

Suboptimality of Asian Executive Options

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

This thesis applies the concept of cost efficiency to the design of executive compensation. In a classical Black-Scholes framework, we are able to express the cost efficient counterpart of the Asian Executive Option explicitly, and design a payoff that has the same distribution as the Asian Executive Indexed Option but comes at a cheaper price. The cost efficient counterpart of the latter option is not analytically tractable, but we are able to simulate its price.

Furthermore, we extend the study of these two types of options in the presence of stochastic interest rates modeled by a Vasicek process. We are able to derive new closed-form pricing formulas for these options. A framework for crafting the state price process is introduced. From here, an explicit expression for the state process is given and its distribution is derived.

Using the pricing formulas and the state price process, we are then able to simulate the prices of the corresponding cost efficient counterparts in a stochastic interest rate environment.

We conclude with some avenues for future research.

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Chapter 1

Introduction

During the recent financial crisis in 2008, executives made millions of dollars in the form of stock options while bankrupting their respective firms. Executives at Bear Stearns and Lehman Brothers made approximately \$1 billion USD from exercising stock options and stock purchases [2]. Would they have engaged in such excessive risk taking had they not been granted stock options? Meanwhile, executive options that were granted when the market bottomed out are now deep in the money due to the market's rebound [52]. Are executives being compensated for their performance, or are they simply riding the wave of recovering market prices?

The main idea of granting executive options is to encourage executives to increase shareholders value. Let us consider the toy example of an executive at Company X with a current stock price of \$100. The executive is given the right, but not the obligation, to purchase 100 units of Company X shares 10 years from now at \$100¹. With this contract in place, the executive has an incentive to work hard to increase the share price as it increases the payoff 10 years later.

If Company X stock price is greater than \$100 10 years from now, the executive receives a positive payoff; otherwise, the payoff is zero. If the stock price is greater (less) than \$100, is it due to the executive's effort, or good (bad) luck? If the executive is unscrupulous, he or she may be tempted to engage in all sorts of financial shenanigans in order to artificially increase the stock price, cash out on the contract, and retire.

¹This is essentially an at-the-money 10 year call option on Company X. It is a highly-simplified example; stock options that are issued in reality have many more bells and whistles. See [32] for some examples of non-traditional executive options.

The efficiency of traditional stock options was questioned by Hall and Murphy [25] where they argue that risk-averse and undiversified executives will actually value the options lower than what they actually cost the firms to grant. The idea of executives discounting the cost granting these options is further investigated by [53]. Despite the lack of consensus on how beneficial stock options actually are [13], their use remains widespread [52]. They have been blamed for the misalignment of incentives and excessive risk taking² as well as incentivizing executives to commit financial fraud [17]. Even though the use of stock options has been named as one of the culprits of the recent crisis, Fahlenbrach and Stulz [21] do not find any evidence of banks actually performing worse than their peers just because they issued more stock options and bigger cash bonuses.

1.1 Indexing and Averaging

In an earlier paper, Johnson and Tian [31] designed an executive option with a strike price that is indexed to a benchmark. The main tenet underlying the use of indexing is that the executive should only be rewarded (penalized) for out-performance (under-performance), and not serendipity. This ties back to our toy example of the executive being rewarded for his or her effort and not luck.

However, the use of indexing alone in executive options is “virtually non-existent” [25] and suboptimal according to Tian [53]. For that reason, the use of averaging is incorporated by [53] and found to be more cost effective (discounted less from their market values by risk averse executives) and incentive effective (stronger incentives to increase stock price) than traditional stock options³. With averaging in place, the unscrupulous executive would have to manipulate the entire stock price path instead of the price at just a particular point in time. This is presumably much harder to achieve than actually doing a good job at increasing shareholder value.

The conclusions drawn by [53] are based on analyses on the certainty equivalent value from an expected utility model. The two classes of options proposed are the Asian Executive Option (AEO), which takes the form of a continuous geometric average Asian call option, and the Asian Executive Indexed Option (AEIO), which takes the form of continuous geometric average Asian exchange option. A more detailed overview of [53] is given in Section 2.1 on page 7.

²“Heads, you become richer than Croesus; tails, you get no bonus, receive instead about four times the national average salary, and may (or may not) have to look for a new job” [11]

³Interestingly enough, the use of averaging in executive compensation is also suggested by Ariely [1] from the perspective of human irrationality.

1.2 Cost Efficiency

The primary focus of this thesis is the application of cost efficiency as prescribed by Bernard, Boyle and Vanduffel [4] to the design of the AEO and AEIO. In doing so, we are able to construct new payoffs that have the same distributions as the original payoffs, but come at a cheaper price. At this point, we want to stress that while Tian’s study of these options is primarily focused on their *incentives*, ours is focused on their *costs* instead.

Building on the work done by Dybvig [19][20], [4] devised an explicit representation of a cost efficient strategy. The notion of cost efficiency will be defined below, but loosely speaking, a strategy is said to be cost efficient if it is the cheapest possible way to achieve a particular probability distribution.

Even though the use of copulas is not new in finance [16] nor actuarial science [41], their novel and brilliant insight is the coupling of the state price process with the payoff itself. This allows the use of techniques from the theory of copulas, including the Fréchet-Hoeffding bounds to prove Theorem 1.2.4 on the following page and results from Tankov [51] to characterize cost efficient payoffs in the presence of state dependent preferences.

Using the same setup as [4], we will make the following assumptions:

1. We assume a Black-Scholes market. It is complete, frictionless and arbitrage free⁴. The risk free rate and volatility are assumed to be constant⁵.
2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the corresponding probability space. Then, there exists a state-price process ξ_t such that $\xi_t S_t$ is a martingale for all traded assets S in this market.
3. There are 2 risky assets in the market i.e. the stock (S_t) and index (I_t). Their price dynamics are driven by 2-dimensional correlated Brownian motions.
4. Agents have preferences that depend only on the terminal distribution of wealth.
5. We assume that all agents agree on the pricing operator. Therefore, the choice of the state price process is fixed.

We also recall the following definitions, and the main theorem that we will be using from [4].

⁴Even Black [9] was fully aware of the deficiencies of the Black-Scholes model. However, in terms of accounting standards, it is still used for expensing executive options [47].

⁵We will later extend this to include stochastic interest rates modeled by a Vasicek process.

Definition 1.2.1. The cost⁶ of a strategy with terminal payoff X_T is given by

$$c(X_T) = E[\xi_T X_T]$$

where the expectation is taken under the physical measure \mathbb{P} .

Definition 1.2.2. A payoff is cost efficient (CE) if any other strategy that generates the same distribution costs at least as much.

Definition 1.2.3. The distributional price of a cdf F is defined as

$$P_D(F) = \min_{\{Y_T | Y_T \sim F\}} c(Y_T)$$

where $\{Y_T | Y_T \sim F\}$ denotes the set of all payoffs that have the same distribution as F . The efficiency loss of a strategy with payoff X_T at maturity T with cdf F is equal to $c(X_T) - P_D(F)$.

Theorem 1.2.4. *Let ξ_T be continuous. Define*

$$Y_T^* = F_{X_T}^{-1}(1 - F_{\xi_T}(\xi_T))$$

as the cost efficient counterpart (CEC) of the payoff X_T . Then, Y_T^ is a CE payoff with the same distribution as X_T and is almost surely unique⁷.*

1.3 Main Results and Outline

The rest of this thesis is organized as follows. In Chapter 2 on page 7 we study the Asian Executive Option and Asian Executive Indexed Option that have been proposed by [53] in the classical Black-Scholes framework. Using the concept of cost efficiency, we are able to construct a new cost efficient counterpart of the Asian Executive Option. We also design a new payoff that is cheaper than the Asian Executive Indexed Option which we call the Power Exchange Executive Option. The cost efficient counterpart of the Asian Executive Indexed Option does not admit a closed form expression, but we are able to simulate its price and investigate the degree of efficiency loss through numerical techniques.

⁶Intuition: ξ_T represents the price of a particular state. When the sample space is discrete, the cost can be interpreted as the average of the outcome of each state weighted by its price.

⁷Intuition: The CEC is achieved by rearranging the outcomes of X_T in each state in reverse order with ξ_T while preserving the original distribution.

Starting from Chapter 3 on page 30 onwards, we will introduce stochastic interest rates modeled by a Vasicek process so that we can study the cost efficiency of the Asian Executive Options with this extra component. In order to do so, we require new pricing formulas for the AEO and AEIO as well as an expression for the state price process in the presence of stochastic interest rates.

The main results of Chapter 3 on page 30 is the derivation of new pricing formulas for the Geometric Asian Option and the Asian Exchange Option when interest rates follow the Vasicek process. As a verification of correctness, we also provide alternative derivations for the European Call Option and European Exchange Option that agree with existing results.

The aim of Chapter 4 on page 49 is a new derivation of the state price process in a market with two risky assets and stochastic interest rates modeled by the Vasicek process. We are able to derive an explicit formula for the state price process, find its distribution, as well as express it as a function of market variables. We also consider special cases with different combinations of number of risky assets and stochastic/constant interest rates, and derive new expressions for their corresponding state price processes in terms of market variables. Whilst we do not utilize these special cases, they can be readily applied to the study of cost efficiency for different classes of options.

Chapter 5 on page 62 uses the results from the previous two chapters to investigate the Asian Executive Option and Asian Executive Indexed Option in a stochastic interest rate environment. Their respective cost efficient counterparts do not admit closed-form expressions, so we rely on numerical methods once again to study the degree of efficiency loss.

To the best of our knowledge, the construction of the cost efficient counterpart for the Asian Executive Option and the design of the Power Exchange Executive Option are new results. In the presence of Vasicek interest rates, the pricing formulas for the Geometric Asian Option and the Asian Exchange Option are our contributions. We also find new formulas for the various cases of the state price process and expressing them in terms of the market variables. The latter is valuable because in a particular state of the world, the state price process itself is not directly observable because it is not a traded asset per se. However, since we have an expression of the state price process in terms of the market variables, we can first observe the market variables and then retrieve the state price process.

Chapter 6 on page 71 ends the thesis with some conclusions and avenues for future research.

Appendix A on page 74 contains some useful identities that are used in our proofs. All

proofs are provided in Appendix B on page 79, and we present a glossary of notations in Appendix C on page 109.

Chapter 2

Asian Executive Compensation

In this chapter, we will use the idea of cost-efficiency as spelled out by Bernard, Boyle and Vanduffel [4] to study the Asian Executive Option (AEO) and Asian Executive Indexed Option (AEIO) as proposed by Tian [53]. Again, we want to reiterate that while Tian's study of these options is primarily focused on their *incentives*, ours is focused on their *costs* instead.

We begin this chapter with an overview of the AEO and AEIO as designed by [53] in Section 2.1. In Section 2.2 on page 9, the framework for our analysis is specified. In Section 2.3 on page 10, we are able to construct the cost efficient counterpart (CEC) for the former option which turns out to be a power option. The latter does not admit an explicit CEC, but we are able to design an option that has the same distribution with a cheaper price in Section 2.4 on page 14. The simulation of the price of the true CEC is also considered, along with some observations about these three options.

The main contributions of this chapter are given in Propositions 2.3.1 on page 11 and 2.4.2 on page 16.

2.1 Overview

This section provides an overview of Tian's work on the use of indexing and averaging in executive options. All of the definitions and results herein are due to [53].

Tian argues that the practice of granting stock options based on terminal stock prices is

suboptimal and firms should be using the average¹ stock prices instead. This work builds on an earlier paper by Johnson and Tian [31] that discusses the use of indexing.

The use of averaging is found to be more cost effective and incentive effective than the traditional stock options. This means that for a given cost of the option grant, risk-averse executives will have a higher subjective value of the Asian option than the European one. On the other hand, incentive effective means that the Asian option provides executives with stronger incentives to increase stock price than the European one.

Tian first assumes that the executive's total wealth comprises of a fixed salary and stock options, and uses the certainty equivalent of the total wealth as a measure of the subjective value of the option grant. The analysis done on this subjective value as a fraction of the actual cost of issuing the option indicates that the Asian option is more cost effective.

A modified version of the pay-performance measure [28] defined as the ratio of the percentage change in the executive's total wealth to the percentage change in stock price is used to compare the incentive effectiveness of the Asian option. Loosely speaking, this involves the partial derivative of the certainty equivalent with respect to the firm's stock price. Using this measure, Tian concludes that the Asian option is indeed more incentive effective as well.

Why is the Asian option more cost effective and incentive effective than the traditional stock option? Firstly, the use of averaging reduces the volatility of the final payoff of the option to the executive, which is more beneficial to risk-averse executives. Secondly, the use of averaging makes it much harder for executives to manipulate the final payoff of the option. Instead of artificially increasing the stock price for a particular point in time, they would have to increase the entire stock price path. Thirdly, averaging also increases the probability of the option expiring in the money. This rectifies the problem of indexed options (indexing alone, without averaging) having lower probabilities of expiring in the money than traditional stock options [25].

Remark 2.1.1. The use of the certainty equivalent in assessing an option from the executive's perspective was proposed by Lambert, Larcker and Verrecchia [36]. The extra constraints imposed on executives (e.g. short selling constraints, restrictions on the sales of the underlying firm's stock, non-transferability of the option itself, and etc.) invalidate the dynamic-hedging arguments. Hence, the *value* of the option to an executive is not necessarily its *cost* à la Black-Scholes.

¹It is pointed out that the geometric average should be used instead of the arithmetic average since the arithmetic average will not penalize a spread that preserves the mean but increases the variance [43]. We opt for the geometric average simply because it is analytically tractable.

2.2 Framework

In this section, we specify the notations as given in [53]. Firstly, we define the \mathbb{P} -dynamics followed by the stock and index as follow:

$$\begin{aligned}\frac{dS_t}{S_t} &= (\mu_S - q_S)dt + \sigma_S dW_t^S \\ \frac{dI_t}{I_t} &= (\mu_I - q_I)dt + \sigma_I dW_t^I \\ dW_t^S dW_t^I &= \rho dt\end{aligned}\tag{2.2.1}$$

(μ and q are respectively the expected return and dividend rate, σ is the volatility and ρ is the correlation. K as used below is understood to be the strike price.)

We then define the geometric averages of the stock and the index as well as the benchmarks:

$$\begin{aligned}\hat{S}_T &= e^{\frac{1}{T} \int_0^T \ln(S_t) dt} & \hat{I}_T &= e^{\frac{1}{T} \int_0^T \ln(I_t) dt} \\ \hat{H}_T &= K(\hat{I}_T/\hat{I}_0)^{\hat{\beta}} \exp(\hat{\eta}T) & H_T &= K(I_T/I_0)^\beta \exp(\eta T)\end{aligned}\tag{2.2.2}$$

where

$$\begin{aligned}\hat{\eta} &= (r - q_S^*) - \hat{\beta}(r - q_I^*) + \frac{1}{2}\hat{\sigma}_I^2\hat{\beta}(1 - \hat{\beta}) \\ \eta &= (r - q_S) - \beta(r - q_I) + \frac{1}{2}\sigma_I^2\beta(1 - \beta) \\ \hat{\beta} &= \frac{\rho\hat{\sigma}_S}{\hat{\sigma}_I} \\ \beta &= \frac{\rho\sigma_S}{\sigma_I}\end{aligned}\tag{2.2.3}$$

and

$$\begin{aligned}q_S^* &= \frac{1}{2} \left(r + q_S + \frac{\sigma_S^2}{6} \right) & q_I^* &= \frac{1}{2} \left(r + q_I + \frac{\sigma_I^2}{6} \right) \\ \hat{q}_S &= \frac{1}{2} \left(\mu_S + q_S + \frac{\sigma_S^2}{6} \right) & \hat{q}_I &= \frac{1}{2} \left(\mu_I + q_I + \frac{\sigma_I^2}{6} \right) \\ \hat{\sigma}_S &= \frac{\sigma_S}{\sqrt{3}} & \hat{\sigma}_I &= \frac{\sigma_I}{\sqrt{3}}\end{aligned}\tag{2.2.4}$$

Remark 2.2.1. The $\hat{\cdot}$ accent is used to denote the geometric average of a particular parameter; omitting $\hat{\cdot}$ refers to the terminal value. The relationship between the terminal and geometric average parameters (e.g. between q_S and \hat{q}_S) is due to the distribution of the geometric averages of the stock and the index. See Kemna and Vorst [34] for more details.

Remark 2.2.2. H_T as defined in (2.2.2) can be thought of as the price of an imaginary asset that tracks the expected performance of the firm's stock, given that the index has no excess return. The idea is that the executive should only be rewarded for firm-specific performance. It was first proposed by [31] and we recall their rationale here.

Firstly they define the excess return on the stock as

$$\alpha = \mu_S - r - \beta(\mu_I - r) \quad (2.2.5)$$

where β is given in (2.2.3) on the previous page. They point out that the motivation behind β is not from the Capital Asset Pricing Model (though it looks identical), but from the fact that including β in (2.2.5) “produces an aggregate performance index (α) on which an optimal sharing rule can be based”.

The benchmark is designed such that the executive is only rewarded for firm-specific performance by conditioning the stock price on the index price assuming the latter has no excess return. They calculate the following conditional expectation:

$$E(S_t|I_t, \alpha = 0) = S_0 \left(\frac{I_t}{I_0} \right)^\beta e^{\eta t} \quad (2.2.6)$$

where η is given in (2.2.3) on the previous page. Based on (2.2.6), they then define H_T as given in (2.2.2) on the previous page.

2.3 Asian Executive Option

With the above framework in place, we can now derive the CEC of the Asian Executive Option. Its payoff at time T is defined as

$$\hat{G}_T = (\hat{S}_T - K)^+ \quad (2.3.1)$$

and the price of \hat{G}_T at time zero, \hat{E}_0 , is simply the price of a continuous geometric Asian option:

$$\begin{array}{l} \hat{E}_0 = S_0 e^{-\frac{1}{2}(r+q_S+\frac{1}{6}\sigma_S^2)T} N(d_1) - K e^{-rT} N(d_2) \\ \text{where} \\ d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \frac{1}{2}\left(r - q_S + \frac{\sigma_S^2}{6}\right)T}{\sigma_S \sqrt{T/3}} \\ d_2 = d_1 - \sigma_S \sqrt{T/3} \end{array} \quad (2.3.2)$$

Proposition 2.3.1 on the following page extends the derivation of [4] and the price of a power option given by Macovschi and Quittard-Pinon [39] to the case of positive dividend yield.

Proposition 2.3.1. *The cost efficient counterpart of the Asian Executive Option (2.3.1) on the previous page is given by*

$$G_T = d \left(S_T^{1/\sqrt{3}} - \frac{K}{d} \right)^+ \quad (2.3.3)$$

where $d = S_0^{1-1/\sqrt{3}} e^{\left(\frac{1}{2}-\frac{1}{\sqrt{3}}\right)\left(\mu_S - q_S - \frac{\sigma_S^2}{2}\right)T}$. This is a power call option which is the simple case of a polynomial option. Its price at time zero is given by

$$E_0 = S_0 e^{\left\{ \left(\frac{1}{\sqrt{3}}-1\right)r + \left(\frac{1}{2}-\frac{1}{\sqrt{3}}\right)\mu_S - \frac{q_S}{2} - \frac{\sigma_S^2}{12} \right\}T} \Phi(d_1) - K e^{-rT} \Phi(d_2) \quad (2.3.4)$$

where

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left[\left(\frac{1}{2}-\frac{1}{\sqrt{3}}\right)(\mu_S - q_S) + \frac{r-q_S}{\sqrt{3}} + \frac{\sigma_S^2}{12}\right]T}{\sigma_S \sqrt{\frac{T}{3}}}$$

$$d_2 = d_1 - \sigma_S \sqrt{\frac{T}{3}}$$

Remark 2.3.2. Note that in this case, we are implicitly assuming that the only risky asset that is traded in the market is the stock (S_T), which gives us a simple expression for the state price process that only depends on one source of randomness. If we assume that the index is also traded in the market, then the state price process, as well as the CEC, will take a rather different form. In fact, the single-asset payoff G_T will be suboptimal in a multidimensional Black-Scholes market [7].

Remark 2.3.3. When $\mu_S = r$, it is clear that $E_0 = \hat{E}_0$ from their respective pricing formulas. In fact, the price of G_T is a decreasing function of the stock yield, and is only cheaper than \hat{G}_T when the stock yield is greater than the risk free rate. We can see this by considering the first derivative of E_0 with respect to the stock yield i.e. $\frac{\partial}{\partial \mu_S} E_0 \leq 0$ (and that \hat{E}_0 does not depend on μ_S). Therefore, G_T is less expensive \hat{G}_T if and only if $\mu_S > r$. This comes also from the fact that the CEC has a different form when $\mu_S < r$, because in that case it is non-increasing with the underlying stock price (see [4]).

Figure 2.3.1 on page 13 plots some prices of \hat{G}_T and G_T across different sets of parameters to illustrate the efficiency loss (recall Definition 1.2.3 on page 4). Each point corresponds to the price for a particular parameter set. We consider the base case given in Table 2.3.1 on the following page, and perturb some of the input parameters as well. We

can see that the degree of efficiency loss is sensitive to all model parameters. The values that are used to plot this figure are given in Table 2.3.2.

Option		Interest		Stock	
T	1	r	6%	S_0	100
K	100			σ_S	30%
				μ_S	12%
				q_S	2%

Table 2.3.1: Base case parameters for sample \hat{G}_T and G_T .

Parameters	G_T	\hat{G}_T	
	E_0	\hat{E}_0	% Eff Loss
Base Case	6.9032	9.2567	34.09%
$K = 80$	20.1972	23.8842	18.25%
$r = 4\%$	6.4431	8.2032	27.32%
$S_0 = 120$	20.9597	25.2216	20.33%
$\mu_S = 8\%$	7.0668	9.2567	30.99%
$\sigma_S = 35\%$	7.8444	10.3265	31.64%
$q_S = 1.5\%$	7.0352	9.4413	34.20%

Table 2.3.2: Prices and efficiency loss of \hat{G}_T compared against G_T across different parameters.

These are used to generate Figure 2.3.1 on the following page and the base case parameters are given in Table 2.3.1.

We can see that across all cases, the price of \hat{G}_T is greater than G_T in accordance with Theorem 1.2.4 on page 4. In the base case the efficiency loss of \hat{G}_T vs G_T is 34.09% (9.2567 vs 6.9032). The degree of efficiency loss is very much a function of the input parameters, with the highest loss of 34.20% (when we perturb q_S to 1.5%) and the lowest loss of 18.25% (when we perturb K to 80).

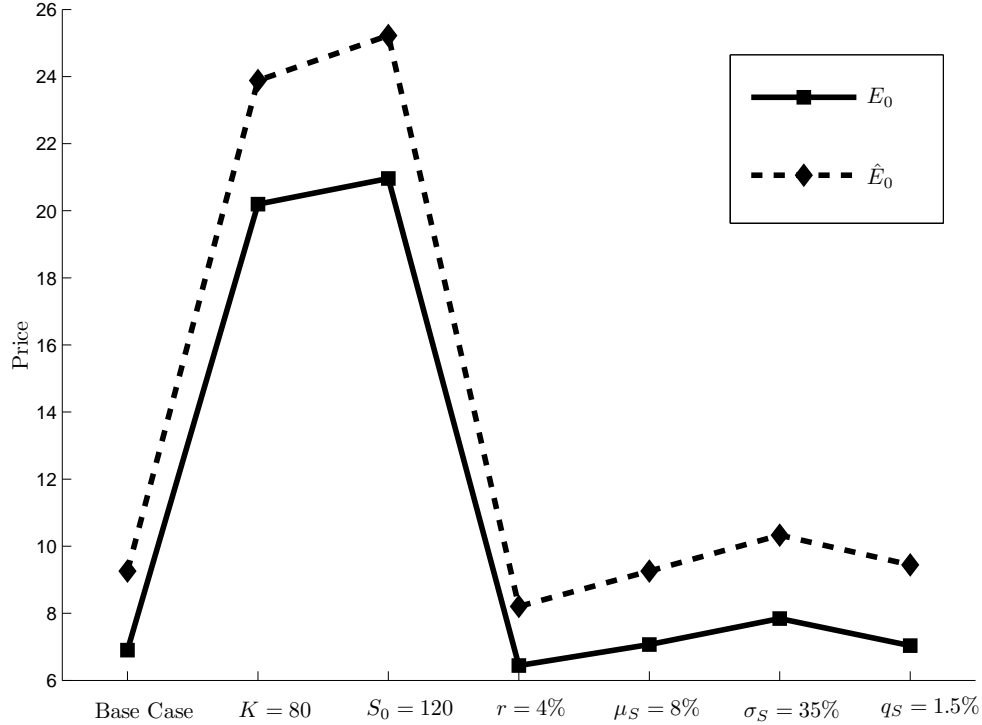


Figure 2.3.1: Sample prices for \hat{G}_T and G_T across different cases

The values used to generate this plot are given in Table 2.3.2 on the previous page.

Note that since G_T is a power call option, we have effectively removed the geometric averaging component from the AEO. By removing this path-dependence, we have created a payoff that has the same distribution as the AEO but comes at a cheaper price. Although the payoff only depends on the terminal stock price, we could argue that this value is dampened by the exponent of $\frac{1}{\sqrt{3}}$ (see (2.3.3) on page 11). This means that a wider swing in the terminal stock price is required for the same dollar impact on the payoff of the power call option as the AEO. Therefore, to a certain degree, the benefits of averaging i.e. reduction in volatility and difficulty in payoff manipulation (see Section 2.1 on page 7) are still retained. In any case, the impact on executives' incentives is an issue that deserves further research.

Figure 2.3.2 plots the prices of \hat{G}_T and G_T vs the risk free rate, r . The parameters used to generate the prices are the same base case parameters in Table 2.3.1 on page 12. The vertical line corresponds to the expected stock return of $\mu_S = 12\%$ in our base case. When $\mu_S > r$, we can see that the price of \hat{G}_T is greater than the price of G_T . When $\mu_S < r$, the opposite is true, with equality holding when $\mu_S = r$. This graph illustrates the observation that we made in Remark 2.3.3 on page 11.

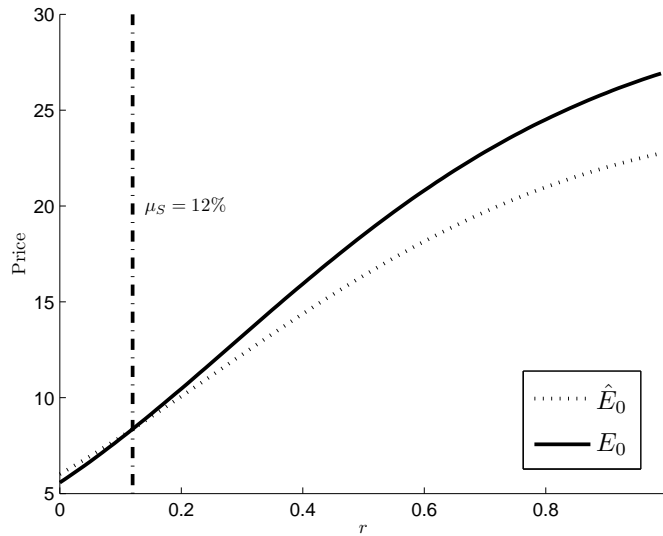


Figure 2.3.2: Prices of \hat{G}_T and G_T vs r

The parameters used to generate this plot are given in Table 2.3.1 on page 12.

2.4 Asian Executive Indexed Option

The main contribution of Tian [53] is the design of the Asian Executive Indexed Option (AEIO) as a form of executive compensation. The analysis shows that the AEIO is more effective than traditional stock options and provide stronger incentives for the executive to increase stock price. Its payoff at time T is given by

$$\hat{A}_T = (\hat{S}_T - \hat{H}_T)^+ \quad (2.4.1)$$

where \hat{S}_T is the average stock price and \hat{H}_T is the non-constant strike price that is linked to the performance of the average benchmark index adjusted for the level of the systematic risk $\hat{\beta}$. The definitions of \hat{H}_T and $\hat{\beta}$ are the geometric average analogues of H_T and β (see Remark 2.2.2 on page 9).

The price of \hat{A}_T at time 0 is given by [53]:

$$\hat{V}_0 = \exp(-q_S^* T) [S_0 \Phi(\hat{d}_1) - K \Phi(\hat{d}_2)] \quad (2.4.2)$$

where

$$\hat{d}_1 = \frac{\ln(S_0/K) + \frac{1}{2} \hat{\sigma}_S^2 (1 - \rho^2) T}{\hat{\sigma}_S \sqrt{(1 - \rho^2) T}}$$

$$\hat{d}_2 = \hat{d}_1 - \hat{\sigma}_S \sqrt{(1 - \rho^2) T}$$

and

$$q_S^* = \frac{1}{2} \left(r + q_S + \frac{\sigma_S^2}{6} \right)$$

$$\hat{\sigma}_S = \frac{\sigma_S}{\sqrt{3}}$$

It also follows from Margrabe's formula for exchange options [40].

2.4.1 Construction of a Cheaper Payoff

Unfortunately, since the payoff \hat{A}_T involves the difference of two lognormal random variables, there is no closed form expression for the payoff CDF (see, e.g. Johnson, Kotz and Balakrishnan [30]). Therefore, Theorem 1.2.4 on page 4 cannot be used directly to find the CEC. One alternative is to find the CEC for \hat{S}_T and \hat{H}_T separately, say S_T^* and H_T^* , and construct the new payoff, which we call the Power Exchange Executive Option (PXEO)

$$A_T^* = (S_T^* - H_T^*)^+ \quad (2.4.3)$$

We are able to show that A_T^* has the same distribution as \hat{A}_T , and yet comes at a cheaper price i.e. we have constructed a *cheaper* payoff.

Remark 2.4.1. We once again emphasize (at the risk of beating a dead horse) that our focus is on the cost of this new option that shares the same distribution as the AEIO while the Tian's focus is on the incentive. The latter is beyond the scope of this thesis and is left for future research.

Proposition 2.4.2 details the construction of such a payoff whereas Corollary 2.4.4 gives its price.

Proposition 2.4.2. *Define respectively S_T^* and H_T^* as follow*

$$S_T^* = d_S S_T^{1/\sqrt{3}} \quad (2.4.4)$$

where $d_S = S_0^{1-1/\sqrt{3}} e^{\left(\frac{1}{2}-\frac{1}{\sqrt{3}}\right)\left(\mu_S - q_S - \frac{\sigma_S^2}{2}\right)T}$ and

$$H_T^* = d_H H_T^{1/\sqrt{3}} \quad (2.4.5)$$

where $d_H = K^{1-1/\sqrt{3}} \exp \left\{ \left(\frac{1}{2} - \frac{1}{\sqrt{3}} \right) (\mu_S - q_S) T + \frac{\sigma_S^2 T}{2} \left[\rho^2 \left(\frac{1}{\sqrt{3}} - \frac{1}{3} \right) - \frac{1}{6} \right] \right\}$. Let

$$A_T^* = (S_T^* - H_T^*)^+ = (d_S S_T^{1/\sqrt{3}} - d_H H_T^{1/\sqrt{3}})^+ \quad (2.4.6)$$

Then \hat{A}_T and A_T^* have the same distribution under the \mathbb{P} measure.

Remark 2.4.3. Note that $A_T^* \in \{Y_T | Y_T \sim F_{\hat{A}_T}\}$ (see Definition 1.2.3 on page 4) but it is not the CEC of \hat{A}_T . This is because it is not a function of the state price process in the 2-dimensional market constituted by the stock and the index (see [4] or [7]).

Corollary 2.4.4. *The price of A_T^* at time 0 is given by*

$$V_0^* = \Upsilon^* [S_0 \Phi(d_1^*) - K \Phi(d_2^*)] \quad (2.4.7)$$

where

$$d_1^* = \frac{\ln\left(\frac{S_0}{K}\right) + \frac{1}{2}\nu^2 T}{\nu\sqrt{T}}$$

$$d_2^* = d_1^* - \nu\sqrt{T}$$

and

$$\nu^2 = \frac{1}{3}\sigma_S^2(1 - \rho^2)$$

$$\Upsilon^* = \exp \left\{ \left[\left(\frac{1}{2} - \frac{1}{\sqrt{3}} \right) \mu_S - \frac{q_S}{2} + \left(\frac{1}{\sqrt{3}} - 1 \right) r - \frac{\sigma_S^2}{12} \right] T \right\}$$

\hat{A}_T is strictly more expensive than A_T^* when $\mu_S > r$.

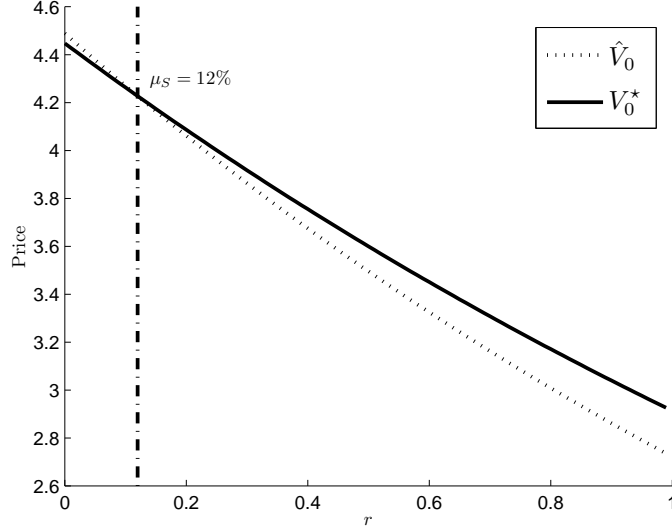


Figure 2.4.1: Prices of \hat{A}_T and A_T^* vs r

The parameters used to generate this plot are given in Table 2.4.1 on page 21.

Figure 2.4.1 plots the prices of \hat{A}_T and A_T^* vs the risk free rate, r . The parameters used to generate the prices are the base case parameters in Table 2.4.1 on page 21. The vertical line corresponds to the expected stock return of $\mu_S = 12\%$ in our base case. When $\mu_S > r$, we can see that the price of \hat{A}_T is greater than the price of A_T^* . When $\mu_S < r$, the opposite is true, with equality holding when $\mu_S = r$. We have seen this relationship between the ordering of the option prices and the ordering of μ_S and r before when we considered \hat{G}_T and G_T . In that case, the ordering is a result of G_T being the CEC of \hat{G}_T (see Remark 2.3.3 on page 11). However, in this case, even though V_T^* is not the CEC of \hat{V}_T , this relationship still holds and it illustrates Corollary 2.4.4 on the previous page.

Remark 2.4.5. Corollary 2.4.4 on the preceding page does not imply an arbitrage opportunity. Even though $\hat{A}_T \sim A_T^*$ (i.e. \hat{A}_T and A_T^* have the same distribution) and $V_0^* < \hat{V}_T$, we do not have state-by-state dominance of A_T^* over \hat{A}_T . However, when initial costs are taken into account, A_T^* does dominate \hat{A}_T in the first-order stochastic sense (see Levy [37]) i.e. $(\hat{A}_T - \hat{V}_0) \prec_{f_{sd}} (A_T^* - V_0^*)$. The proof of this fact is identical to the proof of Proposition 5 in [4] and is omitted.

Remark 2.4.6. Unfortunately, the program applied to \hat{A}_T to construct A_T^* does not always yield a cheaper payoff. For instance consider the option proposed by Kim [35]. Unlike [53],

only the index is averaged - the terminal payoff of the underlying asset is still used. This payoff is given by

$$\hat{C}_T = (S_T - \hat{B}_T)^+ \quad (2.4.8)$$

where

$$\begin{aligned} \hat{B}_T &= \lambda S_0 (\hat{I}_T / I_0)^\beta \exp(\hat{\phi}T) \\ \hat{\phi} &= \frac{1}{2} \left[r - q_S - \beta(r - q_I) - \frac{1}{6}(\sigma_S^2 - (3 - 2\beta)\beta\sigma_I^2) \right] \end{aligned}$$

(λ is an extra parameter introduced to make the design of the option grant more flexible. It is not to be confused with the market price of risk which will be defined and used in Chapters 4 to 5 on pages 49–62.) Following the proof of Proposition 2.4.2 on page 16 and Corollary 2.4.4 on page 16, we can construct the new payoff as

$$C_T^* = (S_T - B_T^*)^+ = \left(S_T - d_B B_T^{1/\sqrt{3}} \right)^+ \quad (2.4.9)$$

where $d_B = (\lambda S_0)^{1-1/\sqrt{3}} \exp \left\{ \left(\frac{1}{2} - \frac{1}{\sqrt{3}} \right) (\mu_S - q_S)T + \frac{\sigma_S^2 T}{2} \left[\rho^2 \left(\frac{1}{\sqrt{3}} - \frac{1}{3} \right) - \frac{1}{6} \right] \right\}$ with $\hat{C}_T \sim C_T^*$.

Its price at time 0 is given by

$$\begin{aligned} P_0^* &= \Upsilon_S \Phi(d_1^*) - \Upsilon_B \Phi(d_2^*) \quad (2.4.10) \\ \text{where} \\ \Upsilon_S &= S_0 \exp(-q_S T) \\ \Upsilon_B &= \lambda S_0 \exp \left\{ \left[\left(\frac{1}{2} - \frac{1}{\sqrt{3}} \right) \mu_S - \frac{q_S}{2} + \left(\frac{1}{\sqrt{3}} - 1 \right) r - \frac{\sigma_S^2}{12} \right] T \right\} \\ \nu^2 &= \sigma_S^2 \left[1 + \rho^2 \left(\frac{1}{3} - \frac{2}{\sqrt{3}} \right) \right] \\ d_1^* &= \frac{-\ln(\lambda) + \left[\left(\frac{1}{\sqrt{3}} - \frac{1}{2} \right) \mu_S + \left(1 - \frac{1}{\sqrt{3}} \right) r - \frac{q_S}{2} + \frac{\sigma_S^2}{12} + \frac{\nu^2}{2} \right] T}{\nu \sqrt{T}} \\ d_2^* &= d_1^* - \nu \sqrt{T} \end{aligned}$$

When $\mu_S = r$, we can check that \hat{C}_T and C_T^* have the same price at time zero. However, when $\mu_S > r$, the latter is actually more expensive by inspecting that $\frac{\partial}{\partial \mu_S} P_0^* > 0$. This means that we have actually constructed a more expensive payoff and goes to show that the program that we applied to \hat{A}_T should only be used with care.

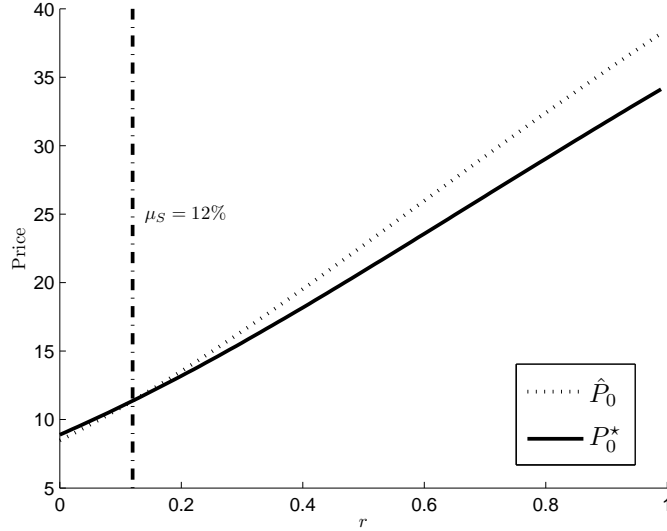


Figure 2.4.2: Prices of \hat{C}_T and C_T^* vs r

The parameters used to generate this plot are given in Table 2.4.1 on page 21.

Figure 2.4.2 plots the prices of \hat{C}_T and C_T^* vs the risk free rate, r . The parameters used to generate the prices are the base case parameters in Table 2.4.1 on page 21. The vertical line corresponds to the expected stock return of $\mu_S = 12\%$ in our base case. When $\mu_S > r$, we can see that the price of \hat{C}_T is *less* than the price of C_T^* . When $\mu_S < r$, the opposite is true, with equality holding when $\mu_S = r$.

2.4.2 The True Cost Efficient Counterpart

We have identified at least one payoff, A_T^* , with the same distribution as the Asian Indexed Option and comes at a cheaper price. The true CEC, which we denote by A_T (and its price at time 0 by V_0) can be estimated using numerical techniques. In order to do so, we need an expression for the state price process for a 2-dimensional market constituted by the stock and the index. To that effect, we follow the convention and results given by [7].

We define the (2×2) matrix Σ as

$$\Sigma = \begin{bmatrix} \sigma_S^2 & \rho\sigma_S\sigma_I \\ \rho\sigma_S\sigma_I & \sigma_I^2 \end{bmatrix}$$

and the drift vector

$$\underline{\mu} = \begin{bmatrix} \mu_S - q_S \\ \mu_I - q_I \end{bmatrix}$$

The constant portfolio is defined as $\underline{\pi}_\rightarrow(t) = \underline{\pi}_\rightarrow = [\pi_S, \pi_I]^T$ where the fractions of the portfolio invested in the stock and index remain constant over time. The terminal value of the security that is constructed using the constant mix $\underline{\pi}_\rightarrow$ is given by:

$$S_T^\pi = \exp \left\{ \left(\mu(\pi) - \frac{\sigma^2(\pi)}{2} \right) T + \sigma(\pi) W_T^\pi \right\}$$

where W_t^π is a standard Brownian motion defined by

$$W_t^\pi = \frac{1}{\sqrt{\underline{\pi}_\rightarrow^T \cdot \Sigma \cdot \underline{\pi}_\rightarrow}} (\pi_S \sigma_S W_t^S + \pi_I \sigma_I W_t^I)$$

and

$$\mu(\pi) = r + \underline{\pi}_\rightarrow^T \cdot (\underline{\mu} - r \cdot \underline{\mathbb{1}}), \quad \sigma^2(\pi) = \underline{\pi}_\rightarrow^T \cdot \Sigma \cdot \underline{\pi}_\rightarrow$$

The market portfolio is given by

$$\underline{\pi}_\rightarrow^* = \frac{\Sigma^{-1} \cdot (\underline{\mu} - r \cdot \underline{\mathbb{1}})}{\underline{\mathbb{1}}^T \cdot \Sigma^{-1} \cdot (\underline{\mu} - r \cdot \underline{\mathbb{1}})} \quad (2.4.11)$$

Remark 2.4.7. The market portfolio given in (2.4.11) is the unique mean-variance efficient portfolio that is fully invested in risky assets (see Proposition 1 in [7]). For the close relations between the market portfolio and the so-called growth optimal portfolio, see [18], [45] or [46].

Let $\theta_* := \theta(\pi_*) = \frac{\mu(\pi_*) - r}{\sigma(\pi_*)}$. Then, the state price process $\xi_*(T)$ is given by

$$\xi_*(T) = \exp \left\{ -rT - \frac{1}{2} \theta_*^2 T - \theta_* W_T^{\pi_*} \right\} \quad (2.4.12)$$

This means that $\xi_*(T) \sim \mathcal{LN}(M_*, \theta_*^2 T)$ where $M_* = -rT - \frac{1}{2} \theta_*^2 T$. The true cost-efficient payoff is given by

$$A_T = F_{\hat{A}_T}^{-1}(1 - F_{\xi_*}(\xi_*(T))) \quad (2.4.13)$$

and its price at time 0 is given by

$$V_0 = E_{\mathbb{P}}[\xi_*(T)A_T] = E_{\mathbb{P}}[\xi_*(T)F_{\hat{A}_T}^{-1}(1 - F_{\xi_*}(\xi_*(T)))] \quad (2.4.14)$$

Now that we have the state price process, it is straightforward to use Monte Carlo simulation to estimate the price of the CEC. This is done by making random draws from the known distribution of $\xi_*(T)$ and evaluating the inverse CDF $F_{\hat{A}_T}^{-1}$ numerically (both under the \mathbb{P} measure).

Figure 2.4.3 on page 24 plots some prices of \hat{A}_T , A^* , A_T across different sets of parameters to illustrate the efficiency loss. Each point corresponds to the price for a particular parameter set. We consider the base case given in Table 2.4.1, and perturb some of the input parameters as well. We can see that the degree of efficiency loss is sensitive to all model parameters. The values that are used to plot this figure are given in Table 2.4.1.

Option		Interest		Stock		Index		Correlation		Monte Carlo	
T	1	r	6%	S_0	100	I_0	100	ρ	0.75	Paths	1,000,000
K	100			σ_S	30%	σ_I	20%				
				μ_S	12%	μ_I	10%				
				q_S	2%	q_I	3%				

Table 2.4.1: Base case parameters for sample \hat{A}_T , A_T^* and A_T .

Parameters	A_T		A_T^*		\hat{A}_T	
	V_0	St Dev	V_0^*	% Eff Loss	\hat{V}_0	% Eff Loss
Base Case	3.2491	0.0075	4.3359	33.45%	4.3561	34.07%
$K = 80$	17.2030	0.0122	19.0787	10.90%	19.1674	11.42%
$r = 4\%$	2.9594	0.0075	4.3727	47.76%	4.3998	48.68%
$S_0 = 120$	17.0242	0.0142	19.2661	13.17%	19.3557	13.70%
$\mu_S = 8\%$	3.9538	0.0074	4.3493	10.00%	4.3561	10.17%
$\mu_I = 13\%$	3.2561	0.0075	4.3359	33.16%	4.3561	33.78%
$\sigma_S = 35\%$	3.9730	0.0090	5.0439	26.95%	5.0673	27.54%
$\sigma_I = 15\%$	3.2579	0.0076	4.3359	33.09%	4.3561	33.71%
$q_S = 1.5\%$	3.2632	0.0076	4.3467	33.20%	4.3670	33.82%
$q_I = 2\%$	3.2598	0.0075	4.3359	33.01%	4.3561	33.63%
$\rho = 0.9$	2.2739	0.0049	2.8582	25.70%	2.8715	26.28%

Table 2.4.2: Prices and efficiency loss of \hat{A}_T and A_T^* and compared against A_T across different parameters.

These are used to generate Figure 2.4.3 on page 24 and the base case parameters are given in Table 2.4.1 on the previous page.

Since A_T^* is a power exchange option, we have effectively removed the geometric averaging component from the AEIO. By removing this path-dependence, we have created a payoff that has the same distribution as the AEIO but comes at a cheaper price. Although the payoff only depends on the terminal stock and benchmark prices, we could argue that these values are dampened by the exponent of $\frac{1}{\sqrt{3}}$ (see (2.4.6) on page 16). This means that a wider swing in the terminal stock and benchmark prices are required for the same dollar impact on the payoff of the power call option as the AEO. Therefore, to a certain degree, the benefits of averaging i.e. reduction in volatility and difficulty in payoff (see Section 2.1 on page 7) manipulation are still retained. The impact on executives' incentives is an area that requires further research.

We can see that across all cases, the price of \hat{A}_T is greater than A_T^* , which is greater than A_T , in accordance with Theorem 1.2.4 on page 4. In the base case the efficiency loss of A_T^* vs A_T is 33.45% (4.3359 vs 3.2491), while the efficiency loss of \hat{A}_T vs A_T is 34.07% (4.3561 vs 3.2491). The degree of efficiency loss is very much a function of the input parameters, with the highest loss when we perturb r to 4% (A_T^* vs A_T of 47.76% and \hat{A}_T vs A_T of 48.68%) and the lowest loss when we perturb K to 80 (A_T^* vs A_T of 10.90% and \hat{A}_T vs A_T of 11.42%)

It is also interesting to note that the prices of A_T^* and \hat{A}_T are very close to each other, with the former still being cheaper than the latter in all cases. This highlights the fact that even though our method of construction yields a cheaper payoff, it is still not the cheapest payoff. For the latter, we still need to rely fully on Theorem 1.2.4 on page 4.

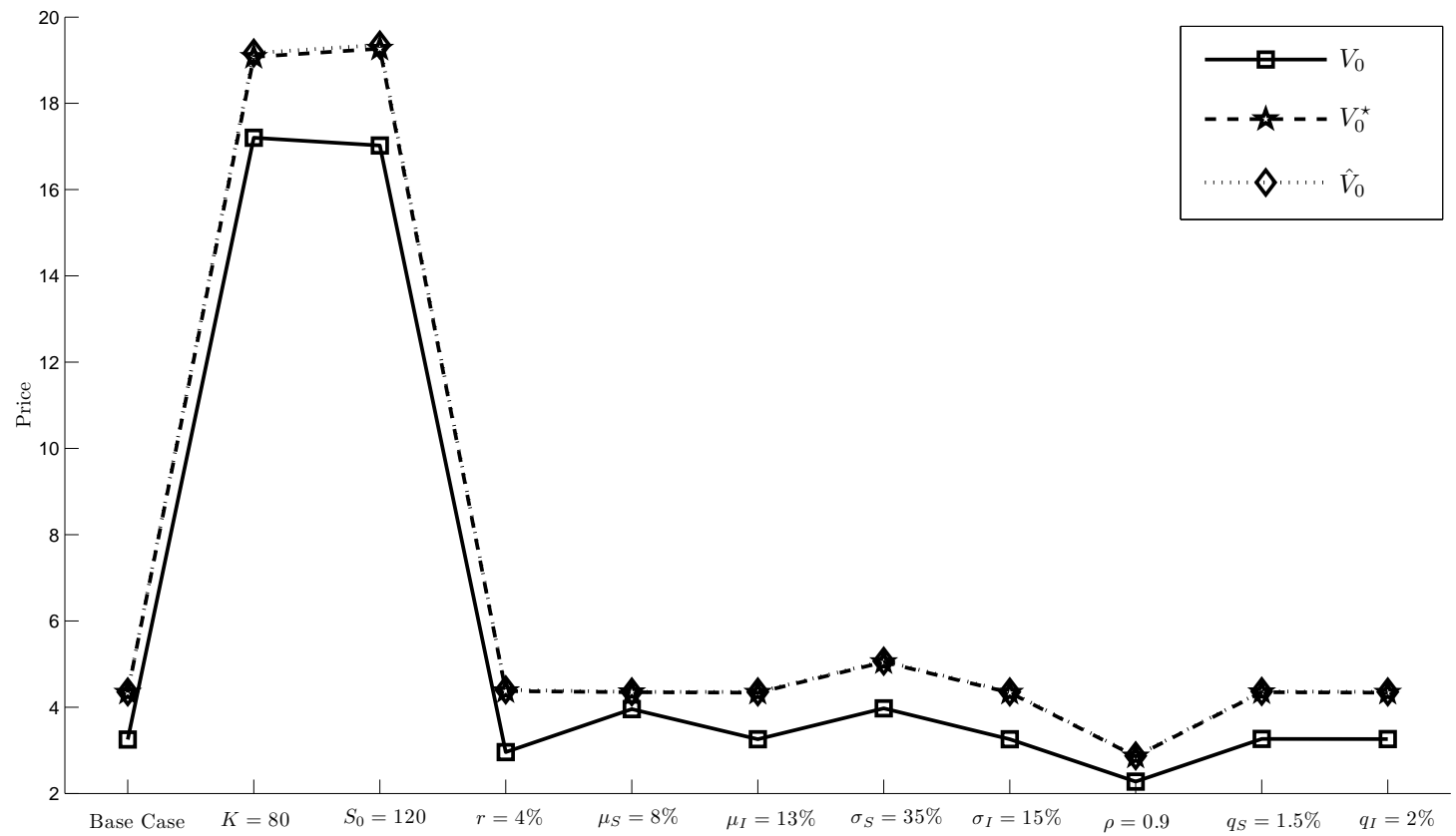


Figure 2.4.3: Sample prices for \hat{V}_T , V_T^* and V_T across different cases

The values used to generate this plot are given in Table 2.4.2 on page 22.

2.4.3 \hat{A}_T vs A_T^* vs A_T

Now that we have all three options in place: the Asian Executive Indexed Option (\hat{A}_T), the Power Exchange Executive Option that we have constructed (A_T^*), and the true cost efficient counterpart (A_T), we can take a closer look at all three together.

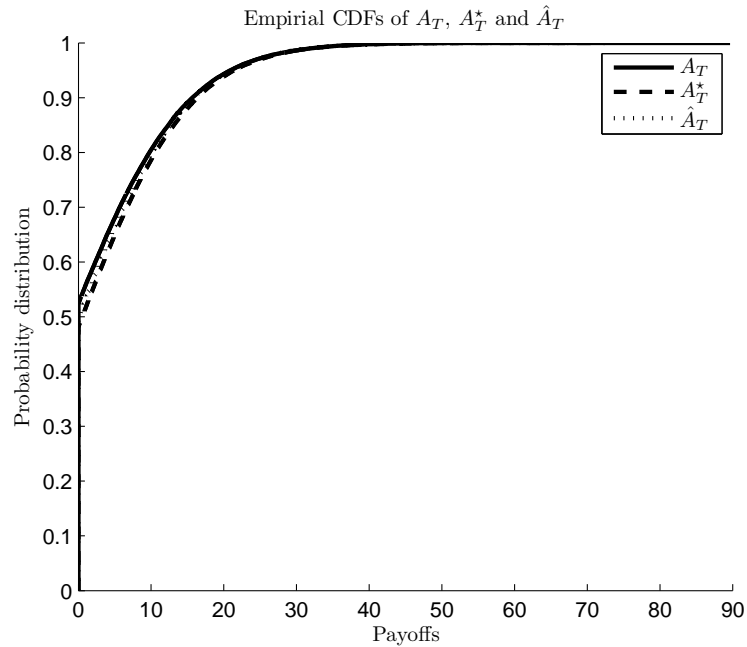


Figure 2.4.4: Empirical CDF of \hat{A}_T , A_T^* and A_T

The parameters used to generate this CDF are given in Table 2.4.1 on page 21.

Figure 2.4.3 shows the empirical CDFs of all three payoffs that have been generated using Monte Carlo simulation. As expected, these three distributions do coincide. According to our assumption that the agent's preference depend only on the terminal distribution of wealth (Assumption 4 on page 3), the executive would be indifferent between all three options. However, the incentives that these three payoffs provide for the executive may be very different.

From the firm's perspective, the cost of issuing these three options are not the same, but the executives are indifferent between all these. Hence, issuing any option other than the

CEC constitutes an efficiency loss. Table 2.4.2 on page 22 displays some sample prices and the loss of efficiency in percentages for our options. In the cases that we have considered, the percentage efficiency loss ranges from a low of 10% to a high of 49%. Even though the degree of efficiency loss varies with the parameters, there is a clear ordering of prices i.e. \hat{A}_T is the most expensive, followed by A^* , with A_T being the cheapest.

Intuitively speaking, the CEC is achieved by reshuffling the outcomes of \hat{A}_T in each state in reverse order with the state price process while still preserving the original distribution. When the state price process is continuous, Theorem 1.2.4 on page 4 provides the method of doing so.

Figures 2.4.5 to 2.4.6 on pages 26–27 illustrate how the outcomes of \hat{A}_T are being reshuffled to A_T^* and A_T respectively. In the former case, we can see a fairly linear relationship between the outcomes of \hat{A}_T and A_T^* , with an empirical correlation is 0.8235. Even though the reshuffling in Proposition 2.4.2 on page 16 is incomplete, we are still able to design a payoff that is cheaper, inherits the desired features of the AEIO, and has a terminal payoff that is highly correlated with the original payoff. When the reshuffling is complete in Figure 2.4.6 on the next page, we have the true CEC. The cheapest price is achieved at the expense of the linear relationship (the empirical correlation drops to 0.3228).

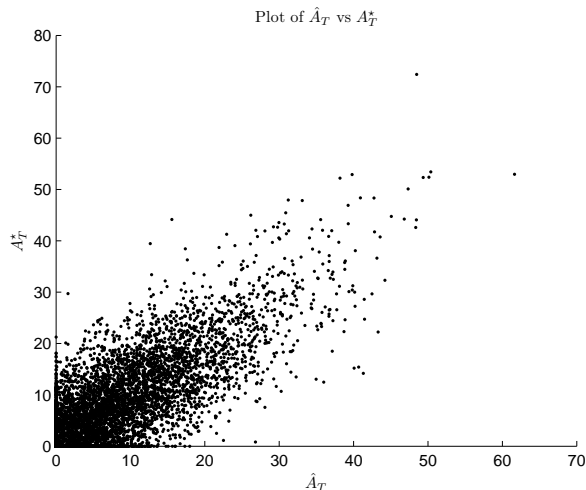


Figure 2.4.5: Reshuffling of outcomes of \hat{A}_T to A_T^*

The parameters used to generate this graph are given in Table 2.4.1 on page 21. Their empirical correlation is 0.8235.

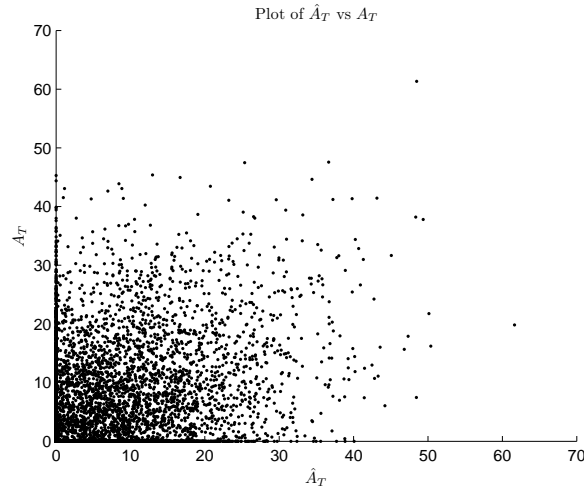


Figure 2.4.6: Reshuffling of outcomes of \hat{A}_T to A_T

The parameters used to generate this graph are given in Table 2.4.1 on page 21. Their empirical correlation is 0.3228.

Remark 2.4.8. If we modify Assumption 4 on page 3 so that agent preferences are state dependent, then the executive will no longer be indifferent between \hat{A}_T , A_T^* , and A_T . In the presence of state dependent preferences, [4] also provides a construction of CEC by using recent developments in the theory of copulas by Tankov [51].

We end this chapter with some graphs that illustrate the connection between outcomes of each of the payoff and the state price process. From Figures 2.4.7 on the next page and 2.4.8 on the following page, we do not see any obvious relationship between ξ_T and \hat{A}_T , and ξ_T and A_T^* . However, in Figure 2.4.9 on page 29, where the payoff is cost-efficient, it becomes evident that A_T is non-increasing with ξ_T . In fact, when ξ_T is continuous, this turns out to be a necessary and sufficient condition for cost efficiency (see Proposition 2, [4]).

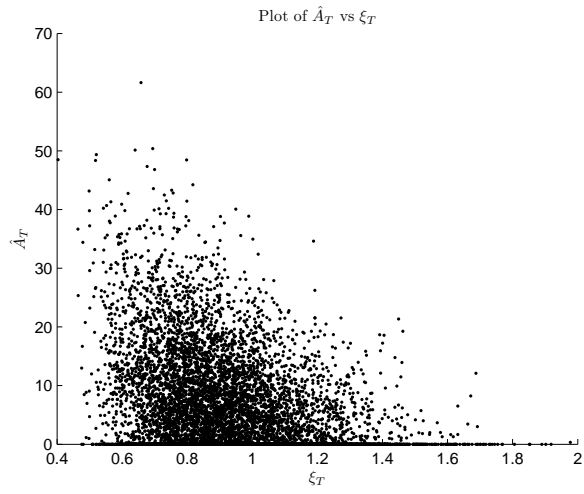


Figure 2.4.7: Outcomes of \hat{A}_T vs ξ_T

The parameters used to generate this graph are given in Table 2.4.1 on page 21.

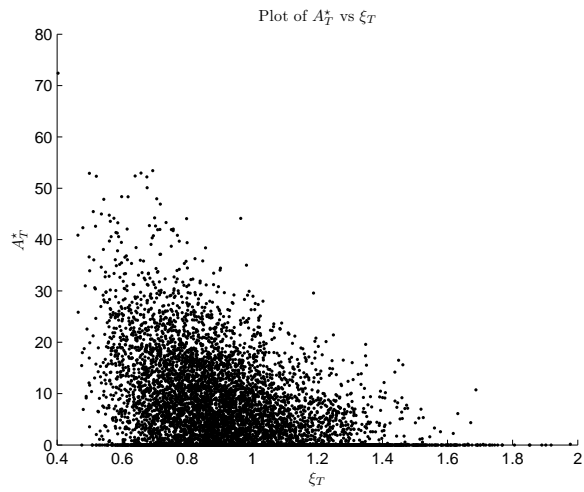


Figure 2.4.8: Outcomes of A_T^* vs ξ_T

The parameters used to generate this graph are given in Table 2.4.1 on page 21.

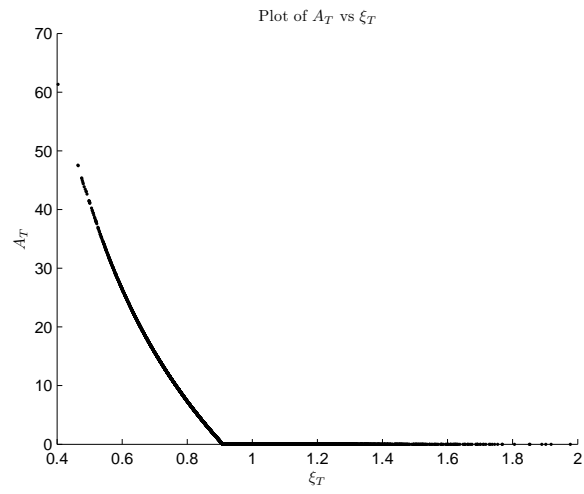


Figure 2.4.9: Outcomes of A_T vs ξ_T

The parameters used to generate this graph are given in Table 2.4.1 on page 21.

Chapter 3

Asian Options with Vasicek Interest Rates

Thus far, we have assumed that interest rates are constant in our model. However, in reality, executive options have much longer maturities which make this assumption unrealistic. In fact, most of these options expire in ten years¹ (see Murphy [44]). One reason for the issuance of long term options is that since the impact of the executive's efforts on firm value typically take longer to surface (as compared to, say, salesmen or factory workers), it is more efficient to issue longer term contracts (see Fudenberg, Holmstrom and Milgrom [22]).

For this reason, we will incorporate stochastic interest rates modeled by a Vasicek [54] process in our study of cost efficiency. Even though the Vasicek model is far from perfect (poor fitting of initial term structure, negative short rates, and etc.), its simplicity eventually leads to analytically tractable pricing formulas². Another reason for using the Vasicek model is that it is a simple enough model that incorporates reversion to a long term mean. At the time of writing, we are in a low interest rates environment and they are expected to rise over the long term³.

The first ingredient required for our study of cost efficiency in the presence of stochastic

¹However, we have decided to use a maturity of $T = 1$ for most of our simulations. Any longer maturity would make the run time too restrictive, and introduces a large amount of time-stepping error. The conclusions that we have drawn are still valid despite this shorter maturity.

²This is presumably desirable for accounting purposes [47].

³On July 20th 2011, the 30 day US Treasury rate is 0.01% while the 30 year rate is 4.25%. Data retrieved from <http://www.treasury.gov/resource-center/data-chart-center/interest-rates/Pages/TextView.aspx?data=yield>.

interest rates are the pricing formulas for the Geometric Asian Option (GAO) and the Asian Exchange Option (AXO), which are the subject of this chapter.

Firstly, we define the dynamics followed by the assets and bond in Section 3.1, which are then used to derive some important distributions under the \mathbb{Q}^T measure in Section 3.2 on page 34. These distributions will allow us to compute (Section 3.3 on page 35) the pricing formulas for the options of interest i.e. GAO and AXO. We are also able to compute the pricing formulas for the European Call Option (ECO) and European Exchange Option (EXO) and show that these agree with existing results. Closed form expressions for certain terms are presented in Section 3.4 on page 40. Section 3.5 on page 42 considers the special case where interest rates are deterministic. Since our pricing formulas are (to the best of our knowledge) new, we will end this chapter with some results from Monte Carlo simulations to verify their correctness in Section 3.6 on page 45.

The key pricing formulas are given in Propositions 3.3.1 on page 35 and 3.3.4 on page 36. Unless otherwise mentioned, the index i is understood to range over 1 and 2 i.e. $i = 1, 2$.

3.1 Assets, Short Rate, and Bond Dynamics

In this section, we begin with setting up the \mathbb{Q} dynamics for the underlying assets and the bond price under the Vasicek short rate model. After that, we will perform a change of numeraire to the T -bond and specify the dynamics under the \mathbb{Q}^T measure - doing so allows us to isolate the stochastic short rate term, $r(T)$. It also greatly simplifies the pricing of our options when we evaluate the expectation of the payoffs. For more details regarding the change of measures and option pricing, see Benninga, Björk, Wiener and Yisra'el [3] or Geman, Karoui and Rochet [23].

This section extends the method of Bernard, Le Courtois and Quittard-Pinon [6] to include two risky assets.

3.1.1 Dynamics Under the Risk Neutral Measure \mathbb{Q}

The Vasicek model short rate dynamics is given by:

$$dr(t) = a(\theta - r(t))dt + \sigma_r dZ_0(t) \tag{3.1.1}$$

where a , θ , and σ_r are constants. We can interpret a as the rate of reversion to the long term mean given by θ . This has the solution of:

$$r(t) = r(0)e^{-at} + \theta(1 - e^{-at}) + \sigma_r \int_0^t e^{-a(t-s)} dZ_0(s) \quad (3.1.2)$$

It has an affine term structure which gives us:

$$P(t, T) = A(t, T)e^{-B(t, T)r(t)} \quad (3.1.3)$$

$$B(t, T) = \frac{1}{a} \{1 - e^{-a(T-t)}\} \quad (3.1.4)$$

$$A(t, T) = \exp \left\{ \left(\theta - \frac{\sigma_r^2}{2a^2} \right) (B(t, T) - T + t) - \frac{\sigma_r^2}{4a} B^2(t, T) \right\} \quad (3.1.5)$$

The bond price dynamics follows:

$$\frac{dP(t, T)}{P(t, T)} = r(t)dt - \sigma_P(t, T)dZ_0(t) \quad (3.1.6)$$

where

$$\sigma_P(t, T) = \frac{\sigma_r}{a} [1 - e^{-a(T-t)}] \quad (3.1.7)$$

By means of the Cholesky decomposition, we can express the dynamics for the underlying assets and bond price using three independent Brownian motions:

$$\frac{dS_1(t)}{S_1(t)} = r(t)dt + \sigma_1 C_{21} dZ_0(t) + \sigma_1 C_{22} dZ_1(t) \quad (3.1.8)$$

$$\frac{dS_2(t)}{S_2(t)} = r(t)dt + \sigma_2 C_{31} dZ_0(t) + \sigma_2 C_{32} dZ_1(t) + \sigma_2 C_{33} dZ_2(t) \quad (3.1.9)$$

$$\frac{dP(t, T)}{P(t, T)} = r(t)dt - \sigma_P(t, T)dZ_0(t) \quad (3.1.10)$$

with the following diffusion terms:

$$\begin{aligned} C_{21} &= \rho_{01} & C_{22} &= \sqrt{1 - \rho_{01}^2} \\ C_{31} &= \rho_{02} & C_{32} &= \frac{\rho_{12} - \rho_{01}\rho_{02}}{\sqrt{1 - \rho_{01}^2}} & C_{33} &= \sqrt{1 - \rho_{02}^2 - \frac{(\rho_{12} - \rho_{01}\rho_{02})^2}{1 - \rho_{01}^2}} \end{aligned} \quad (3.1.11)$$

3.1.2 Dynamics Under the T -Forward Measure \mathbb{Q}^T

Now, we can change the numeraire to the T -bond. Under the \mathbb{Q}^T measure, we want $\frac{S_i(t)}{P(t,T)}$ to be martingales. A straightforward application of Ito's Lemma to $f(S_i(t), P(t, T)) = \frac{S_i(t)}{P(t, T)}$ and the use of Girsanov's Theorem (see e.g. Karatzas and Shreve [33]) yields the following \mathbb{Q}^T dynamics:

$$\frac{dS_1(t)}{S_1(t)} = [r(t) - \sigma_1\sigma_P(t, T)C_{21}] dt + \sigma_1C_{21}dZ_0^T(t) + \sigma_1C_{22}dZ_1^T(t) \quad (3.1.12)$$

$$\begin{aligned} \frac{dS_2(t)}{S_2(t)} &= [r(t) - \sigma_2\sigma_P(t, T)C_{31}] dt + \sigma_2C_{31}dZ_0^T(t) + \sigma_2C_{32}dZ_1^T(t) \\ &\quad + \sigma_2C_{33}dZ_2^T(t) \end{aligned} \quad (3.1.13)$$

$$\frac{dP(t, T)}{P(t, T)} = [r(t) + \sigma_P^2(t, T)] dt - \sigma_P(t, T)dZ_0^T(t) \quad (3.1.14)$$

By Girsanov's Theorem, the T -forward neutral measure \mathbb{Q}^T is defined by its Radon-Nikodym derivative for $u \leq T$

$$\left. \frac{d\mathbb{Q}^T}{d\mathbb{Q}} \right|_u = \exp \left\{ - \int_0^u \sigma_P(s, T)dZ_0(s) - \frac{1}{2} \int_0^u \sigma_P^2(s, T)ds \right\}$$

Lemma 3.1.1 gives the expressions for the assets $S_i(t)$ that do not involve $r(t)$.

Lemma 3.1.1.

$$\begin{aligned} S_1(t) &= \frac{S_1(0)}{P(0, t)} \exp \left\{ \int_0^t \left[\frac{\sigma_P^2(u, t) - \sigma_1^2}{2} - \sigma_P(u, T)(\sigma_1C_{21} + \sigma_P(u, t)) \right] du \right. \\ &\quad + \int_0^t [\sigma_1C_{21} + \sigma_P(u, t)] dZ_0^T(u) \\ &\quad \left. + \int_0^t \sigma_1C_{22}dZ_1^T(u) \right\} \end{aligned} \quad (3.1.15)$$

$$\begin{aligned} S_2(t) &= \frac{S_2(0)}{P(0, t)} \exp \left\{ \int_0^t \left[\frac{\sigma_P^2(u, t) - \sigma_2^2}{2} - \sigma_P(u, T)(\sigma_2C_{31} + \sigma_P(u, t)) \right] du \right. \\ &\quad + \int_0^t [\sigma_2C_{31} + \sigma_P(u, t)] dZ_0^T(u) \\ &\quad \left. + \int_0^t \sigma_2C_{32}dZ_1^T(u) \right. \\ &\quad \left. + \int_0^t \sigma_2C_{33}dZ_2^T(u) \right\} \end{aligned}$$

$$+ \int_0^t \sigma_2 C_{33} dZ_2^T(u) \} \quad (3.1.16)$$

Remark 3.1.2. Henceforth, all calculations are done under the \mathbb{Q}^T measure.

3.2 Some Important Distributions

Now we shift our focus to the distributions of the terminal values and geometric averages of the underlying assets. The distributions of the terminal values are needed to compute the prices for the ECO and EXO, whereas the geometric averages are needed for the GAO and AXO. First, let us define the following quantities:

$$\sigma_{i0}(u, t) := \sigma_i C_{(i+1)i} + \sigma_P(u, t) \quad (3.2.1)$$

$$\sigma_{ij} := \sigma_i C_{(i+1)(j+1)} \quad (3.2.2)$$

$$m_i(u, t) := \frac{\sigma_P^2(u, t) - \sigma_i^2}{2} - \sigma_P(u, T) \sigma_{i0}(u, t) \quad (3.2.3)$$

$$X_1(t) := \int_0^t m_1(u, t) du + \int_0^t \sigma_{10}(u, t) dZ_0^T(u) + \int_0^t \sigma_{11} dZ_1^T(u) \quad (3.2.4)$$

$$X_2(t) := \int_0^t m_2(u, t) du + \int_0^t \sigma_{20}(u, t) dZ_0^T(u) + \int_0^t \sigma_{21} dZ_1^T(u) + \int_0^t \sigma_{22} dZ_2^T(u) \quad (3.2.5)$$

$$\hat{X}_i(T) := \frac{1}{T} \int_0^T \ln S_i(t) dt \quad (3.2.6)$$

Now we can recast the terminal values and geometric averages as follow:

$$S_i(T) = \frac{S_i(0)}{P(0, T)} e^{X_i(T)} \quad (3.2.7)$$

$$\hat{S}_i(T) = e^{\hat{X}_i(T)} \quad (3.2.8)$$

From (3.2.4)–(3.2.6), it is clear that the terminal values and geometric averages are log-normally distributed. If we are able to identify the mean, variance, and covariance of the log of $S_i(T)$ and $\hat{S}_i(T)$, then we are done. These are done in Lemmas 3.2.1 to 3.2.2 on pages 34–35.

Lemma 3.2.1. $S_1(T)$ and $S_2(T)$ form a bivariate lognormal distribution where $S_i(T) \sim \mathcal{LN}(m_i(T), v_i^2(T))$ and their covariance is given by $\text{Cov}[\ln S_1(T), \ln S_2(T)] = v_{12}(T)$. Explicit expressions for $m_i(T)$, $v_i^2(T)$, and $v_{12}(T)$ are given in (3.4.1)–(3.4.3) on page 41.

Lemma 3.2.2. $\hat{S}_1(T)$ and $\hat{S}_2(T)$ form a bivariate lognormal distribution where $\hat{S}_i(T) \sim \mathcal{LN}(\hat{m}_i(T), \hat{v}_i^2(T))$ and their covariance is given by $\text{Cov}[\ln \hat{S}_1(T), \ln \hat{S}_2(T)] = \hat{v}_{12}(T)$. Explicit expressions for $\hat{m}_i(T)$, $\hat{v}_i^2(T)$, and $\hat{v}_{12}(T)$ are given in (3.4.4)–(3.4.6) on page 42.

Remark 3.2.3. Note that we have deliberately left the terms in Lemmas 3.2.1 to 3.2.2 on pages 34–35 unsimplified. This is because the evaluation of these integrals become increasingly laborious, and necessitates the use of a computer algebra system (CAS). Their explicit forms are given in Section 3.4 on page 40 below.

Remark 3.2.4. Even though the statements of Lemmas 3.2.1 to 3.2.2 on pages 34–35 look simple enough, their proofs are in fact rather involved. We have opted to include all the complete (repetitive) details in Appendix B.2 on page 85 for pedagogical reasons, but more importantly, as the pseudo code for evaluating the expressions in a CAS (see Remark 3.2.3).

3.3 Option Pricing Formulas

Now that we have the distributions for the terminal values and geometric averages (Lemmas 3.2.1 to 3.2.2 on pages 34–35), the task of computing the pricing formulas becomes much easier (with the help of Lemmas A.1 on page 74 and A.3 on page 75). We present these formulas in this section which take into account the stochastic interest rates given by the Vasicek process in (3.1.1) on page 31.

The formulas for the Geometric Asian Option and Asian Exchange Option are new results, whereas the ones for the European Call Option and European Exchange Option agree with existing results. The former two will be used later for our study of cost efficiency; the latter two serve as corroboration that our alternative method of derivation is correct.

3.3.1 Geometric Asian Option (GAO)

Proposition 3.3.1. *In the presence of stochastic interest rates given by the Vasicek process in (3.1.1) on page 31, the payoff of the Geometric Asian Option with strike K on the underlying $S_i(T)$ at time T is given by:*

$$GAO = (\hat{S}_i(T) - K)^+ = \left(e^{\frac{1}{T} \int_0^T \ln S_i(t) dt} - K \right)^+ \quad (3.3.1)$$

Its price at time 0 is given by:

$$\begin{aligned}
 \text{Price}(GAO) &= P(0, T) \left[\exp \left\{ \hat{m}_i(T) + \frac{1}{2} \hat{v}_i^2(T) \right\} \Phi(\hat{f}_1(T)) - K \Phi(\hat{f}_2(T)) \right] \\
 &\text{where} \\
 \hat{f}_1(T) &= \frac{\hat{m}_i(T) + \hat{v}_i^2(T) - \ln K}{\hat{v}_i(T)} \\
 \hat{f}_2(T) &= \hat{f}_1(T) - \hat{v}_i(T)
 \end{aligned} \tag{3.3.2}$$

($\hat{m}_i(T)$ and $\hat{v}_i(T)$ are given in (3.4.4)–(3.4.5) on page 42)

Remark 3.3.2. Recall that in the case of constant interest rates, the price of a Geometric Asian Option is of the following form:

$$\begin{aligned}
 \text{Price}(GAO\text{Const}) &= e^{-rT} [S_i(0)e^{rT}\Phi(f_1) - K\Phi(f_2)] \\
 &\text{where} \\
 f_1 &= \frac{\ln \frac{S_i(0)}{K} + (r - \hat{q}_i + \frac{\hat{\sigma}_i}{2}) T}{\hat{\sigma}_i \sqrt{T}} \\
 f_2 &= f_1 - \hat{\sigma}_i \sqrt{T}
 \end{aligned} \tag{3.3.3}$$

We can see that (3.3.2) has the same functional form as (3.3.3), but with different input parameters. The familiar relationship between f_1 and f_2 also holds for $\hat{f}_1(T)$ and $\hat{f}_2(T)$.

Remark 3.3.3. Zhang, Yuan and Wang [50] have in fact derived the price of the GAO under the extended Vasicek model. However, their semi-closed form expression is highly complicated, not intuitive and left largely unsimplified. Their derivation relies on the brute-force integration of the expectation term under the \mathbb{Q} measure, as opposed to our simpler approach of tackling the problem through the \mathbb{Q}^T measure. However, it does seem that the Hull-White model would naturally be the next extension for us to make.

3.3.2 Asian Exchange Option (AXO)

Proposition 3.3.4. *In the presence of stochastic interest rates given by the Vasicek process in (3.1.1) on page 31, the payoff of the Asian Exchange Option on $S_1(T)$ and $S_2(T)$ at time T is given by:*

$$AXO = (\hat{S}_1(T) - \hat{S}_2(T))^+ = \left(e^{\frac{1}{T} \int_0^T \ln S_1(t) dt} - e^{\frac{1}{T} \int_0^T \ln S_2(u) du} \right)^+ \tag{3.3.4}$$

Its price at time 0 is given by

$$\begin{aligned}
\text{Price}(AXO) &= P(0, T) \left[\exp \left\{ \hat{m}_1(T) + \frac{1}{2} \hat{v}_1^2(T) \right\} \Phi(\hat{g}_1(T)) \right. \\
&\quad \left. - \exp \left\{ \hat{m}_2(T) + \frac{1}{2} \hat{v}_2^2(T) \right\} \Phi(\hat{g}_2(T)) \right] \\
&\text{where} \\
\hat{g}_1(T) &= \frac{\hat{m}_1(T) - \hat{m}_2(T) + \hat{v}_1^2(T) - \hat{v}_{12}(T)}{\sqrt{\hat{v}_1^2(T) - 2\hat{v}_{12}(T) + \hat{v}_2^2(T)}} \\
\hat{g}_2(T) &= \hat{g}_1(T) - \sqrt{\hat{v}_1^2(T) - 2\hat{v}_{12}(T) + \hat{v}_2^2(T)}
\end{aligned} \tag{3.3.5}$$

($\hat{m}_i(T)$, $\hat{v}_i^2(T)$, and $\hat{v}_{12}(T)$ are given in (3.4.4)–(3.4.6) on page 42.)

Remark 3.3.5. Recall that in the case of constant interest rates, the price of an Asian Exchange Option is of the following form:

$$\begin{aligned}
\text{Price}(AXO\text{Const}) &= e^{-rT} [S_1(0)e^{rT}\Phi(g_1) - S_2(0)e^{rT}\Phi(g_2)] \\
&\text{where} \\
g_1 &= \frac{\ln \frac{S_1(0)}{S_2(0)} + (\hat{\sigma}_1^2 + \hat{\sigma}_2^2 - 2\hat{\sigma}_1\hat{\sigma}_2\hat{\rho})\frac{T}{2}}{\sqrt{(\hat{\sigma}_1^2 + \hat{\sigma}_2^2 - 2\hat{\sigma}_1\hat{\sigma}_2\hat{\rho})T}} \\
g_2 &= g_1 - \sqrt{(\hat{\sigma}_1^2 + \hat{\sigma}_2^2 - 2\hat{\sigma}_1\hat{\sigma}_2\hat{\rho})T}
\end{aligned} \tag{3.3.6}$$

Again, we can see that (3.3.5) has the same functional form as (3.3.6), and the same relationship between the inputs into the normal cdf terms. It is also interesting to note that the price of the option does not depend on the interest rate.

Remark 3.3.6. Our method generalizes Zhang's [55] approach of computing the EXO under constant interest rates to the AXO under stochastic interest rates.

3.3.3 European Call Option (ECO)

Proposition 3.3.7. *In the presence of stochastic interest rates given by the Vasicek process in (3.1.1) on page 31, the payoff of the European Call Option with strike K on the underlying $S_i(T)$ at time T is given by:*

$$ECO = (S_i(T) - K)^+ \tag{3.3.7}$$

Its price at time 0 is given by:

$$\begin{aligned}
 \text{Price}(ECO) &= P(0, T) \left[\exp \left\{ m_i(T) + \frac{1}{2} v_i^2(T) \right\} \Phi(f_1(T)) - K \Phi(f_2(T)) \right] \\
 \text{where} \\
 f_1(T) &= \frac{m_i(T) + v_i^2(T) - \ln K}{v_i(T)} \\
 f_2(T) &= f_1(T) - v_i(T)
 \end{aligned}
 \tag{3.3.8}$$

($m_i(T)$ and $v_i^2(T)$ are given in (3.4.1)–(3.4.2) on page 41).

Remark 3.3.8. In fact, note that (3.3.8) is really just (3.3.2) on page 36, but with the mean and variance of the terminal values instead of the geometric averages.

Corollary 3.3.9. *The price of the European Call Option given in (3.3.8) is equivalent to the following given by Rabinovitch [48]:*

$$\begin{aligned}
 \text{Price}(ECOR) &= S_i(0) \Phi \left(\frac{\ln \frac{S_i(0)}{KP(0, T)} + \frac{1}{2} v_i^2(0, T)}{v_i(0, T)} \right) \\
 &\quad - KP(0, T) \Phi \left(\frac{\ln \frac{S_i(0)}{KP(0, T)} - \frac{1}{2} v_i^2(0, T)}{v_i(0, T)} \right) \\
 \text{where} \\
 v_i^2(0, T) &= V(0, T) + \sigma_i^2 T + 2\rho_{0i} \frac{\sigma_r \sigma_i}{a} \left[T - \frac{1}{a} (1 - e^{-aT}) \right] \\
 V(0, T) &= \left(\frac{\sigma_r}{a} \right)^2 \left[T + \frac{2}{a} e^{-aT} - \frac{1}{2a} e^{-2aT} - \frac{3}{2a} \right]
 \end{aligned}
 \tag{3.3.9}$$

3.3.4 European Exchange Option (EXO)

Proposition 3.3.10. *In the presence of stochastic interest rates given by the Vasicek process in (3.1.1) on page 31, the payoff of the European Exchange Option on $S_1(T)$ and $S_2(T)$ at time T is given by:*

$$EXO = (S_1(T) - S_2(T))^+
 \tag{3.3.10}$$

Its price at time 0 is given by

$$\begin{aligned}
 \text{Price}(EXO) &= P(0, T) \left[\exp \left\{ m_1(T) + \frac{1}{2} v_1^2(T) \right\} \Phi(g_1(T)) \right. \\
 &\quad \left. - \exp \left\{ m_2(T) + \frac{1}{2} v_2^2(T) \right\} \Phi(g_2(T)) \right] \tag{3.3.11} \\
 \text{where} \\
 g_1(T) &= \frac{m_1(T) - m_2(T) + v_1^2(T) - v_{12}(T)}{\sqrt{v_1^2(T) - 2v_{12}(T) + v_2^2(T)}} \\
 g_2(T) &= g_1(T) - \sqrt{v_1^2(T) - 2v_{12}(T) + v_2^2(T)}
 \end{aligned}$$

($m_i(T)$, $v_i^2(T)$, and $v_{12}(T)$, are given in (3.4.1)–(3.4.3) on page 41.)

Remark 3.3.11. Once again, note that (3.3.11) is really just (3.3.5) on page 37, but with the mean, variance and covariance of the terminal values instead of the geometric averages. Moreover, even though the interest rates are assumed to be stochastic, the pricing formula here does not involve the interest rate (much like Margrabe’s formula [40]).

Corollary 3.3.12. *The price of the European Exchange Option given in (3.3.11) is equivalent to the following given by Bernard and Cui [5]:*

$$\begin{aligned}
 \text{Price}(EXOBC) &= S_1(0)\Phi(g_1) - S_2(0)\Phi(g_2) \tag{3.3.12} \\
 \text{where} \\
 g_1 &= \frac{\ln \frac{S_1(0)}{S_2(0)} + (\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho_{12})\frac{T}{2}}{\sqrt{(\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho_{12})T}} \\
 g_2 &= g_1 - \sqrt{(\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho_{12})T}
 \end{aligned}$$

Remark 3.3.13. In Margrabe’s [40] original pricing of the EXO (with constant interest rates), the second asset is used as the numeraire instead of the T -bond. This method simplifies the problem as we can avoid performing a double integration. In fact, even when interest rates are stochastic, this method still works [5].

However in our case, since the payoffs involve geometric averages, we are unable to use the second asset as the numeraire. This is because the cancellations in the EXO that simplify the integration problem do not happen in the AXO. To see why this is the case,

we follow [5] and define the measure $\hat{\mathbb{Q}}$ (this corresponds to using the second asset as the numeraire) where its Radon-Nikodym derivative takes the following form for $t \leq T$:

$$\frac{d\hat{\mathbb{Q}}}{d\mathbb{Q}} \Big|_t = \exp \left\{ -\frac{1}{2}\sigma_2^2 t + \cdots Z_0(t) + \cdots Z_1(t) + \cdots Z_2(t) \right\}$$

Under $\hat{\mathbb{Q}}$, the dynamics for $S_1(t)$ and $S_2(t)$ would take the following form:

$$\begin{aligned} \frac{dS_1(t)}{S_1(t)} &= r(t)dt + \cdots d\hat{Z}_0(t) + \cdots d\hat{Z}_1(t) \\ \frac{dS_2(t)}{S_2(t)} &= r(t)dt + \cdots d\hat{Z}_0(t) + \cdots d\hat{Z}_1(t) + \cdots d\hat{Z}_2(t) \end{aligned}$$

Then,

$$\begin{aligned} \hat{S}_1(T) &= S_1(0) \exp \left\{ \frac{1}{T} \int_0^T \left[\int_0^t r(s)ds + \int_0^t \cdots d\hat{Z}_0(s) + \int_0^t \cdots d\hat{Z}_1(s) \right] dt \right\} \\ \hat{S}_2(T) &= S_2(0) \exp \left\{ \frac{1}{T} \int_0^T \left[\int_0^t r(s)ds + \int_0^t \cdots d\hat{Z}_0(s) + \int_0^t \cdots d\hat{Z}_1(s) + \int_0^t \cdots d\hat{Z}_2(s) \right] dt \right\} \end{aligned}$$

The price of the AXO is

$$\begin{aligned} \text{Price(AXO)} &= E_{\mathbb{Q}} \left[e^{-\int_0^T r(s)ds} \left(\hat{S}_1(T) - \hat{S}_2(T) \right)^+ \right] \\ &= E_{\mathbb{Q}} \left[e^{-\int_0^T r(s)ds} \hat{S}_2(T) \left(\frac{\hat{S}_1(T)}{\hat{S}_2(T)} - 1 \right)^+ \right] \\ &= E_{\mathbb{Q}} \left[e^{-\int_0^T r(s)ds} S_2(0) \exp \left\{ \frac{1}{T} \int_0^T \left[\int_0^t r(s)ds + \cdots \right] dt \right\} \left(\hat{S}_1(T) - \hat{S}_2(T) \right)^+ \right] \end{aligned}$$

In the case of a EXO, the integral of the short rate terms do cancel out, which lead to a simpler integration problem. However in our case, the integral in the first exponent, $\int_0^T r(s)ds$, does not cancel out with that in the second exponent, $\int_0^t r(s)ds$, and we are still left with a complicated expectation term. Hence, using the second asset as the numeraire is not ideal for the problem at hand - we are better off with using the T -bond instead.

3.4 Closed Form Expressions

We are able to derive closed form Black-Scholes type formulas for the prices, but some of the key terms have been left unsimplified as they involve Riemann integrals that evaluate

to very long and complicated formulas. These expressions have been evaluated with the help of a CAS and are presented in this section.

3.4.1 Closed Form Expressions for the Terminal Values

Recall from Lemma 3.2.1 on page 34 that $S_1(T)$ and $S_2(T)$ form a bivariate lognormal distribution where $S_i(T) \sim \mathcal{LN}(m_i(T), v_i^2(T))$ and their covariance is given by $Cov[\ln S_1(T), \ln S_2(T)] = v_{12}(T)$. The closed form expressions for the mean, variance and covariance of the log of the terminal values are as follow:

$$\begin{aligned}
 & m_i(T) \\
 &= \ln \frac{S_i(0)}{P(0, T)} - \frac{\sigma_i^2 T}{2} + \left(\frac{\sigma_r}{a}\right)^2 \left\{ \frac{1}{ae^{aT}} \left[\frac{1}{4e^{aT}} - 1 \right] + \frac{3}{4a} - \frac{T}{2} \right\} \\
 &+ \frac{\sigma_r \sigma_i \rho_{0i}}{a} \left\{ \frac{1}{a} [1 - e^{-aT}] - T \right\}
 \end{aligned} \tag{3.4.1}$$

$$\begin{aligned}
 & v_i^2(T) \\
 &= \sigma_i^2 T + \left(\frac{\sigma_r}{a}\right)^2 \left\{ \frac{2}{ae^{aT}} \left[1 - \frac{1}{4e^{aT}} \right] + T - \frac{3}{2a} \right\} + \frac{2\sigma_r \sigma_i \rho_{0i}}{a} \left\{ T - \frac{1}{a} [1 - e^{-aT}] \right\}
 \end{aligned} \tag{3.4.2}$$

$$\begin{aligned}
 & v_{12}(T) \\
 &= \sigma_1 \sigma_2 \rho_{12} T + \left(\frac{\sigma_r}{a}\right)^2 \left\{ \frac{2}{ae^{aT}} \left[1 - \frac{1}{4e^{aT}} \right] + T - \frac{3}{2a} \right\} \\
 &+ \frac{\sigma_r}{a} (\sigma_1 \rho_{01} + \sigma_2 \rho_{02}) \left\{ T - \frac{1}{a} [1 - e^{-aT}] \right\}
 \end{aligned} \tag{3.4.3}$$

3.4.2 Closed Form Expressions for the Geometric Averages

Recall from Lemma 3.2.2 on page 35 that $\hat{S}_1(T)$ and $\hat{S}_2(T)$ form a bivariate lognormal distribution where $\hat{S}_i(T) \sim \mathcal{LN}(\hat{m}_i(T), \hat{v}_i^2(T))$ and their covariance is given by $Cov[\ln \hat{S}_1(T), \ln \hat{S}_2(T)] = \hat{v}_{12}(T)$. The closed form expressions for the mean, variance and

covariance of the log of the geometric averages are as follow:

$$\begin{aligned}
& \hat{m}_i(T) \\
&= \ln S_i(0) + \frac{T}{2} \left\{ \theta - \frac{\sigma_1^2}{2} \right\} + \frac{\theta - r_0}{a} \left\{ \frac{1}{aT} [1 - e^{-aT}] - 1 \right\} \\
&+ \left(\frac{\sigma_r}{a} \right)^2 \left\{ \frac{1}{ae^{aT}} \left[\frac{1}{aT} - 1 - \frac{1}{2aTe^{aT}} \right] - \frac{T}{2} + \frac{1}{a} \left[1 - \frac{1}{2aT} \right] \right\} \\
&+ \frac{\sigma_r \sigma_i \rho_{0i}}{a} \left\{ \frac{1}{a^2T} - \frac{T}{2} - \frac{1}{ae^{aT}} \left[\frac{1}{aT} + 1 \right] \right\}
\end{aligned} \tag{3.4.4}$$

$$\begin{aligned}
& \hat{v}_i^2(T) \\
&= \frac{\sigma_i^2 T}{3} + \left(\frac{\sigma_r}{a} \right)^2 \left\{ \frac{T}{3} + \frac{1}{a} \left[\frac{1}{aT} - 1 + \frac{1}{2T^2 a^2} \right] - \frac{2}{a^2 T e^{aT}} \left[1 + \frac{1}{4T a e^{aT}} \right] \right\} \\
&+ \frac{\sigma_r \sigma_i \rho_{0i}}{a} \left\{ \frac{2T}{3} + \frac{1}{a} \left[\frac{2}{a^2 T^2} - 1 \right] - \frac{2}{a^2 T e^{aT}} \left[\frac{1}{aT} + 1 \right] \right\}
\end{aligned} \tag{3.4.5}$$

$$\begin{aligned}
& \hat{v}_{12}(T) \\
&= \frac{\sigma_1 \sigma_2 \rho_{12} T}{3} + \left(\frac{\sigma_r}{a} \right)^2 \left\{ \frac{T}{3} + \frac{1}{a} \left[\frac{1}{aT} - 1 + \frac{1}{2T^2 a^2} \right] - \frac{2}{a^2 T e^{aT}} \left[1 + \frac{1}{4T a e^{aT}} \right] \right\} \\
&+ \frac{\sigma_r}{a} (\sigma_1 \rho_{01} + \sigma_2 \rho_{02}) \left\{ \frac{T}{3} - \frac{1}{a^2 T e^{aT}} \left[\frac{1}{aT} + 1 \right] + \frac{1}{a} \left[\frac{1}{a^2 T^2} - \frac{1}{2} \right] \right\}
\end{aligned} \tag{3.4.6}$$

3.4.3 Closed Form Expressions for the Bond Price

The bond price term under the Vasicek model appears in the various option pricing formulas. We recall the following famous result for completeness (see, e.g. [15]):

$$\begin{aligned}
& P(0, T) \\
&= \exp \left\{ \frac{\theta - r(0)}{a} [1 - e^{-aT}] - T\theta + \left(\frac{\sigma_r}{a} \right)^2 \left[\frac{T}{2} - \frac{3}{4a} + \frac{1}{ae^{aT}} \left(1 - \frac{1}{4e^{aT}} \right) \right] \right\}
\end{aligned} \tag{3.4.7}$$

3.5 Deterministic Interest Rates

Now that we have the pricing formulas for the various options under a Vasicek interest rate model (Section 3.3 on page 35), as well as the explicit expressions for the various input

parameters (Section 3.4 on page 40), we can consider the special case of zero interest rate volatility ($\sigma_r = 0$) i.e. deterministic interest rates. The input parameters into the pricing formulas simplify considerably and we are able to express them in a more explicit fashion.

3.5.1 Setup

When $\sigma_r = 0$, the short rate dynamics becomes

$$dr(t) = a(\theta - r(t))dt \quad (3.5.1)$$

which has the solution of

$$r(t) = r(0)e^{-at} + \theta(1 - e^{-at}) \quad (3.5.2)$$

The bond price becomes:

$$P(0, T) = \exp \left\{ \frac{\theta - r(0)}{a} [1 - e^{-aT}] - T\theta \right\} \quad (3.5.3)$$

The mean, variance and covariance of the terminal values simplify to the following:

$$m_i(T) = \ln \frac{S_i(0)}{P(0, T)} - \frac{\sigma_i^2 T}{2} \quad (3.5.4)$$

$$v_i^2(T) = \sigma_i^2 T \quad (3.5.5)$$

$$v_{12}(T) = \sigma_1 \sigma_2 \rho_{12} T \quad (3.5.6)$$

The mean, variance and covariance of the geometric averages simplify to the following:

$$\hat{m}_i(T) = \ln S_i(0) + \frac{T}{2} \left\{ \theta - \frac{\sigma_i^2}{2} \right\} + \frac{\theta - r_0}{a} \left\{ \frac{1}{aT} [1 - e^{-aT}] - 1 \right\} \quad (3.5.7)$$

$$\hat{v}_i^2(T) = \frac{\sigma_i^2 T}{3} \quad (3.5.8)$$

$$v_{12}(T) = \frac{\sigma_1 \sigma_2 \rho_{12} T}{3} \quad (3.5.9)$$

It is interesting to note that in the case of deterministic interest rates, the variance and covariance for both the terminal values and geometric averages simplify to the familiar results in the case of constant interest rates. However, the mean includes some extra terms. This is expected because the deterministic interest rates do not affect the randomness (volatility) of the assets per se - they only shift their price paths in a deterministic manner.

3.5.2 Deterministic Interest Rates Pricing Formulas

The following propositions detail the prices of the GAO, AXO and ECO when interest rates are deterministic. The price of the EXO is the same as Proposition 3.3.10 on page 38 as we have seen that the interest rate does not enter the pricing formula.

Proposition 3.5.1. *When the short rate dynamics is given by (3.5.1) on the preceding page, the price of the GAO is given by:*

$$\begin{aligned}
 & \text{Price}(GAODeterministic) \\
 &= S_i(0) \exp \left\{ \frac{\theta - r(0)}{a} \left[\frac{1}{aT} - e^{-aT} \left(\frac{1}{aT} + 1 \right) \right] - \frac{T}{2} \left[\theta + \frac{\sigma_i^2}{6} \right] \right\} \\
 & \times \Phi(\hat{f}_1(T)) - K \exp \left\{ \frac{\theta - r(0)}{a} [1 - e^{-aT}] - T\theta \right\} \Phi(\hat{f}_2(T))
 \end{aligned} \tag{3.5.10}$$

where

$$\begin{aligned}
 \hat{f}_1(T) &= \frac{\ln \frac{S_i(0)}{K} + \frac{T}{2} \left[\theta + \frac{\sigma_i^2}{6} \right] + \frac{\theta - r(0)}{a} \left[\frac{1}{aT} (1 - e^{-aT}) - 1 \right]}{\sigma_i \sqrt{T/3}} \\
 \hat{f}_2(T) &= \hat{f}_1(T) - \sigma_i \sqrt{T/3}
 \end{aligned}$$

Proposition 3.5.2. *When the short rate dynamics is given by (3.5.1) on the previous page, the price of the AXO is given by:*

$$\begin{aligned}
 & \text{Price}(AXODeterministic) \\
 &= \exp \left\{ \frac{\theta - r(0)}{a} \left[\frac{1}{aT} (1 - e^{-aT}) - e^{-aT} \right] - \frac{\theta T}{2} \right\} \\
 & \times \left[S_1(0) \exp \left\{ -\frac{\sigma_1^2 T}{12} \right\} \Phi(\hat{g}_1(T)) - S_2(0) \exp \left\{ -\frac{\sigma_2^2 T}{12} \right\} \Phi(\hat{g}_2(T)) \right]
 \end{aligned} \tag{3.5.11}$$

where

$$\begin{aligned}
 \hat{g}_1(T) &= \frac{\ln \frac{S_1(0)}{S_2(0)} + \frac{T}{4} \left[\frac{\sigma_1^2}{3} + \sigma_2^2 \right] - \frac{\sigma_1 \sigma_2 \rho_{12} T}{3}}{\sqrt{\frac{T}{3} (\sigma_1^2 + \sigma_2^2 - 2\sigma_1 \sigma_2 \rho_{12})}} \\
 \hat{g}_2(T) &= \hat{g}_1(T) - \sqrt{\frac{T}{3} (\sigma_1^2 + \sigma_2^2 - 2\sigma_1 \sigma_2 \rho_{12})}
 \end{aligned}$$

Proposition 3.5.3. *When the short rate dynamics is given by (3.5.1) on page 43, the price of the ECO is given by:*

$$\begin{aligned}
 & \text{Price}(ECO_{\text{Deterministic}}) \\
 &= S_i(0)\Phi(f_1(T)) - K \exp \left\{ \frac{\theta - r(0)}{a} (1 - e^{-aT}) - \theta T \right\} \Phi(f_2(T)) \quad (3.5.12)
 \end{aligned}$$

where

$$\begin{aligned}
 f_1(T) &= \frac{\ln \frac{S_i(0)}{K} + \left(\frac{\sigma_i^2}{2} + \theta \right) T - \frac{\theta - r(0)}{a} (1 - e^{-aT})}{\sigma_i \sqrt{T}} \\
 f_2(T) &= f_1(T) - \sigma_i \sqrt{T}
 \end{aligned}$$

3.6 Monte Carlo Simulation

We conclude this chapter with some Monte Carlo simulation to verify the correctness of our pricing formulas given in Section 3.3 on page 35. A simple Euler scheme is used to discretize the interest rates and stock price paths (see e.g., Glasserman [24]). Since we already have the prices for the European Call Option and the European Exchange Option, we can use these two payoffs as control variates in an optimal fashion (see Boyle, Broadie and Glasserman [12]).

The simulation results for the Geometric Asian Option and Asian Exchange Option are presented in Tables 3.6.2 to 3.6.3 on pages 46–47 respectively. We can see that across the board, the pricing formulas agree very closely with the simulation results. In fact, all percentage errors are less than 1% in absolute terms.

The standard deviation for these trials are given in Tables 3.6.4 to 3.6.5 on page 48. We can see that using the ECO and EXO as control variates reduce the standard deviation of our estimates significantly.

Option		Vasicek		Assets		Correlation		Monte Carlo	
T	1	a	0.20	$S_1(0)$	100	ρ_{01}	0.25	Paths	100,000
K	100	θ	0.35	$S_2(0)$	120	ρ_{02}	0.50	Time steps	100,000
		$r(0)$	6%	σ_1	30%	ρ_{12}	0.75		
		σ_r	20%	σ_2	40%				

Table 3.6.1: Base case parameters for sample prices of the GAO on assets 1 and 2, and the AXO.

Parameters	GAO on Asset 1			GAO on Asset 2		
	BS	MC	% Error	BS	MC	% Error
Base case	8.2394	8.2648	-0.3%	23.3099	23.3272	-0.07%
$T = 10$	12.2563	12.2756	-0.16%	13.4416	13.5051	-0.47%
$K = 120$	2.1669	2.1810	-0.65%	12.1107	12.0912	0.16%
$a = 0.4$	8.5059	8.5122	-0.07%	23.5566	23.5221	0.15%
$\theta = 0.15$	8.0393	8.0355	0.05%	23.1622	23.1832	-0.09%
$r(0) = 10\%$	8.9972	8.9705	0.30%	24.1711	24.1223	0.20%
$\sigma_r = 50\%$	8.6084	8.5712	0.43%	22.7620	22.7605	0.01%
$S_2(0) = 100$	8.2394	8.2404	-0.01%	10.0922	10.1097	-0.17%
$\sigma_1 = 60\%$	13.1323	13.1661	-0.26%	23.3099	23.3252	-0.07%
$\sigma_2 = 80\%$	8.2394	8.2307	0.11%	27.3507	27.3214	0.11%
$\rho_{01} = -0.15$	7.9185	7.9282	-0.12%	23.3099	23.3533	-0.19%
$\rho_{02} = -0.5, \rho_{12} = 0.6$	8.2394	8.2431	-0.05%	23.6464	23.5934	0.22%
$\rho_{12} = -0.25$	8.2394	8.2142	0.31%	23.3099	23.3341	-0.10%

Table 3.6.2: Simulated prices of the GAO on assets 1 and 2.

The base case parameters are given in Table 3.6.1.

Parameters	AEO		
	BS	MC	% Error
Base case	1.0274	1.0233	0.40%
$T = 10$	10.1892	10.3010	-1.08%
$K = 120$	1.0274	1.0289	-0.15%
$a = 0.4$	1.0066	1.0123	-0.56%
$\theta = 0.15$	1.0402	1.0451	-0.47%
$r(0) = 10\%$	1.0095	1.0060	0.35%
$\sigma_r = 50\%$	1.1318	1.1315	0.03%
$S_2(0) = 100$	6.2384	6.2011	0.60%
$\sigma_1 = 60\%$	2.5576	2.5492	0.33%
$\sigma_2 = 80\%$	8.6620	8.6637	-0.02%
$\rho_{01} = -0.15$	1.0929	1.0832	0.90%
$\rho_{02} = -0.5, \rho_{12} = 0.6$	1.6374	1.6368	0.04%
$\rho_{12} = -0.25$	6.1666	6.1641	0.04%

Table 3.6.3: Simulated prices of the AXO on assets 1 and 2.

The base case parameters are given in Table 3.6.1 on the preceding page.

Parameters	GAO on Asset 1			GAO on Asset 2		
	R St Dev	C St Dev	% Diff	R St Dev	C St Dev	% Diff
Base case	0.0374	0.0205	82.5%	0.0736	0.0405	81.9%
$T = 10$	0.0732	0.0688	6.4%	0.0793	0.0766	3.5%
$K = 120$	0.0201	0.0126	59.3%	0.0587	0.0325	80.7%
$a = 0.4$	0.0376	0.0206	82.7%	0.0735	0.0400	83.8%
$\theta = 0.15$	0.0371	0.0204	81.3%	0.0742	0.0406	82.6%
$r(0) = 10\%$	0.0381	0.0207	83.7%	0.0732	0.0399	83.4%
$\sigma_r = 50\%$	0.0380	0.0211	79.5%	0.0726	0.0415	74.9%
$S_2(0) = 100$	0.0372	0.0203	82.8%	0.0489	0.0271	80.3%
$\sigma_1 = 60\%$	0.0733	0.0432	69.8%	0.0739	0.0404	83.0%
$\sigma_2 = 80\%$	0.0372	0.0206	80.5%	0.1331	0.0812	64.0%
$\rho_{01} = -0.15$	0.0363	0.0210	72.8%	0.0736	0.0404	81.9%
$\rho_{02} = -0.5, \rho_{12} = 0.6$	0.0370	0.0204	81.7%	0.0752	0.0433	73.9%
$\rho_{12} = -0.25$	0.0371	0.0205	81.3%	0.0735	0.0404	81.9%

Table 3.6.4: Raw and control variate standard deviation of the prices of the GAO.

Parameters	AEO		
	R St Dev	C St Dev	% Diff
Base case	0.0110	0.0079	39.8%
$T = 10$	0.1149	0.1107	3.9%
$K = 120$	0.0110	0.0078	41.7%
$a = 0.4$	0.0108	0.0077	41.0%
$\theta = 0.15$	0.0111	0.0079	40.6%
$r(0) = 10\%$	0.0107	0.0076	40.4%
$\sigma_r = 50\%$	0.0122	0.0089	37.4%
$S_2(0) = 100$	0.0266	0.0157	69.9%
$\sigma_1 = 60\%$	0.0275	0.0187	46.8%
$\sigma_2 = 80\%$	0.0407	0.0261	55.9%
$\rho_{01} = -0.15$	0.0117	0.0084	40.1%
$\rho_{02} = -0.5, \rho_{12} = 0.6$	0.0142	0.0096	47.8%
$\rho_{12} = -0.25$	0.0410	0.0256	60.4%

Table 3.6.5: Raw and control variate standard deviation of the prices of the AXO.

Chapter 4

State Price Process with Vasicek Interest Rates

The second ingredient required for our study of cost efficiency with stochastic interest rates in the context of Asian Executive Compensation is an expression for the state price process (see the definition of the cost efficient counterpart in Theorem 1.2.4 on page 4). More specifically, we require an explicit form for the state price process for a market with two risky assets and stochastic interest rates modeled by a Vasicek process. Even though the literature abounds with option pricing methodologies under environments with much stochasticity, not as much research has been done in the study of state price processes in such frameworks.

Hürlimann [26] gives the state price process for risky assets and interest rates, but assumes that the interest rates and risky assets are driven by the same Brownian motions. Miltersen and Persson [42] provide a general framework for deriving the state price process and give concrete examples when the market has two sources of randomness. However this is insufficient as we need three sources of randomness in our case, and the extension from two to three sources of randomness is non-trivial¹. Jeanblanc, Yor and Chesney [27] provide a fairly general formulation of the state price process, but we want an explicit expression in order to derive its distribution as well.

In Section 4.1 on the following page, we will derive an explicit expression for the state price process and its distribution, and recast it as a function of market variables in Sec-

¹Boyle, Tan and Tian [14] write “Our colleague, Ken Vetzal pointed out that extensions of results from $n = 2$ to $n = 3$ are sometimes not very easy and cited Fermat’s Last Theorem as an illustration”. Fortunately, our problem is trivial compared to Fermat’s Last Theorem.

tion 4.2 on page 53. The latter is needed to retrieve the value of the state price process in a particular state of the world because the state price process itself is not directly observable. Using the same method of derivation, we consider some special cases in Section 4.3 on page 55 which could be of interest for other classes of options.

The formula for the state price process and its expression in terms of the market variables are given in Propositions 4.1.3 on page 52 and 4.2.1 on page 54. A convenient summary of the all the new expressions that we have derived are provided in Section 4.3.5 on page 59.

4.1 State Price Process

In this section we will derive the state price process for a market with two risky assets and stochastic interest rates and then find its distribution.

4.1.1 Setup

Let us consider a market with two risky assets, S_1 and S_2 , with the short rate dynamics being driven by the Vasicek process. Using the setup of Björk [8], we assume the existence of the exogenously given risk free asset B which we call the money market account defined by the \mathbb{P} -dynamics:

$$dB(t) = r(t)B(t)dt \quad (4.1.1)$$

We also assume that there exists a market for zero coupon T -bonds for every value of T . Hence, this market contains an infinite number of bonds plus two the risky assets.

Under the \mathbb{P} measure, the dynamics of the assets and the T -bond (for a fixed choice of T) are:

$$\frac{dS_1(t)}{S_1(t)} = \mu_1(t)dt + \sigma_1 C_{21}dW_0(t) + \sigma_1 C_{22}dW_1(t) \quad (4.1.2)$$

$$\frac{dS_2(t)}{S_2(t)} = \mu_2(t)dt + \sigma_2 C_{31}dW_0(t) + \sigma_2 C_{32}dW_1(t) + \sigma_2 C_{33}dW_2(t) \quad (4.1.3)$$

$$\frac{dP(t, T)}{P(t, T)} = \mu_P(t, T)dt - \sigma_P(t, T)dW_0(t) \quad (4.1.4)$$

where $\mu_1(t)$, $\mu_2(t)$ and $\mu_P(t, T)$ are the drift of the assets and bond price dynamics under the \mathbb{P} measure. The diffusion terms are defined in (3.1.11) on page 32.

We assume that all risky assets traded in the market (i.e. the two risky assets and the bond) share the same market price of risk, λ , that is constant over time. The assumption of the common market price of risk is reasonable, based on the Arbitrage Pricing Theory (see e.g., Ross [49]). The choice of a constant market price of risk (though it is fairly straightforward to introduce time-dependence) is based on the common approach for the Vasicek model [15]. With this in mind, we define the market price of risk as

$$\lambda = \frac{\mu_1(t) - r(t)}{\sigma_1} = \frac{\mu_2(t) - r(t)}{\sigma_2} = \frac{\mu_P(t) - r(t)}{\sigma_P(t, T)} \quad (4.1.5)$$

Combining (4.1.5) and (4.1.2)–(4.1.4) on the preceding page gives us:

$$\frac{dS_1(t)}{S_1(t)} = [r(t) + \lambda\sigma_1]dt + \sigma_1 C_{21}dW_0(t) + \sigma_1 C_{22}dW_1(t) \quad (4.1.6)$$

$$\frac{dS_2(t)}{S_2(t)} = [r(t) + \lambda\sigma_2]dt + \sigma_2 C_{31}dW_0(t) + \sigma_2 C_{32}dW_1(t) + \sigma_2 C_{33}dW_2(t) \quad (4.1.7)$$

$$\frac{dP(t, T)}{P(t, T)} = [r(t) - \lambda\sigma_P(t, T)]dt - \sigma_P(t, T)dW_0(t) \quad (4.1.8)$$

A straightforward application of Ito's Lemma to (4.1.6)–(4.1.8) gives us:

$$\begin{aligned} S_1(t) = S_1(0) \exp \left\{ \int_0^t r(s)ds + \sigma_1 \left(\lambda - \frac{\sigma_1}{2} \right) t \right. \\ \left. + \int_0^t \sigma_1 C_{21}dW_0(s) + \int_0^t \sigma_1 C_{22}dW_1(s) \right\} \end{aligned} \quad (4.1.9)$$

$$\begin{aligned} S_2(t) = S_2(0) \exp \left\{ \int_0^t r(s)ds + \sigma_2 \left(\lambda - \frac{\sigma_2}{2} \right) t \right. \\ \left. + \int_0^t \sigma_2 C_{31}dW_0(s) + \int_0^t \sigma_2 C_{32}dW_1(s) + \int_0^t \sigma_2 C_{33}dW_2(s) \right\} \end{aligned} \quad (4.1.10)$$

$$\begin{aligned} P(t, T) = P(0, T) \exp \left\{ \int_0^t r(s)ds - \lambda \int_0^t \sigma_P(s, T)ds - \frac{1}{2} \int_0^t \sigma_P^2(s, T)ds \right. \\ \left. - \int_0^t \frac{\sigma_r}{a}dW_0(s) + \int_0^t \frac{\sigma_r}{a}e^{-a(T-s)}dW_0(s) \right\} \end{aligned} \quad (4.1.11)$$

Under the \mathbb{P} measure, the short rate has the following dynamics:

$$dr(t) = a \left(\theta + \frac{\lambda\sigma_r}{a} - r(t) \right) dt + \sigma_r dW_0(t) \quad (4.1.12)$$

This has the solution of

$$r(t) = r(0)e^{-at} + \left(\theta + \frac{\lambda\sigma_r}{a} \right) (1 - e^{-at}) + \sigma_r \int_0^t e^{-a(t-s)} dW_0(s) \quad (4.1.13)$$

Remark 4.1.1. Compare the definition of (4.1.12)–(4.1.13) on pages 51–52 to (3.1.1)–(3.1.2) on pages 31–32. The dynamics are usually defined under the \mathbb{Q} measure, and the \mathbb{P} dynamics are retrieved through the introduction of the market price of risk λ [15].

Remark 4.1.2. Note that we have expressed (4.1.11) on the preceding page in this manner intentionally so that the time-dependent stochastic integrand term will cancel out with the one in (4.1.13). It will become apparent later that this makes it possible to express the state price process as a function of market variables.

4.1.2 An Explicit Expression for the State Price Process

Using the setup given in Section 4.1.1 on page 50, Proposition 4.1.3 gives an explicit expression of the state price process while Corollary 4.1.6 on the following page gives its distribution.

Proposition 4.1.3. *Consider a market with two risky assets and stochastic interest rates modeled by the Vasicek process. The state price process is given by*

$$\xi_{S_2}(T) = \exp \left\{ - \int_0^T r(s) ds - \frac{1}{2} (\lambda^2 + \alpha_1^2 + \alpha_2^2) T - \int_0^T \lambda dW_0(s) - \int_0^T \alpha_1 dW_1(s) - \int_0^T \alpha_2 dW_2(s) \right\} \quad (4.1.14)$$

where

$$A = \rho_{01}^2 + \rho_{02}^2 + \rho_{12}^2 - 1 - 2\rho_{01}\rho_{02}\rho_{12} \quad (4.1.15)$$

$$\alpha_1 = \lambda \sqrt{\frac{1 - \rho_{01}}{1 + \rho_{01}}} \quad (4.1.16)$$

$$\alpha_2 = \alpha_1 \cdot \frac{1 + \rho_{01} - \rho_{02} - \rho_{12}}{\sqrt{A}} \quad (4.1.17)$$

Remark 4.1.4. We have implicitly assumed that $A > 0$ in order for (4.1.17) to be well defined. This is also required in order for the systems of equations (B.3.1.5) on page 99 (proof of Proposition 4.1.3) and (B.3.3.5) on page 103 (proof of Proposition 4.2.1) to be well defined.

Remark 4.1.5. To check the correctness of (4.1.14) on the preceding page, we can quickly compute the differentials $d(\xi_{S_2}(t)S_1(t))$, $d(\xi_{S_2}(t)S_2(t))$, and $d(\xi_{S_2}(t)P(t, T))$ using Ito's Lemma and verify that they have zero drift i.e. the quantities are martingales.

Corollary 4.1.6. *Consider a market with two risky assets and stochastic interest rates modeled by the Vasicek process. Under the \mathbb{P} measure, $\xi_{S_2}(T) \sim \mathcal{LN}(m_{\xi_{S_2}}(T), v_{\xi_{S_2}}^2(T))$, where*

$$\begin{aligned}
 m_{\xi_{S_2}}(T) &= -\frac{1}{2}(\lambda^2 + \alpha_1^2 + \alpha_2^2)T \\
 &\quad - \left(\theta + \frac{\lambda\sigma_r}{a}\right)T - \left(r(0) - \theta - \frac{\lambda\sigma_r}{a}\right)\frac{1 - e^{-aT}}{a} \\
 v_{\xi_{S_2}}^2(T) &= (\lambda^2 + \alpha_1^2 + \alpha_2^2)T + \frac{\sigma_r}{a}\left(2\lambda + \frac{\sigma_r}{a}\right)\left(T - \frac{1 - e^{-aT}}{a}\right) \\
 &\quad - \frac{\sigma_r^2}{2a^3}(1 - e^{-aT})^2
 \end{aligned}
 \tag{4.1.18}$$

4.2 State Price Process as a Function of Market Variables

In this section, we express the state price process as a function of market variables. We want this expression because in actuality, the state price process is not a traded asset and we are unable to observe its value directly. If we have such a function, we are able to retrieve the value of the state price process through observable market variables and craft the payoff of the true cost efficient counterpart (CEC).

The key challenge is to express the market variables in such a way that we are able to match the stochastic integrals that arise in the state price process. We will start with matching the stochastic integral terms in the state price process, followed by the integral of the short rate, and finally the constant terms. The program is as follow:

1. The stochastic integrands that appear in the state price process are all constants, so we are unable to match them by simply using the underlying assets and the T -bond since the stochastic integrand in the T -bond is a function of time. However, it turns out that a combination of $S_1(T)$, $S_2(T)$ and $P(0, T) \exp\left\{\frac{r(T)}{a}\right\}$ will suffice (see Remark 4.1.2 on the previous page and (B.3.3.1) on page 103).

2. The integral of the short rate that appears can be matched using the money market account, $B(T)$.
3. The remaining terms are constants and are easily handled. We will introduce a constant, M , as a plug variable.

With these in mind, we are able to give such an expression in Proposition 4.2.1

Proposition 4.2.1. *Consider a market with two risky assets and stochastic interest rates modeled by the Vasicek process. The state price process at time T can be expressed as a function of the market variables in the following manner*

$$\xi_{S_2}^m(T) = \left[P(0, T) \exp \left\{ \frac{r(T)}{a} \right\} \right]^x \left[\frac{S_1(T)}{S_1(0)} \right]^y \left[\frac{S_2(T)}{S_2(0)} \right]^z [B(T)]^b e^M \quad (4.2.1)$$

where

$$\begin{aligned} P(0, T) &= \text{price of a } T\text{-bond at time } 0 \\ S_i(T) &= \text{price of asset } i \text{ at time } T \\ r(T) &= \text{short rate at time } T \\ B(T) &= \text{money market account at time } T \end{aligned}$$

$$x = \frac{a\lambda}{\sigma_r} \cdot \frac{(1 - \rho_{12})(1 + \rho_{12} - \rho_{01} - \rho_{02})}{A} \quad (4.2.2)$$

$$y = \frac{\lambda}{\sigma_1} \cdot \frac{(1 - \rho_{02})(1 + \rho_{02} - \rho_{12} - \rho_{01})}{A} \quad (4.2.3)$$

$$z = \frac{\lambda}{\sigma_2} \cdot \frac{(1 - \rho_{01})(1 + \rho_{01} - \rho_{02} - \rho_{12})}{A} \quad (4.2.4)$$

$$b = x - y - z - 1 \quad (4.2.5)$$

$$\begin{aligned} M = & - \left\{ \frac{1}{2} (\lambda^2 + \alpha_1^2 + \alpha_2^2) T + \frac{x}{a} e^{-aT} \left[r(0) - \theta + \left(\frac{\sigma_r}{a} \right)^2 \left(1 - \frac{1}{4} e^{-aT} \right) \right] \right. \\ & + \frac{x}{a} \left[\theta + \sigma_r \left(\lambda T + \frac{\sigma_r T}{2a} - \frac{3\sigma_r}{4a^2} \right) \right] \\ & \left. + y\sigma_1 \left(\lambda - \frac{\sigma_1}{2} \right) T + z\sigma_2 \left(\lambda - \frac{\sigma_2}{2} \right) T \right\} \quad (4.2.6) \end{aligned}$$

and A as given in (4.1.15) on page 52.

Remark 4.2.2. Even though $r(T)$ is technically a market variable at time T , it is not directly observable per se. However, we can use a short term rate as a proxy, say the monthly rate [15]. In fact, this is how one would go about calibrating the Vasicek model to market data.

4.3 Special Cases

In this section, we will consider some special cases of the state price process as defined in (4.2.1) on the previous page by following the same program given in Section 4.2 on page 53. In fact, thus far, we have the following procedure that can be used to derive state price processes for different combinations of stochastic/constant interest rates and number of risky assets:

1. Setup a system of equations similar to (B.3.1.5) on page 99 (proof of Proposition 4.1.3) and solve for the unknowns to arrive at an expression for the state price process similar to (4.1.14) on page 52. If a particular source of randomness is not present, set the corresponding unknown to zero.
2. Setup a system of equations similar to (B.3.3.5) on page 103 (proof of Proposition 4.2.1) and solve for the unknowns to arrive at an expression for the state price process in terms of market variables, similar to (4.2.1) on the preceding page. If a particular source of randomness is not present, set the corresponding unknown to zero.

As a verification of our procedure, we will see that in the case of constant interest rates with one and two assets, our expressions of the state price process in terms of the market variables agree with existing results given by [4] and [7]. In fact, this must be the case because the market is complete when interest rates are constant i.e. the state price process is unique.

4.3.1 Constant Instant Rates with One Risky Asset

We begin with a market with just one risky asset, S_1 , and constant interest rate. This is the classical Black-Scholes framework.

Proposition 4.3.1. *Consider a market with one risky asset and constant interest rate. The state price process is given by*

$$\xi_{C_1}(T) = \exp \left\{ -rT - \frac{1}{2}\lambda^2 T - \int_0^T \lambda dW_1(s) \right\} \quad (4.3.1)$$

In terms of the market variables, it can be expressed as

$$\xi_{C_1}^m(T) = \left[\frac{S_1(T)}{S_1(0)} \right]^y e^{brT} b e^M \quad (4.3.2)$$

where

$$\begin{aligned} y &= -\frac{\lambda}{\sigma_1} \\ b &= \frac{\lambda}{\sigma_1} - 1 \\ M &= \frac{\lambda}{2}(\lambda - \sigma_1)T \end{aligned}$$

Remark 4.3.2. Note that in the case of constant interest rates, $\int_0^T r(s)ds$ simply evaluates to rT . If interest rates are deterministic (see Section 3.5 on page 42), the expression in Propositions 4.3.1 on the preceding page and 4.3.4 still works with

$$\int_0^T r(s)ds = \frac{1}{a}(r(0) - \theta)(1 - e^{-aT}) + \theta T$$

Corollary 4.3.3. *Our expression of the state price process given in (4.3.2) on the preceding page agrees with the following given by Bernard, Boyle and Vanduffel [4]:*

$$\xi_{BBV}(T) = \gamma \left[\frac{S_1(T)}{S_1(0)} \right]^{-\delta} \quad (4.3.3)$$

where $\theta = \frac{\mu_1 - r}{\sigma_1}$, $\delta = \frac{\theta}{\sigma_1}$, and

$$\gamma = \exp \left\{ \frac{\theta}{\sigma_1} \left(\mu_1 - \frac{\sigma_1^2}{2} \right) T - \left(r + \frac{\theta^2}{2} \right) T \right\}$$

4.3.2 Constant Instant Rates with Two Risky Assets

We will now include a second risky asset, S_2 , but still retain the constant interest rates.

Proposition 4.3.4. *Consider a market with two risky assets and constant interest rate. The state price process is given by*

$$\xi_{C2}(T) = \exp \left\{ -rT - \frac{\lambda^2}{1 + \rho_{12}} T - \int_0^T \lambda dW_1(s) - \int_0^T \lambda \sqrt{\frac{1 - \rho_{12}}{1 + \rho_{12}}} dW_2(s) \right\} \quad (4.3.4)$$

In terms of the market variables, it can be expressed as

$$\xi_{C2}^m(T) = \left[\frac{S_1(T)}{S_1(0)} \right]^y \left[\frac{S_2(T)}{S_2(0)} \right]^z e^{brT} e^M \quad (4.3.5)$$

where

$$\begin{aligned}
 y &= -\frac{\lambda}{\sigma_1(1 + \rho_{12})} \\
 z &= -\frac{\lambda}{\sigma_2(1 + \rho_{12})} \\
 b &= \frac{\lambda}{1 + \rho_{12}} \left(\frac{1}{\sigma_1} + \frac{1}{\sigma_2} \right) - 1 \\
 M &= \frac{\lambda}{1 + \rho_{12}} \left[\lambda - \frac{\sigma_1}{2} - \frac{\sigma_2}{2} \right] T
 \end{aligned}$$

Corollary 4.3.5. *Our expression of the state price process given in (4.3.5) on the previous page agrees with the following given by Bernard, Maj and Vanduffel [7]:*

$$\xi_{BMV}(T) = \gamma \left[\frac{S_\pi(T)^*}{S_\pi^*(0)} \right]^{-\delta} \tag{4.3.6}$$

where $\theta = \frac{\mu(\pi_*) - r}{\sigma(\pi_*)}$, $\delta = \frac{\theta_*}{\sigma(\pi_*)}$, and

$$\gamma = \exp \left\{ \frac{\theta_*}{\sigma(\pi_*)} \left[\mu(\pi_*) - \frac{\sigma^2(\pi_*)}{2} \right] T - \left(r + \frac{\theta^2}{2} \right) T \right\}$$

$\underline{\pi}_\rightarrow^*$ is the market portfolio and S_π^* is the security that is constructed using the constant mix $\underline{\pi}_\rightarrow^*$ as prescribed in Section 2.4.2 on page 19.

Remark 4.3.6. It appears that the formulation of the state price process given by [7] is more succinct than ours. In fact, when the number of assets exceeds two, the market portfolio formulation is more convenient to work with than our approach due to its use of matrix algebra; the expressions in our formulation will quickly become very complicated. However, it is not immediately clear how to extend their approach to incorporate stochastic interest rates.

4.3.3 Stochastic Interest Rates Only

We now consider a market where the short rate is driven by a Vasicek process in (4.1.12) on page 51 and the existence of the money market account $B(t)$ defined in (4.1.1) on page 50.

Proposition 4.3.7. Consider a market stochastic interest rates modeled by a Vasicek process. The state price process is given by

$$\xi_{S_0}(T) = \exp \left\{ - \int_0^T r(s) ds - \frac{1}{2} \lambda^2 T - \int_0^T \lambda dW_0(s) \right\} \quad (4.3.7)$$

In terms of the market variables, it can be expressed as

$$\xi_{S_0}^m(T) = \left[P(0, T) \exp \left\{ \frac{r(T)}{a} \right\} \right]^x [B(T)]^b e^M \quad (4.3.8)$$

where

$$\begin{aligned} x &= -\frac{a\lambda}{\sigma_r} \\ b &= -\frac{a\lambda}{\sigma_r} - 1 \\ M &= -\frac{\lambda^2}{2} T + \frac{\lambda}{\sigma_r} \left\{ e^{-aT} \left[r(0) - \theta + \left(\frac{\sigma_r}{a} \right)^2 \left(1 - \frac{1}{4} e^{-aT} \right) \right] \right. \\ &\quad \left. + \theta + \sigma_r \left(\lambda T + \frac{\sigma_r T}{2a} - \frac{3\sigma_r}{4a^2} \right) \right\} \end{aligned}$$

Remark 4.3.8. This state price process (4.3.7) could be used to investigate the cost efficiency of a bond option.

4.3.4 Stochastic Interest Rates with One Risky Asset

Finally, we add one risky asset, S_1 , to the market described in Section 4.3.3 on the preceding page.

Proposition 4.3.9. Consider a market with one risky asset and stochastic interest rates modeled by a Vasicek process. The state price process is given by

$$\xi_{S_1}(T) = \exp \left\{ - \int_0^T r(s) ds - \frac{\lambda^2}{1 + \rho_{01}} T - \int_0^T \lambda dW_0(s) - \int_0^T \lambda \sqrt{\frac{1 - \rho_{01}}{1 + \rho_{01}}} dW_1(s) \right\} \quad (4.3.9)$$

In terms of the market variables, it can be expressed as

$$\xi_{S_1}^m(T) = \left[P(0, T) \exp \left\{ \frac{r(T)}{a} \right\} \right]^x \left[\frac{S_1(T)}{S_1(0)} \right]^y [B(T)]^b e^M \quad (4.3.10)$$

where

$$\begin{aligned} x &= -\frac{a\lambda}{\sigma_r} \left(\frac{1}{1 + \rho_{01}} \right) \\ y &= -\frac{\lambda}{\sigma_1} \left(\frac{1}{1 + \rho_{01}} \right) \\ b &= \left(\frac{\lambda}{1 + \rho_{01}} \right) \left(\frac{1}{\sigma_1} - \frac{a}{\sigma_r} \right) - 1 \\ M &= -\frac{\lambda^2}{1 + \rho_{01}} T + \frac{\lambda}{\sigma_r(1 + \rho_{01})} \left\{ e^{-aT} \left[r(0) - \theta + \left(\frac{\sigma_r}{a} \right)^2 \left(1 - \frac{1}{4} e^{-aT} \right) \right] \right. \\ &\quad \left. + \theta + \sigma_r \left(\lambda T + \frac{\sigma_r T}{2a} - \frac{3\sigma_r}{4a^2} \right) \right\} \\ &\quad + \frac{\lambda}{1 + \rho_{01}} \left(\lambda - \frac{\sigma_1}{2} \right) T \end{aligned}$$

Remark 4.3.10. This state price process (4.3.9) on the previous page could be used to investigate the cost efficiency of any option written on a single asset in the presence of stochastic interest rates.

4.3.5 Summary

In this section, we present a summary of all the cases considered in this chapter. The general form of the state price process is given by the following:

$$\begin{aligned} \xi_{Ri}(T) &= \exp \left\{ -\int_0^T r(s) ds - \frac{1}{2} (\alpha_0^2 + \alpha_1^2 + \alpha_2^2) T \right. \\ &\quad \left. - \int_0^T \alpha_0 dW_0(s) - \int_0^T \alpha_1 dW_1(s) - \int_0^T \alpha_2 dW_2(s) \right\} \quad (4.3.11) \end{aligned}$$

The R subscript refers to the randomness of the short rate (C for constant or S for stochastic) whereas the i subscript refers to the number of risky assets.

The following table specifies the values taken by the parameters for each of the following cases.

Case	Symbol	Section	α_0	α_1	α_2
Const-1	$\xi_{C1}(T)$	4.3.1	0	λ	0
Const-2	$\xi_{C2}(T)$	4.3.2	0	λ	$\lambda \sqrt{\frac{1-\rho_{12}}{1+\rho_{12}}}$
Stoch-0	$\xi_{S0}(T)$	4.3.3	λ	0	0
Stoch-1	$\xi_{S1}(T)$	4.3.4	λ	$\lambda \sqrt{\frac{1-\rho_{01}}{1+\rho_{01}}}$	0
Stoch-2	$\xi_{S2}(T)$	4.1.2	λ	$\lambda \sqrt{\frac{1-\rho_{01}}{1+\rho_{01}}}$	$\alpha_1 \cdot \frac{1+\rho_{01}-\rho_{02}-\rho_{12}}{\sqrt{A}}$

Table 4.3.1: Parameter values for the state price process.

Const- i and Stoch- i refer to the cases of constant and stochastic interest rates with i assets respectively. A is defined in (4.1.15) on page 52. $\int_0^T r(s)ds$ simplifies to rT and $\frac{1}{a}(r(0) - \theta)(1 - e^{-aT}) + \theta T$ respectively in the constant and deterministic interest rate cases.

When expressed in terms of the market variables, the general form of the state price process is given by the following:

$$\xi_{Ri}^m(T) = \left[P(0, T) \exp \left\{ \frac{r(T)}{a} \right\} \right]^x \left[\frac{S_1(T)}{S_1(0)} \right]^y \left[\frac{S_2(T)}{S_2(0)} \right]^z [B(T)]^b e^M \quad (4.3.12)$$

The R subscript refers to the randomness of the short rate (C for constant or S for stochastic) whereas the i subscript refers to the number of risky assets.

The following table specifies the values taken by the parameters for each of the following cases.

Cases	Symbol	Section	x	y	z	b
Const-1	$\xi_{C1}^m(T)$	4.3.1	0	$-\frac{\lambda}{\sigma_1}$	0	$\frac{\lambda}{\sigma_1} - 1$
Const-2	$\xi_{C2}^m(T)$	4.3.2	0	$-\frac{\lambda}{\sigma_1(1+\rho_{12})}$	$-\frac{\lambda}{\sigma_2(1+\rho_{12})}$	$\frac{\lambda}{1+\rho_{12}} \left(\frac{1}{\sigma_1} + \frac{1}{\sigma_2} \right) - 1$
Stoch-0	$\xi_{S0}^m(T)$	4.3.3	$-\frac{a\lambda}{\sigma_r}$	0	0	$-\frac{a\lambda}{\sigma_r} - 1$
Stoch-1	$\xi_{S1}^m(T)$	4.3.4	$-\frac{a\lambda}{\sigma_r} \left(\frac{1}{1+\rho_{01}} \right)$	$-\frac{\lambda}{\sigma_1} \left(\frac{1}{1+\rho_{01}} \right)$	0	$\left(\frac{\lambda}{1+\rho_{01}} \right) \left(\frac{1}{\sigma_1} - \frac{a}{\sigma_r} \right) - 1$
Stoch-2	$\xi_{S2}^m(T)$	4.2	$\frac{a\lambda}{\sigma_r} \times \frac{1}{A}$	$\frac{\lambda}{\sigma_1} \times \frac{1}{A}$	$\frac{\lambda}{\sigma_2} \times \frac{1}{A}$	$x - y - z - 1$

Table 4.3.2: Parameter values for the state price process expressed in terms of the market variables.

Const- i and Stoch- i refer to the cases of constant and stochastic interest rates with i assets respectively. A is defined in (4.1.15) on page 52. $B(T)$ simplifies to $\exp\{rT\}$ and $\exp\left\{\frac{1}{a}(r(0) - \theta)(1 - e^{-aT}) + \theta T\right\}$ respectively in the constant and deterministic interest rate cases. M can be retrieved by substituting the above parameters into (4.2.6) on page 54.

Chapter 5

Asian Executive Compensation with Vasicek Interest Rates

We are now ready to return to our study of the cost efficiency of the Asian Executive Option (AEO) and Asian Executive Indexed Option (AEIO), but with an added twist of stochastic interest rates. We have derived the pricing formulas for the Geometric Asian Option (GAO) and Asian Exchange Option (AEO) in Chapter 3 on page 30, as well as the required state price process in Chapter 4 on page 49.

Section 5.1 draws some comparisons with the case of constant interest rates whilst Section 5.2 on the following page provides some context for our study. This chapter ends with results from Monte Carlo simulation in Section 5.3 on page 64.

Unless otherwise mentioned, the index i is understood to range over 1 and 2 i.e. $i = 1, 2$.

5.1 Comparison with the Case of Constant Instant Rates

Remark 5.1.1. In the case of constant interest rates, it is possible to construct explicitly the CEC of the AEO as a power option (Proposition 2.3.1 on page 11). However in the presence of stochastic interest rates, it seems like an explicit construction is no longer amenable to Theorem 1.2.4 on page 4, largely due to the extra path-dependence of the interest rates.

Although we were unable to construct the CEC of the AEIO, we were still able to construct the Power Exchange Executive Option (PXEO) (Section 2.4.1 on page 15) that has a cheaper price than the AEIO and yet inherits its desirable features. Once again, this no longer seems possible in the presence of stochastic interest rates. Therefore, we need to resort to Monte Carlo simulation to estimate these CEC's.

Remark 5.1.2. When interest rates are constant, the CEC is not path-dependent. Recall that the CEC of the AEO, G_T (Proposition 2.3.1 on page 11) is a power call option on the terminal stock price S_T , whereas the CEC of the AEIO is a function of the state price process ξ_T which is also not path-dependent. Even the PXEO that we constructed is not path-dependent. In fact, in the Black-Scholes model, path-dependent payoffs are not cost efficient unless $\mu_S = r$ [4].

However, it is interesting to note that once we introduce stochastic interest rates, the CEC is necessarily path-dependent. The CEC is a function of the state price process, and we have seen from Proposition 4.1.3 on page 52 that it is path-dependent.

5.2 Setup

Firstly, we have decided to exclude dividends in order to reduce the number of input parameters. The various pricing formulas are complicated enough already (see Propositions 3.3.1 on page 35 and 3.3.4 on page 36), even without the inclusion of dividends.

Recall from Chapter 2 on page 7 the following definitions of the payoff of the AEO ((2.3.1) on page 10):

$$\hat{G}_T = (\hat{S}_T - K)^+$$

and the AEIO ((2.4.1) on page 14):

$$\hat{A}_T = (\hat{S}_T - \hat{H}_T)^+$$

(Recall also that \hat{H}_T is the non-constant strike price that is linked to the performance of the average benchmark index adjusted for the level of systematic risk $\hat{\beta}$. See Remark 2.2.2 on page 9 also.)

In order to distinguish between the constant and stochastic interest rates cases, we will label the latter in boldface¹ and define their respective payoffs in a slightly different fashion

¹**AEO** and **AEIO** for labels; $\hat{\mathbf{G}}(T)$ and $\hat{\mathbf{A}}(T)$ for payoffs; $\hat{\mathbf{E}}_0$ and $\hat{\mathbf{V}}_0$ for prices; $\mathbf{G}(T)$ and $\mathbf{A}(T)$ for CEC payoffs; \mathbf{E}_0 and \mathbf{V}_0 for CEC prices; $\boldsymbol{\xi}(T)$ for the state price process.

as follow:

$$\hat{\mathbf{G}}(T) = (\hat{S}_i(T) - K)^+ \quad (5.2.1)$$

$$\hat{\mathbf{A}}(T) = (\hat{S}_1(T) - \hat{S}_2(T))^+ \quad (5.2.2)$$

where K is the strike price, and $\hat{S}_1(T)$ and $\hat{S}_2(T)$ are defined in (4.1.9)–(4.1.10) on page 51 respectively. Note that instead of using \hat{H}_T in (5.2.2), we have used $\hat{S}_2(T)$ as a proxy for the benchmark instead. Once again, this is done mainly for simplicity - the key components of indexing and averaging of the AEIO are still preserved.

The prices at time 0 of **AEO** and **AEIO**, $\hat{\mathbf{E}}_0$ and $\hat{\mathbf{V}}_0$, can be calculated directly using our formulas for the GAO (Proposition 3.3.1 on page 35) and the AEO (Proposition 3.3.4 on page 36) respectively.

5.3 Monte Carlo Simulation

We can use the state price process given by Proposition 4.1.3 on page 52 and its distribution in Corollary 4.1.6 on page 53 to simulate the prices of the CEC. A simple Euler scheme is used to discretize the interest rates and stock price paths.

The CEC of the **AEO** and **AEIO** are given by

$$\mathbf{G}(T) = F_{\hat{\mathbf{G}}}^{-1}(1 - F_{\xi}(\xi(T))) \quad (5.3.1)$$

$$\mathbf{A}(T) = F_{\hat{\mathbf{A}}}^{-1}(1 - F_{\xi}(\xi(T))) \quad (5.3.2)$$

Their respective prices are:

$$\mathbf{E}_0 = E_{\mathbb{P}}[\xi(T)\mathbf{G}(T)] \quad (5.3.3)$$

$$\mathbf{V}_0 = E_{\mathbb{P}}[\xi(T)\mathbf{A}(T)] \quad (5.3.4)$$

(5.3.3)–(5.3.4) are estimated by making random draws from the known distribution of $\xi(T)$ and evaluating the inverse CDF's, $F_{\hat{\mathbf{G}}}^{-1}$ and $F_{\hat{\mathbf{A}}}^{-1}$, numerically under the \mathbb{P} measure.

Remark 5.3.1. In the spirit of Section 4.1.1 on page 50, we will specify the value of the market price of risk, λ , as an input parameter instead of the respective expected returns, μ .

Figures 5.3.1 to 5.3.3 on pages 68–70 plot some prices of the **AEO** and **AEIO** and their respective CEC's across different sets of parameters to illustrate the degrees of efficiency

loss. Each point corresponds to the price for a particular parameter set. We consider the base case given in Table 5.3.1, and perturb some of the input parameters as well. We can see that the degree of efficiency loss is sensitive to all model parameters. The values that are used to plot this figure are given in Table 5.3.2 to 5.3.3 on pages 65–66.

Option		Vasicek		Assets		Correlation		Monte Carlo	
T	1	a	0.20	$S_1(0)$	100	ρ_{01}	0.25	Paths	1,000,000
K	100	θ	0.35	$S_2(0)$	120	ρ_{02}	0.50		
		$r(0)$	6%	σ_1	30%	ρ_{12}	0.75		
		σ_r	20%	σ_2	40%				
				λ	0.02				

Table 5.3.1: Base case parameters for sample **AEO** and **AEIO**.

Parameters	AEO on Asset 1				AEO on Asset 2			
	$\hat{\mathbf{E}}_0$	\mathbf{E}_0	% Eff Loss	St Dev	$\hat{\mathbf{E}}_0$	\mathbf{E}_0	% Eff Loss	St Dev
Base case	8.24	7.62	8.1%	0.0140	23.31	22.28	4.6%	0.0285
$K = 120$	2.17	1.97	10.1%	0.0078	12.11	11.47	5.6%	0.0230
$a = 0.4$	8.51	7.91	7.5%	0.0142	23.56	22.62	4.1%	0.0287
$\theta = 0.15$	8.04	7.42	8.3%	0.0137	23.16	22.19	4.4%	0.0281
$r(0) = 10\%$	9.00	8.41	7.0%	0.0149	24.17	23.19	4.2%	0.0294
$\sigma_r = 50\%$	8.61	7.63	12.9%	0.0172	22.76	21.11	7.8%	0.0338
$S_1(0) = 140$	40.40	39.06	3.4%	0.0271	23.31	22.32	4.4%	0.0283
$S_2(0) = 100$	8.24	7.62	8.1%	0.0140	10.09	9.50	6.2%	0.0190
$\sigma_1 = 60\%$	13.13	11.84	11.0%	0.0271	23.31	22.26	4.7%	0.0284
$\sigma_2 = 80\%$	8.24	7.73	6.6%	0.0141	27.35	25.84	5.8%	0.0524
$\lambda = 0.04$	8.24	7.44	10.8%	0.0141	23.31	21.85	6.7%	0.0285
$\rho_{01} = -0.15$	7.92	7.08	11.9%	0.0127	23.31	22.50	3.6%	0.0286
$\rho_{02} = -0.5, \rho_{12} = 0.6$	8.24	7.62	8.1%	0.0139	23.65	20.72	14.1%	0.0239
$\rho_{12} = -0.25$	8.24	7.68	7.2%	0.0140	23.31	22.45	3.8%	0.0285

Table 5.3.2: Prices and efficiency loss of the **AEO** on assets 1 and 2.

These are used to generate Figures 5.3.1 to 5.3.2 on pages 68–69 and the base case parameters are given in Table 5.3.1.

Parameters	AEIO			
	\hat{V}_0	\hat{V}_0	% Eff Loss	St Dev
Base case	1.03	0.73	41.4%	0.0034
$K = 120$	1.03	0.73	40.0%	0.0034
$a = 0.4$	1.01	0.72	40.4%	0.0034
$\theta = 0.15$	1.04	0.72	44.7%	0.0033
$r(0) = 10\%$	1.01	0.72	40.1%	0.0035
$\sigma_r = 50\%$	1.13	0.51	123.8%	0.0032
$S_1(0) = 140$	21.44	19.10	12.3%	0.0165
$S_2(0) = 100$	6.24	5.07	23.2%	0.0086
$\sigma_1 = 60\%$	2.56	1.98	29.1%	0.0096
$\sigma_2 = 80\%$	8.66	6.77	27.9%	0.0127
$\lambda = 0.04$	1.03	0.70	47.0%	0.0034
$\rho_{01} = -0.15$	1.09	0.70	56.1%	0.0032
$\rho_{02} = -0.5, \rho_{12} = 0.6$	1.64	1.56	5.0%	0.0060
$\rho_{12} = -0.25$	6.17	4.92	25.4%	0.0138

Table 5.3.3: Prices and efficiency loss of the **AEIO**.

These are used to generate Figure 5.3.3 on page 70 and the base case parameters are given in Table 5.3.1 on the previous page.

In the base case, the efficiency loss of the **AEO** on assets 1 and 2 are 5.9% and 2.6% respectively. For asset 1, the highest efficiency loss is 12.9% when we perturb σ_r to 50%, while the lowest is 3.4% when we perturb $S_1(0)$ to 140. For asset 2, the highest efficiency loss is 14.1% when we perturb ρ_{02} and ρ_{12} to -0.5 and 0.6 respectively, and the lowest is 3.8% when we perturb ρ_{12} to -0.25. As expected, the efficiency loss is sensitive to all input parameters that pertain to the Vasicek interest rates (a , θ , $r(0)$, and σ_r) i.e. the introduction of stochastic interest rates does have an impact on cost efficiency. The sensitivity of the efficiency loss with respect to the Vasicek parameters is about the same as the sensitivity with respect to the rest of parameters.

It is surprising that the degree of efficiency loss for the **AEIO** is much larger than the **AEO** across all cases, and it fluctuates wildly among the cases considered. For the case where $\sigma_r = 50\%$, the efficiency loss is 123.8%! However, the large relative values could be an artifact of the small absolute values. The base case efficiency loss is 41.4%, and the lowest efficiency loss is at 5.0% when we perturb ρ_{02} and ρ_{12} to -0.5 and 0.6 respectively. It is noteworthy that unlike the **AEO**, the sensitivity of the efficiency loss with respect to

the Vasicek parameters seems to be larger than the sensitivity with respect to the rest of the parameters.

One thing that is certain is that the CEC's of the **AEO** and **AEIO** are all cheaper than their original payoffs, in agreement with Theorem 1.2.4 on page 4. However, we have seen that the degree of efficiency loss is sensitive to the input parameters, as well as the type of payoff itself. This connection is definitely an area ripe for future research.

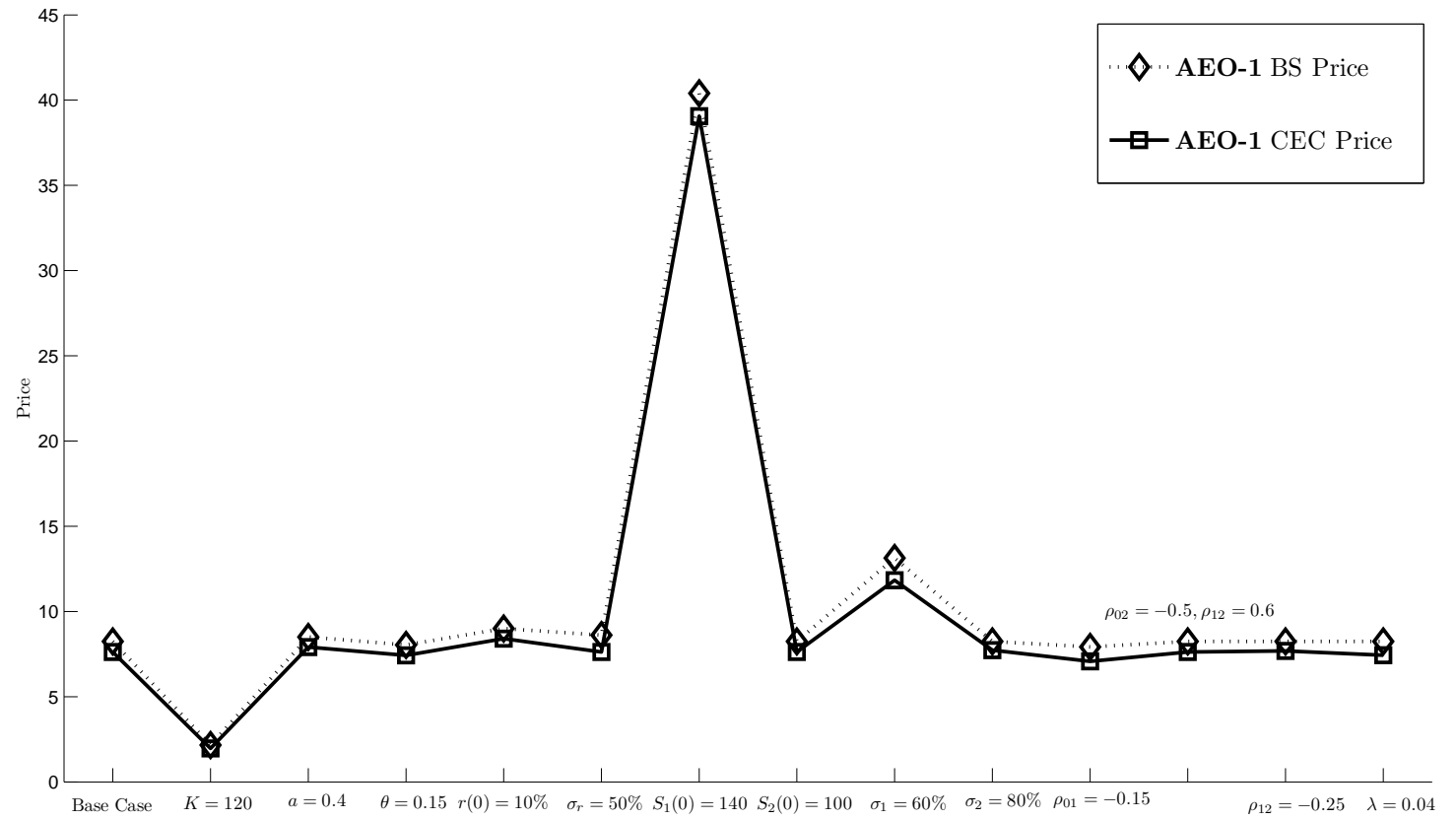


Figure 5.3.1: Sample prices for the **AEO** on asset 1 and its CEC across different cases.

The values used to generate this plot are given in Table 5.3.2 on page 65.

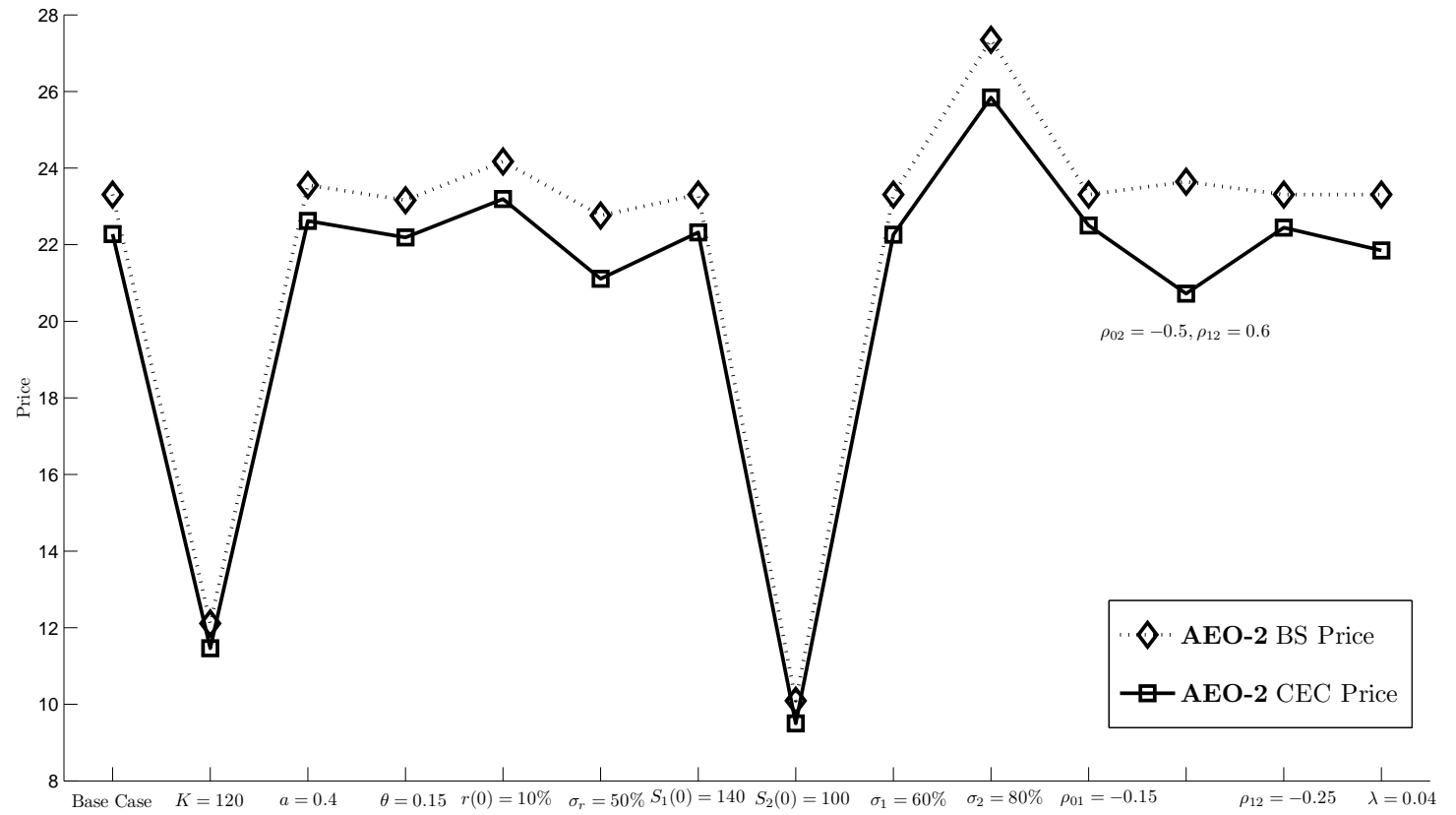


Figure 5.3.2: Sample prices for the **AEO** on asset 2 and its CEC across different cases.

The values used to generate this plot are given in Table 5.3.2 on page 65.

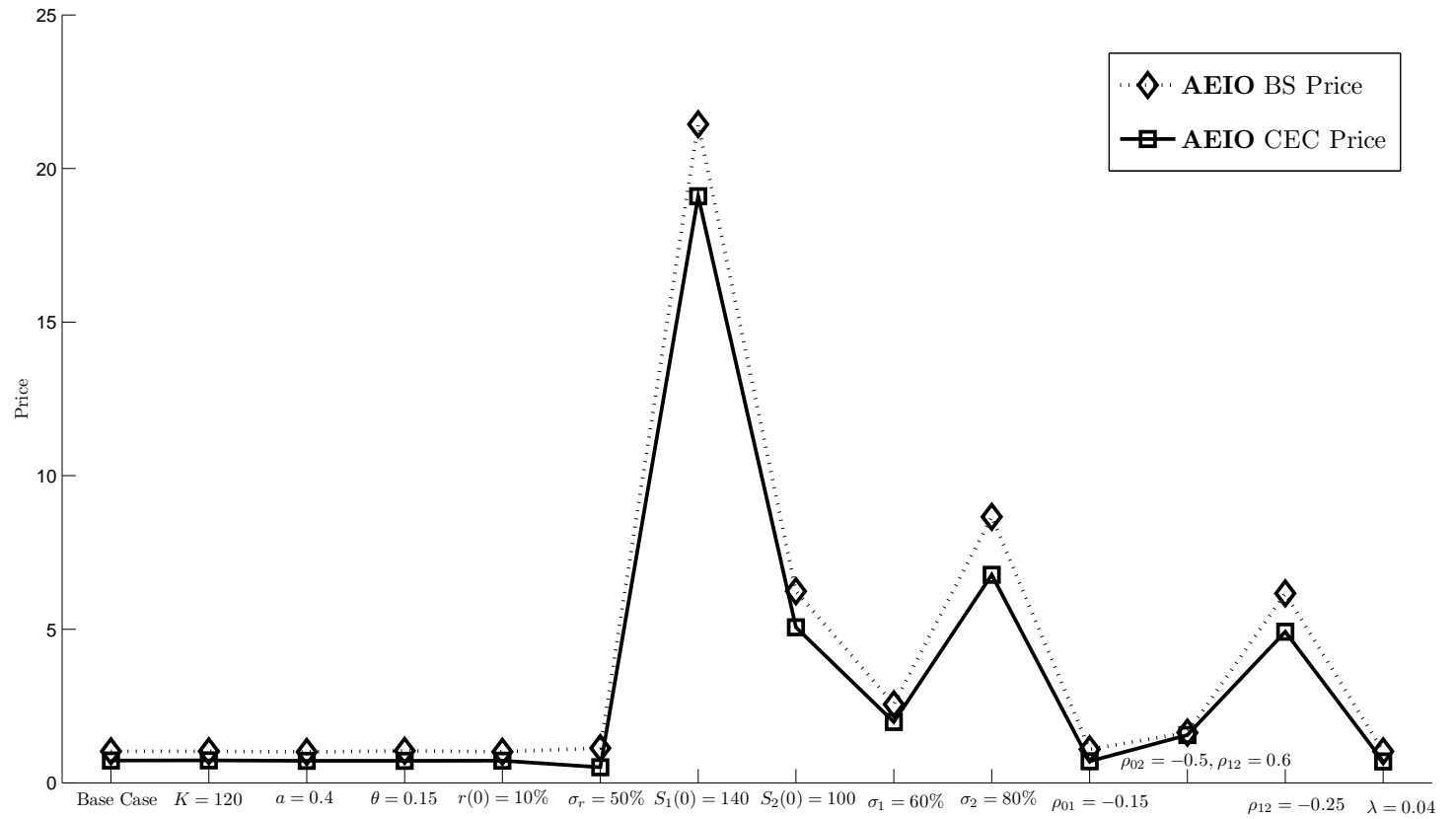


Figure 5.3.3: Sample prices for the **AEIO** and its CEC across different cases.

The values used to generate this plot are given in Table 5.3.3 on page 66.

Chapter 6

Conclusion

In this thesis we apply the concept of cost efficiency as prescribed by Bernard, Boyle and Vanduffel [4] to the Asian Executive Option and Asian Executive Indexed Option designed by Tian [53]. In doing so, we are able to find explicitly the cost efficient counterpart for the Asian Executive Option in the form of a power option, and design a payoff that is *cheaper* than the Asian Executive Indexed Option in the form of the Power Exchange Executive Option. The true cost efficient counterpart of the Asian Executive Indexed Option does not admit a closed form expression - but we are able to simulate its price using the state price process given by Bernard, Maj and Vanduffel [7] and study the degree of efficiency loss.

Given that executive options generally have long maturities, we incorporate stochastic interest rates modeled by a Vasicek process. In order to study the cost efficiency of the Asian Executive Option and Asian Executive Indexed Option, the key requirements are the pricing formulas for the Geometric Asian Option (GAO) and the Asian Exchange Option (AXO), as well as an expression for the state price process in the presence of stochastic interest rates. We are able to meet both requirements and subsequently simulate the prices of the cost efficient counterpart for the Asian Executive Option and Asian Executive Indexed Option.

To the best of our knowledge, all of our results and formulas are new.

Our research in this area is still nascent and here are some questions that remain unanswered:

1. The technique used to construct the Power Exchange Executive Option does not hold in general (Remark 2.4.6 on page 17). What are the conditions that guarantees that our technique will construct a cheaper payoff?

2. How do the incentives of the Power Exchange Executive Option compare with that of the Asian Executive Indexed Option (Remark 2.4.1 on page 15), and what is the incentive structure of its cost efficient counterpart?
3. How can we incorporate state-dependent preferences into the cost efficient counterpart (Remark 2.4.8 on page 27)?
4. Can we extend our pricing formulas to include more robust stochastic interest rate models (Remark 3.3.3 on page 36)?
5. Is there a way to incorporate stochastic interest rates into Bernard, Maj and Vanduffel's [7] formulation of the state price process (Remark 4.3.6 on page 57)?
6. Are there any options that will admit closed form cost efficient counterparts under a stochastic interest rate environment (Remarks 4.3.8 on page 58 and 4.3.10 on page 59)?
7. How can we incorporate Tian's [53] benchmark (\hat{H}_T , (2.2.2) on page 9), into our design of the **AEO** and **AEIO** (Section 5.2 on page 63)?
8. How do the pricing parameters for the Asian Executive Option and Asian Executive Indexed Option impact the degree of efficiency loss (Section 5.3 on page 64)?

We hope that the contributions that we have made thus far will aid in tackling the questions above.

APPENDICES

Appendix A

Useful Identities

In this appendix we will prove some identities that will be useful for deriving option pricing formulas.

Lemma A.1. Let $X \sim \mathcal{N}(\mu, \sigma^2)$. Then

$$E[(e^X - K)^+] = e^{\mu + \frac{\sigma^2}{2}} \Phi\left(\frac{\sigma^2 + \mu - \ln K}{\sigma}\right) - K \Phi\left(\frac{\mu - \ln K}{\sigma}\right)$$

Proof. This is a standard result (see, e.g. [15]). Let $X \sim \mathcal{N}(\mu, \sigma^2)$ and $f_X(x)$ be its corresponding pdf. We are interested in evaluating $E[(e^X - K)^+]$. By definition, we have

$$\begin{aligned} E[(e^X - K)^+] &= \int_{-\infty}^{\infty} (e^x - K)^+ f_X(x) dx \\ &= \int_{\ln K}^{\infty} (e^x - K) f_X(x) dx \\ &= \int_{\ln K}^{\infty} e^x f_X(x) dx - K \int_{\ln K}^{\infty} f_X(x) dx \end{aligned}$$

We consider each integral separately.

$$\begin{aligned} &\int_{\ln K}^{\infty} e^x f_X(x) dx \\ &= \int_{\ln K}^{\infty} e^x \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \end{aligned}$$

$$\begin{aligned}
&= e^{\mu + \frac{\sigma^2}{2}} \int_{\frac{\ln K - \mu}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-\sigma)^2} dy \quad (\text{Let } y = \frac{x-\mu}{\sigma}) \\
&= e^{\mu + \frac{\sigma^2}{2}} \int_{\frac{\ln K - \mu - \sigma^2}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \quad (\text{Let } z = y - \sigma) \\
&= e^{\mu + \frac{\sigma^2}{2}} \text{Prob} \left[Z \geq \frac{\ln K - \mu - \sigma^2}{\sigma} \right] \\
&= e^{\mu + \frac{\sigma^2}{2}} \Phi \left(\frac{\sigma^2 + \mu - \ln K}{\sigma} \right) \tag{A.1}
\end{aligned}$$

$$\int_{\ln K}^{\infty} f_X(x) dx = \text{Prob} \left[Z \geq \frac{\ln K - \mu}{\sigma} \right] = \Phi \left(\frac{\mu - \ln K}{\sigma} \right) \tag{A.2}$$

Combining (A.1) and (A.2) we have:

$$E[(e^X - K)^+] = e^{\mu + \frac{\sigma^2}{2}} \Phi \left(\frac{\sigma^2 + \mu - \ln K}{\sigma} \right) - K \Phi \left(\frac{\mu - \ln K}{\sigma} \right)$$

□

Lemma A.2.

$$\int_{-\infty}^{\infty} \phi(z) \Phi(A + Bz) dz = \Phi \left(\frac{A}{\sqrt{1 + B^2}} \right)$$

Proof. See the Proof of Proposition 1 in Li, Deng and Zhou [38]. □

Lemma A.3. Let $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$, $X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$, and $\text{Cov}[X_1, X_2] = \rho\sigma_1\sigma_2 = \sigma_{12}$. Then

$$E \left[(e^{X_1} - e^{X_2})^+ \right] = e^{\mu_1 + \frac{1}{2}\sigma_1^2} \Phi(d_1) - e^{\mu_2 + \frac{1}{2}\sigma_2^2} \Phi(d_2)$$

where $d_1 = \frac{\mu_1 - \mu_2 + \sigma_1^2 - \sigma_{12}}{\sqrt{\sigma_1^2 - 2\sigma_{12} + \sigma_2^2}}$ and $d_2 = d_1 - \sqrt{\sigma_1^2 - 2\sigma_{12} + \sigma_2^2}$

Proof. This proof is outlined briefly in [55], and the result is formulated differently. We adapt it to our case, and fill in much of the details. Let $f_{12}(u, v)$, $f_1(u)$, $f_2(v)$, $f_{1|2}(u|v)$, and $f_{2|1}(v|u)$ denote the joint pdf, marginal pdfs, and conditional pdfs of X_1 and X_2 . Then, we have

$$\begin{aligned}
&E \left[(e^{X_1} - e^{X_2})^+ \right] \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (e^u - e^v)^+ f_{12}(u, v) du dv
\end{aligned}$$

$$\begin{aligned}
&= \int_{-\infty}^{\infty} \int_{-\infty}^u (e^u - e^v)^+ f_{12}(u, v) dv du \\
&= \int_{-\infty}^{\infty} e^u \int_{-\infty}^u f_{12}(u, v) dv du - \int_{-\infty}^{\infty} e^v \int_v^{\infty} f_{12}(u, v) du dv \\
&= \int_{-\infty}^{\infty} e^u f_1(u) \int_{-\infty}^u f_{2|1}(v|u) dv du - \int_{-\infty}^{\infty} e^v f_2(v) \int_v^{\infty} f_{1|2}(u|v) du dv \quad (\text{A.3})
\end{aligned}$$

We will start with evaluating the first integral. We have the following conditional distribution (see, e.g. [29]):

$$(X_2|X_1 = u) \sim \mathcal{N}\left(\mu_2 + \frac{\sigma_2}{\sigma_1}\rho(u - \mu_1), (1 - \rho^2)\sigma_2^2\right) \quad (\text{A.4})$$

Now, $\int_{-\infty}^u f_{2|1}(v|u) dv$ is just the probability $P[(X_2|X_1 = u) \leq u]$ which evaluates to

$$\Phi\left(\frac{u - \mu_2 - \frac{\sigma_2}{\sigma_1}\rho(u - \mu_1)}{\sqrt{1 - \rho^2}\sigma_2}\right) \quad (\text{A.5})$$

according to (A.4). We can plug (A.5) into our integral of interest to get:

$$\int_{-\infty}^{\infty} e^u \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{1}{2}\left(\frac{u - \mu_1}{\sigma_1}\right)^2} \Phi\left(\frac{u - \mu_2 - \frac{\sigma_2}{\sigma_1}\rho(u - \mu_1)}{\sqrt{1 - \rho^2}\sigma_2}\right) du$$

We perform the change of variable $y = \frac{u - \mu_1}{\sigma_1}$ to get

$$\begin{aligned}
&\int_{-\infty}^{\infty} e^{y\sigma_1 + \mu_1} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} \Phi\left(\frac{y\sigma_1 + \mu_1 - \mu_2 - \sigma_2\rho y}{\sqrt{1 - \rho^2}\sigma_2}\right) dy \\
&= e^{\mu_1 + \frac{1}{2}\hat{v}_1^2(T)} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y - \sigma_1)^2} \Phi\left(\frac{[\mu_1 - \mu_2]/\sigma_2 + [\sigma_1/\sigma_2 - \rho]y}{\sqrt{1 - \rho^2}}\right) dy \\
&= e^{\mu_1 + \frac{1}{2}\hat{v}_1^2(T)} \int_{-\infty}^{\infty} \phi(y - \sigma_1) \Phi\left(\frac{[\mu_1 - \mu_2]/\sigma_2 + [\sigma_1/\sigma_2 - \rho]y}{\sqrt{1 - \rho^2}}\right) dy
\end{aligned}$$

Now we perform another change of variable with $a = y - \mu_1$ to get

$$e^{\mu_1 + \frac{1}{2}\hat{v}_1^2(T)} \int_{-\infty}^{\infty} \phi(a) \Phi(A_1 + B_1 a) da$$

where

$$A_1 = \frac{[\mu_1 - \mu_2]/\sigma_2 + [\sigma_1/\sigma_2 - \rho]\sigma_1}{\sqrt{1 - \rho^2}}$$

$$B_1 = \frac{[\sigma_1/\sigma_2 - \rho]}{\sqrt{1 - \rho^2}}$$

By Lemma A.2 on page 75, we can simplify our integral to the following

$$e^{\mu_1 + \frac{1}{2}\sigma_1^2} \Phi\left(\frac{A_1}{\sqrt{1 + B_1^2}}\right)$$

We will try to simplify $\frac{A_1}{\sqrt{1 + B_1^2}}$. First, the numerator:

$$A_1 = \frac{[\mu_1 - \mu_2]/\sigma_2 + [\sigma_1/\sigma_2 - \rho]\sigma_1}{\sqrt{1 - \rho^2}} = \frac{\mu_1 - \mu_2 + \sigma_1^2 - \sigma_{12}}{\sigma_2 \sqrt{1 - \rho^2}} \quad (\text{A.6})$$

Now, the denominator:

$$\sqrt{1 + B_1^2} = \sqrt{1 + \frac{[\sigma_1/\sigma_2]^2 - 2\sigma_{12}/\sigma_2^2 + [\sigma_{12}/(\sigma_1\sigma_2)]^2}{1 - \rho^2}}$$

$$= \sqrt{\frac{[\sigma_1/\sigma_2]^2 - 2\sigma_{12}/\sigma_2^2 + 1}{1 - \rho^2}} \quad (\text{A.7})$$

Combining (A.6) and (A.7), we get

$$\frac{A_1}{\sqrt{1 + B_1^2}} = \frac{\mu_1 - \mu_2 + \sigma_1^2 - \hat{v}_{12}(T)}{\sigma_2 \sqrt{[\sigma_1/\sigma_2]^2 - 2\hat{v}_{12}(T)/\hat{v}_2^2(T) + 1}}$$

$$= \frac{\mu_1 - \mu_2 + \sigma_1^2 - \hat{v}_{12}(T)}{\sqrt{\sigma_1^2 - 2\hat{v}_{12}(T) + \hat{v}_2^2(T)}}$$

Finally, we have

$$\int_{-\infty}^{\infty} e^u f_1(u) \int_{-\infty}^u f_{2|1}(v|u) dv du = e^{\mu_1 + \frac{1}{2}\sigma_1^2} \Phi(d_1) \quad (\text{A.8})$$

where

$$d_1 = \frac{\mu_1 - \mu_2 + \sigma_1^2 - \sigma_{12}}{\sqrt{\sigma_1^2 - 2\sigma_{12} + \sigma_2^2}} \quad (\text{A.9})$$

The steps to derive the second integral are identical and are omitted. We eventually get

$$\int_{-\infty}^{\infty} e^u f_1(u) \int_{-\infty}^u f_{2|1}(v|u) dv du = e^{\mu_1 + \frac{1}{2}\sigma_1^2} \Phi(d_2) \quad (\text{A.10})$$

where

$$\begin{aligned} d_2 &= \frac{\mu_1 - \mu_2 - \sigma_2^2 + \sigma_{12}}{\sqrt{\sigma_1^2 - 2\sigma_{12} + \sigma_2^2}} \\ &= d_1 - \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} \end{aligned} \quad (\text{A.11})$$

Substituting (A.8) on the previous page and (A.10) into (A.3) on page 76 (together with (A.9) on the preceding page and (A.11)) completes the proof of the identity. \square

Appendix B

Proofs

This appendix contains the proofs of various theorems and propositions.

B.1 Proofs for Chapter 2

B.1.1 Proof of Proposition 2.3.1 on page 11

Proof. From [34], we have

$$\hat{S}_T = S_0 \exp \left\{ \frac{1}{2} \left(\mu_S - q_S - \frac{\sigma_S^2}{2} \right) T + \frac{\sigma_S}{\sqrt{3}} W_T^S \right\}$$

This gives the following cdf of \hat{S}_T under the physical measure \mathbb{P}

$$P(\hat{S}_T \leq s) = \Phi \left(\frac{\ln \left(\frac{s}{S_0} \right) - \frac{1}{2} \left(\mu_S - q_S - \frac{\sigma_S^2}{2} \right) T}{\sigma_S \sqrt{\frac{T}{3}}} \right)$$

Now, if we consider the cdf of the payoff, F_{G_T} , we get

$$F_{G_T}(x) = P(G_T \leq x) = \begin{cases} 0 & \text{if } x < 0 \\ P(\hat{S}_T \leq K + x) = \Phi \left(\frac{\ln \left(\frac{K+x}{S_0} \right) - \frac{1}{2} \left(\mu_S - q_S - \frac{\sigma_S^2}{2} \right) T}{\sigma_S \sqrt{\frac{T}{3}}} \right) & \text{if } x \geq 0 \end{cases}$$

Let $\nu = \Phi \left(\frac{\ln\left(\frac{K}{S_0}\right) - \frac{1}{2}\left(\mu_S - q_S - \frac{\sigma_S^2}{2}\right)T}{\sigma_S \sqrt{\frac{T}{3}}} \right)$, and consider $y \in (0, 1)$. Then we have

$$F_{G_T}^{-1}(y) = \begin{cases} S_0 e^{\Phi^{-1}(y)\sigma_S \sqrt{\frac{T}{3}} + \frac{1}{2}\left(\mu_S - q_S - \frac{\sigma_S^2}{2}\right)T} - K & \text{if } y \geq \nu \\ 0 & \text{if } y < \nu \end{cases}$$

Let $\xi_T = \exp \left\{ -\frac{1}{2}\theta^2 T - \theta W_T^S - (r - q_S)T \right\}$. Then, $\ln(\xi_T) \sim \mathcal{N}(M, \theta^2 T)$, where $M = -\frac{1}{2}\theta^2 T - (r - q_S)T$ and $\theta = \frac{\mu_S - r}{\sigma_S}$. So,

$$F_\xi(x) = \Phi \left(\frac{\ln(x) - M}{\theta \sqrt{T}} \right) \Rightarrow 1 - F_\xi(\xi_T) = \Phi \left(\frac{M - \ln(\xi_T)}{\theta \sqrt{T}} \right)$$

The cost efficient payoff that gives the same distribution as a continuous geometric Asian option is given by $G_T = F_{G_T}^{-1}(1 - F_\xi(\xi_T))$. After some algebra, this leads to the following:

$$G_T = \left(S_0 \exp \left\{ \frac{1}{2} \left(\mu_S - q_S - \frac{\sigma_S^2}{2} \right) T + \sigma_S \sqrt{\frac{T}{3}} \left(\frac{M - \ln(\xi_T)}{\theta \sqrt{T}} \right) \right\} - K \right)^+$$

Let $b = \frac{\theta}{\sigma}$, $a = \exp \left\{ \frac{\theta}{\sigma} \left(\mu_S - q_S - \frac{\sigma_S^2}{2} \right) T - \left(r - q_S + \frac{\theta^2}{2} \right) T \right\}$. Then, we can write $\xi_T = a \left(\frac{S_T}{S_0} \right)^{-b}$. Using this, we can finally express

$$G_T = d \left(S_T^{1/\sqrt{3}} - \frac{K}{d} \right)^+$$

where

$$d = S_0^{1-1/\sqrt{3}} e^{\left(\frac{1}{2} - \frac{1}{\sqrt{3}}\right)\left(\mu_S - q_S - \frac{\sigma_S^2}{2}\right)T}$$

We can see that G_T is a power call option, and we can calculate its price using risk neutral valuation. The price of G_T at time 0, E_0 is given by

$$E_0 = d e^{-rT} E \left[\left(S_T^{1/\sqrt{3}} - \frac{K}{d} \right)^+ \right]$$

It is easy to see that $S_T^{1/\sqrt{3}} = S_0^{1/\sqrt{3}} e^{Y_T}$ where $Y_T \sim \mathcal{N} \left(\frac{1}{\sqrt{3}} \left(r - q_S - \frac{\sigma_S^2}{2} \right) T, \frac{\sigma_S^2}{3} T \right)$. Applying Lemma A.1 on page 74 gives us the desired result. \square

B.1.2 Proof of Proposition 2.4.2 on page 16

Proof. The program of this proof is as follows:

1. Construct the CEC of \hat{S}_T, S_T^* , in isolation of I_T
2. Construct the CEC of \hat{H}_T, H_T^* , in isolation of S_T
3. Demonstrate that $\hat{A}_T \sim A_T^*$

Step 1

First, from [34], we have

$$\hat{S}_T = S_0 \exp \left\{ \frac{1}{2} \left(\mu_S - q_S - \frac{\sigma_S^2}{2} \right) T + \frac{\sigma_S}{\sqrt{3}} W_T^S \right\}$$

Now, if we consider the CDF of the payoff $\hat{S}_T, F_{\hat{S}_T}$, we get

$$F_{\hat{S}_T}(s) = \begin{cases} 0 & \text{if } s < 0 \\ P(\hat{S}_T \leq s) = \Phi \left(\frac{\ln\left(\frac{s}{S_0}\right) - \frac{1}{2} \left(\mu_S - q_S - \frac{\sigma_S^2}{2} \right) T}{\sigma_S \sqrt{\frac{T}{3}}} \right) & \text{if } s \geq 0 \end{cases}$$

We can invert this CDF to get

$$F_{\hat{S}_T}^{-1}(y) = \begin{cases} S_0 e^{\Phi^{-1}(y) \sigma_S \sqrt{\frac{T}{3}} + \frac{1}{2} \left(\mu_S - q_S - \frac{\sigma_S^2}{2} \right) T} & \text{if } 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Now let $\xi_T^S = \exp \left\{ -\frac{1}{2} \theta_S^2 T - \theta_S W_T - (r - q_S) T \right\}$. Then $\ln(\xi_T^S) \sim N(M_S, \theta_S^2 T)$, where $M_S = -\frac{1}{2} \theta_S^2 T - (r - q_S) T$ and $\theta_S = \frac{\mu_S - r}{\sigma_S}$. Then we have

$$F_{\xi^S}(x) = \Phi \left(\frac{\ln(x) - M_S}{\theta_S \sqrt{T}} \right) \Rightarrow 1 - F_{\xi^S}(\xi_T^S) = \Phi \left(\frac{M_S - \ln(\xi_T^S)}{\theta_S \sqrt{T}} \right)$$

This gives us the following cost efficient payoff:

$$S_T^* = F_{\hat{S}_T}^{-1}(1 - F_{\xi^S}(\xi_T^S)) = S_0 \exp \left\{ \frac{1}{2} \left(\mu_S - q_S - \frac{\sigma_S^2}{2} \right) T + \sigma_S \sqrt{\frac{T}{3}} \left(\frac{M_S - \ln(\xi_T^S)}{\theta_S \sqrt{T}} \right) \right\}$$

Let $b_S = \frac{\theta_S}{\sigma_S}$, $a_S = \exp \left\{ \frac{\theta_S}{\sigma_S} \left(\mu_S - q_S - \frac{\sigma_S^2}{2} \right) T - \left(r - q_S + \frac{\theta_S^2}{2} \right) T \right\}$. Then, we can write $\xi_T^S = a_S \left(\frac{S_T}{S_0} \right)^{-b_S}$. Using this, we have $S_T^* = d_S S_T^{1/\sqrt{3}}$ where $d_S = S_0^{1-1/\sqrt{3}} e^{\left(\frac{1}{2} - \frac{1}{\sqrt{3}}\right) \left(\mu_S - q_S - \frac{\sigma_S^2}{2} \right) T}$

Step 2

Firstly, let

$$\hat{\eta}' = (\mu_S - \hat{q}_S) - \hat{\beta}(\mu_I - \hat{q}_I) + \frac{1}{2}\hat{\sigma}_I^2\hat{\beta}(1 - \hat{\beta})$$

and it is easy (but tedious) to check that this is equivalent to $\hat{\eta}$ as defined in (2.2.3) on page 9. Now we can rewrite \hat{H}_T as

$$\hat{H}_T = K(\hat{I}_T/\hat{I}_0)^{\hat{\beta}} \exp(\hat{\eta}'T)$$

From [34], we have

$$\hat{I}_T = I_0 \exp \left\{ \left(\mu_I - \hat{q}_I - \frac{\hat{\sigma}_I^2}{2} \right) T + \hat{\sigma}_I W_T^I \right\}$$

which leads us to

$$\left(\frac{\hat{I}_T}{I_0} \right)^{\hat{\beta}} = \exp \left\{ \hat{\beta}\mu_I T - \hat{\beta}\hat{q}_I T - \frac{\hat{\beta}\hat{\sigma}_I^2}{2} T + \rho\hat{\sigma}_S W_T^I \right\}$$

We also have

$$\exp(\hat{\eta}'T) = \exp \left\{ \mu_S T - \hat{q}_S T - \hat{\beta}\mu_I T + \hat{\beta}\hat{q}_I T + \frac{\hat{\beta}\hat{\sigma}_I^2}{2} - \frac{1}{2}\rho^2\hat{\sigma}_S^2 T \right\}$$

This gives us

$$\hat{H}_T = K \exp \left\{ \frac{1}{2} \left[\mu_S - q_S - \frac{\sigma_S^2}{3} \left(\frac{1}{2} + \rho^2 \right) \right] T + \frac{\rho\sigma_S}{\sqrt{3}} W_T^I \right\}$$

Similarly, let

$$\eta' = (\mu_S - q_S) - \beta(\mu_I - q_I) + \frac{1}{2}\sigma_I^2\beta(1 - \beta)$$

and we can check that is equivalent to η as defined in (2.2.3) on page 9. Now rewrite H_T as

$$H_T = K(I_T/I_0)^\beta \exp(\eta T)$$

Check that

$$\left(\frac{I_T}{I_0} \right)^\beta = \exp \left\{ \beta\mu_I T - \beta q_I T - \frac{\beta\sigma_I^2}{2} T + \rho\sigma_S W_T^I \right\}$$

and

$$\exp(\eta'T) = \exp \left\{ \mu_S T - q_S T - \beta \mu_I T + \beta q_I T + \frac{\beta \sigma_I^2}{2} - \frac{1}{2} \rho^2 \sigma_S^2 T \right\}$$

This gives us

$$H_T = K(I_T/I_0)^\beta \exp(\eta'T) = K \exp \left\{ \left(\mu_S - q_S - \frac{1}{2} \rho^2 \sigma_S^2 \right) T + \rho \sigma_S W_T^I \right\}$$

Now, if we consider the CDF of the payoff \hat{H}_T , $F_{\hat{H}_T}$, we get

$$F_{\hat{H}_T}(h) = \begin{cases} 0 & \text{if } h < 0 \\ P(\hat{H}_T \leq h) = \Phi \left(\frac{\ln(\frac{h}{K}) - \left(\mu_S - \hat{q}_S - \frac{\rho^2 \hat{\sigma}_S^2}{2} \right) T}{\rho \hat{\sigma}_S \sqrt{T}} \right) & \text{if } h \geq 0 \end{cases}$$

We can invert this CDF to get

$$F_{\hat{H}_T}^{-1}(y) = \begin{cases} K e^{\Phi^{-1}(y) \rho \hat{\sigma}_S \sqrt{T} + \left(\mu_S - \hat{q}_S - \frac{\rho^2 \hat{\sigma}_S^2}{2} \right) T} & \text{if } 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Now, we have $\xi_T^I = \exp \left\{ -\frac{1}{2} \theta_I^2 T - \theta_I W_T^I - (r - q_I) T \right\}$ where $\theta_I = \frac{\mu_I - r}{\sigma_I}$. Therefore, $\ln(\xi_T^I) \sim N(M_I, \theta_I^2 T)$, where $M_I = -\frac{1}{2} \theta_I^2 T - (r - q_I) T$. This leads us to

$$F_{\xi_T^I}(x) = \Phi \left(\frac{\ln(x) - M_I}{\theta_I \sqrt{T}} \right) \Rightarrow 1 - F_{\xi_T^I}(\xi_T^I) = \Phi \left(\frac{M_I - \ln(\xi_T^I)}{\theta_I \sqrt{T}} \right)$$

This gives us the following cost efficient payoff:

$$H_T^* = K \exp \left\{ \left(\mu_S - \hat{q}_S - \frac{\rho^2 \hat{\sigma}_S^2}{2} \right) T + \rho \hat{\sigma}_S \left(\frac{M_I - \ln(\xi_T^I)}{\theta_I} \right) \right\}$$

Let $b_I = \frac{\theta_I}{\sigma_I}$, $a_I = \exp \left\{ \frac{\theta_I}{\sigma_I} \left(\mu_I - q_I - \frac{\sigma_I^2}{2} \right) T - \left(r - q_I + \frac{\theta_I^2}{2} \right) T \right\}$. Then, we can write $\xi_T^I = a_I \left(\frac{I_T}{I_0} \right)^{-b_I}$. Define d_H as follows:

$$d_H = K^{1-1/\sqrt{3}} \exp \left\{ \left(\frac{1}{2} - \frac{1}{\sqrt{3}} \right) (\mu_S - q_S) T + \frac{\sigma_S^2 T}{2} \left[\rho^2 \left(\frac{1}{\sqrt{3}} - \frac{1}{3} \right) - \frac{1}{6} \right] \right\}$$

so that we can express the efficient payoff, H_T^* as $H_T^* = d_H H_T^{1/\sqrt{3}}$

Step 3

Note that \hat{A}_T and A_T^* have the same bivariate lognormal distribution since \hat{S}_T and \hat{H}_T have the same distributions as S_T^* and H_T^* respectively. If we can show that they have the same covariances, then we are done. We can check the equality of $Cov[\ln(\hat{S}_T), \ln(\hat{H}_T)]$ and $Cov[\ln(S_T^*), \ln(H_T^*)]$ as these two quantities are more tractable. In fact, we have

$$Cov[\ln(\hat{S}_T), \ln(\hat{H}_T)] = Cov[\ln(S_T^*), \ln(H_T^*)] = \frac{\sigma_S^2 \rho^2}{3} T$$

□

B.1.3 Proof of Corollary 2.4.4 on page 16

Proof. A_T^* is a power exchange option and its price is given by Blenman and Clark [10] as follows:

$$V_0^* = \Upsilon\left(d_S, \frac{1}{\sqrt{3}}, q_S, \sigma_S, S_0, T\right) \Phi(d_1^*) - \Upsilon\left(d_H, \frac{1}{\sqrt{3}}, q_S, \rho\sigma_S, K, T\right) \Phi(d_2^*)$$

where

$$d_1^* = \frac{\ln\left(\frac{d_S S_0^{1/\sqrt{3}}}{d_H K^{1/\sqrt{3}}}\right) + \left[\frac{\sigma_S^2}{2\sqrt{3}}\left(1 - \frac{1}{\sqrt{3}}\right)(\rho^2 - 1) + \frac{1}{2}\nu^2\right]T}{\nu\sqrt{T}}$$

$$d_2^* = \frac{\ln\left(\frac{d_S S_0^{1/\sqrt{3}}}{d_H K^{1/\sqrt{3}}}\right) + \left[\frac{\sigma_S^2}{2\sqrt{3}}\left(1 - \frac{1}{\sqrt{3}}\right)(\rho^2 - 1) - \frac{1}{2}\nu^2\right]T}{\nu\sqrt{T}} = d_1^* - \nu\sqrt{T}$$

and

$$\nu^2 = \frac{1}{3}\sigma_S^2(1 - \rho^2)$$

$$\Upsilon\left(d_S, \frac{1}{\sqrt{3}}, q_S, \sigma_S, S_0, T\right)$$

$$= d_S S_0^{1/\sqrt{3}} \exp\left\{\left[\left(\frac{1}{\sqrt{3}} - 1\right)r - \frac{1}{\sqrt{3}}q_S - \frac{1}{\sqrt{3}}\left(1 - \frac{1}{\sqrt{3}}\right)\frac{\sigma_S^2}{2}\right]T\right\}$$

$$\Upsilon\left(d_H, \frac{1}{\sqrt{3}}, q_S, \rho\sigma_S, K, T\right)$$

$$\begin{aligned}
&= d_H K^{1/\sqrt{3}} \exp \left\{ \left[\left(\frac{1}{\sqrt{3}} - 1 \right) r - \frac{1}{\sqrt{3}} q_S - \frac{1}{\sqrt{3}} \left(1 - \frac{1}{\sqrt{3}} \right) \frac{\rho^2 \sigma_S^2}{2} \right] T \right\} \\
d_S &= S_0^{1-1/\sqrt{3}} \exp \left\{ \left(\frac{1}{2} - \frac{1}{\sqrt{3}} \right) \left(\mu_S - q_S - \frac{\sigma_S^2}{2} \right) T \right\} \\
d_H &= K^{1-1/\sqrt{3}} \exp \left\{ \left(\frac{1}{2} - \frac{1}{\sqrt{3}} \right) (\mu_S - q_S) T + \frac{\sigma_S^2 T}{2} \left[\rho^2 \left(\frac{1}{\sqrt{3}} - \frac{1}{3} \right) - \frac{1}{6} \right] \right\}
\end{aligned}$$

This eventually simplifies to the desired result pricing formula. When $\mu_S = r$ it is easy to see that \hat{V}_0 and V_0^* are equivalent. In fact, the price of A_T^* is a decreasing function of μ_S , and is only cheaper than \hat{A}_T when $\mu_S > r$. We can see this by considering the first derivative of the power exchange option with respect to the stock yield i.e. $\frac{\partial V_0^*}{\partial \mu_S} < 0$. \square

B.2 Proofs for Chapter 3

B.2.1 Proof of Lemma 3.1.1 on page 33

Proof. Consider a different bond price dynamics with $S < T$ under \mathbb{Q}^T

$$\frac{dP(t, S)}{P(t, S)} = [r(t) + \sigma_P(t, S)\sigma_P(t, T)] dt - \sigma_P(t, S)dZ_0^T(t) \quad (\text{B.2.1.1})$$

Applying Ito's Lemma to $g(P(t, T), P(t, S)) = \ln \frac{P(t, T)}{P(t, S)}$, and then replacing S with t , gives us the following expression for the bond price $P(t, T)$ that does not involve $r(t)$:

$$\begin{aligned}
P(t, T) &= \frac{P(0, T)}{P(0, t)} \exp \left\{ \frac{1}{2} \int_0^t [\sigma_P(u, T) - \sigma_P(u, t)]^2 du \right. \\
&\quad \left. - \int_0^t [\sigma_P(u, T) - \sigma_P(u, t)] dZ_0^T(u) \right\} \quad (\text{B.2.1.2})
\end{aligned}$$

Apply Ito's Lemma again to $g(S_1(t), P(t, T))$ and $g(S_2(t), P(t, T))$ to get:

$$\begin{aligned}
\frac{S_1(t)}{P(t, T)} &= \frac{S_1(0)}{P(0, T)} \exp \left\{ -\frac{1}{2} \int_0^t [\sigma_P^2(u, T) + 2\sigma_1\sigma_P(u, T)C_{21} + \sigma_1^2] du \right. \\
&\quad + \int_0^t [\sigma_1 C_{21} + \sigma_P(u, T)] dZ_0^T(u) \\
&\quad \left. + \int_0^t \sigma_1 C_{22} dZ_1^T(u) \right\} \quad (\text{B.2.1.3})
\end{aligned}$$

$$\begin{aligned}
\frac{S_2(t)}{P(t, T)} &= \frac{S_2(0)}{P(0, T)} \exp \left\{ -\frac{1}{2} \int_0^t [\sigma_P^2(u, T) + 2\sigma_2\sigma_P(u, T)C_{31} + \sigma_2^2] du \right. \\
&\quad + \int_0^t [\sigma_2C_{31} + \sigma_P(u, T)] dZ_0^T(u) \\
&\quad + \int_0^t \sigma_2C_{32}dZ_1^T(u) \\
&\quad \left. + \int_0^t \sigma_2C_{33}dZ_2^T(u) \right\} \tag{B.2.1.4}
\end{aligned}$$

Finally, substitute (B.2.1.2) on the previous page into (B.2.1.3)–(B.2.1.4) on pages 85–86. \square

B.2.2 Proof of Lemma 3.2.1 on page 34

Proof. In view of the definition of $S_i(T)$ in (3.2.7) on page 34, we only need to compute the various quantities for $X_i(T)$ - the rest follows immediately. The mean and variance of $S_i(T)$ follow immediately from (3.2.4)–(3.2.5) on page 34 and the properties of the integral of a Brownian motion. By definition:

$$Cov[X_1(T), X_2(T)] = E[X_1(T)X_2(T)] - E[X_1(T)]E[X_2(T)] \tag{B.2.2.1}$$

Now, we are left with evaluating $E[X_1(T)X_2(T)]$. For simplicity let us define the following terms:

$$\begin{aligned}
Z_{10}(a, b, c) &= \int_a^b \sigma_{10}(s, c) dZ_0^T(s) & Z_{11}(a, b) &= \int_a^b \sigma_{11} dZ_1^T(s) \\
Z_{20}(a, b, c) &= \int_a^b \sigma_{20}(s, c) dZ_0^T(s) & Z_{21}(a, b) &= \int_a^b \sigma_{21} dZ_1^T(s) \\
Z_{22}(a, b) &= \int_a^b \sigma_{22} dZ_1^T(s)
\end{aligned} \tag{B.2.2.2}$$

We can rewrite (3.2.4)–(3.2.5) on page 34 as:

$$X_1(T) = M_1(0, T, T) + Z_{10}(0, T, T) + Z_{11}(0, T) \tag{B.2.2.3}$$

$$X_2(T) = M_2(0, T, T) + Z_{20}(0, T, T) + Z_{21}(0, T) + Z_{22}(0, T) \tag{B.2.2.4}$$

Then, we have:

$$\begin{aligned}
E[X_1(T)X_2(T)] &= E[M_1(0, T, T)X_2(T)] + E[Z_{10}(0, T, T)X_2(T)] \\
&\quad + E[Z_{11}(0, T)X_2(T)] \tag{B.2.2.5}
\end{aligned}$$

Now we consider each term in (B.2.2.5) on the preceding page:

$$\begin{aligned}
E[M_1(0, T, T)X_2(T)] &= E[M_1(0, T, T)M_2(0, T, T)] + \underbrace{E[M_1(0, T, T)Z_{20}(0, T, T)]}_{\S} \\
&\quad + \underbrace{E[M_1(0, T, T)Z_{21}(0, T, T)]}_{\S} + \underbrace{E[M_1(0, T, T)Z_{22}(0, T, T)]}_{\S} \\
E[Z_{10}(0, T, T)X_2(T)] &= \underbrace{E[Z_{10}(0, T, T)M_2(0, T, T)]}_{\S} + E[Z_{10}(0, T, T)Z_{20}(0, T, T)] \\
&\quad + \underbrace{E[Z_{10}(0, T, T)Z_{21}(0, T, T)]}_{\P} + \underbrace{E[Z_{10}(0, T, T)Z_{22}(0, T, T)]}_{\P} \\
E[Z_{11}(0, T)X_2(T)] &= \underbrace{E[Z_{11}(0, T)M_2(0, T, T)]}_{\S} + \underbrace{E[Z_{11}(0, T)Z_{20}(0, T, T)]}_{\P} \\
&\quad + E[Z_{11}(0, T)Z_{21}(0, T)] + \underbrace{E[Z_{11}(0, T)Z_{22}(0, T)]}_{\P}
\end{aligned}$$

$\S : E[\text{const} \cdot \int \dots dZ^T] = 0$

$\P : \int \dots dZ_1^T$ is independent of $\int \dots dZ_2^T$

By the properties of the Ito Integral, we have

$$\begin{aligned}
E[Z_{10}(0, T, T)Z_{20}(0, T, T)] &= E \left[\int_0^T \sigma_{10}(s, T) dZ_0^T(s) \cdot \int_0^T \sigma_{20}(s, T) dZ_0^T(s) \right] \\
&= \int_0^T \sigma_{10}(s, T) \sigma_{20}(s, T) ds \\
E[Z_{11}(0, T)Z_{21}(0, T)] &= E \left[\int_0^T \sigma_{11} dZ_1^T(s) \cdot \int_0^T \sigma_{21} dZ_1^T(s) \right] \\
&= \int_0^T \sigma_{11} \sigma_{21} ds \\
&= \sigma_{11} \sigma_{21} T
\end{aligned}$$

Combining all of the above, we finally get

$$E[X_1(T)X_2(T)] = M_1(0, T, T)M_2(0, T, T) + \sigma_{11}\sigma_{21}T + \int_0^T \sigma_{10}(s, T)\sigma_{20}(s, T)ds \quad (\text{B.2.2.6})$$

Finally, we can plug (B.2.2.6) into (B.2.2.1) on page 86 to get the covariance. At this point, a computer algebra system is used to derive the explicit expressions that are given in Section 3.4.1 on page 41. \square

B.2.3 Proof of Lemma 3.2.2 on page 35

Proof. In view of the definition of $\hat{S}_i(T)$ in (3.2.8) on page 34 we only need to compute the various quantities for $\hat{X}_i(T)$ - the rest follows immediately. The program for this proof is to calculate the following terms sequentially:

1. $E[\hat{X}_i(t)]$
2. $E[\hat{X}_i^2(T)]$
3. $E[X_i(t)X_i(u)]$
4. $\int_0^T \int_0^T E[X_i(t)X_i(u)]dudt$
5. $Var[\hat{X}_i(T)]$
6. $E[\hat{X}_1(T)\hat{X}_2(T)]$
7. $E[X_1(t)X_2(u)]$
8. $\int_0^T \int_0^T E[X_1(t)X_2(u)]dudt$
9. $Cov[\hat{X}_1(T), \hat{X}_2(T)]$

After completing the above steps, we use a computer algebra system to arrive at the explicit expressions given in Section 3.4.2 on page 41.

Step 1: $E[\hat{X}_i(t)]$

We begin with considering $E[\hat{X}_i(T)]$

$$\begin{aligned} E[\hat{X}_i(T)] &= E \left[\ln S_i(0) - \frac{1}{T} \int_0^T \ln P(0, t) dt + \frac{1}{T} \int_0^T X_i(t) dt \right] \\ &= \mathcal{S}_i(T) + \mathcal{M}_i(T) := \hat{m}_i(T) \end{aligned} \quad (\text{B.2.3.1})$$

where

$$l(u) = \ln A(0, u) - B(0, u)r(0) \quad (\text{B.2.3.2})$$

$$M_i(a, b, c) = \int_a^b m_i(u, c) du \quad (\text{B.2.3.3})$$

$$\mathcal{S}_i(t) = \ln S_i(0) - \frac{1}{t} \int_0^t l(u) du \quad (\text{B.2.3.4})$$

$$\mathcal{M}_i(t) = \frac{1}{t} \int_0^t M_i(0, u, u) du \quad (\text{B.2.3.5})$$

Step 2: $E[\hat{X}_i^2(T)]$

Note that the derivation of the first moments is slightly easier because we can interchange the expectation and Riemann-integration operators. However, this is not true for the variance operator. Therefore we need to start with finding the second moments and then retrieve the variance.

$$\begin{aligned} E[\hat{X}_i^2(T)] &= E \left[\left\{ \frac{1}{T} \int_0^T \ln \left(\frac{S_i(0)}{P(0, t)} e^{X_i(t)} \right) dt \right\} \left\{ \frac{1}{T} \int_0^T \ln \left(\frac{S_i(0)}{P(0, u)} e^{X_i(u)} \right) du \right\} \right] \\ &= \mathcal{S}_i^2(T) + 2\mathcal{S}_i(T)\mathcal{M}_i(T) + \frac{1}{T^2} \int_0^T \int_0^T E[X_i(t)X_i(u)] du dt \end{aligned} \quad (\text{B.2.3.6})$$

The last equalities come from collecting like terms and Fubini's theorem. Now, we need to compute $E[X_i(t)X_i(u)]$.

Step 3: $E[X_i(t)X_i(u)]$

To compute these expectations, we need to split up the region of integration into 2 cases i.e. Case 1: $t < u$ and Case 2: $u < t$. WLOG we can assume the first case $t < u$ and

retrieve the second case by symmetry. We have (recall (B.2.2.3)–(B.2.2.4) on page 86):

$$\begin{aligned} E[X_1(t)X_1(u)] &= E[M_1(0, t, t)X_1(u)] + E[Z_{10}(0, t, t)X_1(u)] \\ &\quad + E[Z_{11}(0, t)X_1(u)] \end{aligned} \tag{B.2.3.7}$$

$$\begin{aligned} E[X_2(t)X_2(u)] &= E[M_2(0, t, t)X_2(u)] + E[Z_{20}(0, t, t)X_2(u)] \\ &\quad + E[Z_{21}(0, t)X_2(u)] + E[Z_{22}(0, t)X_2(u)] \end{aligned} \tag{B.2.3.8}$$

We will start with considering each term in (B.2.3.7) separately:

$$\begin{aligned} E[M_1(0, t, t)X_1(u)] &= E[M_1(0, t, t)M_1(0, u, u)] + \underbrace{E[M_1(0, t, t)Z_{10}(0, u, u)]}_{\square} \\ &\quad + \underbrace{E[M_1(0, t, t)Z_{11}(0, u)]}_{\square} \\ E[Z_{10}(0, t, t)X_1(u)] &= \underbrace{E[Z_{10}(0, t, t)M_1(0, u, u)]}_{\square} + E[Z_{10}(0, t, t)Z_{10}(0, t, u)] \\ &\quad + \underbrace{E[Z_{10}(0, t, t)Z_{10}(t, u, u)]}_{\diamond} + \underbrace{E[Z_{10}(0, t, t)Z_{11}(0, u)]}_{\triangle} \\ E[Z_{11}(0, t)X_1(u)] &= \underbrace{E[Z_{11}(0, t)M_1(0, u, u)]}_{\square} + \underbrace{E[Z_{11}(0, t)Z_{10}(0, u, u)]}_{\triangle} \\ &\quad + E[Z_{11}(0, t)Z_{11}(0, t)] + \underbrace{E[Z_{11}(0, t)Z_{11}(t, u)]}_{\diamond} \end{aligned}$$

$$\square : E[\text{const} \cdot \int \cdots dZ^T] = 0$$

$$\diamond : \int_0^t \cdots dZ^T \text{ is independent of } \int_t^u \cdots dZ^T$$

$$\triangle : \int \cdots dZ_1^T \text{ is independent of } \int \cdots dZ_2^T$$

By the properties of the Ito Integral, we have

$$E[Z_{10}(0, t, t)Z_{10}(0, t, u)] = E \left[\int_0^t \sigma_{10}(s, t) dZ_0^T(s) \cdot \int_0^t \sigma_{10}(s, u) dZ_0^T(s) \right]$$

$$\begin{aligned}
&= \int_0^t \sigma_{10}(s, t) \sigma_{10}(s, u) ds \\
&:= x_1(t, u) \\
E[Z_{11}(0, t) Z_{11}(0, t)] &= E \left[\int_0^t \sigma_{11} dZ_1^T(s) \cdot \int_0^t \sigma_{11} dZ_1^T(s) \right] \\
&= \int_0^t \sigma_{11}^2 ds \\
&= \sigma_{11}^2 t
\end{aligned}$$

Combining all of the above, we finally get

$$E[X_1(t) X_1(u)] = M_1(0, t, t) M_1(0, u, u) + \sigma_{11}^2 t + x_1(t, u) := E_1(t, u) \quad (\text{B.2.3.9})$$

By symmetry, in Case 2: $u < t$, we get

$$E[X_1(t) X_1(u)] = M_1(0, u, u) M_1(0, t, t) + \sigma_{11}^2 u + x_1(u, t) := E_1(u, t) \quad (\text{B.2.3.10})$$

Now we move on to each term in (B.2.3.8) on the previous page:

$$\begin{aligned}
E[M_2(0, t, t) X_2(u)] &= E[M_2(0, t, t) M_2(0, u, u)] + \underbrace{E[M_2(0, t, t) Z_{20}(0, u, u)]}_{\blacksquare} \\
&\quad + \underbrace{E[M_2(0, t, t) Z_{21}(0, u)]}_{\blacksquare} + \underbrace{E[M_2(0, t, t) Z_{22}(0, u)]}_{\blacksquare} \\
E[Z_{20}(0, t, t) X_2(u)] &= \underbrace{E[Z_{20}(0, t, t) M_2(0, u, u)]}_{\blacksquare} + E[Z_{20}(0, t, t) Z_{20}(0, t, u)] \\
&\quad + \underbrace{E[Z_{20}(0, t, t) Z_{20}(t, u, u)]}_{\clubsuit} + \underbrace{E[Z_{20}(0, t, t) Z_{21}(0, u)]}_{\spadesuit} \\
&\quad + \underbrace{E[Z_{20}(0, t, t) Z_{22}(0, u)]}_{\spadesuit} \\
E[Z_{21}(0, t) X_2(u)] &= \underbrace{E[Z_{21}(0, t) M_2(0, u, u)]}_{\blacksquare} + \underbrace{E[Z_{21}(0, t) Z_{20}(0, u, u)]}_{\spadesuit}
\end{aligned}$$

$$\begin{aligned}
& + E[Z_{21}(0, t)Z_{21}(0, t)] + \underbrace{E[Z_{21}(0, t)Z_{21}(t, u)]}_{\clubsuit} \xrightarrow{0} \\
& + \underbrace{E[Z_{21}(0, t)Z_{22}(0, u)]}_{\spadesuit} \xrightarrow{0} \\
E[Z_{22}(0, t)X_2(u)] & = \underbrace{E[Z_{22}(0, t)M_2(0, u, u)]}_{\blacksquare} \xrightarrow{0} + \underbrace{E[Z_{22}(0, t)Z_{20}(0, u, u)]}_{\spadesuit} \xrightarrow{0} \\
& + \underbrace{E[Z_{22}(0, t)Z_{21}(0, u)]}_{\spadesuit} \xrightarrow{0} + E[Z_{22}(0, t)Z_{22}(0, t)] \\
& + \underbrace{E[Z_{22}(0, t)Z_{22}(t, u)]}_{\clubsuit} \xrightarrow{0}
\end{aligned}$$

■ : $E[\text{const} \cdot \int \dots dZ^T] = 0$

♣ : $\int_0^t \dots dZ^T$ is independent of $\int_t^u \dots dZ^T$

♠ : $\int \dots dZ_1^T$, $\int \dots dZ_2^T$ and $\int \dots dZ_3^T$ are independent

By the properties of the Ito Integral, we have

$$\begin{aligned}
E[Z_{20}(0, t)Z_{20}(0, t, u)] & = E \left[\int_0^t \sigma_{20}(s, t) dZ_0^T(s) \cdot \int_0^t \sigma_{20}(s, u) dZ_0^T(s) \right] \\
& = \int_0^t \sigma_{20}(s, t) \sigma_{20}(s, u) ds \\
& := x_2(t, u)
\end{aligned}$$

$$\begin{aligned}
E[Z_{21}(0, t)Z_{21}(0, t)] & = E \left[\int_0^t \sigma_{21} dZ_1^T(s) \cdot \int_0^t \sigma_{21} dZ_1^T(s) \right] \\
& = \int_0^t \sigma_{21}^2 ds \\
& = \sigma_{11}^2 t
\end{aligned}$$

$$\begin{aligned}
E[Z_{22}(0, t)Z_{22}(0, t)] & = E \left[\int_0^t \sigma_{22} dZ_2^T(s) \cdot \int_0^t \sigma_{22} dZ_2^T(s) \right] \\
& = \int_0^t \sigma_{22}^2 ds
\end{aligned}$$

$$= \sigma_{22}^2 t$$

Combining all of the above, we finally get

$$E[X_2(t)X_2(u)] = M_2(0, t, t)M_2(0, u, u) + (\sigma_{21}^2 + \sigma_{22}^2)t + x_2(t, u) := E_2(t, u) \quad (\text{B.2.3.11})$$

By symmetry, in Case 2: $u < t$, we get

$$E[X_2(t)X_2(u)] = M_2(0, u, u)M_2(0, t, t) + (\sigma_{21}^2 + \sigma_{22}^2)u + x_2(u, t) := E_2(u, t) \quad (\text{B.2.3.12})$$

Step 4: $\int_0^T \int_0^T E[X_i(t)X_i(u)]dudt$

Now we are ready to evaluate these integrals. We split the region of integration into Case 1: $t < u$ and Case 2: $u < t$ to get the following:

$$\begin{aligned} \int_0^T \int_0^T E[X_i(t)X_i(u)]dudt &= \underbrace{\int_0^T \int_0^u E_i(t, u)dtdu}_{\text{Case 1}} + \underbrace{\int_0^T \int_0^t E_i(u, t)dudt}_{\text{Case 2}} \\ &:= \mathcal{E}_i(T) \end{aligned} \quad (\text{B.2.3.13})$$

Step 5: $Var[\hat{X}_i(T)]$

We can plug (B.2.3.13) into (B.2.3.6) on page 89 to get:

$$\begin{aligned} E[\hat{X}_i^2(T)] &= \mathcal{S}_i^2(T) + 2\mathcal{S}_i(T)\mathcal{M}_i(T) + \frac{1}{T^2}\mathcal{E}_i(T) \\ &:= \hat{M}_i(T) \end{aligned} \quad (\text{B.2.3.14})$$

Finally, this gives us the variance:

$$Var[\hat{X}_i(T)] = \hat{M}_i(T) - \hat{m}_i^2(T) := \hat{v}_i^2(T) \quad (\text{B.2.3.15})$$

Step 6: $E[\hat{X}_1(T)\hat{X}_2(T)]$

Note that (similar to the variance operator) we cannot interchange the covariance and Riemann-integration operators. So, we need to resort to the expectation operator that appears in the definition of the covariance term. By definition we have:

$$Cov[\hat{X}_1(T), \hat{X}_2(T)] = E[\hat{X}_1(T)\hat{X}_2(T)] - \hat{m}_1(T)\hat{m}_2(T) \quad (\text{B.2.3.16})$$

Now we are left with $E[\hat{X}_1(T)\hat{X}_2(T)]$.

$$\begin{aligned}
E[\hat{X}_1(T)\hat{X}_2(T)] &= E \left[\left\{ \frac{1}{T} \int_0^T \ln \left(\frac{S_1(0)}{P(0,t)} e^{X_1(t)} \right) dt \right\} \left\{ \frac{1}{T} \int_0^T \ln \left(\frac{S_2(0)}{P(0,u)} e^{X_2(u)} \right) du \right\} \right] \\
&= \mathcal{S}_1(T)\mathcal{S}_2(T) + \mathcal{S}_1(T)\mathcal{M}_2(T) + \mathcal{S}_2(T)\mathcal{M}_1(T) \\
&\quad + \frac{1}{T^2} \int_0^T \int_0^T E[X_1(t)X_2(u)] dudt
\end{aligned} \tag{B.2.3.17}$$

The last equality comes from collecting like terms and Fubini's theorem. We are left with evaluating $E[X_1(t)X_2(u)]$.

Step 7: $E[X_1(t)X_2(u)]$

To compute these expectations, we use the same method presented in Step 3 on page 89. We will split the region of integration into 2 cases i.e. Case 1: $t < u$ and Case 2: $u < t$. Note that the nice symmetry that held for in Step 3 on page 89 does not hold completely in this situation. We cannot blindly swap the orders of t and u in all terms - there are some terms that need to remain unchanged. We start with Case 1: $t < u$

$$E[X_1(t)X_2(u)] = E[M_1(0, t, t)X_2(u)] + E[Z_{10}(0, t, t)X_2(u)] + E[Z_{11}(0, t)X_2(u)] \tag{B.2.3.18}$$

Now we consider each term in (B.2.3.18):

$$\begin{aligned}
E[M_1(0, t, t)X_2(u)] &= E[M_1(0, t, t)M_2(0, u, u)] + \underbrace{E[M_1(0, t, t)Z_{20}(0, u, u)]}_{\dagger} \\
&\quad + \underbrace{E[M_1(0, t, t)Z_{21}(0, u)]}_{\dagger} + \underbrace{E[M_1(0, t, t)Z_{22}(0, u)]}_{\dagger} \\
E[Z_{10}(0, t, t)X_2(u)] &= \underbrace{E[Z_{10}(0, t, t)M_2(0, u, u)]}_{\dagger} + E[Z_{10}(0, t, t)Z_{20}(0, t, u)] \\
&\quad + \underbrace{E[Z_{10}(0, t, t)Z_{20}(t, u, u)]}_{\star} + \underbrace{E[Z_{10}(0, t, t)Z_{21}(0, u)]}_{\bowtie} \\
&\quad + \underbrace{E[Z_{10}(0, t, t)Z_{22}(0, u)]}_{\bowtie}
\end{aligned}$$

$$\begin{aligned}
E[Z_{11}(0, t)X_2(u)] &= \underbrace{E[Z_{11}(0, t)M_2(0, u, u)]}_{\dagger} + \underbrace{E[Z_{11}(0, t)Z_{20}(0, u, u)]}_{\boxtimes} \\
&\quad + E[Z_{11}(0, t)Z_{21}(0, t)] + \underbrace{E[Z_{11}(0, t)Z_{21}(t, u)]}_{\star} \\
&\quad + \underbrace{E[Z_{11}(0, t)Z_{22}(0, u)]}_{\boxtimes}
\end{aligned}$$

$\dagger : E[\text{const} \cdot \int \dots dZ^T] = 0$

$\star : \int_0^t \dots dZ^T$ is independent of $\int_t^u \dots dZ^T$

$\boxtimes : \int \dots dZ_1^T$ is independent of $\int \dots dZ_2^T$

By the properties of the Ito Integral, we have

$$\begin{aligned}
E[Z_{10}(0, t, t)Z_{20}(0, t, u)] &= E \left[\int_0^t \sigma_{10}(s, t) dZ_0^T(s) \cdot \int_0^t \sigma_{20}(s, u) dZ_0^T(s) \right] \\
&= \int_0^t \sigma_{10}(s, t) \sigma_{20}(s, u) ds \\
E[Z_{11}(0, t)Z_{21}(0, t)] &= E \left[\int_0^t \sigma_{11} dZ_1^T(s) \cdot \int_0^t \sigma_{21} dZ_1^T(s) \right] \\
&= \int_0^t \sigma_{11} \sigma_{21} ds \\
&= \sigma_{11} \sigma_{21} t
\end{aligned}$$

Combining all of the above, we finally get

$$\begin{aligned}
E[X_1(t)X_2(u)] &= M_1(0, t, t)M_2(0, u, u) + \sigma_{11}\sigma_{21}t \\
&\quad + \int_0^t \sigma_{10}(s, t)\sigma_{20}(s, u)ds := F(t, u)
\end{aligned} \tag{B.2.3.19}$$

Repeating the above calculations for Case 2: $u < t$ gives

$$\begin{aligned}
E[X_1(t)X_2(u)] &= M_1(0, t, t)M_2(0, u, u) + \sigma_{11}\sigma_{21}u \\
&\quad + \int_0^u \sigma_{10}(s, t)\sigma_{20}(s, u)ds := G(u, t)
\end{aligned} \tag{B.2.3.20}$$

Sanity Check

What if we evaluated $\int_0^T \int_0^T E[X_1(u)X_2(t)]dudt$ instead of $\int_0^T \int_0^T E[X_1(t)X_2(u)]dudt$ for Case 1: $t < u$ and Case 2: $u < t$? We would certainly hope that this does not change our final result, and that is the case. We have derived:

$$\begin{aligned}
& \int_0^T \int_0^T E[X_1(t)X_2(u)]dudt \\
&= \int_0^T \int_0^u \left[M_1(0, t, t)M_2(0, u, u) + \sigma_{11}\sigma_{21}t + \int_0^t \sigma_{10}(s, t)\sigma_{20}(s, u)ds \right] dtdu \\
&+ \int_0^T \int_0^t \left[M_1(0, t, t)M_2(0, u, u) + \sigma_{11}\sigma_{21}u + \int_0^u \sigma_{10}(s, t)\sigma_{20}(s, u)ds \right] dudt \quad (\text{B.2.3.21})
\end{aligned}$$

Repeating the calculations in Step 7 on page 94 would yield:

$$\begin{aligned}
& \int_0^T \int_0^T E[X_1(u)X_2(t)]dudt \\
&= \int_0^T \int_0^u \left[M_1(0, u, u)M_2(0, t, t) + \sigma_{11}\sigma_{21}t + \int_0^t \sigma_{10}(s, u)\sigma_{20}(s, t)ds \right] dtdu \\
&+ \int_0^T \int_0^u \left[M_1(0, u, u)M_2(0, t, t) + \sigma_{11}\sigma_{21}u + \int_0^u \sigma_{10}(s, u)\sigma_{20}(s, t)ds \right] dudt \quad (\text{B.2.3.22})
\end{aligned}$$

Then it is easy to see that (B.2.3.22) is the same as (B.2.3.21), but with the variables u and t swapped.

Step 8: $\int_0^T \int_0^T E[X_1(t)X_2(u)]dudt$

Now we are ready to evaluate these integrals. We split the region of integration into Case 1: $t < u$ and Case 2: $u < t$ to get the following:

$$\begin{aligned}
\int_0^T \int_0^T E[X_1(t)X_2(u)]dudt &= \underbrace{\int_0^T \int_0^u F(t, u)dtdu}_{\text{Case 1}} + \underbrace{\int_0^T \int_0^t G(u, t)dudt}_{\text{Case 2}} \\
&:= \mathcal{E}_{12}(T) \quad (\text{B.2.3.23})
\end{aligned}$$

Step 9: $Cov[\hat{X}_1(T), \hat{X}_2(T)]$

We can plug (B.2.3.23) on the preceding page into (B.2.3.17) on page 94 to get:

$$\begin{aligned} E[\hat{X}_1(T)\hat{X}_2(T)] &= \mathcal{S}_1(T)\mathcal{S}_2(T) + \mathcal{S}_1(T)\mathcal{M}_2(T) \\ &\quad + \mathcal{S}_2(T)\mathcal{M}_1(T) + \frac{1}{T^2}\mathcal{E}_{12}(T) := \hat{M}_{12}(T) \end{aligned} \quad (\text{B.2.3.24})$$

Finally, we can plug (B.2.3.24) into (B.2.3.16) on page 93 to get:

$$Cov[\hat{X}_1(T), \hat{X}_2(T)] = \hat{M}_{12}(T) - \hat{m}_1(T)\hat{m}_2(T) := \hat{v}_{12}(T) \quad (\text{B.2.3.25})$$

□

B.2.4 Proofs of Propositions 3.3.1 on page 35 and 3.3.7 on page 37

Proof. The prices of the GAO and ECO are given by

$$\text{Price(GAO)} = E_{\mathbb{Q}} \left[e^{-\int_0^T r(s)ds} \left(\hat{S}_i(T) - K \right)^+ \right] = P(0, T) E_{\mathbb{Q}^T} \left[\left(\hat{S}_i(T) - K \right)^+ \right] \quad (\text{B.2.4.1})$$

$$\text{Price(ECO)} = E_{\mathbb{Q}} \left[e^{-\int_0^T r(s)ds} (S_i(T) - K)^+ \right] = P(0, T) E_{\mathbb{Q}^T} \left[(S_i(T) - K)^+ \right] \quad (\text{B.2.4.2})$$

Apply Lemma A.1 on page 74 to (B.2.4.1)–(B.2.4.2) above (along with the distributions of $\hat{S}_i(T)$ and $S_i(T)$ given in Lemmas 3.2.1 to 3.2.2 on pages 34–35) to get (3.3.2) on page 36 and (3.3.8) on page 38 respectively. □

B.2.5 Proofs of Propositions 3.3.4 on page 36 and 3.3.10 on page 38

Proof. The prices of the AXO and EXO are given by

$$\begin{aligned} \text{Price(AXO)} &= E_{\mathbb{Q}} \left[e^{-\int_0^T r(s)ds} \left(\hat{S}_1(T) - \hat{S}_2(T) \right)^+ \right] \\ &= P(0, T) E_{\mathbb{Q}^T} \left[\left(\hat{S}_1(T) - \hat{S}_2(T) \right)^+ \right] \end{aligned} \quad (\text{B.2.5.1})$$

$$\text{Price(EXO)} = E_{\mathbb{Q}} \left[e^{-\int_0^T r(s)ds} (S_1(T) - S_2(T))^+ \right]$$

$$= P(0, T)E_{\mathbb{Q}^T} [(S_1(T) - S_2(T))^+] \quad (\text{B.2.5.2})$$

Apply Lemma A.3 on page 75 to (B.2.5.1)–(B.2.5.2) on pages 97–98 above (along with the distributions of $\hat{S}_i(T)$ and $S_i(T)$ given in Lemmas 3.2.1 to 3.2.2 on pages 34–35) to get (3.3.5) on page 37 and (3.3.11) on page 39 respectively. \square

B.2.6 Proof of Corollary 3.3.9 on page 38

Proof. It is easy (but tedious) to check that the following equalities hold:

1. $m_i(T) + \frac{1}{2}v_i^2(T) = \ln \frac{S_i(0)}{P(0, T)}$
2. $m_i(T) + v_i^2(T) - \ln K = \ln \frac{S_i(0)}{KP(0, T)} + \frac{1}{2}v_i^2(0, T)$
3. $v_i^2(T) = v_i^2(0, T)$

The equivalence of (3.3.8)–(3.3.9) on page 38 follows from the above equalities. \square

B.2.7 Proof of Corollary 3.3.12 on page 39

Proof. Check the following equalities:

1. $m_i(T) + \frac{1}{2}v_i^2(T) = \ln \frac{S_i(0)}{P(0, T)}$
2. $m_1(T) - m_2(T) + v_1^2(T) - v_{12}(T) = \ln \frac{S_1(0)}{S_2(0)} + (\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho_{12})\frac{T}{2}$
3. $v_1^2(T) - 2v_{12}(T) + v_2^2(T) = (\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho_{12})T$

The equivalence of (3.3.11)–(3.3.12) on page 39 follows from the above equalities. \square

B.2.8 Proofs of Propositions 3.5.1 to 3.5.3 on pages 44–45

Proof. Use the deterministic interest rate expressions for the mean, variance and covariance of the terminal values ((3.5.4)–(3.5.6) on page 43) and geometric averages ((3.5.7)–(3.5.9) on page 43) in the pricing formulas given by Propositions 3.3.1 on page 35, 3.3.4 on page 36, and 3.3.7 on page 37. \square

B.3 Proofs for Chapter 4

B.3.1 Proof of Proposition 4.1.3 on page 52

Proof. In order to derive the state price process, we want to use the multidimensional Girsanov theorem to change the measure from \mathbb{P} to \mathbb{Q} . First, define the following independent Brownian motions:

$$\begin{aligned} dZ_0(t) &= dW_0(t) + \lambda dt & dZ_1(t) &= dW_1(t) + \alpha_1(t)dt \\ dZ_2(t) &= dW_2(t) + \alpha_2(t)dt \end{aligned} \quad (\text{B.3.1.1})$$

Substitute (B.3.1.1) into (4.1.6)–(4.1.8) on page 51 to get:

$$\begin{aligned} \frac{dS_1(t)}{S_1(t)} &= [r(t) + \lambda\sigma_1]dt + \sigma_1 C_{21}[dZ_0(t) - \lambda dt] + \sigma_1 C_{22}[dZ_1(t) - \alpha_1(t)dt] \\ &= [r(t) + \lambda\sigma_1(1 - C_{21}) - \sigma_1 C_{22}\alpha_1(t)]dt + \sigma_1 C_{21}dZ_0(t) + \sigma_1 C_{22}dZ_1(t) \end{aligned} \quad (\text{B.3.1.2})$$

$$\begin{aligned} \frac{dS_2(t)}{S_2(t)} &= [r(t) + \lambda\sigma_2]dt + \sigma_2 C_{31}[dZ_0(t) - \lambda dt] + \sigma_2 C_{32}[dZ_1(t) - \alpha_1(t)dt] \\ &\quad + \sigma_2 C_{33}[dZ_2(t) - \alpha_2(t)dt] \\ &= [r(t) + \lambda\sigma_2(1 - C_{31}) - \sigma_2 C_{32}\alpha_1(t) - \sigma_2 C_{33}\alpha_2(t)]dt \\ &\quad + \sigma_2 C_{31}dZ_0(t) + \sigma_2 C_{32}dZ_1(t) + \sigma_2 C_{33}dZ_2(t) \end{aligned} \quad (\text{B.3.1.3})$$

$$\frac{dP(t, T)}{P(t, T)} = r(t)dt - \sigma_P(t, T)dZ_0(t) \quad (\text{B.3.1.4})$$

Now, under the \mathbb{Q} measure, we want the drift terms in (B.3.1.2)–(B.3.1.4) to equal the short rate $r(t)$. This amounts to solving the following system of equations for $\alpha_1(t)$ and $\alpha_2(t)$:

$$\begin{cases} \lambda\sigma_1(1 - C_{21}) - \sigma_1 C_{22}\alpha_1(t) = 0 \\ \lambda\sigma_2(1 - C_{31}) - \sigma_2 C_{32}\alpha_1(t) - \sigma_2 C_{33}\alpha_2(t) = 0 \end{cases} \quad (\text{B.3.1.5})$$

$$\Rightarrow \begin{cases} \alpha_1 = \lambda \sqrt{\frac{1 - \rho_{01}}{1 + \rho_{01}}} \end{cases} \quad (\text{B.3.1.6})$$

$$\Rightarrow \begin{cases} \alpha_2 = \alpha_1 \cdot \frac{1 + \rho_{01} - \rho_{02} - \rho_{12}}{\sqrt{A}} \end{cases} \quad (\text{B.3.1.7})$$

where

$$A = \rho_{01}^2 + \rho_{02}^2 + \rho_{12}^2 - 1 - 2\rho_{01}\rho_{02}\rho_{12} \quad (\text{B.3.1.8})$$

(Note that since $\alpha_1(t)$ and $\alpha_2(t)$ do not depend on time t , the argument has been suppressed for simplicity.)

By Girsanov's Theorem, the risk neutral measure \mathbb{Q} is defined by its Radon-Nikodym derivative:

$$\frac{d\mathbb{P}}{d\mathbb{Q}} \Big|_u = \exp \left\{ -\frac{1}{2} (\lambda^2 + \alpha_1^2 + \alpha_2^2) u - \int_0^u \lambda dW_0(s) - \int_0^u \alpha_1 dW_1(s) - \int_0^u \alpha_2 dW_2(s) \right\}$$

Finally, the state price process is given by:

$$\xi_{S2}(T) = e^{-\int_0^T r(s)ds} \frac{d\mathbb{P}}{d\mathbb{Q}} \Big|_T$$

□

B.3.2 Proof of Corollary 4.1.6 on page 53

Proof. From (4.1.13) on page 52 we can see that the integral of the short rate is normally distributed. Then, from (4.1.14) on page 52 it is easy to see that $\xi(T)$ follows a lognormal distribution. Now, define the following quantity

$$\begin{aligned} X(T) = & -\int_0^T r(s)ds - \frac{1}{2} (\lambda^2 + \alpha_1^2 + \alpha_2^2) T \\ & - \int_0^T \lambda dW_0(s) - \int_0^T \alpha_1 dW_1(s) - \int_0^T \alpha_2 dW_2(s) \end{aligned} \quad (\text{B.3.2.1})$$

so that we have

$$\xi_{S2}(T) = e^{X(T)} \sim \mathcal{LN}(E[X(T)], \text{Var}[X(T)])$$

If we can identify $E[X(T)]$ and $\text{Var}[X(T)]$, then we are done. [27] gives the following:

$$\begin{aligned} \int_0^T r(s)ds = & \frac{1}{a} \left[(a\theta + \lambda\sigma_r)T + \left(r(0) - \theta - \frac{\lambda\sigma_r}{a} \right) (1 - e^{-aT}) \right. \\ & \left. - \sigma_r e^{-aT} \int_0^T e^{as} dW_0(s) + \sigma_r W_0(T) \right] \end{aligned} \quad (\text{B.3.2.2})$$

$$E \left[\int_0^T r(s)ds \right] = \left(\theta + \frac{\lambda\sigma_r}{a} \right) T + \left(r(0) - \theta - \frac{\lambda\sigma_r}{a} \right) \frac{1 - e^{-aT}}{a} \quad (\text{B.3.2.3})$$

$$\text{Var} \left[\int_0^T r(s)ds \right] = -\frac{\sigma_r^2}{2a^3} (1 - e^{-aT})^2 + \left(\frac{\sigma_r}{a} \right)^2 \left(T - \frac{1 - e^{-aT}}{a} \right) \quad (\text{B.3.2.4})$$

(B.3.2.2) on the previous page follows from integrating both sides of (4.1.12) on page 51 and substituting for (4.1.13) on page 52. It is easy to see that (B.3.2.3) on the preceding page follows immediately from (B.3.2.2) on the previous page. As for (B.3.2.4) on the preceding page, we can calculate it as follows:

$$\begin{aligned}
& \text{Var} \left[\int_0^T r(s) ds \right] \\
&= \frac{1}{a^2} \text{Var} \left[-\sigma_r e^{-aT} \int_0^T e^{as} dW_0(s) + \sigma_r W_0(T) \right] \\
&= \frac{1}{a^2} \left\{ \sigma_r^2 e^{-2aT} \text{Var} \left[\int_0^T e^{as} dW_0(s) \right] + \sigma_r^2 \text{Var} [W_0(T)] - 2\sigma_r^2 e^{-aT} \text{Cov} \left[\int_0^T e^{as} dW_0(s), W_0(T) \right] \right\} \\
&= \frac{1}{a^2} \left\{ \sigma_r^2 e^{-2aT} \frac{1}{2a} (e^{2aT} - 1) + \sigma_r^2 T - 2\sigma_r^2 e^{-aT} E \left[\int_0^T e^{as} dW_0(s) \cdot \int_0^T dW_0(s) \right] \right\} \\
&= \frac{1}{a^2} \left\{ \frac{\sigma_r^2}{2a} (1 - e^{-2aT}) + \sigma_r^2 T - 2\sigma_r^2 e^{-aT} \int_0^T e^{as} ds \right\} \\
&= \frac{\sigma_r^2}{2a^3} (1 - e^{-2aT}) + \left(\frac{\sigma_r}{a} \right)^2 T - \frac{2\sigma_r^2}{a^3} (1 - e^{aT}) \\
&= -\frac{\sigma_r^2}{2a^3} (1 - e^{-aT})^2 + \left(\frac{\sigma_r}{a} \right)^2 \left(T - \frac{1 - e^{-aT}}{a} \right)
\end{aligned}$$

$E[X(T)]$ is easily calculated as follows:

$$E[X(T)] = -\frac{1}{2}(\lambda^2 + \alpha_1^2 + \alpha_2^2)T - \left(\theta + \frac{\lambda\sigma_r}{a} \right) T - \left(r(0) - \theta - \frac{\lambda\sigma_r}{a} \right) \frac{1 - e^{-aT}}{a}$$

The variance term is more cumbersome:

$$\begin{aligned}
\text{Var}[X(T)] &= \text{Var} \left[\int_0^T r(s) ds \right] + \text{Var} \left[\int_0^T \lambda dW_0(s) \right] + \text{Var} \left[\int_0^T \alpha_1 dW_1(s) \right] \\
&\quad + \text{Var} \left[\int_0^T \alpha_2 dW_2(s) \right] + 2\text{Cov} \left[-\int_0^T r(s) ds, -\int_0^T \lambda dW_0(s) \right] \\
&= -\frac{\sigma_r^2}{2a^3} (1 - e^{-aT})^2 + \left(\frac{\sigma_r}{a} \right)^2 \left(T - \frac{1 - e^{-aT}}{a} \right) + (\lambda^2 + \alpha_1^2 + \alpha_2^2)T \\
&\quad + 2\text{Cov} \left[\int_0^T r(s) ds, \int_0^T \lambda dW_0(s) \right]
\end{aligned}$$

We are left with the covariance term, which evaluates to:

$$\text{Cov} \left[\int_0^T r(s) ds, \int_0^T \lambda dW_0(s) \right]$$

$$\begin{aligned}
&= E \left[\int_0^T r(s) ds \cdot \int_0^T \lambda dW_0(s) \right] \\
&= E \left[\frac{1}{a} \left\{ (a\theta + \lambda\sigma_r)T + \left(r(0) - \theta - \frac{\lambda\sigma_r}{a} \right) (1 - e^{-aT}) \right\} \cdot \int_0^T \lambda dW_0(s) \right] \\
&\quad - \frac{\sigma_r e^{-aT}}{a} E \left[\int_0^T e^{as} dW_0(s) \cdot \int_0^T \lambda dW_0(s) \right] + \frac{\sigma_r}{a} E \left[W_0(T) \int_0^T \lambda dW_0(s) \right]
\end{aligned}$$

The first expectation term is zero, whereas the next two evaluate to the following:

$$\begin{aligned}
E \left[\int_0^T e^{as} dW_0(s) \cdot \int_0^T \lambda dW_0(s) \right] &= \int_0^T e^{as} \lambda ds = \frac{\lambda}{a} (e^{aT} - 1) \\
E \left[W_0(T) \int_0^T \lambda dW_0(s) \right] &= \lambda E[W_0(T)(W_0(T) - W_0(0))] = \lambda T
\end{aligned}$$

Finally, combining all of the above, we have:

$$\text{Var}[X(T)] = (\lambda^2 + \alpha_1^2 + \alpha_2^2)T + \frac{\sigma_r}{a} \left(2\lambda + \frac{\sigma_r}{a} \right) \left(T - \frac{1 - e^{-aT}}{a} \right) - \frac{\sigma_r^2}{2a^3} (1 - e^{-aT})^2$$

□

B.3.3 Proof of Proposition 4.2.1 on page 54

Proof. We first express $\xi_{S_2}^m(T)$ as given by (4.2.1) on page 54:

$$\xi_{S_2}^m(T) = \left[P(0, T) \exp \left\{ \frac{r(T)}{a} \right\} \right]^x \left[\frac{S_1(T)}{S_1(0)} \right]^y \left[\frac{S_2(T)}{S_2(0)} \right]^z [B(T)]^b e^M$$

If we can solve for x , y , z , b , and M , then we are done. For completeness, we will express each term that appears in (4.2.1) on page 54 explicitly. Firstly, we have

$$\begin{aligned}
\int_0^T \sigma_P(s, T) ds &= \int_0^T \frac{\sigma_r}{a} (1 - e^{-a(T-s)}) ds \\
&= \frac{\sigma_r}{a^2} (e^{-aT} + aT - 1) \\
\int_0^T \sigma_P^2(s, T) ds &= \int_0^T \left[\frac{\sigma_r}{a} (1 - e^{-a(T-s)}) \right]^2 ds
\end{aligned}$$

$$= \frac{\sigma_r^2}{2a^3} (4e^{-aT} - e^{-2aT} + 2aT - 3)$$

(4.1.11) on page 51 and (4.1.13) on page 52 give:

$$\begin{aligned} \left[P(0, T) \exp \left\{ \frac{r(T)}{a} \right\} \right]^x &= \exp \left\{ \frac{xr(0)e^{-aT}}{a} + \frac{x}{a} \left(\theta + \frac{\lambda\sigma_r}{a} \right) (1 - e^{-aT}) - x \int_0^T r(s)ds \right. \\ &\quad \left. + \frac{x\lambda\sigma_r}{a^2} (aT + e^{-aT} - 1) + \frac{x\sigma_r^2}{4a^3} (4e^{-aT} - e^{-2aT} + 2aT - 3) \right. \\ &\quad \left. + \int_0^T \frac{x\sigma_r}{a} dW_0(s) \right\} \end{aligned} \quad (\text{B.3.3.1})$$

(4.1.9) on page 51 gives:

$$\begin{aligned} \left[\frac{S_1(T)}{S_1(0)} \right]^y &= \exp \left\{ y \int_0^T r(s)ds + y\sigma_1 \left(\lambda - \frac{\sigma_1}{2} \right) T \right. \\ &\quad \left. + y \int_0^T \sigma_1 C_{21} dW_0(s) + y \int_0^T \sigma_1 C_{22} dW_1(s) \right\} \end{aligned} \quad (\text{B.3.3.2})$$

(4.1.10) on page 51 gives:

$$\begin{aligned} \left[\frac{S_2(T)}{S_2(0)} \right]^z &= \exp \left\{ z \int_0^T r(s)ds + z\sigma_2 \left(\lambda - \frac{\sigma_2}{2} \right) T + z \int_0^T \sigma_2 C_{31} dW_0(s) \right. \\ &\quad \left. + z \int_0^T \sigma_2 C_{32} dW_1(s) + z \int_0^T \sigma_2 C_{33} dW_2(s) \right\} \end{aligned} \quad (\text{B.3.3.3})$$

Finally, we have

$$B(T) = \exp \left\{ b \int_0^T r(s)ds \right\} \quad (\text{B.3.3.4})$$

We will first start with matching the stochastic integral terms. If we compare (B.3.3.1)–(B.3.3.3) with (4.2.1) on page 54 we can see that we need to solve the following system of equations:

$$\begin{cases} \frac{x\sigma_r}{a} + y\sigma_1 C_{21} + z\sigma_2 C_{31} = -\lambda & (\star) \\ y\sigma_1 C_{22} + z\sigma_2 C_{32} = -\alpha_1 \\ z\sigma_2 C_{33} = -\alpha_2 \end{cases} \quad (\text{B.3.3.5})$$

$$\Rightarrow \begin{cases} x = \frac{a\lambda}{\sigma_r} \cdot \frac{(1 - \rho_{12})(1 + \rho_{12} - \rho_{01} - \rho_{02})}{A} & \text{(B.3.3.6)} \\ y = \frac{\lambda}{\sigma_1} \cdot \frac{(1 - \rho_{02})(1 + \rho_{02} - \rho_{12} - \rho_{01})}{A} & \text{(B.3.3.7)} \\ z = \frac{\lambda}{\sigma_2} \cdot \frac{(1 - \rho_{01})(1 + \rho_{01} - \rho_{02} - \rho_{12})}{A} & \text{(B.3.3.8)} \end{cases}$$

Note the curious fact in the numerators of (B.3.3.6)–(B.3.3.8) – we can see that the terms form a cyclic permutation i.e. $\rho_{01} \mapsto \rho_{12} \mapsto \rho_{02} \mapsto \rho_{01}$. The integral of the short rate can be matched by solving the following

$$(y + z - x + b) \int_0^T r(s) ds = - \int_0^T r(s) ds \Rightarrow b = x - y - z - 1 \quad \text{(B.3.3.9)}$$

Finally, we are left with matching the constant terms in the state price process. We want to solve the following:

$$\begin{aligned} & \frac{xr(0)e^{-aT}}{a} + \frac{x}{a} \left(\theta + \frac{\lambda\sigma_r}{a} \right) (1 - e^{-aT}) + \frac{x\lambda\sigma_r}{a^2} (aT + e^{-aT} - 1) \\ & + \frac{x\sigma_r^2}{4a^3} (4e^{-aT} - e^{-2aT} + 2aT - 3) + y\sigma_1 \left(\lambda - \frac{\sigma_1}{2} \right) T + z\sigma_2 \left(\lambda - \frac{\sigma_2}{2} \right) T \\ & + M \\ & = -\frac{1}{2} (\lambda^2 + \alpha_1^2 + \alpha_2^2) T \\ \therefore M & = - \left\{ \frac{1}{2} (\lambda^2 + \alpha_1^2 + \alpha_2^2) T + \frac{x}{a} e^{-aT} \left[r(0) - \theta + \left(\frac{\sigma_r}{a} \right)^2 \left(1 - \frac{1}{4} e^{-aT} \right) \right] \right. \\ & \left. + \frac{x}{a} \left[\theta + \sigma_r \left(\lambda T + \frac{\sigma_r T}{2a} - \frac{3\sigma_r}{4a^2} \right) \right] \right. \\ & \left. + y\sigma_1 \left(\lambda - \frac{\sigma_1}{2} \right) T + z\sigma_2 \left(\lambda - \frac{\sigma_2}{2} \right) T \right\} \quad \text{(B.3.3.10)} \end{aligned}$$

□

B.3.4 Proofs of Propositions 4.3.1 on page 55, 4.3.4 on page 56, 4.3.7 on page 58, and 4.3.9 on page 58

Proof. Follow the program as prescribed in Section 4.3 on page 55. The details are very much in the spirit of the proofs of Propositions 4.1.3 on page 52 and 4.2.1 on page 54, and are omitted. □

B.3.5 Proof of Corollary 4.3.3 on page 56

Proof. We have

$$\xi_{BBV}(T) = \gamma \left[\frac{S_1(T)}{S_1(0)} \right]^{-\delta}$$

where

$$\theta = \frac{\mu_1 - r}{\sigma_1} = \lambda$$

$$\delta = \frac{\theta}{\sigma_1} = \frac{\lambda}{\sigma_1} = -y$$

and

$$\begin{aligned} \gamma &= \exp \left\{ \frac{\theta}{\sigma_1} \left(\mu_1 - \frac{\sigma_1^2}{2} \right) T - \left(r + \frac{\theta^2}{2} \right) T \right\} \\ &= \exp \left\{ \frac{\lambda}{\sigma_1} \left(\sigma_1 \lambda + r - \frac{\sigma_1^2}{2} \right) T - \left(r + \frac{\lambda^2}{2} \right) T \right\} \\ &= \exp \left\{ \frac{\lambda}{2} (\lambda - \sigma_1) T \right\} \exp \left\{ \left(\frac{\lambda}{\sigma_1} - 1 \right) r T \right\} \\ &= e^M [B(T)]^b \end{aligned}$$

Note that we have adapted the definition of the market price of risk, λ , as given in (4.1.5) on page 51 to our case of constant interest rates. Therefore, it is easy to see that

$$\xi_{BBV}(T) = \gamma \left[\frac{S_1(T)}{S_1(0)} \right]^{-\delta} = \left[\frac{S_1(T)}{S_1(0)} \right]^y [B(T)]^b e^M = \xi_{C1}^m(T)$$

(b and M are given in Proposition 4.3.1 on page 55)

□

B.3.6 Proof of Corollary 4.3.5 on page 57

Proof. For simplicity, we will omit the * subscript/superscript notation and denote the market portfolio by $\underline{\pi}$ instead. See also Section 2.4.2 on page 19 for a more complete description of the market portfolio.

We define the (2×2) matrix Σ as

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \rho_{12} \sigma_1 \sigma_2 \\ \rho_{12} \sigma_1 \sigma_2 & \sigma_2^2 \end{bmatrix}$$

and the drift vector

$$\underline{\mu} = \begin{bmatrix} r + \sigma_1 \lambda \\ r + \sigma_2 \lambda \end{bmatrix}$$

by adapting the definition of the market price of risk as given in (4.1.5) on page 51 to our case of constant interest rates. The market portfolio is defined as

$$\underline{\pi} = \frac{\underline{\Sigma}^{-1} \cdot \left(\underline{\mu} - r \cdot \underline{1} \right)}{\underline{1}^T \cdot \underline{\Sigma}^{-1} \cdot \left(\underline{\mu} - r \cdot \underline{1} \right)} = \begin{bmatrix} \frac{\sigma_2}{\sigma_1 + \sigma_2} \\ \frac{\sigma_1}{\sigma_1 + \sigma_2} \end{bmatrix}$$

where the fractions of the portfolio invested in assets 1 and 2 remain constant over time. The drift and volatility of the price process of the security that is constructed using the constant mix $\underline{\pi}$ are given by:

$$\begin{aligned} \mu(\pi) &= r + \underline{\pi}^T \cdot \left(\underline{\mu} - r \cdot \underline{1} \right) = \frac{\sigma_2 r + 2\lambda \sigma_1 \sigma_2 + \sigma_1 r}{\sigma_1 + \sigma_2} \\ \sigma^2(\pi) &= \underline{\pi}^T \cdot \underline{\Sigma} \cdot \underline{\pi} = \frac{2(\rho_{12} + 1)\sigma_1^2 \sigma_2^2}{(\sigma_1 + \sigma_2)^2} \end{aligned}$$

The terminal value of the security that is constructed using the constant mix $\underline{\pi}$ is given by:

$$\begin{aligned} \frac{S_\pi(T)}{S_\pi(0)} &= \exp \left\{ \left(\mu(\pi) - \frac{\sigma^2(\pi)}{2} \right) T + \pi_1 \sigma_1 W_1(T) + \pi_2 \sigma_2 W_2(T) \right\} \\ &= \exp \left\{ \left(\pi_1 \mu_1 + \pi_2 \mu_2 - \frac{\sigma_1^2 \pi_1^2 + \sigma_2^2 \pi_2^2 + 2\rho_{12} \pi_1 \pi_2 \sigma_1 \sigma_2}{2} \right) T + \pi_1 \sigma_1 W_1(T) + \pi_2 \sigma_2 W_2(T) \right\} \\ &= \exp \left\{ \left(\pi_1 \mu_1 - \frac{\pi_1 \sigma_1^2}{2} \right) T + \pi_1 \sigma_1 W_1(T) \right\} \exp \left\{ \left(\frac{\pi_1 \sigma_1^2}{2} - \frac{\pi_1^2 \sigma_1^2}{2} \right) T \right\} \\ &\times \exp \left\{ \left(\pi_2 \mu_2 - \frac{\pi_2 \sigma_2^2}{2} \right) T + \pi_2 \sigma_2 W_2(T) \right\} \exp \left\{ \left(\frac{\pi_2 \sigma_2^2}{2} - \frac{\pi_2^2 \sigma_2^2}{2} \right) T \right\} \\ &\times \exp \{ -\rho_{12} \pi_1 \pi_2 \sigma_1 \sigma_2 T \} \\ &= \left[\frac{S_1(T)}{S_1(0)} \right]^{\pi_1} \left[\frac{S_2(T)}{S_2(0)} \right]^{\pi_2} \exp \left\{ \left[\frac{\pi_1 \sigma_1^2}{2} (1 - \pi_1) + \frac{\pi_2 \sigma_2^2}{2} (1 - \pi_2) - \rho_{12} \pi_1 \pi_2 \sigma_1 \sigma_2 \right] T \right\} \end{aligned}$$

[7] give the following expression of the state price process in terms of the market variables:

$$\xi_{BMV}(T) = \gamma \left[\frac{S_\pi(T)}{S_\pi(0)} \right]^{-\delta}$$

where

$$\theta = \frac{\mu(\pi) - r}{\sigma(\pi)} = \sqrt{\frac{2\lambda^2}{1 + \rho_{12}}}$$

$$\delta = \frac{\theta}{\sigma(\pi)} = \frac{\mu(\pi) - r}{\sigma^2(\pi)} = \frac{(\sigma_1 + \sigma_2)\lambda}{\sigma_1\sigma_2(1 + \rho_{12})}$$

and

$$\begin{aligned} \gamma &= \exp \left\{ \frac{\theta}{\sigma(\pi)} \left[\mu(\pi) - \frac{\sigma^2(\pi)}{2} \right] T - \left(r + \frac{\theta^2}{2} \right) T \right\} \\ &= \exp \left\{ \left[\frac{(\sigma_1 + \sigma_2)\lambda}{\sigma_1\sigma_2(1 + \rho_{12})} \left(\frac{\sigma_2 r + 2\lambda\sigma_1\sigma_2 + \sigma_1 r}{\sigma_1 + \sigma_2} - \frac{(1 + \rho_{12})\sigma_1^2\sigma_2^2}{(\sigma_1 + \sigma_2)^2} \right) - \left(r + \frac{\lambda^2}{1 + \rho_{12}} \right) \right] T \right\} \\ &= \exp \left\{ \left[\frac{(\sigma_1 r + \sigma_2 r + 2\lambda\sigma_1\sigma_2)\lambda - \frac{(1 + \rho_{12})\sigma_1^2\sigma_2^2\lambda}{(\sigma_1 + \sigma_2)^2}}{\sigma_1\sigma_2(1 + \rho_{12})} - r - \frac{\lambda^2}{1 + \rho_{12}} \right] T \right\} \\ &= \exp \left\{ \left[\left(\frac{\lambda}{\sigma_2(1 + \rho_{12})} - \frac{\lambda}{\sigma_1(1 + \rho_{12})} - 1 \right) r + \frac{2\lambda^2}{1 + \rho_{12}} - \frac{\lambda\sigma_1\sigma_2}{\sigma_1 + \sigma_2} - \frac{\lambda^2}{1 + \rho_{12}} \right] T \right\} \\ &= \exp \left\{ \left[\left(\frac{\lambda}{1 + \rho_{12}} \left(\frac{1}{\sigma_1} + \frac{1}{\sigma_2} \right) - 1 \right) r + \lambda \left(\frac{\lambda}{1 + \rho_{12}} - \frac{\sigma_1\sigma_2}{\sigma_1 + \sigma_2} \right) \right] T \right\} \\ &= \exp \{ brT \} \exp \left\{ \left[\lambda \left(\frac{\lambda}{1 + \rho_{12}} - \frac{\sigma_1\sigma_2}{\sigma_1 + \sigma_2} \right) \right] T \right\} \end{aligned}$$

Note that

$$-\pi_1\delta = \frac{\sigma_2}{\sigma_1 + \sigma_2} \cdot \frac{(\sigma_1 + \sigma_2)\lambda}{\sigma_1\sigma_2(1 + \rho_{12})} = -\frac{\lambda}{\sigma_1(1 + \rho_{12})} = y$$

$$-\pi_2\delta = \frac{\sigma_1}{\sigma_1 + \sigma_2} \cdot \frac{(\sigma_1 + \sigma_2)\lambda}{\sigma_1\sigma_2(1 + \rho_{12})} = -\frac{\lambda}{\sigma_2(1 + \rho_{12})} = z$$

Therefore, we have

$$\begin{aligned} &\left[\frac{S_\pi(T)}{S_\pi(0)} \right]^{-\delta} \\ &= \left[\frac{S_1(T)}{S_1(0)} \right]^y \left[\frac{S_2(T)}{S_2(0)} \right]^z \exp \left\{ \left[\frac{y\sigma_1^2}{2}(1 - \pi_1) + \frac{z\sigma_2^2}{2}(1 - \pi_2) + \frac{\rho_{12}(\sigma_1 + \sigma_2)\lambda\pi_1\pi_2}{1 + \rho_{12}} \right] T \right\} \end{aligned}$$

Now, consider the last term in the RHS of the above equality

$$\exp \left\{ \left[\frac{y\sigma_1^2}{2}(1 - \pi_1) + \frac{z\sigma_2^2}{2}(1 - \pi_2) + \frac{\rho_{12}(\sigma_1 + \sigma_2)\lambda\pi_1\pi_2}{1 + \rho_{12}} \right] T \right\}$$

$$\begin{aligned}
&= \exp \left\{ \left[-\frac{\lambda\sigma_1}{2(1+\rho_{12})} \left(1 - \frac{\sigma_2}{\sigma_1 + \sigma_2}\right) - \frac{\lambda\sigma_2}{2(1+\rho_{12})} \left(1 - \frac{\sigma_1}{\sigma_1 + \sigma_2}\right) \right. \right. \\
&\quad \left. \left. + \frac{\rho_{12}\lambda(\sigma_1 + \sigma_2)\frac{\sigma_1\sigma_2}{(\sigma_1 + \sigma_2)^2}}{1 + \rho_{12}} \right] T \right\} \\
&= \exp \left\{ \left[-\frac{\lambda\sigma_1}{2(1+\rho_{12})} - \frac{\lambda\sigma_2}{2(1+\rho_{12})} + \frac{\lambda\sigma_1\sigma_2}{(1+\rho_{12})(\sigma_1 + \sigma_2)} + \frac{\lambda\rho_{12}\sigma_1\sigma_2}{(1+\rho_{12})(\sigma_1 + \sigma_2)} \right] T \right\} \\
&= \exp \left\{ \left[-\frac{\lambda\sigma_1}{2(1+\rho_{12})} - \frac{\lambda\sigma_2}{2(1+\rho_{12})} + \frac{\lambda\sigma_1\sigma_2 + \lambda\rho_{12}\sigma_1\sigma_2}{(1+\rho_{12})(\sigma_1 + \sigma_2)} \right] T \right\} \\
&= \exp \left\{ \left[-\frac{\lambda\sigma_1}{2(1+\rho_{12})} - \frac{\lambda\sigma_2}{2(1+\rho_{12})} + \frac{\lambda\sigma_1\sigma_2}{\sigma_1 + \sigma_2} \right] T \right\}
\end{aligned}$$

Combining all of the above, we finally have

$$\begin{aligned}
&\xi_{BMV}(T) \\
&= \gamma \left[\frac{S_\pi(T)}{S_\pi(0)} \right]^{-\delta} \\
&= \left[\frac{S_1(T)}{S_1(0)} \right]^y \left[\frac{S_2(T)}{S_2(0)} \right]^z \exp \left\{ \left[-\frac{\lambda\sigma_1}{2(1+\rho_{12})} - \frac{\lambda\sigma_2}{2(1+\rho_{12})} + \frac{\lambda\sigma_1\sigma_2}{\sigma_1 + \sigma_2} \right] T \right\} \\
&\times \exp \{brT\} \exp \left\{ \left[\lambda \left(\frac{\lambda}{1+\rho_{12}} - \frac{\sigma_1\sigma_2}{\sigma_1 + \sigma_2} \right) \right] T \right\} \\
&= \left[\frac{S_1(T)}{S_1(0)} \right]^y \left[\frac{S_2(T)}{S_2(0)} \right]^z \exp \left\{ \left[-\frac{\lambda\sigma_1}{2(1+\rho_{12})} - \frac{\lambda\sigma_2}{2(1+\rho_{12})} + \frac{\lambda^2}{1+\rho_{12}} \right] T \right\} \exp \{brT\} \\
&= \left[\frac{S_1(T)}{S_1(0)} \right]^y \left[\frac{S_2(T)}{S_2(0)} \right]^z \exp \left\{ \frac{\lambda}{1+\rho_{12}} \left[\lambda - \frac{\sigma_1}{2} - \frac{\sigma_2}{2} \right] T \right\} \exp \{brT\} \\
&= \left[\frac{S_1(T)}{S_1(0)} \right]^y \left[\frac{S_2(T)}{S_2(0)} \right]^z [B(T)]^b e^M \\
&= \xi_{C_2}^m(T)
\end{aligned}$$

□

Appendix C

Glossary of Notation

This appendix contains some important notation that is used in each chapter.

C.1 Chapter 1

1. ξ_t : State price process in a constant interest rate environment [Assumption 2 on page 3]
2. CE: Cost efficient [Definition 1.2.2 on page 4]
3. CEC: Cost efficient counterpart of a particular payoff [Theorem 1.2.4 on page 4]

C.2 Chapter 2

All notation in this chapter refer to a constant interest rate environment.

1. S_T : Terminal value of the stock at time T [(2.2.1) on page 9]
2. I_T : Terminal value of the index at time T [(2.2.1) on page 9]
3. \hat{S}_T : Continuous geometric average of the stock at time T [(2.2.2) on page 9]
4. \hat{I}_T : Continuous geometric average of the index at time T [(2.2.2) on page 9]
5. H_T : Terminal value of the benchmark at time T [(2.2.2) on page 9]
6. \hat{H}_T : Continuous geometric average of the benchmark at time T [(2.2.2) on page 9]

7. AEO: Asian Executive Option [Section 2.3 on page 10]
8. \hat{G}_T : Payoff of the AEO at time T [(2.3.1) on page 10]
9. \hat{E}_0 : Price of the AEO at time 0 [(2.3.2) on page 10]
10. G_T : Payoff of the CEC of the AEO at time T [(2.3.3) on page 11]
11. E_0 : Price of the CEC of the AEO at time 0 [(2.3.4) on page 11]
12. AEIO: Asian Executive Indexed Option [Section 2.4 on page 14]
13. \hat{A}_T : Payoff of the AEIO at time T [(2.4.1) on page 14]
14. \hat{V}_0 : Price of the AEIO at time 0 [(2.4.2) on page 15]
15. PXEO: Power Exchange Executive Option [Section 2.4.1 on page 15]
16. A_T^* : Payoff of the PXEO at time T [(2.4.6) on page 16]
17. V_0^* : Price of the PXEO at time 0 [(2.4.7) on page 16]
18. A_T : Payoff of the CEC of the AEIO at time T [(2.4.13) on page 20]
19. V_0 : Price of the CEC of the AEIO at time 0 [(2.4.14) on page 21]

C.3 Chapter 3

Unless otherwise mentioned, all notation in this chapter refer to a stochastic interest rate environment modeled by a Vasicek process. The index i is understood to range over 1 and 2 i.e. $i = 1, 2$.

1. Z : Standard Brownian motion under the \mathbb{Q} measure [Section 3.1.1 on page 31]
2. $r(t)$: Short rate [(3.1.1) on page 31]
3. a : Constant rate of the short rate's reversion to the long term mean [(3.1.1) on page 31]
4. θ : Constant long term mean for the short rate [(3.1.1) on page 31]
5. σ_r : Constant short rate volatility [(3.1.1) on page 31]
6. $P(t, T)$: Price of a T -bond at time t [(3.1.3) on page 32]
7. $\sigma_P(t, T)$: Volatility of the T -bond price dynamics [(3.1.7) on page 32]
8. σ_i : Volatility of the price dynamics for asset i [(3.1.8)–(3.1.9) on page 32]
9. ρ_{01} : Correlation between the short rate and asset 1 [(3.1.11) on page 32]

10. ρ_{02} : Correlation between the short rate and asset 2 [(3.1.11) on page 32]
11. ρ_{12} : Correlation between assets 1 and 2 [(3.1.11) on page 32]
12. C_{ij} : Diffusion terms arising from the Cholesky decomposition [(3.1.11) on page 32]
13. Z^T : Standard Brownian motion under the \mathbb{Q}^T measure [Section 3.1.2 on page 33]
14. $S_i(T)$: The terminal value of asset i at time T [(3.2.7) on page 34]
15. $\hat{S}_i(T)$: The continuous geometric average of asset i at time T [(3.2.8) on page 34]
16. GAO: Geometric Asian Option [(3.3.1) on page 35]
17. Price(GAO): Price of the GAO at time 0 [(3.3.2) on page 36]
18. GAOConst: Geometric Asian Option when interest rates are constant [Remark 3.3.2 on page 36]
19. Price(GAOConst): Price of the GAOConst at time 0 [(3.3.3) on page 36]
20. AXO: Asian Exchange Option [(3.3.4) on page 36]
21. Price(AXO): Price of the Asian Exchange Option at time 0 [(3.3.5) on page 37]
22. AXOConst: Asian Exchange Option when interest rates are constant [Remark 3.3.5 on page 37]
23. Price(AXOConst): Price of the AXOConst at time 0 [(3.3.6) on page 37]
24. ECO: European Call Option [(3.3.7) on page 37]
25. Price(ECO): Price of the ECO at time 0 [(3.3.8) on page 38]
26. EXO: European Exchange Option [(3.3.10) on page 38]
27. Price(EXO): Price of the EXO at time 0 [(3.3.11) on page 39]
28. $m_i(T)$: Expected value of the terminal value of asset i at time T i.e. $E[S_i(T)]$ [(3.4.1) on page 41]
29. $v_i^2(T)$: Variance of the terminal value of asset i at time T i.e. $Var[S_i(T)]$ [(3.4.2) on page 41]
30. $v_{12}(T)$: Covariance of the terminal values of assets 1 and 2 at time T i.e. $Cov[S_1(T), S_2(T)]$ [(3.4.3) on page 41]
31. $\hat{m}_i(T)$: Expected value of the geometric average of asset i at time T i.e. $E[\hat{S}_i(T)]$ [(3.4.4) on page 42]
32. $\hat{v}_i^2(T)$: Variance of the geometric average of asset i at time T i.e. $Var[\hat{S}_i(T)]$ [(3.4.5) on page 42]

33. $\hat{v}_{12}(T)$: Covariance of the geometric averages of assets 1 and 2 at time T i.e. $Cov[\hat{S}_1(T), \hat{S}_2(T)]$ [(3.4.6) on page 42]

C.4 Chapter 4

1. W : Standard Brownian motion under the \mathbb{P} measure [Section 4.1.1 on page 50]
2. λ : Market price of risk [(4.1.5) on page 51]
3. $\xi_{S_2}(T)$: State price process at time T in a market with two risky assets and stochastic interest rates modeled by a Vasicek process [(4.1.14) on page 52]
4. $m_\xi(T)$: Expected value of the state price process at time T in a market with two risky assets and stochastic interest rates modeled by a Vasicek process i.e. $E[\xi_{S_2}(T)]$ [(4.1.18) on page 53]
5. $v_\xi^2(T)$: Variance of the state price process at time T in a market with two risky assets and stochastic interest rates modeled by a Vasicek process i.e. $Var[\xi_{S_2}(T)]$ [(4.1.19) on page 53]
6. $\xi_{S_2}^m(T)$: State price process at time T , expressed as a function of the market variables, in a market with two risky assets and stochastic interest rates modeled by a Vasicek process [(4.2.1) on page 54]
7. $\xi_{C_1}(T)$: State price process at time T in a market with one risky asset and constant interest rate [(4.3.1) on page 55]
8. $\xi_{BBV}(T)$: State price process given by Bernard, Boyle and Vanduffel [4] that is equivalent to $\xi_{C_1}(T)$ [(4.3.3) on page 56]
9. $\xi_{C_1}^m(T)$: State price process at time T , expressed as a function of the market variables, in a market with one risky asset and constant interest rate [(4.3.2) on page 55]
10. $\xi_{C_2}(T)$: State price process at time T in a market with two risky assets and constant interest rate [(4.3.4) on page 56]
11. $\xi_{BMV}(T)$: State price process given by Bernard, Maj and Vanduffel [7] that is equivalent to $\xi_{C_2}(T)$ [(4.3.6) on page 57]

12. $\xi_{C2}^m(T)$: State price process at time T , expressed as a function of the market variables, in a market with two risky asset and constant interest rate [(4.3.5) on page 56]
13. $\xi_{S0}(T)$: State price process at time T in a market with stochastic interest rates modeled by a Vasicek process [(4.3.7) on page 58]
14. $\xi_{S0}^m(T)$: State price process at time T , expressed as a function of the market variables, in a market with stochastic interest rates modeled by a Vasicek process [(4.3.8) on page 58]
15. $\xi_{S1}(T)$: State price process at time T in a market with one risky asset and stochastic interest rates modeled by a Vasicek process [(4.3.9) on page 58]
16. $\xi_{S1}^m(T)$: State price process at time T , expressed as a function of the market variables, in a market with one risky asset and stochastic interest rates modeled by a Vasicek process [(4.3.10) on page 59]

C.5 Chapter 5

Unless otherwise mentioned, all notation in this chapter refer to a stochastic interest rate environment modeled by a Vasicek process. The index i is understood to range over 1 and 2 i.e. $i = 1, 2$.

1. **AEO**: Asian Executive Option [Section 5.2 on page 63]
2. $\hat{\mathbf{G}}(T)$: Payoff of the **AEO** at time T [(5.2.1) on page 64]
3. $\hat{\mathbf{E}}_0$: Price of the **AEO** at time 0 [Section 5.2 on page 63]
4. **AEIO**: Asian Executive Indexed Option [Section 5.2 on page 63]
5. $\hat{\mathbf{A}}(T)$: Payoff of the **AEIO** at time T [(5.2.2) on page 64]
6. $\hat{\mathbf{V}}_0$: Price of the **AEIO** at time 0 [Section 5.2 on page 63]
7. $\xi(T)$: State price process at time T [Section 5.3 on page 64]
8. $\mathbf{G}(T)$: Payoff of the CEC of the **AEO** at time T [(5.3.1) on page 64]
9. \mathbf{E}_0 : Price of the CEC of the **AEO** at time 0 [(5.3.3) on page 64]

10. $\mathbf{A}(T)$: Payoff of the CEC of the **AEIO** at time T [(5.3.2) on page 64]
11. \mathbf{V}_0 : Price of the CEC of the **AEIO** at time 0 [(5.3.4) on page 64]

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