

Generalizing Sampling Theory for Time-Varying Nyquist Rates using Self-Adjoint Extensions of Symmetric Operators with Deficiency Indices (1,1) in Hilbert Spaces

by

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Abstract

Sampling theory studies the equivalence between continuous and discrete representations of information. This equivalence is ubiquitously used in communication engineering and signal processing. For example, it allows engineers to store continuous signals as discrete data on digital media.

The classical sampling theorem, also known as the theorem of Whittaker-Shannon-Kotel'nikov, enables one to perfectly and stably reconstruct continuous signals with a constant bandwidth from their discrete samples at a constant Nyquist rate. The Nyquist rate depends on the bandwidth of the signals, namely, the frequency upper bound. Intuitively, a signal's 'information density' and 'effective bandwidth' should vary in time. Adjusting the sampling rate accordingly should improve the sampling efficiency and information storage. While this old idea has been pursued in numerous publications, fundamental problems have remained: How can a reliable concept of time-varying bandwidth be defined? How can samples taken at a time-varying Nyquist rate lead to perfect and stable reconstruction of the continuous signals?

This thesis develops a new non-Fourier generalized sampling theory which takes samples only as often as necessary at a time-varying Nyquist rate and maintains the ability to perfectly reconstruct the signals. The resulting Nyquist rate is the critical sampling rate below which there is insufficient information to reconstruct the signal and above which there is redundancy in the stored samples. It is also optimal for the stability of reconstruction.

To this end, following work by A. Kempf, the sampling points at a Nyquist rate are identified as the eigenvalues of self-adjoint extensions of a simple symmetric operator with deficiency indices $(1, 1)$. The thesis then develops and in a sense completes this theory. In particular, the thesis introduces and studies filtering, and yields key results on the stability and optimality of this new method. While these new results should greatly help in making time-variable sampling methods applicable in practice, the thesis also presents a range of new purely mathematical results. For example, the thesis presents new results that show how to explicitly calculate the eigenvalues of the complete set of self-adjoint extensions of such a symmetric operator in the Hilbert space. This result is of interest in the field of functional analysis where it advances von Neumann's theory of self-adjoint extensions.

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Dedication

This is dedicated to my father.

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Chapter 1

Introduction

Sampling theory is the study of certain spaces of continuous functions in which there is an equivalence between the function and a suitable collection of discretely-taken amplitude samples of the function. Namely, these continuous functions need to be stably reconstructible from their sample values on a discrete set of points.

The classical example of such a function space is the space of Ω -bandlimited functions. A function is said to be Ω -bandlimited if its Fourier transform vanishes outside the closed interval $[-2\pi\Omega, 2\pi\Omega]$. The frequency upper bound Ω is known as the bandlimit and its double is referred to as the bandwidth. The Fourier transform of a function $\phi(t)$, denoted by $\Phi(\omega)$, is defined to be

$$\Phi(\omega) = \mathfrak{F}(\phi(t)) = \int_{t=-\infty}^{+\infty} \phi(t) e^{-i\omega t} dt.$$

The **classical sampling theorem** or the theorem of Whittaker-Shannon-Kotel'nikov (WSK) [1] states that an Ω -bandlimited function $\phi(t)$ can be completely reconstructed for all $t \in \mathbb{R}$ from its amplitudes $\{\phi(t_n)\}_{n=-\infty}^{+\infty}$ on a discrete set of points with an equidistant spacing $t_{n+1} - t_n = \frac{1}{2\Omega}$ by the following reconstruction formula

$$\phi(t) = \sum_{n=-\infty}^{+\infty} G(t, t_n) \phi(t_n). \quad (1.1)$$

The left hand side is a continuous function and the right hand side is a summation over its discrete samples. The function $G(t, t_n)$ in the middle is the so-called reconstruction kernel and it equals $\text{sinc}(2\Omega(t - t_n))$ in the case of the classical sampling theorem.

The classical sampling theorem has been attributed to E. Whittaker and J. Whittaker in their work on interpolation and cardinal functions [3, 4, 5] and to Kotel'nikov in the

Russian literature [6] around the same period of time, in the 1920's. However, it was Shannon who realized the fundamental importance of the classical sampling theorem in communication engineering and information theory in 1949 [1, 2]. So the classical sampling theorem is often simply referred to as the **Shannon sampling theorem**. Ever since, the classical sampling theory has found a tremendous number of practical applications in communication engineering and signal processing. It provides a way of storing and recovering continuous signals (music recordings on compact disks or voice signals on cell phones) from their digital sample data. There is of course also the need for quantization for this purpose, but quantization is secondary for the purposes of this work, and we will not touch upon it further, except for the study of reconstruction stability in Chapter 7.

The classical sampling theory has been generalized in several directions, to include for example, non-uniform sampling, bandpass sampling, interlaced or derivative sampling. For a general review of the classical sampling theorem and its various generalizations, please see the standard textbooks [7, 8, 9] or the classical review papers [10, 11, 12].

The primary goal of this thesis is to develop a new generalization of the classical sampling theory which is adapted to time-varying bandwidths. This non-Fourier generalized sampling theory uses as its mathematical engine the functional analytical theory of self-adjoint extensions of symmetric operators with deficiency indices $(1, 1)$ in Hilbert space.

This thesis is split into two main parts: the mathematical development and the generalized sampling theory. The first part of the thesis reviews the relevant functional analytical methods in Chapter 2 and develops new results on the spectral theory of self-adjoint extensions in Chapter 3. Readers whose main interests are in the generalized sampling theory may skip those two chapters on mathematics and proceed directly on the generalized sampling theory from Chapter 4 to Chapter 8. The second part of the thesis discusses the sampling, reconstruction, filtering, stability and interpolation of the new generalized sampling theory for time-varying bandwidth. This part is designed to be legible also to engineers who may not want to delve deeply into the functional analysis but who may want to make practical use of the generalized sampling theory.

1.1 A New Generalized Sampling Theory for Time-varying Nyquist Rates

Elements of the space of Ω -bandlimited functions possess a constant bandlimit Ω . The classical sampling theory allows perfect and stable reconstruction of continuous Ω -bandlimited signals if the samples of the signal are taken at a rate of $\frac{1}{2\Omega}$. This sampling rate is known as the Nyquist rate [13]. The Nyquist sampling rate is important because it is the critical sampling rate, above which there is redundancy in the taken samples (over-sampling) and

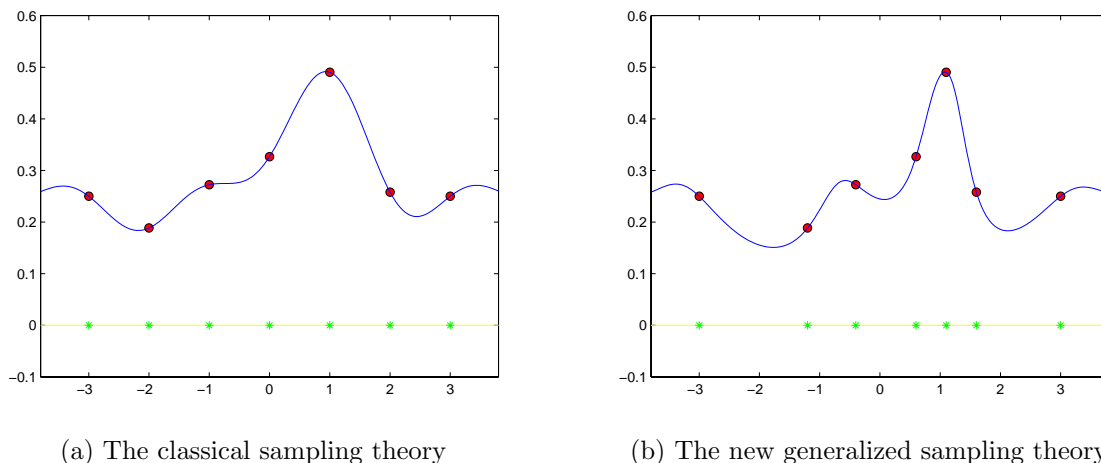


Figure 1.1: Comparison of signals with constant and time-varying Nyquist rates. The red circles indicate the discrete samples. The green dots are the sampling points, which are equidistantly spaced in the classical sampling theory and non-equidistantly spaced in the new generalized sampling theory.

below which one has insufficient samples to recover the continuous signal (under-sampling). It is also the sampling rate which gives most stable reconstruction.

The classical sampling theorem is involved in most analog-digital conversion processes. A typical example is to store a continuous music or voice signal as discrete bits in the digital media and then to reconstruct the continuous signal later. Due to the fact that human ears are sensitive only to the frequencies less than 20 kHz, a music signal can be filtered to retain this bandlimit and remove all the negligible frequencies higher than 20 kHz. Then by the classical sampling theory, the filtered music signal can be completely determined by and perfectly reconstructed from its discrete samples taken at every .025 milliseconds. The frequency range for human speech is roughly from about 50 Hz to 5 kHz. So one often cuts off the bandlimit at frequencies much smaller than 20 kHz and samples at a lower constant rate, e.g. in telephony.

In practice, a signal's 'effective bandwidth' or 'information density' could vary in time. Consider a simple example of a music signal which suddenly oscillates rapidly in a period of time, e.g., when listening to a song with a lower pitch for a piano part and a high pitch and overtone spectrum for a violin. The constant bandlimit of this signal calculated by Fourier transform may be quite high due to the large frequencies needed to resolve the sharp features that occur in that short time interval of high pitched violin activity. These high frequency components determine the overall bandlimit Ω , despite the fact that the signal is slower-varying most of the time. Sampling the signal at this high rate due to the constant bandwidth can be very inefficient. It should be possible to improve sampling

efficiency by adjusting the sampling rate accordingly.

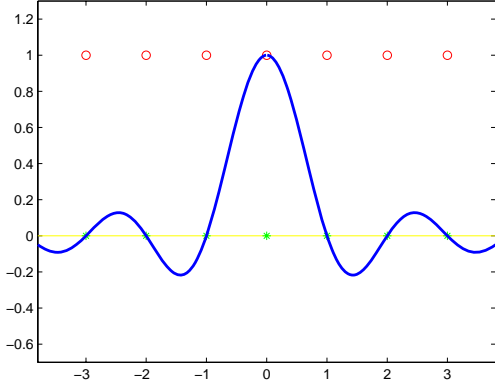
Intuitively, such a signal can be said to obey a ‘time-varying bandwidth’ because in a short period of time the signal contains high frequencies, while at other times, the signal varies slowly and contains only low frequencies. Besides the examples of music signals and speech recordings, the spatial domain for natural images also processes effectively varying frequency contents. If one can sample and reconstruct the signal at a rate according to the signal’s behaviour, one should be able to minimize the redundancy in the samples taken and obtain a more efficient storage of sample values.

The bandwidth of a function is a scalar number defined by Fourier transform. Any attempt to define the bandwidth as a function of time in the framework of Fourier analysis, e.g. via windowed Fourier transform, is difficult and non-unique. However, the Nyquist sampling rate, i.e., the inverse of the bandwidth, if interpreted as the critical sampling rate between over- and under-sampling, can be defined as varying in time. One takes more samples when the signal has higher activity and takes less samples when the signal has slow-varying oscillations, while maintaining the ability to perfectly and stably reconstruct the signal from these discrete samples. The bandwidth defined as the inverse of this time-varying Nyquist rate is then well defined as a time-dependent function. The terms ‘time-varying bandwidth’ and ‘time-varying Nyquist rate’ will be used interchangeably in this sense in this thesis, but the latter is preferred simply because it can be defined from first principles while the term ‘time-varying bandwidth’ is only secondarily defined through it.

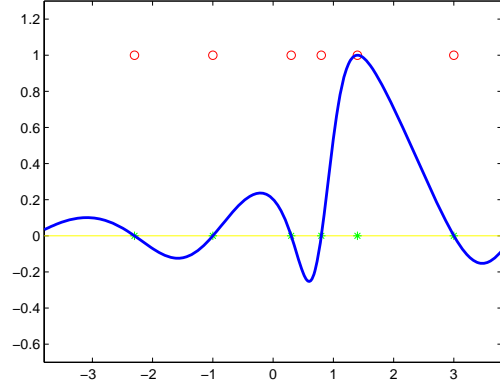
Some of the key questions which have been investigated in this thesis are:

- How can a reliable concept of continuously time-varying bandwidth, or equivalently time-varying Nyquist rate be defined?
- How can a given raw signal be pre-filtered to ensure that it is in the space of functions possessing a desired time-varying Nyquist rate?
- How stable is the reconstruction of a signal from a space of functions of a time-varying Nyquist rate? And, is sampling at this varying rate indeed optimal?
- Using functional analytic techniques, how can one calculate the complete set of all Nyquist rate sampling grids for a given function space. (The result is also of purely functional analytic interest.)
- What is the behavior of the generalized sampling methods in concrete example, such as the Gibbs’ phenomenon?

The Shannon sampling method is usually applied in a four-step-algorithm: (1) Determining the Nyquist rate, (2) filtering, (3) sampling, and (4) reconstruction. In this thesis, I show that also for the generalized sampling method all these steps can be carried out.



(a) An usual sinc reconstruction kernel



(b) A generalized reconstruction kernel

Figure 1.2: Examples of the reconstruction kernels in the classical and generalized sampling theory.

- (1) **Determining the Nyquist rate:** Analyze the frequency contents of the raw signals of interest $\phi^{raw}(t)$ to choose a Nyquist rate of sampling (either constant or time-varying). Then, the Nyquist sampling rate is specified by an increasing and infinite set of sampling points $\{t_n\}_{n=-\infty}^{\infty}$, which will here be referred to as the **Nyquist sampling grid**.
- (2) **Filtering the raw signals:** Filter the raw signal $\phi^{raw}(t)$ to obtain a signal $\phi(t)$ in the function space in which functions possess the pre-specified Nyquist rate. The filter operator P has the following form:

$$\phi(t) = (P\phi^{raw})(t) = \int_{-\infty}^{\infty} \phi^{raw}(\hat{t}) P(t, \hat{t}) \nu(\hat{t}) d\hat{t}. \quad (1.2)$$

- (3) **Taking and storing the discrete samples:** Take and store samples of $\phi(t)$ at the Nyquist rate, namely, $\{\phi(t_n)\}_{n=-\infty}^{\infty}$ on the Nyquist sampling grid $\{t_n\}_{n=-\infty}^{\infty}$;
- (4) **Reconstructing the continuous signals:** Reconstruct $\phi(t)$ for all t from the taken samples $\{\phi(t_n)\}_n$ via the reconstruction formula in Eq. (1.1).

The Shannon sampling method is restricted to the constant Nyquist sampling rate with $t_{n+1} - t_n = \frac{1}{2\Omega}$, but the new generalized sampling method deals with time-varying Nyquist rates. It is important to emphasize that the generalized sampling theory uses different function spaces and different expressions for the reconstruction kernel. Functions in the function space specified by a generally non-equidistant Nyquist sampling grid $\{t_n\}_{n=-\infty}^{\infty}$

are no longer the conventional Fourier-defined bandlimited functions with a constant bandwidth.

Nevertheless, both Shannon and the generalized sampling methods take the same form of reconstruction and filtering as in Eq. (1.1) and (1.2). Only, the expressions of the reconstruction kernel $G(t, t_n)$, filter kernel $P(t, \hat{t})$ and filter measure $\nu(\hat{t})$ are different. See Figure 1.2 for an example of the reconstruction kernel $G(t, t_n)$ in the generalized sampling method. These three functions can be expressed in terms of the pre-specified set of points $\{t_n\}_{n=-\infty}^{\infty}$ in closed form. See Section 5.2.1 for the explicit formulae and an overview of the generalized sampling method.

The second part of the thesis builds the mechanics of Step (2) - (4), namely, the new generalized sampling method for filtering, sampling, and reconstructing signals with time-varying Nyquist rates. The first step of determining the Nyquist sampling grid $\{t_n\}_n$ to specify the frequency contents of the studied signals is dependent on the practical situation at hand and can be analyzed with the help of time-frequency analysis tools like windowed Fourier transform. It will not be the focus of this thesis, but a simple example can be found in Chapter 8.

1.2 Comparison to Existing Sampling Methods

The generalized sampling theory presented in this thesis makes a new addition to the existing sampling methods which deal with non-uniform sampling points or time-varying bandwidth.

While the new generalized sampling theory deals with generally non-equidistant sampling points, it is very different from the conventional non-uniform sampling [14, 15], because the generalized sampling theory reconstruct signals with a time-varying bandwidth. The non-equidistant sampling points at the time-varying Nyquist rate indeed represent the behavior of signals' time-varying frequency contents. Traditional non-uniform sampling tends to reconstruct the Fourier-defined Ω -bandlimited functions (i.e. functions with a constant bandwidth) from non-uniformly distributed samples. In this case, the optimal equidistant sampling points in the case of Shannon are replaced by certain non-equidistant sampling points, and a more complicated corresponding reconstruction kernel is used. However, such reconstruction is possible only if the average sample density, more precisely, the Beurling density, stays the same as Shannon's constant sampling density [16]. Hence, no efficiency is gained.

The corresponding reconstruction kernel functions associated with the non-uniform sampling points tend to be very unstable and computationally expensive, especially given the counterintuitive examples of super-oscillating functions [17, 18]. As concluded in [14],

the tighter sampling points are spaced, the larger the maximum amplitude of the reconstruction kernel function is. The amplitude tends to approach infinitely when the spacing between adjacent points approaches zero. Hence, the non-uniform sampling points are not optimized for the reconstruction of the Ω -bandlimited functions in terms of computational efficiency and stability.

However, the generalized sampling theory here deals with functions that are pre-filtered with a time-varying bandwidth. Hence, the function space in the generalized sampling theory is no longer the space of Fourier-defined Ω -bandlimited functions. It consists of functions already with time-varying frequency content. As a consequence, sampling points spaced at the time-varying Nyquist rate are optimized for the reconstruction of these functions in the sense that if any sampling point deviates from the Nyquist sampling grid, this will deteriorate the stability of reconstruction, and adding/removing a point from the Nyquist sampling grid will lead to redundancy/insufficiency of information for the reconstruction. In other words, similar to the constant Nyquist rate in the classical sampling theory, the time-varying Nyquist rate in the generalized sampling theory is the optimized rate for stability of reconstruction and is the critical rate between over- and under-sampling. These two points will be shown in Section 5.3 and 7.2.

Signals with time-varying frequency contents are usually characterized using the Windowed Fourier Transform (Short-Time Fourier Transform) and wavelets [19, 20, 21]. However, these methods of time-frequency analysis do not naturally result in discrete-time representations of the continuous signals.

Further, the so-found time-dependent frequency content is a local property, because it is obtained by dividing the time domain into many intervals and analyzing the frequency contents of the signal on each interval. However, in this way, windowed Fourier transform and other Fourier-based time-frequency-analysis methods lead to unavoidable windowing artifacts. Moving a sampling point locally will not affect the behavior of the reconstructed signal elsewhere. However, the time-varying bandwidth defined in the new generalized sampling theory is a global property, analogous to the Fourier-defined constant bandwidth for bandlimited functions in the classical sampling theory. The overall bandwidth, while it is a function of time, depends on the behavior of the signal on the whole real line.

Sampling theory for time-varying bandwidths has previously been considered in the literature with two approaches. See [22] for a review of both. The first approach [23] is based on a time-varying low-pass filter. Instead of the conventional sinc function $P(t, \hat{t}) = \text{sinc}(2\Omega(t - \hat{t}))$ in the filter kernel Eq. (1.2), this approach adds a time-dependent term $\omega_c(t)$ in the sinc filter kernel

$$P(t, \hat{t}) = \frac{\sin(2\pi\Omega\omega_c(t)(t - \hat{t}))}{2\pi\Omega(t - \hat{t})} \quad (1.3)$$

to characterize the time-dependent cut-off frequency. This approach leads to a sampling and reconstruction system that does not satisfy the property of perfect reconstruction, although it obeys the consistent re-sampling [24], namely, the amplitudes of the reconstructed function agree with the original function on the set of sampling points, but not generally on the whole real line. Only the special functions of self-similar input signals guarantee perfect reconstruction [25].

Further, the non-equidistant sampling points $\{t_n\}_{n=-\infty}^{\infty}$ in this approach must be found as solutions to $\omega_c(t_n)t_n = n\pi$ for all n . However, the generalized sampling theory in this thesis is directly based on the given Nyquist sampling points obtained in Step (1). For a class of signals to be studied, it is much easier to determine a set of sampling points to represent the time-varying bandwidth of the signals, rather than to determine a continuous function, $\omega_c(t)$, for the frequency contents. For calculational purpose, it is also convenient to deal with a set of sampling points rather than to solve for a continuous function.

Instead of a modification of the sinc function as in Eq. (1.3), the filter kernel $P(t, \hat{t})$ in the generalized sampling theory is written precisely in terms of the sampling points. So it is a direct representation of the pre-specified time-varying Nyquist sampling rate. See Chapter 6 for details. Most importantly, the generalized sampling theory for time-varying bandwidth in this thesis always allows the perfect reconstruction of signals with time-varying bandwidth.

The other existing approach [26, 27] for time-varying bandwidth is based on the so-called time-scaling or time-warping, namely, stretching of the time axis. It considers the signals in a form of $\psi(t) = \phi[\rho(t)]$ where $\phi(t)$ is the conventional Ω -bandlimited function with a constant bandwidth and $\rho(t)$ is the invertible time-warping function. So the function $\psi(t)$ is known as a locally bandlimited signal since it can be sampled according to the local bandwidth which is implied by the warping function $\rho(t)$. Thus, $\psi(t)$ can be perfectly reconstructed from these samples.

The best way to illustrate the difference between this approach and our generalized sampling theory is by the reduction of Gibbs' type of overshoot. Because the approach of locally bandlimited signals by time-warping is only a time-axis-stretching version of Ω -bandlimited function, the amplitudes of the Ω -bandlimited function are not changed, but simply shifted from their locations on the time axis. Hence, it should not be possible to suppress the amplitude of the overshoot occurring in windowing artifacts or ringing artifact. However, the new generalized sampling theory can strongly reduce these type of overshoots. See Chapter 8 for examples.

1.3 Generalizing Principle and Functional Analytical Results

To generalize the classical sampling theory, our starting point is the observation that, in the frequency domain, if a function has a finite support in the open interval $(-2\pi\Omega, 2\pi\Omega)$, then so does its derivative. The finite open support of a function is preserved by the derivative operator $i\frac{d}{d\omega}$. Hence, the Fourier transform of a strictly Ω -bandlimited function is invariant under the action of derivative operator. A function $\phi(t)$ is said to be strictly Ω -bandlimited if its Fourier transform $\Phi(\omega)$ vanishes outside the open interval $(-2\pi\Omega, 2\pi\Omega)$ rather than the closed interval $[-2\pi\Omega, 2\pi\Omega]$. The space of strictly Ω -bandlimited functions is a dense subset of the space of Ω -bandlimited functions. Thus, the space of Fourier transforms of strictly bandlimited functions is invariant under the derivative operator.

In the time domain, the derivative operator $i\frac{d}{d\omega}$ corresponds to the time operator T : $T\phi(t) = t\phi(t)$. The above observation suggests us to define the space of Ω -bandlimited functions as the closure of the invariant domain of T . No matter what bandwidth a class of signals have, the action of the time operator T preserves that bandwidth. This is true for the constant bandwidth in the usual Fourier sense. It should also be true for the time-varying bandwidth if we choose a generic operator T directly in time domain.

How about the sampling and reconstruction property? The operator T in functional analysis is what is called a simple symmetric operator with deficiency indices $(1, 1)$. Such a symmetric operator has a $U(1)$ -family of self-adjoint extensions. Each self-adjoint extension has a set of eigenvalues, which correspond to a set of Nyquist sampling points. Together, the eigenvalues of all self-adjoint extensions of a fixed such symmetric operator T cover the whole real line exactly once. So one is able to recover the signal at any time t . Each eigenvalue is of a multiplicity 1. The eigenvectors of each self-adjoint extension of T form an eigenbasis of the Hilbert space.

The sampling theorem simply states the fact that if a Hilbert space vector is known in the eigenbasis of one self-adjoint extension of T , then the coefficients of the vector in the eigenbases of all other self-adjoint extensions of T are also determined. The reconstruction kernel consists of the matrix elements of the unitary transformations which map in between the eigenbases of the self-adjoint extensions of T .

The set of Nyquist sampling points yields the critical sampling rate because the corresponding eigenbasis is linearly dependent if we add more vectors and is a incomplete basis if we remove any eigenvector from it. Further, the set of Nyquist sampling points is optimal for the stability of reconstruction because the corresponding eigenbasis is orthogonal.

The symmetric operator T in the case of the classical sampling theorem is special because all its self-adjoint extensions have equidistantly spaced eigenvalues. However, as was

first proposed in [28], if one chooses a generic such operator T whose self-adjoint extensions possess generally non-equidistantly spaced eigenvalues, one generalizes the classical sampling theorem for time-varying Nyquist rates.

By developing this generalized sampling theory, further, we will here obtain also new results on the distribution of eigenvalues of self-adjoint extensions. Specifically, given eigenvalues of one self-adjoint extension of the simple symmetric operator T with deficiency indices $(1, 1)$, we obtain a formula to explicitly calculate the eigenvalues of all other self-adjoint extensions of T and the corresponding eigenvectors as well. This provides a constructive completion of the abstract theory of self-adjoint extensions of these operators. In the literature, only the particular example of differential operators on finite intervals has been studied, but not the general case for an abstract simple symmetric operator with deficiency indices $(1, 1)$. See Chapter 3 for the details.

1.4 Physics Motivation

The non-Fourier generalized sampling theory is also motivated by a fundamental physics problem namely the question of whether spacetime is fundamentally continuous or discrete. General relativity is formulated on a smooth, differential manifold, while quantum field theory appears to require the discreteness of spacetime. The messages from these two most important and successful physical theories of our time are so contradictory that this issue spawned a whole range of so-called quantum gravity theories, ranging from string theory to the holographic principle.

As first suggested by Kempf [29], sampling theory could be a very useful mathematical tool to solve this problem: spacetime could be both continuous and differentiable, while all the physical fields of nature, e.g., electro-magnetic fields, are like bandlimited functions, which possess only a finite spatial density of degrees of freedom. Physical fields could be fully captured everywhere if we know their values on a set of discrete points which are sufficiently dense. Therefore, spacetime could be effectively described as both continuous and discrete similar to the equivalence between continuous and discrete representations of information. Sampling theory could provide a mathematical framework in which both general relativity and quantum field theory can coexist.

It has been widely suggested in quantum gravity that there exists a minimum length or volume in nature, of the order of the Planck distance: 10^{-35} meters. This is because when general relativity and quantum theory are considered together, it follows that the notion of distance loses operational meaning at the Planck scale of 10^{-35} meters. If one attempts to resolve a spatial structure with an uncertainty of less than a Planck length, then the uncertainty principle implies a significantly large momentum uncertainty. This momentum uncertainty is large enough to cause a curvature uncertainty which would disturb the very

region in space that it is meant to resolve. This argument implies that there exists a finite lower bound to the achievable uncertainty in spatial localization. Then, Kempf [29] suggests there should be some form of an upper bound to the information density that can be imprinted on physical fields.

This information density upper bound on physical fields is analogous to the bandwidth cutoff of bandlimited functions. This inspires physicists to use sampling theory in quantum gravity. However, the physical fields of nature can not be bandlimited in usual sense because the mathematical representation of the physical fields must be coordinate-system-independent. Hence, the physical fields cannot be bandlimited in the usual Fourier sense in a fixed given coordinate system for the manifold. This is the original motivation to consider a non-Fourier generalization of the classical sampling theory.

The first attempt to mathematically formulate this sampling theory in the context of quantum gravity is by Kempf in [28]. The results in this thesis are rooted from [28], but they are developed in a more engineering-oriented context from a perspective of sampling theory itself. The mathematics is rigorously proven here. For more information on the use of sampling theory in quantum gravity and its generalization to curved manifolds, see [30, 31].

1.5 Outline of the Thesis

Some of the research results reported in this thesis have already appeared in my research papers [32]-[38]. Some sections contain more detailed discussions on the relevant topics. The thesis is organized as follows.

- Chapter 2 reviews the relevant functional analytical definitions and methods for self-adjoint extensions of symmetric operators in Hilbert space. Chapter 3 states the new mathematical results regarding the eigenvalues of these self-adjoint extensions. The generalized sampling theory developed later is based on the mathematical results in Chapter 3. However, readers with interests mostly in the generalized sampling theory can proceed directly from Chapter 4 and may skip these two chapters on functional analysis.
- Chapter 4 recapitulates the basic classical sampling theorem with a proof by Fourier analysis which is formulated in a form suitable for the generalization. Section 4.3 provides a new functional analytical proof of the classical sampling theorem of Whittaker-Shannon-Kotel'nikov.
- Chapter 5 states the main results of the thesis, about the generalized sampling theory. Section 5.2.1 gives an overview of the generalized sampling theory for time-varying

Nyquist rates, followed by more details on sampling, reconstruction and definition of the time-varying Nyquist rate in Section 5.2. Section 5.3 shows that the time-varying Nyquist rate is the critical rate between over- and under-sampling. Section 5.3 recovers the classical sampling theorem as a special case and Section 5.5 is the mathematical justification.

- Chapter 6 introduces the pre-filtering formula for the generalized sampling theory and shows that this filter operator does map the raw signals into the function space with the pre-specified time-varying Nyquist rate. This filter operator is a projection with an uneven measure associated with the time-varying Nyquist rate.
- Chapter 7 discusses the stability issue of the generalized sampling theory. It first finds frame bounds between the l^2 -norm of the discrete samples and the L^2 -norm of the continuous signals. It then shows that the time-varying Nyquist rate is the optimal rate for stability of reconstruction.
- Chapter 8 consists of two parts. First, we modify the reconstruction kernel formula to yield an interpolation method for finitely many non-equidistant points on a finite interval. There are two approaches: either by appending an auxiliary extension of equidistant points or by periodic extensions on the given finite interval. Secondly, we use the new interpolation methods to approximate step functions. We observe that the Gibbs' type of overshoots which appear near a discontinuous jump point can be significantly reduced by adjusting the sampling density locally.
- Chapter 9 includes concluding remarks and open research problems for future work.

Chapter 2

Review of Functional Analytical Methods

This chapter will define the mathematical notations we shall use throughout the thesis and review the relevant functional analytical facts necessary for the mathematical development of this thesis. The definitions and theorems in this chapter can be found in standard textbooks on functional analysis, e.g. [39, 40, 41, 42]. Notice that textbooks of functional analysis cover mostly topics on bounded operators, but the self-adjoint and symmetric operators considered in this thesis are unbounded ones. In particular, we consider the self-adjoint extensions of symmetric operators with deficiency indices $(1, 1)$ in Hilbert space. This class of operators was studied by the Russian mathematician M.G. Krein in 1944 [43, 44, 45] and has been published in English in 1997 [46]. The crucial difference between the results of unbounded and bounded self-adjoint operators will be pointed out.

2.1 Linear Operators in Hilbert Space

We work in a separable complex Hilbert space, denoted by \mathbb{H} , with the inner product denoted by $\langle \cdot, \cdot \rangle$, linear in the first argument and conjugate linear in the second argument. Let c^* (or sometimes \bar{c}) denote the complex conjugate of a complex number $c \in \mathbb{C}$. Then, for any $c_1, c_2 \in \mathbb{C}$ and $\phi_1, \phi_2, \psi \in \mathbb{H}$,

$$\begin{aligned}\langle c_1\phi_1 + c_2\phi_2, \psi \rangle &= c_1 \langle \phi_1, \psi \rangle + c_2 \langle \phi_2, \psi \rangle, \\ \langle \psi, c_1\phi_1 + c_2\phi_2 \rangle &= c_1^* \langle \psi, \phi_1 \rangle + c_2^* \langle \psi, \phi_2 \rangle.\end{aligned}$$

A **separable Hilbert space** \mathbb{H} is a complete inner product space with countable orthonormal basis. An **orthonormal** set means the vectors in it are orthogonal and normalized.

Let $\{e_1, e_2, e_3, \dots\}$ denote an orthonormal basis. Then any such orthonormal basis satisfies

$$\mathbb{H} = \text{span}\{e_1, e_2, e_3, \dots\} \quad \text{and} \quad \langle e_i, e_j \rangle = \delta_{ij}.$$

All infinite dimensional separable Hilbert spaces are isometrically equivalent as shown in the section 4.1, 4.2 of [47]. Examples include $l^2(\mathbb{N})$, $L^2([a, b])$ and $L^2(\mathbb{R})$.

The **orthogonal complement** of a set S in the Hilbert space \mathbb{H} is denoted by \perp and defined by:

$$S^\perp = \mathbb{H} \ominus S = \{\psi \in \mathbb{H} \mid \langle \psi, \phi \rangle = 0, \forall \phi \in S\}.$$

Proposition 4.2 in [47] states that $\mathbb{H} \ominus S$ is closed whether or not S is closed.

A set S is **dense** in \mathbb{H} if the closure of S equals the whole Hilbert space \mathbb{H} , i.e. $\overline{S} = \text{cl}(S) = \mathbb{H}$. It is equivalent to say that the only vector orthogonal to S is the zero vector. Namely, S is dense if and only if $\mathbb{H} \ominus S = \{\psi \in \mathbb{H} \mid \langle \psi, \phi \rangle = 0, \forall \phi \in S\} = \{0\}$.

Definition 1. Let D be a subset of a Hilbert space \mathbb{H} . A map which associates each element $\phi \in D$ to an element $\psi \in \mathbb{H}$ is called an **operator** on \mathbb{H} with **domain** D . The **range** of an operator T is defined as the set consisting of all ψ such that $\psi = T\phi$ for some $\phi \in D$.

For an operator T on \mathbb{H} , we denote $D(T)$ or D_T to be its domain and $R(T)$ or R_T to be its range.

An operator consists of two parts: its domain and its action on the domain. An operator T is said to be **densely defined** if its domain $D(T)$ is a dense subset of \mathbb{H} .

Definition 2. An operator T on \mathbb{H} is **linear** if for any $c_1, c_2 \in \mathbb{C}$ and $\phi, \psi \in D(T)$,

$$c_1 \phi + c_2 \psi \in D(T) \quad \text{and} \quad T(c_1 \phi + c_2 \psi) = c_1 T\phi + c_2 T\psi.$$

For all ϕ, ψ in \mathbb{H} and $c_1, c_2 \in \mathbb{H}$, $c_1 \phi + c_2 \psi \in D(T)$ means the domain $D(T)$ is a subspace of \mathbb{H} . Without explicitly saying, all the operators involved in this thesis are linear.

The norm of a vector ϕ in the Hilbert space \mathbb{H} is denoted by $\|\phi\|$ and is defined to be

$$\|\phi\| = \sqrt{\langle \phi, \phi \rangle}.$$

Definition 3. The **norm** of an operator T (not necessarily linear) is defined as

$$\|T\| := \inf\{K \in \mathbb{R} \mid \|T\phi\| \leq K \|\phi\| \quad \forall \phi \in D(T)\}.$$

The operator T is said to be **bounded** if $\|T\| < \infty$; Otherwise, T is said to be **unbounded**.

For any linear operator T , the above definition is equivalent to

$$\|T\| := \sup \{ \|T\phi\| \mid \phi \in D(T), \|\phi\| \leq 1 \}.$$

A bounded linear operator is always defined everywhere in the Hilbert space \mathbb{H} . At least, its domain can be extended to the whole Hilbert space. However, an unbounded linear operator on an infinitely dimensional Hilbert space \mathbb{H} may only be meaningful on a proper subset of \mathbb{H} .

Theorem 3.13 of [47] states that a linear operator T on \mathbb{H} is continuous if it is bounded on \mathbb{H} . But the continuity does not hold for an unbounded linear operator T . Therefore, to ensure the images of a convergent sequence under T behaving well in terms of convergence, we require a less restrictive condition on the operators:

Definition 4. An operator T (not necessarily linear) is **closed** if

$$\phi_n \in D(T), \quad \lim_{n \rightarrow \infty} \phi_n = \phi, \quad \lim_{n \rightarrow \infty} T\phi_n = \psi$$

imply that

$$\phi \in D(T) \quad \text{and} \quad T\phi = \psi.$$

Clearly a continuous operator is closed, but not the converse. For a closed operator T , the convergence of sequence $\{\phi_1, \phi_2, \phi_3, \dots\}$ does not guarantee the convergence of its images $\{T\phi_1, T\phi_2, T\phi_3, \dots\}$ under T .

Definition 5. Let T, S be two operators on \mathbb{H} . The operator S is said to be an **extension** of T if

$$D(T) \subset D(S) \quad \text{and} \quad S\phi = T\phi, \forall \phi \in D_T.$$

We write $\mathbf{T} \subset \mathbf{S}$ if the above is true.

Definition 6. Let T be an operator on \mathbb{H} . The **closure** of T is defined as the minimal closed extension of T and is denoted by \overline{T} .

In other words, \overline{T} is the closed extension of T which is contained in every closed extension of T .

Example 1: Consider $\mathbb{H} = L^2([a, b])$ with inner product

$$\langle f, g \rangle = \int_a^b f(x) \overline{g(x)} dx.$$

$L^2([a, b])$ is the separable Hilbert space of square integrable, Lebesgue measurable complex-valued functions on a finite interval $[a, b]$, $-\infty < a < b < \infty$.

It is understood that throughout this work we are dealing with Lebesgue measurable functions on $L^2([a, b])$ such that $f = g$ in $L^2([a, b])$ if $f(x) = g(x)$ almost everywhere on $[a, b]$. In other words, if $f, g \in L^2([a, b])$ differ only on a set of measure zero, then $f = g$ in $L^2([a, b])$.

Let us define three operators on $L^2([a, b])$:

$$T_c := i \frac{d}{dx} \text{ on the domain } C_0^1((a, b)), \quad (2.1)$$

$$T := i \frac{d}{dx} \text{ on } D \quad \text{and} \quad T^* := i \frac{d}{dx} \text{ on } D^* \quad (2.2)$$

where

$$D := \{f \in AC([a, b]) \mid f' \in L^2([a, b]), f(a) = 0 = f(b)\}, \quad (2.3)$$

$$D^* := \{g \in AC([a, b]) \mid g' \in L^2([a, b])\} \quad (2.4)$$

The notation $C_0^1((a, b))$ denotes the set of differentiable functions that have continuous derivatives on (a, b) and vanish at the two boundary points. The notation $AC([a, b])$ indicates the space of absolutely continuous functions on $[a, b]$. From analysis, we know that $f \in AC([a, b])$ if and only if there exists a $g \in L^1([a, b])$ such that $f(x) = f(a) + \int_a^x g(t) dt$ ¹ and $f' = g$ almost everywhere. In other words, we can differentiate f in the $L^2([a, b])$ -sense if and only if f is absolutely continuous.

Let us discuss T^* first. Its domain D^* is the largest possible set on which the action $i \frac{d}{dx}$ is properly defined: $i \frac{d}{dx} f$ makes sense in $L^2([a, b])$ if and only if $f \in AC([a, b])$ and $f' \in L^2([a, b])$. f must be absolutely continuous so that we can differentiate it. The vector f' must be in \mathbb{H} so that the image of f under $i \frac{d}{dx}$ still lies in $L^2([a, b])$.

If we restrict the domain D^* to D by imposing the boundary conditions $g(a) = 0 = g(b)$, then we obtain a different operator T . The operators T and T^* have different domains, i.e., $D \subsetneq D^*$. But they have the same action on D , i.e., $T = T^*|_D$. The operator T^* is an extension of T , i.e., $T \subset T^*$.

The domain of T_c , $C_0^1((a, b))$, is a well-known dense subset of $L^2([a, b])$ and it is contained in D . Hence T_c , T , and T^* are all densely defined unbounded linear operators.

We can show that T , T^* are both closed operators, but the operator T_c is not closed. The closure of T_c is indeed T . The proof of this result is long and irrelevant to our topic, hence ignored here. See Section Section 4.13 and 10.3 in [40] on closed operators. \square

¹The integral is a Lebesgue integral

2.1.1 Adjoint Operators

Let A be a bounded linear operator defined on \mathbb{H} . Section 4.4, 4.5 in [47] shows that for any fixed vector $\psi \in \mathbb{H}$, the map $\Gamma(\phi) = \langle A\phi, \psi \rangle$ defines a linear functional in \mathbb{H} . By the Riesz representation theorem, the functional Γ can be expressed in the form $\Gamma(\phi) = \langle \phi, \psi^* \rangle$ for a unique vector $\psi^* \in \mathbb{H}$.

Hence, for any vector $\psi \in \mathbb{H}$, there exist a unique vector $\psi^* \in \mathbb{H}$ such that $\langle A\phi, \psi \rangle = \langle \phi, \psi^* \rangle$, $\forall \phi \in \mathbb{H}$. Then we can define an operator A^* on \mathbb{H} by $A^*\psi = \psi^*$ and call it the adjoint operator of A .

Now consider a general linear operator T , not necessarily bounded. The domain of T may be only a proper subset of \mathbb{H} . For the inner product $\langle T\phi, \psi \rangle$ where ϕ runs through $D(T)$, we can no longer assert that for every $\psi \in \mathbb{H}$ there is a corresponding ψ^* such that

$$\langle T\phi, \psi \rangle = \langle \phi, \psi^* \rangle \quad \forall \phi \in D_T. \quad (2.5)$$

However, there exist some pairs ψ and ψ^* such that (2.5) holds. At least, $\psi = 0 = \psi^*$ does.

Definition 7. For a densely defined linear operator T in \mathbb{H} , let D^* be the set that consists of all $\psi \in \mathbb{H}$ such that there is a pair (ψ, ψ^*) with

$$\langle T\phi, \psi \rangle = \langle \phi, \psi^* \rangle \quad \forall \phi \in D_T,$$

the **adjoint operator** of T , denoted by T^* , is defined to be

$$T^*\psi = \psi^* \quad \text{on the domain } D^*.$$

In other words, the domain of the adjoint operator T^* of T is given by

$$D(T^*) = \{\psi \in \mathbb{H} \mid \exists \psi^* \in \mathbb{H} \text{ such that } \langle T\phi, \psi \rangle = \langle \phi, \psi^* \rangle \quad \forall \phi \in D(T)\}$$

and

$$\langle T\phi, \psi \rangle = \langle \phi, T^*\psi \rangle \quad \forall \phi \in D(T), \psi \in D(T^*). \quad (2.6)$$

It is necessary that T has a dense domain. Otherwise, for a fixed ψ , the existence of such ψ^* is not unique, hence $T^*\psi$ is not well-defined. To see this, suppose $D(T)$ is not dense, so $\mathbb{H} \ominus D(T) \neq \emptyset$. Let (ψ, ψ^*) be a pair for which Eq. (2.5) holds, then for any non-zero vector χ in $\mathbb{H} \ominus D(T)$, $\langle \phi, \chi \rangle = 0$, $(\phi, \psi^* + \chi)$ is another pair for which Eq. (2.5) holds because

$$\langle \phi, \psi^* + \chi \rangle = \langle \phi, \psi^* \rangle + \langle \phi, \chi \rangle = \langle T\phi, \psi \rangle \quad \forall \phi \in D(T).$$

Example 2: Consider the operators T and T^* defined in Eq. (2.2) in Example 1 on $\mathbb{H} = L^2([a, b])$. The interval $[a, b]$ is finite, $-\infty < a < b < +\infty$. We will show that T^* is indeed the adjoint of T . For now, let us temporarily denote the adjoint of T by S^* and show that $T^* = S^*$.

First, for any arbitrary g in D^* , $\forall f \in D$

$$\begin{aligned} \langle g, Tf \rangle &= \int_a^b g(x) \overline{if'(x)} dx \\ &= (-i) \left(g(b)\overline{f(b)} - g(a)\overline{f(a)} \right) - \int_a^b (g'(x)) \overline{if(x)} dx \\ &= (-i) (g(b)0 - g(a)0) + \int_a^b (ig'(x)) \overline{f(x)} dx \\ &= \langle T^*g, f \rangle. \end{aligned}$$

This shows that for any $g \in D^*$, there exists a pair (g, T^*g) satisfying (2.5). Hence, g must be in the domain of the adjoint S^* and $S^*g = T^*g$.

Conversely, suppose that $g \in D(S^*)$ with $g^* = S^*g$, we will show that g is in the domain of T^* , i.e., $g \in D^*$, and $g^* = T^*g$.

Since $g^* \in L^2([a, b])$, we can rewrite for almost everywhere x in $[a, b]$

$$\begin{aligned} g^*(x) &= \frac{d}{dx} \left(\int_a^x g^*(t) dt + C \right) \\ &= -i \frac{d}{dx} \left(\left[i \int_a^x g^*(t) dt \right] + C \right) \\ &= -i \frac{d}{dx} \left(\left[G(x) \right] + C \right) \end{aligned}$$

where $G(x) = i \int_a^x g^*(t) dt \in AC([a, b])$ and C is some arbitrary constant.

Then $\forall f \in D$, we have

$$\begin{aligned} \langle g, Tf \rangle &= \langle S^*g, f \rangle = \langle g^*, f \rangle = \int_a^b g^*(x) \overline{f(x)} dx \\ &= \int_a^b (-i) \frac{d}{dx} (G(x) + C) \overline{f(x)} dx \\ &= \left((-i)(G(x) + C) \overline{f(x)} \right) \Big|_{x=a}^b - \int_a^b (-i)(G(x) + C) \overline{f'(x)} dx \\ &= 0 + \int_a^b \left(-(G(x) + C) \right) \overline{if'(x)} dx \quad \text{since } f(a) = 0 = f(b) \\ &= \langle -(G + C), Tf \rangle. \end{aligned}$$

Hence,

$$\langle g + G + C, Tf \rangle = 0 \quad \forall f \in D_T,$$

namely

$$\int_a^b (g(x) + G(x) + C) \overline{(i f'(x))} dx = 0 \quad \forall f \in D_T. \quad (2.7)$$

Now, choose the constant C such that

$$\int_a^b (g(x) + G(x) + C) dx = 0. \quad (2.8)$$

Let

$$f_0(x) = \int_a^x (g(t) + G(t) + C) dt.$$

Clearly, by the choice of C as in Eq. (2.8), the function f_0 satisfies the boundary conditions $f_0(a) = 0 = f_0(b)$. Hence $f_0 \in D$ and $f_0'(x) = g(x) + G(x) + C$. Substitute into Eq. (2.7) to give

$$\begin{aligned} (-i) \int_a^b (g(x) + G(x) + C) \overline{(g(x) + G(x) + C)} dx &= 0 \\ \implies \int_a^b |g(x) + G(x) + C|^2 dx &= 0 \\ \implies g(x) + G(x) + C &= 0. \end{aligned}$$

Hence

$$g(x) = -G(x) - C = -i \int_a^x g^*(t) dt - C.$$

This shows that $i \frac{d}{dx} g(x) = g^*(x) \in L^2([a, b])$. Further $G(x) \in AC([a, b])$, so $g(x) \in AC([a, b])$ as well. Therefore, one concludes that $g \in D^*$ and

$$T^*g = i \frac{d}{dx} g = g^* = S^*g.$$

Hence $T^* = S^*$, T^* is indeed the adjoint of T . □

Proposition 1. *Let T be a densely defined linear operator in \mathbb{H} and T^* be its adjoint, then*

1. T^* is a linear operator;
2. T^* is always closed, whether or not T is closed;
3. if T has a closure \overline{T} , then $T^* = (\overline{T})^*$;

4. if $T \subset S$, then $T^* \supset S^*$.

Proof of 4. For any $\psi \in D(S^*)$, there exist $\psi^* = S^*\psi$, such that

$$\langle \phi, \psi^* \rangle = \langle \phi, S^*\psi \rangle = \langle S\phi, \psi \rangle \quad \forall \phi \in D(S). \quad (2.9)$$

Because $D(T) \subset D(S)$, Eq. (2.9) holds for all $\phi \in D(T)$ as well, and in this case $S\phi = T\phi$. Hence there is a pair (ψ, ψ^*) such that

$$\langle \phi, \psi^* \rangle = \langle T\phi, \psi \rangle \quad \forall \phi \in D(T). \quad (2.10)$$

Hence $\psi \in D(T^*)$ and $T^*\psi = \psi^* = S^*\psi$. The operator S^* is an extension of T^* . \square

Example 3: Consider the operators T , T_c and T^* defined in Example 1 on $\mathbb{H} = L^2([a, b])$ with $-\infty < a < b < +\infty$. The operator T is the closure of T_c and T^* is the adjoint of T . By the proposition above, $(T_c)^* = (\overline{T_c})^* = T^*$. The operator T^* is also the adjoint of T_c . \square

2.1.2 Symmetric and Self-Adjoint Operators

In finite-dimensional linear algebra, for a matrix A with non-zero determinant, its Hermitian conjugate matrix A^* is defined as the complex conjugate of the transpose of A , namely,

$$(A^*)_{ij} = \overline{A_{ji}}.$$

The square matrix A is called a Hermitian matrix, if

$$A = A^* = \overline{(A)^{tr}}. \quad (2.11)$$

So if A is a $n \times n$ Hermitian matrix, then $\forall u, v \in \mathbb{C}^n$

$$\langle u, Av \rangle = \overline{u^{tr}} Av = \overline{u^{tr}} \overline{(A)^{tr}} v = \overline{(Au)^{tr}} v = \langle Au, v \rangle. \quad (2.12)$$

The converse is also true. Therefore, Eq. (2.12) is used as an equivalent definition of Hermitian matrix in an n -dimensional complex vector space.

Matrices in linear algebra act like operators on the finite dimensional vector space. When the concept of finite dimensional vector spaces is generalized to infinite dimensional Hilbert spaces, adjoint and self-adjoint operators are the corresponding generalization of Hermitian conjugate and Hermitian matrices. But a slight problem occurs: the two equivalent definitions Eq. (2.11) and Eq. (2.12) of Hermitian matrix are no longer equivalent in infinite dimensional Hilbert spaces. Differences arise when dealing with unbounded operators. The concept of Hermitian matrices is generalized to self-adjoint operators by Eq.

(2.11) and is generalized to symmetric operators by Eq. (2.12). Self-adjoint and symmetric operators are the same on finite dimensional Hilbert spaces, but are different in the infinite dimensional case. The precise definitions for both operators are given below, followed by an example to show the connection between the two.

Definition 8. A linear operator T is said to be **symmetric** if

1. its domain $D(T)$ is dense in \mathbb{H} , and

2. for $\phi, \psi \in D(T)$,

$$\langle T\phi, \psi \rangle = \langle \phi, T\psi \rangle.$$

When the domain of a linear operator T is dense, the adjoint T^* is well-defined. The second condition is equivalent to $T \subset T^*$.

Definition 9. A linear operator T is said to be **self-adjoint** if

1. its domain $D(T)$ is dense in \mathbb{H} , and

2. $T = T^*$.

In summary, when $D(T)$ is dense, the linear operator T is **symmetric** if $T \subset T^*$, and T is **self-adjoint** if $T = T^*$. Clearly, self-adjoint operators are always symmetric, but not the converse.

Definition 10. A symmetric operator T is said to be **simple** if there is no subspace invariant under T such that the restriction of T to this subspace is self-adjoint.

We now discuss the concept of the extensions of a symmetric operator. To find the extensions of a symmetric operator, notice first that by definition, the adjoint T^* of a symmetric operator T is an extension of T itself. If both T, S are symmetric and $T \subset S$, then Proposition 1 implies that

$$T \subset S \subset S^* \subset T^*.$$

Namely, when a symmetric operator T is extended to another symmetric operator S , i.e., $T \subset S$, its corresponding adjoint operator that shrinks, i.e., $S^* \subset T^*$. Indeed, it is the domain of the adjoint that shrinks, i.e., $D(S^*) \subset D(T^*)$. This is because the operators T, S and S^* are all restrictions of the adjoint T^* in their respective smaller domains.

As a result, a symmetric extension S of T is the adjoint operator T^* restricted to a smaller domain $D(S) \subset D(T^*)$ such that $\forall \phi, \psi \in D_S$,

$$\langle S\phi, \psi \rangle = \langle \phi, S\psi \rangle.$$

Further, S has the same action as T^* on its domain. We conclude with the following proposition.

Proposition 2. *If T is a symmetric operator in \mathbb{H} and \tilde{D} is a closed subspace such that $D(T) \subset \tilde{D} \subset D(T^*)$, then \tilde{D} is the domain of a closed symmetric extension of T if and only if*

$$\langle T^*\phi, \psi \rangle = \langle \phi, T^*\psi \rangle \quad \forall \phi, \psi \in \tilde{D}. \quad (2.13)$$

Let \tilde{T} denote that symmetric extension of T , then $\tilde{T} = T^*|_{\tilde{D}}$.

When the enlarged symmetric extension S coincides with its adjoint, explicitly when $D(S) = D(S^*)$, the symmetric extension S becomes self-adjoint. Therefore, S is an self-adjoint extension of T if

$$T \subset S = S^* \subset T^*.$$

Example 4: Consider the operators T and T^* defined on $\mathbb{H} = L^2([a, b])$ with $-\infty < a < b < +\infty$ in Example 1. The operator T^* is the adjoint of T . Because $T \subsetneq T^*$, the operator T is symmetric, but not self-adjoint.

Define a new differential operator with a periodic boundary condition:

$$T_p := i \frac{d}{dx} \quad \text{on} \quad D_p := \{\phi \in AC([a, b]) \mid \phi' \in L^2([a, b]), \phi(a) = \phi(b)\}. \quad (2.14)$$

The domain D_p is the restriction of D^* by imposing the periodic boundary condition $\phi(a) = \phi(b)$. Note that $T \subset T_p \subset T^*$ since $D \subset D_p \subset D^*$.

By Proposition 1, $T \subset T_p$ implies $T_p^* \subset T^*$. So for any $g \in D(T_p^*)$, g must be absolutely continuous and $T_p^*g = T^*g = i \frac{d}{dx}g \in L^2([a, b])$. Using this fact and integration by parts, for any $g \in D(T_p^*)$ and $\forall f \in D(T_p)$,

$$\begin{aligned} \langle g, T_p f \rangle &= \int_a^b g(x) \overline{if'(x)} dx \\ &= (-i) \left(g(b) \overline{f(b)} - g(a) \overline{f(a)} \right) + \int_a^b \left(i g'(x) \right) \overline{f(x)} dx \\ &= (-i) \left(g(b) - g(a) \right) \overline{f(a)} + \langle T_p^* g, f \rangle. \end{aligned} \quad (2.15)$$

Because $\langle g, T_p f \rangle = \langle T_p^* g, f \rangle$, it follows that

$$f(a) \left(g(b) - g(a) \right) = 0 \quad \forall f \in D(T_p). \quad (2.16)$$

Since $f(a)$ can be chosen arbitrary, Eq. (4) holds if and only if $g(a) = g(b)$. Hence $g \in D(T_p)$ and $T_p^* \subset T_p$.

Let g be in $D(T_p)$ instead of $D(T_p^*)$, the calculation in Eq. (2.15) shows that

$$\langle g, T_p f \rangle = \langle T_p g, f \rangle \quad \forall f, g \in D(T_p).$$

The operator T_p is symmetric, $T_p \subset T_p^*$. Further, one concludes that $T_p = T_p^*$, the operator T_p is self-adjoint. Therefore, $T \subset T_p = T_p^* \subset T^*$, the new operator T_p is a self-adjoint extension of the unbounded symmetric operator T . \square

The problem of finding a self-adjoint extension of T is to first find a proper domain \tilde{D} such that Eq. (2.13) holds. It follows from the Hellinger-Toeplitz theorem that if T is an unbounded symmetric operator, its domain $D(T)$ can not be the whole Hilbert space \mathbb{H} , but only a dense subset of \mathbb{H} . The dense domain of the symmetric operator T complicates such a direct approach to find an extension of T .

The problem of the domain is a general issue for unbounded operators in Hilbert space. The domains of unbounded operators are usually proper subsets of \mathbb{H} . Hence, when forming the composition operators of two unbounded operators, one always needs to keep track of their domains, to check if the domain of one operator contains the range of the other. This complicates the algebra of unbounded operators. However, the bounded operators do not have this problem. The domain of bounded operators can be at least extended to the whole Hilbert space. This is why the Cayley transform can help one to find the self-adjoint extensions of symmetric operators. The Cayley transform translates symmetric operators into isometric operators and self-adjoint operators into unitary operators. As we will see in the next section, both isometric and unitary operators are bounded operators, so they are much easier to be handle.

2.1.3 Isometric and Unitary Operators

Definition 11. A linear operator U is said to be *isometric* on $D(U)$ if

$$\langle U\phi, U\psi \rangle = \langle \phi, \psi \rangle \quad \forall \phi, \psi \in D(U). \quad (2.17)$$

An isometric operator is also referred to as an *isometry*.

Isometric operators preserve the length, $\|U\phi\| = \|\phi\|$ for all $\phi \in D(U)$. They are bounded with $\|U\| = 1$. The domains of isometric operators are subspaces of \mathbb{H} . Because they are bounded, we are able to extend the domains to be the whole Hilbert space \mathbb{H} . When their domains and ranges are both \mathbb{H} , they become unitary operators.

Definition 12. A linear operator U defined on $D(U) = \mathbb{H}$ is said to be *unitary* if

1. $\langle U\phi, U\psi \rangle = \langle \phi, \psi \rangle \quad \forall \phi, \psi \in \mathbb{H}$,
2. $R(U) = \mathbb{H}$.

Proposition 3. Let U to be an isometric operator in \mathbb{H} , then

- U is bounded and has an isometric inverse,
- U has a closure and its closure \bar{U} is also isometric
- $\dim(D(\bar{U})) = \dim(R(\bar{U}))$

An isometric operator U has the same closed isometric extensions as its closure \bar{U} . Without loss of generality, we only consider closed isometric operators and their closed extension.

Definition 13. The **deficiency indices** (m, n) of an isometric operator U in \mathbb{H} are defined as

$$m = \dim(\mathbb{H} \ominus D(U)) \quad \text{and} \quad n = \dim(\mathbb{H} \ominus R(U)), \quad (2.18)$$

where m, n can be any non-negative integers or infinity.

Definition 14. An isometric operator is **maximal** if it has no proper isometric extension.

By the definition Eq. (2.17), an isometric operator U maps an orthonormal set to another orthonormal set. An isometric operator U is bounded by $\|U\| = 1$. The domain and range of a close symmetric operator U are closed subspaces with the same dimensions. However, this does not imply that the deficiency indices m and n must be identical because we can work within an infinitely dimensional Hilbert space. Consider the right shifting isometric operator $U : (c_1, c_2, c_3, \dots, c_n, \dots) \rightarrow (0, c_1, c_2, \dots, c_{n-1}, \dots)$, which maps which standard basis vector e_i to e_{i+1} . Its domain is the whole l^2 -space, but its range is $l^2 \setminus \{e_1\}$. Both have the same dimension of infinity. In this example, $m = 0$ but $n = 1$.

The method to obtain an isometric extension of U , denoted by \tilde{U} , is to find a closed isometric map U_0 from a subspace in $\mathbb{H} \ominus D(U)$ to $\mathbb{H} \ominus R(U)$ and append U_0 to U to form a larger isometric operator \tilde{U} . Hence,

$$\begin{aligned} D(\tilde{U}) &= D(U) \oplus D(U_0) \quad \text{and} \quad R(\tilde{U}) = R(U) \oplus R(U_0) \\ \text{where } D(U_0) &\subset \mathbb{H} \ominus D(U) \quad \text{and} \quad R(U_0) \subset \mathbb{H} \ominus R(U). \end{aligned}$$

Any vector $\tilde{\phi}$ in $D(\tilde{U})$ can be decomposed into $\tilde{\phi} = \phi + \phi_0$ where $\phi \in D(U)$, $\phi_0 \in D(U_0) \subset \mathbb{H} \ominus D(U)$ and its image under \tilde{U} is given by

$$\tilde{U}\tilde{\phi} = \tilde{U}\phi + \tilde{U}\phi_0 = U\phi + U_0\phi_0.$$

The idea to extend an isometric operator U is to exhaust the orthogonal complements of its domain and range, appending the isometric maps between them to U . This is why the deficiency indices (m, n) , defined to be the dimensions of the orthogonal complements of the domain and range of U , are important.

If $m = n$, then we can find a full isometric map U_0 from $\mathbb{H} \ominus D(U)$ onto $\mathbb{H} \ominus R(U)$. The resulting isometric extension \tilde{U} has both domain and range being \mathbb{H} . So \tilde{U} is a unitary extension of U .

If one of its deficiency indices is zero and the other one is non-zero, then the extra isometric map U_0 does not exist because either $\mathbb{H} \ominus D(U)$ or $\mathbb{H} \ominus R(U)$ is $\{0\}$. The isometric operator U has no non-trivial extension. Hence U is maximal.

Proposition 4. *Let U be a closed isometric operator with deficiency indices (m, n) , then*

- U is unitary if and only if $m = 0 = n$;
- U is maximal if and only if $m = 0 \neq n$ or $m \neq 0 = n$;
- U has non-trivial unitary extensions if and only if $m = n > 0$.

2.1.4 Spectrum of Closed Operators

In functional analysis, the concept of the spectrum of an operator is a generalization of the concept of eigenvalues for matrices. The spectrum of an operator on a finite-dimensional vector space is precisely the set of eigenvalues. However, an operator on an infinite dimensional space may have additional elements in its spectrum, and may have no eigenvalues at all.

Let A be a densely defined closed operator in \mathbb{H} , the **spectrum** of A , denoted by $\sigma(A)$, is defined to be the set of all $\lambda \in \mathbb{C}$ for which $(A - \lambda I)$ does not have a bounded inverse defined on all of \mathbb{H} . The operator I denotes the identity operator.

The resolvent set of A contains the points which are not in the spectrum of A and is denoted as $\rho(A)$. If $\lambda \in \rho(A) = \mathbb{C} \setminus \sigma(A)$, then the resolvent function²

$$R_A(\lambda) = (\lambda I - A)^{-1} \tag{2.19}$$

is a well-defined bounded operator on \mathbb{H} . A point $\lambda \in \mathbb{C}$ is said to be a point of regular type of A if $A - \lambda I$ is bounded below.

The spectrum of the operator A can be divided into three parts, depending on how the resolvent function in Eq. (2.19) fails to be well-defined and bounded:

$$\sigma(A) = \sigma_p(A) \cup \sigma_c(A) \cup \sigma_r(A). \tag{2.20}$$

²This notation is different from the one for the range of an operator. The resolvent function takes a scalar number λ as the input variable and has a subscript of the operator A ; while the notation for the range of an operator A , $R(A)$, takes the operator A as the input variable.

Let $\sigma_p(A)$, $\sigma_c(A)$ and $\sigma_r(A)$ denote the point, continuous, and residual spectrum of A respectively. The **point spectrum** $\sigma_p(A)$ is defined as the set of all **eigenvalues** of A , namely, there exists a non-zero vector ϕ in \mathbb{H} such that $(A - \lambda I)\phi = 0$. The operator $A - \lambda I$ is not injective, hence it has no inverse.

The continuous spectrum $\sigma_c(A)$ is defined as the set of all λ such that $R(A - \lambda I)$ is not closed. The residual spectrum $\sigma_r(A)$ is defined as the set of all λ such that $\lambda \notin \sigma_p(A)$ and $R(A - \lambda I)$ is not dense. For example, in quantum physics, Hamiltonian operator has point spectrum of $\{\hbar\omega(n + \frac{1}{2})\}_{n=0}^{+\infty}$, and the position or momentum operator has purely continuous spectrum of the whole real line.

Self-adjoint operators always possess a purely real spectrum. This includes eigenvalues. The spectra we deal with in Chapter 3 and generally in this thesis are point spectrum, namely, the set of eigenvalues. Symmetric operators may or may not have eigenvalues. If they do, their eigenvalues must be real.

Unitary operators always have a non-vanishing spectrum which lies on the unit circle in the complex plane. Isometric operators may or may not have a non-vanishing spectrum. If they do, it must be on the unit circle in the complex plane.

2.2 The Cayley Transform and Deficiency Indices

In Section 2.1.2, we clarified the difference between symmetric and self-adjoint operators and showed the theoretical way to extend a symmetric operator to be self-adjoint (if possible). Such a direct extension of a symmetric operator is generally unachievable because of the dense domain. The Cayley transform helps to circumvent this issue and is a practical tool to extend symmetric operators. The deficiency indices play crucial roles to characterize the types of the extensions of a symmetric operator. In this section, we will introduce the Cayley transform and the definition of deficiency indices for symmetric operators.

A symmetric operator and its closure have the same closed symmetric extensions. Without loss of generality, we consider only closed symmetric operators. The same holds for isometric operators.

The Cayley transform, defined as³

$$T \mapsto U := (T - i)(T + i)^{-1}, \tag{2.21}$$

maps the unbounded symmetric operators to bounded isometric operators. The Cayley transform has an analogy in the Moebius transform of complex numbers. The Moebius

³It is understood that a scalar number in an expression with operators indicates the multiplication of the scalar number with identity operator, e.g., $T + i = T + iI$ and $T - i = T - iI$.

transform is defined as

$$r \mapsto z := \frac{r - i}{r + i}. \quad (2.22)$$

The Moebius transform maps real numbers r to complex numbers z on the unit circle (excluding point $z = 1$), $|z| = 1$.

Recall that the spectrum of a symmetric operator lies on the real line (if it exists), and that the spectrum of an isometric operator lies on the unit circle in complex plane. Therefore, if the Cayley transform is an extension of the Moebius transform to linear operators, one would expect the Cayley transform in Eq. (2.21) to map a symmetric operator T to an isometric operator U .

The Cayley transform of a symmetric operator T , denoted by U , has domain $D(U) = R(T + i)$ and range $R(U) = R(T - i)$. The Cayley transform U maps $(T + i)\phi$ to $(T - i)\phi$, i.e.,

$$U((T + i)\phi) = (T - i)\phi \quad \forall \phi \in D(T). \quad (2.23)$$

The operator U is isometric since

$$\|T\phi + i\phi\|^2 = \|T\phi\|^2 + \|\phi\|^2 = \|T\phi - i\phi\|^2.$$

Definition 15. Let T be a symmetric operator on \mathbb{H} , then the operator U defined as

$$T \mapsto U := (T - i)(T + i)^{-1} \quad (2.24)$$

with domain $D(U) = R(T + i)$ is called the **Cayley transform** of T .

Let U be an isometric operator on \mathbb{H} with $I - U$ one-to-one, then the **inverse Cayley transform** of U is found to be

$$U \mapsto T := i(I + U)(I - U)^{-1} \quad (2.25)$$

with domain $D(T) = R(I - U)$.

The Cayley transform establishes a one-to-one correspondence between symmetric and isometric operators. The following theorem provides the main features of the Cayley transform.

Theorem 5. Let U be the Cayley transform of a symmetric operator T on \mathbb{H} , then

- U is isometric;
- U is closed if and only if T is closed;

- U is unitary if and only if T is self-adjoint.

Conversely, if U is an isometric operator on \mathbb{H} and $I - U$ one-to-one, then U is the Cayley transform of some symmetric operator T in \mathbb{H} and T is given by the inverse Cayley transform Eq. (2.25) of U .

Recall that the orthogonal complements of the domain and the range of an isometric operator U play a crucial role in determining isometric extensions of U . If U is the Cayley transform of a symmetric operator T , then $D(U) = R(T + i)$ and $R(U) = R(T - i)$. This suggests the definition of deficiency indices of a symmetric operator T to be the dimensions of

$$\mathbb{H} \ominus D(U) = \mathbb{H} \ominus R(T + i) \quad \text{and} \quad \mathbb{H} \ominus R(U) = \mathbb{H} \ominus R(T - i).$$

Such definition of deficiency indices of symmetric operators is consistent with the one of isometric operators via the Cayley transform. Recall the following definition and proposition:

Definition 16. Let T be a linear operator in \mathbb{H} . The kernel of T , denoted by $K(T)$, is the set of vectors which are mapped to 0 under T , namely,

$$K(T) = \{\phi \in \mathbb{H} \mid T\phi = 0\}.$$

Proposition 6. Let A be a densely defined operator, the kernel of the adjoint of A equals the orthogonal complement of the range of A , namely,

$$K(A^*) = \mathbb{H} \ominus \overline{R(A)}.$$

The proof can be found in [39]. This leads to the definition of deficiency spaces and indices of symmetric operators.

Definition 17. Let T be a symmetric operator. We define

$$K_+ = \mathbb{H} \ominus R(T + i) = K(T^* - i), \quad n_+ = \dim(K_+), \quad (2.26a)$$

$$K_- = \mathbb{H} \ominus R(T - i) = K(T^* + i), \quad n_- = \dim(K_-). \quad (2.26b)$$

The subspaces (K_+, K_-) are called the **deficiency spaces** of T and their dimensions (n_+, n_-) are called the **deficiency indices** of T .

Constructing Symmetric Extensions of a Symmetric Operator

Given an unbounded symmetric operator T , we can use the Cayley transform to find a symmetric extension of T . Suppose that \tilde{T} is a symmetric extension of T . Let U and \tilde{U} be their Cayley transform respectively. Then both U and \tilde{U} are isometric. The operator \tilde{U} is an extension of U .

Therefore, to find a symmetric extension \tilde{T} of T , one first needs to find an isometric extension \tilde{U} of U . The operator U is the Cayley transform of T , which is bounded. Its extension \tilde{U} is obtained by appending the isometric maps between the deficiency spaces K_+ and K_- to U . The inverse Cayley transform of \tilde{U} yields the symmetric extension \tilde{T} . In such a way, the extension problem of unbounded operators becomes the problem of bounded ones.

If the two deficiency spaces have the same dimension, i.e., $n_+ = n_-$, then one can exhaust (K_+, K_-) simultaneously so that a unitary extension \tilde{U} of U is obtained. Its inverse Cayley transform, \tilde{T} , is indeed a self-adjoint extension of T .

Of course, if $n_+ = 0 = n_-$, then $K_+ = \emptyset = K_-$, which implies that the domain and range of U are the whole Hilbert space. The Cayley transform U itself is unitary and hence the symmetric operator T is self-adjoint.

However, when one of K_+ and K_- has a smaller dimension, then the smaller deficiency space will be exhausted first. This gives a maximal isometric extension \tilde{U} of U . Hence, there is no unitary extension of U , and consequently, no self-adjoint extension of T either.

The deficiency indices (n_+, n_-) are important to characterize the symmetric extensions of a given symmetric operator T . One concludes that

- T has non-trivial self-adjoint extensions if and only if $n_+ = n_- \neq 0$.
- T is self-adjoint if and only if $n_+ = 0 = n_-$;
- T is maximal if and only if $n_+ = 0 \neq n_-$ or $n_+ \neq 0 = n_-$;

The following are three examples that demonstrate all three cases.

Example 5: Consider the operators T and T^* with domains D and D^* respectively on $\mathbb{H} = L^2([a, b])$ as defined in Eq. (2.2), (2.3) and (2.4) in Example 1, where $[a, b]$ is a finite interval, $-\infty < a < b < +\infty$.

The domain D^* is the largest closed subspace on which $i\frac{d}{dx}$ can be defined, and D is a closed subspace of $D(T^*)$ on which the vanishing boundary condition $\phi(a) = 0 = \phi(b)$ is obeyed. Let us recall some key calculation:

$$\begin{aligned}
\left\langle g, i \frac{d}{dx} f \right\rangle &= \int_a^b g(x) \overline{if'(x)} dx \\
&= -i \left(g(b) \overline{f(b)} - g(a) \overline{f(a)} \right) + \int_a^b (i g'(x)) \overline{f(x)} dx \\
&= -i \left(g(b) \overline{f(b)} - g(a) \overline{f(a)} \right) + \left\langle i \frac{d}{dx} g, f \right\rangle.
\end{aligned} \tag{2.27}$$

Let f be any arbitrary vector in $D(T)$ with boundary condition $f(a) = 0 = f(b)$, we have

$$\left(g(b) \overline{f(b)} - g(a) \overline{f(a)} \right) = 0. \tag{2.28}$$

Then (2.27) implies that for any $f \in D$

$$\left\langle g, i \frac{d}{dx} f \right\rangle = \left\langle i \frac{d}{dx} g, f \right\rangle.$$

Let g be any function in D , the above shows that T is symmetric; Let g in D^* , the above shows that T^* is the adjoint of T . Since $D \subsetneq D^*$, T is not self-adjoint.

The operator T does have non-trivial self-adjoint extensions. In Example 4, we showed that the operator T_p with domain D_p as defined in Eq. (2.14) is a non-trivial self-adjoint extension of T . The functions in D_p obey the periodic boundary condition $f(a) = f(b)$. For any functions f and g in D_p , Eq. (2.28) is clearly obeyed.

Now let us check the deficiency spaces and indices using the definition Eq. (2.26) in which $K_{\pm} = K(T^* \mp i)$. Notice that it is the adjoint operator T^* used to calculate the deficiency vectors of T . The deficiency functions do not obey the vanishing boundary conditions as the functions in the domain of T .

For any $f \in K_+ = K(T^* - i)$,

$$\begin{aligned}
(T^* - i)f(x) &= \left(i \frac{d}{dx} - i \right) f(x) = 0, \\
\frac{d}{dx} f(x) &= f(x), \\
\Rightarrow f(x) &= C_1 e^x, \text{ where } C_1 \text{ is constant.}
\end{aligned} \tag{2.29}$$

Similarly, $\forall g \in K_- = K(T^* + i)$,

$$\frac{d}{dx} g(x) = -g(x) \Rightarrow g(x) = C_2 e^{-x}, \text{ where } C_2 \text{ is constant.} \tag{2.30}$$

Hence,

$$K_+ = \text{span}\{e^x\} \quad \text{and} \quad K_- = \text{span}\{e^{-x}\}.$$

The operator T has deficiency indices $(1, 1)$. This agrees with the fact that T is symmetric and has non-trivial self-adjoint extensions. The operator T_p is one of its self-adjoint extensions and we will show later how to find all self-adjoint extensions of T . \square

Note the calculation above also holds for $a = -\infty$ or $b = +\infty$ or both. In these cases, $f(-\infty)$ and $f(+\infty)$ means

$$f(-\infty) = \lim_{x \rightarrow -\infty} f(x) \quad \text{and} \quad f(+\infty) = \lim_{x \rightarrow +\infty} f(x).$$

Example 6: Consider L^2 -functions on the whole real line \mathbb{R} with $a = -\infty$ and $b = +\infty$. On the Hilbert space $\mathbb{H} = L^2(\mathbb{R}) = L^2((-\infty, +\infty))$, we define

$$T_{\pm\infty} := i \frac{d}{dx} \quad \text{on} \quad \{f \in AC(\mathbb{R}) \mid f' \in L^2(\mathbb{R})\}. \quad (2.31)$$

Because f and f' are square integrable on \mathbb{R} ,

$$\lim_{x \rightarrow -\infty} f(x) = 0 = \lim_{x \rightarrow +\infty} f(x). \quad (2.32)$$

Substitute $a = -\infty$ and $b = +\infty$, the vanishing boundary condition $f(a) = 0 = f(b)$ is automatically obeyed. If the operator T is the one in Example 5 with $[a, b] = (-\infty, +\infty)$, then $T = T_{\pm\infty} = T^* = T_{\pm\infty}^*$. The operator $T_{\pm\infty}$ is self-adjoint. Note that Eq. (2.28) holds for any f and g in $D(T_{\pm\infty})$.

Let us check its deficiency spaces and indices. Following the same calculation as in Eq. (2.29) and (2.30), for any $f \in K_+$, $g \in K_-$,

$$\begin{aligned} f(x) &= C_1 e^x, \\ g(x) &= C_2 e^{-x}, \quad \text{where } C_1, C_2 \text{ are constant,} \end{aligned}$$

but

$$\int_{+\infty}^{-\infty} |e^{\pm x}|^2 dx = +\infty.$$

Hence $C_1 e^x$ and $C_2 e^{-x}$ are not in $\mathbb{H} = L^2(\mathbb{R})$ unless both C_1 and C_2 are zero. Therefore

$$K_+ = \text{span}\{0\} \quad \text{and} \quad K_- = \text{span}\{0\}.$$

The symmetric operator $T_{\pm\infty}$ has deficiency indices $(0, 0)$, which agrees with the fact that $T_{\pm\infty}$ is self-adjoint on $L^2(\mathbb{R})$. \square

Example 7: Consider L^2 -functions on semi-bounded intervals $(-\infty, b]$ and $[a, +\infty)$.

First, consider the interval $[a, b] = (-\infty, b]$ where b bounded. On the Hilbert space $\mathbb{H} = L^2((-\infty, b])$, we define an operator

$$T_{-\infty} := i \frac{d}{dx} \quad \text{on} \quad \{f \in \text{AC}((-\infty, b]) \mid f' \in L^2((-\infty, b]), \phi(b) = 0\}.$$

With $a = -\infty$, the interval is unbounded below. For $f \in L^2((-\infty, b])$, because f and f' are both square integrable,

$$f(a) = \lim_{x \rightarrow -\infty} f(x) = 0.$$

The boundary condition $f(a) = 0$ is automatically obeyed. With $a = -\infty$, the operator $T_{-\infty}$ is the operator T in Example 5. Eq. (2.28) holds $\forall f, g \in D(T_{-\infty})$, so $T_{-\infty}$ is symmetric. Its adjoint $T_{-\infty}^* = T^*$ acts on a domain with the boundary condition $\phi(b) = 0$ removed. Hence $T_{-\infty}$ is not self-adjoint. Further, the operator $T_{-\infty}$ does not have self-adjoint extensions. Because in Eq. (2.27), $g(a)\overline{f(a)} = 0$, Eq. (2.28) holds if and only if $g(b)\overline{f(b)} = 0$, which implies that at least one of $g(b)$ and $f(b)$ equals 0, namely at least one of f and g is in $D(T_{-\infty})$. Hence, there is no domain \tilde{D} such that

$$D(T_{-\infty}) \subsetneq \tilde{D} \subsetneq D(T_{-\infty}^*) \quad \text{and} \quad \langle g, T_{-\infty}^* f \rangle = \langle T_{-\infty}^* g, f \rangle \quad \forall f, g \in \tilde{D}.$$

Hence, the symmetric operator $T_{-\infty}$ is maximal.

Let us check its deficiency spaces and indices of $T_{-\infty}$. Similar to Example 6, by Eq. (2.29), (2.30) and the fact that $(-\infty, b]$ is bounded above,

$$\begin{aligned} & \int_{-\infty}^b |e^x|^2 dx < +\infty, \quad \int_{-\infty}^b |e^{-x}|^2 dx = +\infty \\ \Rightarrow & e^x \in L^2((-\infty, b]), \quad \text{but} \quad e^{-x} \notin L^2((-\infty, b]) \\ \Rightarrow & K_+ = \text{span}\{e^x\}, \quad \text{and} \quad K_- = \{0\}. \end{aligned}$$

Therefore, the symmetric operator $T_{-\infty}$ has deficiency indices $(1, 0)$, which agrees with the fact that $T_{\pm\infty}$ is maximal on $L^2((-\infty, b])$.

Similarly, consider the interval $[a, b] = (a, +\infty]$ where a is bounded. On the Hilbert space $\mathbb{H} = L^2((a, +\infty])$, we define

$$T_{+\infty} := i \frac{d}{dx} \quad \text{on} \quad \{\phi \in \text{AC}(a, +\infty]) \mid \phi' \in L^2((a, +\infty]), \phi(a) = 0\}.$$

A similar calculation shows that the operator $T_{+\infty}$ has deficiency spaces

$$K_+ = \{0\} \quad \text{and} \quad K_- = \text{span}\{e^x\}.$$

Therefore $T_{+\infty}$ has deficiency indices $(0, 1)$. It can be shown that $T_{+\infty}$ is a maximal symmetric operator as well. \square

2.3 Self-Adjoint Extensions of Symmetric Operators

In this section, we will see how the Cayley transform helps to extend a symmetric operator. In brief, the Cayley transform converts the unbounded operator problem of self-adjoint extensions of symmetric operators to the bounded operator problem of unitary extensions of isometric operators.

Let T be a closed densely defined unbounded symmetric operator and S be its Cayley transform. Figure 2.1 illustrates how to extend the symmetric operator T by extending its Cayley transform S , which is an isometry.

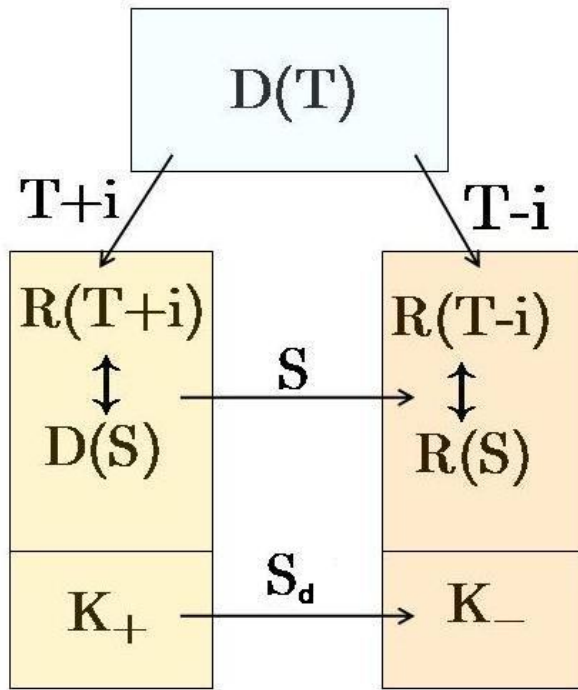


Figure 2.1: Using the Cayley transform to extend a symmetric operator T

An extension of the bounded isometric operator S can be obtained by appending an isometry between K_+ and K_- to S . Specifically, let S_d denote an isometric operator mapping from $D(S_d) \subseteq K_+$ to $R(S_d) \subseteq K_-$, then the direct sum $S \oplus S_d$ is an extension of S : it has a larger domain $D(S) \cup D(S_d)$ and it has the same action as S on the smaller domain $D(S)$. Since $S \oplus S_d$ is isometric, the inverse Cayley transform of $S \oplus S_d$ is symmetric and it is an extension of T .

The extension $S \oplus S_d$ is unitary if and only if $D(S_d) = K_+$ and $R(S_d) = K_-$. For S_d to fully exhaust both K_+ and K_- simultaneously, K_+ and K_- must have the same dimension, namely, T has equal deficiency indices, say $n_+ = D = n_-$, where D can be 0, any positive integer or infinity.

If D is a positive integer, S_d can be any element in a $U(D)$ -group. The unitary group of degree D , denoted $U(D)$, is the group of $D \times D$ unitary matrices with the

group operation that of matrix multiplication. Hence S has a $U(D)$ -family of unitary extensions, which implies that, via the inverse Cayley transform, T has a $U(D)$ -family of self-adjoint extensions.

2.4 The von Neumann Formulas

The Cayley transform will be the primary tool used for the key result on the eigenvalues of self-adjoint extensions in Chapter 3 and the development of the new sampling theory in Chapter 5. The result in this section on the von Neumann Formulas, which can be found in [39], vol. 2, Section 80, will be used only in Section 3.3 to link the example of the differential operators and the classical sampling theorem of Whittaker-Shannon-Kotelnikov.

For a symmetric operator T , although its domain $D(T)$ and its two deficiency spaces K_+ , K_- are neither closed nor orthogonal with respect to the regular inner product $\langle \cdot, \cdot \rangle$ of \mathbb{H} , they are closed and orthogonal with respect to the graph inner product of its adjoint T^* :

$$\langle \cdot, \cdot \rangle_{T^*} = \langle \cdot, \cdot \rangle + \langle T^* \cdot, T^* \cdot \rangle. \quad (2.33)$$

One refers to such closeness and orthogonality as T^* -closed and T^* -orthogonal. The von Neumann Formula states that, for a symmetric operator T , the domain of its adjoint operator T^* has the following representation as the direct sum of three linear subspaces:

$$D(T^*) = D(T) \oplus_{T^*} K_+ \oplus_{T^*} K_-. \quad (2.34)$$

The notation \oplus_{T^*} indicates that the direct sum is with respect to the graph inner product in Eq. (2.33). For any $\phi \in D(T^*)$, it has the following unique decomposition:

$$\phi = \phi_0 + \phi_+ + \phi_-, \quad (2.35)$$

where $\phi_0 \in D(T)$, $\phi_+ \in K_+$, $\phi_- \in K_-$. Further,

$$T^* \phi = T \phi_0 + i \phi_+ - i \phi_-. \quad (2.36)$$

Suppose that T has positive equal deficiency indices $(n_+, n_-) = (D, D)$. The symmetric operator T has a $U(D)$ -family of self-adjoint extensions. Let \tilde{T} be an arbitrary self-adjoint extension of T , and \tilde{U} be the Cayley transform of \tilde{T} . Then $\tilde{U} = S \oplus S_d$, where S is the Cayley transform of T and S_d is some isometric map between K_+ and K_- that uniquely defines the extension \tilde{T} . From the inverse Cayley transform (2.25), the domain of \tilde{T} reads:

$$\begin{aligned} D(\tilde{T}) &= R(I - \tilde{U}) = ((I - S)D(S)) \oplus_{T^*} ((I - S_d)K_+) \\ &= D(T) \oplus_{T^*} ((I - S_d)K_+). \end{aligned} \quad (2.37)$$

Hence, any $\phi \in D(\tilde{T})$ has a unique decomposition

$$\phi = \phi_0 + (I - S_d)\phi_+ = \phi_0 + \phi_+ - S_d\phi_+ \quad (2.38)$$

where $\phi_0 \in D(T)$ and $\phi_+ \in K_+$. Since $S_d\phi_+ \in K_-$, it follows from Eq. (2.35) and (2.36) that the self-adjoint extension \tilde{T} acts as

$$\tilde{T}\phi = T\phi_0 + i\phi_+ + iS_d\phi_+. \quad (2.39)$$

As a result, the von Neumann formulas provides a decomposition of vectors in the domain of the self-adjoint extension \tilde{T} and the action on these elements.

Chapter 3

Eigenvalues of Self-Adjoint Extensions of Simple Symmetric Operators with Deficiency Indices $(1,1)$

This chapter presents the main purely mathematical results of the thesis. One will see precisely how to extend a simple symmetric operator with deficiency indices $(1,1)$ using the Cayley transform and where the eigenvalues of its self-adjoint extensions precisely distribute on the real line. An example of such an operator, the classical differential operator on a finite interval, can be found in Section 3.3. This chapter is based on my paper [38].

The main results in this chapter are the explicit calculation of all of the eigenvalues of all of the self-adjoint extensions. It provides a concrete construction in this area of abstract functional analysis. The other papers also following [28], e.g., [48, 49, 50, 51], focus on the analytical properties of functions in the reproducing kernel Hilbert space associated with these Krein's type of symmetric operators and their relation to the de Branges space.

3.1 Self-Adjoint Extensions of Simple Symmetric Operators with Deficiency Indices $(1,1)$ and the Distribution of their Eigenvalues

In this chapter, we always assume that T is a closed simple symmetric operator with deficiency indices $(1,1)$ unless otherwise stated. It is also referred to as a $(1,1)$ -symmetric

operator. Such a symmetric operator T must be unbounded because a bounded symmetric operator would be self-adjoint and hence would have zero deficiency indices.

Let S be the Cayley transform of T , and ϕ_+ and ϕ_- denote the normalized Hilbert space vectors spanning the one-dimensional deficiency spaces K_+ and K_- :

$$\begin{aligned} K_+ &= \mathbb{H} \ominus D(S) = \mathbb{H} \ominus R(T + i) = K(T^* - i) = \text{span} \{\phi_+\}, \\ K_- &= \mathbb{H} \ominus R(S) = \mathbb{H} \ominus R(T - i) = K(T^* + i) = \text{span} \{\phi_-\}. \end{aligned} \quad (3.1)$$

The unitary extensions of S can be formed by appending any $U(1)$ -map between the two one-dimensional deficiency spaces to S . Since an $U(1)$ -group can be parameterized as $e^{i2\pi\alpha}$, where $0 \leq \alpha < 1$, one writes all the isometries from K_+ onto K_- as:

$$\cdot \mapsto e^{i2\pi\alpha} \langle \cdot, \phi_+ \rangle \phi_- \quad \text{where } 0 \leq \alpha < 1. \quad (3.2)$$

For any vector $C\phi_+$ in K_+ where the coefficient C is a complex number, each isometry in Eq. (3.2) maps $C\phi_+$ to $e^{i2\pi\alpha}C\phi_-$ in K_- . Appending these isometries to S , one obtains all the unitary extensions of S , denoted by $U(\alpha)$ and enumerated by the parameter α with $0 \leq \alpha < 1$:

$$U(\alpha)(\psi) = \begin{cases} S\psi & \text{if } \psi \in D(S), \\ e^{i2\pi\alpha} \langle \psi, \phi_+ \rangle \phi_- & \text{if } \psi \in K_+ = \mathbb{H} \ominus D(S). \end{cases} \quad (3.3)$$

Both the domain and range of $U(\alpha)$ are the whole Hilbert space \mathbb{H} . Hence $U(\alpha)$ is unitary. These unitary extensions have the same action as S on $D(S)$. Their difference is only on the one-dimensional deficiency space K_+ . Therefore, one has

$$(U(\beta) - U(\alpha))\psi = (e^{i2\pi\beta} - e^{i2\pi\alpha}) \langle \psi, \phi_+ \rangle \phi_- \quad \forall \psi \in \mathbb{H}, \quad \forall 0 \leq \alpha, \beta < 1. \quad (3.4)$$

It follows from Eq. (3.4) that the family of unitary extensions $\{U(\alpha) \mid 0 \leq \alpha < 1\}$ is periodically and uniformly continuously in terms of the parameter α . To see the continuity, note the following:

$$\|U(\beta) - U(\alpha)\| = |e^{i2\pi\beta} - e^{i2\pi\alpha}|. \quad (3.5)$$

Clearly, for any non-zero vector $\psi \in \mathbb{H}$,

$$\begin{aligned} \|(U(\beta) - U(\alpha))\psi\| &\leq |e^{i2\pi\beta} - e^{i2\pi\alpha}| |\langle \psi, \phi_+ \rangle| \|\phi_-\| \\ &\leq |e^{i2\pi\beta} - e^{i2\pi\alpha}| \|\psi\| \|\phi_+\| \\ &= |e^{i2\pi\beta} - e^{i2\pi\alpha}| \|\psi\|. \end{aligned} \quad (3.6)$$

The equality holds if $\psi = \phi_+$. The periodicity means that the limit returns back to the one at $\alpha = 0$ as α approaches 1 from below.

For each α , the inverse Cayley transform of $U(\alpha)$, denoted by $T(\alpha)$, is a self-adjoint extension of the symmetric operator T . Since S has a $U(1)$ -family of unitary extensions, then T has a $U(1)$ -family of self-adjoint extensions, enumerated as

$$\{T(\alpha) \mid 0 \leq \alpha < 1\}. \quad (3.7)$$

where α is the parameter in the $U(1)$ -maps $e^{i2\pi\alpha}$ between the two deficiency spaces in Eq. (3.2).

Proposition 7. *Let T be a simple symmetric operator with deficiency indices $(1, 1)$, then T has a $U(1)$ -family of self-adjoint extensions, which can be denoted by $\{T(\alpha) \mid 0 \leq \alpha < 1\}$. If one of the self-adjoint extensions of T has a purely discrete point spectrum, then all the self-adjoint extensions have purely discrete spectra. Furthermore, each real number t is an eigenvalue for one and only one of the self-adjoint extensions of T , and it is of multiplicity 1.*

Proof. First, a self-adjoint operator does not have residual spectrum. Let \tilde{T} denote any self-adjoint extension extension T . If λ is in the residual spectrum of \tilde{T} , then the operator $T - \lambda$ would not have a dense range. There would exist a vector ϕ in $\mathbb{H} \ominus R(T - \lambda)$, which implies that $\langle (T - \lambda)\psi, \phi \rangle = 0$ for all ψ in $D(\tilde{T})$. It follows that $\langle T\psi, \phi \rangle = \langle \psi, \bar{\lambda}\phi \rangle$, $\forall \psi \in D(\tilde{T})$. So $\bar{\lambda}$ is an eigenvalue of the self-adjoint operator \tilde{T} , and it would have to be real. But $\bar{\lambda} = \lambda$ can not be in both point and residual spectrum of \tilde{T} at the same time.

Under the assumption that one of the self-adjoint extensions of T has a purely discrete point spectrum, and the fact that all self-adjoint extensions of T must have the same continuous spectrum (Theorem 1, Section 83 in [39]), one concludes that all the self-adjoint extensions of T must have a purely discrete point spectrum.

The symmetric operator T does not have a continuous spectrum either. If T did, then the continuous spectrum would also be the spectrum of all its self-adjoint extensions. The operator T does not have discrete point spectrum, namely eigenvalues, either. Since if T did, then the eigenspace would be a subspace on which T has a self-adjoint restriction. This would contradict the assumption that T is simple. Therefore, the spectral kernel of the simple symmetric operator T is empty. Each real number is a point of regular type. Then Theorem 3 in Section 83 in [39] states that for each $t \in \mathbb{R}$, there exists one self-adjoint extension of T for which t is an eigenvalue of multiplicity 1.

By Eq. (2.1.2), the adjoint of T , T^* , is an extension of all self-adjoint extensions of T . So each real number t is also an eigenvalue of T^* with multiplicity of at least 1. On the other hand, since T is a symmetric operator with equal deficiency indices $(1, 1)$ and has no eigenvalue, Theorem 4 in Section 83 in [39] implies that the multiplicity of t as an eigenvalue of T^* cannot exceed 1. Hence, each real number t is an eigenvalue of T^*

with a multiplicity exactly 1. Therefore, each $t \in \mathbb{R}$, as an eigenvalue of one of self-adjoint extensions of T , must also have a multiplicity of 1.

If t is an eigenvalue of two self-adjoint extensions of T , say, $T(\alpha)$ and $T(\beta)$, where $0 \leq \alpha \neq \beta < 1$, then they must have the same eigenvector ϕ because t is also the eigenvalue of the adjoint operator T^* with a multiplicity of 1. It follows that $\frac{t-i}{t+i}$ and ϕ are the eigenvalue and eigenvector of their Cayley transforms, $U(\alpha)$ and $U(\beta)$. Eq. (3.4) gives

$$0 = (U(\beta) - U(\alpha)) \phi = (e^{i2\pi\beta} - e^{i2\pi\alpha}) \langle \phi, \phi_+ \rangle \phi_-. \quad (3.8)$$

To show that $\langle \phi, \phi_+ \rangle \neq 0$, one notices that the eigenvector ϕ of the unitary extension can not be in the domain of S because otherwise ϕ would also be an eigenvector of S , thus an eigenvector of T as well, then T would have a self-adjoint restriction on the corresponding eigenspace of ϕ . This would contradict the assumption that T is simple. Therefore, ϕ is not fully in $D(S)$. It is then not orthogonal to the deficiency vector $\phi_+ \in \mathbb{H} \ominus D(S)$. So $\langle \phi, \phi_+ \rangle \neq 0$.

It follows that $e^{i2\pi\beta} - e^{i2\pi\alpha} = 0$. So $\alpha = \beta$. The self-adjoint operators $T(\alpha)$ and $T(\beta)$ are the same one. Therefore, a real number t cannot be the eigenvalue of two distinguished self-adjoint extensions of T . \square

As a consequence, each self-adjoint extension of T has a purely point spectrum, which consists of a set of eigenvalues of multiplicity 1. Together, these eigenvalues cover the real line exactly once. For each fixed α , the eigenvalues of $T(\alpha)$ are real, and can be arranged in an increasing sequence $\{t_n(\alpha)\}_{n \in I}$ where the index set I can be $-\mathbb{N}$ or \mathbb{N} or \mathbb{Z} . The index set could be positive or negative integers because the unbounded operator T can be bounded either below or above, but not in both directions. The operator T can be unbounded in either direction or both. Without loss of generality, we assume that the self-adjoint extension is unbounded in both directions so that we imply $I = \mathbb{Z}$.

Let $\phi_n(\alpha)$ denote the corresponding normalized eigenvector for each eigenvalue $t_n(\alpha)$ of $T(\alpha)$. It is important to notice that one has the freedom to choose the phase of the eigenvector $\phi_n(\alpha)$. For each self-adjoint extension $T(\alpha)$, the set of normalized eigenvectors, $\{\phi_n(\alpha)\}_n$, forms an orthonormal basis of \mathbb{H} . The self-adjoint operator $T_n(\alpha)$ can be written as:

$$T(\alpha) = \sum_{n=-\infty}^{+\infty} t_n(\alpha) \langle \cdot, \phi_n(\alpha) \rangle \phi_n(\alpha). \quad (3.9)$$

Correspondingly, its Cayley transform $U(\alpha)$ in Eq. (3.3) can be expressed in the same eigenbasis as:

$$U(\alpha) = \sum_{n=-\infty}^{+\infty} \frac{t_n(\alpha) - i}{t_n(\alpha) + i} \langle \cdot, \phi_n(\alpha) \rangle \phi_n(\alpha). \quad (3.10)$$

Each eigenvalue of $U(\alpha)$, $\frac{t_n(\alpha)-i}{t_n(\alpha)+i}$, is the Mobius transform of $t_n(\alpha)$, the eigenvalue of $T(\alpha)$. It is a complex number on the unit circle and $\phi_n(\alpha)$ is its corresponding eigenvector. According to the Mobius transform, since the eigenvalues of all $T(\alpha)$, i.e., $\{t_n(\alpha), 0 \leq \alpha < 1, n \in \mathbb{Z}\}$, cover the real line exactly once, the eigenvalues of all $U(\alpha)$ together, i.e., $\{\frac{t_n(\alpha)-i}{t_n(\alpha)+i}, 0 \leq \alpha < 1, n \in \mathbb{Z}\}$, cover the unit circle in the complex plane exactly once except the point $+1$. Further, for each fixed α , because $\{t_n(\alpha)\}_n$ is in an increasing order, the points in $\{\frac{t_n(\alpha)-i}{t_n(\alpha)+i}\}_n$ move counter-clockwise along the unit circle in the complex plane as n increases.

The bounded unitary operators $U(\alpha)$ are uniformly and periodically continuous with respect to the parameter α . Their eigenvalues $\{\frac{t_n(\alpha)-i}{t_n(\alpha)+i}\}_n$ must also be uniformly and periodically continuous on the unit circle with respect to α . Therefore, the eigenvalues $\{t_n(\alpha)\}_n$ are also uniformly and periodically continuous on the real line with respect to α . Further, the norm of $U(\alpha)$ in Eq. (3.5) is differentiable with respect to α , since the following limit exists

$$\lim_{\beta \rightarrow \alpha} \frac{\|U(\beta) - U(\alpha)\|}{|\beta - \alpha|} = \lim_{\beta \rightarrow \alpha} \frac{|e^{i2\pi\beta} - e^{i2\pi\alpha}|}{|\beta - \alpha|} = \lim_{h \rightarrow 0^+} \frac{|e^{i2\pi h} - 1|}{h} = 2\pi$$

Therefore the eigenvalue of $U(\alpha)$, $\frac{t_n(\alpha)-i}{t_n(\alpha)+i}$, is also differentiable with respect to α , and so is $t_n(\alpha)$. Hence one can define the derivative of $t_n(\alpha)$ with respect to α as:

$$t'_n(\alpha) = \frac{dt_n(\alpha)}{d\alpha}. \quad (3.11)$$

The theorem in the next section states that if the eigenvalues of one self-adjoint extension and the corresponding derivatives with respect to α are known, one can explicitly calculate the eigenvalues of all other the self-adjoint extensions of T and obtain an expression for the eigenvector corresponding for any of these eigenvalues.

3.2 Calculation of Eigenvalues and Eigenvectors of the Self-Adjoint Extensions of Simple Symmetric Operators with Deficiency Indices (1,1)

Without loss of generality, one assumes that the given self-adjoint extension of the simple (1,1)-symmetric operator T is $T(0)$, the one at $\alpha = 0$. One can always do so by choosing the phase of the deficiency vector to match the given self-adjoint operator.

Theorem 8. Let T be a simple symmetric operator with deficiency indices $(1, 1)$. Suppose one of its self-adjoint extensions, $T(0)$, at $\alpha = 0$, has purely discrete spectrum with no accumulation point. Each eigenvalue is of multiplicity 1. Let $\{t_n = t_n(0)\}_{n=-\infty}^{\infty}$ and $\{\phi_n = \phi(0)\}_{n=-\infty}^{\infty}$ denote the eigenvalues and the corresponding normalized eigenvectors of $T(0)$. The coefficients of the first deficiency vector ϕ_+ in the eigenbasis $\{\phi_n\}_{n=-\infty}^{\infty}$ are given as $\left\{\frac{f_n}{t_n - i}\right\}_{n=-\infty}^{\infty}$. Then the absolute value of the constant $f_n = f_n(0)$ determines the derivative of the eigenvalue with respect to α :

$$t'_n(0) = \left. \frac{dt_n(\alpha)}{d\alpha} \right|_{\alpha=0} = \pi |f_n(0)|^2. \quad (3.12)$$

The eigenvalues of other self-adjoint extensions $T(\alpha)$ for $0 < \alpha < 1$, denoted by $\{t_n(\alpha)\}_{n=-\infty}^{\infty}$, can be obtained by solving for t in the following equation:

$$\frac{t'_n}{t - t_n} - \sum_{m \neq n} \frac{t'_m (t - t_m)}{(t - t_m)(t_n - t_m)} = \pi \cot(\pi\alpha). \quad (3.13)$$

where α is the parameter in Eq. (3.2) which enumerates the $U(1)$ -map $e^{i2\pi\alpha}$ between the deficiency spaces.

Let ϕ_t denote the normalized eigenvector to an eigenvalue $t \in \mathbb{R}$. Its coefficients in the eigenbasis $\{\phi_n\}_{n=-\infty}^{+\infty}$ are given by:

$$\langle \phi_t, \phi_n \rangle = (-1)^{z(t, t_n)} \frac{\sqrt{t'_n}}{|t - t_n|} \left(\sum_{m=-\infty}^{\infty} \frac{t'_m}{(t - t_m)^2} \right)^{-1/2}. \quad (3.14)$$

where $z(t, \hat{t})$ denote the number of the eigenvalues $\{t_n\}_{n=-\infty}^{\infty}$ between t and \hat{t} exclusively. Further, let $\phi_{\hat{t}}$ denote the normalized eigenvector to another eigenvalue $\hat{t} \in \mathbb{R}$, the inner product of ϕ_t and $\phi_{\hat{t}}$ is given by:

$$\langle \phi_t, \phi_{\hat{t}} \rangle = (-1)^{z(t, \hat{t})} \sum_{n=-\infty}^{\infty} \frac{|t'_n|}{(t - t_n)(\hat{t} - t_n)} \left(\sum_{m=-\infty}^{\infty} \frac{t'_m}{(t - t_m)^2} \right)^{-1/2} \left(\sum_{m=-\infty}^{\infty} \frac{t'_m}{(\hat{t} - t_m)^2} \right)^{-1/2}. \quad (3.15)$$

Note. The significance of Eq. (3.14) and (3.15) is that the inner product of eigenvectors is expressed solely in terms of the given eigenvalues and derivatives, $\{t_n\}_{n=-\infty}^{+\infty}$ and $\{t'_n\}_{n=-\infty}^{+\infty}$. Any real number t is an eigenvalue of some self-adjoint extension of T . One does not need to know to which self-adjoint extension the eigenvector belongs, nor the specification of the eigenvectors. This is important and due to the fact that we are working within an abstract Hilbert space, not tied to any particular representation.

Eq. (3.13) implies that for any fixed $\alpha \in (0, 1)$, one can obtain all the eigenvalues $\{t_n(\alpha)\}_{n=-\infty}^{+\infty}$ of $T(\alpha)$ by solving for t . There is one solution on each open interval (t_n, t_{n+1}) . Conversely, for any real number t , Eq. (3.13) also determines of which self-adjoint extension t is the eigenvalue.

The relation, Eq. (3.12), between the derivatives of the eigenvalues and the coefficients of the first deficiency vector expanded in the eigenbasis of the given self-adjoint extension is true for all self-adjoint extensions.

The pre-determined set of eigenvalues $\{t_n = t_n(0)\}_{n=-\infty}^{+\infty}$ and the derivative $\{t'_n = t'_n(0)\}_{n=-\infty}^{+\infty}$ in the theorem obey the following three conditions to ensure that the operator T is simple symmetric and the first deficiency vector ϕ_+ is normalized:

$$t'_n \neq 0 \quad \forall n, \quad \sum_{n=-\infty}^{+\infty} t'_n \text{ diverges,} \quad \sum_{n=-\infty}^{+\infty} \frac{t'_n}{t_n^2 + 1} = \pi. \quad (3.16)$$

If the given self-adjoint extension $T(0)$ has its eigenvalues $\{t_n\}_{n=-\infty}^{+\infty}$ and the derivatives $\{t'_n\}_{n=-\infty}^{+\infty}$ obeying these conditions, then so do the other self-adjoint extensions.

Proof. Let $\phi_n(\alpha)$ denote the normalized eigenvector to the eigenvalue $t_n(\alpha)$ of the self-adjoint extension $T(\alpha)$ for all $0 \leq \alpha < 1$ and $\phi_n = \phi_n(0)$. The set $\{\phi_n(\alpha)\}_n$, forms an orthonormal eigenbasis of \mathbb{H} . The self-adjoint operator $T(0)$ can be expressed as

$$T(0) = \sum_{n=-\infty}^{+\infty} t_n \langle \cdot, \phi_n \rangle \phi_n. \quad (3.17)$$

Notice that one has the freedom to choose the phase of ϕ_n for all n . Its Cayley transform $U(0)$ is given by

$$U(0) = \sum_{n=-\infty}^{+\infty} \frac{t_n - i}{t_n + i} \langle \cdot, \phi_n \rangle \phi_n. \quad (3.18)$$

Let S denote the Cayley transform of T . It is the restriction of $U(0)$ on $D(S) = \mathbb{H} \ominus K_+$:

$$S = U(0)|_{\mathbb{H} \ominus K_+}. \quad (3.19)$$

where the one-dimensional first deficiency space K_+ is spanned by ϕ_+ . The first deficiency vector ϕ_+ is given by:

$$\phi_+ = \sum_{n=-\infty}^{+\infty} \frac{f_n}{t_n - i} \phi_n. \quad (3.20)$$

The second deficiency vector ϕ_- is the image of ϕ_+ under the $U(0)$:

$$\phi_- = U(0) \phi_+ = \sum_{n=-\infty}^{+\infty} \frac{t_n - i}{t_n + i} \langle \phi_+, \phi_n \rangle \phi_n = \sum_{n=-\infty}^{+\infty} \frac{f_n}{t_n + i} \phi_n. \quad (3.21)$$

All the self-adjoint extensions of T are obtained by the Cayley transform and Eq. (3.3). The parameter α is the one in the $U(1)$ -map $e^{i2\pi\alpha}$ in Eq. (3.2) between K_+ and K_- . With the choice of ϕ_- in Eq. (3.21), the given self-adjoint operator $T(0)$ is indeed the one which corresponds to $\alpha = 0$ in the enumerated one-parameter family of self-adjoint extensions of T . Hence $t_n = t_n(0)$ and $\phi_n = \phi_n(0)$ are consistent for all n .

For each α in $[0, 1)$, the self-adjoint operator $T(\alpha)$ has a set of eigenvalues $\{t_n(\alpha)\}_n$ with multiplicity 1. Let $\phi_n(\alpha)$ denote the normalized eigenvector of the eigenvalue $t_n(\alpha)$. One can expand ϕ_+ in the eigenbasis $\{\phi_n(\alpha)\}_n$ as

$$\phi_+ = \sum_{n=-\infty}^{+\infty} \frac{e^{i2\pi\alpha} f_n(\alpha)}{t_n(\alpha) - i} \phi_n(\alpha), \quad (3.22)$$

where $f_n(\alpha)$ is the constant coefficient to be determined later. The denominator $t_n(\alpha) - i$ and $e^{i2\pi\alpha}$ are for calculation convenience as one will see later. Each $f_n(\alpha)$ has the freedom of the choice of its own phase. At $\alpha = 0$, $f_n = f_n(0)$ for all n . The expansion of ϕ_- in the eigenbasis $\{\phi_n(\alpha)\}_n$ is obtained by substituting $\psi = \phi_+$ into Eq. (3.3) and using Eq. (3.10), (3.22):

$$\begin{aligned} \phi_- &= e^{-i2\pi\alpha} \langle \phi_+, \phi_+ \rangle U(\alpha) \phi_+ = e^{-i2\pi\alpha} U(\alpha) \phi_+ \\ &= e^{-i2\pi\alpha} \sum_n \frac{e^{i2\pi\alpha} f_n(\alpha)}{t_n(\alpha) - i} (U(\alpha) \phi_n(\alpha)) \\ &= \sum_n \frac{f_n(\alpha)}{t_n(\alpha) - i} \left(\frac{t_n(\alpha) - i}{t_n(\alpha) + i} \phi_n(\alpha) \right) \\ &= \sum_n \frac{f_n(\alpha)}{t_n(\alpha) + i} \phi_n(\alpha). \end{aligned} \quad (3.23)$$

For the operator T being simple symmetric and the deficiency vector ϕ_+ to be normalized, the following three conditions are obeyed:

$$f_n \neq 0 \quad \forall n, \quad \sum_{n=-\infty}^{+\infty} |f_n|^2 \text{ diverges}, \quad \sum_{n=-\infty}^{+\infty} \frac{|f_n|^2}{t_n^2 + 1} = 1. \quad (3.24)$$

Notice that the images of vectors in the domain of T under $T + i$ must be orthogonal to $\phi_+ \in K_+$. Hence the domain of T reads:

$$D(T) = \{ \phi \in D(T(0)) \mid \langle (T(0) + i)\phi, \phi_+ \rangle = 0 \}. \quad (3.25)$$

To show the first condition in Eq. (3.24) that $f_n \neq 0$, note that because T is simple, T does not inherit any eigenvector from $T(0)$, i.e., $\phi_n \notin D(T)$. Therefore,

$$0 \neq \langle (T(0) + i)\phi_n, \phi_+ \rangle = (t_n + i) \langle \phi_n, \phi_+ \rangle = f_n^*. \quad (3.26)$$

To show the divergence in the second condition, note that since T is symmetric, $D(T)$ must be dense. If the following summation did not diverge,

$$\sum_n |(t_n - i) \langle \phi_+, \phi_n \rangle|^2 = \sum_n |f_n|^2, \quad (3.27)$$

then there would exist a vector $\psi \in \mathbb{H}$

$$\psi = \sum_n (t_n - i) \langle \phi_+, \phi_n \rangle \phi_n = (T(0) - i) \phi_+, \quad (3.28)$$

so that for any $\phi \in D(T)$,

$$\langle \phi, \psi \rangle = \langle \phi, (T(0) - i) \phi_+ \rangle = \langle (T(0) + i) \phi, \phi_+ \rangle = 0. \quad (3.29)$$

Hence $\psi \in \mathbb{H} \ominus D(T)$. The domain $D(T)$ would not be dense and therefore, the summation in Eq. (3.27) must diverge.

The third condition follows from the normalization of the deficiency vector ϕ_+ and the expansion of ϕ_+ in the eigenbasis $\{\phi_n\}_n$:

$$1 = \langle \phi_+, \phi_+ \rangle = \sum_n \frac{f_n}{t_n - i} \frac{f_n^*}{t_n + i} \langle \phi_n, \phi_n \rangle = \sum_n \frac{|f_n|^2}{t_n^2 + 1}.$$

The three conditions in Eq. (3.24) are true because the operator T is simple symmetric and ϕ_+ is normalized. For exactly the same reason, similar conditions are also true if one replaces $\{t_n\}_n$ and $\{f_n\}_n$ by any other set of eigenvalues and their corresponding coefficients, i.e. $\{t_n(\alpha)\}_n$ and $\{f_n(\alpha)\}_n$ for any fixed α between 0 and 1.

With explicit expansion of the deficiency vectors ϕ_+ and ϕ_- in each eigenbasis $\{\phi_n(t_n)\}_n$ in Eq. (3.22) and Eq. (3.23), one can use Eq. (3.4) to calculate the inner product of eigenvectors $\phi_n(\alpha)$ and $\phi_m(\beta)$ of any two arbitrary self-adjoint extensions $T(\alpha)$ and $T(\beta)$, with $0 \leq \alpha \neq \beta < 1$. First, one applies $U(\beta)$ on the first argument and $U(\alpha)$ on the second argument in the inner product $\langle \phi_m(\beta), \phi_n(\alpha) \rangle$:

$$\begin{aligned} & \langle (U(\beta) - U(\alpha)) \phi_m(\beta), \phi_n(\alpha) \rangle \\ &= \langle U(\beta) \phi_m(\beta), \phi_n(\alpha) \rangle - \langle \phi_m(\beta), U(\alpha)^* \phi_n(\alpha) \rangle \\ &= \left\langle \frac{t_m(\beta) - i}{t_m(\beta) + i} \phi_m(\beta), \phi_n(\alpha) \right\rangle - \left\langle \phi_m(\beta), \frac{t_n(\alpha) + i}{t_n(\alpha) - i} \phi_n(\alpha) \right\rangle \\ &= \left(\frac{t_m(\beta) - i}{t_m(\beta) + i} - \frac{t_n(\alpha) - i}{t_n(\alpha) + i} \right) \langle \phi_m(\beta), \phi_n(\alpha) \rangle \\ &= \frac{2i(t_m(\beta) - t_n(\alpha))}{(t_m(\beta) + i)(t_n(\alpha) + i)} \langle \phi_m(\beta), \phi_n(\alpha) \rangle. \end{aligned} \quad (3.30)$$

On the other hand, apply the right hand side of Eq. (3.4) to $\langle \phi_m(\beta), \phi_n(\alpha) \rangle$:

$$\begin{aligned}
& \langle (e^{i2\pi\beta} - e^{i2\pi\alpha}) \langle \phi_m(\beta), \phi_+ \rangle \phi_- , \phi_n(\alpha) \rangle \\
&= (e^{i2\pi\beta} - e^{i2\pi\alpha}) \langle \phi_m(\beta), \phi_+ \rangle \langle \phi_-, \phi_n(\alpha) \rangle \\
&= (e^{i2\pi\beta} - e^{i2\pi\alpha}) \left(\frac{e^{-i2\pi\beta} f_m^*(\beta)}{t_m(\beta) + i} \right) \left(\frac{f_n(\alpha)}{t_n(\alpha) + i} \right) \\
&= \frac{(1 - e^{i2\pi(\alpha-\beta)}) f_m^*(\beta) f_n(\alpha)}{(t_m(\beta) + i)(t_n(\alpha) + i)}.
\end{aligned} \tag{3.31}$$

Cancel out the common denominator $(t_m(\beta) + i)(t_n(\alpha) + i)$ in the above two equations to give:

$$\langle \phi_m(\beta), \phi_n(\alpha) \rangle = \frac{(1 - e^{i2\pi(\alpha-\beta)}) f_m^*(\beta) f_n(\alpha)}{2i(t_m(\beta) - t_n(\alpha))}. \tag{3.32}$$

Then the trigonometric double angle formula gives:

$$\begin{aligned}
\frac{(1 - e^{i2\pi(\alpha-\beta)})}{2i} &= \frac{1}{2i} (1 - \cos(2\pi(\alpha - \beta)) - i \sin(2\pi(\alpha - \beta))) \\
&= \frac{1}{2i} (2 \sin^2(\pi(\alpha - \beta)) - i 2 \sin(\pi(\alpha - \beta)) \cos(\pi(\alpha - \beta))) \\
&= -\sin(\pi(\alpha - \beta)) e^{i\pi(\alpha-\beta)}.
\end{aligned} \tag{3.33}$$

Therefore, the inner product $\langle \phi_m(\beta), \phi_n(\alpha) \rangle$ in Eq. (3.32) simplifies to

$$\langle \phi_m(\beta), \phi_n(\alpha) \rangle = \frac{\sin(\pi(\beta - \alpha))}{t_m(\beta) - t_n(\alpha)} \left(e^{i\pi\beta} f_m(\beta) \right)^* \left(e^{i\pi\alpha} f_n(\alpha) \right). \tag{3.34}$$

Now one uses the freedom of the phase of $f_n(\alpha)$ for all n and $0 \leq \alpha < 1$ to impose that $(e^{i\pi\alpha} f_n(\alpha))$ is always real. But one is still free to choose it is either positive or negative. Notice that the limit of $e^{-i\pi\alpha}$ goes to -1 as α goes to 1. To ensure the continuity of $f_n(\alpha)$ as α approaches 1, f_n is chosen to have an alternating sign, e.g., $(-1)^n$. In summary, one chooses the phase in the eigenvector $\phi_n(\alpha)$ such that

$$f_n(\alpha) = e^{-i\pi\alpha} (-1)^n |f_n(\alpha)| \quad \forall n \in \mathbb{Z}, 0 \leq \alpha < 1. \tag{3.35}$$

Therefore, Eq. (3.34) becomes:

$$\langle \phi_m(\beta), \phi_n(\alpha) \rangle = \frac{(-1)^{m+n} \sin(\pi(\beta - \alpha))}{t_m(\beta) - t_n(\alpha)} |f_m(\beta)| |f_n(\alpha)|. \tag{3.36}$$

Fix one eigenvector $t_n(\alpha)$ and let $m = n$. Because of the continuity of inner product and the unitary extension in Eq. (3.3), the limit of the inner product $\langle t_n(\beta), t_n(\alpha) \rangle$ as β approaches α should be equal to $\langle t_n(\alpha), t_n(\alpha) \rangle = 1$. Using Eq. (3.36), one has

$$\begin{aligned}
1 &= \langle \phi_n(\alpha), \phi_n(\alpha) \rangle = \lim_{\beta \rightarrow \alpha} \langle \phi_n(\beta), \phi_n(\alpha) \rangle \\
&= (-1)^{2n} |f_n(\alpha)|^2 \lim_{\beta \rightarrow \alpha} \frac{\sin(\pi(\beta - \alpha))}{t_n(\beta) - t_n(\alpha)} \\
&= |f_n(\alpha)|^2 \lim_{\beta \rightarrow \alpha} \frac{\pi \cos(\pi(\beta - \alpha))}{\frac{dt_n(\beta)}{d\beta} \Big|_{\beta=\alpha} - 0} \quad (\text{by l'Hopital's rule}) \\
&= \pi |f_n(\alpha)|^2 \left(\frac{dt_n(\beta)}{d\beta} \Big|_{\beta=\alpha} \right)^{-1}.
\end{aligned} \tag{3.37}$$

Hence, Eq. (3.12) follows as:

$$t'_n(\alpha) = \frac{dt_n(\beta)}{d\beta} \Big|_{\beta=\alpha} = \pi |f_n(\alpha)|^2. \tag{3.38}$$

The three conditions in Eq. (3.16) follow from Eq. (3.38) and (3.24).

One expands $\langle \phi_n(\alpha), \phi_n(\alpha) \rangle = 1$ in the eigenbasis $\{\phi_m = \phi_m(0)\}_m$ of the given self-adjoint operator $T(0)$ to obtain

$$\begin{aligned}
1 &= \sum_m \langle \phi_n(\alpha), \phi_m \rangle \langle \phi_m, \phi_n(\alpha) \rangle \\
&= \sum_m \frac{(-1)^{2(m+n)} \sin^2(\pi\alpha)}{(t_n(\alpha) - t_m)^2} |f_n(\alpha)|^2 |f_m|^2 \\
&= \sin^2(\pi\alpha) |f_n(\alpha)|^2 \sum_m \frac{|f_m|^2}{(t_n(\alpha) - t_m)^2}.
\end{aligned} \tag{3.39}$$

Hence, Eq. (3.38) and (3.39) give

$$\left(\sum_m \frac{|f_m|^2}{(t_n(\alpha) - t_m)^2} \right) \frac{dt_n(\alpha)}{d\alpha} = \frac{\pi}{\sin^2(\pi\alpha)}. \tag{3.40}$$

A further integration with respect to α would yield Eq. (3.13). However, this must be done carefully because of the divergence at $\alpha = 0$. It follows from Eq. (3.38) that $t'_n(\alpha)$ is always positive. So the eigenvalue $t_n(\alpha)$ always moves forward along the real line. Further because of the continuity of $t'_n(\alpha)$ and the fact that all $\{t_n(\alpha)\}_n$ cover the real line exactly

once, for any $\alpha \in (0, 1)$, there is one corresponding $t_n(\alpha)$ in each interval (t_n, t_{n+1}) for all n . Therefore, to integrate, one fixes an arbitrary n , and re-writes Eq. (3.40) as

$$\left(\frac{|f_n|^2}{(t_n(\alpha) - t_n)^2} + \sum_{m \neq n} \frac{|f_m|^2}{(t_n(\alpha) - t_m)^2} \right) \frac{dt_n(\alpha)}{d\alpha} = \frac{\pi}{\sin^2(\pi\alpha)}. \quad (3.41)$$

Under the assumption that the discrete spectrum has no accumulation points, there is a minimum spacing between two adjacent eigenvalues, say $\delta > 0$. Hence $|t_n - t_m| \geq |m - n| \delta$ for all $m \neq n$. Since $t_n(\alpha) - t_m = (t_n(\alpha) - t_n) + (t_n - t_m)$, the summation over the index m , $m \neq n$, converges asymptotically as $\sum_m \frac{1}{(m-n)^2}$. The series is absolutely convergent. Integrate with respect to α from a sufficiently small positive number ϵ to a real number $\alpha \in (0, 1)$, and change the variable on the left hand side, $t_n(\alpha) \in (t_n, t_{n+1})$, to give

$$\begin{aligned} -\frac{|f_n|^2}{t_n(\alpha) - t_n} + \frac{|f_n|^2}{t_n(\epsilon) - t_n} + \sum_{m \neq n} \left(-\frac{|f_m|^2}{t_n(\alpha) - t_m} + \frac{|f_m|^2}{t_n(\epsilon) - t_m} \right) \\ = -\cot(\pi\alpha) + \cot(\pi\epsilon). \end{aligned} \quad (3.42)$$

As ϵ goes to 0^+ , take the Laurent expansion of both sides. For ϵ sufficiently small, $t_n(\epsilon) - t_n \rightarrow \epsilon t'_n$ asymptotically. Since $t'_n = \pi |f_n|^2$ in Eq. (3.38), the second term $\frac{|f_n|^2}{t_n(\epsilon) - t_n}$ on the left hand side asymptotically goes to $\frac{\pi |f_n|^2}{\pi \epsilon t'_n} = \frac{1}{\pi \epsilon}$ as $\epsilon \rightarrow 0^+$. The other terms containing $t_n(\epsilon)$ on the left side in the summation behave well because as $\epsilon \rightarrow 0^+$, the limit $t_n - t_m$ is non-zero for $m \neq n$. On the right hand side, as $\epsilon \rightarrow 0^+$, $\cot(\pi\epsilon)$ asymptotically goes to $\frac{\cos(\pi\epsilon)}{\sin(\pi\epsilon)} = \frac{1}{\pi\epsilon}$. Cancelling out the simple pole $\frac{1}{\pi\epsilon}$ of the Laurent expansion on both sides as $\epsilon \rightarrow 0^+$ gives the following:

$$\frac{|f_n|^2}{t_n(\alpha) - t_n} + \sum_{m \neq n} \left(\frac{|f_m|^2}{t_n(\alpha) - t_m} - \frac{|f_m|^2}{t_n - t_m} \right) = \cot(\pi\alpha). \quad (3.43)$$

Hence

$$\frac{|f_n|^2}{t_n(\alpha) - t_n} - \sum_{m \neq n} \frac{|f_m|^2 (t_n(\alpha) - t_n)}{(t_n(\alpha) - t_m)(t_n - t_m)} = \cot(\pi\alpha). \quad (3.44)$$

Eq. (3.13) follows with $t = t_n(\alpha)$ and $t'_n(\alpha) = \pi |f_n(\alpha)|$.

To show Eq. (3.14), one first uses Eq. (3.39) with $k = n$ to obtain an expression of the $\sin(\pi\alpha) |f_k(\alpha)|$ -term:

$$\sin(\pi\alpha) |f_k(\alpha)| = \left(\sum_m \frac{|f_m|^2}{(t_k(\alpha) - t_m)^2} \right)^{-1/2}. \quad (3.45)$$

Here the sign is always non-negative because $\sin(\pi\alpha) \geq 0$ for all $0 \leq \alpha < 1$. Substitute this into Eq. (3.36) for the inner product of $\phi_k(\alpha)$ and ϕ_n :

$$\begin{aligned} \langle \phi_k(\alpha), \phi_n \rangle &= \frac{(-1)^{k+n} |f_n|}{t_k(\alpha) - t_n} \left(\sin(\pi\alpha) |f_k(\alpha)| \right) \\ &= \frac{(-1)^{k+n} |f_n|}{t_k(\alpha) - t_n} \left(\sum_m \frac{|f_m|^2}{(t_k(\alpha) - t_m)^2} \right)^{-1/2} \\ &= \frac{(-1)^{k+n} \sqrt{t'_n}}{t_k(\alpha) - t_n} \left(\sum_m \frac{t'_m}{(t_k(\alpha) - t_m)^2} \right)^{-1/2} \text{ by Eq. (3.38)}. \end{aligned} \quad (3.46)$$

The sign of the expression depends on two factors: $(-1)^{k+n}$ and $t_n(\alpha) - t_n$. The sign of $(-1)^{k+n}$ is the same as $(-1)^{k-n}$ or $(-1)^{n-k}$. Let $z(t, \hat{t})$ denote the number of the points $\{t_n\}_n$ between t and \hat{t} exclusively. Taking two cases:

$t_k(\alpha) > t_n$: There are exactly $n - k$ points between t_n and $t_k(\alpha)$ excluding the end point t_n . The overall sign is $(-1)^{k+n} = (-1)^{k-n} = (-1)^{z(t_k(\alpha), t_n)}$;

$t_k(\alpha) < t_n$: There are exactly $n - k - 1$ points in $(t_n(\alpha), t_k)$, but notice the sign of $t_n(\alpha) - t_n$ is now negative. The overall sign is still $-(-1)^{k+n} = (-1)^{n-k-1} = (-1)^{z(t_k(\alpha), t_n)}$.

In either case, the sign of the expression agrees with $(-1)^{z(t_k(\alpha), t_n)}$. Hence,

$$\langle \phi_k(\alpha), \phi_n \rangle = (-1)^{z(t_k(\alpha), t_n)} \frac{\sqrt{t'_n}}{|t_k(\alpha) - t_n|} \left(\sum_m \frac{t'_m}{(t_k(\alpha) - t_m)^2} \right)^{-1/2}. \quad (3.47)$$

Replacing $t_k(\alpha)$ by t and its eigenvector $\phi_k(\alpha)$ by ϕ_t gives Eq. (3.14). To calculate the inner product of two eigenvectors ϕ_t and $\phi_{\hat{t}}$ of any eigenvalues t and \hat{t} , one expands in the given eigenbasis $\{\phi_n\}_n$ and uses Eq. (3.14):

$$\begin{aligned} \langle \phi_t, \phi_{\hat{t}} \rangle &= \sum_n \langle \phi_t, \phi_n \rangle \langle \phi_n, \phi_{\hat{t}} \rangle \\ &= \sum_n (-1)^{z(t, t_n) + z(\hat{t}, t_n)} \frac{|t'_n|}{|t - t_n| |\hat{t} - t_n|} \\ &\quad \left(\sum_m \frac{t'_m}{(t - t_m)^2} \right)^{-1/2} \left(\sum_m \frac{t'_m}{(\hat{t} - t_m)^2} \right)^{-1/2}. \end{aligned} \quad (3.48)$$

To determine the sign of n -th term in Eq. (3.48), one takes two cases again:

$(t - t_n)(\hat{t} - t_n) > 0$: when points t and \hat{t} are on the same side of t_n , then the number of points of $\{t_m\}_m$ between t and \hat{t} is the difference of the number of points between t to t_n and \hat{t} to t_n . Hence $z(t, \hat{t}) = z(t, t_n) - z(\hat{t}, t_n)$, and the overall sign is $(-1)^{z(t, \hat{t})} = (-1)^{z(t, t_n) - z(\hat{t}, t_n)} = (-1)^{z(t, t_n) + z(\hat{t}, t_n)}$. The sign of $(t - t_n)(\hat{t} - t_n)$ is the same as $|t - t_n| |\hat{t} - t_n|$

$(t - t_n)(\hat{t} - t_n) < 0$: when points t and \hat{t} are on the opposite sides of t_n , then the number of points of $\{t_m\}_m$ between t and \hat{t} is the sum of $z(t, t_n)$ and $z(\hat{t}, t_n)$ plus the excluded point t_n . Hence the overall sign is $(-1)^{z(t, t_n) + z(\hat{t}, t_n)} = -(-1)^{z(t, \hat{t})}$. The minus sign at the front cancels out the negative sign of $(t - t_n)(\hat{t} - t_n)$.

Therefore, in Eq. (3.48), the sign of the n -th term is determined by the following:

$$\frac{(-1)^{z(t, t_n) + z(\hat{t}, t_n)}}{|t - t_n| |\hat{t} - t_n|} = \frac{(-1)^{z(t, \hat{t})}}{(t - t_n)(\hat{t} - t_n)}. \quad (3.49)$$

Substituting this into Eq. (3.48) gives Eq. (3.15). This completes the proof. \square

3.3 The Example of the Derivative Operator on a Finite Interval

In this section, we will discuss a concrete example of the $U(1)$ -family of self-adjoint extensions of a simple symmetric operator with deficiency indices $(1, 1)$, namely the differential operator on a finite interval with vanishing and periodic boundary conditions. Here, we will not focus on the proof of the example itself, because this classical example has been widely used in the textbooks on the topic of self-adjoint extensions of symmetric operators, but show its relation to the general results established in this chapter.

Specifically, the $U(1)$ -family of self-adjoint extensions in the example of differential operators are conventionally parameterized in terms of the parameter in the periodic boundary conditions: the phase difference between the function values at the two end points has the same form of a parametrization of a $U(1)$ -group, namely, $e^{i2\pi\theta}$, with $0 \leq \theta < 1$. However, it is important to emphasize that the parameter θ is not the same as the parameter α which arises in the $U(1)$ -group between deficiency spaces when using the Cayley transform. Here we will clarify the relation between the parameters θ and α , namely, we will clarify how the conventional boundary condition parametrization of self-adjoint extensions in this particular example relates to the underlying general parametrization through the Cayley transform.

Another reason to look into this example in detail is because this example directly relates to the mathematical foundation of the classical sampling theorem. Note that the

notation of functions in the $L^2([a, b])$ -Hilbert space will be changed to $\Phi(\omega)$ or $\Psi(\omega)$. The frequency variable ω is adopted here because the results will be used later to prove the classical sampling theorem in frequency space by Fourier transform.

The Classical Example of a Differential Operator on a Finite Interval

Example 8: Consider the Hilbert space of $L^2[a, b]$ in Example 5 on a finite interval $[a, b]$. Let $AC[a, b]$ be the set of absolute continuous functions on $[a, b]$, which is dense in $L^2[a, b]$. The set $AC[a, b]$ is the largest set on which the action of differentiation can be defined. Further, the image of the vector in $L^2[a, b]$ under the action of differentiation must also be in the same Hilbert space, namely, $\Phi'(\omega) = \frac{d}{d\omega}\Phi(\omega)$ must be in $L^2[a, b]$.

Let T^* , T , and $T[\theta]$, $0 \leq \theta < 2\pi$ denote the differential operators with the same action $i \frac{d}{d\omega}$, but on different domains:

$$D(T^*) = \{ \Phi(\omega) \in AC[a, b] \mid \Phi'(\omega) \in L^2[a, b] \}, \quad (3.50)$$

$$D(T) = \{ \Phi(\omega) \in AC[a, b] \mid \Phi'(\omega) \in L^2[a, b], \Phi(a) = 0 = \Phi(b) \}, \quad (3.51)$$

$$D(T[\theta]) = \{ \Phi(\omega) \in AC[a, b] \mid \Phi'(\omega) \in L^2[a, b], \Phi(a) = e^{i2\pi\theta} \Phi(b) \}. \quad (3.52)$$

The functions in $D(T^*)$ have arbitrary boundary condition, the ones in $D(T)$ vanish at the two end points, and the functions in $D(T[\theta])$ are subject to a periodic boundary condition up to a phase $e^{i2\pi\theta}$, where $0 \leq \theta < 1$. The parameter θ is used in a square bracket just to distinguish from the parametrization of $T(\alpha)$ which arises from the Cayley transform. Clearly,

$$D(T) \subsetneq D(T[\theta]) \subsetneq D(T^*).$$

From Example 2 and 5, we know that the operator T is a simple symmetric operator with deficiency indices(1, 1), and T^* is the adjoint of T .

Further, each operator $T[\theta]$ is a self-adjoint extension of T . To see these, write out the definition of the adjoint Eq. (2.6) explicitly for any $\Phi(\omega)$ and $\Psi(\omega)$ in $D(T^*)$, an integration by parts gives the following:

$$\begin{aligned} \left\langle i \frac{d}{d\omega} \Phi(\omega), \Psi(\omega) \right\rangle &= \int_a^b i \Phi'(\omega) \Psi^*(\omega) d\omega \\ &= i \Phi(\omega) \Psi^*(\omega) \Big|_{\omega=a}^b + \int_a^b \Phi(\omega) (i \Psi'(\omega))^* d\omega \\ &= i \left(\underline{\Phi(b) \Psi^*(b) - \Phi(a) \Psi^*(a)} \right) + \left\langle \Phi(\omega), i \frac{d}{d\omega} \Psi(\omega) \right\rangle. \end{aligned} \quad (3.53)$$

Hence

$$\left\langle i \frac{d}{d\omega} \Phi(\omega), \Psi(\omega) \right\rangle = \left\langle \Phi(\omega), i \frac{d}{d\omega} \Psi(\omega) \right\rangle$$

if and only if the underlined part equals zero. If $\Phi(\omega)$ vanishes at the two end points, then it is always true, namely,

$$\langle T \Phi(\omega), \Psi(\omega) \rangle = \langle \Phi(\omega), T^* \Psi(\omega) \rangle.$$

If both $\Phi(\omega)$ and $\Psi(\omega)$ have the same periodic boundary condition, i.e., $\Phi(\omega), \Psi(\omega) \in D(T[\theta])$, then

$$\langle T[\theta] \Phi(\omega), \Psi(\omega) \rangle = \langle \Phi(\omega), T[\theta] \Psi(\omega) \rangle$$

because the underlined part equals

$$\Phi(b)\Psi^*(b) - \Phi(a)\Psi^*(a) = \Phi(b)\Psi^*(b) - (e^{i\theta}\Phi(b)) (e^{i\theta}\Psi(b))^* = 0. \quad (3.54)$$

To obtain the eigenvalues $\{t_n[\theta]\}_n$ of each self-adjoint operator $T[\theta]$, one solves the following equation for λ :

$$T[\theta] \Phi(\omega) = i \frac{d}{d\omega} \Phi(\omega) = \lambda \Phi(\omega). \quad (3.55)$$

It gives

$$\Phi(\omega) = C e^{-i\lambda\omega} = \frac{e^{-i\lambda\omega}}{\sqrt{b-a}}, \quad (3.56)$$

where the pre-factor $C = \frac{1}{\sqrt{b-a}}$ is the normalization constant. The boundary condition of $\Phi(\omega)$ in $D(T[\theta])$ gives:

$$\begin{aligned} C e^{-i\lambda a} = \Phi(a) &= e^{i2\pi\theta} \Phi(b) = e^{i2\pi\theta} C e^{-i\lambda b} \\ \implies e^{-i\lambda a} e^{i\lambda b} &= e^{i2\pi\theta} \\ \implies \lambda (b-a) &= 2\pi\theta + 2n\pi = 2\pi(n+\theta). \end{aligned} \quad (3.57)$$

The eigenvalues of each self-adjoint operator $T[\theta]$ are

$$t_n[\theta] = \frac{\theta + 2n\pi}{b-a} \quad \forall n \in \mathbb{Z}, 0 \leq \theta < 2\pi. \quad (3.58)$$

Substituting the eigenvalue into Eq. (3.56) gives the corresponding eigenvector¹

$$\Phi_n^{[\theta]}(\omega) = \frac{e^{-it_n[\theta]\omega}}{\sqrt{b-a}} = \frac{e^{-i\frac{2\pi(n+\theta)\omega}{b-a}}}{\sqrt{b-a}}, \quad a \leq \omega \leq b. \quad (3.59)$$

Each eigenvalue $t_n[\theta]$ is of multiplicity 1.

¹See Chapter 4 for the corresponding representations in time domain.

This $U(1)$ -family of self-adjoint extensions $\{T[\theta]\}_n$ agrees with the fact that the deficiency indices of the simple symmetric operator is $(1, 1)$. In Example 5, we saw that the deficiency spaces of T are spanned by the deficiency vectors Φ_+ and Φ_- :

$$K_{\pm} = K(T^* \mp i) = \text{span} \{\Phi_{\pm}(\omega)\}.$$

Solving for the deficiency vectors, we have

$$\left(i \frac{d}{d\omega} \mp i\right) \Phi_{\pm}(\omega) = 0 \implies \frac{d}{d\omega} \Phi_{\pm}(\omega) = \pm \Phi_{\pm}(\omega). \quad (3.60)$$

To have

$$\begin{aligned} \Phi_+(\omega) &= +C_+ e^{+\omega}, & C_+ &= \sqrt{2/(e^{2b} - e^{2a})}, \\ \Phi_-(\omega) &= -C_- e^{-\omega}, & C_- &= \sqrt{2/(e^{-2a} - e^{-2b})}. \end{aligned} \quad (3.61)$$

where C_+ and C_- are the normalization constants. Note that $C_- = e^{a+b} C_+$.

It is important to point out that the second deficiency vectors $\Phi_-(\omega)$ is chosen to be $-C_- e^{-\omega}$ rather than the positive $C_- e^{-\omega}$ in Example 5. This is because of the consistency with Eq. (3.2), namely,

$$U(0)\Phi_+(\omega) = \Phi_-(\omega). \quad (3.62)$$

Notice that the phase of the second deficiency vector $\Phi_-(\omega)$ is free to be chosen. To obey Eq. (3.62), choose the phase of the second deficiency vector $\Phi_-(\omega)$ to be of the general form of $e^{i\gamma}$. Set $\Phi_-(\omega) = e^{i\gamma} e^{-\omega}$ temporarily. Due to Eq. (2.23), there exist a $\Phi(\omega) \in D(T(0))$ with $\Phi(a) = \Phi(b)$ so that

$$\begin{aligned} (T + i)\Phi(\omega) &= \Phi_+(\omega) \text{ and} \\ (T - i)\Phi(\omega) &= U(0)((T + i)\Phi(\omega)) = U(0)\Phi_+(\omega) = \Phi_-(\omega). \end{aligned}$$

Namely,

$$\begin{aligned} (T + i)\Phi(\omega) &= i\left(\frac{d}{d\omega} + 1\right)\Phi(\omega) = \Phi_+(\omega) = C_+ e^{\omega}, \\ (T - i)\Phi(\omega) &= i\left(\frac{d}{d\omega} - 1\right)\Phi(\omega) = \Phi_-(\omega) = e^{a+b} C_+ (e^{i\gamma} e^{-\omega}). \end{aligned}$$

Subtracting these two equation gives

$$\Phi(\omega) = \frac{C_+}{2i} (e^{\omega} - e^{i\gamma} e^{a+b} e^{-\omega}). \quad (3.63)$$

Since $\Phi(\omega)$ is in the domain of $T[\theta = 0]$, it must obey the boundary condition $\Phi(a) = \Phi(b)$. Substituting $\omega = a$ and $\omega = b$ into Eq. (8), one has

$$\begin{aligned} \frac{C_+}{2i} (e^a - e^{i\gamma} e^b) &= \Phi(a) = \Phi(b) = \frac{C_+}{2i} (e^b - e^{i\gamma} e^a), \\ e^a - e^b &= e^{i\gamma} (e^b - e^a). \end{aligned}$$

Hence $e^{i\gamma} = -1$. The choice of the deficiency vectors is important later in the functional analytical proof of the classical sampling theorem. \square

The Relation to the General α -parametrization of Self-Adjoint Extensions

From Section 3.2, one knows that a general $(1, 1)$ -symmetric operator T has a $U(1)$ -family of self-adjoint extensions $\{T(\alpha) \mid 0 \leq \alpha < 1\}$, where the parameter α arises from the $U(1)$ -map $e^{i\alpha}$ between the deficiency spaces in the Cayley transform. In this particular example of differential operator T , the set of its self-adjoint extensions $\{T[\theta] \mid 0 \leq \theta < 1\}$ are enumerated by the boundary condition parameter θ . These two parameterizations are different. Here we will show a one-to-one map between the two parameters.

The α -parametrization is for general simple symmetric operators with deficiency indices $(1, 1)$, and the θ -parametrization is specified for the differential operator. Although $T(\alpha) \neq T[\theta]$ in general for $\alpha = \theta$, one expects a one-to-one mapping between α and θ because the sets $\{T[\theta] \mid 0 \leq \theta < 1\}$ and $\{T(\alpha) \mid 0 \leq \alpha < 1\}$ describe the same family of self-adjoint extensions of T . More specifically, the θ -parametrization is simply a re-parameterization of the general α -parameterization. Therefore, there should exist a strictly monotonic function $\theta = \theta(\alpha)$ from $[0, 1)$ onto $[0, 1)$ such that

$$T(\alpha) = T[\theta(\alpha)] \quad \forall 0 < \alpha < 1. \quad (3.64)$$

To find the one-to-one correspondence between α and θ , one uses the von Neumann formulas. For a fixed α , suppose that functions in the domain of $T(\alpha)$ are subject to the boundary condition $\Phi(a) = e^{i2\pi\theta} \Phi(b)$ for some θ between 0 and 1. Eq. (3.2), Eq. (3.60) and the von Neumann formulae Eq. (2.38) give that, for any $\Phi(\omega) \in D(T(\alpha))$,

$$\begin{aligned} \Phi(\omega) &= \Phi_0(\omega) + \mu\Phi_+(\omega) - S_d(\mu\Phi_+(\omega)) \\ &= \Phi_0(\omega) + \mu\Phi_+(\omega) - \mu e^{i2\pi\alpha} \Phi_-(\omega) \\ &= \Phi_0(\omega) + \mu C_+ e^\omega + \mu e^{i2\pi\alpha} C_- e^{-\omega} \\ &= \Phi_0(\omega) + \mu C_+ \left(e^\omega + e^{i2\pi\alpha} e^{a+b} e^{-\omega} \right). \end{aligned} \quad (3.65)$$

where $\Phi_0(\omega)$ is in $D(T)$ with vanishing boundary condition and μ is the coefficient of the deficiency vector Φ_+ in the deficiency space K_+ . Evaluate the above expression at the two end points to give

$$\Phi(a) = \mu C_+ \left(e^a + e^{i2\pi\alpha} e^b \right), \quad (3.66)$$

$$\Phi(b) = \mu C_+ \left(e^b + e^{i2\pi\alpha} e^a \right). \quad (3.67)$$

Substituting these into the boundary condition $\Phi(a) = e^{i2\pi\theta}\Phi(b)$, one has

$$e^{i2\pi\theta} = \frac{\Phi(a)}{\Phi(b)} = \frac{e^a + e^{i2\pi\alpha} e^b}{e^b + e^{i2\pi\alpha} e^a}. \quad (3.68)$$

This equation defines a strictly increasing mapping $\theta = \theta(\alpha)$ from $[0, 1)$ onto $[0, 1)$. As α goes from 0 to 1, θ increases from 0 to 1 as well.

Chapter 4

The Classical Sampling Theory

This chapter will give a brief review of the classical sampling theorem of Whittaker-Shannon-Kotel'nikov, followed by a new proof from a functional analytical perspective.

The classical sampling theorem, which is commonly known as the sampling theorem of Whittaker-Shannon-Kotel'nikov (WSK), was originally discovered by E.T. Whittaker and J.M. Whittaker on their study of cardinal series in 1929 [3, 4]. Hence, it is also called the Cardinal Theorem of Interpolation. In the Russian literature, it was also independently studied by Kotel'nikov in 1933 [6]. Some people even think that it goes back to Cauchy [52]. See Chapter 1 of [7] for a great historical review.

Its practical significance was realized by Shannon in his celebrated paper [1] in 1949, which sets the foundation of modern information theory. Therefore, the classical sampling theorem of WSK is well-known as the Shannon Sampling Theorem in communication engineering and information theory. This name will be frequently used in this thesis when we deal with engineering applications as well. Its significance lies in the fact that it provides an equivalence between discrete and continuous representations of information.

Since Shannon's introduction, the classical sampling theorem is widely used in communication engineering and signal processing. It is involved in any analogic-numerical conversion process. Indeed, it is employed ubiquitously in modern engineering applications including CD players and cell phones.

4.1 The Sampling Theorem of Whittaker-Shannon-Kotel'nikov

Definition 18. A function $\phi(t)$ in $L^2(\mathbb{R})$ is said to be Ω -**bandlimited** if and only if its Fourier transform $\Phi(\omega)$ vanishes outside the closed interval $[-2\pi\Omega, 2\pi\Omega]$, namely,

$$\Phi(\omega) = \int_{-\infty}^{+\infty} \phi(t) e^{-i\omega t} dt = 0 \quad \forall \omega \in [-2\pi\Omega, 2\pi\Omega]. \quad (4.1)$$

The scalar number Ω is referred to as the **bandlimit**, and twice of it, 2Ω , is referred to as the **bandwidth** of the function $\phi(t)$.

Theorem 9 (The Theorem of Whittaker-Shannon-Kotel'nikov). A Ω -bandlimited function $\phi(t)$ can be completely determined for all $t \in \mathbb{R}$ from its sample values on a discrete sequence of equidistantly spaced sampling points $\{t_n\}_n$ with $t_{n+1} - t_n = 1/(2\Omega)$ via the following reconstruction formula

$$\phi(t) = \sum_{n=-\infty}^{\infty} G(t, t_n) \phi(t_n). \quad (4.2)$$

The function $G(t, t_n)$ is called the reconstruction kernel and it is the scaled sinc function

$$G(t, t_n) = \text{sinc}(2\Omega(t - t_n)) = \frac{\sin(2\pi\Omega(t - t_n))}{2\pi\Omega(t - t_n)}. \quad (4.3)$$

Definition 19. The constant sampling rate at 2Ω is usually referred to as the **Nyquist sampling rate**.

Definition 20. The set of sampling points at the Nyquist rate will be referred to as the sampling grid at the Nyquist rate or **Nyquist sampling grid** or just simply sampling grid.

Note. The Shannon sampling theorem does not specify the location of the sampling points, but the spacing between two adjacent points in a Nyquist sampling grid must be $1/(2\Omega)$. Therefore, all possible sampling grids can be parameterized by a linear shifting parameter θ , where $0 \leq \theta < 1$, as

$$t_n[\theta] = \frac{n + \theta}{2\Omega}. \quad (4.4)$$

Any set of points obeying the Nyquist rate must be one of the sampling grid $\{t_n[\theta]\}_n$ for some unique θ between 0 and 1. Hence, for an arbitrary fixed θ , the set $\{t_n[\theta]\}_n$ is a Nyquist sampling grid on which the Ω -bandlimited functions can be perfectly reconstructed from its values. Conversely, for any real number t , there exists a unique pair of n and θ such that $t = t_n[\theta]$.

Note. The definition of bandlimited functions are sometimes given for L^1 -functions. Here we restrict the function space to be the Hilbert space of L^2 -functions. The generalization by the functional analytical theory of self-adjoint operators is for Hilbert space. Further, most practical engineering applications deal with signals with finite energy, which are L^2 -functions.

In the remainder of this chapter, one will see the conventional proof of the Shannon sampling theorem by Fourier analysis, as well as a new proof by functional analysis.

4.2 An Elementary Proof by Fourier Analysis

The classical sampling theorem is conventionally derived using Fourier series. Its proof can be found in most standard textbooks on sampling theory, e.g., [9, 7, 8]. However, most of them only prove the sampling theorem on one set of Nyquist sampling points $\{\frac{n}{2\Omega}\}_n$, namely, the sampling grid in Eq. (4.4) with $\theta = 0$. Here we will show a comprehensive proof of the classical sampling theorem on all possible Nyquist sampling grids, $\{t_n[\theta]\}_n$ for all $0 \leq \theta < 1$.

First, let us set the Fourier convention we shall adopt in this thesis and review a few important results from Fourier analysis that we shall use later. The Fourier transform used in this thesis is in term of angular frequency ω in the non-unitary form. The unit of ω is radians per second.

Definition 21 (Fourier transform). *The **Fourier transform** of a function $\phi(t)$ in time t , denoted by $\Phi(\omega)$, is*

$$\Phi(\omega) = \mathfrak{F}(\phi(t)) = \int_{t=-\infty}^{+\infty} \phi(t) e^{-i\omega t} dt. \quad (4.5)$$

The inverse Fourier transform is given by:

$$\phi(t) = \mathfrak{F}^{-1}(\Phi(\omega)) = \frac{1}{2\pi} \int_{\omega=-\infty}^{+\infty} \Phi(\omega) e^{i\omega t} d\omega. \quad (4.6)$$

Let $\phi(t) \rightleftharpoons \Phi(\omega)$ denote a function in the time domain and its Fourier transform in the frequency domain. Here are a few important functional relationships:

$$\phi(t - a) \rightleftharpoons e^{-ia\omega} \Phi(\omega), \quad (4.7)$$

$$|a| \phi(at) \rightleftharpoons \Phi\left(\frac{\omega}{a}\right), \quad (4.8)$$

$$t \phi(t) \rightleftharpoons i \frac{d}{d\omega} \Phi(\omega), \quad (4.9)$$

$$\phi(t) * \psi(t) \rightleftharpoons \Phi(\omega) \Psi(\omega). \quad (4.10)$$

The notation $\phi * \psi$ denotes the convolution of $\phi(t)$ and $\psi(t)$:

$$(\phi * \psi)(t) = \int_{s=-\infty}^{\infty} \phi(t-s)\psi(s) ds.$$

Let $\text{rect}(\omega)$ denote the rectangular function which is 1 on the interval $[-\frac{1}{2}, \frac{1}{2}]$ and zero elsewhere. The scaled rectangular function $\text{rect}(\frac{\omega}{2\pi})$ is the Fourier transform of the sinc function $\text{sinc}(t)$ in time domain. To see this, take the inverse Fourier transform of $\text{rect}(\omega)$ to get

$$\begin{aligned} \mathfrak{F}^{-1}\left(\text{rect}(\omega)\right) &= \frac{1}{2\pi} \int_{\omega=-\infty}^{+\infty} \text{rect}\left(\frac{\omega}{2\pi}\right) e^{i\omega t} d\omega \quad \text{let } u = \frac{\omega}{2\pi} \\ &= \frac{1}{2\pi} \int_{u=-\infty}^{+\infty} \text{rect}(u) e^{i2\pi ut} 2\pi du = \int_{u=-1/2}^{+1/2} 1 e^{i2\pi ut} du \\ &= \frac{1}{i2\pi t} e^{i2\pi ut} \Big|_{u=-1/2}^{1/2} = \frac{1}{i2\pi t} \left(e^{i\pi t} - e^{-i\pi t} \right) \\ &= \frac{1}{i2\pi t} \left(2i \sin(\pi t) \right) = \frac{\sin(\pi t)}{\pi t} = \text{sinc}(t). \end{aligned}$$

Using Eq. (4.8), we have the following important Fourier pair

$$|a| \text{sinc } at \Leftrightarrow \text{rect}\left(\frac{\omega}{2\pi a}\right). \quad (4.11)$$

Definition 22 (Fourier Series). For a periodic function $f(x)$ with period T , $f(x)$ can be expressed as a **Fourier series**,

$$f(x) = \sum_{m=-\infty}^{\infty} c_m e^{i2\pi mx/T}. \quad (4.12)$$

where

$$c_m = \frac{1}{T} \int_T f(x) e^{-i2\pi mx/T} dx. \quad (4.13)$$

The integration is over any one single period of $f(x)$ with length T .

Proof of the Classical Sampling Theorem by Fourier Analysis.

Let $\phi(t)$ be an arbitrary Ω -bandlimited function and $\Phi(\omega)$ be its Fourier transform. In the frequency domain, $\Phi(\omega)$ has a finite support on the closed interval $[-2\pi\Omega, 2\pi\Omega]$.

For any fixed θ between 0 and 1, let $\Phi_T(\omega)$ denote a periodic extension of $\Phi(\omega) e^{i\frac{\theta}{2\Omega}\omega}$ in the frequency domain with a period $T = 4\pi\Omega$, i.e.,

$$\Phi_T(\omega) = \begin{cases} \Phi(\omega) e^{i\frac{\theta}{2\Omega}\omega} & \forall \omega \in [-2\pi\Omega, 2\pi\Omega], \\ \Phi_T(\omega + kT) & \forall \omega \notin [-2\pi\Omega, 2\pi\Omega], k \in \mathbb{Z}. \end{cases} \quad (4.14)$$

A conventional approach would simply extend $\Phi(\omega)$, but it would only prove the sampling theorem for the Nyquist sampling grid $\{t_n[0] = \frac{n}{2\Omega}\}_n$ with the shifting parameter $\theta = 0$. To cover all the possible sampling grids $\{t_n[\theta]\}_n$ in Eq. (4.4) for all $0 \leq \theta < 1$, we need to take the $4\pi\Omega$ -extension of $\Phi(\omega) e^{i\frac{\theta}{2\Omega}\omega}$ instead.

The function $\Phi(\omega)$ is a single period of $\Phi_T(\omega)$ up to a phase. Hence

$$\Phi(\omega) = \Phi_T(\omega) e^{-i\frac{\theta}{2\Omega}\omega} \operatorname{rect}\left(\frac{\omega}{4\pi\Omega}\right). \quad (4.15)$$

Since $\Phi_T(\omega)$ is periodic with $T = 4\pi\Omega$, it can be expressed as a Fourier series

$$\Phi_T(\omega) = \sum_{n=-\infty}^{\infty} c_n e^{-i2\pi n\omega/T} = \sum_{n=-\infty}^{\infty} c_n e^{-i\frac{n}{2\Omega}\omega}. \quad (4.16)$$

The summation index in the Fourier series is replaced by its negative, i.e., $n = -m$, since the periodic function is in the frequency domain rather than the time domain. Hence the sign in the exponential $e^{i2\pi mx/T}$ in the Fourier series in Eq. (4.12) and Fourier coefficients Eq. (4.13) are switched, but it does not change the results of the Fourier series. The Fourier coefficient of $\Phi_T(\omega)$ then reads:

$$\begin{aligned} c_n &= \frac{1}{T} \int_{\omega=-2\pi\Omega}^{2\pi\Omega} \Phi_T(\omega) e^{i2\pi n\omega/T} d\omega \\ &= \frac{1}{4\pi\Omega} \int_{\omega=-2\pi\Omega}^{2\pi\Omega} \left(\Phi(\omega) e^{i\frac{\theta}{2\Omega}\omega} \right) e^{i\frac{n}{2\Omega}\omega} d\omega \quad \text{by Eq. (4.14)} \\ &= \frac{1}{4\pi\Omega} \int_{\omega=-\infty}^{+\infty} \Phi(\omega) e^{i\frac{n+\theta}{2\Omega}\omega} d\omega \quad \text{since } \Phi(\omega) = 0 \text{ outside } [-2\pi\Omega, 2\pi\Omega] \\ &= \frac{1}{2\Omega} \frac{1}{2\pi} \int_{\omega=-\infty}^{+\infty} \Phi(\omega) e^{it_n[\theta]\omega} d\omega = \frac{1}{2\Omega} \phi(t_n[\theta]). \end{aligned} \quad (4.17)$$

The Fourier coefficients of the periodic extension $\Phi_T(\omega)$ in the frequency domain turns out to be the sampling values on a Nyquist sampling grid. The last step is the inverse Fourier transform of $\phi(t)$ at $t = t_n[\theta] = \frac{n+\theta}{2\Omega}$.

Substitute Eq. (4.16) into Eq. (4.15)

$$\begin{aligned}
\Phi(\omega) &= \Phi_T(\omega) e^{-i\frac{\theta}{2\Omega}\omega} \operatorname{rect}\left(\frac{\omega}{4\pi\Omega}\right) \\
&= \left(\sum_{n=-\infty}^{\infty} c_n e^{-i\frac{n}{2\Omega}\omega} \right) e^{-i\frac{\theta}{2\Omega}\omega} \operatorname{rect}\left(\frac{\omega}{4\pi\Omega}\right) \\
&= \sum_{n=-\infty}^{\infty} c_n e^{-i\frac{(n+\theta)}{2\Omega}\omega} \operatorname{rect}\left(\frac{\omega}{4\pi\Omega}\right) \\
&= \sum_{n=-\infty}^{\infty} c_n e^{-it_n[\theta]\omega} \operatorname{rect}\left(\frac{\omega}{4\pi\Omega}\right).
\end{aligned} \tag{4.18}$$

This is an expansion of $\Phi(\omega)$ in the Hilbert space $L^2([-2\pi\Omega, 2\pi\Omega])$. The inverse Fourier transform of Eq. (4.18) gives an expression of the bandlimited function $\phi(t)$ in terms of its sample values

$$\begin{aligned}
\phi(t) &= \mathfrak{F}^{-1}(\Phi(\omega)) = \sum_{n=-\infty}^{\infty} c_n \mathfrak{F}^{-1}\left(e^{-it_n[\theta]\omega} \operatorname{rect}\left(\frac{\omega}{4\pi\Omega}\right)\right) \\
&= \sum_{n=-\infty}^{\infty} c_n 2\Omega \operatorname{sinc}(2\Omega(t - t_n[\theta])) \quad \text{by Eq. (4.7) and (4.11)} \\
&= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2\Omega} \phi(t_n[\theta]) \right) 2\Omega \operatorname{sinc}(2\Omega(t - t_n[\theta])) \quad \text{by Eq. (4.17)} \\
&= \sum_{n=-\infty}^{\infty} \operatorname{sinc}(2\Omega(t - t_n[\theta])) \phi(t_n[\theta]).
\end{aligned} \tag{4.19}$$

This is precisely the reconstruction formula in the classical sampling of Whittaker-Shannon-Kotel'nikov in Eq. (4.2) on a Nyquist sampling grid $\{t_n = t_n[\theta]\}_n$ for any fixed $0 \leq \theta < 1$. □

4.3 A New Proof by Functional Analysis

The underlying mathematics of the classical sampling theorem is also rooted in the spectral theory of self-adjoint extensions of symmetric operators in Hilbert spaces. The classical sampling theorem simply states a functional analytical fact: *if a Hilbert space vector is known in one eigenbasis of a self-adjoint extension, then its coefficients in the eigenbases of all other self-adjoint extensions are determined.*

The coefficients in the eigenbasis coincide with the sampling values of the function, and the Nyquist set of sampling points turns out to be the set of eigenvalues of one self-adjoint operator.

Proof of the Classical Sampling Theorem by Functional Analysis.

Although the proof by functional analysis does not necessarily involve Fourier analysis, it is still easier to understand the proof in frequency space. The Fourier transforms of Ω -bandlimited functions form the function space of $L^2[-2\pi\Omega, 2\pi\Omega]$, which is the function space we studied in Example 8 in Chapter 3 with the boundary points $a = -2\pi\Omega$ and $b = 2\pi\Omega$.

Recall that in Example 8, the differential operator $T = i\frac{d}{d\omega}$ acting on the set of absolutely continuous functions with vanishing boundary conditions is a simple symmetric operator with deficiency indices $(1, 1)$. The operator T has a $U(1)$ -family of self-adjoint extensions, which can be denoted by $\{T[\theta] \mid 0 \leq \theta < 1\}$. Functions in the domain of each self-adjoint extension $T[\theta]$ have a periodic boundary condition $\Phi(-2\pi\Omega) = e^{i2\pi\theta} \Phi(2\pi\Omega)$.

From Eq. (3.58), each self-adjoint extension $T[\theta]$ has a set of equidistantly spaced eigenvalues $\{t_n[\theta] = \frac{n+\theta}{2\Omega}\}_n$. This is precisely the set of Nyquist sampling points. Further, from Eq. (3.59), each eigenvalue $t_n[\theta]$ corresponds to one normalized eigenfunction

$$\Phi_n^{[\theta]}(\omega) = \frac{1}{\sqrt{4\pi\Omega}} e^{-it_n[\theta]\omega}, \quad -2\pi\Omega \leq \omega \leq 2\pi\Omega.$$

For a fixed θ , the set of eigenfunctions $\{\Phi_n^{[\theta]}(\omega)\}_n$ forms an orthonormal eigenbasis of $L^2[-2\pi\Omega, 2\pi\Omega]$. Any function $\Phi(\omega)$ can be expanded in this basis

$$\Phi(\omega) = \sum_{n=-\infty}^{+\infty} \langle \Phi(\omega), \Phi_n^{[\theta]}(\omega) \rangle \Phi_n^{[\theta]}(\omega). \quad (4.20)$$

Since $\Phi(\omega)$ vanishes outside $[-2\pi\Omega, 2\pi\Omega]$, each coefficient reads:

$$\begin{aligned} \langle \Phi(\omega), \Phi_n^{[\theta]}(\omega) \rangle &= \int_{\omega=-2\pi\Omega}^{2\pi\Omega} \Phi(\omega) \Phi_n^{[\theta]}(\omega)^* d\omega \\ &= \int_{\omega=-\infty}^{+\infty} \Phi(\omega) \Phi_n^{[\theta]}(\omega)^* d\omega \\ &= \frac{1}{\sqrt{4\pi\Omega}} \int_{\omega=-\infty}^{+\infty} \Phi(\omega) e^{it_n[\theta]\omega} d\omega \\ &= \frac{2\pi}{\sqrt{4\pi\Omega}} \phi(t_n[\theta]) \quad \text{by inverse Fourier transform.} \end{aligned} \quad (4.21)$$

Hence the coefficients of Fourier transform of an Ω -bandlimited function $\phi(t)$ in the eigenbasis coincide with the function values in time space.

The eigenfunction $\Phi_n^{[\theta]}(\omega)$ in Eq. (4.20) vanishes outside $[-2\pi\Omega, 2\pi\Omega]$. So when one takes its inverse Fourier transform, one needs to multiply it by $\text{rect}(\frac{\omega}{4\pi\Omega})$ first. The eigenfunction in the time domain reads:

$$\begin{aligned}\phi_n^{[\theta]}(t) &= \mathfrak{F}^{-1}\left(\Phi_n^{[\theta]}(\omega)\right) = \mathfrak{F}^{-1}\left(\frac{1}{\sqrt{4\pi\Omega}} e^{-it_n[\theta]\omega} \text{rect}\left(\frac{\omega}{4\pi\Omega}\right)\right) \\ &= \frac{1}{\sqrt{4\pi\Omega}} (2\Omega) \text{sinc}(2\Omega(t - t_n[\theta])).\end{aligned}\tag{4.22}$$

Finally, the inverse Fourier transform of Eq. (4.20) gives exactly the reconstruction formula in Eq. (4.2)

$$\begin{aligned}\phi(t) &= \sum_{n=-\infty}^{+\infty} \left(\frac{2\pi}{\sqrt{4\pi\Omega}}\phi(t_n[\theta])\right) \left(\frac{2\Omega}{\sqrt{4\pi\Omega}} \text{sinc}(2\Omega(t - t_n[\theta]))\right) \\ &= \sum_{n=-\infty}^{+\infty} \phi(t_n[\theta]) \text{sinc}(2\Omega(t - t_n[\theta])).\end{aligned}\tag{4.23}$$

Of course, this formula can also be obtained directly in the time domain without Fourier transform. In the time domain, the simple $(1, 1)$ -symmetric operator T becomes the multiplication operator $T\phi(t) = t\phi(t)$ for any Ω -bandlimited function. Each self-adjoint operator $T[\theta]$ has the same set of eigenvalues $\{t_n[\theta]\}_n$ as in the frequency space, but the corresponding eigenfunction in the time domain becomes the sinc function in Eq. (4.22). The coefficient of $\phi(t)$ in the eigenbasis $\{\phi_n^{[\theta]}\}_n$ is identical to the one obtained in frequency space in Eq. (4.21). Hence, the expansion of $\phi(t)$ in the eigenbasis gives the same result as in Eq. (4.23). \square

Note that the result here holds if one uses the general α -parameter, since in the particular case of the differential operator T in Example 8, the eigenvalues of each self-adjoint extension, represented by either $T[\alpha]$ or $T(\alpha)$, always obey the Nyquist spacing, i.e. $t_{n+1}(\alpha) - t_n(\alpha) = 1/(2\Omega)$.

The parameter θ arises from the periodic boundary condition of L^2 -functions on $[-2\pi\Omega, 2\pi\Omega]$. Hence, it is restricted to the Shannon sampling theorem. However, if we use the general parameter α which arises from the Cayley transform, we can generalize the Shannon sampling theorem. More importantly, in the generalization, we will work directly with the time domain.

Chapter 5

The Generalized Sampling Theory for Time-Varying Nyquist Rates

In this chapter, we further develop a generalized sampling method for time-varying Nyquist rates, which was first proposed by Kempf [28]. Mathematically, it is based on the functional analytical theory of self-adjoint extensions established in Chapter 3. This chapter is organized so that readers without specialized knowledge of functional analysis will understand and be able to use the generalized sampling method in practice. The mathematical proof is arranged to be at the end of the chapter in Section 5.5.

The classical Shannon sampling theory allows sampling and perfect reconstruction of bandlimited signals at a constant Nyquist rate. However, it is clear that the effective bandwidth, or information density of a signal could vary in time. To improve the sampling efficiency, we generalize the classical sampling method, which allows the samples to be taken only as often as necessary according to the behavior of the given signals, namely at a time-varying Nyquist rate, and maintains the ability to perfectly and stably reconstruct the continuous signals from their discrete values on the set of Nyquist sampling points.

First, we will summarize the generalized sampling method of filtering, sampling and reconstructing signals with a time-varying Nyquist rate. Section 5.2 presents the main results on sampling and reconstruction, which includes the calculation of the family of generally non-equidistant Nyquist sampling grids and the explicit formula of the corresponding reconstruction kernels. A precise definition of a time-varying Nyquist rate is given. In Section 5.3, we show that this time-varying rate is indeed the Nyquist rate as the critical sampling rate between over- and under-sampling. Section 5.4 illustrates how the Shannon sampling theorem arises as a special case of the generalized sampling theorem. The pre-filtering will be explored in Chapter 6. The work in this chapter is based on my papers [32, 37].

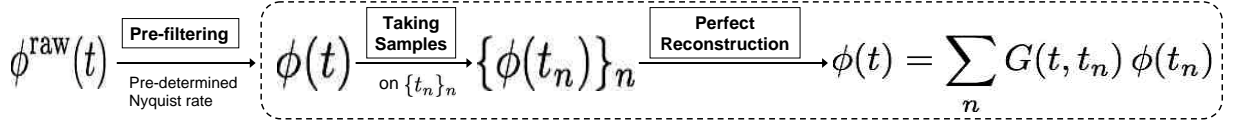


Figure 5.1: The General Scheme of a Sampling Method

5.1 An Overview of the Generalized Sampling Method

Figure 5.1 illustrates the general scheme of sampling theory, which usually consists of four steps. In terms of the Shannon sampling method, each step means the following:

- (S1) Pre-determine a bandlimit Ω , or equivalently, a Nyquist rate based on the frequency analysis of the raw signals of interest $\phi^{raw}(t)$.
- (S2) Filter $\phi^{raw}(t)$ to obtain a function $\phi(t)$ in the desired space of Ω -bandlimited functions. This filtering step can be mathematically represented as a linear operator P :

$$\phi(t) = (P\phi^{raw})(t) = \int_{-\infty}^{\infty} \phi^{raw}(\hat{t}) P(t, \hat{t}) \nu(\hat{t}) d\hat{t}. \quad (5.1)$$

- (S3) Store and/or transmit the samples $\{\phi(t_n)\}_n$ taken at the desired Nyquist rate.
- (S4) Reconstruct $\phi(t)$ for all time t from the discrete samples via the following reconstruction formula

$$\phi(t) = \sum_{n=-\infty}^{\infty} G(t, t_n) \phi(t_n) \quad (5.2)$$

where the function $G(t, t_n)$ is the so-called the reconstruction kernel.

In the Shannon sampling method, a given arbitrary raw signal $\phi^{raw}(t)$ is generally not Ω -bandlimited¹, so one needs to first pre-filter the signal with a bandlimit Ω . The bandlimit Ω is chosen in Step (S1) so that the frequencies of $\phi^{raw}(t)$ larger than Ω are negligible. In Step (S2), the filtered signal $\phi(t)$ gives a good approximation of the raw signal $\phi^{raw}(t)$. In the frequency domain, Shannon's low-pass filter P in Eq. (5.1) multiplies the Fourier transform of $\phi^{raw}(t)$ with the rectangular function which is 1 in $(-2\pi\Omega, 2\pi\Omega)$ and 0 elsewhere. In the time domain, it is equivalent to convolving $\phi^{raw}(t)$ with a sinc function. Hence, in Eq. (5.1), we have

$$P(t, \hat{t}) = \text{sinc}(2\Omega(t - \hat{t})) \quad \text{and} \quad \nu(t) = 2\Omega. \quad (5.3)$$

¹For example, a raw signal with a finite spatial support has no frequency upper bound.

Here $\nu(t)$ indicates the constant Nyquist rate. Details of the pre-filtering step can be found in Chapter 6.

The resulting signal $\phi(t)$ is then Ω -bandlimited. Therefore the sampling theorem, Step (S3) and (S4), in the dashed box in Figure 5.1, applies: the samples $\{\phi(t_n)\}_n$ are taken at the Nyquist rate, namely, on a set of points with an equidistant Nyquist spacing $t_{n+1} - t_n = 1/(2\Omega)$, the continuous Ω -bandlimited signal $\phi(t)$ is perfectly reconstructed for all time t from its discrete sample values $\{\phi(t_n)\}_{n=-\infty}^{\infty}$ via Eq. (5.2). Shannon's reconstruction kernel is the shifted sinc function

$$G(t, t_n) = \text{sinc}(2\Omega(t - t_n)). \quad (5.4)$$

In practice, sampling a signal at a constant rate is clearly not optimally efficient since an arbitrary signal may oscillate in effect in various frequency ranges during different periods of time. It is clear that parts of a realistic signal may possess a lower bandwidth. If one chooses a high constant bandlimit Ω in Step (S1), then sampling the signal at the rate 2Ω leads to wasteful redundancy.

Indeed, intuitively, within the limit of the time-frequency uncertainty relation, an arbitrary signal's 'effective bandwidth' or 'information density' can change in time. This suggests a possibility to improve sampling efficiency by adjusting the sampling rate according to the signal's time-varying effective bandwidth, taking samples of a signal only as frequently as necessary. A corresponding method for sampling and reconstruction at a 'time-varying bandwidth' is clearly desirable.

Therefore, the aim here is to generalize the Shannon sampling theorem, following the sampling scheme of Figure 5.1, but for time-varying bandwidths.

To this end, we note first that the bandwidth as a function of time is ill-defined in the framework of Fourier analysis, because the bandwidth of a signal, as the upper bound of the signal's Fourier transform, is simply time-independent. In principle, however, the Nyquist sampling rate, as the critical sampling rate below which there is insufficient information to recover the signal and above which redundancy exists, can vary in time. In the generalized sampling theory, the bandwidth will be interpreted as the inverse of the Nyquist rate, which can then be time-dependent, and the Fourier analysis is generalized to a new functional analytical method. The new sampling theory allows samples to be taken only as often as necessary at a time-varying Nyquist rate, while maintaining the ability to perfectly and stably reconstruct the signal.

In summary, the generalized sampling method generalizes the four-step sampling algorithm of Shannon:

- (G1) Analyze the frequency content of the raw signals of interest $\phi^{raw}(t)$ to choose a time-varying Nyquist rate, which is to be specified by an increasing and infinite set of sampling points $\{t_n\}_{n=-\infty}^{\infty}$.

- (G2) Filter $\phi^{raw}(t)$ to obtain a signal $\phi(t)$ in the function space which will be reconstructible at the pre-specified time-varying Nyquist rate. The filter operator P has exact the same form as in the case of Shannon in Eq. (5.1):

$$\phi(t) = (P\phi^{raw})(t) = \int_{-\infty}^{\infty} \phi^{raw}(\hat{t}) P(t, \hat{t}) \nu(\hat{t}) d\hat{t}. \quad (5.5)$$

- (G3) Take samples of $\phi(t)$ at the time-varying Nyquist rate, namely, samples $\{\phi(t_n)\}_n$ on the Nyquist sampling grid $\{t_n\}_n$.
- (G4) Reconstruct $\phi(t)$ for all t from the generally non-equidistantly spaced samples $\{\phi(t_n)\}_n$. The reconstruction formula also has the same form as Shannon's sampling method in Eq. (5.2)

$$\phi(t) = \sum_{n=-\infty}^{\infty} G(t, t_n) \phi(t_n). \quad (5.6)$$

In Step (G1), the signal's frequency content can be studied using, for example, windowed Fourier transforms. The part of the raw signal that interests us determines what bandwidth should be kept in a period of time. For example, when wiggles of a signal matter in a period of time, samples are recorded at a denser rate to capture all the details. Although the time-varying bandwidth obtained in Step (G1) is an approximation, the filtering in step (G2) forces the signals to possess this pre-specified time-varying bandwidth or equivalently the time-varying Nyquist rate. Hence the reconstruction in Step (G4) is exact. The choice of $\{t_n\}_n$ in Step (G1) largely determines the quality of the approximation to the raw signal. It is important, but $\{t_n\}_n$ is chosen in accordance with the raw input signals. It strongly depends on the application at hand. Hence, this step will not be further investigated in generalized sampling method. However, see Chapter 8 for a simple example of approximating a step function.

To this end, let us assume that a Nyquist sampling grid, which is an increasing, infinite and generally non-equidistant set of sampling points $\{t_n\}_n$ at the Nyquist rate, has been determined in Step (G1). This set of sampling points represents the desired Nyquist rate. The higher the sampling rate, the denser the sampling points.

Table 5.1 provides a comparison of key features of Shannon and the generalized sampling methods. One notices that a further set of data $\{t'_n\}_n$ arises in Step (G1). The sets $\{t_n\}_n$ and $\{t'_n\}_n$ are closely related. As it will become clear later in Section 5.2.2, the optimal t'_n is difficult to determine, but an intuitive and convenient choice of t'_n , which also works very well in practice, can be specified from the sampling grid $\{t_n\}_n$:

$$t'_n = C_0 \frac{(t_{n+1} - t_{n-1})}{2}. \quad (5.7)$$

Table 5.1: A Comparison of the Shannon and the Generalized Sampling Methods

	Shannon	Generalization
Pre-chosen data	Ω	$\{t_n = t_n(0)\}_n$ and $\{t'_n = t'_n(0)\}_n$
Sampling points	Equidistant	Generally non-equidistant
Bandwidth	Constant	Time-Varying
Both possess a one-parameter family of Nyquist sampling grids		
Notation	$\{t_n(\theta)\}_n$	$\{t_n(\alpha)\}_n$
Parameter	$0 \leq \theta < 1$	$0 \leq \alpha < 1$
Governing Eq.	$t_n(\theta) = \frac{n+\theta}{2\Omega}$	$\frac{t'_n}{t-t_n} - \sum_{m \neq n} \frac{t'_m(t-t_n)}{(t-t_m)(t_n-t_m)} = \pi \cot(\pi\alpha)$
Sampling and perfect reconstruction on each fixed Nyquist grid		
	$\{t_n = t_n(\theta)\}_n, \theta$ fixed	$\{t_n = t_n(\alpha)\}_n$ and $\left\{t'_n = \frac{dt_n(\alpha)}{d\alpha}\right\}_n, \alpha$ fixed
Filtering $P(t, \hat{t})$	$\text{sinc}(2\Omega(t - \hat{t}))$	$\sum_n \frac{(-1)^{z(t, \hat{t}) t'_n }}{(t-t_n)(\hat{t}-t_n)} \left(\sum_m \frac{t'_m}{(t-t_m)^2}\right)^{-\frac{1}{2}} \left(\sum_m \frac{t'_m}{(\hat{t}-t_m)^2}\right)^{-\frac{1}{2}}$
Kernel $G(t, t_n)$	$\text{sinc}(2\Omega(t - t_n))$	$(-1)^{z(t, t_n)} \frac{\sqrt{t'_n}}{ t-t_n } \left(\sum_m \frac{t'_m}{(t-t_m)^2}\right)^{-\frac{1}{2}}$

Here the normalization pre-factor C_0 is given by

$$C_0 = 2\pi \sum_n \frac{t_{n+1} - t_{n-1}}{t_n^2 + 1}. \quad (5.8)$$

Before explaining the details of the generalized sampling theory, let us already state the mechanics of the steps (G2), (G3) and (G4). In Step (G2), the raw signals $\phi^{raw}(t)$ are pre-filtered via Eq. (5.5) with

$$P(t, \hat{t}) = (-1)^{z(t, \hat{t})} \sum_n \frac{t'_n}{(t-t_n)(\hat{t}-t_n)} \left(\sum_m \frac{t'_m}{(t-t_m)^2}\right)^{-1/2} \left(\sum_m \frac{t'_m}{(\hat{t}-t_m)^2}\right)^{-1/2} \quad (5.9)$$

where $z(t, \hat{t})$ is the total number of sampling points in $\{t_n\}_n$ between t and \hat{t} exclusively, and

$$\nu(t) = \frac{d\alpha(t)}{dt}.$$

Here for any $t \in [t_n, t_{n+1})$, the function $\alpha(t)$ is defined by

$$\pi \cot(\pi\alpha) = \frac{t'_n}{t-t_n} - \sum_{m \neq n} \frac{t'_m(t-t_n)}{(t-t_m)(t_n-t_m)}. \quad (5.10)$$

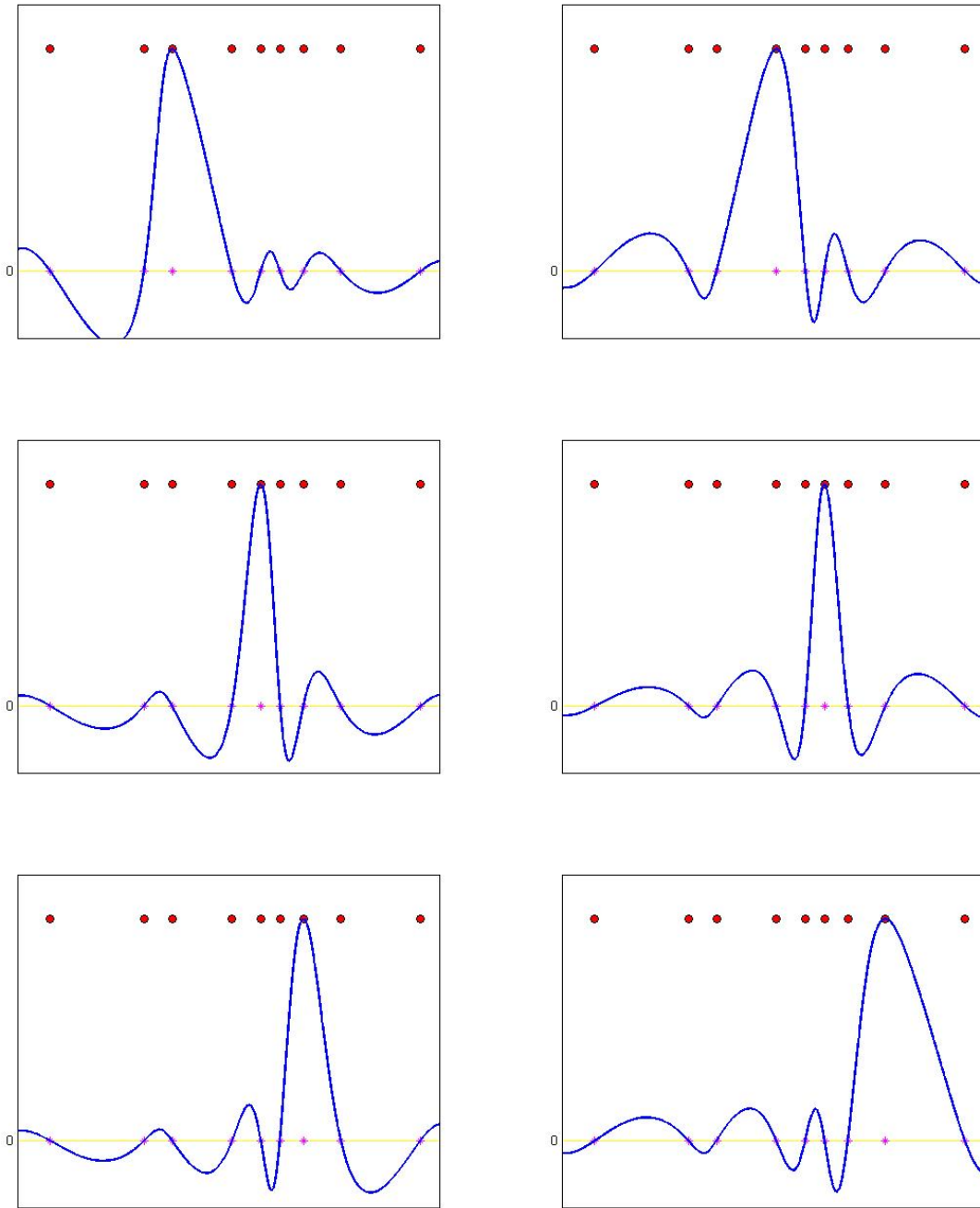


Figure 5.2: Examples of the generalized reconstruction kernel $G(t, t_n)$ as functions in t on a set of non-equidistantly spaced sampling points. Each function $G(t, t_n)$ is ‘centered’ at a different point. The purple dots and red circles indicate the non-equidistant sampling points at the amplitude of 0 and 1 respectively. Notice the non-trivial heights of the side peaks, which differ significantly from those of sinc functions.

When $t = t_n$, the right hand side of Eq. (5.10) diverges. The left hand side diverges if $\alpha = 0$. So $t = t_n$ corresponds to $t_n(\alpha)$ with $\alpha = 0$ as expected. The solution $\alpha(t)$ obeys $0 \leq \alpha(t) < 1$. The resulting signal $\phi(t)$ is in the desired function space of function which can be sampled on the generally non-equidistantly Nyquist sampling grid $\{t_n\}_n$ and perfectly reconstructed from those samples as in Step (G3) and (G4). The reconstruction formula Eq. (5.6) applies with the following generalized reconstruction kernel:

$$G(t, t_n) = (-1)^{z(t, t_n)} \frac{\sqrt{t'_n}}{|t - t_n|} \left(\sum_m \frac{t'_m}{(t - t_m)^2} \right)^{-1/2}. \quad (5.11)$$

The function value at $t = t_m$ is defined to be the limit of $G(t, t_n)$ as t approaches t_m for all integer m . It is straight-forward to show that $G(t_n, t_n) = 1$ and $G(t_m, t_n) = 0$ for all $m \neq n$. Figure 5.2 shows some examples of the generalized reconstruction kernel $G(t, t_n)$ as functions in t on a set of non-equidistantly spaced sampling points. Each function $G(t, t_n)$ is ‘centered’ at a different sampling point. The purple dots and red circles indicate the non-equidistant sampling points at the amplitude of 0 and 1 respectively. See Section 5.2.3 for more examples.

The next section will discuss the details of the sampling and reconstruction steps, i.e., Step (G3) and (G4) in the dashed box in Figure 5.1. The pre-filtering step, i.e., Step (G2), can be found in Chapter 6.

5.2 The Generalized Sampling Theory

The generalized sampling theory, which consists of the sampling and reconstruction steps, i.e., Step (G3) and (G4), is shown below in Section 5.2.2 and 5.2.3 respectively. The proof of these results can be found at the end of this chapter in Section 5.5. First, in Section 5.2.1, we recapitulate the Shannon sampling theorem in a formulation that is convenient for generalization. We list all the key features of the Shannon sampling theorem, which are to be preserved in the generalized sampling theorem.

5.2.1 Key Features of the Shannon Sampling Theorem to Emulate

The Shannon sampling theorem does not specify an initial point in a sampling grid at the constant Nyquist rate, but only specifies the distance between two adjacent points in a grid to be precisely $1/(2\Omega)$. There are many possible Nyquist sampling grids. We can parametrize all possible Nyquist sampling grids as the following

$$t_n(\theta) = \frac{n + \theta}{2\Omega}, \quad 0 \leq \theta < 1. \quad (5.12)$$

For later reference, we note here that this linear θ -parameterization is not the only way to parameterize all possible sampling grids, even for the same parameter range from 0 to 1. Let $\theta = \theta(\gamma)$ be any differentiable and strictly increasing function mapping from $[0, 1)$ onto $[0, 1)$, then

$$t_n(\gamma) = \frac{n + \theta(\gamma)}{2\Omega}, \quad 0 \leq \gamma < 1 \quad (5.13)$$

is also a parameterization of all the Nyquist sampling grids, namely, for any fixed value of the parameter γ , the points in $\{t_n(\gamma)\}_n$ obey the Nyquist rate spacing $1/(2\Omega)$.

In summary, no matter which parameter value one chooses, as the parameter increases from 0 to 1, the points in the sampling grid specified by that parameter simultaneously and continuously move to the right on the real line, and together, the family of these sampling grids covers the real line exactly once.

For each fixed integer n , $t_n(\theta)$ in Eq. (5.12) is clearly differentiable with respect to θ

$$\frac{dt_n(\theta)}{d\theta} = \frac{1}{2\Omega}. \quad (5.14)$$

For a different parameterization $t_n(\theta(\gamma))$, its derivative with respect to the new parameter γ , by the Chain rule, is also proportional to the constant Nyquist rate of Shannon

$$\frac{dt_n(\gamma)}{d\gamma} = \frac{dt_n(\theta)}{d\theta} \frac{d\theta}{d\gamma} = \frac{1}{2\Omega} \frac{d\theta}{d\gamma} \sim \frac{1}{2\Omega}. \quad (5.15)$$

Here the $\left(\frac{d\theta}{d\gamma}\right)$ -term is independent of n . It is a constant on a fixed grid. The derivatives on each fixed sampling grid are proportional to the Nyquist sampling rate. This observation will suggest a definition of ‘time-varying’ Nyquist rate later in Section 5.2.4.

To this end, we recall that the Shannon sampling theorem possesses a natural one-parameter family of Nyquist sampling grids $\{t_n(\theta)\}_n$. Any function $\phi(t)$ in the space of bandlimited functions can be reconstructed from its function values taken on a fixed grid, namely, on $\{t_n = t_n(\theta)\}_n$ with θ fixed. The reconstruction kernel $G(t, t_n(\theta))$ is the scaled and shifted sinc function in Eq. (5.4). In other words, considering $G(t, t_n(\theta))$ as a function in t , let

$$g_n^{(\theta)}(t) = G(t, t_n(\theta)), \quad (5.16)$$

then for each θ , the set of functions $\{g_n^{(\theta)}(t)\}_n$ forms a basis of the space of Ω -bandlimited functions. Most importantly, for each value of the parameter θ from 0 to 1, these bases span the same function space. This seems obvious in the case of Shannon because these basis functions are simply shifted sinc function with the same shape. However, as we will see in the generalization to non-equidistant Nyquist sampling grids, this is not trivial but achievable.

There are two crucial properties of these sinc basis functions. First, on each grid $\{t_n(\theta)\}_n$, θ fixed, the function $g_n^{(\theta)}(t)$ interpolates through all the points in the grid

$$g_n^{(\theta)}(t_m(\theta)) = \delta_{mn}. \quad (5.17)$$

Secondly, the maximum value of each basis function is always 1 at the point it ‘centers’. This is important for the stability of reconstruction.

5.2.2 The Completion of the Sampling Grids

The generalized sampling theory preserves all the properties of Shannon’s sampling theory in the previous section. For example, it possesses a one-parameter family of sampling grids, denoted by $\{t_n(\alpha)\}_n$, where $n \in \mathbb{Z}$ and $0 \leq \alpha < 1$. To distinguish from the linear parameter θ used in the case of Shannon, from now on, we use the parameter α to enumerate all the sampling grids in the generalized sampling theorem. As it will become clear later in the proof, the parameter α is the natural choice of parameter which arises from the theory of self-adjoint extensions in Chapter 2 and 3.

In the following, we first show what yields the one-parameter family of sampling grids and then how the sampling grids satisfy the aforementioned properties. The properties include that as the parameter α increases from 0 to 1, the points smoothly and simultaneously move to the right on the real line, and these points together cover the real line exactly once.

In the case of Shannon, the uniform Nyquist sampling grids $\{t_n(\theta)\}_n$ are generated by only a single number, the bandwidth Ω . The generalized sampling method requires more specification because it has generally non-equidistant Nyquist sampling points.

Intuitively, to determine the time-varying Nyquist rate, we need at least one sampling grid to specify the Nyquist sampling rate. Without loss of generality, one can assume that the known grid is at $\alpha = 0$, denoted by $\{t_n = t_n(0)\}_{n=-\infty}^{\infty}$, arranged in an increasing order.

In the case of Shannon, the constant spacing between adjacent points in one sampling grid is the same for all Nyquist rate grids, namely, $1/(2\Omega)$. This spacing directly relates to the constant bandwidth. In the generalized sampling theorem, however, the local spacing between adjacent points in one Nyquist sampling grid $\{t_n(\alpha)\}_n$ specified by parameter value α might be slightly different from the one of another Nyquist grid $\{t_n(\hat{\alpha})\}_n$ specified by $\hat{\alpha}$. This suggests another set of positive data to specify that variation at each point t_n , and we shall denote it by $\{t'_n\}_n$.

Of course, the sets $\{t_n\}_n$ and $\{t'_n\}_n$ are related. It is reasonable for practical purposes to assume there are minimum and maximum distances between two adjacent points the

sampling grid $\{t_n\}$, namely, there exist two positive real numbers Δ_{\min} and Δ_{\max} such that for all n

$$0 < \Delta_{\min} < \Delta t_n = t_{n+1} - t_n < \Delta_{\max}. \quad (5.18)$$

Further, because the spacing between adjacent sampling points directly relates to t'_n , we also assume a minimum and maximum value for t'_n for all n .

To this end, assume one initial ‘data’ say at $\alpha = 0$, $\{t_n = t_n(0)\}_n$ and $\{t'_n = t'_n(0)\}_n$ are given. Arising from the theory of self-adjoint extensions Eq. 3.16 in Chapter 3, first, these two sets must obey the normalization condition

$$\sum_n \frac{t'_n}{t_n^2 + 1} = \pi. \quad (5.19)$$

Then the whole family of sampling grids $\{t_n(\alpha)\}_n$ is determined by the governing equation

$$\frac{t'_n}{t - t_n} - \sum_{m \neq n} \frac{t'_m (t - t_n)}{(t - t_m)(t_n - t_m)} = \pi \cot(\pi\alpha). \quad (5.20)$$

Specifically, for a fixed value of α between 0 and 1, solving the above equation for t , there is one and only one solution for t in each open interval (t_n, t_{n+1}) for all integers n , which is the sampling point $t = t_n(\alpha)$. Conversely, given any real number t which is not in $\{t_n\}_n$, one can find a unique pair of (n, α) such that $t_n(\alpha) = t$. The integer n is the largest m such that $t_m \leq t$, $t_m \in \{t_m\}_m$, namely, the open interval (t_n, t_{n+1}) contains t . The value of α can be solved by substituting t and n into Eq. (5.20). Hence, for any real number t , one can determine to which sampling grid $\{t_n(\alpha)\}$ it belongs.

In addition, it is also shown in Chapter 3 that the set of positive numbers $\{t'_n\}_n$ turns out to be the set of derivatives of sampling points with respect to the parameter α at $\alpha = 0$. Namely, if we denote the derivative as

$$t'_n(\alpha) = \frac{dt_n(\alpha)}{d\alpha}, \quad (5.21)$$

then $t'_n = t'_n(0)$ for all n . This observation is important. It provides a straight-forward choice of the derivatives $\{t'_n\}_n$ of a given sampling grid $\{t_n\}_n$ in Eq. (5.7). The normalization pre-factor is determined by Eq. (5.19).

The derivative $t'_n(\alpha)$ at other sampling points $t_n(\alpha)$ with $0 < \alpha < 1$ can be obtained by a direct differentiation of $t_n(\alpha)$ with respect to α . It can be shown that $t'_n(\alpha)$ at each $t_n(\alpha)$ can also be calculated from the known sets $\{t_n = t_n(0)\}_n$ and $\{t'_n = t'_n(0)\}_n$ by

$$t'_n(\alpha) = \frac{\pi^2}{\sin^2(\pi\alpha)} \left(\sum_m \frac{t'_m}{(t_n(\alpha) - t_m)^2} \right)^{-1}. \quad (5.22)$$

To summarize, the knowledge of an ‘initial’ set of data, $\{t_n = t_n(0)\}_n$ and $\{t'_n = t'_n(0)\}_n$, at $\alpha = 0$ specifies the whole family of sampling grids $\{t_n(\alpha) | 0 \leq \alpha < 1, n \in \mathbb{Z}\}$. This equivalence is important for practical purposes, because to record and store the samples of a given signal, one only needs to do so on one Nyquist sampling grid.

This family of sampling grids obeys the following properties: as the parameter α increases, the sampling grids $\{t_n(\alpha)\}_n$ move to the right simultaneously and continuously. Together they cover the real line exactly once.

To see this, note that the right hand side of Eq. (5.20), $\cot \pi\alpha$, is a strictly decreasing function. It goes to $+\infty$ at $\alpha = 0^+$, and approaches $-\infty$ at $\alpha = 1^-$. On the left hand side of Eq. (5.20) is also a decreasing function in t on each open interval (t_n, t_{n+1}) . Its derivative with respect to t is $-\sum_{m \in \mathbb{Z}} \frac{t'_m}{(t-t_m)^2}$, which is always negative. Hence the left hand side is strictly decreasing as well. As $t \rightarrow t_n^+$, it goes to $+\infty$ and as $t \rightarrow t_{n+1}^-$, it approaches $-\infty$. Since both sides of Eq. (5.20) are differentiable and strictly decreasing functions from $+\infty$ to $-\infty$, then for each α , there is one and only one t on each interval (t_n, t_{n+1}) such that $t = t_n(\alpha)$. As α increases from 0 to 1, $t = t_n(\alpha)$ increases from t_n to t_{n+1} continuously with

$$\lim_{\alpha \rightarrow 1^-} t_n(\alpha) = t_{n+1}(0). \quad (5.23)$$

As will be explained later in Section 5.5, for each fixed α , the calculated sampling points $\{t_n(\alpha)\}_n$ and the corresponding derivatives $\{t'_n(\alpha)\}_n$ specified by α all automatically obey an analogous normalization condition Eq. (5.19) of $\{t_n\}_n$ and $\{t'_n\}_n$, namely,

$$\sum_n \frac{t'_n(\alpha)}{t_n^2(\alpha) + 1} = \pi. \quad (5.24)$$

5.2.3 The Generalized Reconstruction Kernels

In the Shannon sampling theorem, the constant bandwidth Ω is crucial: it determines the Nyquist sampling grids, the corresponding reconstruction kernel on each grid, and the associated function space. As a comparison, in the generalized sampling theorem for time-varying Nyquist rate, the ‘initial’ sampling grid $\{t_n = t_n(0)\}_n$ and the corresponding derivatives $\{t'_n = t'_n(0)\}_n$ play the role of Ω . The previous section shows how these ‘initial’ data generate the whole family of Nyquist sampling grids $\{t_n(\alpha)\}_n$, $n \in \mathbb{Z}$ and $0 \leq \alpha < 1$. This section will show the reconstruction kernel on each sampling grid and how the function space is spanned by these basis kernel functions.

As we prove in Section 5.5, on any fixed grid, say let $t_n = t_n(\alpha)$, and $t'_n = t'_n(\alpha)$ with α fixed, the reconstruction kernel reads:

$$G(t, t_n) = (-1)^{z(t, t_n)} \frac{\sqrt{t'_n}}{|t - t_n|} \left(\sum_m \frac{t'_m}{(t - t_m)^2} \right)^{-1/2}. \quad (5.25)$$

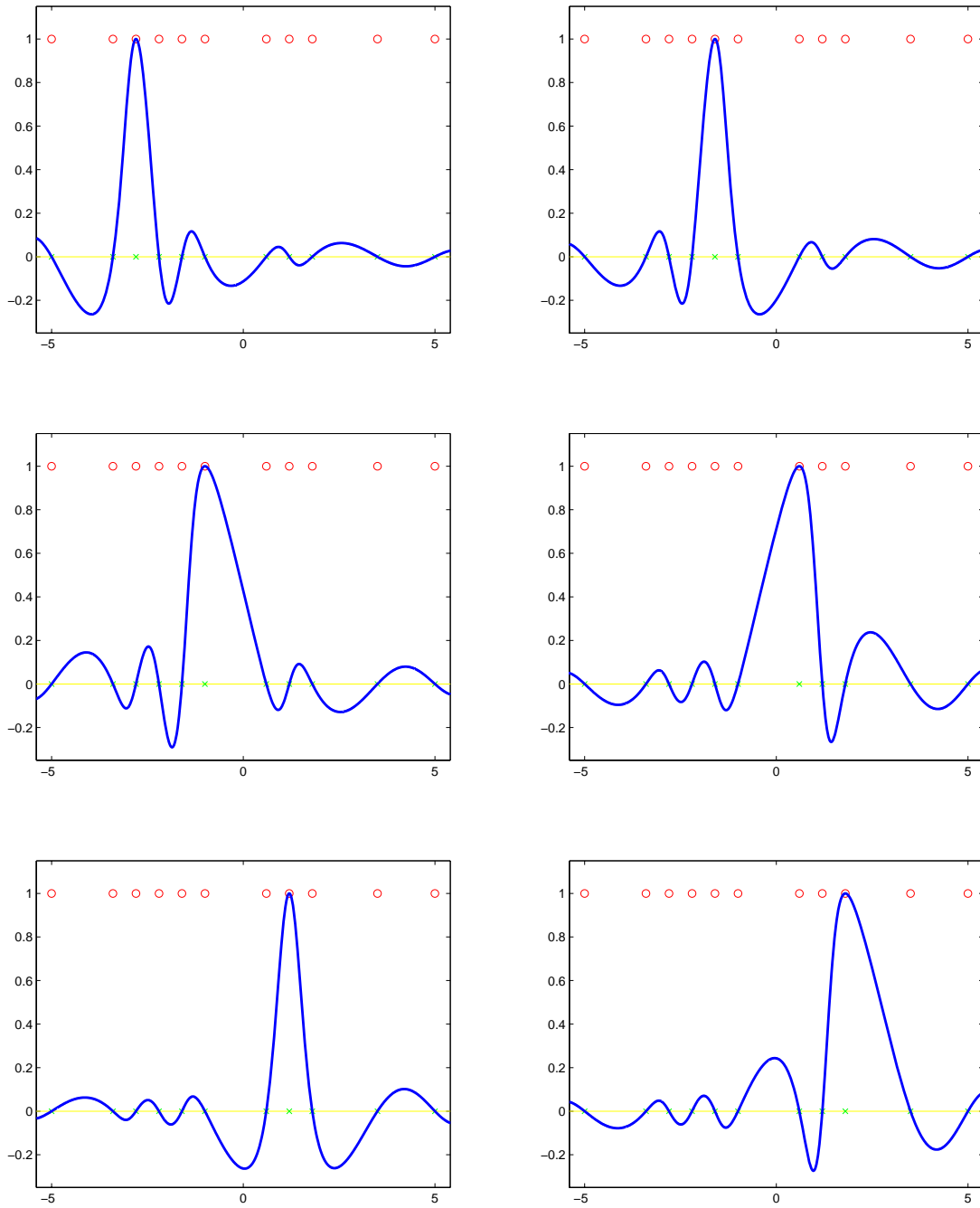


Figure 5.3: Generalized reconstruction kernels as functions in t on a set of arbitrarily chosen sampling points, but ‘centered’ at different points. The green dots and red circles indicate the sampling points at the amplitude of 0 and 1 respectively.

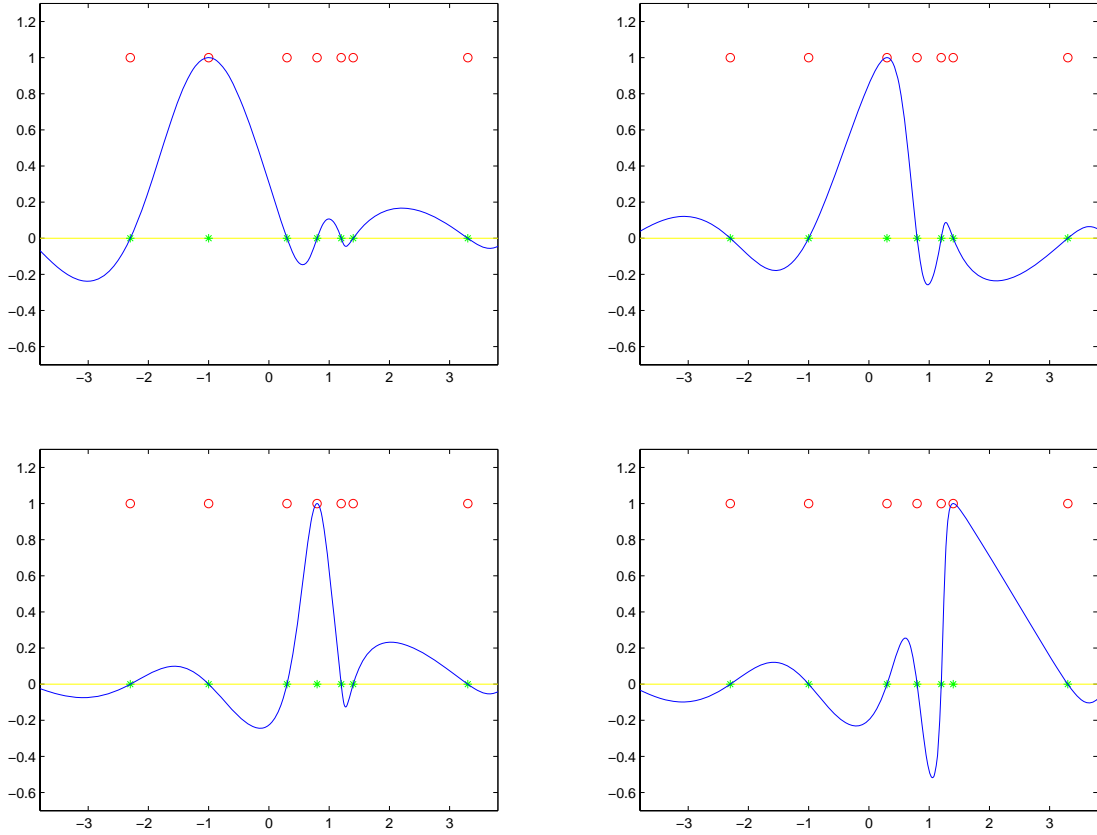


Figure 5.4: Generalized reconstruction kernels as functions in t on another set of arbitrarily chosen sampling points, ‘centered’ at different points.

Here $z(t, t_n)$ is the number of the sampling points $\{t_m\}_m$ between t and t_n exclusively. Figure 5.3 and Figure 5.4 illustrate examples of the reconstruction kernel as a function in t on two different non-equidistant Nyquist sampling grids $\{t_n\}_n$. Each plot is a function $G(t, t_n)$ but centered at a different point t_n .

As a function in t , for a fixed α , the set of functions

$$\{g_n^{(\alpha)}(t) = G(t, t_n(\alpha))\}_n \quad (5.26)$$

form a set of basic composing functions, namely, any function $\phi(t)$ in the function space can be written as a linear combination of these basis functions specified by α via the reconstruction formula Eq. (5.6). The coefficients in this set of basis functions, of course, coincide with the function values on the sampling grid specified by that α .

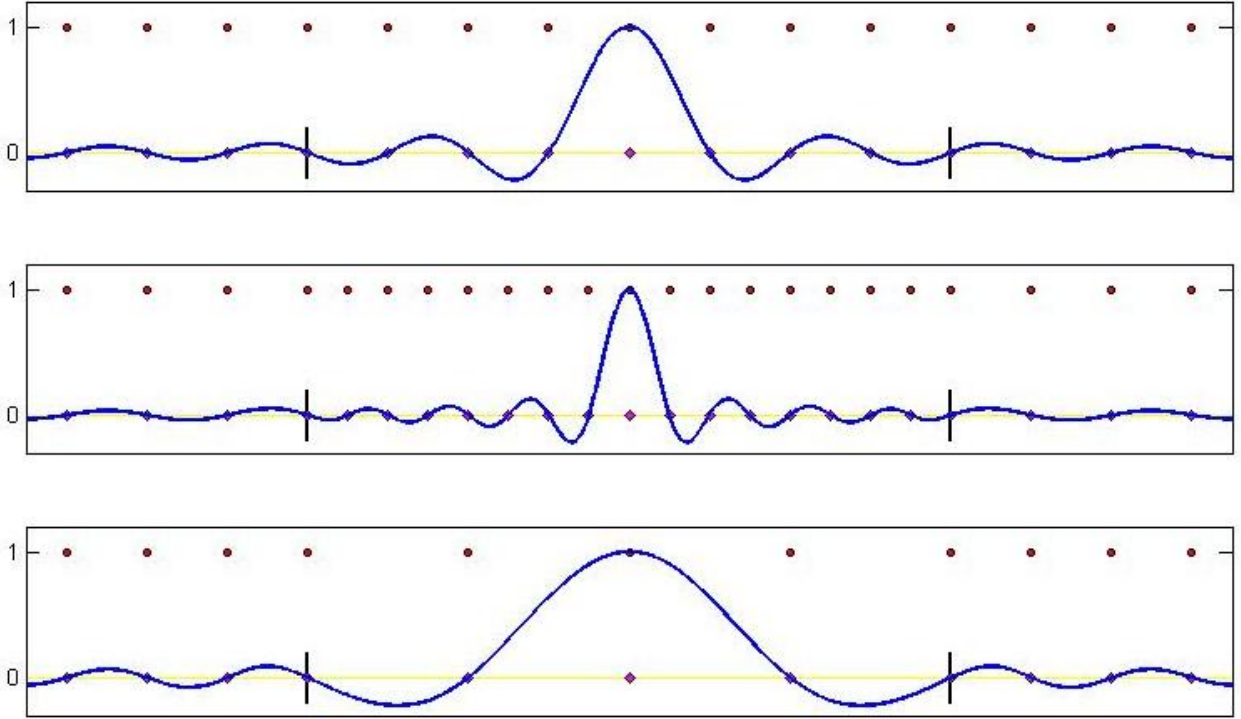


Figure 5.5: Three examples of the generalized sinc function on three different sets of sampling points. The first example is on a set of equidistantly spaced points. It recovers the usual sinc kernel in the classical sampling theorem (see Section 5.4). The second and third example have more and less sampling points respectively in the finite interval in the middle. The maximum value in all three cases is 1.

Notice that on each fixed sampling grid, the reconstruction kernel is explicitly expressed in terms of the points $\{t_n\}_n$ and derivatives $\{t'_n\}_n$ on that grid. Therefore, if one is given an initial sampling grid $\{t_n(0)\}_n$ and their derivatives $\{t'_n(0)\}_n$, one could fully determine the set of basis functions $\{g_n^{(0)}\}$ specified by $\alpha = 0$, which in turn, specifies the function space.

The shape of each basis function $g_n^{(\alpha)}(\alpha)$ is quite non-trivial, see Figure 5.3. Hence it is remarkable that the sets of these basic functions span the same function space for all values of α .

The functions $g_n^{(\alpha)}(t)$ possess many of the properties of the sinc function in the case of

Shannon sampling, for example, the important interpolation property²:

$$g_n^{(\alpha)}(t_m(\alpha)) = G(t_m(\alpha), t_n(\alpha)) = \delta_{mn}. \quad (5.27)$$

Further, $g_n^{(\alpha)}(t)$ is non-zero for all $t \notin \{t_m(\alpha)\}_m$. As t increases along the real line, the sign of $g_n^{(\alpha)}(t)$ changes if and only if it passes a sampling point $t_m(\alpha)$ in the grid specified by α except for $m = n$, in which case, $g_n^{(\alpha)}(t)$ is positive on both intervals $(t_{n-1}(\alpha), t_n(\alpha))$ and $(t_n(\alpha), t_{n+1}(\alpha))$. In addition, $g_n^{(\alpha)}(t)$ always has its maximum value of 1 at $t = t_n(\alpha)$. Since $g_n^{(\alpha)}(t)$ shares these important properties of the sinc function, we will refer to it as the **generalized sinc function**.

See Figure 5.5 for three more examples of the generalized sinc function on three different sets of sampling points. We note that it should be interesting to investigate the analyticity properties of the generalized reconstruction kernel as a function of t given that it is an entire function in the case of Shannon.

A reparametrization changes the enumeration of the Nyquist sampling grids, but it does not change the points in each single grid. So the function space should stay unmodified. Hence, it is expected that the reconstruction kernel remains invariant under such an action of reparametrization.

To see this, let $\{t_n(\gamma) = t_n(\alpha(\gamma))\}_n$ be a reparametrization of $\{t_n(\alpha)\}_n$, where $\alpha = \alpha(\gamma)$ is a differentiable and strictly increasing function mapping from $[0, 1)$ to $[0, 1)$. For each fixed α , there is one and only one γ such that $t_n(\gamma) = t_n(\alpha)$ for all n , i.e., $\{t_n(\alpha)\}_n$ and $\{t_n(\gamma)\}_n$ represent the same sampling grid. The associated derivatives $\left\{t'_n(\alpha) = \frac{dt_n(\alpha)}{d\alpha}\right\}_n$ and $\left\{\frac{dt_n(\gamma)}{d\gamma}\right\}_n$ are different, but only by a constant independent of n , by the chain rule

$$\frac{dt_n(\gamma)}{d\gamma} = \frac{dt_n(\alpha)}{d\alpha} \frac{d\alpha}{d\gamma}. \quad (5.28)$$

Substituting the derivatives $\frac{dt_n(\gamma)}{d\gamma}$ into the reconstruction kernel $G(t, t_n(\gamma))$ in Eq. (5.25), the n -independent constant $\left(\frac{d\alpha}{d\gamma}\right)$ which arises in the $\sqrt{\frac{dt_n(\gamma)}{d\gamma}}$ -term outside the summation cancels out the one in the numerator term $\frac{dt_n(\gamma)}{d\gamma}$ inside the infinite series. Hence, $G(t, t_n(\gamma))$ has exactly the same form of $G(t, t_n(\alpha))$ on $\{t_n(\alpha) = t_n(\alpha(\gamma))\}_n$.

But the results in Eq. (5.20), (5.22), and (5.24) in Section 5.2.2 only hold for the particular parameter α , which directly arises from the theory of self-adjoint extensions as one will see later in Section 5.5.

²This interpolation property follows directly from the expression of the reconstruction kernel $G(t, t_n)$. It is also clear in the proof of the generalized sampling theorem in Section 5.5

Note that in the cancellation argument above, the $(\frac{d\alpha}{d\gamma})$ -term can be replaced by any constant independent of n . Hence, the reconstruction kernel $G(t, t_n)$ on a fixed sampling grid $\{t_n\}_n$ is independent of any scalar multiplication of the derivatives $\{t'_n\}_n$. This is useful in simplifying the calculation of the reconstruction kernel. The complicated normalization constant C_0 needs not be explicitly calculated.

5.2.4 The Definition of Time-Varying Nyquist Rate

As mentioned above, a time-varying Nyquist rate is determined by one sampling grid $\{t_n\}_n$ and its derivatives $\{t'_n\}_n$. Functions in the space are then spanned by the set of generalized sinc functions. In this section, we will provide a precise definition of the time-varying Nyquist rate.

Let us now try to define a function $N(t)$ which shall denote the time-varying Nyquist rate of all the functions in the space. For consistency, $N(t)$ must be the same for all functions in the space and consistent with different parameterizations of the sampling grids. Moreover, $N(t)$ is expected to be directly related to each generally non-equidistant sampling grid.

Similar to the case of Shannon in Eq. (5.15), the time-varying Nyquist rate $N(t)$ must be proportional to the derivatives $\{t'_n(\alpha)\}$ on any fixed sampling grid $\{t_n(\alpha)\}_n$ as

$$N(t_n(\alpha)) \sim t'_n(\alpha) = \frac{dt_n(\alpha)}{d\alpha} \quad \text{on } \{t_n(\alpha)\}_n, \alpha \text{ fixed.} \quad (5.29)$$

Of course, we cannot simply identify $N(t)$ to be the derivative $t'_n(\alpha)$ at a point $t = t_n(\alpha)$, because the derivative $t'_n(\alpha)$ depends on the parametrization. The normalization condition Eq. (5.24) will not even recover the case of Shannon.

A natural question that follows is what quantity can be derived from $t'_n(\alpha)$, which obeys the proportionality in Eq. (5.29) and be reparametrization independent? The integral of $t'_n(\alpha)$ over one period of the parameter α , from $t_n(\alpha)$ to $t_{n+1}(\alpha)$, will do so. To see this, let $\{t_n(\gamma)\}_n$ be an arbitrary reparametrization of $\{t_n(\alpha)\}_n$ where $t_n(\gamma) = t_n(\alpha(\gamma))$. We have

$$\begin{aligned} \int_{\bar{\alpha}=\alpha}^{\alpha+1} t'_n(\bar{\alpha}) d\bar{\alpha} &= \int_{\bar{\alpha}=\alpha}^{\alpha+1} \frac{dt_n(\bar{\alpha})}{d\bar{\alpha}} d\bar{\alpha} \\ &= \int_{\bar{\alpha}=\alpha}^{\alpha+1} \frac{dt_n(\bar{\alpha}(\bar{\gamma}))}{d\bar{\gamma}} \frac{d\bar{\gamma}}{d\bar{\alpha}} d\bar{\alpha} = \int_{\bar{\gamma}=\gamma}^{\gamma+1} \frac{dt_n(\bar{\gamma})}{d\bar{\gamma}} d\bar{\gamma} \\ &= t_{n+1}(\alpha) - t_n(\alpha). \end{aligned} \quad (5.30)$$

For notational convenience, let $t_n(\epsilon + 1) = t_{n+1}(\epsilon)$ and $t_n(\epsilon - 1) = t_{n-1}(\epsilon)$ recursively for all $0 \leq \epsilon < 1$.

We have a reparametrization independent candidate for $N(t)$ at hand i.e., $t_{n+1}(\alpha) - t_n(\alpha)$, but to which t should it be assigned? An obvious answer is either $t = t_n(\alpha + \frac{1}{2})$ or $t = \frac{1}{2}(t_n(\alpha) + t_{n+1}(\alpha))$. The former is not good because t becomes parameter dependent again. The latter is theoretically applicable. The strict monotonicity of $t_n(\alpha)$ and $t_{n+1}(\alpha)$ implies that around each real number t , there is exactly one interval $[t_n(\alpha), t_{n+1}(\alpha)]$ centered at t . But to obtain the value of $N(t)$ at an arbitrary t , it requires a full calculation of $\{t_n(\alpha)\}_n$ for all $0 \leq \alpha < 1$ to find out in which interval t is centered. This is computationally expensive and impractical. Ideally, it should suffice only to possess knowledge of one sampling grid.

This problem is not hard to fix: instead of integrating over one period of α , we will go through two, from $t_{n-1}(\alpha)$ to $t_{n+1}(\alpha)$. On this two-period-interval $[t_{n-1}(\alpha), t_{n+1}(\alpha)]$, $t_n(\alpha)$ becomes a natural symmetric ‘center’ point, and we only need the knowledge of one sampling grid to find out the Nyquist rate at the point in that grid. Specifically, the time-varying Nyquist rate $N(t)$ at any time $t = t_n(\alpha)$ is then defined to be

$$N(t_n(\alpha)) = \frac{1}{2} \int_{\bar{\alpha}=\alpha-1}^{\alpha+1} t'_n(\alpha) d\alpha = \frac{1}{2}(t_{n+1}(\alpha) - t_{n-1}(\alpha)). \quad (5.31)$$

This definition not only keeps the symmetry, but also the independence of reparametrization of sampling grids. The time-varying Nyquist rate at a point t represents the average spacing of the sampling points at the Nyquist rate near t .

5.3 The Time-Varying Nyquist Rates as Critical Sampling Rates

The function space is determined by a given Nyquist sampling grid $\{t_n\}_n$ and the associated derivatives $\{t'_n\}_n$. We also give a mathematical definition of time-varying Nyquist rate possessed by the functions in the space. But what precisely does this Nyquist rate imply?

In the Shannon sampling theorem, the Nyquist rate at the bandwidth is the critical rate, below which one has insufficient information to recover the signal, i.e., one is under-sampling, and above which there is redundancy in the discrete samples, namely, one is over-sampling. In this section, we will show that the time-varying Nyquist rate in the generalization is again the critical sampling rate between under- and over-sampling. Hence, the samples of functions in the function space are indeed taken only as frequently as necessary.

5.3.1 From the Fourier Analysis Perspective

Let us first analyze the bandlimited functions in the Shannon sampling theorem. We started from a Fourier analysis perspective, but the aim here is to find a criterion to distinguish between under- and over-sampling without involving the Fourier transform.

Assume that a given signal of finite energy $\phi(t)$ has a bandlimit Ω_B . The aim is to determine its Nyquist sampling rate, the critical rate between under- and over-sampling. Assume we sample the signal at a rate higher than the Nyquist rate. Namely, the samples are taken on a uniform grid of $\{t_n\}_n$ with $t_{n+1} - t_n = \frac{1}{2\Omega}$ where $\Omega > \Omega_B$. So the sampling points $\{t_n\}_n$ are tighter than the actual Nyquist sampling rate. The variable Ω is always larger than Ω_B .

Let $\Phi(\omega)$ be the Fourier transform of $\phi(t)$. Since $\Omega > \Omega_B$, the Ω_B -bandlimited signal $\phi(t)$ is always Ω -bandlimited. Hence $\Phi(\omega)$ vanishes outside the interval $(-2\pi\Omega, 2\pi\Omega)$. One can consider $\Phi(\omega)$ as a $4\pi\Omega$ -periodic function on the larger interval $(-2\pi\Omega, 2\pi\Omega)$ and expand it in a Fourier series to obtain

$$\Phi(\omega) = \frac{1}{2\Omega} \sum_{n=-\infty}^{\infty} e^{i\omega t_n} \phi(t_n).$$

The samples are taken on an equidistantly spaced grid, so $t_n = \frac{n+\theta}{2\Omega}$ for some fixed θ . At $\omega = 2\pi\Omega$, one has

$$0 = \Phi(2\pi\Omega) = e^{i\pi\theta} \frac{1}{2\Omega} \left(\sum_{n=-\infty}^{\infty} (-1)^n \phi(t_n) \right).$$

Let \tilde{A} denote the alternating sum in the bracket above.

$$\tilde{A} := \sum_n (-1)^n \phi(t_n) \tag{5.32}$$

Then $\tilde{A} = 0$ for any $\Omega > \Omega_B$. Namely, for any rate higher than the Nyquist rate, the alternating sum \tilde{A} must vanish.

However, when $\Omega < \Omega_B$, the Ω_B -bandlimited function $\phi(t)$ may have a bandwidth larger than $(-2\pi\Omega, 2\pi\Omega)$. Forcing the Fourier transform $\Phi(\omega)$ to be $4\pi\Omega$ -periodic will result in overlap near $\omega = \pm 2\pi\Omega$ and hence the above alternating sum \tilde{A} will not be 0 in general.

Therefore, a general strategy to obtain the Nyquist sampling rate is to first sample the signal at a very high rate which guarantees over-sampling ($\Omega \gg \Omega_B$), and lower the sampling rate gradually, while the alternating sum \tilde{A} remains 0. Until Ω hits the fixed threshold bandlimit Ω_B , below this sampling rate, the signal ϕ has Fourier components.

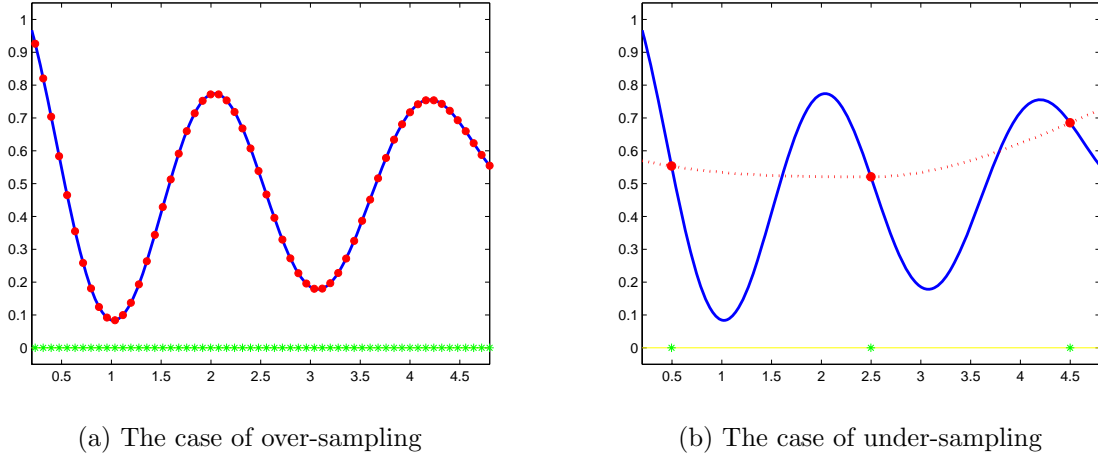


Figure 5.6: Over- and under-sampling of a continuous signal

5.3.2 From a Statistical Perspective

The previous section shows a general strategy to obtain the critical sampling rate between under- and over-sampling using an alternating sum. This section shows that the non-Fourier alternating sum does indeed arise from a statistical perspective to conclude the correlation among samples.

In practice, consider the two extreme cases of the alternating sum of samples taken at equidistant points $\{t_n\}_n$ on a long interval $[t_1, t_N]$

$$\tilde{A}_N := \sum_{n=1}^N (-1)^n \phi(t_n). \quad (5.33)$$

First, when one extremely over-samples, the sampling points are tightly spaced so that nearby sample values are almost the same. The terms in the above alternating sum cancel the adjacent ones, leaving \tilde{A}_N either zero as N goes to infinity or a typical sample value.

In contrast, consider the case that the sampling points are extremely widely spaced, i.e., that one is undersampling. Then, the sample values are essentially uncorrelated. Thus, the alternating sum \tilde{A}_N in Eq. (5.33) is an outcome of the random walk of the signal's amplitudes taken in the time interval $[t_1, t_N]$ and the central limit theorem in statistics indicates that \tilde{A}_N is normal distributed for large N . If the typical amplitude of the signal is say of order one then the typical value for \tilde{A}_N is on the order of \sqrt{N} .

It will be convenient to regularize the divergence $\tilde{A}_N \propto \sqrt{N}$ for $N \rightarrow \infty$ by absorbing

it in a suitable pre-factor $1/\sqrt{N}$. Let

$$\begin{aligned}
A_N &= \frac{\tilde{A}_N}{\sqrt{N}} = \frac{1}{\sqrt{N}} \sum_{n=1}^N (-1)^n \phi(t_n) \\
&= \sum_{n=1}^N (-1)^n \frac{\sqrt{\Delta t_n}}{\sqrt{t_N - t_1}} \phi(t_n) \\
&= \frac{1}{\sqrt{t_N - t_1}} \sum_{n=1}^N (-1)^n \sqrt{\Delta t_n} \phi(t_n).
\end{aligned} \tag{5.34}$$

Here $t_N - t_1 = N \Delta t_n$. The time interval $[t_1, t_N]$ is fixed and Δt_n is a constant sampling spacing. In the generalization, this constant spacing Δt_n is replaced with t'_n which is time-dependent.

When under-sampling, A_N goes to 1. When over-sampling, A_N , as the limit of \tilde{A} divided by \sqrt{N} , is zero. In summary, one identifies the Nyquist sampling rate as the critical sampling rate, below which the alternating sum A_N goes to 1 (in the case of under-sampling), and above which, A_N goes to zero (in the case of over-sampling), as $N \rightarrow \infty$.

5.3.3 From the Functional Analytical Perspective

Replacing the Δt_n with the derivative t'_n in Eq. (5.34), with the immaterial multiplicative constant ignored, the alternating sum

$$A_N = \frac{1}{\sqrt{t_N - t_1}} \sum_{n=1}^N (-1)^n \sqrt{t'_n} \phi(t_n) \tag{5.35}$$

is in a form that is suitable for the generalized sampling theorem. In the generalization, when adding and subtracting amplitudes as a random walk, there is suitably less weight on the amplitudes when they are tighter spaced (i.e., t'_n is small) and more weight on the amplitudes when they are more widely spaced (i.e., t'_n is large).

In the limit as N goes to infinity, the alternating sum Eq. (5.35) does arise from the theory of self-adjoint extensions as one will see in Section 5.5. The alternating sum can be shown to be zero for the functions in the space, similar to the argument in the case of Shannon that the Ω -bandlimited functions are Ω_B -bandlimited for all $\Omega > \Omega_B$.

In conclusion, it implies a criterion of under- and over-sampling as in the case of Shannon, and further suggests an analogous method to obtain a time-varying Nyquist rate for a given signal: one first samples the signal at an extremely high rate which guarantees over-sampling so that A_N approaches 0 as N goes to infinity, and then gradually lowers

the rate, right before the samples start to decorrelate, namely when A_N becomes of order 1. This critical sampling rate obtained is the Nyquist sampling rate of the signal. (This approach by lowering the over-sampled rate to obtain the critical sampling rate of course is not practical. More efficient methods could be developed by using for example windowed Fourier transform, as commented in Step (G1) in Section 5.1.)

Later in Chapter 7, one will also see that the analogy between Shannon's constant Nyquist rate and our time-varying Nyquist rate is not only the critical rate between over- and under-sampling, but also the sampling rate that gives the most stable reconstruction.

5.4 Recovering the Shannon Sampling Theorem

In this section, we will show how the Shannon sampling theorem arises as a special case from our generalized sampling theorem.

First, it is important to point out that there is a non-trivial connection between the α -parametrization in the generalized sampling theorem and linear θ -parametrization. We showed the connection in Chapter 2 in the context of the self-adjoint extensions of symmetric operators with deficiency indices $(1, 1)$. However here we will show this connection from a different perspective, using only the mechanics of the generalized sampling theorem. The calculation here to recover Shannon's case is also a good demonstration of how the results in Section 5.2.2 and 5.2.3 apply.

To recover the Shannon sampling theorem of constant bandlimited functions, we start with a set of sampling points $\{t_n\}_n$ with a constant Nyquist spacing $1/(2\Omega)$, say for $\alpha = 0$,

$$t_n = t_n(0) = \frac{n}{2\Omega}. \quad (5.36)$$

Because of the constancy of Nyquist sampling rate, we expect the corresponding derivatives to be a constant. However, we can not simply assign $t'_n = t'_n(0) = \frac{dt_n(\alpha)}{d\alpha}|_{\alpha=0}$ to be $\frac{1}{(2\Omega)} = \frac{dt_n(\theta)}{d\theta}|_{\theta=0}$, the derivative with respect to θ in the case of Shannon in Eq. (5.14). This choice of t'_n would violate the normalization condition in Eq. (5.19). This non-equivalence implies that when we recover the Shannon's case, the linear θ -parameterization of Shannon in Eq. (5.12) in Section 5.4 is not the specific α -parameterization in the generalized sampling theorem, but it is a re-parametrization of α .

To find $t'_n = t'(0)$ with respect to α , let $t'_n = t'_n(0) = C$ be a constant for all n and

substitute into the normalization condition Eq. (5.19) to give

$$\begin{aligned}
\pi &= \sum_n \frac{C}{t_n^2 + 1} = \sum_n \frac{C}{\left(\frac{n}{2\Omega}\right)^2 + 1} \\
&= C (2\Omega)^2 \sum_n \frac{1}{n^2 + (2\Omega)^2} \\
&= C (2\Omega)^2 \frac{\pi}{(2\Omega)} \coth(2\pi\Omega) \\
&= C \pi 2\Omega \coth(2\pi\Omega).
\end{aligned}$$

Here we used the following trigonometric identity for \coth with $z = 2\Omega$

$$\sum_n \frac{1}{n^2 + z^2} = \frac{\pi}{z} \coth(\pi z).$$

Hence

$$t'_n = t'_n(0) = C = \frac{1}{2\Omega \coth(2\pi\Omega)} = \tanh(2\pi\Omega) \frac{1}{2\Omega}. \quad (5.37)$$

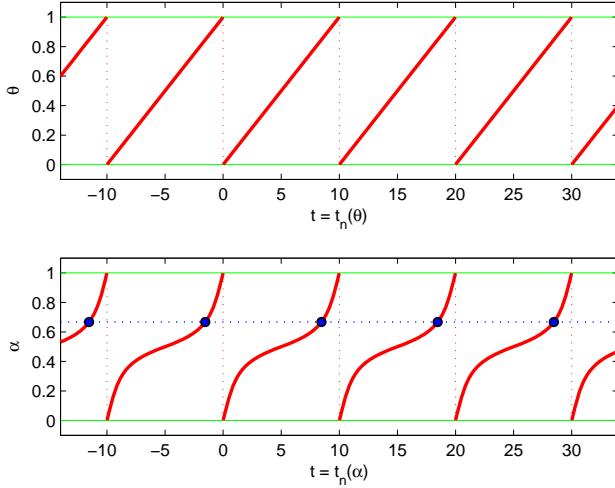


Figure 5.7: The same one-parameter family of sets of equidistant sampling points in the case of Shannon with respect to different parameter θ and α .

With this set of initial data at $\alpha = 0$ in Eq. (5.36) and (5.37), one can use the formulae in Section 5.2.2 to obtain the whole family of sampling grids $\{t_n(\alpha)\}_n$ for $0 < \alpha < 1$.

We expect the following results of $\{t_n(\alpha)\}_n$: the points in each grid specified by a fixed α must have the constant spacing $1/(2\Omega)$ and the resulting α -family of sampling grids $\{t_n(\alpha)\}_n$ is simply a reparametrization of the $\{t_n(\theta)\}_n$ in Eq. (5.12).

To calculate $t = t_n(\alpha)$ for $0 < \alpha < 1$ in each interval (t_n, t_{n+1}) , use the governing equation Eq. (5.20) in Section 5.2.2 to give

$$\begin{aligned}
\pi \cot(\pi\alpha) &= \frac{t'_n}{t - \frac{n}{2\Omega}} - \sum_{m \neq n} \frac{t'_m (t - \frac{n}{2\Omega})}{(t - \frac{m}{2\Omega}) (\frac{n}{2\Omega} - \frac{m}{2\Omega})} \\
&= \tanh(2\pi\Omega) \left(\frac{1}{2\Omega t - n} - \sum_{m \neq n} \frac{2\Omega t - n}{(2\Omega t - m) (n - m)} \right) \\
&= \tanh(2\pi\Omega) \left(\frac{1}{(2\Omega t - n)} - \sum_{k \neq 0} \frac{(2\Omega t - n)}{((2\Omega t - n) + k) k} \right) \\
&= \tanh(2\pi\Omega) \pi \cot(\pi(2\Omega t - n)).
\end{aligned} \tag{5.38}$$

Here we changed the dummy variable $k = m + n$ and used the following trigonometric identity with $z = 2\Omega t - n$

$$\begin{aligned}
\frac{1}{z} - \sum_{k \neq 0} \frac{z}{(z+k)k} &= \frac{1}{z} + \sum_{k \neq 0} \left(\frac{1}{k+z} - \frac{1}{k} \right) \\
&= \frac{1}{z} + \sum_{k=1}^{\infty} \left(\frac{1}{z+k} + \frac{1}{z-k} \right) = \pi \cot(\pi z).
\end{aligned} \tag{5.39}$$

For a fixed α , Eq. (5.38) implies that the term of $(2\Omega t - n)$ on the right hand side is fixed as well. Let it be a constant, say θ , namely, $\theta = 2\Omega t - n$. Then this θ indeed yields a linear parametrization of θ in Eq. (5.12) of the Shannon sampling theorem. Substituting this into Eq. (5.38) gives

$$\tan(\pi\alpha) = \coth(2\pi\Omega) \tan(\pi\theta). \tag{5.40}$$

This equation defines a one-to-one relation between θ and α . The parameter α as a function of θ is a strictly increasing mapping from $[0, 1)$ onto $[0, 1)$. It is the desired re-parametrization function. It implies that α is a reparametrization of θ .

Due to the exponential decay of the term $\coth(2\pi\Omega)$ to 1 in Eq. (5.40), for $\Omega \geq 1$, the parameters α and θ are roughly identical. Hence, for slowly-varying bandwidth, in practice, one can use the linear θ -parametrization to approximate the α -parametrization.

With $\theta = \theta(\alpha)$ defined in Eq. (5.40), the whole family of sampling grids is given by

$$t_n(\alpha) = \frac{n + \theta(\alpha)}{2\Omega}. \tag{5.41}$$

We can differentiate the sampling grids $t_n(\alpha)$ in Eq. (5.41) with respect to α to obtain $t'_n(\alpha)$

$$\frac{t_n(\alpha)}{d\alpha} = \frac{t_n(\theta)}{d\theta} \frac{d\theta}{d\alpha} = \frac{1}{2\Omega} \frac{\cosh(4\pi\Omega) - \cos(2\pi\theta(\alpha))}{\sinh(4\pi\Omega)}. \tag{5.42}$$

Here we used the Chain rule and the following from Eq. (5.40)

$$\begin{aligned}
\frac{d\theta}{d\alpha} &= \tanh(2\pi\Omega) \frac{\sec^2(\pi\alpha)}{\sec^2(\pi\theta)} \\
&= \tanh(2\pi\Omega) \cos^2(\pi\theta) (1 + \coth^2(2\pi\Omega) \tan^2(\pi\theta)) \\
&= \tanh(2\pi\Omega) \cos^2(\pi\theta) + \coth(2\pi\Omega) \sin^2(\pi\theta) \\
&= \frac{\sinh^2(2\pi\Omega) (1 + \cos(2\pi\theta)) + \cosh^2(2\pi\Omega) (1 - \cos(2\pi\theta))}{2 \sinh(2\pi\Omega) \cosh(2\pi\Omega)} \\
&= \frac{\cosh(4\pi\Omega) - \cos(2\pi\theta)}{\sinh(4\pi\Omega)}.
\end{aligned} \tag{5.43}$$

Alternatively, the derivative $t'_n(\alpha)$ can also be calculated directly from the knowledge of the initial ‘data’ $\{t_n = t_n(0)\}$ and $\{t'_n = t'_n(0)\}_n$ using Eq. (5.22) in Section 5.2.2.

$$\begin{aligned}
\frac{dt_n(\alpha)}{d\alpha} &= \frac{\pi^2}{\sin^2(\pi\alpha)} \left(\sum_m \frac{\tanh(2\pi\Omega)/(2\Omega)}{((n + \theta(\alpha))/(2\Omega) - m/(2\Omega))^2} \right)^{-1} \\
&= \frac{\pi^2(1 + \cot^2(\pi\alpha))}{2\Omega \tanh(2\pi\Omega)} \left(\sum_m \frac{1}{((n + \theta(\alpha)) - m)^2} \right)^{-1} \\
&= \frac{1}{2\Omega} \frac{1 + \tanh^2(2\pi\Omega) \cot^2(\pi\theta(\alpha))}{\tanh(2\pi\Omega)(1 + \cot^2(\pi\theta(\alpha)))} \\
&= \frac{1}{2\Omega} \frac{\cosh^2(2\pi\Omega) \sin^2(\pi\theta(\alpha)) + \sinh^2(2\pi\Omega) \cos^2(\pi\theta(\alpha))}{\sinh(2\pi\Omega) \cosh(2\pi\Omega)} \\
&= \frac{1}{2\Omega} \frac{2 \cosh^2(2\pi\Omega) - 2 \cos^2(\pi\theta)}{2 \sinh(2\pi\Omega) \cosh(2\pi\Omega)} \\
&= \frac{1}{2\Omega} \frac{\cosh(4\pi\Omega) - \cos(2\pi\theta(\alpha))}{\sinh(4\pi\Omega)}.
\end{aligned} \tag{5.44}$$

Here Eq. (5.44) used the fact that $\sin^2(z) + \cos^2(z) = 1$ and $\cosh^2(z) - \sinh^2(z) = 1$ in the second last step. The last step of Eq. (5.44) follows from double angle formulas. As expected, the derivative found in Eq. (5.44) agrees with the one in Eq. (5.42).

Finally, the results in Section 5.2.2 are recovered in the special case of Shannon. To this end, we show how the complicated general reconstruction kernel of Eq. (5.25) in Section 5.2.3 simplifies to the sinc kernel in Eq. (5.4) on a fixed uniform sampling grid

$$\{t_n(\alpha)\}_n = \{t_n(\theta)\}$$

$$\begin{aligned}
G(t, t_n(\alpha)) &= (-1)^{z(t, t_n(\theta(\alpha)))} \frac{\sqrt{\frac{dt_n(\theta)}{d\theta} \frac{d\theta}{d\alpha}}}{|t - t_n(\theta(\alpha))|} \left[\sum_m \frac{\left(\frac{dt_m(\theta)}{d\theta}\right) \left(\frac{d\theta}{d\alpha}\right)}{(t - t_m(\theta(\alpha)))^2} \right]^{-1/2} \\
&= (-1)^{z(t, t_n(\theta))} \frac{\sqrt{\frac{1}{2\Omega} \coth(2\pi\Omega)}}{|t - (n + \theta)/(2\Omega)|} \left[\sum_m \frac{\frac{1}{2\Omega} \coth(2\pi\Omega)}{(t - (m + \theta)/(2\Omega))^2} \right]^{-1/2} \\
&= \frac{(-1)^{z(t, t_n(\theta))}}{|2\Omega t - (n + \theta)|} \left[\sum_m \frac{1}{((2\Omega t - \theta) - m)^2} \right]^{-1/2} \\
&= \frac{(-1)^{z(t, t_n(\theta))}}{|2\Omega t - (n + \theta)|} \frac{|\sin(\pi(2\Omega t - (n + \theta)))|}{\pi} \\
&= \text{sinc}(2\Omega t - (n + \theta)) = \text{sinc}(2\Omega(t - t_n(\theta))).
\end{aligned} \tag{5.45}$$

Here, we used the Chain rule and Eq. (5.12), (5.14), (5.40) and (5.43). In the second last step of Eq. (5.45), we made a change of dummy variable $m = n + k$ and used the following trigonometric identity with $z = 2\Omega t - (n + \theta)$

$$\left(\frac{\pi}{\sin(\pi z)}\right)^2 = \sum_{k=-\infty}^{\infty} \frac{1}{(z - k)^2}. \tag{5.46}$$

Therefore, the reconstruction kernel on any sampling grid $\{t_n(\alpha)\}_n$, which is generated from one initial uniform sampling grid in Eq. (5.36), indeed simplifies to the scaled and shifted sinc reconstruction kernel in the case of Shannon. Because the reconstruction kernel is independent of the parameterization, the same result holds by substituting the sampling grids with the θ -parameter.

5.5 Mathematical Proof of the Generalized Sampling Theorem

This section will lay out the mathematical proofs of the generalized sampling theorem using the functional analytical theory of self-adjoint extensions of $(1, 1)$ -symmetric operator introduced in Chapter 3. Most results directly follow from Theorem 7 and 8 in Chapter 3.

Mathematical Motivation

First, we already provided a new functional analytical proof of the classical Shannon sampling theorem in Section 4.3, which can be reformulated as the following: consider the

multiplication operator T in time domain, which acts as

$$T\phi(t) = t\phi(t). \quad (5.47)$$

The set of strictly bandlimited functions is a dense set in the Hilbert space of bandlimited functions. If $\phi(t)$ is a strictly bandlimited function, its Fourier transform $\Phi(\omega)$ has a finite support in the open interval $(-2\pi\Omega, 2\pi\Omega)$. Then so does its derivative $\Phi'(\omega) = \frac{d}{d\omega}\Phi(\omega)$. Because the multiplication operator T in time domain is the derivative operator $i\frac{d}{d\omega}$ in frequency domain, the fact that the derivative $\Phi'(\omega)$ also has a finite support in $(-2\pi\Omega, 2\pi\Omega)$ implies that $T\phi(t)$ is strictly bandlimited as well. In other words, whatever the bandwidth of the class of signals which we may consider, the action of T preserves that bandwidth. This suggests to define the class of signals of bandlimited functions as the closure of the invariant domain of the operator T .

Specifically, we saw in the frequency space that the differential operator $T = i\frac{d}{d\omega}$ acting on the set of absolutely continuous functions with vanishing boundary conditions is a simple symmetric operator with deficiency indices $(1, 1)$. Each self-adjoint extension of T possesses a set of equidistant spaced eigenvalues $\{t_n\}_n$ at the constant Nyquist rate, which is the Nyquist sampling grid. Each eigenvalue has an eigenvector. The eigenbasis of each self-adjoint extension forms a basis of the function space. The reconstruction formula is simply a decomposition of the function in that basis. Coincidentally, the coefficients of the function in the eigenbasis turn out to be the function values at the corresponding eigenvalues.

In functional analytical terms, the sampling theorem simply states the fact that if a Hilbert space vector is known in the Hilbert eigenbasis of one self-adjoint extension of T , then the vector's coefficients in the Hilbert space eigenbasis of all other self-adjoint extensions of T are also determined.

For the generalization, notice first that the usual notion of Fourier theory, which derives the Shannon sampling theorem, does not lead to the generalized sampling method with a time-varying Nyquist rate, because the Fourier series possess an equidistant nature. Hence in the generalization, we completely replace the conventional Fourier analysis by the use of the theory of self-adjoint extensions. So we work directly in time domain.

Recall that the functional analytical proof the Shannon sampling theorem in Section 4.3 is carried out in frequency domain with Fourier transform. We took the Fourier transform and worked in the frequency space because the concrete example of the differential operator on $L^2([-2\pi\Omega, 2\pi\Omega])$ is more well-known than the abstract theory of self-adjoint extensions of $(1, 1)$ -symmetric operator. However, all the calculations can be carried out without the Fourier transform. We can simply work directly in time domain with the multiplication operator T .

Nevertheless, the multiplication operator T in the proof of the Shannon sampling theorem is special because its self-adjoint extensions have equidistantly spaced eigenvalues.

By considering a general $(1,1)$ -symmetric operator T , whose self-adjoint extensions have generally non-equidistantly spaced eigenvalues, we expect the Shannon sampling theorem to be generalized for time-varying Nyquist sampling rates.

General Setup of the Multiplication Operator T

In an abstract separable complex Hilbert space \mathbb{H} , let T be the multiplication operator

$$T\phi = t\phi. \quad (5.48)$$

where the Hilbert space vector ϕ is simply an abstract Hilbert space vector. It is not yet a function of t . The definition of ϕ as a function in t will be made clear later.

We know that from Section 5.2.2, two sets of predetermined numbers $\{t_n = t_n(0)\}_n$ and $\{t'_n = t'_n(0)\}_n$ are given. The former set $\{t_n\}_n$ is a set of sampling points at the Nyquist rate, which are predetermined in Step (G1) depending on what Nyquist rate we require functions in function space to possess. We assume that there is minimum and maximum spacing between two adjacent points as in Eq. (5.18).

The set $\{t'_n\}_n$ is independent of $\{t_n\}_n$. It specifies the density of the sampling points $\{t_n\}$ and it is proportional to be the Nyquist rate. However, from the theory of self-adjoint extensions, we know that t'_n is the derivative, it determines how fast the sampling point $t_n(\alpha)$ is moving with respect to α . Hence, t'_n can be best approximated by the average spacing between adjacent points at t_n in Eq. (5.7). This not only provides a practical determination of $\{t'_n\}_n$ from the knowledge of the given Nyquist sampling grid $\{t_n\}_n$, but also agrees with the definition of the Nyquist rate in Section 5.2.4.

Such choices of $\{t_n\}_n$ and $\{t'_n\}_n$ satisfy the three conditions in Eq. (3.16) in Theorem 8 in Chapter 3. Hence, we can build the simple symmetric operator T with deficiency indices exactly as in Theorem 8: let $T(0)$ be the self-adjoint operator with eigenvalues $\{t_n\}_n$

$$T(0) = \sum_n t_n \langle \cdot, \phi_n(0) \rangle \phi_n(0) \quad (5.49)$$

where $\{\phi_n(0)\}_n$ can be any orthonormal basis of the Hilbert space \mathbb{H} . The vector $\phi_n(0)$ is the normalized eigenvector of $t_n = t_n(0)$. The specification of these vectors is not important, its representation as a function of time will be determined later.

Let $U(0)$ be the Cayley transform of $T(0)$. Using the set $\{t'_n\}$ to restrict the domain of $U(0)$ to obtain an isometric operator S as in Eq. (3.18) and (3.19). The inverse Cayley transform of S is the desired symmetric operator T .

The symmetric operator T is simple and has deficiency indices $(1,1)$. The $(1,1)$ -symmetric operator T has a $U(1)$ -family of self-adjoint extensions of symmetric operators,

which can be denoted as $\{T(\alpha)\}_n$ with $0 \leq \alpha < 1$. The self-adjoint extension specified by $\alpha = 0$ is indeed the self-adjoint operator $T(0)$ we started with in Eq. (5.49). The self-adjoint operator $T(\alpha)$ has an infinite set of eigenvalues, denoted by $\{t_n(\alpha)\}$. This is also a Nyquist sampling grid.

Proposition 7 states that, together, for all α , the set of points $\{t_n(\alpha) \mid 0 \leq \alpha < 1, n \in \mathbb{Z}\}$ covers the real line exactly once. For any real number t , there is a unique pair (n, α) in $\mathbb{Z} \times [0, 1)$ such that $t = t_n(\alpha)$. Each eigenvalue $t = t_n(\alpha)$ has a normalized eigenvector denoted by $\phi_t = \phi_n(\alpha)$.

Therefore, each Hilbert space vector ϕ defines a scalar-valued function $\phi(t)$, $t \in \mathbb{R}$. The function value at each $t \in \mathbb{R}$ is evaluated to be the inner product of Hilbert space vector ϕ and the corresponding eigenvector ϕ_t of the real number t :

$$\phi(t) = \langle \phi_t, \phi \rangle = \langle \phi_n(\alpha), \phi \rangle \quad \text{where } t = t_n(\alpha). \quad (5.50)$$

Proof of the results in Section 5.2

The reconstruction formula on the given sampling grid $\{t_n\}_n$ can be obtained by expanding the definition of Eq. (5.50) in the eigenbasis $\{\phi_n = \phi_{t_n}\}_n$

$$\begin{aligned} \phi(t) &= \langle \phi_t, \phi \rangle = \sum_n \langle \phi_t, \phi_n \rangle \langle \phi_n, \phi \rangle \\ &= \sum_n G(t, t_n) \phi(t_n). \end{aligned} \quad (5.51)$$

The left hand side is the continuous scalar-valued function $\phi(t)$ for all t , and the right hand side has the discrete samples $\{\phi(t_n)\}_n$ on $\{t_n\}_n$. The reconstruction kernel $G(t, t_n)$ is indeed the inner product of the eigenvectors in Eq. (3.14), which can be expressed in terms of predetermined sets $\{t_n\}_n$ and $\{t'_n\}_n$:

$$G(t, t_n) = \langle \phi_t, \phi_n \rangle = (-1)^{z(t, t_n)} \frac{\sqrt{t'_n}}{|t - t_n|} \left(\sum_m \frac{t'_m}{(t - t_m)^2} \right)^{-1/2}. \quad (5.52)$$

It is important to notice that the function $G(t, t_n)$ is real-valued in t . This is what one expects: if the signal's sample values are all real, the reconstructed signal must be real at all t .

The set of reconstruction kernels $\{G(t, t_n)\}_n$ form a basis of the function space. The fact that $G(t, t_n)$ equals 0 at all sampling points except t_n makes it easy to construct functions in the space. The function values on $\{t_n\}_n$ are free to choose. Once they are determined, one can use Eq. (5.51) to construct all the functions in the space.

Therefore, given the two predetermined sets $\{t_n\}_n$ and $\{t'_n\}_n$, the function space is mathematically well defined. For practical purposes, an engineer would usually obtain the functions in the space by filtering, which will be proved later.

To find out the one-parameter family of sampling grids $\{t_n(\alpha)\}_n$ for $0 \leq \alpha < 1$, note that for a fixed α , the set of sampling points at the Nyquist rate specified by the parameter α is the set of eigenvalues of the self-adjoint operator $T(\alpha)$. Hence, the governing equation of the Nyquist sampling points Eq. (5.10) or Eq. (5.20) in Section 5.2.2 is exactly the governing equation of eigenvalues of self-adjoint extensions Eq. (3.13) in Theorem 8.

On each fixed sampling grid, $\{t_n(\alpha)\}_n$ with α fixed, functions in the function space can be perfectly reconstructed from their values on the set $\{t_n(\alpha)\}_n$. Similar to Eq. (5.51), the functions defined in Eq. (5.50) can also be expanded in the eigenbasis $\{\phi_n(\alpha)\}_n$ of the self-adjoint extension $T(\alpha)$

$$\begin{aligned}\phi(t) &= \langle \phi_t, \phi \rangle = \sum_n \langle \phi_t, \phi_n(\alpha) \rangle \langle \phi_n(\alpha), \phi \rangle \\ &= \sum_n G(t, t_n(\alpha)) \phi(t_n(\alpha))\end{aligned}\tag{5.53}$$

where the reconstruction kernel $G(t, t_n(\alpha))$ is the inner product of two eigenvectors $\phi_m(\beta)$ and $\phi_n(\alpha)$ with $t = t_m(\beta)$ for some unique m and β . Theorem 8 in Chapter 3 shows that this inner product can be expressed in a similar form as in Eq. (5.52), which is Eq. (5.11) or Eq. (5.25) in Section 5.2.3.

The normalization equations Eq. (5.19) and Eq. (5.24) in Section 5.2.2 follow from the third condition in Eq. (3.16).

Proof of the results in Section 5.3

To prove the alternating sum criterion Eq. (5.34) in Section 5.3, we use the results in the proof of Theorem 8. First, note that the domain of T is a dense set of the Hilbert space. For any function in the function space, there must be a sequence of functions in the domain of T arbitrary close to it. Any vector in $D(T)$ must obey Eq. (3.25), which implies that

$$\langle (T(0) + i)\phi, \phi_+ \rangle = \langle \phi, (T(0) - i)\phi_+ \rangle = 0\tag{5.54}$$

where ϕ_+ is the first deficiency vector space which can be expressed in the eigenbasis $\{\phi_n\}_n$ as in Eq. (3.20). The square of its coefficients f_n is the square root of the derivative t'_n divided by π as in Eq. (3.38). Hence, substitute Eq. (3.35) and Eq. (3.38) into Eq. (3.20)

to get

$$\begin{aligned}
\phi_+ &= \sum_n \frac{f_n}{t_n - i} \phi_n = \sum_n \frac{(-1)^n |f_n|}{t_n - i} \phi_n \\
&= \sum_n \frac{(-1)^n \sqrt{\frac{t'_n}{\pi}}}{t_n - i} \phi_n = \frac{1}{\sqrt{\pi}} \sum_n \frac{(-1)^n \sqrt{t'_n}}{t_n - i} \phi_n.
\end{aligned} \tag{5.55}$$

Substitute this into Eq. (5.54) to get

$$\begin{aligned}
&\frac{1}{\sqrt{\pi}} \left\langle \phi, (T(0) - i) \left(\sum_n \frac{(-1)^n \sqrt{t'_n}}{t_n - i} \phi_n \right) \right\rangle = 0 \\
\Rightarrow &\sum_n \left\langle \phi, \frac{(-1)^n \sqrt{t'_n}}{t_n - i} (T(0) - i) \phi_n \right\rangle = 0 \\
\Rightarrow &\sum_n (-1)^n \sqrt{t'_n} \left\langle \phi, \frac{1}{t_n - i} (t_n - i) \phi_n \right\rangle = 0 \\
\Rightarrow &\sum_n (-1)^n \sqrt{t'_n} \langle \phi, \phi_n \rangle = \sum_n (-1)^n \sqrt{t'_n} \phi(t_n) = 0.
\end{aligned} \tag{5.56}$$

This is the vanishing alternating sum up to a constant in Eq. (5.35) in Section 5.3.

Chapter 6

Filtering in the Generalized Sampling Method

So far, we have illustrated the sampling and reconstruction parts of the generalized sampling method, namely, Step (G3) and (G4) in the dashed part in Figure 5.1. In this chapter, we will introduce the pre-filtering part, i.e. Step (G2) in the generalized sampling method.

Although the function space with time-varying Nyquist rate is well-defined as spanned by the generalized sinc functions, the function space is not automatically available for practical purposes. Here we will show how arbitrary raw signals can be filtered into the space of functions that can be perfectly represented by their non-equidistantly spaced Nyquist samples. The results on the filter operator P here are based on my paper [36].

6.1 The Time-Varying Filter Operator

The challenge to generalize the conventional low-pass filter with constant bandwidth to a time-varying bandwidth is that the filtering operation in Shannon sampling is done in the frequency domain with Fourier transform, but Fourier analysis is not applicable in the generalized sampling method.

Hence, let us first examine the low-pass filter directly in the time domain. We will then give an intuitive derivation of the generalized version of the filter operator and then show that this resulting filter operator does have the correct properties. Let P denote the filter operator.

In the Shannon sampling method, since arbitrary given raw signals $\phi^{\text{raw}}(t)$ are generally not bandlimited, one usually passes the signals through a low-pass filter, which removes all the frequencies above the bandlimit Ω . The bandlimit Ω is chosen so that the disregarded

frequencies are negligible. The resulting signal $\phi(t)$ approximates the raw signal $\phi^{raw}(t)$ and is in the space of Ω -bandlimited functions.

Let $\phi(t)$ and $\Phi(\omega)$ denote a signal and its Fourier transform. The filtering operation $\phi(t) = (P\phi^{raw})(t)$ in frequency space is

$$\Phi(\omega) = \Phi^{raw}(\omega) \operatorname{rect}\left(\frac{\omega}{4\pi\Omega}\right).$$

In the time domain, this is equivalent to the convolution of $\phi^{raw}(t)$ with the sinc function

$$\begin{aligned} \phi(t) &= (P\phi^{raw})(t) = \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} \Phi^{raw}(\omega) \operatorname{rect}\left(\frac{\omega}{4\pi\Omega}\right) d\omega \\ &= \phi^{raw}(t) * (2\Omega \operatorname{sinc}(2\Omega t)) \\ &= \int_{\hat{t}=-\infty}^{\infty} \phi^{raw}(\hat{t}) \left(2\Omega \operatorname{sinc}(2\Omega(t - \hat{t})) \right) d\hat{t}. \end{aligned}$$

Hence, the filter operator P can be written in the form of Eq. (5.5)

$$\phi(t) = (P\phi^{raw})(t) = \int_{-\infty}^{\infty} \phi^{raw}(\hat{t}) P(t, \hat{t}) \nu(\hat{t}) d\hat{t} \quad (6.1)$$

where $P(t, \hat{t}) = \operatorname{sinc}(2\Omega(t - \hat{t}))$ and $\nu(t) = 2\Omega$ as in Eq. (5.3).

Notice that $\operatorname{sinc}(2\Omega(t - \hat{t}))$ is the sinc reconstruction kernel of t centered at \hat{t} . To generalize, one replaces the sinc term with the generalized reconstruction kernel $G(t, \hat{t})$. Recall that for any real number \hat{t} , there exists an \hat{n} and $\hat{\alpha} \in [0, 1)$ such that $\hat{t} = t_{\hat{n}}(\hat{\alpha})$. Hence, for the generalized sampling method, this suggests the following

$$P(t, \hat{t}) = G(t, t_{\hat{n}}(\hat{\alpha})). \quad (6.2)$$

Notice that $P(t, \hat{t}) = G(t_n(\alpha), t_{\hat{n}}(\hat{\alpha}))$ is commutative, i.e., $P(t, \hat{t}) = P(\hat{t}, t)$, because the underlying definition of $G(t, \hat{t})$ in Eq. (5.51) is the inner product of two Hilbert space eigenvectors associated to the eigenvalues t and \hat{t} and the inner product is commutative.

On a fixed sampling grid, say $\{t_n = t_n(\alpha)\}$ with α fixed, since $G(t, t_{\hat{n}}(\hat{\alpha}))$ is a function in t and it is in the function space, one can expand $G(t, t_{\hat{n}}(\hat{\alpha}))$ by the reconstruction formula Eq. (5.6) and (5.11) to obtain an expression of $P(t, \hat{t})$ in terms of one Nyquist

sampling grid $\{t_n\}$ and $\{t'_n\}_n$:

$$\begin{aligned}
P(t, \hat{t}) &= G(t, t_{\hat{n}}(\hat{\alpha})) = \sum_n G(t, t_n) G(t_n, t_{\hat{n}}(\hat{\alpha})) \\
&= \sum_n G(t, t_n) G(\hat{t}, t_n) \\
&= \sum_n \left[\left((-1)^{z(t, t_n)} \frac{\sqrt{t'_n}}{|t - t_n|} \left(\sum_m \frac{t'_m}{(t - t_m)^2} \right)^{-1/2} \right) \right. \\
&\quad \left. \left((-1)^{z(\hat{t}, t_n)} \frac{\sqrt{t'_n}}{|\hat{t} - t_n|} \left(\sum_m \frac{t'_m}{(\hat{t} - t_m)^2} \right)^{-1/2} \right) \right] \\
&= \sum_n \left[\frac{(-1)^{z(t, t_n) + z(\hat{t}, t_n)} t'_n}{|t - t_n| |\hat{t} - t_n|} \right] \left(\sum_m \frac{t'_m}{(t - t_m)^2} \right)^{-1/2} \left(\sum_m \frac{t'_m}{(\hat{t} - t_m)^2} \right)^{-1/2} \\
&= (-1)^{z(t, \hat{t})} \left(\sum_n \frac{t'_n}{(t - t_n)(\hat{t} - t_n)} \right) \left(\sum_m \frac{t'_m}{(t - t_m)^2} \right)^{-1/2} \left(\sum_m \frac{t'_m}{(\hat{t} - t_m)^2} \right)^{-1/2}.
\end{aligned} \tag{6.3}$$

Recall that $z(t, \hat{t})$ is defined to be the number of sampling points in $\{t_m\}_m$ between t and \hat{t} exclusively. If both t and \hat{t} are on the same side of t_n on the real line, then $(-1)^{z(t, t_n) + z(\hat{t}, t_n)} = (-1)^{z(t, \hat{t})}$ and $(t - t_n)(\hat{t} - t_n)$ is positive; else, if both t and \hat{t} are on the opposite of t_n on the real line, $(-1)^{z(t, t_n) + z(\hat{t}, t_n)}$ is the negative of $(-1)^{z(t, \hat{t})}$, but $(t - t_n)(\hat{t} - t_n)$ is negative as well. Therefore, the overall sign of the n -th term above is simplified to $(-1)^{z(t, \hat{t})}$ multiplied by the sign of $(t - t_n)(\hat{t} - t_n)$ in the denominator. This is Eq. (5.9) in Section 5.1.

Now, let us consider the term $\nu(t) = 2\Omega$ in Shannon's filter operator Eq. (6.1). Recall that in Shannon sampling, $(2\Omega)^{-1}$ is the constant Nyquist sampling rate which controls the spacing between the equidistant Nyquist sampling points. For the generalized sampling method, $t'_n(\alpha) = \frac{dt_n(\alpha)}{d\alpha}$ plays the role of $1/(2\Omega)$. Hence, its inverse $\frac{d\alpha}{dt}$, which is a continuous function in $t = t_n(\alpha)$, can replace the (2Ω) -term in the case of Shannon. In the generalized filter operator Eq. (6.1)

$$\nu(t) = \frac{d\alpha}{dt} = (t'_n(\alpha))^{-1}. \tag{6.4}$$

The function $\alpha(t)$ is defined by the governing equation Eq. (5.20) of the Nyquist sampling grids in Section 5.2.2.

Analogous to Shannon's low-pass filter, we obtained a formula for the filter operator P in Eq. (5.5) for time-varying Nyquist rate. The derivation is heuristic, but intuitive.

In the next section, we will verify that such a filter operator P with $P(t, \hat{t})$ given in Eq. (6.3) and $\nu(t)$ given in Eq. (6.4) indeed maps any given raw signal into the desired

function space with a pre-specified Nyquist rate, and if one applies the filter operator on an already-filtered signal, it results the same signal, namely, $P^2 = P$.

It needs to be pointed out that although the filter operator P maps functions into the function space, it is not a projection operator in the L^2 -space with the usual flat measure. In Section 6.3, we will see that the operator P is a projection operator with an uneven measure $\mu(t) = \frac{d\alpha}{dt}$.

6.2 Verification of the Result on Filtering

To show that the filter operator P defined in Eq. (5.5) obeys the properties of a filter operator, namely, P maps any given raw signal into the function space of the predetermined time-varying Nyquist rate and $P^2 = P$, we need to use the discrete and continuous resolution of the identity in the Hilbert space. Some results from the theory of self-adjoint extensions in Chapter 3 and Section 5.5 will be used in this and the next section.

The discrete resolution of the identity can be obtained by an expansion of the identity operator in any eigenbasis. Specifically, for any fixed α , the normalized eigenvectors $\{\phi_n(\alpha)\}_n$ corresponding to the eigenvalues $\{t_n(\alpha)\}_n$ of the self-adjoint operator $T(\alpha)$ form an eigenbasis. The expansion of the identity operator in this basis yields the discrete resolution of the identity

$$\begin{aligned} P(t, \hat{t}) &= \langle \phi_t, \phi_{\hat{t}} \rangle = \sum_n \langle \phi_t, \phi_n(\alpha) \rangle \langle \phi_n(\alpha), \phi_{\hat{t}} \rangle \\ &= \sum_{n=-\infty}^{\infty} G(t, t_n(\alpha)) G(\hat{t}, t_n(\alpha)). \end{aligned} \quad (6.5)$$

As α runs from 0 to 1, the point $t_n(\alpha)$ runs continuously from $t_n(0)$ to $t_n(1) = t_{n+1}(0)$ for all n . Together, for all n and α , $\{t_n(\alpha)\}$ covers the whole real line exactly once with no gap. For each real number t , there is a unique α and a unique integer n such that $t = t_n(\alpha)$. Hence, one can define $\alpha = \alpha(t)$ as a continuous function of t in \mathbb{R} . Integrating Eq. (6.5) with respect to α from 0 to 1, one has

$$\begin{aligned} P(t, \hat{t}) &= P(t, \hat{t}) \int_{\alpha=0}^1 1 d\alpha = \int_{\alpha=0}^1 P(t, \hat{t}) d\alpha \\ &= \int_{\alpha=0}^1 \left(\sum_{n=-\infty}^{\infty} G(t, t_n(\alpha)) G(\hat{t}, t_n(\alpha)) \right) d\alpha \\ &= \sum_{n=-\infty}^{\infty} \left(\int_{\alpha=0}^1 G(t, t_n(\alpha)) G(\hat{t}, t_n(\alpha)) d\alpha \right). \end{aligned} \quad (6.6)$$

For each n in \mathbb{Z} , change the integrating variable from α to \bar{t} . As α goes from 0 to 1, \bar{t} runs from $t_n(0)$ to $t_n(1) = t_{n+1}(0)$. Hence,

$$\begin{aligned} P(t, \hat{t}) &= \sum_{n=-\infty}^{\infty} \left(\int_{\bar{t}=t_n(0)}^{t_{n+1}(0)} G(t, \bar{t}) G(\hat{t}, \bar{t}) \frac{d\alpha}{d\bar{t}} d\bar{t} \right) \\ &= \int_{\bar{t}=-\infty}^{\infty} P(t, \bar{t}) P(\bar{t}, \hat{t}) \frac{d\alpha}{d\bar{t}} d\bar{t}. \end{aligned} \quad (6.7)$$

As discussed in Section 6.1, due to the commutative property of inner product, the kernel $P(t, \hat{t})$ is also commutative, namely,

$$P(t, \hat{t}) = G(t, \hat{t}) = \langle \phi_t, \phi_{\hat{t}} \rangle = \langle \phi_{\hat{t}}, \phi_t \rangle = P(\hat{t}, t). \quad (6.8)$$

Now, using (6.5) and (6.7), it is straightforward to show that the linear operator P maps $L^2(\mathbb{R})$ -signals into the generalized sampling function space.

To this end, recall that functions in the generalized sampling space are spanned by the set of generalized sinc functions in $G(t, t_n)$ on any fixed grid $\{t_n\}_n$. Their coefficients in this expansion are their function values on $\{t_n\}_n$.

Let $\phi^{raw}(t)$ be an arbitrary $L^2(\mathbb{R})$ -function and $\phi(t) = P\phi^{raw}(t)$. To see if $\phi(t)$ is indeed in the function space, notice that the summation of its function value at each t_n multiplied by the basis function $G(t, t_n)$ is in the function space, because each $G(t, t_n)$ is in the function space already. We simply need check if this summation turns out to be $\phi(t)$ or not. Using $\phi(t_n) = P\phi^{raw}(t_n)$ by its definition in Eq. (5.5), it follows that

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \phi(t_n) G(t, t_n) &= \sum_n (P\phi^{raw})(t_n) G(t, t_n) \\ &= \sum_n \left(\int_{-\infty}^{\infty} \phi^{raw}(\hat{t}) P(t_n, \hat{t}) \nu(\hat{t}) d\hat{t} \right) G(t, t_n) \\ &= \int_{-\infty}^{\infty} \phi^{raw}(\hat{t}) \left(\sum_n P(t_n, \hat{t}) G(t, t_n) \right) \nu(\hat{t}) d\hat{t} \\ &= \int_{-\infty}^{\infty} \phi^{raw}(\hat{t}) \left(P(t, \hat{t}) \right) \nu(\hat{t}) d\hat{t} \quad \text{using Eq. (6.5)} \\ &= (P\phi^{raw})(t) = \phi(t). \end{aligned}$$

The filtered function $\phi(t)$ is a linear combination of the generalized sinc functions. Hence P indeed maps functions into the desired function space. Further, from Eq. (6.7),

$$PG(t, t_n) = \int_{\hat{t}=-\infty}^{\infty} G(\hat{t}, t_n) P(t, \hat{t}) \frac{d\alpha}{d\hat{t}} d\hat{t} = G(t, t_n).$$

Hence P maps each generalized sinc function $G(t, t_n)$ to itself. Therefore, if $\phi(t)$ is a linear combination of $G(t, t_n)$, by linearity, this also holds for $\phi(t)$, namely, $P\phi(t) = \phi(t)$. The range of P is indeed the desired generalized function space and $P^2 = P$.

6.3 Projection with the Time-Varying Measure $\mu(t) = \frac{d\alpha}{dt}$

Although the filter operator P maps the $L^2(\mathbb{R})$ -functions into the pre-specified function, P is not a projection operator in the bigger $L^2(\mathbb{R})$ -function space with the usual flat measure. It is straight forward to show that in general

$$\langle P\psi^{\text{raw}}(t), \phi^{\text{raw}}(t) \rangle \neq \langle \psi^{\text{raw}}(t), P\phi^{\text{raw}}(t) \rangle. \quad (6.9)$$

This is due to the time-varying bandwidth of the filtered signals. The functions in the function space pre-specified by the Nyquist grid $\{t_n\}_n$ and its derivatives $\{t'_n\}$ are scalar-valued L^2 -functions, but they are the representations of the underlying abstract Hilbert space vectors. This function space is subject to an uneven measure $\mu(t) = \frac{d\alpha}{dt}$.

To show that P is a projection operator with this uneven measure, it suffices to show the self-adjointness of P with the following inner product

$$\langle \psi(t), \phi(t) \rangle_{\frac{d\alpha}{dt}} = \int_{t=-\infty}^{\infty} \psi^*(t) \phi(t) \frac{d\alpha}{dt} dt. \quad (6.10)$$

This inner product has an uneven measure $\mu(t) = \frac{d\alpha}{dt}$. Then

$$\begin{aligned} \langle P\psi(t), \phi(t) \rangle_{\frac{d\alpha}{dt}} &= \int_{t=-\infty}^{\infty} (P\psi(t))^* \phi(t) \frac{d\alpha}{dt} dt \\ &= \int_{t=-\infty}^{\infty} \left(\int_{\hat{t}=-\infty}^{\infty} \psi^*(\hat{t}) G(t, \hat{t}) \frac{d\alpha}{d\hat{t}} d\hat{t} \right)^* \phi(t) \frac{d\alpha}{dt} dt \\ &= \int_{\hat{t}=-\infty}^{\infty} \psi^*(\hat{t}) \left(\int_{t=-\infty}^{\infty} G(t, \hat{t}) \phi(t) \frac{d\alpha}{dt} dt \right)^* \frac{d\alpha}{d\hat{t}} d\hat{t} \\ &= \int_{t=-\infty}^{\infty} \psi^*(t) (P\phi(t)) dt \\ &= \langle \psi(t), P\phi(t) \rangle_{\frac{d\alpha}{dt}}. \end{aligned} \quad (6.11)$$

Since $P^2 = P$, the fact that P is self-adjoint concludes that the operator P is a projection operator onto the function space with the uneven measure $\mu(t) = \frac{d\alpha}{dt}$ which arises from the pre-specified time-varying Nyquist rate.

Chapter 7

The Stability of the Generalized Sampling Theorem

A crucial question in sampling and interpolation theory is the question of stability. There are two types of stability that are important to the generalized sampling theorem for time-varying Nyquist rates.

First, we need to find upper and lower bounds between the L^2 -norm of continuous signals with the time-varying Nyquist rate and the l^2 -norm of the discrete samples taken at the Nyquist rate¹. This stability issue is addressed in my paper [33] and will be discussed in Section 7.1.

Secondly, we will show that the time-varying Nyquist rate of the function space inherits an important stability property of the constant Nyquist rate in the Shannon sampling theorem, namely, sampling at the Nyquist rate gives the most stable reconstruction. Any sample point deviated from the Nyquist grid will deteriorate the stability of reconstruction. This result is discussed in Section 7.2 and can also be found in my paper [37].

7.1 Stability: Bounds on Reconstruction Errors

The stability of reconstruction is measured by the ratio between the L^2 -norm of the constructed continuous signal $\phi(t)$ and the l^2 -norm of the discrete samples $\{\phi(t_n)\}_n$. For example, sample quantization necessarily introduces l^2 -errors. In the case of Shannon sampling theory, the L^2 -norm of a Ω -bandlimited function $\phi(t)$ is $\frac{1}{2\Omega}$ multiplied by the

¹The ordered set of discrete sample values $\{\phi(t_n)\}_{n=-\infty}^{+\infty}$ are treated as a sequence vector in l^2 Hilbert space with the standard l^2 -norm.

l^2 -norm of its discrete samples $\{\phi(t_n)\}_n$

$$\frac{1}{2\Omega} \|\phi(t_n)\|_{l^2}^2 = \|\phi(t)\|_{L^2}^2. \quad (7.1)$$

The boundedness of the relation between the L^2 -norm and the l^2 -norm is a safeguard for the reconstructed signals. A finite bound between the two norms ensures the stability of reconstruction: a square-summable set of samples will yield a square-integrable function. Further any error in the signal's sample leads to a bounded error in the reconstructed continuous signals.

In the case of the generalized sampling method, the frame bounds are not tight, but in the following, we will find the upper and lower bounds between the norms of the continuous signal and its discrete samples.

First, we note that the l^2 -norm of the samples at a Nyquist sampling rate is unambiguously defined. The l^2 -norms of samples on different Nyquist sampling grids are the same because they are the norm of the function vector in the abstract Hilbert space on which the underlying self-adjoint operators act. The samples at the Nyquist rate are the coefficients of the Hilbert space vector spanned in an eigenbasis, their l^2 -norm equals the norm of the Hilbert space vector.

Let $\mu(\alpha)$ be a probability density function, $0 \leq \alpha < 1$, such that

$$0 \leq \mu(\alpha) \leq 1 \text{ and } \int_0^1 \mu(\alpha) d\alpha = 1. \quad (7.2)$$

Then

$$\begin{aligned} \|\phi(t_n(\alpha))\|_{l^2}^2 &= \sum_n |\phi(t_n(\alpha))|^2 \quad \forall 0 \leq \alpha < 1 \\ &= \int_{\alpha=0}^1 \mu(\alpha) \sum_n |\phi(t_n(\alpha))|^2 d\alpha \\ &= \int_{\alpha=0}^1 \mu(\alpha) \sum_n |\phi(t_n(\alpha))|^2 \frac{1}{t'_n(\alpha)} \frac{dt_n(\alpha)}{d\alpha} d\alpha. \end{aligned} \quad (7.3)$$

Here we multiplied and divided by $t'_n(\alpha) = \frac{dt_n(\alpha)}{d\alpha}$. Now, for a fixed α between 0 and 1, we define that

$$\begin{aligned} \Delta_{\min}(\alpha) &= \min_{n \in \mathbb{Z}} t'_n(\alpha) \\ \Delta_{\max}(\alpha) &= \max_{n \in \mathbb{Z}} t'_n(\alpha). \end{aligned} \quad (7.4)$$

Using a different parametrization of α , we can make $\Delta_{\min}(\alpha)$ and $\Delta_{\max}(\alpha)$ arbitrarily small and arbitrarily large for some particular values of α . In order to have a parametrization

independent measures of $t'_n(\alpha)$, we integrate $\Delta_{\min}(\alpha)$ and $\Delta_{\max}(\alpha)$ for all $0 \leq \alpha < 1$. We therefore define² that

$$\begin{aligned}\Delta_{\min} &= \int_0^1 \Delta_{\min}(\alpha) d\alpha = \int_0^1 \left(\min_{n \in \mathbb{Z}} t'_n(\alpha) \right) d\alpha \\ \Delta_{\max} &= \int_0^1 \Delta_{\max}(\alpha) d\alpha = \int_0^1 \left(\max_{n \in \mathbb{Z}} t'_n(\alpha) \right) d\alpha.\end{aligned}\tag{7.5}$$

These two quantities are independent of re-parametrization. To see this, let $\gamma = \gamma(\alpha)$ be a re-parametrization of α , then, by the Chain rule,

$$\Delta_{\min} = \int_0^1 \min_{n \in \mathbb{Z}} \left(\frac{t_n(\alpha)}{d\alpha} \right) d\alpha = \int_0^1 \min_{n \in \mathbb{Z}} \left(\frac{t_n(\gamma)}{d\gamma} \frac{d\gamma}{d\alpha} \right) d\alpha.$$

Since $\frac{d\gamma}{d\alpha}$ is independent of n , we can pull out $\frac{d\gamma}{d\alpha}$ and change the variable of integration to obtain

$$\begin{aligned}\Delta_{\min} &= \int_0^1 \left[\min_{n \in \mathbb{Z}} \left(\frac{t_n(\gamma)}{d\gamma} \right) \right] \frac{d\gamma}{d\alpha} d\alpha \\ &= \int_0^1 \left[\min_{n \in \mathbb{Z}} \left(\frac{t_n(\gamma)}{d\gamma} \right) \right] d\gamma = \int_0^1 \Delta_{\min}(\gamma) d\gamma.\end{aligned}$$

Similarly Δ_{\max} is independent of reparametrization as well.

The definition of $\Delta_{\min}(\alpha)$ and $\Delta_{\max}(\alpha)$ in Eq. (7.5) implies the following inequality:

$$\Delta_{\min}(\alpha) \leq t'_n(\alpha) \leq \Delta_{\max}(\alpha) \quad \text{for all } n.\tag{7.6}$$

Substituting this into Eq. (7.3), we have

$$\begin{aligned}\|\phi(t_n)\|_{l^2}^2 &\geq \int_{\alpha=0}^1 \mu(\alpha) \sum_n |\phi(t_n(\alpha))|^2 \frac{1}{\Delta_{\max}(\alpha)} \frac{dt_n(\alpha)}{d\alpha} d\alpha \\ &= \int_{\alpha=0}^1 \frac{\mu(\alpha)}{\Delta_{\max}(\alpha)} \sum_n |\phi(t_n(\alpha))|^2 \frac{dt_n(\alpha)}{d\alpha} d\alpha\end{aligned}\tag{7.7}$$

and

$$\begin{aligned}\|\phi(t_n)\|_{l^2}^2 &\leq \int_{\alpha=0}^1 \mu(\alpha) \sum_n |\phi(t_n(\alpha))|^2 \frac{1}{\Delta_{\min}(\alpha)} \frac{dt_n(\alpha)}{d\alpha} d\alpha \\ &= \int_{\alpha=0}^1 \frac{\mu(\alpha)}{\Delta_{\min}(\alpha)} \sum_n |\phi(t_n(\alpha))|^2 \frac{dt_n(\alpha)}{d\alpha} d\alpha.\end{aligned}\tag{7.8}$$

²The assumption of the minimum and maximum values of $t'_n(\alpha)$ guarantees that Δ_{\min} is positive and Δ_{\max} is finite. See Eq. (5.18)

Here both $\Delta_{\min}(\alpha)$ and $\Delta_{\max}(\alpha)$ are independent of the summation index n , but are α -dependent. Now choose the probability density function $\mu(\alpha)$ to be

$$\mu(\alpha) = \frac{1}{\Delta_{\max}} \Delta_{\max}(\alpha), \quad 0 \leq \alpha < 1$$

so that it satisfies $0 \leq \mu(\alpha) \leq 1$ and $\int_0^1 \mu(\alpha) d\alpha = 1$. Then we can further simplify Eq. (7.7) to get

$$\begin{aligned} \|\phi(t_n)\|_{l^2}^2 &\geq \int_{\alpha=0}^1 \frac{\mu(\alpha)}{\Delta_{\max}(\alpha)} \sum_n |\phi(t_n(\alpha))|^2 \frac{dt_n(\alpha)}{d\alpha} d\alpha \\ &= \int_{\alpha=0}^1 \frac{1}{\Delta_{\max}} \sum_n |\phi(t_n(\alpha))|^2 \frac{dt_n(\alpha)}{d\alpha} d\alpha \\ &= \frac{1}{\Delta_{\max}} \int_{t_n(\alpha)=t_n(0)}^{t_{n+1}(0)} \sum_n |\phi(t_n(\alpha))|^2 dt_n(\alpha) \\ &= \frac{1}{\Delta_{\max}} \sum_n \left[\int_{t=t_n(0)}^{t_{n+1}(0)} |\phi(t)|^2 dt \right] \quad \text{let } t = t_n(\alpha) \\ &= \frac{1}{\Delta_{\max}} \int_{t=-\infty}^{+\infty} |\phi(t)|^2 dt \\ &= \frac{1}{\Delta_{\max}} \|\phi(t)\|_{L^2}^2 \end{aligned} \tag{7.9}$$

by a change of variable and using the fact that the sampling grids $\{t_n(\alpha)\}_n$ together cover the real line exactly once. Similarly, if we using another density function

$$\mu(\alpha) = \frac{1}{\Delta_{\min}} \Delta_{\min}(\alpha), \quad 0 \leq \alpha < 1$$

in Eq. (7.8), we can also find a lower bound in a similar manner

$$\|\phi(t_n)\|_{l^2}^2 \leq \frac{1}{\Delta_{\min}} \|\phi(t)\|_{L^2}^2. \tag{7.10}$$

Therefore, on any fixed Nyquist sampling grid, Eq. (7.9) and (7.10) give

$$\frac{1}{\Delta_{\max}} \|\phi(t)\|_{L^2}^2 \leq \|\phi(t_n)\|_{l^2}^2 \leq \frac{1}{\Delta_{\min}} \|\phi(t)\|_{L^2}^2 \tag{7.11}$$

or equivalently,

$$\Delta_{\min} \|\phi(t_n)\|_{l^2}^2 \leq \|\phi(t)\|_{L^2}^2 \leq \Delta_{\max} \|\phi(t_n)\|_{l^2}^2. \tag{7.12}$$

Therefore, we have found upper and lower bounds of the L^2 -norm of the continuous signal and the l^2 -norm of the discrete samples. These bounds ensure that the reconstruction is stable. Any error in the measurements of the signals' samples leads to a bounded error in the reconstructed continuous signals. The size of the errors is bounded by the average of the maximum values of $t'_n(\alpha)$.

In the case of Shannon, the frame bounds are tight. We expect the constant bounds $\Delta_{\min} = \frac{1}{2\Omega} = \Delta_{\max}$ as in Eq. (7.1). We have shown above that the formula of Δ_{\min} and Δ_{\max} are both parametrization independent, so we can use the linear θ -parametrization $\{t_n(\theta) = \frac{n+\theta}{2\Omega}\}_n$

$$\begin{aligned} \Delta_{\min} &= \int_{\alpha=0}^1 \Delta_{\min}(\alpha) d\alpha = \int_{\theta=0}^1 \Delta_{\min}(\theta) d\theta \\ &= \int_{\alpha=0}^1 \left(\min_n \frac{dt_n(\theta)}{d\theta} \right) d\theta = \int_{\theta=0}^1 \left(\min_n \frac{1}{2\Omega} \right) d\theta = \frac{1}{2\Omega}. \end{aligned} \tag{7.13}$$

The same result holds for Δ_{\max} . Thus, we have a tight bound in the case of Shannon as expected.

7.2 The Time-Varying Nyquist Rates yield the Most Stable Reconstruction in their Function Space

In the Shannon sampling theorem, the constant Nyquist rate not only happens to be the critical sampling rate between over- and under-sampling. It also is the rate with the most stable reconstruction. If any sampling point deviates from the Nyquist sampling grid, it will deteriorate the stability of the reconstruction. In this section, we will show that this is also true for the new time-varying Nyquist rate.

In the Shannon sampling theorem, one can also reconstruct the signal from a set of non-uniform samples as long as the average sampling density is at least the constant Nyquist rate. Specifically, let $\{t_n\}_n$ be a set of Nyquist rate sampling points with $t_{n+1} - t_n = \frac{1}{2\Omega}$, and let $\{\tau_n\}_n$ be a set of sampling points with the same average (Beurling) density as $\{t_n\}_n$, see [16]. Any Ω -bandlimited signal $\phi(t)$ can be reconstructed for all t from the samples taken on both $\{t_n\}_n$ or $\{\tau_n\}_n$ via

$$\begin{aligned} \phi(t) &= \sum_n \phi(t_n) \operatorname{sinc}(2\Omega(t - t_n)) \\ &= \sum_n \phi(\tau_n) S_n(t). \end{aligned} \tag{7.14}$$

Both the set of shifted sinc reconstruction kernel and the set of $S_n(t)$ are bases for the function space. As demonstrated in Yen's paper [14], the bunching of non-uniform sampling points in the traditional framework yields a high amplitude of the basis functions and thus leads to an unstable reconstruction. To understand the cause of the instability, one could decompose the reconstruction of $\phi(t)$ from non-Nyquist samples $\{\phi(\tau_n)\}_n$ into two steps, namely,

$$\{\phi(\tau_n)\}_n \rightarrow \{\phi(t_n)\}_n \rightarrow \phi(t). \quad (7.15)$$

In the second step, $\{\phi(t_n)\}_n \rightarrow \phi(t)$, the reconstruction from the Nyquist samples is known to possess a constant norm bound $1/2\Omega$. Therefore, the stability of the reconstruction from the non-Nyquist sampling grid $\{\tau_n\}_n$ depends on the first step: converting the sample values from the non-Nyquist sampling grid $\{\tau_n\}_n$ to the Nyquist grid $\{t_n\}_n$. Let L denote such an operator and M be its inverse.

$$\begin{aligned} L &: \{\phi(\tau_n)\}_n \rightarrow \{\phi(t_n)\}_n \\ M = L^{-1} &: \{\phi(t_n)\}_n \rightarrow \{\phi(\tau_n)\}_n \end{aligned} \quad (7.16)$$

The map L can be considered as an infinite square matrix, acting on the column vector with entries $\{\phi(\tau_n)\}_n$ and mapping to the column vector with entries $\{\phi(t_n)\}_n$. In functional analysis, it is equivalent to mapping from one normalized but non-orthogonal basis to an orthonormal basis. The orthonormal basis corresponds to the discrete samples on $\{t_n\}_n$ at the Nyquist rate, while the samples on $\{\tau_n\}_n$ are associated with a normalized but non-orthogonal basis.

The stability of the operator L is measured by its condition number, denoted by $\kappa(L)$. The condition number of L is the same as the one of its inverse $M = L^{-1}$ because

$$\kappa(L) = \|L\| \|L^{-1}\| = \|L\| \|M\| = \kappa(M). \quad (7.17)$$

The condition number of a linear operator L is always greater than or equal to 1. It equals to 1 when L is unitary, namely, when the set $\{\tau_n\}_n$ is another Nyquist sampling grid. Therefore, the stability of reconstruction from $\{\phi(\tau_n)\}_n$ to $\{\phi(t_n)\}_n$ deteriorates if $\kappa(L)$, or equivalently $\kappa(M)$, gets larger. When $\kappa(M) > 1$, the stability of reconstruction from $\{\phi(\tau_n)\}_n$ is worse than the reconstruction at the constant Nyquist rate.

Let us now show that this is true for the generalized sampling theory for time-varying Nyquist rates. To this end, fix a Nyquist sampling grid $\{t_n(\alpha)\}_n$ with a set of corresponding derivatives $\{t'_n(\alpha)\}_n$. This determines the function space with a fixed time-varying Nyquist rate.

As the simplest case, let us assume that only one point deviates from the Nyquist rate. Namely, let $\{t_n\}_n$ be an arbitrary but fixed Nyquist sampling grid and t^* be a single point

not in $\{t_n\}_n$. The new non-Nyquist sampling grid $\{\tau_n\}_n$ consists of the points $\{t_n\}_n$ with a point, say t_0 , replaced by t^* :

$$\tau_n = t_n \quad \forall n \neq 0 \quad \text{and} \quad \tau_0 = t^* \neq t_0 \quad \forall n. \quad (7.18)$$

Evaluate $\phi(t)$ at the point $t = t^*$ to obtain

$$\begin{aligned} \phi(t^*) &= \sum_n G(t^*, t_n) \phi(t_n) \\ &= G(t^*, t_0) \phi(t_0) + \sum_{n \neq 0} G(t^*, t_n) \phi(t_n). \end{aligned} \quad (7.19)$$

Then the operator M in Eq. (7.16) can be written in the matrix form

$$\{\phi(\tau_n)\}_n = \begin{bmatrix} \vdots \\ \phi(\tau_{-1}) \\ \phi(\tau_0) \\ \phi(\tau_1) \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ \phi(t_{-1}) \\ \phi(t^*) \\ \phi(t_1) \\ \vdots \end{bmatrix} = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots \\ \cdots & G(t^*, t_{-1}) & G(t^*, t_0) & G(t^*, t_1) & \cdots \\ & 0 & 0 & 1 & \\ & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ \phi(t_{-1}) \\ \phi(t_0) \\ \phi(t_1) \\ \vdots \end{bmatrix} \quad (7.20)$$

with the entries:

$$\begin{aligned} M_{0,n} &= G(t^*, t_n) \quad \forall n, \\ M_{m,n} &= \delta_{mn} \quad \forall n \quad \forall m \neq 0. \end{aligned} \quad (7.21)$$

One can show that the operator M is normal by calculating the explicit expression of $M^T M v$ and $M M^T v$ in l^2 -space and showing that $M^T M v = M M^T v$ for any l^2 -Hilbert space vector $v = \{v_i\}_{i=-\infty}^{+\infty}$. Because the operator M is normal, its condition number $\kappa(M)$ equals the modulus of the ratio between the maximal and minimal eigenvalues of M . Hence, one needs to find the eigenvalues of M .

Solving the characteristic equation $\det(M - \lambda I) = 0$ for λ , one obtains two eigenvalues:

$$\lambda_{\min} = M_{0,0} = G(t^*, t_0) \quad \text{and} \quad \lambda_{\max} = 1. \quad (7.22)$$

The first eigenvalue $\lambda_{\min} = G(t^*, t_0)$ is of multiplicity of 1. It has a simple eigenvector with entries $\phi(\tau_i) = \delta_{0i}$. The second eigenvalue $\lambda_{\max} = 1$ has multiplicity of infinity. Its n -th eigenvector, $n \neq 0$, has the following entry: $\phi(\tau_i) = \delta_{ni}$, $\forall i \neq 0$ and $\phi(\tau_0) = \frac{M_{0,0}}{1 - M_{0,n}}$. Because the reconstruction kernel $G(t^*, t_0) \leq 1$, the first eigenvalue is the minimum one and the second eigenvalue is the maximum one.

Therefore, the condition number of M is

$$\kappa(M) = \left| \frac{\lambda_{\max}}{\lambda_{\min}} \right| = \frac{1}{|G(t^*, t_0)|}. \quad (7.23)$$

This condition number, $\kappa(M)$, gives a direct measure of the stability of reconstruction from the samples on a non-Nyquist sampling grid: the larger the value of $\kappa(M)$ is, the worse the stability of reconstruction is.

By inspection of the behavior of the reconstruction kernel $G(t, t_0)$ centered at $t = t_0$, we observe the following:

- If $t^* = t_0$, then the denominator $G(t^*, t_0) = 1$ and $\lambda(M) = 1$. This is the case where we did not move the Nyquist sampling point t_0 at all. That is, it was the reconstruction from the original Nyquist sampling grid. Hence, we have the minimum condition number 1 and the reconstruction is the most stable one. Notice that since $G(t, t_0) = 1$ if and only if $t = t_0$ and the maximum absolute value of $G(t, t_0)$ is 1, the reconstruction from the Nyquist sampling grid is indeed the most stable one, and shifting one point to anywhere will worsen the stability of reconstruction.
- The point t^* is near t_0 . The value of $G(t, t_0)$ is close to 1 only in a close region around to $t = t_0$. Hence the less that t^* deviates from $t = t_0$, the better the stability of reconstruction is.
- If $t^* = t_n$, $n \neq 0$, then the denominator $G(t^*, t_0) = 0$ and the condition number approaches infinity. This is the case when we have one sample short, i.e., one sampling point in a Nyquist grid is missing. We have insufficient information to reconstruct the signal and so the condition number becomes unbounded.
- The point t^* is very close to another Nyquist sampling point t_n , $n \neq 0$. The denominator $G(t^*, t_0)$ is very small, close to zero. So the condition number is enormously large and the reconstruction becomes extraordinary unstable. This agrees with the case of Shannon as in [14]: the bunching of two sampling points destroys the stability of reconstruction.
- Because the kernel $G(t, t_0)$ decays like $1/t$ as t goes to both negative and positive infinity, one could conclude that in general the stability of reconstruction gets worse proportional to t as t goes infinity in both directions on the real line.

Chapter 8

Interpolation and Reducing Gibbs' Overshoot

The generalized sampling theorem for time-varying Nyquist rates can be used to interpolate sets of non-equidistant points in a continuous and smooth manner. The interpolation formula can be modified for finite sets of non-equidistant points or periodic sets of non-equidistant points. The interpolation formulae themselves in both cases can be simplified to finite sums.

The interpolation method for non-equidistant points can be used to reduce the Gibbs' overshoot. As is well-known, when using conventional Whittaker-Shannon interpolation, the Gibbs' overshoot in approximating a step function is not reducible [53]. However, using the generalized sampling method, we choose the set of sampling points adapted to the behavior of the function and we can show that, numerically, the new interpolation method allows one to strongly reduce Gibbs' overshoot. In two concrete examples of approximating the step function and the periodic step function, the amplitude of the overshoot is reduced by roughly 70%. The results in this Chapter are based on my papers [34] and [35].

8.1 Interpolation Method for Infinitely Many Non-Equidistant Points

The generalized sampling theorem can be used to construct a continuous time-varying bandlimited function interpolating samples $\{\phi(t_n)\}_n$ on an infinite set of points $\{t_n\}_n$. Combining the reconstruction formula Eq. (5.6) and the kernel Eq. (5.11), the resulting

interpolant function $\phi(t)$ is given by

$$\begin{aligned}\phi(t) &= \sum_{n=-\infty}^{+\infty} G(t, t_n) \phi(t_n) \\ &= \sum_{n=-\infty}^{+\infty} (-1)^{z(t, t_n)} \frac{\sqrt{t'_n}}{|t - t_n|} \left(\sum_{m=-\infty}^{+\infty} \frac{t'_m}{(t - t_m)^2} \right)^{-1/2} \phi(t_n).\end{aligned}\tag{8.1}$$

The value of the derivative t'_n can be calculated from the points $\{t_n\}_n$ using Eq. (5.7)

$$t'_n = C_0 \frac{(t_{n+1} - t_{n-1})}{2}\tag{8.2}$$

with a normalization constant C_0 given in Eq. (5.8).

For numerical computation, it is more convenient to use the following form of the generalized reconstructional kernel: if $t \in (t_k, t_{k+1})$ for any fixed integer k , then

$$\begin{aligned}G(t, t_n) &= (-1)^{z(t, t_n)} \frac{\sqrt{t'_n}}{|t - t_n|} \left(\sum_{m=-\infty}^{+\infty} \frac{t'_m}{(t - t_m)^2} \right)^{-1/2} \\ &= (-1)^{k-n} \frac{\sqrt{t'_n}}{t - t_n} \left(\sum_{m=-\infty}^{+\infty} \frac{t'_m}{(t - t_m)^2} \right)^{-1/2}.\end{aligned}\tag{8.3}$$

This form of the formula simply carries out the function $z(t, t_n)$ which denotes the number of sampling points $\{t_m\}_m$ between t and t_n exclusively. Without the absolute value over $(t - t_n)$, the sign of the expression changes whenever t passes a sampling point t_n . This is easier in computational implementation. See Eq. (3.46) in Chapter 3 for the proof.

There are two advantages of this new interpolation method. First, different from wavelets and polynomial interpolations, the resulting interpolant function $\phi(t)$ is continuous and smooth. Second, different from the Whittaker-Shannon interpolation, the infinite set of samples in this interpolation can be generally non-equidistantly spaced.

However, these are two major weakness as well. First, it is generally too computationally expensive because the interpolation formula in Eq. (8.1) involves two infinite sums. Second, in practical situations, only finitely many samples can be involved. People are only interested in the interpolation of finitely many samples on a finite interval.

In the following two sections, we will show how to modify the interpolation method in Eq. (8.1) for a set of finitely many points by appending equidistant sets of points on both sides of the concerned interval. Such a modification also simplifies the interpolation formula to finite sum. In addition, a similar interpolation method is obtained for periodic samples.

8.2 Interpolation Method for Finitely Many Non-Equidistant Points using Auxiliary Equidistant Extension

Given finitely many samples, say $\{(t_k, \phi(t_k))\}_{k=1}^K$ on a finite interval, say $[a, b]$ with $a \leq t_1 < t_2 < \dots < t_K \leq b$, we are interested in finding a smooth interpolation through these samples. The interpolation formula in Eq. (8.1) is not suitable for such finite interpolation. But one can fix this problem by auxiliary extensions of samples on equidistantly spaced points towards both directions outside the interval.

The idea is that a set of equidistant points can simplify the reconstruction kernel to sinc functions as in the recovery of Shannon's sampling theorem. A finite set of non-equidistant points with equidistant auxiliary sampling points outside the interval can be considered as a set of equidistant points along the real line, subtracting the equidistant ones in the given finite interval and adding the given non-equidistant ones. The resulting sum in the reconstruction kernel should also be simplified to finite as well.

Specifically, assume the samples $\{t_k\}_{n=1}^K$ are in a closed interval $[-M\Delta s, M\Delta s]$, where M is some positive integer and Δs is the unit distance between two adjacent points outside the given interval. One is free to choose Δs . One can choose Δs as the average spacing for the sampling points on the interval, or the average sampling spacing for samples that behave regularly. The latter case means to exclude the intervals on which the signal has extremely oscillatory activity so that one must pay special attention to, e.g., sampling at a much higher density to capture all the sharp and unusual features.

If the given interval $[a, b]$ is not symmetric about $t = 0$ on the real line, one can always simultaneously shift all the samples to make the interval symmetric. Such a horizontal moving of samples does not change the shape of the resulting interpolating function and one can easily obtain the interpolating function through the original samples by horizontally shifting the interpolating function back.

Further, if the boundary of the interval is not an integer multiplication of Δs , one simply enlarges the interval further to ensure the boundaries match an integer multiplication of Δs . This assumption is to simplify calculation. Of course, the calculation can be carried out for general boundary points a, b without any of these assumptions, but the result will not be in a closed form. It will be an expression of the derivative of the Gamma function, which can always be carried out numerically using function `psi` in Matlab. However, for demonstration purposes and the example of reducing Gibbs' overshoot later, we will carry out the calculation with the assumptions that the boundary points are integer multiplication of Δs .

In such a way, one will obtain infinitely many sampling points and yield a interpolating

function $\phi(t)$ on the whole real line. We are only interested in the part of the signal which interpolates the given finitely many samples. So the outer summation in the interpolation formula Eq. (8.1) is finite. Further, due to the periodic pattern of the samples outside the given finite interval, the inner summation in Eq. (8.1) also simplifies to be finite.

To this end, the finitely many given samples $\{(t_k, \phi(t_k))\}_{k=1}^K$ on $[-M\Delta s, M\Delta s]$ and the auxiliary samples $\{s_m\}_{|m|>M}$, where we let

$$s_m = m \Delta s \quad \forall m \in \mathbb{Z},$$

form a Nyquist sampling grid Λ , denoted as

$$\Lambda = \{\tilde{t}_n\}_n = \{s_m\}_{m=-\infty}^{-(M+1)} \cup \{t_k\}_{k=1}^K \cup \{s_m\}_{m=M+1}^{\infty}. \quad (8.4)$$

For computational convenience, we also assume that $t_1 = s_{-M}$ and $t_K = s_M$. If t_1 and t_K are not, one can add two extra auxiliary end points s_{-M} and s_M to the original finite set of samples, and make them the new starting and ending points for $\{t_k\}_k$. This simplifies the derivatives

$$s'_m = C_0 \Delta s \quad \forall |m| > M \quad (8.5)$$

and

$$t'_k = C_0 \Delta t_k \quad \text{where} \quad \Delta t_k = \frac{1}{2} (t_{k+1} - t_{k-1}) \quad \forall 1 \leq k \leq K. \quad (8.6)$$

For notational convenience, t_0 denotes $s_{-(M+1)}$ and t_{K+1} denotes s_{M+1} . The normalization constant C_0 is given by Eq. (5.8). Recall that C_0 will be eventually canceled out in the reconstruction kernel.

With the sampling points given in Eq. (8.4) and their derivatives in Eq. (8.5), (8.6), the generalized reconstruction kernel $G(t, \tilde{t}_n)$ in Eq. (5.11) reads:

$$G(t, \tilde{t}_n) = (-1)^{z(t, \tilde{t}_n)} \frac{\sqrt{\tilde{t}'_n}}{|t - \tilde{t}_n|} [S(t)]^{-1/2} = (-1)^{z(t, \tilde{t}_n)} \frac{\sqrt{\tilde{t}'_n}}{t - \tilde{t}_n} \frac{1}{\sqrt{S(t)}} \quad (8.7)$$

where the infinite summation $S(t)$ can be simplified to

$$\begin{aligned} S(t) &= \sum_n \frac{\tilde{t}'_n}{(t - \tilde{t}_n)^2} = \sum_{k=1}^K \frac{t'_k}{(t - t_k)^2} + \sum_{|m|>M} \frac{s'_m}{(t - s_m)^2} \\ &= \sum_{k=1}^K \frac{C_0 \Delta t_k}{(t - t_k)^2} + \sum_{|m|>M} \frac{C_0 \Delta s}{(t - m \Delta s)^2} \\ &= C_0 \sum_{k=1}^K \frac{\Delta t_k}{(t - t_k)^2} + C_0 \left[\sum_{m=-\infty}^{\infty} \frac{\Delta s}{(t - m \Delta s)^2} - \sum_{m=-M}^M \frac{\Delta s}{(t - m \Delta s)^2} \right] \\ &= C_0 \left[\sum_{k=1}^K \frac{\Delta t_k}{(t - t_k)^2} + \frac{\pi^2}{\Delta s} \sin^{-2}\left(\frac{\pi t}{\Delta s}\right) - \sum_{m=-M}^M \frac{\Delta s}{(t - m \Delta s)^2} \right]. \end{aligned} \quad (8.8)$$

Later in Eq. (8.10), the constant C_0 later cancelled out with the one in $\sqrt{t'_k} = \sqrt{C_0 \Delta t_k}$.

Here we used the trigonometric identity Eq. (5.46) in the recovery of the sinc kernel in Shannon's sampling theorem

$$\sum_{m=-\infty}^{\infty} \frac{\Delta s}{(t - m\Delta s)^2} = \frac{1}{\Delta s} \sum_{m=-\infty}^{\infty} \frac{1}{\left(\frac{t}{\Delta s} - m\right)^2} = \frac{\pi^2}{\Delta s} \operatorname{sinc}^{-2}\left(\pi \left(\frac{t}{\Delta s}\right)\right). \quad (8.9)$$

Now substitute the formula of the reconstruction kernel Eq. (8.7) and Eq. (8.8) into the reconstruction formula Eq. (5.6), we can obtain the interpolating function $\phi(t)$. Remember that since we are only interested in the interpolation through the finitely many points on $\{t_k\}_{k=1}^K$, we only need to sum up the reconstruction kernel $G(t, t_k)$ for all k from 1 to K . So

$$\begin{aligned} \phi(t) &= \sum_{k=1}^K G(t, t_k) \phi(t_k) = \sum_{k=1}^K (-1)^{z(t, t_k)} \frac{\sqrt{t'_k}}{|t - t_k|} \left[S(t) \right]^{-1/2} \phi(t_k) \\ &= \sum_{k=1}^K (-1)^{z(t, t_k)} \frac{\sqrt{\Delta t_k}}{|t - t_k|} \left[\frac{\pi^2}{\Delta s} \operatorname{sinc}^{-2}\left(\frac{\pi t}{\Delta s}\right) + \sum_{l=1}^K \frac{\Delta t_l}{(t - t_l)^2} - \sum_{m=-M}^M \frac{\Delta s}{(t - m\Delta s)^2} \right]^{-\frac{1}{2}} \phi(t_k) \end{aligned} \quad (8.10)$$

or in the form of Eq. (8.3) for computational convenience, if $t \in (t_{\hat{k}}, t_{\hat{k}+1})$ for any fixed integer \hat{k} , we have

$$\phi(t) = \sum_{k=1}^K (-1)^{k-\hat{k}} \frac{\sqrt{\Delta t_k}}{t - t_k} \left[\frac{\pi^2}{\Delta s} \operatorname{sinc}^{-2}\left(\frac{\pi t}{\Delta s}\right) + \sum_{l=1}^K \frac{\Delta t_l}{(t - t_l)^2} - \sum_{m=-M}^M \frac{\Delta s}{(t - m\Delta s)^2} \right]^{-\frac{1}{2}} \phi(t_k). \quad (8.11)$$

This function $\phi(t)$ interpolates the given finite set of samples $\{(t_k, \phi(t_k))\}_{k=1}^K$. Notice that the summations in $\phi(t)$ are now all finite. This makes the computation of the interpolation function applicable in practice.

Plots of the generalized sinc function in Chapter 5 are computed using this method where only finitely many samples are non-equidistant. Unlike polynomial interpolation, in which case the interpolant goes unbounded outside the considered finite interval, the interpolating function $\phi(t)$ behaves reasonably outside the interpolation interval, it decays gradually as $\frac{1}{t}$ as $t \rightarrow \pm\infty$.

The singularities in the summation $S(t)$ in Eq. (8.8) may need to be specially handled. The summation $S(t)$ has singularities on the set of sampling points, say denoted by Λ . This can be fixed by directly assigning the values of 1 or 0 to $G(t, t_k)$ at the sampling points without using the formula in Eq. (8.7), because we know

$$G(\tilde{t}_n, t_k) = \delta(\tilde{t}_n - t_k) \quad \forall \tilde{t}_n \in \Lambda.$$

In addition, the trick of removing equidistant points and adding non-equidistant points in the finite interval introduces new singular points at $t = n\Delta s$ for $-M \leq n \leq M$. In this case, we should not use the removing-then-adding-tricks, but directly carry out $\sum_{|m|>M} \frac{s'_m}{t-m\Delta s}$ in $S(t)$ in the first line of Eq. (8.8).

$$\begin{aligned}
\sum_{|m|>M} \frac{s'_m}{(t-m\Delta s)^2} \Big|_{t=n\Delta s} &= \frac{C_0}{\Delta s} \sum_{|m|>M} \frac{1}{(n-m)^2} \\
&= \frac{C_0}{\Delta s} \left[\sum_{m=M+1}^{\infty} \frac{1}{(n-m)^2} + \sum_{m=M+1}^{\infty} \frac{1}{(n+m)^2} \right] \\
&= \frac{C_0}{\Delta s} \left[\sum_{k=M-n+1}^{\infty} \frac{1}{k^2} + \sum_{k=M+n+1}^{\infty} \frac{1}{k^2} \right] \\
&= \frac{C_0}{\Delta s} \left[\frac{\pi^2}{3} - \sum_{k=1}^{M-n} \frac{1}{k^2} - \sum_{k=1}^{M+n} \frac{1}{k^2} \right]
\end{aligned} \tag{8.12}$$

This summation eventually simplifies to finite as well. Note here for any positive integer L ,

$$\sum_{k=L+1}^{\infty} \frac{1}{k^2} = \sum_{k=1}^{\infty} \frac{1}{k^2} - \sum_{k=1}^L \frac{1}{k^2} = \frac{\pi^2}{6} - \sum_{k=1}^L \frac{1}{k^2}.$$

These two special cases may become trivial for practical computation because Matlab will simply ignore these singular points.

8.3 Interpolation Method for Finitely Many Non-Equidistant Points using Periodic Extension

Another approach to interpolating samples on a finite interval is to make a periodic extension of the samples along the whole real line. The resulting interpolating function will be a periodic function. In this case the infinite summations in both reconstruction kernel Eq. (5.11) and the reconstruction formula Eq. (5.6) will simplify to finite ones as well. Hence, we have a closed form expression of the final interpolant $\phi(t)$.

We are given N samples $\{(\tau_k, \phi(\tau_k))\}_{n=1}^N$ on a finite interval $[0, T)$. Suppose the sampling points $\{\tau_1, \tau_2, \dots, \tau_N\} \subseteq [0, T)$ are all arranged in an increasing order. We are looking for a function $\phi(t)$ on $[0, T)$ to interpolate these samples.

To extend the samples with a period T , we define an infinite set of sampling points $\{t_n\}_n$

$$t_{nN+k} = nT + \tau_k, \quad \forall 1 \leq k \leq N, n \in \mathbb{Z} \tag{8.13}$$

and the function values on them

$$\phi(t_{nN+k}) = \phi(t_k) \quad \forall 1 \leq k \leq N, n \in \mathbb{Z}. \quad (8.14)$$

The derivatives are periodic as well

$$t'_{nN+k} = \tau'_k = C_0 \Delta \tau_k \quad \text{where } \Delta \tau_k = \frac{1}{2}(t_{k+1} - t_{k-1}) \quad \forall 1 \leq k \leq N, n \in \mathbb{Z}. \quad (8.15)$$

The normalization constant C_0 can be calculated by Eq. (5.8), but is canceled out eventually in the calculation of reconstruction formula.

Using Eq. (8.3) to calculate the reconstruction kernel, we have for any $t \in (t_{\hat{k}}, t_{\hat{k}+1})$, \hat{k} fixed,

$$\begin{aligned} G(t, t_{nN+k}) &= \frac{(-1)^{z(t, t_{nN+k})} \sqrt{t'_k}}{|t - t_{nN+k}|} \left[\sum_{m=-\infty}^{+\infty} \sum_{l=1}^N \frac{t'_{mN+l}}{(t - t_{mN+l})^2} \right]^{-1/2} \\ &= \frac{(-1)^{\hat{k} - (nN+k)} \sqrt{\tau'_k}}{(t - t_{nN+k})} \left[\sum_{l=1}^N \sum_{m=-\infty}^{+\infty} \frac{\tau'_l}{(t - \tau_l - mT)^2} \right]^{-1/2} \\ &= \frac{(-1)^{\hat{k} - k - nN} \sqrt{\tau'_k}}{(t - t_{nN+k})} \left[\sum_{l=1}^N \tau'_l \left(\sum_{m=-\infty}^{+\infty} \frac{1}{((t - \tau_l) - mT)^2} \right) \right]^{-1/2} \\ &= \frac{(-1)^{\hat{k} - k} (-1)^{nN} \sqrt{\tau'_k}}{(t - t_{nN+k})} \frac{T}{\pi} \left[\sum_{l=1}^N \tau'_l \sin^{-2} \left(\frac{\pi}{T} (t - \tau_l) \right) \right]^{-1/2}. \end{aligned} \quad (8.16)$$

In the last step, we used the sine trigonometric identity Eq. (5.46) to simplify the summation in the square bracket

$$\sum_{m=-\infty}^{+\infty} \frac{1}{((t - \tau_l) - mT)^2} = \frac{1}{T^2} \sum_{m=-\infty}^{+\infty} \frac{1}{\left(\frac{t - \tau_l}{T} - m\right)^2} = \frac{\pi^2}{T^2} \sin^{-2} \left(\pi \frac{t - \tau_l}{T} \right). \quad (8.17)$$

Substitute this into the interpolation formula Eq. (8.1) to obtain the periodic interpolant $\phi(t)$, for any $t \in (t_{\hat{k}}, t_{\hat{k}+1})$, \hat{k} fixed,

$$\begin{aligned} \phi(t) &= \sum_{n=-\infty}^{+\infty} \sum_{k=1}^N G(t, t_{nN+k}) \phi(t_{nN+k}) = \sum_{k=1}^N \sum_{n=-\infty}^{+\infty} G(t, t_{nN+k}) \phi(\tau_k) \\ &= \sum_{n=-\infty}^{+\infty} \sum_{k=1}^N \left[\frac{(-1)^{\hat{k} - k} (-1)^{nN} \sqrt{\tau'_k}}{(t - t_{nN+k})} \frac{T}{\pi} \left[\sum_{l=1}^N \tau'_l \sin^{-2} \left(\frac{\pi}{T} (t - \tau_l) \right) \right]^{-1/2} \right] \phi(\tau_k) \\ &= \frac{T}{\pi} \sum_{k=1}^N \left[\sum_{n=-\infty}^{+\infty} \frac{(-1)^{nN}}{(t - t_{nN+k})} \right] \left((-1)^{\hat{k} - k} \sqrt{\tau'_k} \phi(\tau_k) \right) \left[\sum_{l=1}^N \tau'_l \sin^{-2} \left(\frac{\pi}{T} (t - \tau_l) \right) \right]^{-1/2}. \end{aligned} \quad (8.18)$$

Using the trigonometric identity for cotangent Eq. (5.39) with $z = \frac{t-\tau_k}{T}$, the summation over index n in the second square bracket can be further simplified. We assume that the number of sampling points in each period, N , is even. So $(-1)^{nN} = 1$ and

$$\begin{aligned} \sum_{n=-\infty}^{+\infty} \frac{(-1)^{nN}}{(t - t_{nN+k})} &= \sum_{n=-\infty}^{+\infty} \frac{1}{t - \tau_k - nT} \\ &= \frac{1}{T} \sum_{n=-\infty}^{+\infty} \frac{1}{\left(\frac{t-\tau_k}{T} - n\right)} = \frac{\pi}{T} \cot\left(\frac{\pi}{T}(t - \tau_k)\right). \end{aligned} \quad (8.19)$$

If N is odd, this infinite sum can be simplified to a finite sum in the form of sigma function, which can always be numerically determined. However, its expression is lengthy and tedious. Hence, the case for N is odd is ignored here.

As a result, for any $t \in (\tau_{\hat{k}}, \tau_{\hat{k}+1})$, \hat{k} fixed, the interpolant periodic $\phi(t)$ is given by:

$$\begin{aligned} \phi(t) &= \frac{T}{\pi} \sum_{k=1}^N \left[\frac{\pi}{T} \cot\left(\frac{\pi}{T}(t - \tau_k)\right) \right] \left((-1)^{\hat{k}-k} \sqrt{\tau'_k} \phi(\tau_k) \right) \left[\sum_{l=1}^N \tau'_l \sin^{-2}\left(\frac{\pi}{T}(t - \tau_l)\right) \right]^{-1/2} \\ &= \left[\sum_{k=1}^N (-1)^{\hat{k}-k} \sqrt{\tau'_k} \cot\left(\frac{\pi}{T}(t - \tau_k)\right) \phi(\tau_k) \right] \left[\sum_{l=1}^N \tau'_l \sin^{-2}\left(\frac{\pi}{T}(t - \tau_l)\right) \right]^{-1/2}. \end{aligned} \quad (8.20)$$

This function $\phi(t)$ interpolates a finite set of samples $\{(\tau_k, \phi(\tau_k))\}_{k=1}^N$ on a finite interval $[0, T)$ provided that N is even. For t outside the interval $[0, T)$, the function $\phi(t)$ is periodic with a period T .

8.4 Reducing Gibbs' Overshoot in Approximating a Step Function

The Shannon sampling theorem is not only used for the perfect reconstruction of functions in the space of Ω -bandlimited functions, but has also been widely used to approximate non-bandlimited functions. However, the phenomenon of Gibbs' overshoot occurs whenever using Shannon's sinc kernel to approximate a function with discontinuous jump points. This leads to ringing artifacts in signal processing and image compression.

The simplest example to illustrate this type of overshoot is approximating the step function $H(t)$. See Figure 8.1.

$$H(t) = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases} \quad (8.21)$$

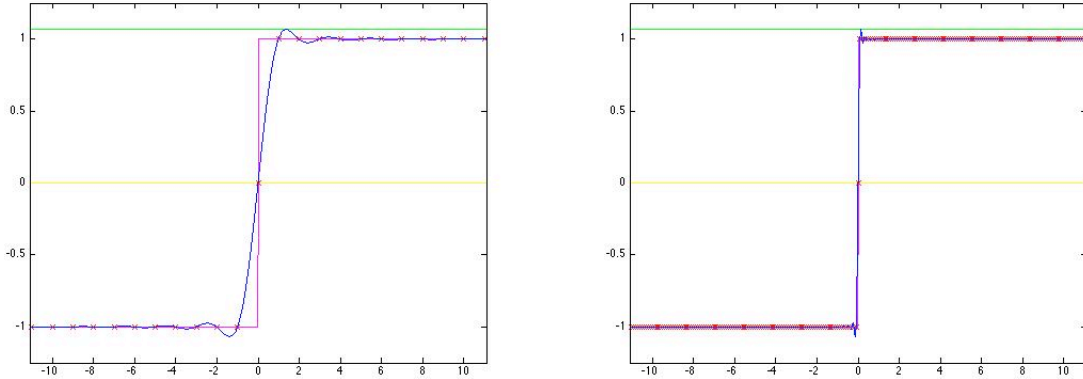


Figure 8.1: Approximations of the step function by Shannon sampling. The left panel uses a wider sampling spacing of 1.0, while the right panel uses 0.1. The green solid line on the top indicates the amplitude of Gibbs' overshoot near $t = 0$, which is the same in both cases.

In Figure 8.1, the step function $H(t)$ is approximated using Whittaker-Shannon interpolation on a set of equidistant points $\{s_n = n\Delta s = \frac{n}{2\Omega}\}_n$

$$\phi_{\text{shannon}}(t) = \sum_{n=-\infty}^{+\infty} \text{sinc}\left(2\Omega(t - s_n)\right) H(s_n) = \sum_{n=-\infty}^{+\infty} \text{sinc}\left(\frac{t - n\Delta s}{\Delta s}\right) H(n\Delta s). \quad (8.22)$$

The approximating function is a sum of equidistantly shifted sinc functions. The equidistant sampling points are at integer multiples of the constant spacing $\Delta s = 1/(2\Omega)$.

The Gibbs' overshoot does not die out as the frequency (or bandwidth Ω) increases, but approaches a finite limit. Figure 8.1 shows two Whittaker-Shannon interpolations using $\Delta s = 0.1$ (on the right) and $\Delta s = 1.0$ (on the left). Although the sampling density on the right ($\Delta s = 0.1$) is ten times tighter than the one on the left ($\Delta s = 1.0$), the maximum values of both approximating functions are about 1.0664 with an error of 0.001. We used 1000 terms in (5.6) in both cases. As is well known, the 6.64% difference to the original step function $H(t)$, i.e., Gibbs' overshoot, can not be further reduced when using Shannon sampling, even when increasing the bandwidth.

Next, we use the generalized sampling method to approximate $H(t)$ on a set of non-equidistant points adapted to the behavior of the function. The reduction of Gibbs' overshoot using the new interpolation is overwhelming. See Figure 8.2.

In Figure 8.2, we approximated the step function $H(t)$ using the generalized sampling method. Outside the interval $[-10, 10]$, the sampling points have the same constant spacing $\Delta s = 1.0$ as the ones on the left in Figure 8.1. But in the neighborhood interval $[-10, 10]$

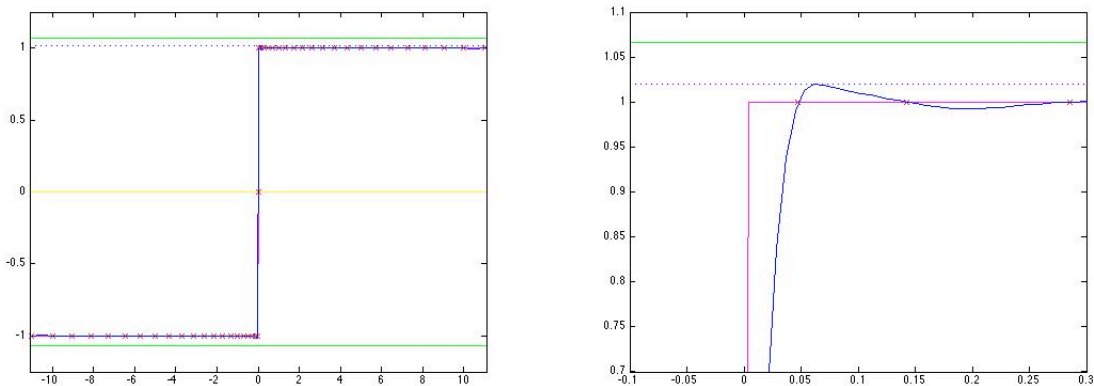


Figure 8.2: Approximating the step function by the generalized sampling method with non-equidistant sampling points. The right plot zooms in near the jump point. The dashed line indicates the maximum amplitude of the approximating function.

of the jump point, we adjusted the sampling density with 20 extra sampling points. As a result, the maximum value is reduced to 1.0193, which is a 70.9% reduction in the amplitude of Gibbs' type of overshoot. The amplitude is subject to a numeric error of 0.001, which implies an error of 0.1% in the reduction percentage.

The plot on the right in Figure 8.2 is a zoom-in of the plot on the left near the jump point. The solid line on the top is the amplitude of the Gibbs' overshoot on the uniform grid in the case of Shannon (which is 1.0664), and the dashed line indicates the amplitude of maximum value with the new generalized sampling theorem (which is 1.0193). This indicates that the new method could be very useful, for example, to reduce Gibbs ringing in image compression.

To use our generalized sampling method to approximate a function, the strategy is:

1. Find a set of points $\{t_n\}_n$ to form a sampling grid which depends on the behavior of the function to be approximated. The sampling points have a minimum and maximum spacing as in Eq. (5.18);
2. Set the corresponding derivatives $\{t'_n\}_n$ from $\{t_n\}_n$ as in Eq. (5.7);
3. Construct the approximating function using the interpolation formula Eq. (8.1).

The third step uses the corresponding interpolation method, which is Eq. (8.10) in this case as we will see. The second step follows from the first step naturally. Hence the difficult

task is the first step: given a function, how does one choose a suitable set of sampling points?

In the specific problem of approximating the step function $H(t)$, we have seen that the uniform grid is not optimal, because the step function has a sudden change at the jump point, i.e. a large Gibbs' overshoot. Intuitively $H(t)$ has infinitely large bandwidth at $t = 0$, while it has zero bandwidth elsewhere. A windowed Fourier transform shows a regularized behavior of high bandwidth at the step and decreasing bandwidth away from the step. Recall that in the Shannon case, the constant derivative $t'_n \sim \frac{1}{2\Omega}$ is inversely proportional to the bandwidth Ω . Hence, the constant derivatives on a uniform grid are not matched up with the jump in $H(t)$.

When we take samples, the derivative, which controls the sampling density, must match the varying bandwidth. Hence, we expect to take

- more samples (at a higher rate) when the function oscillates faster, and
- less samples (at a lower rate) in the period of time with less oscillations.

Thus, for $H(t)$, the spacing between sampling points should be small near the jump point $t = 0$ where the function has a high bandwidth, and gradually increases to some constant spacing when we sample far away from $t = 0$.

The simplest gradually increasing spacing is linear. Hence, we use the sampling grid with linearly increasing spacing in approximating the step function $H(t)$. We expect that this non-equidistant sampling grid will yield a better approximation than the Whittaker-Shannon interpolation because now the sampling density matches better with the behavior of the step function $H(t)$.

Because $H(t)$ is anti-symmetric about $t = 0$, the sampling points should also be symmetric with respect to $t = 0$. Let $t_0 = 0$ be the center sampling point, then $t_{-k} = -t_k$ for all $k \geq 1$. From now, let us focus on finding the positive sampling points.

Assume that outside the interval $[-M\Delta s, M\Delta s]$, we have equidistant sampling points with a constant spacing Δs

$$s_m = m\Delta s \quad \forall |m| > M$$

where M is some positive integer. Inside this interval, we have non-equidistant sampling points with linearly increasing spacing. Assume that we have K such positive non-equidistant sampling points. Let $\Delta r = t_1 - t_0$ be the first spacing between t_0 and t_1 , and

δ be the linear spacing increment. Then

$$\begin{aligned}
t_1 &= t_0 + \Delta r = \Delta r, \\
t_2 &= t_1 + (\Delta r + \delta) = 2\Delta r + \delta, \\
t_3 &= t_2 + (\Delta r + 2\delta) = 3\Delta r + 3\delta, \\
&\vdots \\
t_{K-1} &= t_{K-2} + (\Delta r + (K-2)\delta) = (K-1)\Delta r + \frac{1}{2}(K-1)(K-2)\delta, \\
t_K &= t_{K-1} + (\Delta r + (K-1)\delta) = K\Delta r + \frac{1}{2}K(K-1)\delta.
\end{aligned}$$

The chosen sampling grid is

$$\{s_m\}_{m=-\infty}^{-(M+1)} \cup \{t_k\}_{k=-K}^K \cup \{s_m\}_{m=M+1}^{\infty}.$$

The largest non-equidistant sampling point t_K must match the equidistant sampling point $s_M = M\Delta s$, and the spacing must gradually increase to the constant spacing Δs . Hence

$$t_K = K\Delta r + \frac{1}{2}K(K-1)\delta = M\Delta s, \quad (8.23)$$

$$(t_K - t_{K-1}) + \delta = \Delta r + K\delta = \Delta s. \quad (8.24)$$

The difference of 2 times Eq. (8.24) and $(K-1)$ times Eq. (8.24) gives

$$(K+1)\Delta r = (2M - K + 1)\Delta s. \quad (8.25)$$

Notice that Δr is (and should be much) smaller than Δs . By Eq. (8.25), this means that

$$(K+1) \geq (2M - K + 1) \implies M \leq K.$$

The smaller Δr , the better approximation. So we want to choose K as large as possible. Re-write Eq. (8.25) as

$$K = \frac{(2M+1)\Delta s - \Delta r}{\Delta s + \Delta r} = (2M+1) - \frac{2(M+1)}{\Delta s + \Delta r}\Delta r. \quad (8.26)$$

All the terms are positive, so $K < 2M + 1$. The maximum possible value of K is $2M$. With $K = 2M$, from Eq. (8.26) we have

$$\frac{2(M+1)}{\Delta s + \Delta r}\Delta r = 1 \implies \Delta s = (2M+1)\Delta r.$$

Substituting this and $K = 2M$ to (8.23), we have

$$\Delta r + (2M)\delta = (2M+1)\Delta r \implies \delta = \Delta r.$$

In summary, to approximate the step function $H(t)$ using this linearly increasing spacing method, we pick an interval $[-M\Delta s, M\Delta s]$ from the uniform grid with a constant spacing Δs , and replace the uniform sampling points in that interval by a set of points whose adjacent spacing is linearly increasing. There will be twice as many points as with the uniform grid, i.e. $K = 2M$, and $\delta = \Delta r = \frac{1}{2M+1}\Delta s$. In other words, M and Δs are free to choose.

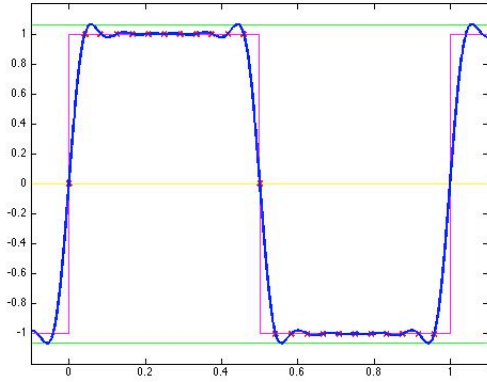
Notice that this set of sampling points is same as the one in Section 8.2, a finite set of points with auxiliary equidistant extension. Hence, once the set of sampling points are determined, we can use the interpolation kernel Eq. (8.7) in Section 8.2. Because of the simplified finite sum in Eq. (8.7), the truncation error does not arise from the reconstruction kernel in Eq. (5.25)

In Figure 8.2, the step function $H(t)$ is approximated by this method with $M = 10$ and $\Delta s = 1.0$. Comparing this grid with the uniform grid with same Δs , we simply replaced the equidistant sampling points on $[-10, 10]$ by a set of non-equidistant point, whose adjacent spacing linearly decreases toward the jump point $t = 0$. The maximum value of the new approximation is reduced to 1.0193 with an error of 0.001. This gives about 70% of reduction in Gibbs' overshoot.

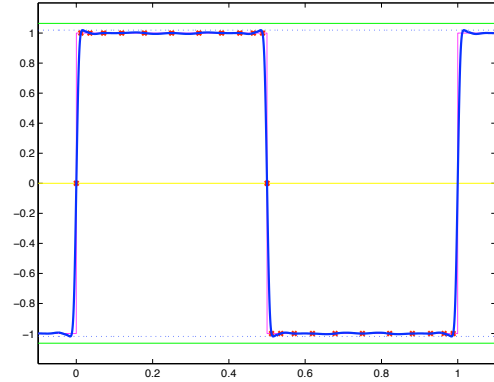
Naturally, the question arises whether one can do better with a differently spaced non-equidistant sampling grid. Of course, we do not expect that the linear change in sampling density yields the optimal grid spacing to match the behavior of a step function. A more abrupt increase in sampling density toward the jump point $t = 0$ should match the behaviour of the step function better, hence resulting in a better approximation. For example, one could consider a quadratic change in sampling spacing rather than linear. But the calculation involved in quadratic change might be very computationally expensive. (I attempted but my computer could not carry out the numerical calculation. A faster and more powerful computer might do.) Nevertheless, using a non-equidistant grid with a linear change in sampling density shows a significant reduction of 70% in Gibbs' overshoot. This already demonstrates an advantage of the generalized sampling method.

8.5 Reducing Gibbs' Overshoot in Approximating a Periodic Step Function

The conventional Gibbs' phenomenon arises in Fourier series of a piecewise continuously differentiable periodic function with a jump discontinuity. Even as the number of terms in the partial sum of Fourier series increases, the maximum amplitude of the partial sum near the jump point does not die out, but approaches a finite limit over the function value.



(a) Using Fourier series



(b) Using the generalized sampling method

Figure 8.3: Approximating the periodic step function with $N = 24$ sampling points. Samples are denoted by \times . The solid lines on the top and bottom indicate the maximum and minimum values of Gibbs' overshoot. The dashed lines indicate the maximum and minimum values of the approximation by the generalized sampling theory.

In this section, we will demonstrate the interpolation method for the periodic step function $H_p(t)$. Surprisingly, using the same linearly increasing spacing distribution as in the approximation of regular step function $H(t)$ in Section 8.4, about 70% of Gibbs' overshoot is reduced as well.

$$H_p(t) = \begin{cases} 1 & t \in (0, \frac{1}{2}) \\ 0 & t = \frac{1}{2} \\ -1 & t \in (\frac{1}{2}, 1) \\ H(t+1) & \forall t \in \mathbb{R} \end{cases} \quad (8.27)$$

The periodic interpolation formula is given in Eq. (8.20) in Section 8.3 in a closed form with finite sum. Hence, the only task left is to find a finite set of interpolation points $\{\tau_k\}_{k=1}^N$ in one period $[0, 1)$ adapted to the behavior of $H_p(t)$.

Similar to the case of the step function in Section 8.4, the reason that Fourier series on equidistant points failed is because of, intuitively, the sudden change in the amplitude of a step function $H_p(t)$ at its jump points $t = 0, \frac{1}{2}$ and 1. The function can be considered to suddenly oscillate at an 'infinite' frequency in a sufficiently small neighborhood at the jump points, namely to have an 'infinite' bandwidth at $t = 0, \frac{1}{2}$ and 1. A uniform grid implies uniform bandwidth. Intuitively, the uniform grid in the case of Whittaker-Shannon inter-

polation is therefore not matched with the increase of bandwidth in a small neighborhood of jump points.

We therefore choose N sampling points with non-equidistant spacings so that the smallest spacing, (the highest bandwidth) occurs near the jump points at $t = 0, \frac{1}{2}, 1$, and the spacing gradually increases away from the jump points (the bandwidth decreases). We use the simplest such increasing change in spacing, which is linear.

Due to the symmetry of the jump points at $t = 0, \frac{1}{2}, 1$, we divide one period $[0, 1)$ into four equal subintervals with length $\frac{1}{4}$. On the first subinterval, $[0, \frac{1}{4})$, we choose K points so that their adjacent spacing is linearly increasing. Let δ be the linear increment in spacing, then

$$\begin{aligned} \tau_1 &= 0, \tau_2 = \delta, \tau_3 = 3\delta, \dots \\ \tau_K &= \frac{1}{2}K(K-1)\delta. \end{aligned} \tag{8.28}$$

The $(K+1)^{\text{th}}$ point is $\frac{1}{4}$. The sampling points on $(\frac{1}{4}, \frac{1}{2}]$ are a mirror image of the points on $[0, \frac{1}{4})$ with respect to $t = \frac{1}{4}$, and the points on $[\frac{1}{2}, 1)$ repeat the ones on $[0, \frac{1}{2})$. Therefore, we have in total $N = 4K$ points on $[0, 1)$.

Choosing $K = 6$ and using the interpolation formula in Eq. (8.20) in Section 8.3, we numerically obtain the approximation of $H_p(t)$ in Figure 8.3b. It has the same total number of sampling points ($N = 24$) on $[0, 1)$ as in Figure 8.3a using Shannon's method. But its maximum amplitude is 1.0193, which is about 70% of reduction to the overshoot.

In Figure 8.3a, $H_p(t)$ is approximated using Shannon's shifted sinc reconstruction kernel with $N = 24$ sampling points on one periodic interval $[0, 1)$. Samples are denoted by \mathbf{x} in the plot, and the solid line at the top indicates the maximum value of the approximating function, which is 1.0640. Within an error of 0.003, the 6.40% overshoot beyond the maximum amplitude 1 of the step function $H_p(t)$ can not be further reduced even if we increase the sampling density.

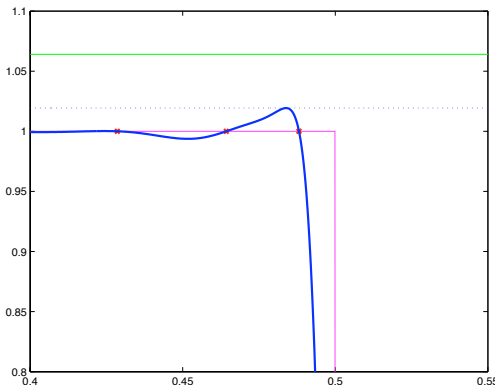


Figure 8.4: This is a zoom-in of Figure 8.3b. The solid line on the top indicates the amplitude of the Gibbs' overshoot. The dashed line indicates the maximum amplitude of the approximation by the new generalized sampling theory. Roughly a 70% reduction of Gibbs' overshoot is observed by using same number of sampling points.

In Figure 8.3b, we use the same number of points $N = 24$ in one period, but we choose the sampling points to match the behaviour of the step function. Intuitively, the jump in the step function contains high frequencies. Thus more samples are taken near the jump points $t = 0, \frac{1}{2},$ and 1 . In this example, the maximum value of the approximation is reduced to 1.0074 with an error of 0.0003 . This is roughly a 70% reduction of Gibbs' overshoot without increasing the number of samples, but only varying the local sample density.

Figure 8.4 is a zoom-in of Figure 8.3b near the jump point. The dashed line on the top indicates the maximum values of the approximating function using the generalized sampling, while the solid line indicates the overshoot in the case of Shannon.

Chapter 9

Conclusions and Outlook

This thesis has been concerned with a generalized sampling theory which allows one to sample a continuous signal only as often as necessary while maintaining the ability to perfectly and stably reconstruct the signal from these discrete samples. The overarching idea is that adjusting to a time-varying bandwidth will improve the sampling and information storage efficiency. We expect that this generalized sampling theory opens up a wide range of applications in signal processing and quantum gravity.

To this end, this thesis systematically develops and in some respects completes a new non-Fourier generalization of the classical sampling theory for time-varying Nyquist rates. The thesis defines and explains what we mean by ‘a time-varying Nyquist rate’ or equivalently ‘a time-varying bandwidth’. Various issues regarding the sampling theory for such time-varying Nyquist rates are studied. In particular, the thesis presents for the first time the full algorithm of filtering, sampling and reconstruction. It also clarifies the stability of the sampling method for time-varying Nyquist rates.

In this chapter, we will present some possible future research opportunities. Notice that the ideas described in this chapter are immature and have not been fully investigated. So far they are only my speculations. I welcome any suggestions from the readers on how these ideas could be carried further.

9.1 Limit of Lagrange Polynomial Interpolation

In sampling and interpolation theory, an interesting and important question is whether the sampling theory involving infinitely many sampling points contains, as a special case, the case of finite interpolation. Indeed, it has been shown in [54] that the reconstruction formula in the classical sampling theorem arises as a formal limiting case of the finite Lagrange formula on a set of equidistantly-spaced sampling points.

A natural question which then follows is what is the limiting case of the finite Lagrange formula on a general set of non-equidistantly spaced sampling points? Will the reconstruction formula in the generalized sampling theorem presented in this thesis covers this case as a special case?

First, let us recapitulate how the sinc reconstruction kernel in the classical sampling theorem arises as the limit of Lagrange interpolation on a set of equidistantly spaced points. Notice that in the case of the classical sampling theorem, the sinc reconstruction kernel $G(t, t_n) = \text{sinc}(2\Omega(t - t_n))$ can be rewritten in the following form

$$G(t, t_n) = \frac{H(t)}{H'(t_n)(t - t_n)} \quad (9.1)$$

where

$$H(t) = \frac{\sin(2\pi\Omega t)}{2\pi\Omega}. \quad (9.2)$$

To see why this is true on a set of equidistantly Nyquist points $\{t_n = \frac{n}{2\Omega}\}_{n=-\infty}^{+\infty}$, one uses the fact that $\sin(n\pi) = 0$ and $\cos(n\pi) = (-1)^n$.

$$\begin{aligned} \frac{H(t)}{H'(t_n)} &= \frac{\left(\frac{\sin(2\pi\Omega t)}{2\pi\Omega}\right)}{\cos(2\pi\Omega t_n)} = \frac{1}{2\pi\Omega} \frac{\sin(2\pi\Omega t)}{\cos(n\pi)} = \frac{1}{2\pi\Omega} \frac{\sin(2\pi\Omega t)}{(-1)^n} \\ &= \frac{1}{2\pi\Omega} \left(\sin(2\pi\Omega t)(-1)^n \right) = \frac{1}{2\pi\Omega} \left(\sin(2\pi\Omega t)(-1)^n - \cos(2\pi\Omega t) 0 \right) \\ &= \frac{1}{2\pi\Omega} \left(\sin(2\pi\Omega t) \cos(n\pi) - \cos(2\pi\Omega t) \sin(n\pi) \right) \\ &= \frac{1}{2\pi\Omega} \left(\sin(2\pi\Omega t) \cos(2\pi\Omega t_n) - \cos(2\pi\Omega t) \sin(2\pi\Omega t_n) \right) \\ &= \frac{1}{2\pi\Omega} \sin(2\pi\Omega(t - t_n)). \end{aligned} \quad (9.3)$$

Hence,

$$\frac{H(t)}{H'(t_n)(t - t_n)} = \frac{\sin(2\pi\Omega(t - t_n))}{2\pi\Omega(t - t_n)} = \text{sinc}(2\Omega(t - t_n)) = G(t, t_n).$$

Then using the Euler product representation with $z = 2\Omega t$

$$\frac{\sin(\pi z)}{\pi z} = \prod_{k=1}^{+\infty} \left(1 - \frac{z^2}{k^2}\right)$$

one can obtain $H(t)$ as the limit of Lagrange formula on the infinite set of points $\{t_n = \frac{n}{2\Omega}\}_{n=-\infty}^{+\infty}$,

which is the right hand side of the following equation:

$$\begin{aligned}
H(t) &= \frac{\sin(2\pi\Omega t)}{2\pi\Omega} = t \prod_{k=1}^{+\infty} \left(1 - \frac{(2\Omega t)^2}{k^2}\right) = t \prod_{k=1}^{+\infty} \left(1 - \frac{t^2}{(k/(2\Omega))^2}\right) \\
&= t \prod_{k=1}^{+\infty} \left(1 - \frac{t^2}{t_k^2}\right) = (t - t_0) \prod_{k=1}^{+\infty} \left(1 - \frac{t}{t_k}\right) \left(1 + \frac{t}{t_k}\right) \\
&= (t - t_0) \prod_{k=1}^{+\infty} \left(1 - \frac{t}{t_k}\right) \left(1 - \frac{t}{t_{-k}}\right) = (t - t_0) \prod_{k \neq 0} \left(\frac{t - t_k}{t_0 - t_k}\right).
\end{aligned} \tag{9.4}$$

Interestingly, the generalized reconstruction kernel $G(t, t_n)$ can also be written in the form of Eq. (9.1). Let $\{t_n\}_{n=-\infty}^{+\infty}$ denote an infinite set of non-equidistantly sampling points with associated derivatives $\{t'_n\}_{n=-\infty}^{+\infty}$. For any real number t in $[t_n, t_{n+1}]$, $n \in \mathbb{Z}$, let

$$H(t) = (-1)^n \left[\sum_{m=-\infty}^{+\infty} \frac{t'_m}{(t - t_m)^2} \right]^{-1/2}. \tag{9.5}$$

Then

$$\begin{aligned}
H'(t) &= (-1)^n \left[\sum_{m=-\infty}^{+\infty} \frac{t'_m}{(t - t_m)^2} \right]^{-3/2} \left(-\frac{1}{2}\right) \left[\sum_{m=-\infty}^{+\infty} \frac{t'_m}{(t - t_m)^3} \right] (-2) \\
&= (-1)^n \frac{\sum_m \frac{t'_m}{(t - t_m)^3}}{\left[\sum_m \frac{t'_m}{(t - t_m)^2} \right]^{3/2}}.
\end{aligned}$$

When the term $H'(t)$ is evaluated at $t = t_n$, the n -th term in both summations in the numerator and denominator diverge. In the limit $t \rightarrow t_n$, the n -th term dominates the other terms in both summations. Hence

$$\begin{aligned}
H'(t_n) &= \lim_{t \rightarrow t_n} H'(t) = (-1)^n \lim_{t \rightarrow t_n} \frac{\frac{t'_n}{(t - t_n)^3}}{\left(\frac{t'_n}{(t - t_n)^2}\right)^{3/2}} = \\
&= (-1)^n \lim_{t \rightarrow t_n} \frac{\frac{t'_n}{(t - t_n)^3}}{\frac{(t'_n)^{3/2}}{(t - t_n)^3}} = (-1)^n \lim_{t \rightarrow t_n} (t'_n)^{-1/2} = \frac{(-1)^n}{\sqrt{t'_n}}
\end{aligned}$$

Therefore, for any t in $[t_k, t_{k+1})$, $k \in \mathbb{Z}$,

$$\begin{aligned} G(t, t_n) &= \frac{H(t)}{H'(t_n)(t-t_n)} = \frac{(-1)^k \left[\sum_{m=-\infty}^{+\infty} \frac{t'_m}{(t-t_m)^2} \right]^{-1/2}}{\frac{(-1)^n}{\sqrt{t'_n}} (t-t_n)} \\ &= (-1)^{k-n} \frac{\sqrt{t'_n}}{t-t_n} \left[\sum_{m=-\infty}^{+\infty} \frac{t'_m}{(t-t_m)^2} \right]^{-1/2} \end{aligned}$$

This is another equivalent form of the reconstruction kernel Eq. (5.11) without the function $z(t, t_n)$ and without the absolute value sign over $(t-t_n)$ in the denominator. See Eq. (3.46) in Chapter 3 or Eq. (8.3) in Chapter 8 for the equivalence. With the reconstruction kernel $G(t, t_n)$ rewritten in the form of Eq. (9.1), we ask whether the reconstruction kernel in the generalized sampling theorem provides, as a special case, the limiting case of the Lagrange interpolation over an infinite set of sampling points $\{t_n\}_{-\infty}^{+\infty}$?

Given finitely many points $\{(t_n, \phi(t_n))\}_{n=-N}^N$, the Lagrange interpolation polynomial reads

$$L(t) = \sum_{n=-N}^N L_n(t) \phi(t_n), \quad (9.6)$$

where each Lagrange basis polynomial is

$$L_n(t) = \prod_{k \neq n} \frac{t-t_k}{t_n-t_k} = \frac{w(t)}{w_n(t-t_n)}. \quad (9.7)$$

It is straight-forward to see that $L_n(t_m) = \delta_{nm}$. The Lagrange basic polynomial expressed in the form of $\frac{w(t)}{w_n(t-t_n)}$ is called the barycentric interpolation formula [55], in which case,

$$w(t) = \prod_{k=-N}^N (t-t_k) \quad (9.8)$$

and w_n is a constant defined by

$$w_n = \prod_{\substack{-N \leq k \leq N \\ k \neq n}} (t_n - t_k). \quad (9.9)$$

Comparing the form of Eq. (9.1) and Eq. (9.7), we ask if the following equation is true when we let the number of the points in $\{t_n\}_{n=-N}^N$ approach infinity:

$$\frac{H(t)}{H'(t_n)} = \lim_{N \rightarrow +\infty} \frac{w(t)}{w_n} \quad (9.10)$$

This might be an ambitious question, but a very important and interesting one. If it is true, we find the limit of Lagrange interpolation over finitely many non-equidistantly spaced points

In the case of the classical sampling theorem, it is true. Notice that in Eq. (9.4), what we really showed is

$$\frac{H(t)}{H'(t_0)} = \frac{\sin(2\pi\Omega(t - t_0))}{2\pi\Omega} = \frac{w(t)}{w_0} \quad (9.11)$$

This is indeed the special case with $n = 0$ and $t_0 = 0$. Here $H'(t_0) = 1$. Changing the equidistant set of points from $\{t_n = \frac{n}{2\Omega}\}_n$ to $\{t_n[\theta] = \frac{n+\theta}{2\Omega}\}_n$ with any fixed θ between 0 and 1, the calculation in Eq. (9.3) will not obviously carry through. But if Eq. (9.10) is true, then this follows immediately.

A more precise question to ask: given an infinite set of points $\{t_n\}_{n=-\infty}^{+\infty}$, is there a collection of derivatives $\{t'_n\}_{n=-\infty}^{+\infty}$ so that Eq. (9.10) holds? Notice that the left hand side of Eq. (9.10), which corresponds to the reconstruction kernel in the generalized sampling theory, depends on two independent sets of numbers, $\{t_n\}$ and $\{t'_n\}_n$. In contrast, the right hand side of Eq. (9.10), which involves the Lagrange interpolation, is expressed solely in terms of the sampling points $\{t_n\}_n$. If there exists a collection of $\{t'_n\}_n$ so that Eq. (9.10) is true, then conversely, this set of $\{t'_n\}_n$ is the ‘natural’ choice of the derivatives in the generalized sampling theory.

To try to prove Eq. (9.10), one should start with simple cases, for example, one could consider a set of equidistant points $\{t_n = \frac{n}{2\Omega}\}_n$ with the middle point t_0 shifted between t_{-1} and t_1 and see if Eq. (9.10) can still hold. Alternatively, one could also consider a set of points with a periodic pattern. One could start with the one with a minimum number of points in each period, i.e., two points in each period. In both cases, the expression of both sides of Eq. (9.10) should be greatly simplified.

9.2 On Sturm-Liouville Operators

The first-order differential operator $i\frac{d}{dt}$ on a finite interval is a simple symmetric operator with deficiency indices $(1, 1)$. Such a symmetric operator has a $U(1)$ -family of self-adjoint extensions. Eigenvalues of each self-adjoint extension constitute a set of sampling points and their corresponding eigenvectors form an eigenbasis, which allows the reconstruction of functions in the space. These facts yield the generalized sampling theory in this thesis.

What is the situation with the second-order differential operator $-\frac{d^2}{dt^2}$? This operator is a simple symmetric with deficiency indices $(2, 2)$. Hence it has a $U(2)$ -family of self-adjoint extensions. Eigenvectors of each self-adjoint extension form an eigenbasis. Hence, one would also expect that a sampling and reconstruction method arises from such a

second-order differential operator using the theory of self-adjoint extensions of symmetric operators.

More generally, we can consider the Sturm-Liouville operator

$$L^* = -\frac{d}{dt} p(t) \frac{d}{dt} + q(t) \quad (9.12)$$

with the following domain

$$D(L^*) = \{\psi \in AC([a, b]) \mid p\psi' \in AC([a, b]), -(p\psi')' + q\psi \in L^2([a, b])\}. \quad (9.13)$$

Here $p(t)$, $q(t)$ are real-valued functions and $p(t)$, $q(t)$, $\frac{1}{p(t)}$ are Lebesgue measurable on $[a, b]$. The set $D(L^*)$ is the largest set in $L^2([a, b])$ on which the Sturm-Liouville operator in Eq. (9.12) can be defined. We can restrict the domain of L^* with vanishing boundary conditions to yield a symmetric operator

$$L = -\frac{d}{dt} p(t) \frac{d}{dt} + q(t) \quad (9.14)$$

with the domain

$$D(L) = \{\phi \in D(L^*) \mid \phi(a) = p(a)\phi'(a) = 0 = p(b)\phi'(b) = \phi(b)\}. \quad (9.15)$$

One can show that this second-order differential operator L is a symmetric operator with equal deficiency indices (n, n) where $n = 0, 1$ or 2 and the operator L^* is its adjoint.

To see that L^* is the adjoint operator of L , we evaluate the following

$$\begin{aligned} \langle \psi, L\phi \rangle - \langle L^*\psi, \phi \rangle &= \int_{t=a}^b \bar{\psi} \left(-(p\phi')' + q\phi \right) - \left((-p\bar{\psi}')' + q\bar{\psi} \right) \phi \, dt \\ &= \int_{t=a}^b -\bar{\psi}(t) (p(t) \phi'(t))' + (p(t) \bar{\psi}'(t))' \phi(t) \, dt \\ &= \left[-\bar{\psi}(t) p(t) \phi'(t) \right]_{t=a}^b + \int_{t=a}^b \bar{\psi}'(t) p(t) \phi'(t) \, dt \\ &\quad + \left[p(t) \bar{\psi}'(t) \phi(t) \right]_{t=a}^b - \int_{t=a}^b p(t) \bar{\psi}'(t) \phi'(t) \, dt \\ &= \left[(p(t) \bar{\psi}'(t)) \phi(t) - \bar{\psi}(t) (p(t) \phi'(t)) \right]_{t=a}^b = 0 \end{aligned} \quad (9.16)$$

If $\phi(t)$ is in $D(L)$, then both $\phi(t)$ and $p(t)\phi'(t)$ vanish at the end points a, b . The above equation always holds for any $\psi(t)$ in $D(L^*)$. This is only a heuristic verification that the operator L^* is the adjoint of L . A rigorous proof can be found in [56], Section 17.

The fact that the symmetric operator L has equal deficiency indices follows from Theorem 9.14 in [42]. To show the value of the deficiency indices n , one calculates the deficiency space $K(L^* - i) = 0$. Namely, solve for $\phi(t)$ such that

$$(L^* - i)\phi(t) = -(p(t)\phi'(t))' + (q(x) - i)\phi(t) = 0 \quad (9.17)$$

The deficiency index n is the number of linearly independent solutions in $L^2([a, b])$ to this second-order differential equation. The existence-uniqueness theorem says that for any point $a < t_0 < b$ and two arbitrary constants c_1, c_2 , this differential equation on $[a, b]$ has one and only one solution subject to the initial condition that $\phi(t_0) = c_1$ and $p(t_0)\phi'(t_0) = c_2$. So there are exactly two linearly independent solutions to Eq. (9.17). But they may not all belong to $L^2([a, b])$. Hence there are at most two solutions in $L^2([a, b])$. Therefore, the deficiency indices of the symmetric operator L is (n, n) and the non-negative integer n is at most 2. When $n = 2$, there is a one-to-one correspondence between the $U(2)$ -family of maps between the deficiency spaces of the symmetric operator and the general boundary conditions which make the Sturm-Liouville operator self-adjoint [57].

The Sturm-Liouville boundary value problem has been used to yield kernels for the sampling theory of Kramer's type [58] (another generalization of the classical sampling theorem) and it has been shown that such Kramer's sampling methods possesses a Lagrange-type interpolation expansion [59, 60]. It could be very interesting to investigate if one can draw a connection between the Kramer's sampling theory and the sampling theory from the consideration of the same Sturm-Liouville operator with the same boundary conditions but which uses the theory of self-adjoint extensions of symmetric operators.

To construct a sampling-reconstruction method using self-adjoint extensions of a symmetric operator L with deficiency indices $(2, 2)$, one can start with extending the domain of L under L^* to obtain the symmetric extensions of L with deficiency indices $(1, 1)$ and then follow the same treatment in this thesis to find all the self-adjoint extensions of the resulting symmetric operator with deficiency indices $(1, 1)$. The eigenvalues of any self-adjoint extension constitute a set of sampling points and the expansion in the corresponding eigenbasis yields the corresponding reconstruction formula. Hence a sampling and reconstruction method is indeed expected.

Specifically, for example, for a Sturm-Liouville operator L in Eq. (9.14), assume that there are two linearly independent solution to Eq. (9.17), both in $L^2([a, b])$. So the operator L has a deficiency indices $(2, 2)$. Define an α -family of symmetric extensions of L , denoted by L_α , $0 \leq \alpha < 1$, with the same action as $\frac{d}{dt}(p(t)\frac{d}{dt}) + q(t)$, but on the following domains

$$D(L_\alpha) = \{\phi \in D(L^*) \mid \cos(\pi\alpha)\phi(a) = \sin(\pi\alpha)p(a)\phi'(a), \phi(b) = 0 = p(b)\phi'(b)\}. \quad (9.18)$$

Theorem 10.17 in [42] has shown that each operator L_α is a symmetric operator with deficiency indices $(1, 1)$. Further, for each fixed α between 0 and 1, we can define a family

of self-adjoint extensions $\{L_{\alpha,\beta} \mid 0 \leq \beta < 1\}$ where

$$D(L_{\alpha,\beta}) = \{ \phi \in D(L^*) \mid \begin{aligned} \cos(\pi\alpha)\phi(a) &= \sin(\pi\alpha)p(a)\phi'(a), \\ \cos(\pi\beta)\phi(b) &= \sin(\pi\beta)p(b)\phi'(b) \end{aligned} \}. \quad (9.19)$$

For any fixed α , it can be shown that the eigenvalues of $L_{\alpha,\beta}$, denoted by $\{t_n[\alpha, \beta]\}_n$, cover the real line exactly once for all $0 \leq \beta < 1$.

Understanding the connection between Kramer's sampling method which arises from Sturm-Liouville boundary value problem and the self-adjoint extensions of Sturm-Liouville operator should give direct access to the distribution of the sampling points. It might also lead to the construction of Kramer's sampling kernel directly from the desired set of sampling points rather than from a specific Sturm-Liouville boundary value problem. Namely, we construct the symmetric operator with deficiency indices $(2, 2)$ by restricting a self-adjoint operator with a pre-specified set of eigenvalues. In Kramer's sampling method rooted from a Sturm-Liouville boundary value problem, the sampling points are determined by the given boundary conditions, one can not freely control the sampling points.

This study of spectra of self-adjoint extensions of second-order differential operators should be very useful also in other applications. For example, in differential geometry, the first-order differential operator $i\frac{d}{dt}$ is not covariant on curved manifolds, but the second-order Laplacian operator $\Delta = -\frac{d^2}{dt^2}$ can be generalized to operate covariantly on functions defined on a Riemannian manifold. For works on how this is used to generalize sampling theory for curved manifolds, see R.T.W. Martin's thesis [61].

Further, the Hamiltonian operator $H = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x)$ on a finite one-dimensional interval is also a second-order differential operator. Subject to certain boundary conditions, it can have self-adjoint extensions. The eigenvalues of these self-adjoint extensions constitute the energy eigenvalues of the system. For example, in a crystal, Bloch discovered that the periodicity of the potential function $V(x)$ implies the periodicity of the eigen-wave function $\phi(x)$ of an electron, up to a phase (See Chapter 9 of [62]). Solving the Schrodinger equation for an electron is equivalent to solving for the eigenfunctions of the Hamiltonian H_α on a single interval $[0, L]$ with the periodic boundary condition $\psi(L) = e^{i2\pi\alpha}\psi(0)$. Here α is called the Bloch momentum. This α -family of self-adjoint extensions is a subset of the $U(2)$ -family of self-adjoint extensions of the Hamiltonian operator with vanishing boundary conditions. The set of eigenvalues of these self-adjoint extensions constitutes the energy bands of electrons. Because these eigenvalues do not cover the whole real line, the energy bands of electrons have gaps, which are important for explaining the difference between insulators, semiconductors and conductors.

9.3 Interpolation of Functions with Discontinuities and Spectrum with Accumulated Points

In Chapter 8, we saw that using a set of non-equidistant sampling points with sampling density linearly increasing toward the discontinuous point at $t = 0$, one achieves a significant 70% reduction of the Gibbs' overshoot in the approximation of a step function. The Gibbs' reduction is useful, for example, in image compression to reduce 'ringing' near discontinuities in the images. Linear change in the sampling density probably does not yield the optimal sampling grid in terms of reducing the Gibbs' overshoot. One expects that a more abrupt increasing in sampling density toward the discontinuous point $t = 0$ matches the behavior of the step function better. Hence it results in a better approximation and a further reduction of the Gibbs' overshoot, e.g., a quadratic increasing in sampling density should provide a reduction more than 70%. A natural question to follow would be how far we can ultimately reduce the Gibbs' overshoot?

The best result would be an absolute elimination of the overshoot. But this may not be achievable since the generalized sampling method assumes a finite minimum spacing between two adjacent points. This is rooted from the fact that the self-adjoint extension of the symmetric operator we consider here does not have accumulated eigenvalues. It sets a lower bound on the distance between two eigenvalues of a self-adjoint extension.

However, if one considers a symmetric operator T such that eigenvalues of its self-adjoint extension has the accumulated point at $t = r$, then one should be able to reduce the Gibbs' overshoot near the discontinuous point $t = r$ to an arbitrarily small amount. In this case, one can let the eigenvalues be denoted by the union of two increasing sets as the following

$$\{t_n\}_{n=-\infty}^{+\infty} \cup \{\hat{t}_n\}_{n=-\infty}^{+\infty}, \text{ where } t_m < r < \hat{t}_n, t_n < t_{n+1}, \hat{t}_n < \hat{t}_{n+1}, \forall m, n \in \mathbb{Z} \quad (9.20)$$

and

$$\lim_{n \rightarrow -\infty} t_n = -\infty, \quad \lim_{n \rightarrow +\infty} t_n = r^-, \quad \lim_{n \rightarrow -\infty} \hat{t}_n = r^+, \quad \lim_{n \rightarrow +\infty} \hat{t}_n = +\infty. \quad (9.21)$$

As long as this collection of eigenvalues obeys the conditions in Eq. (3.16), the underlying mathematics still holds except all the formulae need to be expressed in terms of a union of two sets of sampling points instead of one and all the summations need to be treated with extra care on the singularity at $t = r$.

As a consequence, any further generalization of the generalized sampling theory should be accompanied by the further development of the mathematical theory of self-adjoint extensions of simple symmetric operators in Hilbert space.

For another example, to improve the convergence of the reconstruction kernel, one needs to use an over-complete set of basis in the Hilbert space rather than the eigenbasis of a self-adjoint extension. This may not only be interesting from an engineer's perspective, but also be of its own interest in terms of functional analytical theory. What could be an expression of a self-adjoint extension in terms of an over-complete basis? What type of operator is it if the operator can be decomposed in terms of an over-complete basis of eigenvectors from different self-adjoint extensions? How to express such an operator solely in terms of the self-adjoint extensions?

As a consequence of such a development in functional analysis, one expects to be able to increase the convergence rate, for example, from the asymptotic convergence as $1/t$ to one with $1/t^2$ or $1/t^n$, $n \geq 2$. This should be achievable at a cost of increased redundancy in the samples. The signal is sampled at a rate higher than the desired Nyquist rate. The advantage is that we not only obtain a higher convergence rate in the reconstruction, but also have the ability to recover lost data.

There are also other directions in which one can further extend the mathematical results on self-adjoint extensions of symmetric operator with deficiency indices $(1, 1)$, although their immediate consequences in terms of sampling theory may not be clear. For example, it could be interesting to see what if the symmetric operator is not simple? What if the symmetric operator has continuous spectrum, in which case all its self-adjoint extensions will inherit the continuous spectrum?

9.4 Sampling for Vector-Valued Functions and Symmetric Operators with higher Deficiency Indices

In this thesis, we develop a generalized sampling theory for scalar-valued functions $\phi(t)$ by considering the symmetric operators with deficiency indices $(1, 1)$. A further generalization to vector-valued functions could be possible by considering the symmetric operators with high deficiency indices (N, N) , $N \geq 2$. This is important in quantum gravity because all the observed fundamental physical fields, like electromagnetic fields, are vector-valued functions.

Recall that by the Cayley transform, a simple symmetric operator T with deficiency indices (N, N) has a $U(N)$ -family of self-adjoint extensions. Each element in the $U(N)$ -group of maps between the two N -dimensional deficiency spaces extends T to a self-adjoint operator. This generalization is suggested by the fact that the $U(N)$ -group is richer than N copies of the $U(1)$ -group.

$$U(1) \otimes U(1) \otimes \dots \otimes U(1) \subsetneq U(N)$$

Indeed, a naive sampling theory for vector-valued functions is to treat the N -dimensional vector-valued function as a cartesian product of N scalar-valued functions and to accommodate the sampling theory for scalar-valued functions on each of its components. However, this treatment ignores the fact the the components of a vector-valued function may be correlated and one should be able to improve sampling efficiency by taking this into account. Therefore, using the symmetric operators with deficiency indices (N, N) , we expect to be able to generalize the sampling theory for classes of N -dimensional vector-valued functions which not only possess a time-varying Nyquist rate, but also prefer a direction of sampling, varying in time.

To be precise, let us look at the case of the generalized sampling theory for scalar-valued functions in this thesis. Let T denote the simple symmetric operator T with deficiency indices $(1, 1)$. Each real number t is an eigenvalue of its adjoint T^* with multiplicity of 1. So there is a normalized eigenvector ϕ_t associated with each eigenvalue t . Of course, each real number t is also an eigenvalue for one of the self-adjoint extensions of T . We define the scalar-valued function $\phi(t)$ for each vector ϕ in the Hilbert space as

$$\phi(t) = \langle \phi_t, \phi \rangle \quad \forall t \in \mathbb{R}. \quad (9.22)$$

If a self-adjoint extension of T has a set of eigenvalues $\{t_n\}_n$, then their corresponding eigenvectors $\{\phi_{t_n}\}_n$ form an eigenbasis and the decomposition in this eigenbasis gives the reconstruction formula

$$\phi(t) = \langle \phi_t, \phi \rangle = \sum_{n=-\infty}^{+\infty} \langle \phi_t, \phi_{t_n} \rangle \langle \phi_{t_n}, \phi \rangle = \sum_{n=-\infty}^{+\infty} \langle \phi_t, \phi_{t_n} \rangle \phi(t_n). \quad (9.23)$$

This is a discrete representation of the scalar-valued function $\phi(t)$ on the set of sampling points $\{t_n\}_n$.

By analogue, let T now be a simple symmetric operator with deficiency indices (N, N) . Its adjoint T^* has each real number t as its eigenvalue with multiplicity N . Of course, one can also find a self-adjoint extension of T which has the real number t as its eigenvalue with multiplicity of N . So for each real number t , there is a corresponding N -dimensional eigenspace, denoted by E_t . Let the following be an orthonormal basis of E_t

$$E_t = \text{span} \{ \phi_t^1, \dots, \phi_t^k, \dots, \phi_t^N \}. \quad (9.24)$$

Then for any vector ϕ in the Hilbert space, we can define a vector-valued function $\vec{\phi}(t)$ as

$$\vec{\phi}(t) = \begin{bmatrix} \phi^1(t) \\ \vdots \\ \phi^k(t) \\ \vdots \\ \phi^N(t) \end{bmatrix} = \begin{bmatrix} \langle \phi_t^1, \phi \rangle \\ \vdots \\ \langle \phi_t^k, \phi \rangle \\ \vdots \\ \langle \phi_t^N, \phi \rangle \end{bmatrix} \quad (9.25)$$

If a self-adjoint extension of T has a set of eigenvalues $\{t_n\}_n$ and each eigenvalue is of multiplicity N , then the decomposition in the eigenbasis associated to this self-adjoint extension yields the reconstruction of $\vec{\phi}(t)$ from its vector values on $\{t_n\}_n$. The k^{th} -component of $\vec{\phi}(t)$ is

$$\begin{aligned}\phi^k(t) = \langle \phi_t^k, \phi \rangle &= \sum_{n=-\infty}^{+\infty} \sum_{j=1}^N \langle \phi_t^k, \phi_{t_n}^j \rangle \langle \phi_{t_n}^j, \phi \rangle \\ &= \sum_{n=-\infty}^{+\infty} \sum_{j=1}^N \langle \phi_t^k, \phi_{t_n}^j \rangle \phi^j(t_n)\end{aligned}\tag{9.26}$$

Here $\phi^j(t_n)$ is the j^{th} -component of the function value at the sampling points t_m . The inner product $\langle \phi_t^k, \phi_{t_n}^j \rangle$ is the reconstruction kernel. This is a discrete representation of the vector-valued function $\vec{\phi}(t)$ on the set of sampling points $\{t_n\}_n$. A potentially very useful sampling-reconstruction method is expected.

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