An introduction to Gerber-Shiu analysis

by

Mirabelle Huynh

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

A valuable analytical tool to understand the event of ruin is the Gerber-Shiu discounted penalty function. It acts as a unified means of identifying ruin-related quantities which may help insurers understand their vulnerability ruin. This thesis provides an introduction to the basic concepts and common techniques used for the Gerber-Shiu analysis.

Chapter 1 introduces the insurer's surplus process U_t in the ordinary Sparre Andersen model along with key quantities such as the time of ruin T, the surplus immediately prior to ruin U_{T-}, the deficit at ruin $|U_T|$, and the infinite ruin probability. Defective renewal equations, the Dickson-Hipp transform, and Lundberg's fundamental equation are reviewed.

Chapter 2 introduces the classical Gerber-Shiu discounted penalty function $m_{\delta,12}(u)$. The function's value as a unified means of identifying ruin-related quantities is examined by considering special cases of the function for various penalty functions and values of δ. Two framework equations are derived by conditioning on the first drop in surplus below its initial value, and by conditioning on the time and amount of the first claim. A detailed discussion is provided for each of these conditioning arguments. By conditioning on the first drop in surplus below its initial level, we show that $m_{\delta,12}(u)$ satisfies a defective renewal equation; this result gives insight into the mathematical structure of the Gerber-Shiu function and provides guidance in our analysis. To determine a relationship between $m_{\delta,12}(u)$ with the interclaim time density and claim size density, we condition on the time and amount of the first claim to derive an integral equation satisfied by $m_{\delta,12}(u)$. The classical Poisson model (where interclaim times are exponentially distributed) is then considered and $m_{\delta,12}(u)$ is solved in full generality. In addition, we consider an exponential claim size density and an arbitrary interclaim time and determine the form of the solution of $m_{\delta,12}(u)$ up to an unknown density, which is derived in Chapter 4.

Chapter 3 introduces the delayed renewal model which allows the time until the first claim to be distributed differently than subsequent interclaim times. A brief discussion is made of a special case of the model called the stationary renewal model where the density of the time to the first claim is the equilibrium density of the density of subsequent interclaim times. Next, the Gerber-Shiu function in the delayed renewal model is considered and denoted as $m^d_{\delta,12}(u)$. Since the delayed renewal model reverts to the ordinary model after the first claim, Gerber-Shiu analysis in the delayed renewal model aims to determine the relationship between $m_{\delta,12}(u)$ and $m^d_{\delta,12}(u)$ such that we can extend results found in the ordinary model to the delayed model. We determine a functional relationship between $m_{\delta,12}(u)$ and $m^d_{\delta,12}(u)$ for a class of first interclaim time densities which includes the equilibrium density for the stationary renewal model, and the exponential density.

To conclude, Chapter 4 introduces a generalized Gerber-Shiu function where the penalty function includes two additional random variables : the minimum surplus level before ruin X_T, and the surplus immediately after the claim before the claim causing ruin R_{N_T-1}. This generalized Gerber-Shiu function, denoted as $m_\delta(u)$, allows for the study of random variables such as the last interclaim time and the last ladder height, which otherwise could not be studied using the classical definition of the function. Additionally, it is assumed that the size of a claim is dependant on the interclaim time that precedes it. As is done in Chapter 2, a detailed discussion of each of the two conditioning arguments is provided. Interestingly, despite containing a 4-variable penalty function, by conditioning on the first drop in surplus, it is demonstrated that $m_\delta(u)$ satisfies a defective renewal equation that is only dependent on the density of the 3 variables U_{T-}, $|U_T|$, and R_{N_T-1} which does not involve X_T. Using the uniqueness property of Laplace transforms, the form of the joint defective discounted densities of the 4 variables (U_{T-}, $|U_T|$, X_T, R_{N_T-1}), as well as the last ladder height are determined. The classical Poisson model is revisited and $m_\delta(u)$ is solved in full generality when the penalty function does not involve X_T. This is used to derive the joint defective discounted density of (U_{T-}, $|U_T|$, R_{N_T-1}) and the joint defective density of the last interclaim time before ruin and the claim causing ruin. Also revisited is the exponential claim size assumption where a penalty function is assumed such that $m_\delta(u)$ becomes the Laplace transform of the joint defective discounted density of (U_{T-}, $|U_T|$, X_T, R_{N_T-1}). This is solved in full generality and used to obtain the proper density of the last interclaim time as well as the joint defective discounted density of (U_{T-}, $|U_T|$).

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And where would I be without my loving and supportive friends and family?
I'd be "ruined".

Dedication

This is dedicated to my mother, father, and sister, for all their unconditional love and support throughout the course of my studies.

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Chapter 1

Background

1.1 Introduction

To provide sound risk management, insurers must adequately understand the risk of their financial obligations. This entails understanding worst case scenarios such as when the insurer's surplus process goes below 0 which we define as "ruin". Since insurers usually manage large amounts of capital, failure to properly calculate the risk of ruin can result in serious financial consequences. However, even in the simplest cases, an insurer's surplus process is very complicated to understand. A valuable analytical tool to understand the event of ruin is the Gerber-Shiu discounted penalty function. It acts as a unified means of identifying ruin-related quantities which may help insurers understand their vulnerability to the event of ruin. This may explain why after its introduction in the well-known paper [Gerber and Shiu (1998)], a considerable amount of ruin theory research has been devoted to study this function. The purpose of this thesis is to provide an introduction to the basic concepts and common techniques used in the so-called Gerber-Shiu analysis.

This function was first studied in the classical Poisson model where interclaim times are assumed to be exponentially distributed. In this model, claim sizes and interclaim times are also assumed to be independent. Research then began on the Gerber-Shiu function in the ordinary Sparre Andersen model where the interclaim times are instead assumed to follow an arbitrary density. Because the Gerber-Shiu function satisfies a defective renewal equation in the ordinary Sparre Andersen model, we are able to use the mathematical machinery behind defective renewal equations in our analysis. However, in this model, there is an implicit assumption that a claim occurs at time 0 which may not be true in settings we wish to consider. Thus, the Gerber-Shiu function in the delayed renewal model has also been considered where the first interclaim time is assumed to follow a (possibly) different density than the common density followed by subsequent interclaim times as is discussed in [Willmot, G.E. (2004b)] and [Dickson and Willmot (2003)].

Generalizations of the Gerber-Shiu function have also been studied, for example [Cheung et al. (2010a)], which will be discussed further in Chapter 4. A Gerber-Shiu function that has been generalized to a more general "cost" function is considered in [Cai et al. (2009)] and the Gerber-Shiu function at absolute ruin is considered in [Cai, J. (2007)].

There has also been a considerable amount of research on models where the size of a claim is assumed to be dependant on the interclaim time preceding it such as in [Willmot and Woo (2011)] and [Badescu et al. (2007)] but this topic is beyond the scope of the present work.

The thesis will proceed via four chapers. Chapter 1 introduces the insurer's surplus process U_t in the ordinary Sparre Andersen model and provides some mathematical preliminaries.

Chapter 2 introduces the classical Gerber-Shiu discounted penalty function $m_{\delta,12}(u)$. To obtain an expressions satisfied by $m_{\delta,12}(u)$, conditioning arguments are used. We commonly condition on the first drop in surplus below the initial value u and/or the time and amount of the first claim. In the classical Poisson model, results are needed only from conditioning on the time and amount of the first claim in order to solve for $m_{\delta,12}(u)$ in full generality. However, except for a few assumptions for the interclaim time density i.e. exponential or Coxian-2, we almost always require both conditioning arguments. After introducing the joint density of the time of ruin, surplus immediately prior to ruin, and deficit at ruin in Section 2.2, Sections 2.3 and 2.4 will examine each of the conditioning arguments in detail to obtain expressions satisfied by $m_{\delta,12}(u)$. Following this, two examples are considered. First, where interclaim times are exponentially distributed for an arbitrary claim size distribution (i.e. the classical Poisson model) and second where claim sizes are assumed to be exponentially distributed for an arbitrary interclaim time distribution.

Chapter 3 considers the Gerber-Shiu function in the delayed renewal model.

Finally, Chapter 4 considers an extension of the classical Gerber-Shiu function by generalizing the penalty function. This generalized Gerber-Shiu function will allow for analysis of ruin-related quantities that could not be studied using the classical definition of the function.

1.2 Ordinary Sparre Andersen model

Consider an insurer's surplus process defined by

$$U_t = u + ct - \sum_{i=1}^{N_t} Y_i, \tag{1.2.1}$$

where $u \geq 0$ is the initial surplus and $c > 0$ is the premium rate. The number of claims process $\{N_t, t \geq 0\}$ is assumed to be a renewal process defined by a sequence of positive, independent and identically distributed (i.i.d.) interclaim times $\{V_i\}_{i=1}^{\infty}$. That is, V_1 is the time of the first claim and V_i for $i = 2, 3, ...$ is the time between the $(i-1)$th and ith claim. For V an arbitrary V_i, define the interclaim time distribution function (d.f.) to be $K(t) = 1 - \overline{K}(t) = \Pr(V \leq t)$ and its probability density function (p.d.f.) to be $k(t) = K'(t)$ for $t > 0$. Also, we assume $\{Y_i\}_{i=1}^{\infty}$ is an i.i.d. sequence of positive claim size random variables (r.v.'s) and is independent of $\{V_i\}_{i=1}^{\infty}$. For Y an arbitrary claim size Y_i, let $P(y) = \Pr(Y \leq y)$ be its d.f. and $p(y) = P'(y)$ for $y > 0$ be its p.d.f.

We define the time of ruin T to be the time in which the surplus level first falls below zero i.e. $T = inf\{t > 0 : U_t < 0\}$ and we let $T = \infty$ if $U_t > 0$ for all $t > 0$. Thus, the N_Tth claim is the claim causing ruin. The surplus immediately prior to ruin U_{T-} and the deficit at ruin $|U_T|$ are 2 key r.v.'s in the so-called Gerber-Shiu analysis to be introduced in the next chapter. We also define the infinite ruin probability or simply, ruin probability, to be $\psi(u) = \Pr(\mathrm{T} < \infty | U_0 = u)$. Consider the following graphical representation of a sample path of the surplus process U_t.

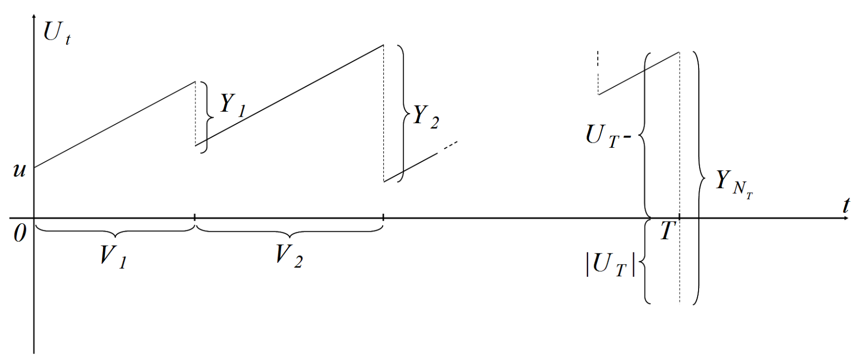

Figure 1.1: Sample path of the surplus process U_t

Note that the claim causing ruin Y_{N_T} is given by $U_{T-} + |U_T|$. Additionally, we assume the premium rate c satisfies the positive security loading condition

$$c\mathrm{E}[V] > \mathrm{E}[Y], \tag{1.2.2}$$

such that $\psi(u) < 1$ [Asmussen and Albrecher (2010), Section 3.3] and we often let $c = (1+\theta)E[Y]/E[V]$ where $\theta > 0$.

Also, we define the Laplace transform of a function $f(x)$ by

$$\tilde{f}(s) = \int_0^\infty e^{-sx} f(x) dx.$$

1.3 Preliminaries

Before introducing the so-called Gerber-Shiu analysis in the next section, we first introduce a few important results which will be very useful in the sequel.

1.3.1 Dickson-Hipp transform

The Dickson-Hipp transform of an integrable real-valued function $f(x)$ [Dickson and Hipp (2001)] is given by

$$\begin{aligned} T_r f(x) &= e^{rx} \int_x^\infty e^{-ry} f(y) dy \\ &= \int_0^\infty e^{-ry} f(x+y) dy, \end{aligned} \tag{1.3.1}$$

where r satisfies $|\tilde{f}(r)| < \infty$.

It can easily be shown that the Dickson-Hipp transform is a linear operator i.e. for constants a_i and integrable real-valued functions $f_i(x)$ where $i = 1, 2, ..., n$

$$T_r \left(\sum_{i=1}^{n} a_i\, f_i(x) \right) = \sum_{i=1}^{n} a_i\, T_r\big(f_i(x)\big). \tag{1.3.2}$$

Also, note that $\tilde{f}(s) = T_s f(0)$ and thus, the Dickson-Hipp transform is a "generalized" Laplace transform. Additionally, note that we can write the tail $\overline{F}(x) = \int_x^\infty f(t)dt$ as $T_0 f(x)$.

The properties of this transform are discussed in detail in [Li and Garrido (2004)] and understanding these properties is very useful in connection with Gerber-Shiu analysis. This is particularly true when interclaim times are assumed to be Coxian-n distributed as is done in [Willmot and Woo (2010)] and [Willmot, G.E. (2011)]. For our purposes, we consider the Laplace transform of $T_r f(x)$ given by

$$\begin{aligned}
\int_0^\infty e^{-sx}(T_r f(x))dx &= \int_0^\infty e^{-(s-r)x}\int_x^\infty e^{-ry}f(y)dydx \\
&= \int_0^\infty e^{-ry}\left(\int_0^y e^{-(s-r)x}dx\right)f(y)dy \\
&= \int_0^\infty e^{-ry}\left(\frac{1-e^{-(s-r)y}}{s-r}\right)f(y)dy \\
&= \frac{\int_0^\infty e^{-ry}f(y)dy - \int_0^\infty e^{-sy}f(y)dy}{s-r} \\
&= \frac{\tilde{f}(r)-\tilde{f}(s)}{s-r}, \qquad (1.3.3)
\end{aligned}$$

for $s \neq r$.

Note that the Laplace transform of $\overline{F}(x)$ is given by (1.3.3) if we let $r = 0$ and thus

$$\begin{aligned}
\int_0^\infty e^{-sx}\overline{F}(x)dx &= \int_0^\infty e^{-sx}(T_0 f(x))dx \\
&= \frac{1-\tilde{f}(s)}{s}. \qquad (1.3.4)
\end{aligned}$$

Now let Y be a r.v. with p.d.f. $f(y) = F'(y)$ and let $F_x(y) = F(x+y)/\overline{F}(x)$ be its excess loss distribution. For $r \in \mathbb{R}$, consider the following distribution which we will refer to as the generalized equilibrium distribution of $F(y)$

$$F_{ge}(y;r) = \frac{\int_0^\infty e^{-rx}\overline{F}(x)F_x(y)dx}{\int_0^\infty e^{-rx}\overline{F}(x)dx},$$

which is a mixture of $F_x(y)$ over x. In [Willmot, G.E. (2011)], it is shown that if $F(y)$ is a mixed Erlang d.f., then $F_{ge}(y;r)$ is a different mixture of the same Erlangs. It is also shown that if $F(y)$ is a Coxian-n d.f., then $F_{ge}(y;r)$ is also a Coxian-n d.f. Note that if $r=0$, then

$$\begin{aligned} F_{ge}(y;0) &= \frac{\int_0^\infty F(x+y)dx}{\int_0^\infty \overline{F}(x)dx} \\ &= \frac{\int_y^\infty F(x)dx}{\mathrm{E}[Y]}, \end{aligned}$$

which is the equilibrium distribution of $F(y)$. Now let the generalized equilibrium density of $f(y)$ be defined as

$$\begin{aligned} f_{ge}(y;r) &= F'_{ge}(y;r) \\ &= \frac{e^{ry}\int_y^\infty e^{-rx}f(x)dx}{\int_0^\infty e^{-rx}\overline{F}(x)dx}, \end{aligned} \tag{1.3.5}$$

and since the numerator is the Dickson-Hip transform of $f(x)$ and the denominator is the Laplace transform of $\overline{F}(x)$, then using (1.3.4)

$$f_{ge}(y;r) = \frac{T_r f(y)}{\frac{1-\tilde{f}(r)}{r}},$$

and using (1.3.3), its Laplace transform is given by

$$\tilde{f}_{ge}(s;r) = \frac{r}{s-r}\frac{\tilde{f}(r)-\tilde{f}(s)}{1-\tilde{f}(r)}, \tag{1.3.6}$$

which is a useful result in later analysis.

1.3.2 Lundberg's fundamental equation

Consider Lundberg's fundamental equation given by

$$1 - \mathrm{E}\left[e^{-sY-(\delta-cs)V}\right] = 0, \tag{1.3.7}$$

for Y and V an arbitrary claim size r.v. and interclaim time r.v., respectively. If Y and V are independent r.v.'s, then (1.3.7) is equivalent to

$$1 - \tilde{p}(s)\tilde{k}(\delta - cs) = 0. \tag{1.3.8}$$

Lundberg's fundamental equation and in particular, its root(s) with positive real parts have an important role in connection with the Gerber-Shiu analysis as will become evident in later sections. Note that if $\delta = 0$, then $s = 0$ is clearly a non-negative root.

1.3.3 Laplace transform of an important integral function

In Gerber-Shiu analysis, we often come across a function that is of the form

$$\eta_\delta(u) = \int_0^\infty e^{-\delta t}\omega_t(u+ct)k(t)dt, \tag{1.3.9}$$

and we are interested in its Laplace transform which is given by

$$\begin{aligned}
\tilde{\eta}_\delta(s) &= \int_0^\infty e^{-su}\left(\int_0^\infty e^{-\delta t}\omega_t(u+ct)k(t)dt\right)du \\
&= \int_0^\infty e^{-(\delta-cs)t}\left(\int_0^\infty e^{-s(u+ct)}\omega_t(u+ct)du\right)k(t)dt \\
&= \int_0^\infty e^{-(\delta-cs)t}\left(\tilde{\omega}_t(s) - \int_0^{ct} e^{-sx}\omega_t(x)dx\right)k(t)dt \\
&= \int_0^\infty e^{-(\delta-cs)t}\,\tilde{\omega}_t(s)k(t)dt - \tilde{\omega}^*_\delta(\delta-cs),
\end{aligned} \tag{1.3.10}$$

where

$$\tilde{\omega}^*_\delta(s) = \int_0^\infty \int_0^{ct} e^{-\frac{1}{c}\{\delta x + s(ct-x)\}}\omega_t(x)k(t)dxdt. \tag{1.3.11}$$

If $\omega_t(x) = \omega(x)$ i.e. a function independent of t, then

$$\tilde{\eta}_\delta(s) = \tilde{\omega}(s)\tilde{k}(\delta-cs) - \tilde{\omega}^*_\delta(\delta-cs), \tag{1.3.12}$$

where

$$\tilde{\omega}^*_\delta(s) = \int_0^\infty \int_0^{ct} e^{-\frac{1}{c}\{\delta x + s(ct-x)\}}\omega(x)k(t)dxdt.$$

1.3.4 Defective renewal equations

Suppose $m(x)$ satisfies the defective renewal equation

$$m(x) = \phi\int_0^x m(x-y)f(y)dy + v(x), \qquad x \geq 0 \tag{1.3.13}$$

where $\phi \in (0,1)$ and $f(y) = F'(y)$ is a p.d.f. with $F(0) = 0$. We assume $v(x)$ is a locally bounded function i.e. $|v(x)| < \infty$ for $x < \infty$.

To find a general solution for $m(x)$, we begin by taking the Laplace transform of (1.3.13) to obtain

$$\tilde{m}(s) = \phi\, \tilde{m}(s)\tilde{f}(s) + \tilde{v}(s),$$

and solving for $\tilde{m}(s)$, we obtain

$$\tilde{m}(s) = \frac{\tilde{v}(s)}{1 - \phi \tilde{f}(s)}. \tag{1.3.14}$$

Now consider a compound r.v. with secondary p.d.f. $f(x)$ and primary distribution a geometric with parameter ϕ (i.e. GEO(ϕ)). We call this an "associated" compound geometric r.v. It has a mass point of $1 - \phi$ at 0 and for $x > 0$, if its p.d.f. is given by

$$g(x) = \sum_{n=1}^{\infty} (1-\phi)(\phi)^n f^{*n}(x), \tag{1.3.15}$$

where $f^{*n}(x)$ is the n-fold convolution of $f(x)$ with itself, then the d.f. of this compound geometric r.v. can be written as

$$G(y) = 1 - \overline{G}(y) = 1 - \phi + \int_0^y g(x)dx. \tag{1.3.16}$$

Consider the Laplace transform of $g(x) = -\overline{G}'(x)$ given by

$$\begin{aligned} \tilde{g}(s) &= \int_0^{\infty} e^{-sx} g(x)dx \\ &= \sum_{n=1}^{\infty} (1-\phi)(\phi)^n \{\tilde{f}(s)\}^n \\ &= \frac{1-\phi}{1-\phi\tilde{f}(s)} - (1-\phi), \end{aligned} \tag{1.3.17}$$

using properties of an infinite geometric series. Substitution into (1.3.14) yields

$$\begin{aligned} \tilde{m}(s) &= \frac{\tilde{v}(s)}{1-\phi}(1 - \phi + \tilde{g}(s)) \\ &= \tilde{v}(s) + \frac{\tilde{v}(s)\tilde{g}(s)}{1-\phi}, \end{aligned}$$

and inversion leads to

$$m(x) = v(x) + \frac{1}{1-\phi}\int_0^x v(y)g(x-y)dy, \tag{1.3.18}$$

which is a general solution to the defective renewal equation (1.3.13). Thus, we note a close relationship between defective renewal equations and its "associated" compound geometric distribution $G(x)$.

Now consider when $v(x)$ is differentiable and recall that $\overline{G}(0) = \phi$ from (1.3.16). Then using integration by parts, we obtain

$$\begin{aligned}\int_0^x v(y)g(x-y)dy &= v(y)\overline{G}(x-y)\Big]_{y=0}^{x} - \int_0^x \overline{G}(x-y)v'(y)dy \\ &= \phi\, v(x) - v(0)\overline{G}(x) - \int_0^x \overline{G}(x-y)v'(y)dy,\end{aligned}$$

and substitution into (1.3.18) yields

$$\begin{aligned}m(x) &= v(x) + \frac{1}{1-\phi}\left(\phi\, v(x) - v(0)\overline{G}(x) - \int_0^x \overline{G}(x-y)v'(y)dy\right) \\ &= \frac{1}{1-\phi}\left(v(x) - v(0)\overline{G}(x) - \int_0^x \overline{G}(x-y)v'(y)dy\right), \end{aligned}\tag{1.3.19}$$

which is an alternate form of the general solution for $m(x)$ in (1.3.18) if $v(x)$ is differentiable. This result will be useful in Section 4.3 when we determine a density of interest.

Using (1.3.17), the Laplace transform of $\overline{G}(x) = \phi - \int_0^x g(y)dy$ from (1.3.16) is given by

$$\begin{aligned}\tilde{\overline{G}}(s) &= \frac{\phi}{s} - \frac{\tilde{g}(s)}{s} \\ &= \frac{\phi}{s} - \frac{1}{s}\left(\frac{1-\phi}{1-\phi\tilde{f}(s)} - (1-\phi)\right) \\ &= \frac{1}{s}\left(1 - \frac{1-\phi}{1-\phi\tilde{f}(s)}\right) \\ &= \frac{1}{s}\left(\frac{1-\phi\tilde{f}(s)-(1-\phi)}{1-\phi\tilde{f}(s)}\right) \\ &= \frac{\phi}{s}\frac{1-\tilde{f}(s)}{1-\phi\tilde{f}(s)},\end{aligned}$$

which we rearrange to obtain

$$\big(1-\phi\tilde{f}(s)\big)\tilde{\overline{G}}(s) = \phi\,\frac{1-\tilde{f}(s)}{s},$$

or equivalently

$$\tilde{\overline{G}}(s) = \phi\tilde{\overline{G}}(s)\tilde{f}(s) + \phi\,\frac{1-\tilde{f}(s)}{s}.$$

Its inversion leads to

$$\overline{G}(x) = \phi\int_0^x \overline{G}(x-y)f(y)dy + \phi\overline{F}(x), \qquad x \geq 0, \tag{1.3.20}$$

and thus, the tail of the "associated" compound geometric r.v. also follows a defective renewal equation given by (1.3.20). And since a defective renewal equation uniquely defines a function, given that an arbitrary function satisfies a defective renewal equation with a form given by (1.3.20), then the function is equal to the tail of a compound GEO(ϕ) r.v. with secondary p.d.f $f(x)$. Note that we can also express $\overline{G}(x)$ as

$$\overline{G}(x) = \sum_{n=1}^{\infty}(1-\phi)(\phi)^n\overline{F}^{*n}(x),$$

where $\overline{F}^{*n}(x) = \int_x^{\infty} f^{*n}(y)dy$.

Chapter 2

The classical Gerber-Shiu function

2.1 Introduction

Since we are interested in identifying various ruin-related quantities to understand the risk of ruin, we focus our attention to a Gerber-Shiu discounted penalty function [Gerber and Shiu (1998)] given by

$$m_{\delta,12}(u) = \mathrm{E}[e^{-\delta T} w_{12}(U_{T^-}, |U_T|) I(T < \infty) | U_0 = u], \tag{2.1.1}$$

which is a function of u where $\delta \geq 0$ is often interpreted as a force of interest. The so-called "penalty function" $w_{12}(x, y)$, is assumed to be an integrable function for $x > 0$ and $y > 0$. We define the indicator function $I(A)$ to equal 1 if the event A occurs and equal to 0 if the event A does not occur. Thus, since $I(T < \infty)$ is included in $m_{\delta,12}(u)$, we are generally only concerned with cases where ruin is inevitable.

With carefully chosen functions for $w_{12}(x, y)$ and values of δ, the Gerber-Shiu function becomes an expression for various ruin-related quantities of interest i.e. moments, d.f.'s, Laplace transforms of densities we are interested in, the ruin probability, etc. Some special cases of the Gerber-Shiu function and the ruin-related quantities that it becomes for various functions of $w_{12}(x, y)$ and values of δ are given in the table on the next page.

Table 2.1: Special Cases of the Gerber-Shiu Function: Ruin-Related Quantities

MOMENTS		
$w_{12}(x,y)$	δ	$m_{\delta,12}(u)$
$x^j y^k$	0	$E[(U_{T-})^j(\lvert U_T\rvert)^k I(T<\infty)\lvert U_0=u]$, joint jth moment of the surplus immediately prior to ruin and kth moment of the deficit at ruin
x^j	0	$E[(U_{T-})^j I(T<\infty)\lvert U_0=u]$, jth moment of the surplus immediately prior to ruin
y^k	0	$E[(\lvert U_T\rvert)^k I(T<\infty)\lvert U_0=u]$, kth moment of the deficit at ruin
y^k	>0	$E[e^{-\delta T}(\lvert U_T\rvert)^k I(T<\infty)\lvert U_0=u]$, discounted kth moment of the deficit at ruin
$(x+y)^j$	0	$E[(U_{T-}+\lvert U_T\rvert)^j I(T<\infty)\lvert U_0=u]$, jth moment of the claim causing ruin

DISTRIBUTION FUNCTIONS		
$w_{12}(x,y)$	δ	$m_{\delta,12}(u)$
$I(X\le x, Y\le y)$	0	$E[I(U_{T-}\le x,\lvert U_T\rvert\le y)I(T<\infty)\lvert U_0=u]$, joint d.f. of the surplus immediately prior to ruin and the deficit at ruin
$I(Y\le y)$	0	$E[I(\lvert U_T\rvert\le y)I(T<\infty)\lvert U_0=u]$, d.f. of the deficit at ruin
$I(X\le x)$	0	$E[I(U_{T-}\le x)I(T<\infty)\lvert U_0=u]$, d.f. of the surplus immediately prior to ruin
$I(X+Y\le z)$	0	$E[I(U_{T-}+\lvert U_T\rvert\le z)I(T<\infty)\lvert U_0=u]$, d.f. of the claim causing ruin

LAPLACE TRANSFORM OF DENSITIES		
$w_{12}(x,y)$	δ	$m_{\delta,12}(u)$
$e^{-s_1x-s_2y}$	>0	$E[e^{-\delta T - s_1 U_{T-} - s_2\|U_T\|} I(T<\infty)\|U_0=u]$, Laplace transform of the joint density of the time of ruin, the surplus immediately prior to ruin, and the deficit at ruin
$e^{-s(x+y)}$	0	$E[e^{-s(U_{T-}+\|U_T\|)} I(T<\infty)\|U_0=u]$, Laplace transform of the density of the claim causing ruin
1	>0	$E[e^{\delta T} I(T<\infty)\|U_0=u]$, Laplace transform of the density of the time of ruin
RUIN PROBABILITY		
$w_{12}(x,y)$	δ	$m_{\delta,12}(u)$
1	0	$E[I(T<\infty)\|U_0=u]=\psi(u)$, ruin probability

And since, $m_{\delta,12}(u)$ can become expressions for various ruin-related quantities of interest depending on our choice of the penalty function, it is easy to see that the Gerber-Shiu function serves as a valuable analytical tool to understand the event of ruin and acts as a unified means of identifying ruin-related quantities. Note that in the case where $w_{12}(x,y)=1$ and $\delta>0$, $m_{\delta,12}(u)$ becomes the Laplace transform of the density of the time of ruin. We will later show that in this case, $m_{\delta,12}(u)$ satisfies a defective renewal equation and is also equal to the tail of a compound geometric r.v. Obtaining the density of the time of ruin T by inverting this special case of the Gerber-Shiu function allows for the calculation of finite ruin probabilities i.e. $\Pr(T\le t|U_0=u)$ for $t>0$. In [Dickson and Willmot (2005)], an expression for the density of the time of ruin is derived using this method in the classical Poisson model where interclaim times are exponentially distributed and finite ruin probabilities for mixed Erlang claim sizes are calculated. Determining an explicit solution for this density is not easy (even in the classical Poisson model) and is beyond the scope of this thesis.

Under certain assumptions for $p(y)$ and/or $k(t)$ (such as in the classical Poisson model where $k(t)$ is assumed to follow an exponential distribution), we are able to explicitly state $m_{\delta,12}(u)$ entirely in terms of known quantities. Otherwise, in some cases of $p(y)$ and/or $k(t)$, $m_{\delta,12}(u)$ can only be solved for certain choices of the penalty function.

To obtain an expressions satisfied by $m_{\delta,12}(u)$, we use conditioning arguments. We commonly condition on the first drop in surplus below the initial value u and/or the time and amount of the first claim. After introducing the joint density of $(T, U_{T-}, |U_T|)$ in Section 2.2, Sections 2.3 and 2.4 will examine each of the conditioning arguments in detail to obtain expressions satisfied by $m_{\delta,12}(u)$.

2.2 The joint density of T, U_{T-}, and $|U_T|$

As a forewarning, the notation introduced in this section as well as some later sections may seem somewhat daunting at first but it is actually quite simple. Nonetheless, tips pointing out what some of the trickier superscripts and subscripts are denoting will be provided.

In this section, we wish to consider $h_{12}(t, x, y|u)$, the joint density of $(T, U_{T-}, |U_T|)$ at (t, x, y). A subscript "1" indicates a quantity involving U_{T-} and a subscript "2" indicates a quantity involving $|U_T|$. Note that because we are assuming that ruin occurs (i.e. $T < \infty$), the joint and marginal densities of T, U_{T-}, and $|U_T|$ integrate to the probability of ruin (a quantity less than 1) and are thus, defective densities.

First, consider when the surplus immediately prior to ruin and the time of ruin are given and equal to x and t, respectively. Thus, to guarantee that ruin occurs from surplus level x at time t, this is equivalent to a claim exceeding x immediately occurring which has probability $\overline{P}(x)$. And in particular, if we want a deficit at ruin of size y, this claim must be of size $x + y$ which has density $p(x + y)$. Therefore, given $U_{T-} = x$ and $T = t$, the conditional density of $|U_T|$ is given by

$$p_x(y) = \frac{p(x+y)}{\overline{P}(x)}, \qquad y > 0,\, x > 0, \tag{2.2.1}$$

which is the excess loss density of $p(y)$ and does not depend on the time of ruin $T = t$.

Now consider when ruin occurs on the first claim. If an insurer starts with an initial surplus of u and collects premium at rate c up to time t when the first claim that causes ruin occurs, then the surplus immediately prior to ruin is $x = u + ct$. Or equivalently, ruin occurs at time $t = \frac{x-u}{c}$. A deficit of y is obtained when the first claim is of size $x + y$. Therefore, the joint distribution of $(T, U_{T-}, |U_T|)$ at (t, x, y) for ruin occurring on the first claim is $k(t)p(x + y)$ where $t = \frac{x-u}{c}$. And with a change of variable from t to x, this joint defective density is given by

$$h^*_{12}(x, y|u) \quad = \quad \frac{1}{c} k\left(\frac{x-u}{c}\right) p(x+y), \tag{2.2.2}$$

where $t = \frac{x-u}{c}$, $x > u$, and $y > 0$. A superscript " $*$ " that *appears on an "h"* indicates a density for ruin occurring on the first claim. Since $p_x(y)$ was determined to be the conditional density of $|U_T|$ given $U_{T-} = x$ and the time of ruin, we can also write $h^*_{12}(x, y|u)$ as

$$h^*_{12}(x, y|u) \quad = \quad h^*_1(x|u)p_x(y), \tag{2.2.3}$$

where using (2.2.1) and (2.2.2), $h^*_1(x|u) = \frac{1}{c}k\left(\frac{x-u}{c}\right)\overline{P}(x)$ is the marginal defective density of U_{T-} for ruin occurring on the first claim.

Next, consider when ruin occurs on claims subsequent to the first. Then there is no direct relationship between t, x, and y. We only know that x must be less than $u + ct$. Let's denote $h^{**}_{12}(t, x, y|u)$ to be the joint defective density of $(T, U_{T-}, |U_T|)$ given that ruin occurs on claims subsequent to the first. A superscript " $**$" indicates a density for ruin occurring on subsequent claims.

Therefore, the joint defective density of $(T, U_{T-}, |U_T|)$ which we denote as $h_{12}(t, x, y|u)$ is defined differently depending on whether ruin occurs on the first claim ($x = u + ct$) or subsequent claims ($x < u + ct$) and can be summarized as follows

$$h_{12}(t, x, y|u) = \begin{cases} h^*_{12}(x, y|u) = \frac{1}{c}k\left(\frac{x-u}{c}\right)p(x + y), & t = \frac{x-u}{c},\ x > u,\ y > 0 \\ \\ h^{**}_{12}(t, x, y|u), & t > 0,\ 0 < x < u + ct,\ y > 0. \end{cases}$$

Now we introduce a class of densities which we refer to as "discounted" densities because we appear to be "discounting" the density of $(T, U_{T-}, |U_T|)$ from the time of ruin using δ. These discounted densities do not have any meaningful probabilistic interpretations and in fact, to make these types of interpretations, we often need to let $\delta = 0$. However, these discounted densities will become very useful in the following sections. First, using (2.2.2) and (2.2.3), let the joint defective discounted density of the surplus immediately prior to ruin and the deficit at ruin for ruin occurring on the first claim be given by

$$\begin{aligned} h^*_{\delta,12}(x, y|u) &= e^{-\delta\frac{x-u}{c}}h^*_{12}(x, y|u) \\ &= e^{-\delta\frac{x-u}{c}}\frac{1}{c}k\left(\frac{x-u}{c}\right)p(x + y) \\ &= h^*_{\delta,1}(x|u)p_x(y), \end{aligned} \tag{2.2.4}$$

where

$$h^*_{\delta,1}(x|u) = e^{-\delta\frac{x-u}{c}}\frac{1}{c}k\left(\frac{x-u}{c}\right)\overline{P}(x),$$

is the defective discounted marginal density of U_{T-} for ruin occurring on the first claim. A subscript "δ" that *appears on an "h"* indicates that it is a discounted density. Also, let the joint defective discounted density of the surplus immediately prior to ruin and the deficit at ruin for ruin occurring on subsequent claims be given by

$$h^{**}_{\delta,12}(x,y|u) \;=\; \int_0^\infty e^{-\delta t} h^{**}_{12}(t,x,y|u)dt. \tag{2.2.5}$$

Note that, similarly to (2.2.3),

$$h^{**}_{12}(t,x,y|u) = h^{**}_{1}(t,x|u)p_x(y), \tag{2.2.6}$$

where $h^{**}_1(t,x|u)$ is the joint defective density of T and U_{T-} on claims subsequent to the first. Then using (2.2.5)

$$h^{**}_{\delta,12}(x,y|u) \;=\; h^{**}_{\delta,1}(x|u)p_x(y),$$

where

$$h^{**}_{\delta,1}(x|u) = \int_0^\infty e^{-\delta t} h^{**}_1(t,x|u)dt,$$

is the marginal defective discounted density of U_{T-} for ruin occurring on claims subsequent to the first.

Next, using (2.2.3) and (2.2.6), let

$$\begin{aligned} h_{\delta,12}(x,y|u) &= h^{*}_{\delta,12}(x,y|u) + h^{**}_{\delta,12}(x,y|u) && (2.2.7)\\ &= h^{*}_{\delta,1}(x|u)p_x(y) + h^{**}_{\delta,1}(x|u)p_x(y) \\ &= h_{\delta,1}(x|u)p_x(y), && (2.2.8) \end{aligned}$$

be the joint defective discounted density of $(U_{T-}, |U_T|)$ where

$$h_{\delta,1}(x|u) \;=\; h^{*}_{\delta,1}(x|u) + h^{**}_{\delta,1}(x|u),$$

is the marginal defective discounted density of U_{T-}.

Since the Gerber-Shiu function $m_{\delta,12}(u)$ given by (2.1.1) is essentially an expectation of $e^{-\delta T} w_{12}(U_{T-}, |U_T|)I(T<\infty)$, we can write $m_{\delta,12}(u)$ as a sum of contributions from ruin on the first claim $\left(t = \frac{x-u}{c},\ x>u,\ y>0\right)$, ruin on subsequent claims $(t>0,\ 0<x<u+ct,\ y>0)$, and 0 otherwise as follows

$$
\begin{aligned}
m_{\delta,12}(u) &= \int_0^\infty \int_u^\infty e^{-\delta\left(\frac{x-u}{c}\right)} w_{12}(x,y) h^*_{12}(x,y|u)\,dxdy \\
&+ \int_0^\infty \int_0^\infty \int_0^{u+ct} e^{-\delta t} w_{12}(x,y) h^{**}_{12}(x,y|u)\,dxdydt \\
&= \int_0^\infty \int_0^\infty w_{12}(x,y) \Big(h^*_{\delta,12}(x,y|u) + h^{**}_{\delta,12}(x,y|u) \Big) dxdy \\
&= \int_0^\infty \int_0^\infty w_{12}(x,y) h_{\delta,12}(x,y|u)\,dxdy, \qquad (2.2.9)
\end{aligned}
$$

using (2.2.4), (2.2.5), and (2.2.7).

2.3 Conditioning on the first drop in surplus

In this section, by conditioning on the first drop in surplus below its initial level, we determine an important equation satisfied by $m_{\delta,12}(u)$. Consider the following graphical representation of a sample path followed by a surplus process with initial level u where the first drop in surplus below its initial level is of size y and occurs at time t.

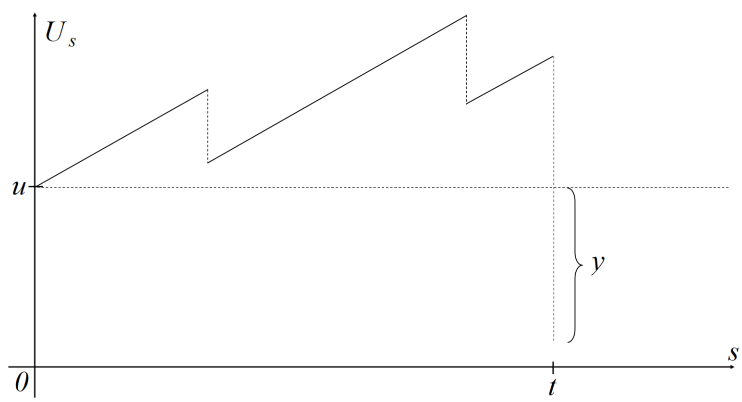

Figure 2.1: A sample path of U_t showing the first drop in surplus

For convenience, we will refer to a drop in surplus below its initial level as simply, a drop in surplus. Now consider $h_{12}(t,x,y|u)$, the joint density of $(T, U_{T-}, |U_T|)$ at (t,x,y) and simply let $u = 0$. Then $h_{12}(t,x,y|0)$ is the density of a surplus process starting at level 0, being at level x above 0 at time t without ruin first occurring and then a claim of size $x + y$ immediately occurring such that ruin occurs with a deficit of y and a surplus immediately prior to ruin of x. A sample path of this scenario is shown in the following figure.

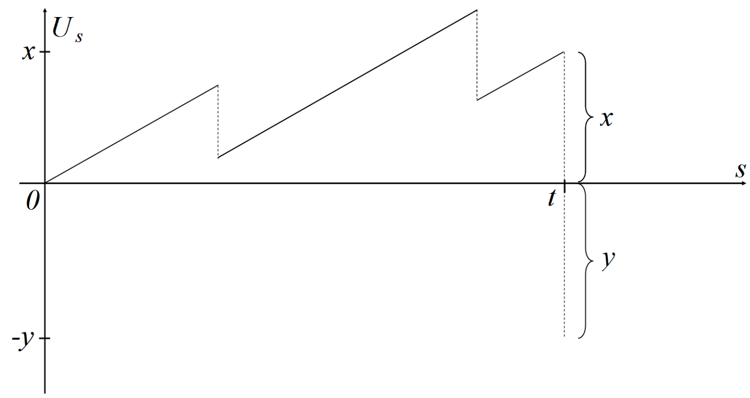

Figure 2.2: A sample path of U_t showing ruin occurring when $u = 0$

Now consider an almost parallel scenario where a surplus process starting at level u is at an amount x above u at time t without a drop in surplus first occurring and then a claim of size $x + y$ immediately occurring such that the first drop in surplus occurs and is of size y and the surplus immediately prior to the drop is $u + x$. Note that if $x = u + ct$, then the first drop in surplus occurs on the first claim, and if $x < u + ct$, then the first drop occurs on subsequent claims. A sample path of this scenario is shown in the following figure.

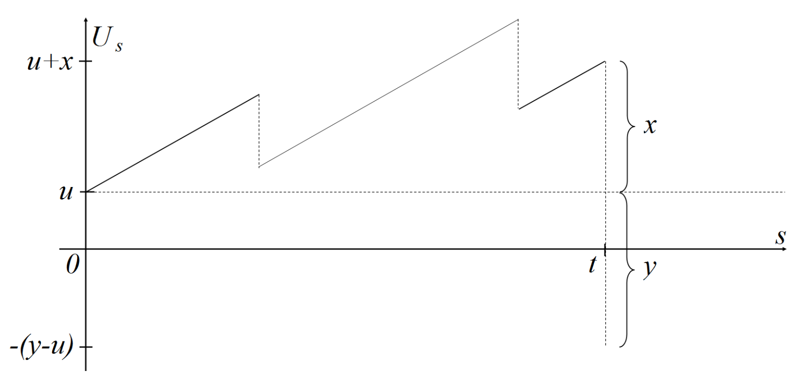

Figure 2.3: A sample path of U_t showing a drop in surplus causing ruin

Referring to Figure 2.2 and 2.3, we can observe that the sample path in Figure 2.2 is simply the sample path of Figure 2.3 translated up an amount u and thus, it is easy to see that both scenarios must share the same density $h_{12}(t, x, y|0)$. Therefore, for a surplus process starting at level u, $h_{12}(t, x, y|0)$ is the density of the first drop in surplus of size y from a surplus level $u + x$ at time t. Also, depending on the size y of the drop in surplus, ruin could occur ($y > u$) or not occur ($y < u$).

Now, we wish to determine an expression for $m_{\delta,12}(u)$ by conditioning on the first drop in surplus. To do this, we consider the following 4 cases for which the first drop in surplus occurs:

Case 1: first drop occurs on the first claim and causes ruin
Case 2: first drop occurs on the first claim and does not cause ruin
Case 3: first drop occurs on a subsequent claim and causes ruin
Case 4: first drop occurs on a subsequent claim and does not cause ruin

Let's now consider each of these cases in detail and outline the contribution that each case makes to $m_{\delta,12}(u) = E[e^{-\delta T} w_{12}(U_{T^-}, |U_T|)I(T < \infty)|U_0 = u]$. Make note of the corresponding figures which help to illustrate each scenario.

Case 1: first drop occurs on the first claim and causes ruin

The first drop occurs on the first claim and causes ruin when the surplus process, starting at level u, accumulates at rate c until it reaches an amount $x = ct$ above u when the first claim occurs (at time $t = \frac{x}{c}$) and causes a drop in surplus below u of size y where y must exceed u to cause ruin. Then the surplus immediately prior to ruin is $u + x$ and the deficit at ruin is $y - u$.

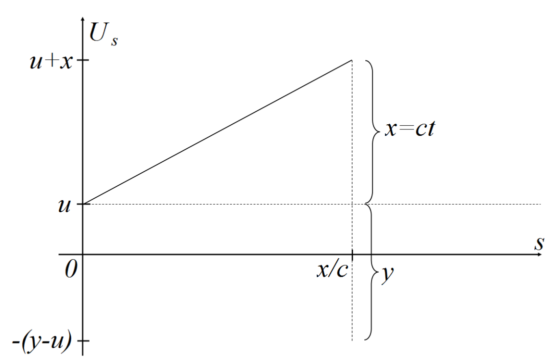

Figure 2.4: Conditioning on the first drop in surplus - Case 1

This occurs with density $h_{12}^*(x, y|0)$ where $x > 0$ and $y > u$. Thus, using (2.2.4), the contribution to $m_{\delta,12}(u)$ for this case is

$$\begin{aligned} &= \int_u^\infty \int_0^\infty e^{-\delta \frac{x}{c}} w_{12}(u+x, y-u) h_{12}^*(x, y|0) dx dy \\ &= \int_u^\infty \int_0^\infty w_{12}(u+x, y-u) h_{\delta,12}^*(x, y|0) dx dy. \end{aligned} \tag{2.3.1}$$

Case 2: first drop occurs on the first claim and does not cause ruin

This case is satisfied by the scenario presented in Case 1 except that the drop in surplus below u of size y must be less than u for ruin not to occur. This also occurs with density $h_{12}^*(x, y|0)$ but for $x > 0$ and $y < u$. Since ruin does not occur and because we have assumed the surplus process renews after every claim, the process is said to renew with an initial surplus of $u - y$ where an amount of time $t = \frac{x}{c}$ has passed.

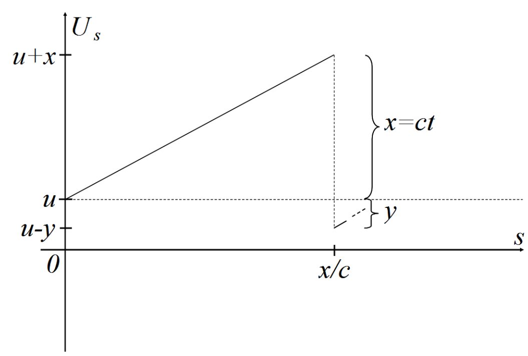

Figure 2.5: Conditioning on the first drop in surplus - Case 2

Thus, again using (2.2.4), the contribution to $m_{\delta,12}(u)$ for this case is

$$\begin{aligned} &= \int_0^u \int_0^\infty e^{-\delta \frac{x}{c}} m_{\delta,12}(u-y) h_{12}^*(x, y|0) dx dy \\ &= \int_0^u m_{\delta,12}(u-y) \left(\int_0^\infty h_{\delta,12}^*(x, y|0) dx \right) dy. \end{aligned} \tag{2.3.2}$$

Case 3: first drop occurs on a subsequent claim and causes ruin

The first drop occurs on a subsequent claim and causes ruin when the surplus process, starting at initial level u, is an amount x above u at time t when a claim (not the first) causes a drop in surplus for the first time of size $y > u$ for ruin to occur. Again, the surplus prior to ruin is $u + x$ and the deficit at ruin is $y - u$.

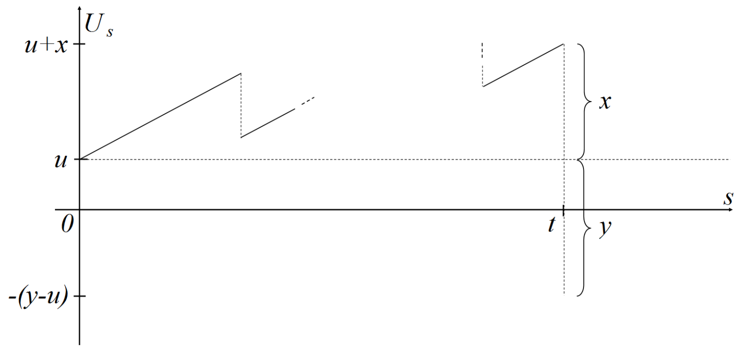

Figure 2.6: Conditioning on the first drop in surplus - Case 3

This occurs with density $h^{**}_{12}(t, x, y|0)$ where $t > 0$, $x > 0$ and $y > u$. Thus, using (2.2.5), the contribution to $m_{\delta,12}(u)$ for this case is

$$\begin{aligned} &= \int_u^\infty \int_0^\infty \int_0^\infty e^{-\delta t} w_{12}(u + x, y - u) h^{**}_{12}(t, x, y|0) dt dx dy \\ &= \int_u^\infty \int_0^\infty w_{12}(u + x, y - u) h^{**}_{\delta,12}(x, y|0) dx dy. \end{aligned} \tag{2.3.3}$$

Case 4: first drop occurs on a subsequent claim and does not cause ruin

This case is satisfied by the scenario presented in Case 3 except that the drop in surplus below u of size y must be less than u for ruin not to occur. This also occurs with density $h^{**}_{12}(t, x, y|0)$ but for $t > 0$, $x > 0$ and $y < u$. The process is said to renew with an initial surplus of $u - y$ where an amount of time t has passed.

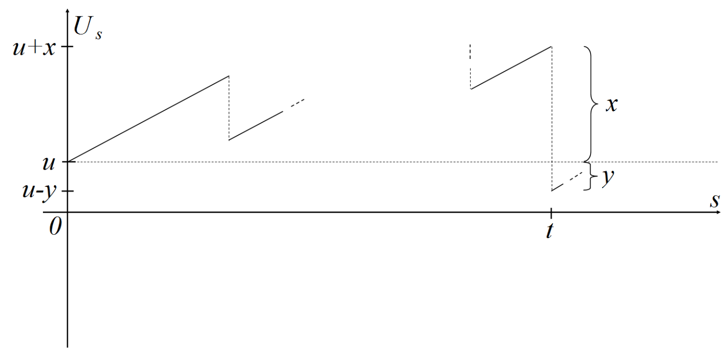

Figure 2.7: Conditioning on the first drop in surplus - Case 4

Thus, again using (2.2.5), the contribution to $m_{\delta,12}(u)$ for this case is

$$\begin{aligned} &= \int_0^u \int_0^\infty \int_0^\infty e^{-\delta t} m_{\delta,12}(u-y) h_{12}^{**}(t,x,y|0)dtdxdy \\ &= \int_0^u m_{\delta,12}(u-y)\left(\int_0^\infty h_{\delta,12}^{**}(x,y|0)dx\right)dy. \end{aligned} \tag{2.3.4}$$

Recall that the Gerber-Shiu function $m_{\delta,12}(u)$ is an expectation of a function involving $e^{-\delta T}$ which explains why we multiply by $e^{-\delta\frac{x}{c}}$ and $e^{-\delta t}$ in (2.3.2) and (2.3.4), respectively, and that is to account for the amount of time that has passed until the time of ruin. By summing the contributions to $m_{\delta,12}(u)$ from all 4 cases (2.3.1), (2.3.2), (2.3.3), and (2.3.4), and using (2.2.7), we obtain the following expression for $m_{\delta,12}(u)$

$$\begin{aligned} m_{\delta,12}(u) &= \int_0^u m_{\delta,12}(u-y)\left\{\int_0^\infty h_{\delta,12}^{*}(x,y|0)dx + \int_0^\infty h_{\delta,12}^{**}(x,y|0)dx\right\}dy \\ &+ \int_u^\infty \int_0^\infty w_{12}(u+x,y-u)\left\{h_{\delta,12}^{*}(x,y|0) + h_{\delta,12}^{**}(x,y|0)\right\}dxdy \\ &= \int_0^u m_{\delta,12}(u-y)\left\{\int_0^\infty h_{\delta,12}(x,y|0)dx\right\}dy \\ &+ \int_u^\infty \int_0^\infty w_{12}(u+x,y-u)h_{\delta,12}(x,y|0)dxdy. \end{aligned} \tag{2.3.5}$$

Now define the positive constant

$$
\begin{aligned}
\phi_\delta &= \int_0^\infty \int_0^\infty h_{\delta,12}(x,y|0)dydx \qquad (2.3.6)\\
&= \int_0^\infty h_{\delta,1}(x|0)\left(\int_0^\infty p_x(y)dy\right)dx\\
&= \int_0^\infty h_{\delta,1}(x|0)dx, \qquad (2.3.7)
\end{aligned}
$$

using (2.2.8). Note from (2.2.9) that $m_{\delta,12}(0)=\phi_\delta$ when $w_{12}(x,y)=1$. Then using (2.1.1),

$$
\begin{aligned}
0<\phi_\delta &= \mathrm{E}[e^{-\delta T}I(T<\infty)|U_0=0]\\
&\leq \mathrm{E}[I(T<\infty)|U_0=0]=\Pr(T<\infty|U_0=0)=\psi(0)\\
&< 1. \qquad (2.3.8)
\end{aligned}
$$

Also note that if $\delta=0$, then $\phi_0=\psi(0)$ which is the probability of ruin when the initial surplus $u=0$ and can also be interpreted as the probability of a drop in surplus.

Now we define the ladder height

$$
f_\delta(y) = \frac{1}{\phi_\delta}\int_0^\infty h_{\delta,12}(x,y|0)dx, \qquad y>0, \qquad (2.3.9)
$$

which using (2.3.6), is a proper density. And using (2.2.8), we can also express this as

$$
f_\delta(y)=\int_0^\infty p_x(y)\left\{\frac{h_{\delta,1}(x|0)}{\phi_\delta}\right\}dx, \qquad (2.3.10)
$$

and thus, the ladder height is a mixture over x of the p.d.f. $p_x(y)$ with the mixing weights $\frac{h_{\delta,1}(x|0)}{\phi_\delta}$. And therefore, in many cases, $f_\delta(u)$ is of the same family of distributions as $p(y)$ i.e. if claim sizes are assumed to be mixed Erlang, then $f_\delta(u)$ is a different mixture of the same Erlangs [Willmot and Lin (2011)]. Note that when $\delta=0$, $f_0(y)$ can be interpreted as the density of the size of the first drop in surplus given the drop occurs.

Thus, using (2.3.5) and (2.3.9), we can write $m_{\delta,12}(u)$ as the following defective renewal equation

$$
m_{\delta,12}(u)=\phi_\delta\int_0^u m_{\delta,12}(u-y)f_\delta(y)dy+v_{\delta,12}(u), \qquad (2.3.11)
$$

where

$$
v_{\delta,12}(u)=\int_u^\infty\int_0^\infty w_{12}(u+x,y-u)h_{\delta,12}(x,y|0)dxdy. \qquad (2.3.12)
$$

Recall from (1.3.18) of the Preliminaries section that the general solution to $m_{\delta,12}(u)$ can be expressed as

$$m_{\delta,12}(u) = v_{\delta,12}(u) + \frac{1}{1-\phi_\delta}\int_0^u v_{\delta,12}(y) g_\delta(u-y)dy, \tag{2.3.13}$$

where

$$g_\delta(u) = -\overline{G}'_\delta(u) = \sum_{n=1}^{\infty}(1-\phi_\delta)(\phi_\delta)^n f_\delta^{*n}(u), \tag{2.3.14}$$

is the p.d.f. of an "associated" compound geometric r.v. for $u > 0$ where ϕ_δ is defined by (2.3.6) and $f_\delta^{*n}(y)$ is the n-fold convolution of $f_\delta(y)$ defined by (2.3.9) with itself.

We are able to simplify $m_{\delta,12}(u)$ by considering special cases for $w_{12}(x,y)$. For example, if $w_{12}(x,y) = w_2(y)$, then (2.3.11) simplifies to

$$m_{\delta,2}(u) = \phi_\delta \int_0^u m_{\delta,2}(u-y) f_\delta(y)dy + v_{\delta,2}(u), \tag{2.3.15}$$

where using (2.3.12) and (2.3.9)

$$\begin{aligned} v_{\delta,2}(u) &= \int_u^\infty \int_0^\infty w_2(y-u) h_{\delta,12}(x,y|0)dxdy \\ &= \phi_\delta \int_u^\infty w_2(y-u) f_\delta(y)dy, \end{aligned} \tag{2.3.16}$$

which is a considerable simplification since unlike $v_{\delta,12}(u)$, $v_{\delta,2}(u)$ is a function of ϕ_δ and the ladder height $f_\delta(y)$ and does not depend on $h_{\delta,12}(x,y|0)$. Also, using (2.3.13), the general solution to $m_{\delta,2}(u)$ simplifies to

$$m_{\delta,2}(u) = v_{\delta,2}(u) + \frac{1}{1-\phi_\delta}\int_0^u v_{\delta,2}(y) g_\delta(u-y)dy, \tag{2.3.17}$$

where $g_\delta(y)$ is given by (2.3.14).

Now, consider when $w_{12}(x,y) = w_2(y) = 1$. Then $m_{\delta,12}(u) = \mathrm{E}[e^{-\delta T} I(T<\infty)|U_0 = u]$ and using (2.3.16), $v_{\delta,2}(u)$ simplifies to $\phi_\delta \overline{F}_\delta(u)$ where $\overline{F}_\delta(u) = \int_u^\infty f_\delta(y)dy$. Then it follows from (1.3.20) of the Preliminaries section that $m_{\delta,12}(u)$ is given by the tail of a compound geometric distribution

$$\overline{G}_\delta(u) = \sum_{n=1}^{\infty}(1-\phi_\delta)(\phi_\delta)^n \overline{F}_\delta^{*n}(u),$$

where $\overline{F}_\delta^{*n}(u) = \int_u^\infty f_\delta^{*n}(y)dy$. And thus (2.3.15) simplifies to

$$\overline{G}_\delta(u) = \phi_\delta \int_0^u \overline{G}_\delta(u-y) f_\delta(y)dy + \phi \overline{F}_\delta(u). \tag{2.3.18}$$

Then since $\overline{G}_\delta(u) = \mathrm{E}[e^{-\delta T} I(T < \infty)|U_0 = u]$, for $\delta = 0$, $\overline{G}_0(u) = \psi(u)$. Also, using (2.3.8), note that $\phi_\delta = \mathrm{E}[e^{-\delta T} I(T < \infty)|U_0 = 0] = \overline{G}_\delta(0)$.

We wish to obtain a relationship between the unknown quantities $h_{\delta,12}(x, y|0)$, ϕ_δ, and $f_\delta(y)$ in the solution of $m_{\delta,12}(u)$ given by (2.3.13) with the interclaim time density $k(t)$ and the claim size density $p(y)$ so that we can solve $m_{\delta,12}(u)$ in full generality. However, the analysis presented so far does not yield information on this relationship (except in (2.2.2), of course). To obtain this information, we usually condition on the time and amount of the first claim to obtain an integral equation satisfied by $m_{\delta,12}(u)$ in terms of $k(t)$ and $p(y)$. We discuss this method of conditioning in the next section. By conditioning on the time and amount of the first claim, letting $w_{12}(x, y) = 1$ to obtain $\overline{G}_\delta(u)$, and using (2.3.18), we are generally able to identify ϕ_δ and $f_\delta(y)$ which solves $m_{\delta,2}(u)$ given by (2.3.17). Therefore, the analysis presented thus far gives us insight into the mathematical structure of $m_{\delta,12}(u)$ and provides guidance in our analysis but does not provide an explicit solution for $m_{\delta,12}(u)$ in terms of our known p.d.f.'s $k(t)$ and $p(y)$. It is also important to note that even though $m_{\delta,12}(u)$ and $m_{\delta,2}(u)$ are both functions of u, if we are able to identify $h_{\delta,12}(x, y|0)$, then using (2.3.6) and (2.3.9), we can determine ϕ_δ and $f_\delta(y)$, respectively, and thus solve for both $m_{\delta,12}(u)$ and $m_{\delta,2}(u)$ in full generality. In principle, we can obtain $h_{\delta,12}(x, y|0)$ from $m_{\delta,12}(0)$ which can be obtained by $\lim s \to \infty\ s\tilde{m}_{\delta,12}(s)$ from the initial value theorem for Laplace transforms.

2.4 Conditioning on the time and amount of the first claim

We now determine an integral expression satisfied by $m_{\delta,12}(u)$ by conditioning on the time t and amount y of the first claim. To do this, we consider the values of t and y that lead to the following 2 cases:

Case 1: first claim causes ruin
Case 2: first claim does not cause ruin

Let's now consider each of these 2 cases in detail and outline the contribution that each case makes to $m_{\delta,12}(u)$. Consider each of the corresponding figures which help to illustrate each scenario and note that the first claim of size y occurs at time t with density $p(y)k(t)$.

Case 1: first claim causes ruin

For the case where the first claim causes ruin, consider an insurer with an initial surplus of u that collects premiums at rate c up to time t when the first claim occurs such that the surplus immediately prior to the first claim is $u + ct$. For ruin to occur on the first claim, the size of the first claim y must exceed $u+ct$ for the surplus to fall below zero. This leads to ruin occurring at time t, a surplus immediately prior to ruin of $u + ct$, and a deficit at ruin of $y - u - ct$.

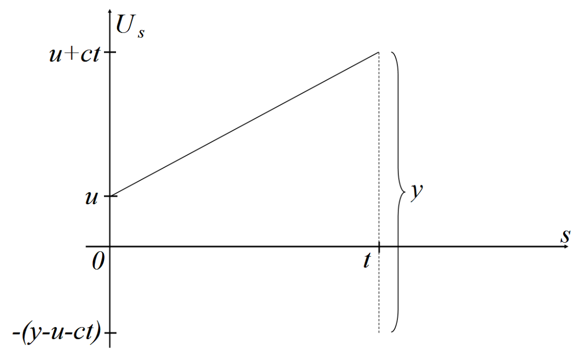

Figure 2.8: Conditioning on the time and amount of the first claim - Case 1

Thus, the contribution to $m_{\delta,12}(u)$ for this case is denoted as $\beta_{\delta,12}(u)$ and is given by

$$\begin{aligned} \beta_{\delta,12}(u) &= \int_0^\infty \int_{u+ct}^\infty e^{-\delta t} w_{12}(u+ct, y-u-ct)p(y)k(t)dydt \\ &= \int_0^\infty e^{-\delta t} \alpha_{12}(u+ct)k(t)dt, \end{aligned} \quad (2.4.1)$$

where

$$\alpha_{12}(x) = \int_x^\infty w_{12}(x, y-x)p(y)dy. \quad (2.4.2)$$

Case 2: first claim does not cause ruin

For the case where the first claim does not cause ruin, recall from Case 1 that the surplus immediately prior to the first claim is $u + ct$. Thus, for ruin not to occur on the first claim, the size of the first claim y must be less than $u + ct$ for the surplus to remain above 0. The process is then said to renew with an initial surplus of $u + ct - y$ where an amount of time t has passed.

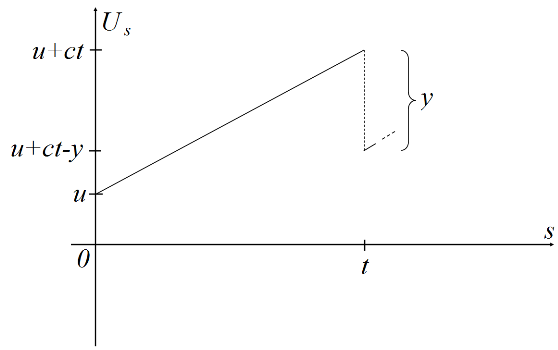

Figure 2.9: Conditioning on the time and amount of the first claim - Case 2

Thus, the contribution to $m_{\delta,12}(u)$ for this case is given by

$$\begin{aligned} &= \int_0^\infty \int_0^{u+ct} e^{-\delta t} m_{\delta,12}(u + ct - y)p(y)k(t)dydt \\ &= \int_0^\infty e^{-\delta t} \sigma_{\delta,12}(u + ct)k(t)dt, \end{aligned} \tag{2.4.3}$$

where

$$\sigma_{\delta,12}(x) = \int_0^x m_{\delta,12}(x - y)p(y)dy. \tag{2.4.4}$$

Note that since $\sigma_{\delta,12}(x)$ is a convolution of $m_{\delta,12}(x)$ and $p(x)$, its Laplace transform is given by

$$\tilde{\sigma}_{\delta,12}(s) = \tilde{m}_{\delta,12}(s)\tilde{p}(s). \tag{2.4.5}$$

By summing the contributions (2.4.1) and (2.4.3) to $m_{\delta,12}(u)$ from each of the 2 cases, we obtain the following expression

$$m_{\delta,12}(u) = \int_0^\infty e^{-\delta t} \sigma_{\delta,12}(u + ct)k(t)dt + \beta_{\delta,12}(u). \tag{2.4.6}$$

We note that the first term on the right hand side of (2.4.6) is of the form (1.3.9) from the Preliminaries section with $\omega_t(x)$ replaced by $\sigma_{\delta,12}(x)$ and thus, since $\sigma_{\delta,12}(x)$ is not a function of t, using (1.3.12) and (2.4.5), its Laplace transform is given by

$$\begin{aligned} & \int_0^\infty e^{-su}\int_0^\infty e^{-\delta t}\sigma_{\delta,12}(u+ct)k(t)dtdu & (2.4.7)\\ = \;& \tilde{\sigma}_{\delta,12}(s)\tilde{k}(\delta-cs)-\widetilde{\sigma}^*_{\delta,12}(\delta-cs) & (2.4.8)\\ = \;& \tilde{m}_{\delta,12}(s)\tilde{p}(s)\tilde{k}(\delta-cs)-\widetilde{\sigma}^*_{\delta,12}(\delta-cs), & \end{aligned}$$

where

$$\tilde{\sigma}^*_{\delta,12}(s)=\int_0^\infty\int_0^{ct}e^{-\frac{1}{c}\{\delta x+s(ct-x)\}}\sigma_{\delta,12}(x)k(t)dxdt. \tag{2.4.9}$$

Therefore, the Laplace transform of (2.4.6) is given by

$$\tilde{m}_{\delta,12}(s)=\tilde{m}_{\delta,12}(s)\tilde{p}(s)\tilde{k}(\delta-cs)-\tilde{\sigma}^*_{\delta,12}(\delta-cs)+\tilde{\beta}_{\delta,12}(s),$$

and solving for $\tilde{m}_{\delta,12}(s)$ we obtain

$$\Big(1-\tilde{p}(s)\tilde{k}(\delta-cs)\Big)\tilde{m}_{\delta,12}(s)=\tilde{\beta}_{\delta,12}(s)-\tilde{\sigma}^*_{\delta,12}(\delta-cs). \tag{2.4.10}$$

Note that the coefficient of $\tilde{m}_{\delta,12}(s)$ equals 0 when $s=\rho_\delta$, where ρ_δ is a root of Lundberg's fundamental equation given by (1.3.8) in the Preliminaries section. Furthermore, if $\mathbb{R}(\rho_\delta)\geq 0$, the left hand side of (2.4.10) is zero and we are left with

$$\tilde{\sigma}^*_{\delta,12}(\delta-c\rho_\delta)=\tilde{\beta}_{\delta,12}(\rho_\delta), \tag{2.4.11}$$

and since $\tilde{\beta}_{\delta,12}(\rho_\delta)$ is a known quantity in terms of $p(y)$ and $k(t)$, the above equality allows us to identify unknown quantities in $\tilde{\sigma}^*_{\delta,12}(\delta-cs)$ which is generally needed to invert $\tilde{m}_{\delta,12}(s)$ using (2.4.10) (numerically or analytically) under additional assumptions for $p(y)$ and/or $k(t)$.

2.5 Example 1: classical Poisson model

The classical Poisson model is characterized by an ordinary Sparre Andersen model where the claim size density $p(y)$ is assumed to be arbitrary and the interclaim time density $k(t)$ is assumed to be exponential [Willmot, G.E. (2011)]. Thus, we now consider when

$k(t) = \lambda e^{-\lambda t}$ which has Laplace transform $\tilde{k}(s) = \frac{\lambda}{\lambda+s}$ and using (2.4.9) we obtain

$$\begin{aligned}
\tilde{\sigma}^*_{\delta,12}(s) &= \int_0^\infty \int_0^{ct} e^{-\frac{1}{c}\{\delta x + s(ct-x)\}} \sigma_{\delta,12}(x) \lambda e^{-\lambda t} dx dt && (2.5.1)\\
&= \int_0^\infty e^{-\frac{1}{c}(\delta - s)x} \sigma_{\delta,12}(x) \left(\int_{\frac{x}{c}}^\infty \lambda e^{-(\lambda+s)t} dt \right) dx \\
&= \int_0^\infty e^{-\frac{1}{c}(\delta - s)x} \sigma_{\delta,12}(x) \frac{\lambda}{\lambda+s} e^{-\frac{1}{c}(\lambda+s)x} dx \\
&= \frac{\lambda}{\lambda+s} \int_0^\infty e^{-\left(\frac{\lambda+\delta}{c}\right)x} \sigma_{\delta,12}(x) dx \\
&= \frac{\lambda}{\lambda+s} \tilde{\sigma}_{\delta,12}\left(\frac{\lambda+\delta}{c}\right). && (2.5.2)
\end{aligned}$$

Then (2.4.10) becomes

$$\left(1 - \tilde{p}(s) \frac{\lambda}{\lambda+\delta-cs}\right) \tilde{m}_{\delta,12}(s) = \tilde{\beta}_{\delta,12}(s) - \frac{\lambda}{\lambda+\delta-cs} \tilde{\sigma}_{\delta,12}\left(\frac{\lambda+\delta}{c}\right). \tag{2.5.3}$$

Theorem 1. *For $\delta > 0$, there exists one real and positive root $s = \rho_\delta$ to Lundberg's fundamental equation given by*

$$1 - \tilde{p}(s) \frac{\lambda}{\lambda+\delta-cs} = 0, \tag{2.5.4}$$

and if $\delta = 0$, then $s = \rho_0 = 0$ is the only real and non-negative root.

Proof. Lundberg's fundamental equation given by (2.5.4) can be rewritten as

$$\lambda + \delta - cs = \lambda \tilde{p}(s). \tag{2.5.5}$$

Now let

$$\begin{aligned}
Y_1(s) = \lambda + \delta - cs \implies \quad & Y_1(0) = \lambda + \delta, \\
& Y_1'(0) = -c < 0.
\end{aligned}$$

That is, $Y_1(s)$ is a decreasing linear function in s with a vertical intercept of $\lambda + \delta$. Also, let

$$\begin{aligned}
Y_2(s) = \lambda \tilde{p}(s) = \lambda\, \mathrm{E}[e^{-sY}] \implies \quad & Y_2(0) = \lambda \tilde{p}(0) = \lambda, \\
& Y_2'(s) = -\lambda\, \mathrm{E}[Y e^{-sY}] < 0 \implies Y_2'(0) = -\lambda\, \mathrm{E}[Y], \\
& Y_2''(s) = \lambda \mathrm{E}[Y^2 e^{-sY}] > 0.
\end{aligned}$$

That is, $Y_2(s)$ is a decreasing and concave-up function with a vertical intercept of λ. Then (2.5.5) is equivalent to

$$Y_1(s) = Y_2(s).$$

For $\delta > 0$, consider the following graphical representation of $Y_1(s)$ and $Y_2(s)$.

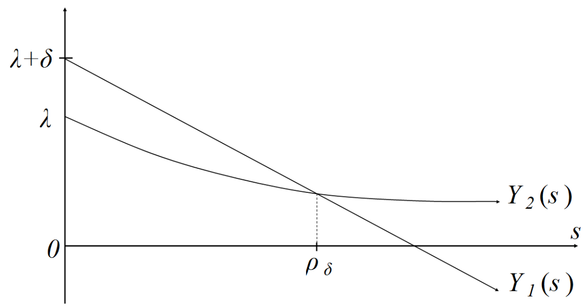

Figure 2.10: Example 1 - $Y_1(s) = Y_2(s)$ for $\delta > 0$

It is clear from the graph that for $\delta > 0$, there is one real and positive root $s = \rho_\delta$ to Lundberg's fundamental equation. Now we consider the case when $\delta = 0$. Note that since $c\frac{1}{\lambda} > \mathrm{E}[Y] \implies c > \lambda \mathrm{E}[Y]$ by the positive security loading condition (1.2.2), $Y_1'(0) = -c$ is more negative than $Y_2'(0) = -\lambda\,\mathrm{E}[Y]$. Thus, $Y_1(s)$ has a steeper negative slope than $Y_2(s)$ at $s = 0$. And when $\delta = 0$, we can graph $Y_1(s)$ and $Y_2(s)$ as follows

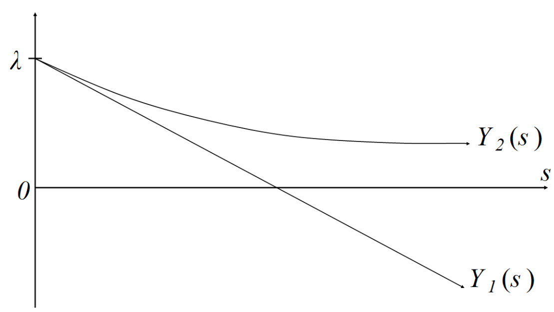

Figure 2.11: Example 1 - $Y_1(s) = Y_2(s)$ for $\delta = 0$

Then from the graph, it is clear that for $\delta = 0$, $s = \rho_0 = 0$ is the only real and non-negative root to Lundberg's fundamental equation.

$\square$

Then from (2.4.11) and using (2.5.2), it follows that

$$\frac{\lambda}{\lambda+\delta-c\rho_\delta}\tilde{\sigma}_{\delta,12}\left(\frac{\lambda+\delta}{c}\right)=\tilde{\beta}_{\delta,12}(\rho_\delta),$$

where ρ_δ is the one real and non-negative root to Lundberg's fundamental equation and after re-arranging, we obtain

$$\tilde{\sigma}_{\delta,12}\left(\frac{\lambda+\delta}{c}\right)=\frac{\lambda+\delta-c\rho_\delta}{\lambda}\tilde{\beta}_{\delta,12}(\rho_\delta),$$

and substitution into (2.5.3) leads to

$$\left(1-\tilde{p}(s)\frac{\lambda}{\lambda+\delta-cs}\right)\tilde{m}_{\delta,12}(s)=\tilde{\beta}_{\delta,12}(s)-\frac{\lambda+\delta-c\rho_\delta}{\lambda+\delta-cs}\tilde{\beta}_{\delta,12}(\rho_\delta),$$

and by multiplying both sides by $-\left(\frac{\lambda+\delta-cs}{c}\right)$, we obtain

$$\left(s-\frac{\lambda+\delta}{c}+\frac{\lambda}{c}\tilde{p}(s)\right)\tilde{m}_{\delta,12}(s)=\frac{1}{c}\Big((\lambda+\delta-c\rho_\delta)\tilde{\beta}_{\delta,12}(\rho_\delta)-(\lambda+\delta-cs)\tilde{\beta}_{\delta,12}(s)\Big). \quad (2.5.6)$$

Using (2.4.1), the Laplace transform of $\beta_{\delta,12}(u)$ is given by

$$\tilde{\beta}_{\delta,12}(s) \;=\; \int_0^\infty e^{-su}\int_0^\infty e^{-\delta t}\alpha_{12}(u+ct)\lambda e^{-\lambda t}dtdu,$$

and we recognize this form as being the same as (2.4.7) with $\sigma_{\delta,12}(x)$ replaced by $\alpha_{12}(x)$ and $k(t)$ replaced by $\lambda e^{-\lambda t}$. Then using (2.4.8) and (2.4.9), it follows that

$$\tilde{\beta}_{\delta,12}(s)=\tilde{\alpha}_{12}(s)\frac{\lambda}{\lambda+\delta-cs}-\tilde{\alpha}^*_{\delta,12}(\delta-cs), \quad (2.5.7)$$

where

$$\tilde{\alpha}^*_{\delta,12}(s)=\int_0^\infty\int_0^{ct}e^{-\frac{1}{c}\{\delta x+s(ct-x)\}}\alpha_{12}(x)\lambda e^{-\lambda t}dxdt,$$

which we note has the same form as (2.5.1) with $\sigma_{\delta,12}(x)$ replaced by $\alpha_{12}(x)$ and thus, it follows that

$$\tilde{\alpha}^*_{\delta,12}(s)=\frac{\lambda}{\lambda+s}\tilde{\alpha}_{12}\left(\frac{\lambda+\delta}{c}\right).$$

And thus, using (2.5.7), we obtain

$$\tilde{\beta}_{\delta,12}(s) = \frac{\lambda}{\lambda+\delta-cs}\left(\tilde{\alpha}_{12}(s) - \tilde{\alpha}_{12}\left(\frac{\lambda+\delta}{c}\right)\right),$$

or equivalently,

$$(\lambda+\delta-cs)\tilde{\beta}_{\delta,12}(s) = \lambda\left(\tilde{\alpha}_{12}(s) - \tilde{\alpha}_{12}\left(\frac{\lambda+\delta}{c}\right)\right),$$

and thus, the right-hand side of (2.5.6) is given by

$$\begin{aligned} & \frac{1}{c}\Big((\lambda+\delta-c\rho_\delta)\,\tilde{\beta}_{\delta,12}(\rho_\delta) - (\lambda+\delta-cs)\,\tilde{\beta}_{\delta,12}(s)\Big) \\ = {} & \frac{\lambda}{c}\left(\tilde{\alpha}_{12}(\rho_\delta) - \tilde{\alpha}_{12}\left(\frac{\lambda+\delta}{c}\right) - \tilde{\alpha}_{12}(s) + \tilde{\alpha}_{12}\left(\frac{\lambda+\delta}{c}\right)\right) \\ = {} & \frac{\lambda}{c}\big(\tilde{\alpha}_{12}(\rho_\delta) - \tilde{\alpha}_{12}(s)\big). \end{aligned} \tag{2.5.8}$$

Now recall Lundberg's fundamental equation given by (2.5.4) with $s = \rho_\delta$ which we can rearrange to obtain $\frac{\lambda+\delta}{c} = \rho_\delta + \frac{\lambda}{c}\tilde{p}(\rho_\delta)$ such that the coefficent of $\tilde{m}_{\delta,12}(s)$ in (2.5.6) can be rewritten as

$$\begin{aligned} s - \frac{\lambda+\delta}{c} + \frac{\lambda}{c}\tilde{p}(s) &= s - \rho_\delta - \frac{\lambda}{c}\tilde{p}(\rho_\delta) + \frac{\lambda}{c}\tilde{p}(s) \\ &= (s-\rho_\delta)\left(1 - \frac{\lambda}{c}\frac{\tilde{p}(\rho_\delta) - \tilde{p}(s)}{s-\rho_\delta}\right) \\ &= (s-\rho_\delta)\left(1 - \left\{\frac{\lambda}{c}\frac{1-\tilde{p}(\rho_\delta)}{\rho_\delta}\right\}\left\{\frac{\rho_\delta}{s-\rho_\delta}\frac{\tilde{p}(\rho_\delta)-\tilde{p}(s)}{1-\tilde{p}(\rho_\delta)}\right\}\right). \end{aligned} \tag{2.5.9}$$

Now, let

$$\begin{aligned} \phi_\delta &= \frac{\lambda}{c}\frac{1-\tilde{p}(\rho_\delta)}{\rho_\delta} \\ &= \frac{\lambda}{c}\int_0^\infty e^{-\rho_\delta y}\overline{P}(y)dy, \end{aligned} \tag{2.5.10}$$

and since $\int_0^\infty \overline{P}(y)dy = \mathrm{E}[Y]$, we know that $\int_0^\infty e^{-\rho_\delta y}\overline{P}(y)dy < \mathrm{E}[Y]$ and thus, $\phi_\delta < \frac{\lambda}{c}\mathrm{E}[Y]$ which is a quantity less than 1 by the positive security loading condition (1.2.2). We also let

$$\tilde{f}_\delta(s) = \frac{\rho_\delta}{s-\rho_\delta}\frac{\tilde{p}(\rho_\delta)-\tilde{p}(s)}{1-\tilde{p}(\rho_\delta)},$$

which using (1.3.6) of the Preliminaries section, is the Laplace transform of the generalized equilibrium density of $p(y)$. And thus, using (1.3.5), we obtain

$$f_\delta(y) = \frac{e^{\rho_\delta y}\int_y^\infty e^{-\rho_\delta x}p(x)dx}{\int_0^\infty e^{-\rho_\delta x}\overline{P}(x)dx}. \tag{2.5.11}$$

Note that if $\delta = 0$, then as shown earlier, $\rho_0 = 0$ and $f_0(y) = \overline{P}(y)/\mathrm{E}[Y]$ which is the equilibrium density of $p(y)$. Using (2.5.9), we obtain

$$s - \frac{\lambda+\delta}{c} + \frac{\lambda}{c}\tilde{p}(s) = (s-\rho_\delta)(1-\phi_\delta\tilde{f}_\delta(s)), \tag{2.5.12}$$

such that together with (2.5.8), (2.5.6) becomes

$$(s-\rho_\delta)(1-\phi_\delta\tilde{f}_\delta(s))\tilde{m}_{\delta,12}(s) = \frac{\lambda}{c}(\tilde{\alpha}_{12}(\rho_\delta) - \tilde{\alpha}_{12}(s)),$$

which we rearrange to obtain

$$\tilde{m}_{\delta,12}(s) = \phi_\delta\tilde{m}_{\delta,12}(s)\tilde{f}_\delta(s) + \frac{\lambda}{c}\frac{\tilde{\alpha}_{12}(\rho_\delta) - \tilde{\alpha}_{12}(s)}{s-\rho_\delta}.$$

Using (1.3.3), inversion leads to

$$m_{\delta,12}(u) = \phi_\delta\int_0^u m_{\delta,12}(u-y)f_\delta(y)dy + \frac{\lambda}{c}T_{\rho_\delta}\alpha_{12}(u),$$

which is a defective renewal equation satisfied by $m_{\delta,12}(u)$ and thus, using (2.3.13), its general solution is given by

$$m_{\delta,12}(u) = \frac{\lambda}{c}T_{\rho_\delta}\alpha_{12}(u) + \frac{1}{1-\phi_\delta}\int_0^u \frac{\lambda}{c}T_{\rho_\delta}\alpha_{12}(y)g_\delta(u-y)dy,$$

where

$$g_\delta(y) = \sum_{n=1}^{\infty}(1-\phi_\delta)(\phi_\delta)^n f_\delta^{*n}(y), \tag{2.5.13}$$

with ϕ_δ and $f_\delta(y)$ given by (2.5.10) and (2.5.11), respectively. Thus, note that when interclaim times are assumed to be exponential, we only needed results from conditioning on the time and amount of the first claim in order to solve for $m_{\delta,12}(u)$ in full generality. We will revisit the classical Poisson model again in Section 4.5 where we will consider a generalized Gerber-Shiu function.

2.6 Example 2: exponential claim sizes

We now consider when claim sizes are exponentially distributed and the interclaim time distribution $k(t)$ is assumed to be arbitrary [Willmot, G.E. (2011)]. That is, let $p(y) = \beta e^{-\beta y}$ where $\tilde{p}(s) = \frac{\beta}{\beta+s}$ and its excess loss density is given by

$$p_x(y) = \frac{p(x+y)}{\overline{P}(x)} = \frac{\beta e^{-\beta(x+y)}}{e^{-\beta x}} = \beta e^{-\beta y}. \tag{2.6.1}$$

Using (2.3.10) and (2.3.7), the ladder height is given by

$$\begin{aligned} f_\delta(y) &= \int_0^\infty \beta e^{-\beta y} \left\{ \frac{h_{\delta,1}(x|0)}{\phi_\delta} \right\} dx \\ &= \beta e^{-\beta y} \int_0^\infty \left\{ \frac{h_{\delta,1}(x|0)}{\phi_\delta} \right\} dx \\ &= \beta e^{-\beta y}, \end{aligned} \tag{2.6.2}$$

and is thus, also exponentially distributed.

Recall that when $w_{12}(x,y) = 1$, $m_{\delta,12}(u)$ is given by $\overline{G}_\delta(u)$, the tail of a compound geometric distribution which using (2.3.18) and (2.6.2), satisfies

$$\overline{G}_\delta(u) = \phi_\delta \int_0^u \overline{G}_\delta(u-y)\beta e^{-\beta y} dy + \phi_\delta e^{-\beta u}. \tag{2.6.3}$$

Taking its Laplace transform, it follows that

$$\tilde{\overline{G}}_\delta(s) = \phi_\delta \tilde{\overline{G}}_\delta(s) \frac{\beta}{\beta+s} + \frac{\phi_\delta}{\beta+s},$$

and solving for $\tilde{\overline{G}}_\delta(s)$ yields

$$\tilde{\overline{G}}_\delta(s) = \frac{\phi_\delta}{\beta(1-\phi_\delta)+s},$$

which we invert to obtain

$$\overline{G}_\delta(u) = \phi_\delta e^{-\beta(1-\phi_\delta)u}. \tag{2.6.4}$$

To determine ϕ_δ, we first consider the expression for $m_{\delta,12}(u)$ given by (2.4.6) that was obtained by conditioning on the time and amount of the first claim . If we let $w_{12}(x,y) = 1$ such that $m_{\delta,12}(u)$ is given by $\overline{G}_\delta(u)$, then (2.4.4) becomes

$$\sigma_{\delta,12}(x) = \int_0^x \overline{G}_\delta(x-y)\beta e^{-\beta y} dy,$$

and (2.4.2) becomes

$$\alpha_{12}(x) = \int_x^\infty \beta e^{-\beta y} dy = e^{-\beta x}.$$

Therefore, using (2.4.6) and (2.4.1), it follows that

$$\overline{G}_\delta(u) = \int_0^\infty e^{-\delta t} \left\{ \int_0^{u+ct} \overline{G}_\delta(u+ct-y)\beta e^{-\beta y} dy + e^{-\beta(u+ct)} \right\} k(t)dt. \tag{2.6.5}$$

And from (2.6.3), by replacing u with $u + ct$ and dividing both sides by ϕ_δ, we obtain

$$\int_0^{u+ct} \overline{G}_\delta(u+ct-y)\beta e^{-\beta y} dy + e^{-\beta(u+ct)} = \frac{\overline{G}_\delta(u+ct)}{\phi_\delta}.$$

We recognize this as the bracketed term in (2.6.5) and substitution into (2.6.5) results in

$$\phi_\delta \overline{G}_\delta(u) = \int_0^\infty e^{-\delta t} \overline{G}_\delta(u+ct) k(t)dt, \tag{2.6.6}$$

and using (2.6.4), $\overline{G}_\delta(u+ct) = \phi_\delta e^{-\beta(1-\phi_\delta)(u+ct)} = \overline{G}_\delta(u) e^{-\beta(1-\phi_\delta)ct}$ which we substitute into (2.6.6) to obtain

$$\phi_\delta \overline{G}_\delta(u) = \int_0^\infty e^{-\delta t} \overline{G}_\delta(u) e^{-\beta(1-\phi_\delta)ct} k(t)dt,$$

and by cancelling $\overline{G}_\delta(u)$ on both sides, it follows that ϕ_δ satisfies

$$\phi_\delta = \tilde{k}(\delta + c\beta(1-\phi_\delta)). \tag{2.6.7}$$

and recall from (2.3.8) that $0 < \phi_\delta < 1$.

Theorem 2. *There exists one real root $x = \phi_\delta \in (0, 1)$ to the equation*

$$x = \tilde{k}(\delta + c\beta(1-x)), \tag{2.6.8}$$

and in fact, $0 < \phi_\delta < \tilde{k}(\delta)$.

Proof. To begin, it is convenient to make a change of variable, $t = \beta(1-x)$. Then (2.6.8) becomes

$$1 - \frac{t}{\beta} = \tilde{k}(\delta + ct), \tag{2.6.9}$$

Since $x = 1 - \frac{t}{\beta}$, then for $x \in (0, 1)$, we require $t \in (0, \beta)$. Thus, we are only concerned with values of $t \in (0, \beta)$ satisfying (2.6.9).

Now let $Y_1(t) = 1 - t/\beta$ and $Y_2(t) = \tilde{k}(\delta + ct) = \mathrm{E}[e^{-(\delta+ct)V}]$ and thus, (2.6.9) is equivalent to $Y_1(t) = Y_2(t)$. Note that $Y_1(t)$ is a decreasing linear function with a vertical intercept of 1 and horizontal intercept β. Now consider

$$\begin{aligned} Y_2(t) &> 0, \\ Y_2'(t) &= -c\,\mathrm{E}[Ve^{-(\delta+ct)V}] < 0, \\ Y_2''(t) &= c^2\,\mathrm{E}[V^2e^{-(\delta+ct)V}] > 0, \end{aligned}$$

and thus $Y_2(t)$ is a positive, decreasing and concave-up function with a vertical intercept of $\tilde{k}(\delta)$. Additionally, consider when $t = \beta(1 - \tilde{k}(\delta))$ then

$$Y_1\big(\beta(1 - \tilde{k}(\delta))\big) = 1 - \frac{\beta(1 - \tilde{k}(\delta))}{\beta} = \tilde{k}(\delta).$$

For $\delta > 0$, consider the following graphical representation of $Y_1(t)$ and $Y_2(t)$ where $t = \kappa_\delta$ is the root of $Y_1(t) = Y_2(t)$.

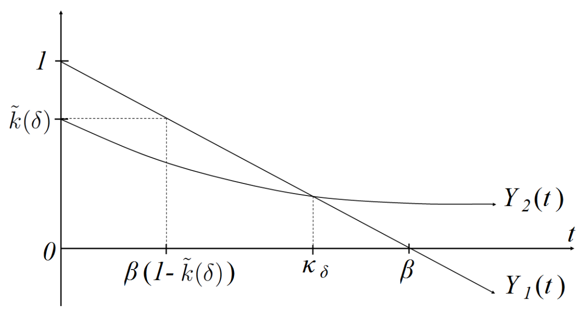

Figure 2.12: Example 2 - $Y_1(t) = Y_2(t)$ for $\delta > 0$

Now consider when $\delta = 0$, then

$$\begin{aligned} &Y_1(0) = Y_2(0) = 1, \\ &Y_2(t) = \tilde{k}(ct) = \mathrm{E}[e^{-ctV}] \implies Y_2'(0) = -c\mathrm{E}[V], \\ &Y_1(t) = 1 - \frac{t}{\beta} \implies Y_1'(0) = -\frac{1}{\beta} = -\mathrm{E}[Y], \end{aligned}$$

and since $c\mathrm{E}[V] > \mathrm{E}[Y]$ by the positive security loading condition, $Y_2'(0) < Y_1'(0)$ which implies $Y_2(t)$ has a steeper negative slope than $Y_1(t)$ at $t = 0$. Consider the following graphical representation of $Y_1(t)$ and $Y_2(t)$ where $t = \kappa_0$ is the root of $Y_1(t) = Y_2(t)$. Note that $0 < \kappa_0 < \beta$.

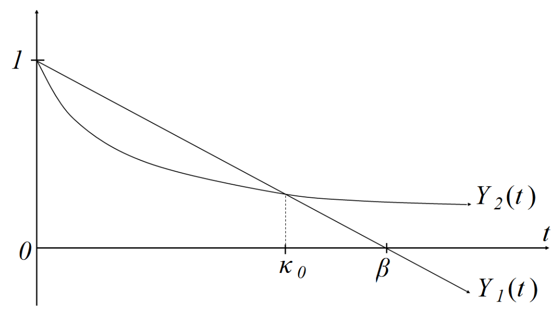

Figure 2.13: Example 2 - $Y_1(t) = Y_2(t)$ for $\delta = 0$

And thus, it is obvious from Figure 2.12 and 2.13 that for $\delta \geq 0$, there exists one real and positive root $t = \kappa_\delta$ to (2.6.9) and $\beta(1 - \tilde{k}(\delta)) < \kappa_\delta < \beta$. Thus, from (2.6.9)

$$1 - \frac{\kappa_\delta}{\beta} = \tilde{k}(\delta + c\kappa_\delta),$$

or equivalently

$$\frac{\beta}{\beta - \kappa_\delta}\,\tilde{k}(\delta + c\kappa_\delta) = 1. \tag{2.6.10}$$

Now recall Lundberg's fundamental equation given by (1.3.8) when $p(y) = \beta e^{-\beta y}$ which we can rearrange to obtain

$$\frac{\beta}{\beta + s}\tilde{k}(\delta - cs) = 1, \tag{2.6.11}$$

and comparing (2.6.10) and (2.6.11), we note that $s = -\kappa_\delta$ is the unique and positive root to Lundberg's fundamental equation.

Now recall the change of variables we made earlier. To reverse our change of variables, let $\kappa_\delta = \beta(1 - \phi_\delta)$. Thus, it follows that there exists one real root $x = \phi_\delta \in (0, 1)$ to (2.6.8) and since $\beta(1 - \tilde{k}(\delta)) < \kappa_\delta < \beta$, then $0 < \phi_\delta < \tilde{k}(\delta)$. $\square$

And using (2.3.11), the defective renewal equation for $m_{\delta,12}(u)$ when claim sizes are exponential is given by

$$m_{\delta,12}(u) = \phi_\delta \int_0^u m_{\delta,12}(u - y)\beta e^{-\beta y} dy + v_{\delta,12}(u), \tag{2.6.12}$$

where, from (2.3.12) and (2.2.8),

$$v_{\delta,12}(u) = \int_u^\infty \beta e^{-\beta y} \int_0^\infty w_{12}(u+x, y-u) h_{\delta,1}(x|0) dx dy,$$

and thus, the general solution to $m_{\delta,12}(u)$ is given by

$$m_{\delta,12}(u) = v_{\delta,12}(u) + \frac{1}{1-\phi_\delta} \int_0^u v_{\delta,12}(y) g_\delta(u-y) dy,$$

where from (2.6.4), $g_\delta(u) = -\overline{G}'_\delta(u) = \beta\phi_\delta(1-\phi_\delta)e^{-\beta(1-\phi_\delta)u}$. In Section 4.6, we will again consider the exponential claim size assumption but for a generalized Gerber-Shiu function. We will also discuss a method to identify $h_{\delta,1}(x|0)$ and an explicit solution for $m_{\delta,12}(u)$ when $w_{12}(x, y) = e^{-s_1 x - s_2 y}$ will be derived.

Chapter 3

The delayed renewal model

3.1 Introduction

In the ordinary Sparre Andersen model which we will refer to as simply, the ordinary model, there is an implicit assumption that a claim occurs at time 0. This may not be true in settings we wish to consider i.e. when time 0, the time we begin observing the insurer's surplus process, is the time between the occurrence of 2 claims. To avoid making this assumption, the delayed renewal model allows V_1 to follow a (possibly) different p.d.f. $k_d(t)$ than the common density $k(t)$ followed by $\{V_2, V_3, ...\}$. That is, the time until the first claim may be distributed differently than subsequent interclaim times. Otherwise, in all other respects, the delayed renewal model is identical to the ordinary model. i.e. $\{V_2, V_3, ...\}$ is still an i.i.d. sequence of positive r.v.'s with common p.d.f. $k(t)$ and V_1 is still assumed to be independent of $\{V_2, V_3, ...\}$ only now it has p.d.f. $k_d(t)$.

We denote T_d to be the time of ruin and we consider the same surplus process defined by (1.2.1) in the delayed renewal model. Let the corresponding Gerber-Shiu function be defined as

$$m^d_{\delta,12}(u) = E[e^{-\delta T_d} w_{12}(U_{T_d-}, |U_{T_d}|) I(T_d < \infty)|U_0 = u],$$

where U_{T_d-} and $|U_{T_d}|$ is the surplus immediately prior to ruin and the deficit at ruin in the delayed renewal model, respectively. Otherwise, all assumptions for the Gerber-Shiu function under the ordinary risk model still apply i.e. $\delta \geq 0$, $u \geq 0$ and $w_{12}(x, y)$ is an integrable function for $x > 0$ and $y > 0$.

Since the delayed renewal model and ordinary model differ only by how the time until the first claim V_1 is distributed, it is easy to see that after the first claim occurs, the delayed renewal risk process reverts back to an ordinary Sparre Andersen process. And thus, the close relationship between these two models often allow for results in the delayed renewal model to be expressed in terms of results in the ordinary model. And when doing analysis in the delayed model, we are mostly interested in determining this relationship for various distributional assumptions for the time until the first claim $k_d(t)$ such that we can extend results found in the ordinary model to the delayed model. To determine the relationship between $m^d_{\delta,12}(u)$ and $m_{\delta,12}(u)$, we begin by conditioning on the time and amount of the first claim. Recall from (2.4.6) and (2.4.1) that under the ordinary risk model,

$$\begin{aligned} m_{\delta,12}(u) &= \int_0^\infty e^{-\delta t}\sigma_{\delta,12}(u+ct)k(t)dt + \beta_{\delta,12}(u) \\ &= \int_0^\infty e^{-\delta t}\big\{\sigma_{\delta,12}(u+ct) + \alpha_{12}(u+ct)\big\}k(t)dt \\ &= \int_0^\infty e^{-\delta t}\gamma_\delta(u+ct)k(t)dt, \end{aligned} \tag{3.1.1}$$

where

$$\gamma_\delta(x) = \sigma_{\delta,12}(x) + \alpha_{12}(x). \tag{3.1.2}$$

Since the delayed renewal process reverts back to an ordinary model after the first claim, we only need to replace $k(t)$ with $k_d(t)$ in (3.1.1) to obtain an expression for $m^d_{\delta,12}(u)$ which is given by

$$m^d_{\delta,12}(u) = \int_0^\infty e^{-\delta t}\gamma_\delta(u+ct)k_d(t)dt,$$

and with a change of variable, it follows that

$$m^d_{\delta,12}(u) = \frac{1}{c}\int_u^\infty e^{-\delta\left(\frac{t-u}{c}\right)}\gamma_\delta(t)k_d\left(\frac{t-u}{c}\right)dt, \tag{3.1.3}$$

and differentiation yields

$$m'^d_{\delta,12}(u) = \frac{\delta}{c}m^d_{\delta,12}(u) - \frac{k_d(0)}{c}\gamma_\delta(u) - \frac{1}{c^2}\int_u^\infty e^{-\delta\left(\frac{t-u}{c}\right)}\gamma_\delta(t)k'_d\left(\frac{t-u}{c}\right)dt. \tag{3.1.4}$$

Recall that time 0 is now assumed to be a time between the occurrence of 2 claims. In renewal theory, the time to the next claim as measured from an arbitrary time x is called a “forward recurrence time”. Thus, it would be ideal for $k_d(t)$ to equal the distribution of the forward recurrence time from time 0 which often depends on the time of the last claim before time 0. But since we only begin observing the process at time 0, the time of the last claim before time 0 is unknown. Thus, as a solution to this dilemma, we consider the limiting distribution of the forward recurrence time as measured from time x where $x \to \infty$ which from [Willmot, G.E. (2004a)] equals $k_e(t) = \overline{K}(t)/\mathrm{E}[V]$, the equilibrium density of the interclaim time density $k(t)$ and does not depend on the time of the last claim. Thus, we are often interested in choosing $k_d(t)$ equal to $k_e(t)$. Interestingly, if we let $k_d(t) = k_e(t)$, the forward recurrence time of the delayed surplus process turns out to equal to $k_e(t)$ [Willmot, G.E. (2004a)] and thus, we are not required to know the time of the last claim occurring before time 0, as desired. This special case of the delayed model is referred to as the stationary model.

If the time between the claim occurring before time 0 and the claim occurring after time 0 is exponentially distributed, then by the memoryless property, the conditional distribution at time 0 of the time to the next claim is the same exponential distribution and is thus independent of the last claim before time 0. Therefore, another choice of $k_d(t)$ that we are often interested in is the exponential distribution.

Refer to [Kim, S.Y. (2007)] where $m^d_{\delta,12}(u)$ is studied for $k_d(t)$ belonging to either the exponential, combination of exponentials, or Coxian class. In the next example, we will consider $m^d_{\delta,12}(u)$ for a special class of $k_d(t)$ [Willmot, G.E. (2004b)].

3.2 Example 3: $m^d_{\delta,12}(u)$ for a special class of $k_d(t)$

We now consider $m^d_{\delta,12}(u)$ when the density of the first interclaim time $k_d(t)$ is a weighted average of a generalized equilibrium density of $k(t)$ similar to (1.3.5), and an exponential distribution [Willmot, G.E. (2004b)]. That is, let

$$k_d(t) = q\frac{e^{-rt}\int_t^\infty e^{ry}k(y)dy}{\int_0^\infty e^{ry}\overline{K}(y)dy} + (1-q)re^{-rt}, \qquad t \geq 0, \tag{3.2.1}$$

where r satisfies $\tilde{k}(r) < \infty$ and if $0 \leq q < 1$, then $r > 0$, and if $q = 1$, then $-\infty < r < \infty$.

Note that when $q = 1$ and $r = 0$, $k_d(t) = k_e(t)$ and the delayed renewal process becomes the stationary process. Also, when $q = 0$, $k_d(t)$ is an exponential density. Added flexibility is obtained when $0 < q < 1$. For example, if $k(t)$ is a mixture of n exponentials, then $k_d(t)$ is a mixture of $n+1$ exponentials [Willmot, G.E. (2004b)]. Or if $k(t)$ is a mixed Erlang density, then the generalized equilibrium density is a different mixture of the same Erlang densities such that $k_d(t)$ is the density of a mixed Erlang and an exponential as is shown in [Willmot and Lin (2011)]. Also, $k_d(t)$ may be evaluated analytically if $k(t)$ belongs to the class of power functions (i.e. $k(t) \propto t^c$ where c is a non-negative integer) or, if $k(t)$ is of phase-type, then there exists a closed form expression for $k_d(t)$ (see [Willmot, G.E. (2004b)] and references therein).

Using integration by parts, it follows that

$$\int_0^\infty e^{ry}\overline{K}(y)dy = -\frac{1}{r} + \frac{1}{r}\int_0^\infty e^{ry}k(y)dy$$

then using (3.2.1), we can express $k_d(0)$ as

$$\begin{aligned} k_d(0) &= q\frac{1 + r\int_0^\infty e^{ry}\overline{K}(y)dy}{\int_0^\infty e^{ry}\overline{K}(y)dy} + (1-q)r \\ &= r + \frac{q}{\int_0^\infty e^{ry}\overline{K}(y)dy}, \end{aligned}$$

such that $k_d(t)$ can be rewritten as

$$k_d(t) = (k_d(0) - r)e^{-rt}\int_t^\infty e^{ry}k(y)dy + (1-q)re^{-rt}, \tag{3.2.2}$$

and after differentiation, we obtain

$$k_d'(t) = \big(r - k_d(0)\big)k(t) - rk_d(t), \tag{3.2.3}$$

which is a result that will be useful in the following.

Now consider the integral term on the right side of (3.1.4) and using (3.2.3), (3.1.1), and (3.1.3), it follows that

$$\begin{aligned} &\frac{1}{c^2}\int_u^\infty e^{-\delta(\frac{t-u}{c})}\gamma_\delta(t)k_d'\left(\frac{t-u}{c}\right)dt \\ =\ &\frac{1}{c^2}\int_u^\infty e^{-\delta(\frac{t-u}{c})}\gamma_\delta(t)\left[\big(r - k_d(0)\big)k\left(\frac{t-u}{c}\right) - rk_d\left(\frac{t-u}{c}\right)\right]dt \\ =\ &\frac{(r - k_d(0))}{c}m_{\delta,12}(u) - \frac{r}{c}m^d_{\delta,12}(u). \end{aligned}$$

Then using (3.1.2), (3.1.4) becomes

$$\begin{aligned} m'^d_{\delta,12}(u) &= \frac{\delta}{c}m^d_{\delta,12}(u) - \frac{k_d(0)}{c}\gamma_\delta(u) - \frac{(r - k_d(0))}{c}m_{\delta,12}(u) + \frac{r}{c}m^d_{\delta,12}(u) \\ &= \frac{r+\delta}{c}m^d_{\delta,12}(u) - \frac{r}{c}m_{\delta,12}(u) - \frac{k_d(0)}{c}\alpha_{12}(u) \\ &+ \frac{k_d(0)}{c}\big(m_{\delta,12}(u) - \sigma_{\delta,12}(u)\big). \end{aligned} \tag{3.2.4}$$

We wish to determine the Laplace transform of (3.2.4). But first, let's just consider the Laplace transform of its last term and using (2.4.5), this is given by

$$\frac{k_d(0)}{c}\big(\tilde{m}_{\delta,12}(s) - \tilde{\sigma}_{\delta,12}(s)\big) = \frac{k_d(0)}{c}\,\tilde{m}_{\delta,12}(s)(1 - \tilde{p}(s)). \tag{3.2.5}$$

Now consider a function defined as

$$\sigma^e_{\delta,12}(u) = \int_0^u m_{\delta,12}(u-y)p_e(y)dy, \tag{3.2.6}$$

where $p_e(y) = \overline{P}(y)/\mathrm{E}[Y]$ is the equilibrium density of $p(y)$. Taking its Laplace transform yields

$$\widetilde{\sigma}^e_{\delta,12}(s) = \tilde{m}_{\delta,12}(s)\,\frac{1-\tilde{p}(s)}{sE[Y]}. \tag{3.2.7}$$

Therefore, using (3.2.5) and (3.2.7), the Laplace transform of (3.2.4) is given by

$$s\tilde{m}^d_{\delta,12}(s) - m^d_{\delta,12}(0) = \frac{r+\delta}{c}\tilde{m}^d_{\delta,12}(s) - \frac{r}{c}\tilde{m}_{\delta,12}(s) - \frac{k_d(0)}{c}\tilde{\alpha}_{12}(s) + \frac{k_d(0)E[Y]}{c}\,s\,\widetilde{\sigma}^e_{\delta,12}(s),$$

and after rearranging, we obtain

$$\left(s - \frac{r+\delta}{c}\right)\tilde{m}^d_{\delta,12}(s) = m^d_{\delta,12}(0) - \frac{r}{c}\tilde{m}_{\delta,12}(s) - \frac{k_d(0)}{c}\tilde{\alpha}_{12}(s) + \frac{k_d(0)E[Y]}{c}\,s\,\widetilde{\sigma}^e_{\delta,12}(s). \tag{3.2.8}$$

And when $s = \frac{r+\delta}{c}$, the left side becomes 0 and it follows that

$$m^d_{\delta,12}(0) = \frac{r}{c}\tilde{m}_{\delta,12}\left(\frac{r+\delta}{c}\right) + \frac{k_d(0)}{c}\tilde{\alpha}_{12}\left(\frac{r+\delta}{c}\right) - \frac{k_d(0)E[Y]}{c}\left(\frac{r+\delta}{c}\right)\tilde{\sigma}^e_{\delta,12}\left(\frac{r+\delta}{c}\right),$$

and substitution into (3.2.8) yields

$$\begin{aligned}
\left(s - \frac{r+\delta}{c}\right)\tilde{m}^d_{\delta,12}(s) \;\; = \;\; & \frac{r}{c}\left[\tilde{m}_{\delta,12}\left(\frac{r+\delta}{c}\right) - \tilde{m}_{\delta,12}(s)\right] + \frac{k_d(0)}{c}\left[\tilde{\alpha}_{12}\left(\frac{r+\delta}{c}\right) - \tilde{\alpha}_{12}(s)\right] \\
- \;\; & \frac{k_d(0)E[Y]}{c}\left(\frac{r+\delta}{c}\right)\left[\tilde{\sigma}^e_{\delta,12}\left(\frac{r+\delta}{c}\right) - \tilde{\sigma}^e_{\delta,12}(s)\right] \\
+ \;\; & \frac{k_d(0)E[Y]}{c}\left(s - \frac{r+\delta}{c}\right)\tilde{\sigma}^e_{\delta,12}(s).
\end{aligned}$$

Dividing both sides by $\left(s - \frac{r+\delta}{c}\right)$ and rearranging, we obtain

$$\begin{aligned}
\tilde{m}_{\delta,12}^{d}(s) &= \frac{k_d(0)E[Y]}{c}\tilde{\sigma}_{\delta,12}^{e}(s) + \frac{r}{c}\left[\frac{\tilde{m}_{\delta,12}\left(\frac{r+\delta}{c}\right) - \tilde{m}_{\delta,12}(s)}{s - \frac{r+\delta}{c}}\right] + \frac{k_d(0)}{c}\left[\frac{\tilde{\alpha}_{12}\left(\frac{r+\delta}{c}\right) - \tilde{\alpha}_{12}(s)}{s - \frac{r+\delta}{c}}\right] \\
&- \frac{k_d(0)E[Y]}{c}\left(\frac{r+\delta}{c}\right)\left[\frac{\tilde{\sigma}_{\delta,12}^{e}\left(\frac{r+\delta}{c}\right) - \tilde{\sigma}_{\delta,12}^{e}(s)}{s - \frac{r+\delta}{c}}\right]. \qquad (3.2.9)
\end{aligned}$$

Recalling the form of the Laplace transform of Dickson-Hipp operators (1.3.3) together with (3.2.6), we can invert (3.2.9) to obtain

$$m_{\delta,12}^{d}(u) = \frac{k_d(0)E[Y]}{c}\int_0^u m_{\delta,12}(u-y)p_e(y)dy + v_{\delta,12}^{d}(u), \qquad (3.2.10)$$

where $p_e(y) = \overline{P}(y)/\mathrm{E}[Y]$ and using (1.3.2), we obtain

$$v_{\delta,12}^{d}(u) = T_{\frac{r+\delta}{c}}\left(\frac{r}{c}m_{\delta,12}(u) + \frac{k_d(0)}{c}\alpha_{12}(u) - \frac{k_d(0)E[Y]}{c}\left(\frac{r+\delta}{c}\right)\sigma_{\delta,12}^{e}(u)\right). \qquad (3.2.11)$$

Note that (3.2.10) shows the integral relationship between $m_{\delta,12}^{d}(u)$ and $m_{\delta,12}(u)$, as desired. As previously mentioned, we are often interested in the stationary model where $k_d(t) = k_e(t)$, the equilibrium density of $k(t)$. Using (3.2.2), we note that if $k_d(0) = 1/\mathrm{E}[V]$ and $r = 0$, then $k_d(t) = k_e(t)$ and (3.2.10) simplifies to

$$m_{\delta,12}^{s}(u) = \frac{1}{1+\theta}\int_0^u m_{\delta,12}(u-y)p_e(y)dy + v_{\delta,12}^{s}(u),$$

where using (3.2.11), we obtain

$$v_{\delta,12}^{s}(u) = T_{\frac{\delta}{c}}\left(\frac{1}{c\mathrm{E}[V]}\alpha_{12}(u) - \frac{1}{1+\theta}\left(\frac{\delta}{c}\right)\sigma_{\delta,12}^{e}(u)\right),$$

and using the security loading condition $c\mathrm{E}[V] = (1+\theta)\mathrm{E}[Y]$ from (1.2.2).

Chapter 4

A generalized Gerber-Shiu function

4.1 Introduction

In this section, we consider a generalized Gerber-Shiu function [Cheung et al. (2010a)] and we begin by introducing 2 random variables of interest. First, let X_t be the minimum surplus level before time t and is thus defined as $X_t = \inf_{0\leq s<t} U_s$. Therefore, X_T is the minimum surplus level before ruin occurs. By incorporating X_T into our analysis, we are able to study the last ladder height which is given by $X_T + |U_T|$. Note that if ruin occurs on the first claim, then $X_T = u$. Second, let $R_n = u + \sum_{i=1}^{n}(cV_i - Y_i)$ for $n = 1, 2, ...$ and $R_0 = u$. Then for $n \geq 1$, it is easy to see that R_n is the surplus level immediately after the nth claim. Note that the $(N_T - 1)$th claim is the claim before the claim causing ruin which will be referred to as the second last claim before ruin. Therefore, R_{N_T-1} is the surplus immediately after this claim and incorporating this quantity into our analysis allows for the study of the last interclaim time given by $(U_{T-} - R_{N_T-1})/c$. Therefore, a "generalized" Gerber-Shiu function including the two r.v.'s X_T and R_{N_T-1} will allow greater insight into the surplus process and thus, a better understanding of the event of ruin. And thus, we generalize the Gerber-Shiu function as follows

$$m_\delta(u) = E[e^{-\delta T} w(U_{T-}, |U_T|, X_T, R_{N_T-1}) I(T < \infty) | U_0 = u], \tag{4.1.1}$$

which is identical to (2.1.1) in all respects except that we incorporate X_T and R_{N_T-1} into our analysis by extending the 2-variable penalty function $w_{12}(x, y)$ to a 4-variable function $w(x, y, z, v)$ where $w(x, y, z, v)$ is assumed to be an integrable function for $x > 0$, $y > 0$, $z > 0$, and $v > 0$. Like $m_{\delta,12}(u)$, this generalized Gerber-Shiu function can also be written as a defective renewal equation as the following sections will show. But before that, we introduce the joint defective density of $(T, U_{T-}, |U_T|, R_{N_T-1})$ at (t, x, y, v) and other useful densities. The following 2 sections will examine each of the conditioning arguments previously discussed in detail (i.e. conditioning on the first drop in surplus and conditioning on the time and amount of the first claim) to obtain expressions satisfied by $m_\delta(u)$. Most of the techniques used to analyze $m_\delta(u)$ are identical to those used previously to analyze the classical Gerber-Shiu function $m_{\delta,12}(u)$. Note that a subscript "3" will indicate a quantity involving X_T and a subscript "4" will indicate a quantity involving R_{N_T-1}.

Now let's assume that Y is not independent of V i.e. the size of a claim now depends on the interclaim time before the claim occurred. Thus, let $P(y|t) = \Pr(Y \leq y|V = t)$ be the conditional distribution of the claim size given the interclaim time preceding the claim is of length t. We specify the joint density function of (V_i, Y_i) for $i = 1, 2, ...$ by $p(y|t)k(t)$ where $p(y|t) = P'(y|t)$ and we assume the pairs $\{(V_i, Y_i); i = 1, 2, ...\}$ are i.i.d. such that the Sparre Andersen random walk structure of the surplus process is conserved.

First, when ruin occurs on the first claim. Then $N_T = 1$ and $R_{N_T-1} = R_0 = u$. Using the same arguments used to derive (2.2.2) when ruin occurs on the first claim, we know that $t = \frac{x-u}{c}$. Therefore, the joint defective density of $(T, U_{T-}, |U_T|, R_{N_T-1})$ at (t, x, y, v) when ruin occurs on the first claim is equal to (2.2.2) with $p(y)$ replaced by $p(y|t)$ and is thus, given by

$$h_{12}^*(x, y|u) = \frac{1}{c} k\left(\frac{x-u}{c}\right) p\left(x + y \middle| \frac{x-u}{c}\right), \tag{4.1.2}$$

where $t = \frac{x-u}{c}$, $x > u$, $y > 0$, and $v = u$.

Otherwise, if ruin occurs occurs on claims subsequent to the first, there is still no direct relationship between t, x, y and v. We only know that $x < u + ct$ and that $v < x$ since the surplus increases from v to x by a rate c during the last interclaim time. Let's denote the joint defective density of $(T, U_{T-}, |U_T|, R_{N_T-1})$ at (t, x, y, v) given that ruin occurs on claims subsequent to the first as $h_{124}^{**}(t, x, y, v|u)$. Therefore, $h_{124}(t, x, y, v|u)$, the joint defective density of $(T, U_{T-}, |U_T|, R_{N_T-1})$ at (t, x, y, v) can be summarized as

$$h_{124}(t, x, y, v|u) = \begin{cases} h_{12}^*(x, y|u), & t = \frac{x-u}{c},\ x > u,\ y > 0,\ v = u \\ \\ h_{124}^{**}(t, x, y, v|u), & t > 0,\ v < x < u + ct,\ y > 0,\ v > 0, \end{cases} \tag{4.1.3}$$

and again, note that there is a different density for whether ruin occurs on the first claim ($t = \frac{x-u}{c}$, $x > u$, $y > 0$, $v = u$) or subsequent claims ($t > 0$, $v < x < u + ct$, $y > 0$, $v > 0$).

Also, recall from (2.2.4) with $p(y)$ replaced by $p(y|t)$ that

$$\begin{aligned} h^*_{\delta,12}(x,y|u) &= e^{-\delta\frac{x-u}{c}} h^*_{12}(x,y|u) \qquad (4.1.4) \\ &= e^{-\delta\frac{x-u}{c}} \frac{1}{c} k\left(\frac{x-u}{c}\right) p\left(x+y \middle| \frac{x-u}{c}\right), \end{aligned}$$

and similarly to (2.2.5), let

$$h^{**}_{\delta,124}(x,y,v|u) = \int_0^\infty e^{-\delta t} h^{**}_{124}(t,x,y,v|u)dt. \qquad (4.1.5)$$

Then the discounted joint density of $(U_{T^-}, |U_T|)$ is given by

$$h_{\delta,12}(x,y|u) = h^*_{\delta,12}(x,y|u) + \int_0^x h^{**}_{\delta,124}(x,y,v|u)dv. \qquad (4.1.6)$$

And as was done in the classical case, we let

$$\phi_\delta = \int_0^\infty \int_0^\infty h_{\delta,12}(x,y|0)dxdy, \qquad (4.1.7)$$

and

$$f_\delta(y) = \frac{1}{\phi_\delta} \int_0^\infty h_{\delta,12}(x,y|0)dx. \qquad (4.1.8)$$

Note that in the independent case i.e. if we assume the claim sizes are independent of the interclaim time preceding it, then $p(y|t) = p(y)$ and from (2.2.1), the density of $|U_T|$ at y given $U_{T-} = x$ is $p_x(y)$. Then it follows from (4.1.4) that

$$h^*_{\delta,12}(x,y|u) = h^*_{\delta,1}(x|u)p_x(y),$$

where

$$h^*_{\delta,1}(x|u) = e^{-\delta\frac{x-u}{c}} \frac{1}{c} k\left(\frac{x-u}{c}\right) \overline{P}(x),$$

is the marginal defective discounted density of $U_{T-} = x$ for ruin occurring on the first claim. And from (4.1.5)

$$h^{**}_{\delta,124}(x,y,v|u) = h^{**}_{\delta,14}(x,v|u)p_x(y), \qquad (4.1.9)$$

where $h^{**}_{\delta,14}(x,v|u) = \int_0^\infty e^{-\delta t} h^{**}_{14}(t,x,v|u)dt$ and $h^{**}_{14}(t,x,v|u)$ is the joint defective density of
(T, U_{T-}, R_{N_T-1}) at (t,x,v) for ruin occurring on subsequent claims. And thus, from (4.1.6),

$$h_{\delta,12}(x,y|u) = h_{\delta,1}(x|u)p_x(y), \tag{4.1.10}$$

where $h_{\delta,1}(x|u) = h^*_{\delta,1}(x|u) + \int_0^x h^{**}_{\delta,14}(x,v|u)dv$ is the marginal defective discounted density of U_{T-}.

Using the densities and related quantities introduced in this section and letting $u = 0$, we are able to condition on the first drop in surplus to obtain a defective renewal equation satisfied by $m_\delta(u)$ as was done in the classical case for $m_{\delta,12}(u)$ and will be discussed in the next section.

4.2 Conditioning on the first drop in surplus revisited

Recall from Section 2.3 when we discussed the classical Gerber-Shiu function, that we can determine the density of a drop in surplus below its initial level by using $h_{12}(t,x,y|u)$ and simply letting $u = 0$. Now consider a surplus that starts with an initial value of u and falls to level $u + v$ immediately after the claim before the claim causing the first drop in surplus. If the surplus then rises to an amount x above u at time t when a claim of size $x + y$ occurs and causes the first drop in surplus, then using similar arguments to those used in Section 2.3, this scenario occurs with density $h_{124}(t,x,y,v|0)$ given by (4.1.3). Furthermore, if $t = \frac{x}{c}$, then $v = u$, the first drop occurs on the first claim, and $h_{124}(t,x,y,v|0) = h^*_{12}(x,y|0) = \frac{1}{c}k\left(\frac{x}{c}\right)p\left(x+y\middle|\frac{x}{c}\right)$. Otherwise, if $t > \frac{x}{c}$, then $v < x$, the first drop occurs on a claim subsequent to the first, and $h_{124}(t,x,y,v|0) = h^{**}_{124}(t,x,y,v|0)$.

As will be shown, despite the fact that $m_\delta(u)$ contains a penalty function involving X_T, we only require the joint defective discounted density of $(U_{T-}, |U_T|, R_{N_T-1})$, which is independent of X_T, to obtain a defective renewal equation for $m_\delta(u)$. To express $m_\delta(u)$ as a defective renewal equation, we condition on the first drop in surplus below the initial value of u. The method of conditioning is identical to that in outlined Section 2.3 for the classical definition of the Gerber-Shiu function $m_{\delta,12}(u)$ since we will condition on the same 4 cases:

Case 1: first drop occurs on the first claim and causes ruin
Case 2: first drop occurs on the first claim and does not cause ruin
Case 3: first drop occurs on a subsequent claim and causes ruin
Case 4: first drop occurs on a subsequent claim and does not cause ruin

However, we make a few adjustments to the contributions (2.3.1), (2.3.2), (2.3.3), and (2.3.4) that each case makes to $m_{\delta,12}(u)$ to account for the additional variables X_T and R_{N_T-1} that are now included in $m_\delta(u)$. Let's now consider these adjustments in detail and outline the contribution that each case makes to $m_\delta(u)$. Make note of each of the corresponding figures which help to illustrate each scenario.

Case 1: first drop occurs on the first claim and causes ruin

When the first drop occurs on the first claim and causes ruin, $N_T = 1$ and $R_{N_T-1} = R_0 = u$. Also, the minimum surplus level before ruin X_T is u.

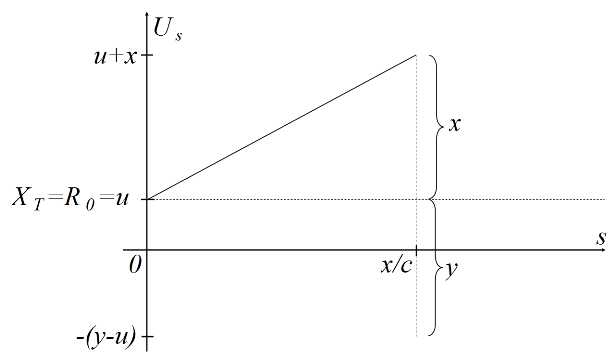

Figure 4.1: Conditioning on the first drop in surplus - Case 1 revisited

Thus, by making adjustments to (2.3.1), the contribution to $m_\delta(u)$ for this case is

$$\begin{aligned} &= \int_u^\infty \int_0^\infty e^{-\delta \frac{x}{c}} w(u+x, y-u, u, u) h_{12}^*(x, y|0) dx dy \\ &= \int_u^\infty \int_0^\infty w(u+x, y-u, u, u) h_{\delta,12}^*(x, y|0) dx dy. \qquad (4.2.1) \end{aligned}$$

Case 2: first drop occurs on the first claim and does not cause ruin

When the first drop of size y occurs on the first claim and doesn't cause ruin, then $y < u$ and the surplus process is said to restart with an initial surplus of $u - y$.

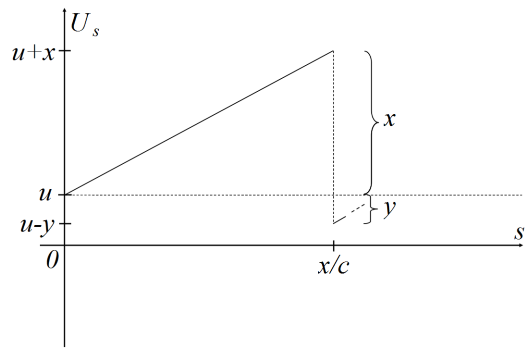

Figure 4.2: Conditioning on the first drop in surplus - Case 2 revisited

Thus, the contribution to $m_\delta(u)$ for this case is essentially identical to (2.3.2) and is given by

$$
\begin{aligned}
&= \int_0^u \int_0^\infty e^{-\delta \frac{x}{c}} m_\delta(u-y) h_{12}^*(x,y|0)dxdy \\
&= \int_0^u m_\delta(u-y) \left(\int_0^\infty h_{\delta,12}^*(x,y|0)dx \right) dy. \qquad (4.2.2)
\end{aligned}
$$

Case 3: first drop occurs on a subsequent claim and causes ruin

When the first drop occurs on a subsequent claim and causes ruin, the minimum surplus level before ruin is u and the surplus immediately after the second last claim before ruin must be above u by some amount, say v, where $v < x$.

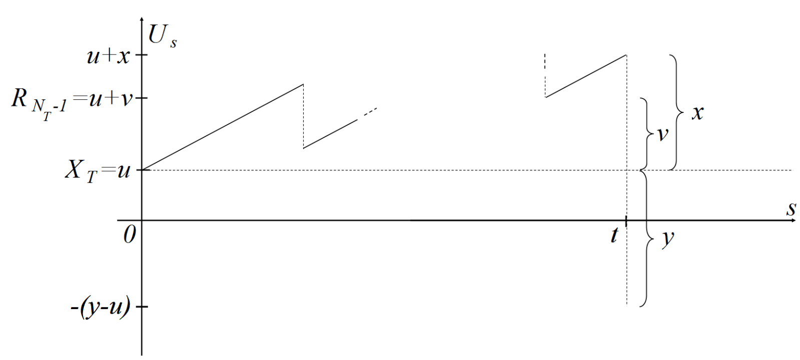

Figure 4.3: Conditioning on the first drop in surplus - Case 3 revisited

Thus, by making adjustments to (2.3.3), the contribution to $m_\delta(u)$ for this case is

$$\begin{aligned} &= \int_u^\infty \int_0^\infty \int_0^x \int_0^\infty e^{-\delta t} w(u+x, y-u, u, u+v) h_{124}^{**}(t,x,y,v|0)dtdvdxdy \\ &= \int_u^\infty \int_0^\infty \int_0^x w(u+x, y-u, u, u+v) h_{\delta,124}^{**}(x,y,v|0)dvdxdy. \end{aligned} \tag{4.2.3}$$

Case 4: first drop occurs on a subsequent claim and does not cause ruin

When the first drop of size y occurs on a subsequent claim and does not cause ruin, then $y < u$ and the surplus process is said to restart with an initial surplus of $u - y$. The surplus immediately after the second last claim before ruin must be above u by some amount, say v, where $v < x$.

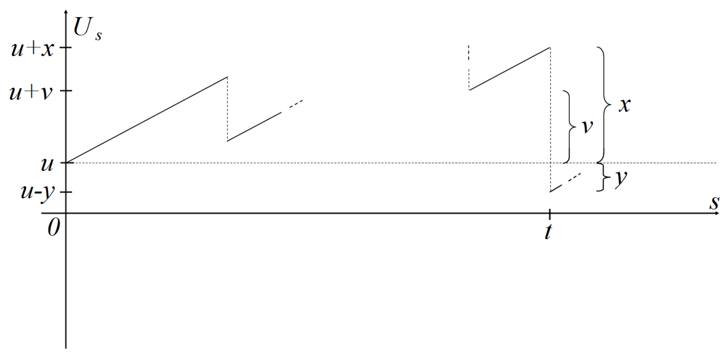

Figure 4.4: Conditioning on the first drop in surplus - Case 4 revisited

Thus, by making adjustments to (2.3.4), the contribution to $m_\delta(u)$ for this case is

$$\begin{aligned} &= \int_0^u \int_0^\infty \int_0^x \int_0^\infty e^{-\delta t} m_\delta(u-y) h_{124}^{**}(t,x,y,v|0)dtdvdxdy \\ &= \int_0^u m_\delta(u-y) \int_0^\infty \int_0^x h_{\delta,124}^{**}(x,y,v|0)dvdxdy. \end{aligned} \tag{4.2.4}$$

By summing the contributions to $m_\delta(u)$ from all 4 cases (4.2.1), (4.2.2), (4.2.3), and (4.2.4), we obtain the following expression

$$m_\delta(u) = \int_0^u m_\delta(u-y) \left\{ \int_0^\infty h_{\delta,12}^{*}(x,y|0)dx + \int_0^\infty \int_0^x h_{\delta,124}^{**}(x,y,v|0)dvdx \right\} dy + v_\delta(u), \tag{4.2.5}$$

where

$$\begin{aligned}
v_\delta(u) &= \int_u^\infty \int_0^\infty w(u+x, y-u, u, u) h^*_{\delta,12}(x, y|0) dx dy \\
&+ \int_u^\infty \int_0^\infty \int_0^x w(u+x, y-u, u, u+v) h^{**}_{\delta,124}(x, y, v|0) dv dx dy \qquad (4.2.6) \\
&= \int_0^\infty e^{-\delta t} \left(\int_0^\infty w(u+ct, y, u, u) p(u+ct+y|t) dy \right) k(t) dt \\
&+ \int_0^\infty \int_0^\infty \int_0^x w(u+x, y, u, u+v) h^{**}_{\delta,124}(x, u+y, v|0) dv dx dy, \qquad (4.2.7)
\end{aligned}$$

by using (4.1.4) and change of variables. Then using (4.1.6), (4.2.5) can be expressed as

$$m_\delta(u) = \int_0^u m_\delta(u-y) \left\{ \int_0^\infty h_{\delta,12}(x, y|0) dx \right\} dy + v_\delta(u),$$

and thus, using (4.1.7) and (4.1.8), we can write $m_\delta(u)$ as the following defective renewal equation

$$m_\delta(u) = \phi_\delta \int_0^u m_\delta(u-y) f_\delta(y) dy + v_\delta(u). \qquad (4.2.8)$$

And thus, it follows from (1.3.18) from the Preliminaries section that the general solution to $m_\delta(u)$ is given by

$$m_\delta(u) = v_\delta(u) + \frac{1}{1-\phi_\delta} \int_0^u v_\delta(y) g_\delta(u-y) dy, \qquad (4.2.9)$$

where using (1.3.15), $g_\delta(u) = \sum_{n=1}^\infty (1-\phi_\delta)(\phi_\delta)^n f_\delta^{*n}(u)$ with ϕ_δ and $f_\delta(y)$ given by (4.1.7) and (4.1.8), respectively. We note that despite the fact that $m_\delta(u)$ is a function of u and contains a 4-variable penalty function involving X_T, its general solution only depends on the joint defective discounted density of the 3 variables U_{T-}, $|U_T|$, and R_{N_T-1} given $u=0$.

Now let's consider special cases of $m_\delta(u)$ for various penalty functions. If $w(x, y, z, v) = w_{124}(x, y, v)$, then (4.2.8) simplifies to

$$m_{\delta,124}(u) = \phi_\delta \int_0^u m_{\delta,124}(u-y) f_\delta(y) dy + v_{\delta,124}(u),$$

where using (4.2.6)

$$\begin{aligned}
v_{\delta,124}(u) &= \int_u^\infty \int_0^\infty w_{124}(u+x, y-u, u) h^*_{\delta,12}(x, y|0) dx dy \\
&+ \int_u^\infty \int_0^\infty \int_0^x w_{124}(u+x, y-u, u+v) h^{**}_{\delta,124}(x, y, v|0) dv dx dy. \quad (4.2.10)
\end{aligned}$$

And if $w(x, y, z, v) = w_3(z)w_{124}(x, y, v)$, then (4.2.8) simplifies to

$$m_{\delta,3,124}(u) = \phi_\delta \int_0^u m_{\delta,3,124}(u - y)f_\delta(y)dy + v_{\delta,3,124}(u), \tag{4.2.11}$$

where using (4.2.6) and (4.2.10),

$$v_{\delta,3,124}(u) = w_3(u)v_{\delta,124}(u). \tag{4.2.12}$$

And if $w(x, y, z, v) = w_{123}(x, y, z)$, then (4.2.8) simplifies to

$$m_{\delta,123}(u) = \phi_\delta \int_0^u m_{\delta,123}(u - y)f_\delta(y)dy + v_{\delta,123}(u), \tag{4.2.13}$$

where, using (4.2.6) and (4.1.6)

$$\begin{aligned} v_{\delta,123}(u) &= \int_u^\infty \int_0^\infty w_{123}(u + x, y - u, u) \left\{ h^*_{\delta,12}(x, y|0) + \int_0^x h^{**}_{\delta,124}(x, y, v|0)dv \right\} dxdy \\ &= \int_u^\infty \int_0^\infty w_{123}(u + x, y - u, u)h_{\delta,12}(x, y|0)dxdy, \end{aligned} \tag{4.2.14}$$

which is a considerable simplification considering that it no longer depends on $h^{**}_{\delta,124}(x, y, v|0)$ which is often very difficult to identify in terms of $k(t)$ and $p(y)$. Instead, $v_{\delta,123}(u)$ only depends on $h_{\delta,12}(x, y|0)$, the joint defective discounted density of the surplus immediately prior to ruin and the deficit at ruin given an initial surplus of 0 which under certain assumptions for $k(t)$ and/or $p(y)$ can be obtained using the classical Gerber-Shiu function $m_{\delta,12}(0)$ when $w_{12}(x, y) = e^{-s_1x-s_2y}$. And once $h_{\delta,12}(x, y|0)$ has been identified, using (4.1.7) and (4.1.8), $m_{\delta,123}(u)$ can be solved in full generality.

Now, consider when $w(x, y, z, v) = w_{23}(y, z)$ which allows for the analysis of the last ladder height $X_T + |U_T|$. Then (4.2.13) simplifies to

$$m_{\delta,23}(u) = \phi_\delta \int_0^u m_{\delta,23}(u - y)f_\delta(y)dy + v_{\delta,23}(u), \tag{4.2.15}$$

where using (4.1.7) and (4.1.8), (4.2.14) simplifies to

$$\begin{aligned} v_{\delta,23}(u) &= \int_u^\infty w_{23}(y - u, u) \left(\int_0^\infty h_{\delta,12}(x, y|0)dx \right) dy \\ &= \phi_\delta \int_u^\infty w_{23}(y - u, u)f_\delta(y)dy, \end{aligned} \tag{4.2.16}$$

which only depends on ϕ_δ and $f_\delta(y)$ and thus, we note that the distribution of the last ladder height may be determined from the generic ladder height, $f_\delta(y)$. In particular, consider when $w_{23}(y,z) = w_5(y+z)$ i.e. a function of the last ladder height, then (4.2.15) becomes

$$m_{\delta,5}(u) = \phi_\delta \int_0^u m_{\delta,5}(u-y) f_\delta(y)dy + v_{\delta,5}(u), \tag{4.2.17}$$

where using (4.2.16)

$$v_{\delta,5}(u) \;\; = \;\; \phi_\delta \int_u^\infty w_5(y) f_\delta(y) dy. \tag{4.2.18}$$

Note that when $w(x,y,z,v) = w_2(y)$, then we obtain (2.3.15) and if $w(x,y,z,v) = 1$ then we obtain (2.3.18) from Section 2.3.

Also, since the special cases of $m_\delta(u)$ given by (4.2.8), (4.2.11), (4.2.13), (4.2.15) and (4.2.17) satisfy defective renewal equations, we are again able to use (1.3.18) of the Preliminaries section to determine general solutions to these special cases. And in principle, we may be able to obtain $h^{**}_{\delta,124}(x,y,v|0)$ from the solution of $m_{\delta,124}(0)$ when $w_{124}(x,y,v) = e^{-s_1x-s_2y-s_4v}$ which can be used together with (4.1.2), (4.1.6), (4.1.7), and (4.1.8) to obtain $h_{\delta,12}(x,y|0)$, $f_\delta(y)$, and ϕ_δ. This would allow the special cases of $m_\delta(u)$ presented in this section to be solved in full generality.

Again, the analysis presented thus far provides insight into the mathematical structure of $m_\delta(u)$ but does not yield information on its relationship with the interclaim time density $k(t)$ and the claim size density $p(y)$. And as described previously, to obtain this information, we usually also need to condition on the time and amount of the first claim. However, because $m_\delta(u)$ now contains a penalty function involving X_T, it is not easy to use this conditioning argument to obtain an expression satisfied by $m_\delta(u)$. But recall the defective renewal equation for $m_\delta(u)$ given by (4.2.8) can be expressed in terms of the joint discounted density of $(U_{T-}, |U_T|, R_{N_T-1})$ given $u=0$ which does not involve X_T. Thus, it suffices to consider conditioning on the time and amount of the first claim when $w(x,y,z,v) = w_{124}(x,y,v)$ which will be examined in Section 4.4. We can sometimes identify ϕ_δ and $f_\delta(y)$ using this approach when $w_{124}(x,y,v) = 1$ for certain $k(t)$ and/or $p(y)$ and thus, would allow for $m_{\delta,23}(u)$ in (4.2.15) to be solved.

In the next section, we consider the mathematical structure of the joint defective discounted density of $(U_{T-}, |U_T|, X_T, R_{N_T-1})$.

4.3 Associated defective densities

To determine the joint defective discounted density of $(U_{T-}, |U_T|, X_T, R_{N_T-1})$ from $m_\delta(u)$, we choose the penalty function $w(x,y,z,v) = e^{-s_1x-s_2y-s_3z-s_4v}$ which can be rewritten as $w(x,y,z,v) = w_3(z)w_{124}(x,y,v)$ where $w_3(z) = e^{-s_3z}$ and $w_{124}(x,y,v) = e^{-s_1x-s_2y-s_4v}$. Thus, $m_\delta(u)$ follows a defective renewal equation given by (4.2.11) and from (4.2.12), $v_{\delta,3,124}(u) = e^{-s_3u}v_{\delta,124}(u)$, where using (4.2.10)

$$\begin{aligned} v_{\delta,124}(u) &= \int_u^\infty \int_0^\infty e^{-s_1(u+x)-s_2(y-u)-s_4u} h^*_{\delta,12}(x,y|0)dxdy \\ &+ \int_u^\infty \int_0^\infty \int_0^x e^{-s_1(u+x)-s_2(y-u)-s_4(u+v)} h^{**}_{\delta,124}(x,y,v|0)dvdxdy, \end{aligned}$$

and with a change of variables, it follows that

$$\begin{aligned} v_{\delta,124}(u) &= \int_0^\infty \int_u^\infty e^{-s_1x-s_2y-s_4u} h^*_{\delta,12}(x-u,u+y|0)dxdy \\ &+ \int_0^\infty \int_u^\infty \int_u^x e^{-s_1x-s_2y-s_4v} h^{**}_{\delta,124}(x-u,u+y,v-u|0)dvdxdy. \end{aligned} \tag{4.3.1}$$

And thus, using (4.2.9), the general solution of $m_\delta(u)$ when $w(x,y,z,v) = e^{-s_1x-s_2y-s_3z-s_4v}$ is given by

$$m_{\delta,3,124}(u) = e^{-s_3u}v_{\delta,124}(u) + \frac{1}{1-\phi_\delta}\int_0^u e^{-s_3z}v_{\delta,124}(z)g_\delta(u-z)dz,$$

and substitution of (4.3.1) and rearranging results in

$$\begin{aligned} m_{\delta,3,124}(u) &= \int_0^\infty \int_u^\infty e^{-s_1x-s_2y-s_3u-s_4u} h^*_{\delta,12}(x-u,u+y|0)dxdy \\ &+ \int_0^\infty \int_u^\infty \int_u^x e^{-s_1x-s_2y-s_3u-s_4v} h^{**}_{\delta,124}(x-u,u+y,v-u|0)dvdxdy \\ &+ \int_0^u \int_0^\infty \int_z^\infty e^{-s_1x-s_2y-s_3z-s_4z}\left(h^*_{\delta,12}(x-z,z+y|0)\frac{g_\delta(u-z)}{1-\phi_\delta}\right)dxdydz \\ &+ \int_0^u \int_0^\infty \int_z^\infty \int_z^x e^{-s_1x-s_2y-s_3z-s_4v} \\ &\times \left(h^{**}_{\delta,124}(x-z,z+y,v-z|0)\frac{g_\delta(u-z)}{1-\phi_\delta}\right)dvdxdydz. \end{aligned} \tag{4.3.2}$$

Note that if $w(x, y, z, v) = e^{-s_1 x - s_2 y - s_3 z - s_4 v}$, then recall from (4.1.1) that $m_\delta(u)$ is given by

$$m_{\delta,3,124}(u) = \mathrm{E}[e^{-\delta T - s_1 U_{T-} - s_2 |U_T| - s_3 X_T - s_4 R_{N_T - 1}} I(T < \infty) | U_0 = u],$$

which is equal to the Laplace transform of the joint defective discounted density of $(U_{T-}, |U_T|, X_T, R_{N_T-1})$ i.e. if $h_\delta(x, y, z, v|u)$ is the joint defective discounted density of $(U_{T-}, |U_T|, X_T, R_{N_T-1})$ at (x, y, z, v), then

$$m_{\delta,3,124}(u) = \int_0^\infty \int_0^\infty \int_0^u \int_0^x e^{-s_1 x - s_2 y - s_3 z - s_4 v} h_\delta(x, y, z, v|u) dv dz dy dx,$$

and thus, using the uniqueness of Laplace transforms, we can determine $h_\delta(x, y, z, v|u)$ from (4.3.2). Note that $h_\delta(x, y, z, v|u)$ consists of different densities on different subspaces of (x, y, z, v) where the subspace reveals information on how ruin occurs. We summarize the different densities that make up $h_\delta(x, y, z, v)$ on the following page.

$$h_\delta(x,y,z,v|u) = \begin{cases} h^*_{\delta,12}(x-u, u+y|0), & x>u,\ y>0,\ z=u,\ v=u \\ & \text{(corresponding to ruin occurring} \\ & \text{on the first claim)} \\ h^{**}_{\delta,124}(x-u, u+y, v-u|0), & x>u,\ y>0,\ z=u,\ u<v<x \\ & \text{(corresponding to ruin occurring} \\ & \text{on the first drop in surplus} \\ & \text{caused by a claim subsequent to} \\ & \text{the first)} \\ h^*_{\delta,12}(x-z, z+y|0)\frac{g_\delta(u-z)}{1-\phi_\delta}, & x>z,\ y>0,\ 0<z<u,\ v=z \\ & \text{(corresponding to a drop in} \\ & \text{surplus to its lowest level } z \\ & \text{without ruin first occurring,} \\ & \text{followed by a claim causing ruin)} \\ h^{**}_{\delta,124}(x-z, z+y, v-z|0)\frac{g_\delta(u-z)}{1-\phi_\delta}, & z<v<x,\ y>0,\ 0<z<u \\ & \text{(corresponding to a drop in} \\ & \text{surplus to its lowest level } z \\ & \text{without ruin first occurring,} \\ & \text{followed by ruin occurring but} \\ & \text{not on the next claim).} \end{cases}$$

Note that when $\delta = 0$, we can interpret $\frac{g_0(u-z)}{1-\phi_\delta}$ as the density of the surplus process dropping to its lowest level z without ruin first occurring.

Now consider when the penalty function is given by $w(x,y,z,v) = w_{123}(x,y,z) = e^{-s_1x-s_2y-s_3z}$, then $m_\delta(u)$ satisfies a defective renewal equation given by (4.2.13) and from (4.2.14)

$$\begin{aligned} v_{\delta,123}(u) &= \int_u^\infty \int_0^\infty e^{-s_1(u+x)-s_2(y-u)-s_3u} h_{\delta,12}(x,y|0)dxdy \\ &= \int_0^\infty \int_u^\infty e^{-s_1x-s_2y-s_3u} h_{\delta,12}(x-u,u+y|0)dxdy, \end{aligned}$$

and using (4.2.9), substitution into the general solution for $m_{\delta,123}(u)$ yields

$$\begin{aligned} m_{\delta,123}(u) &= v_{\delta,123}(u) + \frac{1}{1-\phi_\delta}\int_0^u v_{\delta,123}(z)g_\delta(u-z)dz \\ &= \int_0^\infty \int_u^\infty e^{-s_1x-s_2y-s_3u} h_{\delta,12}(x-u,u+y|0)dxdy \\ &+ \int_0^u \int_0^\infty \int_z^\infty e^{-s_1x-s_2y-s_3z}\left(h_{\delta,12}(x-z,z+y|0)\frac{g_\delta(u-z)}{1-\phi_\delta}\right)dxdydz, \end{aligned}$$

and again, using the uniqueness of Laplace transforms, we can determine $h_{\delta,123}(x,y,z|u)$, the joint defective discounted density of $(U_{T-},|U_T|,X_T)$ at (x,y,z) which is summarized below

$$h_{\delta,123}(x,y,z|u) = \begin{cases} h_{\delta,12}(x-u,u+y|0), & x>u,\ y>0,\ z=u \\ & \text{(corresponding to ruin occurring on the} \\ & \text{first drop below } u\text{)} \\ & \\ h_{\delta,12}(x-z,z+y|0)\frac{g_\delta(u-z)}{1-\phi_\delta}, & x>z,\ y>0,\ 0<z<u \\ & \text{(corresponding to ruin occurring but not} \\ & \text{on the first drop in surplus).} \end{cases}$$

We are able to use the same method to identify the density of the last ladder height $X_T+|U_T|$ by choosing the penalty function $w(x,y,z,v) = w_5(y+z) = e^{-s_5(y+z)}$ and using (4.2.17). Then since (4.2.18) is differentiable, it is convenient to use the form of the general solution given by (1.3.19) from the Preliminaries section which becomes

$$\begin{aligned} m_{\delta,5}(u) &= \frac{1}{1-\phi_\delta}\left\{\phi_\delta \int_u^\infty e^{-s_5 y} f_\delta(y)dy - \left(\phi_\delta \int_0^\infty e^{-s_5 y} f_\delta(y)dy\right)\overline{G}_\delta(u)\right. \\ &+ \left.\int_0^u \overline{G}_\delta(u-y)\phi_\delta e^{-s_5 y} f_\delta(y)dy\right\} \\ &= \frac{\phi_\delta}{1-\phi_\delta}\left\{\int_u^\infty e^{-s_5 y}\big[1-\overline{G}_\delta(u)\big]f_\delta(y)dy + \int_0^u e^{-s_5 y}\big[\overline{G}_\delta(u-y)-\overline{G}_\delta(u)\big]f_\delta(y)dy\right\}, \end{aligned}$$

then by the uniqueness of Laplace transforms, the last ladder height has a defective discounted density given by

$$f_\delta(u,y) = \begin{cases} \frac{\phi_\delta}{1-\phi_\delta}\big[\overline{G}_\delta(u-y)-\overline{G}_\delta(u)\big]f_\delta(y), & y<u \\ \frac{\phi_\delta}{1-\phi_\delta}\big[1-\overline{G}_\delta(u)\big]f_\delta(y), & y>u. \end{cases}$$

It can be shown from [Cheung et al. (2010a)] that the last ladder height before ruin is stochastically larger than the generic ladder heights with density $f_\delta(y)$, as we would expect.

4.4 Conditioning on the time and amount of the first claim revisited

To determine an equation satisfied by $m_{\delta,124}(u)$ by conditioning on the time t and amount y of the first claim, we use a method very similar to that of Section 2.4 for $m_{\delta,12}(u)$. That is, we consider the same 2 cases:

Case 1: first claim causes ruin
Case 2: first claim does not cause ruin

And thus, we generalize (2.4.6) by making adjustments to the contributions (2.4.1) and (2.4.3) that each case makes to $m_{\delta,12}(u)$ to account for the dependency between V and Y and for the additional variable R_{N_T-1} now included in the penalty function. Let's now consider these adjustments in detail and outline the contribution that each case makes to $m_{\delta,124}(u)$. Make note of each of the corresponding figures which help to illustrate each scenario.

Case 1: first claim causes ruin

If ruin occurs on the first claim, then $N_T = 1$ and $R_{N_T-1} = R_0 = u$ and as was described in Section 2.4, the size of the first claim y is greater than $u+ct$ with the surplus immediately prior to ruin equal to $u+ct$ and the deficit at ruin equal to $u+ct-y$.

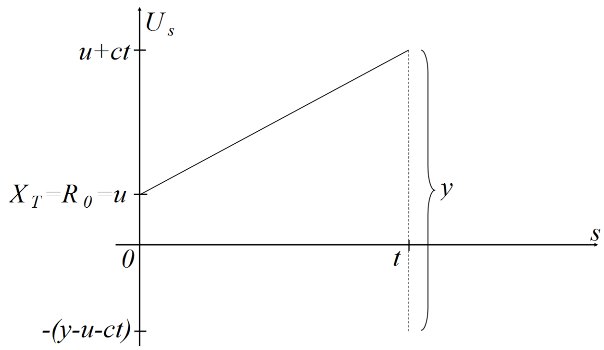

Figure 4.5: Conditioning on the time and amount of the first claim - Case 1

Thus, the contribution for this case can obtained by generalizing (2.4.1) as follows

$$\beta_{\delta,124}(u) = \int_0^\infty e^{-\delta t}\alpha_{t,124}(u+ct,u)k(t)dt \tag{4.4.1}$$

$$= \frac{1}{c}\int_u^\infty e^{-\delta\left(\frac{x-u}{c}\right)}\alpha_{t,124}(x,u)k\left(\frac{x-u}{c}\right)dx, \tag{4.4.2}$$

where

$$\alpha_{t,124}(x,u) = \int_x^\infty w_{124}(x,y-x,u)p(y|t)dy. \tag{4.4.3}$$

Case 2: first claim does not cause ruin

If ruin does not occur on the first claim, then the size of the first claim y is less than $u+ct$ and the process is said to renew with an initial surplus of $u+ct-y$ given an amount of time t as passed.

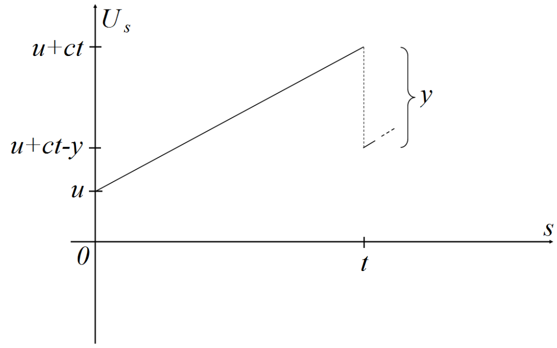

Figure 4.6: Conditioning on the time and amount of the first claim - Case 2

Thus, the contribution for this case can be obtained by generalizing (2.4.3) as follows

$$= \int_0^\infty e^{-\delta t}\sigma_{t,\delta,124}(u+ct)k(t)dt, \tag{4.4.4}$$

where

$$\sigma_{t,\delta,124}(x) = \int_0^x m_{\delta,124}(x-y)p(y|t)dy. \tag{4.4.5}$$

And therefore, summing the contributions (4.4.2) and (4.4.4) that each case makes to $m_{\delta,124}(u)$, we obtain

$$m_{\delta,124}(u) = \int_0^\infty e^{-\delta t}\sigma_{t,\delta,124}(u+ct)k(t)dt + \beta_{\delta,124}(u), \tag{4.4.6}$$

which we note has the same form as (2.4.6) with $\sigma_{\delta,12}(x)$ replaced by $\sigma_{t,\delta,124}(x)$ and $\alpha_{12}(x)$ replaced by $\alpha_{t,124}(x,u)$. To take the Laplace transform of the first term of (4.4.6), we use (1.3.10) of the Preliminaries section to obtain

$$\int_0^\infty e^{-su}\int_0^\infty e^{-\delta t}\sigma_{t,\delta,124}(u+ct)k(t)dtdu = \int_0^\infty e^{-(\delta-cs)t}\tilde{\sigma}_{t,\delta,124}(s)k(t)dt - \tilde{\sigma}^*_{\delta,124}(\delta-cs),$$

where using (1.3.11), we obtain

$$\widetilde{\sigma}^*_{\delta,124}(s) = \int_0^\infty\int_0^{ct} e^{-\frac{1}{c}\{\delta x+s(ct-x)\}}\sigma_{t,\delta,124}(x)k(t)dxdt. \tag{4.4.7}$$

Since $\tilde{\sigma}_{t,\delta,124}(s) = \tilde{m}_{\delta,124}(s)\tilde{p}(s|t)$, the Laplace transform of (4.4.6) is given by

$$\tilde{m}_{\delta,124}(s) = \tilde{m}_{\delta,124}(s) \int_0^\infty e^{-(\delta-cs)t}\tilde{p}(s|t)k(t)dt - \widetilde{\sigma}^*_{\delta,124}(\delta - cs) + \tilde{\beta}_{\delta,124}(s), \qquad (4.4.8)$$

and because $\int_0^\infty e^{-(\delta-cs)t}\,\tilde{p}(s|t)\,k(t)dt = \mathrm{E}[e^{-sY-(\delta-cs)Y}]$, we can rearrange to obtain

$$\left(1 - \mathrm{E}\left[e^{-sY-(\delta-cs)Y}\right]\right)\tilde{m}_{\delta,124}(s) = \tilde{\beta}_{\delta,124}(s) - \widetilde{\sigma}^*_{\delta,124}(\delta - cs), \qquad (4.4.9)$$

and again, since the left hand side of (4.4.9) is 0 when $s = \rho_\delta$ where ρ_δ is any root with positive real part to Lundberg's fundamental equation (1.3.7), then we are able to identify unknown constants in $\widetilde{\sigma}^*_{\delta,124}(\delta - cs)$ using

$$\widetilde{\sigma}^*_{\delta,124}(\delta - c\rho_\delta) = \tilde{\beta}_{\delta,124}(\rho_\delta),$$

and as mentioned in Section 2.4 for the classical case, this is usually needed to invert the Laplace transform of $\tilde{m}_{\delta,124}(s)$ under additional assumptions for $k(t)$ and/or $p(y)$.

Different dependency models have been considered for (V, Y). i.e. [Willmot and Woo (2011)], [Woo, J.K. (2011)], etc. However, for the following examples, we will assume that the claim sizes Y and interclaim times V are independent. And thus, we replace the conditional density $p(y|t)$ with $p(y)$ such that (4.4.6) becomes

$$m_{\delta,124}(u) = \int_0^\infty e^{-\delta t}\sigma_{\delta,124}(u + ct)k(t)dt + \beta_{\delta,124}(u),$$

where $\sigma_{\delta,124}(x)$ is equal to $\sigma_{t,\delta,124}(x)$ with $p(y|t)$ replaced by $p(y)$.

4.5 Example 4: classical Poisson model revisited

We now revisit the classical Poisson model considered in Example 1 which assumes the interclaim time is exponentially distributed i.e. $k(t) = \lambda e^{-\lambda t}$ [Cheung et al. (2010b)]. In Section 4.2, we found out that despite containing a 4-variable penalty function involving X_T, $m_\delta(u)$ can be expressed in terms of the joint defective discounted density of $(U_{T-}, |U_T|, R_{N_T-1})$ which does not involve X_T. And thus, it suffices to consider when $w(x, y, z, v) = w_{124}(x, y, v)$.

First, we note that $\widetilde{\sigma}^*_{\delta,124}(s)$ given by (4.4.7) is equal to $\widetilde{\sigma}^*_{\delta,12}(s)$ given by (2.5.1) with $\sigma_{\delta,12}(x)$ replaced by $\sigma_{t,\delta,124}(x)$ which in this example where Y and V are assmed to be independent, is equal to $\sigma_{\delta,124}(x)$. And thus, using (2.5.2), it follows that

$$\widetilde{\sigma}^*_{\delta,124}(s) = \frac{\lambda}{\lambda + s}\tilde{\sigma}_{\delta,124}\left(\frac{\lambda+\delta}{c}\right),$$

and substituting this into (4.4.9), we obtain

$$\left\{1-\tilde{p}(s)\frac{\lambda}{\lambda+\delta-cs}\right\}\tilde{m}_{\delta,124}(s)=\tilde{\beta}_{\delta,124}(s)-\frac{\lambda}{\lambda+\delta-cs}\tilde{\sigma}_{\delta,124}\left(\frac{\lambda+\delta}{c}\right), \qquad (4.5.1)$$

where using (4.4.2),

$$\beta_{\delta,124}(u)=\frac{\lambda}{c}\int_u^\infty e^{-\left(\frac{\lambda+\delta}{c}\right)(x-u)}\alpha_{124}(x,u)dx, \qquad (4.5.2)$$

where $\alpha_{124}(x,u)$ and $\sigma_{\delta,124}(s)$ is given by (4.4.3) and (4.4.5), respectively, with $p(y|t)$ replaced by $p(y)$. And since, (4.5.1) is equal to (2.5.3) with $\tilde{\beta}_{\delta,12}(s)$ replaced by $\tilde{\beta}_{\delta,124}(s)$, $\tilde{m}_{\delta,12}(s)$ replaced by $\tilde{m}_{\delta,124}(s)$ and $\tilde{\sigma}_{\delta,12}(s)$ replaced by $\tilde{\sigma}_{\delta,124}(s)$, it follows from (2.5.6) that

$$\begin{aligned}
\left\{s-\frac{\lambda+\delta}{c}+\frac{\lambda}{c}\tilde{p}(s)\right\}\tilde{m}_{\delta,124}(s) &= \frac{1}{c}\left\{(\lambda+\delta-c\rho_\delta)\,\tilde{\beta}_{\delta,124}(\rho_\delta)-(\lambda+\delta-cs)\tilde{\beta}_{\delta,124}(s)\right\}\\
&= \left(\frac{\lambda+\delta}{c}-\rho_\delta\right)\tilde{\beta}_{\delta,124}(\rho_\delta)-\left(\frac{\lambda+\delta}{c}-s\right)\tilde{\beta}_{\delta,124}(s)\\
&= (s-\rho_\delta)\tilde{\beta}_{\delta,124}(s)+\left(\frac{\lambda+\delta}{c}-\rho_\delta\right)\left(\tilde{\beta}_{\delta,124}(\rho_\delta)-\tilde{\beta}_{\delta,124}(s)\right),
\end{aligned}$$

where in Theorem 1 of Example 1, we showed that ρ_δ is the one real and positive root to Lundberg's fundamental equation. Using (2.5.12) we can express the above equation as follows

$$(s-\rho_\delta)(1-\phi_\delta\tilde{f}_\delta(s))\tilde{m}_{\delta,124}(s)=(s-\rho_\delta)\tilde{\beta}_{\delta,124}(s)+\left(\frac{\lambda+\delta}{c}-\rho_\delta\right)\left(\tilde{\beta}_{\delta,124}(\rho_\delta)-\tilde{\beta}_{\delta,124}(s)\right),$$

where ϕ_δ and $f_\delta(y)$ are given by (2.5.10) and (2.5.11), respectively. And rearranging, we obtain

$$\tilde{m}_{\delta,124}(s)=\phi_\delta\tilde{m}_{\delta,124}(s)\tilde{f}_\delta(s)+\tilde{\beta}_{\delta,124}(s)+\left(\frac{\lambda+\delta}{c}-\rho_\delta\right)\frac{\tilde{\beta}_{\delta,124}(\rho_\delta)-\tilde{\beta}_{\delta,124}(s)}{s-\rho_\delta},$$

which we invert to obtain the following defective renewal equation

$$m_{\delta,124}(u)=\phi_\delta\int_0^u m_{\delta,124}(u-y)f_\delta(y)dy+v_{\delta,124}(u),$$

where using (1.3.3) and (1.3.1)

$$v_{\delta,124}(u)=\beta_{\delta,124}(u)+\left(\frac{\lambda+\delta}{c}-\rho_\delta\right)\int_u^\infty e^{-\rho_\delta(v-u)}\beta_{\delta,124}(v)dv. \qquad (4.5.3)$$

Then the general solution for $m_{\delta,124}(u)$ is given by

$$m_{\delta,124}(u) \quad = \quad v_{\delta,124}(u) + \frac{1}{1-\phi_\delta}\int_0^u v_{\delta,124}(v)g_\delta(u-v)dv,$$

where $g_\delta(y)$ is given by (2.5.13) and substitution of (4.5.3) results in

$$\begin{aligned} m_{\delta,124}(u) \quad = \quad & \beta_{\delta,124}(u) + \left(\frac{\lambda+\delta}{c} - \rho_\delta\right)\int_u^\infty e^{-\rho_\delta(v-u)}\beta_{\delta,124}(v)dv \\ & + \frac{1}{1-\phi_\delta}\int_0^u \beta_{\delta,124}(v)g_\delta(u-v)dv \\ & + \frac{1}{1-\phi_\delta}\left(\frac{\lambda+\delta}{c} - \rho_\delta\right)\int_0^u\left(\int_v^\infty e^{-\rho_\delta(t-v)}\beta_{\delta,124}(t)dt\right)g_\delta(u-v)dv, \end{aligned} \tag{4.5.4}$$

where after a change of variables, the integral in the last term becomes

$$\begin{aligned} & \int_0^u\left(\int_v^\infty e^{-\rho_\delta(t-v)}\beta_{\delta,124}(t)dt\right)g_\delta(u-v)dv \\ & = \int_0^u \beta_{\delta,124}(t)\int_0^t e^{-\rho_\delta(t-v)}g_\delta(u-v)dvdt + \int_u^\infty \beta_{\delta,124}(t)\int_0^u e^{-\rho_\delta(t-v)}g_\delta(u-v)dvdt. \end{aligned}$$

It is convenient in the following analysis to replace v by t, and replace t by v in the above equation to obtain

$$\begin{aligned} & \int_0^u\left(\int_t^\infty e^{-\rho_\delta(v-t)}\beta_{\delta,124}(v)dv\right)g_\delta(u-t)dt \\ & = \int_0^u \beta_{\delta,124}(v)\int_0^v e^{-\rho_\delta(v-t)}g_\delta(u-t)dtdv + \int_u^\infty \beta_{\delta,124}(v)\int_0^u e^{-\rho_\delta(v-t)}g_\delta(u-t)dtdv, \end{aligned}$$

and therefore, substitution into (4.5.4) and rearranging yields

$$\begin{aligned} m_{\delta,124}(u) \quad = \quad & \beta_{\delta,124}(u) \\ + \quad & \frac{1}{1-\phi_\delta}\int_0^u \beta_{\delta,124}(v)\left\{g_\delta(u-v) + \left(\frac{\lambda+\delta}{c} - \rho_\delta\right)\int_0^v e^{-\rho_\delta(v-t)}g_\delta(u-t)dt\right\}dv \\ + \quad & \left(\frac{\lambda+\delta}{c} - \rho_\delta\right)\int_u^\infty \beta_{\delta,124}(v)\left\{e^{-\rho_\delta(v-u)} + \frac{1}{1-\phi_\delta}\int_0^u e^{-\rho_\delta(v-t)}g_\delta(u-t)dt\right\}dv \\ = \quad & \beta_{\delta,124}(u) + \int_0^\infty \beta_{\delta,124}(v)\tau_\delta(u,v)dv, \end{aligned} \tag{4.5.5}$$

where

$$\tau_\delta(u,v) = \begin{cases} \frac{1}{1-\phi_\delta}\left\{g_\delta(u-v) + \left(\frac{\lambda+\delta}{c} - \rho_\delta\right)\int_0^v e^{-\rho_\delta(v-t)} g_\delta(u-t)dt\right\}, & v < u \\ \\ \left(\frac{\lambda+\delta}{c} - \rho_\delta\right)\left\{e^{-\rho_\delta(v-u)} + \frac{1}{1-\phi_\delta}\int_0^u e^{-\rho_\delta(v-t)} g_\delta(u-t)dt\right\}, & v > u. \end{cases} \tag{4.5.6}$$

Note that if $\delta = 0$, then it was shown in Example 1 that $\rho_0 = 0$, and recall that $g_0(u) = -\overline{G}_0'(u) = -\psi'(u)$, and $\phi_0 = \overline{G}_0(0) = \psi(0)$. Then for $\delta = 0$, (4.5.6) becomes

$$\tau_0(u,v) = \begin{cases} \frac{1}{1-\psi(0)}\{\frac{\lambda}{c}[\psi(u-v) - \psi(u)] - \psi'(u-v)\}, & v < u \\ \frac{\lambda}{c}\frac{1-\psi(u)}{1-\psi(0)}, & v > u. \end{cases} \tag{4.5.7}$$

We now consider when $w_{124}(x,y,v) = e^{-s_1x-s_2y-s_4v}$ such that

$$m_{\delta,124}(u) = \mathrm{E}[e^{-\delta T - s_1U_{T-} - s_2|U_T| - s_4R_{N_T-1}} I(T<\infty)|U_0 = u)], \tag{4.5.8}$$

is the Laplace transform of $h_{\delta,124}(x,y,v|u)$, the joint discounted density of $(U_{T-}, |U_T|, R_{N_T-1})$. Then using (4.4.3) with $p(y|t)$ replaced with $p(y)$, $\alpha_{124}(x,u)$ becomes

$$\begin{aligned} \alpha_{124}(x,u) &= \int_x^\infty e^{-s_1x-s_2(y-x)-s_4u} p(y)dy \\ &= e^{-s_1x-s_4u}\int_0^\infty e^{-s_2y} p(x+y)dy. \end{aligned} \tag{4.5.9}$$

And thus, using (4.1.2) with $p(y|t)$ replaced with $p(y)$, (4.5.2) becomes

$$\begin{aligned} \beta_{\delta,124}(u) &= \frac{\lambda}{c}\int_u^\infty e^{-\left(\frac{\lambda+\delta}{c}\right)(x-u)} e^{-s_1x-s_4u}\int_0^\infty e^{-s_2y}p(x+y)dydx \\ &= \int_u^\infty\int_0^\infty e^{-\delta\left(\frac{x-u}{c}\right)-s_1x-s_2y-s_4u}\left(\frac{\lambda}{c}e^{-\lambda\left(\frac{x-u}{c}\right)}p(x+y)\right)dydx \\ &= \int_u^\infty\int_0^\infty e^{-\delta\left(\frac{x-u}{c}\right)-s_1x-s_2y-s_4u} h_{12}^*(x,y|u)dydx. \end{aligned} \tag{4.5.10}$$

It follows from (4.5.8) that $m_{\delta,124}(u)$ is essentially an expectation of the function $e^{-\delta T - s_1U_{T-} - s_2|U_T| - s_4R_{N_T-1}} I(T<\infty)$. Then we can write $m_{\delta,124}(u)$ as a sum of contributions from ruin on the first claim (where $t = \frac{x-u}{c}$, $x > u$, $y > 0$ and $R_{N_T-1} = R_0 = u$) and ruin on subsequent claims ($t > 0$, $v < x < u + ct$, $y > 0$ and $v > 0$). And using the density of $(T, U_{T-}, |U_T|, R_{N_T-1})$ summarized in (4.1.3), $m_{\delta,124}(u)$ can be written as follows

$$\begin{aligned} m_{\delta,124}(u) &= \int_u^\infty \int_0^\infty e^{-\delta\left(\frac{x-u}{u}\right)-s_1x-s_2y-s_4u} h_{12}^*(x,y|u)dydx \\ &+ \int_0^\infty \int_0^x \int_0^\infty \int_0^\infty e^{-\delta t-s_1x-s_2y-s_4v} h_{124}^{**}(t,x,y,v|u)dtdydvdx \\ &= \beta_{\delta,124}(u) + \int_0^\infty \int_0^x \int_0^\infty e^{-s_1x-s_2y-s_4v} h_{\delta,124}^{**}(x,y,v|u)dydvdx, \quad (4.5.11) \end{aligned}$$

using (4.5.10) and (4.1.5). We rearrange to solve for the second term on the right side and using (4.5.5), we obtain

$$\begin{aligned} &\int_0^\infty \int_0^x \int_0^\infty e^{-s_1x-s_2y-s_4v} h_{\delta,124}^{**}(x,y,v|u)dydvdx \\ &= m_{\delta,124}(u) - \beta_{\delta,124}(u) \\ &= \int_0^\infty \beta_{\delta,124}(v)\tau_\delta(u,v)dv \\ &= \int_0^\infty \tau_\delta(u,v) \int_v^\infty \int_0^\infty e^{-s_1x-s_2y-s_4v} h_{\delta,12}^*(x,y|v)dydxdv \\ &= \int_0^\infty \int_0^\infty \int_0^x e^{-s_1x-s_2y-s_4v} \Big(\tau_\delta(u,v)h_{\delta,12}^*(x,y|v)\Big)dvdydx, \qquad (4.5.12) \end{aligned}$$

using (4.5.10) and (4.1.4) with $p(y|t)$ replaced by $p(y)$.

And thus, substituting (4.5.10) and (4.5.12) into (4.5.11) yields

$$\begin{aligned} m_{\delta,124}(u) &= \int_u^\infty \int_0^\infty e^{-s_1x-s_2y-s_4u} h_{\delta,12}^*(x,y|u)dydx \\ &+ \int_0^\infty \int_0^\infty \int_0^x e^{-s_1x-s_2y-s_4v} \Big(\tau_\delta(u,v)h_{\delta,12}^*(x,y|v)\Big)dvdydx, \end{aligned}$$

then using the uniqueness of Laplace transforms, the joint defective discounted density of $(U_{T-}, |U_T|, R_{N_T-1})$ at (x, y, v) is given by

$$h_{\delta,124}(x,y,v|u) = \begin{cases} h_{\delta,12}^*(x,y|u), & x > v,\ y > 0,\ v = u \\ \tau_\delta(u,v)h_{\delta,12}^*(x,y|v), & x > v,\ y > 0,\ v \neq u, \end{cases}$$

where $h_{\delta,12}^*(x,y|u) = \frac{\lambda}{c}e^{-\left(\frac{\lambda+\delta}{c}\right)(x-u)}p(x+y)$ and $\tau_\delta(u,v)$ is given by (4.5.6). We are also able to determine various marginal discounted densities of U_{T-}, $|U_T|$, and R_{N_T-1} by integrating over certain variables as is done in [Cheung et al. (2010b)].

Now, we wish to identify the joint density of the last interclaim before ruin V_{N_T} and the claim causing ruin Y_{N_T}. Since $V_{N_T} = (U_{T-} - R_{N_T-1})/c$ and $Y_{N_T} = U_{T-} + |U_T|$,

$$e^{-zV_{N_T}-sY_{N_T}} = e^{-\left(s+\frac{z}{c}\right)U_{T-}-s|U_T|+\frac{z}{c}R_{N_T-1}}.$$

And thus, $m_{\delta,124}(u)$ is equal to $E\left[e^{-zV_{N_T}-sY_{N_T}}I(T<\infty)|U_0=u\right]$, if $s_1 = \left(s+\frac{z}{c}\right)$, $s_2 = s$, $s_4 = -\frac{z}{v}$, and $\delta = 0$. Then $\alpha_{124}(x,u)$ from (4.5.9) becomes

$$\begin{aligned}\alpha_{124}(x,u) &= e^{-\left(s+\frac{z}{c}\right)x+\frac{z}{c}u}\int_0^\infty e^{-sy}p(x+y)dy \\ &= e^{-\frac{z}{c}(x-u)}\int_x^\infty e^{-sy}p(y)dy.\end{aligned}$$

And thus, for $\delta = 0$, (4.5.2) becomes

$$\begin{aligned}\beta_{0,124}(u) &= \frac{\lambda}{c}\int_u^\infty e^{-\left(\frac{\lambda+z}{c}\right)(x-u)}\int_x^\infty e^{-sy}p(y)dydx \\ &= \lambda\int_0^\infty e^{-(\lambda+z)t}\int_{u+ct}^\infty e^{-sy}p(y)dydt \\ &= \int_0^\infty\int_{u+ct}^\infty e^{-zt-sy}\left(\lambda e^{-\lambda t}p(y)\right)dydt,\end{aligned}$$

using a change of variables. Then substitution into (4.5.5) yields

$$\begin{aligned}E[e^{-zV_{N_T}-sY_{N_T}}I(T<\infty)|U_0=u] &= \int_0^\infty\int_{u+ct}^\infty e^{-zt-sy}\left(\lambda e^{-\lambda t}p(y)\right)dydt \\ &+ \int_0^\infty\left(\int_0^\infty\int_{v+ct}^\infty e^{-zt-sy}\lambda e^{-\lambda t}p(y)dydt\right)\tau_0(u,v)dv \\ &= \int_0^\infty\int_{u+ct}^\infty e^{-zt-sy}\left(\lambda e^{-\lambda t}p(y)\right)dydt \\ &+ \int_0^\infty\int_{ct}^\infty e^{-zt-sy}\left(\lambda e^{-\lambda t}p(y)\int_0^{y-ct}\tau_0(u,v)dv\right)dydt,\end{aligned}$$

and thus, by the uniqueness of Laplace transforms, it follows that the joint density of the last interclaim time and the claim causing ruin (V_{N_T}, Y_{N_T}) is given by

$$h_6(t,y|u) = \begin{cases}\lambda e^{-\lambda t}p(y)\left(1+\int_0^{y-ct}\tau_0(u,v)dv\right), & y > u+ct \\ \lambda e^{-\lambda t}p(y)\int_0^{y-ct}\tau_0(u,v)dv, & ct < y < u+ct \\ 0, & y < ct,\end{cases}$$

where $\tau_0(u.v)$ is given by (4.5.7).

4.6 Example 5: exponential claim sizes revisited

We will continue to assume that V and Y are independent i.e. $p(y|t) = p(y)$ and consider when claims are exponentially distributed i.e. $p(y) = \beta e^{-\beta y}$ which were assumptions made in Example 2 when analyzing $m_{\delta,12}(u)$ containing a 2-variable penalty function [Cheung et al. (2010a)]. But now, we consider a generalized penalty function $w(x, y, z, v)$ and in particular, we let $w(x, y, z, v) = e^{-s_1x-s_2y-s_3z-s_4v}$ such that $m_\delta(u)$ becomes

$$m_\delta(u) = \mathrm{E}[e^{-\delta T}e^{-s_1U_{T-}-s_2|U_T|-s_3X_T-s_4R_{N_T-1}}I(T < \infty)|U_0 = u], \tag{4.6.1}$$

which as was previously mentioned, is the Laplace transform of the joint defective discounted density of $(U_{T-}, |U_T|, X_T, R_{N_T-1})$.

Recall from (4.1.9) and (2.6.1) that

$$\begin{aligned} h^{**}_{\delta,124}(x, u + y, v|0) &= p_x(u + y)h^{**}_{\delta,14}(x, v|0) \\ &= \beta e^{-\beta(u+y)}h^{**}_{\delta,14}(x, v|0), \end{aligned}$$

and thus, using (4.2.7), we obtain

$$\begin{aligned} v_\delta(u) &= \int_0^\infty e^{-\delta t}\left(\int_0^\infty e^{-s_1(u+ct)-s_2y-s_3u-s_4u}\beta e^{-\beta(u+ct+y)}dy\right)k(t)dt \\ &\quad + \int_0^\infty\int_0^\infty\int_0^x e^{-s_1(u+x)-s_2y-s_3u-s_4(u+v)}\beta e^{-\beta(u+y)}h^{**}_{\delta,14}(x, v|0)dvdxdy \\ &= e^{-(\beta+s_1+s_3+s_4)u}\left(\int_0^\infty \beta e^{-(\beta+s_2)y}dy\right) \\ &\quad \times\left(\int_0^\infty e^{-(\delta+cs_1+c\beta)t}k(t)dt + \int_0^\infty\int_0^x e^{-s_1x-s_4v}h^{**}_{\delta,14}(x, v|0)dvdx\right) \\ &= \frac{\beta e^{-(\beta+s_1+s_3+s_4)u}}{\beta + s_2}\left(\tilde{k}(\delta + cs_1 + c\beta) + \tilde{h}^{**}_{\delta,14}(s_1, s_4|0)\right), \end{aligned}$$

where $\tilde{h}^{**}_{\delta,14}(s_1, s_4|0)$ is the bivariate Laplace transform of $h^{**}_{\delta,14}(x, y|0)$. Now let $\xi_\delta(s_1, s_4) = \tilde{k}(\delta + cs_1 + c\beta) + \tilde{h}^{**}_{\delta,14}(s_1, s_4|0)$ such that

$$v_\delta(u) = \frac{\beta\xi_\delta(s_1, s_4)}{\beta + s_2}e^{-(\beta+s_1+s_3+s_4)u}. \tag{4.6.2}$$

Since the ladder height $f_\delta(y)$ is derived independently from the choice of penalty function, using (2.6.2), $f_\delta(y) = \beta e^{-\beta y}$. And because $m_\delta(u)$ is known to satisfy a defective renewal equation, then using (4.2.8) we have

$$m_\delta(u) = \phi_\delta \int_0^u m_\delta(u-y)\beta e^{\beta y} dy + v_\delta(u),$$

where $v_\delta(u)$ is given by (4.6.2) and from Theorem 2, ϕ_δ is the one real root $\in (0,1)$ satisfying (2.6.7). Taking the Laplace transform of this defective renewal equation yields

$$\tilde{m}_\delta(z) = \phi_\delta \tilde{m}_\delta(z)\frac{\beta}{\beta+z} + \frac{\beta\xi_\delta(s_1,s_4)}{\beta+s_2}(\beta+s_1+s_3+s_4+z)^{-1},$$

and solving for $\tilde{m}_\delta(z)$, we obtain

$$\tilde{m}_\delta(z) = \frac{\beta\xi_\delta(s_1,s_4)}{\beta+s_2}\frac{(\beta+s_1+s_3+s_4+z)^{-1}}{1-\phi_\delta\beta(\beta+z)^{-1}},$$

After some algebra, we have

$$\tilde{m}_\delta(z) = \frac{\beta\xi_\delta(s_1,s_4)}{(\beta+s_2)(\phi_\delta\beta+s_1+s_3+s_4)}\left\{\frac{s_1+s_3+s_4}{\beta+s_1+s_3+s_4+z} + \frac{\phi_\delta\beta}{\beta(1-\phi_\delta)+z}\right\},$$

which using (2.6.4) we invert to obtain

$$m_\delta(u) = \frac{\beta\xi_\delta(s_1,s_4)}{(\beta+s_2)(\phi_\delta\beta+s_1+s_3+s_4)}\left\{(s_1+s_3+s_4)e^{-(\beta+s_1+s_3+s_4)u} + \beta\overline{G}_\delta(u)\right\}. \quad (4.6.3)$$

To express $\xi_\delta(s_1,s_4)$ in terms of $\tilde{k}(s)$, we consider the expression for $m_\delta(u)$ given by (4.4.6) which was derived by conditioning on the time and amount of the first claim. But first, since $\xi_\delta(s_1,s_4)$ is not a function of s_2 or s_3, we let $s_2 = s_3 = 0$ and thus $\tilde{m}_\delta(z)$ from (4.6.3) simplifies to

$$m_{\delta,14}(u) = \frac{\xi_\delta(s_1,s_4)}{\phi_\delta\beta+s_1+s_4}\left\{(s_1+s_4)e^{-(\beta+s_1+s_4)u} + \beta\overline{G}_\delta(u)\right\}. \quad (4.6.4)$$

Then (4.4.6) simplifies to

$$m_{\delta,14}(u) = \int_0^\infty e^{-\delta t}\sigma_{\delta,14}(u+ct)k(t)dt + \beta_{\delta,14}(u), \quad (4.6.5)$$

where from (4.4.1) and (4.4.3)

$$\begin{aligned}
\beta_{\delta,14}(u) &= \int_0^\infty e^{-\delta t}\left(\int_{u+ct}^\infty e^{-s_1(u+ct)-s_4u}\beta e^{-\beta y}dy\right)k(t)dt \\
&= \int_0^\infty e^{-\delta t-(\beta+s_1)(u+ct)-s_4u}k(t)dt \\
&= e^{-(\beta+s_1+s_4)u}\tilde{k}(\delta+c\beta+cs_1), \quad (4.6.6)
\end{aligned}$$

and using (4.4.5) and (4.6.4)

$$\begin{aligned}\sigma_{\delta,14}(x) &= \int_0^x m_{\delta,14}(x-y)\beta e^{-\beta y}dy.\\ &= \frac{\xi_\delta(s_1,s_4)}{\phi_\delta\beta+s_1+s_4}\Big\{(s_1+s_4)\int_0^x e^{-(\beta+s_1+s_4)(x-y)}\beta e^{-\beta y}dy+\beta\int_0^x \overline{G}_\delta(x-y)\beta e^{-\beta y}dy\Big\}.\end{aligned}$$

Recall from (2.6.3) that

$$\int_0^x \overline{G}_\delta(x-y)\beta e^{-\beta y}dy=\frac{\overline{G}_\delta(x)}{\phi_\delta}-e^{-\beta x},$$

and thus, we have

$$\begin{aligned}\sigma_{\delta,14}(x) &= \frac{\beta\xi_\delta(s_1,s_4)}{\phi_\delta\beta+s_1+s_4}\left\{e^{-\beta x}\left(1-e^{-(s_1+s_4)x}\right)+\left(\frac{\overline{G}_\delta(x)}{\phi_\delta}-e^{-\beta x}\right)\right\}\\ &= \frac{\beta\xi_\delta(s_1,s_4)}{\phi_\delta\beta+s_1+s_4}\left\{\frac{\overline{G}_\delta(x)}{\phi_\delta}-e^{-(\beta+s_1+s_4)x}\right\}. \qquad (4.6.7)\end{aligned}$$

Therefore, using (4.6.6), (4.6.7), and (2.6.6), (4.6.5) becomes

$$\begin{aligned}m_{\delta,14}(u) &= \frac{\beta\xi_\delta(s_1,s_4)}{\phi_\delta\beta+s_1+s_4}\left\{\frac{1}{\phi_\delta}\int_0^\infty e^{-\delta t}\overline{G}_\delta(u+ct)k(t)dt-\int_0^\infty e^{-\delta t-(\beta+s_1+s_4)(u+ct)}k(t)dt\right\}\\ &\quad +e^{-(\beta+s_1+s_4)u}\tilde{k}(\delta+c\beta+cs_1)\\ &= \frac{\beta\xi_\delta(s_1,s_4)}{\phi_\delta\beta+s_1+s_4}\left\{\overline{G}_\delta(u)-e^{-(\beta+s_1+s_4)u}\tilde{k}(\delta+c\beta+cs_1+cs_4)\right\}\\ &\quad +e^{-(\beta+s_1+s_4)u}\tilde{k}(\delta+c\beta+cs_1). \qquad (4.6.8)\end{aligned}$$

And thus, when equating (4.6.4) and (4.6.8), the terms involving $\overline{G}_\delta(u)$ cancel out and dividing both sides by $e^{-(\beta+s_1+s_4)u}$ results in

$$\frac{\xi_\delta(s_1,s_4)}{\phi_\delta\beta+s_1+s_4}(s_1+s_4)=\tilde{k}(\delta+c\beta+cs_1)-\frac{\beta\xi_\delta(s_1,s_4)}{\phi_\delta\beta+s_1+s_4}\tilde{k}(\delta+c\beta+cs_1+cs_4),$$

and solving for $\xi_\delta(s_1,s_4)$, we obtain

$$\xi_\delta(s_1,s_4)=\frac{(\phi_\delta\beta+s_1+s_4)\tilde{k}(\delta+c\beta+cs_1)}{s_1+s_4+\beta\tilde{k}(\delta+c\beta+cs_1+cs_4)},$$

which is a function in terms of $\tilde{k}(s)$. Therefore, using (4.6.3)

$$m_\delta(u) = C_\delta(s_1, s_2, s_3, s_4)\left\{(s_1 + s_3 + s_4)e^{-(\beta+s_1+s_3+s_4)u} + \beta\overline{G}_\delta(u)\right\}, \qquad (4.6.9)$$

where

$$C_\delta(s_1, s_2, s_3, s_4) = \frac{\beta(\phi_\delta\beta + s_1 + s_4)\tilde{k}(\delta + c\beta + cs_1)}{(\beta + s_2)(\phi_\delta\beta + s_1 + s_3 + s_4)(s_1 + s_4 + \beta\tilde{k}(\delta + c\beta + cs_1 + cs_4))},$$

and $\overline{G}_\delta(u)$ is given by (2.6.4).

Now recall that if we want to study the density of the last interclaim time before ruin given by $V_{N_T} = (U_{T-} - R_{N_T-1})/c$, then we choose $w(x, y, z, v) = e^{-s\left(\frac{x-v}{c}\right)} = e^{-\frac{s}{c}x+\frac{s}{c}v}$. Thus, if we denote the density of the last interclaim time before ruin as $k_T(t|u) = -\overline{K}'_T(t|u)$, then the Laplace transform of $k_T(t|u)$ is given by $\mathrm{E}[e^{-sV_{N_T}}I(T < \infty)|U_0 = u]$ which can be obtained from (4.6.9) with $\delta = 0$, $s_1 = s/c$, $s_2 = s_3 = 0$, and $s_4 = -s/c$. And thus, recalling that $\overline{G}_0(u) = \psi(u)$, we obtain

$$\mathrm{E}[e^{-sV_{N_T}}I(T < \infty)|U_0 = u] = \frac{\tilde{k}(c\beta + s)}{\tilde{k}(c\beta)}\psi(u),$$

which we invert to obtain the density of the last interclaim time

$$k_T(t|u) = \frac{e^{-c\beta t}k(t)}{\tilde{k}(c\beta)}\psi(u).$$

The proper distribution of $V_{N_T}|T < \infty$ is given by

$$\frac{k_T(t|u)}{\psi(u)} = \frac{e^{-c\beta t}k(t)}{\tilde{k}(c\beta)},$$

which we note does not depend on the initial surplus u and is easily evaluated for many forms of $k(t)$. It is shown in [Cheung et al. (2010a)] that this last interclaim time V_{N_T} is stochastically dominated by the generic interclaim time V i.e. $\overline{K}_T(t|u) \le \overline{K}(t)$, as is consistent with our intuition. This is because a shorter interclaim time means less time for premium to be collected which increases the chances of a claim occuring that is large enough to cause ruin.

Now consider when $s_3 = s_4 = 0$, such that $m_\delta(u)$ simplifies to $m_{\delta,12}(u)$ and using (4.6.9)

$$m_{\delta,12}(u) = \frac{\beta\tilde{k}(\delta + c\beta + cs_1)}{(\beta + s_2)(s_1 + \beta\tilde{k}(\delta + c\beta + cs_1))}\left\{s_1e^{-(\beta+s_1)u} + \beta\overline{G}_\delta(u)\right\}, \qquad (4.6.10)$$

which is an explicit solution to (2.6.12) in terms of $\tilde{k}(s)$ when $w_{12}(x,y) = e^{-s_1x-s_2y}$.

We are now interested in determining $h_{\delta,12}(x,y|u)$, the joint defective discounted density of the surplus immediately prior to ruin and the deficit at ruin [Willmot, G.E. (2011)]. And since $h_{\delta,12}(x,y|u) = h_{\delta,1}(x|u)\beta e^{\beta y}$ from (4.1.10) where $h_{\delta,1}(x|u)$ is the defective discounted density of the surplus immediately prior to ruin, we only need to focus our attention to identifying $h_{\delta,1}(x|u)$ to determine $h_{\delta,12}(x,y|u)$. Thus, we let $s_2 = s_3 = s_4 = 0$ such that (4.6.1) becomes

$$\begin{aligned} m_\delta(u) &= \mathrm{E}[e^{-\delta T - s_1 U_{T^-}} I(T<\infty)|U_0 = u] \\ &= \tilde{h}_{\delta,1}(s_1|u), \end{aligned}$$

where $\tilde{h}_{\delta,1}(s_1|u)$ is the Laplace transform of $h_{\delta,1}(x|u)$. Then letting $s_2 = 0$ and using (4.6.10), we obtain

$$\tilde{h}_{\delta,1}(s_1|u) = \frac{\tilde{k}(\delta + c\beta + cs_1)}{s_1 + \beta\tilde{k}(\delta + c\beta + cs_1)}\big\{s_1 e^{-(\beta+s_1)u} + \beta\overline{G}_\delta(u)\big\},$$

and since

$$\tilde{h}_{\delta,1}(s_1|0) = \frac{\tilde{k}(\delta + c\beta + cs_1)}{s_1 + \beta\tilde{k}(\delta + c\beta + cs_1)}(s_1 + \beta\phi_\delta), \tag{4.6.11}$$

then we can write

$$\tilde{h}_{\delta,1}(s_1|u) = \frac{\tilde{h}_{\delta,1}(s_1|0)}{s_1 + \beta\phi_\delta}\big\{s_1 e^{-(\beta+s_1)u} + \beta\overline{G}_\delta(u)\big\}.$$

And by adding and subtracting $\beta\phi_\delta e^{-(\beta+s_1)u}$, we obtain

$$\begin{aligned} \tilde{h}_{\delta,1}(s_1|u) &= \frac{\tilde{h}_{\delta,1}(s_1|0)}{s_1 + \beta\phi_\delta}\big(\,(s_1 + \beta\phi_\delta)e^{-(\beta+s_1)u} + \beta\overline{G}_\delta(u) - \beta\phi_\delta e^{-(\beta+s_1)u}\,\big) \\ &= \tilde{h}_{\delta,1}(s_1|0)e^{-(\beta+s_1)u} + \frac{\beta\tilde{h}_{\delta,1}(s_1|0)}{s_1 + \beta\phi_\delta}\big(\overline{G}_\delta(u) - \phi_\delta e^{-(\beta+s_1)u}\,\big). \end{aligned} \tag{4.6.12}$$

For an arbitrary function $h(x)$ and constant a, consider the Laplace transform $e^{-as}\tilde{h}(s)$ which is given by

$$\begin{aligned} e^{-as}\tilde{h}(s) &= e^{-as}\int_0^\infty e^{-sx}h(x)dx \\ &= \int_0^\infty e^{-(x+a)s}h(x)dx \\ &= \int_a^\infty e^{-sx}h(x-a)dx, \end{aligned}$$

and thus $e^{-as}\tilde{h}(s)$ is the Laplace transform of $h(x-a)\mathrm{I}(x>a)$.

Thus, we invert (4.6.12) to obtain

$$\begin{aligned} h_{\delta,1}(x|u) &= e^{-\beta u}h_{\delta,1}(x-u|0)I(x>u)+\beta\overline{G}_{\delta}(u)\int_0^x e^{-\beta\phi_\delta(x-y)}h_{\delta,1}(y|0)dy \\ &\quad +e^{-\beta u}\beta\phi_\delta\int_0^{x-u} e^{-\beta\phi_\delta(x-u-y)}h_{\delta,1}(y|0)dy\, I(x>u), \end{aligned} \tag{4.6.13}$$

which is an equation for $h_{\delta,1}(x|u)$ in terms of $h_{\delta,1}(x|0)$. And thus, we can further refine our focus to identifying $h_{\delta,1}(x|0)$. Recall from (4.6.11) with s_1 replaced by s that

$$\begin{aligned} \tilde{h}_{\delta,1}(s|0) &= \frac{\tilde{k}(\delta+c\beta+cs)}{s+\beta\tilde{k}(\delta+c\beta+cs)}(s+\beta\phi_\delta) \\ &= \frac{\tilde{k}(\delta+c\beta+cs)}{s+\beta-\beta(1-\tilde{k}(\delta+c\beta+cs))}(s+\beta-\beta(1-\phi_\delta)), \end{aligned}$$

and recall from Example 2 that we let $\kappa_\delta=\beta(1-\phi_\delta)$ and showed that $-\kappa_\delta$ is the unique and positive root to Lundberg's fundamental equation and thus, satisfies $\kappa_\delta=\beta(1-\tilde{k}(\delta+c\kappa_\delta))$. Then by adding and subtracting κ_δ, it follows that

$$\begin{aligned} \tilde{h}_{\delta,1}(s-\beta|0) &= \frac{\tilde{k}(\delta+cs)}{s-\beta(1-\tilde{k}(\delta+cs))}(s-\kappa_\delta) \\ &= \frac{\tilde{k}(\delta+cs)}{s-\kappa_\delta-\beta(1-\tilde{k}(\delta+cs))+\beta(1-\tilde{k}(\delta+c\kappa_\delta))}(s-\kappa_\delta) \\ &= \frac{\tilde{k}(\delta+cs)}{1-\beta\left\{\frac{\tilde{k}(\delta+c\kappa_\delta)-\tilde{k}(\delta+cs)}{s-\kappa_\delta}\right\}}. \end{aligned} \tag{4.6.14}$$

Now define the proper density

$$n_\delta(t)=\frac{e^{-\delta\frac{t}{c}}k\left(\frac{t}{c}\right)}{c\tilde{k}(\delta)}, \tag{4.6.15}$$

which has the Laplace transform

$$\tilde{n}_\delta(s)=\frac{\tilde{k}(\delta+cs)}{\tilde{k}(\delta)}.$$

Then (4.6.14) can be written as

$$\tilde{h}_{\delta,1}(s-\beta|0) = \frac{\tilde{k}(\delta)\tilde{n}_\delta(s)}{1-\beta\tilde{k}(\delta)\left\{\frac{\tilde{n}_\delta(\kappa_\delta)-\tilde{n}_\delta(s)}{s-\kappa_\delta}\right\}}.$$

Since $\kappa_\delta = \beta(1-\tilde{k}(\delta+c\kappa_\delta)) \implies \beta = \frac{\kappa_\delta}{1-\tilde{k}(\delta+c\kappa_\delta)}$, substituting this result into $\tilde{h}_{\delta,1}(s-\beta|0)$ yields

$$\begin{aligned}
\tilde{h}_{\delta,1}(s-\beta|0) &= \frac{\tilde{k}(\delta)\tilde{n}_\delta(s)}{1-\frac{\kappa_\delta}{1-\tilde{k}(\delta+c\kappa_\delta)}\tilde{k}(\delta)\left\{\frac{\tilde{n}_\delta(\kappa_\delta)-\tilde{n}_\delta(s)}{s-\kappa_\delta}\right\}} \\
&= \frac{\tilde{k}(\delta)\tilde{n}_\delta(s)}{1-\frac{\tilde{k}(\delta)}{1-\tilde{k}(\delta+c\kappa_\delta)}(1-\tilde{n}_\delta(\kappa_\delta))\left\{\frac{\kappa_\delta}{s-\kappa_\delta}\frac{\tilde{n}_\delta(\kappa_\delta)-\tilde{n}_\delta(s)}{1-\tilde{n}_\delta(\kappa_\delta)}\right\}} \\
&= \frac{\tilde{v}_\delta^*(s)}{1-\phi_\delta^*\tilde{f}_\delta^*(s)},
\end{aligned} \tag{4.6.16}$$

where

$$\tilde{v}_\delta^*(s) = \tilde{k}(\delta)\tilde{n}_\delta(s),$$

$$\begin{aligned}
\phi_\delta^* &= \frac{\tilde{k}(\delta)}{1-\tilde{k}(\delta+c\kappa_\delta)}(1-\tilde{n}_\delta(\kappa_\delta)) \\
&= \frac{\tilde{k}(\delta)-\tilde{k}(\delta+c\kappa_\delta)}{1-\tilde{k}(\delta+c\kappa_\delta)},
\end{aligned}$$

where it is easy to see that $\phi_\delta^* \in (0,1)$ for $\delta > 0$ and $\phi_0^* = 1$, and

$$\tilde{f}_\delta^*(s) = \frac{\kappa_\delta}{s-\kappa_\delta}\frac{\tilde{n}_\delta(\kappa_\delta)-\tilde{n}_\delta(s)}{1-\tilde{n}_\delta(\kappa_\delta)}. \tag{4.6.17}$$

And from (1.3.6), (4.6.17) is the Laplace transform of the generalized equilibrium density of $n_\delta(y)$ and thus, using (1.3.5), it follows that

$$f_\delta^*(y) = \frac{e^{\kappa_\delta y}\int_y^\infty e^{-\kappa_\delta x}n_\delta(x)dx}{\int_0^\infty e^{-\kappa_\delta y}\overline{N}_\delta(y)dy},$$

where from (1.3.1)

$$\begin{aligned}\overline{N}_\delta(y) &= \int_y^\infty n_\delta(x)dx = \int_y^\infty \frac{e^{-\delta \frac{x}{c}} k\left(\frac{x}{c}\right)}{c\tilde{k}(\delta)}dx \\ &= \frac{e^{-\delta\frac{y}{c}} T_\delta k\left(\frac{y}{c}\right)}{\tilde{k}(\delta)}.\end{aligned}$$

And thus, by rearranging (4.6.16), we obtain

$$\tilde{h}_{\delta,1}(s-\beta|0) = \phi_\delta^* \tilde{h}_{\delta,1}(s-\beta|0)\tilde{f}_\delta^*(s) + v_\delta^*(s),$$

which we invert to obtain

$$e^{\beta x} h_{\delta,1}(x|0) = \phi_\delta^* \int_0^x e^{-\beta(x-y)} h_{\delta,1}(x-y|0) f_\delta^*(y)dy + \tilde{k}(\delta) n_\delta(x),$$

which is a defective (proper) renewal equation for $e^{\beta x} h_{\delta,1}(x|0)$ for $\delta > 0$ ($\delta = 0$). And thus, using (1.3.18) of the Preliminaries section, the general solution to this defective renewal equation for $\delta > 0$ is given by

$$e^{\beta x} h_{\delta,1}(x|0) = \tilde{k}(\delta) n_\delta(x) + \frac{1}{1-\phi_\delta^*}\int_0^x \tilde{k}(\delta) n_\delta(y) g_\delta^*(x-y)dy, \tag{4.6.18}$$

where if $(f_\delta^*(y))^{*n}$ is the n-fold convolution of $f_\delta^*(y)$ with itself, then

$$g_\delta^*(y) = \sum_{n=1}^\infty (1-\phi_\delta^*)(\phi_\delta^*)^n (f_\delta^*(y))^{*n}.$$

Therefore, multiplying both sides of (4.6.18) by $e^{-\beta x}$ and using (4.6.15), we obtain the following equation for $h_{\delta,1}(x|0)$

$$\begin{aligned}h_{\delta,1}(x|0) &= e^{\beta x}\left\{\tilde{k}(\delta) n_\delta(x) + \frac{\tilde{k}(\delta)}{1-\phi_\delta^*}\int_0^x n_\delta(y) g_\delta^*(x-y)dy\right\} \\ &= e^{\beta x}\left\{\frac{e^{-\delta\frac{x}{c}}\tilde{k}\left(\frac{x}{c}\right)}{c} + \frac{1}{c(1-\phi_\delta^*)}\int_0^x e^{-\delta\left(\frac{y}{c}\right)} k\left(\frac{y}{c}\right) g_\delta^*(x-y)dy\right\}.\end{aligned}$$

And thus, using (4.6.13), (4.1.10), and (2.6.1), we can determine $h_{\delta,12}(x,y|u)$, the joint defective discounted density of U_{T-} and $|U_T|$ when claim sizes are exponentially distributed.

Chapter 5

Summary and Highlights

The summary and highlights for each chapter are given in the following subsections.

Chapter 2 The classical Gerber-Shiu Function

- In Section 2.1, the Gerber-Shiu function's value as a unified means of identifying ruin-related quantities was examined by considering special cases of the function for various penalty functions and values of δ.

- In Section 2.2, the form of the joint defective density of $(T, U_{T-}, |U_T|)$ was introduced and was determined to be defined differently depending on whether ruin occurs on the first claim or subsequent claims. In addition, discounted densities which do not have any meaningful probabilistic interpretations but are useful in our analysis were introduced.

- In Section 2.3, by conditioning on the first drop in surplus below its initial level, $m_{\delta,12}(u)$ was shown to satisfy a defective renewal equation; this result gives insight into the mathematical structure of the Gerber-Shiu function but does not yield information on the relationship between $h_{\delta,12}(x, y|0)$, ϕ_δ, and $f_\delta(y)$ with the interclaim time denstiy $k(t)$ or claim size density $p(y)$. A general solution to $m_{\delta,12}(u)$ was obtained which was shown to be closely related to an "associated" compound geometric r.v. Furthermore, we showed that the tail of this "associated" compound geometric r.v. (which is equal to the Laplace transform of the density of the time of ruin with $w_{12}(x, y) = 1$), also satisfies a defective renewal equation. When $w_{12}(x, y) = w_2(y)$,

$m_{\delta,12}(u)$ simplifies considerably to $m_{\delta,2}(u)$ since unlike $m_{\delta,12}(u)$, it only depends on ϕ_δ and the ladder height $f_\delta(y)$ and does not depend on $h_{\delta,12}(x,y|0)$, which is often more difficult to obtain. It was also shown that even though $m_{\delta,12}(u)$ and $m_{\delta,2}(u)$ are both functions of u, if we are able to identify $h_{\delta,12}(x,y|0)$ (i.e. from $m_{\delta,12}(0)$), then we can solve for both $m_{\delta,12}(u)$ and $m_{\delta,2}(u)$ in full generality.

- In Section 2.4, to determine a relationship between $m_{\delta,12}(u)$ with $k(y)$ and $p(y)$, we conditioned on the time and amount of the first claim to derive an integral equation satisfied by $m_{\delta,12}(u)$. It was shown that Lundberg's fundamental equation, and in particular its roots with positive real parts allows for the identification of unknown quantities which is generally needed to invert $m_{\delta,12}(u)$ (either numerically or analytically) under additional assumptions for $k(t)$ and $p(y)$.

- In Section 2.5, the classical Poisson model was then considered and by conditioning on the time and amount of the first claim, $m_{\delta,12}(u)$ was solved in full generality. We note that in this case, conditioning on the first drop in surplus was not needed.

- In Section 2.6, we considered an exponential claim size density and an arbitrary interclaim time to determine the form of the solution of $m_{\delta,12}(u)$ up to an unknown density which was derived in Chapter 4.

Chapter 3 The delayed renewal model

- In Section 3.1, the delayed renewal model was introduced which allows the time until the first claim to follow a (possibly) different p.d.f. $k_d(t)$ than the common density $k(t)$ followed by subsequent interclaim times. It solves the implicit assumption in the ordinary Sparre Andersen model that a claim occurs at time 0 which may not be true in settings we wish to consider. A brief discussion was provided on a special case of the delayed renewal model called the stationary renewal model where $k_d(t)$ is equal to the limiting distribution of the forward recurrence time which turns out to equal the equilibrium density of $k(t)$.

- In Section 3.2, since Gerber-Shiu analysis in the delayed renewal model aims to determine the relationship between $m_{\delta,12}(u)$ and $m^d_{\delta,12}(u)$, a functional relationship between $m_{\delta,12}(u)$ and $m^d_{\delta,12}(u)$ was determined for a class of first interclaim time densities which includes the equilibrium density and the exponential distribution. This is done using differential equations and Laplace transforms.

Chapter 4 A generalized Gerber-Shiu function

- In Section 4.1, we introduced a generalized Gerber-Shiu function where the penalty function includes 2 additional r.v.'s: the minimum surplus level before ruin X_T, and the surplus immediately after the second last claim before ruin R_{N_T-1} in the dependant Sparre Andersen model. This generalized Gerber-Shiu function allows for the study of r.v.'s such as the last interclaim time and the last ladder height which otherwise could not be studied using the classical definition of the function.

- In Section 4.2, similar to what was done Chapter 2, the form of the joint defective density of $(T, U_{T-}, |U_T|, R_{N_T})$ was introduced and was determined to be defined differently depending on whether ruin occurs on the first claim or subsequent claims. And again, corresponding discounted densities were introduced. Interestingly, despite containing a 4-variable penalty function, by conditioning on the first drop in surplus, $m_\delta(u)$ is shown to satisfy a defective renewal equation that is only dependent on the density of the 3 variables U_{T-}, $|U_T|$ and R_{N_T} (which does not involve X_T). We also considered the simplification of defective renewal equations satisfied by special cases of $m_\delta(u)$ for various choices of the penalty function. For example, if $w(x, y, z, v) = w_{23}(y, z)$, then the general solution to $m_{\delta,23}(u)$ only depends on ϕ_δ and $f_\delta(y)$ and thus the last ladder height may be determined from the generic ladder height $f_\delta(y)$.

- In Section 4.3, using the uniqueness property of Laplace transforms and the general solution for $m_\delta(u)$, the form of the joint defective discounted densities of $(T, U_{T-}, |U_T|, X_T, R_{N_T-1})$, $(T, U_{T-}, |U_T|, X_T)$, as well as the last ladder height were determined.

- In Section 4.4, the method of conditioning on the time and amount of the first claim was considered. Since the analysis presented from conditioning on the first drop in surplus provides insight into the mathematical structure of $m_\delta(u)$ but does not yield information on its relationship with $k(t)$ and $p(y)$, we usually condition on the time and amount of the first claim to obtain this information. However, because $m_\delta(u)$ now contains a penalty function involving X_T, it is not easy to use this conditioning argument. But since Section 4.2 demonstrated that $m_\delta(u)$ can be expressed in terms of the joint defective discounted density of $(T, U_{T-}, |U_T|, R_{N_T-1})$ given $u = 0$ which does not involve X_T, it suffices to consider conditioning on the time and amount of the first claim when $w(x, y, z, v) = w_{124}(x, y, v)$. And again, it was shown that Lundberg's fundamental equation, and in particular its roots with positive real parts allows for the identification of unknown quantities which is generally needed to invert $m_\delta(u)$ under additional assumptions for $k(t)$ and $p(y)$.

- In Section 4.5, the classical Poisson model was revisited and $m_\delta(u)$ was solved in full generality when the penalty function does not involve X_T. Using the uniqueness property of Laplace transforms, this was used to derive the joint defective discounted density of $(T, U_{T-}, |U_T|, R_{N_T-1})$ and the joint defective density of the last interclaim time before ruin and the claim causing ruin.

- In Section 4.6, the exponential claim size assumption was revisited and $m_\delta(u)$ was solved in full generality when a penalty function is assumed such that $m_\delta(u)$ becomes the Laplace transform of the joint defective discounted density of $(T, U_{T-}, |U_T|, X_T, R_{N_T-1})$. This was used to solve for the density of the last interclaim time as well as the joint defective discounted density of $(U_{T-}, |U_T|)$.

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