Inventory Decisions for the Price Setting Retailer: Extensions to the EOQ Setting

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Practical inventory settings often include multiple generations of the same product on hand. New products often arrive before old stock is exhausted, but most inventory models do not account for this. Such a setting gives rise to the possibility of inter-generational substitution between products. We study a retailer that stocks two product generations and we show that from a cost perspective the retailer is better off stocking only one generation. We proceed with a profit scheme and develop a price-setting profit maximization model, proving that in one and two generation profit models there exists a unique solution. We use the profit model to show that there are cases where it is more profitable to stock two generations. We discuss utility and preference extensions to the profit model and present the general n-product case.

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Chapter 1

Introduction

The Economic Order Quantity (EOQ) model is one of the building blocks in inventory management. It can be traced back to 1913 when Ford Whitman Harris, an engineer for Westinghouse Electric, first presented the model [4]. The basic premise is to try to balance the cost of ordering inventory against the cost of holding inventory over multiple periods in a fixed time horizon. The model has been studied extensively and has a rich literature surrounding it, with research into perishable goods and inventory systems with stockouts.

Each extension to the framework is aimed at making the model match more closely with what happens in a real inventory system. However, the literature falls short of addressing inventory systems that stock multiple generations of the same product; a major feature of real inventory systems. Upon receiving a new batch of goods, retailers often possess older products from the previous batch. The presence of both new and old goods creates an opportunity for inter-generational substitution, however the literature is largely silent on this. We see multiple generations of the same products all the time at the grocery store (milk with different expirations) on car lots (this years' model and last years' model of the same type of car), with fashion retailers, and the list goes on. Hence, an important question arises: when is it preferable for a retailer to hold two generations of a product?

In considering multiple generations, we touch upon another important topic: product pricing. Most EOQ related settings neglect this area and consider the retailer as a price taker. These settings are concerned with the total cost of the inventory system and leave product pricing to revenue management. In view of an inventory system with multiple generations of products, one salient question is: how should we price them? That is, how will our pricing scheme impact revenue and profit. There are other important questions: how will the optimal order quantity be affected in the presence of multiple product generations, and how many cycles will there be?

We present an EOQ model in which the retailer can hold two product generations and set prices, then arguing that under this setting, profit, rather than cost, is a more appropriate optimization criterion. Our approach can also be extended to include consumer utility, which we believe establishes an important link between consumers, prices and inventory quantities. In the following sections we breifly discuss the historical origins of the model and some existing EOQ extensions; we consider some of the work already undertaken for price setting retailers, lay the framework of our approach, and quickly review the major results.

1.1 Existing EOQ Variants in Inventory Management

In 1918, E.W. Taft published the next major EOQ development after Harris [4]. His contribution is recognized today as the Economic Production Quantity (EPQ) model and incorporates the production rate of the item held in inventory. Following Taft, a number of practitioners would publish (without citation) models so similar to the EOQ and EPQ that a comprehensive review by Fairfield E. Raymond in 1931 was necessary to determine the origins of the model [4]. Raymond also wrote a standard exposition for the model that closely resembles Harris'.

In the decades since Harris and Taft, the EOQ model has been developed extensively. Many relaxations to the model assumptions have been made, and many extensions to the model framework have been undertaken. Zhang et al. [24] recognized that in the presence of a stockout of one product, there might be adverse impacts on related products. They modify the partial backordering framework to include correlated demand caused by cross-selling. Under their framework they present a major item (which allows back ordering) and minor item (without stock outs) with identical order cycle times. The major item has an independent demand, while the minor item has demand correlated with the major item. They go on to show the effect of lost sales of the minor item due to back ordering of the major item. This work represents an important step in linking the demand of one product with another. Although our framework is not concerned with backorders, we recognize the importance of a model with cross demand effects and include it in our work.

A great deal of work has been done in the area of perishable goods. Weiss [23] approached the problem by using non-linear holding costs to approximate the increasing costs of holding a product in inventory for longer periods of time. Ferguson et al. [5] applied a variation of the Weiss model in a numerical study to show how the parameters in the model could be estimated, and demonstrated the cost reductions over the classical EOQ model. Fujiwara et al. [6] approach the same problem by applying linear and exponential penalty costs (with second order approximations). Their approach leads naturally to closed form solutions for perishable items, can be extended to included backordering, and the authors claim it can easily be extended to the EPQ framework. These works are important steps in dealing with products that have a life expectancy, but neglect a very real occurrence: two generations of a product available for purchase at the same time. Our framework employs two distinct generations of the same product with separate but related demands for each.

Imperfect items form another important class of EOQ considerations. Salameh and Jaber [16] give a framework for imperfect and poor quality items, showing the effects on inventory costs. They assume a fixed fraction of incoming inventory is imperfect, account for the screening time associated with finding the imperfect inventory, and enforce a condition that the quantity ordered minus the imperfect items would be enough to meet demand over the period. Khan et al. [8] review a number of extensions to Salameh and Jaber's framework by other authors. Extensions include the EPQ model, shortages and backordering, and inclusions into supply chains. We recognize the importance of this research, and although our framework cannot directly address these questions, we believe a similar framework that considers the price of perfect and imperfect products would change the inventory decisions and should also be considered.

There are a number of other extensions to the EOQ setting including fuzzy (uncertain) parameters, restocking decisions, production capacity limits and multi-period models. Excellent summaries can be found in Zipkin [25] as well as in Chan et al. [2].

1.2 Pricing Strategies in Inventory Management

Pricing decisions form a major part of a retailer or supplier's decisions. Recently there has been some interest in the inclusion of prices into the EOQ model. Including the concept of "price" can take many forms. Matsuyama [10] introduces two relaxations independently; namely that there is a discount to purchase prices for larger orders (similar to bulk discounts) and a new set up cost that depends on the size of the order. These inclusions are aimed at addressing a more realistic ordering and inventory cost scheme, however they still assume the retailer is a price taker. By contrast, we do not.

Wang and Tung [22] propose a pricing model for products that are gradually becoming obsolete (technology or clothing). Their framework employees a time and price dependent demand rate on a single product. Analysis with their model suggests that multiple discounts provide higher profit than a single discount. This is an important contribution for two reasons: it uses a profit criterion and recognizes the impact of price on demand. However, their framework does not address holding multiple generations of the same product.

Mishra and Mishra [11] also consider deteriorating products in a market with perfect competition. Their framework employs an inventory dependent demand rate and a linear demand rate, switching definitions at a predetermined point in the period. They then employ an algorithm for computing prices which involves computing the marginal revenue and marginal cost at equilibrium. Their economic framework also assumes the retailer is a price taker.

By specifying demand as a decreasing quadratic function of selling price, Sana [17] considers an EOQ model with perishable items and price sensitive demand. This framework employs n non-identical cycles in the unit time, during which the product quality deteriorates and the demand is a function of price in each period. The goal is to choose prices in each period so as to maximize revenue over the entire cycle. The framework addresses the question of optimal order quantities and optimal prices, however it does not consider holding multiple generations of the same product.

Other important price setting work includes Rosenberg's [15] optimal price and inventory decisions for a monopolistic firm compared against return on inventory investment and Porteus' [14] work with reducing inventory set up costs in an EOQ framework to reduce inventory operating cost. Ladany and Sternlieb [9] study the interaction between EOQ and marketing policies with product selling price playing a role as a driver for demand. Other work has been done on multi-period models and lot sizing, however it is generally unrelated to our work as this body of literature is concerned with methods to deal with variable and uncertain demand.

1.3 Proposed EOQ Extensions

We extend the EOQ framework in two important ways. First we consider a setting in which there are two generations of the same product: old and new. We show how to explicitly solve for the optimal order quantities and then prove that when using a total cost criterion, we never prefer to have two generations. We switch from a total cost criterion to a profit maximization criterion and try to maximize profit over the cycle thereby allowing retailers to set order quantity and price. To prevent retailers from setting prices arbitrarily high, we make demand a function of price. We solve the model for a single generation, provide conditions for the existence of a unique optimal solution, and explain why these conditions are satisfied by real inventory systems.

We then formulate the framework for two products, allowing the retailer to set prices for both generations and the order quantity optimally. The general demand setup allows for own and cross price effects on the demand for products. We provide conditions for the existence of a unique optimal solution, and explain why these conditions are realistic in practise. When we compare the profit maximization framework across the one and two generation models, we find that when demand substitution between generations is high, single generation retailers are able to capture more of the demand for older products. The higher prices charged for products coupled with a better inventory cost structure make the the single generation setting the preferred choice in the presence of high demand substitution. We find that when demand substitution is low it is more profitable to have two generations instead of one. Lower demand substitution implies that the single generation retailer neglects the revenue available from old product, whereas that revenue is captured fully by the two generation setting.

We consider the impact of demand substitution on product prices, order quantities, and cycle length. The comparison of one and two generation settings is sensitive to the choice of price for old products, and we explain the consequences of this sensitivity. We show how to extend the model to incorporate utility functions and thus connect the link between consumer utility for products with the optimal prices and order quantities of that product.

The rest of this thesis is organized as follows: in Chapter 2, we first review the classical EOQ model and then develop the two generation EOQ model. We then compare the two models and show definitively that we prefer a single generation under the total cost criterion. In Chapter 3, we develop the profit maximization model with price setting for one generation and then extend the model to a two generation setting. We then use a numerical example to compare the two models across demand substitution and the price of old products. In Chapter 4, we show how to extend the two generation price setting model to incorporate utility models, specifically the Dixit-Stiglitz model and the Mussa-Rosen model, and provide a general extension to our model from one to n products. Chapter 5 concludes and offers some remarks on the possible direction of future research with our framework.

Chapter 2

Classical EOQ and Two Product EOQ

In this chapter we first review the classical single generation EOQ model. This EOQ setting assumes that there is only a single type of product in inventory available for consumers. We challenge this notion by observing that real inventory systems often have two or more generations of the same product (e.g. multiple years of the same model of car on a car lot). We introduce a two generation EOQ model that allows the retailer to have two generations of the same product: an old product and a new product. We compare the classical EOQ model to the new two generation EOQ model to find conditions under which retailers might prefer two generations of products over one and vice versa.

2.1 Classical EOQ

The classical EOQ model deals with a single identical product and tries to minimize the total cost of the inventory system over a unit time (e.g. per year). To accomplish this, the model assumes the following [18]:

1. The demand rate is constant and deterministic.
2. The order quantity need not be an integral number of units, and there are no minimum or maximum restrictions on size.
3. The unit variable cost does not depend on the replenishment quantity; in particular, there are no discounts in either the unit purchase cost or the unit transportation cost.
4. The cost factors do not change appreciably with time; in particular, inflation is at a low level.
5. The item is treated entirely independently of other items; that is, benefits from joint review or replenishment do not exist or are simply ignored.

6 The replenishment lead time is of zero duration, extension to a known nonzero duration creates no problem.

7. No shortages are allowed.
8. The entire order quantity is delivered at the same time.
9. The planning horizon (unit time) is very long. In other words, we assume that all parameters will continue at the same values for a long time.

Figure 2.1 shows the inventory position $I(t)$, where q units of inventory are ordered, arrive instantaneously, and are then consumed at demand rate D. Thus the inventory position is $I(t) = q - Dt$. As soon as all of the inventory is depleted, the next batch of inventory arrives and replenishes the stock. The time between successive inventory arrivals is denoted by the cycle T with inventory replenishment occurring at T, $2T$, $3T$ and so on, since all the cycles in the planning horizon are identical.

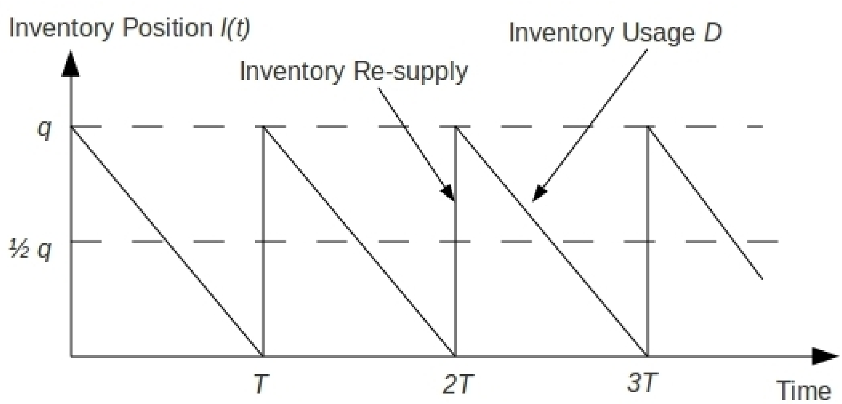


Figure 2.1: Inventory Position $I(t)$ for a single product.

In a simple inventory system the goal is the minimize the total cost of three factors: the cost of ordering inventory, the average cost of holding inventory and the purchase cost of inventory. The cost of ordering inventory is the order cost K multiplied by the frequency of orders during the unit time $\overline{OF}$. The average inventory holding cost is given by the unit cost of holding inventory h multiplied by the average inventory over the cycle $\overline{I}$. The purchase cost of inventory is simply the cost per unit of inventory c multiplied by the demand rate during the unit time D. Thus the total cost over a single cycle is $TC = K\overline{OF} + h\overline{I} + Dc$.

The order frequency is $\overline{OF} = \frac{1}{T}$ and is the number of cycles during the unit time. Each cycle ends when all of the inventory has been exhausted, or when $I(T) = 0$. This implies that $I(T) = q - DT = 0$, such that $T = \frac{q}{D}$. Hence the order frequency is $\overline{OF} = \frac{D}{q}$. The average inventory cost over the cycle is given by the integral $\overline{I} = \frac{1}{T}\int_0^T I(t)dt$, with result $\overline{I} = \frac{q}{2}$. Therefore the total cost TC is given by

$$TC(q) = \frac{KD}{q} + \frac{hq}{2} + Dc \tag{2.1}$$

where

D	= demand rate, per unit time
c	= cost per unit, per unit time
h	= holding cost per unit, per unit time
K	= ordering cost per order
q	= order quantity

We can examine the behaviour of each component of (2.1). The order cost for the unit time decreases in q since fewer orders need to be made if larger quantities are ordered. The inventory holding cost increases in q, since larger orders obviously imply a larger inventory. The purchase cost is a necessary cost to the inventory system, but plays no direct role in determining the order quantity in this basic EOQ model. The order cost and the inventory holding cost shown in Figure 2.2 graphed against the order quantity. The variable portion of the total cost, which is the sum of the order cost and the inventory holding cost, follows a convex U-shape with a defined minimum. The point at which the order cost and the inventory holding cost lines cross is also the point, and associated order quantity q^*, which minimizes the total cost.

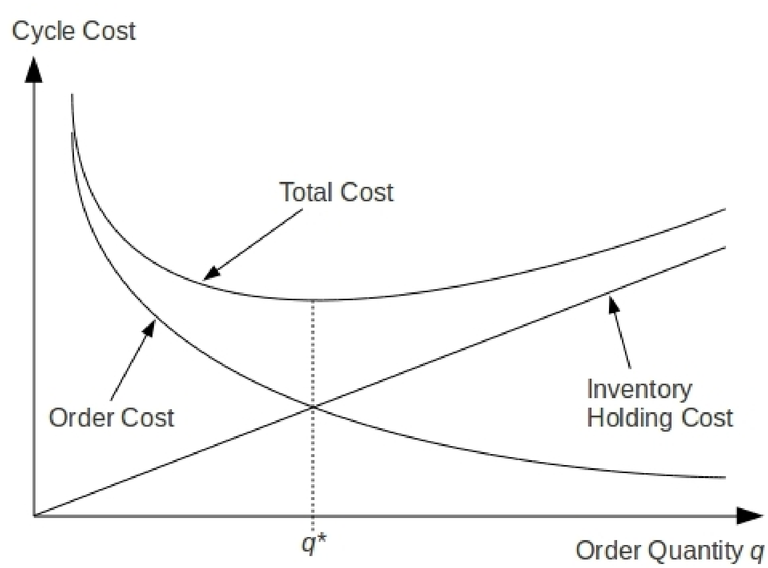


Figure 2.2: Total cost vs. order quantity.

Solving (2.1) entails taking the first derivative $\frac{dTC}{dq}$, setting the result to zero, and solving for q. The value of q which minimizes total cost is then

$$q^* = \sqrt{\frac{2KD}{h}}. \tag{2.2}$$

Substituting q^* back into (2.1) and simplifying gives the minimum total cost

$$TC^* = \sqrt{2hKD} + Dc. \tag{2.3}$$

The existence and uniqueness of the minimum can be verified by way of the second order condition

$$\frac{d^2TC}{dq^2} = \frac{KD}{q^3}. \tag{2.4}$$

Equation (2.4) is strictly positive for all values of $q > 0$ which implies that $\frac{dTC}{dq}$ is strictly increasing and that function in (2.1) achieves a minimum at q^*.

In the next section we develop the two generation EOQ model and follow the same procedure employed in the development of the classical EOQ model. The notions of order cost, inventory holding cost and total cost are important to both models. Further, the shapes of the graphs of these functions and their functional forms turn out to be remarkably similar between the two models.

2.2 Two Generation EOQ

The motivation for the two generation EOQ model comes from observing that in practise retailers often stock more than one generation of the same product. Retailers order a batch of a product, and before that product is entirely depleted they may order a new batch of products. It may be beneficial for a retailer to hold new products and old products. We develop a framework to investigate these choices.

In this model, two generations are considered; a "new" product q_1 which is ordered at the beginning of the current cycle and is consumed at a rate D_1 leaving behind some amount at the end of the current cycle, and an "old" product q_2 which is the remaining product from the previous cycle consumed at a rate D_2 to be exhausted at the end of the current cycle. The situation is depicted in Figure 2.3, and considers an infinite time horizon as in the classical EOQ. Exhaustion at the end of the cycle establishes a clear end cycle condition as in the classical EOQ model, and ensures that the retailer has two generations throughout each cycle during the entire unit time for consumers to choose from.

The inventory position for new products is $I_1(t) = q_1 - D_1 t$. The decision to be made is how much of q_1 to order at the beginning of the cycle T . At the end of T there are q_2

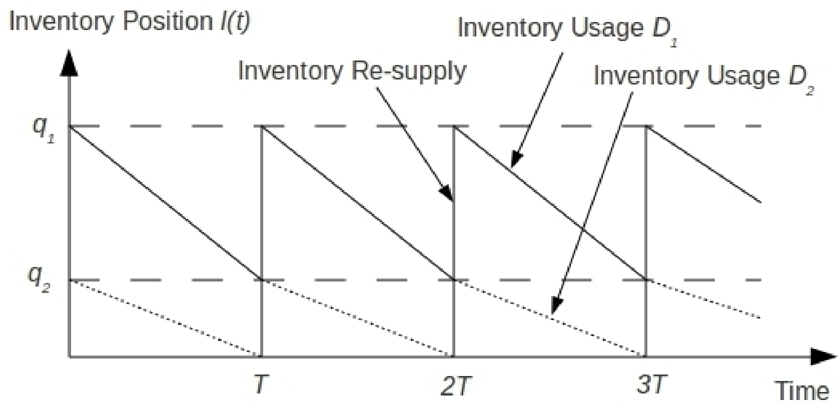


Figure 2.3: Inventory Positions for two products.

units of new product left, which implies an end cycle condition $I_1(T) = q_2 = q_1 - D_1T$. Thus the number of new products to order at the beginning of the period is $q_1 = q_2 + D_1T$.

Next, the inventory position for old products is given by $I_2(t) = q_2 - D_2t$. The value q_2 is determined by the constraint that all old products must be exhausted by the end of the cycle which implies the end cycle condition $I_2(T) = 0 = q_2 - D_2T$. Using the old product end cycle condition, we find that the cycle length is given by $T = \frac{q_2}{D_2}$. We can then rewrite $q_1 = q_2m$ where $m = (1 + \frac{D_1}{D_2})$.

As with the classical EOQ, total cost is given by the sum of order costs, inventory holding costs and purchase cost during the unit time: $TC = K\overline{OF} + h\overline{I} + Dc$. The order frequency is $\overline{OF} = \frac{1}{T} = \frac{D_2}{q_2}$. The average inventory is $\overline{I} = \overline{I_1} + \overline{I_2}$, where $\overline{I_1} = \frac{q_1+q_2}{2}$ and $\overline{I_2} = \frac{q_2}{2}$. Upon substituting for q_1, we have $\overline{I} = \frac{q_2(m+2)}{2}$. Therefore the total cost TC is given by

$$TC(q_2) = \frac{KD_2}{q_2} + \frac{hq_2(m+2)}{2} + (D_1 + D_2)c. \tag{2.5}$$

where

D_1 = demand rate per unit time for new products
D_2 = demand rate per unit time for old products
q_1 = order quantity for new products
q_2 = period starting quantity quantity of old products
m = $(1 + \frac{D_1}{D_2})$

Equations (2.5) and (2.1) are remarkably similar in their form. In fact, if we were to graph the order cost and inventory holding cost components of (2.5) against quantity q_2 we would have a picture similar to Figure 2.2. Taking the derivative with respect to q_2 and applying the first order condition $\frac{dTC}{dq} = 0$ yields

$$q_2^* = \sqrt{\frac{2KD_2}{h}}\sqrt{\frac{1}{m+2}}. \tag{2.6}$$

Solving for q_2^* in (2.6) allows us to determine the true quantities of interest; the optimal order quantity of new products q_1^* and the optimal total cost TC^* which are

$$q_1^* = \sqrt{\frac{2KD_2}{h}}\sqrt{\frac{m^2}{m+2}}, \tag{2.7}$$

$$TC^* = \sqrt{2hKD_2}\sqrt{m+2} + (D_1 + D_2)c. \tag{2.8}$$

Equations (2.6) through (2.8) have a similar form to equations (2.2) and (2.3).

The existence and uniqueness of the minimum can be verified by way of the second order condition

$$\frac{d^2TC}{dq_2^2} = \frac{KD_2}{q_2^3}. \tag{2.9}$$

Equation (2.9) is strictly positive for all values of $q_2 > 0$ which implies that $\frac{dTC}{dq}$ is strictly increasing and that function in (2.5) achieves a minimum at q_2^*. We also note that this is true for all $D_2 > 0$.

There are two interesting cases with the two generation EOQ model that are worth mentioning: the case where $D_1 = 0$ and the case where $D_2 = 0$. When $D_1 = 0$, the model specifies the ordering of q_2 new products at the beginning of the cycle and holding them for the entire cycle so that there are q_2 old products at the beginning of the next cycle. Clearly, ordering new products and holding them when there is no demand for them is a sub-optimal policy, but we question the usefulness of employing a two generation model if there is only demand for a single generation. The other interesting case is when $D_2 = 0$. In this case, we take $\lim_{D_2 \to 0}$ of equations (2.6) through (2.8). Equation (2.6) reduces to $q_2 = 0$, while (2.7) and (2.8) become (2.2) and (2.3) respectively. Hence, in the presence of zero demand for old products, the two generation EOQ model neatly reduces to the classical EOQ model.

2.3 Classical vs. Two Generation EOQ

Having developed a framework under which a retailer is able to stock two different generations of the same product, we pose the following question: under what conditions would a retailer choose such a strategy? In the two generation setting there is demand for new and old goods. In the single generation setting, there is only one good, the new good. How does the one generation model compare to the two generation model when there is demand for two goods? The goal is to compare the total cost of having a single generation against the total cost of having two generations.

We begin by assuming that in the presence of only one new good some fraction of demand for old goods would be shifted to new goods. Specifically, let $D = D_1 + \alpha D_2$, where $0 \leq \alpha \leq 1$ is the fraction of old demand that would shift in a one generation setting ($\alpha = 1$ corresponds to full substitution of demand). By constructing the ratio $\frac{TC^*_{One}}{TC^*_{Two}}$, which is (2.3) divided by (2.8), we find that if $\frac{TC^*_{One}}{TC^*_{Two}} < 1$ we prefer a one generation setting. Otherwise we prefer a two generation setting. This ratio is

$$\frac{TC^*_{One}}{TC^*_{Two}} = \frac{\sqrt{2hK(D_1 + \alpha D_2)} + (D_1 + \alpha D_2)c}{\sqrt{2hK(D_1 + 3D_2)} + (D_1 + D_2)c}. \tag{2.10}$$

From (2.10) it is clear by inspection that $\frac{TC^*_{One}}{TC^*_{Two}} < 1$ for all values of α, and only when $D_2 = 0$ does $\frac{TC^*_{One}}{TC^*_{Two}} = 1$. This result implies that from an inventory cost perspective, we always prefer a single generation setting and that a retailer has no incentive to stock multiple generations of the same product.

Our motivation for developing a two generation EOQ model was to explore why retailers in practise offer two generations of the same product. If the total cost over the unit time of holding two generations is at best the same as holding only one generation, and otherwise worse, we suspect that total cost might not be the right criterion for comparing these two settings. In the next chapter, we move from a cost minimizing framework to a profit maximizing framework to explore the problem further. Not only will the retailer be able to control the optimal order quantity, but they will also be able to control product prices.

Chapter 3

The Price Setting Retailer

In the previous chapter, we discussed the classical EOQ model and extended it to the two generation EOQ model to explore the conditions under which a retailer would prefer to stock two generations of the same product. We found that from an inventory cost perspective, a retailer never prefers to stock two generations. Yet, we observe retailers in practise stocking multiple generations of a product. Thus, we suspect that total inventory cost during the unit time might not be the right criterion to consider and we move to a profit maximization setting.

First, we examine a retailer that stocks one generation and we create a profit function that considers the profit during the unit time from a single generation and the inventory cost during the unit time given by the classical EOQ model. We then repeat the formulation for a retailer stocking two generations by considering the revenue during the unit time when there are two generations and the inventory cost during the unit time given by the two generation EOQ model. We conclude by comparing the two models and show the conditions under which a retailer would prefer one versus two generations.

3.1 One Generation Price Setting Retailer

Consider a retailer who has to make two decisions: what quantity of inventory to hold to minimize the cost of holding inventory and what price to charge for the inventory to maximize revenue. We need to create a profit function that increases from higher prices. However, we also need to account for the fact that higher prices negatively impact demand and reduces the profit function. Thus we make demand a function of price.

For ease of use we will assume a linear demand rate as commonly assumed in the economic literature. Let $D = \delta - \phi p$, where $\delta \geq 0$ is the base demand (or intercept) and

$\phi \geq 0$ is the rate of substitution between price and demand (or slope). This simple set up penalizes higher prices by lowering demand. We require that $D > 0$ to maintain a realistic interpretation of demand, and this implies that $0 \leq p < \frac{\delta}{\phi}$.

The profit during the unit time is naturally broken into revenue and cost such that $\pi(p,q) = R(p) - TC(p,q)$. The revenue during the unit time is given by $R(p) = pD = p(\delta - \phi p)$, while the cost is simply given by the classical EOQ formula (which assumes $q > 0$), but with a price dependent demand rate such that $TC(p,q) = \frac{K(\delta-\phi p)}{q} + \frac{hq}{2} + (\delta - \phi p)c$. The profit function is then

$$\pi(p,q) = (p-c)(\delta - \phi p) - \frac{K(\delta - \phi p)}{q} - \frac{hq}{2}. \tag{3.1}$$

The gradient of (3.1) is

$$\nabla \pi(p,q) = \begin{bmatrix} \delta - \phi(2p - c) + \frac{K\phi}{q} \\ \frac{K(\delta-\phi p)}{q^2} - \frac{h}{2}. \end{bmatrix} \tag{3.2}$$

Enforcing the first order condition $\nabla \pi(p^*, q^*) = 0$ and solving the two gradient equations gives values for p^* and q^*. Solving the second gradient equation for p gives $p^* = \frac{2K\delta - hq^{*2}}{2K\phi}$, and substituting this into the first equation leads to a third order polynomial $hq^3 - (\delta - \phi c)Kq + K^2\phi = 0$. One of the three roots is real (the other two are a complex conjugate pair). The structure of the real root contains higher order radicals and evaluating it with the problem parameters can also yield a complex solution. We instead seek a computational solution to the gradient equations.

An optimal solution (extreme point) is guaranteed to exist only if certain second order conditions hold. Specifically, the determinant of the Hessian must be positive and the top left entry must be negative.

Proposition 1: *Consider the one generation price setting retailer with demand function $D = \delta - \phi p$. There exists a unique solution (p^*, q^*) that maximizes the profit function (3.1).*

Proof. The Hessian of the profit function $H[\pi(p,q)]$ is given by

$$H[\pi(p,q)] = \begin{bmatrix} -2\phi & -\frac{K\phi}{q^2} \\ -\frac{K\phi}{q^2} & -\frac{2K(\delta-\phi p)}{q^3} \end{bmatrix}. \tag{3.3}$$

Since $\phi > 0$ we are guaranteed that the top left entry is negative. The determinant of the Hessian is

$$|H[\pi(p,q)]| = -\frac{K\phi(K\phi - 4q(\delta - \phi p))}{q^4}. \tag{3.4}$$

As long as $|H[\pi(p,q)]| > 0$, we are guaranteed that there exists an optimal solution, and this can be achieved if $K\phi < 4q(\delta - \phi p)$. Equivalently, in (3.4) we require

$$(\delta - \phi p) > \frac{K\phi}{4q}. \tag{3.5}$$

The above inequality does not specify a solution: it cannot be solved for q^* when given p^*; similarly it cannot be solved for p^* when given q^*. Inequality (3.5) states that for a unique solution to exist, demand must be greater than 0 (since $q > 0$), and consequently $p < \frac{\delta}{\phi}$. Positive demand necessarily exists for real inventory systems (without returns) and is assumed by our model. Thus, there will always exist an optimal solution (p^*, q^*) to the one generation price setting retailer problem over the set $\{q > 0 \cup 0 \leq p < \frac{\delta}{\phi}\}$ that maximizes the profit function (3.1). □

We have now developed a price setting model that a retailer could employ when stocking a single generation. In the next section, we develop a price setting model that a retailer would face when stocking two generations of the same product.

3.2 Two Generation Price Setting Retailer

Having developed a price setting model applicable to a retailer stocking only a single generation, we repeat the process for a retailer stocking two generations. Our goal is to compare these models and determine which strategy retailers prefer.

The retailer now has to maximize profit with two generations present. In establishing a relationship between prices and demand, we need to ensure that increasing the price of a product results in a decrease in the demand of that product. We might also like that increasing the price of new products results in an increase in demand for old products.[1] Let demand for new products be $D_1(p_1, p_2) = \delta_1 - \phi_1 p_1 + \phi_{12} p_2$ and demand for old products be $D_2(p_1, p_2) = \delta_2 - \phi_2 p_2 + \phi_{21} p_1$. The parameters $\phi_i \geq 0$ and $\phi_{ij} \geq 0$ allow flexibility in specifying the slopes of the demand functions and have necessary dimensions, while δ_1 and δ_2 are simply intercepts specified as a base "demand rate per unit time". For simplicity of notation, we occasionally refer to the demand functions $D_1(p_1, p_2)$ and $D_2(p_1, p_2)$ simply as D_1 and D_2 respectively. We also require that $D_1(p_1, p_2) > 0$ and $D_2(p_1, p_2) > 0$ to maintain a realistic interpretation of demand. This bounds the prices as follows: $0 \leq p_1 < \frac{\delta_1 + \phi_{12} p_2}{\phi_1}$ and $0 \leq p_2 < \frac{\delta_2 + \phi_{21} p_1}{\phi_2}$.

[1] We ignore pathological examples such as "Giffen" goods, however the framework can be easily modified to accommodate them.

The profit during the unit time is $\pi(p_1,p_2,q_2) = R(p_1,p_2) - TC(p_1,p_2,q_2)$. The revenue contribution is $R(p_1,p_2) = p_1D_1 + p_2D_2$ and the cost contribution is the two product EOQ model (2.5) with price dependent demand rates. Thus the full profit function is

$$\begin{aligned}\pi(p_1,p_2,q_2) = (p_1-c)(\delta_1-\phi_1p_1+\phi_{12}p_2) + (p_2-c)(\delta_2-\phi_2p_2+\phi_{21}p_1) \\ -\tfrac{K(\delta_2-\phi_2p_2+\phi_{21}p_1)}{q_2} - \tfrac{hq_2}{2}\left(3+\tfrac{(\delta_1-\phi_1p_1+\phi_{12}p_2)}{(\delta_2-\phi_2p_2+\phi_{21}p_1)}\right).\end{aligned} \tag{3.6}$$

The gradient equations are

$$\nabla\pi(p_1,p_2,q_2) = \begin{bmatrix} D_1-(p_1-c)\phi_1+(p_2-c)\phi_{21}-\frac{K\phi_{21}}{q_2}+\frac{hq_2}{2}\left(\frac{k_1}{D_2}+\frac{\phi_{21}D_1}{D_2^2}\right) \\ D_2-(p_2-c)\phi_2+(p_1-c)\phi_2-\frac{K\phi_2}{q_2}-\frac{hq_2}{2}\left(\frac{k_{12}}{D_2}+\frac{\phi_2D_1}{D_2^2}\right) \\ \frac{KD_2}{q_2^2}-\frac{h}{2}\left(3+\frac{D_1}{D_2}\right) \end{bmatrix}. \tag{3.7}$$

Enforcing the first order condition $\nabla\pi(p_1^*,p_2^*,q_2^*) = 0$ and solving the three gradient equations computationally gives the values for p_1^*, p_2^* and q_2^*. We can then solve for the value of π^* and q_1^* which is the order quantity at the beginning of the cycle.

Establishing general conditions for optimality in the two product case is more challenging than in the one product case, since there is no natural analog to the process we used in the one product case. But it would be nice to find conditions under which an optimal solution is guaranteed to exist. One strategy might be to compute the Hessian $H[\pi(p_1,p_2,q_2)]$. If the diagonal entries of the Hessian are negative ($H[\pi_{ii}] < 0$), which they are, and diagonal dominance of the Hessian holds ($|H[\pi_{ii}]| > \sum_{j\neq i}|H[\pi_{ij}]|$) then all of the eigenvalues of $H[\pi]$ are negative. This is a convenient result, but ensuring diagonal dominance of the Hessian turns out to be incredibly difficult.

Our approach will be to consider the profit function and apply the following theorem:

Theorem ([1], pg. 54): *A non-negative linear combination of concave functions is also a concave function. That is, if $f^i(\cdot): X \to \mathfrak{R}, i = 1,2,...,m$, are concave functions on a convex subset $X \subset \mathfrak{R}^n$, then $f(\boldsymbol{x}) \equiv \sum_{i=1}^m \alpha_i f^i(\boldsymbol{x})$ where $\alpha_i \in \mathfrak{R}_+, i = 1,2,...,m$, is also a concave function on $X \subset \mathfrak{R}^n$.*

To find conditions for optimality, we return to the profit function $\pi(p_1,p_2,q_2)$ and break it into its separate parts $R(p_1,p_2)$ and $TC(p_1,p_2,q_2)$. We then establish concavity conditions for each part, and consequently the whole profit function.

Proposition 2: *Consider the two generation price setting retailer with demand functions $D_1(p_1,p_2)$ and $D_2(p_1,p_2)$. If $\phi_1 > 0$ and $4\phi_1\phi_2 > (\phi_{12}+\phi_{21})^2$, then there exists a unique solution (p_1^*,p_2^*,q_2^*) that maximizes the profit function (3.6).*

Proof. The contribution from revenue is $R(p_1, p_2) = p_1(\delta_1 - \phi_1 p_1 + \phi_{12} p_2) + p_2(\delta_2 - \phi_2 p_2 + \phi_{21} p_1)$. The Hessian is

$$H[R(p_1, p_2)] = \begin{bmatrix} -2\phi_1 & \phi_{12} + \phi_{21} \\ \phi_{12} + \phi_{21} & -2\phi_2 \end{bmatrix}. \tag{3.8}$$

The determinant of the Hessian is $|H[R(p_1, p_2)]| = 4\phi_1\phi_2 - (\phi_{12} + \phi_{21})^2$. For concavity of the revenue contribution $R(p_1, p_2)$ we require $-2\phi_1 < 0$ and $|H[R(p_1, p_2)]| > 0$, or equivalently

$$\begin{gathered} \phi_1 > 0, \\ 4\phi_1\phi_2 > (\phi_{12} + \phi_{21})^2. \end{gathered} \tag{3.9}$$

The cost contribution is

$$TC(p_1, p_2, q_2) = \frac{KD_2(p_1, p_2)}{q_2} + \frac{hq_2\left(3 + \frac{D_1(p_1,p_2)}{D_2(p_1,p_2)}\right)}{2} + (D_1(p_1, p_2) + D_2(p_1, p_2))c. \tag{3.10}$$

$TC(p_1, p_2, q_2)$ includes the demand functions $D_1(p_1, p_2)$ and $D_2(p_1, p_2)$. We have already specified that $D_1(p_1, p_2) > 0$ and $D_2(p_1, p_2) > 0$. Once selected, the demands are constant from the point of view of $TC(p_1, p_2, q_2)$, which then simply becomes $TC(q_2)$ from (2.5). The choices of p_1 and p_2 can be thought of as defining a family of $TC(q_2)$ functions. Every function in this family exhibits similar behaviour; they are all convex (as shown before with (2.9)). Among them, there will be a p_1^* and p_2^* (optimal to $\pi(p_1, p_2, q_2)$) that define a particular instance of $TC(q_2)$. Therefore positive demand guarantees strict convexity of $TC(p_1, p_2, q_2)$, and strict concavity of $-TC(p_1, p_2, q_2)$.

We conclude by applying the theorem: the profit function $\pi(p_1, p_2, q_2) = R(p_1, p_2) + (-TC(p_1, p_2, q_2))$ is concave if (3.9) holds. Thus, there will always exist an optimal solution (p_1^*, p_2^*, q_2^*) to the two generation price setting retailer problem over the set $\{q_2 > 0 \cup 0 \leq p_1 < \frac{\delta_1 + \phi_{12} p_2}{\phi_1} \cup 0 \leq p_2 < \frac{\delta_2 + \phi_{21} p_1}{\phi_2}\}$ that maximizes the profit function (3.6). ☐

The price setting model allows us to set prices, but it does not require us to. We have the flexibility to consider a setup where the price of new products is known and the retailer can then choose the price of old products. Or we can fix the price of old products and allow the retailer to choose the price of new products. In the next section we will compare both the one and two generation price setting models and see whether or not retailers prefer to stock one or two generations of the same product.

3.3 One vs. Two Generation Price Setting Retailers - Numerical Illustration

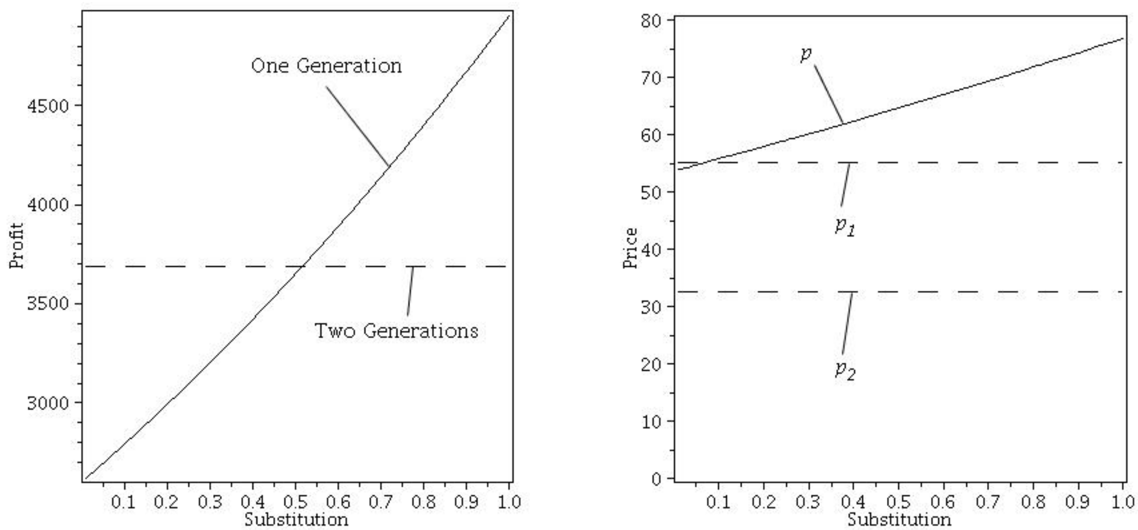
Now that we have both one and two generation price setting models that consider the profit criterion, we can begin to shed some light on our original observation: why in practise do retailers sometimes stock two generations of the same product?

We will assume, as we did when comparing EOQ models, that in the presence of only one new good, some fraction of demand for old goods will shift to new goods. Specifically, let $D(p_1) = D_1(p_1, p_2) + \alpha D_2(p_1, p_2)$ with $0 \leq \alpha \leq 1$. Depending on how much demand for old products is captured when the retailer only stocks new products, it may or may not be more profitable to have two product generations; the choice of α will affect the decision. Additionally, the price of old products cannot be determined by the single generation setting, so we must set it as a parameter for the single generation model; the choice of p_2 will affect the decision. Finally, our analysis is valid only if a retailer selects one of the two strategies for the unit time; we will not permit the retailer to switch inventory strategies over the planning horizon.

Let $K = 100$, $h = 0.2$, $c = 2$, $\delta_1 = 100$, $\delta_2 = 80$, $\phi_1 = 1$, $\phi_2 = 1.5$, $\phi_{12} = 0.1$ and $\phi_{21} = 0.15$. The optimal solution to the two generation price setting retailer problem is $p_1^* = 55.05$, $p_2^* = 32.52$, $q_1^* = 216.38$, $q_2^* = 91.82$ and $\pi^* = 3684.22$. Now we consider the one generation price setting model with the same parameters. We shall investigate how the choice α changes the profit, price, and order quantity, all while taking $p_2 = p_2^* = 32.52$ from the two generation setting.

Figure 3.1(a) shows how profit changes as α changes. As the substitution α increases, we observe the profit in the one generation setting increases as well, since increasing α implies that more of the demand for old products can be captured by the retailer even when there are no old products offered for sale. When α is greater than about 0.52, it is more profitable to have only new products. If the substitution is high, the profit in the one generation setting surpasses the two generation setting for two reasons. First, more consumers are purchasing the new product, which has a higher price than the old product, and thus provides more revenue per unit sold. Second, the inventory costs are lower for the one generation retailer, because they are not holding new products during the current cycle so that they have old products to sell in the next cycle. The combination of these two effects eventually allows the one generation retailer to surpass the profit of the two generation retailer when demand substitution is high.

We observe that, in this scenario, when α is less than about 0.52, it is more profitable to have both old and new products. If demand substitution is low, the profit from the one generation setting is lower than the profit from the two generation setting. This is due to the fact that the single generation retailer neglects much of the revenue possible from the demand for old products. Despite the higher inventory costs, the two generation setting captures all the revenue available from the demand for both products, and is the preferred strategy when demand substitution is low. Note that the profit from two generations is not a function of α and remains constant at $\pi^* = 3684.22$.



(a) Profit vs. Substitution α.

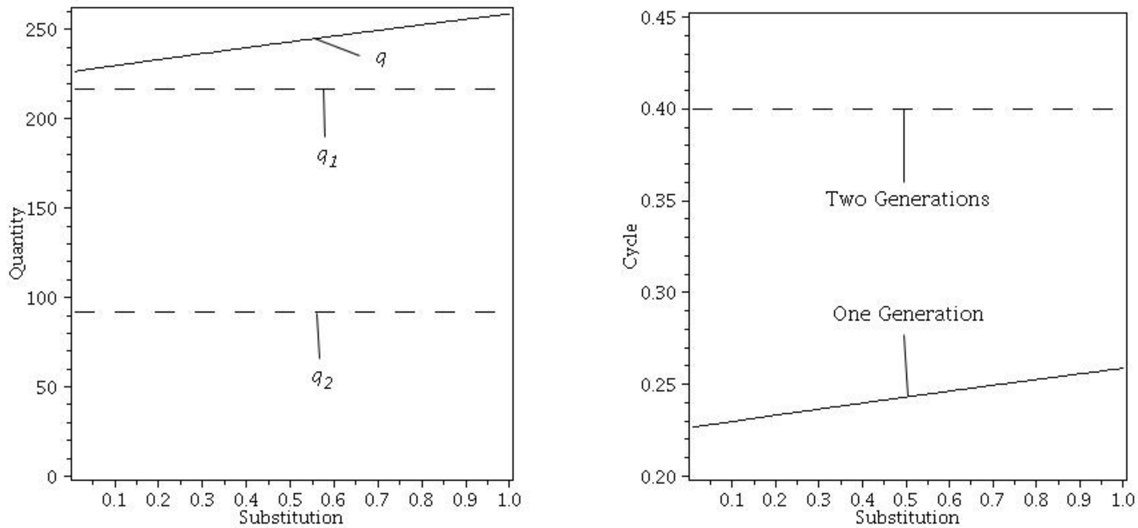
(b) Prices vs. Substitution α.

Figure 3.1: Profit and Prices vary with Substitution, $p_2 = 32.52$, $K = 100$, $h = 0.2$, $c = 2$, $\delta_1 = 100$, $\delta_2 = 80$, $\phi_1 = 1$, $\phi_2 = 1.5$, $\phi_{12} = 0.1$ and $\phi_{21} = 0.15$.

Figure 3.1(b) shows how the price varies in the one generation model as substitution α changes. The price in the single generation setting is almost always higher than both of the prices in the two generation setting, primarily because substitution means the retailer can afford to charge higher prices that reduce demand but improve profit. Note that even when all of the demand is captured, the price in the one generation setting is not the sum of the prices in the two generation setting. Ever higher prices eventually destroy demand and reduce profit, so there is a trade-off when increasing the price. Additionally, the model cannot violate the positive demand condition given by (3.5).

Figure 3.2(a) shows how the quantity varies in the one generation model as substitution α changes. Again, note that even when all of the demand is captured, the quantity in the one generation setting is not the sum of the quantities in the two generation setting. This is because the rising order quantity incurs higher inventory holding costs, so the model adjusts by increasing the order frequency (reducing the cycle) to balance the total cost during the unit time, which is shown in Figure 3.2(b).

Now we consider the one generation price setting model with the same initial parameters. We shall investigate how the choice p_2 changes the profit, price, and order quantity,



(a) Quantity vs. Substitution α, $p_2 = 32.52$.

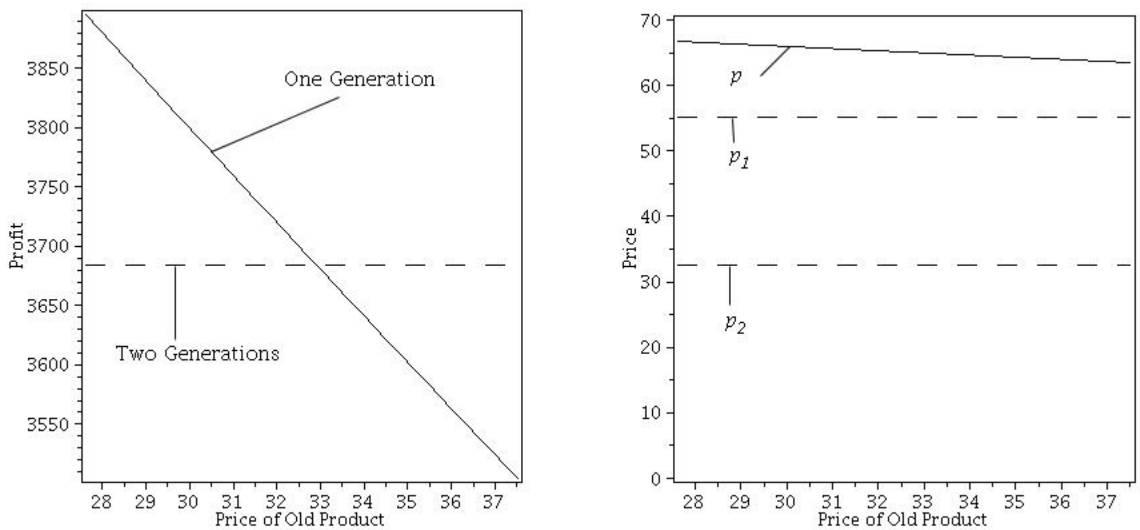
(b) Cycle vs. Substitution α, $p_2 = 32.52$.

Figure 3.2: Quantities and Cycle vary with Substitution, $p_2 = 32.52$, $K = 100$, $h = 0.2$, $c = 2$, $\delta_1 = 100$, $\delta_2 = 80$, $\phi_1 = 1$, $\phi_2 = 1.5$, $\phi_{12} = 0.1$ and $\phi_{21} = 0.15$

all while taking $\alpha = 0.52$, the approximate indifference point in the previous analysis. We expect that when $p_2 = 32.52$ we will be indifferent to either setting. We observe in Figure 3.3(a) that the profit is sensitive to p_2; lower prices for older products result in a preference toward stocking only new products, while higher prices for older products result in a preference for stocking both generations. This result is intuitive: as the price of the older product increases, it becomes more profitable to stock it.

Figure 3.3(b) shows that as the price of old products increases, the price the retailer can charge for new product also declines. This result is counter intuitive, and stems from the construction of the combined single generation demand function: $D(p_1) = (\delta_1 + \alpha\delta_2) - (\phi_1 - \alpha\phi_{21})p_1 - (\alpha\phi_2 - \phi_{12})p_2$. Because of the form of the demand function, increases in the price of old products actually cause demand for new products to decline as well.[2] To counteract this, the retailer must charge lower prices for the new product to maintain demand. Consequently, $D = D_1 + \alpha D_2$ may not be the best construction to explore how the change in the the price of old products affects the price of new products, since we would expect the increase in price of old products to increase demand for new products. The order quantity in the single generation setting declines as the price increases. This is because the higher prices are reducing demand, and so smaller orders (Figure 3.4(a)) are

[2]This is not true for all α. If $\alpha > \frac{\phi_{12}}{\phi_2}$ then increases in p_2 cause decreases in $D(p_1)$. However, if $\alpha < \frac{\phi_{12}}{\phi_2}$ increases in p_2 cause increases in $D(p_1)$. If $\alpha = \frac{\phi_{12}}{\phi_2}$ changes in p_2 have no effect on $D(p_1)$.



(a) Profit vs. Price of Old Product p_2.

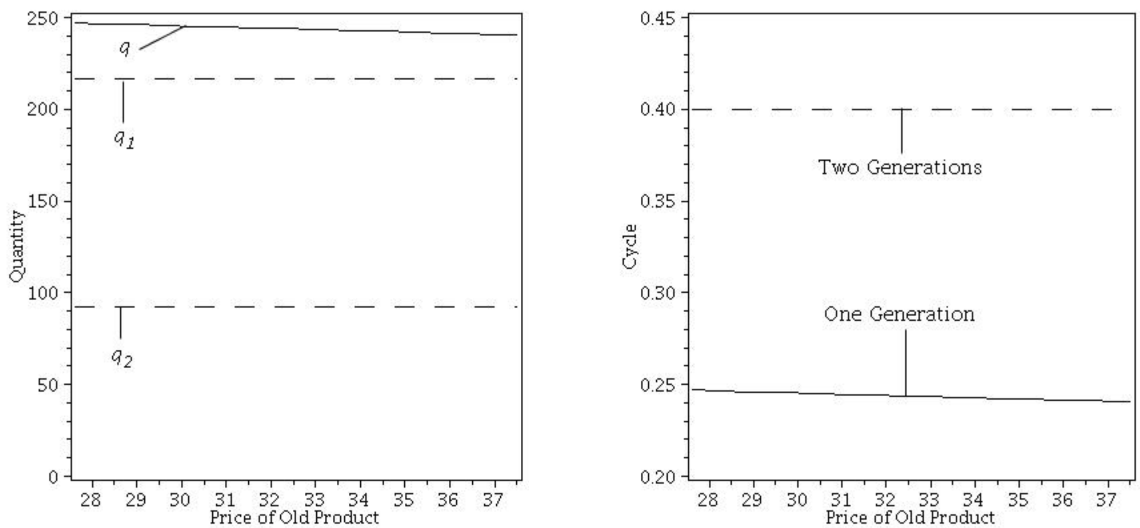
(b) Prices vs. Price of Old Product p_2.

Figure 3.3: Profit and Prices vary with Price of Old Product, $\alpha = 0.52$, $K = 100$, $h = 0.2$, $c = 2$, $\delta_1 = 100$, $\delta_2 = 80$, $\phi_1 = 1$, $\phi_2 = 1.5$, $\phi_{12} = 0.1$ and $\phi_{21} = 0.15$.

made less frequently (Figure 3.4(b)).

In summary, the choice of α and p_2 clearly affect the results when comparing the two price setting models. Increasing demand substitution α eventually makes holding only a single generation a more profitable choice. When consumers of old products are unwilling to purchase new products, the two generation setting is the preferred choice. Increasing the price of old products p_2 eventually makes holding two generations the more profitable choice. Methods for estimating α and justifications for using a certain p_2 when comparing these two frameworks are outside the scope of our discussion; we simply state that care must be taken in choosing these parameters.

We have demonstrated that in a profit maximization setting when retailers are allowed to choose product prices, it may be more profitable for a retailer to stock two generations of a product instead of just one. In the final chapter, we will consider a further extension to the two generation price setting model that uses consumer utility as the force behind demand, and we will generalize the two generation model to allow for more than one product type.



(a) Quantity vs. Price of Old Product p_2.

(b) Cycle vs. Price of Old Product p_2.

Figure 3.4: Quantities and Cycle vary with Price of Old Product, $\alpha = 0.52$, $K = 100$, $h = 0.2$, $c = 2$, $\delta_1 = 100$, $\delta_2 = 80$, $\phi_1 = 1$, $\phi_2 = 1.5$, $\phi_{12} = 0.1$ and $\phi_{21} = 0.15$.

Chapter 4

Utility Function Extensions and the General Case

The motivation for the previous chapters was to find out why in practise retailers may choose to stock two generations of the same product. Now that we have shown clearly that there are cases when a retailer will prefer to stock two generations, we further extend the model for the two generation price setting retailer. In this chapter we show how to use consumer utility to derive the demand for consumers. We view this as the next natural step in the process of linking inventory quantities to consumer behaviour. We then present the general n-product two generation profit maximization model - the natural extension to to the two generation price setting retailer - and give conditions for the existence and uniqueness of the solution.

4.1 Price Setting Retailer with Dixit-Stiglitz Utility Functions

Demand for retail goods ultimately stems from consumer utility for those goods. If we can connect consumer utility to demand, and then demand to inventory quantities, then we can better understand the link between consumer preference and inventory decisions. Using the price setting model, we will employ utility functions proposed by Dixit and Stiglitz to generate consumer demand.

The Dixit-Stiglitz [3] consumer problem employs a two variable quadratic function $U(d_1, d_2)$ that forms a concave surface specified by the parameters α_i, β_i and γ. The parameters α_i and β_i are valuation parameters for product i, while γ is a substitution parameter between products i and j. Each consumer faces the following constrained maximization problem when attempting to maximize his/her utility subject to a budget constraint:

$$\begin{array}{ll} \text{Maximize} & U(d_1, d_2) = \alpha_1 d_1 + \alpha_2 d_2 - \frac{1}{2}\beta_1 d_1^2 - \frac{1}{2}\beta_2 d_2^2 - \gamma d_1 d_2 \\ \text{subject to} & p_1 d_1 + p_2 d_2 = m \end{array} \tag{4.1}$$

where $\alpha_i > 0$, $\beta_i > 0$, $\gamma > 0$ and $m > 0$.

The method for solving this problem is to form the Lagrangian function $\mathcal{L}(d_1, d_2, \lambda) = U(d_1, d_2) + \lambda(m - p_1 d_1 - p_2 d_2)$, and solve the gradient equations $\nabla\mathcal{L}(d_1, d_2, \lambda) = 0$, which leads to

$$\begin{aligned} \tfrac{\partial L}{\partial d_1} &= \alpha_1 - \beta_1 d_1 - \gamma d_2 - \lambda p_1 = 0, \\ \tfrac{\partial L}{\partial d_2} &= \alpha_2 - \beta_2 d_1 - \gamma d_1 - \lambda p_2 = 0, \\ \tfrac{\partial L}{\partial \lambda} &= m - p_1 d_1 - p_2 d_2 = 0. \end{aligned} \tag{4.2}$$

The first two equations can be re-arranged and divided into the form $\frac{\partial U/\partial q_1}{\partial U/\partial q_2} = \frac{p_1}{p_2}$, which relates the marginal rate of substitution for the consumer to the slope of the budget constraint.[3] This ratio leads to the inverse demand curves

$$\begin{aligned} p_1 &= \alpha_1 - \beta_1 d_1 - \gamma d_2, \\ p_2 &= \alpha_2 - \beta_2 d_1 - \gamma d_1. \end{aligned} \tag{4.3}$$

The equations in (4.3) can be solved for the demands d_1 and d_2 to give

$$\begin{aligned} d_1 &= \tfrac{\alpha_1\beta_2 - \alpha_2\gamma}{\beta_1\beta_2 - \gamma^2} - \tfrac{\beta_2}{\beta_1\beta_2 - \gamma^2} p_1 + \tfrac{\gamma}{\beta_1\beta_2 - \gamma^2} p_2, \\ d_2 &= \tfrac{\alpha_2\beta_1 - \alpha_1\gamma}{\beta_1\beta_2 - \gamma^2} - \tfrac{\beta_1}{\beta_1\beta_2 - \gamma^2} p_2 + \tfrac{\gamma}{\beta_1\beta_2 - \gamma^2} p_1. \end{aligned} \tag{4.4}$$

The form of the equations in (4.4) follows the same format as the demand functions we used in the two product price setting model. We can use that same model once we make the following substitutions

$$\begin{aligned} \delta_1 &= \tfrac{\alpha_1\beta_2 - \alpha_2\gamma}{\beta_1\beta_2 - \gamma^2}, \quad \phi_1 = \tfrac{\beta_2}{\beta_1\beta_2 - \gamma^2}, \quad \phi_{12} = \tfrac{\gamma}{\beta_1\beta_2 - \gamma^2}, \\ \delta_2 &= \tfrac{\alpha_2\beta_1 - \alpha_1\gamma}{\beta_1\beta_2 - \gamma^2}, \quad \phi_2 = \tfrac{\beta_1}{\beta_1\beta_2 - \gamma^2}, \quad \phi_{21} = \tfrac{\gamma}{\beta_1\beta_2 - \gamma^2}. \end{aligned} \tag{4.5}$$

As long as $\beta_1\beta_2 > \gamma^2$, then equations (4.5) satisfy the optimality conditions (3.9) and are valid for use in the two generation price setting model. These expressions provide a concrete way of linking consumer preference to the inventory decisions of a retailer without creating any new mathematical complications. We believe this to be an area of further interesting research.

[3]Omitting the budget constraint merely removes the link between utility and a particular budget, but we still retain the information about consumer preference.

4.2 Price Setting Retailer with Mussa Rosen Income Utility Functions

In the previous section we showed how Dixit-Stiglitz utility functions could be used to generate demand for the two generation price setting retailer. We now show how to adopt the Mussa Rosen Income Utility framework.

The Mussa Rosen [12] framework assumes that consumers lie on an income distribution $0 \leq i \leq \Phi$. Each product has a utility curve, which depends on each consumer's income. Specifically, let $u_1 = i\alpha_1 - p_1$ and $u_2 = i\alpha_2 - p_2$ for new and old products respectively, where we assume that $\alpha_1 > \alpha_2$ and $p_1 > p_2$. We are interested in the situation depicted in Figure 4.1. For those consumers who have positive utility generated from either product, they demand the product with greater utility. Accordingly d_1 and d_2 denote the demand for new and old products respectively.

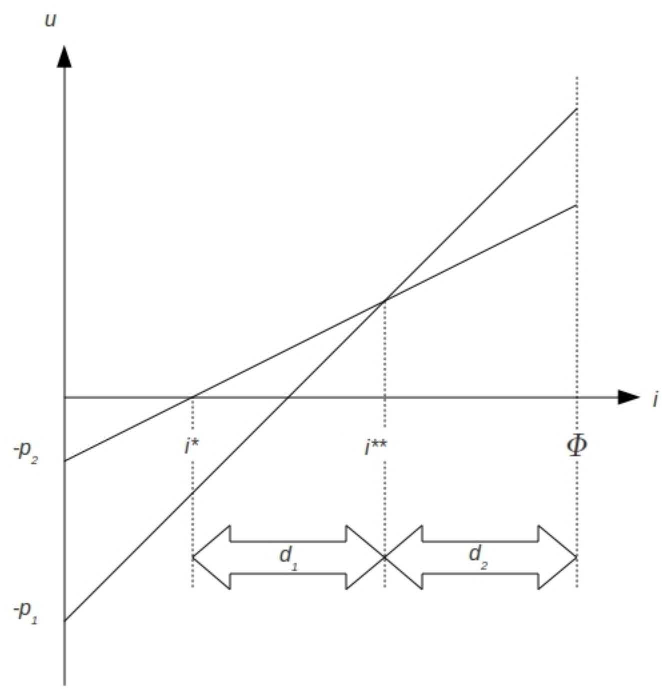


Figure 4.1: Mussa-Rosen Income-Utility curve for two products

We can solve for d_1 and d_2 explicitly in terms of the prices, since $d_1 = 1 - i^{**}$ and $d_2 = i^{**} - i^*$, by finding the intercept $i^* = \frac{p_2}{\alpha_2}$ (consumers who are indifferent to consuming old products or no product) and the crossing $i^{**} = \frac{p_1 - p_2}{\alpha_1 - \alpha_2}$ (consumers who are indifferent to consuming new products or old products). Thus, the demands are

$$\begin{aligned} d_1 &= \Phi - \tfrac{p_1-p_2}{\alpha_1-\alpha_2}, \\ d_2 &= \tfrac{p_1-p_2}{\alpha_1-\alpha_2} - \tfrac{p_2}{\alpha_2}. \end{aligned} \tag{4.6}$$

The expressions in (4.6) can be re-arranged into the following

$$\begin{aligned} d_1 &= \Phi - \tfrac{1}{\alpha_1-\alpha_2}p_1 + \tfrac{1}{\alpha_1-\alpha_2}p_2, \\ d_2 &= -\tfrac{\alpha_1}{(\alpha_1-\alpha_2)\alpha_2}p_2 + \tfrac{1}{\alpha_1-\alpha_2}p_1. \end{aligned} \tag{4.7}$$

We can then match the parameters from (4.7) to the demand parameters in the two product price setting model as follows

$$\begin{aligned} &\delta_1 = \Phi, \quad \phi_1 = \tfrac{1}{\alpha_1-\alpha_2}, \quad \phi_{12} = \tfrac{1}{\alpha_1-\alpha_2}, \\ &\delta_2 = 0, \quad \phi_2 = \tfrac{\alpha_1}{(\alpha_1-\alpha_2)\alpha_2}, \quad \phi_{21} = \tfrac{1}{\alpha_1-\alpha_2}. \end{aligned} \tag{4.8}$$

The parameters in (4.8) always satisfy the optimality conditions (3.9), so they are valid for use in the two generation price setting model. The expressions differ dramatically from what one might expect for demand parameters. For example, $\delta_2 = 0$ implies there is no base demand for old products in the model. Further, three of the four elasticities are identical. Nonetheless, the formulation satisfies the mathematical requirements of our model. Expressions (4.8) provide a concrete way to link consumer utility and income to inventory prices and quantities. If a retailer can estimate consumer income and utility for new and old products, it can determine the precise prices to set and quantities to hold to maximize profit.

4.3 n-Product Two Generation Price Setting Retailer

In Chapter 3 we introduced a price setting retailer that could choose between stocking one or two generations of the same product type (e.g., two different years of the same car model). In this section, we present a general formulation that allows a retailer to stock two generations of n different products. The goal of this formulation is to capture how interaction between demands for different types of products and across generations affect the quantities of those products that the retailer stocks.

The general situation can become complicated quite quickly. Some of the different product types may come from the same supplier, so replenishment of those products could be linked. We assume that each of the products is shipped from an independent supplier; the cost to order and replenishment of different products is independent.

The demand for each product is affected by its own price and the price of all the other product types and generations. The general demand function for product i of age j is

$$D_{ij}(p_{11}, p_{12}, p_{21}, p_{22}, \ldots, p_{n1}, p_{n2}) = \delta_{ij} + \sum_{k,l} \phi_{ijkl} p_{kl}. \tag{4.9}$$

where ϕ_{ijkl} is the elasticity or substitution between product i of age j and product k of age l. We assume $D_{ij}(p_{11}, p_{12}, p_{21}, p_{22}, \ldots, p_{n1}, p_{n2}) > 0$, with base demand $\delta_{ij} > 0$ and own price elasticity $\phi_{ijij} < 0$. The cross elasticities ϕ_{ijkl} for $kl \neq ij$ could be either positive or negative if products are substitutes (then $\phi_{ijkl} > 0$), and if products are complementary (then $\phi_{ijkl} < 0$). Each price is bounded: $0 \leq p_{ij} < \frac{\delta_{ij} - \sum_{(k,l)\neq(i,j)} \phi_{ijkl}}{\phi_{ijij}}$.

Individual profit functions for each product type follow the form of (3.6) and are $\pi_i(p_{i1}, p_{i2}, q_{i2}) = R_i(p_{11}, p_{12}, p_{21}, p_{22}, \ldots, p_{n1}, p_{n2}) - TC_i(p_{i1}, p_{i2}, q_{i2})$. The retailer's full profit function over the unit time is the sum of each individual profit function for each product

$$\pi = \sum_{i=1}^{n} \pi_i(p_{i1}, p_{i2}, q_{i2}). \tag{4.10}$$

Equation (4.10) is simply a composition of revenue functions and cost functions, all linked together by demand functions. From this, we note that the revenue contribution is

$$R(p_{11}, p_{12}, p_{21}, p_{22}, \ldots, p_{n1}, p_{n2}) = \sum_{i=1}^{n} R_i = \sum_{i=1}^{n} p_{i1} D_{i1} + p_{i2} D_{i2}. \tag{4.11}$$

Equation (4.10) can be solved by enforcing the first order condition $\nabla \pi = 0$ and solving $3n$ gradient equations for p^*_{i1}, p^*_{i2} and q^*_{i2}, $i = 1, ..., n$. We now provide conditions for the existence and uniqueness of the optimal solution.

Proposition 3: *Consider the n-product two generation price setting retailer with demand functions $D_{ij}(p_{11}, p_{12}, p_{21}, p_{22}, \ldots, p_{n1}, p_{n2})$ for $i = 1, ..., n$. If the Hessian matrix $H[R(p_{11}, p_{12}, p_{21}, p_{22}, \ldots, p_{n1}, p_{n2})]$ is negative definite, then there exists a unique solution $(p^*_{i1}, p^*_{i2}, q^*_{i2})$ for all $i = 1, ..., n$ that maximizes the profit function (4.10).*

Proof. Recall the Theorem ([1], pg. 54) from Section 3.2. We can prove that the profit function (4.10) is concave and has a unique solution if it is composed of concave functions.

The Hessian matrix $H[R(p_{11}, p_{12}, p_{21}, p_{22}, \ldots, p_{n1}, p_{n2})]$ of revenue contribution (4.11) is

$$H[R] = \begin{bmatrix} 2\phi_{1111} & \phi_{1112} + \phi_{1211} & \phi_{1121} + \phi_{2111} & \ldots & \phi_{11nn} + \phi_{nn11} \\ \phi_{1112} + \phi_{1211} & 2\phi_{1212} & \phi_{1221} + \phi_{2112} & \ldots & \phi_{12nn} + \phi_{nn12} \\ \phi_{1121} + \phi_{2111} & \phi_{1221} + \phi_{2112} & 2\phi_{2121} & \ldots & \phi_{21nn} + \phi_{nn21} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi_{11nn} + \phi_{nn11} & \phi_{12nn} + \phi_{nn12} & \phi_{21nn} + \phi_{nn21} & \ldots & 2\phi_{nnnn} \end{bmatrix} \tag{4.12}$$

$H[R]$ is composed entirely of known parameters, so determining whether the matrix is negative definite can be done computationally. If the Hessian $H[R]$ is negative definite, then the function $R(p_{11}, p_{12}, p_{21}, p_{22}, \ldots, p_{n1}, p_{n2})$ is strictly concave. Since we have specified that the demand functions $D_{i1} > 0$ and $D_{i2} > 0$, each total cost function $TC_i(p_{i1}, p_{i2}, q_{i2})$ reduces to an instance of $TC_i(q_{i2})$ (depending on the particular value of the demands), all of which must be strictly convex (as shown before with (2.9)). Thus each $-TC_i(p_{i1}, p_{i2}, q_{i2})$ is strictly concave.

We conclude by applying the Theorem ([1], pg. 54): the profit function $\pi = R(p_{11}, p_{12}, p_{21}, p_{22}, \ldots, p_{n1}, p_{n2}) - \sum_{i=1}^{n} TC_i(p_{i1}, p_{i2}, q_{i2})$ is concave if the Hessian $H[R]$ from (4.12) is negative definite. Thus, there will always exist an optimal solution (p_{ij}^*, q_{i2}^*) to the n-product two generation price setting retailer problem over the set $\{q_{i2} > 0 \cup 0 \leq p_{ij} < \frac{\delta_{ij} - \sum_{(k,l) \neq (i,j)} \phi_{ijkl}}{\phi_{ijij}}\}$ for all $i = 1, ..., n$ and $j = 1, 2$. □

The n-product two generation model allows the retailer to examine the interaction between different product types and product generations, and how it effects optimal prices and order quantities. A retailer may be a price taker for some products and have the opportunity to set prices for others; this framework is flexible enough to consider a variety of such situations and to undertake a study of how changes between product types and product ages affect overall profit and inventory decisions. We believe this extension to be an area of further interesting research.

Chapter 5

Conclusions

Our work is motivated by the observation that retailers in practise often stock multiple generations of the same product. We considered the relevant inventory management literature and concluded it did not address this specific situation. Using the classical EOQ model as a starting point, we developed a two generation EOQ model that allowed retailers to simultaneously hold new and old products over the entire planning horizon. We then showed that from a total inventory cost perspective over the planning horizon, having two generations is always more expensive than having one, and proposed to look at the problem from a total profit view point.

The profit maximization model allowed for retailers to set order quantities as well as prices, and we showed that under such a scheme there are clearly cases where a retailer would prefer to stock two generations instead of one generation and vice versa – they pick the more profitable strategy. But the more profitable strategy depends on how much demand for old products the one generation price setting retailer can capture. We showed that in situations where consumers of old products are unwilling to purchase new products, the two generation setting is the superior choice.

With a better understanding of why retailers in practise prefer two generations, we then considered what motivates demand for a product. The goal here was to link inventory decisions with consumer preferences. We showed how this might be done using Dixit-Stiglitz utility functions, as well as the Mussa Rosen income-utility framework. Within these frameworks, if a retailer can estimate consumer utility (and income), then optimal prices and quantities can be found that maxmimize the retailer's profit. Our model is robust, and we expect there are other utilty frameworks that can be adapted to it. We also considered the general n-product two generation model with price setting extension, and gave conditions for the existence and uniqueness of a solution. We think these extensions are important areas of further research.

The two generation EOQ model we developed can easily be extended to include more product generations for further investigation. We expect that a price setting model with greater than two generations would exhibit similar behaviour to the two generation price setting retailer model we discussed. Interesting questions begin to arise: are three generations better than two, and where does demand for the third product go if there is no third product – to the first generation, second, or perhaps both? We can also extend the model in other important ways to study and quantify the effect of end period scrap on total cost and profit. The classical EOQ setting is an important building block in inventory management, an we believe it can still be used to find new insight into existing problems.

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