Optimal Portfolio Execution Strategies: Uncertainty and Robustness

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Optimal investment decisions often rely on assumptions about the models and their associated parameter values. Therefore, it is essential to assess suitability of these assumptions and to understand sensitivity of outcomes when they are altered. More importantly, appropriate approaches should be developed to achieve a robust decision. In this thesis, we carry out a sensitivity analysis on parameter values as well as model specification of an important problem in portfolio management, namely the *optimal portfolio execution problem*. We then propose more robust solution techniques and models to achieve greater reliability on the performance of an optimal execution strategy.

The *optimal portfolio execution problem* yields an execution strategy to liquidate large blocks of assets over a given execution horizon to minimize the mean of the execution cost and risk in execution. For large-volume trades, a major component of the execution cost comes from price impact. The optimal execution strategy then depends on the market price dynamics, the execution price model, the price impact model, as well as the choice of the risk measure.

In this study, first, sensitivity of the optimal execution strategy to estimation errors in the price impact parameters is analyzed, when a deterministic strategy is sought to minimize the mean and variance of the execution cost. An upper bound on the size of change in the solution is provided, which indicates the contributing factors to sensitivity of an optimal execution strategy. Our results show that the optimal execution strategy and the efficient frontier may be quite sensitive to perturbations in the price impact parameters.

Motivated by our sensitivity results, a regularized robust optimization approach is devised when the price impact parameters belong to some uncertainty set. We first illustrate that the classical robust optimization might be unstable to variation in the uncertainty set. To achieve greater stability, the proposed approach imposes a regularization constraint on the uncertainty set before being used in the minimax optimization formulation. Improvement in the stability of the robust solution is discussed and some implications of the regularization on the robust solution are studied.

Sensitivity of the optimal execution strategy to market price dynamics is then investigated. We provide arguments that jump diffusion models using compound poisson processes naturally model uncertain price impact of other large trades. Using stochastic dynamic programming, we derive analytical solutions for minimizing the expected execution cost under jump diffusion models and compare them with the optimal execution strategies obtained from a diffusion process.

A jump diffusion model for the market price dynamics suggests the use of Conditional Value-at-Risk (CVaR) as the risk measure. Using Monte Carlo simulations, a smoothing technique, and a parametric representation of a stochastic strategy, we investigate an approach to minimize the mean and CVaR of the execution cost. The devised approach can further handle constraints using a smoothed exact penalty function.

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Chapter 1

Introduction

1.1 Overview

On May 6, 2010, over about 30 minutes, the price of US stock market indices, stock-index futures, and options suddenly dropped by more than 5%, and rebounded rapidly. This short period of extreme intraday volatility (see plot (a) in Figure 1.1) is referred to as the *Flash Crash* (Kirilenko et al., 2010). In the course of this extreme market volatility, a large sell program was executed in the *June 2010 E-mini S&P 500* futures contract (see plot (b) in Figure 1.1). This trade has been considered a main trigger for the Flash Crash.

According to the joint report of Commodity Futures Trading Commission (CFTC) and Securities and Exchange Commission (SEC) (see e.g., (Kirilenko et al., 2010)) at 2:32 p.m. (EDT) on May 6, 2010, the mutual fund complex Waddell & Reed initiated a sell order of $75,000$ E-mini S&P 500 futures contracts (valued at approximately $4.1 billion) in order to hedge an existing equity position. The trader chose to execute this unusually large sell program via an automated execution algorithm. This algorithm was programmed to feed orders to target an execution rate of 9% of the trading volume calculated over the previous minute, with no regard to price or time. As a result, the sell program was executed extremely rapidly in just 20 minutes. Consequently, it incurred a large price impact and triggered a sharp price decline. In order to prevent further price change, at 2:45 p.m. (EDT), trading on the E-Mini paused for five seconds. After the market resumed trading, prices fluctuated for a few seconds. However, after that, price of the E-mini began a rapid ascent until the market got to the same price level where it was at 2:32 p.m. (EDT) when the rapid sell-off began.

As Kirilenko et al. (2010) report, during the early moments of the sell program's execution, some of the other investors provided liquidity. However, a few minutes later, they aggressively sold contracts and competed for liquidity with the selling program. In this way, they further amplified the price impact of this program. The combined price pressure from the sell program and other trades drove the price of the E-Mini down.

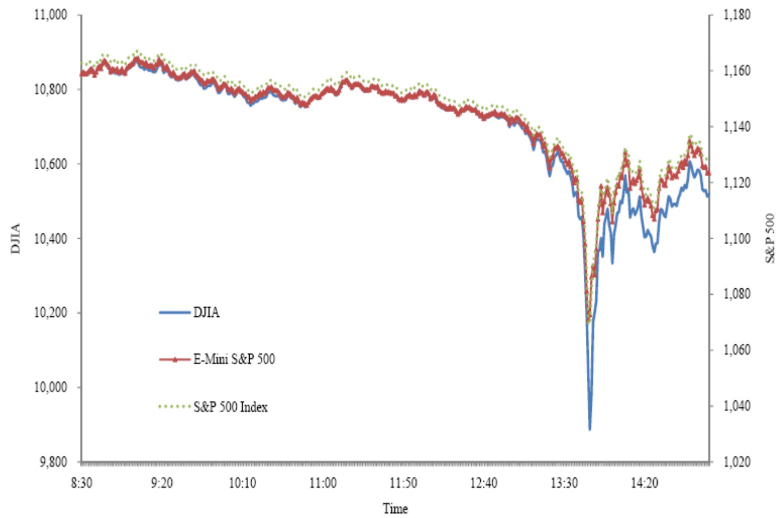
The Flash Crash incident clearly demonstrates the effect of the price impact from a trade execution. It also indicates the importance of an execution algorithm in cost control and the desirability of the execution algorithm to adapt to price changes.

Price impact often represents the largest portion of the total transaction cost in large trades. It is the difference between the execution price and what the market price would have been in the absence of the transaction (Torre and Ferrari, 1997). Price impact mainly consists of liquidity costs and information effects transmitted by the investor's own trade. Liquidity costs include additional prices an investor pays for immediate execution of the trade; this is often called the *temporary price impact* and only affects the execution price at the moment of trading. Furthermore the imbalance between supply and demand, due to the investor's trade, usually transmits information to the market which can move the future market price. For example, selling a large block of an asset may suggest that the seller believes the asset is overvalued. The effect of this information is often referred to as the *permanent price impact*. The total price impact is the sum of the temporary and permanent price impact.

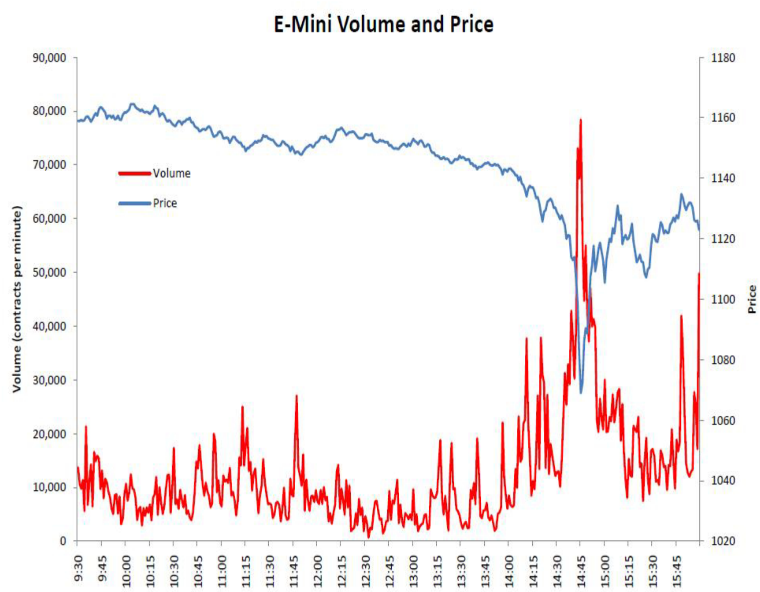
Characteristics of the temporary and permanent price impact, as well as their distinctions, have been addressed broadly in the literature, see, e.g., (Holthausen et al., 1990; Chan and Lakonishok, 1993, 1995; Almgren and Chriss, 2000/2001; Huberman and Stanzl, 2004) and the references herein. A common result of these studies is that the magnitude of the price impact is a function of the trade size. This function is called the *price impact function*. The expected price impact function is typically estimated through a linear or nonlinear regression based on the available historical transaction data, see, e.g., (Almgren et al., 2005).

Due to the dependence of the price impact on the trading volume, portfolio managers usually split a large trade into smaller partial orders, called *packages*. They then submit these partial orders gradually over several periods. Such a sequence of trades submitted over a finite trading horizon is called an *execution strategy*. The size of each package can still be large enough to induce a significant price change (Gabaix et al., 2006). There are many possible execution strategies to complete a desired trade, each of which is associated with an execution cost. Execution strategies which suggest to trade quickly over first periods incur a large execution cost and low risk. In contrast, more evenly paced execution strategies yield a smaller execution cost and higher risk. Therefore, a delicate balance must be struck between risk and cost. Given a permanent and temporary price impact model, along with the market price dynamics and execution price model, the problem of controlling risk and cost of executing a trade is called the *optimal portfolio execution problem*. Estimating and controlling execution cost and risk is essential to portfolio management, particularly for institutional traders, who often trade in large sizes.

Since the market price in the absence of an execution is not observable and must be predicted, price impact cannot be directly measured. Furthermore, future price impact needs to be forecasted based on historical execution prices. As a result estimating the price impact and the execution cost associated with an execution strategy is challenging, see, e.g., (Torre and Ferrari, 1997; Bessembinder, 2003). For example, to obtain a reliable estimate,



(a) U.S. equity indices on May 6, 2010



(b) June 2010 E-mini S&P 500 futures contracts: Trading volume and price

Figure 1.1: The Flash Crash incidence on May 6, 2010 (Kirilenko et al., 2010).

large statistical samples over an extended time horizon are required. However, as investors typically avoid costly trades, they are not included in the trade data. Therefore estimates from the trade data may systematically underestimate price impacts. Moreover, the level of information available for assets is unequal; information about heavily traded assets is more available than that of thinly traded assets. Furthermore, cross effect of price impacts among different assets should be regarded.

In addition, simultaneous estimation of the expected temporary and permanent price impact of concurrent trades make the estimation process complex. Thus modeling and estimating price impact of the investor's own trade as well as that of concurrent trades is a very difficult task. Therefore, the obtained estimations are likely to be erroneous. These estimation errors and their effect on the optimality of an obtained execution strategy must be investigated and taken into account when seeking an optimal strategy. The present thesis investigates this issue.

1.2 Contributions

In this thesis, we study sensitivity of an optimal execution strategy and its performance to variations in the price impact of the investor's own trade and the market price specification. We then propose more robust solution techniques and models to achieve greater reliability on the performance of the optimal strategy. The main contributions of the thesis are threefold.

Firstly, we analyze sensitivity of the optimal portfolio execution strategy and the efficient frontier to estimation errors in the impact matrices. Here, we assume that the execution strategy is deterministic and variance is used to measure the execution risk. Furthermore, permanent and temporary price impact functions are assumed to be linear, which are defined by *permanent* and *temporary impact matrices*. Our work in this area makes the following contributions, presented in Chapter 3 and also in (Moazeni et al., 2010):

- We analyze several mathematical properties of the optimal portfolio execution problem. In particular, we show that, instead of depending on the permanent and temporary impact matrices individually, the optimal execution strategy is determined by a combined impact matrix, which is a linear combination of the two matrices. We also prove that the minimum expected execution cost strategy is the naive execution strategy, when the permanent impact matrix is symmetric and the Hessian of the objective function is positive definite. Thus perturbations which maintain symmetry in the permanent impact matrix and positive definiteness in the Hessian of the objective function do not change the optimal strategy.

- We then provide upper bounds on the size of change in the optimal execution strategy in a more general setting. These upper bounds are in terms of change in the impact matrices, the eigenvalues of a block tridiagonal matrix defined by the combined impact matrix, the risk aversion parameter, and the covariance matrix. These upper bounds suggest the contributing factors to sensitivity of an optimal execution strategy.

- The change in the efficient frontier increases as the risk aversion parameter decreases for asymmetric perturbations. We consistently observe that imposing no-buying constraints for a sell execution or maintaining symmetry in the permanent impact matrix decreases sensitivity of the optimal execution strategy and the efficient frontier to perturbations.

Secondly, to take into account of estimation errors in the impact matrices and to obtain a more robust optimal execution strategy, we adopt a regularized robust optimization. Our contributions in this area, presented in Chapter 4 and also in (Moazeni et al., 2011b), are as follows:

- One of the concerns in robust optimization is determining the uncertainty set and potential instability of the approach to small changes in the uncertainty set. We illustrate potential instability of the robust optimization approach to perturbation in the uncertainty set for the optimal portfolio execution problem when impact matrices are uncertain.

- We then propose a *regularized robust optimization* approach for the optimal portfolio execution problem which offers better stability properties than the classical robust solution. Given any compact and convex uncertainty set for price impact matrices, we construct a *regularized uncertainty set* by including a *regularization constraint*. The regularization constraint is a linear matrix inequality defined by the Hessian of the objective function and a *regularization parameter*.

- Several implications of this regularization on the regularized robust solution and the robust efficient frontier are proved and computationally illustrated.

Thirdly, we analyze how the optimal execution strategy changes with the market price model specification, under the assumption that an optimal strategy can be dynamically adjusted. In particular, to obtain a more accurate estimation for the execution cost, we propose a jump model for the market price to capture permanent price impact of other concurrent large trades. A jump diffusion process for the market price can capture the characteristics of tail distributions due to price impact from institutional trades. Our main contributions in this area, which are presented in Chapter 5 and also in (Moazeni et al., 2011a), are as below:

- We provide arguments that compound jump diffusion processes naturally model uncertain price impact of other large trades. We explicitly model the jump component, using two compound Poisson processes where random jump amplitudes capture uncertain permanent price impact of other large buy and sell trades.

- Using stochastic dynamic programming, we derive analytical solutions for minimizing the expected execution cost under jump diffusion models. Our results indicate that, when the expected market price change is nonzero, likely due to large trades, assumptions on the market price model can have significant impact on the optimal execution strategy.

- We analyze qualitative and quantitative differences of the expected execution cost and risk between optimal execution strategies, determined under a multiplicative jump diffusion model and an additive jump diffusion model.

In addition, under a jump diffusion process for the market price, a risk measure which captures the fat tail characteristics of the execution cost distribution is more appropriate. However, computing optimal stochastic portfolio execution strategies under an appropriate risk consideration presents many computational challenges. To obtain a stochastic (market price dependent) execution strategy under a mean-CVaR objective, we devise a computational technique based on smoothing and parametric rules. Our main contributions in this area, presented in Chapter 6 and also in (Moazeni et al., 2011c), are summarized as below:

- We apply a smooth and parametric approach to minimize mean and Conditional Value-at-Risk (CVaR) of the execution cost, using Monte Carlo simulations. The proposed approach reduces computational complexity by smoothing the nondifferentiability arising from the simulation discretization and by employing a parametric representation of a stochastic strategy.

- We further handle constraints using a smoothed exact penalty function.

- Using a downside risk as an example, we illustrate that the proposed approach can be generalized to other risk measures. In addition, we analyze the effect of including risk consideration on the optimal execution strategy.

1.3 Structure of the Thesis

The thesis is structured as follows. Chapter 2 presents the mathematical formulation of the optimal portfolio execution problem and reviews several main results from the literature. Chapter 3 analyzes in detail sensitivity of the optimal execution strategy and its cost and risk to estimation errors in parameters of the price impact functions. Our proposed regularized robust optimization approach is described in Chapter 4. Sensitivity of the optimal execution strategy to the market price model is discussed in Chapter 5, where a jump model is also proposed to model uncertain price impact of other concurrent large trades. In Chapter 6 we propose a method for solving the stochastic mean-CVaR optimal portfolio execution problem, using smoothing and parametric rules. Main conclusions of the thesis and some directions for future work are presented in Chapter 7.

Chapter 2

The Optimal Portfolio Execution Problem

This chapter presents the mathematical formulation for the optimal portfolio execution problem. Since the mathematical analysis for buying and selling are similar, without loss of generality, here we assume that the investor's goal is to liquidate blocks of assets. Orders are placed before price changes are known, and only market orders are considered.

2.1 Mathematical Formulation

Assume that an investor plans to liquidate his holdings in m assets during N periods in the time horizon T, $t_0 = 0 < t_1 < \cdots < t_N = T$, where $\tau \stackrel{\text{def}}{=} t_k - t_{k-1} = \frac{T}{N}$ for $k = 1, 2, \ldots, N$. The investor's position at time t_k is denoted by the m-vector $x_k = (x_{1k}, x_{2k}, \ldots, x_{mk})^T$, where x_{ik} is the investor's holding in the ith asset at period k. The investor's initial position is $x_0 = \bar{S}$, and his final position x_N equals 0, which guarantees complete liquidation by time T. The amount of trading in the kth period, denoted by an m-vector n_k, is the difference between positions at two consecutive times t_{k-1} and t_k, where

$$n_k = x_{k-1} - x_k, \qquad k = 1, 2, \ldots, N. \tag{2.1.1}$$

Negative n_{ik} implies that the ith asset is bought between t_{k-1} and t_k. We refer to a sequence $\{n_k\}_{k=1}^N$ satisfying $\sum_{k=1}^N n_k = \bar{S}$ as an *execution strategy*.

Let $\tilde{P}_k$ be the execution price of one unit of assets at time t_k for $k = 1, 2, \ldots, N$. Due to price volatility and price impacts, $\tilde{P}_k$ is not deterministic over the execution horizon. Given the temporary price impact function $h(\cdot)$, we assume that the m-vector unit execution price $\tilde{P}_k$ is given by

$$\tilde{P}_k = P_{k-1} - h\left(\frac{n_k}{\tau}\right), \quad k = 1, 2, \ldots, N, \tag{2.1.2}$$

where the m-vector P_k denotes market price per share at time t_k. The deterministic initial market price is denoted by P_0. Similar to (Almgren and Chriss, 2000/2001), we assume that the permanent price impact of the decision maker's trade is a deterministic function of the trading rate, denoted by $g(\cdot)$, and

$$P_k = \mathcal{F}_{k-1}(P_{k-1}) - \tau g\left(\frac{n_k}{\tau}\right), \quad k = 1, 2, \ldots, N-1, \tag{2.1.3}$$

where $\mathcal{F}_{k-1}(P_{k-1})$ represents the market price at time t_k when the decision maker does not trade in $(t_{k-1}, t_k]$, e.g., $\mathcal{F}_{k-1}(P_{k-1}) = P_{k-1} + \Sigma\xi_k$ with ξ_k a multi-variate standard normal, and Σ the volatility matrix of the asset prices. Random price $\mathcal{F}_{k-1}(P_{k-1})$ at time t_k can also be specified by other models which can correspond to a jump diffusion model (Merton, 1976) or a stochastic volatility model (Heston, 1993). The permanent price impact function captures the permanent price impact per unit of time and multiplying by τ in (2.1.3) represents the permanent impact within $(t_{k-1}, t_k]$.

The functions $g(\cdot)$ and $h(\cdot)$ measure the expected permanent price impact and temporary price impact, respectively. There are only a few studies about the structure of the price impact functions. Theoretical analysis in (Brown et al., 2004) indicates that the temporary price impact function is monotonically increasing in size, but it is approximately a square root over small trades and logarithmic for large trades. Huberman and Stanzl (2004) show that when the price impact of trades is assumed to be time-independent, only linear permanent price impact functions can support viable market prices; nonlinear permanent price impact functions can give rise to a sequence of trades that generates infinite expected profits per unit of risk. However, the temporary price impact function can be of a more general form. It has been frequently stated in the literature that the temporary price impact function should be concave, see, e.g., (Almgren et al., 2005) and the references therein. Moreover, empirical studies have argued that permanent price impact costs for buying and selling are different, and there is an asymmetry in the overall impact of buys and sells; the permanent price impact of buys is larger than that of sells, see, e.g., (Chan and Lakonishok, 1993; Saar, 2001) and the references therein.

Literature on the optimal portfolio execution problem frequently assumes a linear price impact model, see, e.g., (Bertsimas and Lo, 1998; Bertsimas et al., 1999; Almgren and Chriss, 2000/2001; Huberman and Stanzl, 2004). Throughout the thesis, we consider linear time-independent price impact functions in which price impacts at each interval $(t_{k-1}, t_k]$ are assumed to be proportional to the trading rate $v = \frac{n_k}{\tau}$:

$$\begin{aligned} g(v) &= Gv, \\ h(v) &= Hv, \end{aligned} \tag{2.1.4}$$

where H and G are m-by-m matrices, referred to as the temporary and permanent impact matrices. These impact matrices H and G are the expected price depressions caused by trading assets at a unit rate.

We assume model (2.1.4) mainly for the following reasons. First, it captures both the permanent and temporary price impacts of large trades, while being simple enough to allow for a mathematical analysis. Second, it has been used in many theoretical studies, e.g., (Almgren and Chriss, 2000/2001; Almgren and Lorenz, 2007; Carlin et al., 2007; Schied and Schöoneborny, 2008).

Using the impact matrices H and G, we define a matrix Θ, frequently used later on:

$$\Theta \overset{\text{def}}{=} \frac{1}{\tau}\left(H + H^T\right) - G. \tag{2.1.5}$$

Subsequently, we refer to Θ as the *combined impact matrix*.

Given an execution strategy $\{n_k\}_{k=1}^N$, the total amount received at the end of the time horizon T equals $\sum_{k=1}^N n_k^T \tilde{P}_k$. The difference between this quantity and the value of an ideal benchmark trade is the *execution cost* (Almgren, 2010). The benchmark is commonly taken as the value of the portfolio at the arrival price P_0. Thus, the *execution cost* of a trade is often defined as

$$P_0^T \bar{S} - \sum_{k=1}^N n_k^T \tilde{P}_k.$$

Market volatility and uncertainty in price impact of trades over the trading horizon T make predicting the exact value of the execution cost corresponding to an execution strategy impossible. Whence, the execution cost is uncertain. To decrease risk one can trade more rapidly, which will increase the execution cost due to the limited liquidity of markets. Thus, there is a tradeoff between risk and the cost of trading. This tradeoff can be expressed in a mean and risk setting with a risk aversion parameter $\mu \geq 0$, as below:

$$\begin{aligned} \min_{n_1, n_2, \ldots, n_N} \quad & \mathbf{E}\left(P_0^T \bar{S} - \sum_{k=1}^N n_k^T \tilde{P}_k\right) + \mu \cdot \Psi\left(P_0^T \bar{S} - \sum_{k=1}^N n_k^T \tilde{P}_k\right) \\ \text{s.t.} \quad & \sum_{k=1}^N n_k = \bar{S}, \end{aligned} \tag{2.1.6}$$

where $\mathbf{E}(\cdot)$ denotes the expectation of a random variable and $\Psi(\cdot)$ is some risk measure. The risk aversion parameter μ expresses the tolerance level of the decision maker to risk. A large value of μ corresponds to the investor's small tolerance to risk.

Alternatively problem (2.1.6) can be formulated in terms of the positions $\{x_k\}_{k=0}^N$ as in the following:

$$\begin{aligned} \min_{x_0, x_1, \ldots, x_N} \quad & \mathbf{E}\left(P_0^T \bar{S} - \sum_{k=1}^N (x_{k-1} - x_k)^T \tilde{P}_k\right) + \mu \cdot \Psi\left(P_0^T \bar{S} - \sum_{k=1}^N (x_{k-1} - x_k)^T \tilde{P}_k\right) \\ \text{s.t.} \quad & x_0 = \bar{S}, \\ & x_N = 0. \end{aligned} \tag{2.1.7}$$

We refer to solutions of problems (2.1.6) and (2.1.7) as an *optimal execution strategy* and *optimal positions*, respectively.

Problems (2.1.6) or (2.1.7) may include some other constraints to reflect regulation constraints or the investor's preferences. For example, to forbid purchases in a sell execution, the constraints $n_k \geq 0$ or $x_{k-1} \geq x_k$ can be included in problems (2.1.6) and (2.1.7), respectively.

The most obvious strategy is to sell at a constant rate over the whole liquidation period. We refer to this strategy as the *naive (execution) strategy*,

$$n_k = \frac{\bar{S}}{N}, \quad k = 1, 2, \cdots, N. \tag{2.1.8}$$

This execution strategy is an important benchmark since it is optimal under some simple model assumptions.

The optimal portfolio execution problem, with variance as the risk measure, shares a similar mathematical structure with the traditional multi-period mean-variance portfolio optimization problem when a transaction cost is associated with rebalancing the portfolio. In both problems, given some fixed number of investment periods and the initial portfolio, the goal is to produce a sequence of trades that maximizes some expected utility of the final wealth and minimizes risk. However, in the classical multi-period mean-variance portfolio optimization problem, permanent price impacts of the trades are not modeled. Moreover, the optimal portfolio execution problem includes a specific constraint on the investor's portfolio position at the end of the trading time horizon.

The similarity of the optimal portfolio execution problem to the mean-variance portfolio optimization problem motivates the notion of an efficient frontier in the context of the optimal portfolio execution problem. A feasible execution strategy is *efficient* if it has the least expected execution cost among all execution strategies with the same variance of the execution cost. The collection of efficient execution strategies yield the *(mean-variance) efficient frontier* of the execution strategy universe. The notion of efficient frontier can also be extended to other risk measures.

Next we review the recent literature on the optimal portfolio execution problem.

2.2 Related Literature

There is a large body of literature on the optimal portfolio execution problem, see, e.g., (Huberman and Stanzl, 2004) or (Almgren, 2010), and references therein.

The effect of large trades on a market have been studied empirically by several authors starting with (Kraus and Stoll, 1972), who utilized data from the New York Stock Exchange, see also (Chan and Lakonishok, 1993, 1995; Keim and Madhavan, 1996, 1997; Chiyachantana et al., 2004). They generally find that execution of large orders exerts both permanent and temporary price impact.

Bertsimas and Lo (1998) study minimizing the expected execution cost of acquiring large amounts of a single asset for several price impact models. Using stochastic dynamic programming, they provide closed form representations for the optimal execution strategy. Their work is extended to portfolios of assets in (Bertsimas et al., 1999).

More advanced execution strategies under a risk consideration are presented in (Almgren and Chriss, 1999, 2000/2001), where a weighted sum of mean and variance of the execution cost is minimized. Almgren and Chriss (2000/2001) assume that permanent and temporary price impact functions are linear and market price evolves according to an additive diffusion process. Their approach assumes that the optimal execution strategy does not depend on the market price and is deterministic. This work is extended to nonlinear price impact functions in (Almgren, 2003). Linear price impact functions and the optimization framework presented in (Almgren and Chriss, 2000/2001) are adopted and further studied by Jondeaua et al. (2007). They use the tick-by-tick data from Paris Stock Exchange to estimate (deterministic but time-varying) temporary and permanent price impact functions. They then obtain the optimal execution strategy and study some characteristics of the solution as a function of the permanent price impact, the temporary price impact, and standard deviation of the underlying market price process.

A potential disadvantage of the Almgren-Chriss approach is that trading positions do not dynamically depend on the asset price movement. Assuming a slightly different price dynamics than (Almgren and Chriss, 2000/2001), Huberman and Stanzl (2005) provide a closed-form solution for the dynamic execution strategy which minimizes mean and variance of the execution cost. The use of dynamic programming to minimize mean and variance of the execution cost is further studied in (Moazeni, 2011). There, a sufficient condition on the existence of the dynamic programming equation for the optimal portfolio execution problem with a mean-variance objective is provided.

Optimal execution in a continuous-time framework has also been studied, mostly to trade a single asset. Related work includes (Subramanian and Jarrow, 2001; He and Mamaysky, 2005; Vath et al., 2007; Forsyth, 2010; Draviam et al., 2010). (Ting et al., 2007) extends the work (Bertsimas and Lo, 1998) to the continuous-time setting by providing closed form solution for minimizing the expected execution cost through solving a Hamilton-Jacobi-Bellman equation. They solely consider temporary price impact and assume that market price follows a Brownian motion. Their optimal execution strategies confirm that greater liquidity requires more gradual liquidation. Infinite-horizon optimal execution cost problem in the continuous-time setting, with a linear price impact model and an additive price dynamics, is considered in (Schied and Schöoneborny, 2008). They assume that the investor seeks to maximize the expected utility of her cash position after liquidation of holdings in an asset. A wide range of utility functions have been considered. To determine the adaptive execution strategy, a stochastic control approach is used to characterize the value function and a solution. A mean-variance framework of the problem in the continuous-time setting to execute a single asset in the absence of permanent price impact is addressed by Almgren (2009). In this work, variance is approximated with an expected value, which then allows the use of stochastic dynamic programming. A similar setting with a diffusion market price process and a linear temporary price impact function is considered by Almgren and Lorenz (2007). They propose an adaptive strategy to minimize mean and variance of the execution cost.

The derivation of the adaptive strategy relies on a parameter which depends on the price impact parameter, risk aversion parameter, and volatility. At each time, the static strategy to minimize mean and variance of the execution cost is computed and its objective function value is evaluated. Depending on the interval which contains the objective function value, the parameter is readjusted and the optimal strategy corresponding to the new parameter value is computed.

To take into account price impact of other market participants' trading activities, Almgren and Lorenz (2006) present a single asset market price model for the optimal execution cost problem. The model is a Brownian motion with drift, where the drift factor follows a normal distribution and is to capture trading targets of other market participants. They then provide a closed form representation for the optimal execution strategy to minimize the expected execution cost.

Optimal execution by modeling the underlying limit order book has also been studied previously, see, e.g., (Alfonsi et al., 2010) and the references therein.

In the aforementioned literature, different models and parameter values are assumed. It is important to analyze sensitivity of the optimal execution strategy and its performance to these assumptions. It is further beneficial to devise methods to alleviate this sensitivity. This is indeed the focus of this thesis.

Chapter 3

Optimal Execution Strategies and Sensitivity to Price Impact Matrices

3.1 Introduction

Sensitivity of mean-variance efficient portfolios to estimation errors in the expected returns and the covariance matrix has been widely studied in the literature, see, e.g., (Jobson and Korkie, 1980; Kallberg and Ziemba, 1984; Frost and Savarino, 1988; Michaud, 1989; Best and Grauer, 1991; Broadie, 1993; Chopra and Ziemba, 1993; Chen and Zhao, 2003). However, to the best of our knowledge, sensitivity of the optimal execution strategy and efficient frontier to estimation errors in the permanent and temporary price impact functions has not been addressed yet. For the mean-variance portfolio optimization, the difficulty in accurate estimation of the expected rate of returns is well known. As discussed in Chapter 1, estimating price impacts is rather challenging. Furthermore, literature dealing with ways to improve the estimation of price impact functions is scarce. Therefore, it is important to understand the sensitivity of an optimal execution strategy and the efficient frontier to errors in parameters of the price impact functions. Recognizing the effect of estimation errors may provide more realistic expectations about the future performance of a chosen execution strategy.

A common approach to investigate the effect of estimation errors is to interpret the errors as perturbations to the data and to perform a sensitivity analysis on the solution. Sensitivity discussions are essential in model validation. In this chapter, we carry out sensitivity analysis to study the effect of estimation errors in the impact matrices on the optimal execution strategy and the efficient frontier.

In addition to the impact matrices, the structure of the optimal portfolio execution problem in the mean-variance setting depends on the covariance matrix of asset prices. Estimation errors can also occur in the covariance matrix. However, in contrast to the impact matrices, there is an extensive literature on techniques to improve the estimation of the covariance matrix, see, e.g., (Disatnik and Benninga, 2007) and the references therein. Furthermore, most of the recent literature on addressing estimation risks in the mean-variance portfolio optimization focus exclusively on the impact of estimation errors in the mean return by taking the covariance matrix as known, see, e.g., (TerHorst et al., 2006; Garlappi et al., 2007; Antoine, 2008). Therefore, in this study, we assume that the covariance matrix is given, and we mainly focus on sensitivity of the optimal execution strategy and the efficient frontier to perturbations in the impact matrices. As in (Almgren and Chriss, 2000/2001), our discussion in this chapter assumes that optimal strategy is deterministic and market price evolves according to an additive diffusion process.

We first show that the optimal execution strategy depends on the combined impact matrix Θ given in (2.1.5), rather than H and G individually. This suggests that one may want to estimate Θ directly in order to determine an optimal execution strategy. In addition, we prove that when the permanent impact matrix is symmetric and the combined impact matrix is positive definite, a unique optimal execution strategy exists for any positive risk aversion parameter.

We discuss some cases in which the optimal execution strategy is insensitive to perturbations in the impact matrices. In particular, we prove that, for any symmetric permanent impact matrix and positive definite matrix Θ, the naive execution strategy minimizes the expected execution cost. Therefore, as long as the symmetry of the permanent impact matrix G is maintained, the minimum expected execution cost strategy is not sensitive to perturbations.

We then analyze sensitivity of the optimal execution strategy when the risk aversion parameter is positive or the permanent impact matrix is asymmetric. Since the impact matrices appear both in the Hessian matrix and the linear coefficient of the quadratic objective function for the optimal portfolio execution problem, the optimal execution strategy in general may be quite sensitive to their perturbations. We provide upper bounds on the size of change in the optimal execution strategy in terms of the change in the impact matrices and a magnification factor. These upper bounds explicitly specify which factors may magnify the effect of estimation errors on the optimal execution strategy. For example, following the established upper bounds, it can be easily seen that the change in the optimal execution strategy decreases when a large risk aversion parameter is chosen. In general, upper bounds for the magnification factors depend on the eigenvalues of the block tridiagonal Hessian matrix defined by the covariance matrix, the impact matrices, and the risk aversion parameter. The upper bounds can be simplified when the permanent impact matrix and its perturbation are symmetric. Under these assumptions and the additional assumption of a positive risk aversion parameter, the magnification factor becomes small when the minimum eigenvalue of *either* the covariance matrix *or* the combined impact matrix Θ is large. When both of these minimum eigenvalues are small, the optimal execution strategy may be very sensitive to the estimation errors. These results implicitly evince that the optimal execution strategy for trading a single asset is expected to be less sensitive than the optimal execution strategy for trading portfolios.

We also illustrate sensitivity of the efficient frontier to perturbations in the impact matrices through simulations. Our computational results demonstrate that, when buying is prohibited for a sell execution, the optimal execution strategy and efficient frontier are less sensitive than the case when buying is permitted. Indeed, when buying is allowed, the efficient frontier can be quite sensitive to perturbations in the impact matrices. In particular, changes in the efficient frontier can become very large for a small risk aversion parameter if perturbations in the permanent impact matrix are asymmetric. We also observe that, for the minimum variance execution cost strategies, estimation errors can lead to large variations in the expected execution cost. Finally, we compare the effect of estimation errors in the covariance matrix on the optimal execution strategy and the efficient frontier with their sensitivity to perturbations in the impact matrices. Our simulations indicate that perturbations in the impact matrices affect both the optimal execution strategy and the efficient frontier more prominently than perturbations in the covariance matrix, particularly when the risk aversion parameter is small. Our sensitivity analysis is restricted to the optimal portfolio execution problems and perturbations for which both the original problem and perturbed problems have unique solutions.

The presentation of this chapter is as follows. The mathematical formulation of the assumed optimal portfolio execution problem is described in §3.2. We discuss, in §3.3, sensitivity of the optimal execution strategy to perturbations in the impact matrices and provide upper bounds on the size of its change. Simulations are carried out in §3.4 to illustrate the effect of perturbations in the impact matrices on the efficient frontier and optimal execution strategy. Concluding remarks are given in §3.5.

3.2 Mean-Variance Optimal Execution Strategy

In this section, we study the mean-variance formulation of the optimal portfolio execution problem, i.e., variance $\mathbf{Var}(\cdot)$ is used as the risk measure $\Psi(\cdot)$ in problems (2.1.6) and (2.1.7). We further assume that $\mathcal{F}_{k-1}(P_{k-1})$ in equation (2.1.3) follows a discrete arithmetic random walk:

$$\mathcal{F}_{k-1}(P_{k-1}) = P_{k-1} + \tau^{1/2}\Sigma\xi_k.$$

Here $\xi_k = (\xi_{1k}, \xi_{2k}, ..., \xi_{lk})^T$ represents an l-vector of independent standard normals and Σ is an $m \times l$ volatility matrix of the asset prices. Thus the market price is given by:

$$P_k = P_{k-1} + \tau^{1/2}\Sigma\xi_k - \tau g\left(\frac{n_k}{\tau}\right). \tag{3.2.1}$$

For any execution strategy $\{n_k\}_{k=1}^N$ and its associated positions $\{x_k\}_{k=0}^N$, applying the execution price model (2.1.2) and market price dynamics (3.2.1), we obtain

$$\sum_{k=1}^{N} n_k^T \tilde{P}_k = \bar{S}^T P_0 + \sum_{k=1}^{N} \tau^{\frac{1}{2}} x_k^T \Sigma\xi_k - \tau \sum_{k=1}^{N} x_k^T g\left(\frac{n_k}{\tau}\right) - \sum_{k=1}^{N} n_k^T h\left(\frac{n_k}{\tau}\right). \tag{3.2.2}$$

Thus the variance of the execution cost equals (recall that n_k is assumed to be deterministic)

$$\mathbf{Var}\left(P_0^T\bar{S}-\sum_{k=1}^{N}n_k^T\tilde{P}_k\right)=\tau\sum_{k=1}^{N}x_k^TCx_k, \tag{3.2.3}$$

where $C=\Sigma\Sigma^T$ is the $m\times m$ symmetric positive semidefinite covariance matrix of asset prices. Notice that the variance of the execution cost, under the assumptions of this section, does not depend on the impact matrices.

From (3.2.2), when the price impact model (2.1.4) is applied, the expected execution cost can be expressed as below:

$$\begin{aligned}
&\mathbf{E}\left(P_0^T\bar{S}-\sum_{k=1}^{N}n_k^T\tilde{P}_k\right)=\sum_{k=1}^{N}x_k^TGn_k+\sum_{k=1}^{N}n_k^TH\frac{n_k}{\tau}\\
&=\sum_{k=1}^{N}x_k^TG(x_{k-1}-x_k)+\sum_{k=1}^{N}\frac{1}{\tau}\left(x_k-x_{k-1}\right)^T H\left(x_k-x_{k-1}\right)\\
&=\sum_{k=1}^{N}x_k^T\left(\frac{H}{\tau}-G\right)x_k+\sum_{k=1}^{N}x_k^T\left(G-\frac{H}{\tau}\right)x_{k-1}-\sum_{k=1}^{N}x_{k-1}^T\frac{H}{\tau}x_k+\sum_{k=2}^{N}x_{k-1}^T\frac{H}{\tau}x_{k-1}+x_0^T\frac{H}{\tau}x_0\\
&=\frac{1}{\tau}\bar{S}^TH\bar{S}-\frac{1}{\tau}x_N^THx_N+\sum_{k=1}^{N}x_k^T\left(\frac{2}{\tau}H-G\right)x_k+\sum_{k=1}^{N}x_k^T\left(G-\frac{1}{\tau}\left(H+H^T\right)\right)x_{k-1}.
\end{aligned} \tag{3.2.4}$$

Using equation (2.1.5), we obtain

$$\begin{aligned}
&\mathbf{E}\left(P_0^T\bar{S}-\sum_{k=1}^{N}(x_{k-1}-x_k)^T\tilde{P}_k\right)+\mu\cdot\mathbf{Var}\left(P_0^T\bar{S}-\sum_{k=1}^{N}(x_{k-1}-x_k)^T\tilde{P}_k\right)\\
&=\frac{1}{\tau}\bar{S}^TH\bar{S}-\frac{1}{\tau}x_N^THx_N+\sum_{k=1}^{N}x_k^T\left(\frac{2}{\tau}H-G+\mu\tau C\right)x_k-\sum_{k=1}^{N}x_k^T\Theta x_{k-1}\\
&=\frac{1}{\tau}\bar{S}^TH\bar{S}-\frac{1}{\tau}x_N^THx_N+\sum_{k=1}^{N}\frac{1}{2}x_k^T\left(\Theta+\Theta^T+2\mu\tau C\right)x_k-\sum_{k=1}^{N}\frac{1}{2}\left(x_k^T\Theta x_{k-1}+x_{k-1}^T\Theta^Tx_k\right).
\end{aligned}$$

Eliminating the constant term $\frac{1}{\tau}\bar{S}^TH\bar{S}$ from the objective function and explicitly imposing $x_N=0$ and $x_0=\bar{S}$, problem (2.1.7) is reduced to the following quadratic minimization problem:

$$\min_{z\in\mathcal{R}}\ \frac{1}{2}z^TW(H,G,\mu)z+b^T(H,G)z. \tag{3.2.5}$$

The $m(N-1)\times m(N-1)$ symmetric tridiagonal block matrix $W(H,G,\mu)$ and the $m(N-1)$-

vectors $b(H,G)$ are defined as follows:

$$W(H,G,\mu) \stackrel{\text{def}}{=} \begin{pmatrix} L & -\Theta^T & 0 & \dots & 0 \\ -\Theta & L & -\Theta^T & \dots & 0 \\ 0 & -\Theta & L & & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & L \end{pmatrix}, \qquad b(H,G) \stackrel{\text{def}}{=} \begin{pmatrix} -\Theta\bar{S} \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

where $L = (\Theta + \Theta^T) + 2\mu\tau C$. The $m(N-1)$-vector z of decision variables is as below:

$$z \stackrel{\text{def}}{=} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{N-1} \end{pmatrix}.$$

In problem (3.2.5), $\mathcal{R}$ denotes a subset of $\mathbb{R}^{m(N-1)}$ corresponding to feasible execution strategies.

When purchasing is allowed during selling and no other constraint is imposed, $\mathcal{R} = \mathcal{R}_0 \stackrel{\text{def}}{=} \mathbb{R}^{m(N-1)}$. Thus an optimal execution strategy can be obtained from the following unconstrained quadratic programming problem:

$$\min_{z \in \mathcal{R}_0} \quad \frac{1}{2} z^T W(H,G,\mu) z + b^T(H,G) z. \tag{3.2.6}$$

When $W(H,G,\mu)$ is not positive semidefinite, problem (3.2.6) has no local minima and is unbounded below. If $W(H,G,\mu)$ is positive semidefinite but singular, problem (3.2.6) has either no solution or infinitely many solutions. Problem (3.2.6) has a unique minimizer if and only if $W(H,G,\mu) \succ 0$. The unique minimizer in this case is $z^* = -W^{-1}(H,G,\mu)b(H,G)$.

The set $\mathcal{R}$ may include constraints on the asset positions. For example, a liquidation plan may prohibit purchasing over the trading horizon. In this case, the feasible set $\mathcal{R} = \mathcal{R}_c$, where

$$\mathcal{R}_c \stackrel{\text{def}}{=} \left\{ \left(x_1^T, \cdots, x_{N-1}^T\right) : \ \bar{S} \geq x_1, \ x_{k-1} \geq x_k \ \text{ for } \ k = 2, \cdots, N-1, \ x_{N-1} \geq 0 \right\}. \tag{3.2.7}$$

To simplify the representation of these constraints, we introduce the sequence of square matrices $\{J_k\}_{k=1}^{N-1}$, where $J_1 = (1)$ and

$$J_2 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \ J_3 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \ J_k = \begin{pmatrix} J_{k-1} & \begin{pmatrix} 0 \\ \cdots \\ 0 \\ -1 \end{pmatrix} \\ 0 & 1 \end{pmatrix} \quad \text{for } k \geq 2. \tag{3.2.8}$$

Thus in the presence of these constraints, the optimal portfolio execution problem is reduced to the following problem:

$$\begin{aligned} \min_{z \in \mathbb{R}^{m(N-1)}} \quad & \frac{1}{2} z^T W(H, G, \mu) z + b^T(H, G) z, \\ \text{s.t.} \quad & (-e_1^T \otimes I_m) z \geq -\bar{S}, \\ & (J_{N-1} \otimes I_m) z \geq 0. \end{aligned} \tag{3.2.9}$$

Here $\otimes$ denotes the Kronecker product of two matrices.

The quadratic programming problem (3.2.9) is convex if and only if $W(H, G, \mu) \succeq 0$. Since the set of feasible strategies $\mathcal{R}_c$ is compact, the Weierstrass Theorem along with the continuity of the objective function of problem (3.2.9), implies that problem (3.2.9) has a global minimizer. Moreover, positive definiteness of $W(H, G, \mu)$ guarantees that the global minimizer is unique.

Similar to the mean-variance portfolio optimization problem, the optimal portfolio execution problem (3.2.9) is a quadratic programming problem. However, in contrast to the mean-variance portfolio optimization problem in which the expected return appears only in the linear term of the quadratic objective, in the optimal portfolio execution problem (3.2.9) the impact matrices appear in both the quadratic term and the linear term of the objective function in a structured fashion. Therefore sensitivity analysis restricted to perturbations in the linear term of the quadratic objective function, see, e.g., (Best and Grauer, 1991), is not applicable in this context. It is necessary to explicitly analyze the effect of estimation errors in the impact matrices for the optimal portfolio execution problem.

Since one expects a unique optimal execution strategy under a reasonable price impact model (whether buying is permitted or not), assuming $W(H, G, \mu)$ is positive definite seems appropriate. This assumption guarantees that both problems (3.2.6) and (3.2.9) have unique solutions.

The representation of problem (3.2.9) indicates that the optimal execution strategy only depends on the combined impact matrix Θ rather than matrices H and G individually.

In the following lemma, we show that when the permanent impact matrix G is symmetric, positive definiteness of Θ is a necessary and sufficient condition for the positive definiteness of $W(H, G, 0)$. Symmetric permanent impact matrices, e.g., diagonal matrices, have been used in the literature on the optimal portfolio execution problem, see, e.g., (Almgren and Chriss, 2000/2001; Almgren et al., 2005).

Lemma 3.2.1. *Let the permanent impact matrix G be symmetric. Then $W(H, G, 0) \succeq 0$ if and only if $\Theta \succeq 0$. In particular, $W(H, G, 0) \succ 0$ if and only if $\Theta \succ 0$.*

Proof. When G is symmetric, $\Theta = \Theta^T$. For any real $m(N-1)$-vector $h = (h_1^T, h_2^T, ..., h_{N-1}^T)^T$,

$$\begin{aligned} h^T W(H,G,0)h &= -\sum_{i=1}^{N-2} h_{i+1}^T \Theta h_i + \sum_{i=1}^{N-1} h_i^T \Theta h_i + \sum_{i=1}^{N-1} h_i^T \Theta h_i - \sum_{i=1}^{N-2} h_i^T \Theta h_{i+1} \\ &= h_1^T \Theta h_1 - \sum_{i=1}^{N-2} h_{i+1}^T \Theta (h_i - h_{i+1}) + \sum_{i=1}^{N-2} h_i^T \Theta (h_i - h_{i+1}) + h_{N-1}^T \Theta h_{N-1} \\ &= h_1^T \Theta h_1 + \sum_{i=1}^{N-2} (h_{i+1} - h_i)^T \Theta (h_{i+1} - h_i) + h_{N-1}^T \Theta h_{N-1}. \end{aligned} \tag{3.2.10}$$

If $\Theta \succeq 0$, each term in (3.2.10) is nonnegative. Thus $h^T W(H,G,0)h \geq 0$ and consequently $W(H,G,0) \succeq 0$. The other direction of the statement follows from the fact that 2Θ is a leading principle submatrix of $W(H,G,0)$. Therefore (symmetric) positive semidefiniteness of $W(H,G,0)$ implies $\Theta \succeq 0$. □

For any $\mu \geq 0$, $W(H,G,\mu) = 2\mu\tau I_{N-1} \otimes C + W(H,G,0)$. Hence, when the permanent impact matrix G is symmetric and $\Theta \succeq 0$, Lemma 3.2.1 implies that the matrix $W(H,G,\mu) \succeq 0$ for any $\mu \geq 0$.

Thus, whether buying is permitted or not, the uniqueness of the optimal execution strategy for any risk aversion parameter $\mu \geq 0$ is guaranteed when $\Theta \succ 0$ and $G = G^T$. In the next proposition, we show that under these assumptions, the execution strategy that minimizes the expected execution cost ($\mu = 0$) can be found explicitly. Surprisingly, it does not depend on the values of the impact matrices.

Proposition 3.2.1. *Let the permanent impact matrix G be symmetric. Then the naive execution strategy $n_k^* = \frac{\bar{S}}{N}$, $k = 1, 2, \ldots, N$, is the unique execution strategy that minimizes the expected execution cost (solves problems* (3.2.6) *and* (3.2.9) *with $\mu = 0$) if and only if $\Theta \succ 0$.*

Proof. The assumption $G^T = G$ implies $\Theta^T = \Theta$. Consequently $W(H,G,0) = (J_{N-1} + J_{N-1}^T) \otimes \Theta$. Firstly we consider the situation when buying is allowed, i.e., problem (3.2.6) with $\mu = 0$. A direct use of Lemma 3.2.1 implies that problem (3.2.6) with $\mu = 0$ has a unique solution if and only if $\Theta \succ 0$. Applying properties of the Kronecker product, this unique solution equals

$$\begin{aligned} z^* = -W^{-1}(H,G,0)b(H,G) &= \left(\left(J_{N-1} + J_{N-1}^T\right)^{-1} \otimes \Theta^{-1} \right) \left(e_1 \otimes \Theta \bar{S}\right) \\ &= \left(\left(J_{N-1} + J_{N-1}^T\right)^{-1} e_1 \right) \otimes \bar{S}. \end{aligned}$$

Recall that $e_1 = (1, 0, \cdots, 0)^T$. Applying the explicit representation of the inverse of the tridiagonal matrix $J_{N-1} + J_{N-1}^T$, see, e.g., (da Fonseca, 2007), we have

$$\left(J_{N-1} + J_{N-1}^T\right)^{-1} e_1 = \left(\frac{N-1}{N}, \frac{N-2}{N}, \cdots, \frac{1}{N} \right)^T.$$

Hence, the minimum expected execution cost position equals $\{x_k^*\}_{k=0}^N = \left\{\frac{(N-k)}{N}\bar{S}\right\}_{k=0}^N$, which corresponds to the naive execution strategy $n_k^* = \frac{\bar{S}}{N}$, $k = 1, \cdots, N$. Since this solution satisfies the constraints of problem (3.2.9), it is also the unique minimum expected cost execution strategy when buying is prohibited. □

Therefore, the minimum expected execution cost strategy is always the naive execution strategy of trading a constant number of shares per period, under the assumptions that the permanent impact matrix G is symmetric and Θ is positive definite. Thus, the minimum expected execution cost strategy is insensitive to any change in the impact matrices as long as the perturbation maintains the strict convexity of the objective function and the symmetry in the permanent impact matrix.

When buying is permitted, the optimal execution strategy is the solution to a linear system with the coefficient matrix $W(H, G, \mu)$. Thus it is important to analyze the condition number of this matrix. Moreover, as we show subsequently in §3.3, sensitivity of the optimal execution strategy to perturbations in the impact matrices depends on the eigenvalues of $W(H, G, \mu)$. In the rest of this section, we analyze the condition number and the minimum eigenvalue of the matrix $W(H, G, \mu)$. We apply the following result from (Kulkarni et al., 1999) in our discussion.

Lemma 3.2.2. *Let $N \geq 2$. Then the eigenvalues $\lambda_i\left(J_{N-1} + J_{N-1}^T\right)$ equal $2\left(1 - \cos\left(\frac{i\pi}{N}\right)\right)$ for $i = 1, 2, \ldots, N-1$.*

A direct consequence of Lemma 3.2.2 is

$$\kappa_2\left(J_{N-1} + J_{N-1}^T\right) = \frac{\lambda_{\max}\left(J_{N-1} + J_{N-1}^T\right)}{\lambda_{\min}\left(J_{N-1} + J_{N-1}^T\right)} = \frac{1 - \cos\left(\frac{(N-1)\pi}{N}\right)}{1 - \cos\left(\frac{\pi}{N}\right)} = \cot^2\left(\frac{\pi}{2N}\right). \tag{3.2.11}$$

Since $W(H, G, \mu) = 2\mu\tau I_{N-1} \otimes C + W(H, G, 0)$ and the matrices C and $W(H, G, 0)$ are symmetric, the Courant-Fischer Theorem (see, e.g., Theorem 8.1.5 in (Golub and Loan, 1996), which is also provided in Theorem (A.1.1) of Appendix A) implies that

$$\lambda_{\min}(W(H, G, \mu)) \geq 2\mu\tau\lambda_{\min}(C) + \lambda_{\min}(W(H, G, 0)). \tag{3.2.12}$$

When G is symmetric, this lower bound can be stated explicitly in terms of the combined impact matrix Θ:

Corollary 3.2.1. *Let $N \geq 2$ and the permanent impact matrix G be symmetric. Then*

$$\lambda_{\min}\left(W(H, G, \mu)\right) \geq 2\mu\tau\lambda_{\min}(C) + 4\sin^2\left(\frac{\pi}{2N}\right)\lambda_{\min}(\Theta).$$

In addition, the equality holds when $\mu = 0$.

Proof. When G is symmetric, $W(H,G,0)$ can be represented as the Kronecker product of the matrices $\left(J_{N-1}+J_{N-1}^T\right)$ and Θ. Thus, using properties of Kronecker matrix products (see, e.g., Appendix A or Property IX in section 2.3 of (Graham, 1981)), we obtain

$$\lambda_{\min}(W(H,G,0)) = \lambda_{\min}\left(J_{N-1}+J_{N-1}^T\right)\lambda_{\min}(\Theta) = 2\left(1-\cos\left(\frac{\pi}{N}\right)\right)\lambda_{\min}(\Theta).$$

This result, along with inequality (3.2.12), completes the proof. □

In the next proposition, we investigate how $\kappa_2(W(H,G,\mu))$ depends on the condition number of the covariance matrix and the combined impact matrix Θ.

Proposition 3.2.2. *Let $W(H,G,0)\succ 0$ and $N\geq 2$. Then*

(a) $\kappa_2(W(H,G,0)) \geq \cot^2\left(\frac{\pi}{2N}\right)\cdot\kappa_2(\Theta+\Theta^T)$.

(b) If, in addition, G is symmetric, then

$$\kappa_2(W(H,G,0)) = \cot^2\left(\frac{\pi}{2N}\right)\cdot\kappa_2(\Theta). \tag{3.2.13}$$

(c) Assume $C\succ 0$. Then $\kappa_2(W(H,G,\mu)) \leq \kappa_2(C)+\kappa_2(W(H,G,0))$ for any $\mu\geq 0$.

Proof. (a) To prove part (a), we use the fact that the matrices $W(H,G,0)$ and $W(H,G^T,0)$ have identical eigenvalues. More precisely, $(r_1^T, r_2^T, ..., r_{N-1}^T)^T$ is an eigenvector of $W(H,G,0)$ associated with the eigenvalue λ if and only if $(r_{N-1}^T, r_{N-2}^T, ..., r_1^T)^T$ is an eigenvector of $W(H,G^T,0)$ for the same eigenvalue. In particular, we have

$$\lambda_{\max}\left(W(H,G,0)\right) = \lambda_{\max}\left(W(H,G^T,0)\right), \quad \lambda_{\min}\left(W(H,G,0)\right) = \lambda_{\min}\left(W(H,G^T,0)\right).$$

Now the corollary of the Courant-Fischer Theorem along with the assumption $W(H,G,0)\succ 0$, results in

$$\frac{\lambda_{\max}\left(W(H,G,0)+W(H,G^T,0)\right)}{\lambda_{\min}\left(W(H,G,0)+W(H,G^T,0)\right)} \leq \frac{2\lambda_{\max}\left(W(H,G,0)\right)}{2\lambda_{\min}\left(W(H,G,0)\right)} = \kappa_2\left(W(H,G,0)\right).$$

Consequently $\kappa_2\left(W(H,G,0)+W(H,G^T,0)\right) \leq \kappa_2\left(W(H,G,0)\right)$. This inequality, along with the expression of $W(H,G,0)+W(H,G^T,0)$ as the Kronecker product of the matrices $\left(J_{N-1}+J_{N-1}^T\right)$ and $(\Theta+\Theta^T)$, yields

$$\kappa_2\left(W(H,G,0)\right) \geq \kappa_2\left(W(H,G,0)+W(H,G^T,0)\right) = \kappa_2\left(\left(J_{N-1}+J_{N-1}^T\right)\otimes\left(\Theta+\Theta^T\right)\right). \tag{3.2.14}$$

Thus

$$\kappa_2(W(H,G,0)) \geq \kappa_2\left(J_{N-1}+J_{N-1}^T\right)\kappa_2\left(\Theta+\Theta^T\right) = \cot^2\left(\frac{\pi}{2N}\right)\kappa_2(\Theta+\Theta^T), \tag{3.2.15}$$

which completes the proof of part (a).

(b) When Θ is symmetric, $W(H,G,\mu) = W(H,G^T,\mu)$. Therefore

$$\kappa_2\left(W(H,G,0) + W(H,G^T,0)\right) = \kappa_2\left(2W(H,G,0)\right) = \kappa_2(W(H,G,0)).$$

Hence, equality holds in (3.2.14) and (3.2.15). This completes the proof of part (b).

(c) To prove part (c), let $\mu \geq 0$ be given. Using $W(H,G,\mu) = 2\mu\tau I_{N-1} \otimes C + W(H,G,0)$, we have

$$\begin{aligned}
\kappa_2\left(W(H,G,\mu)\right) &= \frac{\lambda_{\max}\left(W(H,G,0) + 2\mu\tau I_{N-1} \otimes C\right)}{\lambda_{\min}\left(W(H,G,0) + 2\mu\tau I_{N-1} \otimes C\right)} \\
&\leq \frac{\lambda_{\max}\left(W(H,G,0)\right)}{\lambda_{\min}\left(W(H,G,0)\right)} + \frac{\lambda_{\max}(2\mu\tau I_{N-1} \otimes C)}{\lambda_{\min}(2\mu\tau I_{N-1} \otimes C)} \qquad (3.2.16) \\
&= \kappa_2(W(H,G,0)) + \kappa_2(C),
\end{aligned}$$

where inequality (3.2.16) comes from the fact that $\lambda_{\min}(W(H,G,0)) > 0$ and $\lambda_{\min}(2\mu\tau I_{N-1} \otimes C) > 0$. □

Proposition 3.2.2 shows that the condition number of the matrix $W(H,G,\mu)$ can be large when the condition number of either the covariance matrix C or the Hessian matrix $W(H,G,0)$ is large. However, $\kappa_2(W(H,G,0))$ is at least as large as $\cot^2\left(\frac{\pi}{2N}\right)$ times the condition number of the matrix $(\Theta + \Theta^T)$. Proposition 3.2.2 also implies that, in the single asset trading, the condition number of the obtained matrix $W(H,G,0)$ depends only on the number of periods N.

In the next section, we investigate sensitivity of the optimal execution strategy to perturbations in the impact matrices H and G.

3.3 Sensitivity of the Optimal Execution Strategy

In this section, we derive some upper bounds for the change in the optimal execution strategy in terms of the change in the impact matrices and eigenvalues of the Hessian of the objective function. Such analysis indicates under what conditions the optimal execution strategy is insensitive to perturbations and when it may become very sensitive. Throughout, we denote perturbations in the temporary and permanent impact matrices as ΔH and ΔG, respectively. In subsequent discussions, we assume that $W(H,G,\mu) \succ 0$. Therefore, for sufficiently small perturbations of ΔH and ΔG, the matrix $W(H+\Delta H, G+\Delta G, \mu)$ is symmetric positive definite. Consequently the optimal execution strategy after perturbation remains unique.

Given the perturbed impact matrices $H + \Delta H$ and $G + \Delta G$, the perturbed optimal portfolio execution problem (3.2.9) is

$$\begin{aligned}
\min_{z \in \mathbb{R}^{m(N-1)}} \quad & \frac{1}{2} z^T W(H+\Delta H, G+\Delta G, \mu) z + b^T(H+\Delta H, G+\Delta G) z, \qquad (3.3.1) \\
\text{s.t.} \quad & (-e_1^T \otimes I_m) z \geq -\bar{S}, \\
& (J_{N-1} \otimes I_m) z \geq 0,
\end{aligned}$$

where the matrix J_{N-1} is defined in (3.2.8). Problems (3.2.9) and (3.3.1) have the same set of feasible points. Applying the properties

$$
\begin{aligned}
b(H+\Delta H, G+\Delta G) &= b(H,G)+\Delta b, & \Delta b &\stackrel{\text{def}}{=} b(\Delta H, \Delta G),\\
W(H+\Delta H, G+\Delta G, \mu) &= W(H,G,\mu)+\Delta W, & \Delta W &\stackrel{\text{def}}{=} W(\Delta H, \Delta G, 0), \quad (3.3.2)
\end{aligned}
$$

we may restate the objective function of problem (3.3.1):

$$\frac{1}{2}z^T W(H,G,\mu)z + b^T(H,G)z + \frac{1}{2}z^T \Delta W z + \Delta b^T z. \tag{3.3.3}$$

Quantities ΔW and Δb are determined by $\Delta\Theta = \frac{1}{\tau}(\Delta H + \Delta H^T) - \Delta G$. Thus the solution of problem (3.3.1) and consequently the optimal execution strategy depends on the perturbation in the combined impact matrix $\Delta\Theta$ rather than ΔH or ΔG individually. Therefore, all of the perturbations in the impact matrices that produce the same $\Delta\Theta$ affect the optimal execution strategy identically. In particular, when $\Delta\Theta = 0$, we have $\Delta W = 0$ and $\Delta b = 0$. Therefore, problems (3.3.1) and (3.2.9) have identical solutions. Hence the optimal execution strategy is insensitive to this special perturbation of the impact matrices ΔH and ΔG, when $\Delta\Theta = 0$.

Furthermore, as we discussed in Proposition 3.2.1, the minimum expected execution cost strategy is also insensitive to any perturbation in the impact matrices as long as the perturbed permanent impact matrix $G + \Delta G$ remains symmetric and $W(H + \Delta H, G + \Delta G, 0) \succ 0$. Specifically, when trading a single asset and $\mu = 0$, the optimal execution strategy is not sensitive to any change in the impact matrices, assuming that the minimum expected execution cost problem has a unique solution.

Therefore, in the aforementioned cases the optimal execution strategy and the variance of the execution cost remain the same. However, in both cases, the expected value of the corresponding execution cost changes as the impact matrices are perturbed. When $\Delta\Theta = 0$, the change in the mean of the execution cost is $\frac{1}{\tau}\bar{S}^T \Delta H \bar{S}$. In the second case, in which $\mu = 0$, $\Theta = \Theta^T$ and $\Delta\Theta = \Delta\Theta^T$ are considered, the variation in the mean of the execution cost equals $\frac{1}{2}z^T \Delta W z + \Delta b^T z + \frac{1}{\tau}\bar{S}^T \Delta H \bar{S}$, where $z = (N-1, N-2, \cdots, 1)^T \frac{\bar{S}}{N}$. However, when ΔG is asymmetric, our simulation study in §3.4 shows that the change in the mean and particularly the variance of the execution cost can become very significant for small values of μ, especially when $\mu = 0$.

In the rest of this section, we analyze sensitivity of the optimal execution strategy to more general perturbations in the impact matrices. Firstly, we note that the Euclidean distance between any two execution strategies $n^* = \{n_k^*\}_{k=1}^N$ and $\bar{n} = \{\bar{n}_k\}_{k=1}^N$ is related to the change between corresponding positions $x^* = \{x_k^*\}_{k=0}^N$ and $\bar{x} = \{\bar{x}_k\}_{k=0}^N$:

$$\begin{aligned}
\|n^* - \bar{n}\|_2^2 &= \sum_{k=1}^N \|n_k^* - \bar{n}_k\|_2^2 = \sum_{k=1}^N \left\|x_{k-1}^* - x_k^* - (\bar{x}_{k-1} - \bar{x}_k)\right\|_2^2 \\
&= \sum_{k=1}^N \left\|x_{k-1}^* - \bar{x}_{k-1}\right\|_2^2 + \sum_{k=1}^N \|x_k^* - \bar{x}_k\|_2^2 - 2\sum_{k=1}^N (x_{k-1}^* - \bar{x}_{k-1})^T (x_k^* - \bar{x}_k) \\
&\leq 2\sum_{k=1}^N \left\|x_{k-1}^* - \bar{x}_{k-1}\right\|_2^2 + 2\sum_{k=1}^N \|x_k^* - \bar{x}_k\|_2^2 = 4\sum_{k=1}^{N-1} \|x_k^* - \bar{x}_k\|_2^2 .
\end{aligned}$$

This result can be summarized as

$$\|n^* - \bar{n}\|_2 \leq 2\,\|x^* - \bar{x}\|_2 . \tag{3.3.4}$$

We start our sensitivity discussion with the strategy on which no constraint is imposed. In the following theorem, we exploit the explicit representation of the solution of problem (3.2.6) to determine the exact change in the optimal execution strategy. For notational simplicity, abbreviate $W(H, G, \mu)$ as W when there is no confusion.

Theorem 3.3.1. *Consider the optimal portfolio execution problem* (3.2.6)*. Assume* $W(H, G, \mu) \succ 0$ *and* $W(H + \Delta H, G + \Delta G, \mu) \succ 0$*. Denote the unique solutions of problem* (3.2.6) *before and after perturbation with* z^* *and* $\bar{z}$ *respectively. Then*

$$z^* - \bar{z} = W^{-1}(H + \Delta H, G + \Delta G, \mu)\left(\Delta b - \Delta W W^{-1}(H, G, \mu) b(H, G)\right). \tag{3.3.5}$$

Furthermore, let $n^* = \{n_k^*\}_{k=1}^N$ *and* $\bar{n} = \{\bar{n}_k\}_{k=1}^N$ *be the optimal execution strategies corresponding to the solutions* z^* *and* $\bar{z}$ *respectively. Then, there exists a magnification factor* $\vartheta > 0$ *such that:*

$$\|n^* - \bar{n}\|_2 \leq 2\|z^* - \bar{z}\|_2 \leq 2\vartheta\|\bar{S}\|_2\left(1 + 4\sqrt{m}\vartheta\|\Theta\|_2\right)\|\Delta\Theta\|_2, \tag{3.3.6}$$

where $\vartheta \leq \frac{1}{\min\{\lambda_{\min}(W), \lambda_{\min}(W+\Delta W)\}}$.

Proof. Positive definiteness of W guarantees problem (3.2.6) has the unique solution $z^* = -W^{-1}b(H, G)$. Similarly, under the assumption $W + \Delta W \succ 0$, problem (3.2.6), with the perturbed impact matrices $H + \Delta H$ and $G + \Delta G$, has a unique solution, namely $\bar{z} = -(W + \Delta W)^{-1}(b(H, G) + \Delta b)$. Therefore

$$\begin{aligned}
(W + \Delta W)(z^* - \bar{z}) &= (W + \Delta W)[-W^{-1}b(H, G) - (-(W + \Delta W)^{-1}(b(H, G) + \Delta b))] \\
&= -b(H, G) - \Delta W W^{-1} b(H, G) + b(H, G) + \Delta b \\
&= -\Delta W W^{-1} b(H, G) + \Delta b,
\end{aligned}$$

which proves (3.3.5). Thus

$$\begin{aligned}
\|z^* - \bar{z}\|_2 &= \left\|(W + \Delta W)^{-1}\left(\Delta b - \Delta W W^{-1} b(H, G)\right)\right\|_2 \\
&\leq \|(W + \Delta W)^{-1}\|_2 \left(\|\Delta b\|_2 + \|\Delta W\|_2 \|W^{-1}\|_2 \|b(H, G)\|_2\right).
\end{aligned}$$

Since $W + \Delta W$ and W are symmetric positive definite, the above inequality is reduced to

$$\|z^* - \bar{z}\|_2 \leq \frac{1}{\lambda_{\min}(W + \Delta W)} \left(\|\Delta b\|_2 + \frac{\|\Delta W\|_2}{\lambda_{\min}(W)} \|b(H, G)\|_2 \right) \tag{3.3.7}$$

$$\leq \frac{1}{\lambda_{\min}(W + \Delta W)} \left(\|\Delta \Theta\|_2 + \frac{\|\Delta W\|_2}{\lambda_{\min}(W)} \|\Theta\|_2 \right) \|\bar{S}\|_2. \tag{3.3.8}$$

Since ΔW is symmetric, $\|\Delta W\|_1 = \|\Delta W\|_\infty$ and $\|\Delta W\|_2 \leq \sqrt{\|\Delta W\|_1 \|\Delta W\|_\infty} = \|\Delta W\|_1$. Thus

$$\begin{aligned} \|\Delta W\|_2 &\leq \|\Delta \Theta\|_1 + \|\Delta \Theta + \Delta \Theta^T\|_1 + \|\Delta \Theta^T\|_1 \\ &\leq 2\|\Delta \Theta\|_1 + 2\|\Delta \Theta^T\|_1 \\ &= 2\|\Delta \Theta\|_1 + 2\|\Delta \Theta\|_\infty \\ &\leq 4\sqrt{m}\|\Delta \Theta\|_2. \end{aligned} \tag{3.3.9}$$

Substituting inequality (3.3.9) in (3.3.8) and using inequality (3.3.4) complete the proof of (3.3.6). □

Inequality (3.3.7) is valid for any unconstrained quadratic minimization problem with the Hessian matrix W and the linear coefficient $b(H, G)$. If the perturbation of the Hessian matrix ΔW is sufficiently small, relative to the change in the linear coefficient Δb, the upper bound is dominated by $\vartheta\|\Delta b\|$, which is linear in ϑ. This is particularly the case in the traditional mean-variance portfolio optimization since the covariance (Hessian) matrix can in general be estimated more accurately than the mean rate of return (linear coefficient). However, in the optimal portfolio execution problem, the change in the combined impact matrix $\Delta\Theta$ appears in both the linear coefficient $b(H, G)$ and the Hessian matrix W. Therefore, when the magnification factor ϑ is sufficiently large, the upper bound is dominated by the term $\vartheta^2\|\Delta W\|_2$, which is quadratic in ϑ. Thus, the effect of estimation errors in the impact matrices can potentially be more significant than the effect of perturbations in the mean rate of return in the traditional mean-variance portfolio optimization.

The upper bound in (3.3.6) illustrates the main factors which can magnify the effect of estimation errors in the impact matrices on the optimal execution strategy. This effect is described through the magnification factor ϑ. When the upper bound of ϑ is small, the optimal execution strategy is not so sensitive to perturbations in the impact matrices. On the other hand, the optimal execution strategy may be sensitive to the perturbation $\Delta\Theta$ when this upper bound is large.

The provided upper bound for ϑ in Theorem 3.3.1 depends only on the minimum eigenvalues of W and $W+\Delta W$. When both of these eigenvalues are large, the magnification factor becomes small. Consequently the optimal execution strategy does not change significantly due to perturbations in the impact matrices. When $\mu > 0$ and $C \succ 0$, according to inequality (3.2.12), $\lambda_{\min}(W)$ (and similarly $\lambda_{\min}(W + \Delta W)$) increases as $\mu\lambda_{\min}(C)$ increases, which implies that the magnification factor ϑ becomes small. This result indicates that, when the risk aversion parameter is nonzero and $\lambda_{\min}(C)$ is large (or equivalently $\kappa_2(C)$ is small), the optimal execution strategy is not very sensitive to the perturbations. Furthermore, assuming $C \succ 0$, variation in the optimal execution strategy due to the perturbations diminishes as $\mu \to +\infty$. This result is entirely expected; since as $\mu \to +\infty$ the objective function of problem (3.2.6) is dominated by the variance of the execution cost which depends only on the covariance matrix C. In these two cases the investor may not need to be concerned about the effect of estimation errors in the impact matrices.

On the other hand, when the minimum eigenvalue of $C \succ 0$ is small, the influence of estimation errors on the optimal execution strategy may become more prominent for a small risk aversion parameter. This dependence of the effect of estimation errors on the risk aversion parameter is analogous to the traditional mean-variance portfolio optimization, see, e.g., (Chopra and Ziemba, 1993).

When the permanent impact matrices G and $G + \Delta G$ are symmetric, using Corollary 3.2.1, the upper bound for ϑ presented in Theorem 3.3.1 can be stated in terms of the minimum eigenvalues of Θ and $\Theta + \Delta\Theta$.

Corollary 3.3.1. *Let the assumptions in Theorem 3.3.1 hold. In addition, assume that G and ΔG are symmetric, $\lambda_{\min}(\Theta) \geq 0$ and $\lambda_{\min}(\Theta + \Delta\Theta) \geq 0$. Then there exists a magnification factor $\vartheta_0 > 0$ such that*

$$\|n^* - \bar{n}\|_2 \leq 2\|z^* - \bar{z}\|_2 \leq \vartheta_0 \|\bar{S}\|_2 \left(1 + 2\sqrt{m}\vartheta_0\|\Theta\|_2\right) \|\Delta\Theta\|_2,$$

where

$$\vartheta_0 \leq \frac{1}{\mu\tau\lambda_{\min}(C) + 2\sin^2\left(\frac{\pi}{2N}\right) \cdot \min\{\lambda_{\min}(\Theta), \lambda_{\min}(\Theta + \Delta\Theta)\}}. \tag{3.3.10}$$

For a sufficiently small perturbation, the upper bound of ϑ_0 in (3.3.10) becomes small if and only if *either* $\mu\lambda_{\min}(C)$ *or* $\lambda_{\min}(\Theta)$ is large. Either case results in a small sensitivity of the optimal execution strategy to perturbations. However, for a given positive risk aversion parameter, when both $\lambda_{\min}(C)$ and $\lambda_{\min}(\Theta)$ are small, the change in the optimal execution strategy can potentially be large relative to the perturbation $\Delta\Theta$. Note that, based on Proposition 3.2.1, the minimum expected execution cost strategy is insensitive to perturbations when matrices G and $G + \Delta G$ are symmetric and Θ and $\Theta + \Delta\Theta$ are positive definite.

Now consider the optimal portfolio execution problem when buying is not permitted. Denote the coefficient matrix of the no-buying constraints in problem (3.2.9) with A, i.e., $A = Y \otimes I_m$, where

$$Y \stackrel{\text{def}}{=} \begin{pmatrix} -e_1^T \\ J_{N-1} \end{pmatrix}. \tag{3.3.11}$$

The following property of the bidiagonal matrix J_{N-1} is proved in (da Fonseca, 2007).

Proposition 3.3.1. *For every integer $k \geq 1$, $\lambda_i\left(J_k J_k^T\right) = 2\left(1-\cos\left(\frac{2i-1}{2k+1}\pi\right)\right)$ for $i = 1,2,\ldots,k$. Particularly, $\lambda_{\min}\left(J_k J_k^T\right) = 2\left(1-\cos\left(\frac{\pi}{2k+1}\right)\right) = 4\sin^2\left(\frac{\pi}{4k+2}\right)$.*

Therefore, $\lambda_{\min}(J_k J_k^T)$ is a decreasing function of k, i.e., for $k \geq 1$, $\lambda_{\min}(J_k J_k^T) \geq \lambda_{\min}(J_{k+1} J_{k+1}^T)$. Moreover, $1 \geq \lambda_{\min}(J_k J_k^T) > 0$ and consequently the matrix $J_k J_k^T$ is symmetric positive definite.

Next, we present a property of the coefficient matrix of the binding constraints in problem (3.2.9) at a feasible point. Throughout, for a given subset J of the row indices of A, we let A_J denote the submatrix of A consisting of the rows with indices in J. The submatrix of A consisting of those columns with indices in a subset J of column indices is denoted by A^J.

Lemma 3.3.1. *Consider the coefficient matrix A of the constraints in problem* (3.2.9)*, i.e., $A = Y \otimes I_m$, where Y is defined in* (3.3.11)*. Let z^* be a feasible point of problem* (3.2.9) *and J be the set of indices of the binding constraints at z^*. Then*

$$\lambda_{\min}\left(A_J A_J^T\right) \geq \lambda_{\min}\left(J_{N-1} J_{N-1}^T\right) = 4\sin^2\left(\frac{\pi}{4N-2}\right). \tag{3.3.12}$$

Proof. Applying properties of the Kronecker product, we have

$$A_J A_J^T = (Y\otimes I_m)_J\,(Y\otimes I_m)_J^T = \left((Y\otimes I_m)\,(Y\otimes I_m)^T\right)_J^J = \left(\left(YY^T\right)\otimes I_m\right)_J^J. \tag{3.3.13}$$

Let M be the permutation matrix such that

$$[1,2,\ldots,N,1,2,\ldots,N,\ldots,1,2,\ldots,N]M^T = [1,\ldots,1,2,\ldots,2,\ldots,N,\ldots,N].$$

Therefore, corresponding to the index set J, we can find an index set J' such that

$$M_J^J\left(\left(\left(YY^T\right)\otimes I_m\right)_J^J\right)(M_J^J)^T = \left(I_m \otimes \left(YY^T\right)\right)_{J'}^{J'}.$$

Thus

$$\begin{aligned}\lambda_{\min}\left(\left(\left(YY^T\right)\otimes I_m\right)_J^J\right) &= \lambda_{\min}\left(M_J^J\left(\left(\left(YY^T\right)\otimes I_m\right)_J^J\right)(M_J^J)^T\right)\\ &= \lambda_{\min}\left(\left(I_m\otimes\left(YY^T\right)\right)_{J'}^{J'}\right)\\ &\geq \min_{i=1,2,\ldots,m}\lambda_{\min}\left(Y_{(i)}Y_{(i)}^T\right).\end{aligned}$$

Here $Y_{(i)}$ is the submatrix of Y with the row indices equal to $j - N(i-1)$ where

$$j \in \left(J' \cap \{N(i-1)+1,\ldots,N(i-1)+N\}\right).$$

This result, along with equality (3.3.13), yields

$$\lambda_{\min}\left(A_J A_J^T\right) \geq \min_{i=1,2,\ldots,m}\lambda_{\min}\left(Y_{(i)}Y_{(i)}^T\right). \tag{3.3.14}$$

In the rest of the proof, we show that for every $i = 1, 2, \ldots, m$,

$$\lambda_{\min}\left(Y_{(i)}Y_{(i)}^T\right) \geq \lambda_{\min}\left(J_{N-1}J_{N-1}^T\right). \tag{3.3.15}$$

Denote positions associated with the feasible point z^* with $\{x_k^*\}_{k=0}^N$. Note that, for each asset i, a positive number of shares must be sold in at least one of the periods. Thus, for every asset $i = 1, 2, \ldots, m$, there is at least some $j \in \{1, 2, ..., N\}$ so that the constraint corresponding to the jth row of Y is not active. For any asset i, there are two cases to consider: either $\bar{S}_i > x_{i1}^*$ or $\bar{S}_i = x_{i1}^*$. In the first case, $\bar{S}_i > x_{i1}^*$, the rows of $Y_{(i)}$ are a subset of the rows of J_{N-1}. Let $Y_{(-i)}$ denote the submatrix consisting of rows of J_{N-1} that are not in $Y_{(i)}$. We then have

$$\begin{aligned}
\lambda_{\min}\left(J_{N-1}J_{N-1}^T\right) &= \min_{z\neq 0} \frac{z^T\left(J_{N-1}J_{N-1}^T\right)z}{z^Tz} \qquad (3.3.16)\\
&= \min_{(z_1,z_2)\neq 0} \frac{z_1^TY_{(i)}Y_{(i)}^Tz_1 + z_2^TY_{(-i)}Y_{(i)}^Tz_1 + z_1^TY_{(i)}Y_{(-i)}^Tz_2 + z_2^TY_{(-i)}Y_{(-i)}^Tz_2}{z_1^Tz_1 + z_2^Tz_2}\\
&\leq \min_{(z_1,0)\neq 0} \frac{z_1^T\left(Y_{(i)}Y_{(i)}^T\right)z_1}{z_1^Tz_1} = \lambda_{\min}\left(Y_{(i)}Y_{(i)}^T\right),
\end{aligned}$$

which proves inequality (3.3.15).

Now let $\bar{S}_i = x_{i1}^*$. In this case, there must be at least some $j \in \{2, 3, \ldots, N\}$, such that $x_{i(j-1)}^* > x_{ij}^*$. When $N = 2$, the second constraint must be inactive, which implies that the only row of $Y_{(i)}$ is the first row of Y. Therefore, $\lambda_{\min}\left(Y_{(i)}Y_{(i)}^T\right) = \lambda_{\min}\left(J_{N-1}J_{N-1}^T\right)$.

When $N \geq 3$, at least one of the rows of J_{N-1} corresponds to an inactive constraint. Let this row be the jth row of J_{N-1} where $j \in \{1, 2, \ldots, N-1\}$. When $j = N-1$, $Y_{(i)}$ does not include the last row of J_{N-1}. Since the matrix Y after eliminating its last row equals $-J_{N-1}^T$, $Y_{(i)}$ is a submatrix of $-J_{N-1}^T$. Thus similar to (3.3.16), we can show that

$$\lambda_{\min}\left(Y_{(i)}Y_{(i)}^T\right) \geq \lambda_{\min}\left(J_{N-1}^TJ_{N-1}\right) = \lambda_{\min}\left(J_{N-1}J_{N-1}^T\right),$$

where the last equality comes from the fact that $J_{N-1}^TJ_{N-1}$ and $J_{N-1}J_{N-1}^T$ have identical eigenvalues. This result proves inequality (3.3.15) for this case.

When $j \in \{1, 2, \ldots, N-2\}$, the rows of $Y_{(i)}$ are a subset of the rows of the following matrix

$$\begin{pmatrix} -J_j^T & 0 \\ 0 & J_{N-j-1} \end{pmatrix}.$$

Note that

$$\begin{pmatrix} -J_j^T & 0 \\ 0 & J_{N-j-1} \end{pmatrix}\begin{pmatrix} -J_j & 0 \\ 0 & J_{N-j-1}^T \end{pmatrix} = \begin{pmatrix} J_j^TJ_j & 0 \\ 0 & J_{N-j-1}J_{N-j-1}^T \end{pmatrix}.$$

Therefore

$$\lambda_{\min}\left(Y_{(i)}Y_{(i)}^T\right) \geq \min\left\{\lambda_{\min}\left(J_j^T J_j\right), \lambda_{\min}\left(J_{N-j-1}J_{N-j-1}^T\right)\right\} \geq \lambda_{\min}\left(J_{N-1}J_{N-1}^T\right),$$

where the last inequality comes from the facts that $\lambda_{\min}\left(J_j^T J_j\right) = \lambda_{\min}\left(J_j J_j^T\right)$ and $\lambda_{\min}\left(J_j J_j^T\right)$ is a decreasing function of j. Thus, for every $i = 1, 2, \ldots, m$, inequality (3.3.15) holds. Applying inequalities (3.3.14), (3.3.15), and Proposition 3.3.1 completes the proof. □

Our analysis for sensitivity of the optimal execution strategy, when buying is prohibited, is based on a result in (Hager, 1979). He proves that for a linearly constrained quadratic programming problem, under some conditions on the Hessian of the objective function and the Jacobian matrix of the binding constraints, both the solution and the Lagrange multipliers are Lipschitz continuous functions of the problem data. An estimate for the Lipschitz constant is discussed in §3 of (Hager, 1979); this result is summarized in the following theorem. Note that the upper bound presented in the following theorem is slightly tighter than the bound in Lemma 3.2 of (Hager, 1979); but the result essentially follows from the same proof.

Theorem 3.3.2. *Consider the following quadratic programming problem with the data $d = (Q, b, A, c)$*

$$\begin{aligned} \min_{x\in\mathbb{R}^n} \quad & \frac{1}{2}x^T Q x + b^T x \\ s.t. \quad & Ax + c \leq 0. \end{aligned} \tag{3.3.17}$$

Let $\mathcal{D}$ be a convex set of data so that for every $d \in \mathcal{D}$, the above problem has a unique solution, denoted by $x(d)$, and a unique Lagrange multiplier, denoted by $u(d)$. Let $J(d)$ be the set of indices corresponding to the binding constraints at $x(d)$. Assume that there exist some parameters $v_1 < +\infty$, $v_2 < +\infty$, $\beta > 0$, and $\alpha > 0$ so that for every $d = (Q, b, A, c)$ in $\mathcal{D}$:

(a) $\|Q\|_2 \leq v_1$,

(b) $\left\|A_{J(d)}^T\right\|_2 \leq v_2$,

(c) $\left\|A_{J(d)}^T u(d)\right\|_2 \geq \beta \|u(d)\|_2$,

(d) $x^T Q x \geq \alpha \|x\|_2^2$, *for every x such that* $A_{J(d)}x = 0$.

Then there exists a positive constant $\varrho < +\infty$ such that for every $d_1, d_2 \in \mathcal{D}$:

$$\begin{aligned} \|x(d_1) - x(d_2)\|_2 \leq \; & \varrho\left(\|b(d_1) - b(d_2)\|_2 + \|c(d_1) - c(d_2)\|_2\right) + \varrho^2\left(\max_{i\in\{1,2\}}\left(\|b(d_i)\|_2 + \|c(d_i)\|_2\right)\right) \\ & \left(\|Q(d_1) - Q(d_2)\|_2 + \|A(d_1) - A(d_2)\|_2 + \left\|A^T(d_1) - A^T(d_2)\right\|_2\right), \end{aligned}$$

where $\varrho \leq \frac{1}{\alpha} + \frac{1}{\beta}\left(\frac{v_1}{\alpha} + 1\right)\left(v_2 + \frac{v_2 v_1}{\beta} + 1\right)$.

In the optimal portfolio execution problem, the impact matrices only appear in the objective function. Therefore, perturbations in the impact matrices do not affect the constraints of problem (3.2.9). When the constraints in Problem (3.3.17) do not change for any $d \in \mathcal{D}$, a tighter upper bound for ϱ can be obtained as in the following corollary:

Corollary 3.3.2. *Let the assumptions in Theorem 3.3.2 hold. In addition, assume that $A(d)$ and $c(d)$ are constant on $\mathcal{D}$, i.e., $A(d) = A$ and $c(d) = c$ for every $d \in \mathcal{D}$. Let $d_1 \in \mathcal{D}$ be given. Then there exists a positive constant $\varrho_0 < +\infty$ such that for every $d_2 \in \mathcal{D}$*

$$\|x(d_1) - x(d_2)\|_2 \leq$$
$$\varrho_0 \left\|b(d_1) - b(d_2)\right\|_2 + \varrho_0^2 \left(\max_{i\in\{1,2\}} \|b(d_i)\|_2 + \max\{1, \|Q(d_1)\|_2\} \|c\|_2 \right) \left\|Q(d_1) - Q(d_2)\right\|_2,$$

where $\varrho_0 \leq \frac{1}{\alpha} + \frac{1}{\beta \max\{1,\|Q(d_1)\|_2\}} \left(\frac{v_1}{\alpha} + 1\right) \left(v_2 + \frac{v_2 v_1}{\beta \max\{1,\|Q(d_1)\|_2\}} + 1\right).$

Proof. Consider problem (3.3.17) with the input data $d = (Q, b, A, c)$ with the corresponding parameters v_1, v_2, β, and α given. Clearly, for any $d \in \mathcal{D}$, problem (3.3.17) and the following problem have an identical solution:

$$\begin{aligned} \min_{x\in\mathbb{R}^n} \quad & \frac{1}{2} x^T \bar{Q} x + \bar{b}^T x \\ \text{s.t.} \quad & Ax + c \leq 0, \end{aligned} \tag{3.3.18}$$

where $\bar{Q} = rQ$, $\bar{b} = rb$, and $r = \frac{1}{\max\{1,\|Q(d_1)\|_2\}}$. Applying Theorem 3.3.2 to problem (3.3.18) and using the fact that $A(d)$ and $c(d)$ are constant on $\mathcal{D}$, there exists a positive constant $\hat{\varrho} < +\infty$ such that for every $d_2 \in \mathcal{D}$:

$$\begin{aligned} \|x(d_1) - x(d_2)\|_2 \; & \leq \hat{\varrho} \left(\left\|\bar{b}(d_1) - \bar{b}(d_2)\right\|_2 + \|c(d_1) - c(d_2)\|_2\right) + \hat{\varrho}^2 \left(\max_{i\in\{1,2\}} (\|\bar{b}(d_i)\|_2 + \|c(d_i)\|_2) \right) \\ & \left(\left\|\bar{Q}(d_1) - \bar{Q}(d_2)\right\|_2 + \|A(d_1) - A(d_2)\|_2 + \left\|A^T(d_1) - A^T(d_2)\right\|_2\right) \\ & = \hat{\varrho} \left\|rb(d_1) - rb(d_2)\right\|_2 + \hat{\varrho}^2 \left(\max_{i\in\{1,2\}} \|rb(d_i)\|_2 + \|c\|_2 \right) \|rQ(d_1) - rQ(d_2)\|_2 \\ & = r\hat{\varrho} \left\|b(d_1) - b(d_2)\right\|_2 + r^2\hat{\varrho}^2 \left(\max_{i\in\{1,2\}} \|b(d_i)\|_2 + \frac{1}{r}\|c\|_2 \right) \left\|Q(d_1) - Q(d_2)\right\|_2, \end{aligned}$$

where $\hat{\varrho} \leq \frac{1}{r\alpha} + \frac{1}{\beta} \left(\frac{rv_1}{r\alpha} + 1\right) \left(v_2 + \frac{rv_2v_1}{\beta} + 1\right)$ or equivalently

$$r\hat{\varrho} \leq \frac{1}{\alpha} + \frac{r}{\beta} \left(\frac{v_1}{\alpha} + 1\right) \left(v_2 + \frac{rv_2v_1}{\beta} + 1\right).$$

The result follows by defining $\varrho_0 = r\hat{\varrho}$ and substituting $r = \frac{1}{\max\{1,\|Q(d_1)\|_2\}}$ into the above inequality. This completes the proof. □

The following theorem establishes an upper bound on the size of change in the optimal execution strategy, due to perturbation in the impact matrices, when buying is not permitted.

Theorem 3.3.3. *Assume for the given risk aversion parameter $\mu \geq 0$, $W(H,G,\mu) \succ 0$ and $W(H+\Delta H, G+\Delta G, \mu) \succ 0$. Denote the unique solutions of problems (3.2.9) and (3.3.1) with z^* and $\bar{z}$ respectively. Then there exists some $\varsigma > 0$ such that*

$$\begin{aligned} \|n^* - \bar{n}\|_2 &\leq 2\,\|z^* - \bar{z}\|_2 \\ &\leq 2\varsigma \left\|\bar{S}\right\|_2 \left(1 + 4\varsigma\sqrt{m}\,(\max\{1, \|W\|_2\} + \|\Theta\|_2 + \|\Delta\Theta\|_2)\right) \|\Delta\Theta\|_2, \end{aligned} \tag{3.3.19}$$

where $n^* = \{n_k^*\}_{k=1}^N$ and $\bar{n} = \{\bar{n}_k\}_{k=1}^N$ are the optimal execution strategies associated with z^* and $\bar{z}$ respectively, and

$$\varsigma \leq \frac{1}{\underline{\lambda}} \left(1 + \frac{1}{2\sin^2\left(\frac{\pi}{4N-2}\right)} \left(\frac{\bar{\lambda} + \underline{\lambda}}{\max\{1, \lambda_{\max}(W)\}}\right) \left(\frac{\bar{\lambda}}{\max\{1, \lambda_{\max}(W)\}} + 3\sin\left(\frac{\pi}{4N-2}\right)\right)\right),$$

with $\bar{\lambda} = \max\limits_{\eta\in[0,1]} \lambda_{\max}(W + \eta\Delta W)$ and $\underline{\lambda} = \min\limits_{\eta\in[0,1]} \lambda_{\min}(W + \eta\Delta W)$.

Proof. For the given perturbations ΔG and ΔH of the impact matrices, define

$$\mathcal{D} \stackrel{\text{def}}{=} \{d(\eta) = (H + \eta\Delta H, G + \eta\Delta G) : \ \eta \in [0,1]\}.$$

Clearly, $\mathcal{D}$ is a convex set. Since $W(H,G,\mu) \succ 0$ and $W(H+\Delta H, G+\Delta G, \mu) \succ 0$ for any $\eta \in [0,1]$, we have

$$W + \eta\Delta W = W(H + \eta\Delta H, G + \eta\Delta G, \mu) = (1-\eta)W(H,G,\mu) + \eta W(H+\Delta H, G+\Delta G, \mu) \succ 0.$$

Therefore, for any $\eta \in [0,1]$, problem (3.2.9) with the impact matrices $H + \eta\Delta H$ and $G + \eta\Delta G$ has a unique solution, denoted as $z(\eta)$. For a given $\eta \in [0,1]$, let $J(\eta)$ denote the set of indices corresponding to the binding constraints of problem (3.2.9) at $z(\eta)$. Lemma 3.3.1 implies that $\lambda_{\min}\left(A_{J(\eta)}A_{J(\eta)}^T\right) > 0$ and consequently $A_{J(d)}A_{J(d)}^T$ is invertible. Thus the rows of $A_{J(d)}$ are linearly independent and problem (3.2.9) has a unique Lagrange multiplier $u(\eta)$.

Define

$$\bar{\lambda} \stackrel{\text{def}}{=} \max\{\lambda_{\max}(W + \eta\Delta W) : \eta \in [0,1]\}, \quad \underline{\lambda} \stackrel{\text{def}}{=} \min\{\lambda_{\min}(W + \eta\Delta W) : \eta \in [0,1]\}.$$

Since for any $\eta \in [0,1]$, $W + \eta\Delta W$ is symmetric positive definite, $\bar{\lambda}$ and $\underline{\lambda}$ are positive. In addition,

$$\|W + \eta\Delta W\|_2 = \lambda_{\max}(W + \eta\Delta W) \leq \bar{\lambda} \quad \forall\eta \in [0,1]. \tag{3.3.20}$$

Using the Courant-Fischer Theorem, we have $\lambda_{\max}(W + \eta\Delta W) \leq \lambda_{\max}(W) + \eta\lambda_{\max}(\Delta W)$, which implies $\bar{\lambda} < +\infty$. Furthermore, for any $\eta \in [0,1]$, $W + \eta\Delta W$ is symmetric. The Courant-Fischer Theorem yields

$$z^T(W + \eta\Delta W)z \geq \lambda_{\min}(W + \eta\Delta W)\|z\|_2^2 \geq \underline{\lambda}\|z\|_2^2 \quad \forall\eta \in [0,1]. \tag{3.3.21}$$

Applying the definitions of 1-norm, $\|.\|_1$, and ∞-norm, $\|.\|_\infty$, for the matrix $A^T_{J(\eta)}$, we get

$$\left\|A^T_{J(\eta)}\right\|_1 = \max_{i\in J(\eta)} \sum_{j=1}^{N-1} |a_{ij}| \leq \max_{i=1,2,\ldots,N} \sum_{j=1}^{N-1} |a_{ij}| = \|A\|_\infty\,, \quad \forall \eta \in [0,1]$$

$$\left\|A^T_{J(\eta)}\right\|_\infty = \max_{j=1,2,\ldots,N-1} \sum_{i\in J(\eta)} |a_{ij}| \leq \max_{j=1,2,\ldots,N-1} \sum_{i=1}^{N} |a_{ij}| = \|A\|_1\,, \quad \forall \eta \in [0,1],$$

where a_{ij} is the entry of A in the ith row and jth column. Hence

$$\left\|A^T_{J(\eta)}\right\|_2 \leq \sqrt{\left\|A^T_{J(\eta)}\right\|_1 \left\|A^T_{J(\eta)}\right\|_\infty} \leq \sqrt{\|A\|_1 \|A\|_\infty} \leq \sqrt{\|(Y\otimes I_m)\|_1 \|(Y\otimes I_m)\|_\infty} \leq 2,$$

where the last inequality follows from $\|Y\otimes I_m\|_\infty = 2$ and $\|Y\otimes I_m\|_1 = 2$. Therefore,

$$\left\|A^T_{J(\eta)}\right\|_2 \leq 2 \quad \forall \eta \in [0,1]. \tag{3.3.22}$$

For any $\eta \in [0,1]$ and the associated unique optimal Lagrange multiplier $u(\eta)$, the Courant-Fischer Theorem yields

$$\left\|A^T_{J(\eta)}u(\eta)\right\|_2^2 = u(\eta)^T A_{J(\eta)}A^T_{J(\eta)}u(\eta) \geq \lambda_{\min}\left(A_{J(\eta)}A^T_{J(\eta)}\right)\|u(\eta)\|_2^2 \geq \lambda_{\min}\left(J_{N-1}J^T_{N-1}\right)\|u(\eta)\|_2^2\,, \tag{3.3.23}$$

where the last inequality comes from Lemma 3.3.1. Hence

$$\left\|A^T_{J(\eta)}u(\eta)\right\|_2 \geq 2\sin\left(\frac{\pi}{4N-2}\right)\|u(\eta)\|_2 \quad \forall \eta \in [0,1]. \tag{3.3.24}$$

Inequalities (3.3.20), (3.3.21), (3.3.22), and (3.3.24) show that the assumptions of Theorem 3.3.2 are satisfied on the convex data set $\mathcal{D}$ for

$$v_1 \stackrel{\text{def}}{=} \bar{\lambda}, \quad v_2 \stackrel{\text{def}}{=} 2, \quad \beta \stackrel{\text{def}}{=} 2\sin\left(\frac{\pi}{4N-2}\right) > 0, \quad \alpha \stackrel{\text{def}}{=} \underline{\lambda} > 0.$$

Applying Corollary 3.3.2 to problem (3.2.9), there exists some ς such that

$$\begin{aligned}\|z^* - \bar{z}\|_2 &\leq \varsigma\|\Delta b\|_2 + \varsigma^2\|\Delta W\|_2(\max\{\|b(H,G)\|_2, \|b(H+\Delta H, G+\Delta G)\|_2\} \\ &\quad + \max\{1,\|W\|_2\}\|\bar{S}\|_2) \\ &\leq \varsigma\|\Delta\Theta\bar{S}\|_2 + \varsigma^2\|\Delta W\|_2\left(\|\Theta\bar{S}\|_2 + \|\Delta\Theta\bar{S}\|_2 + \max\{1,\|W\|_2\}\|\bar{S}\|_2\right) \\ &\leq \varsigma\|\bar{S}\|_2\left(\|\Delta\Theta\|_2 + \varsigma\|\Delta W\|_2\left(\|\Theta\|_2 + \|\Delta\Theta\|_2 + \max\{1,\|W\|_2\}\right)\right),\end{aligned}$$

where $\varsigma \leq \frac{1}{\underline{\lambda}}\left(1 + \frac{1}{2\sin^2\left(\frac{\pi}{4N-2}\right)}\left(\frac{\bar{\lambda}+\underline{\lambda}}{\max\{1,\lambda_{\max}(W)\}}\right)\left(\frac{\bar{\lambda}}{\max\{1,\lambda_{\max}(W)\}} + 3\sin\left(\frac{\pi}{4N-2}\right)\right)\right)$. Applying inequality (3.3.9), i.e., $\|\Delta W\|_2 \leq 4\sqrt{m}\|\Delta\Theta\|_2$, and inequality (3.3.4) completes the proof. $\square$

Theorem 3.3.3 provides an upper bound for the size of the change in the optimal execution strategy, when buying is not permitted. For a given N, the upper bound of the magnification factor ς depends, at least asymptotically (as $\Delta W \to 0$), only on the eigenvalues of the Hessian matrix W.

Similar to Theorem 3.3.1, a small value of ς guarantees that the optimal execution strategy is not very sensitive to the perturbation in the combined impact matrix $\Delta\Theta$. As $\Delta W \to 0$, we have $\bar{\lambda} \approx \lambda_{\max}(W)$. Hence, the term

$$\left(1 + \frac{1}{2\sin^2\left(\frac{\pi}{4N-2}\right)} \left(\frac{\bar{\lambda} + \underline{\lambda}}{\max\{1, \lambda_{\max}(W)\}}\right) \left(\frac{\bar{\lambda}}{\max\{1, \lambda_{\max}(W)\}} + 3\sin\left(\frac{\pi}{4N-2}\right)\right)\right), \tag{3.3.25}$$

is bounded by a constant which depends only on N. Therefore, for the fixed number of periods N, asymptotically (as $\Delta W \to 0$), the upper bound for the magnification factor ς is small when $\lambda_{\min}(W)$ is large.

The eigenvalues $\lambda_{\max}(W)$, $\underline{\lambda}$ and $\bar{\lambda}$ increase with the same rate as the risk aversion parameter μ increases, and consequently, all the terms in (3.3.25) are bounded as $\mu \to +\infty$. However, when $C \succ 0$, $\frac{1}{\underline{\lambda}}$ approaches zero as $\mu \to +\infty$. Therefore, as the risk aversion parameter μ increases, the upper bound for ς becomes small. Hence, when the covariance matrix is positive definite, $\|z^* - \bar{z}\|_2 \to 0$ as $\mu \to +\infty$. This result indicates that, similar to the case that buying is allowed, sensitivity of the optimal execution strategy to perturbation in the impact matrices diminishes as the risk aversion parameter μ increases.

We can express the upper bound for the magnification factor ς provided in Theorem 3.3.3 in terms of the eigenvalues of C and Θ, when the permanent impact matrix G and its perturbation ΔG are symmetric. Under these assumptions, the Courant-Fischer Theorem can be applied, and we have

$$\begin{aligned}
\underline{\lambda} &= \min_{\eta\in[0,1]} \lambda_{\min}(W + \eta\Delta W) \geq 2\mu\tau\lambda_{\min}(C) + \lambda_{\min}(J_{N-1} + J_{N-1}^T) \min_{\eta\in[0,1]} \lambda_{\min}(\Theta + \eta\Delta\Theta),\\
\bar{\lambda} &= \max_{\eta\in[0,1]} \lambda_{\max}(W + \eta\Delta W) \leq 2\mu\tau\lambda_{\max}(C) + \lambda_{\max}(J_{N-1} + J_{N-1}^T) \max_{\eta\in[0,1]} \lambda_{\max}(\Theta + \eta\Delta\Theta).
\end{aligned}$$

Applying these inequalities, the upper bound for ς, obtained in Theorem 3.3.3, can be simplified as follows:

Corollary 3.3.3. *Let the assumptions in Theorem 3.3.3 hold. In addition, assume that the matrices G and ΔG are symmetric. Then there exists a magnification factor ς_0 such that*

$$\|n^* - \bar{n}\|_2 \leq 2\|z^* - \bar{z}\|_2 \leq \varsigma_0\|\bar{S}\|_2 \left(1 + 2\varsigma_0\sqrt{m}\left(\max\{1, \|W\|_2\} + \|\Theta\|_2 + \|\Delta\Theta\|_2\right)\right)\|\Delta\Theta\|_2,$$

where

$$\begin{aligned}
\varsigma_0 \;\leq\; & \left(\frac{1}{\mu\tau\lambda_{\min}(C) + 2\sin^2\left(\frac{\pi}{2N}\right)\min_{\eta\in[0,1]} \lambda_{\min}(\Theta + \eta\Delta\Theta)}\right) \\
& \left(1 + \frac{1}{2\sin^2\left(\frac{\pi}{4N-2}\right)} \left(\frac{\bar{\lambda} + \underline{\lambda}}{\max\{1, \lambda_{\max}(W)\}}\right) \left(\frac{\bar{\lambda}}{\max\{1, \lambda_{\max}(W)\}} + 3\sin\left(\frac{\pi}{4N-2}\right)\right)\right).
\end{aligned} \tag{3.3.26}$$

Inequality (3.3.26) indicates that when the permanent impact matrices G and $G+\Delta G$ are symmetric, the magnification factor ς_0 asymptotically (as $\Delta W \to 0$) depends on $\mu\tau\lambda_{\min}(C) + 2\sin^2\left(\frac{\pi}{2N}\right)\lambda_{\min}(\Theta)$. In this case, for a given positive risk aversion parameter, ς_0 becomes small when *either* the minimum eigenvalue of the covariance matrix *or* the minimum eigenvalue of the combined impact matrix Θ is large. However, when both $\mu\lambda_{\min}(C)$ and $\lambda_{\min}(\Theta)$ are small, the upper bound for ς_0 in (3.3.26) becomes large, which suggests pronounced sensitivity of the optimal execution strategy to estimation errors in the impact matrices.

Both upper bounds in inequalities (3.3.5) and (3.3.19) indicate that the change in the optimal execution strategy increases proportionally with respect to the size of the initial portfolio holding $\bar{S}$. Next, we precisely analyze the dependence of the optimal execution strategy on the initial portfolio holding.

Lemma 3.3.2. *Consider the optimal portfolio execution problem* (3.2.9) *with the impact matrices H and G where $W(H,G,\mu) \succ 0$. Let $z^* = ((x_1^*)^T, \cdots, (x_{N-1}^*)^T)^T$ be the solution with the initial portfolio holding $\bar{S}$. Then, for every $\alpha \geq 0$, αz^* is the solution of problem* (3.2.9) *with the initial portfolio holding $\alpha\bar{S}$.*

Proof. First note that z is a feasible point of problem (3.2.9) with the initial portfolio holding $\bar{S}$ if and only if αz is a feasible point of problem (3.2.9) with the initial portfolio holding $\alpha\bar{S}$. Since z^* is the solution, for every feasible point z of problem (3.2.9), we have

$$\frac{1}{2}(z^*)^T W(H,G,\mu) z^* - (\Theta\bar{S})^T x_1^* \leq \frac{1}{2} z^T W(H,G,\mu) z - (\Theta\bar{S})^T x_1,$$

where x_1^* and x_1 are positions in the first period corresponding to z^* and z respectively. Therefore, multiplying the above inequality by α^2, we get

$$\frac{1}{2}(\alpha z^*)^T W(H,G,\mu)(\alpha z^*) - (\alpha\Theta\bar{S})^T(\alpha x_1^*) \leq \frac{1}{2}(\alpha z)^T W(H,G,\mu)(\alpha z) - (\alpha\Theta\bar{S})^T(\alpha x_1).$$

This result yields αz^* is the solution of problem (3.2.9) with the initial portfolio holding $\alpha\bar{S}$. $\square$

In the next section, we use simulations to illustrate sensitivity of the optimal execution strategy and the efficient frontier to perturbations in the impact matrices.

3.4 Computational Investigation

In this section, we use simulations to computationally investigate the influence of perturbations in the impact matrices on the optimal execution strategy and efficient frontier. With exception of §3.4.3, we assume that the covariance matrix is accurately given. The simulations are done using MATLAB Version 6.5.

Consider an investor who holds a portfolio of three different assets with the initial holding $\bar{S}_i = 10^5,\ i = 1, 2, 3$. The goal is to liquidate the holdings in five days by trading daily, i.e., $T = 5$, $N = 5$, and $\tau = 1$. Let the true daily asset price covariance matrix be[1]

$$C = \begin{pmatrix} 0.324625 & 0.022983 & 0.420395 \\ 0.022983 & 0.049937 & 0.019247 \\ 0.420395 & 0.019247 & 0.764097 \end{pmatrix} \times 1\%, \tag{3.4.1}$$

Note that $\lambda_{\min}(C) = 0.00045310$, $\lambda_{\max}(C) = 0.01019550$, and $\kappa_2(C) = 22.50175986$. The price impact model (2.1.4) assumes that the price impacts are proportional to the trading rate. Assume that the median daily trading volume of each asset is one million shares. For the temporary impact matrix, we suppose that for each 10% of the daily volume traded, the price impact equals the daily variance. In addition, we assume that selling 20% of the daily volume incurs a permanent price depression equal to the daily variance. In other words,

$$\begin{aligned} H &= \frac{C}{0.10 \times 10^6} = 10^{-5} \cdot C \qquad \$ \text{ per share}^2, \\ G &= \frac{C}{0.20 \times 10^6} = (0.5 \times 10^{-5}) \cdot C \qquad \$ \text{ per day per share}^2. \end{aligned} \tag{3.4.2}$$

Note that $W(H, G, 0) \succ 0$ and $\lambda_{\min}(W(H, G, 0)) = 2.5960 \times 10^{-9}$. Throughout this section, we refer to H and G as the *true* impact matrices, and to the corresponding optimal execution strategy as the *true optimal execution strategy*.

In our simulation investigation, we assume that perturbations in the impact matrices have independent normal distributions. Specifically,

$$\Delta H = \rho \max \{\|He_i\|_\infty,\ i = 1, 2, 3\}\, \phi, \quad \Delta G = \rho \max \{\|Ge_i\|_\infty,\ i = 1, 2, 3\}\, \varphi, \tag{3.4.3}$$

where ϕ and φ are 3×3 random matrices whose elements are independent zero-mean Gaussian random variables with unit variance. We use the randn command in MATLAB to generate ϕ and φ. The parameter $\rho \in [0, 1]$ indicates the size of the relative perturbation.

In order to ensure that the optimal execution strategy corresponding to the perturbed impact matrices, $H + \Delta H$ and $G + \Delta G$, is unique, we only consider perturbation with $W(H + \Delta H, G + \Delta G, 0) \succ 0$. We refer to the optimal execution strategy determined from a pair of perturbed impact matrices as the *estimated optimal execution strategy*. We use the convex quadratic optimization solver mskqpopt in the software package MOSEK Version 4.0 to compute the optimal execution strategies, both in the presence and absence of no-buying constraints.

In §3.4.1, we illustrate sensitivity of the optimal execution strategy to perturbations in the impact matrices and present plots of true and estimated optimal execution strategies. Sensitivity of the efficient frontier is demonstrated in §3.4.2. In §3.4.3, the effect of estimation errors in the covariance matrix C on the optimal execution strategy and the efficient frontier is compared with the effect of perturbations in the impact matrices.

[1]This is the covariance matrix of three risky assets used in Rockafellar and Uryasev (2000).

3.4.1 Sensitivity of The Optimal Execution Strategy

In this section, we investigate effect of the risk aversion parameter μ and no-buying constraints on the sensitivity of the optimal execution strategy. For illustration, we focus on the cases when the risk aversion parameter $\mu = 0$, which corresponds to minimizing the expected execution cost, and $\mu = 10^{-5}$. Following Proposition 3.2.1, the true optimal execution strategy, which minimizes the expected execution cost, is the naive execution strategy $n_k = \frac{1}{5}\bar{S}$ for $k = 1, \ldots, 5$, since in our assumed setting G is symmetric and $\Theta \succ 0$. On the other hand, the perturbed impact matrices $H + \Delta H$ and $G + \Delta G$ from (3.4.3) are typically asymmetric. For each simulation study, a relative perturbation $\rho = 0.05$ is assumed.

Figure 3.1 plots the true optimal execution strategy when $\mu = 0$ (the naive execution strategy) against optimal execution strategies corresponding to 50 simulated perturbed impact matrices. The left plots are generated under the assumption that buying is allowed. For the plots on the right, it is assumed that buying is not permitted. Graphs in Figure 3.1 demonstrate that the optimal execution strategy in this case is quite sensitive to perturbations in the impact matrices. In addition, these plots illustrate that imposing no-buying constraints on the problem significantly decreases the sensitivity of the optimal execution strategy to perturbations. Note that the range in the number of shares traded (vertical axis), when buying is allowed, is much larger than the range after imposing no-buying constraints.

For the risk aversion parameter $\mu = 10^{-5}$, the true optimal execution strategy and estimated optimal execution strategies associated with the same set of perturbed impact matrices are plotted in Figure 3.2. Similar to the previous case, the left plots are generated under the assumption that buying is allowed. For the plots on the right, it is assumed that buying is not permitted. Comparing Figure 3.2 with Figure 3.1, it is clear that, sensitivity of the optimal execution strategy to perturbations in the impact matrices is decreased when the risk aversion parameter $\mu = 10^{-5}$. Moreover, in Figure 3.2, there is little difference in sensitivity of the optimal execution strategy to perturbations whether no-buying constraints are imposed or not.

In addition, for each asset, we compute ratio of the average difference, between the true and estimated optimal execution strategies, to the initial holding, i.e.,

$$\varepsilon_i(\mu) \stackrel{\text{def}}{=} \frac{1}{M\bar{S}_i} \sum_{\ell=1}^{M} \|n_i^{(\ell)} - n_i^*\|_1, \qquad i = 1, 2, 3, \tag{3.4.4}$$

where the vector $n_i^{(\ell)}$ is the estimated optimal execution strategy of the ith asset in the ℓth simulation. Table 3.1 presents the values of $\varepsilon_i(\mu)$ for various choices of μ for $M = 5000$ simulations. From Table 3.1, we observe that, whether buying is allowed or not, the relative average error $\varepsilon_i(\mu)$ decreases as the risk aversion parameter μ increases. For example while the relative average error in asset 2 is 37.3619% for $\mu = 10^{-5}$, it becomes less than 0.1% for $\mu \geq 0.05$. This observation is consistent with our analytical result that the change in the optimal execution strategy decreases as the risk aversion parameter increases. Table 3.1 also confirms that the optimal execution strategy when buying is prohibited is less sensitive than

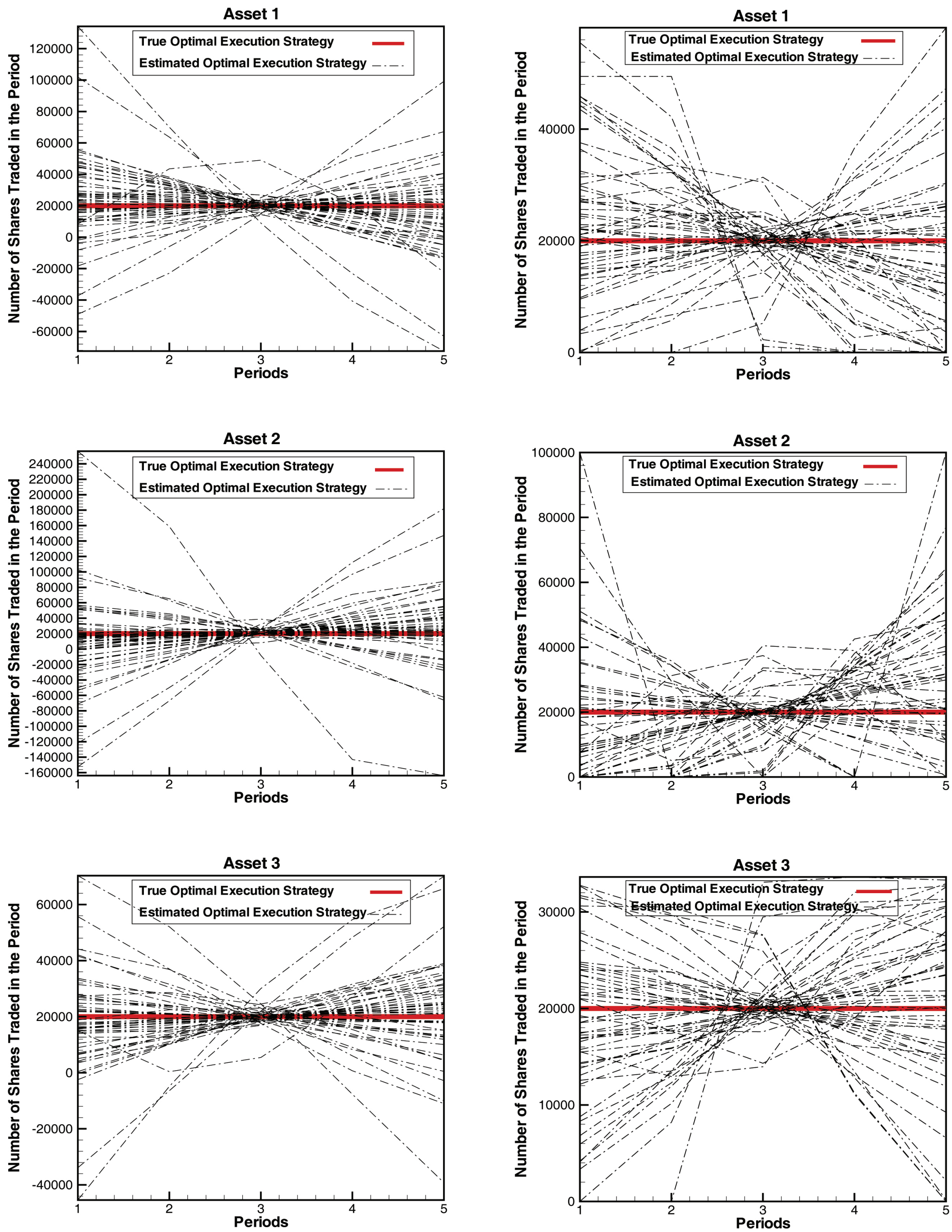


Figure 3.1: Optimal execution strategies for $\mu = 0$ with 5% relative perturbations in the impact matrices for 50 simulations. For plots on the left, buying is allowed. Buying is prohibited for plots on the right.

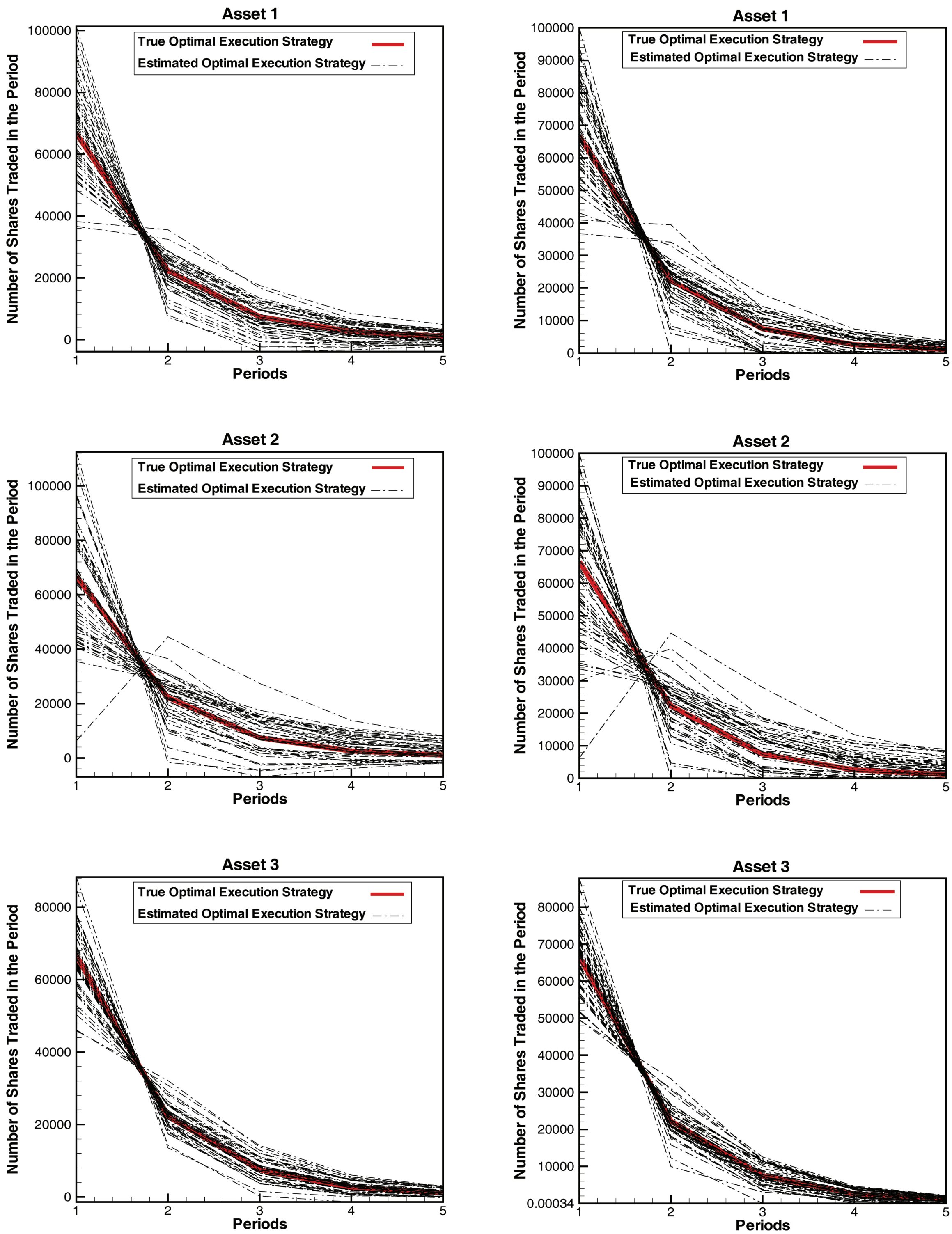


Figure 3.2: Optimal execution strategies for $\mu = 10^{-5}$ with 5% relative perturbations in the impact matrices for 50 simulations. For plots on the left, no constraint is imposed. Buying is prohibited for plots on the right.

μ	Buying is allowed			Buying is prohibited		
	Asset 1	Asset 2	Asset 3	Asset 1	Asset 2	Asset 3
0	154.3614	229.6044	229.6111	43.7554	53.4204	53.4303
10^{-8}	122.0550	169.4105	74.2133	43.6052	53.2656	26.8415
10^{-5}	24.0294	37.3619	14.1912	22.7091	34.4506	13.5175
0.05	0.0196	0.0349	0.0115	0.0177	0.0317	0.0105
0.5	0.0020	0.0035	0.0012	0.0018	0.0032	0.0011
1	0.0010	0.0017	0.0006	0.0009	0.0016	0.0005

Table 3.1: Relative average error $\varepsilon_i(\mu)$ (percentage) with 5% relative perturbations in the impact matrices, with general (likely asymmetric) perturbations in the permanent impact matrix, based on 5000 simulations.

μ	Buying is allowed			Buying is prohibited		
	Asset 1	Asset 2	Asset 3	Asset 1	Asset 2	Asset 3
0	9.9306×10^{-10}	1.4253×10^{-9}	1.4254×10^{-9}	0.0005	0.0008	0.0008
10^{-8}	0.9364	1.3388	0.5476	0.9181	1.3250	0.5382
10^{-5}	21.3055	34.2984	12.5860	20.6919	32.2731	12.2386
0.05	0.0187	0.0339	0.0111	0.0173	0.0311	0.0103
0.5	0.0019	0.0034	0.0011	0.0017	0.0031	0.0010
1	0.0009	0.0017	0.0006	0.0009	0.0016	0.0005

Table 3.2: Relative average error $\varepsilon_i(\mu)$ (percentage) with 5% relative perturbations in the impact matrices, with symmetric perturbations in the permanent impact matrix, based on 5000 simulations.

the one obtained when buying is allowed. This difference is more striking for small values of μ. As μ increases, the difference between the two cases almost diminishes.

Given $G = G^T$ in our example, Proposition 3.2.1 implies that, when perturbation in the permanent impact matrix satisfies $\Delta G^T = \Delta G$, the unique minimum expected execution cost strategy is the naive execution strategy. In the settings of our computation, the only possible violation is the asymmetric perturbation in the permanent impact matrix. This evinces the importance of maintaining symmetry in estimating the permanent impact matrix if it is known or assumed to be symmetric. To illustrate how restricting to symmetric perturbations affects sensitivity of the optimal execution strategy, we compute $\varepsilon_i(\mu)$ for estimated optimal execution strategies with the perturbed impact matrices $H + \Delta H$ and $G + \frac{1}{2}(\Delta G + \Delta G^T)$, where ΔH and ΔG are determined according to (3.4.3). Table 3.2 presents these values.

Comparing Table 3.2 with Table 3.1, we observe that the relative average error $\varepsilon_i(\mu)$, with symmetric perturbations in the permanent impact matrix, is smaller than the relative average error $\varepsilon_i(\mu)$ in Table 3.1 with asymmetric perturbations. This difference is more significant for small values of μ, particularly $\mu < 10^{-5}$. For $\mu \geq 10^{-5}$ there is little difference in the relative average errors in Table 3.2 with those in Table 3.1. Thus, when the permanent impact matrix G is known to be symmetric and this property is maintained with its estimate, an investor who wants to minimize only the expected execution cost need not to worry about the effect of estimation errors in the impact matrices on the optimal execution strategy.

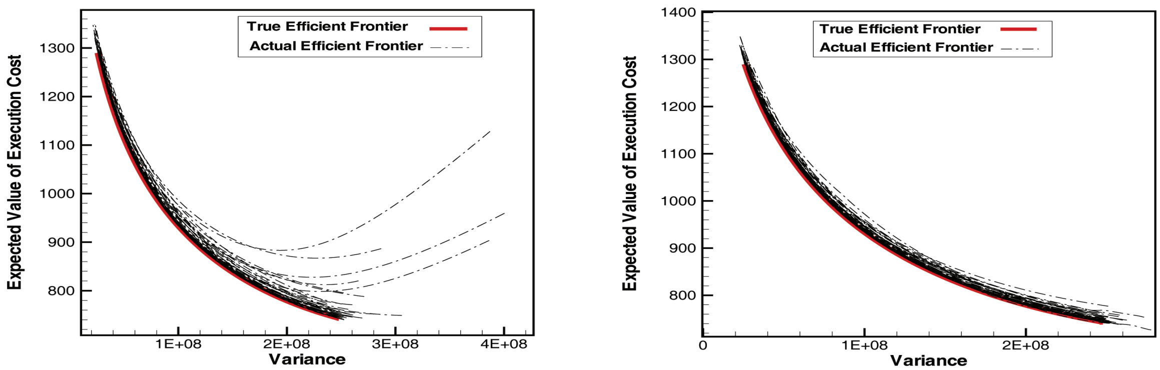
3.4.2 Sensitivity of The Efficient Frontier

In this section, we illustrate the effect of perturbations in the impact matrices on the efficient frontier in the space of the variance and the expected execution cost. For a given pair of perturbed impact matrices $H + \Delta H$ and $G + \Delta G$, we compute the following three efficient frontiers for $\mu \in [0, 10^{-5}]$:

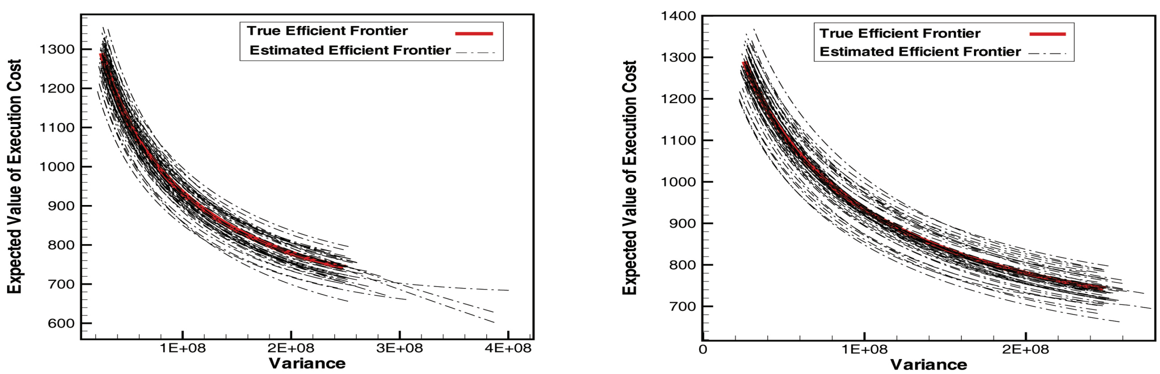
- The *true (efficient) frontier* is the efficient frontier computed from the true values of the impact matrices H and G.
- The *actual (efficient) frontier* is the curve of the true mean and variance of the execution cost of the optimal execution strategy determined from the perturbed impact matrices $H + \Delta H$ and $G + \Delta G$. The actual frontier depicts the true performance of estimated optimal execution strategies for various values of μ.
- The *estimated (efficient) frontier* is the efficient frontier computed from the perturbed impact matrices $H + \Delta H$ and $G + \Delta G$.

The notions of the actual frontier and estimated frontier have been used in (Broadie, 1993) to investigate the effect of estimation errors in mean return and the covariance matrix in the traditional mean-variance portfolio optimization. As mentioned in (Broadie, 1993), the estimated frontier is what appears to be the case based on estimated input data, but the actual frontier is what really occurs based on the true values of the data. Since the true values of the data are unknown, the true and actual frontiers are unobservable in practice. Note that actual frontiers can never be below the true efficient frontier as the optimal portfolio execution problem is a minimization problem. However, estimated frontiers can be either above or below the true and actual frontiers.

When perturbation in the permanent impact matrix is asymmetric, the effect of perturbations in the impact matrices on the efficient frontier is demonstrated in Figure 3.3. Figure 3.3 (a) illustrates deviations of actual frontiers from the true efficient frontier for $\mu \in [0, 10^{-5}]$. From the plot on the left generated under the assumption that buying is allowed, it can be observed that large deviations of actual frontiers can occur, particularly when the risk aversion parameter is very small. Moreover, the lengths of the actual frontier from different simulations vary drastically; the lengths of some actual frontiers a great deal differ from the length of the true frontier. Figure 3.3 (a) also demonstrates that, similar to the sensitivity of the optimal execution strategy, the change in the efficient frontier decreases as μ increases.

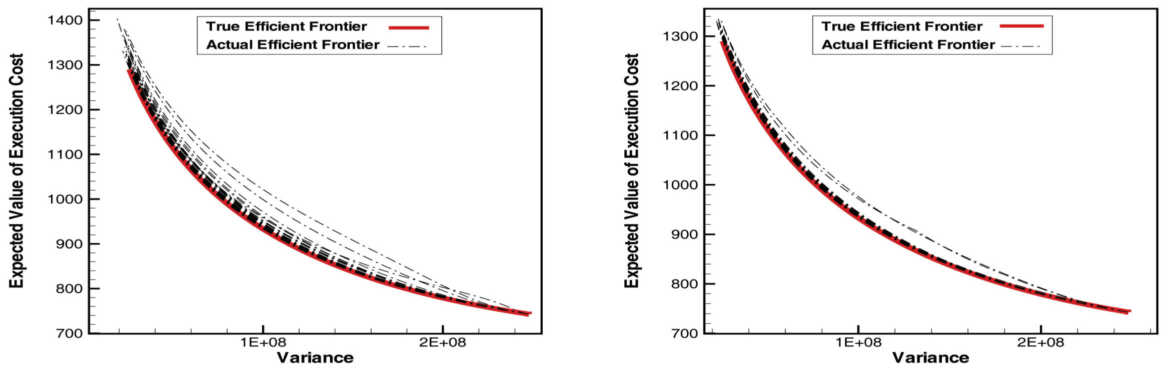


(a) Actual Frontiers versus the True Frontier

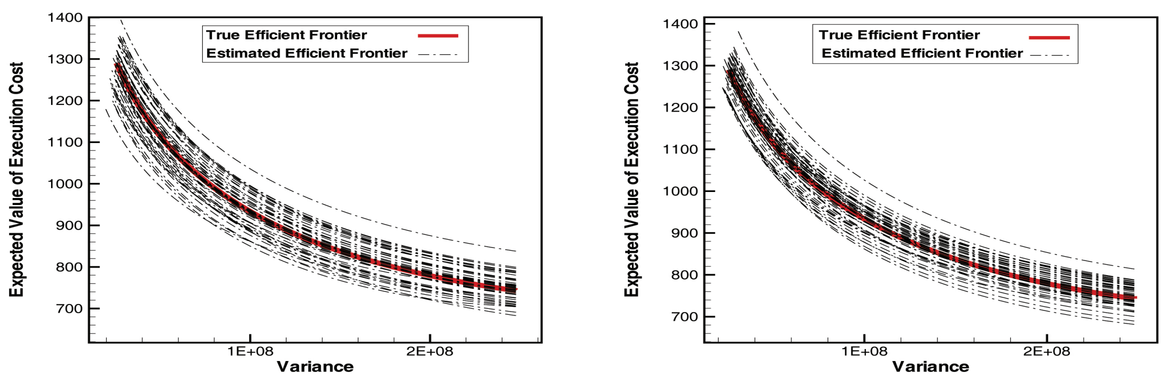


(b) Estimated Frontiers versus the True Frontier

Figure 3.3: Actual and estimated frontiers with 5% relative perturbations in the impact matrices, with general (likely asymmetric) perturbations in the permanent impact matrix, for 50 simulations. Buying is allowed for plots on the left. Buying is prohibited for plots on the right.



(a) Actual Frontiers versus the True Frontier



(b) Estimated Frontiers versus the True Frontier

Figure 3.4: Actual and estimated frontiers with 5% relative perturbations in the impact matrices, with symmetric perturbations in the permanent impact matrix, for 50 simulations. Buying is allowed for plots on the left. Buying is prohibited for plots on the right.

Therefore, an execution strategy that minimizes the variance of the execution cost can be estimated more accurately than the one that minimizes the mean of the execution cost. This is consistent with our theoretical results.

Comparing the right plot to the left plot in Figure 3.3 (a), we observe that deviations of actual frontiers from the true frontier are significantly reduced in the right plot in which buying is prohibited. There is also less variation in the length of actual frontiers. Thus, imposing no-buying constraints significantly decreases the effect of estimation errors in impact matrices. Similar phenomena has been reported in the mean-variance portfolio optimization, see, e.g., (Frost and Savarino, 1988; Best and Grauer, 1991; Jagannathan and Ma, 2003).

Figure 3.3 (b) depicts deviations of estimated frontiers from the true frontier. Comparing to plots in Figure 3.3 (a), there seems to be less difference in the deviations for different risk aversion parameters, whether no-buying constraints are imposed or not. In addition, for a large risk aversion parameter, we observe larger deviations in the estimated frontiers than in the actual frontiers. On the other hand, deviations of estimated frontiers are smaller than those of actual frontiers for a small risk aversion parameter.

Let perturbed impact matrices be $H+\Delta H$ and $G+\frac{1}{2}(\Delta G+\Delta G^T)$ where ΔH and ΔG are defined as in (3.4.3); thus the permanent impact matrix perturbation is symmetric. For these symmetric perturbations in the permanent impact matrix, Figure 3.4 illustrates differences between actual frontiers and the true efficient frontier are significantly reduced when the risk aversion parameter is small. In particular, when the risk aversion parameter is near zero, actual frontiers are very close to the true frontier. Maintaining symmetry does not seem to affect the sensitivity at the left end of the frontier for a large value of μ. In addition, we note that the assumption of symmetry in the permanent impact matrix has little effect on estimated frontiers.

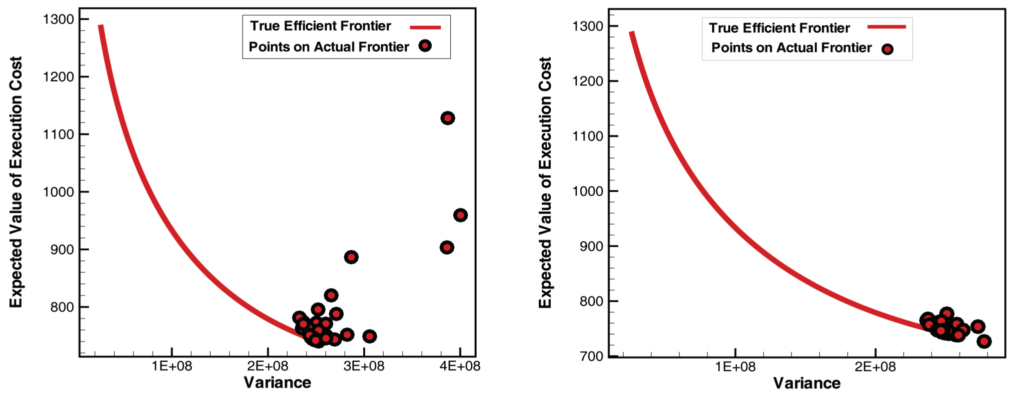
We now compare sensitivity of the mean of the execution cost with sensitivity of the variance of the execution cost. Figure 3.5 displays (mean,variance) points on the actual frontiers for $\mu = 0$ and $\mu = 10^{-5}$. The left plots are generated when buying is permitted. For the plots on the right, buying is prohibited. This figure suggests that, when $\mu = 0$, variation in the variance of the execution cost is relatively larger than the variation in the mean of the execution cost. For $\mu = 10^{-5}$, on the other hand, the relative variation in the mean is larger than the variance of the execution cost.

To quantify the relative variations in the mean and variance of the actual execution cost, for a given risk aversion parameter μ, a relative perturbation ρ and a finite set of simulations M, we consider the following measures

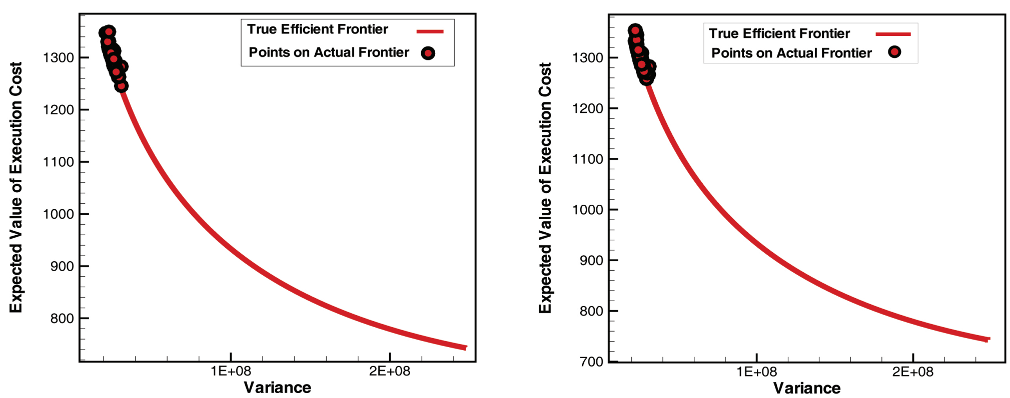
$$\varepsilon_{\text{var}}(\mu,\rho) \stackrel{\text{def}}{=} \frac{\max_{j=1}^{M} \text{var}_j(\mu) - \min_{j=1}^{M} \text{var}_j(\mu)}{|\text{true variance for } (\mu=0) - \text{ true variance for } (\mu=10^{-5})|}, \quad (3.4.5)$$

$$\varepsilon_{\text{mean}}(\mu,\rho) \stackrel{\text{def}}{=} \frac{\max_{j=1}^{M} \text{mean}_j(\mu) - \min_{j=1}^{M} \text{mean}_j(\mu)}{|\text{true mean for } (\mu=0) - \text{true mean for } (\mu=10^{-5})|},$$

where $(\text{var}_j(\mu), \text{mean}_j(\mu))$ is the coordinate of the actual frontier for the jth simulation. Table 3.3 displays values of $\varepsilon_{\text{var}}(\mu,\rho)$ and $\varepsilon_{\text{mean}}(\mu,\rho)$ using $M = 50$ simulations. These results



(a) Risk Aversion Parameter $\mu = 0$



(b) Risk Aversion Parameter $\mu = 10^{-5}$

Figure 3.5: Points on the actual frontier for $\mu = 0$ and $\mu = 10^{-5}$ with 5% relative perturbations in the impact matrices, with general (likely asymmetric) perturbations in the permanent impact matrix, for 50 simulations. Buying is allowed for plots on the left. For plots on the right, buying is prohibited.

ρ	Buying is allowed			
	$\mu = 0$		$\mu = 10^{-5}$	
	ε_{var}	$\varepsilon_{\text{mean}}$	ε_{var}	$\varepsilon_{\text{mean}}$
0.02	0.16026	0.09950	0.01271	0.05570
0.05	0.75538	0.70920	0.04420	0.18996
0.08	3.83520	3.05705	0.05605	0.26419
0.10	45.16636	33.01414	0.08737	0.36430

ρ	Buying is prohibited			
	$\mu = 0$		$\mu = 10^{-5}$	
	ε_{var}	$\varepsilon_{\text{mean}}$	ε_{var}	$\varepsilon_{\text{mean}}$
0.02	0.04643	0.03685	0.02038	0.08873
0.05	0.18433	0.09185	0.03951	0.17412
0.08	0.29582	0.18370	0.06779	0.33724
0.10	0.39132	0.14567	0.06600	0.27287

Table 3.3: Relative variation in variance and mean of the execution cost, $\varepsilon_{\text{var}}(\mu, \rho)$ and $\varepsilon_{\text{mean}}(\mu, \rho)$, due to perturbations in the impact matrices with general (likely asymmetric) perturbations in the permanent impact matrix, based on 50 simulations.

illustrate that, whether buying is allowed or not, the relative variation $\varepsilon_{\text{var}}(0, \rho)$ is larger than the relative variation in mean $\varepsilon_{\text{mean}}(0, \rho)$. On the other hand, the relative variation in variance $\varepsilon_{\text{var}}(10^{-5}, \rho)$ is less than the relative variation in mean $\varepsilon_{\text{mean}}(10^{-5}, \rho)$. In addition, Table 3.3 demonstrates that the difference in both mean and variance increases quickly and nonlinearly as the relative perturbation ρ increases. The fast increase is particularly prominent when buying is permitted.

3.4.3 Sensitivity to Perturbations in the Covariance Matrix

So far we have assumed that an accurate estimation for the covariance matrix is given, and we only considered the effect of perturbations in the impact matrices. This is in accordance with the recent literature on addressing estimation risks in the mean-variance portfolio optimization which focuses exclusively on the impact of sensitivity in the mean return by taking the covariance matrix as known, see, e.g., (TerHorst et al., 2006; Garlappi et al., 2007; Antoine, 2008). For the purpose of completeness, in this section, we compare the effect of estimation errors in the covariance matrix on the optimal execution strategy and the efficient frontier with the effect of perturbations in the impact matrices. In the sequel, we refer to the covariance matrix C, defined in (3.4.1), as the true covariance matrix. To simulate estimated covariance matrices, we use the perturbed matrix $C + \frac{1}{2}\left(\Delta C + \Delta C^T\right)$, where perturbations

μ	Buying is allowed			Buying is prohibited		
	Asset 1	Asset 2	Asset 3	Asset 1	Asset 2	Asset 3
0	0	0	0	0	0	0
10^{-8}	0.1980	0.3515	0.1167	0.1995	0.3519	0.1176
10^{-5}	17.7192	29.4778	10.4594	17.0635	28.3482	10.0912
0.05	0.0820	0.1398	0.0467	0.0188	0.0435	0.0111
0.5	0.0115	0.0179	0.0064	0.0019	0.0044	0.0011
1	0.0060	0.0092	0.0034	0.0009	0.0022	0.0006

Table 3.4: Relative average error $\varepsilon_i(\mu)$ (percentage) with 5% relative perturbations in the covariance matrix based on 5000 simulations.

ΔC have independent normal distributions. Specifically,

$$\Delta C = \frac{1}{2}\rho \max\{\|Ce_i\|_\infty, i = 1, 2, 3\}\,(\phi + \phi^T),$$

where ϕ is a 3×3 random matrix whose elements are independent zero-mean Gaussian random variables with unit variance. The parameter ρ indicates the size of the relative perturbation. To ensure that the perturbed matrix is positive definite and symmetric, we only consider perturbations with $C + \frac{1}{2}\left(\Delta C + \Delta C^T\right) \succ 0$. Throughout §3.4.3, we assume that accurate values of the impact matrices are given, and the true impact matrices defined in (3.4.2) are used.

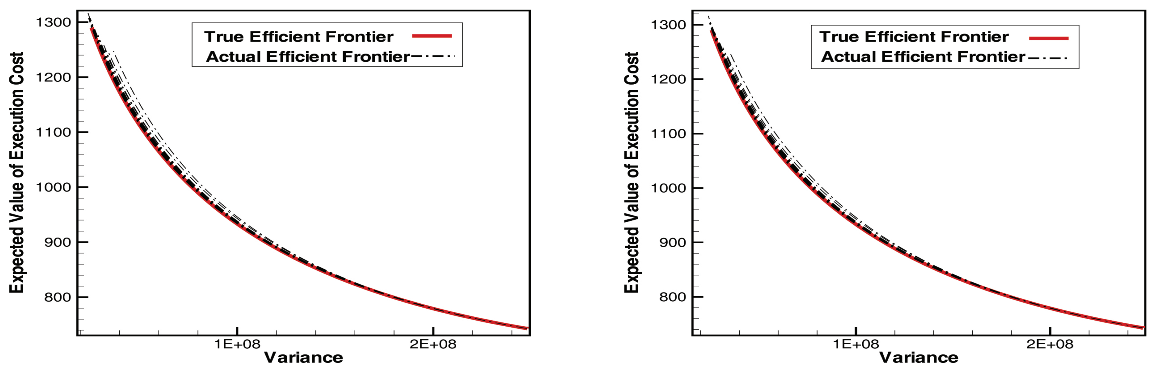
When the risk aversion parameter $\mu = 0$, any perturbation in the covariance matrix has no effect on the optimal execution strategy. However, when $\mu > 0$, perturbations in the covariance matrix might be influential. To assess the effect of perturbations in the covariance matrix on the optimal execution strategy, for every asset i, we compute the ratio of the average difference $\varepsilon_i(\mu)$, defined in equation (3.4.4), between the optimal execution strategy n_i^* corresponding to the true covariance matrix C, and the optimal execution strategy $n_i^{(\ell)}$ obtained from the perturbed covariance matrix $C + \frac{1}{2}\left(\Delta C + \Delta C^T\right)$ in the ℓth simulation. Table 3.4 illustrates these values for various choices of μ for $M = 5000$ simulations with $\rho = 5\%$ relative perturbations in the covariance matrix C. Similar to the perturbations in the impact matrices, Table 3.4 suggests that the effect of estimation errors in the covariance matrix on the optimal execution strategy depends on the choice of the risk aversion parameter μ. Table 3.4 maintains the same trend as in Table 3.2. However, for smaller values of μ, ($\mu \leq 10^{-5}$), the relative average error $\varepsilon_i(\mu)$ in Table 3.4 are notably smaller than those in Table 3.2. However, as μ increases the values of $\varepsilon_i(\mu)$ in Table 3.4 get slightly larger (less than 0.1%) than the values of the relative average error presented in Table 3.2. This indicates that an investor should concern more on the effect of estimation errors in the impact matrices on the optimal execution strategy than the influence of estimation errors in the covariance matrix, specially an investor who chooses small values for μ.

To investigate the effect of perturbation in the covariance matrix on the efficient frontier, we use the notions of true, estimated, and actual frontiers in a similar sense as defined in §3.4.2. Here ΔH and ΔG are zero and the covariance matrix C is perturbed. Figure 3.6 (a) depicts the effect of perturbations in the covariance matrix on the actual frontiers. Comparing the plots in Figure 3.6 (a) with Figure 3.4 (a), we observe that the difference between actual frontiers and the true frontier in Figure 3.6 (a) are less than the change between them in Figure 3.4 (a). This suggests that 5% perturbation in the impact matrices, even when symmetry is maintained, is more influential on the actual efficient frontier than perturbations of the same magnitute in the covariance matrix. Similar to the plots in Figure 3.4 (a), graphs in Figure 3.6 (a) demonstrate that differences between actual frontiers and the true efficient frontier are significantly reduced when the risk aversion parameter μ is small. Particularly, when the risk aversion parameter approaches zero, at the right end of the frontiers, actual frontiers converge to the true frontier. This behaviour is expected, as for $\mu = 0$, the variance of the execution cost and consequently the covariance matrix do not play any significant role in finding the optimal execution strategy. Deviations of the actual frontiers from the true frontier at the left end are almost similar to Figure 3.4 (a).

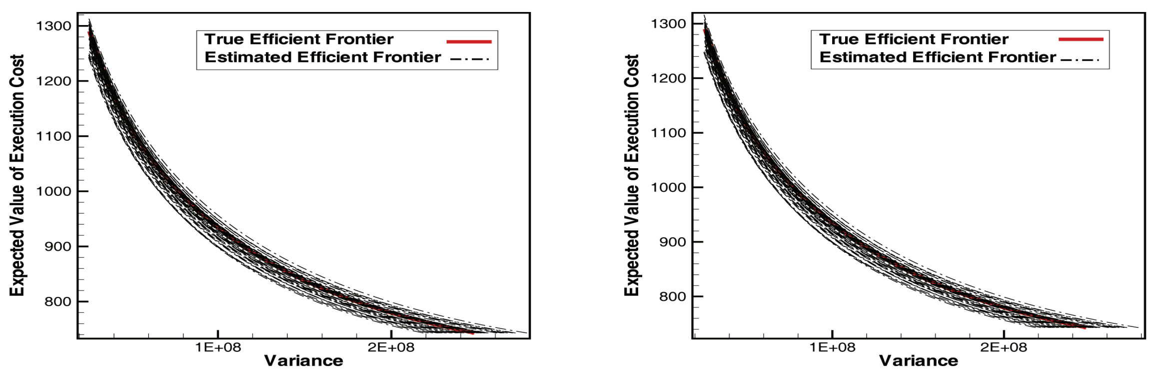
Deviations of estimated frontiers from the true frontier are depicted in Figure 3.6 (b). Comparing to plots in Figure 3.4 (b), there seems to be less difference in the deviations of the estimated frontier in 3.6 (b) from the true frontier for different risk aversion parameters. Similar to Figure 3.4, for a large risk aversion parameter, we observe larger deviations in the estimated frontiers than in the actual frontiers. When the risk aversion parameter $\mu = 0$, the optimal execution strategy for all of the perturbed matrices $C + \frac{1}{2}\left(\Delta C + \Delta C^T\right)$ is identical. Since the impact matrices are fixed in this case, the optimal expected execution cost is unchanged for all of the perturbations ΔC, while they incur different estimations for the variance of the execution cost. This explains the behavior of the right end of the estimated frontiers in 3.6 (b).

In summary, our computational investigation suggests that the effect of estimation errors in impact matrices on the execution strategy and efficient frontiers can be quite large in general. Moreover, the effect of these errors varies with the risk aversion parameter. For a large risk aversion parameter, the difference between the true frontier and the actual frontier is small. In addition, we consistently observe that imposing no-buying constraints decreases the effect of estimation errors on both the optimal execution strategy and the efficient frontier. Moreover, when appropriate, maintaining symmetry in the permanent impact matrix decreases the effect of estimation errors.

In our simulations, we have also noticed that it is possible for a small perturbation in impact matrices to make the Hessian matrix indefinite. As the number of assets grow, this issue becomes even more pronounced. When the Hessian matrix is indefinite, the optimal portfolio execution problem (3.2.6), in which buying is permitted, no longer has a solution since the objective function becomes unbounded below. This is another evidence of potentially large sensitivity of the optimal execution strategy to perturbations in the impact matrices.



(a) Actual Frontiers versus the True Frontier



(b) Estimated Frontiers versus the True Frontier

Figure 3.6: Actual and estimated frontiers with 5% relative perturbations in the covariance matrix for 50 simulations. Buying is allowed for plots on the left. Buying is prohibited for plots on the right.

3.5 Concluding Remarks

Specification and estimation of the price impact function in the optimal portfolio execution problem inevitably have errors. Therefore it is important to analyze how sensitive the optimal execution strategy and the efficient frontier are to these estimation errors. In this chapter, we consider linear price impact functions and study the effect of perturbations in the parameters of the price impact function.

We first show that the optimal execution strategy is determined from the combined impact matrix $\Theta = \frac{1}{\tau}\left(H + H^T\right) - G$.

We discuss some cases in which the optimal execution strategy is insensitive to the estimation errors in the impact matrices. For example, the optimal execution strategy, which minimizes the expected execution cost, is the naive execution strategy as long as the permanent impact matrix and its perturbation are symmetric and the combined impact matrices Θ and $\Theta + \Delta\Theta$ are positive definite. In other words, when symmetry is maintained for the permanent impact matrices and positive definiteness is maintained for the combined impact matrices, the minimum expected execution cost strategy is not sensitive to changes in the impact matrices.

We provide upper bounds for the size of change in the optimal execution strategy in terms of the change in the impact matrices and some magnification factor. In general, the magnification factor is defined by the minimum eigenvalue of the block tridiagonal Hessian matrix W. This matrix W is determined by the covariance matrix C, the combined impact matrix Θ, and the risk aversion parameter μ. From the established upper bounds, it can be concluded that the change in the optimal execution strategy diminishes as the risk aversion parameter increases. However, for a small risk aversion parameter, estimation errors may significantly affect the optimal execution strategy and efficient frontiers.

Our computational investigation confirms the importance of accurate specification of the impact matrices. We demonstrate that, in addition, maintaining symmetry of the permanent impact matrix also reduces the effect of estimation errors. Moreover, our simulations suggest that adding appropriate constraints, such as no-buying constraints, can significantly alleviate the effect of estimation errors in the impact matrices. Consistent with our theoretical results, the computational investigation shows large sensitivity of the optimal execution strategy and the efficient frontier for a small risk aversion parameter μ, when the permanent impact matrix is asymmetric. Specially, this change becomes more significant in the absence of no-buying constraints. This result also coincides with the observation that, for the traditional mean-variance portfolio optimization, the effect of estimation errors in the mean and covariance matrix reduces when no-buying constraints are imposed on the problem.

In summary, our theoretical and computational results indicate that the optimal execution strategy can potentially be very sensitive to estimation errors in the impact matrices. This is particularly the case if the permanent impact matrix is asymmetric, the risk aversion parameter is small, and buying is permitted.

Chapter 4

Regularized Robust Optimization

4.1 Introduction

Sensitivity of the optimal execution strategy and the efficient frontier to estimation errors in the impact matrices motivates us to devise an optimization approach in which this estimation risk is explicitly taken into account.

Indeed, uncertainty is inevitable in any real world decision making problem. An optimization problem formulation often relies on model parameters which must be estimated. This presents challenges in the precise notion of optimality and computation of an optimal decision. Several approaches to account for data uncertainty in optimization problems have been proposed in the literature. In particular, *robust optimization* has gained much interest over the last decade, see, e.g., (Beyer and Sendhoff, 2007; Ben-Tal et al., 2009). In robust optimization, parameter uncertainty is modeled deterministically through an *uncertainty set*, which includes all or most possible parameter values. The approach then offers a solution which has the best worst objective value when parameters belong to the uncertainty set. In this chapter, we consider a robust optimization technique for the optimal portfolio execution problem to handle uncertainty in price impact.

The current robust optimization methodology, however, has shortcomings. Firstly, it can be conservative in the sense that a robust solution may have poor objective values for many realizations of the data including the nominal one, see, e.g., (Bienstock, 2007). Shrinking the uncertainty set using a scaling factor has been a typical technique to alleviate this issue, see, e.g., (Ben-Tal and Nemirovski, 2000; Bertsimas and Sim, 2004). An additional problem, which has not been addressed in the current robust optimization literature, is the potential instability of the robust solution to variation in the uncertainty set.

Specifically we consider the optimal portfolio execution problem with uncertain price impact matrices. Similar to Chapter 3, we assume a deterministic strategy and an additive market price dynamics.

Firstly, we use simulation to illustrate the potential instability of the classical robust optimization method for the optimal portfolio execution problem, with respect to the uncertainty set for price impact parameters. In particular, for an interval uncertainty set, we show that sensitivity of the robust solution and the robust efficient frontier to perturbations in the boundaries of the uncertainty set can be larger than sensitivity of the nominal solution and the nominal efficient frontier to changes in the nominal price impact parameters. Next we show that, for a convex and compact uncertainty set and convex set of feasible execution strategies, a robust optimal execution strategy uniquely exists, when the Hessian of the objective function is positive definite for every realization of price impact parameters in the uncertainty set. Under this assumption, the unique robust solution can be computed via solving a convex programming problem which yields a worst case realization of the price impact parameters, and optimal Lagrange multipliers. These values are then used to determine the robust optimal execution strategy.

To improve stability of the robust optimization, we propose the following *regularized robust optimization* approach for the optimal portfolio execution problem. Given any convex compact uncertainty set, a *regularization constraint* is included to construct a *regularized uncertainty set*. This regularized uncertainty set is then used in the minimax formulation to yield a regularized robust solution.

For the optimal portfolio execution problem with uncertain parameters in a linear price impact model, the regularization constraint is a lower bound constraint on the minimum eigenvalue of the Hessian of the objective function. We refer to the fixed lower bound as the *regularization parameter*. The regularization constraint using the eigenvalue function retains convexity of a convex uncertainty set. Varying eigenvalues of some design matrix to enhance stability properties is fairly common in engineering problems, see, e.g., (Lewis and Overton, 1996).

The intuition behind the proposed regularization constraint comes from the results obtained in Chapter 3: variation in the solution can be large, as price impact parameters are perturbed, when the minimum eigenvalue of the Hessian of the objective function corresponding to the pair of price impact parameters is small. By imposing the regularization constraint, we prevent potential instability of the robust solution by excluding elements, which may result in unstable solutions, from the uncertainty set.

We make two main contributions in this chapter. Firstly, we study sensitivity of the classical robust optimization to changes in the uncertainty set. Secondly, we propose a regularized robust optimization approach for the optimal portfolio execution problem with uncertain price impact matrices. The regularized robust solution is unique and can be obtained by an efficient method based on convex optimization for a positive regularization parameter. We illustrate that including the regularization constraint in the uncertainty set improves stability of the robust solution. We formally show that the change in the regularized robust optimal execution strategy is bounded above by the change in the worst case price impact parameters over the regularized uncertainty set. In addition, the change in the regularized robust solution converges to zero when the variation in the uncertainty set approaches zero. We then investigate some implications of the regularization on the regularized robust solution and its robust objective function value. Since the regularized uncertainty set is a subset of the original uncertainty set, it controls conservatism of the robust solution by offering a better optimal objective function value. The regularized robust mean-variance efficient frontier with a smaller regularization parameter dominates that with a larger regularization parameter. We also establish a bound on the distance between the regularized robust optimal execution strategy and the naive strategy, which suggests that this distance decreases as the regularization parameter increases.

This chapter is organized as follows. The classical robust optimization approach is described in §4.2, where we also discuss potential instability of the robust solution to variation in the uncertainty set. Derivation of the robust solution under the assumption that the Hessian of the objective function is positive definite over the uncertainty set is presented in §4.3. We propose the regularized robust optimization approach for the optimal portfolio execution problem in §4.4. Stability of the approach is discussed in §4.5. Several implications of regularization on the regularized robust solution and its objective function value are addressed in §4.6. Concluding remarks are given in §4.7.

4.2 Classical Robust Optimization

Robust optimization has been broadly used in various fields (Beyer and Sendhoff, 2007; Ben-Tal et al., 2009), with portfolio management as one of its main applications, see, e.g., (Ben-Tal et al., 2000; Ghaoui et al., 2003; Ceria and Stubbs, 2006; Bienstock, 2007; Kim and Boyd, 2007; Garlappi et al., 2007; Fabozzi et al., 2007; Takeda et al., 2008; Lu, 2008; Demiguel and Nogales, 2009) and the references therein. Robust optimization is an alternative method to stochastic programming to deal with parameter uncertainty in mathematical programming. Unlike stochastic programming, no distribution on uncertain parameters is assumed and possible values are equally important. In this methodology, data uncertainty is described by an uncertainty set, which hopefully includes all or most possible realizations of the uncertain input parameters.

Given a nonempty, convex, and compact uncertainty set $\mathcal{U}$, robust optimization yields a solution that optimizes the worst-case performance when the input data belongs to $\mathcal{U}$. The minimax robust optimization methodology consists of the following two steps:

- constructing an uncertainty set $\mathcal{U}$,
- solving the minimax problem.

An uncertainty set is typically specified by a confidence interval associated with a statistical method to estimate the parameters based on historical data, see, e.g., (Goldfarb and Iyengar, 2003). Its specification may depend on the desired level of robustness and assumptions about modeling errors. The choice of the uncertainty set also contributes to tractability and conservativeness of the approach. Intervals and ellipsoids have typically been used in the literature on robust optimization to describe an uncertainty set.

We explore here the usefulness of the robust optimization for the optimal portfolio execution problem with uncertain impact matrices, henceforth denoted by $\tilde{H}$ and $\tilde{G}$. Subsequently, $(\tilde{H}, \tilde{G})$ denotes a vector in $\mathbb{R}^{2m^2}$, obtained by stacking the columns of the matrices $\tilde{H}$ and $\tilde{G}$ on top of one another. Since the covariance matrix can be estimated relatively more accurately, in comparison to the impact matrices, we continue to assume that the covariance matrix C is accurately given.

Let $\mathcal{U} \subseteq \mathbb{R}^{2m^2}$ denote a compact uncertainty set for impact matrices, a robust optimal execution strategy can be obtained by solving the following *robust counterpart problem*:

$$RC(\mathcal{U}): \qquad \inf_{z \in \mathcal{R}} \max_{(\tilde{H}, \tilde{G}) \in \mathcal{U}} \frac{1}{\tau} \bar{S}^T \tilde{H} \bar{S} + \frac{1}{2} z^T W(\tilde{H}, \tilde{G}, \mu) z + b^T(\tilde{H}, \tilde{G}) z.$$

Compactness of $\mathcal{U}$ implies that the optimal value of the inner maximization problem is attained and the use of max rather than sup is justified. Notice that here the uncertainty only affects the objective function.

As the size of the uncertainty set $\mathcal{U}$ increases, the objective value at a robust solution is likely to increase. This drawback of robust optimization has been frequently referred in the literature as the conservativeness of the methodology. Ben-Tal and Nemirovski (1998, 1999, 2000), El-Ghaoui and Lebret (1997), and El-Ghaoui et al. (1998) suggest to rectify the over-conservatism of robust solutions by specifying an interval uncertainty set to be an ellipsoid of a smaller size. Bertsimas and Sim (2004) propose the use of a different subset of the uncertainty set to control the level of conservatism in the robust solution.

In addition to being a conservative approach, specification of an uncertainty set is arbitrary to a large degree, and an uncertainty set built on the historical data may not be able to accurately explain future scenarios. Robust optimization can be viewed as a black box which takes the uncertainty set as its input and produces a robust solution as an output. Thus, it is important to understand how stable the robust solution is with respect to variation in the uncertainty set. This issue, further discussed below, has not been considered in the robust optimization literature to date.

We say a general robust optimization scheme or a robust solution is *stable* with respect to the uncertainty set, if a small variation in the uncertainty set produces a small change in the robust solution. Next we use a robust optimal portfolio execution example to illustrate potential instability of the robust solution with respect to change in the uncertainty set. We use here an interval uncertainty set in our example due to its simplicity.

Consider an interval uncertainty set $\mathcal{U} = \mathcal{U}_H \times \mathcal{U}_G$, where

$$\begin{aligned} \mathcal{U}_H &\stackrel{\text{def}}{=} [\underline{H}, \overline{H}] = \left\{ \tilde{H} \in \mathbb{R}^{m^2} : \quad \underline{H}_{ij} \leq \tilde{H}_{ij} \leq \overline{H}_{ij} \right\}, \\ \mathcal{U}_G &\stackrel{\text{def}}{=} [\underline{G}, \overline{G}] = \left\{ \tilde{G} \in \mathbb{R}^{m^2} : \quad \underline{G}_{ij} \leq \tilde{G}_{ij} \leq \overline{G}_{ij} \right\}. \end{aligned} \tag{4.2.1}$$

For the interval uncertainty set $\mathcal{U} = \mathcal{U}_H \times \mathcal{U}_G$ in (4.2.1), the inner maximization problem in $RC(\mathcal{U})$ becomes [1]

$$\begin{aligned}
\max_{\tilde{H},\tilde{G}} \quad & \sum_{k=1}^{N}\sum_{i,j=1}^{m} \tilde{G}_{ij}(x_k)_i(x_{k-1}-x_k)_j + \frac{1}{\tau}\sum_{k=1}^{N}\sum_{i,j=1}^{m} \tilde{H}_{ij}\,(x_k-x_{k-1})_i\,(x_k-x_{k-1})_j + \mu\tau\sum_{k=1}^{N} x_k^T C x_k \\
\text{s.t.} \quad & \underline{H}_{ij} \le \tilde{H}_{ij} \le \overline{H}_{ij}, \quad i=1,2,...,m,\ j=1,2,...,m, \qquad (4.2.2)\\
& \underline{G}_{ij} \le \tilde{G}_{ij} \le \overline{G}_{ij}, \quad i=1,2,...,m,\ j=1,2,...,m,
\end{aligned}$$

which is a linear optimization problem in terms of the variables $\tilde{H}_{ij}$ and $\tilde{G}_{ij}$ with box constraints. At the solution, each variable equals either its upper bound or lower bound, depending on the sign of its coefficient in the objective function. Whence a robust solution of problem $RC(\mathcal{U})$ solves the following problem:

$$\begin{aligned}
\inf_{z=(x_1,\cdots,x_{N-1})\in\mathcal{R}} \quad & \frac{1}{\tau}\bar{S}^T\overline{H}\bar{S} + \frac{1}{2}z^T W(\overline{H},\overline{G},\mu)z + b^T(\overline{H},\overline{G})z \\
& + \sum_{i,j}\left(\overline{G}_{ij}-\underline{G}_{ij}\right)\max\left\{0, \sum_{k=1}^{N-1}(x_k)_i(x_k-x_{k-1})_j\right\} \\
& + \frac{1}{\tau}\sum_{i,j}\left(\overline{H}_{ij}-\underline{H}_{ij}\right)\max\left\{0, -\sum_{k=1}^{N}(x_k-x_{k-1})_i(x_k-x_{k-1})_j\right\}. \qquad (4.2.3)
\end{aligned}$$

This problem can be formulated as minimizing a quadratic function subject to quadratic constraints; an optimization method is not guaranteed to yield a global solution in general.

Our objective in the rest of this section is to understand sensitivity of a robust optimal execution strategy to variation in the interval uncertainty set. To this end, we conduct a sensitivity analysis based on simulations; this technique has been previously used for the Markowitz mean variance portfolio optimization, see, e.g., (Broadie, 1993).

We assume that there exists an uncertainty set $\mathcal{U}$ which yields a robust strategy with the desired properties; we refer to this as the original uncertainty set. Suppose this uncertainty set is unknown; some perturbed uncertainty set $\overline{\mathcal{U}}$ is instead applied by the decision maker.

The performance of a strategy is represented by a mean-variance efficient frontier. An *original efficient frontier* depicts the performance of the strategy with respect to the original data. An *actual frontier* describes the actual performance of (using the original data) a strategy determined using perturbed data.

For given nominal impact matrices H and G, we refer to the solution of problem (3.2.5) as the nominal optimal execution strategy. The *original nominal frontier* is the curve of the original mean and variance of the execution cost of the nominal optimal execution strategy when the risk aversion parameter μ varies in $(0,\infty)$. For the perturbed impact matrices $H+\Delta H$ and $G+\Delta G$, the *actual nominal frontier* is the curve of mean and variance of the

[1] The second summation of the objective function in problem (4.2.2), at $k=1$, yields the term $\frac{1}{\tau}\bar{S}^T\tilde{H}\bar{S}$.

execution cost computed from the original nominal impact matrices H and G for the optimal execution strategy determined from the perturbed impact matrices $H + \Delta H$ and $G + \Delta G$.

Similarly we can consider a robust efficient frontier of the robust solution with respect to an uncertainty set $\mathcal{U}$; it is the curve of the worst case mean and variance of the execution cost of the robust solution. This notion of robust efficient frontier is described in (Kim and Boyd, 2007). We also extend the notions of *original* and *actual* (mean-variance) efficient frontier to the robust frontier. The *original robust frontier* corresponds to the worst case mean and variance of the execution cost with respect to the original uncertainty set $\mathcal{U}$ for the robust solution obtained from $\mathcal{U}$. An *actual robust frontier* for the perturbed uncertainty set $\overline{\mathcal{U}}$ is the curve of the worst case mean and variance with respect to the original uncertainty set $\mathcal{U}$ for the robust solution computed from a perturbed uncertainty set $\overline{\mathcal{U}}$.

Using simulations, we consider a three asset robust optimal portfolio execution problem with respect to an interval uncertainty set to illustrate sensitivity of the robust solution to the uncertainty set specification; the details are described in Example 4.2.1. In our simulation study, we use the open-source solver Gloptipoly3 (Henrion et al., 2009) to compute a solution for problem (4.2.3). Gloptipoly3 returns a flag, indicating whether the obtained solution is global or not. Simulated perturbations are selected when Gloptipoly3 has indeed obtained a global solution for the robust optimization problem. This example has been used in §3.4 to illustrate sensitivity of the nominal execution strategy to the impact matrices; we also include nominal efficient frontiers and nominal solutions here to compare them with the robust efficient frontiers and robust solutions.

Example 4.2.1. *Consider liquidation of three assets with the initial holding $\bar{S}_i = 10^5$ shares, $i = 1, 2, 3$, in five days by trading daily, i.e., $T = 5$, $N = 5$, and $\tau = 1$. We assume that there is no constraint on the execution strategy, i.e., $\mathcal{R} = \mathcal{R}_0$. The assets are currently traded at price $P_0 = 50\$/share$. Let the daily asset price covariance matrix be as in* (3.4.1). *The nominal permanent and temporary impact matrices H and G are assumed to be as in* (3.4.2). *Note that $\lambda_{\min}(W(H, G, 0)) = 2.5960 \times 10^{-9}$.*

For simplicity we assume that the temporary impact matrix is accurately given, i.e., $\overline{H} = \underline{H} = H$, and only the permanent impact matrix is uncertain with $\overline{G} = 3 \cdot G$ and $\underline{G} = 0.2 \cdot G$, i.e.,

$$\mathcal{U}_H = \{H\}, \qquad \mathcal{U}_G = [0.2 \cdot G,\ 3 \cdot G] \tag{4.2.4}$$

Notice that $\lambda_{\min}(W(\overline{H}, \overline{G}, 0)) = 8.6534 \times 10^{-10}$, which is smaller than $\lambda_{\min}(W(H, G, 0))$.

We now add 5% perturbation $\Delta G^{(\ell)}$ and $\Delta \overline{G}^{(\ell)}$ to the nominal permanent impact matrix G and the upper bound of the original uncertainty set $\mathcal{U}_G$ as follows:

$$\Delta G^{(\ell)} = 5\% \cdot \max_{i,j} \{|G_{ij}|\} \cdot \phi^{(\ell)}, \quad \Delta \overline{G}^{(\ell)} = 5\% \cdot \max_{i,j} \{|\overline{G}_{ij}|\} \cdot \phi^{(\ell)}, \tag{4.2.5}$$

where $\phi^{(\ell)}$ is a standard normal random sample (computed using randn *in MATLAB). A sample $\phi^{(\ell)}$ is selected only if the nominal solution corresponding to the perturbed permanent* *impact matrix* $G + \Delta G^{(\ell)}$ *uniquely exits (the matrix* $W(H, G + \Delta G^{(\ell)}, 0)$ *is positive definite), the perturbed uncertainty set* $\overline{\mathcal{U}}_G = [\underline{G},\ \overline{G} + \Delta\overline{G}^{(\ell)}]$ *is a valid interval (all entries of the matrix* $\overline{G} + \Delta\overline{G}^{(\ell)} - \underline{G}$ *are nonnegative), and Gloptipoly3 obtains a global solution for the robust optimization problem* (4.2.3) *with* $\overline{\mathcal{U}} = \mathcal{U}_H \times \overline{\mathcal{U}}_G$.

The original robust frontier and actual robust frontiers corresponding to 50 *perturbations* $\overline{\mathcal{U}}_G$ *to the original uncertainty set* $\mathcal{U} = \mathcal{U}_H \times \mathcal{U}_G$ *(with* $\mu \in [0, 10^{-5}]$*) are graphed in the left plot in Figure 4.1. We observe large deviations of the actual robust frontiers from the original robust frontier. This indicates that the robust frontier can be unstable to perturbations in the uncertainty set. For comparison, the right plot in Figure 4.1 graphs the original nominal frontier and* 50 *actual nominal frontiers corresponding to* 50 *perturbed nominal impact matrices* $G + \Delta G^{(\ell)}$. *This plot shows that sensitivity of the robust frontiers to perturbations in the uncertainty set may be larger than sensitivity of the nominal frontiers to perturbations in the nominal impact matrices.*

In addition, it can be observed from Figure 4.1 that deviations of actual frontiers from the original ones are more prominent for small risk aversion parameters. We further examine variation in the optimal execution strategy when $\mu = 0$. *Figure 4.2 illustrates sensitivity of the robust optimal execution strategy for* $\mu = 0$ *to perturbations in* $\mathcal{U}_G$ *(left plots) and compares it to sensitivity of the nominal solution to perturbations in the nominal permanent impact matrix* G *(right plots). Significant variation in the robust optimal execution strategy can be observed from the left plots; variation is more severe in comparison to variation in the nominal optimal execution strategy depicted in the right plots. Note that both the original nominal solution and the original robust solution in this case are the naive strategy since the matrices* G *and* $\overline{G}$ *are symmetric (see Proposition 3.2.1).*

Example 4.2.1 clearly illustrates that the robust optimal execution strategy can be unstable with respect to variation in the uncertainty set. This can also be seen when the set of feasible execution strategies is $\mathcal{R}_c$. In this case, for every $z = (x_1, \cdots, x_{N-1}) \in \mathcal{R}_c$, $x_{k-1} \geq x_k \geq 0$. Hence, $(\overline{H}, \overline{G})$ is the solution to problem (4.2.2) and the worst case realization of impact matrices is the same regardless which execution strategy is adopted. Therefore, when $W(\overline{H}, \overline{G}, 0)$ is positive definite, the (global) robust solution of problem $RC(\mathcal{U})$ with respect to the interval uncertainty set $\mathcal{U} = \mathcal{U}_H \times \mathcal{U}_G$ can be obtained simply by solving the following convex quadratic programming problem:

$$\min_{z \in \mathcal{R}_c} \quad \frac{1}{\tau} \bar{S}^T \bar{H} \bar{S} + \frac{1}{2} z^T W(\overline{H}, \overline{G}, \mu) z + z^T b(\overline{H}, \overline{G}). \tag{4.2.6}$$

Consequently, applying the robust optimization approach to obtain a robust execution strategy, we end up with a nominal optimal portfolio execution problem with the impact matrices replaced by the upper bounds of the uncertainty set. Hence, sensitivity of the robust solution to variation in the uncertainty set $\mathcal{U} = \mathcal{U}_H \times \mathcal{U}_G$ is the same as sensitivity of the solution to perturbation in the impact matrices $\overline{H}$ and $\overline{G}$. Applying Theorem 3.3.1 for the optimal portfolio execution problem (4.2.6) implies that the robust solution may be very sensitive

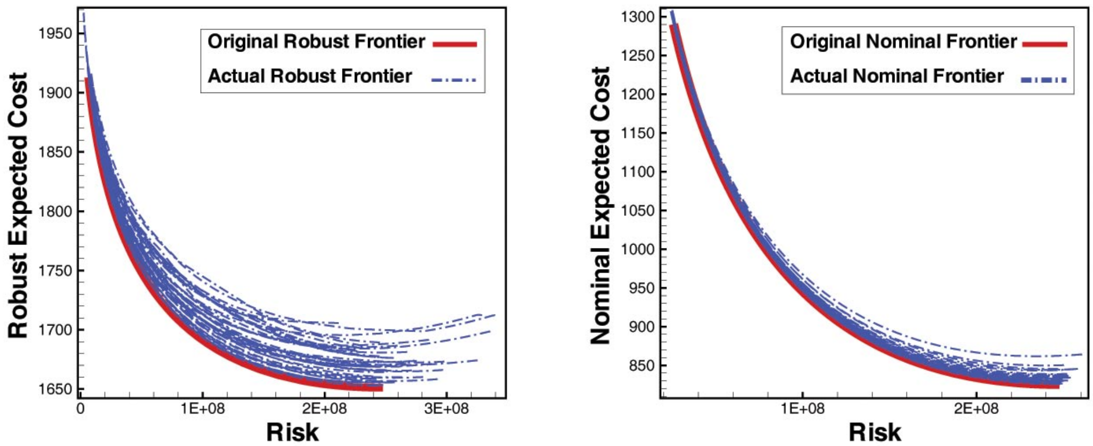


Figure 4.1: Comparing sensitivity of the robust efficient frontier to 5% perturbation in the upper bound of the uncertainty set $\mathcal{U}_G$ with sensitivity of the nominal efficient frontier to 5% perturbation in the nominal permanent impact matrix G.

to change in the uncertainty set $\mathcal{U}$ if $\lambda_{\min}(W(\overline{H}, \overline{G}, \mu))$ is sufficiently small. Indeed, this sensitivity may be larger than the sensitivity of the nominal optimal execution strategy to the nominal impact matrices (H, G) when $\lambda_{\min}\big(W(\overline{H}, \overline{G}, \mu)\big) \le \lambda_{\min}(W(H, G, \mu))$.

Next we show that the robust optimal execution strategy can be computed by semidefinite programming when the Hessian $W(\tilde{H}, \tilde{G}, \mu)$ is positive definite for every $(\tilde{H}, \tilde{G}) \in \mathcal{U}$. Indeed, this method will also be used for our proposed regularized robust optimization described in §4.4.

4.3 Robust Optimal Execution Strategy

For simplicity, we denote the objective function of $RC(\mathcal{U})$ by $\Upsilon(z, \tilde{H}, \tilde{G})$, i.e.,

$$\Upsilon(z, \tilde{H}, \tilde{G}) \stackrel{\text{def}}{=} \frac{1}{\tau}\bar{S}^T \tilde{H} \bar{S} + \frac{1}{2} z^T W(\tilde{H}, \tilde{G}, \mu) z + b^T(\tilde{H}, \tilde{G}) z. \tag{4.3.1}$$

The function $\Upsilon(z, \tilde{H}, \tilde{G})$ is linear in $(\tilde{H}, \tilde{G})$ and quadratic in z. The function $\Upsilon(\cdot, \tilde{H}, \tilde{G})$ is in general non-convex, as the uncertainty set $\mathcal{U}$ may include scenarios $(\tilde{H}, \tilde{G})$ where the matrix $W(\tilde{H}, \tilde{G}, \mu)$ is not positive semidefinite. Thus robust problem $RC(\mathcal{U})$ is NP-hard in general [2].

When $W(\tilde{H}, \tilde{G}, \mu)$ is positive semidefinite, for every $(\tilde{H}, \tilde{G}) \in \mathcal{U}$, $\Upsilon(\cdot, \tilde{H}, \tilde{G})$ is a convex quadratic function. Using Theorem 5.5 of Rockafellar (1996), $\max_{(\tilde{H}, \tilde{G}) \in \mathcal{U}} \Upsilon(\cdot, \tilde{H}, \tilde{G})$ is

[2]Note that the problem of minimizing a non-convex quadratic function is known to be NP-hard (Pardalos and Vavasis, 1991).

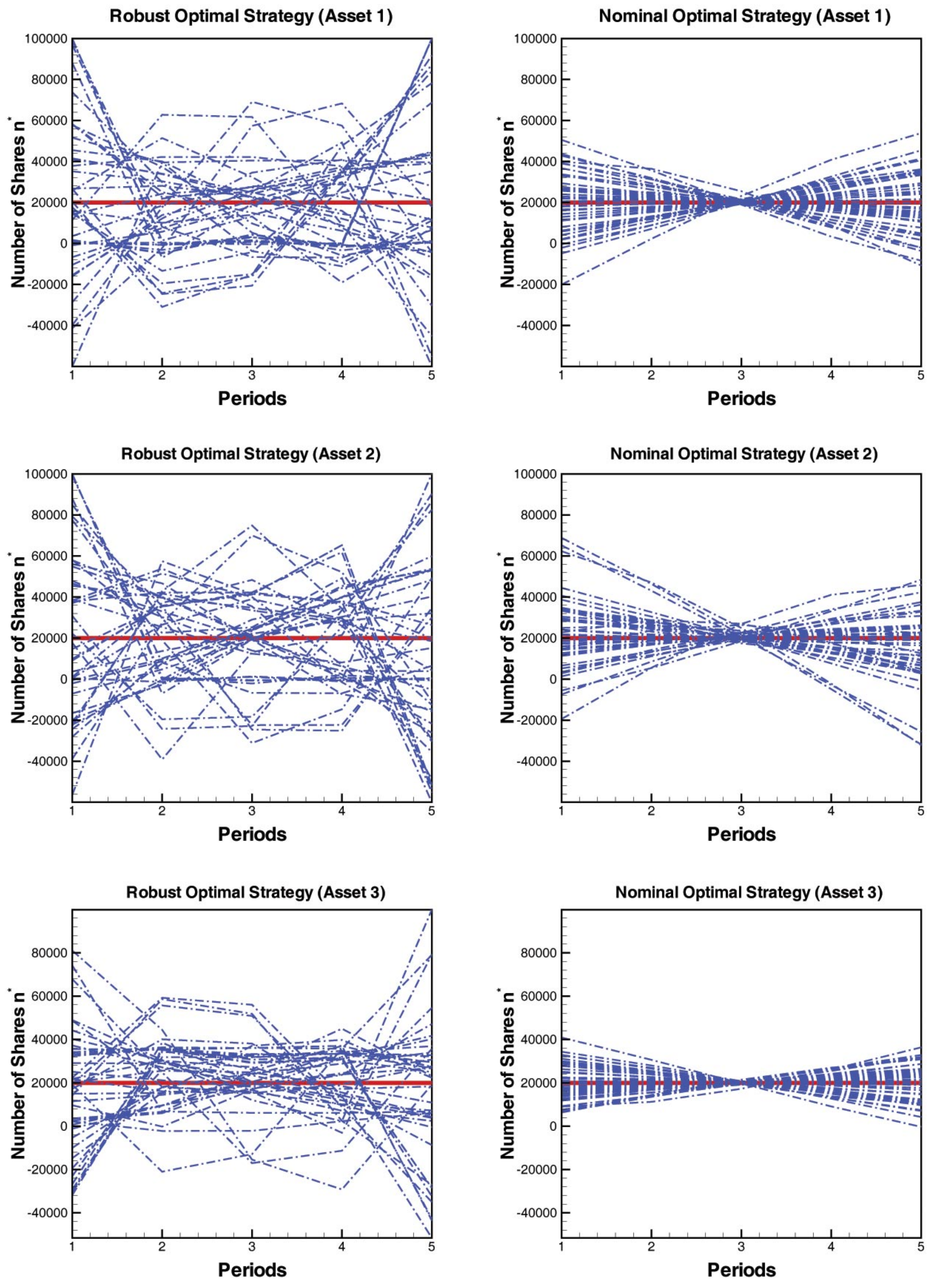


Figure 4.2: Comparing sensitivity of the (classical) robust solution to 5% perturbation in the upper bound of the uncertainty set $\mathcal{U}_G$ with sensitivity of the nominal solution to 5% perturbation in the nominal permanent impact matrix G. Risk aversion parameter is $\mu = 0$ and $\mathcal{R} = \mathcal{R}_0$.

convex. Thus, the problem $\left(\inf_{z\in\mathcal{R}} \left(\max_{(\tilde{H},\tilde{G})\in\mathcal{U}} \Upsilon(z,\tilde{H},\tilde{G})\right)\right)$ is a convex optimization problem for a convex feasible set $\mathcal{R}$. The following proposition shows the existence and uniqueness of the saddle point for the minimax problem $RC(\mathcal{U})$ under convexity assumption.

Proposition 4.3.1. *Let $\mathcal{R}$ be nonempty, convex, and closed, and the uncertainty set $\mathcal{U}$ be nonempty, convex, and compact. For a given risk aversion parameter $\mu \geq 0$, assume that the matrix $W(\tilde{H},\tilde{G},\mu)$ is positive definite, for every $(\tilde{H},\tilde{G}) \in \mathcal{U}$. Then the minimax problem $RC(\mathcal{U})$ has a saddle point (z_u, H_u, G_u), i.e.,*

$$\Upsilon\left(z_u,\tilde{H},\tilde{G}\right) \leq \Upsilon\left(z_u,H_u,G_u\right) \leq \Upsilon\left(z,H_u,G_u\right), \quad \forall\, (\tilde{H},\tilde{G}) \in \mathcal{U}, \quad \forall\, z \in \mathcal{R}. \quad (4.3.2)$$

Moreover, for every two saddle points $(z^{(1)},H^{(1)},G^{(1)})$ and $(z^{(2)},H^{(2)},G^{(2)})$, we have $z^{(1)} = z^{(2)}$, i.e., the robust optimal execution strategy from problem $RC(\mathcal{U})$ is unique.

Proof. From the convexity of $\mathcal{R}$ and $\mathcal{U}$, compactness of $\mathcal{U}$, and $\Upsilon(z,\tilde{H},\tilde{G})$ being strictly convex in z and linear in $(\tilde{H},\tilde{G})$, we have

$$\inf_{z\in\mathcal{R}} \max_{(\tilde{H},\tilde{G})\in\mathcal{U}} \Upsilon\left(z,\tilde{H},\tilde{G}\right) = \max_{(\tilde{H},\tilde{G})\in\mathcal{U}} \inf_{z\in\mathcal{R}} \Upsilon\left(z,\tilde{H},\tilde{G}\right), \quad (4.3.3)$$

see, e.g., Theorem 3 of (Simons, 1995), which is provided in Theorem A.3.2.

Let $(H_u,G_u) \in \mathcal{U}$ be an optimal point for the outer maximization problem below

$$\max_{(\tilde{H},\tilde{G})\in\mathcal{U}} \inf_{z\in\mathcal{R}} \Upsilon\left(z,\tilde{H},\tilde{G}\right).$$

Thus,

$$\left(\inf_{z\in\mathcal{R}} \Upsilon\left(z,H_u,G_u\right)\right) = \left(\max_{(\tilde{H},\tilde{G})\in\mathcal{U}} \inf_{z\in\mathcal{R}} \Upsilon\left(z,\tilde{H},\tilde{G}\right)\right). \quad (4.3.4)$$

Since $(H_u,G_u) \in \mathcal{U}$, $W(H_u,G_u,\mu)$ is positive definite. Thus $\Upsilon\left(z,H_u,G_u\right)$ is a strictly convex quadratic function. Since $\mathcal{R}$ is closed, there exists $z_{(H_u,G_u)} \in \mathcal{R}$ at which $\inf_{z\in\mathcal{R}} \Upsilon\left(z,H_u,G_u\right)$ is uniquely attained (see, e.g., Proposition 2.5 of (Dostál, 2009) which is presented in Proposition A.2.1). Thus

$$\Upsilon\left(z_{(H_u,G_u)},H_u,G_u\right) = \inf_{z\in\mathcal{R}} \Upsilon\left(z,H_u,G_u\right) = \max_{(\tilde{H},\tilde{G})\in\mathcal{U}} \inf_{z\in\mathcal{R}} \Upsilon\left(z,\tilde{H},\tilde{G}\right) \quad (4.3.5)$$

and

$$\Upsilon\left(z_{(H_u,G_u)},H_u,G_u\right) \leq \Upsilon\left(z,H_u,G_u\right), \quad \forall\, z \in \mathcal{R}. \quad (4.3.6)$$

From (4.3.5) and (4.3.3), we get

$$\inf_{z\in\mathcal{R}} \max_{(\tilde{H},\tilde{G})\in\mathcal{U}} \Upsilon\left(z,\tilde{H},\tilde{G}\right) = \Upsilon\left(z_{(H_u,G_u)},H_u,G_u\right).$$

Therefore, $z_{(H_u,G_u)}$ is a solution of the outer infimum on the left problem of equation (4.3.3). Thus

$$\Upsilon\left(z_{(H_u,G_u)}, H_u, G_u\right) = \inf_{z\in\mathcal{R}} \max_{(\tilde{H},\tilde{G})\in\mathcal{U}} \Upsilon\left(z, \tilde{H}, \tilde{G}\right) = \max_{(\tilde{H},\tilde{G})\in\mathcal{U}} \Upsilon\left(z_{(H_u,G_u)}, \tilde{H}, \tilde{G}\right)$$

Hence,

$$\Upsilon\left(z_{(H_u,G_u)}, \tilde{H}, \tilde{G}\right) \leq \Upsilon\left(z_{(H_u,G_u)}, H_u, G_u\right), \quad \forall\ (\tilde{H},\tilde{G}) \in \mathcal{U}. \tag{4.3.7}$$

Inequalities (4.3.6) and (4.3.7) imply that $(z_{(H_u,G_u)}, H_u, G_u)$ is a saddle point.

For the uniqueness, note that positive definiteness of $W(\tilde{H},\tilde{G},\mu)$ for every $(\tilde{H},\tilde{G}) \in \mathcal{U}$ yields strict convexity of $\Upsilon(\cdot,\tilde{H},\tilde{G})$. In particular, $\Upsilon(\cdot, H^{(1)}, G^{(1)})$ is strictly convex, and the problem $\min_{z\in\mathcal{R}} \Upsilon(z, H^{(1)}, G^{(1)})$ has a unique solution. Hence, if $z^{(1)} \neq z^{(2)}$,

$$\Upsilon\left(z^{(1)}, H^{(1)}, G^{(1)}\right) < \Upsilon\left(z^{(2)}, H^{(1)}, G^{(1)}\right) \leq \Upsilon\left(z^{(2)}, H^{(2)}, G^{(2)}\right).$$

This contradicts to the fact that both $(z^{(1)}, H^{(1)}, G^{(1)})$ and $(z^{(2)}, H^{(2)}, G^{(2)})$ are saddle points and consequently $\Upsilon(z^{(1)}, H^{(1)}, G^{(1)}) = \Upsilon(z^{(2)}, H^{(2)}, G^{(2)})$. Therefore, $z^{(1)} = z^{(2)}$. □

Proposition 4.3.1 indicates that the robust optimal execution strategy is unique, when the matrix $W(\tilde{H},\tilde{G},\mu)$ is positive definite for every $(\tilde{H},\tilde{G}) \in \mathcal{U}$.

A typical approach to obtain a robust solution is to find a semidefinite programming (SDP) representation for the robust counterpart problem $RC(\mathcal{U})$. Ben-Tal and Nemirovski (1998) show that the robust counterpart of an uncertain convex quadratically constrained quadratic programming problem, with separate ellipsoidal uncertainty sets for the Hessian and linear term of the objective function can be explicitly modeled as a linear semidefinite programming. As is explained in (Halldorsson and Tütüncü, 2003), the model in (Ben-Tal and Nemirovski, 1998) places the uncertainty description on the square root of the Hessian, whence, every matrix in the uncertainty set is positive semidefinite. However, when one has an uncertainty description for only the Hessian, transferring that into an uncertainty description on the Cholesky-like factors can be difficult. Ben-Tal and Nemirovski (1998) further discuss that a more general uncertainty for the Hessian and linear term of the quadratic objective function leads to an NP-hard robust counterpart problem.

Here, we apply semidefinite programming to solve problem $RC(\mathcal{U})$, when the matrix $W(\tilde{H},\tilde{G},\mu)$ is positive definite for every $(\tilde{H},\tilde{G}) \in \mathcal{U}$. However, in contrast to the typical approach, see, e.g., (Ben-Tal et al., 2009), in which the dual of the inner maximization problem is taken, similar to (Kim and Boyd, 2008), we first switch the order of min and max; then we take the dual of the minimization problem and show that it is SDP representable. We summarize our discussion in the following proposition. Below, we consider that the set of feasible execution strategies $\mathcal{R}$ is defined by linear inequality constraints

$$\mathcal{R} = \{z \in \mathbb{R}^{m(N-1)} : Az \leq c\}, \tag{4.3.8}$$

where c is an r-vector and A is a $r \times m(N-1)$ matrix. This representation of $\mathcal{R}$ lets us treat any linear inequality constraint such as nonnegativity constraints or bound constraints on execution strategies, in a unified manner. Furthermore, since an equality constraint can be represented using two inequality constraints, it can also be used when linear equality constraints are imposed on an execution strategy.

Proposition 4.3.2. *Let the uncertainty set $\mathcal{U}$ be nonempty, convex, and compact, and the matrix $W(\tilde{H}, \tilde{G}, \mu)$ be positive definite for every $(\tilde{H}, \tilde{G}) \in \mathcal{U}$. Furthermore, assume the nonempty feasible set $\mathcal{R}$ is as in (4.3.8). Then the robust solution to $RC(\mathcal{U})$ equals*

$$z_u = -W(H_u, G_u, \mu)^{-1} \left(b(H_u, G_u) + A^T \lambda_u \right), \tag{4.3.9}$$

where (H_u, G_u) and $\lambda_u \in \mathbb{R}^r_+$ constitute a solution of the following problem:

$$P(\mathcal{U}): \quad \max_{(\tilde{H},\tilde{G}) \in \mathcal{U},\ \lambda \in \mathbb{R}_+,\ v \in \mathbb{R}} \quad v$$
$$s.t. \quad \begin{bmatrix} \frac{2}{\tau}\bar{S}^T \tilde{H} \bar{S} - 2c^T\lambda - 2v & \left(b(\tilde{H}, \tilde{G}) + A^T\lambda \right)^T \\ b(\tilde{H}, \tilde{G}) + A^T\lambda & W(\tilde{H}, \tilde{G}, \mu) \end{bmatrix} \succeq 0. \tag{4.3.10}$$

When no constraint is imposed, i.e., $\mathcal{R} = \mathcal{R}_0$, the robust solution of $RC(\mathcal{U})$ is

$$z_u = -W(H_u, G_u, \mu)^{-1} b(H_u, G_u), \tag{4.3.11}$$

where (H_u, G_u) constitutes an optimal point of the following problem:

$$\max_{(\tilde{H},\tilde{G}) \in \mathcal{U},\ v \in \mathbb{R}} \quad v \tag{4.3.12}$$
$$s.t. \quad \begin{bmatrix} \frac{2}{\tau}\bar{S}^T \tilde{H} \bar{S} - 2v & b(\tilde{H}, \tilde{G})^T \\ b(\tilde{H}, \tilde{G}) & W(\tilde{H}, \tilde{G}, \mu) \end{bmatrix} \succeq 0.$$

Proof. The given assumptions and Proposition 4.3.1 imply that the infimum is attained. Furthermore, problem $RC(\mathcal{U})$ equals:

$$\begin{aligned} &\max_{(\tilde{H},\tilde{G}) \in \mathcal{U}} \ \min_{z \in \mathcal{R}} \ \frac{1}{\tau}\bar{S}^T \tilde{H} \bar{S} + \frac{1}{2} z^T W(\tilde{H}, \tilde{G}, \mu) z + b^T(\tilde{H}, \tilde{G}) z \\ = &\max_{(\tilde{H},\tilde{G}) \in \mathcal{U}} \ \frac{1}{\tau}\bar{S}^T \tilde{H} \bar{S} + \min_{z \in \mathcal{R}} \left(\frac{1}{2} z^T W(\tilde{H}, \tilde{G}, \mu) z + b^T(\tilde{H}, \tilde{G}) z \right). \end{aligned} \tag{4.3.13}$$

The Lagrangian function of the inner minimization problem in (4.3.13) is:

$$\begin{aligned} L(z, \lambda) &= \frac{1}{2} z^T W(\tilde{H}, \tilde{G}, \mu) z + b^T(\tilde{H}, \tilde{G}) z + \lambda^T (Az - c) \\ &= \frac{1}{2} z^T W(\tilde{H}, \tilde{G}, \mu) z + \left(b(\tilde{H}, \tilde{G}) + A^T\lambda \right)^T z - c^T \lambda. \end{aligned}$$

Since $L(z,\lambda)$ is a strictly convex quadratic function of z, the Lagrange dual problem is:

$$\begin{aligned}
&\max_{\lambda\in\mathbb{R}^r_+} \left(\min_{z\in\mathbb{R}^{m(N-1)}} L(z,\lambda) \right) \\
&= \max_{\lambda\in\mathbb{R}^r_+} \left(-\frac{1}{2}\left(b(\tilde{H},\tilde{G})+A^T\lambda\right)^T W(\tilde{H},\tilde{G},\mu)^{-1}\left(b(\tilde{H},\tilde{G})+A^T\lambda\right) - c^T\lambda \right).
\end{aligned}$$

Here $\mathbb{R}^r_+$ denotes the nonnegative orthant. Since $\mathcal{R}$ is defined by linear inequalities, Slater's condition and consequently strong duality hold for the inner minimization problem of (4.3.13). Thus:

$$\min_{z\in\mathcal{R}} \left(\frac{1}{2} z^T W(\tilde{H},\tilde{G},\mu) z + b^T(\tilde{H},\tilde{G}) z \right) = \max_{\lambda\in\mathbb{R}^r_+} \left(\min_{z\in\mathbb{R}^{m(N-1)}} L(z,\lambda) \right).$$

Thus problem (4.3.13) is reduced to:

$$\max_{(\tilde{H},\tilde{G})\in\mathcal{U},\lambda\in\mathbb{R}^r_+} \frac{1}{\tau}\bar{S}^T\tilde{H}\bar{S} - \frac{1}{2}\left(b(\tilde{H},\tilde{G})+A^T\lambda\right)^T W^{-1}(\tilde{H},\tilde{G},\mu)\left(b(\tilde{H},\tilde{G})+A^T\lambda\right) - c^T\lambda. \quad (4.3.14)$$

Problem (4.3.14) can be reformulated as:

$$\begin{aligned}
&\max_{(\tilde{H},\tilde{G})\in\mathcal{U},\ \lambda\in\mathbb{R}^r_+,\ v\in\mathbb{R}} \quad v \\
&\text{s.t.} \quad \frac{1}{\tau}\bar{S}^T\tilde{H}\bar{S} - \frac{1}{2}\left(b(\tilde{H},\tilde{G})+A^T\lambda\right)^T W(\tilde{H},\tilde{G},\mu)^{-1}\left(b(\tilde{H},\tilde{G})+A^T\lambda\right) - c^T\lambda \geq v.
\end{aligned}$$

Since $W(\tilde{H},\tilde{G},\mu)$ is positive definite, using the Schur complement, inequality

$$\frac{1}{\tau}\bar{S}^T\tilde{H}\bar{S} - \frac{1}{2}\left(b(\tilde{H},\tilde{G})+A^T\lambda\right)^T W(\tilde{H},\tilde{G},\mu)^{-1}\left(b(\tilde{H},\tilde{G})+A^T\lambda\right) - c^T\lambda \geq v, \quad (4.3.15)$$

holds if and only if the linear matrix inequality

$$\begin{bmatrix} \frac{2}{\tau}\bar{S}^T\tilde{H}\bar{S} - 2c^T\lambda - 2v & \left(b(\tilde{H},\tilde{G})+A^T\lambda\right)^T \\ b(\tilde{H},\tilde{G})+A^T\lambda & W(\tilde{H},\tilde{G},\mu) \end{bmatrix} \succeq 0,$$

holds, with strict positive definiteness in the last constraint if and only if strict inequality holds in inequality (4.3.15).

Therefore, a solution of the inner maximization problem in $RC(\mathcal{U})$ can be obtained by solving the maximization problem $P(\mathcal{U})$. Let the pair (H_u, G_u) and $\lambda_u \in \mathbb{R}^r_+$ be a solution of problem $P(\mathcal{U})$, then the robust solution equals (4.3.9).

When no constraint is imposed, i.e, $\mathcal{R} = \mathcal{R}_0$, problem (4.3.13) is reduced to the following problem:

$$\max_{(\tilde{H},\tilde{G})\in\mathcal{U}} \frac{1}{\tau}\bar{S}^T\tilde{H}\bar{S} - \frac{1}{2}\, b^T(\tilde{H},\tilde{G}) W^{-1}(\tilde{H},\tilde{G},\mu) b(\tilde{H},\tilde{G}). \quad (4.3.16)$$

A similar discussion then implies that the robust solution becomes (4.3.11) where (H_u, G_u) is an optimal point of problem (4.3.12). □

At an optimal point of $P(\mathcal{U})$, optimal objective value v represents the execution cost corresponding to the robust optimal execution strategy. Convexity of the objective function, the uncertainty set $\mathcal{U}$, and the linear matrix inequality constraints imply that problem $P(\mathcal{U})$ is a convex programming problem. When $\mathcal{U}$ is defined by linear matrix inequalities, problem $P(\mathcal{U})$ is a linear semidefinite programming problem; this problem can be solved using high-quality open-source solvers, e.g., SEDUMI (Sturm, 2001) or SDPT3 (Toh et al., 2002).

An advantage of the above derivation for the robust solution is that this approach does not depend on any specific structure (e.g., interval or ellipsoidal) of the uncertainty set. We also adopt this derivation for the regularized uncertainty set introduced in §4.4. It is worth mentioning that formulation $P(\mathcal{U})$ also allows us to include the constraint $\tilde{G} = \tilde{G}^T$ in the uncertainty set specification, when there is some evidence that the permanent impact matrix is symmetric.

4.4 Regularized Robust Optimization

Example 4.2.1 illustrates that a robust execution strategy can be sensitive to the uncertainty set specification. Now we propose a regularized robust optimization formulation to address this issue.

For the nominal optimal portfolio execution problem (3.2.6), sensitivity of the optimal execution strategy and the efficient frontier has been studied in Chapter 3. This analysis shows that, when the minimum eigenvalue of the Hessian of the objective function $W(H, G, \mu)$ is small, the optimal point may vary significantly when the impact matrices change slightly. This result suggests that excluding those elements, which yield a small minimum eigenvalue for $W(\tilde{H}, \tilde{G}, 0)$, from the uncertainty set $\mathcal{U}$ may prevent an unstable solution. This idea is also related to the well known regularization technique in which prior information is included in the problem formulation to stabilize the solution. The most common form of regularization for ill-posed least square problems is Tikhonov regularization, see, e.g., (Engl, 1993; Fierro et al., 1997; Tikhonov and Arsemin, 1997), where a two-norm bound constraint is included. Here we propose to regularize uncertainty set to obtain more stable robust solutions.

Let $\mathcal{U} \subseteq \mathbb{R}^{2m^2}$ be a nonempty, convex, and compact uncertainty set for the impact matrices. Given $\mathcal{U}$ and a positive constant $\rho > 0$, we impose the regularization constraint $\lambda_{\min}(W(\tilde{H}, \tilde{G}, 0)) \geq \rho$ on the uncertainty set:

$$\mathcal{V}(\mathcal{U}, \rho) \overset{\text{def}}{=} \left\{ (\tilde{H}, \tilde{G}) \in \mathcal{U} \mid \lambda_{\min}(W(\tilde{H}, \tilde{G}, 0)) \geq \rho \right\}. \tag{4.4.1}$$

We refer to the parameter ρ and the set $\mathcal{V}(\mathcal{U}, \rho)$ as the *regularization parameter* and the *regularized uncertainty set*, respectively. The regularization constraint $\lambda_{\min}(W(\tilde{H}, \tilde{G}, 0)) \geq \rho$ is equivalent to the matrix inequality constraint $W(\tilde{H}, \tilde{G}, 0) \succeq \rho I_{m(N-1)}$ where $I_{m(N-1)}$ is the $m(N-1) \times m(N-1)$ identity matrix. Figure 4.3 illustrates how the regularized uncertainty set $\mathcal{V}(\mathcal{U}, \rho)$ compares with $\mathcal{U}$ for two values of ρ in a single asset execution.

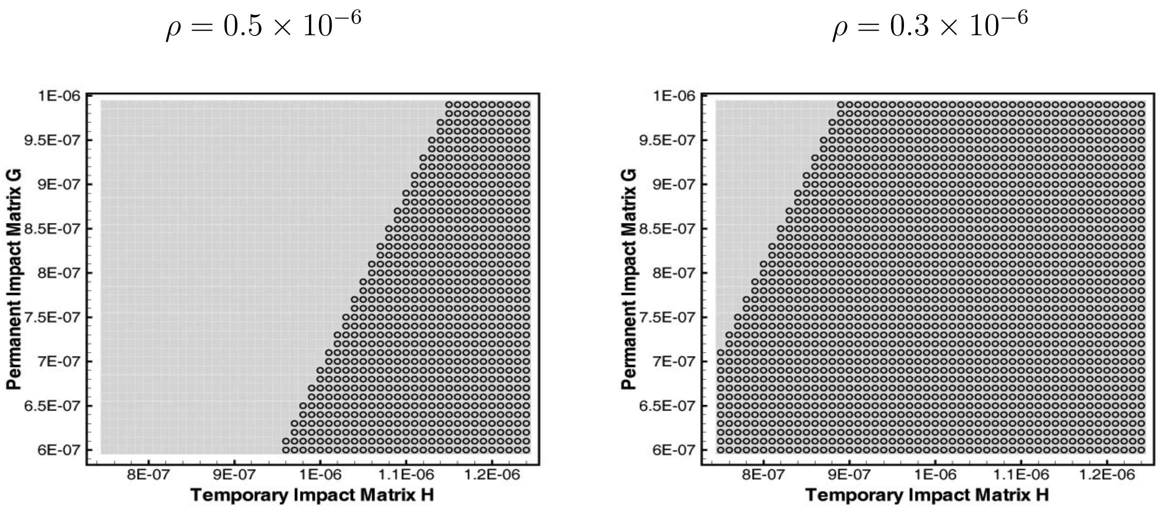


Figure 4.3: Regularized uncertainty set $\mathcal{V}(\mathcal{U}, \rho)$ versus uncertainty set $\mathcal{U}$. Here, $m = 1$, $N = 5$, and the nominal impact matrices are $H = 10^{-6}$ and $G = 8 \times 10^{-7}$. The uncertainty set is $\mathcal{U} = \{(\tilde{H}, \tilde{G}) \in \mathcal{U}_H \times \mathcal{U}_G : \lambda_{\min}(W(\tilde{H}, \tilde{G}, 0)) \geq 0\}$ where $\mathcal{U}_H = [0.75 \cdot H,\ 1.25 \cdot H]$ and $\mathcal{U}_G = [0.75 \cdot G,\ 1.25 \cdot G]$. The grey area denotes the original uncertainty set and circle pattern denotes the regularized uncertainty set $\mathcal{V}(\mathcal{U}, \rho)$.

We note that convexity of $\mathcal{U}$ and convexity of the regularization constraint imply convexity of $\mathcal{V}(\mathcal{U}, \rho)$. Moreover, since the function $\lambda_{\min}(\cdot)$ is a continuous function, closeness of $\mathcal{U}$ implies closeness of $\mathcal{V}(\mathcal{U}, \rho)$.

For every $(\tilde{H}, \tilde{G}) \in \mathcal{V}(\mathcal{U}, \rho)$ and $\mu \geq 0$, the Courant-Fischer Theorem yields

$$\lambda_{\min}(W(\tilde{H}, \tilde{G}, \mu)) = \lambda_{\min}\left(W(\tilde{H}, \tilde{G}, 0) + 2\mu\tau I_N \otimes C\right) \geq \rho + 2\mu\tau\lambda_{\min}(C), \quad (4.4.2)$$

where $\otimes$ denotes the Kronecker product of two matrices. Thus by imposing the specified regularization constraint, we ensure that the minimum eigenvalue of the Hessian of the objective function at the worst case impact matrices is positive and not very small.

The regularization parameter value ρ affects the size of the regularized uncertainty set $\mathcal{V}(\mathcal{U}, \rho)$. If ρ increases, the size of the uncertainty set decreases, i.e., implicitly one is demanding robustness with respect to a smaller set of parameter values. As a result, the resulting robust strategy will be less conservative.

To ensure that the set $\mathcal{V}(\mathcal{U}, \rho)$ is nonempty, the regularization parameter ρ needs to be chosen carefully. In particular, when $\lambda_{\min}(W(H, G, 0)) > 0$ for the nominal impact matrices $(H, G) \in \mathcal{U}$, the regularized uncertainty set $\mathcal{V}(\mathcal{U}, \rho)$ is nonempty for any $\rho \leq \lambda_{\min}(W(H, G, 0))$. Thus the regularization parameter can be proportional to this value. If the regularization parameter ρ is strictly greater than $\max_{(\tilde{H}, \tilde{G}) \in \mathcal{U}} \lambda_{\min}(W(\tilde{H}, \tilde{G}, 0))$, the regularized uncertainty set becomes empty.

Given an uncertainty set $\mathcal{U}$ and a positive regularization parameter ρ, the regularized robust optimization formulation is given below:

$$\Phi_\mu(\rho) \stackrel{\text{def}}{=} \min_{z\in\mathcal{R}} \max_{(\tilde{H},\tilde{G})\in\mathcal{V}(\mathcal{U},\rho)} \frac{1}{\tau}\bar{S}^T\tilde{H}\bar{S} + \frac{1}{2}z^T W(\tilde{H},\tilde{G},\mu)z + b^T(\tilde{H},\tilde{G})z. \quad (4.4.3)$$

When z^* constitutes a solution of problem (4.4.3), it is subsequently called a *regularized robust optimal execution strategy*. Including a regularization constraint also allows us to compute a regularized robust solution using problem $P(\mathcal{V}(\mathcal{U},\rho))$ and equation (4.3.9). This is the case even for an interval uncertainty set $\mathcal{U}$ for which, computing a robust solution can be NP-hard in the absence of a regularization constraint. For an interval uncertainty set $\mathcal{U}$, problem $P(\mathcal{V}(\mathcal{U},\rho))$ is a linear semidefinite programming problem which can be solved efficiently. Note that $\Phi_\mu(\rho) = v^*$, where v^* is the optimal value of problem $P(\mathcal{V}(\mathcal{U},\rho))$.

Now we illustrate the effect of regularization on stability of the robust solution with an interval uncertainty using the portfolio execution described in Example 4.2.1. We use the same $M = 50$ perturbations $\Delta\bar{G}^{(\ell)}$, used in Figures 4.1 and 4.2. Regularized robust solutions are computed in MATLAB 7.9, using (4.3.11) and (4.3.12). Problem (4.3.12) is solved using `CVX`, a package for specifying and solving convex programs (Grant and Boyd, 2009) in MATLAB.

Figure 4.4 illustrates sensitivity of the actual robust efficient frontier corresponding to the regularized robust execution strategy to perturbation in the uncertainty set. The actual robust frontier for the regularized robust solutions is the worst case mean and variance with respect to the original uncertainty set $\mathcal{U}$. Comparing Figure 4.4 with Figure 4.1, we observe clear improvement in stability of the regularized robust solution. Furthermore, Figure 4.4 indicates that increasing the regularization parameter ρ reduces variation in the actual robust frontiers.

Figure 4.5 illustrates stability of the regularized robust optimal execution strategy when $\mu = 0$ for two regularization parameter values ρ. Comparing the left plots with the right plots in Figure 4.5 indicates that the sensitivity is larger for a smaller regularization parameter ρ. In addition, comparison between Figure 4.5 and Figure 4.2 indicates that the regularized robust optimal execution strategy has a more stable behavior to perturbation in the upper bound of the interval uncertainty set $\mathcal{U}_G$ than the classical robust optimal execution strategy. Note that, for both regularization parameter values, the worst case original permanent impact matrix G_u for $P(\mathcal{V}(\mathcal{U},\rho))$ is symmetric; thus the regularized robust optimal execution strategy is the naive strategy, which follows from Proposition 3.2.1. For a perturbed uncertainty set $\overline{\mathcal{U}} = \mathcal{U}_H \times \overline{\mathcal{U}}_G$, the worst case permanent impact matrix G_u from problem $P(\mathcal{V}(\overline{\mathcal{U}},\rho))$ is typically not symmetric; thus the strategy can differ significantly from the naive strategy.

Next we formally analyze stability of the regularized robust solution.

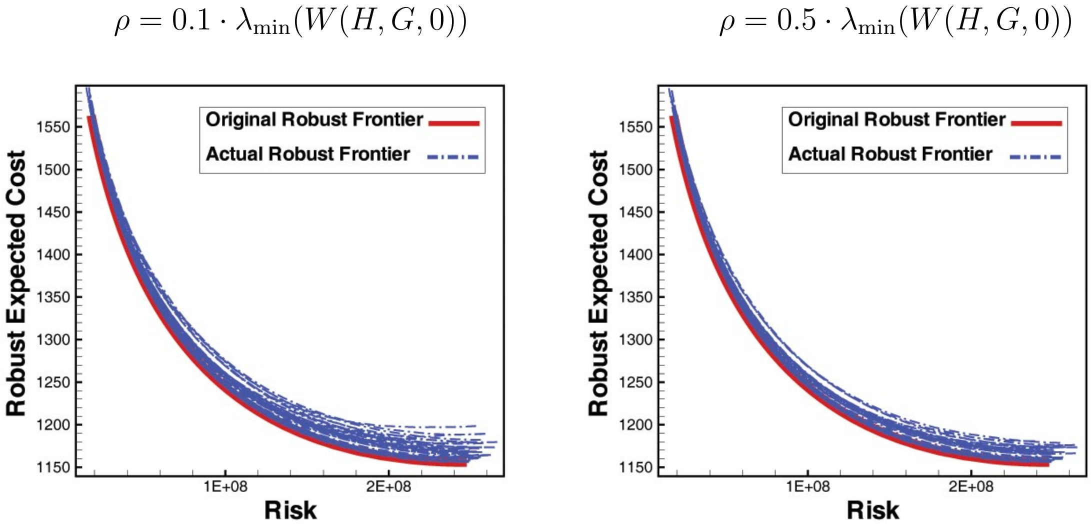


Figure 4.4: Sensitivity of the robust efficient frontier for the regularized robust optimal execution strategy to 5% perturbation in the upper bound of the uncertainty set $\mathcal{U}_G$, for liquidating the three assets in Example 4.2.1.

4.5 Stability of the Regularized Robust Optimal Execution Strategy

In this section, we establish a bound on the change in the regularized robust optimal execution strategy, when the uncertainty set is perturbed. This bound explicitly indicates how the regularization parameter ρ affects sensitivity of the regularized robust solution to variation in the uncertainty set. In addition, we show that the change in the regularized robust solution converges to zero when the change in the uncertainty set $\mathcal{U}$ converges to zero.

We measure perturbation in the uncertainty set by the Hausdorff distance (Hausdorff, 1962), which quantifies how far two subsets in a metric space are from each other. Given a metric space $(\mathcal{X}, d)$, the Hausdorff distance between two subsets $\mathcal{S}, \mathcal{T} \subseteq \mathcal{X}$ is defined by:

$$\mathbf{Haus}_d(\mathcal{S}, \mathcal{T}) \stackrel{\text{def}}{=} \max \left\{ \sup_{s \in \mathcal{S}} \ \inf_{t \in \mathcal{T}} \ d(s,t), \ \sup_{t \in \mathcal{T}} \ \inf_{s \in \mathcal{S}} \ d(s,t) \right\},$$

see, e.g., Remark 4.40 of (Bonnans and Shapiro, 2000) for a more detailed discussion.

When both subsets $\mathcal{S}$ and $\mathcal{T}$ are bounded, $\mathbf{Haus}_d(\mathcal{S}, \mathcal{T})$ is finite. The Hausdorff distance, defined on a metric space $(\mathcal{X}, d)$, is a metric on the set of all *non-empty compact* subsets of $\mathcal{X}$, see, e.g., Proposition 4.1.8 of (Papadopoulos, 2005). This metric has been previously used to measure perturbation to a set, see, e.g., (Alvoni and Papini, 2005). Here, we define the Hausdorff metric induced by the metric d, below, on $\mathbb{R}^{2m^2}$:

$$d\left((H_1, G_1), (H_2, G_2)\right) \stackrel{\text{def}}{=} \frac{2}{\tau}\|H_1 - H_2\|_2 + \|G_1 - G_2\|_2. \tag{4.5.1}$$

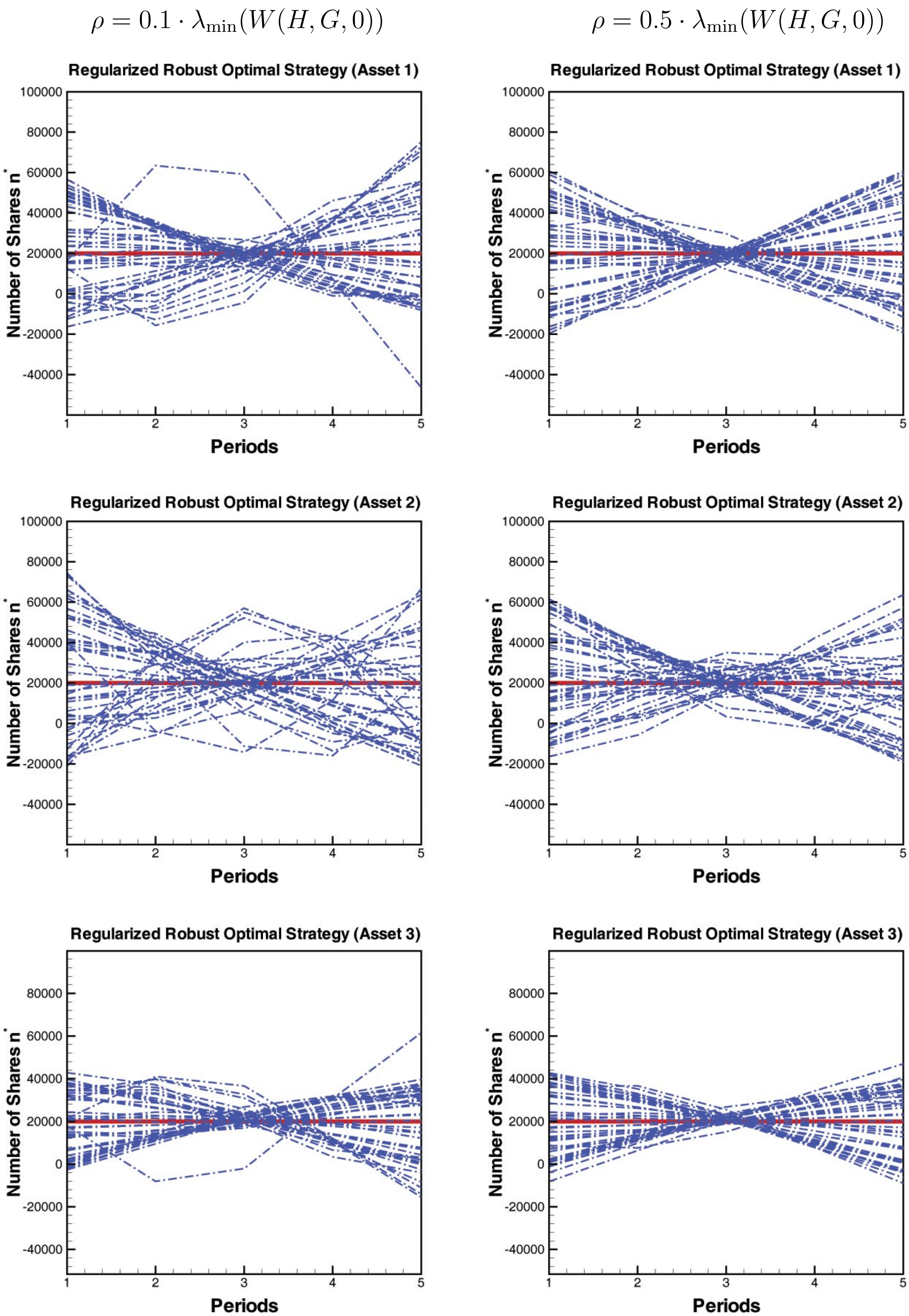


Figure 4.5: Sensitivity of the regularized robust optimal execution strategy ($\mu = 0$) to perturbation in the upper bound of the uncertainty set $\mathcal{U}_G$, for Example 4.2.1.

The norm $\|\cdot\|_2$ here denotes the matrix 2-norm.

Measuring perturbation in the uncertainty set using $\mathbf{Haus}_d$, we show next that, as $\mathbf{Haus}_d(\mathcal{U},\overline{\mathcal{U}}) \to 0$, the distance of the regularized robust strategies corresponding to $\mathcal{U}$ and $\overline{\mathcal{U}}$ also approaches zero. Our analysis mainly relies on results in Chapter 3 and (Fiacco, 1974). Below, $\lambda_{\max}(\cdot)$ denotes the maximum eigenvalue.

Theorem 2.1 in (Fiacco, 1974), presented in Theorem A.1.2, implies when the sequence $\{\mathcal{S}_k\}$ of closed subsets of a compact set $\mathcal{T}$ of a metric space $(\mathcal{X}, d)$ approaches a compact set $\mathcal{S} \subseteq \mathcal{X}$, i.e., $\mathbf{Haus}_d(\mathcal{S}_k, \mathcal{S}) \to 0$, then any sequence of solutions of minimizing a continuous function f over $\mathcal{S}_k$ contains at least one convergent subsequence and all cluster points are solutions of $\min_{x\in\mathcal{S}} f(x)$.

We precede the stability analysis for the regularized robust optimal execution strategy by the following auxiliary lemma. This result is used when Theorem 2.1 in (Fiacco, 1974) is applied for the portfolio execution cost problem with $\mathcal{R} = \mathcal{R}_c$, using (4.3.14).

Lemma 4.5.1. *Let $\mathcal{R} = \mathcal{R}_c$ and a nonempty, convex, compact uncertainty set $\mathcal{U}$ be given. Assume that the regularization parameter $\rho > 0$ is chosen such that $\mathcal{V}(\mathcal{U}, \rho)$ is nonempty. Then problem* (4.3.14)*, applied for the regularized uncertainty set $\mathcal{V}(\mathcal{U}, \rho)$, shares the same set of solutions with the following problem:*

$$\begin{array}{ll} \max & \frac{1}{\tau}\bar{S}^T\tilde{H}\bar{S} - \frac{1}{2}\left(b(\tilde{H},\tilde{G}) + A^T\lambda\right)^T W^{-1}(\tilde{H},\tilde{G},\mu)\left(b(\tilde{H},\tilde{G}) + A^T\lambda\right) - c^T\lambda \\ s.t. & \|\lambda\|_2 \le \lambda_u \\ & (\tilde{H},\tilde{G}) \in \mathcal{V}(\mathcal{U},\rho), \quad \lambda \in \mathbb{R}^r_+, \end{array} \tag{4.5.2}$$

where

$$\lambda_u = \frac{4\sqrt{m}\Lambda_u + 2\mu\tau\lambda_{\max}(C)}{(\rho + 2\mu\tau\lambda_{\min}(C))\cdot\sin\left(\frac{\pi}{4N-2}\right)}\left(1 + \frac{4\sqrt{m}\Lambda_u + 2\mu\tau\lambda_{\max}(C)}{2\sin^2\left(\frac{\pi}{4N-2}\right)}\right)(\Lambda_u + 1)\,\|\bar{S}\|_2.$$

Here, $\Lambda_u = \max_{(\tilde{H},\tilde{G})\in\mathcal{U}} \|\tilde{\Theta}\|_2$ where $\tilde{\Theta}$ is the combined impact matrix corresponding to $\tilde{H}$ and $\tilde{G}$. Furthermore, the set of feasible points of problem (4.5.2) *is compact.*

Proof. Recall that problem (4.3.14) includes the Lagrange dual problem of the inner minimization problem in (4.3.13), which is given below

$$\min_{z\in\mathcal{R}_c} \quad \frac{1}{2}z^TW(\tilde{H},\tilde{G},\mu)z + b^T(\tilde{H},\tilde{G})z. \tag{4.5.3}$$

Let J be the set of indices of binding constraints defining $\mathcal{R}_c$ at the solution of problem (4.5.3). Lemma 3.3.1 yields

$$\lambda_{\min}\left(A_JA_J^T\right) \ge 4\sin^2\left(\frac{\pi}{4N-2}\right), \tag{4.5.4}$$

Furthermore, using equation (3.3.22) we have

$$\|A_J^T\|_2 \leq 2. \tag{4.5.5}$$

Proposition A.2.2 implies that the Lagrange multiplier λ of the constraints defining $\mathcal{R}_c$ satisfies

$$\begin{aligned}
\|\lambda\|_2 &\leq \frac{2\lambda_{\max}(W(\tilde{H},\tilde{G},\mu))}{\lambda_{\min}(W(\tilde{H},\tilde{G},\mu))\cdot\sqrt{\lambda_{\min}(A_J A_J^T)}}\left(1+\frac{\lambda_{\max}(W(\tilde{H},\tilde{G},\mu))\cdot\|A_J^T\|_2}{\lambda_{\min}(A_J A_J^T)}\right)(\|\tilde{\Theta}\bar{S}\|_2+\|\bar{S}\|_2)\\
&\leq \frac{\lambda_{\max}(W(\tilde{H},\tilde{G},\mu))}{(\rho+2\mu\tau\lambda_{\min}(C))\cdot\sin\left(\frac{\pi}{4N-2}\right)}\left(1+\frac{\lambda_{\max}(W(\tilde{H},\tilde{G},\mu))}{2\sin^2\left(\frac{\pi}{4N-2}\right)}\right)(\Lambda_u+1)\|\bar{S}\|_2, \qquad (4.5.6)
\end{aligned}$$

where inequalities (4.5.4), (4.5.5), and (4.4.2) are used to derive inequality (4.5.6). Notice that Λ_u is finite as $\mathcal{U}$ is compact.

For every $(\tilde{H},\tilde{G})\in\mathcal{V}(\mathcal{U},\rho)$, the matrix $W(\tilde{H},\tilde{G},0)$ is symmetric. Thus

$$\begin{aligned}
\|W(\tilde{H},\tilde{G},0)\|_2 \leq \|W(\tilde{H},\tilde{G},0)\|_1 &\leq \|\tilde{\Theta}\|_1+\|\tilde{\Theta}+\tilde{\Theta}^T\|_1+\|\tilde{\Theta}^T\|_1\\
&\leq \|\tilde{\Theta}\|_1+\|\tilde{\Theta}\|_1+\|\tilde{\Theta}^T\|_1+\|\tilde{\Theta}^T\|_1\\
&= 2\|\tilde{\Theta}\|_1+2\|\tilde{\Theta}\|_\infty \leq 4\sqrt{m}\Lambda_u. \qquad (4.5.7)
\end{aligned}$$

Therefore,

$$\lambda_{\max}(W(\tilde{H},\tilde{G},\mu)) \leq \|W(\tilde{H},\tilde{G},0)\|_2+2\mu\tau\lambda_{\max}(C) \leq 4\sqrt{m}\Lambda_u+2\mu\tau\lambda_{\max}(C). \tag{4.5.8}$$

Using inequalities (4.5.7) and (4.5.8) in inequality (4.5.6) we get,

$$\|\lambda\|_2 \leq \frac{4\sqrt{m}\Lambda_u+2\mu\tau\lambda_{\max}(C)}{(\rho+2\mu\tau\lambda_{\min}(C))\cdot\sin\left(\frac{\pi}{4N-2}\right)}\left(1+\frac{4\sqrt{m}\Lambda_u+2\mu\tau\lambda_{\max}(C)}{2\sin^2\left(\frac{\pi}{4N-2}\right)}\right)(\Lambda_u+1)\,\|\bar{S}\|_2. \tag{4.5.9}$$

Thus optimal points of problem (4.3.14) satisfy inequality (4.5.9). Whence problems (4.3.14) and (4.5.2) have the same set of solutions.

The upper bound in inequality (4.5.9) depends on $\mathcal{U}$ and the constants ρ, N, and $\bar{S}$. Therefore, it is finite for any compact uncertainty set $\mathcal{U}\subseteq\mathbb{R}^{2m^2}$. Thus when $\mathcal{U}$ is nonempty and compact, the set of feasible points of problem (4.5.2) is closed and bounded, and consequently compact. □

Lemma 4.5.1 is used in part (c) of the following theorem.

Theorem 4.5.1. *Let the risk aversion parameter $\mu\geq 0$ and a nonempty convex compact uncertainty set $\mathcal{U}$ be given. Assume that the regularization parameter $\rho>0$ is chosen such that $\mathcal{V}(\mathcal{U},\rho)$ is nonempty. Denote a solution to the regularized robust problem* (4.4.3) *with respect to the uncertainty set $\mathcal{U}$ with (z_u,H_u,G_u). Let $\overline{\mathcal{U}}$ be any nonempty convex compact uncertainty set such that $\mathcal{V}(\overline{\mathcal{U}},\rho)$ is nonempty, and $(z_{\bar{u}},H_{\bar{u}},G_{\bar{u}})$ be a solution to problem* (4.4.3) *with respect to $\overline{\mathcal{U}}$. Denote the combined impact matrices corresponding to (H_u,G_u) and $(H_{\bar{u}},G_{\bar{u}})$ with Θ_u and $\Theta_{\bar{u}}$, respectively. Define $\Lambda_u=\max_{(\tilde{H},\tilde{G})\in\mathcal{V}(\mathcal{U},\rho)}\|\tilde{\Theta}\|_2$. Then the following hold:*

(a) *When the execution strategy is unconstrained, i.e.,* $\mathcal{R} = \mathcal{R}_0$,

$$\frac{\|z_u - z_{\bar{u}}\|_2}{\|\bar{S}\|_2} \le \frac{1}{\beta_\rho}\left(1 + \frac{4\sqrt{m}}{\beta_\rho}\Lambda_u\right)\|\Theta_u - \Theta_{\bar{u}}\|_2, \tag{4.5.10}$$

where $\beta_\rho = \rho + 2\mu\tau\lambda_{\min}(C)$.

(b) *When buying is prohibited in the sell execution strategy, i.e.,* $\mathcal{R} = \mathcal{R}_c$, *there exists* $\varsigma_{u,\bar{u}} > 0$ *such that*

$$\frac{\|z_u - z_{\bar{u}}\|_2}{\|\bar{S}\|_2} \le \varsigma_{u,\bar{u}}\left(1 + 4\sqrt{m}\varsigma_{u,\bar{u}}\left(\max\{1,\beta_u\} + \Lambda_u + \|\Theta_u - \Theta_{\bar{u}}\|_2\right)\right)\|\Theta_u - \Theta_{\bar{u}}\|_2, \tag{4.5.11}$$

where $\beta_u = 4\sqrt{m}\Lambda_u + 2\mu\tau\lambda_{\max}(C)$ *and*

$$\varsigma_{u,\bar{u}} \le \frac{1}{\beta_\rho}\left(1 + \frac{\beta_u + 4\sqrt{m}\|\Theta_u - \Theta_{\bar{u}}\|_2}{\sin^2\left(\frac{\pi}{4N-2}\right)\beta_\rho}\left(\frac{\beta_u + 4\sqrt{m}\|\Theta_u - \Theta_{\bar{u}}\|_2}{\beta_\rho} + 3\sin\left(\frac{\pi}{4N-2}\right)\right)\right). \tag{4.5.12}$$

(c) *In addition, for any uncertainty set* $\overline{\mathcal{U}}$ *with* $\mathbf{Haus}_d(\overline{\mathcal{U}},\mathcal{U}) \to 0$, *we have* $\|z_u - z_{\bar{u}}\|_2 \to 0$, *when* $\mathcal{R}$ *equals either* $\mathcal{R}_c$ *or* $\mathcal{R}_0$, *and the metric* $d(\cdot,\cdot)$ *is defined in* (4.5.1).

Proof. First we note that, since $\mathcal{U}$ and consequently $\mathcal{V}(\mathcal{U},\rho)$ are compact, Λ_u is finite. Furthermore, since $(H_u, G_u) \in \mathcal{V}(\mathcal{U},\rho)$ and $(H_{\bar{u}}, G_{\bar{u}}) \in \mathcal{V}(\overline{\mathcal{U}},\rho)$, the matrices $W(H_u, G_u, \mu)$ and $W(H_{\bar{u}}, G_{\bar{u}}, \mu)$ are both positive definite. Whence, Proposition 4.3.1 implies that the corresponding regularized robust strategies are unique.

For notational simplicity, denote

$$W_u \overset{\text{def}}{=} W(H_u, G_u, \mu), \qquad W_{\bar{u}} \overset{\text{def}}{=} W(H_{\bar{u}}, G_{\bar{u}}, \mu).$$

Since $(H_u, G_u) \in \mathcal{V}(\mathcal{U},\rho)$ and $(H_{\bar{u}}, G_{\bar{u}}) \in \mathcal{V}(\overline{\mathcal{U}},\rho)$, inequality (4.4.2) yields

$$\min\{\lambda_{\min}(W_u), \lambda_{\min}(W_{\bar{u}})\} \ge \beta_\rho, \tag{4.5.13}$$

$$\hat{\lambda} \overset{\text{def}}{=} \max\{1, \lambda_{\max}(W_u)\} \ge \max\{1, \rho + 2\mu\tau\lambda_{\min}(C)\} \ge \beta_\rho. \tag{4.5.14}$$

Since the regularized robust solutions z_u and $z_{\bar{u}}$ solve the nominal optimal portfolio execution problem with the impact matrices (H_u, G_u) and $(H_{\bar{u}}, G_{\bar{u}})$, respectively, Theorem 3.3.1 yields

$$\|z_u - z_{\bar{u}}\|_2 \le \frac{\|\bar{S}\|_2\|\Theta_u - \Theta_{\bar{u}}\|_2}{\min\{\lambda_{\min}(W_u), \lambda_{\min}(W_{\bar{u}})\}}\left(1 + \frac{4\sqrt{m}}{\min\{\lambda_{\min}(W_u), \lambda_{\min}(W_{\bar{u}})\}}\|\Theta_u\|_2\right), \tag{4.5.15}$$

when $\mathcal{R} = \mathcal{R}_0$.

Applying inequality (4.5.13) in (4.5.15), and the fact that $\|\Theta_u\|_2 \le \Lambda_u$, we obtain inequality (4.5.10) and the proof of part (a) is completed.

Next, let $\mathcal{R} = \mathcal{R}_c$. Theorem 3.3.3 implies that

$$\|z_u - z_{\bar{u}}\|_2 \leq \varsigma_{u,\bar{u}} \left\|\bar{S}\right\|_2 \left(1 + 4\varsigma_{u,\bar{u}}\sqrt{m}\left(\hat{\lambda} + \|\Theta_u\|_2 + \|\Theta_u - \Theta_{\bar{u}}\|_2\right)\right) \|\Theta_u - \Theta_{\bar{u}}\|_2, \quad (4.5.16)$$

where

$$\varsigma_{u,\bar{u}} \leq \frac{1}{\underline{\lambda}}\left(1 + \frac{(\bar{\lambda} + \underline{\lambda})}{2\sin^2\left(\frac{\pi}{4N-2}\right)\hat{\lambda}}\left(\frac{\bar{\lambda}}{\hat{\lambda}} + 3\sin\left(\frac{\pi}{4N-2}\right)\right)\right), \quad (4.5.17)$$

with $\bar{\lambda} = \max_{\eta\in[0,1]} \lambda_{\max}(W_u + \eta(W_{\bar{u}} - W_u))$, $\underline{\lambda} = \min_{\eta\in[0,1]} \lambda_{\min}(W_u + \eta(W_{\bar{u}} - W_u))$, and $\hat{\lambda}$ is as in (4.5.14).

The Courant-Fischer Theorem yields

$$\begin{aligned}\underline{\lambda} &= \min_{\eta\in[0,1]} \lambda_{\min}\left(W_u + \eta(W_{\bar{u}} - W_u)\right) \quad (4.5.18)\\ &\geq \min_{\eta\in[0,1]} \left(\eta\lambda_{\min}(W_{\bar{u}}) + (1-\eta)\lambda_{\min}(W_u)\right) \geq \beta_\rho,\end{aligned}$$

where the last inequality comes from inequality (4.4.2).

Since the matrix $W_u - W_{\bar{u}}$ is symmetric, we have $\|W_u - W_{\bar{u}}\|_1 = \|W_u - W_{\bar{u}}\|_\infty$. Hence,

$$\|W_u - W_{\bar{u}}\|_2 \leq \sqrt{\|W_u - W_{\bar{u}}\|_1 \|W_u - W_{\bar{u}}\|_\infty} = \|W_u - W_{\bar{u}}\|_1.$$

Therefore we have

$$\begin{aligned}\|W_u - W_{\bar{u}}\|_2 &\leq \|W_u - W_{\bar{u}}\|_1\\ &\leq \|\Theta_u - \Theta_{\bar{u}}\|_1 + \left\|\Theta_u - \Theta_{\bar{u}} + (\Theta_u - \Theta_{\bar{u}})^T\right\|_1 + \left\|(\Theta_u - \Theta_{\bar{u}})^T\right\|_1\\ &\leq 2\|\Theta_u - \Theta_{\bar{u}}\|_1 + 2\left\|(\Theta_u - \Theta_{\bar{u}})^T\right\|_1\\ &= 2\|\Theta_u - \Theta_{\bar{u}}\|_1 + 2\|\Theta_u - \Theta_{\bar{u}}\|_\infty \quad (4.5.19)\\ &\leq 4\sqrt{m}\,\|\Theta_u - \Theta_{\bar{u}}\|_2.\end{aligned}$$

This result along with the Courant-Fischer Theorem imply that

$$\begin{aligned}\bar{\lambda} &= \max_{\eta\in[0,1]} \lambda_{\max}\left(W_u + \eta(W_{\bar{u}} - W_u)\right)\\ &\leq \max_{\eta\in[0,1]} \left(\lambda_{\max}(W_u) + \eta\lambda_{\max}(W_{\bar{u}} - W_u)\right)\\ &\leq \lambda_{\max}(W_u) + \max_{\eta\in[0,1]} \eta\|W_{\bar{u}} - W_u\|_2\\ &\leq \lambda_{\max}(W_u) + \|W_{\bar{u}} - W_u\|_2\\ &\leq \lambda_{\max}(W_u) + 4\sqrt{m}\|\Theta_{\bar{u}} - \Theta_u\|_2. \quad (4.5.20)\end{aligned}$$

Since $(H_u, G_u) \in \mathcal{U}$, inequality (4.5.8) yields $\lambda_{\max}(W_u) \leq \beta_u$. Using this inequality in inequality (4.5.20), we get

$$\begin{aligned}&\bar{\lambda} \leq \beta_u + 4\sqrt{m}\|\Theta_{\bar{u}} - \Theta_u\|_2,\\ &\bar{\lambda} + \underline{\lambda} \leq 2\bar{\lambda} \leq 2(\beta_u + 4\sqrt{m}\|\Theta_{\bar{u}} - \Theta_u\|_2).\end{aligned}$$

Applying these inequalities, along with inequalities (4.5.14) and (4.5.18), in (4.5.17) yields inequality (4.5.12). Furthermore, using inequalities $\|\Theta_u\|_2 \leq \Lambda_u$ and $\|W_u\|_2 = \lambda_{\max}(W_u) \leq \beta_u$ in (4.5.16), inequality (4.5.11) is obtained. This completes the proof of part (b).

The proof of part (c) relies on the result in (Fiacco, 1974) (see Theorem A.1.2 and Corollary A.1.1) for problems (4.3.16) and (4.5.2). Since the matrix $W(\tilde{H}, \tilde{G}, \mu)$ is positive definite over $\mathcal{V}(\mathcal{U}, \rho)$, the entries of the inverse matrix $W^{-1}(\tilde{H}, \tilde{G}, \mu)$ are continuous functions of the entries of the matrices $\tilde{H}$ and $\tilde{G}$ (see, e.g., (Basener, 2006)). Whence, the objective functions of problems (4.3.16) and (4.5.2) are continuous with respect to elements of $\tilde{H}$, $\tilde{G}$, and λ.

First consider the case when the set of feasible execution strategies is $\mathcal{R}_0$. Suppose $\mathbf{Haus}_d(\bar{\mathcal{U}}, \mathcal{U}) \to 0$ and $\|z_u - z_{\bar{u}}\|_2 \not\to 0$. Thus there exists some $\epsilon > 0$ such that for every k there exists some $\overline{\mathcal{U}}_k \subseteq \mathbb{R}^{2m^2}$ with $\mathbf{Haus}_d(\overline{\mathcal{U}}_k, \mathcal{U}) < \frac{1}{k}$ and $\|z_u - z_{\bar{u}_k}\|_2 > \epsilon$. Here $z_{\bar{u}_k}$ is the regularized robust solution corresponding to the uncertainty set $\{overline\mathcal{U}_k\}$. Let $\{(H_{\bar{u}_k}, G_{\bar{u}_k})\}_k$ be a sequence of solutions of problem (4.3.16) with the uncertainty sets $\overline{\mathcal{U}}_k$. Corollary A.1.1 yields there exists a subsequence $\{(H_{\bar{u}_{k_i}}, G_{\bar{u}_{k_i}})\}_i$ of the sequence $\{(H_{\bar{u}_k}, G_{\bar{u}_k})\}_k$ that approaches to a solution (H_u, G_u) of problem (4.3.16) with the uncertainty set $\mathcal{U}$. Thus, for i sufficiently large, $d((H_u, G_u), (H_{\bar{u}_{k_i}}, G_{\bar{u}_{k_i}})) \to 0$. Consequently, $\|\Theta_u - \Theta_{\bar{u}_{k_i}}\|_2 \to 0$, because $\|\Theta_u - \Theta_{\bar{u}_{k_i}}\|_2 \leq d((H_u, G_u), (H_{\bar{u}_{k_i}}, G_{\bar{u}_{k_i}}))$. Using inequality (4.5.10) and the fact that the regularized robust solutions z_u and $z_{\bar{u}_{k_i}}$ are unique, we get $\|z_u - z_{\bar{u}_{k_i}}\|_2 \to 0$, for i large enough. This result is in contradiction to $\|z_u - z_{\bar{u}_{k_i}}\|_2 > \epsilon$. Whence, $\|z_u - z_{\bar{u}}\|_2 \to 0$ as $\mathbf{Haus}_d(\bar{\mathcal{U}}, \mathcal{U}) \to 0$.

Now, let $\mathcal{R} = \mathcal{R}_c$. Recall that (H_u, G_u) solves problem (4.3.10) or equivalently problem (4.3.14). Furthermore, Lemma 4.5.1 indicates that (H_u, G_u) constitutes a solution of problem (4.5.2) in which the set of feasible points is compact. Therefore, Corollary A.1.1 is applicable to problem (4.5.2). A similar discussion, as in the previous case, through Corollary A.1.1 and inequalities (4.5.11) and (4.5.12) completes the proof of part (c), when $\mathcal{R} = \mathcal{R}_c$. □

Theorem 4.5.1 implies that small variations in the uncertainty set $\mathcal{U}$ result in small changes in the regularized robust solution. In other words, the regularized robust solution is asymptotically stable with respect to change in the uncertainty set.

4.6 Implications of Regularization

In this section, we discuss additional implications of the proposed regularization on the robust solution, the robust optimal value, and the efficient frontier.

4.6.1 Implications on the Optimal Execution Strategy

Here, we analyze how the regularization parameter affects some characteristics of the regularized robust optimal execution strategy.

For every $(\tilde{H}, \tilde{G}) \in \mathcal{V}(\mathcal{U}, \rho)$, inequality (4.4.2) yields

$$\left\|W^{-1}(\tilde{H}, \tilde{G}, \mu)\right\|_2 = \lambda_{\max}\left(W^{-1}(\tilde{H}, \tilde{G}, \mu)\right) = \frac{1}{\lambda_{\min}(W(\tilde{H}, \tilde{G}, \mu))} \le \frac{1}{\rho + 2\mu\tau\lambda_{\min}(C)}. \quad (4.6.1)$$

The following proposition shows that the regularized robust solution satisfies a Tikhonov-type regularization constraint, when $\mathcal{R} = \mathcal{R}_0$.

Proposition 4.6.1. *Let $\mathcal{R} = \mathcal{R}_0$, the risk aversion parameter $\mu \ge 0$, and the nonempty convex compact uncertainty set $\mathcal{U}$ be given. Assume that the regularization parameter ρ is chosen such that $\mathcal{V}(\mathcal{U}, \rho)$ is nonempty. Then the regularized robust solution $z_u \in \mathcal{R}_0$ of problem* (4.4.3) *satisfies:*

$$\frac{\|z_u\|_2}{\|\bar{S}\|_2} \le \frac{\Lambda_u}{\rho + 2\mu\tau\lambda_{\min}(C)}, \quad (4.6.2)$$

where $\Lambda_u = \max_{(\tilde{H}, \tilde{G}) \in \mathcal{V}(\mathcal{U}, \rho)} \|\Theta_u\|_2$.

Proof. When the set of feasible execution strategies is given by $\mathcal{R}_0$, the regularized robust solution is determined by equation (4.3.9). Applying inequality (4.6.1), we get:

$$\begin{aligned} \|z_u\|_2 &= \left\|-W^{-1}(H_u, G_u, \mu)\ b(H_u, G_u)\right\|_2 \\ &\le \left\|W^{-1}(H_u, G_u, \mu)\right\|_2 \|b(H_u, G_u)\|_2 \\ &\le \frac{\|b(H_u, G_u)\|_2}{\rho + 2\mu\tau\lambda_{\min}(C)}. \end{aligned}$$

Using $\|b(H_u, G_u)\|_2 = \|-\Theta_u\bar{S}\|_2 \le \|\Theta_u\|_2\|\bar{S}\|_2 \le \Lambda_u\|\bar{S}\|_2$ in the above inequality completes the proof of inequality (4.6.2). □

Proposition 4.6.1 indicates that including the regularization constraint in the uncertainty set implicitly offers a solution that satisfies a two-norm constraint on the execution strategy.

In addition, the regularization parameter also affects the Euclidean distance between the regularized robust optimal execution strategy and the naive strategy. The naive strategy can be used as a benchmark since it has been proved to be optimal in special circumstances. For example, the naive strategy is always the solution for the nominal optimal portfolio execution problem (3.2.6), when $\mu = 0$, the permanent impact matrix G is symmetric, and Θ is positive definite (see, e.g., Proposition 3.2.1). Furthermore, when the permanent impact matrix $\tilde{G}$ is symmetric for every element in $\mathcal{U}$, which holds in a single asset case, the robust optimal execution strategy is the naive strategy, regardless of the choice of the uncertainty set. A bound on the distance between the regularized robust optimal execution strategy and the naive strategy benchmark is established in the next proposition.

Proposition 4.6.2. *Let $\mathcal{R} = \mathcal{R}_0$, the risk aversion parameter $\mu \ge 0$, and the nonempty convex compact uncertainty set $\mathcal{U}$ be given. Assume the regularization parameter ρ is chosen* *such that $\mathcal{V}(\mathcal{U},\rho)$ is nonempty. Then, the regularized robust optimal execution strategy z_u of problem (4.4.3) satisfies:*

$$\frac{\|z_u - z_n\|_2}{\|\bar{S}\|_2} \leq \frac{\Lambda_u}{4\sin^2\left(\frac{\pi}{2N}\right)\rho}\left(1 + \frac{4\sqrt{m}\Lambda_u + 2\mu\tau\lambda_{\max}(C)}{\beta_\rho}\right), \tag{4.6.3}$$

where $\beta_\rho = \rho + 2\mu\tau\lambda_{\min}(C)$ and z_n represents the naive strategy $x_k = \left(\frac{N-k}{N}\right)\bar{S}$ for $k = 1, 2, \cdots, N$.

Proof. Let $(H_u, G_u) \in \mathcal{V}(\mathcal{U},\rho)$ be a solution of problem (4.3.12) with the uncertainty set $\mathcal{V}(\mathcal{U},\rho)$. The unique regularized robust solution from problem (4.4.3) is then

$$z_u = -W^{-1}(H_u, G_u, \mu)\; b(H_u, G_u).$$

For simplicity, denote

$$W_u \stackrel{\text{def}}{=} W(H_u, G_u, \mu), \quad b_u \stackrel{\text{def}}{=} b(H_u, G_u).$$

Notice that $\|b_u\|_2 \leq \|\Theta_u\|_2\|\bar{S}\|_2 \leq \Lambda_u\|\bar{S}\|_2$. Whence

$$\|b(H_u^T, G_u^T)\|_2 = \|\Theta_u^T\bar{S}\|_2 \leq \|\Theta_u^T\|_2\|\bar{S}\|_2 = \|\Theta_u\|_2\|\bar{S}\|_2 \leq \Lambda_u\|\bar{S}\|_2. \tag{4.6.4}$$

Since the naive strategy minimizes the expected execution cost for any symmetric permanent impact matrix where the combined impact matrix is positive definite (see, e.g., Proposition 3.2.1), it is the unique optimal execution strategy corresponding to the impact matrices $H_u + H_u^T$ and $G_u + G_u^T$, i.e., $z_n = -W_n^{-1}b_n$, where

$$W_n \stackrel{\text{def}}{=} W\left(H_u + H_u^T, G_u + G_u^T, 0\right), \quad b_n \stackrel{\text{def}}{=} b\left(H_u + H_u^T, G_u + G_u^T\right).$$

Therefore

$$W_n(z_u - z_n) = W_n\left(-W_u^{-1}b_u - (-W_n^{-1}b_n)\right) = -W_nW_u^{-1}b_u + b_n = (W_u - W_n)W_u^{-1}b_u - b_u + b_n.$$

Using $b_n - b_u = b\left(H_u^T, G_u^T\right)$ and $W_u - W_n = W(-H_u^T, -G_u^T, \mu)$, we have:

$$\begin{aligned}
\|z_u - z_n\|_2 &= \|W_n^{-1}\left((W_u - W_n)W_u^{-1}b_u - b_u + b_n\right)\|_2 \\
&\leq \|W_n^{-1}\|_2\left(\|W_u - W_n\|_2\|W_u^{-1}\|_2\|b_u\|_2 + \|-b_u + b_n\|_2\right) \\
&= \|W_n^{-1}\|_2\left(\|W(-H_u^T, -G_u^T, \mu)\|_2\|W_u^{-1}\|_2\|b_u\|_2 + \|b\left(H_u^T, G_u^T\right)\|_2\right) \\
&= \frac{1}{\lambda_{\min}(W_n)}\left(\|W(-H_u^T, -G_u^T, \mu)\|_2\frac{\|b_u\|_2}{\lambda_{\min}(W_u)} + \|b\left(H_u^T, G_u^T\right)\|_2\right) \\
&\leq \frac{1}{\lambda_{\min}(W_n)}\left(\|W(-H_u^T, -G_u^T, \mu)\|_2\frac{\Lambda_u\|\bar{S}\|_2}{\rho + 2\mu\tau\lambda_{\min}(C)} + \Lambda_u\|\bar{S}\|_2\right) \qquad (4.6.5) \\
&\leq \frac{1}{\lambda_{\min}(W_n)}\left(\left(\|W(-H_u^T, -G_u^T, 0)\|_2 + 2\mu\tau\lambda_{\max}(C)\right)\frac{\Lambda_u\|\bar{S}\|_2}{\beta_\rho} + \Lambda_u\|\bar{S}\|_2\right). \qquad (4.6.6)
\end{aligned}$$

where inequality (4.6.5) comes from (4.4.2) and (4.6.4). Inequality (4.6.6) comes from the Courant-Fischer theorem.

Corollary 3.2.1 applied to W_n implies that

$$\lambda_{\min}\left(W_n\right) = 4\sin^2\left(\frac{\pi}{2N}\right)\lambda_{\min}\left(\Theta_u + \Theta_u^T\right).$$

Since $(H_u, G_u) \in \mathcal{V}(\mathcal{U}, \rho)$, we have $\lambda_{\min}(W(H_u, G_u, 0)) \geq \rho$. This yields $\lambda_{\min}(\Theta_u + \Theta_u^T) \geq \rho$, as the matrix $\Theta_u + \Theta_u^T$ is a leading principle submatrix of $\lambda_{\min}(W(H_u, G_u, 0))$. Thus we get

$$\lambda_{\min}\left(W_n\right) \geq 4\sin^2\left(\frac{\pi}{2N}\right)\rho. \tag{4.6.7}$$

Furthermore, a vector $(a_1^T, a_2^T, ..., a_{N-1}^T)^T$ is an eigenvector of $W(H_u, G_u, 0)$ associated with the eigenvalue λ if and only if the vector $(-a_{N-1}^T, -a_{N-2}^T, ..., -a_1^T)^T$ is an eigenvector of $W(-H_u^T, -G_u^T, 0)$ for the same eigenvalue. Thus,

$$\left\|W(-H_u^T, -G_u^T, 0)\right\|_2 = \left\|W(H_u, G_u, 0)\right\|_2 \leq 4\sqrt{m}\Lambda_u, \tag{4.6.8}$$

where the inequality comes from (4.5.7) for (H_u, G_u). Using inequalities (4.6.7) and (4.6.8) in (4.6.6) completes the proof of (4.6.3). □

Table 4.1 computationally shows this property for liquidating the three assets in Example 4.2.1. It indicates that the Euclidean distance between the naive strategy and the regularized robust solution decreases as the regularization parameter increases. Figure 4.6 further illustrates the impact of the regularization parameter ρ on the regularized robust optimal execution strategy. We observe that, as the regularization parameter increases, trading for the first and second assets becomes more even while trading for the third asset becomes slightly more uneven. Plots in Figure 4.6 further depict the difference between the regularized robust optimal execution strategies (for $\rho_0 = 0.8, 1, 1.3$) and the (classical) robust solution (the thin solid line).

Proposition 4.6.2 indicates that if ρ increases, the upper bound on the difference between the regularized robust optimal execution strategy and the naive strategy decreases. This property demonstrates a difference between the regularization parameter ρ and the risk aversion parameter μ. When a large risk aversion parameter μ is chosen, the optimal execution strategy becomes close to the strategy of liquidating the entire holding in the first period. We note that Proposition 4.6.2 can be extended for more general feasible sets $\mathcal{R}$.

4.6.2 Implications on the Efficient Frontier

A mean-variance efficient frontier clearly depicts the performance of a strategy in terms of the cost and risk. Here, we study impact of regularization on the efficient frontier. Under the assumed model, following (3.2.3), variance of the execution cost does not depend on the

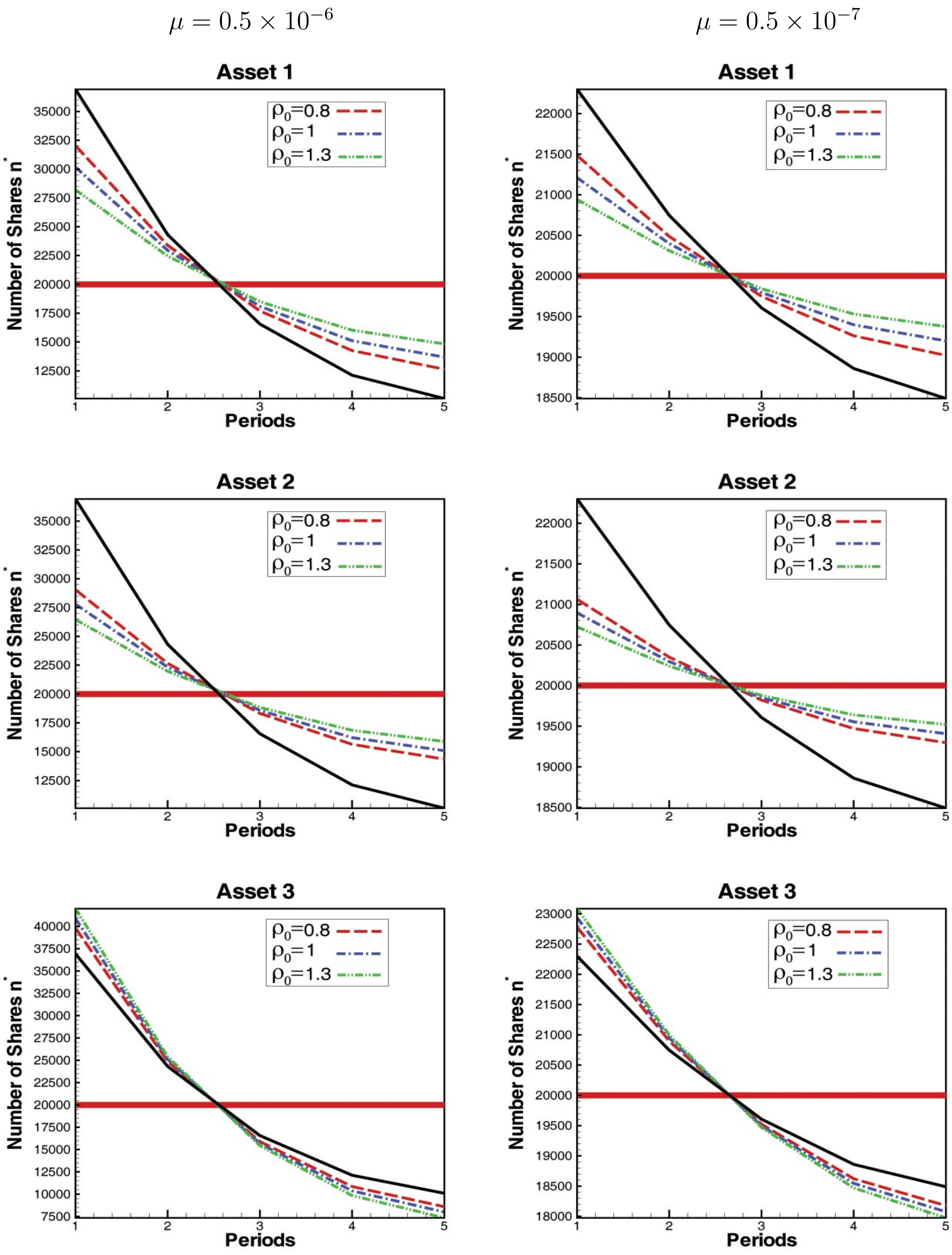


Figure 4.6: Execution strategies for liquidating three assets in Example 4.2.1 with the uncertainty set as in (4.2.4) and $\rho = \rho_0 \cdot \lambda_{\min}(W(H, G, 0))$. The feasible set is $\mathcal{R} = \mathcal{R}_0$. The thick solid line depicts the naive strategy. The thin solid line represents the (unregularized) robust solution.

ρ_0	$\|z_u - z_n\|_2/\|\bar{S}\|_2$	
	$\mu = 0.5 \times 10^{-6}$	$\mu = 0.5 \times 10^{-7}$
0.1	0.587858	0.084620
0.8	0.501393	0.070671
1	0.492655	0.070255
1.3	0.488592	0.070077

Table 4.1: Difference between the regularized robust solution and the naive strategy, for liquidating the three assets in Example 4.2.1. Here the regularization parameter equals $\rho = \rho_0 \cdot \lambda_{\min}(W(H, G, 0))$.

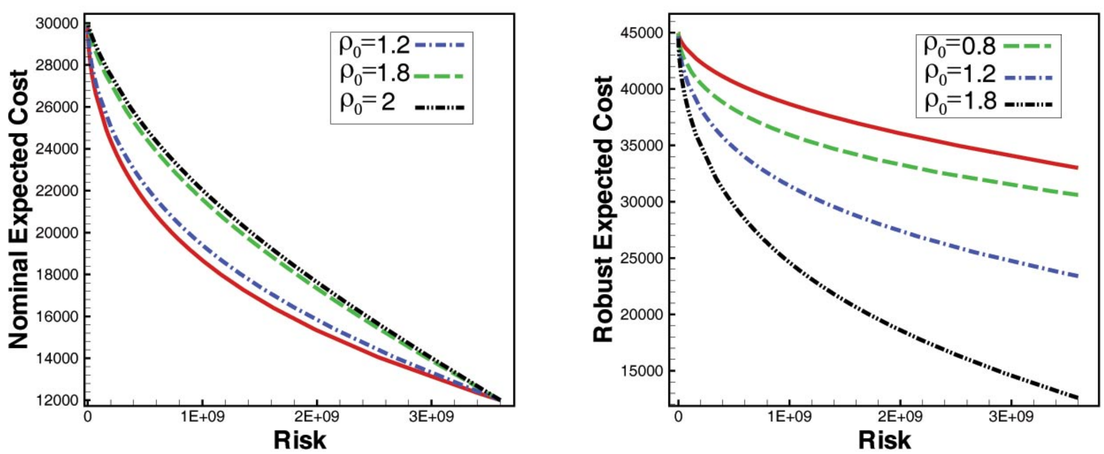


Figure 4.7: A single asset trading with $C = 0.003$, $H = 10^{-5} \cdot C$, $G = 0.5 \times 10^{-5} \cdot C$. The uncertainty set is $\mathcal{U} = \mathcal{U}_H \times \mathcal{U}_G$, where $\mathcal{U}_H = [0.5 \cdot H,\ 1.5 \cdot H]$ and $\mathcal{U}_G = [0.5 \cdot G,\ 4 \cdot G]$. Frontiers are for $\mu \in [0, 10^{-3}]$ and the feasible set of execution strategies is $\mathcal{R}_0$. The regularization parameter equals $\rho = \rho_0 \cdot \lambda_{\min}(W(H, G, 0))$. Right plot illustrates robust frontier (with respect to $\mathcal{U}$) of nominal solutions (depicted by solid line), and robust frontiers (with respect to $\mathcal{V}(\mathcal{U}, \rho)$) of regularized solutions for several choices of ρ. Left plot illustrates the nominal efficient frontier of nominal solution (depicted by solid line) and nominal frontier of regularized robust solutions.

impact matrices. Whence, robust optimization problem $RC(\mathcal{U})$ minimizes the weighted sum of the worst case mean of the execution cost and the variance of the execution cost.

Firstly we compare the nominal mean-variance performance of the nominal optimal execution strategy with that of the regularized robust optimal execution strategy. Every robust solution is a feasible point for the nominal optimal portfolio execution problem with nominal impact matrices. Thus, the nominal efficient frontier of nominal solutions is always below the nominal efficient frontier of the robust solution with respect to any uncertainty set $\mathcal{U}$. The nominal frontier of the robust solution is the curve of the nominal mean and variance points corresponding to the robust optimal execution strategy.

We consider here a single asset execution example to illustrate. The left plot in Figure 4.7 compares the nominal frontier of the nominal solution with the nominal frontier of the regularized robust solution. At the left end of the frontiers (corresponding to $\mu \to 10^{-3}$), all of the nominal frontiers converge to a single point, which corresponds to the optimal execution strategy of minimizing the variance of the execution cost, $n_1^* = \bar{S}$ and $n_i^* = 0$ for $i = 2, ..., N$. As μ increases, more weight is given to minimizing the expected cost and the difference among the frontiers becomes more prominent. The difference increases as the regularization parameter increases. Since here the nominal permanent impact matrices and worst-case permanent impact matrix are symmetric, the naive strategy is optimal for the nominal and robust problems, when $\mu = 0$. Hence, the frontiers also converge to a single point at the right end. An interesting observation from the left plot in Figure 4.7, is that the nominal frontier of the regularized robust solution does not intersect with the nominal frontier of the nominal solution, except at its ends (comparing it with Figure 1 in (Zhu et al., 2009)). This suggests that a regularized robust solution cannot be obtained simply by adjusting μ in the nominal optimization framework. Hence, in general (for general uncertainty sets) there is no correspondence between the risk aversion parameter μ and the regularization parameter ρ.

Next we assess robust performance by examining the robust frontier. The robust frontier (with respect to an uncertainty set $\mathcal{U}$) of the nominal solution is the worst case mean and variance of the nominal solution. Since the solution of the nominal optimal portfolio execution problem is feasible for problem $RC(\mathcal{U})$, the variance and worst case mean of its corresponding execution cost are no smaller than those of the robust solution. Therefore, the robust frontier of the nominal optimal execution strategy is always above the robust efficient frontier of the robust optimal execution strategy. This has also been computationally observed in (Tütüncü and Koenig, 2004) for a single period traditional portfolio optimization, when only the mean return is subject to uncertainty.

Conservativeness of the regularized robust optimal execution strategy can be adjusted through the regularization parameter. As the regularization parameter ρ increases, the size of the regularized uncertainty set decreases. Hence, $\Phi_\mu(\rho_1) \geq \Phi_\mu(\rho_2)$, when $\rho_1 \leq \rho_2$. Here, $\Phi_\mu(\cdot)$ is the robust optimal value defined in (4.4.3). Hence, the regularized robust solution becomes less conservative. In particular, $\Phi_\mu(\rho) \leq \Phi_\mu(0)$, for every $\rho \geq 0$.

Let the feasible region $\mathcal{R}$ be closed and convex, and the uncertainty set $\mathcal{U}$ be nonempty, convex, and compact. Assume ρ_1 and ρ_2 are two regularization parameters where $0 \le \rho_1 \le \rho_2$, and the sets $\mathcal{V}(\mathcal{U}, \rho_1)$ and $\mathcal{V}(\mathcal{U}, \rho_2)$ are nonempty. Then the (mean-variance) robust frontier with respect to $\mathcal{V}(\mathcal{U}, \rho_2)$ of the regularized robust solutions corresponding to ρ_2 is always below the mean-variance robust frontier with respect to $\mathcal{V}(\mathcal{U}, \rho_1)$ of the regularized robust solution corresponding to ρ_1. The property is depicted in the right plot in Figure 4.7.

In regularized robust optimization, when the regularization parameter ρ increases, the robust frontier with respect to the regularized uncertainty set of the regularized robust solution is pushed down. The following discussion illustrates this result.

Let the regularized robust optimal execution strategy, corresponding to the risk aversion parameter μ and the regularized uncertainty set $\mathcal{V}(\mathcal{U}, \rho_1)$, be z_{ρ_1}. Denote the variance and robust mean of the execution cost corresponding to z_{ρ_1} with $\mathbf{V}_1$ and $\mathbf{E}_1$, respectively:

$$\begin{aligned}
\mathbf{V}_1 &\stackrel{\text{def}}{=} \tau z_{\rho_1}^T (I \otimes C) z_{\rho_1}, \\
\mathbf{E}_1 &\stackrel{\text{def}}{=} \max_{(\tilde{H}, \tilde{G}) \in \mathcal{V}(\mathcal{U}, \rho_1)} \frac{1}{\tau} \bar{S}^T \tilde{H} \bar{S} + \frac{1}{2} z_{\rho_1}^T W(\tilde{H}, \tilde{G}, 0) z_{\rho_1} + b^T(\tilde{H}, \tilde{G}) z_{\rho_1}.
\end{aligned}$$

Let z_{ρ_2} be the regularized robust optimal execution strategy obtained from the regularized uncertainty set $\mathcal{V}(\mathcal{U}, \rho_2)$ and the risk aversion parameter $\hat{\mu}$ such that $\tau z_{\rho_2}^T (I \otimes C) z_{\rho_2} = \mathbf{V}_1$. Denote the robust expected execution cost corresponding to z_{ρ_2} with $\mathbf{E}_2$, i.e.,

$$\mathbf{E}_2 \stackrel{\text{def}}{=} \max_{(\tilde{H}, \tilde{G}) \in \mathcal{V}(\mathcal{U}, \rho_2)} \frac{1}{\tau} \bar{S}^T \tilde{H} \bar{S} + \frac{1}{2} z_{\rho_2}^T W(\tilde{H}, \tilde{G}, 0) z_{\rho_2} + b^T(\tilde{H}, \tilde{G}) z_{\rho_2}.$$

Using Sion's convex-concave minimax theorem (see, e.g., Theorem 3 in (Simons, 1995), which is presented in Theorem A.3.2) and the fact that both $\mathcal{V}(\mathcal{U}, \rho_2)$ and $\mathcal{R}$ are convex, and the uncertainty set $\mathcal{V}(\mathcal{U}, \rho_2)$ is compact, we have:

$$\begin{aligned}
\mathbf{E}_2 + \hat{\mu}\mathbf{V}_1 = \Phi_{\hat{\mu}}(\rho_2) &= \max_{(\tilde{H}, \tilde{G}) \in \mathcal{V}(\mathcal{U}, \rho_2)} \min_{z \in \mathcal{R}} \frac{1}{\tau} \bar{S}^T \tilde{H} \bar{S} + \frac{1}{2} z^T W(\tilde{H}, \tilde{G}, \hat{\mu}) z + b^T(\tilde{H}, \tilde{G}) z \quad (4.6.9) \\
&\le \max_{(\tilde{H}, \tilde{G}) \in \mathcal{V}(\mathcal{U}, \rho_2)} \frac{1}{\tau} \bar{S}^T \tilde{H} \bar{S} + \frac{1}{2} z_{\rho_1}^T W(\tilde{H}, \tilde{G}, \hat{\mu}) z_{\rho_1} + b^T(\tilde{H}, \tilde{G}) z_{\rho_1} \\
&\le \max_{(\tilde{H}, \tilde{G}) \in \mathcal{V}(\mathcal{U}, \rho_1)} \frac{1}{\tau} \bar{S}^T \tilde{H} \bar{S} + \frac{1}{2} z_{\rho_1}^T W(\tilde{H}, \tilde{G}, \hat{\mu}) z_{\rho_1} + b^T(\tilde{H}, \tilde{G}) z_{\rho_1} \quad (4.6.10) \\
&= \mathbf{E}_1 + \hat{\mu}\mathbf{V}_1,
\end{aligned}$$

where inequality (4.6.10) comes from the assumption $\rho_1 \le \rho_2$, which yields $\mathcal{V}(\mathcal{U}, \rho_2) \subseteq \mathcal{V}(\mathcal{U}, \rho_1)$.

Thus we obtained $\mathbf{E}_2 + \hat{\mu}\mathbf{V}_1 \le \mathbf{E}_1 + \hat{\mu}\mathbf{V}_1$ and consequently $\mathbf{E}_2 \le \mathbf{E}_1$. In other words, the point $(\mathbf{V}_1, \mathbf{E}_2)$ on the robust frontier corresponding to $\mathcal{V}(\mathcal{U}, \rho_2)$ is below the point $(\mathbf{V}_1, \mathbf{E}_1)$. Hence the robust frontier of $\mathcal{V}(\mathcal{U}, \rho_2)$ is below the robust frontier of $\mathcal{V}(\mathcal{U}, \rho_1)$.

The main property, used in the above argument, is the fact that the variance of the execution cost does not depend on the impact matrices and uncertainty sets. This property does not hold in the traditional portfolio optimization with an uncertain covariance matrix whose uncertainty set is non-separable from the mean uncertainty set. In such cases, equalities (4.6.9) and (4.6.10) fail.

4.7 Concluding Remarks

To address estimation risk in impact matrices, we consider robust optimization for the optimal portfolio execution problem. The minimax robust optimization can provide an optimal worst case performance guarantee. Effectiveness of the robust optimization, however, depends on specification of the uncertainty set, which is often imprecise. An uncertainty set with a large size can yield an overly conservative solution.

In addition, we illustrate that the robust execution strategy can be sensitive to the specification of the uncertainty set. Specifically, sensitivity of the robust execution strategy to the upper bound of an interval uncertainty set for the permanent impact matrix can be more severe than the sensitivity of the nominal execution strategy to the nominal impact matrices.

Motivated by the sensitivity analysis for the optimal execution strategy for the nominal optimal portfolio execution problem in the previous chapter, we propose a regularized robust optimization framework for the optimal portfolio execution problem. By imposing a regularization constraint to bound the minimum eigenvalue of the Hessian of the objective function in the problem, we show both mathematically and computationally that sensitivity of the regularized robust execution strategy is significantly improved.

We propose an efficient method based on convex optimization for the regularized robust execution problem. While a robust execution strategy may not be easily computed for a general uncertainty set, the regularized robust execution strategy can be efficiently derived using convex programming. Finally we analyze mathematically and computationally several implications of regularization on the execution strategy and its corresponding cost.

Chapter 5

Optimal Execution Under Jump Models For Uncertain Price Impact

5.1 Introduction

In the previous chapters, we have addressed sensitivity and robustness of the deterministic optimal execution strategy, obtained under a mean-variance objective and an additive market price dynamics, to changes in the parameters of linear price impact functions.

However, the typical assumption of a Brownian motion model for the market price dynamics is questionable. In the context of the optimal portfolio execution problem, it fails to capture the impact of large trades from other institutions concurring during the course of the execution. Analogous to the fact that one's own large trade causes a discrete market price change, a large trade from others also induces a permanent (uncertain) price impact on the market price. These uncertain permanent price impact of other large trades should be modeled appropriately in the market price dynamics when seeking an optimal execution strategy and evaluating the risk associated with an execution strategy. Unfortunately current quantitative analysis of the execution cost does not explicitly model this source of price depression; only the permanent price impact of the decision maker's own trade is explicitly considered. Indeed, the normal distribution assumption contradicts the well recognized empirical evidence that the short term (a day or less) asset return probability distribution function typically has fat tails, see, e.g., (Campbell et al., 1996; Pagan, 1996; Cont, 2001).

There are relatively few studies on the optimal portfolio execution problem under a model which accounts for price impact of other concurrent large trades. Carlin et al. (2007) develop a repeated game of complete information to model repeated interaction of price impact of large investors who attempt to minimize the expected execution cost. This model however relies on the assumptions that participants are strategic, and their execution strategies and their overall trading target are common knowledge. In (Almgren and Lorenz, 2006), a Bayesian approach is proposed to introduce information on other large trades based on the observed price. This approach models the market price through normal distributions. Thus risk assessment under this model, particularly the tail risk, is likely to be inaccurate. Note that, in both studies, no risk consideration is given in devising an optimal execution strategy. In (Alfonsi et al., 2010), optimal execution strategies in order books are considered; the authors also mentioned, without any explicit discussion, that perhaps jump models for the market price should be considered.

In this chapter, we make no assumption about the decision maker's knowledge of other institutions' trading targets or their execution strategies. Thus, arrivals and price impact of other large trades are uncertain. We investigate reasonable models for this uncertainty and their effect on the optimal execution strategy and execution risk. The main contributions of this chapter include the following:

- Following the methodology in market microstructure theory in which uncertainty in order arrivals over time are modeled by Poisson processes, see, e.g., (Garman, 1976), we explicitly model uncertain permanent price impact of other large trades using compound Poisson processes. Jump events, in this model, represent uncertain arrivals of other large trades and random jump amplitudes represent their uncertain permanent price impact. In the proposed model the market price evolution is defined by the summation of a continuous diffusion process (for "normal" trades) and two compound Poisson processes for permanent price impact of large buy and sell trades. Our proposed model accounts for discrete large changes in the market price to better capture the fat tails in the probability distribution of the price due to concurrent large trades by other institutions.

- Since the first concern in portfolio execution is the expected cost, we derive explicit formulae for optimal execution strategies to minimize the expected execution cost (*optimal risk neutral execution strategies*), under discrete additive jump diffusion models as well as multiplicative jump diffusion models with linear price impact functions. The additive diffusion model *without uncertain jumps* has been used previously in the literature, see, e.g., (Almgren and Chriss, 2000/2001). Since the stock price is typically modeled by a multiplicative model, we also consider multiplicative models with jumps. We analyze implications of model assumptions on the optimal execution strategies, execution cost, and execution risk. In addition we apply a computational method to determine the optimal execution strategy which minimizes the CVaR of the execution cost, assuming a strategy is deterministic.

- We compare the execution cost distribution and risk values for the optimal risk neutral execution strategy under a mean and volatility-adjusted Brownian diffusion model and the jump diffusion model. We illustrate that for quantitative assessment of risk, model assumptions can make a significant difference, particularly with respect to the assessment of extreme risk. Therefore, using an appropriate model is crucial in evaluating the risk exposure associated with an execution strategy, even for a risk neutral investor seeking a strategy which solely minimizes the expected execution cost. Furthermore, when a risk measure such as CVaR is minimized, the solutions under the two models are different and the execution risk can be underestimated by a Brownian diffusion process with no jump.

Our theoretical and computational investigation also establishes the following result and observations. Firstly, under an additive diffusion market price model and with linear price impact functions, it has been noted that, see, e.g., (Bertsimas and Lo, 1998), when the expected market price change is zero, the optimal risk neutral execution strategy is the naive execution strategy of trading an equal amount in each period. We generalize this result by proving that, when the expected market price change aside from the permanent price impact of the decision maker's own trade is zero, the optimal risk neutral execution strategy derived from stochastic dynamic programming is always static, unrelated to the specification of the market price evolution. Moreover, for stationary linear price impact functions this static strategy is reduced to the naive strategy. *Unless otherwise stated explicitly, in this chapter, we simply refer to the expected market price change aside from the permanent price impact of the decision maker's own trade as the expected market price change.*

Secondly, when the expected market price change is nonzero, specification of the market price evolution matters and the optimal execution strategy derived under each model can be significantly different from the naive strategy. The optimal risk neutral execution strategy obtained under the additive jump diffusion model is static and independent of the asset price volatility. In contrast, the optimal risk neutral execution strategy under the multiplicative jump diffusion model is dynamic and depends on the market price realization. Hence, this execution strategy adjusts the trading size according to the trading impact of other investors realized during the previous periods. In addition, the optimal risk neutral execution strategy under the multiplicative jump diffusion model depends on the covariance matrix.

Finally, we investigate the degree of suboptimality of both the naive strategy and the optimal risk neutral execution strategy under the additive jump diffusion model in terms of the expected execution cost. We observe that the expected execution cost associated with the optimal risk neutral execution strategy obtained under the multiplicative jump model can be significantly less than the expected execution cost of the naive strategy. Moreover, its expected execution cost can be notably smaller than that of the execution strategy optimal under the additive jump model with comparable expected market price change and volatility. This is particularly true as the asset return volatility or the trading horizon increases.

This chapter is organized as follows. In §5.2, we motivate and describe the proposed jump process to capture uncertain permanent price impact of other large institutions. In §5.3, we provide closed-form expressions for the optimal execution strategies under an additive jump diffusion model and a multiplicative jump diffusion model. The computational method to minimize the CVaR of the execution cost in described in §5.4. In §5.5, simulations are carried out to compare different execution strategies and model assumptions in terms of the expected execution cost and risk assessment. Concluding remarks are presented in §5.6.

Similar to (Bertsimas and Lo, 1998; Almgren and Chriss, 2000/2001; Huberman and Stanzl, 2004), our presentation mainly follows the discrete time framework since the analytic formula for optimal risk neutral execution strategy is presented under a discrete time model. We also analyze the execution risk in the discrete time setting. We note that the continuous time optimal execution problem for the single asset has also been widely studied, see, e.g., (Forsyth, 2010). The rational for the jump process can also be appreciated in contrast to a continuous time Brownian model.

5.2 Jump Processes for Uncertain Price Impact of Large Trades

For the optimal portfolio execution problem, the random variable $\mathcal{F}_{k-1}(P_{k-1})$ in the market price dynamics (2.1.3) is typically characterized by a normal random variable corresponding to an increment of a Brownian motion process. When the expected market price change is zero, the optimal risk neutral execution strategy is the naive strategy under many market price dynamics, see, e.g., (Bertsimas and Lo, 1998; Huberman and Stanzl, 2005). This observation may suggest that one needs not be concerned with the specification of the market price dynamics (or price impact functions). However, secondary to the expected execution cost, the risk of the execution cost is another main concern for investors. Accurate assessment of the execution risk associated with an execution strategy needs an accurate model for the market price. In addition, based on 15 minutes returns of 1000 largest U.S. assets in several international indices, Gabaix et al. (2006) show that trades of large institutions cause nonzero expected short term market price changes.

Furthermore, empirical evidence indicates that the distribution of the short term asset return typically has fat tails, see, e.g., (Campbell et al., 1996; Pagan, 1996; Cont, 2001). One likely reason for the fat tail distribution is the price impact of trades from institutions. Gabaix et al. (2006) show that trades of large institutions generate excess asset price volatility.

There is an additional contradiction in modeling market price dynamics as a Brownian motion; this contradiction can be better seen in the context of a continuous time framework. When the market price is modeled by a continuous process, permanent price impact of the decision maker's own trade causes a discrete change in the market price while the impact of large trades from other institutions maintains price continuity.

In this chapter we assume that the arrival time of large trades from other institutions as well as their impact are unknown to the decision maker. Following the approach proposed in (Garman, 1976), we model these uncertain arrivals using Poisson processes with constant arrival rates. The arrival of each trade induces an unknown permanent price impact and causes a jump in the market price. We use a random jump size to model the uncertain impact; the jump size is assumed to follow a known distribution. Combining this with uncertain arrivals, the uncertain price impact of uncertain trades from other institutions are modeled by *compound Poisson processes.* Including compound Poisson processes in the market price dynamics yields a price distribution with fatter tails than that of a normal distribution. The proposed model is likely to be a more accurate representation for the trading activities of institutional investors.

To further distinguish buys from sells, we assume that arrivals of buy and sell trades are independent Poisson processes with deterministic arrival rates. For simplicity, we first consider a single asset trading, and then generalize the model to trading of multiple assets. Let $\{X_t : t \in [0, T]\}$ be a Poisson process in the execution horizon $[0, T]$ with a constant arrival rate $\lambda_x \geq 0$. The process $\{X_t\}$ models uncertain arrivals of sell trades from other institutions. Similarly, a Poisson process $\{Y_t : t \in [0, T]\}$ with a constant arrival rate $\lambda_y \geq 0$ represents arrivals of buy trades. Processes $\{X_t\}$ and $\{Y_t\}$, respectively, count the number of sell and buy events during the time period $[0, t)$. Initially $X_0 = 0$ and $Y_0 = 0$. We assume that processes $\{X_t\}$ and $\{Y_t\}$ are independent of each other.

Using the Poisson processes $\{X_t\}$ and $\{Y_t\}$, we model uncertain permanent price impact of trades by other institutions in $[t_{k-1}, t_k)$ as below:

$$\mathcal{J}(k) \stackrel{\text{def}}{=} \sum_{\ell=1}^{Y_{t_k} - Y_{t_{k-1}}} \chi_\ell(k) - \sum_{\ell=1}^{X_{t_k} - X_{t_{k-1}}} \pi_\ell(k), \tag{5.2.1}$$

where $\chi_\ell(k)$ and $\pi_\ell(k)$ are random variables with known distributions. When the upper limit of a summation in (5.2.1) is zero, the summation itself is zero.

For every period k, the random variable $\pi_\ell(k)$ represents the permanent price impact of the ℓth sell trade in $[t_{k-1}, t_k)$. We assume that the random variables $\{\pi_\ell(k)\}$ are independently distributed with the mean $\mu_x(k)$ and standard deviation $\sigma_x(k)$. Similarly, the random variable $\chi_\ell(k)$ captures the permanent price impact of the ℓth buy trade in period k. The random variables $\{\chi_\ell(k)\}$ are assumed to be independently distributed with mean and standard deviation $\mu_y(k)$ and $\sigma_y(k)$, respectively.

Using two separate compound Poisson processes in equation (5.2.1) provides the flexibility to choose different arrival rates and distributional characteristics for permanent price impact of buys and sells from other institutions. Distinguishing permanent price impact of sell trades and buy trades by their arrival rates or distributions for the jump sizes is similar to the *double jump diffusion* process for modeling asset price dynamics, see, e.g., (Ramezani and Zeng, 2007) and references therein. Furthermore, empirical studies on institutional trades indicate that market reacts differently to buy and sell orders: buys have larger permanent price impact than sells, see, e.g., (Saar, 2001) and references therein. Employing two compound Poisson processes allows us to set $\mu_y(k) \geq \mu_x(k)$ to capture this market behavior.

The proposed jump diffusion model can be extended to a portfolio of m assets. For each asset $i = 1, 2, \ldots, m$, we similarly define two independent Poisson processes $\{X_t^{(i)}\}$ and $\{Y_t^{(i)}\}$

with constant arrival rates $\lambda_x^{(i)}$ and $\lambda_y^{(i)}$, respectively. In this case, $\mathcal{J}(k)$ is the m-vector

$$\mathcal{J}(k) \stackrel{\text{def}}{=} \begin{pmatrix} \displaystyle\sum_{\ell=1}^{Y_{t_k}^{(1)}-Y_{t_{k-1}}^{(1)}} \chi_\ell^{(1)}(k) - \sum_{\ell=1}^{X_{t_k}^{(1)}-X_{t_{k-1}}^{(1)}} \pi_\ell^{(1)}(k) \\ \vdots \\ \displaystyle\sum_{\ell=1}^{Y_{t_k}^{(m)}-Y_{t_{k-1}}^{(m)}} \chi_\ell^{(m)}(k) - \sum_{\ell=1}^{X_{t_k}^{(m)}-X_{t_{k-1}}^{(m)}} \pi_\ell^{(m)}(k) \end{pmatrix} \tag{5.2.2}$$

We further note that, if necessary, the compound Poisson processes of different assets can be allowed to include correlations to capture cross-asset relations observed.

For simplicity we assume subsequently that, for every period k, random jump sizes for sell trades at period k are independent of random jump sizes for buy trades at period k. In addition, we assume that the jump amplitudes are independent of the Poisson processes, and the compound Poisson processes are independent of the Brownian motion process used to model *normal* market price changes.

Below we incorporate jumps in two specifications for the market price dynamics, namely, additive model and multiplicative model. The additive diffusion process has been used frequently in the literature on the optimal portfolio execution problem, see, e.g., (Almgren and Chriss, 2000/2001); this is mainly due to the simplicity of the additive model which leads to determination of the optimal execution strategy in the early literature. In practice, a multiplicative model is more accurate in modeling the stock price and it has been more widely adopted in the finance literature for asset price modeling.

Additive Jump Diffusion Models. Here we assume that the change in the market price comes from a Brownian increment and a jump $\mathcal{J}^{\mathbf{a}}(k)$, which represents permanent price impact of other large trades:

$$\mathcal{F}_{k-1}(P_{k-1}) = P_{k-1} + \tau^{1/2}\Sigma^{\mathbf{a}} Z_k + \tau\alpha_0^{\mathbf{a}} + \mathcal{J}^{\mathbf{a}}(k). \tag{5.2.3}$$

The m-vector $\tau\alpha_0^{\mathbf{a}}$ can be interpreted as the expected price change due to small trades, which is likely to be negligible. The random vector Z_k is an l-vector of independent standard normals, and $\Sigma^{\mathbf{a}}$ is an $m \times l$ volatility matrix of the asset price changes. Based on high frequency financial price data, it has been noted in (McCulloch and Tsay, 2001) that significant percentages of trades lead to no price change. Similarly, we decompose the market price change into random shocks which lead to no expected price change, and jump events that cause a nonzero expected price change. Notice that we have used the superscript $\mathbf{a}$ to distinguish the model parameters in the *additive* model (5.2.3) from those of *multiplicative* model subsequently presented. Throughout this chapter bold superscripts of matrices and vectors should not be considered as exponents.

Together with the price impact of the decision maker's own trade, the market price dynamics is:

$$P_k = P_{k-1} + \tau^{1/2}\Sigma^{\mathbf{a}}\xi_k + \tau\alpha_0^{\mathbf{a}} + \mathcal{J}^{\mathbf{a}}(k) - \tau g\left(\frac{n_k}{\tau}\right), \qquad \text{where} \tag{5.2.4}$$

$$\mathcal{J}^{\mathbf{a}}(k) = \sum_{j=1}^{Y_{t_k}-Y_{t_{k-1}}} \chi_j^{\mathbf{a}}(k) - \sum_{j=1}^{X_{t_k}-X_{t_{k-1}}} \pi_j^{\mathbf{a}}(k), \quad \text{for } k = 1, 2, \ldots, N.$$

We use $\mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k)$ and $\mathbf{Cov}_{\mathcal{J}}^{\mathbf{a}}(k)$ to refer to $\mathbf{E}(\mathcal{J}^{\mathbf{a}}(k))$ and $\mathbf{Cov}(\mathcal{J}^{\mathbf{a}}(k))$, respectively. In the additive market price dynamics (5.2.4), the total market price change is decomposed into two components, one due to small trades, captured by $\tau\alpha_0^{\mathbf{a}} + \tau^{1/2}\Sigma^{\mathbf{a}}\xi_k$, and the other due to the permanent price impact of large trades, modeled by $\mathcal{J}^{\mathbf{a}}(k)$. Whence, the total expected market price change in each trading interval becomes $\tau\alpha_0^{\mathbf{a}} + \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k)$. Since ξ_k and $\mathcal{J}^{\mathbf{a}}(k)$ are assumed to be independent, the covariance of the total market price change in the kth period equals $\Sigma^{\mathbf{a}}(\Sigma^{\mathbf{a}})^T + \mathbf{Cov}_{\mathcal{J}}^{\mathbf{a}}(k)$.

Multiplicative Jump Diffusion Models. In practice, often one explicitly models return rather than price change; here we incorporate jump in such a model. Let the market return, aside from the permanent price impact of the decision maker's trades, be characterized by a normal distribution plus uncertain permanent price impact of other large trades. In the single asset trading context, this corresponds to

$$\frac{\mathcal{F}_{k-1}(P_{k-1}) - P_{k-1}}{P_{k-1}} = \tau\alpha_0^{\mathbf{m}} + \tau^{1/2}\Sigma^{\mathbf{m}}\xi_k + \mathcal{J}^{\mathbf{m}}(k),$$

or equivalently

$$\mathcal{F}_{k-1}(P_{k-1}) = P_{k-1}\left(1 + \tau\alpha_0^{\mathbf{m}} + \tau^{1/2}\Sigma^{\mathbf{m}}\xi_k + \mathcal{J}^{\mathbf{m}}(k)\right). \tag{5.2.5}$$

Similarly, the multiplicative jump diffusion model for m assets, together with the price impact of the decision maker's own trade, can be described as below:

$$P_k = \mathrm{Diag}(P_{k-1}) \cdot \left(e + \tau\alpha_0^{\mathbf{m}} + \tau^{1/2}\Sigma^{\mathbf{m}}\xi_k + \mathcal{J}^{\mathbf{m}}(k)\right) - \tau g\left(\frac{n_k}{\tau}\right), \quad \text{where} \tag{5.2.6}$$

$$\mathcal{J}^{\mathbf{m}}(k) \stackrel{\text{def}}{=} \sum_{j=1}^{Y_{t_k}-Y_{t_{k-1}}} \left(\chi_j^{\mathbf{m}}(k) - e\right) - \sum_{j=1}^{X_{t_k}-X_{t_{k-1}}} \left(\pi_j^{\mathbf{m}}(k) - e\right). \tag{5.2.7}$$

Here, e is the m-vector of all ones and $\mathrm{Diag}(P_{k-1})$ is a diagonal matrix with the m-vector P_{k-1} as its diagonal. The components of the l-vector ξ_k are independent standard normals and $\Sigma^{\mathbf{m}}$ is an $m \times l$ volatility matrix of the asset returns. The term $\tau\alpha_0^{\mathbf{m}}$ can be interpreted as the expected return due to small trades. Here, the superscript $\mathbf{m}$ emphasizes parameters in the multiplicative jump model. Jump amplitudes $\pi_j^{\mathbf{m}}(k)$ and $\chi_j^{\mathbf{m}}(k)$ represent uncertain permanent price impacts, and are assumed to be drawn from known distributions. We denote the expected value and covariance matrix of $\mathcal{J}^{\mathbf{m}}(k)$ with $\mathbf{E}_{\mathcal{J}}^{\mathbf{m}}(k)$ and $\mathbf{Cov}_{\mathcal{J}}^{\mathbf{m}}(k)$, respectively.

5.3 Dynamic Optimal Mean Execution Cost Strategy

The optimal portfolio execution problem in the generic form is described in (2.1.6). Since the main objective of the decision maker is to minimize the expected execution cost, we first consider here the optimal risk neutral execution strategy under jump models when purchasing is allowed, i.e.,

$$\min_{n_1,\cdots,n_N} \mathbf{E}\left(P_0^T\bar{S}-\sum_{k=1}^{N}n_k^T\tilde{P}_k\right) \quad \text{s.t.} \quad \sum_{k=1}^{N}n_k=\bar{S}. \tag{5.3.1}$$

We will also analyze properties of the optimal risk neutral execution strategy in terms of both the expected execution cost and execution risk.

Stochastic dynamic programming has been used to determine the optimal execution strategy when the market price evolves according to a Brownian motion, see, e.g., (Bertsimas and Lo, 1998; Bertsimas et al., 1999; Huberman and Stanzl, 2005). The key ingredients of the stochastic dynamic programming for problem (5.3.1) are described below.

Let the optimal-value function at t_{k-1} corresponding to problem (5.3.1) be

$$V_k^*(P_{k-1},x_{k-1})=\min_{n_k,\cdots,n_N}\mathbf{E}\left(P_0^T\bar{S}-\sum_{j=k}^{N}n_j^T\tilde{P}_j\;\middle|\;P_{k-1},\ x_{k-1}\right), \quad \text{s.t.} \quad \sum_{j=k}^{N}n_j=x_{k-1}.$$

Here, $n_k,\cdots,n_N$ are over the set of $\mathbb{R}^m$-valued functions of the system state, namely current asset holdings x_{k-1} and current market price P_{k-1}.

For $k=N$, $n_N^*=x_{N-1}$ since there is no choice but to execute the entire remaining order x_{N-1}. Whence, for the model (2.1.2), the optimal-value function for the last period becomes

$$\begin{aligned} V_N^*(P_{N-1},x_{N-1}) &= \min_{n_N,\ x_N=0}\mathbf{E}\left(P_0^T\bar{S}-n_N^T\tilde{P}_N \mid P_{N-1},\ x_{N-1}\right) \\ &= P_0^T\bar{S}-x_{N-1}^T\left(P_{N-1}-h\left(\frac{x_{N-1}}{\tau}\right)\right). \end{aligned} \tag{5.3.2}$$

For the linear temporary price impact function (2.1.4), we have

$$V_N^*(P_{N-1},x_{N-1})=P_0^T\bar{S}-x_{N-1}^TP_{N-1}+\frac{1}{2}x_{N-1}^T\frac{H+H^T}{\tau}x_{N-1}. \tag{5.3.3}$$

Now assume that n_{k+1}^* and $V_{k+1}^*(P_k,x_k)$ have been determined. The optimal execution n_k^* and the optimal-value function $V_k^*(P_{k-1},x_{k-1})$ can be determined from the Bellman's principle of optimality which relates, recursively backwards in time, the optimal-value function in period k to the optimal-value function in period $k+1$:

$$V_k^*(P_{k-1},x_{k-1})=\min_{n_k}\ \mathbf{E}\left(-n_k^T\tilde{P}_k+V_{k+1}^*(P_k,x_k)\mid P_{k-1},\ x_{k-1}\right).$$

Next we present the optimal risk neutral execution strategies under three different model assumptions: when the expected market price change is zero, additive jump diffusion models, as well as multiplicative jump diffusion models.

5.3.1 Effect of a Zero Expected Market Price Change

An optimal execution strategy in general depends on the market price dynamics, i.e., $\mathcal{F}_k(\cdot)$ in (2.1.3). For a single asset execution under linear price impact functions and an additive diffusion model with zero expected market price change, the optimal risk neutral execution strategy is the naive strategy. The assumption that the expected market price change is zero may be reasonable in the absence of large institutional trades.

We now generalize this result to more general model assumptions for the portfolio case in Theorem 5.3.1.

Theorem 5.3.1. *Let the market price dynamics and the execution price model be given by equations* (2.1.3) *and* (2.1.2)*, respectively. In addition, assume that*

$$\mathbf{E}\left(\mathcal{F}_{k-1}\left(P_{k-1}\right) \mid P_{k-1}\right) = P_{k-1}, \;\; k = 1, 2, \cdots, N-1. \tag{5.3.4}$$

Assume further that the price impact functions $g(\cdot)$ and $h(\cdot)$ are deterministic functions of the trading rate $\frac{n_k}{\tau}$ and do not depend on the market price. Then the unique optimal risk neutral execution strategy for the optimal portfolio execution problem (5.3.1)*, when it exists, is static (state independent). Furthermore, for the linear price impact functions* (2.1.4) *with constant impact matrices, symmetric permanent impact matrix G and positive definite combined impact matrix Θ, the optimal risk neutral execution strategy $\{n_k^*\}_{k=1}^N$ is the naive strategy.*

This result highlights the important role of the expected market price change in the optimal execution strategy. Note that the results hold without specific assumption on the market price dynamics $\mathcal{F}_k(\cdot)$. The proof of Theorem 5.3.1 is provided in Appendix B.1.

In general, the expected market price change in each period is nonzero, likely due to institutional trades. We will show that, in this case, the model assumptions and the expected market price change can significantly affect the optimal execution strategy.

In §5.3.2 and §5.3.3, we focus on two specifications of the market price model (2.1.3) that include the jump process $\mathcal{J}(k)$.

5.3.2 Additive Jump Diffusion Market Price Models

We now present a closed-form expression for the optimal risk neutral execution strategy with respect to the additive jump diffusion model (5.2.4).

Theorem 5.3.2. *Assume that the $m \times m$ symmetric matrices $\{A_k\}_{k=1}^N$, specified by the following recursive equation:*

$$A_k = A_{k+1} - (A_{k+1} - \Theta^T)A_{k+1}^{-1}(A_{k+1} - \Theta^T)^T, \quad k = 1, 2, \ldots, N-1, \tag{5.3.5}$$

with $A_N = \Theta^T + \Theta$, *are positive definite. Moreover, let* m*-vectors* $\{b_k\}_{k=1}^N$ *and scalars* $\{c_k\}_{k=1}^N$ *be defined as follows:*

$$\begin{aligned} b_k = \;& b_{k+1} + (\Theta^T - A_{k+1})A_{k+1}^{-1}\left(b_{k+1} - \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k+1) + \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k)\right) - \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k+1) \\ & + 2\mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k) + \tau\alpha_0^{\mathbf{a}}, \\ c_k = \;& c_{k+1} + \frac{1}{2}\left(b_{k+1} - \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k+1) + \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k)\right)^T A_{k+1}^{-1}\left(b_{k+1} - \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k+1) + \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k)\right), \end{aligned} \tag{5.3.6}$$

with $b_N = \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(N) + \tau\alpha_0^{\mathbf{a}}$ *and* $c_N = 0$. *Then the unique optimal risk neutral execution strategy* $n^* = \{n_k^*\}_{k=1}^N$ *of problem* (5.3.1) *under the additive jump model* (5.2.4) *is:*

$$\begin{aligned} n_k^* &= -A_{k+1}^{-1}\left(b_{k+1} - \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k+1) + \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k) + (\Theta^T - A_{k+1})^T x_{k-1}^*\right), k = 1, \ldots, N-1, \\ n_N^* &= \bar{S} - \sum_{k=1}^{N-1} n_k^*, \end{aligned} \tag{5.3.7}$$

where $x_0^* = \bar{S}$ *and* $x_k^* = x_{k-1}^* - n_k^*$ *for* $k = 1, 2, \ldots, N-2$. *Furthermore, the optimal expected execution cost equals:*

$$V_1^*(P_0, x_0) = P_0^T \bar{S} - \frac{1}{2}\bar{S}^T(\Theta^T - A_1 - G)\bar{S} - \left(P_0 + b_1 - \mathbf{E}_{\mathcal{J}}^{a}(1) - \tau\alpha_0^{\mathbf{a}}\right)^T \bar{S} - c_1.$$

A proof for Theorem 5.3.2 is given in Appendix B.2.

Theorem 5.3.2 indicates that the optimal risk neutral execution strategy under the additive model (5.2.4) does not depend on the market price realization. In addition, volatility $\Sigma^{\mathbf{a}}$ and covariance $\mathbf{Cov}_{\mathcal{J}}^{\mathbf{a}}(k)$ play no role in determining the optimal risk neutral execution strategy (5.3.7). However, the expected permanent price impact of other large trades, $\mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k)$, affects the optimal execution strategy. This can be seen more clearly from Proposition 5.3.1 below under an additional symmetry assumption.

Proposition 5.3.1. *Let the permanent impact matrix* G *be symmetric and the combined impact matrix* Θ *be positive definite. Moreover, assume for every* $k = 1, 2, \ldots, N$, $\mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k) = \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}$ *for some constant* $\mathbf{E}_{\mathcal{J}}^{\mathbf{a}}$. *Then the unique optimal risk neutral execution strategy is*

$$n_k^* = \frac{\bar{S}}{N} - \frac{(N+1-2k)}{2}\Theta^{-1}\left(\mathbf{E}_{\mathcal{J}}^{\mathbf{a}} + \tau\alpha_0^{\mathbf{a}}\right), \quad k = 1, 2, \ldots, N. \tag{5.3.8}$$

Note that the symmetry assumption holds, for example, when permanent impact matrix is a diagonal matrix; this is also assumed in the literature, see, e.g., (Almgren and Chriss, 2000/2001). We provide a proof for Proposition 5.3.1 in Appendix B.2.

In contrast to the naive strategy, the optimal execution strategy (5.3.8) now depends on the impact matrices and varies over time as a linear function of $\Theta^{-1}(\tau\alpha_0^{\mathbf{a}} + \mathbf{E}_{\mathcal{J}}^{\mathbf{a}})$. While the naive strategy never buys for a sell execution, the optimal risk neutral execution strategy (5.3.8) may suggest buying in some periods during liquidation. Note that the solution (5.3.8) reduces to the naive strategy when the total expected market price change $\tau\alpha_0^{\mathbf{a}} + \mathbf{E}_{\mathcal{J}}^{\mathbf{a}} = 0$.

This result is consistent with Theorem 5.3.1. When $\Theta^{-1}(\tau\alpha_0^{\mathbf{a}} + \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}) < 0$, the optimal risk neutral execution strategy is a strictly decreasing linear function of k. Specifically the decision maker trades more than $\frac{\bar{S}}{N}$ shares in the periods $1, 2, \ldots, \lceil \frac{N-1}{2} \rceil$, while, in the periods $\lfloor \frac{N+3}{2} \rfloor, \ldots, N$, he trades less than $\frac{\bar{S}}{N}$ shares per period. Similarly, when $\Theta^{-1}(\tau\alpha_0^{\mathbf{a}} + \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}) > 0$, the optimal risk neutral execution strategy is a strictly increasing function of the time period k. Figure 5.1 illustrates the optimal risk neutral execution strategy n^* versus the naive strategy $\bar{n}$ in liquidating a single asset using the parameters presented in Table 5.1.

We further examine what parameters $E_{\mathcal{J}}^{\mathbf{a}}$ depends on. Let the jump sizes $\pi_j^{\mathbf{a}}(k)$ and $\chi_j^{\mathbf{a}}(k)$ be normally distributed with means $\mu_x^{\mathbf{a}}(k)$ and $\mu_y^{\mathbf{a}}(k)$, and standard deviations $\sigma_x^{\mathbf{a}}(k)$ and $\sigma_y^{\mathbf{a}}(k)$, respectively. Hence, for the single asset execution, we have, see, e.g., Theorem 9.1 in (Karlin and Taylor, 1981):

$$\begin{aligned} \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k) &= \tau\lambda_y \mathbf{E}\left(\chi_j^{\mathbf{a}}(k)\right) - \tau\lambda_x \mathbf{E}\left(\pi_j^{\mathbf{a}}(k)\right) = \tau\left(\lambda_y \mu_y^{\mathbf{a}}(k) - \lambda_x \mu_x^{\mathbf{a}}(k)\right), \quad (5.3.9) \\ \mathbf{Cov}_{\mathcal{J}}^{\mathbf{a}}(k) &= \tau\lambda_x \left(\mathbf{Var}(\pi_j^{\mathbf{a}}) + \left(\mathbf{E}(\pi_j^{\mathbf{a}})\right)^2\right) + \tau\lambda_y \left(\mathbf{Var}(\chi_j^{\mathbf{a}}) + \left(\mathbf{E}(\chi_j^{\mathbf{a}})\right)^2\right) \\ &= \tau\lambda_x \left((\sigma_x^{\mathbf{a}}(k))^2 + (\mu_x^{\mathbf{a}}(k))^2\right) + \tau\lambda_y \left(\left(\sigma_y^{\mathbf{a}}(k)\right)^2 + \left(\mu_y^{\mathbf{a}}(k)\right)^2\right). \quad (5.3.10) \end{aligned}$$

Under the assumptions in Proposition 5.3.1, we observe that buy and sell arrival rates and the expected permanent price impacts directly affect the expected market price change and consequently the optimal risk neutral execution strategy. When $\lambda_x = \lambda_y$ and $\mu_x^{\mathbf{a}}(k) = \mu_y^{\mathbf{a}}(k)$, $\mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k) = 0$ while $\mathbf{Cov}_{\mathcal{J}}^{\mathbf{a}}(k)$ is strictly positive when either $\sigma_x^{\mathbf{a}}(k)\lambda_x$ is positive or $\sigma_y^{\mathbf{a}}(k)\lambda_y$ is positive. In this case, trades increase the volatility without causing a direction in the market price change.

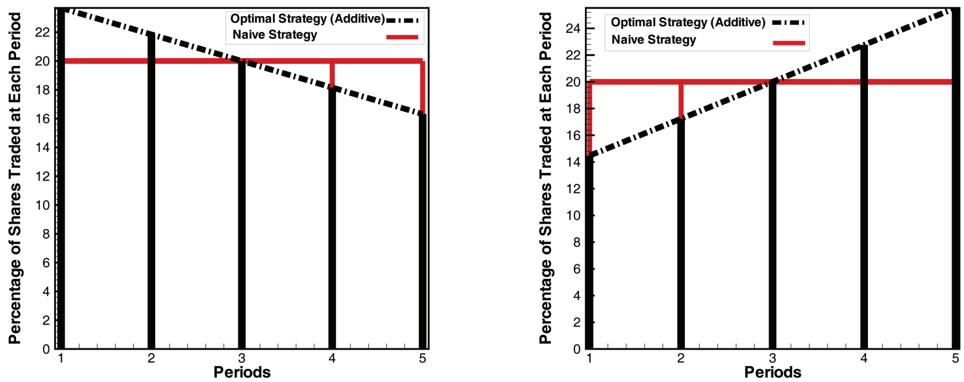
5.3.3 Multiplicative Jump Diffusion Market Price Models

The simplicity of the additive jump diffusion model (5.2.3) leads to a static optimal risk neutral execution strategy. However, from a practical perspective, the additive model (5.2.3) has limitations. For example, its optimal strategy is static and therefore cannot adapt to the price information revealed during the course of trading.

Theorem 5.3.3 presents the optimal risk neutral execution strategy from problem (5.3.1) when market price dynamics and execution price model are given by the multiplicative jump diffusion model (5.2.6) and (2.1.2), respectively. Subsequently we denote the $m \times m$ identity matrix with I_m. Moreover, we use $A \circ B$ to denote the componentwise (Hadamard) product of the matrices A and B.

Theorem 5.3.3. *Assume that the sequence of deterministic symmetric matrices $\{D_k\}_{k=1}^N$, defined by*

$$D_k = -2G^T A_k G + \frac{H + H^T}{\tau} - (G^T B_k + B_k^T G) - 2C_k, \quad (5.3.11)$$



(a) $\mathbf{E}^{\mathbf{a}}_{\mathcal{J}} = -8.750 \times 10^{-2}$ (b) $\mathbf{E}^{\mathbf{a}}_{\mathcal{J}} = 1.3125 \times 10^{-1}$

Figure 5.1: The optimal execution strategy n^* in equation (5.3.8), versus the naive strategy $\bar{n}$ for selling $\bar{S} = 10^6$ shares within $T = 5$ days. Here $N = 5$, and $\alpha_0^{\mathbf{a}} = 0$, $\mu_x^{\mathbf{a}}(k) = 0.2125$, $\mu_y^{\mathbf{a}}(k) = 0.2250$ for $k = 1, 2, \ldots, N$. For the left plot, permanent price impact of other large trades causes a negative expected market price change (arrival rates $(\lambda_x, \lambda_y) = (2, 1.5)$). For the right plot permanent price impact of other large trades causes a positive expected market price change (arrival rates are $(\lambda_x, \lambda_y) = (1.5, 2)$).

are positive definite, where the deterministic matrix B_k and the symmetric matrices A_k and C_k are derived from

$$\begin{aligned}
A_{k-1} &= A_k \circ Q_{k-1} + L_{k-1} A_k L_{k-1} + \frac{1}{2} \left(I_m - L_{k-1}(2A_k G + B_k)\right) D_k^{-1} \left(I_m - L_{k-1}(2A_k G + B_k)\right)^T, \\
B_{k-1} &= L_{k-1} B_k - \left(I_m - L_{k-1}(B_k + 2A_k G)\right) D_k^{-1} \left(2C_k + G^T B_k\right), \qquad (5.3.12) \\
C_{k-1} &= C_k + \frac{1}{2} \left(2C_k + B_k^T G\right) D_k^{-1} \left(2C_k + G^T B_k\right).
\end{aligned}$$

Here $L_{k-1} = \mathrm{Diag}\left(e + \tau \alpha_0^{\boldsymbol{m}} + \mathbf{E}^{\boldsymbol{m}}_{\mathcal{J}}(k-1)\right)$, $Q_{k-1} = \tau \Sigma^{\boldsymbol{m}} (\Sigma^{\boldsymbol{m}})^T + \mathbf{Cov}^{\boldsymbol{m}}_{\mathcal{J}}(k-1)$, and $A_N = 0$, $B_N = I_m$, and $C_N = -\frac{H + H^T}{2\tau}$. Then the unique optimal risk neutral execution strategy $n^* = \{n_k^*\}_{k=1}^N$ is given by

$$\begin{aligned}
n_k^* &= D_{k+1}^{-1} \left(I_m - \left(B_{k+1}^T + 2G^T A_{k+1}\right) L_k\right) P_{k-1} - D_{k+1}^{-1} \left(2C_{k+1} + G^T B_{k+1}\right) x_{k-1}, \qquad (5.3.13) \\
k &= 1, \ldots, N-1, \\
n_N^* &= \bar{S} - \sum_{k=1}^{N-1} n_k^*.
\end{aligned}$$

Furthermore, the optimal expected execution cost becomes

$$V_1^*(P_0, x_0) = P_0^T \bar{S} - P_0^T A_1 P_0 - P_0^T B_1 x_0 - x_0^T C_1 x_0. \qquad (5.3.14)$$

The proof of Theorem 5.3.3 is given in Appendix B.3.

The optimal risk neutral execution strategy (5.3.13), derived under the multiplicative jump diffusion model (5.2.6), is significantly different from the optimal execution strategy (5.3.7) under the additive jump diffusion model (5.2.4). Firstly, the optimal risk neutral execution strategy (5.3.13) does depend on the covariance matrices $\Sigma^{\mathbf{m}}(\Sigma^{\mathbf{m}})^T$ and $\mathbf{Cov}^{\mathbf{m}}_{\mathcal{J}}$. In addition, strategy (5.3.13), obtained under the multiplicative model (5.2.6), is stochastically dynamic and depends on the future market price realization P_{k-1}, when $\tau\alpha_0^{\mathbf{m}} + \mathbf{E}^{\mathbf{m}}_{\mathcal{J}}(k) \neq 0$. When $\tau\alpha_0^{\mathbf{m}} + \mathbf{E}^{\mathbf{m}}_{\mathcal{J}}(k)$ is zero for every period k, Theorem 5.3.1, applied to the price dynamics (5.2.6), implies that the solution (5.3.13) becomes static.

Assume that the jump amplitude is log-normally distributed, i.e., $\log \pi_j^{\mathbf{m}}(k)$ and $\log \chi_j^{\mathbf{m}}(k)$ have normal distributions with means $\mu_x^{\mathbf{m}}(k)$ and $\mu_y^{\mathbf{m}}(k)$, and standard deviations $\sigma_x^{\mathbf{m}}(k)$ and $\sigma_y^{\mathbf{m}}(k)$, respectively. For a single asset trading ($m = 1$), we have, see, e.g., (Karlin and Taylor, 1975) page 268:

$$
\begin{aligned}
\mathbf{E}\left(\pi_j^{\mathbf{m}}(k)\right) &= \exp\left(\mu_x^{\mathbf{m}}(k) + \frac{1}{2}\left(\sigma_x^{\mathbf{m}}(k)\right)^2\right),\\
\mathbf{Var}\left(\pi_j^{\mathbf{m}}(k)\right) &= \left(\exp\left((\sigma_x^{\mathbf{m}}(k))^2\right) - 1\right)\exp\left(2\mu_x^{\mathbf{m}}(k) + (\sigma_x^{\mathbf{m}}(k))^2\right),\\
\mathbf{E}\left(\chi_j^{\mathbf{m}}(k)\right) &= \exp\left(\mu_y^{\mathbf{m}}(k) + \frac{1}{2}\left(\sigma_y^{\mathbf{m}}(k)\right)^2\right),\\
\mathbf{Var}\left(\chi_j^{\mathbf{m}}(k)\right) &= \left(\exp\left((\sigma_y^{\mathbf{m}}(k))^2\right) - 1\right)\exp\left(2\mu_y^{\mathbf{m}}(k) + \left(\sigma_y^{\mathbf{m}}(k)\right)^2\right).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathbf{E}^{\mathbf{m}}_{\mathcal{J}}(k) &= \tau\lambda_y\left(\exp\left(\mu_y^{\mathbf{m}}(k) + \frac{(\sigma_y^{\mathbf{m}}(k))^2}{2}\right) - 1\right) - \tau\lambda_x\left(\exp\left(\mu_x^{\mathbf{m}}(k) + \frac{(\sigma_x^{\mathbf{m}}(k))^2}{2}\right) - 1\right),\\
\mathbf{V}^{\mathbf{m}}_{\mathcal{J}}(k) &= \tau\lambda_x\left(\mathbf{Var}\left(\pi_j^{\mathbf{m}}(k)\right) + \left(\mathbf{E}\left(\pi_j^{\mathbf{m}}(k)\right) - 1\right)^2\right) + \tau\lambda_y\left(\mathbf{Var}\left(\chi_j^{\mathbf{m}}(k)\right) + \left(\mathbf{E}\left(\chi_j^{\mathbf{m}}(k)\right) - 1\right)^2\right).
\end{aligned}
$$

In contrast to $\mathbf{E}^{\mathbf{a}}_{\mathcal{J}}$, now the volatility of the permanent price impact affects the expected market price change. Notice that other distributions can also be considered for the jump amplitudes. For example, Pareto and Beta distributions have been considered for jump amplitudes in a double exponential jump diffusion process to model asset price evolution in the literature, see, e.g., (Kou, 2002; Ramezani and Zeng, 2007).

5.4 Assessing and Controlling the Execution Risk

The optimal risk neutral execution strategy under the multiplicative jump diffusion model (5.2.6), given the values of the two state variables P_k and x_k, depends only on the expected market return $\mathbf{E}^{\mathbf{m}}_{\mathcal{J}}$ and the covariance $\mathbf{Cov}^{\mathbf{m}}_{\mathcal{J}}$ (see Theorem 5.3.3). Thus, the optimal risk neutral execution strategy is identical to the optimal strategy obtained under the following adjusted model *without jump* for the market price

$$P_k = P_{k-1} + \mathrm{Diag}(P_{k-1})\left(\tau\left(\alpha_0^{\mathbf{m}} + \tau^{-1}\mathbf{E}_{\mathcal{J}}^{\mathbf{m}}\right) + \tau^{1/2}\left(\Sigma^{\mathbf{m}} + \tau^{-1/2}\mathbf{Cov}_{\mathcal{J}}^{\mathbf{m}1/2}\right)\xi_k\right) - Gn_k. \quad (5.4.1)$$

While the market price in model (5.4.1) is normally distributed and has no jump, the market price P_k of models (5.2.6) and (5.4.1) share the same first and second moments. Hence, for the purpose of determining the optimal risk neutral execution strategy, one does not need to differentiate between model (5.4.1) without jump and model (5.2.6) with jumps.

However, in addition to the expected execution cost, one needs to be concerned about the execution risk which can be assessed from the execution cost distribution. For risk management purposes, it is important to quantitatively measure and manage the execution risk. The multiplicative jump model (5.2.6) and the adjusted normal model (5.4.1) clearly leads to distinctively different execution cost distributions. We illustrate the difference computationally in §5.5.

If one also wants to control execution risk when choosing an execution strategy, then the stochastic programming problem (2.1.6) needs to be solved with an appropriate risk measure $\Psi(\cdot)$ for the execution cost. Under the jump model, the distribution of the execution cost is asymmetric and the variance is not appropriate since it treats the cost and profit equally. Alternative to variance, *Value-at-risk* (VaR) is a standard benchmark for a firm-wide measure of risk. For a given time horizon $\bar{t}$ and confidence level β, the value-at-risk of a portfolio is the loss in the portfolio's market value over the time horizon $\bar{t}$ that is exceeded with probability $1-\beta$. However, as a risk measure, VaR has recognized limitations. For example it lacks subadditivity and convexity, see, e.g., (Artzner et al., 1997, 1999). The CVaR risk measure, also known as the *mean excess loss*, *mean shortfall* or *tail VaR*, is an attractive alternative to VaR. For a given time horizon $\bar{t}$ and confidence level β, CVaR is the conditional expectation of the loss above VaR for the time horizon $\bar{t}$ and the confidence level β. It has been shown that CVaR is a coherent risk measure and has many attractive properties including convexity, see, e.g., (Artzner et al., 1999). In addition, minimizing CVaR typically leads to a portfolio with a small VaR. Using CVaR for the optimal portfolio execution problem seems appropriate as short term asset returns have fat tails and trading impact leads to price jumps.

Denote the execution cost by the random variable $L \stackrel{\text{def}}{=} \left(P_0^T\bar{S} - \sum_{k=1}^N n_k^T\tilde{P}_k\right)$. For a given confidence level $\beta \in (0,1)$, CVaR_β is given below

$$\mathrm{CVaR}_\beta(L) = \min_{\alpha\in\mathbb{R}}\left(\alpha + (1-\beta)^{-1}\mathbf{E}\left([L-\alpha]^+\right)\right), \quad (5.4.2)$$

where $[z]^+ = \max(z,0)$, see, e.g., (Rockafellar and Uryasev, 2000). With the CVaR risk measure, the optimal portfolio execution problem (2.1.6) becomes

$$\begin{aligned}\min_{\substack{n_1,\cdots,n_N\in\mathbb{R}^m\\ \alpha\in\mathbb{R}}} \quad & \mathbf{E}\left(P_0^T\bar{S} - \sum_{k=1}^N n_k^T\tilde{P}_k\right) + \mu\cdot\left(\alpha + \frac{1}{1-\beta}\mathbf{E}\left([P_0^T\bar{S} - \sum_{k=1}^N n_k^T\tilde{P}_k - \alpha]^+\right)\right) \quad (5.4.3)\\ \text{s.t.} \quad & \sum_{k=1}^N n_k = \bar{S}.\end{aligned}$$

This is a multi-stage stochastic nonlinear programming problem. In particular, the execution cost depends nonlinearly on n_k due to the permanent price impact. Solving this problem is computationally challenging; we devise a computational method to approximate the solution in Chapter 6.

In the rest of this section, similar to (Almgren and Chriss, 2000/2001), we assume that the strategy $\{n_k\}_{k=1}^N$ is deterministic. We use the following computational method to obtain an optimal static execution strategy under the CVaR risk measure. Since there is no analytic expression for the CVaR evaluation, Monte Carlo simulation is required to discretize a CVaR minimization problem. Unfortunately, under a discretization with M simulations, the objective function in (5.4.3) includes the sum of M piecewise nonlinear functions:

$$\begin{aligned} \min_{\substack{n_1,\cdots,n_N \in \mathbb{R}^m \\ \alpha\in\mathbb{R}}} \quad & \frac{1}{M}\sum_{j=1}^{M}\left(P_0^T\bar{S} - \sum_{k=1}^{N} n_k^T\tilde{P}_k^{(j)}\right) + \mu\cdot\left(\alpha + \frac{1}{M(1-\beta)}\sum_{j=1}^{M}\left[P_0^T\bar{S} - \sum_{k=1}^{N} n_k^T\tilde{P}_k^{(j)} - \alpha\right]^+\right) \\ \text{s.t.} \quad & \sum_{k=1}^{N} n_k = \bar{S}, \end{aligned}$$

where the superscript (j) denotes the jth simulation.

The CVaR risk measure is typically continuously differentiable (Rockafellar and Uryasev, 2000). Since nondifferentiability here arises from simulation discretization, we apply a smoothing technique in (Alexander et al., 2006) for the single period CVaR optimization problem. The convergence property of this smoothing method is established in (Xu and Zhang, 2009). We approximate the nonsmooth piecewise linear function $[z]^+$ by a continuously differentiable piecewise quadratic function $\rho_\epsilon(z)$ for some small resolution parameter ϵ:

$$\rho_\epsilon(z) = \begin{cases} z & \text{if } \ z > \epsilon \\ \frac{z^2}{4\epsilon} + \frac{1}{2}z + \frac{1}{4}\epsilon & \text{if } \ -\epsilon \le z \le \epsilon \\ 0 & \text{if } z < -\epsilon \end{cases} \tag{5.4.4}$$

In particular, the execution strategy which minimizes the CVaR_β of the execution cost can be computed from the following minimization problem:

$$\min_{\alpha\in\mathbb{R}, n_1,\cdots,n_N\in\mathbb{R}^m} \alpha + \frac{1}{(1-\beta)M}\sum_{j}^{M}\rho_\epsilon\left(P_0^T\bar{S} - \sum_{k=1}^{N} n_k^T\tilde{P}_k^{(j)} - \alpha\right) \quad \text{s.t.} \quad \sum_{k=1}^{N} n_k = \bar{S}. \tag{5.4.5}$$

5.5 Performance Comparison

We now present our computational investigation of the potential effect of the model assumption on the optimal risk neutral execution strategy. We evaluate trading performance in terms of the expected execution cost, execution risk, and more generally execution cost distribution.

Because of a more accurate characterization for the short term asset return, the multiplicative jump diffusion model (5.2.6) with known model parameters is assumed for the future market price. Since trading impact of large institutions is likely to cause a nonzero change in the expected market price and return, we assume that the expected change in the market price is nonzero. Based on the assumed model, we then compare the following three strategies:

- Strategy$_M$: optimal risk neutral execution strategy under the assumed multiplicative jump model (5.2.6).

- Strategy$_A$: optimal risk neutral execution strategy under the additive jump diffusion model (5.2.4) with comparable means and covariances set as below

  $$\mathbf{E}^{\mathbf{a}}_{\mathcal{J}} = P_0 \mathbf{E}^{\mathbf{m}}_{\mathcal{J}}, \qquad \tau \Sigma^{\mathbf{a}} (\Sigma^{\mathbf{a}})^T + \mathbf{Cov}^{\mathbf{a}}_{\mathcal{J}} = P_0^2 \left(\tau \Sigma^{\mathbf{m}} (\Sigma^{\mathbf{m}})^T + \mathbf{Cov}^{\mathbf{m}}_{\mathcal{J}} \right).$$

  We denote the total volatility $(\tau \Sigma^{\mathbf{m}} (\Sigma^{\mathbf{m}})^T + \mathbf{Cov}^{\mathbf{m}}_{\mathcal{J}})^{1/2}$ by σ_{tot}. Note that Strategy$_A$ does not depend on the covariance $\mathbf{Cov}^{\mathbf{a}}_{\mathcal{J}}$ and volatility $\Sigma^{\mathbf{a}}$.

- Strategy$_N$: the naive strategy which is optimal when the expected total market price change is zero, the permanent impact matrix G is symmetric, and the combined impact matrix Θ is positive definite. The naive strategy is used as the performance benchmark; the comparison illustrates the importance of accurate modeling of the market price dynamics in determining an optimal execution strategy.

We conduct computational investigations for a single asset trading example. The expected market price change due to small trades is assumed to be zero, i.e., $\alpha_0^{\mathbf{a}} = 0$ (\$/share)/day and $\alpha_0^{\mathbf{m}} = 0$ (1/day). We also assume that variance $\tau \Sigma^{\mathbf{m}} (\Sigma^{\mathbf{m}})^T$ (due to *normal* trading) constitutes 10% of the total variance σ_{tot}^2. Specifically, we consider selling $\bar{S}$ shares over T days. Unless otherwise stated, the parameter values in Table 5.1 are used.

Parameters	Values
Number of Periods	$N = T$
Interval Length	$\tau = T/N = 1$ day
Temporary Impact Matrix	$H = 2.5 \times (10^{-6})$ (\$ $\cdot$ day)/share^2
Permanent Impact Matrix	$G = 2.5 \times (10^{-7})$ \$/share^2
Initial Asset Price	$P_0 = 50$ \$/share

Table 5.1: Parameter values for the single asset execution example.

In addition parameters λ_x and λ_y are trading arrival rates per day. We assume that the jump amplitudes $\pi_j^{\mathbf{m}}$ and $\chi_j^{\mathbf{m}}$ are log-normally distributed, and $\mathbf{E}^{\mathbf{a}}_{\mathcal{J}}(k) = \mathbf{E}^{\mathbf{a}}_{\mathcal{J}}$ and $\mathbf{E}^{\mathbf{m}}_{\mathcal{J}}(k) = \mathbf{E}^{\mathbf{m}}_{\mathcal{J}}$ for some constants $\mathbf{E}^{\mathbf{a}}_{\mathcal{J}}$ and $\mathbf{E}^{\mathbf{m}}_{\mathcal{J}}$, $\mathbf{Cov}^{\mathbf{m}}_{\mathcal{J}}(k) = \mathbf{Cov}^{\mathbf{m}}_{\mathcal{J}}$ and $\mathbf{Cov}^{\mathbf{a}}_{\mathcal{J}}(k) = \mathbf{Cov}^{\mathbf{a}}_{\mathcal{J}}$ for some constants $\mathbf{Cov}^{\mathbf{m}}_{\mathcal{J}}$ and $\mathbf{Cov}^{\mathbf{a}}_{\mathcal{J}}$.

Furthermore, the market price dynamics is determined by the following parameters $\Sigma^{\mathbf{m}}$, $\mu_x^{\mathbf{m}}$, $\mu_y^{\mathbf{m}}$, $\sigma_x^{\mathbf{m}}$, $\sigma_y^{\mathbf{m}}$, λ_x, and λ_y. In subsequent computational results, we have simply assigned reasonable parameter values for illustrative purposes; we also choose these parameter values so that the magnitudes of trading impact represented by $\mathbf{E}\left(\pi_j^{\mathbf{m}}(k)\right) - 1$ and $\mathbf{E}\left(\chi_j^{\mathbf{m}}(k)\right) - 1$ are reasonable. In addition, since in general the permanent price impact of buying is larger than selling, we choose larger values for means of jump amplitude for buys than for sells, i.e., $\mu_y^{\mathbf{m}} \geq \mu_x^{\mathbf{m}}$.

5.5.1 Comparison of the Execution Risk

We assess the difference in execution risk under the multiplicative jump diffusion model (5.2.6), denoted as Model_M, and the adjusted model (5.4.1) without jump, denoted by Model_A.

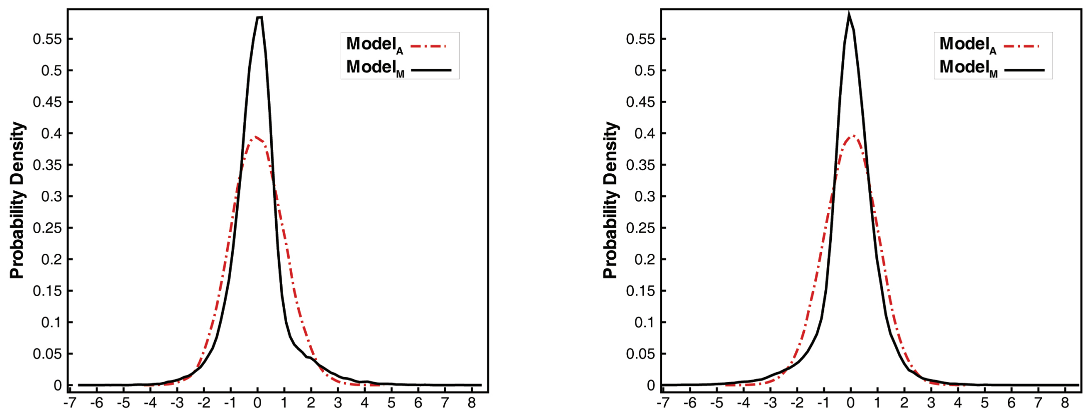
The market price model (5.4.1) leads to a normal distribution for the market price P_k which can underestimate the tail risk (likely due to large trades of other institutions). However, the multiplicative jump model (5.2.6), in which permanent price impact of other institutional trades are modeled by compound Poisson processes, is capable of better characterizing the short term asset returns and describing the fat tails.

In subplot (a) of Figure 5.2, probability density functions of the market price P_1 under the models (5.2.6) and (5.4.1) are compared. Subplot (b) compares the execution cost distribution associated with Model_M and Model_A. Under the proposed jump model (5.2.6), compared with the normal model (5.4.1), the execution cost has larger probability of small costs and higher probability of extreme costs. Using the model (5.4.1), it is possible to significantly underestimate the execution risk.

Figure 5.3 compares the risk measured in standard deviation and VaR for the execution strategies Strategy_M, Strategy_A, and Strategy_N, under the assumed multiplicative jump diffusion model (5.2.6). Figure 5.3 illustrates that the risk values are quite different between the naive strategy and Strategy_M or Strategy_A. We note that at $\mathbf{E}_{\mathcal{J}}^{\mathbf{m}} = 0$ the risk measure values are identical since the three execution strategies Strategy_N, Strategy_M and Strategy_A coincide at this point.

Figure 5.3 also illustrates that including an appropriate risk measure in problem (2.1.6) is important in determining the optimal execution strategy. Under the proposed jump model, the coherent risk measure CVaR may be more appropriate.

Assume that an execution strategy $\{n_k\}_{k=1}^N$ is deterministic, we compute the minimum $\text{CVaR}_{95\%}$ strategies under Model_M and Model_A. Table 5.2 illustrates the difference between the optimal static (price-independent) execution strategies to minimize $\text{CVaR}_{95\%}$ of the execution cost computed under the two models Model_M and Model_A. Table 5.2 indicates that, although Model_M and Model_A share the same optimal risk neutral execution strategy, they yield different optimal execution strategies when $\text{CVaR}_{95\%}$ of the execution cost is minimized. The difference in $\text{CVaR}_{95\%}$ values is about 3.7%. While the strategy to minimize variance



(a) Standardized Market Price Distribution (with zero mean and unit variance)

(b) Standardized Execution Cost Distribution (with zero mean and unit variance)

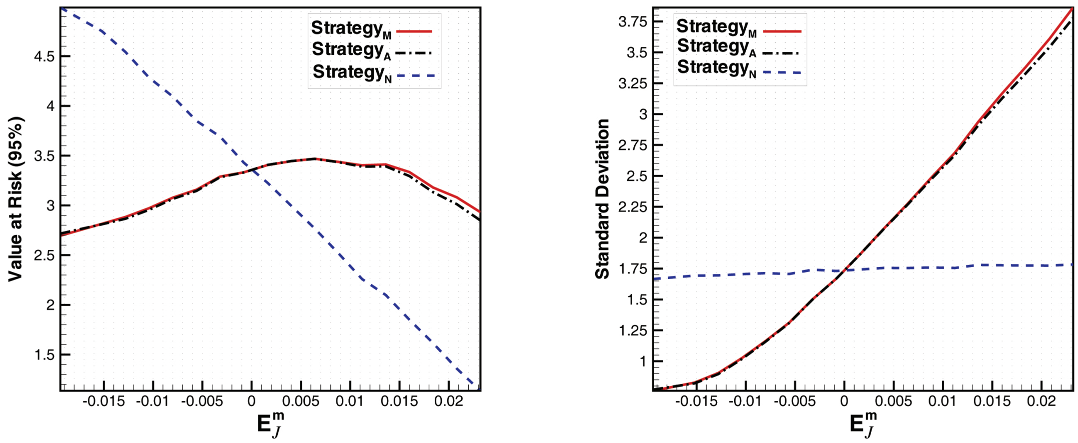
Figure 5.2: Probability density functions of Model_M and Model_A for $M = 50,000$ simulations. The kurtosis of P_1 for Model_M is 7.03 while for Model_A is 3.03. The kurtosis of the execution cost per share for Model_M is 7.50 while for Model_A is 3.04. Initial holding is $\bar{S} = 10^6$ shares. The jump parameters are $\lambda_x = 1$, $\mu_x^{\mathbf{m}} = 9.901 \times 10^{-3}$, $(\sigma_x^{\mathbf{m}})^2 = 9.802 \times 10^{-5}$, $\lambda_y = 0.2$, $\mu_y^{\mathbf{m}} = 1.049 \times 10^{-2}$, $(\sigma_y^{\mathbf{m}})^2 = 2.873 \times 10^{-3}$. These values yield $\Sigma^{\mathbf{m}} = 9.045 \times 10^{-3}$, $\mathbf{E}_{\mathcal{J}}^{\mathbf{m}} = -0.0076$, and $\mathbf{Cov}_{\mathcal{J}}^{\mathbf{m}} = 8.182 \times 10^{-4}$.

of the execution cost suggests to liquidate the entire holding immediately, the strategies for minimizing $\text{CVaR}_{95\%}$ under both models sell in the first couple of periods aggressively and purchases are made in the last periods. Although here minimizing $\text{CVaR}_{95\%}$ strategies share a similar pattern under both Model_M and Model_A, there is significant difference in the amount of trading in these two strategies.

5.5.2 Comparison of Optimal Execution Strategies

When the expected total market price change is nonzero, Strategy_M is dynamic while the optimal Strategy_A is static. However, since the initial price P_0 and the initial holding x_0 are known, the optimal execution n_1^* for both Strategy_M and Strategy_A are deterministic. Figure 5.4 compares Strategy_M and Strategy_A for the first period as a function of $\mathbf{E}_{\mathcal{J}}^{\mathbf{m}}$. As is illustrated in Figure 5.4, the difference in Strategy_M and Strategy_A increases as $\mathbf{E}_{\mathcal{J}}^{\mathbf{m}}$ moves away from zero.

Given a fixed $\mathbf{E}_{\mathcal{J}}^{\mathbf{m}}$, Figure 5.5 illustrates the optimal execution strategies Strategy_M and Strategy_A from period 2 to period 5 for $M = 1000$ simulations of the jump amplitudes and pricing shocks ξ_k. These plots clearly illustrate the significant difference of these execution



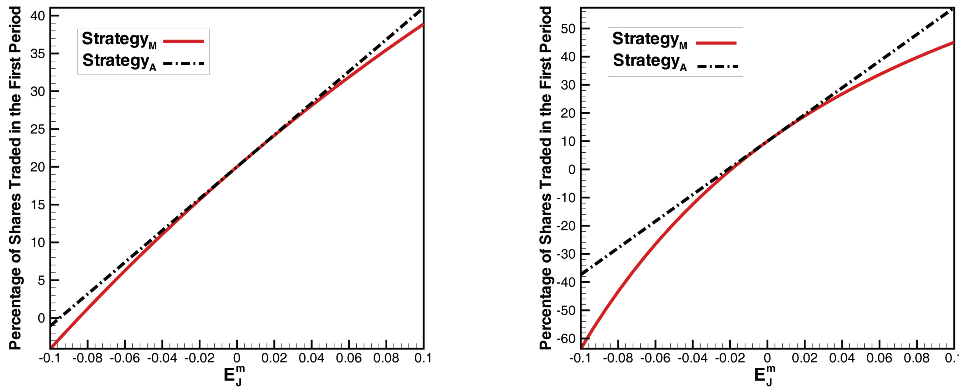
(a) VaR

(b) Standard Deviation

Figure 5.3: Risk measures of the execution costs for Strategy$_M$, Strategy$_A$, and Strategy$_N$ for $M = 40,000$ simulations. Initial holding is $\bar{S} = 10^6$ shares. Time horizon is $T = 5$. The jump parameter values are $\lambda_x = 2$, $\mu_x^{\mathbf{m}} = 9.9013 \times 10^{-3}$, $\sigma_x^{\mathbf{m}} = 9.9007 \times 10^{-3}$, and $\lambda_y \in [0.05, 3.6]$. These values yield $\Sigma^{\mathbf{m}} = 9.6484 \times 10^{-3}$, $\sigma_{tot} = 0.032$, and $\mathbf{Cov}^{\mathbf{m}}_{\mathcal{J}} = 9.3091 \times 10^{-4}$.

	Model$_M$	Model$_A$
CVaR$_{95\%}$	2.50×10^6	2.59×10^6
n_1	9.83×10^5	9.61×10^5
n_2	1.91×10^4	1.12×10^5
n_3	7.11×10^3	-5.87×10^4
n_4	-1.13×10^3	-4.55×10^3
n_5	-8.59×10^3	-9.04×10^3

Table 5.2: Optimal execution strategies which minimize CVaR$_{95\%}$ under Model$_M$ and Model$_A$, and the corresponding optimal values using $M = 50,000$ simulations in executing a single asset. Parameters are as in Table 5.1, and $T = 5$ days, and $\bar{S} = 10^6$ shares. The jump parameter values equal $\lambda_x = 3$, $\mu_x^{\mathbf{m}} = 9.5 \times 10^{-3}$, $\sigma_x^{\mathbf{m}} = 10^{-2}$, $\lambda_y = 0.5$, $\mu_y^{\mathbf{m}} = 6.9 \times 10^{-3}$, and $\sigma_y^{\mathbf{m}} = 3.2 \times 10^{-2}$. These values yield $\Sigma^{\mathbf{m}} = 0.009$, $\mathbf{E}^{\mathbf{m}}_{\mathcal{J}} = -0.025050$, and $\mathbf{Cov}^{\mathbf{m}}_{\mathcal{J}} = 1.106555 \times 10^{-3}$.



(a) $T = 5$ days

(b) $T = 10$ days

Figure 5.4: Comparison of the optimal execution n_1^* under the multiplicative jump model and under the additive jump model. Initial holding is $\bar{S} = 10^7$ shares. The total volatility equals $\sigma_{tot} = 0.03$.

strategies. While the naive strategy Strategy$_N$ suggests to trade an equal amount in each period, Strategy$_A$ is time varying but independent of the market price realized at the beginning of each period. In contrast, Strategy$_M$ is stochastic and varies with the realized market prices. In comparison to the naive strategy, both execution strategies Strategy$_M$ and Strategy$_A$ suggest to sell more aggressively initially in order to take advantage of the expected impact of large trades $\mathbf{E}^m_{\mathcal{J}} < 0$.

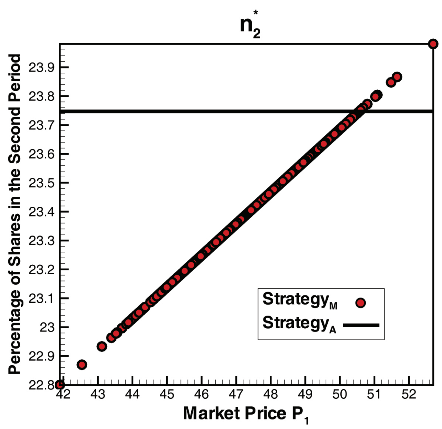
According to Theorems 5.3.2 and 5.3.3, another main difference between Strategy$_M$ and Strategy$_A$ is that the execution strategy Strategy$_M$ depends on the covariance of the market return, while the execution strategy Strategy$_A$ does not depend on the covariance of market prices. We illustrate implications of this property next.

5.5.3 Comparison of Expected Execution Costs

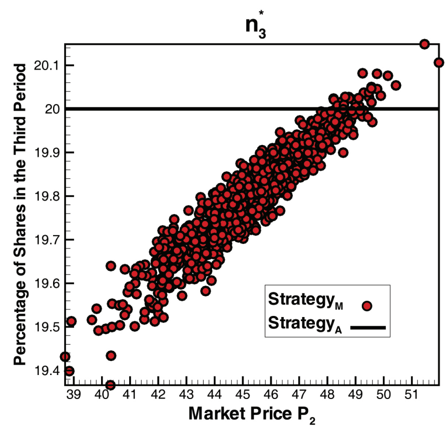
We now compare expected execution costs associated with strategies Strategy$_N$, Strategy$_M$, and Strategy$_A$, presented in cents per share. We quantify the average execution cost difference per period for a single asset trading using the following measure,

$$\mathbf{D}^{\mathbf{am}} \stackrel{\text{def}}{=} \frac{1}{NM\bar{S}} \sum_{k=1}^{N} \sum_{i=1}^{M} \left| n_k^{(\mathbf{m})}(i) \tilde{P}_k^{(\mathbf{m})}(i) - n_k^{(\mathbf{a})} \tilde{P}_k^{(\mathbf{a})}(i) \right|, \tag{5.5.1}$$

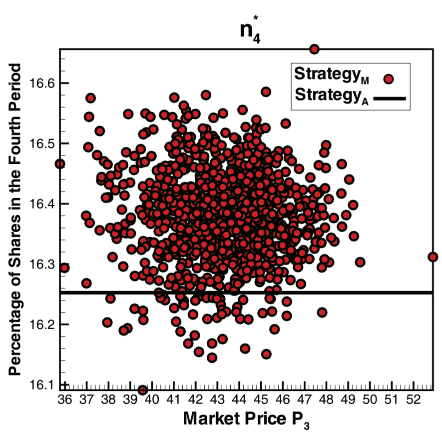
where the number M is the total number of simulations, $n_k^{(\mathbf{m})}(i)$ is the optimal risk neutral execution under the multiplicative jump diffusion model for the kth period in the ith simulation, $n_k^{(\mathbf{a})}$ is the optimal execution for the kth period derived under the additive jump



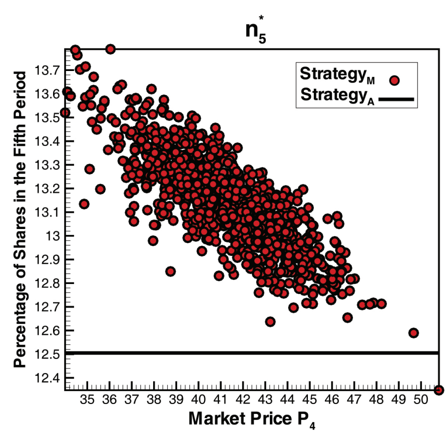
(a) Strategy at the second period



(b) Strategy at the third period



(c) Strategy at the fourth period



(d) Strategy at the fifth period

Figure 5.5: Optimal execution strategies n_2^*, n_3^*, n_4^* and n_5^* under the multiplicative jump model and the additive jump model for $M = 1000$ simulations. The trading horizon is $T = 5$ days. Initial holding is $\bar{S} = 10^7$ shares. The parameter values are $\lambda_x = 3.8$, $\mu_x^{\mathbf{m}} = 9.901 \times 10^{-3}$, $\sigma_x^{\mathbf{m}} = 9.901 \times 10^{-3}$, $\lambda_y = 0.2$, $\mu_y^{\mathbf{m}} = 1.186 \times 10^{-2}$, and $\sigma_y^{\mathbf{m}} = 1.198 \times 10^{-2}$. These values yield $\Sigma^{\mathbf{m}} = 9.045 \times 10^{-3}$, $\mathbf{E}_{\mathcal{J}}^{\mathbf{m}} = -3.560 \times 10^{-2}$, and $\mathbf{Cov}_{\mathcal{J}}^{\mathbf{m}} = 8.182 \times 10^{-4}$.

σ_{tot}	$\mu_y^{\mathbf{m}}$	$\sigma_y^{\mathbf{m}}$	$\Sigma^{\mathbf{m}}$	Strategy$_M$	Strategy$_N$	Strategy$_A$	$\mathbf{D}^{\mathbf{am}}(\%)$
0.020	0.00590	0.0125	6.03×10^{-3}	21.40	292.29	22.73	86.26
0.025	0.00540	0.0341	7.54×10^{-3}	21.19	293.08	22.66	89.10
0.030	0.00478	0.0490	9.05×10^{-3}	21.04	293.90	22.60	92.48
0.035	0.00406	0.0621	10.55×10^{-3}	20.97	293.31	22.65	95.77
0.040	0.00322	0.0744	12.06×10^{-3}	20.58	293.43	22.62	100.41
0.045	0.00227	0.0862	13.57×10^{-3}	20.32	292.01	22.59	104.61
0.050	0.00121	0.0977	15.08×10^{-3}	20.13	292.78	22.59	108.71

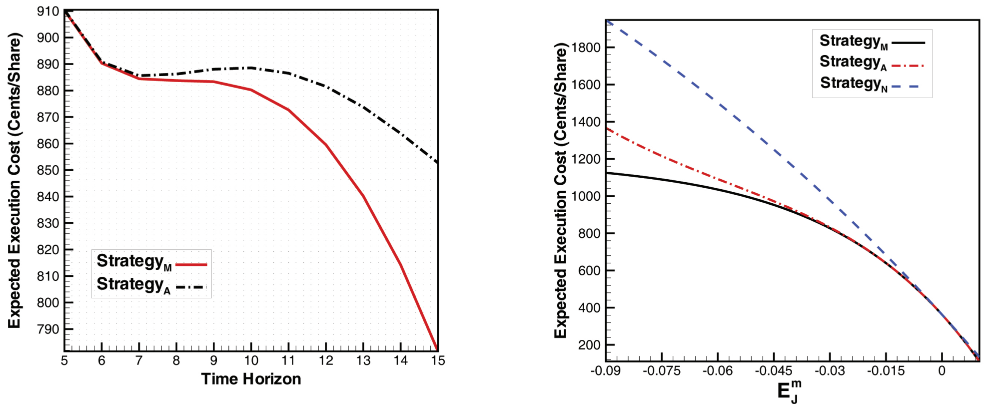
Table 5.3: Average expected execution cost (cents per share) and $\mathbf{D}^{\mathbf{am}}$ (percentage) for $M = 100,000$ simulations. Trading horizon is $T = 10$ days. Initial holding is $\bar{S} = 10^6$ shares. Jump parameters equal $\lambda_x = 2.6$, $\lambda_y = 0.2$, $\mu_x^{\mathbf{m}} = 4.938 \times 10^{-3}$ and $\sigma_x^{\mathbf{m}} = 9.950 \times 10^{-3}$. Thus, $\mathbf{E}_{\mathcal{J}}^{\mathbf{m}} = -1.180 \times 10^{-2}$.

diffusion model. The prices $\tilde{P}_k^{(\mathbf{m})}(i)$ and $\tilde{P}_k^{(\mathbf{a})}(i)$ are the execution prices at period k in the ith simulation, corresponding to the execution strategies $n^{(\mathbf{m})}(i)$ and $n^{(\mathbf{a})}$, respectively. The market price is assumed to follow a multiplicative jump diffusion model.

Using simulation, we compute $\mathbf{D}^{\mathbf{am}}$ measure for $T = 10$ days and various values of σ_{tot}. These quantities are reported in Table 5.3 which also includes the averaged execution costs of the three execution strategies Strategy$_M$, Strategy$_A$, and Strategy$_N$. As Table 5.3 indicates, the average relative difference $\mathbf{D}^{\mathbf{am}}(\%)$ can be quite significant. Moreover, the value of $\mathbf{D}^{\mathbf{am}}(\%)$ increases as σ_{tot} increases. Notice that, as Strategy$_A$ and Strategy$_N$ do not depend on the asset price volatility, their corresponding expected execution costs are constant as σ_{tot} changes; the slight variations for Strategy$_A$ and Strategy$_N$ are due to Monte Carlo simulations.

Proposition B.4.1 provides an analytical formula for the expected execution cost of the optimal execution strategy obtained under the additive jump diffusion model. For Strategy$_M$, which is optimal under the multiplicative jump diffusion model, the expected execution cost of Strategy$_M$ decreases as σ_{tot} increases. This is due to the fact that, under the multiplicative model, the solution is truly stochastically dynamic; thus it is capable of capturing price variations. In contrast, Strategy$_A$ is static and its execution cost do not depend on σ_{tot}.

Subplot (a) in Figure 5.6 depicts dependence of the expected execution cost on the trading horizon T. As Figure 5.6 demonstrates, when the time horizon increases, the expected execution cost of the execution strategy Strategy$_A$ becomes much higher than the expected execution cost of Strategy$_M$. Subplot (b) in Figure 5.6 illustrates how the expected execution cost associated with each of these three execution strategies varies as $\mathbf{E}_{\mathcal{J}}^{\mathbf{m}}$ changes, focusing when $\mathbf{E}_{\mathcal{J}}^{\mathbf{m}} < 0$. This figure clearly shows that in the depicted range, as $\mathbf{E}_{\mathcal{J}}^{\mathbf{m}}$ deviates from zero, the expected execution cost of Strategy$_N$ becomes significantly higher than the expected execution costs of Strategy$_M$ and Strategy$_A$. Moreover, as it is expected, the expected execution cost corresponding to Strategy$_A$ is also greater than that of Strategy$_M$. This difference becomes more prominent as $\mathbf{E}_{\mathcal{J}}^{\mathbf{m}}$ moves away from zero.



(a) Execution cost as time horizon varies

(b) Execution cost as $\mathbf{E}_{\mathcal{J}}^{\mathbf{m}}$ varies

Figure 5.6: Comparison in the expected execution cost (cents per share). The expected costs in subplot (b) are computed using Theorem 5.3.3 for Strategy$_M$ and Proposition B.4.1 for Strategy$_A$. Initial holding is $\bar{S} = 10^7$ shares. The values specified for the model parameters yield $\Sigma^{\mathbf{m}} = 9.045 \times 10^{-3}$. For Subplot (a), $\lambda_x = 3.8$, $\mu_x^{\mathbf{m}} = 9.901 \times 10^{-3}$, $\sigma_x^{\mathbf{m}} = 9.901 \times 10^{-3}$, $\lambda_y = 0.2$, $\mu_y^{\mathbf{m}} = 1.186 \times 10^{-2}$, $\sigma_y^{\mathbf{m}} = 1.198 \times 10^{-2}$, $\mathbf{E}_{\mathcal{J}}^{\mathbf{m}} = -3.560 \times 10^{-2}$, $\mathbf{Cov}_{\mathcal{J}}^{\mathbf{m}} = 8.182 \times 10^{-4}$.

5.6 Concluding Remarks

Current literature on the optimal portfolio execution problem typically assumes that the market return (or price change) has a normal distribution. There are two main problems with this assumption. Firstly, the empirical study indicates that the short term return distribution often has fat tails, possibly due to permanent price impact of institutional trades. Such fat tails cannot be described by normal distributions. Secondly, while the permanent price impact of a decision maker's own trade causes a discrete price depression, it is not reasonable to model permanent price impact of other concurrent large trades by a continuous Brownian motion.

In this chapter, we suggest using jump processes to capture uncertain permanent price impact of trades by other institutions. The proposed model includes two compound Poisson processes corresponding to buy and sell trades, respectively. Using stochastic dynamic programming, we provide an analytical solution to the risk neutral execution strategy which minimizes the expected execution cost under the proposed jump diffusion models for the evolution of the market price. This solution is static (state independent) when the expected market price change is zero. However, when the expected market price change is nonzero, the optimal execution strategy derived under the multiplicative jump diffusion model is stochastic and dynamic. In addition the optimal execution strategy depends on the asset return volatility. In contrast, under an additive jump diffusion model, the optimal execution strategy does not depend on the asset price realization or the volatility, even when the expected market price change is nonzero.

Under the proposed jump diffusion model, more accurate assessment of the execution risk can be made. When the market price change is modeled by normal distributions, the tail execution risk can be significantly underestimated. Using simulations, we illustrate that the execution cost distribution associated with the naive strategy, optimal risk neutral strategy under the additive jump diffusion model, and the optimal risk neutral strategy under the multiplicative jump diffusion model are qualitatively different. This highlights the importance of using an appropriate model to determine an optimal execution strategy. In addition, we assess differences in the optimal execution strategies derived under different model assumptions. Assuming that the market price dynamics is characterized by a multiplicative jump diffusion model, we show that the execution strategy optimal under an additive jump diffusion model (with comparable mean and standard deviation) can perform notably suboptimally when the asset return volatility or the trading horizon increases.

Our main focus, in this chapter, was on investigating model assumptions and the resulting optimal execution strategies. We considered the optimal execution strategy for minimizing the expected execution cost. There are many possible objective functions which are of interest to institutional investors. In particular, a natural additional criterion to include is some measure of risk, e.g., variance, VaR or CVaR of the execution cost. While the inclusion of a risk measure into the objective function is conceptually straightforward and probably desirable, an analytical expression for the optimal execution strategy is not available except in very special cases (Bertsimas and Lo, 1998). Including risk measures in the objective function might make the optimal-value function non-separable in the sense of stochastic dynamic programming, see, e.g., (Yao et al., 2003). Therefore, the presence of some nonlinear risk measure makes solving the stochastic dynamic programming very challenging. We further investigate this in the next chapter.

Chapter 6

Smoothing and Parametric Rules for Stochastic Mean-CVaR Optimal Execution Strategy

6.1 Introduction

Since trading takes time and the permanent price impact of a trade can affect the future asset price, the optimal portfolio execution problem is fundamentally a stochastic dynamic programming problem, see, e.g., (Bertsimas and Lo, 1998). In a single asset case, Almgren and Lorenz (2007) provide an optimal adaptive strategy. Stochastic (adaptive) execution strategies can explicitly recognize market price change during the trading horizon. In addition it has been shown in (Almgren and Lorenz, 2007) that a significant improvement over static strategies can be achieved through stochastic execution strategies.

When no risk is considered, analytical solutions have been found for the stochastic dynamic programming problem which minimizes the expected execution cost under several price models, see, e.g., (Bertsimas and Lo, 1998; Bertsimas et al., 1999). Under a specific additive market price model with a deterministic price impact model and volatility, Huberman and Stanzl (2005) have obtained a closed-form solution for minimizing the mean and variance of the execution cost.

In addition to the expected execution cost, one is often interested in controlling the risk in execution, e.g., including minimizing variance of the execution cost as an objective. Unfortunately, under general price models, the mean-variance objective formulations for the optimal portfolio execution problem are not amenable to stochastic dynamic programming techniques; the dynamic programming equation may not exist. When this occurs, a time-consistent dynamic solution cannot be determined using a stochastic dynamic programming technique. Even when a dynamic programming equation exists, obtaining a closed-form solution in general may not be possible, particularly when constraints are included.

In Chapter 5, a model is proposed which explicitly characterizes uncertain arrivals of other large trades by including jump processes to the market price dynamics. The proposed jump diffusion model includes two compound Poisson processes, with random jump amplitudes capturing uncertain permanent price impact of other large buy and sell trades respectively. Since the execution cost distribution is now asymmetric and may have fat tails, CVaR or a downside risk measure is more appropriate. In addition, CVaR is a coherent risk measure which can measure extreme events/execution costs and has attractive properties such as convexity, see, e.g., (Artzner et al., 1997; Rockafellar and Uryasev, 2000).

In (Shapiro, 2008), dynamic programming equation is applied to dynamically coherent risk measures; however no computational result is provided. In general, when the objective function includes a risk measure such as CVaR, numerical methods are required to compute stochastic dynamic programming solutions. When a portfolio of risky assets are involved, solving a multi-stage optimal portfolio execution problem is computationally challenging, since computational complexity grows exponentially in the number of state variables. Thus computing a stochastic dynamic programming solution is often computationally intractable in practice; this is known as the *curse of dimensionality*. As discussed in (Shapiro, 2008), while two stage *linear* stochastic programming problems can be solved with a reasonable accuracy, computational complexity in solving multistage stochastic programming problems grows quickly with the increase of the number of stages. Many approximation algorithms in the literature have been considered to obtain approximations to stochastic programming solutions, see, e.g., (de Farias and Roy, 2003; Powell, 2007). Solving a multi-stage stochastic programming problem is even more challenging when there are inequality constraints (Haugh and Lo, 2001).

The goal of this chapter is to propose a tractable computational approach to obtain an approximate stochastic dynamic programming solution for the optimal portfolio execution problem when mean and some risk measure of the execution cost are minimized. To achieve optimality at each time period k, a new stochastic strategy can be computed by considering optimality conditional on the information set $\mathcal{I}_k$ at time k. In particular, our approach relies on Monte Carlo simulations, where simulation price paths are generated by iid samples for the random variables in the decision time horizon. Compared to the backward iteration in the dynamic programming approach, methods based on forward simulation paths have attractive features. While backward dynamic programming approaches to multi-stage stochastic programming problems suffer the curse of dimensionality when applied to problems with high dimensional state spaces, the use of a forward simulation base approach for multi-stage multi-asset stochastic optimization problem does not incur exponential growth in computational complexity. Simulation based approximation solution approaches have been previously applied successfully in (Longstaff and Schwartz, 2001) to solve a stochastic dynamic programming for pricing an American option. Coleman et al. (2007) also use a similar method for the total risk minimization with a quadratic objective. In this case, they observe that this approach is capable of achieving relatively good accuracy comparing to the analytic solution. In (Coleman et al., 2007), decision variables are approximated using cubic splines.

There are, however, additional computational challenges in solving the multi-stage multi-asset optimal portfolio execution problem based on simulations. Firstly, if a strategy is allowed to be an arbitrarily path dependent, the number of variables in the simulation optimization problem is proportional to the number of scenarios which is very large in general. Furthermore, unlike the single period simulation CVaR optimization problem, the multi-period simulation optimal portfolio execution problem is piecewise nonlinear rather than piecewise linear due to the presence of permanent price impact. This can be problematic since solving a general nonlinear programming problem is more difficult than solving a linear programming problem. Moreover, if constraints, e.g., bound constraints, are imposed, the number of corresponding constraints in the simulation optimization problem also becomes proportional to the number of simulations.

In this chapter, we propose techniques to overcome these computational challenges for the simulation approach to multi-stage CVaR execution cost minimization. To reduce the number of variables, we first represent execution strategies using a parametric model with unknown parameters. Different parametric forms can be used. In this chapter, we assume that an execution strategy depends linearly on the price and the trading accomplished thus far; this parametric form is motivated by the analytic formula for the optimal risk neutral execution strategy derived in Chapter 5. To alleviate the piecewise nonlinearity in the objective function arising from the simulation discretization to the CVaR measure, we apply the smoothing technique proposed in (Alexander et al., 2006) for a single period CVaR optimization problem. The motivation behind the smoothing is the same as in the single period case: the piecewise nature in the simulation CVaR optimization problem arises from simulation discretization but the CVaR risk measure in the continuous model is in fact continuously differentiable. To handle constraints, we first apply the exact penalty function and then use smoothing to alleviate the piecewise nature of the exact penalty function. Indeed, our proposed smoothing method of the exact penalty function corresponds to applying a new penalty function which is piecewise quadratic but continuously differentiable. The new penalty function can be regarded as a combination of the quadratic and exact penalty functions.

Using the proposed parametric representation and smoothing method, we obtain a static nonlinear optimization problem with a potentially nonlinear objective function. We then use the trust region algorithm in (Coleman and Li, 1996) to solve this problem. The first and second derivatives of the objective function can be computed using automatic differentiation, see, e.g., (Coleman and Verma, 2000). We further note that our proposed computational approach is quite general and it can be applied to alternative risk measures other than CVaR.

The presentation is organized as follows. In §6.2, we explain mean-CVaR stochastic optimal portfolio execution problem. Our smoothing and parametric rules are explained in §6.3. In §6.3.3, we describe handling constraints using a smoothed exact penalty function. Our computational investigation is provided in §6.4. Concluding remarks are given in §6.5.

6.2 Stochastic Mean-CVaR Optimal Strategy

While the main objective of the decision maker is to minimize the expected execution cost, he may be concerned with the execution risk, i.e., uncertainty in the total amount that will be received from the trade implementation. In particular, as we discussed in the previous chapter, a jump-diffusion model with two compound Poisson processes is more appropriate since they explicitly model the uncertain impact of uncertain arrivals of other large buy and sell trades. Under this model assumption, the execution cost distribution is asymmetric and may have fat tails. This provides rationale for using a CVaR risk measure in the optimal portfolio execution problem formulation.

Since trading takes time and the permanent price impact affects the future market price, optimal portfolio execution problem is a multi-stage stochastic programming problem. The solution to this multi-stage stochastic programming problem can potentially yield a solution which adapts to market price and the impact of other large trades.

When execution risk is considered, the stochastic programming formulation for the optimal portfolio execution problem is:

$$\begin{aligned} \min_{\substack{n_1,\ldots,n_N \\ n_k:\mathcal{I}_k\text{-measurable} \\ k=1,\ldots,N}} \quad & \mathbf{E}\left(P_0^T\bar{S} - \sum_{k=1}^{N} n_k^T\tilde{P}_k\right) + \mu\cdot\Psi\left(P_0^T\bar{S} - \sum_{k=1}^{N} n_k^T\tilde{P}_k\right) \\ \text{s.t.} \quad & \sum_{k=1}^{N} n_k = \bar{S}, \end{aligned} \tag{6.2.1}$$

where $\Psi(\cdot)$ is a risk measure for the execution cost and $\mu \geq 0$ is a risk aversion parameter. Here $\mathcal{I}_k$ denote the information set observable at time t_k.

Stochastic dynamic programming has been previously used to minimize the expected execution cost when the market price evolves according to a Brownian motion, and the permanent price impact of the decision maker's trade makes a discrete price change, see, e.g., (Bertsimas and Lo, 1998; Bertsimas et al., 1999). However, when a risk measure such as variance or CVaR is included in problem (6.2.1) with a positive risk aversion parameter, the multi-stage stochastic programming problem becomes significantly more complex. Moreover, when a dynamic programming equation cannot be found, a solution $\{n_k\}_{k=1}^N$ of the stochastic programming problem (6.2.1) computed at the initial time t_0 does not necessarily have the time consistency property. In other words, n_k from problem (6.2.1) is not optimal at time t_k, i.e., it may not solve the following problem:

$$\begin{aligned} \min_{\substack{n_k,\ldots,n_N \\ n_j:\mathcal{I}_j\text{-measurable} \\ j=k,\ldots,N}} \quad & \mathbf{E}\left(P_0^T\bar{S} - \sum_{i=1}^{N} n_i^T\tilde{P}_i \,\middle|\, \mathcal{I}_k\right) + \mu\cdot\Psi\left(P_0^T\bar{S} - \sum_{i=1}^{N} n_i^T\tilde{P}_i \,\middle|\, \mathcal{I}_k,\right) \\ \text{s.t.} \quad & \sum_{k=1}^{N} n_k = \bar{S}. \end{aligned} \tag{6.2.2}$$

Here it is assumed that $n_1, \cdots, n_{k-1}$ are given.

Given that problems (6.2.1) and (6.2.2) yield different solutions at time t_k, $k \geq 2$, the decision maker has two different ways to implement an execution strategy through the multi-stage stochastic programming formulations. The first possibility is to compute the optimal strategy $\{n_k\}_{k=1}^N$ at the initial time based only on problem (6.2.1). Then at time t_k, the amount n_k, computed from (6.2.1), is implemented even though it may not be optimal from t_k perspective. Alternatively, to ensure conditional optimality at time t_k, the decision maker can ignore the already computed strategy for t_k from problem (6.2.1) and adopts the strategy for time t_k by solving a conditional stochastic programming problem (6.2.2) to determine trading amount for this period.

No matter which method the decision maker adopts for execution, she needs to solve one of the multi-stage stochastic programming problems (6.2.1) and (6.2.2). Computing solutions to these problems is a daunting task. In the remaining part of this chapter, we focus on developing a tractable computational technique applicable to both problems (6.2.1) or problem (6.2.2), and we are not concerned with which one is preferred. Since our proposed computational method can be applied to both problems (6.2.1) and (6.2.2), without loss of generality, we present our proposed approach for problem (6.2.1).

Notice that problem (6.2.1) or (6.2.2) may have additional constraints, such as a no-buying requirement while selling. In this case, even when a dynamic programming equation exists, computational methods cannot easily handle constraints since the value function from the dynamic programming under constraints becomes nondifferentiable, see, e.g., (Bertsimas and Lo, 1998).

Different risk measures can be included in the objective function of problem (6.2.1). Typical choices of the risk measure $\Psi(\cdot)$ are variance, VaR, or CVaR. In this chapter, we focus on CVaR risk measure since we believe that the short horizon return is far from a normal distribution and it is important to properly capture the tail risk. We note however that our proposed computational approach is applicable to other risk measures including variance and downside risk.

CVaR is frequently defined based on VaR. In the context of the optimal portfolio execution problem, let X denote the execution cost in the given time horizon. For a given confidence level β, VaR is the smallest cost over the time horizon that is exceeded with probability no greater than $1-\beta$, i.e., $\mathrm{VaR}_\beta(X) = \inf\{x \in \mathbb{R} : \Pr(X \leq x) \geq \beta\}$. Using VaR, CVaR can be defined as

$$\mathrm{CVaR}_\beta(X) = \mathbf{E}\left(X : X \geq \mathrm{VaR}_\beta(X)\right).$$

Without reference to VaR, a more direct way of defining CVaR is:

$$\mathrm{CVaR}_\beta(X) = \min_\alpha \left(\alpha + \frac{1}{1-\beta}\mathbf{E}\left([X-\alpha]^+\right)\right), \tag{6.2.3}$$

where $[z]^+ = \max(z, 0)$. When the random cost X has a strictly increasing and continuous probability distribution function, these two definitions are equivalent. However the latter definition yields a coherent risk measure even when the distribution is discontinuous. In addition, formulation (6.2.3) directly leads to linear or nonlinear programming formulations under simulation discretizations. It then can be solved by available linear programming or nonlinear programming optimization techniques.

The mean-CVaR optimal portfolio execution problem with the risk aversion parameter $\mu \geq 0$ is then given as below

$$\begin{array}{cc} \min\limits_{\substack{n_1, n_2, ..., n_N \\ n_k : \mathcal{I}_k - \text{measurable}}} & \mathbf{E}\left(P_0^T \bar{S} - \sum_{k=1}^{N} n_k^T \tilde{P}_k\right) + \mu \cdot \text{CVaR}_\beta \left(P_0^T \bar{S} - \sum_{k=1}^{N} n_k^T \tilde{P}_k\right) \\ \text{s.t.} & \sum_{k=1}^{N} n_k = \bar{S}. \end{array} \quad (6.2.4)$$

Using CVaR definition (6.2.3), formulation (6.2.4) is reduced to the following problem:

$$\begin{array}{cc} \min\limits_{\substack{\alpha \in \mathbb{R}, n_1, n_2, ..., n_N \\ n_k : \mathcal{I}_k - \text{measurable}}} & \mathbf{E}\left(P_0^T \bar{S} - \sum_{k=1}^{N} n_k^T \tilde{P}_k\right) + \mu\alpha + \frac{\mu}{1-\beta}\mathbf{E}\left(\left[P_0^T \bar{S} - \sum_{k=1}^{N} n_k^T \tilde{P}_k - \alpha\right]^+\right), \\ \text{s.t.} & \sum_{k=1}^{N} n_k = \bar{S}. \end{array} \quad (6.2.5)$$

Additional $n_k \geq 0$ constraints can also be incorporated in problem (6.2.5).

We note that, when the objective of the optimal portfolio execution problem is to minimize only the variance of the execution cost, i.e.,

$$\min_{n_1,\ldots,n_N} \mathbf{Var}\left(P_0^T \bar{S} - \sum_{k=1}^{N} n_k^T \tilde{P}_k\right) \quad \text{s.t} \quad x_N = 0, \quad (6.2.6)$$

the optimal execution strategy is the strategy of liquidating the entire holding in the first period:

$$n_1 = \bar{S}, \quad n_k = 0 \quad k \geq 2. \quad (6.2.7)$$

This can be easily seen since the variance of the execution cost associated with this strategy equals zero. The CVaR of the execution cost associated with the execution strategy (6.2.7) is

$$\text{CVaR}_\beta \left(P_0^T \bar{S} - \sum_{k=1}^{N} n_k^T \tilde{P}_k\right) = \frac{1}{\tau}\bar{S}^T h\left(\frac{\bar{S}}{\tau}\right).$$

We note that the strategy (6.2.7) for minimizing variance is in general not the strategy for minimizing CVaR.

In the next section, we describe our proposed smoothing and parametric approach to obtain an approximate solution of problem (6.2.5) efficiently.

6.3 The Proposed Smoothing and Parametric Method

Since CVaR risk measure does not have an analytic expression in general, Monte Carlo (MC) simulation is typically applied to discretize the CVaR optimization problem. For the optimal portfolio execution problem, the discretized problem is more complex since the price path changes when the trading amount changes due to permanent price impact. Assume that the market price dynamics in the kth time period is given by $\mathcal{F}(P_{k-1}, \xi_k)$ where ξ_k is a random vector. We generate M random paths $\{\xi_1, \cdots, \xi_{N-1}\}$ and these sample values are fixed for simulation CVaR problems even when price paths change with the trading amount $\{n_k\}$. For any given $\{n_k\}_{k=1}^N$, let $\{P_k\}_{k=1}^N$ denote market price path corresponding to $\{\xi_1, \cdots, \xi_{N-1}\}$, we obtain a discretized stochastic optimization problem for problem (6.2.5):

$$\begin{aligned} \min_{\substack{n_1, n_2, \ldots, n_N, \alpha \\ n_k : \mathcal{I}_k - \text{measurable}}} \quad & \frac{1}{M} \sum_{j=1}^{M} \left(P_0^T \bar{S} - \sum_{k=1}^{N} n_k^T \tilde{P}_k^{(j)} \right) + \mu\alpha + \frac{\mu}{M(1-\beta)} \sum_{j=1}^{M} \left[P_0^T \bar{S} - \sum_{k=1}^{N} n_k^T \tilde{P}_k^{(j)} - \alpha \right]^+ \\ \text{s.t.} \quad & \sum_{k=1}^{N} n_k = \bar{S}. \end{aligned} \tag{6.3.1}$$

The superscript (j) indicates the jth scenario. Note that for each k and j, $\tilde{P}_k^{(j)}$ is a $m \times 1$ vector, where m is the number of assets in the portfolio. The continuously differentiable nonlinear objective function in problem (6.2.5) now becomes a piecewise nonlinear objective function. Each simulation corresponds to one nonlinear function piece; here the nonlinearity arises from the iterative dependence due to the permanent price impact. Using a standard technique of replacing the piecewise function $[\cdot]^+$ with a set of constraints, this piecewise nonlinear minimization problem can be formulated as a nonlinear programming problem with the number of nonlinear constraints proportional to the number of Monte Carlo simulations M. Solving such a large scale nonlinear programming problem is computationally expensive, as the number of scenarios M is typically very large. Therefore, as the first step, we use a smoothing method to avoid dealing with a very large number of constraints; this is described in §6.3.1.

6.3.1 Eliminating Non-differentiability

To reduce computational complexity of problem (6.3.1), we use a smoothing technique, proposed by Alexander et al. (2006) for a single period CVaR optimization problem. The basic idea is to approximate the piecewise linear function $[z]^+$ with a continuously differentiable piecewise quadratic function $\rho_\epsilon(z)$ with a small resolution parameter ϵ:

$$\rho_\epsilon(z) = \begin{cases} z & \text{if } \ z > \epsilon \\ \frac{z^2}{4\epsilon} + \frac{1}{2}z + \frac{1}{4}\epsilon & \text{if } \ -\epsilon \le z \le \epsilon \\ 0 & \text{if } z < -\epsilon \end{cases} \tag{6.3.2}$$

Note that $\rho_\epsilon(z) \geq 0$ for every ϵ and z. Using (6.3.2), problem (6.3.1) is then reduced to the following continuously differentiable nonlinear minimization problem:

$$\begin{aligned}
\min_{\substack{n_1,n_2,\dots,n_N,\alpha \\ n_k:\mathcal{I}_k-\text{measurable}}} \quad & -\frac{1}{M}\sum_{j=1}^{M}\sum_{k=1}^{N} n_k^T \tilde{P}_k^{(j)} + \mu\alpha + \frac{\mu}{M(1-\beta)}\sum_{j=1}^{M}\rho_\epsilon\left(P_0^T\bar{S} - \sum_{k=1}^{N} n_k^T\tilde{P}_k^{(j)} - \alpha\right) \\
\text{s.t.} \quad & \sum_{k=1}^{N} n_k = \bar{S}.
\end{aligned} \tag{6.3.3}$$

In problem (6.3.3), the objective function is actually continuously differentiable, since each simulation no longer introduces a nonlinear function piece. Therefore, there is no need to include an additional constraint for each simulation to avoid non-differentiability.

6.3.2 Using Parametric Trading Rules

To obtain a stochastic execution strategy which adapts to the market price, one can let n_k freely depend on each price scenario, i.e.,

$$\begin{aligned}
\min_{\substack{n_1^{(j)},\dots,n_N^{(j)},\alpha \\ n_k^{(j)}:\mathcal{I}_k-\text{measurable}}} \quad & -\frac{1}{M}\sum_{j=1}^{M}\sum_{k=1}^{N} (n_k^{(j)})^T \tilde{P}_k^{(j)} + \frac{\mu}{M(1-\beta)}\sum_{j=1}^{M}\rho_\epsilon\left(P_0^T\bar{S} - \sum_{k=1}^{N} (n_k^{(j)})^T\tilde{P}_k^{(j)} - \alpha\right) \\
& +\mu\alpha \\
\text{s.t.} \quad & \sum_{k=1}^{N} n_k^{(j)} = \bar{S}, \quad j = 1,2,\dots,M.
\end{aligned} \tag{6.3.4}$$

The number of decision variables in the nonlinear minimization problem (6.3.4) is of order $M \cdot N$, where M is the number of scenarios and N is the number of periods. Hence, solving problem (6.3.4) directly is computationally expensive as the number of scenarios M is typically large.

In addition we need to ensure that the execution strategy is *non-anticipatory*, n_k is $\mathcal{I}_k$-measurable. More precisely, execution strategy at stage k must only depend on the information available up to time t_k.

To resolve these two issues, we explicitly require that the execution strategy to have a parametric representation as below:

$$n_k = f_k\left(P_{k-1}, x_{k-1}\right), \qquad k = 1, 2, \dots, N-1. \tag{6.3.5}$$

Here f_k is a deterministic function of P_{k-1} and x_{k-1}, where P_{k-1} represents the market price at time t_{k-1} and x_{k-1} quantifies the total number of shares to be sold. This explicitly restricts the strategy to be non-anticipatory.

Applying the decision rule (6.3.5) in problem (6.3.4), we arrive at:

$$\begin{aligned}
\min_{\substack{n_1^{(j)},n_2^{(j)},\ldots,n_N^{(j)},\alpha \\ n_k^{(j)}=f_k\left(P_{k-1}^{(j)},x_{k-1}^{(j)}\right)}} \quad & -\frac{1}{M}\sum_{j=1}^{M}\sum_{k=1}^{N}(n_k^{(j)})^T\tilde{P}_k^{(j)} + \frac{\mu}{M(1-\beta)}\sum_{j=1}^{M}\rho_\epsilon\left(P_0^T\bar{S} - \sum_{k=1}^{N}(n_k^{(j)})^T\tilde{P}_k^{(j)} - \alpha\right) \\
& +\mu\alpha \\
\text{s.t.} \quad & \sum_{k=1}^{N} n_k^{(j)} = \bar{S}, \quad j = 1, 2, \ldots, M. \qquad (6.3.6)
\end{aligned}$$

Assuming that the parametric function f_k depends on a small number of parameters, the number of unknown variables in the optimization problem (6.3.6) is then significantly reduced. The equality constraint can also be eliminated by an explicit variable substitution. Thus problem (6.3.6) can be represented as an unconstrained continuously differentiable nonlinear minimization problem with a total of $O(l \times (N-1))$ variables, where l denotes the number of parameters in the definition of f_k.

Now, we describe a specific linear trading rule used in our computational investigation for approximating the optimal execution strategy. This parametric representation is motivated by the explicit formula derived in Chapter 5 for minimizing the expected execution cost under a multiplicative jump-diffusion model.

Specifically we assume the following linear parametric model for a stochastic optimal execution strategy:

$$\begin{aligned}
n_k &= Y_k P_{k-1} + Z_k x_{k-1} + c_k, \quad k = 1, 2, ..., N-1, \qquad (6.3.7) \\
n_N &= \bar{S} - \sum_{k=1}^{N-1} n_k,
\end{aligned}$$

where Y_k and Z_k are $m \times m$ unknown matrix parameters, and c_k is an m unknown parameter vector. The m vector P_{k-1} represents the market price in the previous period and x_{k-1} is the m vector of shares remaining to be sold.

Indeed the optimal execution strategy for minimizing the expected execution cost has exactly this linear parametric representation. Thus the computed optimal execution strategy based on (6.3.7), when $\mu = 0$ and no constraint is included, attains minimum execution cost (i.e. no loss of optimality). When a positive risk aversion parameter is used, the parametric model assumption (6.3.7) may lead to a suboptimal solution. When $Y_k = 0$ and $Z_k = 0$, the strategy is a static execution strategy. One may further assume that $Y_1 = 0$ and $Z_1 = 0$, to reduce parameter redundancy since the strategy at $k = 1$ is deterministic and n_1 can be determined solely by c_1.

Using representation (6.3.7) for n_k, problem (6.3.6) is reduced to computing

$$c_1, Y_2, Z_2, c_2, \ldots, Y_{N-1}, Z_{N-1}, c_{N-1}, \quad \text{and} \quad \alpha,$$

from the following problem:

$$\begin{aligned}
\min \quad & -\frac{1}{M}\sum_{j=1}^{M}\sum_{k=1}^{N}(n_k^{(j)})^T\tilde{P}_k^{(j)} + \frac{\mu}{M(1-\beta)}\sum_{j=1}^{M}\rho_\epsilon\left(P_0^T\bar{S} - \sum_{k=1}^{N}(n_k^{(j)})^T\tilde{P}_k^{(j)} - \alpha\right) \\
& +\mu\alpha \qquad (6.3.8) \\
\text{s.t.} \quad & \sum_{k=1}^{N} n_k^{(j)} = \bar{S}, \quad j = 1, 2, ..., M, \\
& n_1^{(j)} = c_1, \\
& n_k^{(j)} = Y_k P_{k-1}^{(j)} + Z_k x_{k-1}^{(j)} + c_k, \quad k = 2, 3, ..., N-1.
\end{aligned}$$

After eliminating the decision variables $n_k^{(j)}$, the number of decision variables in this problem equals $(N-2)(2m^2+m)+m+1$ which does not depend on the number of simulations M.

6.3.3 Handling Inequality Constraints Using Penalty Functions

In an optimal portfolio execution problem, one may want to impose additional inequality constraints, for example, no buying during a selling order execution. Handling inequality constraints in stochastic dynamic programming is in general challenging, see, e.g., (Grossman and Vila, 1992; Bertsimas and Lo, 1998). Using the simulation approach as in problem (6.3.6), the number of constraints becomes proportional to the number of simulations, since there exists a constraint corresponding to each future scenario. Thus computational complexity becomes prohibitive, particularly when the objective function is nonlinear due to permanent price impact.

Penalty functions are well established methods for handling constraints in nonlinear optimization, see, e.g., (Nocedal and Wright, 2000). Quadratic penalty functions, exact penalty functions, and barrier functions are frequently used in practice. While barrier functions typically require a strictly feasible point to start with, the quadratic penalty function and exact penalty function achieve feasibility in the optimization process. One attractive property of the exact penalty function, in comparison to the quadratic penalty function, is the existence of a finite penalty parameter (under suitable assumptions) using which a minimizer of the penalized optimization problem is a minimizer of the original optimization problem. If a quadratic penalty function is used, the penalized optimization yields a solution of the constrained optimization problem asymptotically as the penalty parameter converges to $+\infty$.

Consequently we prefer to use the exact penalty function. To illustrate this technique, assume that we want to include the following set of L constraints in optimization problem (6.2.4):

$$a_\ell(n_1, \ldots, n_N) \le 0, \qquad \ell = 1, 2, \ldots, L.$$

the simulation problem corresponding to (6.3.6) will have the following $M \cdot L$ constraints:

$$a_\ell\left(n_1^{(j)}, \ldots, n_N^{(j)}\right) \le 0, \qquad j = 1, 2, \ldots, M, \quad \ell = 1, 2, \ldots, L.$$

When the number of simulations M increases, the number of constraints increases accordingly. Consequently the computational cost for solving the corresponding nonlinear optimization problem can quickly become prohibitive. Using the exact penalty function $\max\left\{0, a_\ell\left(n_1^{(j)}, \ldots, n_N^{(j)}\right)\right\}$ for the inequality $a_\ell(n_1^{(j)}, \ldots, n_N^{(j)}) \leq 0$ and a large enough penalty parameter $\vartheta > 0$, we arrive at the following penalty optimization problem:

$$\begin{array}{ll} \min\limits_{\substack{\alpha\in\mathbb{R}, n_1^{(j)}, n_2^{(j)}, \ldots, n_N^{(j)} \\ n_k^{(j)} = f_k\left(P_{k-1}^{(j)}, x_{k-1}^{(j)}\right) \\ k=1,2,\ldots,N-1}} & -\dfrac{1}{M}\displaystyle\sum_{j=1}^{M}\sum_{k=1}^{N}(n_k^{(j)})^T\tilde{P}_k^{(j)} + \dfrac{\mu}{M(1-\beta)}\sum_{j=1}^{M}\rho_\epsilon\left(P_0^T\bar{S} - \sum_{k=1}^{N}(n_k^{(j)})^T\tilde{P}_k^{(j)} - \alpha\right) \\ & +\mu\alpha + \vartheta \cdot \displaystyle\sum_{\ell=1}^{L}\sum_{j=1}^{M}\max\left\{0, a_\ell\left(n_1^{(j)}, \ldots, n_N^{(j)}\right)\right\} \\ \text{s.t.} & \displaystyle\sum_{k=1}^{N} n_k^{(j)} = \bar{S}, \quad j = 1, 2, ..., M, \end{array} \tag{6.3.9}$$

Unfortunately the above penalty optimization problem is piecewise differentiable due to the use of the exact penalty function, with the number of function pieces proportional to the number of simulations. Once again, computational cost for solving the penalty optimization problem can quickly become prohibitive. Instead of resorting to the quadratic penalty, we choose to smooth the exact penalty function, given its similarity to nondifferentiability in the CVaR risk measure. Using smoothing based on the function $\rho_\epsilon(\cdot)$ defined in (6.3.2), we approximate the penalty optimization problem (6.3.9) by the following smooth unconstrained minimization problem:

$$\begin{array}{ll} \min\limits_{\substack{\alpha\in\mathbb{R}, n_1^{(j)}, n_2^{(j)}, \ldots, n_N^{(j)} \\ n_k^{(j)} = f_k\left(P_{k-1}^{(j)}, x_{k-1}^{(j)}\right) \\ k=1,2,\ldots,N-1}} & -\dfrac{1}{M}\displaystyle\sum_{j=1}^{M}\sum_{k=1}^{N}(n_k^{(j)})^T\tilde{P}_k^{(j)} + \dfrac{\mu}{M(1-\beta)}\sum_{j=1}^{M}\rho_\epsilon\left(P_0^T\bar{S} - \sum_{k=1}^{N}(n_k^{(j)})^T\tilde{P}_k^{(j)} - \alpha\right) \\ & +\mu\alpha + \vartheta . \displaystyle\sum_{\ell=1}^{L}\sum_{j=1}^{M}\rho_\epsilon\left(a_\ell\left(n_1^{(j)}, \ldots, n_N^{(j)}\right)\right) \\ \text{s.t.} & \displaystyle\sum_{k=1}^{N} n_k^{(j)} = \bar{S}, \quad j = 1, 2, ..., M. \end{array} \tag{6.3.10}$$

Here we can regard the smoothed function $\rho_\epsilon(\cdot)$ as a new penalty function; it is a hybrid of the quadratic penalty function and the exact penalty function. Indeed this new penalty function can be regarded as an exact penalty function with a resolution determined by the parameter ϵ. This parameter ϵ can be different from that in the smoothed function for CVaR and it can vary with the constraints. We are currently investigating theoretical properties of this new penalty function.

The objective function of problem (6.3.10) is continuously differentiable but quite nonlinear due to smoothing of piecewise functions as well as the existence of the permanent price impact. Optimization methods for minimizing a continuously differentiable objective function typically require derivative calculations to achieve a good computational performance. In our subsequent computational investigation, we use the trust region method in (Coleman and Li, 1996) with the derivative evaluations using automatic differentiation; for further discussion on automatic differentiation we refer the reader to (Griewank and Corliss, 1991; Coleman and Verma, 2000; Nocedal and Wright, 2000) and references therein.

6.4 Computational Results

This section presents several computational examples to illustrate feasibility and efficacy of our proposed smoothing and parametric representation approach for approximating optimal stochastic execution strategies. In addition we assess performance of the computed stochastic execution strategy. The objective of our computational investigation is to demonstrate

- Accuracy of the computed execution strategies by comparing them to the strategies from analytic formulae when they exist;
- Capability of the proposed technique to handle inequality constraints;
- Applicability of the technique to alternative risk measures. This also allows us to study the effect of the choice of a risk measure on the optimal execution strategy.

Specifically, we approximate the optimal execution strategy by solving problem (6.3.8). We assume that the market price follows a jump diffusion process as in (5.2.6) with $\alpha_0^{\mathbf{m}} = 0$. In addition, we assume linear time-independent price impact functions as in (2.1.4).

We assume that the jump amplitudes are log-normally distributed and identically distributed over period, i.e., $\log \pi_\ell^{(i)}(k)$ and $\log \chi_\ell^{(i)}(k)$ have normal distributions for all i and k, with means $\mu_x^{\mathbf{m}}$ and $\mu_y^{\mathbf{m}}$, and standard deviations $\sigma_x^{\mathbf{m}}$ and $\sigma_y^{\mathbf{m}}$, respectively. We further assume that the arrival rates $\lambda_x^{(i)}$ and $\lambda_y^{(i)}$ of different assets in the portfolio are equal to λ_x and λ_y, respectively.

In summary, the execution price model and market price dynamics are as follows:

$$\tilde{P}_k = P_{k-1} - \frac{H}{\tau} n_k, \tag{6.4.1}$$

$$P_k = \text{Diag}(P_{k-1})(e_m + \tau^{1/2}\Sigma^{\mathbf{m}}\xi_k + \mathcal{J}^{\mathbf{m}}(k)) - G n_k. \tag{6.4.2}$$

Unless otherwise stated, our computation generates $M = 12,000$ sample paths of random variables $\{(\xi_1, \mathcal{J}_1), \cdots, (\xi_{N-1}, \mathcal{J}_{N-1})\}$. We use automatic differentiation in ADMAT: Automatic Differentiation Toolbox (Coleman and Verma, 2000) to compute gradients. The Hessian is then computed using the finite difference method.

The optimal execution strategy in general differs with the choice of the risk measure. For example, it can be shown that the variance of the execution cost, under our assumed model, does not depend on the impact matrices. However, CVaR of the execution cost depends on the impact matrix.

The proposed computational method can be applied to other downside risk measures such as *Semi-standard deviation*, see, e.g., (Fabozzi et al., 2007) page 59:

$$\begin{aligned}\Psi\left(P_0^T\bar{S} - \sum_{k=1}^{N} n_k^T\tilde{P}_k\right) &\stackrel{\text{def}}{=} \mathbf{E}\left(\left[\bar{S}^T P_0 - \sum_{k=1}^{N} n_k^T\tilde{P}_k\right]^+\right) \\ &\approx \frac{1}{M}\sum_{j=1}^{M}\rho_\epsilon\left(\bar{S}^T P_0 - \sum_{k=1}^{N}(n_k^{(j)})^T\tilde{P}_k^{(j)}\right). \qquad (6.4.3)\end{aligned}$$

To assess accuracy and effect of risk measures, we compare the following execution strategies:

- Strategy$_M$: strategy which minimizes the expected execution cost, i.e., $\mu = 0$ in problem (6.3.8).
- Strategy$_C$: strategy which minimizes CVaR$_{95\%}$, without considering the expected execution cost.
- Strategy$_S$: strategy which minimizes the variance (or standard deviation) of the execution cost.
- Strategy$_N$: the naive strategy, $n_k = \frac{\bar{S}}{N}, \quad k = 1, 2, \ldots, N$.
- Strategy$_D$: strategy which minimizes the semi-standard deviation risk measure (see equation (6.4.3)).

6.4.1 Accuracy of the Computational Approach

To illustrate accuracy of the proposed computational approach, we compare the computed execution strategy from (6.3.8) and its performance with the exact optimal execution strategy for minimizing the expected execution cost only and for minimizing variance of the execution cost only, since an analytic solution exists for both cases. Strategy$_S$ is obtained by solving problem (6.3.8) with the objective function replaced by variance of the execution cost:

$$\mathbf{Var}\left(P_0^T\bar{S} - \sum_{k=1}^{N} n_k^T\tilde{P}_k\right) \approx \frac{1}{M}\sum_{j=1}^{M}\left(\left(P_0^T\bar{S} - \sum_{k=1}^{N}(n_k^{(j)})^T\tilde{P}_k^{(j)}\right) - \left(P_0^T\bar{S} - \frac{1}{M}\sum_{j=1}^{M}\sum_{k=1}^{N}(n_k^{(j)})^T\tilde{P}_k^{(j)}\right)\right)^2.$$

Let Strategy$^*_M$ and Strategy$^*_S$ denote the exact strategies from the analytic formulae to minimize mean and variance of the execution cost, respectively.

Parameters	Values
Trading Horizon	$T = 5$ days
Number of Periods	$N = 5$
Interval Length	$\tau = T/N = 1$ day
Initial Portfolio Price	$P_0 = 50e_3$ \$/share
Initial Holdings	$\bar{S} = 10^6 e_3$ shares
CVaR Confidence Level	$\beta = 0.95$

Table 6.1: Parameter values used in the simulations.

We consider an execution problem for a portfolio of three assets with the parameter setting described in Table 6.1.

We assume that the daily asset return covariance matrix is as in (3.4.1). We further assume:

$$H = 0.5 \times 10^{-4} \cdot C, \qquad G = 0.5 \times 10^{-5} \cdot C, \quad \Sigma = (0.001 \cdot C)^{1/2}.$$

We let arrival rates and jump amplitudes be identical for the three assets:

$$\lambda_x = 0.5, \quad \mu_x^{\mathbf{m}} = 10^{-4}, \quad \sigma_x^{\mathbf{m}} = 10^{-3}, \quad \lambda_y = 2, \quad \mu_y^{\mathbf{m}} = 10^{-4}, \quad \sigma_y^{\mathbf{m}} = 10^{-3}.$$

When only variance of the execution cost is minimized, the exact optimal execution strategy is given in (6.2.7) for which the optimal objective value equals zero. Furthermore, when only the expected execution cost is minimized, an analytical formula for the optimal execution strategy obtained from the stochastic dynamic programming is provided in Chapter 5. We use these two cases as benchmarks to illustrate accuracy of the proposed technique.

Table 6.2 compares the expected execution cost, standard deviation, and CVaR of the computed execution strategies with those of the optimal execution strategies using explicit formulae. Comparing Strategy$_M$ with Strategy$^*_M$, we observe approximately five significant digits of accuracy in the expected execution cost and three significant digits in standard deviation. The variance of the Strategy$_S$ is about 10^{-3} compared to zero for Strategy$^*_S$; however the expected execution cost agrees in about 6 significant digits.

To examine the difference in the execution strategy, we quantify the percentage difference between the exact optimal execution strategy and the computed execution strategy using the following measure:

$$\varepsilon(i,k) \stackrel{\text{def}}{=} \left(100/\|\bar{S}\|_\infty\right) \times \max_{1\le j\le M} \left| n_*^{(k)}(i,j) - \hat{n}^{(k)}(i,j) \right|, \quad i = 1, \cdots, m, \quad k = 1, \cdots, N,$$

where, for asset i in simulation j, $n_*^{(k)}(i,j)$ and $\hat{n}^{(k)}(i,j)$ are the analytical solution and the computed solution at period k, respectively. Values of $\varepsilon(i,k)$ are reported in Table 6.3 for $M = 12,000$ simulations. The results indicate that the computed solutions are relatively

	Mean	Standard Deviation	CVaR
Strategy_S	1.0319544883×10^6	0.0338536247	1.0319549344×10^6
Strategy_S^*	1.0319545000×10^6	0	1.0319545000×10^6
Strategy_M	1.9618159824×10^5	3.2645688653×10^5	8.6482124208×10^5
Strategy_M^*	1.9618206384×10^5	3.2634725096×10^5	8.6451163525×10^5

Table 6.2: Mean, $\text{CVaR}_{95\%}$, and Standard deviation of the execution cost corresponding to each strategy.

Percentage Difference $\varepsilon(i,k)$ Corresponding to Strategy_M					
Asset	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$
1	0.0215	1.4905	0.3609	-0.0056	-0.9441
2	-0.2181	-0.7944	1.4332	0.0572	0.2588
3	-0.0183	-0.7452	-0.0056	0.3408	0.6913
Percentage Difference $\varepsilon(i,k)$ Corresponding to Strategy_S					
Asset	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$
1	1.9109×10^{-6}	2.8269×10^{-4}	3.3402×10^{-4}	1.6659×10^{-4}	3.4062×10^{-4}
2	2.3535×10^{-6}	3.1935×10^{-4}	1.2011×10^{-4}	2.7644×10^{-4}	2.7478×10^{-4}
3	-2.5138×10^{-6}	2.5517×10^{-4}	2.0356×10^{-4}	1.7203×10^{-4}	2.7208×10^{-4}

Table 6.3: Comparisons to benchmark strategies Strategy_M^* and Strategy_S^*.

close to the exact ones, and the maximum difference between them is at most 1.5% which most likely comes from computational errors.

For minimizing CVaR, there is no analytic solution. Table 6.4 presents mean and $\text{CVaR}_{95\%}$ of the execution cost corresponding to the computed solution of problem (6.3.8) for different choices of μ. Even though we cannot explicitly assess the accuracy in this case, we do observe that, for the computed strategy, the expected execution cost increases while $\text{CVaR}_{95\%}$ decreases, when the risk aversion parameter μ increases.

μ	CVaR(95%)	Expected Execution Cost
0	0.86482	0.19620
1	0.77781	0.20439
10	0.77485	0.20505
$+\infty$	0.77464	0.20512

Table 6.4: Mean and $\text{CVaR}_{95\%}$ of the execution cost in dollars per share.

Improvements in the objective function value by the optimization solver over iterations are presented in Figure 6.1. These plots demonstrate that for the portfolio example of three assets considered, around 40 to 50 iterations in the optimization solver are enough to obtain a near optimal solution. The computational times are reported below each graph. The computations were done in MATLAB 7.9.0 on a Pentium 3.00GHz running Windows XP with 0.99 GB RAM.

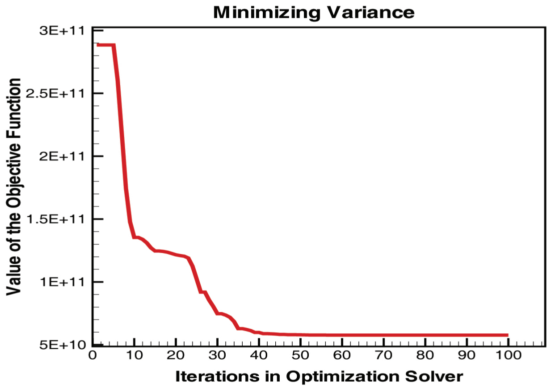
6.4.2 Handling Constraints

We now illustrate effectiveness of the smoothed penalty function to handle constraints. We also investigate effect of constraints $n_k \geq 0$ on the computed optimal execution strategy and the corresponding objective function value. We consider liquidation of $\bar{S} = 10^6$ shares of a single asset whose initial market price is $P_0 = 50$ dollars per share. Permanent and temporary price impact values are assumed to be $G = 2.5 \times 10^{-7}$ and $H = 2.5 \times 10^{-6}$, respectively, and $\Sigma = 0.009$. Jump parameters are as follows:

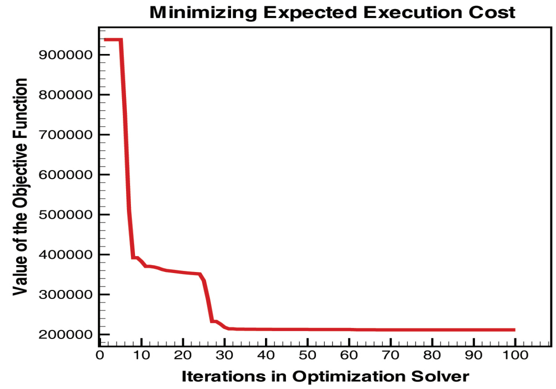
$$\lambda_x = 3, \quad \mu_x = 9.5 \times 10^{-3}, \quad \sigma_x = 10^{-2}, \quad \lambda_y = 0.5, \quad \mu_y = 6.9 \times 10^{-4}, \quad \sigma_y = 3.2 \times 10^{-2}.$$

CVaR and mean of the execution costs corresponding to the optimal execution strategies with and without the constraint $n_k \geq 0$ are presented in Table 6.5. The risk aversion parameter is $\mu = 100$.

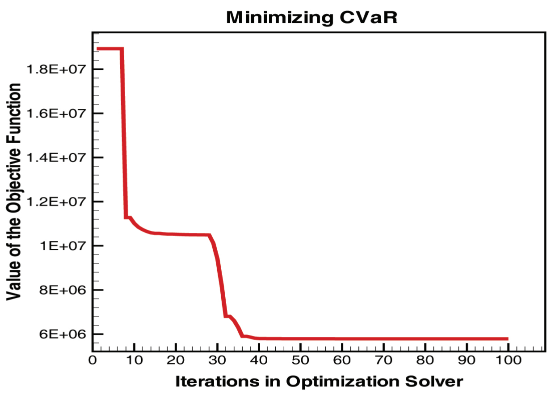
Figure 6.2 depicts the optimal execution strategy for minimizing mean and CVaR of the execution cost with the risk aversion parameter $\mu = 100$ in the presence of the non-negativity constraints $n_k \geq 0$. These plots show that the computed optimal execution strategy using



25 mins 44.10 seconds



25 mins 51.38 seconds



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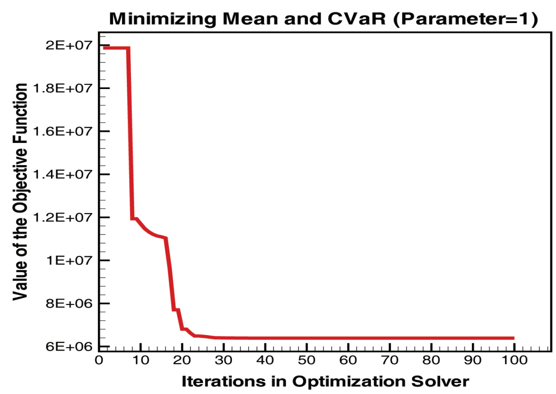
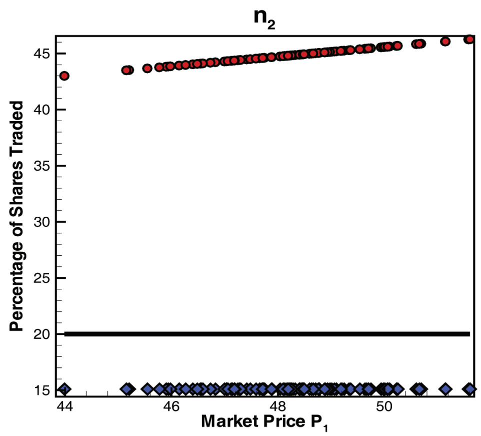


60 mins 24.23 seconds

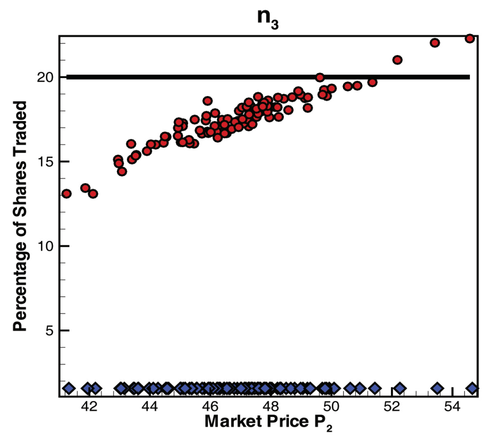
Figure 6.1: Progress of the optimization solver over iterations.

Execution Strategies	$\text{CVaR}_{95\%}$	Expected Execution Cost
n_k unconstrained	3.37896	1.42706
$n_k \geq 0$	5.40572	2.51237

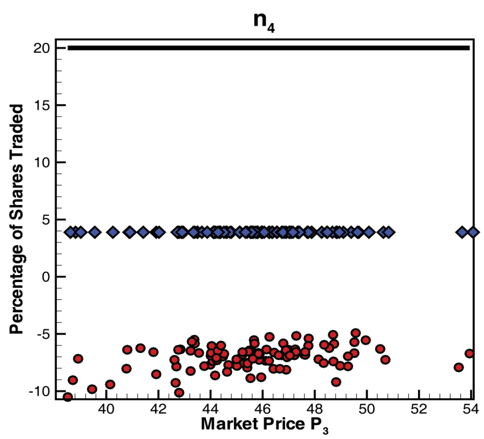
Table 6.5: Effect of constraints $n_k \geq 0$: cost and risk values in dollars per share



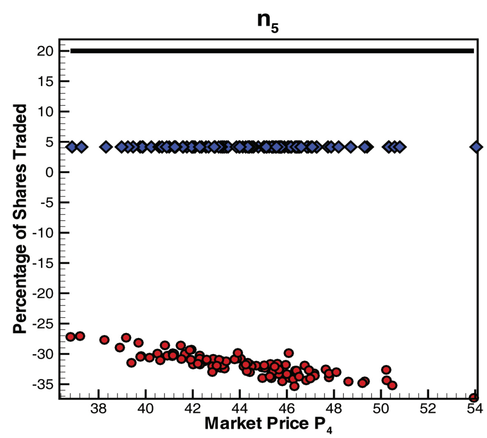
(a) Strategy at the second period



(b) Strategy at the third period



(c) Strategy at the fourth period



(d) Strategy at the fifth period

$\lambda_x = 3, \quad \mu_x^{\mathbf{m}} = 9.5 \times 10^{-3}, \quad \sigma_x^{\mathbf{m}} = 10^{-2}, \quad \lambda_y = 0.5, \quad \mu_y^{\mathbf{m}} = 6.9 \times 10^{-4}, \quad \sigma_y^{\mathbf{m}} = 3.2 \times 10^{-2}$

$\bar{S} = 10^6, \quad P_0 = 50, \quad H = 2.5 \times 10^{-6}, \quad G = 2.5 \times 10^{-7}, \quad \Sigma = 0.009.$

Figure 6.2: The 100 realizations of the computed optimal execution strategies as functions of the market price for a single asset trading with and without non-negativity constraints. A circle shows an execution strategy when buying is allowed, and a diamond represents a strategy when buying is prohibited. The line in each graph indicates Strategy$_N$. Strategies have been computed using the penalty parameter $\vartheta = 10^4$ and the risk aversion parameter $\mu = 100$. In the first period when buying is allowed, $n_1^* = 76.64867\%$ of the initial holding and when buying is prohibited, $n_1^* = 75.28936\%$ of the initial holding.

the penalty parameter $\vartheta = 10^4$ indeed satisfies $n_k \geq 0$. In particular, while the execution strategy when n_k is not bound constrained suggests to sell more in the first period and buy in the last periods ($k = 4, 5$); the execution strategy computed under $n_k \geq 0$ is more conservative and the strategy does not seem to vary with the asset price significantly.

6.4.3 Applicability to Other Risk Measures

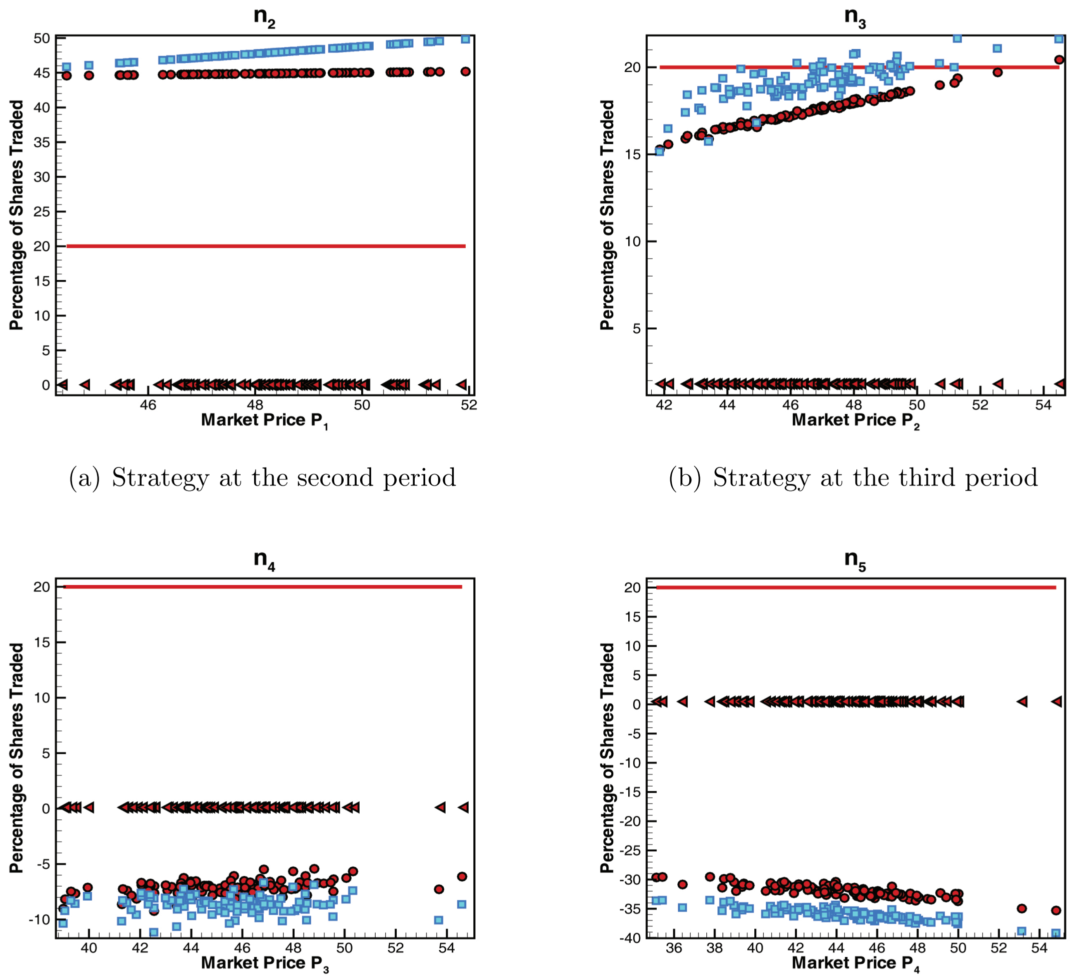
Here we illustrate use of the proposed approach for the semi-standard deviation risk measure when trading a single asset. The setting is as in Section 6.4.2.

Figure 6.3 demonstrates that Strategy$_D$ is very similar to Strategy$^*_S$. Furthermore, Strategy$_M$ is more aggressive comparing to Strategy$_C$, i.e., it suggests to trade more in the first periods and buy in the last periods.

It is worth mentioning that the results provided in this section depend on our assumed linear parametric representation in equation (6.3.7). If we choose other representations, the configuration of the computed optimal execution strategies might differ. We leave investigating properties of the solutions under different parametric representations for the execution strategy for future research.

6.5 Concluding Remarks

In this chapter, we use smoothing and parametric rules to attain a stochastic price-dependent execution strategy when a nonlinear risk measure is included in the objective function for the optimal portfolio execution problem. This formulation yields the exact solution of the stochastic dynamic programming when only expected cost is minimized. Our computational approach can handle multi-asset cases as well as constraints with no prominent additional computational cost.



(a) Strategy at the second period

(b) Strategy at the third period

(c) Strategy at the fourth period

(d) Strategy at the fifth period

$\lambda_x = 3, \quad \mu_x^{\mathbf{m}} = 9.5 \times 10^{-3}, \quad \sigma_x^{\mathbf{m}} = 10^{-2}, \quad \lambda_y = 0.5, \quad \mu_y^{\mathbf{m}} = 6.9 \times 10^{-4}, \quad \sigma_y^{\mathbf{m}} = 3.2 \times 10^{-2},$
$\bar{S} = 10^6, \quad P_0 = 50, \quad H = 2.5 \times 10^{-6}, \quad G = 2.5 \times 10^{-7}, \quad \Sigma = 0.009.$

Figure 6.3: The 100 realizations of the optimal execution strategies Strategy$_M$ (squares), Strategy$_C$ (circles), and Strategy$_D$ (triangles) as functions of market price for a single asset trading when no constraint is imposed. The line in each graph indicates the naive strategy Strategy$_N$. In the first period, Strategy$_N$ suggests to sell $n_1^* = 20.00\%$, Strategy$_M$ suggests to sell $n_1^* = 77.44418\%$, and Strategy$_C$ suggests to sell $n_1^* = 76.63369\%$, and Strategy$_D$ suggests to sell $n_1^* = 97.57436\%$ of the total holding.

Chapter 7

Conclusions and Future Work

Given the market price dynamics, the execution price model, and the price impact model, an optimal execution strategy minimizes the mean and risk of the execution cost when liquidating large blocks of assets over a given execution horizon. In this thesis, we analyze sensitivity of the optimal portfolio execution strategies to model parameters as well as models themselves, and we propose robust strategies.

We first analyze sensitivity of the optimal execution strategy and the efficient frontier to errors in estimating the price impact parameters. Here, the execution risk is measured using the variance and the optimal execution strategy is assumed to be deterministic. Furthermore, it is assumed that the market price has a normal distribution. We identify some cases in which the optimal execution strategy is insensitive to the estimation errors in the price impact parameters. Specifically, the optimal execution strategy which minimizes the expected execution cost is the naive execution strategy as long as the permanent impact matrix and its perturbation are symmetric, and the corresponding combined impact matrices are positive definite. We provide an upper bound on the size of change in the optimal execution strategy. Our theoretical and computational results indicate that the optimal execution strategy may potentially be very sensitive to estimation errors in the price impact parameters. This is particularly the case if the permanent impact matrix is asymmetric, the risk aversion parameter is small, and no constraint is imposed on a solution.

Motivated by the sensitivity analysis, we consider the robust optimization to address uncertainty in the impact matrices. Potential instability of the classical robust optimization to variation in the uncertainty set is illustrated through an example. To achieve greater stability, the proposed approach imposes a regularization constraint on the uncertainty set before being used in the minimax optimization formulation. Improvement in the stability of the robust solution is discussed both theoretically and experimentally. Some implications of the regularization on the robust solution and the mean-variance efficient frontier are investigated.

The optimal execution strategy should be stochastic to adapt to market conditions. We further investigate implications of market price model assumption on the stochastic optimal execution strategy. In particular, we suggest using jump diffusion processes for the market price dynamics to capture uncertain permanent price impact of other large trades. The proposed model includes two compound Poisson processes corresponding to buy and sell trades, respectively. Using stochastic dynamic programming, we provide analytical solutions for minimizing the expected execution cost under discrete jump diffusion models and compare them with the optimal execution strategies obtained from a continuous diffusion process.

A jump diffusion model for the market price dynamics suggests the use of Conditional Value-at-Risk (CVaR) as the risk measure. However, solving multi-stage stochastic programming problem is a daunting task, particularly when there are constraints. Under both temporary and permanent price impact, the objective function of the optimal portfolio execution problem can be quite nonlinear when a risk measure for the execution cost is included. We propose a tractable computational approach to compute an optimal execution strategy. The approach employs Monte Carlo simulations, a smoothing technique, and parametric rules for the optimal strategy. The smoothing technique alleviates the problem associated with the nondifferentiability arising from the CVaR risk measure for each simulation. The parametric rule allows a strategy to be stochastic and reduces the number of optimization variables at the same time. In particular, a linear parametric representation permits the exact representation of the execution strategy for minimizing the expected cost. The approach then yields a stochastic execution strategy which depends on the price and holdings at trading time. The computational complexity of the resulting method does not depend on the number of simulations. While we focus on CVaR risk measure, the proposed computational method is applicable to different risk measures, e.g., downside risk as well as variance. In addition, a smoothed exact penalty function is applied to handle stochastic constraints.

Some directions for future research are:

- It may be desirable to extend the idea of regularization to other robust formulations, especially for robust quadratic programming.
- Another interesting issue to look into in regularized robust optimization is how to reduce dimensionality of the semidefinite programming representation. One approach can be to replace the regularization constraint with $\tilde{\Theta} \succeq \rho I$.
- In this thesis, we do not address how to estimate the parameters of the proposed jump diffusion model. It will be interesting to investigate techniques to estimate the parameters.
- Empirical performance assessment of the proposed jump diffusion model using real world data suggests another future research direction.
- It may be interesting to study some properties of the resulting nonlinear differentiable optimization problem, in terms of convexity or uniqueness of the solution. It may be helpful to employ the specific structure of the problem in the optimization solver to speed up the code. We also would like to extend our method to solve other similar dynamic stochastic programming problems in finance.

APPENDICES

Appendix A

Mathematical Preliminaries

A.1 Linear Algebra and Optimization

The matrix p-norm of the real matrix $A \in \mathbb{R}^{m\times n}$ is the matrix norm induced by the vector p-norm:

$$\|A\|_p = \sup_{x\neq 0} \frac{\|Ax\|_p}{\|x\|_p}. \tag{A.1.1}$$

For $p = 1$ and ∞, we have

$$\|A\|_1 = \max_{1\le j\le n} \sum_{i=1}^{m} |a_{ij}|,$$
$$\|A\|_\infty = \max_{1\le i\le m} \sum_{j=1}^{n} |a_{ij}|.$$

The p-norms satisfy certain inequalities that are frequently used:

$$\frac{1}{\sqrt{n}}\|A\|_\infty \le \|A\|_2 \le \sqrt{m}\|A\|_\infty,$$
$$\frac{1}{\sqrt{m}}\|A\|_1 \le \|A\|_2 \le \sqrt{n}\|A\|_1,$$
$$\|A\|_2 \le \sqrt{\|A\|_1\|A\|_\infty}.$$

A symmetric matrix A is said to be *positive semidefinite*, denoted by $A \succeq 0$, if $y^TAy \ge 0$ for every vector y of the appropriate dimension. Equivalently, all eigenvalues of A are non-negative. If $y^TAy > 0$ for every nonzero vector y, then A is said to be *positive definite*, $A \succ 0$. Equivalently, all eigenvalues of A are positive. If A has some positive and some negative eigenvalues, then A is said to be an *indefinite matrix*. The sum of positive semidefinite (respectively positive definite) matrices is also positive semidefinite (positive definite). For a matrix $A \in \mathbb{R}^{m\times n}$, the matrix A^TA is symmetric and is also positive semidefinite since $y^T(A^TA)y = (Ay)^T(Ay) = \|Ay\|_2^2 \geq 0$, for every $y \in \mathbb{R}^n$.

Let $\lambda_1(A), \lambda_2(A), \cdots$ denote the distinct eigenvalues of a symmetric matrix A. We usually order them

$$\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A).$$

Let λ be an eigenvalue of A^TA. By definition, $A^TAx = \lambda x$ for some eigenvector x, so $\|Ax\|_2^2 = x^TA^TAx = x^T\lambda x = \lambda\|x\|_2^2$. It follows from (A.1.1) with $p = 2$ that

$$\|A\|_2^2 = \max_i \ \lambda_i\left(A^TA\right).$$

Therefore, when A is symmetric, the matrix 2-norm equals:

$$\|A\|_2 = \max_i |\lambda_i(A)|.$$

We denote the minimum and maximum eigenvalues of a matrix A having all real eigenvalues by $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$, respectively. The next theorem presents well-known cases of the *Courant-Fischer-Weyl Theorem*, see, e.g., Theorem 8.1.5 in (Golub and Loan, 1996):

Theorem A.1.1. *Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times n}$ be symmetric. Then*

$$\lambda_{\min}(A) = \min_{y\in\mathbb{R}^n\setminus\{0\}} \frac{y^TAy}{y^Ty}, \qquad \lambda_{\max}(A) = \max_{y\in\mathbb{R}^n\setminus\{0\}} \frac{y^TAy}{y^Ty}.$$

Furthermore,

$$\begin{aligned}\lambda_{\max}(A) + \lambda_{\min}(B) &\leq \lambda_{\max}(A+B) \leq \lambda_{\max}(A) + \lambda_{\max}(B),\\ \lambda_{\min}(A) + \lambda_{\min}(B) &\leq \lambda_{\min}(A+B) \leq \lambda_{\min}(A) + \lambda_{\max}(B).\end{aligned}$$

Suppose now that A is nonsingular and has all real eigenvalues. Then its eigenvalues are nonzero, so $Ax = \lambda x$ implies $\lambda^{-1}x = A^{-1}x$. Hence

$$\lambda_{\max}(A^{-1}) = 1/\lambda_{\min}(A).$$

The following result, well-known as the *Schur Complement*, is used in the thesis:

Lemma A.1.1. *Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{m\times m}$ be symmetric. Further assume that B is positive definite. Then*

$$M \stackrel{\text{def}}{=} \begin{pmatrix} B & U^T \\ U & A \end{pmatrix} \succeq 0 \Longleftrightarrow A - UB^{-1}U^T \succeq 0. \tag{A.1.2}$$

Moreover, $M \succ 0$ if and only if $A - UB^{-1}U^T \succ 0$.

The *Kronecker product* of two matrices $A_{m\times n}$ and $B_{p\times q}$, denoted by $A\otimes B$, is the $mp\times nq$ block matrix

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}. \tag{A.1.3}$$

For the properties of the Kronecker product, a reader is referred to §4.5.5 in (Golub and Loan, 1996). In particular,

$$\begin{aligned} &(A\otimes B)(C\otimes D) = AC\otimes BD, \\ &(A\otimes B)^{-1} = A^{-1}\otimes B^{-1}, \\ &(A\otimes B)^{T} = A^{T}\otimes B^{T}. \end{aligned}$$

Furthermore, let A and B be two square matrices of size n and m, respectively. Denote the eigenvalues of A and B by $\alpha_1, \cdots, \alpha_n$ and $\beta_1, \cdots, \beta_m$, listed according to multiplicity. Then the eigenvalues of $A\otimes B$ are $\alpha_i\beta_j$ for $i = 1, \cdots, n$ and $j = 1, \cdots, m$ (see, e.g., Property IX in section 2.3 of (Graham, 1981)).

For two matrices A and B of the same dimensions, the *Hadamard product*, also known as the *entrywise product*, denoted by $A\circ B$, is a matrix of the same dimension with elements given by

$$(A\circ B)_{ij} = A_{ij}B_{ij}.$$

The interior of a set $S\subseteq \mathbb{R}^n$, denoted by $\mathrm{int}(S)$, is given by

$$\mathrm{int}(S) = \{x\in S|(x+\epsilon B)\subseteq S \text{ for some } \epsilon > 0\},$$

where $B = \{x\in\mathbb{R}^n : \|x\|\leq 1\}$ for some vector norm $\|\cdot\|$ on $\mathbb{R}^n$. If $\mathrm{int}(S) = S$, then S is said to be *open*.

The closure of $S\subseteq\mathbb{R}^n$, denoted by $cl(S)$, is the set of all points $x\in\mathbb{R}^n$ where there exists a sequence of points in S converging to x. If $cl(S) = S$, then S is said to be *closed*.

A subset S of a metric space $\mathcal{X}$ is said to be *compact* if for every collection $\{\mathcal{R}_\alpha\}_\alpha$ of open subsets of $\mathcal{X}$ such that $S\subseteq\cup_\alpha\mathcal{R}_\alpha$, there exists a finite number of $\mathcal{R}_{\alpha_i}$, $i = 1, 2, \cdots, M$ such that $S\subseteq\cup_{i=1}^{M}\mathcal{R}_{\alpha_i}$. Compact subsets of metric spaces are closed, and closed subsets of compact sets are compact. A set S is called *sequentially compact* if every sequence $\{x_k\}_k$ in S has a point c such that given an open neighborhood $\mathcal{N}$ of c, there exist infinitely many k such that $x_k\in\mathcal{N}$. For a subset S of a metric space $\mathcal{X}$, S is compact if and only if S is sequentially compact. A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

A useful tool in optimization on a compact feasible set is the *Weierstrass (Extreme Value) Theorem*, e.g., see (Bertsekas, 1996). It states that a continuous function on a closed and bounded subset of $\mathbb{R}^n$ attains its maximum and minimum values, each at least once.

The following theorem is a major result on which our discussion in Chapter 4 relies. A topological space $\mathcal{X}$ is called *first-countable* if for every point $x \in \mathcal{X}$ there exists a sequence $\{\mathcal{A}_k\}_k$ of open neighbourhoods of x such that for any open neighbourhood $\mathcal{B}$ of x there exists a $\mathcal{A}_k$ contained in $\mathcal{B}$. A Hausdorff space is a topological space in which distinct points have disjoint neighbourhoods.

Theorem A.1.2. *[Theorem 2.1 of (Fiacco, 1974)] Let $\mathcal{X}$ be a first-countable Hausdorff space and T is a sequentially compact nonempty subset of $\mathcal{X}$. Consider the following problems*

$$\min_{x \in \mathcal{R} \cap T} \quad f(x), \qquad\qquad \min_{x \in \mathcal{R}_k \cap T} \quad f_k(x),$$

where

- *f and f_k are real-valued functions defined on $\mathcal{X}$,*
- *the subsets $\mathcal{R} \subseteq \mathcal{X}$ and $\mathcal{R}_k \subseteq \mathcal{X}$ are closed for all k,*
- *$\mathcal{R} \cap T$ is nonempty,*
- *$\mathcal{R}_k \cap T \to \mathcal{R} \cap T$,*
- *$f_k(x)$ is continuous on an open set containing $\mathcal{R} \cap T$, for k large,*
- *$f_k \to f$ uniformly on some open set containing $\mathcal{R} \cap T$.*

Then for k large, there exists $x_k \in \mathcal{R}_k \cap T$ such that $f_k(x_k) = \min_{x \in \mathcal{R}_k \cap T} f_k(x)$. Furthermore, any (global) minimizing sequence $\{x_k\}$ contains at least one convergent subsequence and all cluster points are (global) minimizing points of $f(x)$ in $\mathcal{R} \cap T$.

When in Theorem A.1.2, $f_k = f$ for all k, the following corollary can be derived:

Corollary A.1.1. *Let $(\mathcal{X}, d)$ be a metric space and $f : \mathcal{X} \to \mathbb{R}$ be continuous on $\mathcal{X}$. Consider the following problem:*

$$Q(\mathcal{R}): \quad \max_{x \in \mathcal{R}} \ f(x),$$

where $\mathcal{R} \subseteq \mathcal{X}$ is nonempty and compact. Then for any sequence of nonempty compact subsets of $\mathcal{X}$, $\{\mathcal{R}_k\}_k$, with $\mathbf{Haus}_d(\mathcal{R}, \mathcal{R}_k) \to 0$, and for any maximizing sequence $\{x_k\}_k$ of problems $Q(\mathcal{R}_k)$, there exists at least one convergent subsequence and all cluster points of $\{x_k\}_k$ are maximizing points of problem $Q(\mathcal{R})$.

Proof. First note that every metric space is first-countable and Hausdorff. Since $\mathcal{R}$ is compact, it is a bounded subset of the metric space $(\mathcal{X}, d)$. Thus, it is contained in a ball of finite radius, i.e. there exists $x_0 \in \mathcal{X}$ and $M > 0$ such that $d(x_0, x) < M$, for all $x \in \mathcal{R}$.

Since $\mathbf{Haus}_d(\mathcal{R}, \mathcal{R}_k) \to 0$, there exists K_0 such that for every $k \geq K_0$, $\mathbf{Haus}_d(\mathcal{R}, \mathcal{R}_k) \leq \epsilon_0$, and consequently $\sup_{x \in \mathcal{R}_k} \inf_{y \in \mathcal{R}} d(x, y) \leq \epsilon_0$. Therefore for every $x \in \mathcal{R}_k$, there exists some $x^{(k)} \in \mathcal{R}$ such that $d(x, x^{(k)}) \leq \epsilon_0$. Hence, $d(x, x_0) \leq d(x, x^{(k)}) + d(x^{(k)}, x_0) \leq \epsilon_0 + M$. Thus x is in the ball $B_{M+\epsilon_0}(x_0) = \{x \in \mathcal{X} : d(x, x_0) \leq \epsilon_0 + M\}$. Consequently, $\mathcal{R}_k \subseteq B_{M+\epsilon_0}(x_0)$, for every $k \geq K_0$. Denote the closure of $B_{M+\epsilon_0}(x_0)$ by T. Thus T is a compact subset of $\mathcal{X}$ and $\mathcal{R}_k \subseteq T$, for all $k \geq K_0$. The result then follows from Theorem A.1.2 using the defined set T.

□

The following result is very useful to prove convexity of some functions. A proof follows from the fact that the intersection of a collection of convex sets is convex.

Theorem A.1.3 (Theorem 5.5 of (Rockafellar, 1996))**.** *The pointwise supremum of an arbitrary collection of convex functions is convex.*

A.2 Quadratic Programming Problem

Consider the quadratic function $f : \mathbb{R}^n \to \mathbb{R}$, given by

$$f(x) = \frac{1}{2} x^T Q x + b^T x,$$

with Q is a symmetric matrix and $b \in \mathbb{R}^n$. The function f is convex if and only if $Q \succeq 0$ (and concave if and only if $Q \preceq 0$). The function f is strictly convex if and only if $Q \succ 0$.

The following proposition provides a sufficient condition when infimum of a function $f(x)$ over $x \in \Omega$ is attained:

Proposition A.2.1. *[Proposition 2.5 of Dostál (2009)] Let f be a quadratic function defined on a nonempty closed convex set $\mathcal{S} \subseteq \mathbb{R}^n$. Then the following statements hold:*

1. *If f is strictly convex, then a global solution of $\min_{x \in \mathcal{S}} f(x)$ exists and is necessarily unique.*

2. *A global solution of $\min_{x \in \mathcal{S}} f(x)$ exists if and only if f is bounded from below on $\mathcal{S}$.*

A primal problem and its Lagrangian dual are linked through the Lagrangian function. Consider the following quadratic programming problem with linear constraints:

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & \frac{1}{2} x^T Q x + b^T x \\ \text{s.t.} \quad & Ax + c \leq 0. \end{aligned} \tag{A.2.1}$$

The Lagrangian function associated with the problem (A.2.1) is as below:

$$L(x, \lambda) = \frac{1}{2}x^T Qx + b^T x + \lambda^T (Ax + c),$$

when λ is the *Lagrange multiplier* associated with the inequality constraints. Equation (3.15) in (Hager, 1979) provides a bound on $\|\lambda\|_2$ in terms of Q, b, A, and c. This result is summarized in the following proposition:

Proposition A.2.2. *In problem* (A.2.1)*, let Q be symmetric positive definite, and the feasible set is nonempty. Let $\lambda_{\min}(A_J A_J^T) > 0$, when J is the set of indices of binding constraints at the solution of problem* (A.2.1)*. Then any optimal Lagrange multiplier λ is bounded by:*

$$\|\lambda\|_2 \leq \frac{2\lambda_{\max}(Q)}{\lambda_{\min}(Q) \cdot \sqrt{\lambda_{\min}(A_J A_J^T)}} \left(1 + \frac{\lambda_{\max}(Q) \cdot \|A_J^T\|_2}{\lambda_{\min}(A_J A_J^T)}\right) (\|b\|_2 + \|c\|_2)\,. \tag{A.2.2}$$

Below, we provide the proof from (Hager, 1979) for the convenience of the reader.

Proof. Positive definiteness of Q along with the assumption that the feasible region is nonempty implies that problem (A.2.1) has a unique solution x. Let J denote the indices corresponding to binding constraints for x. Let λ be an optimal Lagrange multiplier. The first order necessary condition for problem (A.2.1) can be expressed in the form:

$$\begin{pmatrix} Q & A_J^T \\ A_J & 0 \end{pmatrix} \begin{pmatrix} x \\ \lambda_J \end{pmatrix} = \begin{pmatrix} -b \\ -c_J \end{pmatrix}. \tag{A.2.3}$$

Express $x = x^p + x^\perp$, where $A_J x^p = 0$ and $x^\perp$ is perpendicular to the null space of A_J. Since $x^\perp$ lies in the range space of A_J^T, there exists y such that $x^\perp = A_J^T y$. From the second equation in (A.2.3), we have $A_J x = -c_J$. Thus

$$\|c\|_2 \geq \|c_J\|_2 = \|A_J x\|_2 = \|A_J x^\perp\|_2 = \|A_J A_J^T y\|_2 \geq \lambda_{\min}(A_J A_J^T)\|y\|_2 \geq \lambda_{\min}(A_J A_J^T)\frac{\|x^\perp\|_2}{\|A_J^T\|_2}.$$

Thus we have

$$\frac{\|A_J^T\|_2}{\lambda_{\min}(A_J A_J^T)}\|c\|_2 \geq \|x^\perp\|_2. \tag{A.2.4}$$

On the other hand, multiplying the first equation of (A.2.3) by $(x^p)^T$ from left gives us $(x^p)^T Qx + (x^p)^T A_J^T \lambda_J = -(x^p)^T b$. Note that $(x^p)^T A_J^T \lambda_J = 0$. Thus

$$\|x^p\|_2\|b\|_2 \geq \|(x^p)^T b\|_2 = \|(x^p)^T Q x^p + (x^p)^T Q x^\perp\|_2 \geq \lambda_{\min}(Q)\|x^p\|_2^2 - \|x^p\|_2\|Q\|_2\|x^\perp\|_2.$$

Dividing by $\|x^p\|_2$ and applying (A.2.4) in the above inequality yields:

$$\|b\|_2 \geq \lambda_{\min}(Q)\|x^p\|_2 - \|Q\|_2\|x^\perp\|_2 \geq \lambda_{\min}(Q)\|x^p\|_2 - \|Q\|_2\frac{\|A_J^T\|}{\lambda_{\min}(A_J A_J^T)}\|c\|_2.$$

Thus we have

$$\frac{1}{\lambda_{\min}(Q)}\left(\|b\|_2+\|Q\|_2\frac{\|A_J^T\|}{\lambda_{\min}(A_JA_J^T)}\|c\|_2\right)\geq \|x^p\|_2. \tag{A.2.5}$$

The first equality of (A.2.3) and the assumption $\lambda_{\min}(A_JA_J^T)>0$ yield

$$\sqrt{\lambda_{\min}(A_JA_J^T)}\|\lambda\|_2\leq \|A_J^T\lambda\|_2\leq \|b\|_2+\|Qx\|_2\leq \|b\|_2+\lambda_{\max}(Q)(\|x^p\|_2+\|x^\perp\|_2). \tag{A.2.6}$$

Applying inequalities (A.2.4) and (A.2.5) in inequality (A.2.6), we arrive at

$$\begin{aligned}\|\lambda\|_2 &\leq \frac{1}{\sqrt{\lambda_{\min}(A_JA_J^T)}}\left(\|b\|_2+\lambda_{\max}(Q)(\|x^p\|_2+\|x^\perp\|_2)\right)\\ &\leq \frac{1}{\sqrt{\lambda_{\min}(A_JA_J^T)}}\left(1+\frac{\lambda_{\max}(Q)}{\lambda_{\min}(Q)}\right)\left(\|b\|_2+\lambda_{\max}(Q)\frac{\|A_J^T\|_2}{\lambda_{\min}(A_JA_J^T)}\|c\|_2\right)\\ &\leq \frac{1}{\sqrt{\lambda_{\min}(A_JA_J^T)}}\left(2\frac{\lambda_{\max}(Q)}{\lambda_{\min}(Q)}\right)\left(1+\lambda_{\max}(Q)\frac{\|A_J^T\|_2}{\lambda_{\min}(A_JA_J^T)}\right)(\|b\|_2+\|c\|_2),\end{aligned}$$

where the last inequality comes from the facts that $\frac{\lambda_{\max}(Q)}{\lambda_{\min}(Q)}\geq 1$ and $\max\{\|b\|_2,\|c\|_2\}\leq \|b\|_2+\|c\|_2$. This completes the proof. □

A.3 Minimax Optimization Problem

Rockafellar (1996) defines a saddle-point as below:

Definition A.3.1. *[page 380 of Rockafellar (1996)] A point $(\bar{u},\bar{v})$ is a saddle-point of K with respect to maximizing over C and minimizing over D if $(\bar{u},\bar{v})\in C\times D$ and*

$$K(u,\bar{v})\leq K(\bar{u},\bar{v})\leq K(\bar{u},v),\qquad \forall u\in C,\quad \forall v\in D.$$

This means that the function $K(\bar{u},\cdot)$ attains its infimum over D at $\bar{v}$, while $K(\cdot,\bar{v})$ attains its supremum over C at $\bar{u}$. The relationship between saddle-points and saddle-values is as follows:

Theorem A.3.1 (Lemma 36.2 of Rockafellar (1996))**.** *Let K be any function from a non-empty product set $C\times D$ to $[-\infty,\infty]$. A point $(\bar{u},\bar{v})$ is a saddle-point of K (with respect to maximizing over C and minimizing over D) if and only if the supremum in the expression $\sup_{u\in C}\inf_{v\in D}\ K(u,v)$ is attained at $\bar{u}$, the infimum in the expression $\inf_{v\in D}\sup_{u\in C}\ K(u,v)$ is attained at $\bar{v}$, and these two extrema are equal. If $(\bar{u},\bar{v})$ is a saddle-point, the saddle-value of K is $K(\bar{u},\bar{v})$.*

The following theorem presents some sufficient condition when the infimum and supremum in the minimax problem can switch their order:

Theorem A.3.2. *[Theorem 3 in (Simons, 1995)] Let X be a compact convex subset of a linear topological space, Y be a convex subset of a linear topological space, and $f : X \times Y \to \mathbb{R}$ be upper semicontinuous on X and lower semicontinuous on Y. Suppose that, for all $y \in Y$ and $\lambda \in \mathbb{R}$, $\{x \in X : f(x,y) \geq \lambda\}$ is convex and for all $x \in X$ and $\lambda \in \mathbb{R}$, the set $\{y \in Y : f(x,y) \leq \lambda\}$ is convex. Then*

$$\inf_{y \in Y} \max_{x \in X} f(x,y) = \max_{x \in X} \inf_{y \in Y} f(x,y).$$

Appendix B

Proofs

B.1 Effect of a Zero Expected Market Price Change

In this section, we present the proof of Theorem 5.3.1.

Proof. We first prove by induction on k that, when $\mathbf{E}\left(\mathcal{F}_{k-1}\left(P_{k-1}\right)|P_{k-1}\right) = P_{k-1}$ holds and (deterministic) price impact functions are independent of the market prices, optimal execution n_k^* does not depend on P_{k-1}, and for $k = 1, 2, \ldots, N$, the optimal-value function is given by

$$V_k^*(P_{k-1}, x_{k-1}) = P_0^T \bar{S} - x_{k-1}^T P_{k-1} + R_{k-1}(x_{k-1}), \tag{B.1.1}$$

where $R_{k-1}(\cdot)$ is a deterministic function independent of P_{k-1}.

For $k = N$, optimal execution n_N^* equals x_{N-1} and from equation (5.3.2) the optimal-value function in the last period becomes

$$V_N^*(P_{N-1}, x_{N-1}) = P_0^T \bar{S} - x_{N-1}^T P_{N-1} + x_{N-1}^T h\left(\frac{x_{N-1}}{\tau}\right).$$

This confirms the correctness of (B.1.1) for $k = N$ with $R_{N-1}(x_{N-1}) = x_{N-1}^T h\left(\frac{x_{N-1}}{\tau}\right)$. Assume that in the period $(k+1)$, optimal execution n_{k+1}^* only depends on x_k and the optimal-value function at time period $k+1$ is

$$V_{k+1}^*(P_k, x_k) = P_0^T \bar{S} - x_k^T P_k + R_k(x_k),$$

where $R_k(x_k)$ does not depend on P_k. The Bellman's principle of optimality in the kth step yields:

$$\begin{aligned} V_k^*(P_{k-1}, x_{k-1}) &= \min_{n_k \in \mathbb{R}^m} \mathbf{E}\left[-n_k^T \tilde{P}_k + V_{k+1}^*(P_k, x_k) \mid P_{k-1},\ x_{k-1}\right] \\ &= \min_{n_k \in \mathbb{R}^m} \mathbf{E}\left[-n_k^T \left(P_{k-1} - h\left(\frac{n_k}{\tau}\right)\right) + P_0^T \bar{S} - x_k^T P_k + R_k(x_k) \mid P_{k-1},\ x_{k-1}\right]. \end{aligned}$$

Applying the market price dynamics (2.1.3), equation (2.1.1) and assumption (5.3.4), the optimal-value function $V_k^*(P_{k-1}, x_{k-1})$ becomes

$$\begin{aligned}
&\min_{n_k \in \mathbb{R}^m} \ \mathbf{E}[-n_k^T \left(P_{k-1} - h\left(\frac{n_k}{\tau}\right)\right) + P_0^T \bar{S} - (x_{k-1} - n_k)^T \left(\mathcal{F}_{k-1}(P_{k-1}) - \tau g\left(\frac{n_k}{\tau}\right)\right) \\
&\qquad\qquad + R_k(x_{k-1} - n_k) \mid P_{k-1}, x_{k-1}] \\
&= \min_{n_k \in \mathbb{R}^m} \ \left(-n_k^T \left(P_{k-1} - h\left(\frac{n_k}{\tau}\right)\right) + P_0^T \bar{S} - (x_{k-1} - n_k)^T \left(P_{k-1} - \tau g\left(\frac{n_k}{\tau}\right)\right) + R_k(x_{k-1} - n_k)\right) \\
&= P_0^T \bar{S} - x_{k-1}^T P_{k-1} + \min_{n_k \in \mathbb{R}^m} \ \left(n_k^T h\left(\frac{n_k}{\tau}\right) + (x_{k-1} - n_k)^T \tau g\left(\frac{n_k}{\tau}\right) + R_k(x_{k-1} - n_k)\right). \qquad \text{(B.1.2)}
\end{aligned}$$

The objective function of the minimization problem in (B.1.2) does not depend on P_{k-1} and is only in terms of x_{k-1} and specifications of the price impact functions $h(\cdot)$ and $g(\cdot)$. Hence, optimal execution n_k^* does not depend on P_{k-1} and consequently is static. Moreover, the optimal objective value of the minimization problem in (B.1.2) becomes

$$V_k^*(P_{k-1}, x_{k-1}) = P_0^T \bar{S} - x_{k-1}^T P_{k-1} + R_{k-1}(x_{k-1}),$$

where

$$R_{k-1}(x_{k-1}) = \min_{n_k \in \mathbb{R}^m} \ \left(n_k^T h\left(\frac{n_k}{\tau}\right) + (x_{k-1} - n_k)^T \tau g\left(\frac{n_k}{\tau}\right) + R_k(x_{k-1} - n_k)\right).$$

This proves the correctness of equation (B.1.1) for k. Thus, for $k = 1, 2, \ldots, N$, the optimal-value function is as in equation (B.1.1), and the optimal execution n_k^* is independent of P_{k-1} and consequently is static.

Now, let the price impact functions be given by (2.1.4) where the permanent impact matrix G is symmetric and the matrix Θ is positive definite. By induction on k, we prove that for $k = 1, 2, \ldots, N$,

$$\begin{aligned}
n_k^* &= \frac{1}{N-k+1} x_{k-1}, \qquad \text{(B.1.3)} \\
V_k^*(P_{k-1}, x_{k-1}) &= P_0^T \bar{S} - P_{k-1}^T x_{k-1} + \frac{1}{2} x_{k-1}^T \left(\frac{\Theta}{N-k+1} + G\right) x_{k-1}.
\end{aligned}$$

From (5.3.3), the optimal execution n_N^* equals x_{N-1}, and the optimal-value function becomes

$$\begin{aligned}
V_N^*(P_{N-1}, x_{N-1}) &= P_0^T \bar{S} - x_{N-1}^T P_{N-1} + \frac{1}{2} x_{N-1}^T \frac{H + H^T}{\tau} x_{N-1} \\
&= P_0^T \bar{S} - x_{N-1}^T P_{N-1} + \frac{1}{2} x_{N-1}^T (\Theta + G) x_{N-1}.
\end{aligned}$$

This confirms the correctness of (B.1.3) for $k = N$. Now assume (B.1.3) is true for $k+1$. Therefore,

$$V_{k+1}^*(P_k, x_k) = P_0^T \bar{S} - P_k^T x_k + \frac{1}{2} x_k^T \left(\frac{\Theta}{N-k} + G\right) x_k.$$

Using this assumption, we show that (B.1.3) is true for k. The Bellman's principle of optimality in the kth step becomes:

$$
\begin{aligned}
&V_k^*(P_{k-1}, x_{k-1}) = \min_{n_k \in \mathbb{R}^m} \mathbf{E}\left[-n_k^T \tilde{P}_k + V_{k+1}^*(P_k, x_k) \mid P_{k-1},\ x_{k-1}\right] \\
&= \min_{n_k \in \mathbb{R}^m} \mathbf{E}\left[-n_k^T P_{k-1} + n_k^T \frac{H}{\tau} n_k + \left(P_0^T \bar{S} - P_k^T x_k + \frac{1}{2} x_k^T \left(\frac{\Theta}{N-k} + G\right) x_k\right) \mid P_{k-1},\ x_{k-1}\right] \\
&= \min_{n_k \in \mathbb{R}^m} \left(-n_k^T P_{k-1} + n_k^T \frac{H}{\tau} n_k + P_0^T \bar{S} - \mathbf{E}\left[P_k^T x_k | P_{k-1}, x_{k-1}\right] + (x_{k-1} - n_k)^T \frac{A}{2} (x_{k-1} - n_k)\right),
\end{aligned}
$$

where $A = \frac{\Theta}{N-k} + G$. Applying the market price dynamics (2.1.3) and equation (2.1.1), the expected value in the above statement can be stated in terms of P_{k-1} and x_{k-1}:

$$
\begin{aligned}
\mathbf{E}\left[P_k^T x_k | P_{k-1}, x_{k-1}\right] &= \mathbf{E}\left[(\mathcal{F}_{k-1}(P_{k-1}) - G n_k)^T (x_{k-1} - n_k) \mid P_{k-1},\ x_{k-1}\right] \\
&= (P_{k-1} - G n_k)^T (x_{k-1} - n_k),
\end{aligned}
$$

where the last equality comes from the assumption $\mathbf{E}\left[\mathcal{F}_{k-1}(P_{k-1}) \mid P_{k-1},\ x_{k-1}\right] = P_{k-1}$. Hence, after some algebraic manipulation, the optimal-value function $V_k^*(P_{k-1}, x_{k-1})$ equals

$$
P_0^T \bar{S} - P_{k-1}^T x_{k-1} + \frac{1}{2} x_{k-1}^T A x_{k-1} + \min_{n_k \in \mathbb{R}^m} \left(\frac{1}{2} n_k^T \left(\frac{N-k+1}{N-k} \Theta\right) n_k - \left(\frac{\Theta}{N-k} x_{k-1}\right)^T n_k\right).
$$

When the matrix $\Theta + \Theta^T = 2\Theta$ is positive definite, the unique solution of the above minimization problem becomes

$$
n_k^* = \frac{1}{(N-k+1)} x_{k-1}. \tag{B.1.4}
$$

Therefore, the optimal value function $V_k^*(P_{k-1}, x_{k-1})$ equals

$$
V_k^*(P_{k-1}, x_{k-1}) = P_0^T \bar{S} - P_{k-1}^T x_{k-1} + \frac{1}{2} x_{k-1}^T \left(\frac{\Theta}{N-k+1} + G\right) x_{k-1}.
$$

This completes the induction. Using equation (2.1.1) and $x_0 = \bar{S}$, it can be shown that n_k^* obtained in (B.1.4) equals $\frac{\bar{S}}{N}$, which is the naive strategy.

□

B.2 Optimal Execution Strategy Under Additive Jump Market Price Models

Below, we provide a proof for Theorem 5.3.2.

Proof. We prove by induction on k that the optimal execution and the optimal-value function are given by:

$$V_k^*(P_{k-1}, x_{k-1}) = P_0^T \bar{S} - \frac{1}{2} x_{k-1}^T \left(\Theta^T - A_k - G\right) x_{k-1} - \left(P_{k-1} + b_k - \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k) - \tau \alpha_0^{\mathbf{a}}\right)^T x_{k-1} - c_k,$$
$$n_{k-1}^* = A_k^{-1} \left(b_k - \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k) + \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k-1) + \Theta^T - A_k^T x_{k-2}\right), \quad k = 2, 3, \ldots, N, \tag{B.2.1}$$

where A_k, b_k and c_k are defined as in equations (5.3.5) and (5.3.6), and the matrix A_k is symmetric.

For $k = N$, optimal execution n_N^* equals x_{N-1}. From equation (5.3.3), the optimal-value function in the last period becomes

$$V_N^*(P_{N-1}, x_{N-1}) = P_0^T \bar{S} - P_{N-1}^T x_{N-1} + \frac{1}{2} x_{N-1}^T \frac{H + H^T}{\tau} x_{N-1}, \tag{B.2.2}$$

Hence, equation (B.2.1) holds for $k = N$ with $A_N = \Theta^T + \Theta$, $b_N = \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(N) + \tau \alpha_0^{\mathbf{a}}$, and $c_N = 0$. Notice that the matrix A_N is symmetric. Assume that the statement (B.2.1) holds for $k+1$, particularly:

$$\begin{aligned} &V_{k+1}^*(P_k, x_k) = \\ &P_0^T \bar{S} - \frac{1}{2} x_k^T \left(\Theta^T - A_{k+1} - G\right) x_k - \left(P_k + b_{k+1} - \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k+1) - \tau \alpha_0^{\mathbf{a}}\right)^T x_k - c_{k+1}, \end{aligned} \tag{B.2.3}$$

where A_{k+1} is symmetric. We will prove the correctness of (B.2.1) for k. Applying Bellman's principle of optimality in the kth step yields

$$V_k^*(P_{k-1}, x_{k-1}) = \min_{n_k \in \mathbb{R}^m} \mathbf{E}\left[-n_k^T \tilde{P}_k + V_{k+1}^*(P_k, x_k) \mid P_{k-1},\ x_{k-1}\right]. \tag{B.2.4}$$

Substituting (B.2.3) into equation (B.2.4), the objective function in (B.2.4) becomes

$$\begin{aligned} &\mathbf{E}[-n_k^T \tilde{P}_k + P_0^T \bar{S} - \frac{1}{2} x_k^T (\Theta^T - A_{k+1} - G) x_k - (P_k + b_{k+1} - \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k+1) - \tau \alpha_0^{\mathbf{a}})^T x_k \\ &\quad - c_{k+1} \mid P_{k-1},\ x_{k-1}] \\ &= P_0^T \bar{S} - \frac{1}{2} x_{k-1}^T (\Theta^T - A_{k+1} - G) x_{k-1} - \left(P_{k-1} + b_{k+1} - \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k+1) + \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k)\right)^T x_{k-1} \\ &\quad + \frac{1}{2} n_k^T A_{k+1} n_k + \left(b_{k+1} - \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k+1) + \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k) + (\Theta^T - A_{k+1})^T x_{k-1}\right)^T n_k - c_{k+1}. \end{aligned}$$

Note that this function to be minimized is quadratic in n_k. Moreover, from the induction hypothesis the matrix A_{k+1} is symmetric. In addition, the matrix A_{k+1} is positive definite by assumption, and consequently the objective function is convex. It is straightforward to verify that the solution is attained at

$$n_k^* = -A_{k+1}^{-1}\left(b_{k+1} - \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k+1) + \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k) + (\Theta^T - A_{k+1})^T x_{k-1}\right).$$

Whence the optimal-value function $V_k^*(P_{k-1}, x_{k-1})$ equals

$$\begin{aligned}
&P_0^T \bar{S} - \frac{1}{2} x_{k-1}^T \left((\Theta^T - A_{k+1})A_{k+1}^{-1}(\Theta^T - A_{k+1})^T + \Theta^T - A_{k+1} - G\right) x_{k-1} \qquad \text{(B.2.5)}\\
&- \left(P_{k-1} + (\Theta^T - A_{k+1})A_{k+1}^{-1}\left(b_{k+1} - \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k+1) + \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k)\right) + b_{k+1} - \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k+1) + \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k)\right)^T x_{k-1}\\
&- \frac{1}{2}\left(b_{k+1} - \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k+1) + \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k)\right)^T A_{k+1}^{-1}\left(b_{k+1} - \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k+1) + \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k)\right) - c_{k+1}.
\end{aligned}$$

Substituting equations (5.3.5) and (5.3.6) in (B.2.5) yields the correctness of equation (B.2.1) for k. Furthermore, equation (5.3.5) and the symmetry assumption of A_{k+1} yield the matrix A_k is symmetric. This completes the induction. □

We now prove the statement in Proposition 5.3.1.

Proof. By a simple induction we can prove that, when $G = G^T$ and $\mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k) = \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}$ for $k = 1, 2, \ldots, N$, equations (5.3.5) and (5.3.6) yield

$$A_k = \left(\frac{N+2-k}{N+1-k}\right)\Theta, \quad b_k = \frac{N-k+2}{2}\left(\mathbf{E}_{\mathcal{J}}^{\mathbf{a}} + \tau\alpha_0^{\mathbf{a}}\right). \qquad \text{(B.2.6)}$$

Positive definiteness of Θ implies that the matrix A_k is positive definite, for every $k = 1, 2, \ldots, N$. Hence, the assumption in Theorem 5.3.2 is satisfied and stochastic dynamic programming offers a unique solution. Substituting equations (B.2.6) in (5.3.7), we get:

$$n_k^* = \frac{1}{N-k+1} x_{k-1}^* - \frac{(N-k)}{2}\Theta^{-1}\left(\mathbf{E}_{\mathcal{J}}^{\mathbf{a}} + \tau\alpha_0^{\mathbf{a}}\right), \quad k = 1, 2, \ldots, N-1, \qquad \text{(B.2.7)}$$

or equivalently

$$x_{k-2}^* = (N-k+2)\left(n_{k-1}^* + \frac{(N-k+1)}{2}\Theta^{-1}\left(\mathbf{E}_{\mathcal{J}}^{\mathbf{a}} + \tau\alpha_0^{\mathbf{a}}\right)\right), \quad k = 2, 3, \ldots, N. \qquad \text{(B.2.8)}$$

Applying equation (B.2.7), equation (B.2.8) and equation (2.1.1), we get:

$$\begin{aligned}
n_k^* &= \frac{1}{N-k+1}(x_{k-2}^* - n_{k-1}^*) - \frac{(N-k)}{2}\Theta^{-1}\left(\mathbf{E}_{\mathcal{J}}^{\mathbf{a}} + \tau\alpha_0^{\mathbf{a}}\right)\\
&= \frac{1}{N-k+1}\left((N-k+2)\left(n_{k-1}^* + \frac{(N-k+1)}{2}\Theta^{-1}(\mathbf{E}_{\mathcal{J}}^{\mathbf{a}} + \tau\alpha_0^{\mathbf{a}})\right) - n_{k-1}^*\right)\\
&\quad - \frac{(N-k)}{2}\Theta^{-1}\left(\mathbf{E}_{\mathcal{J}}^{\mathbf{a}} + \tau\alpha_0^{\mathbf{a}}\right)\\
&= n_{k-1}^* + \Theta^{-1}(\mathbf{E}_{\mathcal{J}}^{\mathbf{a}} + \tau\alpha_0^{\mathbf{a}}), \qquad k = 2, 3, \ldots, N-1. \qquad \text{(B.2.9)}
\end{aligned}$$

Now, we use equations (B.2.7) and (B.2.9) to prove (5.3.8) by induction on $k \leq N-1$. Equation (B.2.7) for $k=1$ directly implies the correctness of (5.3.8) for $k=1$. Assuming that equation (5.3.8) holds for $k-1$, we will prove it for k. Using equation (B.2.9), we have

$$n_k^* = \frac{\bar{S}}{N} - \frac{(N+3-2k)}{2}\Theta^{-1}\left(\mathbf{E}_{\mathcal{J}}^{\mathbf{a}} + \tau\alpha_0^{\mathbf{a}}\right) + \Theta^{-1}\left(\mathbf{E}_{\mathcal{J}}^{\mathbf{a}} + \tau\alpha_0^{\mathbf{a}}\right) = \frac{\bar{S}}{N} - \frac{N+1-2k}{2}\Theta^{-1}\left(\mathbf{E}_{\mathcal{J}}^{\mathbf{a}} + \tau\alpha_0^{\mathbf{a}}\right),$$

which proves the correctness of equation (5.3.8) for $k = 2, 3, \ldots, N-1$, and the induction is complete.

Since $\sum_{k=1}^{N} n_k^* = \bar{S}$, for $k = N$ we must have

$$\begin{aligned} n_N^* = \bar{S} - \textstyle\sum_{k=1}^{N-1} n_k^* &= \bar{S} - (N-1)\frac{\bar{S}}{N} + \frac{1}{2}\Theta^{-1}\left(\mathbf{E}_{\mathcal{J}}^{\mathbf{a}} + \tau\alpha_0^{\mathbf{a}}\right)\sum_{k=1}^{N-1}(N+1-2k) \\ &= \frac{\bar{S}}{N} + \frac{(N-1)}{2}\Theta^{-1}\left(\mathbf{E}_{\mathcal{J}}^{\mathbf{a}} + \tau\alpha_0^{\mathbf{a}}\right), \end{aligned}$$

which shows the correctness of (5.3.8) for $k = N$. □

B.3 Optimal Execution Strategy Under Multiplicative Jump Price Models

In this Appendix, we prove Theorem 5.3.3. Recall that $\mathbf{E}_{\mathcal{J}}^{\mathbf{m}}(k)$ and $\mathbf{Cov}_{\mathcal{J}}^{\mathbf{m}}(k)$ denote the expected value and covariance matrix of $\mathcal{J}^{\mathbf{m}}(k)$. Moreover, we refer to the $m \times m$ identity matrix, and the $m \times m$ zero matrix as I and 0, respectively. Moreover, we denote the m-vector of all ones with e.

Proof. By backward induction on k, we prove that optimal execution is given by (5.3.13), matrices A_k and C_k are symmetric, and the optimal-value function is given by equation (5.3.14).

For $k = N$, the constraint $x_N = 0$ yields the optimal execution n_N^* must equal x_{N-1}. Using equation (5.3.3), the optimal-value function in the last period becomes

$$V_N^*(P_{N-1}, x_{N-1}) = P_0^T \bar{S} - P_{N-1}^T x_{N-1} + \frac{1}{2} x_{N-1}^T \frac{H + H^T}{\tau} x_{N-1}, \tag{B.3.1}$$

which is obtained from substitution $A_N = 0$, $B_N = I$ and $C_N = -\frac{H+H^T}{2\tau}$ in equation (5.3.14). Note that matrices A_N and C_N are symmetric.

Assume that statement (5.3.14) holds for $k+1$, i.e., the optimal-value function $V_{k+1}^*(P_k, x_k)$ is given by

$$V_{k+1}^*(P_k, x_k) = P_0^T \bar{S} - P_k^T A_{k+1} P_k - P_k^T B_{k+1} x_k - x_k^T C_{k+1} x_k, \tag{B.3.2}$$

with A_{k+1} and C_{k+1} are symmetric. We now prove the correctness of equation (5.3.14) for k. Bellman's principle of optimality implies

$$V_k^*(P_{k-1}, x_{k-1}) = \min_{n_k \in \mathbb{R}^m} \ \mathbf{E}\left[-n_k^T \tilde{P}_k + V_{k+1}^*(P_k, x_k) \mid P_{k-1},\ x_{k-1} \right]. \tag{B.3.3}$$

Substituting equation (B.3.2) into equation (B.3.3), we obtain:

$$\begin{aligned} &V_k^*(P_{k-1}, x_{k-1}) = \\ &\min_{n_k \in \mathbb{R}^m} \ \mathbf{E}\left[P_0^T \bar{S} - n_k^T \tilde{P}_k - P_k^T A_{k+1} P_k - P_k^T B_{k+1} x_k - x_k^T C_{k+1} x_k \mid P_{k-1},\ x_{k-1} \right]. \end{aligned} \tag{B.3.4}$$

Given P_{k-1} and x_{k-1}, equation $x_k = x_{k-1} - n_k$ and the execution price model (5.2.6), the terms $n_k^T \tilde{P}_k$ and $x_k^T C_{k+1} x_k$ in the objective function of the minimization problem in (B.3.4) are deterministic. Hence:

$$\mathbf{E}\left[n_k^T \tilde{P}_k \mid P_{k-1},\ x_{k-1} \right] = n_k^T \tilde{P}_k = n_k^T \left(P_{k-1} - \frac{H}{\tau} n_k \right) = n_k^T P_{k-1} - n_k^T \frac{H}{\tau} n_k, \tag{B.3.5}$$

$$\mathbf{E}\left[x_k^T C_{k+1} x_k \mid P_{k-1},\ x_{k-1} \right] = (x_{k-1} - n_k)^T C_{k+1} (x_{k-1} - n_k). \tag{B.3.6}$$

Define $\mathcal{L}_k = e + \tau\alpha_0^{\boldsymbol{m}} + \mathcal{J}^{\boldsymbol{m}}(k) + \tau^{1/2}\Sigma^{\boldsymbol{m}} Z_k$. Using market price dynamics (5.2.6), we get:

$$\mathbf{E}\left[P_k^T B_{k+1} x_k \mid P_{k-1},\ x_{k-1}\right] = P_{k-1}^T L_k B_{k+1}(x_{k-1} - n_k) - n_k^T G^T B_{k+1}(x_{k-1} - n_k), \quad \text{(B.3.7)}$$

where $L_k = \text{Diag}\left(\mathbf{E}(\mathcal{L}_k)\right)$. Similarly, using the market price dynamics (5.2.6), the term $P_k^T A_{k+1} P_k$ is stated as:

$$\begin{aligned} &P_k^T A_{k+1} P_k = && \text{(B.3.8)} \\ &P_{k-1}^T \text{Diag}\left(\mathcal{L}_k\right) A_{k+1} \text{Diag}\left(\mathcal{L}_k\right) P_{k-1} - 2P_{k-1}^T \text{Diag}\left(\mathcal{L}_k\right) A_{k+1} G n_k + n_k^T G^T A_{k+1} G n_k. \end{aligned}$$

In addition

$$\begin{aligned} &\mathbf{E}\left[P_{k-1}^T \text{Diag}\left(\mathcal{L}_k\right) A_{k+1} \text{Diag}\left(\mathcal{L}_k\right) P_{k-1} \mid P_{k-1},\ x_{k-1}\right] = \\ &P_{k-1}^T \left(A_{k+1} \circ \mathbf{E}\left[\mathcal{L}_k \mathcal{L}_k^T \mid P_{k-1},\ x_{k-1}\right]\right) P_{k-1}, \end{aligned}$$

where $\circ$ denotes the Hadamard product.

Since $\mathbf{E}(Z_k) = 0$ and the random vectors Z_k and $\mathcal{J}^{\boldsymbol{m}}(k)$ are independent, we obtain

$$\mathbf{E}\left[\mathcal{L}_k \mathcal{L}_k^T\right] = \mathbf{E}[\mathcal{L}_k]\mathbf{E}[\mathcal{L}_k]^T + \tau\Sigma^{\boldsymbol{m}}(\Sigma^{\boldsymbol{m}})^T + \mathbf{Cov}_{\mathcal{J}}^{\boldsymbol{m}}(k).$$

Hence,

$$\begin{aligned} &\mathbf{E}\left[P_{k-1}^T \text{Diag}\left(\mathcal{L}_k\right) A_{k+1} \text{Diag}\left(\mathcal{L}_k\right) P_{k-1} \mid P_{k-1},\ x_{k-1}\right] && \text{(B.3.9)} \\ &= P_{k-1}^T \left(L_k A_{k+1} L_k + \left(\tau\Sigma^{\boldsymbol{m}}(\Sigma^{\boldsymbol{m}})^T + \mathbf{Cov}_{\mathcal{J}}^{\boldsymbol{m}}(k)\right) \circ A_{k+1}\right) P_{k-1}. \end{aligned}$$

Taking expectation from (B.3.8) and substituting equation (B.3.9), we have

$$\begin{aligned} \mathbf{E}\left[P_k^T A_{k+1} P_k \mid P_{k-1},\ x_{k-1}\right] = \ & P_{k-1}^T \left(A_{k+1}.*\left(\tau\Sigma^{\boldsymbol{m}}(\Sigma^{\boldsymbol{m}})^T + \mathbf{Cov}_{\mathcal{J}}^{\boldsymbol{m}}(k)\right) + L_k A_{k+1} L_k\right) P_{k-1} \\ & -2P_{k-1}^T L_k A_{k+1} G n_k + n_k^T G^T A_{k+1} G n_k. \quad \text{(B.3.10)} \end{aligned}$$

Substituting equations (B.3.5), (B.3.6), (B.3.7) and (B.3.10) into equation (B.3.4), the objective function of the minimization problem in (B.3.4) is reduced to:

$$\begin{aligned} &P_0^T \bar{S} - x_{k-1}^T C_{k+1} x_{k-1} - P_{k-1}^T L_k B_{k+1} x_{k-1} && \text{(B.3.11)} \\ &-P_{k-1}^T \left(A_{k+1}.*\left(\tau\Sigma^{\boldsymbol{m}}(\Sigma^{\boldsymbol{m}})^T + \mathbf{Cov}_{\mathcal{J}}^{\boldsymbol{m}}(k)\right) + L_k A_{k+1} L_k\right) P_{k-1} \\ &+\left(x_{k-1}^T \left(2C_{k+1} + B_{k+1}^T G\right) + P_{k-1}^T \left(-I + L_k B_{k+1} + 2L_k A_{k+1} G\right)\right) n_k + \frac{1}{2} n_k^T D_{k+1} n_k, \end{aligned}$$

where D_{k+1} is as in equation (5.3.11). Hence, the minimization problem in (B.3.4) is quadratic in n_k. Since D_{k+1} is assumed to be positive definite, the unique solution is attained at

$$n_k^* = -D_{k+1}^{-1} \left(x_{k-1}^T \left(2C_{k+1} + B_{k+1}^T G\right) + P_{k-1}^T \left(-I + L_k B_{k+1} + 2L_k A_{k+1} G\right)\right)^T. \quad \text{(B.3.12)}$$

Substituting n_k^* into (B.3.11) and after some algebraic manipulation, the optimal-value function $V_k^*(P_{k-1}, x_{k-1})$ becomes

$$V_k^*(P_{k-1}, x_{k-1}) = P_0^T \bar{S} - P_{k-1}^T A_k P_{k-1} - P_{k-1}^T B_k x_{k-1} - x_{k-1}^T C_k x_{k-1},$$

where the matrices A_k, B_k and C_k are given by equations (5.3.12). Notice that when A_{k+1} and C_{k+1} are symmetric, equations (5.3.12) indicate that A_k and C_k are also symmetric. This completes the induction. □

B.4 Analytical Formulae for Expected Execution Costs

Assuming that the true model for the market price is the multiplicative model (5.2.6), closed-form expressions for the expected execution costs of the naive strategy and the execution strategy (5.3.8) can be easily derived. The following proposition presents these formulae.

Proposition B.4.1. *Let the true model for the market price be the multiplicative model* (5.2.6)*, and for every k, $\mathbf{E}^{\boldsymbol{m}}_{\mathcal{J}}(k) = \mathbf{E}^{\boldsymbol{m}}_{\mathcal{J}}$, for some constant $\mathbf{E}^{\boldsymbol{m}}_{\mathcal{J}}$. Assume that the permanent impact matrix is symmetric and the matrix Θ is positive definite. Then the expected (true) execution cost of the execution strategy n^*, optimal under the additive jump diffusion model, given in equation* (5.3.8)*, equals*

$$\mathbf{E}\left[P_0^T\bar{S} - \sum_{k=1}^{N}\tilde{P}_k^T n_k^*\right] = \bar{S}^T P_0 + \bar{S}^T\frac{H}{N\tau}\bar{S} + \frac{\bar{S}^T}{N}\sum_{k=1}^{N} L^{k-1}\left(\frac{N-k}{N}G\bar{S} - P_0\right) \tag{B.4.1}$$
$$+\sum_{k=0}^{N-1}\left(\mathbf{E}^{\mathbf{a}}_{\mathcal{J}} + \tau\alpha^{\mathbf{a}}_0\right)^T\Theta^{-1}\left(\frac{k(N-k)}{2N}\left(L^{k-1}G - GL^{k-1}\right)\bar{S} + \frac{(N-2k-1)}{2}L^k P_0\right)$$
$$+\left(\mathbf{E}^{\mathbf{a}}_{\mathcal{J}} + \tau\alpha^{\mathbf{a}}_0\right)^T\Theta^{-1}\left(\frac{N(N^2-1)}{12\tau}H + \sum_{k=1}^{N-1}\frac{N-k}{12}\left(N^2-1-2k(k+N)\right)L^{k-1}G\right)$$
$$\Theta^{-1}\left(\mathbf{E}^{\mathbf{a}}_{\mathcal{J}} + \tau\alpha^{\mathbf{a}}_0\right),$$

where $L \stackrel{\text{def}}{=} \operatorname{Diag}\left(e + \tau\alpha^{\boldsymbol{m}}_0 + \mathbf{E}^{\boldsymbol{m}}_{\mathcal{J}}\right)$. Here the superscript k in the term L^k is the exponent of L.

Proof. Let the market price evolves according to the multiplicative jump diffusion model in (5.2.6). Whence, following the optimal execution strategy n^* given in equation (5.3.8), the total amount received at the end of the time horizon equals

$$\sum_{k=1}^{N}\tilde{P}_k^T n_k^* = \sum_{k=1}^{N}\left(P_{k-1} - \frac{H}{\tau}n_k^*\right)^T n_k^* = \sum_{k=1}^{N}P_{k-1}^T\left(\frac{\bar{S}}{N} - \frac{(N+1-2k)}{2}\Theta^{-1}(\tau\alpha^{\mathbf{a}}_0 + \mathbf{E}^{\mathbf{a}}_{\mathcal{J}})\right)$$
$$-\sum_{k=1}^{N}\left(\frac{\bar{S}}{N} - \frac{(N+1-2k)}{2}\Theta^{-1}(\tau\alpha^{\mathbf{a}}_0 + \mathbf{E}^{\mathbf{a}}_{\mathcal{J}})\right)^T\frac{H}{\tau}\left(\frac{\bar{S}}{N} - \frac{(N+1-2k)}{2}\Theta^{-1}(\tau\alpha^{\mathbf{a}}_0 + \mathbf{E}^{\mathbf{a}}_{\mathcal{J}})\right).$$

After simplifying the expression in the right-hand-side of the above equation and using the equalities $\sum_{k=1}^{N}(N+1-2k)^2 = \frac{N(N^2-1)}{3}$ and $\sum_{k=1}^{N}(N+1-2k) = 0$, we arrive at

$$\sum_{k=1}^{N}\tilde{P}_k^T n_k^* = \sum_{k=1}^{N}\frac{\bar{S}^T}{N}P_{k-1} - \sum_{k=1}^{N}\frac{(N+1-2k)}{2}\left(\mathbf{E}^{\mathbf{a}}_{\mathcal{J}} + \tau\alpha^{\mathbf{a}}_0\right)^T\Theta^{-1}P_{k-1}$$
$$-\bar{S}^T\frac{H}{N\tau}\bar{S} - \frac{N(N^2-1)}{12\tau}\left(\mathbf{E}^{\mathbf{a}}_{\mathcal{J}} + \tau\alpha^{\mathbf{a}}_0\right)^T\Theta^{-1}H\Theta^{-1}\left(\mathbf{E}^{\mathbf{a}}_{\mathcal{J}} + \tau\alpha^{\mathbf{a}}_0\right).$$

Therefore, the expected value of the execution cost becomes

$$\mathbf{E}\left[P_0^T\bar{S}-\sum_{k=1}^{N}\tilde{P}_k^Tn_k^*\right] = P_0^T\bar{S}-\frac{\bar{S}^T}{N}\sum_{k=1}^{N}\mathbf{E}[P_{k-1}]+\left(\mathbf{E}_{\mathcal{J}}^{\mathbf{a}}+\tau\alpha_0^{\mathbf{a}}\right)^T\Theta^{-1}\sum_{k=1}^{N}\frac{(N+1-2k)}{2}\mathbf{E}[P_{k-1}] + \bar{S}^T\frac{H}{N\tau}\bar{S}+\frac{N(N^2-1)}{12\tau}\left(\mathbf{E}_{\mathcal{J}}^{\mathbf{a}}+\tau\alpha_0^{\mathbf{a}}\right)^T\Theta^{-1}H\Theta^{-1}\left(\mathbf{E}_{\mathcal{J}}^{\mathbf{a}}+\tau\alpha_0^{\mathbf{a}}\right). \quad \text{(B.4.2)}$$

Since the random variables Z_k and $\mathcal{J}^{\boldsymbol{m}}(k)$ are independent of P_{k-1}, using the conditional expectation theorem, see, e.g., (Varadhan, 2001), we get

$$\begin{aligned}\mathbf{E}\left[P_k\right] &= \mathbf{E}\left[\mathbf{E}\left[P_k|P_{k-1}\right]\right] \quad \text{(B.4.3)}\\ &= \mathbf{E}\left[\mathrm{Diag}(P_{k-1})\left(e+\tau\alpha_0^{\boldsymbol{m}}+\tau^{1/2}\Sigma^{\boldsymbol{m}}Z_k-\mathcal{J}^{\boldsymbol{m}}(k)\right)\right]-Gn_k^* = L\mathbf{E}\left[P_{k-1}\right]-Gn_k^*,\end{aligned}$$

where $L=\mathrm{Diag}(e+\tau\alpha_0^{\boldsymbol{m}}+\mathbf{E}_{\mathcal{J}}^{\boldsymbol{m}})$. Therefore,

$$\sum_{k=1}^{N}\mathbf{E}\left[P_{k-1}\right]=P_0+\sum_{k=2}^{N}\left(L\mathbf{E}\left[P_{k-2}\right]-Gn_{k-1}^*\right)=P_0+LP_0+\sum_{k=1}^{N-2}L\mathbf{E}\left[P_k\right]-\sum_{k=1}^{N-1}Gn_k^*.$$

Using equation (B.4.3), the summation $\sum_{k=1}^{N-2}\mathbf{E}[P_k]$ can be further simplified and we get

$$\sum_{k=1}^{N}\mathbf{E}\left[P_{k-1}\right]=\left(\sum_{k=1}^{N}L^{k-1}\right)P_0-\sum_{k=1}^{N-1}\sum_{i=1}^{N-k}L^{k-1}Gn_i^*. \quad \text{(B.4.4)}$$

Applying the expression for n^* in equation (5.3.8), and the equation

$$\sum_{i=1}^{N-k}(N+1-2i)=k(N-k),$$

we get

$$\sum_{k=1}^{N}\mathbf{E}[P_{k-1}]=\sum_{k=1}^{N}L^{k-1}\left(P_0-\frac{N-k}{N}G\bar{S}\right)+\sum_{k=1}^{N-1}\frac{k(N-k)}{2}L^{k-1}G\Theta^{-1}\left(\mathbf{E}_{\mathcal{J}}^{\boldsymbol{m}}+\tau\alpha_0^{\mathbf{a}}\right).$$

Similarly by using equation (B.4.3), we may show that

$$\sum_{k=1}^{N}(N+1-2k)\mathbf{E}[P_{k-1}]=\sum_{k=0}^{N-1}\left(N-2k-1\right)L^kP_0-\sum_{k=1}^{N-1}\sum_{i=1}^{N-k}\left(N-2(k+i)+1\right)L^{k-1}Gn_i^*.$$

Applying the expression of n^* in equation (5.3.8) and the equations

$$\begin{aligned}&\sum_{i=1}^{N-k}\left(N-2\left(k+i\right)+1\right)=-k(N-k),\\ &\sum_{i=1}^{N-k}(N-2(k+i)+1)(N+1-2i)=\frac{(N-k)}{3}(N^2-1-2k(k+N)),\end{aligned}$$

the value of $\sum_{k=1}^{N}(N+1-2k)\mathbf{E}[P_{k-1}]$ equals

$$\begin{aligned}
&\sum_{k=0}^{N-1}(N-2k-1)\,L^k P_0 \\
&-\sum_{k=1}^{N-1}\sum_{i=1}^{N-k}(N-2\,(k+i)+1)\,L^{k-1}G\left(\frac{\bar{S}}{N}-\frac{(N+1-2i)}{2}\Theta^{-1}(\tau\alpha_0^{\mathbf{a}}+\mathbf{E}_{\mathcal{J}}^{\mathbf{a}})\right) \\
&=\sum_{k=0}^{N-1}(N-2k-1)\,L^k P_0+\sum_{k=1}^{N-1}\frac{k(N-k)}{N}L^{k-1}G\bar{S} \\
&+\sum_{k=1}^{N-1}\frac{(N-k)}{6}(N^2-1-2k(k+N))L^{k-1}G\Theta^{-1}(\tau\alpha_0^{\mathbf{a}}+\mathbf{E}_{\mathcal{J}}^{\mathbf{a}}).
\end{aligned}$$

We use this quantity to compute the third term in the right-hand-side of equation (B.4.2). Therefore, the expected execution cost in (B.4.2) is reduced to (B.4.1). □

When $(\mathbf{E}_{\mathcal{J}}^{\mathbf{a}}+\tau\alpha_0^{\mathbf{a}})=0$, the execution strategy (5.3.8) is reduced to the naive strategy. Therefore, equation (B.4.1) with $(\mathbf{E}_{\mathcal{J}}^{\mathbf{a}}+\tau\alpha_0^{\mathbf{a}})=0$ yields the expected execution cost of the naive strategy.

Corollary B.4.1. *The (true) expected execution cost of the naive strategy $\bar{n}$ equals*

$$\mathbf{E}\left[P_0^T\bar{S}-\sum_{k=1}^{N}\tilde{P}_k^T\bar{n}_k\right]=\bar{S}^T P_0+\bar{S}^T\frac{H}{N\tau}\bar{S}+\frac{\bar{S}^T}{N}\sum_{k=1}^{N}L^{k-1}\left(\frac{N-k}{N}G\bar{S}-P_0\right). \quad \text{(B.4.5)}$$

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