortality data. Li et al. (2008) has developed a threshold life table. To generalize the threshold life table methodology to a stochastic version, it is challenging to guarantee the smoothness of mortality distribution.

- Risk management of longevity risk in deep-deferred annuity

In a world of low levels of voluntary annuitization rates amongst the elderly, deep-deferred annuities offer a new way of boosting lifetime retirement income. In stead of paying a large lump sum at retirement for an immediate annuity, participants will only pay a small amount for a deep-deferred annuity and receive income at older ages. Deep-deferred annuities offer an efficient protection against longevity risk. However, the fundamental supply problem with deep deferred annuities relates to longevity risk, especially the upper tail risk.

Modeling and managing the involved longevity risk are challenging, especially in the case of joint lives. One potential research direction is to design a longevity risk backed security for risk management of deep-deferred annuities. A deferred longevity bond or a longevity risk swaption is a possible choice for transferring the risk to the capital market. The proposed securitization option should avoid early coupon payments that have very low longevity risk attached to them, overcoming the problem of being capital-intensive.

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