

On Infinitesimal Inverse Spectral Geometry  
by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Eduardo Dos Santos Lobo Brandao

# Abstract

Spectral geometry is the field of mathematics which concerns relationships between geometric structures of manifolds and the spectra of canonical differential operators. Inverse Spectral Geometry in particular concerns the geometric information that can be recovered from the knowledge of such spectra.

A deep link between inverse spectral geometry and sampling theory has recently been proposed. Specifically, it has been shown that the very shape of a Riemannian manifold can be discretely sampled and then reconstructed up to a cutoff scale. In the context of Quantum Gravity, this means that, in the presence of a physically motivated ultraviolet cutoff, spacetime could be regarded as simultaneously continuous and discrete, in the sense that information can.

In this thesis, we look into the properties of the Laplace-Beltrami operator on a compact Riemannian manifold with no boundary. We discuss the behaviour of its spectrum regarding a perturbation of the Riemannian structure. Specifically, we concern ourselves with infinitesimal inverse spectral geometry, the inverse spectral problem of locally determining the shape of a Riemannian manifold. We discuss the recently presented idea that, in the presence of a cutoff, a perturbation of a Riemannian manifold could be uniquely determined by the knowledge of the spectra of natural differential operators. We apply this idea to the specific problem of determining perturbations of the two dimensional flat torus through the knowledge of the spectrum of the Laplace-Beltrami operator.

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# Chapter 1

## Introduction

A simple gedanken experiment shows that there are problems reconciling General Relativity (GR) and the uncertainty principle. Resolving spacetime distances beyond the Planck length would lead to high momentum uncertainties. These would lead to large uncertainties in the curvature, via the Einstein equation; which would in turn imply uncertainties in knowing the distances, larger than the Planck length. Therefore it should not be possible to resolve spacetime distances beyond the Planck scale.

This is an example of the conceptual difficulties in considering the short-distance structure of spacetime, that results from trying to combine results from two very successful scientific theories, General relativity and Quantum Theory (QT) (see e.g. [20] and the references therein for a more precise discussion of this and other thought-experiments that lead to the existence of a minimum length).

Although sophisticated attempts at combining these two theories have been made, this fascinating problem has remained unresolved. For an historical perspective, see, e.g. [62] and the references therein.

In this context, the natural question thus arises as to the nature of spacetime: is it fundamentally discrete, like a lattice with finite spacing  $l$ , for which the notion of distances smaller than  $l$  have no physical meaning? Or can it be described as a continuous manifold, and some mechanism renders distances smaller than the Plank length operationally meaningless?

A third possibility has recently been proposed [38]: that spacetime could actually be *simultaneously* discrete and continuous, in the information theoretic sense, to be specified below.

To understand the claim, we begin by recalling a central result in sampling theory, Shannon's sampling theorem [61]. It states that any function  $f$  whose highest frequency is bounded by  $\Omega$  can be recovered from samples of its amplitudes  $f(t_n)$  taken at equidistant sampling points  $t_n$ , provided the distance between these,  $t_{n+1} - t_n$  is at most  $(2\Omega)^{-1}$ , via the following formula

$$f(t) = \sum_{n=-\infty}^{+\infty} \frac{\sin \pi \Omega(2(t - t_n))}{\pi \Omega(2(t - t_n))} f(t_n) \quad (1.1)$$

It states that a bandlimited function can be completely reconstructed from the knowledge of its amplitudes at a discrete set of points.

Shannon's Theorem, which is used everyday in engineering applications, can be seen under two different lights:

On the one hand it states that the set of functions that can be described using a finite density of degrees of freedom in momentum space can also be described using a finite density of degrees of freedom in position space. So in this sense, the theorem is not surprising at all.

But perhaps more interestingly, it provides a *sufficient condition* for a set of vectors in position space to be a 'good' basis for a particular set of functions.

Immediately, we are lead to ask how the reconstruction argument depends on the underlying space. And that there should be some information to be gained on the underlying space by the conditions we need to impose for the reconstruction to be possible (if it is at all).

Generalizations of Shannon's theorem for  $\mathbb{R}^n$  are straightforward and see everyday use in, e.g., image processing.

Generalizing the theorem to functions on non-Euclidean spaces should be where the most interesting results lie. We also expect it to be far less trivial, given the previous argument.

Indeed, Fourier analysis, which provided the duality between position and momentum representations in the Euclidean setting does not extend well to curved spaces.

Moreover, it does not extend well to non-equidistant sampling, which is needed to covariantly define scalars on a manifold via sampling.

The key insight consists in looking for a covariant operator that provides a duality between position and momentum spaces in the setting of Riemannian geometry, an analogue to the classical Fourier setting.

A natural choice is the Laplace-Beltrami operator, which, besides being a natural operator (which implies that it is 'local' and its action on elements of its domain commutes with that of the induced by diffeomorphisms; a precise definition can be found in [40]), has a discrete and complete spectral decomposition.

The generalization of the space of bandlimited functions, then, can be carried through by analogy with the Fourier setting: it is the projection of the space of square summable fields onto the space spanned by the elements of the basis of the eigenfunctions of the Laplacian corresponding to eigenvalues up to a certain cutoff.

The program of generalizing sampling theory has seen remarkable success:



Techniques that allow a generalization of Shannon's theorem to bandlimited fields in non-Euclidean Riemannian manifolds were first provided in [33]. They were further developed in [34] and [35], where finally a generalization of 1.1 for the Riemannian setting was shown.

Which means that in the information theoretical sense of Shannon's theorem, bandlimited fields on general Riemannian manifolds can be seen as simultaneously continuous and discrete.

The possibility of generalization for the pseudo-Riemannian case is a natural question, and of obvious physical interest. There are considerable difficulties in defining bandlimitation in this setting, however: specifically in choosing a natural differential operator for which a discrete spectral decomposition theorem exists and for which the eigenfunctions are a complete basis.

We have argued above that perhaps the most interesting aspect of Shannon's theorem lies in the insight it provides about the underlying space via the requirements it imposes on a set of vectors in position space for the reconstruction to be possible.

This insight is indeed even deeper than that. In fact, it was first suggested in [38] that spacetime *itself* could have this sampling property. An explicit formulation of what this means was then carried out in [37]. We will address it thoroughly in Chapter 4, and investigate some of the suggestions in [37].

For the moment, we simply mention that, provided an ultraviolet cutoff is imposed on the spectrum of the Laplacian, that intuitively limits the resolution in position space, a manifold can be given a description in terms of the *covariant* spectrum of the Laplacian up to a cutoff.

What sort of geometric information this sequence effectively contains is of course the obvious question. Specifically, the crucial matter of the sequence of eigenvalues of the Laplacian identifying *uniquely* a manifold in the equivalence class of manifolds up to a cutoff.

Briefly, the fact that one can describe uniquely metrics on manifolds by a diffeomorphism-invariant sequence of numbers would provide us with a diffeomorphism invariant measure on the space of all metrics, naturally solving the issue of 'overcounting' when evaluating integrals in the path integral formulation of Euclidean Quantum Gravity.

Although Quantum Gravity is a motivation for the problems being considered in this thesis, it is well outside its scope. In fact, so as to not replace an 'overcounting' problem with an 'undercounting' one, we must carefully address the issue of the sequence of eigenvalues of the Laplacian uniquely identifying a metric tensor, up to the diffeomorphisms.

This leads us to a question that was posed in the 1960's by Mark Kac [31]: can one hear the shape of a drum?

Kac's question evolved into its own field of mathematics: inverse spectral geometry. Inverse spectral geometry deals precisely with the matter of recovering geometrical and topological information from the knowledge of the

spectra of natural differential operators on a manifold.

The question has proven to be notoriously difficult. For special classes of manifolds, it has been answered in both the affirmative and the negative. For general manifolds, the very existence of natural differential operators for which the spectrum is discrete is a non-trivial one. Boundaries are an added complication.

In more rigorous and general terms, Kac's question can be formulated as follows:

Does the knowledge of the spectrum of the Laplace operator on functions on a Riemannian  $n$ -dimensional, smooth, compact manifold with or without boundary, suffice to recover the knowledge of the metric tensor, up to a diffeomorphism?

In the sequel, we aim to provide an overview of the several attempts and techniques that have been developed to answer this question, as described below; we shall also concern ourselves with applying and/or identifying possible solution paths that make use of the information theoretical insights provided in [36].

We begin by defining the Laplace operator on scalars, and motivate its definition as generalization of the familiar Euclidean Laplacian.

We then proceed to show that a spectral decomposition of the Laplacian exists for closed, connected Riemannian manifolds: its eigenvalues are discrete and its eigenfunctions are a basis for  $L^2(M)$ .

We choose to derive this result as part of the proof of a useful expansion, known as Minakshisundaram-Pleijel (MP) asymptotic formula, that expresses the behaviour of the trace of the kernel of the heat operator for small times.

In Chapter 3 we present a collection of results that, in the same spirit, establish a link between geometric and spectral information.

Specifically, we present the Minakshisundaram-Pleijel expansion, and how it provides a link between the knowledge of the spectrum of the Laplacian and the underlying geometry; in particular, we present the famous Weyl's formula as a corollary. We present rigidity results, that show that certain classes of manifolds are completely characterized by the spectrum of the Laplace operator.

We then discuss the nature of the difficulties of the problem of inverting the spectrum. We present a few negative results, which have been the subject of much research. Although important, most of the results, fortunately, lie outside the domain of our concern.

In Chapter 4 we outline the program first suggested in [36] and discuss the strategy in the context of sampling theory.

In brief, we look into the first order perturbation of both the manifold and the spectrum of the Laplace operator on scalars.

In the context of sampling theory and for perturbations of the metric that are described by a scalar, we look into the problem of inverting the

spectrum as one of inverting the finite-dimensional matrix of the derivative of the spectral function with respect to metric.

We also try to provide some insight on the geometrical information that can be expressed as a cutoff spectrum of the Laplace operator.

The well-posedness of this problem and the appropriate setup for actual computations is our next concern: it is the subject of chapter 5, in which I also show that a known decomposition for covariant two-tensors in dimension three also holds for arbitrary dimensions.

In Chapter 6, we discuss some properties of the spectrum of the Laplacian under variations of the metric. Specifically, we show that the spectral map is a uniformly continuous map between Fréchet spaces, among other regularity results. This allows us to show an important result, known as Uhlenbeck's Theorem, that states the generality of non-degenerate spectra.

I use these results to state and prove a new stability result in 6.2.7.

Chapter 7 consists in the application of the program above to the two-dimensional flat torus. Though a particularly simple manifold, it is one of the very few for which the spectrum can be calculated explicitly.

I show that, in this case, the first derivative is not invertible. I discuss how this problem could be resolved in the light of the results in chapters 6 and 4 and clarify that there is no contradiction with the known rigidity of flat tori in arbitrary dimension.

We finish our presentation by discussing our results and providing an outlook on future avenues of research in the context of the program outlined in [37].

## Chapter 2

# Laplace operator

### 2.1 Definitions of the Laplace Operator

We present two generalizations of the Laplace operator in Euclidean space. The first, the Laplace-Beltrami operator is perhaps the more straightforward: it is the operator on functions on a manifold that defined via replacing the familiar definitions of gradient and divergence in Euclidean space by the corresponding generalizations in differential geometry. The second one, the Laplace - de Rham operator, is defined on differential forms of arbitrary degree via an inner product. Both reduce to the familiar Euclidean Laplace operator on functions (zero forms).

#### 2.1.1 A generalization of the Euclidean Laplace operator

We begin by recalling the following definition (see e.g. [58]):

**Definition 2.1.1 (Directional Derivative)** *Let  $f : M \rightarrow M$  be a  $C^1$  function on a differentiable manifold  $M$  and  $\xi$  a vector field. For  $p \in M$  we define the derivative of  $f$  in the direction  $\xi(p)$  as*

$$\xi f(p) = (f \circ \gamma)'(0)$$

where  $\gamma$  is any curve on  $M$  such that its tangent vector at  $p$  is  $\xi(p)$ .

From which the following straightforward generalization follows:

**Definition 2.1.2 (Gradient)** *Let  $(M, g)$  be a differentiable manifold with metric  $g$  and  $f \in C^1(M)$ . For  $f : M \rightarrow M$  in  $C^1(M)$  we define the gradient of  $f$  as the vector field  $\nabla f$  such that*

$$g(\nabla f, \xi) = \xi f$$

for any vector field  $\xi$ . In coordinates, then, we have

$$(\nabla f)^a = f_{,b} g^{ab} \quad (2.1)$$

where comma denotes differentiation with respect to the labelled coordinate function.

Next, much as in Euclidean geometry, we investigate the change in the volume of some connected domain  $\Omega$ , bounded by  $\partial\Omega$ , in  $M$  an orientable manifold, under the flow of a vector field  $\xi$ . For  $\omega$  a volume form, we have [60]

$$\begin{aligned} \int_{\Omega} \omega + \epsilon \mathcal{L}_{\xi} \omega + O(\epsilon^2) - \int_{\Omega} \omega &= \epsilon \int_{\Omega} \mathcal{L}_{\xi} \omega \\ &= \epsilon \int_{\Omega} d(\omega(\xi)) \end{aligned}$$

Where as we noted  $\omega(\xi) := \omega(\xi, \dots)$  and used the identity [60],  $\mathcal{L}_{\xi} \omega = d(\omega(\xi)) + (d\omega)(\xi)$  (valid for forms of arbitrary degree) and the fact that, if  $\omega$  is a volume form on  $M$  a manifold of dimension  $n$ , then  $d\omega$  is of degree  $n+1$  and vanishes identically. The argument above then motivates the following definition:

**Definition 2.1.3 (Divergence)** *Let  $\xi$  be a  $C^1(M)$  vector field and  $\omega$  a volume form. Then, we define the divergence of  $\xi$  with respect to  $\omega$  as*

$$\operatorname{div}_{\omega}(\xi)\omega = d(\omega(\xi))$$

Now, by Stoke's theorem we have

$$\epsilon \int_{\Omega} d(\omega(\xi)) = \epsilon \int_{\partial\Omega} \omega(\xi)$$

and if  $\omega$  is the volume form induced by the metric  $g$  on  $(M, g)$ , that is, if, for some choice of coordinates we have

$$\omega = \sqrt{g} dx^1 \wedge \dots \wedge dx^n$$

then it follows that

$$\begin{aligned} \omega(\xi) &= \sqrt{g} (\xi^1 dx^2 \wedge dx^3 \wedge \dots - \xi^2 dx^1 \wedge dx^3 \wedge \dots + \dots) \Rightarrow \\ d(\omega(\xi)) &= (\sqrt{g} \xi^1)_{,1} dx^1 \wedge \dots + (\sqrt{g} \xi^2)_{,2} dx^1 \wedge \dots + \dots \\ &= \frac{1}{\sqrt{g}} (\sqrt{g} \xi^i)_{,i} \omega \end{aligned}$$

And so, in coordinates, 2.1.3, for  $\omega$  the metric induced volume form, reads:

$$\operatorname{div}_\omega \xi = \frac{1}{\sqrt{g}} (\sqrt{g} \xi^i)_{,i} \quad (2.2)$$

And we see that the familiar definition can be recovered in Euclidean space.

Definitions 2.1.2 and 2.1.3 allow us to define

**Definition 2.1.4 (Laplace Beltrami Operator)** *Let  $(M, g)$  be a smooth manifold with metric  $g$  and induced volume form  $\omega$ . For  $\phi : M \rightarrow \mathbb{R}$  we define the Laplace Beltrami operator  $\Delta_B(g) : C^\infty(M) \rightarrow C^\infty(M)$*

$$\Delta_B(g)\phi = -\operatorname{div}_\omega(\nabla\phi) \quad (2.3)$$

Using 2.2 and 2.1.2, we get the following expression in coordinates:

$$\Delta_B(g)\phi = -\frac{1}{\sqrt{g}} (\sqrt{g} \phi_{,i} g^{ij})_{,j} \quad (2.4)$$

## 2.1.2 The Laplace De Rham operator

We begin by recalling the following definition:

**Definition 2.1.5 (Hodge Dual)** *For  $\omega$  a  $k$ -form, with components  $\omega_{i_1 \dots i_k}$ , define the  $(n-k)$  form  $*\omega$ , the Hodge Dual of  $\omega$ , in the following way:*

$$(*\omega)_{i_1 i_2 \dots i_{n-k}} = \frac{1}{k!} \omega^{j_1 \dots j_k} \sqrt{|\det g|} \epsilon_{j_1 \dots j_k i_1 \dots i_{n-k}}$$

From which it follows that

$$**\omega = (-1)^{k(n-k)} s\omega$$

where  $s$  is the signature of the metric. Consequently, we have

$$*^{-1}\omega = (-1)^{k(n-k)} s* \quad (2.5)$$

In what follows,  $\Omega^k(M)$  denotes the set of smooth differential forms of degree  $k$  on  $(M, g)$  a manifold of  $\dim M = n$ ,  $d$  is the exterior derivative operator, and  $\delta$  the codifferential, defined as follows:

**Definition 2.1.6 (Codifferential)**

$$\delta = (-1)^{n(k+1)+1} * d*$$

Using 2.5, we get

$$\delta = (-1)^k *^{-1} d*$$

We can then define the following inner product on  $\Omega^k(M)$ :

**Definition 2.1.7**

$$\langle \phi, \psi \rangle_g = \int_M \phi \wedge * \psi$$

(note that the subscript  $g$  is needed in defining  $\langle \cdot, \cdot \rangle_g$  because of the definition of Hodge Dual 2.1.5).

It can be shown that  $\delta$  is the formal adjoint of  $d$  with respect to this inner product. Explicitly, let  $\omega, \theta \in \Omega^k(M)$  with  $M$  an  $n$  dimensional manifold. Then, of course, we have that  $\omega \wedge * \theta$  is an  $n$ -form and, by definition of the exterior derivative, we have that  $d(\omega \wedge * \theta) = 0$ . The claim then follows from Stokes' Theorem:

$$\begin{aligned} 0 &= \int_M d(\omega \wedge * \theta) \\ &= \int_M d\omega \wedge * \theta - (-1)^{k+1} \omega \wedge d * \theta \\ &= \int_M d\omega \wedge * \theta - (-1)^{k+1} \omega \wedge * *^{-1} d * \theta \\ &= \int_M d\omega \wedge * \theta - \omega \wedge * \delta \theta \\ &= \langle d\omega, \theta \rangle_g - \langle \omega, \delta \theta \rangle_g \end{aligned}$$

We are now ready to define the Laplace-de Rham operator (also Hodge Laplacian):

**Definition 2.1.8 (Laplace de Rham Operator)** *Let  $(M, g)$  be a smooth manifold with metric  $g$  and induced volume form  $\omega$ . For  $\theta \in \Omega^k(M)$  we define the Laplace de Rham operator  $\Delta_{dR}(g) : \Omega^k(M) \rightarrow \Omega^k(M)$*

$$\Delta_{dR}(g)\theta = (\delta d + d\delta)\theta \tag{2.6}$$

**2.1.3 Equivalence of the definitions for zero-forms**

We now show that the two definitions coincide for zero-forms.

Because  $\delta\theta = 0$  for  $\theta \in \Omega^1(M)$ , we have

$$\begin{aligned} \Delta_{dR}(g)\theta &= (\delta d + d\delta)\theta \\ &= \delta d\theta \end{aligned}$$

and so, using the results above, we have

$$\begin{aligned}
\langle \phi, \delta d\psi \rangle_g &= \langle d\phi, d\psi \rangle_g \\
&= \int_M d\phi \wedge *d\psi \\
&= \int_M d\phi \wedge *(\psi_{,i} dx^i) \\
&= \int_M \sqrt{g} \phi_{,k} dx^k \wedge (\psi_{,j} g^{ji} \epsilon_{iJ} dx^J) \\
&= \int_M \phi_{,k} \psi_{,j} g^{kj} \sqrt{g} dx \\
&= - \int_M \frac{1}{\sqrt{g}} (\psi_{,j} g^{kj} \sqrt{g})_{,k} \sqrt{g} dx \\
&= \int_M \phi \Delta_B \psi \sqrt{g} dx \\
&= (\phi, \Delta_B \psi)_g
\end{aligned}$$

where the second-to-last line follows by integration by parts.

In particular, given  $\phi, \psi$  elements of a basis for  $\Omega^1(M)$  the above expression is true and shows equality of the matrix elements of  $\Delta_{dR}(g)$  and  $\Delta_B(g)$ , which shows the claim.

Since we have the equivalence of the two definitions, we drop the subscripts and denote either Laplacian on  $(M, g)$  by  $\Delta(g)$ .

## 2.2 Heat Kernel: Existence and Properties

In this section we present a number of properties of the Laplace operator on functions (at least  $C^2$ ) on a smooth, connected closed Riemannian manifold  $M$ : specifically, those that can be derived by what is known in the literature as "Heat Kernel Techniques". This presentation will be used as an illustration of these techniques. We shall discuss extensions of these techniques and their limitations in Chapter 4.

We begin by defining the Heat operator and its kernel; we then proceed by showing that, if the Heat Kernel exists, the spectrum of the Laplace operator is discrete, has no accumulation points other than infinity and that its eigenfunctions are a basis of  $L^2(M)$ ; we proceed to establishing the existence of the Heat kernel for closed manifolds by way of a recursion formula by Minakshisundaram-Pleijel; finally, we show a namesake asymptotic formula and discuss some geometric information encoded therein.

### 2.2.1 Preliminaries: Functional Analysis Review

In this section, we recall a few definitions concerning linear operators, in which we follow [11]. We remark that although all vector spaces considered



below are over the reals, generalization to the complex case is straightforward (and in any case, not necessary for our purposes).

We begin by settling some notation:

Given  $E$  a Banach space, we denote the space of continuous linear forms on  $E$  by  $E'$ .  $E'$  is given the operator norm, which we denote by  $\| \cdot \|_{E'}$ .

For  $f \in E'$  and  $x \in E$  we write  $\langle f, x \rangle := f(x)$ ; we call  $\langle \cdot, \cdot \rangle$  the inner product in the duality  $E, E'$ ; a subscript shall be used whenever confusion could arise.

**Definition 2.2.1 (Unbounded Linear Operator)** *Let  $E, F$  be two Banach spaces. We call any linear operator  $A : D(A) \subset E \rightarrow F$  defined on the vector subspace  $D(A) \subset E$  an unbounded linear operator. We note that, under this terminology, an unbounded linear operator may very well be bounded; one is cautioned to read this definition as that of a ‘not-necessarily-bounded’ linear operator.*

**Definition 2.2.2 (Adjoint Operator)** *Let  $A$  be an unbounded linear operator (as above). Let  $D(A)$  be dense in  $E$ , that is  $\overline{D(A)} = E$ . We define  $A^* : D(A^*) \subset F' \rightarrow E'$  as follows:*

1. We begin by defining the domain of  $A^*$ , a vector subspace of  $F'$ :

$$D(A^*) = \{v \in F'; \exists c \geq 0 : |(v, Au)| \leq c\|u\| \ \forall u \in D(A)\}$$

2. Let now  $v \in D(A^*)$ . We consider the map  $g : D(A) \rightarrow \mathbb{R}$  defined by

$$g(u) = \langle v, Au \rangle, \quad u \in D(A)$$

*From the definition of  $D(A^*)$  we have that  $|g(u)| \leq c\|u\|$  for all  $u \in D(A)$ , from which it follows, by the Hahn-Banach Theorem, that  $g$  can be prolonged to a linear map  $f$  defined on the entire space,  $f : E \rightarrow \mathbb{R}$  such that, for all  $u \in E$*

$$|f(u)| \leq c\|u\|$$

*That is,  $f$  is continuous, thus in  $E'$ . From which it follows that, since  $f, g$  coincide in  $D(A)$  and  $D(A)$  is dense,  $f$  is unique. The following is thus a sound definition of a linear operator  $A^* : D(A^*) \subset F' \rightarrow E'$ , the adjoint of  $A$ :*

$$A^*v := f$$

We note that, by definition, the following relation holds for all  $u \in D(A)$  and  $v \in D(A^*)$ :

$$(v, Au)_{F',F} = (A^*v, u)_{E',E}$$

Let now  $A : D(A) \subset H \rightarrow H$  be an unbounded linear operator with  $D(A)$  dense in  $H$  a real Hilbert space with inner product  $(\cdot, \cdot)$ . By the Riezs Representation Theorem, for every continuous linear form  $f$  in  $H'$ , there is a unique  $v$  element in  $H$  such that  $f(u) = (v, u)$  for all  $u \in H$ . This provides us with an isomorphism that allows us to identify  $H$  and its dual  $H'$ . With this identification, we can consider  $A^*$  as an unbounded linear operator on  $H$ . We can now write the following definition:

**Definition 2.2.3 (Self-Adjoint Operator)** *An unbounded operator  $A$  as above is called self-adjoint if*

$$A^* = A, \quad D(A^*) = D(A)$$

Before we restrict ourselves to the more familiar setting of bounded linear operators we recall the following definition,

**Definition 2.2.4 (Symmetric Operator)** *Let  $A$  be an unbounded linear operator, defined as above. We say that  $A$  is symmetric if  $(u, Av) = (v, Au)$  for all  $u, v \in D(A)$ .*

We note that, by the last remark in the definition 2.2.2, if  $A$  is self-adjoint, then it is necessarily symmetric. And again by definition, if  $A$  is bounded, then inspection of the definition of  $D(A^*)$  2.2.2 suffices to establish that  $D(A^*) = D(A)$ . Symmetry and self-adjointness, for the case of bounded linear operators, are equivalent (as they should be, since they are generalizations of the familiar definition in the finite-dimensional setting).

The matter is more subtle when  $A$  is not a bounded linear operator (as is the case that concern us, the Laplace Operator on  $L^2(M)$ , with  $M$  a compact, connected Riemannian with no boundary):  $A$  is symmetric iff  $A \subset A^*$ , from which it follows that  $D(A) \subset D(A^*)$ . The two operators coincide in  $D(A)$ , but it could very well be the case that the inclusion is strict. We shall address this matter more carefully later on.

In order to state an important theorem, we settle some more terminology. We begin with the following

**Definition 2.2.5 (Compact Operator)** *A bounded linear operator  $C : E \rightarrow F$  is compact if the closure of the image under  $C$  of a bounded subset  $B \subset E$  is compact (where 'bounded subset' and 'compact' refer to the norms in the appropriate spaces: respectively,  $\|\cdot\|_E$  and  $\|\cdot\|_F$ )*

We proceed with defining

**Definition 2.2.6 (Resolvent Set)** *Given a bounded linear operator  $T : E \rightarrow E$  we define*

$$\rho(T) = \{\lambda \in \mathbb{R} : (T - \lambda I) \text{ is bijective on } E\} \quad (2.7)$$

and call  $\rho(T)$  the resolvent set of  $T$ .

We call  $\lambda$  an **eigenvalue** of  $T$  if the kernel of the operator  $(T - \lambda I)$  is not trivial. If that is the case, we call the kernel of this operator the **eigenspace** associated with the eigenvalue  $\lambda$ .

We take a moment to clarify terminology. It may be somewhat confusing that the set above is called the ‘resolvent’: consider the eigenvalue equation for  $T$ ,  $(T - \lambda I)v = 0$ . If  $\lambda$  is in  $\rho(T)$  then  $(T - \lambda I)$  is bijective, by definition. Since  $T$  is continuous, it follows that it is continuous as well. But then, by the closed graph theorem (Banach-Schauder Theorem (cf. [56], [11])) that  $(T - \lambda I)^{-1}$  exists and is bounded. Since, for continuous operators with continuous inverses, the operator norm of the inverse is the inverse of the operator norm, we have that if  $\lambda$  is in  $\rho(T)$  then the norm of  $(T - \lambda I)$  cannot be zero (thus, cannot be a solution to the eigenvalue equation): otherwise the norm of  $(T - \lambda I)^{-1}$  would not be bounded, a contradiction. The ‘resolvent’ set is, then, exactly the set of numbers that are *not* solutions to the eigenvalue equation  $(T - \lambda I)v = 0$ . The following should then clarify the ‘issue’:

Note that if, given  $\lambda \in \mathbb{R}$  and  $f \in E$ , we seek solutions  $u$  of

$$(T - \lambda I)u = f$$

then  $u$  is given by, should the inverse exist

$$u = (T - \lambda I)^{-1}f$$

From the preceding discussion, the resolvent is then the set of  $\lambda$ s that assure the existence of a bounded linear inverse of  $(T - \lambda I)$ , hence for which a solution to the equation above can be found in this fashion (a well-behaved solution, in the sense that it is in  $E$  since the inverse is linear and bounded).

Bearing this in mind, we can now proceed to defining

**Definition 2.2.7 (Spectrum)** *Let  $T : E \rightarrow E$  be a bounded linear operator. We define the spectrum of  $T$  as the set*

$$\sigma(T) = \mathbb{R} \setminus \rho(T)$$

Regarding this set, we present two important results[11]:

**Proposition 2.2.8** *Let  $T : E \rightarrow E$  be a bounded linear operator. Then, the set  $\sigma(T)$  is compact and*

$$\sigma(T) \subset [-\|T\|, \|T\|]$$

For *compact* operators on an infinite dimensional space  $T : E \rightarrow E$  we have:

**Proposition 2.2.9** *Let  $T : E \rightarrow E$  be a compact linear operator on a infinite dimensional space  $E$ . Then we have that  $0 \in \sigma(T)$ . Also  $\lambda \in \mathbb{R} \setminus \{0\}$  is an eigenvalue of  $T$  if and only if it is in  $\sigma(T) \setminus \{0\}$ . Furthermore, one and only one of the following holds:*

1.  $\sigma(T) = \{0\}$
2.  $\sigma(T) \setminus \{0\}$  is finite
3.  $\sigma(T) \setminus \{0\}$  is a sequence that converges to zero.

The last claim follows from the following result (for compact operators  $T : E \rightarrow E$  on an infinite dimensional space), together with the compactness of the spectrum:

**Proposition 2.2.10** *Any convergent sequence of distinct elements of  $\sigma(T) \setminus \{0\}$ ,  $\{\lambda_n\}$ , must converge to zero.*

From which it follows that elements of  $\sigma(T) \setminus \{0\}$  are *isolated points* (since there are no repetitions in  $\sigma(T)$ ).

The proof is quite interesting and can be found in [11].

Another matter is the *multiplicity* of each eigenvalue, the dimension of the eigenspace associated with  $\lambda$ : Given a compact linear operator  $T : E \rightarrow E$  and  $\lambda \in \mathbb{R} \setminus \{0\}$  it can be shown that the dimension of the kernel of the operator  $(T - \lambda I)$  is finite dimensional, see [41]. The claim follows essentially from recalling that the Kernel of a linear operator is a vector subspace and showing that the unit ball in that space is compact (which implies finite dimension, by Riesz's Theorem). In particular, the claim holds for  $\lambda \in \sigma(T) \setminus \{0\}$ .

Finally, we state the following

**Theorem 2.2.11 (Spectral Theorem)** *Let  $T$  be a compact self-adjoint operator on a Hilbert space  $E$ . Then, there is an orthonormal basis of  $E$  consisting of eigenvectors of  $T$ .*

### 2.2.2 Motivation

The Laplace operator is obviously a linear symmetric operator in  $L^2(M)$  ( $M$  compact, no boundary); for every  $\phi, \psi \in \text{Dom}(\Delta)$  we have:

$$\begin{aligned} (\phi, \Delta\psi)_g &= \langle \phi, \Delta\psi \rangle_g \\ &= \langle d\phi, d\psi \rangle_g \\ &= \langle d\psi, d\phi \rangle_g \end{aligned}$$

Since it is also a differential operator, it is easy to check that it is unbounded (see [41]), which, since by a corollary of the Closed Graph Theorem (the Hellinger-Toeplitz Theorem, cf. [41]) a symmetric everywhere defined operator is bounded, means that it cannot be everywhere defined (its domain cannot be  $L^2(M)$ ).

In fact, being an unbounded operator, 2.2.11 cannot be used directly.

To establish the basic facts on the properties of the spectrum and eigenvalues of the Laplace operator, one approach would be to generalize theorem 2.2.11 and 2.2.9 for the non-compact case. This is the approach that can be found in [22], which, although we will not cover here, we summarize briefly:

Although, as mentioned, the Laplacian is not everywhere defined, it can be shown that it is densely defined, and it does have self-adjoint extensions; indeed, it can be shown that it has a unique self-adjoint extension, and it is therefore an essentially self-adjoint operator. This, together with its ellipticity, allows one to prove the basic facts on the properties of its spectrum and eigenfunctions.

An alternative, and this is the path we shall take, is to use integral operator methods, essentially following [14]. Although outside the scope of this thesis, we mention that this method can be generalized for non-symmetric operators [9].

For an approach that follows essentially the same treatment as we present here, but applied to bounded Euclidean domains with smooth boundary, cf. [18]. We found it to be a very useful introduction to these techniques.

### 2.2.3 Summary of the Proof

We begin by defining the *heat operator*  $L$ , a differential operator that describes the propagation of heat on a given manifold. We then define a *fundamental solution* to the heat equation  $Lu = -F$ . We prove *Duhamel's Principle* to show that, should it exist, this fundamental solution is *unique*, and every solution to the heat equation can be recovered from it.

Should the fundamental solution exist, we then show that the integral operator that has it as its kernel is compact and self-adjoint; this allows us to apply the spectral theorem and establish the existence of a discrete

basis for  $L^2(M)$  composed of its eigenfunctions. After showing that the integral operator also has what we shall call a semigroup property, we show the existence of a discrete basis of  $L^2(M)$  composed of the eigenfunctions of the Laplacian.

It remains to show existence of the fundamental solution, which we will do, for closed manifolds.

## 2.2.4 The heat equation

**Definition 2.2.12 (Heat Operator)** *We define the operator  $L$ , the Heat operator, acting on functions  $u(x, t)$  in  $C^0(M \times (0, \infty))$  that are  $C^2$  in the ‘space’ variable  $x$  and  $C^1$  in the ‘time’ variable  $t$ :*

$$L = \Delta - \frac{\partial}{\partial t}$$

## 2.2.5 Fundamental solutions

We define the Green’s function for the Heat operator:

**Definition 2.2.13 (Fundamental Solution)** *Define a fundamental solution  $G$  of  $Lu = 0$  as a function  $G(x, y, t)$ ,  $C^2$  in the space variables  $x, y$  and  $C^1$  in time  $t$  such that*

1.  $G$  is in the kernel of  $L$  for fixed  $y$ . That is, for fixed  $y \in M$ , we have

$$\forall y \in M, L_x G(x, y, t) = 0$$

(the index  $x$  indicates to which variable  $L$  applies to)

2.  $\lim_{t \rightarrow 0} G(x, y, t) = \delta(x - y)$ , where  $\delta$  is the Dirac delta. By definition, this implies

$$\lim_{t \rightarrow 0} \int_M G(x, y, t) f(x) \sqrt{g} dx = f(y)$$

## 2.2.6 Duhamel’s Principle

We show a result that will serve to establish the symmetry and uniqueness of  $G$

**Proposition 2.2.14 (Duhamel’s Principle)** *Let  $u, v$  be continuous functions on  $M \times (0, \infty)$ ,  $C^2$  on the space variable and  $C^1$  on the time variable. Then, for all  $[\alpha, \beta] \subseteq (0, t)$  we have*

$$\int_M [u(x, t - \beta) v(x, \beta) - u(x, t - \alpha) v(x, \alpha)] \sqrt{g} dx = \int_{\alpha}^{\beta} d\tau \int_M [(Lu)(x, t - \tau) v(x, \tau) - u(x, t - \tau) (Lv)(x, \tau)] \sqrt{g} dx$$

**Proof:** By definition of  $L$ ,

$$\begin{aligned} (Lu)(x, t - \tau)v(x, \tau) - u(x, t - \tau)(Lv)(x, \tau) = \\ (\Delta u)(x, t - \tau)v(x, \tau) - u(x, t - \tau)(\Delta v)(x, \tau) \\ + \frac{\partial}{\partial \tau}(u(x, t - \tau)v(x, \tau)) \end{aligned}$$

and from the symmetry of the Laplace operator, it follows that

$$\int_M [(\Delta u)(x, t - \tau)v(x, \tau) - u(x, t - \tau)(\Delta v)(x, \tau)] \sqrt{g} dx = 0$$

and so

$$\begin{aligned} \int_\alpha^\beta d\tau \int_M [(Lu)(x, t - \tau)v(x, \tau) - u(x, t - \tau)(Lv)(x, \tau)] \sqrt{g} dx = \\ \int_\alpha^\beta d\tau \int_M \left[ \frac{\partial}{\partial \tau}(u(x, t - \tau)v(x, \tau)) \right] \sqrt{g} dx = \\ \int_M [u(x, t - \beta)v(x, \beta) - u(x, t - \alpha)v(x, \alpha)] \sqrt{g} dx \end{aligned}$$

## 2.2.7 Uniqueness and symmetry of fundamental solutions

It will prove convenient to show that solutions to the heat equation, if they exist, are unique (it is, of course, important in itself). To do so, we start by noting the following proposition:

**Proposition 2.2.15** *Let  $u$  be a solution to the homogeneous heat equation. Then*

1. *The following function of  $t$  is a constant:*

$$M_1(t) = \int_M u(x, t) \sqrt{g} dx$$

2. *The following function of  $t$  is decreasing:*

$$M_2(t) = \int_M u^2(x, t) \sqrt{g} dx$$

**Proof:** The proof follows straightforwardly because the time derivative can be carried under the integral in both cases; it is then a matter of ‘replacing’ time derivatives with the Laplace operator, because  $Lu = 0$  using the symmetry of the Laplace operator and the positive-definiteness of  $g$ ; in 1 we

get

$$\frac{d}{dt}M_1(t) = -(\Delta u, 1)_g = (u, \Delta 1)_g = 0$$

and for 2 we get

$$\begin{aligned} \frac{d}{dt}M_2(t) &= -2(u, \Delta u)_g \\ &= -2\langle du, du \rangle_g \\ &< 0 \end{aligned}$$

where the inequality in the last line is strict because we discard the trivial solution.

From which it follows the following corollary:

**Corollary 2.2.16** *Let  $f, F$  in 2.8 be continuous. Then, there is at most one continuous function  $u : M \times [0, \infty)$  that is a solution to the heat equation and that satisfies  $u(x, 0) = f(x)$*

**Proof:** If  $u_1, u_2$  are solutions to 2.8 then their difference  $v$  is a solution to the heat equation and  $v(x, 0) = f(x) - f(x) = 0$ . Since  $V(t) = \int_M v^2(x, t)\sqrt{g}dx$  is decreasing, by 2.2.15, and  $V(0) = 0$ , we have the claim.

We are now ready to show the following

**Theorem 2.2.17** *Assume there is a  $G$  as in 2.2.13. Then*

1. *Any fundamental solution of the Heat equation on  $M$  is **symmetric** in the two space variables.*
2. *The fundamental solution is **unique**.*
3. *Given bounded continuous functions  $f : M \rightarrow \mathbb{R}$ ,  $F : M \times (0, \infty) \rightarrow \mathbb{R}$ , the solution to the initial value problem (IVP):*

$$Lu(x, t) = -F(x, t), \quad u(x, 0) = f(x) \tag{2.8}$$

*should it exist, is given by*

$$u(x, t) = \int_M G(x, y, t)\sqrt{g}dy + \int_0^t d\tau \int_M G(x, y, t - \tau)F(y, \tau)\sqrt{g}dy \tag{2.9}$$

4. *In particular, on all  $M \times (0, \infty)$  we have*

$$\int_M G(x, y, t)\sqrt{g}dx = 1 \tag{2.10}$$



**Proof:** We begin by showing 1 and 2, symmetry and uniqueness of the fundamental solution. Take  $G_1, G_2$  fundamental solutions to the heat equation on  $M$ . We apply Duhamel's principle setting

$$u(z, \tau) = G_1(z, x, \tau), \quad v(z, \tau) = G_2(z, y, \tau)$$

And noting that  $LG_1 = LG_2 = 0$  we get

$$\int_M [G_1(z, x, t - \beta) G_2(z, y, \beta) - G_1(z, x, t - \alpha) G_2(z, y, \alpha)] \sqrt{g} dz = 0$$

Letting  $\alpha \rightarrow 0$  and  $\beta \rightarrow t$  we get, because  $\lim_{t \rightarrow 0} G_i(z, x, t) = \delta(z - x)$

$$G_2(x, y, t) = G_1(y, x, t) \tag{2.11}$$

and so applying the same reasoning to a single fundamental solution  $G$  we get

$$G(x, y, t) = G(y, x, t) \tag{2.12}$$

for all  $x, y \in M$ , and we have that any fundamental solution is symmetric in the space variables. But then, given two fundamental solutions, applying this symmetry to one of them, say  $G_1$ , we get

$$\begin{aligned} G_2(x, y, t) &= G_1(y, x, t) \\ G_2(x, y, t) &= G_1(x, y, t) \end{aligned}$$

which shows uniqueness. To show the last two claims, simply take *the* (by 2.2.16) fundamental solution  $G$ ; setting  $v(z, \tau) = G(x, z, \tau)$ , applying Duhamel's principle and letting  $\alpha \rightarrow 0$  and  $\beta \rightarrow t$  as above, we get 3. The last claim immediately follows.

### 2.2.8 Semigroup property

Assume existence of a fundamental solution to 2.8. Then, given a continuous function  $f : M \rightarrow \mathbb{R}$  the following defines an integral operator for fixed  $t > 0$ :

$$\mathbb{B}_t f = \int_M G(x, y, t) f(y) \sqrt{g} dy \tag{2.13}$$

That is, the operator that, given initial conditions  $f$ , returns the solution to the IVP 2.8 at time  $t$ .

We examine the product of two such operators:

$$\begin{aligned}
\mathbb{B}_{t_2}\mathbb{B}_{t_1}f &= \\
& \int_M G(x, y, t_2)(\mathbb{B}_{t_1}f)(y)\sqrt{g}dy = \\
& \int_M G(x, y, t_2) \left( \int_M G(y, z, t_1)f(z)\sqrt{g}dz \right) \sqrt{g}dy = \\
& \int_M \left( \int_M G(z, y, t_1)G(x, y, t_2)\sqrt{g}dy \right) f(z)\sqrt{g}dz
\end{aligned}$$

But using the definition 2.2.13, it follows easily that the term in parenthesis is a fundamental solution:

$$\int_M G(z, y, t_1)G(x, y, t_2)\sqrt{g}dy = G(x, z, t_1 + t_2) \quad (2.14)$$

and so

$$\mathbb{B}_{t_2}\mathbb{B}_{t_1} = \mathbb{B}_{t_1+t_2} \quad (2.15)$$

So for each  $t > 0$  we have a continuous integral operator (bounded and linear) with a symmetric kernel is compact and self adjoint, and we can apply the Spectral Theorem 2.2.11 to it.

Also, we note that 2.15 implies that the integral operator 2.13 is also **positive** for  $t > 0$ :

$$(\mathbb{B}_t f, f)_g = (\mathbb{B}_{t/2+t/2} f, f)_g \quad (2.16)$$

$$= (\mathbb{B}_{t/2} f, \mathbb{B}_{t/2} f)_g \geq 0 \quad (2.17)$$

## 2.2.9 Eigenfunctions are a discrete basis for $L^2(M)$

In this section we show, combining the results above, that if the existence of the fundamental solution is guaranteed, the spectrum of the Laplacian is discrete and there is a basis for functions composed of eigenfunctions of the Laplacian:

**Theorem 2.2.18 (Sturm-Liouville Decomposition)** *Let  $M$  be compact. Then, the following holds:*

1. *There is a complete, discrete orthonormal basis of  $L^2(M)$  consisting of the eigenfunctions of  $\Delta$ :  $\{\phi_i\}_{i \in \mathbb{N}}$ , such that  $\Delta\phi_i = \lambda_i\phi_i$ . The eigenval-*

ues satisfy

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots \uparrow \infty \quad (2.18)$$

and each eigenvalue has finite multiplicity.

2. Also, each eigenfunction is as smooth as  $G$ , the fundamental solution to the heat equation.

3. Finally,

$$G(x, y, t) = \sum_{i=0}^{\infty} e^{-\lambda_i t} \phi_i(x) \phi_i(y) \quad (2.19)$$

with convergence absolute and uniform for every  $t > 0$ . In particular, the trace of the Heat kernel is

$$\int_M G(x, x, t) \sqrt{g} dx = \sum_{i=0}^{\infty} e^{-\lambda_i t} \quad (2.20)$$

**Proof:** Since for each  $t > 0$  the operator  $\mathbb{B}_t$  is compact and self-adjoint, we have, by the spectral theorem 2.2.11, that its eigenfunctions  $\{\psi_i(t)\}_{i \in \mathbb{N}}$  are a complete orthonormal basis for  $L^2(M)$  and its associated eigenvalues satisfy

$$\lambda_0(t) \geq \lambda_1(t) \geq \dots \downarrow 0 \quad (2.21)$$

Using 2.15, we see, for  $k \in \mathbb{N} \setminus \{0\}$ , the following relation between the  $j$ th eigenfunction and eigenvalue of the operators  $\mathbb{B}_t$  and  $\mathbb{B}_{kt}$

$$\begin{aligned} \mathbb{B}_t \phi_j^{(t)} &= \lambda_j^{(t)} \phi_j^{(t)} \Rightarrow \\ (\mathbb{B}_t)^k \left( \phi_j^{(t)} \right)^k &= \left( \lambda_j^{(t)} \right)^k \left( \phi_j^{(t)} \right)^k \Rightarrow \\ \mathbb{B}_{kt} \left( \phi_j^{(t)} \right)^k &= \left( \lambda_j^{(t)} \right)^k \left( \phi_j^{(t)} \right)^k \end{aligned}$$

from which it follows that

$$\lambda_j^{(kt)} = \left( \lambda_j^{(t)} \right)^k \quad (2.22)$$

Using this result, we find

$$\begin{aligned} \mathbb{B}_t \phi_j^{(t)} &= \lambda_j^{(t)} \phi_j^{(t)} \Rightarrow \mathbb{B}_t \mathbb{B}_t \phi_j^{(t)} = (\lambda_j^{(t)})^2 \phi_j^{(t)} \\ &\Rightarrow \dots \\ (\mathbb{B}_t)^k \phi_j^{(t)} &= (\lambda_j^{(t)})^k \phi_j^{(t)} \Rightarrow \mathbb{B}_{kt} \phi_j^{(t)} = (\lambda_j^{(kt)}) \phi_j^{(t)} \end{aligned}$$

That is,

$$\phi_j^{(kt)} = \phi_j^{(t)} \quad (2.23)$$

From which it immediately follows that 2.22 and 2.23 are valid for positive rational  $k$  and, since  $\mathbb{B}_t$  is continuous with respect to  $t$ , we have 2.22 and 2.23 for all positive  $k$ . Which means that

$$\lambda_j^{(t)} = \left(\lambda_j^{(1)}\right)^t = \exp\left(t \log \lambda_j^{(1)}\right) \text{ and } \phi_j^{(t)} = \phi_j^{(1)}$$

Since, as seen in 2.2.15,  $(\mathbb{B}_t f, f)_g$  is decreasing in  $t$ , we have that the argument of the exponential must be negative.

If we set

$$\lambda_j = -\ln \lambda_j^{(1)}, \text{ and } \phi_j = \phi_j^{(1)}$$

the eigenvalue equation reads

$$\begin{aligned} \mathbb{B}_t \phi_j^t &= \lambda_j^{(t)} \phi_j^t \\ \mathbb{B}_t \phi_j &= e^{-\lambda_j t} \phi_j \end{aligned}$$

for all non-negative integer  $j$ .

In particular, we have that  $\phi_j$  is  $C^k(M)$  if and only if  $G$  is. Finally, since  $\mathbb{B}_t \phi_j$  is a solution to the heat equation we have

$$\begin{aligned} 0 = L(\mathbb{B}_t \phi_j) &= e^{-\lambda_j t} \Delta \phi_j - \phi_j \frac{d}{dt} e^{-\lambda_j t} \\ &= e^{-\lambda_j t} (\Delta \phi_j + \lambda_j \phi_j) \end{aligned}$$

and  $\phi_j$  is an eigenfunction of  $\Delta$  with eigenvalue  $\lambda_j$ . The behaviour of  $\lambda_j$  as  $j \rightarrow \infty$  follows from the fact that the  $\lambda_j^{(1)}$  are increasing with respect to  $j$  and  $\lambda_j = -\ln \lambda_j^{(1)}$ .

The expansion 2.19 is valid on  $L^2(M \times M)$  by the Hilbert-Schmidt theorem (cf. [54]), which states precisely that an integral operator with a symmetric kernel in  $L^2(M \times M)$  obeys such decomposition. Since we have that

$$\int_M \int_M G(x, y, t) G(x, y, t) dx dy = \int_M G(x, x, 2t) dx \quad (2.24)$$

$$= 1 \quad (2.25)$$

where the last step follows from the last claim in 2.2.17, which holds for all  $y \in M$  and  $t \in (0, \infty)$ , the kernel is square summable. Since  $G$  is a continuous (2.2.13), symmetric (by the first claim in 2.2.17) and non-negative definite kernel (cf. 2.16), we have that the expansion 2.19 is valid with respect to

absolute and uniform convergence via Mercer's theorem, which we state for completeness (for details, see [30]):

**Theorem 2.2.19 (Mercer's Theorem)** *Let  $K(x, y)$  be a continuous, symmetric kernel such that the associated integral operator is non-negative; then 2.19 holds and the series converges uniformly and absolutely.*

The claim about the trace is straightforward.

It remains to show the existence of the kernel of the heat equation, which will be treated in the following section, for closed manifolds.

### 2.2.10 Existence for closed Manifolds

We now proceed to showing how to construct a fundamental solution for closed manifolds, which suffices to satisfy the conditions in 2.2.18 and the claim therein. The proof is rather technical and long, and we merely provide a summary of it. For a complete exposition see [14].

#### Summary of the Proof

The general idea of the proof is to construct the fundamental solution using as a model that of Euclidean space:

$$\mathcal{E}(x, y, t) = (4\pi t)^{-\frac{n}{2}} e^{-d^2(x,y)/4t} \quad (2.26)$$

where  $d$  is the Euclidean distance and  $n$  is the dimension. Intuitively, we expect, for small enough  $t$ , this to be a good model: after all, a Riemannian manifold is locally Euclidean.

So we begin by introducing Riemann normal coordinates, wherever they can be used. We assume that *every* point has a neighbourhood that is well described by these coordinates. To make matters more precise, define [58]

**Definition 2.2.20 (Injectivity radius)** *Let  $M$  be a geodesically complete manifold. We define the injectivity radius at  $x$ ,  $inj_x(M)$  as the supremum of the radii of the balls such that the exponential map is a diffeomorphism. We define the injectivity radius of  $M$  as*

$$inj(M) = \inf_{x \in M} \{inj_x(M)\}$$

We now assume that the injectivity radius of  $M$  is *strictly* positive (that is, normal coordinates can be used on a neighbourhood of every point).

We use 2.26 as a first order approximation, after cutting it off with a step function  $\eta$ , that vanishes outside the injectivity radius:

$$H_0 = \eta \mathcal{E}$$

where the Euclidean distance in 2.26 is replaced by the Riemannian distance.

To define higher order approximations, we introduce functions  $u_i$ :

$$u_i(\cdot, y) : \mathbf{B}_{\text{inj}(M)}(y) \rightarrow \mathbb{R}$$

and look for an approximate solution by modulating the Euclidean kernel with a power series in  $t$

$$H_k = \mathcal{E} \sum_{i=0}^k u_i(x, y) t^i$$

where the  $u_i$  are defined using the following induction formula:

$$L_x H_k = \mathcal{E} t^k \Delta_x u_k$$

Since these formulas cannot be expected to hold globally, one again cuts the  $H_k$  using the step function  $\eta$  defined above:

$$\mathbb{H}_k = \eta H_k$$

These functions, that approximate a kernel, are called *parametrics*; they share a few of the kernel's properties (they are symmetric and are asymptotically delta functions for small times).

The final step in the construction of the kernel is done via a convolution product thus defined:

**Definition 2.2.21 (Convolution Product)** *Given functions  $A, B \in C^0(M \times M \times [0, \infty))$  we define their convolution by*

$$(A * B)(x, y, t) = \int_0^t d\tau \int_M A(x, z, \tau) G(z, y, t - \tau) \sqrt{g} dx$$

The idea is to construct a kernel of the form

$$K = \mathbb{H}_k + \mathbb{H}_k * F$$

That is, finding  $K$  such that  $L_x K = 0$ . Using the ansatz above this is

$$L_x (\mathbb{H}_k + \mathbb{H}_k * F) = 0$$

and this can be seen [14] to be equivalent to finding  $F$  such that

$$L_x \mathbb{H}_k = F - (L_x \mathbb{H}_k)$$

from which one has

$$F = \sum_{i=0}^{\infty} (L_x \mathbb{H}_k)^{*i} \quad (2.27)$$

It remains to investigate the convergence of the series 2.27. It can be shown [14] that it does, for  $k > n/2 + 2$ .

And so we have that, for  $k > n/2 + 2$ ,

$$K(x, y, t) = \mathbb{H}_k + (L_x \mathbb{H}_k * F)(x, y, t) \quad (2.28)$$

$$= \frac{e^{-d^2(x,y)/4t}}{(4\pi t)^{n/2}} \left( \rho(d(x, y)) \sum_{i=0}^k t^i u_i(x, y) + O(t^{k+1}) \right) \quad (2.29)$$

is a fundamental solution for the heat equation on  $M$ . Furthermore, the fundamental solution is  $C^\infty$  on  $M \times M \times (0, \infty)$ , which implies, as seen above in 2.2.18, that all the eigenfunctions of the Laplace operator are  $C^\infty$ .

Finally, combining 2.2.18 with 2.28 we get the following expression for the trace of the fundamental solution, known as the **Minakshisundaram-Pleijel asymptotic formula** (or, in the physics literature, as the **Seeley-de Witt** formula):

$$\sum_{i=0}^{\infty} e^{-\lambda_i t} = \int_M K(x, x, t) \sqrt{g} dx \quad (2.30)$$

$$= \frac{1}{(4\pi t)^{n/2}} \left( \sum_{i=0}^k t^i \int_M u_i(x, x) \right) + O(t^{k+1-n/2}) \quad (2.31)$$

In particular, we have, setting  $\int u_i := a_i$ , as  $t \rightarrow 0$

$$\sum_{i=0}^{\infty} e^{-\lambda_i t} \sim (4\pi t)^{-n/2} (a_0 + a_1 t + \dots) \quad (2.32)$$

The  $a_i$  can be seen to be expressible as integrals of polynomials of the curvature tensor and its successive covariant derivatives [14]. For a derivation of the formula for small deviations from a flat metric, see [48] chap. 13.

We will have some more to say about the  $a_i$  further on. For now it suffices to mention that they promise a link between two seemingly distinct objects: the spectrum of the Laplacian, on the one hand, and the geometry of  $M$ , on

the other.

### **Manifolds with boundary**

The structure of the proof of the existence of a discrete basis for  $L^2(M)$  given in the section above carries largely over to the case of manifolds with boundary: the governing equation for the diffusion problem being the same, while Dirichlet or Neumann boundary conditions (or mixed) need to be set. For compact manifolds with boundary, the existence of the kernel of the heat equation with Dirichlet boundary conditions follows rather the same program; the proof can be found in [14], for regular domains. For domains with non-smooth boundaries there is a plethora of similar results (with added difficulty, of course) which we will not present here. For a review of some of the results see, e.g. [14][70][71].

Although the proof of existence of the Heat Kernel we presented is dependent on the existence of a non-negative injectivity radius and positive sectional curvature, these are requirements that, as explained, have specifically to do with the use of the Euclidean kernel as a model for our Riemannian kernel and the use of Riemann normal coordinates. There are alternative proofs of existence of the heat kernel that do not impose these constraints: e.g., using the theory of pseudo-differential operators with parameter, [39], [22]. Another approach, that can be generalized to non-elliptic operators can be found in an unpublished review by Greiser [26].

### **Pseudo-Riemannian Manifolds**

There are considerable difficulties in adapting the results from the Riemannian to the pseudo-Riemannian case (i.e. to indefinite metrics). This specifically has to do with the Laplacian no longer being an elliptic operator.

Although outside the scope of this thesis, we briefly mention that the common approach, for Lorentzian metrics, is to Wick-rotate the time coordinate, that is, performing a coordinate transformation to ‘imaginary time’,  $t \rightarrow it$ . This effectively turns a Lorentzian metric into a Riemannian one, and one can then proceed to employing the techniques that are valid in the Riemannian setting, e.g. the existence of a discrete and complete spectral decomposition and the expansion 2.32.

There are many applications of heat kernel techniques to physical problems in general and to quantum gravity in particular, see e.g. [21].



## Chapter 3

# Inverse Spectral Geometry

### 3.1 Motivation

The birth of the field of inverse spectral geometry can be traced back to the 1960's, when Mark Kac posed the question: "Can one hear the shape of a drum?"[31]. The question has been answered, for particular cases, many times since, in the positive and in the negative, for drums and their generalizations (manifolds). Conditions under which the question has a positive answer have been established and methods to construct 'drums' that answer it in the negative have been created.

In this chapter, we present a few results concerning geometric information that can be derived via the knowledge of the spectrum of the Laplacian. It is meant an illustration of the relation that exists between the spectrum and the underlying geometry and as such, it is brief.

For clarity, we divide the results in "positive" and "negative" in our presentation.

### 3.2 Positive Results

Most of the positive results have been established via the study of geometric invariants that are obtained as the coefficients of expansions of kernels of operators involving the Laplacian [70]. In establishing the existence of the Heat Kernel for closed Riemannian manifolds, we have encountered one such expansion, the Minakshisundaram-Pleijel recursion formula.

We list some of the coefficients of this expansion below and refer geometric information that can be obtained via the study of these coefficients. Specifically, we discuss 'rigidity' results in which certain classes of manifolds can be shown to be completely identified by their spectrum.

### 3.2.1 Heat Invariants Results

In this section, we present a list of heat invariants and mention a few positive results that can be derived from their knowledge.

Although the coefficients of the Minakshisundaram-Pleijel asymptotic formula ( $a_i = \int_M u_i(x, x) \sqrt{g} dx$ , where the  $u_i$  are defined in 2.28) are in principle computable to arbitrary order, concrete expressions for high-order coefficients are difficult to give. The first coefficient is easy enough, because  $u_0(x, x) = 1$  and so  $a_0 = \text{vol}(M)$ . For completeness, we list the first three:

$$\begin{aligned} a_0 &= \text{vol}(M) \\ a_1 &= \frac{1}{6} \int_M R \sqrt{g} dx \\ a_2 &= \frac{1}{360} \int_M (2R^{ijkl} R_{ijkl} - 2\text{Ric}^{ij} \text{Ric}_{ij} + 5R^2) \sqrt{g} dx \end{aligned}$$

where the terms in the last integrand are, respectively, the Riemann curvature tensor, the Ricci tensor and the scalar curvature.

We shall be interested in particular in the case where  $\dim(M) = 2$ , for which a straightforward computation shows that the integrand in  $a_2$  is  $6R^2$ .

It should be mentioned that the coefficients have been computed to higher orders although the calculations quickly become very cumbersome and explicit expressions are too large to present here. Historically, the third order coefficient was found quite early on [59], whereas explicit formulae for the fourth coefficient were only found some twenty years later [1] [2]). For an overview on the different methods used in calculating these coefficients, see [4].

Although outside the scope of this thesis, we mention that, in the particular case of symmetric spaces, however, it is even possible to find a generating function for the *entire* sequence of heat invariants [3].

### 3.2.2 Weyl's asymptotic formula

Weyl's asymptotic formula is a surprising result that provides a connection between the knowledge of the spectrum of the Laplace operator on a given manifold (or equivalently, of the trace of the Heat kernel) and the volume of said manifold. We state it below:

**Proposition 3.2.1 (Weyl's asymptotic formula)** *Let  $N(\lambda)$ , the counting function, denote the number of eigenvalues of the Laplacian below  $\lambda$ . Then, as  $\lambda \rightarrow \infty$*

$$\text{Vol}(M, g) \sim (4\pi)^{n/2} \Gamma\left(\frac{n}{2} + 1\right) \frac{N(\lambda)}{\lambda^{n/2}} \quad (3.1)$$

Although not historically how it was found [18], Weyl's formula can be shown to be a corollary of the expansion 2.32, as we do below, following [18] closely, for completeness. To do so, we first state the Hardy-Littlewood-Karamata (HLK) Theorem:

**Theorem 3.2.2 (HLK Theorem)** *Let  $N(\lambda)$  be a non-decreasing function such that*

1.  $f(t) = \int_0^\infty e^{-\lambda t} dN(\lambda)$  converges for all  $t > 0$ .
2.  $f(t) \sim \frac{A}{t^\alpha}$  as  $t \rightarrow 0^+$  for some  $\alpha > 0$

Then, as  $\lambda \rightarrow \infty$

$$N(\lambda) \sim \frac{A\lambda^\alpha}{\Gamma(\frac{\alpha}{2} + 1)}$$

To apply this theorem, we note that 2.32 implies, in particular, as  $t \rightarrow 0$  that:

$$\sum_{k=0}^{\infty} e^{-\lambda_k t} \sim \frac{\text{vol}(M)}{(4\pi t)^{n/2}}$$

We can write the summation on the LHS of this expression as an integral with the discrete measure  $dN(\lambda)$ , where  $N(\lambda)$  is the counting function introduced in 3.2.1:

$$\sum_{k=0}^{\infty} e^{-\lambda_k t} = \int_0^\infty e^{-\lambda t} dN(\lambda) \tag{3.2}$$

And a straightforward application of the HLK theorem yields the formula 3.1.

### Wave Invariants

The *wave invariants* are the coefficients in the trace of the wave kernel, the kernel of the operator that models the propagation of waves on  $M$ :

The study of the kernel for this operator is rather more difficult than that of the heat kernel (in fact, the kernel of the wave operator is a distribution [71]). Although outside the scope of this thesis, we remark that the study of the invariants of its trace has yielded some encouraging results: for a review on this subject and a list of results see e.g. [70][71].

### 3.2.3 Rigidity Results

In this section, we provide a concise review of a result in [42] concerning the rigidity of flat tori with respect to what the author calls "infinitesimal

deformations". Informally, the author establishes that any sufficiently small deformation of the flat torus, in any dimension, will not have the spectrum of the flat torus. In this sense, flat tori in any dimension are completely identified (locally) by their spectra.

We follow [42] closely and begin with settling some terminology:

**Definition 3.2.3 (Infinitesimal Deformation)** *We call a smooth, one-parameter family of Riemannian metrics  $g(t)$  on  $M$ , such that  $g(0) = g$ , a deformation of the metric  $g$ . An infinitesimal deformation of  $g$  is the symmetric covariant two tensor  $h$  such that  $h = g'(0)$ .*

A deformation is said to be isospectral if, for any two elements of the family  $g(t)$ , the spectra of the respective Laplace operators on functions coincide.

The paper begins by decomposing the perturbation to the metric tensor, much in the same way as will be presented in Chapter 5:

$$h = \tilde{h} + \mathcal{L}_X(g) \tag{3.3}$$

with  $\nabla^i \tilde{h}_{ij} = 0$ . That a decomposition of this kind exists will be shown in 5.5.1 (indeed we show a stronger result, cf. 5.32).

Following the article, we call an infinitesimal perturbation *trivial* if  $\tilde{h} = 0$  or, in the language of 5.4, if there is a choice of gauge such that the components of  $h$  are zero.

The main result in [42] is the following theorem, valid in any dimension:

**Theorem 3.2.4 (Rigidity of Flat Tori)** *There is no non-trivial isospectral deformation of a flat Torus.*

The central idea of the proof is to study the variation of the coefficients in the MP expansion 2.32 with respect to the infinitesimal deformations defined above.

The argument goes as follows: for any isospectral deformation  $g(t)$ , since the spectrum of the Laplace operator is a geometric invariant (recall that the Laplace operator is a natural operator) the coefficients in the MP expansion, seen as functions of the parameter  $t$ , must be constants. In the notation in 2.32,  $a'_i(0) = 0$  for all  $i$ .

A computation that we shall not present shows that, for the flat torus, this implies that infinitesimal deformations that do not change the spectrum are necessarily trivial, i.e., all metrics on a neighbourhood of the flat torus in any dimension do not have the flat torus' spectrum.

**Remark** Because I shall look into the flat torus in two dimensions in Chapter 7 in the context of perturbation theory, I would like to make the following clarification: there is nothing perturbative in the statement ' $g$  is isospectral to  $\tilde{g}$ '. It means that the spectra of the respective Laplace operators coincide.

In perturbation theoretic terms, the isospectrality statement means that the two spectra coincide to **all** orders.

Rigidity results exist for much larger classes of manifolds. As an example (the largest class, in fact), we state the following result [16]

**Theorem 3.2.5** *A compact negatively curved Riemannian manifold is spectrally rigid.*

### 3.3 Negative Results

We present below a brief overview of some results that state that some classes of manifolds cannot be distinguished from the knowledge of certain of their spectra. There are many examples of such results, although this reflects perhaps more the difficulty of the inverse problem than anything else. The appearance of methods that allow one to construct families of manifolds with the same spectrum has also contributed to the vast number of examples.

We refer to the papers listed below for a more thorough overview on this subject. By ‘ $g$  and  $\tilde{g}$  are isospectral’ we mean that they have the same spectrum of the Laplacian on functions.

A complete description of these techniques is outside the scope of this thesis because, although the ‘size’ of the class of isospectral non-isometric manifolds in the set of all manifolds is unknown, the difficulty in finding examples for which the spectral map is not invertible and the particularity of such examples suggests that it is perhaps a small subset (i.e. a ‘finite-dimensional’ subset, in some sense). To our knowledge, there is no indication that this is not the case.

The first example of isospectral, non-isometric Riemannian manifolds, a pair of 16 dimensional flat tori was provided by Milnor in 1964 [47].

In 1985 T. Sunada provided a number theoretic inspired technique to explicitly construct isospectral manifolds [65]; several generalizations quickly surfaced; the first of which [17] by C. Gordon et al., constructs manifolds that are indeed also strongly isospectral, i.e., the spectra of the Laplace de Rham operator on forms of arbitrary degree coincides. Other generalizations of Sunada’s method, [6][7][50] are based on representation theoretic techniques and provide isospectral manifolds that are locally isometric, that is:

There is a *local diffeomorphism*, a map  $f : M \rightarrow N$  between the two differentiable manifolds of the same dimension,  $(M, g)$  and  $(N, h)$ , such that for every point in  $M$  there is an open set  $U$  that contains it whose image under  $f(U)$  is open and for which the restriction of  $f|_U : U \rightarrow f(U)$  is a diffeomorphism; and such that the metric induced by  $f$  via pullback,  $f^*h = g$  (cf. [58]).

A different method, that produces isospectral manifolds with different local geometry, and some examples of its application, is described in [25]; we

refer to the discussion within on the particular geometrical aspects in which the families of isospectral manifolds constructed in such a way differ. As an example of further applications, see [24] and references therein.

A natural question, given the apparent abundance of examples is the following: can all isospectral non-isometric manifolds be generated by these techniques? The question is difficult and remains unanswered.

There is a partial answer, a 'generic' converse of Sunada's theorem, which was established by H.Pesce in [53] (see [52] for details; also [51]). Without going into details, this converse provides conditions for two manifolds that are known to be isometric to be "Sunada Pairs", i.e., the result of the application of Sunada's technique.

For a clear and succinct exposition, on this matter and on the technique itself and a converse for graphs, see [13].

## Chapter 4

# Infinitesimal Inverse Spectral Geometry

Sampling theoretical methods, Shannon's sampling theorem in particular, are of everyday use. In the following section, we outline the method proposed to generalize sampling theory to fields on manifolds and manifolds themselves first outlined in [38] and detailed in [37], which we follow closely.

### 4.1 Helmholtz Argument

We begin by noting the following argument by Einstein, who credits Helmholtz:

The shape of a manifold does not manifest itself solely by the non-triviality of the parallel transport of tensors. One can also think of the shape of the manifold in terms of the non-triviality of mutual distances between points.

To that effect, consider  $m$  points on  $n$ -dimensional flat space. In Cartesian coordinates, these points can be represented by  $mn$  coordinate functions, and we write the  $k$ th coordinate of the  $i$ th point as  $x_k^i$ . The mutual distances between these points can be recorded in a symmetric traceless  $m \times m$  matrix with  $(m^2 - m)/2$  non trivial entries  $a_{ij}$ .

By Pythagoras' theorem, we have the same number of equations  $a_{ij}^2 = \sum_{k=1}^n (x_k^i - x_k^j)^2$ . If we have more equations than coordinates, i.e.,  $m > 2n+1$ , we can eliminate the coordinates from these equations, leaving  $m(m-1)/2 - mn$  non trivial relations between the mutual distances, that express the fact that we are in Euclidean space, where Pythagoras' theorem holds. If the manifold has curvature, these equations that relate the  $a_{ij}$  are violated.

One can then think of the shape of a given manifold as being given by this table of mutual distances between its points. Since the number of points is finite, one can of course not hope to have a complete characterization of the manifold in such a way. However, it does provide a 'skeleton' of the shape, that should serve as a good description at large scales [37].

The key idea now is to adapt Helmholtz argument to a different notion of

distance, one that is more suited to quantum field theory. Instead of taking the table of geodesic distances, one takes the amplitude of the correlator  $G(x, y)$  between two points as a measure of distance between those two points.

To argue that this is indeed a good measure of distance we cite the following result by Varadhan [68], for the heat equation in particular:

$$\lim_{t \rightarrow 0^+} t \log G(x, y, t) = -d^2(x, y) \quad (4.1)$$

where  $G$  is the heat kernel,  $d$  is the geodesic distance and the formula holds within the injectivity radius (cf.2.2.20).

## 4.2 Sampling Fields and Manifolds

Before we proceed to describing the extension of sampling theory to both manifolds and fields, we begin with settling some notation that will prove useful in this exposition. In what follows, all manifolds are Riemannian, smooth, connected and closed, and we have shown in 2.2.18 that in this case, the spectrum of the scalar Laplacian on  $(M, g)$  is discrete and positive, with no accumulation points. The metric tensor  $g$  is assumed to be smooth as well.

**Definition 4.2.1 (Spectral Map)** *Given a manifold  $(M, g)$  we define the spectral map  $\sigma$  as the map that assigns it the partially ordered set of the eigenvalues of its Laplace operator on functions (with multiplicities). We will often refer to  $\sigma[(M, g)]$  as the spectrum of the manifold  $(M, g)$ . Where there is no ambiguity, we denote it  $\sigma(g)$ .*

**Definition 4.2.2 ( $\Lambda$ -Isospectral Manifolds)** *We call the set*

$$\sigma_\Lambda[(M, g)] = \sigma[(M, g)] \cap [0, \Lambda)$$

*the  $\Lambda$ -cutoff spectrum of the manifold  $(M, g)$ . Given two manifolds  $(M, g)$  and  $(\tilde{M}, \tilde{g})$ , we say that they are  $\Lambda$ -isospectral if*

$$\sigma_\Lambda[(M, g)] = \sigma_\Lambda[(\tilde{M}, \tilde{g})]$$

*We say that two manifolds are isospectral if they are  $\Lambda$ -isospectral for all  $\Lambda > 0$ . We denote  $\Sigma_\Lambda[(M, g)]$  the equivalence class of manifolds that are  $\Lambda$ -isospectral to  $(M, g)$ . Where there is no ambiguity, we denote it  $\Sigma_\Lambda[g]$ .*

Finally, we provide the following covariant generalization of the notion of bandlimited function in Shannon's sampling theorem 1.1:

**Definition 4.2.3 (Bandlimited Field)** *We call  $\phi \in C^\infty(M)$   $\Lambda$ -bandlimited if it is in the span of the eigenfunctions of the Laplace operator on  $(M, g)$*



whose eigenvalues are smaller than  $\Lambda$ . We denote this set  $\mathcal{F}_\Lambda$ . We remark that, since the eigenfunctions of the Laplace operator are a basis for  $L^2(M)$  (cf. 2.2.18) we have  $\mathcal{F}_\infty = L^2(M)$ .

Consider now some spacetime  $(M, g)$  where a physically motivated cutoff  $\Lambda$  was imposed, in the sense that in the path integral formulation only the  $\Lambda$ -bandlimited subset of fields is considered. Letting  $\langle \phi | \psi \rangle$  denote the inner product with respect to the measure induced by  $g$  as in 6.2 (with the usual generalization for complex fields), we note that, for  $\phi \in \mathcal{F}_\Lambda$  we have  $\phi(x) = \langle x | \phi \rangle = \langle x | P_\Lambda | \phi \rangle$ .

This intuitively expresses a minimum length uncertainty principle, for the point-localized fields  $|x\rangle$  are, as far as a theory restricted to  $\mathcal{F}_\Lambda$  is concerned, indistinguishable from the fields  $P_\Lambda |x\rangle$ .

Take now  $N$  generic points in  $M$ ,  $\{x_1, x_2, \dots, x_N\}$ , to which correspond, as we have seen above, the  $N$  kets  $\{P_\Lambda |x_1\rangle, \dots, P_\Lambda |x_N\rangle\}$ . Assume that the correlators for, e.g., a free scalar for all these points are explicitly known

$$a_{ij} = \langle x_i | P_\Lambda (\Delta + m^2)^{-1} P_\Lambda |x_j \rangle \quad (4.2)$$

Since  $P_\Lambda = \sum_{\lambda_i < \Lambda} |\phi_i\rangle \langle \phi_i|$ , where  $\Delta(g)\phi_i = \lambda_i \phi_i$ , we get, plugging in 4.2

$$a_{ij} = \sum_{m,n=1}^N \langle x_i | \phi_m \rangle (\langle \phi_m | (\Delta + m^2)^{-1} | \phi_n \rangle) \langle \phi_n | x_j \rangle \quad (4.3)$$

$$= b_{im} \tilde{a}_{mn} b_{nj}^\dagger \quad (4.4)$$

where summation over repeated indices is implied. Since the Laplacian and  $(\Delta + m^2)^{-1}$  are diagonal in the same basis, the matrix  $\tilde{a}$  is diagonal and knowledge of its spectrum is equivalent to that of the first  $N$  eigenvalues of the Laplacian. The matter now is to show that the matrix  $a$  can be diagonalized. To that effect, note that

$$\tilde{a} = b^{-1} a (b^\dagger)^{-1} \quad (4.5)$$

which means that, since  $\tilde{a}$  is known, its explicit diagonalization amounts to the knowledge of the  $b_{im} = \langle x_i | \phi_m \rangle$ s.

To proceed, we assume that the inner products between the vectors  $P_\Lambda |x_j\rangle$  are known for all  $i, j$ . That is, that we have, in  $\mathcal{F}_\Lambda$

$$c_{ij} := \langle x_i | P_\Lambda^2 |x_j \rangle \quad (4.6)$$

$$= \langle x_i | \text{Id}_{\mathcal{F}_\Lambda} |x_j \rangle \quad (4.7)$$

$$= \sum_{k=1}^N \langle x_i | \phi_k \rangle \langle \phi_k | x_j \rangle \quad (4.8)$$

$$= b_{ik} b_{kj}^\dagger \quad (4.9)$$

From which it follows that  $(b^\dagger)^{-1} = c^{-1}b$ . Inserting these back in 4.5, we get

$$\tilde{a} = b^{-1}ac^{-1}b \quad (4.10)$$

Since we have assumed knowledge of both  $a$  and  $c$ , we can diagonalize it. And since, by the last expression above,  $\tilde{a}$  and  $ac^{-1}$  are similar matrices, they have the same characteristic polynomial, and so we recover the eigenvalues of  $\tilde{a}$  from the diagonalization of the known matrix  $ac^{-1}$ . Since  $\tilde{a}$  is diagonal in the eigenbasis of the Laplacian, we see that we can recover the eigenvalues of the Laplacian from this diagonalization, which in turn allows us to recover the matrix  $b_{im} = \langle x_i | \phi_m \rangle$ .

To sum up:

1. We assumed the presence of an ultraviolet cutoff  $\Lambda$  on the spectrum of the Laplacian, such that elements of  $\Sigma_\lambda[(M, g)]$  are indistinguishable and that elements of  $\mathcal{F}_\Lambda$  are indistinguishable.
2. Within these equivalence classes, we have assumed known a description of the manifold  $(M, g)$  in terms of a matrix of correlators sampled at  $N$  generic points.
3. We further assumed the knowledge of the matrix of the position overlaps,  $\langle x_i | x_j \rangle$  and that it is invertible (again, within the equivalence class).
4. We conclude that the assumptions above amount to the knowledge of  $\sigma_\Lambda[(M, g)]$ . We have thus obtained a description of the equivalence class  $\Sigma_\lambda[(M, g)]$  in terms of  $\sigma_\Lambda[(M, g)]$

The third assumption above can be somewhat relaxed, or rather, replaced with the sampling of the correlator for a scalar field with different mass  $M^2 \neq m^2$ . The third assumption can then be seen a special case, for  $\lim_{M \rightarrow \infty} \langle x_i | (\Delta + M^2)^{-1} | x_j \rangle = \langle x_i | x_j \rangle$ . As in the first case, the matrix for this second correlator, in order that the cutoff spectrum be recoverable, is also required to be invertible. For details, see [37].

As a matter of interpretation, we discuss the invertibility assumption further. To assume that the  $N \times N$  matrix of position overlaps is invertible amounts to assuming that the set  $\{P_\Lambda | x_1 \rangle, \dots, P_\Lambda | x_N \rangle\}$  is a basis for the  $N$ -dimensional space  $\mathcal{F}_\Lambda$ .

But since one is considering projections onto an  $N$ -dimensional vector space of general vectors that live on an infinite dimensional space, the invertibility assumption can be expected to hold generally.

It remains to show that a reconstruction of a field  $\Psi$  is also possible. To see that, we note that, if we have the samples  $\langle x_i | \Psi \rangle$ , we have

$$\langle x_i | \Psi \rangle = \sum_{k=1}^N \langle x_i | \phi_k \rangle \langle \phi_k | \Psi \rangle \Leftrightarrow \quad (4.11)$$

$$\Psi_i = b_{ik} \tilde{\Psi}_k \quad (4.12)$$

and since the matrix  $b$  is known and invertible, we can recover the coefficients of  $\Psi$  in the eigenbasis of the Laplacian in the cutoff space  $\mathcal{F}_\Lambda$ .

As claimed in the introduction, it is then possible to establish an equivalence between discrete and continuous representations of both manifolds and fields, in the equivalence classes  $\Sigma_\Lambda[(M, g)]$  and  $\mathcal{F}_\Lambda$ .

### 4.3 Finite part of the spectrum

In the previous sections we showed, following [37], that a description of the geometric information about a given Riemannian manifold  $M$  that can be recovered via Helmholtz argument, can be written in terms of cutoff spectrum of the Laplacian. The following two questions are then natural:

1. We know by 2.2.18 that the set  $\sigma_\Lambda[(M, g)]$  is a finite set of positive real numbers. Are there any other constraints?
2. We detailed how the finite part of the spectrum below a given cutoff can be recovered from a discrete sampling of the correlator. Intuitively, one should not expect to be able to resolve geometric information for scales smaller than the sampling density. Thus, the finite part of the spectrum should not contain information on scales smaller than the cutoff scale. But what does this mean exactly? If we fix the volume and the dimension, say, can we expect to see, e.g. a bound on curvature?

The answer to the first question is provided by the following theorem [15]:

**Theorem 4.3.1** *Given a compact manifold  $M$  of dimension no smaller than 3 and a **finite**  $X \subset \mathbb{R}^+$ , there is a Riemannian metric  $g$  on  $M$  for which the first eigenvalues are  $X$ . That is, for some  $\Lambda > 0$ , we have  $\sigma_\Lambda[(M, g)] = X$ .*

As for the second question, an obvious place to look for answers is the asymptotic expansion 2.32, because it relates the *full* knowledge of the spectrum with geometric invariants.

It is natural to ask if one can recover the coefficients in said expansion if a sufficiently large but finite number of eigenvalues are known. The following two theorems, that can be found in [44] provide an answer to this question:

**Theorem 4.3.2** *Let  $\mathcal{M}^d$  denote the set of symmetric positive-definite covariant 2-tensors on an  $d$ -dimensional manifold  $M$ . Given a sequence of real numbers  $U = \{u_n\}_{n \in \mathbb{N}}$  denote  $U_k = \{u_i \in U : i \leq k\}$ .*

*The following then holds:*

*For every sequence of real numbers  $\Lambda = \{0 < \lambda_1 \leq \dots\}$  and  $V \in (0, \infty)$  and  $S \in \mathbb{R}$  there is a sequence of elements of  $\mathcal{M}^d$ ,  $\{g_n\}_{n \in \mathbb{N}}$  such that*

1. *For  $k \leq m$*

$$\sigma_{\lambda_{k+1}}[g_m] = \Lambda_k \quad (4.13)$$

2. *Letting  $\sigma$  denote the scalar curvature, we have, for all  $m$*

$$\text{Vol}(M, g_m) = V \quad (4.14)$$

$$\int_M \sigma \sqrt{g_m} dx = S \quad (4.15)$$

3. *Letting  $a_i(M, g)$  denote the coefficients of the MP-asymptotic expansion 2.32, we have, for  $k \geq 1$*

$$\lim_{m \rightarrow \infty} a_{2k}(M, g_m) = +\infty \quad (4.16)$$

$$\lim_{m \rightarrow \infty} a_{2k+1}(M, g_m) = -\infty \quad (4.17)$$

If we relax the constraint on the sectional curvature, we have

**Theorem 4.3.3** *For every sequence of real numbers  $\Lambda = \{0 < \lambda_1 \leq \dots\}$  and  $V \in (0, \infty)$  there is a sequence of elements of  $\mathcal{M}^d$ ,  $\{g_n\}_{n \in \mathbb{N}}$  such that*

1. *For  $k \leq m$*

$$\sigma_{\lambda_{k+1}}[g_m] = \Lambda_k \quad (4.18)$$

2. *For all  $m$ ,*

$$\text{Vol}(M, g_m) = V \quad (4.19)$$

$$(4.20)$$

3. *Letting  $\sigma$  denote the scalar curvature, we have, for all  $m$*

$$\sigma(g_m) < -m^2 \quad (4.21)$$

To sum up:

We saw in 4.3.1 that the finite part of the spectrum of the scalar Laplacian, for compact, connected Riemannian manifolds in dimension greater than two can be arbitrarily prescribed. That is, given a compact connected manifold

of dimension larger than two, and an arbitrary finite set of positive numbers (and zero), there is always a metric tensor for which the first eigenvalues of the Laplacian consists of those numbers.

We mention that a similar result exists for the finite spectra of Hodge Laplacians on forms of any degree [27]. The result holds for dimension greater than one, but there is a constraint on the multiplicity. Additionally, the spectra of all the Hodge Laplacians prescribed in such fashion are the same: they cannot be prescribed independently.

In the language of the previous section, we say that, given a manifold  $M$ , every finite set of non-negative set of numbers identifies an equivalence class  $\Sigma_\Lambda[(M, g)]$ .

Theorems 4.3.2 and 4.3.3 show that the finite part of the spectrum can only encode coarse geometric information on the underlying manifold and provide a striking parallel with the idea that the matrix of samples of the correlator at a finite number of points can only provide a skeleton of the shape of the manifold.

## 4.4 Strategy Outline

Despite the caveats in the previous section, we expect, as argued in Chapter 3, the knowledge of the full spectrum of the Laplacian on functions,  $\sigma_\infty[(M, g)]$  to encode a great deal of geometric information.

In particular, we expect  $\sigma_\Lambda[(M, g)]$  to provide geometric information in at least the ‘coarse’ sense explained in 4.1 and 4.3. In this sense, we saw how a member that defines the equivalence class  $\Sigma_\Lambda[(M, g)]$  can be recovered by sampling the correlator at sufficiently many points. Also, we saw how, if the amplitudes of a scalar field are sampled at said points, it can be recovered from the knowledge of those samples.

To extend the program further, we need to understand the following question: if  $g \neq \tilde{g}$  does it imply that  $\sigma_\infty[g] \neq \sigma_\infty[\tilde{g}]$ ? The answer is important in what concerns us: because if not true, it would mean that our characterization of the set of all metrics on  $M$  in terms of its spectrum would not be one-to-one.

As outlined in 3 the answer is notoriously difficult. Following [37], we look into it in a slightly simplified form:

Consider manifolds  $(M, g(t))$  that can be described as a scalar perturbation of  $(M, g(0))$ . These can in turn be described by a finite number of samples, as seen above, say  $N$ . Which, following the method outlined in the previous section, allow us to recover its  $N$  coefficients in the eigenbasis of the cutoff Laplacian.

To this small change in the metric will correspond, with caveats discussed at length in Chapter 5, a sequence of perturbations to the first  $N$  eigenvalues of the Laplacian below the cutoff.

The idea now is to study the invertibility of this linear approximation to  $\sigma_\Lambda[(M, g(t))]$  at  $t = 0$ . Since the map is linear and from  $\mathbb{R}^N$  into  $\mathbb{R}^N$ , we expect this to be generally possible.

Of course, as we shall see in detail in 5, only a small subset of the set of perturbations to a given metric can be described by a scalar perturbation.

But, since we have just seen that a general scalar perturbation should be locally well described by the  $N$  eigenvalues of the Laplacian below the cutoff  $\Lambda$ , the set  $\sigma_\Lambda[(M, g)]$  cannot possibly describe a general perturbation to the metric tensor.

However, one expects [37] that, in the same way as for the case of scalar perturbations, the knowledge of the coefficients of the vector and tensor perturbations in the eigenbasis of the Laplacian on vectors and tensors would suffice to describe the manifold in terms of the respective spectra.

## Chapter 5

# First-Order Perturbation of the Laplacian

In what follows  $(M, g)$  is a closed Riemannian manifold, for which we have shown the existence of an orthonormal basis for the Hilbert space  $L^2(M)$  consisting of eigenfunctions of the scalar Laplacian in 2.2.18.

We wish to determine the spectrum of the Laplace operator  $\Delta(g + h)$ , where  $h$  is a small perturbation to the metric (to be made precise below).

We begin by finding an explicit expression for the Laplace operator for such a small perturbation, thus defining the perturbation operator as a function of  $h$ .

We proceed with some results on perturbation theory, which can be found in Quantum Mechanics books, e.g [46], and it will suffice for our purposes. A more precise formulation exists, see e.g. [32]. We shall use Dirac's notation for the inner product defined in 6.2. The subscript indicates the metric.

### 5.1 First order perturbation of the Laplace operator

In this section, we set out to find the first order perturbation of the scalar Laplacian. For convenience, we use its representation in some basis, where, because  $\partial M = \emptyset$ , the following holds:

$$\int_M \sqrt{\tilde{g}} dx \phi_{,i} \psi_{,j} \tilde{g}^{ij} = \langle \phi | \Delta(\tilde{g}) \psi \rangle_{\tilde{g}} \quad (5.1)$$

Our goal is to express this operator in terms of  $\Delta(g)$ , assuming that  $\tilde{g} = g + h$ , with  $h$  small. To that aim, we expand the expression above and keep terms up to first order in perturbation.

We begin by computing the variation of the determinant with respect to

perturbations of the metric. Recall that, for finite dimensional matrices

$$\det A = \exp(\text{Tr} \log(A)) \quad (5.2)$$

where  $\text{Tr}$  denotes the trace. If  $A(s)$  is a one-parameter family of such matrices, then

$$\frac{d}{ds} \exp(\text{Tr} \log(A(s))) = \det(A) \text{Tr} \left( A^{-1} \frac{d}{ds} A \right) \quad (5.3)$$

Upon choosing coordinates, set  $A := g_{\mu\nu}$  and, for fixed  $\alpha, \beta$ , set  $s := g_{\alpha\beta}$ . In the sequel, we note  $g$  the determinant of the metric tensor. Substituting in 5.3, we get

$$\begin{aligned} \frac{\partial g}{\partial s} &= \frac{\delta g}{\delta g_{\alpha\beta}} \\ &= g g^{\mu\nu} \frac{\partial g_{\mu\nu}}{\partial g_{\alpha\beta}} \\ &= g g^{\alpha\beta} \end{aligned}$$

which, because  $\frac{\partial g_{\mu\nu}}{\partial g_{\alpha\beta}} = \delta_{\mu}^{\alpha} \delta_{\nu}^{\beta}$  is

$$\frac{\delta g}{\delta g_{\alpha\beta}} = g g^{\alpha\beta} \quad (5.4)$$

And since the determinant of the inverse is the inverse of the determinant, we find, taking the derivative of  $g g^{-1}$ ,

$$\frac{\delta g}{\delta g^{\alpha\beta}} = -g g_{\alpha\beta} \quad (5.5)$$

We now expand  $\sqrt{\tilde{g}}$  in a power series

$$\sqrt{\tilde{g}} = \sqrt{g} + \frac{1}{2} \frac{1}{\sqrt{g}} \frac{\delta \tilde{g}}{\delta g^{\alpha\beta}} \delta g^{\alpha\beta} + \dots \quad (5.6)$$

which, using 5.5 is, to first order,

$$\sqrt{\tilde{g}} = \sqrt{g} \left( 1 - \frac{1}{2} g_{\alpha\beta} \delta g^{\alpha\beta} \right) \quad (5.7)$$



Using these results, we can expand 5.1, getting

$$\begin{aligned}
\int_M \sqrt{\tilde{g}} \phi_{,i} \psi_{,j} \tilde{g}^{ij} dx &= \int_M (\sqrt{g+h}) \phi_{,i} \phi_{,j} (g^{ij} + h^{ij}) dx \\
&= \int_M \sqrt{g} (1 - \frac{1}{2} g_{kl} h^{kl}) (g^{ij} + h^{ij}) \phi_{,i} \phi_{,j} dx \\
&= \int_M \sqrt{g} \phi_{,i} \psi_{,j} g^{ij} dx + \int_M \sqrt{g} \phi_{,i} \psi_{,j} h^{ij} dx \\
&\quad - \frac{1}{2} \int_M \sqrt{g} (g_{kl} h^{kl}) \phi_{,i} \psi_{,j} g^{ij} dx
\end{aligned}$$

which can be written as an inner product with respect to the measure on  $(M, \tilde{g})$ :

$$\begin{aligned}
&\int_M \sqrt{g} \phi_{,i} \psi_{,j} g^{ij} dx - \int_M \sqrt{g} \phi_{,i} \psi_{,j} h^{ij} dx - \frac{1}{2} \int_M \sqrt{g} (g_{kl} h^{kl}) \phi_{,i} \psi_{,j} g^{ij} dx = \\
&\int_M \phi \frac{1}{\sqrt{g}} (\sqrt{g} \psi_{,j} g^{ij})_{,i} \sqrt{g} dx - \int_M \phi \frac{1}{\sqrt{g}} (\sqrt{g} \psi_{,j} h^{ij})_{,i} \sqrt{g} dx \\
&+ \int_M \phi \frac{1}{2\sqrt{g}} (\sqrt{g} (g_{kl} h^{kl}) \psi_{,j} g^{ij})_{,i} \sqrt{g} dx
\end{aligned}$$

Inverting 5.7 we get  $\frac{1}{\sqrt{g}} = (1 + \frac{1}{2} g_{kl} h^{kl}) \frac{1}{\sqrt{\tilde{g}}}$ . Thus, setting  $Tr(h) = g_{kl} h^{kl}$ , we arrive at the following expression, for the first term in the expression above ( $:= A$ ):

$$\begin{aligned}
A &= - \int_M \phi \frac{1}{\sqrt{g}} (\sqrt{g} \psi_{,j} g^{ij})_{,i} \sqrt{g} dx - \int_M \phi \frac{Tr(h)}{2\sqrt{g}} (\sqrt{g} \psi_{,j} g^{ij})_{,i} \sqrt{g} dx \\
&= \langle \phi | (1 + \frac{Tr(h)}{2}) \Delta(g) | \psi \rangle_{\tilde{g}} \\
&:= \langle \phi | \Delta(g) | \psi \rangle_{\tilde{g}} + \langle \phi | \tilde{A}(g, h) | \psi \rangle_{\tilde{g}}
\end{aligned}$$

As for the second ( $:= B$ ) and third ( $:= C$ ) terms, ignoring contributions higher than first order means we can just drop the tilde, getting

$$\begin{aligned}
B &= - \int_M \phi \frac{1}{\sqrt{g}} (\sqrt{g} \psi_{,j} h^{ij})_{,i} \sqrt{g} dx \\
&:= \langle \phi | \tilde{B}(g, h) | \psi \rangle_{\tilde{g}} \\
C &= \int_M \phi \frac{1}{2\sqrt{g}} (\sqrt{g} (g_{kl} h^{kl}) \psi_{,j} g^{ij})_{,i} \sqrt{g} dx \\
&:= \langle \phi | \tilde{C}(g, h) | \psi \rangle_{\tilde{g}}
\end{aligned}$$

Collecting all the terms, and calling the resulting operator  $\Delta(g) + \tilde{\Delta}(g, h)$  we

get the following expression

$$\langle \phi | \Delta(g) + \tilde{\Delta}(g, h) | \psi \rangle_{\tilde{g}} = \langle \phi | \Delta(g) + \frac{Tr(h)}{2} \Delta(g) + \tilde{B}(g, h) + \tilde{C}(g, h) | \psi \rangle_{\tilde{g}}$$

From which we read the perturbation operator

$$\tilde{\Delta}(g, h) = \tilde{A}(g, h) + \tilde{B}(g, h) + \tilde{C}(g, h) \quad (5.8)$$

## 5.2 First-Order Perturbation Theory

### 5.2.1 Non-degenerate eigenvalues

We assume knowledge of both the spectrum and the eigenfunctions of the Laplacian on  $(M, g)$ . We recall that these are a basis for  $C^\infty(M)$ , orthogonal with respect to the inner product  $\langle \cdot | \cdot \rangle_g$ . We further assume that they are normalized, that is, for any two eigenfunctions of the Laplacian,  $\phi_n, \phi_m$  we have  $\langle \phi_n | \phi_m \rangle_g = \delta_{mn}$ . To calculate the spectrum of the perturbed operator  $\tilde{\Delta}(g, h)$ , we assume that the perturbed eigenfunctions and eigenvalues of  $\Delta(g + h)$ , if  $h$  is small, are ‘close’ to those of  $\Delta(g)$ . That is, we assume that if

$$\Delta(g)\phi_n = \lambda_n\phi_n \quad (5.9)$$

then the following holds, for ‘small’  $h$ :

$$\Delta(g+h)(\phi_n + \delta\phi_n + \delta^2\phi_n + \dots) = (\lambda_n + \delta\lambda_n + \delta^2\lambda_n + \dots)(\phi_n + \delta\phi_n + \delta^2\phi_n + \dots) \quad (5.10)$$

where we assumed that the perturbed operator also has a discrete spectrum.

### First-Order corrections to the Eigenvalues

Expanding the expression 5.10 and keeping terms up to first order, using  $\Delta(g + h) = \Delta(g) + \tilde{\Delta}(h, g)$ , where the second term is the operator defined in 5.8, we get

$$\begin{aligned} \Delta(g + h)(\phi_n + \delta\phi_n) &= \lambda_n\phi_n + \delta\lambda_n\phi_n + \lambda_n\delta\phi_n \Rightarrow \\ \Delta(g)\phi_n + \Delta(g)\delta\phi_n + \tilde{\Delta}(h, g)\phi_n &= \lambda_n\phi_n + \delta\lambda_n\phi_n + \lambda_n\delta\phi_n \Rightarrow \\ \Delta(g)\delta\phi_n + \tilde{\Delta}(h, g)\phi_n &= \delta\lambda_n\phi_n + \lambda_n\delta\phi_n \end{aligned}$$

Expanding  $\delta\phi_n = \sum a_k\phi_k$  and taking the inner product of the expression above with  $\phi_n$ , we get

$$\begin{aligned} \sum \lambda_k a_k \langle \phi_n | \phi_k \rangle_g + \langle \phi_n | \tilde{\Delta}(h, g) | \phi_n \rangle_g &= \\ \delta\lambda_n + \lambda_n \sum a_k \langle \phi_n | \phi_k \rangle_g & \end{aligned}$$

which, recalling the orthonormality of the eigenfunctions, means that the perturbation to the  $n$ -th eigenvalue is

$$\delta\lambda_n = \langle \phi_n | \tilde{\Delta}(h, g) | \phi_n \rangle_g \quad (5.11)$$

To get the expression for the  $i$ th coefficient of the perturbed eigenfunctions, we take the inner product of the first-order term expression above with  $\phi_i$ :

$$\lambda_i \langle \phi_i | \delta\phi_n \rangle + \langle \phi_i | \tilde{\Delta}(h, g) | \phi_n \rangle = \delta\lambda_n \langle \phi_i | \phi_n \rangle + \lambda_n \langle \phi_i | \delta\phi_n \rangle$$

The case  $n = i$  is that considered above, which leads to 5.11. For  $i \neq n$ , the expression above is

$$\langle \phi_i | \delta\phi_n \rangle = \frac{\langle \phi_i | \tilde{\Delta}(g, h) | \phi_n \rangle}{\lambda_n - \lambda_i}$$

And so we get the following expression for the perturbation of the eigenfunctions

$$|\delta\phi_n\rangle = \sum_{i \neq n} \frac{\langle \phi_i | \tilde{\Delta}(g, h) | \phi_n \rangle}{\lambda_n - \lambda_i} |\phi_i\rangle \quad (5.12)$$

### Second-Order corrections

Of course one can continue the procedure above, in principle, to all orders in perturbation theory. In particular, equating second order terms in 5.10, we get

$$\Delta(g)\delta^2\phi_n + \tilde{\Delta}(g, h)\delta\phi_n = \lambda_n\delta^2\phi_n + \delta\lambda_n\delta\phi_n + \delta^2\lambda_n\phi_n$$

Again noting that the perturbations to the eigenfunctions can be written in the eigenbasis of the eigenfunctions of the Laplace operator, that is, that there are  $\alpha_i, \beta_i$  such that  $\delta\phi_n = \sum \alpha_i\phi_i$  and  $\delta^2\phi_i = \sum \beta_i\phi_i$ , we get, plugging in above

$$\sum_i \lambda_i\beta_i\phi_i + \sum_i \alpha_i\tilde{\Delta}(g, h)\phi_i = \lambda_n \sum_i \beta_i\phi_i + \delta\lambda_n \sum_i \alpha_i\phi_i + \delta^2\lambda_n\phi_n$$

Taking the inner product with  $\phi_n$  and recalling the orthonormality of the eigenfunctions we get

$$\lambda_n \beta_n + \sum_i \alpha_i \langle \phi_n | \tilde{\Delta}(g, h) | \phi_i \rangle_g = \lambda_n \beta_n + \delta \lambda_n \alpha_n + \delta^2 \lambda_n$$

Inserting 5.11, we get

$$\begin{aligned} \sum_i \alpha_i \langle \phi_n | \tilde{\Delta}(g, h) | \phi_i \rangle_g &= \langle \phi_n | \tilde{\Delta}(h, g) | \phi_n \rangle_g \alpha_n + \delta^2 \lambda_n \\ \delta^2 \lambda_n &= \sum_{i \neq n} \alpha_i \langle \phi_n | \tilde{\Delta}(g, h) | \phi_i \rangle_g \end{aligned}$$

Inserting the coefficients of the first order perturbation to the eigenfunctions that were found in 5.12 we get, for the second order correction to the eigenvalues

$$\delta^2 \lambda_n = \sum_{i \neq n} \frac{|\langle \phi_n | \tilde{\Delta}(g, h) | \phi_i \rangle_g|^2}{\lambda_n - \lambda_i} \quad (5.13)$$

## 5.2.2 Degenerate Eigenvalues

The expression 5.11 obviously only holds as such if the eigenspace of the eigenvalue  $\lambda_n$  is one-dimensional.

Assume now that  $\lambda_n$  is degenerate. Since we have seen in 2.2.18 that the degeneracy of each eigenspace of the Laplace operator is finite, we shall assume that its associated eigenspace is, without loss of generality,  $j$ -dimensional and represent each basis vector by  $\{\phi_{ni}\}_{i=0 \dots j}$ .

The unperturbed eigenstate  $\phi_n$  corresponding to the eigenvalue  $\lambda_n$  is an unknown (but fixed) linear combination of the generators of the eigenspace:

$$\phi_n = \sum_i \alpha_i \phi_{ni}$$

As above, assume that the eigenvalue equation holds approximately for the perturbed Laplacian; equating first-order terms in 5.10 we get

$$\Delta(g) \delta \phi_n + \tilde{\Delta}(h, g) \phi_n = \delta \lambda_n \phi_n + \lambda_n \delta \phi_n$$

Again using that the perturbation can be written in the original eigenbasis, substitute  $\delta \phi_n = \sum_{ki} a_{ki} \phi_{ki}$  and  $\phi_n = \sum_j \alpha_j \phi_{nj}$  and take the inner product with  $\phi_{nj}$  to get

$$\begin{aligned} \sum_{kj} \lambda_k a_{kj} \langle \phi_{ni} | \phi_{kj} \rangle_g + \sum_j \alpha_j \langle \phi_{ni} | \tilde{\Delta}(g, h) | \phi_{nj} \rangle_g = \\ \delta \lambda_n \sum_j \alpha_j \langle \phi_{ni} | \phi_{nj} \rangle_g + \lambda_n \sum_{kj} a_{kj} \langle \phi_{ni} | \phi_{kj} \rangle_g \end{aligned}$$

Using orthogonality of the eigenfunctions we get

$$\lambda_n a_{ni} + \sum_j \alpha_j \langle \phi_{ni} | \tilde{\Delta}(g, h) | \phi_{nj} \rangle_g = \delta \lambda_n \alpha_i + \lambda_n a_{ni}$$

And solving for  $\delta \lambda_n$  we arrive at the following expression for the perturbation of the eigenvalue  $\lambda_n$

$$\delta \lambda_n = \frac{1}{\alpha_i} \sum_j \alpha_j \langle \phi_{ni} | \tilde{\Delta}(h, g) | \phi_{nj} \rangle_g \quad (5.14)$$

Of course the perturbation comes in terms of the unknown coefficients  $\alpha_j$ . But, since the perturbation operator is known, then so are the matrix elements in the expression. Collecting the  $j$ -many contractions with  $\phi_{ni}$ , we get a system of as many equations as the degeneracy of the  $n$ th eigenspace, from which one can eliminate the  $\alpha_i$ , getting a  $j$ th degree polynomial in  $\delta \lambda_n$ , the zeros of which are the desired perturbations.

As for the perturbation to the eigenfunctions, the same argument as in the non-degenerate case shows that, for  $n \neq k$

$$\langle \phi_{kl} | \delta \phi_n \rangle = \frac{\langle \phi_{kl} | \tilde{\Delta}(g, h) | \phi_n \rangle}{\lambda_n - \lambda_k}$$

Of course, once the  $\delta \lambda_n$  are determined, the  $\alpha_i$  in 5.14 are known. We can thus write explicitly  $|\phi_n\rangle = \sum_i \alpha_i |\phi_{ni}\rangle$ , getting

$$\langle \phi_{kl} | \delta \phi_n \rangle = \sum_i \alpha_i \frac{\langle \phi_{kl} | \tilde{\Delta}(g, h) | \phi_{ni} \rangle}{\lambda_n - \lambda_k} \quad (5.15)$$

Summing over  $k \neq n$  and  $l$  we get

$$|\delta \phi_n\rangle = \sum_{k \neq n} \sum_{i, l} \alpha_i \frac{\langle \phi_{kl} | \tilde{\Delta}(g, h) | \phi_{ni} \rangle}{\lambda_n - \lambda_k} |\phi_{kl}\rangle \quad (5.16)$$

## Second-Order Corrections

An argument such as for the non-degenerate case then shows that the second-order correction to the eigenvalues are given by the following expression:

$$\delta^2 \lambda_n = \sum_{k \neq n} \frac{|\langle \phi_k | \tilde{\Delta}(g, h) | \phi_n \rangle_g|^2}{\lambda_n - \lambda_k} \quad (5.17)$$

Where now the  $\phi_i$  are  $d$ -fold degenerate and so the expression above should be regarded as a set of equations for the  $\delta^2 \lambda_n$ .

## 5.3 Inaudible Perturbations

This section deals with a specific class of necessarily ‘inaudible’ perturbations of a given metric, by which we mean perturbations that do not change the spectrum of the Laplace operator. As in 3.2.3, these will be seen to be trivial.

In the first two sections, we discuss isometries at length. We begin by reviewing some ideas from differential geometry and then attempt to present (somewhat exhaustively, perhaps), the idea of general covariance and its consequences in what concerns the spectrum of the Laplace operator.

The remaining sections are devoted to presenting two different approaches, used in the theory of cosmological perturbations to define what one means by a perturbed manifold, and pointing out the similarities between the two.

### 5.3.1 Diffeomorphisms

Let  $M$  be a manifold and  $X$  a vector tangent to the curve  $\gamma : ]-a, a[ \rightarrow M$  at the point  $\gamma(0) = p$ . That is, by definition, given  $f : M \rightarrow \mathbb{R}$ ,

$$Xf = \frac{d}{dt}(f \circ \gamma)(t)|_{t=0}$$

In coordinates  $x : M \rightarrow \mathbb{R}^n$ , this reads

$$\begin{aligned} Xf &= \frac{d}{dt}(f \circ x^{-1} \circ x \circ \gamma)(t)|_{t=0} \\ &= \nabla(f \circ x^{-1})|_{x(\gamma(0))} \cdot (x \circ \gamma)'(t)|_{t=0} \end{aligned}$$

Of course the value does not depend on the coordinates chosen, because

$$\begin{aligned} Xf &= \frac{d}{dt}(f \circ y^{-1} \circ y \circ x^{-1} \circ x \circ y^{-1} \circ y \circ \gamma)(t)|_{t=0} \\ &= \nabla(f \circ y^{-1})J(y \circ x^{-1}) \cdot J(x \circ y^{-1})(y \circ \gamma)'(t)|_{t=0} \\ &= \nabla(f \circ y^{-1})|_{y(\gamma(0))} \cdot (x \circ \gamma)'(t)|_{t=0} \end{aligned}$$

Here, we introduced the obvious notation for the Jacobian of the transformation. From this expression we read off the familiar coordinate transformation law for the components of a vector field at  $p$ :

$$\frac{d}{dt}(x^i \circ \gamma)(t)|_{t=0} = J(x \circ y^{-1})^i_j \frac{d}{dt}(y^j \circ \gamma)(t)|_{t=0}$$

Or, in a more manageable notation

$$v^i(x)(p) = \frac{\partial x^i}{\partial y^j} v^j(y)(p)$$

The crucial point now is that one can use this transformation law to *define* a new vector field:

$$\begin{aligned} v^i(x)(p) &= \frac{\partial x^i}{\partial y^j} v^j(y)(p) \\ v^i(y \circ y^{-1} \circ x)(p) &= \frac{\partial x^i}{\partial y^j} v^j(y)(p) \\ \tilde{v}^i(y)(y^{-1} \circ x)(p) &= \frac{\partial x^i}{\partial y^j} v^j(y)(p) \end{aligned}$$

That is, the field that has as components in coordinates  $y$  at the point  $y^{-1} \circ x(p)$  the same as the vector field  $v$  would have in coordinates  $x$  at  $p$ .

We note that  $v$  and  $\tilde{v}$  are entirely different fields. In fact, if the former is the field tangent to the congruence  $\gamma$ , the latter is the field tangent to the congruence  $y^{-1} \circ x \circ \gamma$ . To show this, we calculate it at the point  $y^{-1} \circ x \circ \gamma(0) = y^{-1} \circ x(p)$ :

$$\begin{aligned} X_{(y^{-1} \circ x \circ \gamma)} f &= \frac{d}{dt}(f \circ y^{-1} \circ x \circ \gamma)(t)|_{t=0} \\ &= \frac{d}{dt}(f \circ y^{-1} \circ x \circ y^{-1} \circ y \circ \gamma)(t)|_{t=0} \\ &= \nabla(f \circ y^{-1}) \cdot J(x \circ y^{-1})(y \circ \gamma)'(t)|_{t=0} \end{aligned}$$

The new vector field so defined is called the *push-forward* vector field by  $\Phi : y^{-1} \circ x$  and the coordinate transformation  $\Phi : M \rightarrow M$  induces a map  $\Phi_* : T_p M \rightarrow T_{\Phi(p)} M$ , the action of which was defined above.

It is straightforward to generalize the pushforward for higher order contravariant tensors: it is the tensor field  $\tilde{t} = \Phi_* t$  that has as components in the coordinate system  $y$  at  $y^{-1} \circ x(p)$  the same as the tensor field  $t$  would have in coordinates  $x$  at  $p$ . The corresponding map for covariant tensors is the *pullback*, and we denote it by  $\Phi^*$ .

Again, we stress that these are entirely different objects.

We can further generalize the pushforward operation by relaxing the requirement that  $\Phi$  is a coordinate transformation: one can, given a vector field

on  $M$  tangent to the congruence  $\gamma$  and  $\Phi : M \rightarrow N$  an immersion, define the new vector field that is tangent to the congruence  $\Phi \circ \gamma$ .

### General covariance

Now suppose we have a covariant differential equation on the components of  $n$  tensors fields on  $M$ ,  $\phi_i$ , which we represent succinctly as

$$D[\phi_1(x), \dots, \phi_n(x)] = 0$$

Since this equation is of the same variance throughout, then obviously

$$D[\phi_1(x), \dots, \phi_n(x)] = 0 \Rightarrow D[\phi_1(y), \dots, \phi_n(y)] = 0 \quad (5.18)$$

in any coordinate system  $y$ .

But since  $\phi_i(y)$  are the components of the vector field  $\Phi_*\phi_i$  in the  $x$  coordinate system, we have that

$$D[\phi_i(x), \dots] = 0 \Rightarrow D[(\Phi_*\phi_i)(x), \dots] = 0 \quad (5.19)$$

for any diffeomorphism  $\Phi$ , provided we replace every tensor field in the equation with the one induced by the diffeomorphism.

We thus say that the theory defined by the differential equation above is *invariant under diffeomorphisms*.

Now, as we have said in the previous section, one can use coordinate transformations to define these new fields. For this reason, it is often said that the invariance of a theory with respect to diffeomorphisms is nothing but invariance under coordinate transformations; we hope to have made clear in the exposition above both why they are so easily mistaken for the same concept and why they are in fact, very much different.

Recall that whereas the expression (5.18) just states that a solution does not depend on the particular choice of coordinates, the expression (5.19) actually provides *new* solutions to the differential equation: at a point  $p \in M$ , we have that  $\phi_i(p) \neq (\Phi_*\phi_i)(p)$ .

Given two sets of component functions that solve the differential equation above, how can one tell if they are the components of the same tensor in different coordinate systems or two different tensors, one the diffeomorphism-induced by the other?

Well, one would compare their values at some point, and find that they are generally different. But this requires that one has a way to define a point on the manifold that is *extrinsic* to the manifold structure, and physicists do not have that privilege. This matter is what we deal with in the next section.



### Einstein's hole argument

Let  $\Phi^*g$  be a solution to the Einstein equation. As we've seen, then all the metrics induced by diffeomorphisms are solutions to the same equation. In particular, so is  $g$ . But we can tell the two solutions apart because their components do not coincide at some point on  $M$ . Since the metric determines the curvature tensor, if there is a point where the curvature is zero, say  $p \in M$ , then it is possible to find a solution  $\Phi^*g$  where the curvature is zero at the point  $\Phi(p)$  (because the components of the latter at  $\Phi(p)$  are the transformed components of  $g$  and zero is zero in any coordinate system).

But then the Einstein equations *are not deterministic*, an obviously undesirable conclusion.

This argument just presented is known as "Einstein's hole argument". In Rovelli [55], we find the following solution for this conundrum:

If we use intersections of geodesics to define points on the manifold, the two solutions,  $\Phi^*g$  and  $g$  agree. To see this, we note that if a curve  $\gamma$  is the solution to the geodesic equation on  $(M, \Phi^*g)$ , then the curve  $\Phi \circ \gamma$  is the solution to the geodesic equation on  $(M, g)$ :

$$\begin{aligned} \delta \int_a^b \Phi^*g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))dt &= 0 \Rightarrow \\ \delta \int_a^b g_{(\Phi \circ \gamma)(t)}([D\Phi]\dot{\gamma}(t), [D\Phi]\dot{\gamma}(t))dt &= 0 \Rightarrow \\ \delta \int_a^b g_{(\Phi \circ \gamma)(t)}\left(\frac{d}{dt}(\Phi \circ \gamma)(t), \frac{d}{dt}(\Phi \circ \gamma)(t)\right)dt &= 0 \end{aligned}$$

And so, the components of the metric tensor  $\Phi^*g$  in coordinates  $x$  at the point  $\gamma_1 \cap \gamma_2$ , where  $\gamma$  is a geodesic on  $(M, \Phi^*g)$  are the same as the components of  $g$  at the point corresponding to the induced geodesics  $\Phi(\gamma_1) \cap \Phi(\gamma_2)$  in the coordinate system induced by the diffeomorphism  $\Phi$ . Which means that if one cannot tell the points  $\gamma_1 \cap \gamma_2$  and  $\Phi(\gamma_1) \cap \Phi(\gamma_2)$  apart, then they would seem to be the components of the same tensor in different coordinates.

And can one tell the two points apart? As a mathematician, one can always think of the manifold as a subset of some ambient, large-dimensional euclidean space, with the induced differential structure, which in physics would correspond to assuming the existence of an absolute coordinate system. In this coordinate system, the two points do not coincide.

In physics, we are bound to choosing coordinates in an intrinsic fashion. Is there an intrinsic way to distinguish the two points above?

Say we want to test if we are in a solution  $g$  or  $\Phi^*g$ . Take two geodesics and measure the curvature at the point they intersect. Whether we are on one or the other, we will measure components of the metric tensor that are related by a coordinate transformation, and would not be able to tell them

apart.

And so, since there is no absolute frame of reference, given a solution to the Einstein equations, any two elements in the set of solutions defined by the action of diffeomorphisms on a given solution (the orbit of the diffeomorphism group) is indistinguishable: the theory is deterministic.

### Diffeomorphisms and the spectrum

Since the Laplace equation is generally covariant, it is also invariant under diffeomorphisms. Which, as we've seen, means that the Laplace equation does not change when every field is replaced with the corresponding diffeomorphism induced field:

$$\begin{aligned} L[g, f] = 0 &\Rightarrow L[\Phi^*g, \Phi^*f] = 0 \\ &\Leftrightarrow \\ \Delta_g f + \lambda f = 0 &\Rightarrow \Delta_{\Phi^*g}(f \circ \Phi) + \lambda(f \circ \Phi) = 0 \end{aligned}$$

So what about an isometry,  $\Phi : \Phi^*g = g$ ? Let the Laplace equation be solved and try to solve the Laplace equation for the manifold that has the same structure but induced metric. Let that metric be the same as in the original manifold. Then

$$\begin{aligned} L[g, f] = 0 &\Rightarrow L[\Phi^*g, \Phi^*f] = 0 \\ &\Rightarrow L[g, \Phi^*f] = 0 \end{aligned}$$

And so isometries generate new eigenfunctions to the same Laplacian (if a function is an eigenfunction of the Laplacian and if there is an isometry  $\Phi$  then  $f \circ \Phi$  is an eigenfunction of the Laplacian).

So isometries imply degeneracy of the spectrum.

The converse statement can be found in [5]:

**Proposition 5.3.1** *Let  $M$  be a compact connected  $C^\infty$  manifold of dimension no less than two. If the spectrum of the Laplace operator on functions is simple, i.e., there are no repeated eigenvalues, then the group of all isometries of  $(M, g)$  is discrete.*

We present the proof in [5] for completeness.

**Proof:** Let  $\{\phi_i\}_{i \in \mathbb{N}}$  denote a complete, orthonormal basis of  $C^\infty(M)$  composed of eigenfunctions of the Laplace operator. We denote the eigenvalues in the following way:

$$0 < \lambda_1 = \dots = \lambda_{j_1} < \lambda_{j_1+1} = \dots = \lambda_{j_2} < \dots$$

and call  $V_k$  the eigenspace associated with the eigenvalue  $\lambda_{j_k}$ . Now take a sufficiently large number  $N = 1 + j_1 + \cdots + j_r$  of eigenfunctions (note that  $N = \dim V_r + \dim V_{r-1} + \cdots + \dim V_1 + 1$ ) such that

$$f : M \rightarrow \mathbb{R}^N$$

$$f(x) = (\phi_0(x), \phi_1(x), \dots, \phi_{N-1}(x))$$

is an embedding (note that  $M$  is compact). As seen above, the action of an element  $\Psi$  of the group of all isometries of  $(M, g)$  on functions on  $M$ ,  $Iso(M)$  is given by  $\Psi^* f = f \circ \Psi^{-1}$ . The elements of the group are maps of  $C^\infty(M)$  into itself; as seen above,  $(\Psi^* f, \Psi^* h)_g = (\Psi^* f, \Psi^* h)_{\Psi^* g} = (f, h)_g$  and, by definition,  $(\Psi \circ \Phi)^* f = f \circ (\Psi \circ \Phi)^{-1} = f \circ \Phi^{-1} \circ \Psi^{-1} = \Psi^* \Phi^* f$ . Since, as seen above, the Laplacian commutes with isometries, we have that the isometries map each eigenspace  $V_k$  into itself. We then have a Lie group homomorphism  $f^*$  of  $Iso(M)$  into the orthogonal group of the Euclidean space  $(\sum_{k=0}^r V_k, (\cdot, \cdot))$ . Since  $f$  is an embedding, it is one-to-one and so is  $f^*$ . If each of the eigenspaces is one-dimensional, the Lie subgroup  $f^*(G)$  of the orthogonal group  $O(\sum_{k=0}^r V_k)$  is discrete, which, since  $f^*$  is one-to-one, implies in turn that  $G$  is discrete.

## 5.4 Perturbation of manifolds

In this section, we are interested in defining precisely what is meant by a *deformation* of a Riemannian manifold  $(M, g)$ .

In this presentation, we follow the structure of the review [45], which divides the approaches to the problem of perturbing spacetimes in a covariant way in two categories: an intrinsically covariant (because coordinate independent) formulation by Sachs [57] Stewart and Walker [64] and Stewart [63]; and a widely used, coordinate dependent (but covariant) description by Mukhanov, Feldman and Brandenberger [10].

Both the formulations fix the topology, in the sense that, if  $(M', g')$  is a deformation of  $(M, g)$ ,  $M$  is diffeomorphic to  $M'$ .

The problem consists, then, in *defining* a metric space structure in the set of positive definite metric tensors on  $M$ , which we denote  $\mathcal{M}$ .

Informally, in both cases, the idea consists in thinking of perturbations of a given metric tensor  $g$  as tangent vectors to paths on  $\mathcal{M}$  that go through it. One then, either through a clever construction or directly, takes the path to be defined by a symmetric 2-tensor field  $h$  on  $M$ .

Although outside the scope of this thesis, we mention that an interesting discussion on the great deal of freedom in defining what one means by a perturbation to a manifold can be found in [64].

Although different approaches, we will find that both yield essentially the same results.

### 5.4.1 Coordinate Dependent Description

As mentioned in the introduction, in this approach [10], the problem is one of considering perturbative solutions to the Einstein equations in a cosmological setting. In this particular setting, we should mention that the metric is non-Riemannian (FRW), although the arguments in the sequel do not depend on the signature of the metric nor its particular form (other than sufficient smoothness).

Given some atlas on  $M$ , we define a perturbation to a metric tensor  $g^{(0)}$  as

$$g_{\mu\nu}(x^\alpha) = g_{\mu\nu}^{(0)}(x^\alpha) + \epsilon h_{\mu\nu}(x^\alpha) \quad (5.20)$$

We investigate the change in this tensor induced by the flow along the integral lines of a  $C^\infty$  vector field  $\xi$  on  $M$ . Since the component functions are functions on  $M$ , one can expand them in a Taylor series along the integral lines of the vector field

$$g_{\mu\nu}(x^\alpha + \epsilon\xi^\alpha) = g_{\mu\nu}(x^\alpha) + \epsilon g_{\mu\nu,\beta}\xi^\beta(x^\alpha) + \dots \quad (5.21)$$

As discussed in detail in 5.3.1 one can look at this expression as defining a new tensor  $\tilde{g}$ , whose components in coordinates  $x^\alpha + \epsilon\xi^\alpha$  are given by the expression on the right-hand side of 5.21. But if it is to define a tensor, its components should transform under a coordinate transformation  $x^\alpha + \epsilon\xi^\alpha \rightarrow x^\alpha$  as

$$\begin{aligned} \tilde{g}_{\mu\nu}(x^\alpha + \epsilon\xi^\alpha) &= \frac{\partial x^i}{\partial(x^\mu + \epsilon\xi^\mu)} \frac{\partial x^j}{\partial(x^\nu + \epsilon\xi^\nu)} \tilde{g}_{ij}(x^\alpha) \\ &= (\delta_\mu^i - \epsilon\xi_{,\mu}^i)(\delta_\nu^j - \epsilon\xi_{,\nu}^j)\tilde{g}_{ij}(x^\alpha) \\ &= \tilde{g}_{\mu\nu} - \epsilon\tilde{g}_{\mu j}\xi_{,\nu}^j - \epsilon\tilde{g}_{\nu i}\xi_{,\mu}^i \end{aligned}$$

Where we kept terms up to first order. Substituting back in 5.21 and keeping terms up to first order in perturbation, we get

$$\tilde{g}_{\mu\nu} - g_{\mu\nu} = \epsilon(g_{\mu\nu,\alpha}\xi^\alpha + \tilde{g}_{\mu\alpha}\xi_{,\nu}^\alpha + \tilde{g}_{\nu\alpha}\xi_{,\mu}^\alpha) \quad (5.22)$$

And since the difference between the ‘tilde’ and ‘untilde’ tensors is of first order, we can just drop the tildes on the RHS and recognize the component expression for the Lie derivative of the tensor  $g$  with respect to the vector field  $\epsilon\xi$ :

$$\tilde{g} = g + \epsilon\mathcal{L}_\xi g \quad (5.23)$$

Which, for a small perturbation of the metric, to first order, implies

$$\tilde{g}^{(0)} + \epsilon \tilde{h} = g^{(0)} + \epsilon h + \epsilon \mathcal{L}_\xi g$$

Since, as we have seen, a generally covariant theory is invariant under diffeomorphisms, the tilde and ‘untilde’ metrics are physically indistinguishable to first order.

Which means that the perturbations  $h$  and  $h + \mathcal{L}_\xi g$  are not physically distinguishable: one can always go to a diffeomorphic manifold in which the perturbed metric is written in more manageable form.

This allows us to define an equivalence class of perturbations; given  $h$  a symmetric 2-tensor, a perturbation of  $g$ , we say that  $\tilde{h}$  is an equivalent perturbation if  $\tilde{h} = h + \mathcal{L}_\xi g$ , for some  $\xi$ . We denote equivalence in this sense as  $\tilde{h} \sim h$  and define the set of perturbations to  $g$  equivalent to  $h$  as

$$E(g, h) = \{\tilde{h} \in S(M) : \tilde{h} \sim h\} \quad (5.24)$$

where  $S(M)$  denotes the set of smooth symmetric covariant 2-tensors on  $M$ . Of course  $E(g, h) = E(g, h + \mathcal{L}_\xi g)$ , and we can choose any element of the set to define it. A choice of element is a choice of  $\xi$ , and we call it a *choice of gauge*.

## 5.4.2 Coordinate Independent Description

The motivation for this formulation is twofold: to define perturbations of manifolds in a coordinate-free way, while at the same time keeping the description close to the intuition of a perturbation of a manifold as being a family of manifolds in some sense ‘close’ to it. The idea is to define a perturbation of a given manifold as a manifold that is "almost diffeomorphic" to it [63][64]. More precisely, a manifold  $(\tilde{M}, \tilde{g})$  is defined to be a perturbation of  $(M, g)$  if there is a diffeomorphism  $\Phi : M \rightarrow \tilde{M}$  such that  $\Phi^* \tilde{g} - g$  is ‘small’.

This formulation is arguably closer to the intuition of a perturbation of a spacetime as being another spacetime, rather than a tensor field. The perturbed manifolds, though, are constructed in a such a way as to make the distinction immaterial as far as perturbations of the metric tensor are concerned, as we shall see in the sequel.

Consider a one-parameter family of manifolds  $(M(\epsilon), g(\epsilon))$  of the same dimension  $n$ , such that  $(M(0), g(0)) = (M, g)$  and require that the following holds:

1. There is a smooth, separable,  $n + 1$  dimensional manifold  $\mathcal{M}$  of which the  $M(\epsilon)$  are smooth, non-intersecting submanifolds, properly embedded (so that the topology is preserved).
2. The parameter  $\epsilon$  defines a smooth function on  $\mathcal{M}$ , whose value at a point is the label of the submanifold it is in.

3.  $d\epsilon \neq 0$ , and the integral surfaces of  $d\epsilon$  are the  $M(\epsilon)$
4.  $g(\epsilon)$  defines a smooth tensor field  $\tilde{g}$  on  $\mathcal{M}$  of signature  $(0, s(g))$ .
5. The singular hypersurfaces of  $\tilde{g}$ ,  $\tilde{g}^{\alpha\beta}d\epsilon_\alpha = 0$  are the  $M(\epsilon)$  and  $\tilde{g}$  induces the metric  $g(\epsilon)$  on each hypersurface.

That is, we think of  $(M, g)$  as being the ‘basis’ for the family of perturbed manifolds, which we model on  $(M, g)$  in a way that preserves topology and ‘collect’ as a family of non-intersecting surfaces of a higher-dimensional manifold, each being surfaces of constant  $\epsilon$  and define the metric on the higher dimensional manifold in such a way that the inner product of vectors tangent to some integral surface of  $d\epsilon$  coincides with that given by  $g(\epsilon)$ .

Now take a vector field  $\xi$  on  $\mathcal{M}$ , smooth and everywhere transversal to each of the submanifolds. Then we can think of it as a way to identify points in each of the submanifolds: we think of two points on  $\mathcal{M}$  as being the same point if they are in the same integral line of  $\xi$ . We restrict our attention to vector fields parametrized in such a way that  $\Phi_\epsilon(M) = M(\epsilon)$ , where  $\Phi_\epsilon$  is the map that takes  $p \in \mathcal{M}$  to the point a parameter distance  $\epsilon$  along the integral line of the vector field  $\xi$  that goes through it.

It is now straightforward to make precise the idea of ‘almost’ diffeomorphic manifolds:  $(M(\epsilon), g(\epsilon))$  is a perturbation of  $(M, g)$  if

$$\Phi_\epsilon^*g(\epsilon) - g(0) = O(\epsilon)$$

when  $\epsilon \rightarrow 0$ . But this is just the definition of the Lie derivative

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon}(\Phi_\epsilon^*g(\epsilon) - g(0)) &= \mathcal{L}_\xi g(\epsilon)|_{\epsilon=0} \\ &\Rightarrow \\ \Phi_\epsilon^*g(\epsilon) &= g(0) + \epsilon \mathcal{L}_\xi g(\epsilon)|_{\epsilon=0} + O(\epsilon^2) \end{aligned}$$

The choice of point identification map is arbitrary and is called a *choice of gauge*. Given two choices of gauge (two vector fields,  $\xi_1, \xi_2$  to which correspond point identification maps  $\Phi$  and  $\Psi$ ), we have

$$\begin{aligned} \Phi_\epsilon^*g(\epsilon) - \Psi_\epsilon^*g(\epsilon) &= \epsilon \mathcal{L}_{\xi_1 - \xi_2} g(\epsilon)|_{\epsilon=0} + O(\epsilon^2) \\ &= \mathcal{L}_{\epsilon(\xi_1 - \xi_2)} g(\epsilon)|_{\epsilon=0} + O(\epsilon^2) \end{aligned}$$

Because of the way we chose to parametrize the vector fields, by construction, the vector  $\epsilon(\xi_1 - \xi_2)$  must be tangent to  $M(\epsilon)$  ( $d/d\epsilon$  is a good coordinate vector, by construction, and both vector fields have components  $\epsilon$  at  $M(\epsilon)$  in that direction, also by construction). In particular, for  $\epsilon \rightarrow 0$ , we have that the difference between the two perturbations is the Lie derivative of a vector field on  $M$ , in accordance with the results in the last section, that is, they are elements of the same equivalence class 5.24.

Since we've seen that the action of a diffeomorphism cannot be distinguished from a change of coordinates, the two manifolds  $(M, g_1)$  and  $(M, g_2)$  (with the perturbed metrics we arrive at by using the fields  $\xi_1$  and  $\xi_2$ ) are, to first order, physically indistinguishable - although, as was shown, the particular expression for the perturbation part depends on the chosen coordinates.

As a side-note, we remark that the only metrics for which there are *gauge independent* perturbations are the everywhere zero or euclidean metrics: for these are the only metrics for which the difference between two choices of gauge,  $\mathcal{L}_{(\xi_1 - \xi_2)}$  is zero for *any*  $\xi_1 - \xi_2$  on  $M$ .

This statement can be generalized to any other tensor field on  $M$ , since the construction of perturbed metrics above carries through: there are no gauge independent perturbations of geometrical quantities, other than constant scalar, identically zero tensor fields and linear combinations of Kronecker deltas (sometimes referred to as Sach's lemma).

Finally, in light of the discussion above on diffeomorphism invariance, it is interesting to note that, using this definition, two diffeomorphic manifolds are a perturbation of each other (diffeomorphic implies quasi-diffeomorphic).

## 5.5 Decomposition of covariant two tensors

In what follows, we present a result by York, following [69] closely. I provide some of the omitted derivations, which I carry through for clarity. I also generalize it to arbitrary dimension.

**Proposition 5.5.1** *Let  $T$  be a symmetric contravariant two-tensor field on  $M$ . Then, there is a vector field  $Y$  on  $M$  such that:*

$$T^{ab} = T_{tt}^{ab} + \frac{1}{N} T^{cd} g_{cd} g^{ab} + (LY)^{ab} \quad (5.25)$$

where  $N = \dim(M)$ , and  $T_{tt}^{ab}$  is such that, with  $\nabla$  the Levi-Civita connection,

$$\begin{aligned} T_{tt}^{ab} &= T_{tt}^{ba} \\ T_{tt}^{cd} g_{cd} &= 0 \\ \nabla_a T_{tt}^{ab} &= 0 \end{aligned}$$

And

$$(LY)^{ab} = \nabla^a Y^b + \nabla^b Y^a - \frac{2}{N} (\nabla_c Y^c) g^{ab} \quad (5.26)$$

The vector field  $Y$  is the solution to

$$\nabla_a (LY)^{ab} = \nabla_a (T^{ab} - \frac{1}{N} T^{cd} g_{cd} g^{ab}) \quad (5.27)$$

and is unique up to the kernel of the linear operator defined by the RHS of

the expression above, i.e.

$$A : Y^b \rightarrow \nabla_a(LY)^{ab}$$

### 5.5.1 Preliminaries

We shall establish the existence of solutions to 5.27 by showing that the operator on the LHS is elliptic and formally self-adjoint and that its kernel is orthogonal to that of the RHS of equation 5.26.

We will briefly explain why these conditions suffice to guaranty the existence of solutions to the equation above:

Let  $A$  be an unbounded linear operator, acting on the elements of some Banach space  $E$ . Recall that the ‘standard’ way to establish the existence of solutions  $u \in E$  to the equation (such as 5.26)

$$Au = f \tag{5.28}$$

is to multiply both sides of 5.28 with a ‘test’ function  $v$  and to look for solutions in some (complete, separable) inner product space. That is, we look for solutions of

$$(v, A\tilde{u}) = (v, f) \tag{5.29}$$

where  $\tilde{u}$  is obviously restricted to the subset of  $E$  for which the inner product above is finite for *all*  $v$  in the inner product space. If a solution to this equation exists, we call it a *weak solution*. Theorems that guaranty the existence and uniqueness of such solutions often require  $A$  to be symmetric and positive definite. Having established the existence of a weak solution, the problem consists in showing that the unique weak solution is in fact a *strong solution*, a solution on  $E$ .

If  $A$  is symmetric, we have

$$\begin{aligned} (v, A\tilde{u}) &= (v, f) \\ (\tilde{u}, Av) &= (v, f) \end{aligned}$$

Since we require this to hold for *all*  $v$  in the inner product space, it should, in particular, hold for  $v$  in the kernel of  $A$ . Which implies that, for all  $v \in \ker(A)$ ,  $(v, f) = 0$ . So, for symmetric operators, existence of a weak solution can only be established using the ansatz above if the kernel of  $A$  is orthogonal to the ‘source term’  $f$ .

The ellipticity requirement, in a rather simplified manner, is one of assuring that the resolvent of the equation above, an integral operator of the type encountered when establishing the existence of a basis for  $L^2(M)$  composed of eigenfunctions of the Laplacian, is of Fredholm type and thus compact (recall that the domain of integration is bounded). The symmetry of  $A$  implies



that the kernel is symmetric, which in turn implies that the integral operator is compact and self-adjoint.

A detailed exposition can be found in the preparations to the proof of Theorem 5.1 in [49]. Although the theorem concerns domains with boundary and Dirichlet BC, a generalization to other BC and boundary-less domains is claimed to exist.

### Proof of Proposition 5.5.1

We now present in detail the proof of Proposition 5.5.1 in [69].

To check ellipticity, it proves to be convenient to rewrite the operator on the LHS of 5.27:

$$\begin{aligned}\nabla_a(LY)^{ab} &= \nabla_a(\nabla^a Y^b + \nabla^b Y^a - \frac{2}{N}\nabla_c Y^c g^{ab}) \\ &= \Delta Y^b + \nabla_a \nabla^b Y^a - \frac{2}{N}\nabla^b \nabla_a Y^a\end{aligned}$$

Which, since

$$R_{.cba}^d X_d = \nabla_a \nabla_b X_c - \nabla_b \nabla_a X_c$$

can be written as

$$\nabla_a(LY)^{ab} = \Delta Y^b + R_a^b Y^a + (1 - \frac{2}{N})\nabla^b \nabla_a Y^a \quad (5.30)$$

A differential operator is elliptic if its symbol is invertible. The symbol of the operator above ( $:=A$ ) is obtained by replacing every highest-order occurrence of covariant derivatives by an arbitrary non vanishing vector field  $V$ . It is the operator defined by its action on vector fields  $W$  as follows

$$[\sigma(A)]_a^b W^a = (1 - \frac{2}{N})V_a V^b W^a + V_c V^c \delta_a^b W^a \quad (5.31)$$

and invertibility follows from choosing e.g.  $V^b = (1, 0, \dots)$  in local coordinates. It is also strongly elliptic, since all its eigenvalues are positive (recall that  $N \geq 2$ ).

As for formal self-adjointness, only the third term in 5.30 is non-trivial. We wish to show that, for arbitrary  $X, Y$

$$\int_M X_b \nabla^b (\nabla_a Y^a) \sqrt{g} dx = \int_M Y_a \nabla^a (\nabla_b X^b) \sqrt{g} dx$$

To do so, we begin by showing that in this expression, in the case where the connection is the Levi Civita connection (as it is presently), one can do integration by parts with covariant derivatives, as one would with regular

derivatives. To see this, take

$$\partial_c \sqrt{g} = \frac{1}{2\sqrt{g}} \frac{\partial g}{\partial g_{ab}} g_{ab,c}$$

Using 5.3, we have

$$\begin{aligned} \partial_c \sqrt{g} &= \frac{1}{2} \sqrt{g} g^{ab} g_{ab,c} \\ &= \sqrt{g} \frac{1}{2} g^{ab} (g_{ab,c} + g_{ac,b} - g_{bc,a}) \\ &= \sqrt{g} \Gamma_{ac}^a \end{aligned}$$

where we've used the symmetry of the metric and the definition of the Levi Civita connection. Which means that, for a vector field  $v^i$  and a function  $f$ ,

$$\begin{aligned} \int_M (v^i \nabla_i f) \sqrt{g} dx &= \int_M (v^i \partial_i f) \sqrt{g} dx \\ &= - \int_M \partial_i (\sqrt{g} v^i) f dx \\ &= - \int_M (\partial_i (\sqrt{g}) v^i + \sqrt{g} \partial_i v^i) f dx \\ &= - \int_M (\nabla_i v^i) f \sqrt{g} dx \end{aligned}$$

where the last line just follows from the definition of the covariant derivative. It now follows directly that the third term in 5.30 is formally self-adjoint: For arbitrary  $X, Y$ , we have, integrating by parts twice,

$$\begin{aligned} \int_M X_b \nabla^b (\nabla_a Y^a) \sqrt{g} dx &= - \int_M (\nabla^b X_b) (\nabla_a Y^a) \sqrt{g} dx \\ &= \int_M Y_a \nabla^a (\nabla_b X^b) \sqrt{g} dx \end{aligned}$$

Now, to find the kernel of 5.30 we note the rather obvious fact that if  $C^a$  is such that

$$(LC)^{ab} = 0$$

then  $\nabla_b (LC)^{ab} = 0$ . Conversely, if  $C^a$  is in the kernel, then for all  $X$

$$\int_M X_a \nabla_b (LC)^{ab} \sqrt{g} dx = 0$$

In particular, then, we have

$$\begin{aligned}
0 &= \int_M C_a \nabla_b (LC)^{ab} \sqrt{g} dx \\
&= - \int_M (\nabla_b C_a) (LC)^{ab} \sqrt{g} dx \\
&= - \frac{1}{2} \int_M (\nabla_b C_a + \nabla_a C_b) (LC)^{ab} \sqrt{g} dx
\end{aligned}$$

since  $(LC)^{ab}$  is symmetric.

Finally, since  $(LC)^{ab}$  is traceless

$$\int_M (\nabla^d C_d) g_{ab} (LC)^{ab} \sqrt{g} dx = 0$$

and so we have

$$\begin{aligned}
0 &= - \frac{1}{2} \int_M (\nabla_b C_a + \nabla_a C_b - \frac{2}{N} g_{ab} \nabla^d C_d) (LC)^{ab} \sqrt{g} dx \\
&= - \frac{1}{2} \int_M (LC)_{ab} (LC)^{ab} \sqrt{g} dx
\end{aligned}$$

And since  $(LC)_{ab} (LC)^{ab} \geq 0$ , we have that the integral being zero implies  $(LC)^{ab} = 0$  everywhere and so it follows that the kernel is exactly  $K = \{C : (LC)^{ab} = 0\}$ . This will in general be a set with only the null vector, because  $C \in K$  implies that it is a killing vector to a metric conformal to  $g$  [69] and, as we shall see in Section 6.3, the set of metrics that have killing vectors is a ‘small’ subset of the set of all metrics.

Finally, we show that the RHS of 5.27 is orthogonal to  $K$ , which follows straightforwardly from the calculations above. For  $C \in K$ ,

$$\begin{aligned}
\int_M C_a \nabla_b (T^{ab} - \frac{1}{N} T g^{ab}) \sqrt{g} dx &= - \frac{1}{2} \int_M (LC)_{ab} (T^{ab} - \frac{1}{N} T g^{ab}) \sqrt{g} dx \\
&= 0
\end{aligned}$$

And we have the decomposition 5.25, unique up to the kernel  $K = \{C : (LC)^{ab} = 0\}$ .

## 5.6 Audible perturbations of the spectrum

Let  $g + \epsilon h$  be a perturbation to  $g$ . We have shown in the last section that, given  $h$  a symmetric 2-tensor, there is a  $Y$  such that the following decomposition holds:

$$h^{ab} = h_{tt}^{ab} + \frac{1}{N} h^{cd} g_{cd} g^{ab} + \nabla^a Y^b + \nabla^b Y^a - \frac{2}{N} (\nabla_c Y^c) g^{ab} \quad (5.32)$$

Because in any expression on the components of tensors involving Lie derivatives, one can replace all derivatives with covariant derivatives (because taking the Lie derivative is, by definition, a covariant operation)

$$\begin{aligned}
\mathcal{L}_\xi g_{ij} &= g_{ij,k} \xi^k + g_{ik} \xi_{,j}^k + g_{kj} \xi_{,i}^k \\
&= g_{ij;k} \xi^k + g_{ik} \xi_{,j}^k + g_{kj} \xi_{,i}^k \\
&= g_{ik} \xi_{,j}^k + g_{kj} \xi_{,i}^k \\
&= \xi_{i;j} + \xi_{j;i}
\end{aligned}$$

(where we used the compatibility of the connection).

Using this result in (5.32), we get

$$h^{ab} = h_{tt}^{ab} + \frac{1}{N}(h^{cd} g_{cd} - 2(\nabla_c Y^c))g^{ab} + \mathcal{L}_Y g^{ab} \quad (5.33)$$

It is now apparent that a convenient choice of the element that defines the equivalence class  $E(g, h)$  as in (5.24) can be made, since

$$g + \epsilon h \sim (g + \epsilon h - \epsilon \mathcal{L}_Y g)$$

For with this choice, we have, by 5.33

$$h^{ab} = h_{tt}^{ab} + \frac{1}{N}(h^{cd} g_{cd} - 2(\nabla_c Y^c))g^{ab}$$

which we write in a compact fashion as

$$h = h_{tt} + \Omega g$$

and so

$$g(\epsilon) = g + \epsilon(h_{tt} + \Omega g) \quad (5.34)$$

is our choice for expressing *any* arbitrary (since  $Y$  was shown to exist) perturbation.

Finally, two remarks:

1. Counting degrees of freedom: the perturbation  $h$ , the LHS of the expression 5.34 is parametrized by the components of a symmetric two-tensor in  $N$  dimensions, to which correspond  $N(N + 1)/2$  functional degrees of freedom.

The expression on the RHS is parametrized by two parts: the first are the components of a symmetric, transverse tensor in  $N$  dimensions, to which correspond  $-N + N(N + 1)/2$  degrees of freedom; the traceless

constraint subtracts 1 degree of freedom. The second part is a function, which adds 1 degree of freedom. Since this parametrization assumes a choice of gauge, we have a ‘hidden’  $+N$  degrees of freedom. Adding all the contributions, we see that the degrees of freedom in the RHS and the LHS agree, as they should.

2. Perturbations for which, in 5.34,  $h_{tt} = 0$ , are conformal and so, for any choice of  $\Omega$ , they leave the Weyl tensor invariant. This decomposition can be seen, then, as parametrizing changes in the metric in a Weyl-invariant ( $h_{tt} = 0$ ) and non-Weyl invariant way.

## Chapter 6

# Behaviour of Spectrum under variations of the metric

### 6.1 Motivation

The present chapter consists mostly of a review of [5]. I also carry through some of the omitted derivations and provide intuition for some crucial points, for clarity.

As detailed in Section 4.4 we are interested in the behaviour of the spectrum of the Laplace operator on a neighbourhood of a given metric tensor.

It has long been established [8] that the spectrum of the Laplace operator varies continuously with respect to the  $C^\infty$  topology (to be defined) on the set of metric tensors on  $M$ . But we are interested in the behaviour of the spectrum in the presence of a cutoff, and for that reason we are interested in showing *uniform continuity*.

Unless it proves to be a *truly* universal constant, the cutoff is only to be prescribed up to some precision. We would hope to have the finite set of eigenvalues of the Laplacian below said cutoff to, in a sense, be ‘coordinates’ for the set of ‘physical’ spacetimes (adapting the terminology of [37]).

Also, given a manifold  $M$ , if a metric  $g$  on  $M$  is an element of this set of ‘physical’ spacetimes, one would expect that, for some reasonable topology on the space of metrics on  $M$ , so would all metrics in some neighbourhood of  $g$ .

Suppose no uniform bound on changes of the eigenvalues with respect to some change on the metric could be found. Then one could not hope to even define dimension in the ‘spectral space’: there could be a metric  $\tilde{g}$  in a neighbourhood of  $g$  for which the number of eigenvalues below the cutoff would be dramatically different from those of  $g$ .

This concern about the robustness of this characterization of the ‘physical’ spacetimes is particularly relevant for the program outlined in Section 4.4.

I will prove a new result that establishes precisely that the dimension of

the cutoff space remains constant for small perturbations of the metric in 6.2.7.

Our ultimate goal would be to study the spectral function on paths on the space of metrics on some given manifold  $M$ . As we shall see explicitly in 7, there are considerable difficulties with regard to degeneracies of the spectrum. We would like to have a result that shows that metrics for which the spectrum are degenerate are so rare that one can always find a path on the space of metric tensors that can avoid them.

That fundamental result is known as Uhlenbeck's Theorem, which states that metrics for which the spectrum of the Laplace operator is degenerate are, in some sense to be made precise further on, 'rare'.

### 6.1.1 Statement of Theorems and Structure of the Proof

For clarity, we first state precisely the two main results in [5] without proof. In what follows,  $\mathcal{M}$  is the set of all  $C^\infty$  Riemannian metric tensors on  $M$  and  $\rho$  is a to-be-defined complete distance on  $\mathcal{M}$  which gives the  $C^\infty$  topology. The eigenvalues of the Laplace operator on  $(M, g)$  are denoted  $\{\lambda_k(g)\}_{k \in \mathbb{N}}$ . Throughout this chapter,  $M$  is smooth, connected, compact and without boundary.

We begin with the statement of the first theorem:

**Theorem 6.1.1 (Uniform continuity of the Spectrum)** *Let  $M$  be a manifold as above and  $\dim(M) = n$ . Then, for all  $\delta > 0$  and  $g, g' \in \mathcal{M}$ ,*

$$\rho(g, g') < \delta \Rightarrow e^{-(n+1)\delta} \leq \frac{\lambda_k(g)}{\lambda_k(g')} \leq e^{(n+1)\delta}$$

*for each  $k \in \mathbb{N} \setminus \{0\}$ . The word "uniform" in the title refers to this last statement.*

The second result, and we shall explain in more detail the sequel, can be interpreted as stating that manifolds for which the spectrum of the Laplacian is degenerate are 'rare'. We state it below:

**Theorem 6.1.2 (Uhlenbeck's Theorem)** *Let  $M$  be a compact connected  $C^\infty$  manifold,  $\dim(M) \geq 2$ . Then, the set*

$$\mathcal{S} = \{g \in \mathcal{M} : \text{all eigenvalues have multiplicity one}\}$$

*is a countable intersection of open dense subsets in the complete metric space  $(\mathcal{M}, \rho)$*

## 6.2 Uniform continuity of the Spectrum

### 6.2.1 Structure of the Proof

The first problem consists in defining a complete distance that gives the  $C^\infty$  topology on the set  $\mathcal{M}$  of symmetric, positive definite two-tensors on  $M$ .

There is a canonical way to do so, using the machinery of Fréchet spaces, that crucially depends on the set on which we define the topology being a vector space, a condition which  $\mathcal{M}$  obviously does not satisfy.

To surmount this difficulty, we begin by constructing a distance  $\rho'$  in the set of symmetric two tensors on  $M$ , the vector space  $S(M)$ , wherein it gives the  $C^\infty$  topology, in this canonical fashion (to be detailed below). The metric space  $(S(M), \rho')$  is known to be complete.

We then define another distance,  $\rho''$ , only now on the space of *positive definite* symmetric two tensors  $\mathcal{M}$ . We show that the distance thus defined turns  $(\mathcal{M}, \rho'')$  into a complete metric space.

Finally, we define yet another distance,  $\rho$ , the sum of the two, which gives the  $C^\infty$  topology (because every open  $\rho$ -ball contains an open  $\rho'$ -ball). We show that the metric space  $(\mathcal{M}, \rho)$  is complete.

The remaining part of the proof of the theorem (6.1.1) is straightforward, once a "max-min"-type characterization of the eigenvalues of the Laplace operator is shown (which we present, for completeness).

### 6.2.2 Fréchet Spaces: a concise Review

We begin by recalling a few fundamental topological concepts and results; we proceed with the definition of Fréchet space; we then present a set of sufficient conditions for a space to be Fréchet, which will prove to be more useful in constructing a Fréchet space. It can be shown that these conditions are also necessary.

In this brief exposition, we follow [56] and [43]. A more detailed account on the subject of Fréchet spaces, which also proves the Nash-Moser theorem (sufficient conditions for invertibility of maps between Fréchet spaces) and presents many of its applications can be found in [28].

#### Review of some Topological Results

We recall a few fundamental facts on topology:

Let  $X$  be a set. A **topology** on  $X$  is a set  $\tau$  of subsets of  $X$  such that  $X, \emptyset \in \tau$ ,  $V_1, V_2 \in \tau \Rightarrow V_1 \cap V_2 \in \tau$  and the any union of elements of  $\tau$  is in  $\tau$ .

We call a subset of  $X$  **open** if it is in  $\tau$  and **closed** if its complement is in  $\tau$ .

We call a set to which we have assigned a topology a **topological space** and denote it  $(X, \tau)$ .



We say that  $\tau' \subset \tau$  is a **base** for the topology  $\tau$  if every element of  $\tau$  can be written as the union of elements in  $\tau'$ .

Given  $x \in X$ , a **neighbourhood** of  $x$  is an element of  $\tau$  that contains  $x$ .

If  $\nu$  is a subset of the neighbourhoods of  $x$  such that any neighbourhood of  $x$  contains an element of  $\nu$ , we call it a **local base** at  $x$ .

Given a vector space  $X$  over some field and a topology  $\tau$ , we say that  $(X, \tau)$  is a **topological vector space** (TVS) if  $\{x\}$  is closed for any  $x \in X$  and the vector space operations are continuous. That is, for  $x_1, x_2 \in X$ , given  $V_x \in \tau$  a neighbourhood of  $x = x_1 + x_2$ , there are  $V_{x_1}, V_{x_2}$  neighbourhoods of, respectively,  $x_1$  and  $x_2$  such that  $V_{x_1} + V_{x_2} \subset V_x$  (and similarly for the multiplication by scalars).

We say that a topological space  $(X, \tau)$  is **separable** or **Hausdorff** if distinct points have distinct neighbourhoods.

If  $(X, \tau)$  is a TVS, it can easily be shown that it is separable and that  $V \in \tau$  if and only if  $a + V \in \tau$  for all  $a \in X$ . For a TVS, then, a local base of, say, zero, suffices to specify the topology.

A topological space  $X$  is **metrizable** if there is a metric on  $X$  compatible with the topology. If the compatible metric is such that  $d(x + z, y + z) = d(x, y)$  for all  $x, y, z \in X$  we say that the metric is **translation invariant**.

A metric space  $(X, d)$  is **complete** if every Cauchy sequence in  $X$  converges in  $X$ .

If  $(X, \tau)$  is a TVS, then the notion of Cauchy sequence can be defined in a manner that does not require the notion of metric: a sequence  $\{x_i\} \subset X$  is Cauchy if, given a local base  $\mathcal{B}$ , for any element  $V$  of that local base an  $N$  can be found such that for  $m, n \geq N$  we have  $x_m - x_n \in V$ .

If the metric  $d$  that induces the topology  $\tau$  on  $X$  a TVS is **translation invariant**, then it can be shown that a sequence is Cauchy with respect to  $d$  if and only if it is Cauchy with respect to the topology.

Hence, if  $d_1, d_2$  are two **invariant** metrics on  $X$  a TVS that induce  $\tau$ , then their Cauchy sequences coincide and  $(X, d_1)$  is complete if and only if  $(X, d_2)$  is.

Finally, we have the following

**Definition 6.2.1** *A TVS with a convex local base, whose topology is induced by a translation invariant norm and complete is called a Fréchet space.*

### Constructing a Fréchet space

In what follows, we shall present a canonical way to construct a Fréchet space. We begin with recalling yet another definition:

**Definition 6.2.2** *A seminorm over a vector space  $X$  is a real-valued function  $p$  such that:*

1.  $p(x + y) \leq p(x) + p(y)$ , for all  $x, y$  in  $X$

2.  $p(\alpha x) = |\alpha|p(x)$  for all  $x$  in  $X$  and scalar  $\alpha$

It is clear that a seminorm is a norm if, in addition,

3.  $p(x) = 0 \Rightarrow x = 0$

A set of seminorms  $P$  over  $X$  is said to be **separating** if  $x \in X$  and  $x \neq 0$  implies that there is at least one  $p \in P$  such that  $p(x) \neq 0$ .

A separating set of seminorms over  $X$ ,  $P$  can be used to define a local base on  $X$ , in the following way:

For each  $p \in P$  and  $n \in \mathbb{N}$  we construct the set

$$V(p, n) = \left\{ x \in X : p(x) < \frac{1}{n} \right\}$$

and call  $\mathcal{B}$  the set of all finite intersections of  $V(p, n)$ . Then, it can be shown that  $\mathcal{B}$  is a convex balanced local base for a topology  $\tau$  on  $X$  which turns it into a locally convex space. Furthermore, with respect to this topology, each seminorm is continuous.

It can be shown that if a TVS  $(X, \tau)$  has a *countable* local base, then it is metrizable, the metric is translation invariant and its open balls are convex and so it follows that if  $(X, \tau)$  with such a topology is complete, it is a Fréchet space.

Regarding completeness, we recall a few facts:

Let  $\tau$  be the topology generated by a countable set of separating seminorms on  $X$ ,  $\{\| \cdot \|_k\}_{k \in \mathbb{N}}$ . Then, it can be shown that the following is a metric and that it induces the same topology on  $X$ :

$$d(x, y) = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{\|x - y\|_k}{1 + \|x - y\|_k} \quad (6.1)$$

Since the first statement presents no difficulties, we show the second statement:

Since each of the seminorms is continuous with respect to  $\tau$ , so is each term in the series.

Since  $0 < \frac{\|x - y\|_k}{1 + \|x - y\|_k} < 1$ , the series converges uniformly and  $d(\cdot, \cdot)$  is continuous. But then the inverse image of an open set is open and so the balls

$$B_\delta(0) = \{y \in X : d(x, 0) < \delta\}$$

are open.

And they are in fact a (local, but  $X$  is a TVS) basis for the topology induced by the seminorms, for every neighbourhood of zero  $V_0$  contains one such balls: for a local base at 0 is, as defined above, the set of all finite

intersections of sets of the form

$$V(\| \cdot \|_k, n_k) = \left\{ x \in X : \|x\|_k < \frac{1}{n_k} \right\}$$

and  $V_0$ , being a neighbourhood, must contain one such sets, say  $W$ .

The bound  $\delta$  on the series implies at least the same bound on each of the terms:

$$f(\|x\|_k) \equiv \frac{\|x\|_k}{1 + \|x\|_k} < 2^k \delta$$

And since  $f$  is a monotonically increasing bounded function,

$$\|x\|_k < f^{-1}(2^k \delta)$$

and one can certainly choose  $\delta > 0$  such that the RHS is smaller than each of the finitely many  $1/n_k$ . So  $B_\delta \subset W$  and the topologies coincide.

And so we have a canonical method to construct a Fréchet space from a vector space  $X$ :

1. Define a countable family of separating seminorms
2. Construct a local base as above, which is guaranteed to be convex. Since  $X$  is a vector space, the topology defined by the local base is Hausdorff.
3. Since the local base is countable, the TVS it defines is metrizable and the metric is invariant (we have constructed a compatible metric explicitly above).
4. Since, as stated above, completeness can be established using *any* invariant metric compatible with a given topology, it suffices to establish convergence of Cauchy sequences with respect to the metric defined above.

### 6.2.3 The Fréchet space $(S(M), \rho)$

The author follows [19] and [23] in the construction of a complete distance that induces the  $C^\infty$  topology, which is a straightforward adaptation, for the component functions of symmetric two-tensors on a manifold, of the construction which is customary for functions on Euclidean space. Details can be found in Functional Analysis textbooks, e.g. [56] [43].

Let  $M$  be a compact  $n$ -dimensional  $C^\infty$  manifold with no boundary. Fix  $A = \{U_\lambda\}_{\lambda \in \Lambda}$  a finite cover of  $M$  such that the closure of every  $U_\lambda \in A$  is contained in the open coordinate neighbourhood  $V_\lambda$  (a finite cover is guaranteed to exist, since  $M$  is compact), with coordinates  $x_i$ .

Let now  $h \in S(M)$  (a vector space) and  $h_{ij}$  its components. Set

$$|h|_{\lambda,k} = \sup_{x \in U_\lambda} \sum_{|\alpha| \leq k} \sum_{i,j} \left| \frac{\partial^{|\alpha|}}{\partial(x_1)^{\alpha_1} \dots \partial(x_n)^{\alpha_n}} h_{ij}(x) \right|$$

Where  $|\alpha| = \alpha_1 + \dots + \alpha_n$ , with  $\alpha_i \in \mathbb{N}$ .  $|h|_{k,\lambda}$  is a seminorm on  $U_\lambda$  for every  $k$  and setting  $\|h\|_k = \sum_{\lambda \in \Lambda} |h|_{k,\lambda}$  defines a seminorm on  $S(M)$  for every  $k$  (the only possible issue being the convergence of the sum in  $\lambda$ ; but, as we mentioned, since  $M$  is compact,  $\Lambda$  is a finite set).

It is immediate that this family of seminorms is separating and, by construction, countable and so, using the ‘prescription’ above, they induce a locally convex metrizable Hausdorff topology  $\tau$  on  $S(M)$ , called the  $C^\infty$  topology, compatible with the following complete distance on  $S(M)$  [56]:

$$\rho'(h_1, h_2) = \sum_{k=0}^{\infty} 2^{-k} \|h_1 - h_2\|_k (1 + \|h_1 - h_2\|_k)^{-1}$$

Which, in short, means that  $(S(M), \tau)$  is a Fréchet space.

## 6.2.4 The complete metric space $(\mathcal{M}, \rho)$

Let now  $\mathcal{M}$  be the set of Riemannian metrics on  $M$ .

### The first distance

Let  $x \in M$  and  $P_x$  be the set of symmetric positive definite bilinear forms on  $T_x M \times T_x M$ . We start by defining a distance on  $P_x$  by

$$\rho_x''(\phi, \psi) = \inf_{\delta > 0} \left\{ e^{-\delta} \phi < \psi < e^{\delta} \phi \right\}$$

Where  $A < B$  is shorthand for ‘ $A - B$  is positive definite’ on the tangent space it is being evaluated at.

It can be shown that this distance makes  $(P_x, \rho_x'')$  into a complete metric space.

Following [5], we organize this and some useful results in a lemma.

**Lemma 6.2.3** *Let  $G_x$  denote the group of invertible linear mappings of  $T_x M$  onto itself. Set, for  $A \in G_x$  and  $\phi \in S_x$ ,  $\phi_A(u, v) = \phi(Au, Av)$ , for  $u, v \in T_x M$ .*

*In the usual manner, choose a basis for the tangent space and identify  $S_x$  with the set of real symmetric matrices of degree  $n$ ,  $S(n)$ : the  $i, j$  entry of the matrix identified with the form  $\phi$  being  $\phi(e_i, e_j)$ . Call this identification  $\Phi : S_x \rightarrow S(n)$ . Then we have:*

1.  $\rho_x''(\phi_A, \psi_A) = \rho_x''(\phi, \psi)$  for every  $\phi, \psi \in P_x$  and  $A \in G_x$ .

2. Let  $\phi_0$  be the element of  $P_x$  such that  $\Phi(\phi_0) = Id$ . Then

$$\rho_x''(\phi, \phi_0) = \|\log \Phi(\phi)\|, \quad \phi \in P_x$$

Where  $\log$  is the inverse image of the matrix exponential and  $\|\cdot\|$  is the operator norm.

3. The metric space  $(P_x, \rho_x'')$  is complete.

4. If  $\{\phi_j\}$  is a convergent sequence in  $P_x$  with respect to the  $\rho_x''$  norm, with limit  $\phi \in P_x$ , then

$$\lim_{j \rightarrow \infty} \phi_j(u, v) = \phi(u, v)$$

for all  $u, v \in T_x M$

Finally, we define a distance on  $\mathcal{M}$  as follows:

**Definition 6.2.4**

$$\rho''(g_1, g_2) = \sup_{x \in M} \rho_x''(g_1(x), g_2(x)), \quad g_1, g_2 \in \mathcal{M}$$

**The second distance**

We now define a complete distance that generates the  $C^\infty$  topology on  $\mathcal{M}$

$$\rho(g_1, g_2) = \rho'(g_1, g_2) + \rho''(g_1, g_2), \quad g_1, g_2 \in \mathcal{M}$$

That  $\rho$  is a distance on  $\mathcal{M}$  is straightforward. That it generates the  $C^\infty$  topology also, because every  $\rho$  ball contains a  $\rho'$  ball and the latter generates the  $C^\infty$  topology on  $M$ . It remains to show:

**Proposition 6.2.5** *The metric space  $(\mathcal{M}, \rho)$  is complete.*

**Proof:** : Let  $\{g_n\}_{n \in \mathbb{N}}$  be a Cauchy sequence in  $(\mathcal{M}, \rho)$ . Since, by construction, if  $\rho(g_i, g_j) < \delta$  then both  $\rho'(g_i, g_j) < \delta$  and  $\rho''(g_i, g_j) < \delta$ , it is also a Cauchy sequence in  $(\mathcal{M}, \rho')$  and, since  $\mathcal{M} \subset S(M)$ , also in  $(S(M), \rho')$ .

But  $(S(M), \rho')$  is complete, and so there is a  $g \in S(M)$  such that  $\lim_{n \rightarrow \infty} \rho'(g_n, g) = 0$ .

We have in particular, for each  $x \in M$  and  $u, v \in T_x M$

$$\lim_{n \rightarrow \infty} (g_n)_x(u, v) = g_x(u, v)$$

To show this, note that

$$\begin{aligned}
\left| (g^{(n)} - g)_y(u, v) \right| &= \left| \sum_{a,b} u^a v^b (g^{(n)} - g)_y(\partial_a, \partial_b) \right| \\
&\leq \sum_{a,b} \left| u^a v^b \right| \left| (g^{(n)} - g)_y(\partial_a, \partial_b) \right| \\
&\leq M \sum_{a,b} \left| (g^{(n)} - g)_y(\partial_a, \partial_b) \right|
\end{aligned}$$

For  $y \in U_\lambda$ , we have, then

$$\begin{aligned}
\left| (g^{(n)} - g)_y(u, v) \right| &\leq M \sum_{a,b} \left| (g^{(n)} - g)_y(\partial_a, \partial_b) \right| \\
&\leq \sup_{U_\lambda} M \sum_{a,b} \left| (g^{(n)} - g)_y(\partial_a, \partial_b) \right| \\
&= M \|g^{(n)} - g\|_{\lambda,0} \\
&\leq (\max \{M, 1\}) \sum_{\lambda \in \Lambda} \|g^{(n)} - g\|_{\lambda,0} \\
&= (\max \{M, 1\}) \|g^{(n)} - g\|_0
\end{aligned}$$

On the other hand, since  $\rho'(g^{(n)}, g) \rightarrow 0$ , for all  $\delta'$  there is an  $N \in \mathbb{N}$  such that, for  $n \geq N$  we have  $\|g^{(n)} - g\| < \delta'$ . That is, for  $n \geq N$  we have

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{1}{2^k} \frac{\|g^{(n)} - g\|_k}{1 + \|g^{(n)} - g\|_k} &< \delta' \Rightarrow \\
\frac{1}{2^k} \frac{\|g^{(n)} - g\|_k}{1 + \|g^{(n)} - g\|_k} &< \delta', \text{ for all } k \in \mathbb{N} \Rightarrow \\
\frac{\|g^{(n)} - g\|_0}{1 + \|g^{(n)} - g\|_0} &< \delta'
\end{aligned}$$

But, as we've seen above, given any  $\delta'' = \frac{\delta}{(\max \{M, 1\})}$ , we can choose a  $\delta'$  such that  $\frac{|u|}{|u|+1} < \delta' \Rightarrow |u| < \delta''$ . Combining this with the results above, we have the claim.

On the other hand, since  $\{g^{(n)}\}_{n \in \mathbb{N}}$  is Cauchy with respect to  $\rho''$ , for every  $\delta > 0$  there is an  $N' \in \mathbb{N}$  such that  $n, m \geq N'$  implies

$$\rho''_x(g_x^{(n)}, g_x^{(m)}) \leq \sup_{x \in M} \rho''_x(g_x^{(n)}, g_x^{(m)}) < \delta$$

and so the sequence  $\{g_x^{(n)}\}_{n \in \mathbb{N}}$  is also Cauchy in  $(P_x, \rho''_x)$  for every  $x \in M$ . Since  $(P_x, \rho''_x)$  is complete, the sequence converges to, say,  $\bar{g}_x \in P_x$ . Using

the Lemma above, we have, then

$$\lim_{n \rightarrow \infty} (g_x^{(n)}) (u, v) = \bar{g}_x(u, v)$$

for all  $x \in M$  and  $u, v \in T_x M$ . Combining this with the result above, we conclude that  $g = \bar{g} \in \mathcal{M}$ . Now it remains to show that it converges with respect to  $\rho''$ . To see that, we note that for  $m, n \geq N'$  the inequalities above hold; which means that for  $m$  fixed, we have

$$\lim_{n \rightarrow +\infty} \rho_x''(g_x^{(n)}, g_x^{(m)}) \leq \delta$$

But this means

$$\begin{aligned} \rho_x''(g_x, g_x^{(m)}) &\leq \delta, \text{ for all } x \in M \Rightarrow \\ \sup_{x \in M} \rho_x''(g_x, g_x^{(m)}) &\leq \delta \Leftrightarrow \\ \rho''(g, g^{(m)}) &\leq \delta \end{aligned}$$

and we have shown the claim.

### 6.2.5 Max-min theorem

We now show a useful characterization of the spectrum of the Laplace operator.

We recall (cf. chapter 2) the definitions of the following inner products,

#### Definition 6.2.6

$$(\phi, \psi)_g = \int_M \phi \psi \sqrt{g} dx, \quad \phi, \psi \in C^\infty(M) \quad (6.2)$$

and

#### Definition 6.2.7

$$(\omega, \theta)_g = \int_M g^{-1}(\omega, \theta) \sqrt{g} dx, \quad \omega, \theta \in \Omega^1(M) \quad (6.3)$$

where  $\Omega^k(M)$  is the space of smooth differential forms of degree  $k$  on  $M$ .

We denote both norms induced by these inner products by  $\|a\|_g^2 = (a, a)_g$ , where it is clear to which of the inner products it refers.

We assume  $M$  is a smooth closed connected manifold of dimension  $n$ , as we do throughout this chapter, and denote the  $k$ th eigenvalue of the Laplace operator on functions on  $(M, g)$  as  $\lambda_k(g)$ .

The following result then holds

**Theorem 6.2.8** (*Max-Min Theorem*) Let  $L_k$  be some  $k$ -dimensional subspace of  $C^\infty(M)$ . Define

$$\Lambda_g(L_k) = \sup_{0 \neq \phi \in L_k} \left\{ \frac{\|d\phi\|_g^2}{\|\phi\|_g^2} \right\}$$

Then we have

$$\lambda_k(g) = \inf_{L_{k+1}} \Lambda_g(L_{k+1})$$

where the infimum is taken over all  $k+1$  dimensional subspaces of  $C^\infty(M)$

**Proof:** Take as a complete orthonormal basis of  $C^\infty(M)$  the set  $\{\phi_n\}_{n \in \mathbb{N}}$  of eigenfunctions of the Laplace operator,  $\Delta\phi_k = \lambda_k(g)\phi_k$ . Then, denoting the  $k+1$  dimensional subspace of  $C^\infty(M)$  generated by  $\{\phi_n\}_{n=0 \dots k}$  by  $L_{k+1}^0$ ,

$$\begin{aligned} \Lambda_g(L_{k+1}^0) &= \sup_{0 \neq \phi \in L_{k+1}^0} \left\{ \left( \frac{\|d\phi\|_g}{\|\phi\|_g} \right)^2 \right\} \\ &= \sup_{0 \neq \phi \in L_{k+1}^0} \left\{ \left( \frac{(\phi \Delta \phi)_g}{\|\phi\|_g} \right)^2 \right\} \end{aligned}$$

But this is just the operator norm of the quadratic form induced by the Laplace operator on the vector space generated by its first  $k+1$  eigenfunctions:

$$\|\Delta\| = \sup\{(\phi, \Delta\phi); \phi \in L_{k+1}^0, (\phi, \phi) = 1\}$$

Since for any  $\phi \in L_{k+1}^0$ ,  $(\phi, \phi) = 1$  we have

$$\begin{aligned} (\phi, \Delta\phi) &= \sum_{i=0}^k \lambda_i \alpha_i^2 \\ &\leq \lambda_k \end{aligned}$$

where we denoted  $\alpha_i = (\phi, \phi_i)$ . In particular, for  $\phi = \phi_k$  we have  $(\phi_k, \Delta\phi_k) = \lambda_k(g)$ , and so it follows that  $\Lambda_g(L_{k+1}^0) = \lambda_k(g)$  (i.e., the operator norm of the Laplace operator on the vector space generated by its first  $k$  eigenfunctions is the spectral radius of its restriction to that space). But then we have that  $\lambda_k(g) = \Lambda_g(L_{k+1}^0) \geq \inf_{L_{k+1}} \Lambda(L_{k+1})$ . Now it remains to show equality:

With that aim, assume that  $\lambda_k(g) > \inf_{L_{k+1}} \Lambda(L_{k+1})$ . Then there is a  $k+1$  subspace of  $C^\infty(M)$ , say  $L_{k+1}$  such that  $\lambda = \Lambda_g(L_{k+1}) < \lambda_k(g)$ . Then,



by definition, for all  $\phi \in L_{k+1}$  we have

$$\begin{aligned} \lambda \geq \frac{\sum_{i=0}^{\infty} \lambda_i(g) \alpha_i^2}{\sum_{i=0}^{\infty} \alpha_i^2} &\Leftrightarrow \\ \sum_{i=0}^{\infty} (\lambda - \lambda_i) \alpha_i^2 &\geq 0 \Leftrightarrow \\ \sum_{\lambda \geq \lambda_i} (\lambda - \lambda_i) \alpha_i^2 + \sum_{\lambda < \lambda_i} (\lambda - \lambda_i) \alpha_i^2 &\geq 0 \end{aligned}$$

which implies that

$$\sum_{\lambda \geq \lambda_i} (\lambda - \lambda_i) \alpha_i^2 \geq \sum_{\lambda < \lambda_i} (\lambda_i - \lambda) \alpha_i^2 \quad (6.4)$$

Now let  $m = \max \{i \in \mathbb{N} : \lambda_i \leq \lambda\}$ . Define the following linear map  $\Phi : L_{k+1} \rightarrow C^\infty(M)$

$$\Phi(\phi) = \sum_{i=0}^m (\phi, \phi_i) \phi_i, \text{ for } \phi \in L_{k+1}$$

Then it follows that the dimension of  $\Phi(L_{k+1})$  is less than  $k+1$ : because

$$\begin{aligned} i < m &\Rightarrow \lambda_i \leq \lambda < \lambda_k \\ \Rightarrow \dim \Phi(L_{k+1}) &\leq m+1 < k+1 \end{aligned}$$

But then there must be a *non-zero* element of  $L_{k+1}$ , say  $\psi$ , which is in the kernel of  $\Phi$ . By definition, this means that

$$(\phi_i, \psi) = 0 \text{ for all } 0 \leq i \leq m \quad (6.5)$$

Since the inequality (6.4) holds for all elements of  $L_{k+1}$ , we have, for  $\psi$  in particular, that the LHS is zero. But then, because on the RHS we sum over  $\lambda < \lambda_i$ , all the terms on the RHS are zero, which in turn implies that  $(\Psi, \phi_i) = 0$  for all  $i$  such that  $\lambda_i > \lambda$ . Combining this with (6.5) we have

$$\psi = \sum_{i=0}^{\infty} (\psi, \phi_i) \phi_i = 0$$

which is a contradiction, and we have the claim.

### 6.2.6 Proof of theorem

The proof of Theorem 6.1.1 is now straightforward, combining the results (6.2.8) and (6.2.5) above.

**Proof:** Set local coordinates on  $U \subset M$  open. Then, by definition of the distance  $\rho''$ , for each  $\delta > 0$  the following relation holds between the matrices of the components in these coordinates of every element  $g' \in \mathcal{M}$  of the open ball  $\rho''(g, g') < \delta$ :

$$e^{-\delta} [g'_{ab}] < [g_{ab}] < e^{\delta} [g'_{ab}] \quad (6.6)$$

where, as above,  $A < B$  stands for  $A - B$  is positive definite. The matrices being symmetric, it follows that

$$e^{-\delta} [g'^{ab}] < [g^{ab}] < e^{\delta} [g'^{ab}] \quad (6.7)$$

and also

$$e^{-n\delta/2} \sqrt{g'} < \sqrt{g} < e^{n\delta/2} \sqrt{g'} \quad (6.8)$$

where again,  $n = \dim(M)$ . And so, using the definitions for the inner products on functions and forms as in (6.2, 6.3) we get, for all  $\phi \in C^\infty(M)$  with support contained in  $U$

$$e^{-n\delta/2} \|\phi\|_{g'}^2 \leq \|\phi\|_g^2 \leq e^{n\delta/2} \|\phi\|_{g'}^2 \quad (6.9)$$

and for all  $\omega \in \Omega(M)$  with support contained in  $U$

$$e^{-\left(\frac{n}{2}+1\right)\delta} \|\omega\|_{g'}^2 \leq \|\omega\|_g^2 \leq e^{\left(\frac{n}{2}+1\right)\delta} \|\omega\|_{g'}^2 \quad (6.10)$$

Using the partition of unity, we have (6.9,6.10) for every  $\phi \in C^\infty(M)$  and  $\omega \in \Omega(M)$ . Combining the two, we get in particular

$$e^{-(n+1)\delta} \frac{\|d\phi\|_{g'}^2}{\|\phi\|_{g'}^2} \leq \frac{\|d\phi\|_g^2}{\|\phi\|_g^2} \leq e^{(n+1)\delta} \frac{\|\phi\|_{g'}^2}{\|\phi\|_{g'}^2}$$

for every non-zero  $\phi \in C^\infty(M)$ , which implies the claim, as promised, by a straightforward application of Theorem 6.2.8.

## 6.2.7 Discussion and some applications

I shall now state and prove the new regularity result that was hoped for, as mentioned in the introduction to this chapter (cf. 6.1), as a consequence of Theorem 6.1.1:

**Proposition 6.2.9** *Let  $g \in \mathcal{M}$  and  $\Gamma$  a cutoff of the spectrum of the Laplace operator. Let  $\lambda$  be the largest eigenvalue of  $\Delta(g)$  for which  $\lambda < \Gamma$ .*

*Then, there is  $V_\delta(g)$  a neighbourhood of  $g$  such that, for any  $\tilde{g} \in V_\delta(g)$ , the dimension of the cutoff space, i.e., the number of eigenvalues below  $\Gamma$ , is constant.*

**Proof:** Since by 2.2.11, the spectrum is discrete, positive and has no accumulation points, we have

$$\begin{aligned} \inf_{k \in \mathbb{N}} \{\lambda_{k+1}(g) - \lambda_k(g)\} &= \min_{k \in \mathbb{N}} \{\lambda_{k+1}(g) - \lambda_k(g)\} \\ &:= \mu \end{aligned}$$

By Theorem 6.1.1, the following holds for all positive integers  $k$ , for  $\tilde{g}$  such that  $\rho(g, \tilde{g}) < \delta$

$$e^{-m\delta} \lambda_k(g) \leq \lambda_k(\tilde{g}) \leq e^{m\delta} \lambda_k(g)$$

where we set  $m := n + 1$ . In particular

$$e^{-m\delta} \lambda_{k+1}(g) \leq \lambda_{k+1}(\tilde{g}) \leq e^{m\delta} \lambda_{k+1}(g)$$

Take now  $\lambda_q$ , the highest eigenvalue strictly below  $\Gamma$ . Requiring that the the number of eigenvalues below the cutoff remains constant for metrics  $\tilde{g}$  in  $\rho(g, \tilde{g}) < \delta$  means that the following must hold

$$e^{m\delta} \lambda_q(g) < \Gamma < e^{-m\delta} \lambda_{q+1}(g)$$

Or equivalently, that

$$\lambda_{q+1} - \lambda_q > \Gamma \sinh(m\delta)$$

which will hold for an *arbitrary* cutoff if

$$\sinh(m\delta) < \frac{\mu}{\Gamma}$$

The claim then follows, because  $\sinh$  is monotonically increasing.

To our knowledge, the only other result on the continuity of the spectrum is Berger's [8], which we will present in the proof of Uhlenbeck's theorem; it does not guaranty a uniform bound on the spectrum and as such would not suffice to show Proposition 6.2.9. We would like to add that, in the case that interests us, of very high dimensional (but finite) function spaces, being able to guaranty the conservation of dimension of the function space in question upon variations of the metric by checking a *single* bound is of obvious interest.

### 6.3 Uhlenbeck's theorem

One cannot hope to generically have an inverse of the spectral map at a point  $g \in \mathcal{M}$  of which the spectrum is degenerate: for then, there could certainly

be variations of the metric that leave some (or all, cf. the constructions in [25] and the discussion in section 3.3) eigenvalue unchanged.

Since we are concerned with the existence of a local inverse, we aim to investigate the behaviour of the multiplicities in the spectrum of the Laplace operator with respect to variations of the metric.

To that effect, we present a result known as Uhlenbeck's theorem [67].

We begin with recalling the following definition of a 'small' subset of a topological set [29]:

**Definition 6.3.1 (Meagre set)** *A subset of a topological set is called meagre if it is the countable union of nowhere dense subsets.*

A meagre set is also called a *set of the first category* (Baire). A non-meagre set is called a *set of the second category*. The notion of meagre set allows us to define its dual:

**Definition 6.3.2 (Residual set)** *Given a topological space  $X$ ,  $S \subset X$  is said to be residual if  $S$  is the countable intersection of open dense subsets of  $X$  (equivalently,  $S$  is the complement of a meagre subset of  $X$ , i.e. comeagre).*

With this terminology, we are ready to state a theorem that, in short, tells us that the set of manifolds on which the spectrum of the Laplace operator is non-degenerate is 'large':

**Theorem 6.3.3 (Uhlenbeck's Theorem)** *Let  $M$  be a compact, connected  $C^\infty$  manifold of dimension not less than two. Let  $\mathcal{M}$  be the set of  $C^\infty$  metrics on  $M$  and  $\rho$  the complete distance above. Define*

$$\mathcal{S} = \{g \in \mathcal{M} : \lambda_0(g) < \lambda_1(g) < \cdots < \lambda_k(g) < \cdots\},$$

*the set of all metrics in  $\mathcal{M}$  for which all the eigenvalues of the Laplacian are non-degenerate. Then  $\mathcal{S}$  is a residual set in  $(\mathcal{M}, \rho)$*

We sketch the proof as in [5] and refer to [66] and [67] for different versions.

**Proof:** We note that the set  $\mathcal{S}$  defined above is the intersection of the following sets:

$$\mathcal{S}_k = \{g \in \mathcal{M} : \lambda_0(g) < \lambda_1(g) < \cdots < \lambda_k(g)\}$$

We note that  $\mathcal{M} = \mathcal{S}_1$  and that  $\mathcal{S}_{k+1} \subset \mathcal{S}_k$  for all  $k$ . If we show that all the  $\mathcal{S}_k$  are open, the theorem then follows by induction if we show  $\mathcal{S}_{k+1}$  is dense in  $\mathcal{S}_k$  for all  $k$  (since  $\mathcal{M}$  is obviously dense in  $\mathcal{M}$ ).

To show that the  $\mathcal{S}_k$  are open in  $(\mathcal{M}, \rho)$  is a straightforward application of theorem 6.1.1: one wishes to show that, given a metric  $g \in \mathcal{M}$ , there is a  $\delta > 0$  such that all the  $g'$  such that  $\rho(g, g') < \delta$  are in  $\mathcal{S}_k$ . An educated guess

for such  $\delta > 0$  is that one for which  $\epsilon(2\lambda_k(g))^{-1} > \exp((n+1)\delta) - 1$ , where  $\epsilon$  is the minimum of the differences between successive eigenvalues up to  $k$ . By construction and the triangle inequality it then follows that

$$\begin{aligned}\epsilon &\leq \lambda_{j+1}(g) - \lambda_j(g) \\ &\leq |\lambda_{j+1}(g) - \lambda_{j+1}(g')| + |\lambda_{j+1}(g') - \lambda_j(g')| + |\lambda_j(g') - \lambda_j(g)|\end{aligned}$$

and applying theorem 6.1.1 to the first and last terms we have

$$\epsilon \leq 2\lambda_k(g)(\exp((n+1)\delta) - 1) + |\lambda_{j+1}(g') - \lambda_j(g')|$$

From which it follows that

$$0 < \epsilon - 2\lambda_k(g)(\exp((n+1)\delta) - 1) \leq |\lambda_{j+1}(g') - \lambda_j(g')|$$

That is, for all  $i = 0, 1, \dots, k-1$  all the eigenvalues of the Laplace operator on  $(M, g')$  are different, and  $g'$  is in  $\mathcal{S}_k$ .

It then remains to show density of  $\mathcal{S}_{k+1}$  in  $\mathcal{S}_k$ . The idea of the proof is to show that, given a metric  $g \in \mathcal{S}_k$ , there is a deformation that is in  $\mathcal{S}_{k+1}$ . To do so, one uses the following result by Berger[5]:

**Proposition 6.3.4** *Let  $g \in \mathcal{M}$ . Define a one-parameter deformation of  $g$  in the direction  $h$ :  $g(t) = g + th$ ,  $|t| < \epsilon$ . Let  $\lambda$  be an eigenvalue of the Laplacian on  $(M, g)$  with multiplicity  $m$ . Then, there are  $\Lambda_i(t) \in \mathbb{R}$  and  $\phi_i(t) \in C^\infty(M)$ ,  $i = 1, \dots, m$  such that*

1. For all  $i = 1, \dots, m$ ,  $\Lambda_i(0) = \lambda$ .
2. For all  $|t| < \epsilon$  and  $i = 1, \dots, m$ ,  $\Lambda_i(t)$  and  $\phi_i(t)$  are, respectively, an eigenvalue and associated eigenfunction of  $\Delta(g + th)$
3. For all  $i = 1, \dots, m$  both  $\Lambda_i(t)$  and  $\phi_i(t)$  depend real analytically on  $t$ , for  $|t| < \epsilon$
4. Furthermore, for each  $t : |t| < \epsilon$ , the  $\phi_i(t)$  are orthonormal with respect to the inner product 6.2 defined on  $(M, g(t))$

In brief, the proposition above establishes that one can smoothly and orthonormally deform both the eigenfunctions and eigenvalues of the Laplacian on  $(M, g)$  to match a those of  $(M, g + th)$ .

It can be shown that there is a deformation of the metric such that, if  $\lambda$  is an  $n$ -degenerate eigenvalue, at least two of the perturbed eigenvalues are first-order distinct (the first order terms in the expansion of  $\Lambda(t)$  are distinct). The theorem then follows by a clever application of theorem 6.1.1 (for details, see [5]).

### 6.3.1 Discussion

As stated in the introduction to this section, we do not expect in general to have an inverse of the spectral map (even a local inverse) if there are degeneracies in the spectrum. Uhlenbeck's theorem, then, is of critical importance, because it states that, in the sense we made precise above, 'most' manifolds have a non-degenerate spectrum. This result can be further refined, using the following corollary of Theorem 6.1.1 [5]:

**Corollary 6.3.5** *The multiplicity of each eigenvalue depends upper semi-continuously on  $g \in \mathcal{M}$ , with respect to the distance  $\rho$  defined in 6.2.4: For each  $g \in \mathcal{M}$  and  $k \in \mathbb{N}$ , there is a  $\delta > 0$  such that, if  $\rho(g, g') < \delta$  then the multiplicity of  $\lambda_k(g')$  is not greater than that of  $\lambda_k(g)$ .*

Combining this result with Uhlenbeck's theorem, we have that, if the spectrum of the Laplacian on  $(M, g)$  is simple (no degeneracies), then there is a  $\delta > 0$  such that for *all*  $g'$  in  $\rho(g, g') < \delta$ , the spectrum of  $\Delta(g')$  is simple.

Which potentially opens the door to the existence of a *local* inverse of the spectral map, as conjectured in Section 4.4: if indeed a local inverse of the spectral map is only possible for manifolds with simple spectra, Uhlenbeck's theorem guarantees that there is a neighbourhood of these for which the spectral map is potentially invertible.

## Chapter 7

# Explicit Computations

In this chapter we apply some of the results above to the two-dimensional flat torus. As discussed above, this is not the ideal setting for the program outlined in Chapter 4. However, it is one of the very few examples for which the spectrum and eigenfunctions can be explicitly calculated (and in any dimension), and this, as seen in Chapter 5, is a requirement for perturbation theory. We shall have more to say on this in Chapter 8.

### 7.1 The Flat Torus

#### 7.1.1 Spectrum of the Laplacian

We intend to calculate the perturbation of the spectrum of the Laplacian on the flat, two-dimensional torus. In order to do so, we first calculate its spectrum. The metric is  $g_{ij} = \delta_{ij}$ , and the Laplacian is thus

$$\Rightarrow \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

For a function  $f$  on the torus the following boundary conditions hold (BC)

$$\begin{aligned} f(x, 2\pi L_2) &= f(x, 0) \\ f(2\pi L_1, y) &= f(0, y) \end{aligned}$$

The eigenfunctions of the Laplacian are solutions of the Laplace equation:

$$\Delta f + \lambda f = 0 \wedge BC \tag{7.1}$$

We look for solutions of the form  $f = e^{ikx} e^{i\tilde{k}y}$ . Plug in above to get

$$(-k^2 - \tilde{k}^2 + \lambda)f = 0 \Rightarrow \lambda = k^2 + \tilde{k}^2$$

Using the BC, which separate also:

$$\begin{aligned} e^0 = e^{ik2\pi L_1} &\Rightarrow 2\pi L_1 k = 0 + 2\pi n, n \in \mathbb{Z} \\ &\Rightarrow k = \frac{n}{L_1}, n \in \mathbb{Z} \end{aligned}$$

and the other boundary condition implies

$$\tilde{k} = \frac{m}{L_2}, m \in \mathbb{Z}$$

And so eigenfunctions of this form with eigenvalue  $\lambda$  are

$$f(x, y) = e^{i(\frac{n}{L_1}x + \frac{m}{L_2}y)}, \lambda = \frac{n^2}{L_1^2} + \frac{m^2}{L_2^2}, n, m \in \mathbb{Z}$$

We note that

$$\begin{aligned} \frac{n^2}{L_1^2} + \frac{m^2}{L_2^2} = \frac{n'^2}{L_1^2} + \frac{m'^2}{L_2^2} &\Rightarrow \\ n = \pm n', m = \pm m' \vee \frac{n^2 - n'^2}{m^2 - m'^2} = \frac{L_1^2}{L_2^2} \end{aligned}$$

And since, as discussed above, degeneracies are precisely the special case which we wish to avoid, we require  $\frac{L_1^2}{L_2^2}$  to be irrational. We now take the real and imaginary parts of the complex solution to get real solutions:

$$\begin{aligned} \cos\left(\frac{n}{L_1}x + \frac{m}{L_2}y\right), \sin\left(\frac{n}{L_1}x + \frac{m}{L_2}y\right) \\ \lambda = \frac{n^2}{L_1^2} + \frac{m^2}{L_2^2}, \text{ with } n, m \in \mathbb{Z} \end{aligned}$$

which we normalize and denote

$$\Psi(n, m, p, s) \equiv \begin{cases} \frac{1}{2\pi^2 L_1 L_2} \cos\left(\frac{n}{L_1}x + (-1)^s \frac{m}{L_2}y\right), & \text{if } p = 0 \\ \frac{1}{2\pi^2 L_1 L_2} \sin\left(\frac{n}{L_1}x + (-1)^s \frac{m}{L_2}y\right), & \text{if } p = 1 \end{cases}$$

where now  $n, m \in \mathbb{N}$ . Since the Wronskian of these is not zero, they are independent and span the space of solution space to the partial differential equation (PDE)7.1. Since the PDE happens to be the Laplace equation, the solution space is  $L^2(T)$ . So, any function  $f$  on the torus can be written as

$$f = \sum_{n,m=0}^{\infty} \sum_{p,s=0}^1 \alpha(n, m, p, s) \Psi(n, m, p, s)$$



where

$$\begin{aligned}\Delta\Psi(n, m, p, s) &= \left(\frac{n^2}{L_1^2} + \frac{m^2}{L_2^2}\right)\Psi(n, m, p, s) \\ &\equiv \lambda(n, m)\Psi(n, m, p, s)\end{aligned}$$

We note that  $\lambda(m, n) \neq \lambda(n, m)$  and that each eigenvalue is, at most, fourfold degenerate: fourfold degenerate for positive  $m, n$ , twofold if either  $m$  or  $n$  are zero and non-degenerate for  $n, m$  zero.

### 7.1.2 The $tt$ decomposition on the flat torus

As we have seen above,  $\forall h \exists y$  :

$$h^{ij} = h_{tt}^{ij} + \frac{1}{2}hg^{ij} + \nabla^i y^j + \nabla^j y^i - (\nabla_k y^k)g^{ij} \quad (7.2)$$

In particular, if  $g + h$  is a small perturbation to  $g$ , we can choose a gauge  $\xi = -y$  where

$$h^{ij} = h_{tt}^{ij} + \frac{1}{2}hg^{ij} - \nabla_k y^k g^{ij}$$

Here,  $h_{tt}$  is symmetric, traceless and transverse. The first two conditions mean that we can write its components in terms of functions  $a, b$ :

$$h_{tt}^{ij} = \begin{bmatrix} a & b \\ b & -a \end{bmatrix} \quad (7.3)$$

And the transverse conditions mean that

$$\nabla_i h_{tt}^{ij} = \begin{cases} \frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} = 0 \\ \frac{\partial b}{\partial x} - \frac{\partial a}{\partial y} = 0 \end{cases} \quad (7.4)$$

which are just the Cauchy-Riemann equations, which means that  $a, b$  are the real and imaginary parts of a holomorphic function on the torus, which means that, since the torus is compact and simply connected, that they are both constants . One can make the same statement by showing that these conditions imply that both  $a, b$  are harmonic functions, simply by taking derivatives of the expressions above, which, since the torus has no boundary, means that they must be constant. That is:

$$\begin{aligned}\frac{\partial^2 a}{\partial x^2} + \frac{\partial^2 b}{\partial x \partial y} &= 0 \\ \frac{\partial^2 b}{\partial y \partial x} - \frac{\partial^2 a}{\partial y^2} &= 0\end{aligned}$$

Subtracting these two expressions we get  $\Delta a = 0$ , and similarly one would find  $\Delta b = 0$ . But since

$$\begin{aligned}\Delta f = 0 &\Rightarrow \int_M f \Delta f \sqrt{g} dx = 0 \\ &\Rightarrow \int_M f_{,i} f_{,j} g^{ij} \sqrt{g} dx = 0\end{aligned}$$

And since  $g$  is positive definite, this means that  $f_{,i} = 0$  everywhere, hence a constant. Yet another way to show the claim is to use the fact that we have a basis for functions on the torus, and  $a, b$  are in the zeroth eigenspace; since the zeroth eigenspace is spanned by  $\{1\}$ , they must be constant.

### 7.1.3 Perturbation of the Spectrum

For convenience, we denote

$$\langle nmps | n'm'p's' \rangle = \int_0^{2\pi L_1} \int_0^{2\pi L_2} \Psi(n, m, p, s) \Psi(n', m', p', s') dx dy$$

And by construction

$$\langle nmps | n'm'p's' \rangle = \delta_{nn'} \delta_{mm'} \delta_{pp'} \delta_{ss'}$$

We wish to find the first order perturbation to the eigenvalue  $\lambda(n, m)$ .

We look into the case  $m, n > 0$ , for which we have, using 5.11, the four equations

$$\delta\lambda(n, m) = \frac{1}{\beta(p', s')} \sum_{p, s} \beta(p, s) \langle nmp's' | \tilde{\Delta}(g, h) | nmps \rangle$$

where the operator  $\tilde{\Delta}(g, h)$  is

$$\tilde{\Delta}(g, h) = \tilde{A}(g, h) + \tilde{B}(g, h) + \tilde{C}(g, h) \tag{7.5}$$

where, as shown in 5.8

$$\tilde{A}(g, h) = \frac{Tr(h)}{2} \Delta(g)$$

and

$$\begin{aligned}\tilde{B}(g, h)\psi &= \frac{1}{\sqrt{g}} (\sqrt{g} \psi_{,j} h^{ij})_{,i} \\ \tilde{C}(g, h)\psi &= \frac{1}{\sqrt{g}} (\sqrt{g} (g_{kl} h^{kl}) \psi_{,j} g^{ij})_{,i}\end{aligned}$$

We decompose the symmetric perturbation tensor as above

$$h^{ij} = h_{tt}^{ij} + \left(\frac{1}{2}Tr(h) - \nabla_k y^k\right)g^{ij}$$

and rewrite it conveniently as the sum of a conformal perturbation and a transverse-traceless perturbation

$$h^{ij} = h_{tt}^{ij} + \Omega g^{ij}$$

And linearity implies that

$$\begin{aligned}\tilde{\Delta}(g, h) &= \tilde{\Delta}(g, h_{tt}) + \tilde{\Delta}(g, \Omega g) \\ &= \tilde{A}(g, h_{tt}) + \tilde{B}(g, h_{tt}) + \tilde{C}(g, h_{tt}) \\ &\quad + \tilde{A}(g, \Omega g) + \tilde{B}(g, \Omega g) + \tilde{C}(g, \Omega g)\end{aligned}$$

#### 7.1.4 The conformal perturbation

##### The A term

We calculate first the perturbation on the  $n, m$  eigenvalue for the first term in 7.5, for the conformal part of the perturbed metric; that is, the perturbation corresponding to the  $\tilde{A}(g, \Omega g)$  term above;

$$\begin{aligned}\delta\lambda(n, m) &= \frac{1}{\beta(p', s')} \sum_{p, s} \beta(p, s) \langle nmp' s' | \tilde{A}(g, \Omega g) | nmps \rangle \\ &= \frac{1}{\beta(p', s')} \sum_{p, s} \beta(p, s) \langle nmp' s' | \Omega \Delta | nmps \rangle \\ &= \frac{\lambda(n, m)}{\beta(p', s')} \sum_{p, s} \beta(p, s) \langle nmp' s' | \Omega | nmps \rangle\end{aligned}$$

We now focus our attention on the matrix elements  $\langle nmp' s' | \Omega | nmps \rangle$ ; we expand  $\Omega$  in the eigenbasis of the Laplace operator:

$$\begin{aligned}\Omega &= \alpha_0 + \sum_{\tilde{m}} \sum_{\tilde{p}} \alpha(\tilde{m}, 0, \tilde{p}, 0) \Psi(\tilde{m}, 0, \tilde{p}, 0) \\ &\quad + \sum_{\tilde{n}} \sum_{\tilde{p}} \alpha(0, \tilde{n}, \tilde{p}, 0) \Psi(0, \tilde{n}, \tilde{p}, 0) \\ &\quad + \sum_{\tilde{n}, \tilde{m} > 0} \sum_{\tilde{p}, \tilde{s}} \alpha(\tilde{n}, \tilde{m}, \tilde{p}, \tilde{s}) \Psi(\tilde{n}, \tilde{m}, \tilde{p}, \tilde{s})\end{aligned}$$

getting for the matrix elements

$$\begin{aligned}
\langle nmp's' | \Omega | nmps \rangle &= \alpha_0 \int \Psi(n, m, p', s') \Psi(n, m, p, s) \\
&+ \sum_{\tilde{m}} \sum_{\tilde{p}} \alpha(\tilde{m}, 0, \tilde{p}, 0) \int \Psi(n, m, p', s') [\Psi(\tilde{m}, 0, \tilde{p}, 0) \Psi(n, m, p, s)] \\
&+ \sum_{\tilde{n}} \sum_{\tilde{p}} \alpha(0, \tilde{n}, \tilde{p}, 0) \int \Psi(n, m, p', s') [\Psi(0, \tilde{n}, \tilde{p}, 0) \Psi(n, m, p, s)] \\
&+ \sum_{\tilde{n}\tilde{m}} \sum_{\tilde{p}\tilde{s}} \alpha(\tilde{n}, \tilde{m}, \tilde{p}, \tilde{s}) \int \Psi(n, m, p', s') [\Psi(\tilde{n}, \tilde{m}, \tilde{p}, \tilde{s}) \Psi(n, m, p, s)]
\end{aligned}$$

In order to compute these integrals, we further expand the factors in square brackets in terms of the eigenfunctions of the Laplacian. A straightforward computation shows that

$$\begin{aligned}
\cos(ax + by) \sin(a'x + b'y) &= \frac{1}{2} [\sin((a + a')x + (b + b')y) - \sin((a - a')x + (b - b')y)] \\
\sin(ax + by) \sin(a'x + b'y) &= \frac{1}{2} [\cos((a - a')x + (b - b')y) - \cos((a + a')x + (b + b')y)] \\
\cos(ax + by) \cos(a'x + b'y) &= \frac{1}{2} [\cos((a - a')x + (b - b')y) + \cos((a + a')x + (b + b')y)]
\end{aligned}$$

And since the non-zero integrals are the ones for which

$$\Psi(m, n, p, s) \Psi(\tilde{m}, \tilde{n}, \tilde{p}, \tilde{s}) = K \Psi(m, n, p', s')$$

we get the following four equations:

$$\begin{aligned}
\beta_{00} \frac{\delta\lambda(m, n)}{\lambda(m, n)} &= \beta_{00} \left[ \alpha_0 + \frac{1}{2} \alpha(2m, 2n, 0, 0) \right] \\
&+ \beta_{01} \left[ \frac{1}{2} (\alpha(2m, 0, 0, 0) + \alpha(0, 2n, 0, 0)) \right] \\
&+ \beta_{11} \left[ \frac{1}{2} (\alpha(2m, 0, 1, 0) - \alpha(0, 2n, 1, 0)) \right] \\
&+ \beta_{10} \left[ \frac{1}{2} \alpha(2m, 2n, 1, 0) \right]
\end{aligned}$$

and

$$\begin{aligned}\beta_{01} \frac{\delta\lambda(m, n)}{\lambda(m, n)} &= \beta_{00} \left[ \frac{1}{2} (\alpha(2m, 0, 0, 0) + \alpha(0, 2n, 0, 0)) \right] \\ &+ \beta_{01} \left[ \alpha_0 + \frac{1}{2} \alpha(2m, 2n, 0, 1) \right] \\ &+ \beta_{10} \left[ \frac{1}{2} (\alpha(2m, 0, 1, 0) + \alpha(0, 2n, 1, 0)) \right] \\ &+ \beta_{11} \left[ \frac{1}{2} \alpha(2m, 2n, 1, 1) \right]\end{aligned}$$

and

$$\begin{aligned}\beta_{10} \frac{\delta\lambda(m, n)}{\lambda(m, n)} &= \beta_{00} \left[ \frac{1}{2} (\alpha(2m, 2n, 1, 0)) \right] \\ &+ \beta_{01} \left[ \frac{1}{2} (\alpha(2m, 0, 1, 0)) \right] \\ &+ \beta_{10} \left[ \alpha_0 - \frac{1}{2} (\alpha(0, 2n, 0, 0) - \alpha(2m, 2n, 0, 0)) \right] \\ &+ \beta_{11} \left[ \frac{1}{2} (\alpha(0, 2n, 0, 0) - \alpha(2m, 0, 0, 0)) \right]\end{aligned}$$

and

$$\begin{aligned}\beta_{11} \frac{\delta\lambda(m, n)}{\lambda(m, n)} &= \beta_{00} \left[ -\frac{1}{2} (\alpha(0, 2n, 1, 0) + \alpha(2m, 0, 1, 0)) \right] \\ &+ \beta_{01} \left[ \frac{1}{2} (\alpha(2m, 2n, 1, 1)) \right] \\ &+ \beta_{10} \left[ -\frac{1}{2} (\alpha(2m, 0, 0, 0)) \right] \\ &+ \beta_{11} \left[ \alpha_0 - \frac{1}{2} (\alpha(2m, 2n, 0, 1)) \right]\end{aligned}$$

### The B and C terms

As for the two remaining terms, because in two dimensions we have  $g_{cd}g^{cd} = \dim(M) = 2$ , the matrix element  $\langle mnp's' | (\tilde{B}(g, \Omega g) + \tilde{C}(g, \Omega g)) | mnps \rangle$  is zero:

$$\begin{aligned}\langle mnp's' | (\tilde{B}(g, \Omega g) + \tilde{C}(g, \Omega g)) | mnps \rangle &= \\ \int_M \sqrt{g} [(\Psi(m, n, p', s'))_{,i} (\Psi(m, n, p, s))_{,j} \Omega g^{ij} & \\ - \frac{1}{2} (\Omega g_{cd} g^{cd}) (\Psi(m, n, p', s'))_{,i} (\Psi(m, n, p, s))_{,j} g^{ij}] &= 0\end{aligned}$$

And so the *conformal* part of the perturbation of the eigenvalue  $\lambda$  is just given by the  $\tilde{A}$  term in 7.5.

### 7.1.5 The non-conformal perturbation

#### The A and C terms

Since  $h_{tt}$  is traceless, all the matrix elements for the  $\tilde{A}$  operator in (7.5) are zero. The  $\tilde{C}$  term in (7.5) is also proportional to the trace and so it also vanishes. That leaves us with the  $\tilde{B}$  term, which we calculate below:

#### The B term

$$\begin{aligned} \langle mnp's' | \tilde{B}(g, h_{tt}) | nmpps \rangle &= \int (\Psi(m, n, p', s')_{,i} (\Psi(m, n, p, s)_{,j} h_{tt}^{ij} dx \\ &= \\ &\int a(\Psi(p', s')_{,x} \Psi(p, s)_{,x} - \Psi(p', s')_{,y} \Psi(p, s)_{,y}) + b(\Psi(p', s')_{,x} \Psi(p, s)_{,y} + \Psi(p', s')_{,y} \Psi(p, s)_{,x}) \end{aligned}$$

Where we have omitted the  $m, n$  for convenience. We note the following

$$\begin{aligned} \Psi(m, n, p, s)_{,x} &= \frac{m}{L_1} (-1)^{p+1} \Psi(m, n, \bar{p}, s) \\ \Psi(m, n, p, s)_{,y} &= \frac{n}{L_2} (-1)^{p+s+1} \Psi(m, n, \bar{p}, s) \end{aligned}$$

where  $\bar{p}$  stands for ‘not  $p$ ’. Substituting above, we get

$$\begin{aligned} \langle mnp's' | \tilde{B}(g, h_{tt}) | nmpps \rangle &= \\ \left[ a \left( \frac{m^2}{L_1^2} (-1)^{p+p'} + \frac{n^2}{L_2^2} (-1)^{p+p'+s+s'+1} \right) + b \frac{mn}{L_1 L_2} (-1)^{p+p'} \left( (-1)^s + (-1)^{s'} \right) \right] \delta_{pp'} \delta_{ss'} \end{aligned}$$

Finally, substituting this in

$$\delta\lambda(n, m) = \frac{1}{\beta(p', s')} \sum_{p, s} \beta(p, s) \langle mnp's' | \tilde{B}(g, h_{tt}) | nmpps \rangle$$

we get the two perturbations

$$\delta\lambda(n, m) = \left( b \frac{2mn}{L_1 L_2} (-1)^{s'} + a \left( \frac{m^2}{L_1^2} - \frac{n^2}{L_2^2} \right) \right) \quad (7.6)$$

### 7.1.6 Putting the two together

Finally, we combine the results in

$$\delta\lambda(n, m) = \frac{1}{\beta(p', s')} \sum_{p, s} \beta(p, s) \langle nmp' s' | \tilde{A}(g, \Omega g) + \tilde{B}(g, h_{tt}) | nmps \rangle$$

Getting the following system of four equations

$$\begin{aligned} & \beta_{00} \left[ \alpha_0 + \frac{1}{2} \alpha(2m, 2n, 0, 0) + \frac{1}{\lambda(m, n)} \left( b \frac{2mn}{L_1 L_2} + a \left( \frac{m^2}{L_1^2} - \frac{n^2}{L_2^2} \right) \right) - \frac{\delta\lambda(m, n)}{\lambda(m, n)} \right] \\ & + \beta_{01} \left[ \frac{1}{2} (\alpha(2m, 0, 0, 0) + \alpha(0, 2n, 0, 0)) \right] \\ & + \beta_{11} \left[ \frac{1}{2} (\alpha(2m, 0, 1, 0) - \alpha(0, 2n, 1, 0)) \right] \\ & + \beta_{10} \left[ \frac{1}{2} \alpha(2m, 2n, 1, 0) \right] \\ & = 0 \end{aligned}$$

and

$$\begin{aligned} & \beta_{00} \left[ \frac{1}{2} (\alpha(2m, 0, 0, 0) + \alpha(0, 2n, 0, 0)) \right] \\ & + \beta_{01} \left[ \alpha_0 + \frac{1}{2} \alpha(2m, 2n, 0, 1) + \frac{1}{\lambda(m, n)} \left( -b \frac{2mn}{L_1 L_2} + a \left( \frac{m^2}{L_1^2} - \frac{n^2}{L_2^2} \right) \right) - \frac{\delta\lambda(m, n)}{\lambda(m, n)} \right] \\ & + \beta_{10} \left[ \frac{1}{2} (\alpha(2m, 0, 1, 0) + \alpha(0, 2n, 1, 0)) \right] \\ & + \beta_{11} \left[ \frac{1}{2} \alpha(2m, 2n, 1, 1) \right] \\ & = 0 \end{aligned}$$

and

$$\begin{aligned} & \beta_{00} \left[ \frac{1}{2} (\alpha(2m, 2n, 1, 0)) \right] \\ & + \beta_{01} \left[ \frac{1}{2} (\alpha(2m, 0, 1, 0)) + \frac{1}{\lambda(m, n)} \left( b \frac{2mn}{L_1 L_2} + a \left( \frac{m^2}{L_1^2} - \frac{n^2}{L_2^2} \right) \right) - \frac{\delta\lambda(m, n)}{\lambda(m, n)} \right] \\ & + \beta_{10} \left[ \alpha_0 - \frac{1}{2} (\alpha(0, 2n, 0, 0) - \alpha(2m, 2n, 0, 0)) \right] \\ & + \beta_{11} \left[ \frac{1}{2} (\alpha(0, 2n, 0, 0) - \alpha(2m, 0, 0, 0)) \right] \\ & = 0 \end{aligned}$$

and

$$\begin{aligned}
& \beta_{00} \left[ -\frac{1}{2} (\alpha(0, 2n, 1, 0) + \alpha(2m, 0, 1, 0)) \right] \\
& + \beta_{01} \left[ \frac{1}{2} (\alpha(2m, 2n, 1, 1)) \right] \\
& + \beta_{10} \left[ -\frac{1}{2} (\alpha(2m, 0, 0, 0)) \right] \\
& + \beta_{11} \left[ \alpha_0 - \frac{1}{2} (\alpha(2m, 2n, 0, 1)) + \frac{1}{\lambda(m, n)} \left( -b \frac{2mn}{L_1 L_2} + a \left( \frac{m^2}{L_1^2} - \frac{n^2}{L_2^2} \right) \right) - \frac{\delta\lambda(m, n)}{\lambda(m, n)} \right] \\
& = 0
\end{aligned}$$

We collect the system in the symmetric matrix:

$$\begin{bmatrix} a_{00} - \delta\tilde{\lambda} & a_{01} & a_{02} & a_{03} \\ a_{01} & a_{11} - \delta\tilde{\lambda} & a_{12} & a_{13} \\ a_{02} & a_{12} & a_{22} - \delta\tilde{\lambda} & a_{23} \\ a_{03} & a_{13} & a_{23} & a_{33} - \delta\tilde{\lambda} \end{bmatrix} \begin{bmatrix} \beta_{00} \\ \beta_{01} \\ \beta_{10} \\ \beta_{11} \end{bmatrix} = 0$$

where the  $a_{ij}$  can be read from the four equations above and we have set  $\frac{\delta\lambda(m, n)}{\lambda(m, n)} \equiv \delta\tilde{\lambda}$ . Setting  $\mathbf{A} \equiv [a_{ij} - (\delta\tilde{\lambda})\delta_{ij}]$ , we have

$$\{\mathbf{A}\bar{\beta} = 0, \forall \bar{\beta}\} \Leftrightarrow \det \mathbf{A} = 0$$

Which can be written explicitly:

$$\det \mathbf{A} = 0 \Leftrightarrow (\delta\tilde{\lambda})^4 + C_1(\delta\tilde{\lambda})^3 + C_2(\delta\tilde{\lambda})^2 + C_3(\delta\tilde{\lambda}) + C_4 = 0$$



Where the  $C_i$  are given by the following expressions

$$C_1 = -(a_{00} + a_{11} + a_{22} + a_{33})$$

$$C_2 = a_{00}a_{11} + a_{00}a_{22} + a_{00}a_{33} + a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} \\ - (a_{01}^2 + a_{02}^2 + a_{03}^2 + a_{12}^2 + a_{13}^2 + a_{23}^2)$$

$$C_3 = -a_{00}a_{11}a_{22} - 2a_{01}a_{12}a_{02} - a_{11}a_{22}a_{33} - a_{00}a_{11}a_{33} - a_{00}a_{22}a_{33} \\ + a_{01}^2a_{33} + a_{00}a_{23}^2 - 2a_{12}a_{13}a_{23} + a_{02}^2a_{33} + a_{11}a_{03}^2 \\ + a_{00}a_{13}^2 - 2a_{02}a_{03}a_{23} + a_{13}^2a_{22} + a_{01}^2a_{22} + a_{11}a_{23}^2 \\ + a_{03}^2a_{22} + a_{11}a_{02}^2 + a_{00}a_{12}^2 - 2a_{01}a_{13}a_{03} + a_{12}^2a_{33}$$

$$C_4 = a_{03}^2a_{12}^2 + a_{00}a_{11}a_{22}a_{33} - a_{11}a_{03}^2a_{22} - 2a_{02}a_{13}a_{03}a_{12} \\ - a_{01}^2a_{22}a_{33} + 2a_{00}a_{12}a_{13}a_{23} - a_{11}a_{02}^2a_{33} - a_{00}a_{11}a_{23}^2 \\ + 2a_{01}a_{12}a_{02}a_{33} + a_{01}^2a_{23}^2 - 2a_{01}a_{12}a_{03}a_{23} - 2a_{01}a_{13}a_{02}a_{23} \\ + 2a_{01}a_{13}a_{03}a_{22} + a_{02}^2a_{13}^2 + 2a_{02}a_{11}a_{03}a_{23} - a_{00}a_{12}^2a_{33} - a_{00}a_{13}^2a_{22}$$

Solving it for  $\delta\tilde{\lambda}$ , then, amounts to finding the roots of this fourth-degree polynomial which, incidentally, is the highest degree for which this can be done by radicals. The quartic formulas were found by Lodovico Ferrari in the 1500's and are too unwieldy to write here. They can be found in e.g. [12].

### 7.1.7 Discussion of First-Order results

It immediately comes to our attention the existence of ‘flat’ directions in our perturbation expression: a conformal perturbation for which  $\Omega$  above only has odd coefficients is inaudible to first order.

This is by no means in contradiction with the rigidity results discussed above 3.2.3, which state that there is no non-trivial perturbation of the flat torus (in any dimension, and in particular, in dimension two), as I clarified in 3.2.3: one is, however, forced to go further in perturbation order to investigate the invertibility of the spectral function, since the first-order approximation is clearly not one-to-one.

The next section will deal with this matter.

### 7.1.8 Second-Order Terms

We begin by recalling Eq.5.17, for the second order correction to the eigenvalues in the degenerate case

$$\delta^2 \lambda_n = \sum_{k \neq n} \frac{|\langle \phi_k | \tilde{\Delta}(g, h) | \phi_n \rangle_g|^2}{\lambda_n - \lambda_k}$$

We write this explicitly for the conformal part, for the flat torus, using the notation above

$$\delta^2 \lambda(n, m) = \sum_{n', m' \neq n, m} \frac{|\langle nm | \Omega | n' m' \rangle|^2}{\lambda(n, m) - \lambda(n', m')} \quad (7.7)$$

We now look more closely to the matrix elements in the expression. Projecting  $\Omega$  onto the eigenspaces of the Laplacian we have

$$\Omega = \sum \alpha(n'', m'') \Psi(n'', m'') \quad (7.8)$$

Which means that

$$\langle nm | \Omega | n' m' \rangle = \sum_{n'' m''} \alpha(n'', m'') \int_M \Psi(n, m) \Psi(n'', m'') \Psi(n', m')$$

where  $\Psi$  and  $\Psi'$  are not in the same eigenspace. But, as seen above, the eigenfunctions on the torus have the interesting property that their product is a linear combination of eigenfunctions corresponding to the eigenvalues  $\lambda(m + m', n + n')$  and  $\lambda(m - m', n - n')$ . Orthogonality then implies that only integrals for which  $m'' = m \pm m', n'' = n \pm n'$  can be not-zero: each of the matrix elements will ‘select’ the coefficients of the perturbation corresponding to these eigenspaces. Since the sum in 7.7 is over all  $m, n \neq m', n'$  and there is no natural number that is not either the sum or the difference of two different natural numbers (two being the only number that requires a difference), we expect *all* of the  $\alpha$ 's will appear in 7.7.

We are of course not claiming that there are no ‘flat directions’ as was the case for the first-order terms, for which, as seen above, any conformal perturbation with zero *even* coefficients will not change the eigenvalues to first order.

We know, however, from Theorem 3.2.4 that there must be an order of perturbation such that there are no flat directions.

## Chapter 8

# Outlook

We noted in 7.1.7 that there are first-order conformal perturbations of the flat torus in two dimensions that leave the first order corrections to its spectrum invariant. The approximation proposed in 4.4 does not work, in this case.

However, because of the rigidity Theorem 3.2.4 for flat tori, there must be an order  $n$  such that there are no perturbations of the flat torus for which all the corrections to its spectrum up to order  $n$  are zero.

We conjecture that the reason for this difficulty is the high symmetry of the manifold we wish to perturb.

It is then crucial to understand if it is the case that, if  $g$  has an isometry, then linear the approximation to the spectral function  $\sigma(g)$  is *not* invertible.

Recall that, for the flat torus in 2 dimensions, the conformal perturbation coefficients  $\alpha_i$  in the eigenbasis of the Laplacian that did not affect the eigenvalues to first order were the odd ones. And that this was the case because, for the flat torus the integrals of the eigenfunctions obey  $\int_M \phi_i^2 \phi_j = \delta_{2j,i}$ .

For a general Riemannian manifold, with simple spectrum, the first-order perturbation to the spectrum given by a conformal perturbation  $\Omega = \sum_i \alpha_i \phi_i$  (in the eigenbasis of the Laplacian) is

$$\frac{\delta \lambda_n}{\lambda_n} = \sum_i \alpha_i \int_M \phi_n^2 \phi_i \quad (8.1)$$

A necessary condition for invertibility is that all the  $\alpha_i$  should appear in this expression. So, for every  $i$  there should be at least one  $n$  such that

$$\int_M \phi_n^2 \phi_i \neq 0 \quad (8.2)$$

A necessary condition for the invertibility of the linear approximation to the spectral function is, then, that the squares of eigenfunctions should span  $L^2(M)$ .

Overall, this seems to be a very interesting question, for which we would like to provide an answer. We also found no literature on the subject.

If it is indeed possible to show that, for symmetric spaces, the necessary condition fails to hold, one could, of course consider an arbitrary Riemannian manifold and try to establish local invertibility. By Uhlenbeck's theorem, it has a simple spectrum. But the problem is that for arbitrary manifolds the computation of the spectrum and the eigenfunctions is highly non-trivial. This, in turn, makes it impossible to apply perturbation theory, since the eigenfunctions will only be known to some precision. This could perhaps be controlled if one were to consider as a starting point for our perturbation strategy a manifold that is itself a perturbation of a symmetric manifold. In the general case, one would expect that its spectrum, via Uhlenbeck's theorem, should be simple. This would perhaps allow us to perform computations, while removing the degeneracy that is the conjectured reason for the failure of the invertibility.

As another avenue of research, also outlined in [37] it would be interesting to study the characterization of metrics by the spectra of other natural differential operators, such as higher order Laplacians on forms, in the context of the program outlined in Chapter 4.

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