The Fourier algebra of a locally trivial groupoid

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

The goal of this thesis is to define and study the Fourier algebra $A(G)$ of a locally compact groupoid G. If G is a locally compact group, its Fourier-Stieltjes algebra $B(G)$ and its Fourier algebra $A(G)$ were defined by Eymard in 1964. Since then, a rich theory has been developed. For the groupoid case, the algebras $B(G)$ and $A(G)$ have been studied by Ramsay and Walter (borelian case, 1997), Renault (measurable case, 1997) and Paterson (locally compact case, 2004). In this work, we present a new definition of $A(G)$ in the locally compact case, specially well suited for studying locally trivial groupoids.

Let G be a locally compact proper groupoid. Following the group case, in order to define $A(G)$, we consider the closure under certain norm of the span of the left regular G-Hilbert bundle coefficients. With the norm mentioned above, the space $A(G)$ is a commutative Banach algebra of continuous functions of G vanishing at infinity. Moreover, $A(G)$ separates points and it is also a $B(G)$-bimodule. If, in addition, G is compact, then $B(G)$ and $A(G)$ coincide. For a locally trivial groupoid G we present an easier to handle definition of $A(G)$ that involves "trivializing" the left regular bundle.

The main result of our work is a decomposition of $A(G)$, valid for transitive, locally trivial groupoids with a "nice" Haar system. The condition we require the Haar system to satisfy is to be compatible with the Haar measure of the isotropy group G_u^u at a fixed unit u. If the groupoid is transitive, locally trivial and unimodular, such a Haar system always can be constructed. For such groupoids, our theorem states that $A(G) \simeq C_0(G^u) \overset{h}{\otimes} A(G_u^u) \overset{h}{\otimes} C_0(G^u)$, where $\overset{h}{\otimes}$ denotes the Haagerup tensor product of operator spaces. This decomposition provides an operator space structure for $A(G)$ and makes this algebra a completely contractive Banach algebra.

If the locally trivial groupoid G has more than one transitive component, say $G = \sqcup_i G_i$, since these components are also topological components, there is a correspondence between G-Hilbert bundles and families of G_i-Hilbert bundles. Thanks to this correspondence, the Fourier-Stieltjes and Fourier algebra of G can be written as sums of the algebras of the G_i

components. Thus, the decomposition can be stated as $A(G) \simeq c_0 - \oplus_i C_0(G_i^0) \overset{h}{\otimes} A(G_{u_i}^{u_i}) \overset{h}{\otimes} C_0(G_i^0)$, if $u_i \in G_i$ for all i.

The theory of operator spaces is the main tool used in our work. In particular, the many properties of the Haagerup tensor product $\overset{h}{\otimes}$ are of vital importance.

Our decomposition can be applied to (trivially) locally trivial groupoids of the form $X \times X$ and $X \times H \times X$, for a locally compact space X and a locally compact group H. It can also be applied to transformation group groupoids $X \times H$ arising from the action of a Lie group H on a locally compact space X and to the fundamental groupoid $\Pi(X)$ of a path-connected manifold X.

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During most of my time as a PhD student at UW I had the best boyfriend of the world. It has been only two months since I have the best husband of the world. Dear Peyman, thanks very much for your love, help, care and support.

Dedication

This work is dedicated to the memory of my father, Arturo José Martí Simeto.
I am sure he would have loved to read it.

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Chapter 1

Introduction

"[. . .] The author hopes the approach adopted on the book will contribute to an appreciation of the intrinsic beauty and importance of groupoids, and help in overcoming a not uncommon psychological aversion to the concept that he himself initially experienced!"

Alan L. T. Paterson, [32].

Let me use this introduction to tell you a story [1], the story of the genesis and the process of writing this thesis: how the topic was selected, how the questions (and sometimes the answers) arose, what were the tools used and the goals achieved.

For some obscure reason that I cannot explain, contradicting the quote from Paterson above, I was already interested in groupoids when I started my PhD. at University of Waterloo. Professors Nico Spronk and Brian Forrest agreed to supervise me [2] and since they study Fourier algebras of groups, the topic "Fourier algebras of groupoids" looked like a good idea.[3] At this point, a copy of Paterson's paper "The Fourier algebra for locally

[1]As a kid, the author of this thesis wanted to be writer, not a mathematician.

[2]However, they expressed on many occasions that I was free to switch to Number Theory if so I wished.

[3]There were other ideas, like "Fourier algebras of Fell bundles"... maybe I will have a grad student looking at that some day.

compact groupoids" ([33]) magically appeared in my mailbox. I do not know the meaning of such an apparition in other "mathematical families", but in mine, that means you have to read it. So I did. Or at least I tried. The "psychological aversion" mentioned above was experienced as I pushed my way through the pages of the article. It was a bit overwhelming, but it was definitely fun, despite my lack of intuition as far as groupoids were concerned. At least my previous exposure to Fell bundles helped me understand the Hilbert bundles involved. We had a seminar running back then, a weekly opportunity for me to try to understand a bit more while presenting Paterson's work to my supervisors and my fellow grad students.[4] The topic of the thesis was still very vague. My supervisors suggested I could concentrate on transformation group groupoids, but I could not make much progress. A few weeks later they suggested I should concentrate on finite groupoids. Being completely honest, I was not thrilled by the suggestion. I wanted to work on something more grandiose! It is time to deeply thank Professor Ebrahim Samei, back then a postdoctoral fellow in our department, who convinced me of doing what I had to do.

Before we keep going with our story, let me briefly and informally discuss the definitions and properties of the Fourier algebras of locally compact groups, as well as the definition of groupoids.

If H is a locally compact group, two commutative Banach algebras can be associated to it: the Fourier-Stieltjes algebra $B(H)$ and the Fourier algebra $A(H)$. This rich theory started with Eymard's work "L'algèbre de Fourier d'un groupe localement compact", [18]. The definitions of $B(H)$ and $A(H)$ are included in the first section of Chapter 4. Suppose $\pi : H \to \mathcal{U}(\mathcal{H})$ is a continuous homomorphism, where $\mathcal{H}$ is a Hilbert space and $\mathcal{U}(\mathcal{H})$ denotes the group of unitary operators on $\mathcal{H}$. If ξ and η are elements of $\mathcal{H}$, let $(\xi, \eta)_\pi$ denote the map from H to $\mathbb{C}$, called a coefficient map, defined by $(\xi, \eta)_\pi(h) = \langle \pi(h)\xi, \eta \rangle$. The Fourier-Stieltjes algebra of H is the space of coefficient functions of all equivalence classes of unitary, continuous representations of H. It is a unital subspace of $C_b(H)$, the space of continuous and bounded functions on H. It is a commutative algebra with point-

[4] Many thanks to Mahya, Mike, Cam and Elcim!

wise product. Moreover, it is a Banach algebra: if φ is an element of $B(H)$, the norm of φ is defined by

$$\|\varphi\| = \inf_{\varphi=(\xi,\eta)_\pi} \|\xi\|\|\eta\|,$$

where π is a continuous unitary representation of H and ξ, η are elements of $\mathcal{H}_\pi$. Alternatively, the Fourier-Stieltjes algebra can be defined as the span of the continuous positive definite functions of H, and also as the Banach space dual of $C^*(H)$, the C^*-algebra of H.

Let m be the left Haar measure on H. The Fourier algebra of H is obtained by considering only coefficients of the left regular representation $\lambda : H \to \mathcal{U}(L^2(H, m))$, $\lambda(h)(f)(h') = f(h^{-1}h')$,

$$A(H) = \{(\xi, \eta)_\lambda : \xi, \eta \in L^2(H, m)\}.$$

This space is included in $C_0(H)$, the space of continuous functions vanishing at infinity.[5] Moreover, with the norm of $B(H)$, this space is a closed ideal. It also admits other equivalent definitions: it is the predual of the von Neumann algebra of the group $VN(H)$ and can be obtained as a quotient of a projective tensor product of L^2 spaces.

Since they were defined by Eymard, these algebras had been well studied and many properties had been proved (see for instance, the discussion about amenability at the end of Section 4.1). Considering that groups are particular cases of groupoids, we may wonder if it is possible to develop a theory of Fourier algebras of groupoids.

We refer to the beginning of Chapter 2 for the formal definition of groupoids. For now, we just say that intuitively, groupoids are very much like groups, with the difference that the product is not defined everywhere and that we have many identities instead of just one. We can think about a graph and the operation of concatenation. We can concatenate two edges of the graph only if the first of them has as range the source of the second one (that is, the first one ends where the second one starts). Groupoids also have an inverse operation that works like the inverse on a group. The identities or units of the graph

[5] A function $f : X \to Y$, where X is a topological space and Y is a normed space, vanishes at infinity if for all $\varepsilon > 0$ there exists a compact subset K of X such that $\|f(x)\| < \varepsilon$ if $x \in K^c$.

example are the vertices. If G is a groupoid, its space of units is denoted by G^0. The loops at a vertex u of the graph form a group, called isotropy group and denoted by G_u^u.

Groups are examples of groupoids, and so are union of groups, sets and equivalence relations. An equivalence relation is a groupoid with the following structure: suppose u, v and w are elements of a set X; if u is related to v and v is related to w, then we are allowed to multiply (u, v) and (v, w) and the result is (u, w). The inverse of a pair (u, v) is the pair (v, u).

There is one more example of a class of groupoids I wish to mention here, and it is the fundamental groupoid of a topological space. The fundamental group of a space X is defined by fixing a base point x of the space and considering the continuous loops at x up to homotopical equivalence. The operation on this group is concatenation, the unit is the constant loop and the inverse of a loop is the loop travelled in the opposite direction. The elements of the fundamental groupoid are equivalence classes of continuous paths on the space, with any starting and ending point. In this case, homotopical equivalence with fixed end points is considered. Fundamental groupoids are useful to allow change of base points for the fundamental group.

We are interested in topological locally compact groupoids, that is, groupoids with a locally compact, Hausdorff, second countable topology that makes the operations continuous. The family of groupoids is a very broad one. In most cases, when studying groupoids, we restrict ourselves to certain families of them that verify some given properties (for instance, principal, étale or proper groupoids). In our work we concentrate on locally trivial groupoids. Fundamental groupoids of spaces with reasonable topological properties are examples of locally trivial groupoids.

For a survey of the history of groupoids and their application to different areas of mathematics, we refer to Brown's article "From groups to groupoids: a brief survey", [8]. We just mention here that groupoids were introduced by Brandt in 1927, see [7]. During the following decades they were used in Galois Theory. Ehresmann's work from the 50's (see Charles Ehresmann's "Oeuvres complètes et commentées") helped popularize them and they started to be used in bundle theory, differential geometry, differential topology and foliation theory.

Getting back to our story, the first step towards understanding the Fourier algebra of a finite groupoid is analyzing the full equivalence groupoid on n elements, $I_n = \{1, 2, \cdots, n\} \times \{1, 2, \cdots, n\}$. This example is included in Paterson's paper. As a space, $A(G) = B(G)$ and consist of functions on n^2 elements, that can be represented as $n \times n$ matrices. The point-wise product of the Fourier algebras is the Schur product of matrices and the norm is the Schur multipliers norm. This space can also be represented as $(\mathbb{C}^n, \|\cdot\|_\infty) \overset{h}{\otimes} (\mathbb{C}^n, \|\cdot\|_\infty)$, and here the Haagerup tensor product of operator spaces, one of the main tools of this work, made its first appearance.[6]

Let me say a few words about operator spaces and their Haagerup tensor product. The third Chapter is dedicated to this topic. A concrete operator space $\mathcal{V}$ is a linear subspace of $\mathcal{B}(\mathcal{H})$, the space of bounded operators on a Hilbert space $\mathcal{H}$. On each space $M_n(\mathcal{V})$ a norm can be considered, since there is a natural algebraic homomorphism $M_n(\mathcal{B}(\mathcal{H})) \simeq \mathcal{B}(\mathcal{H}^n)$. An (abstract) operator space is a vector space $\mathcal{V}$ together with a family of norms $\{\|\cdot\|_n\}$ satisfying some compatibility properties, see Section 3.1. Every operator space can be represented as a concrete operator space.

If $\mathcal{V}$ and $\mathcal{W}$ are operator spaces, then, as in the theory of Banach spaces, various norms can be considered on the algebraic tensor product $\mathcal{V} \otimes \mathcal{W}$. The Haagerup tensor norm is one of them, and the completion of $\mathcal{V} \otimes \mathcal{W}$ with respect of this norm is the Haagerup tensor product denoted by $\mathcal{V} \overset{h}{\otimes} \mathcal{W}$. This tensor product does not have an equivalent in the theory of Banach spaces. This is not the only surprising fact about $\overset{h}{\otimes}$: in fact, it has many very interesting properties, such as being injective and projective simultaneously and its remarkable behaviour when tensoring Hilbert spaces. [7]

[6] At this point I felt Professor Nico Spronk was persuaded of the beauty of groupoids.

[7] You may have thought that I am brave because I wanted to study objects so unappealing at first sight as groupoids... I have to confess I was terrified when I realized the Haagerup tensor product of operator spaces was going to be a main ingredient of my thesis. I had a hard time understanding Professor Spronk's enthusiasm, but it did not take me long to agree that the Haagerup tensor product is a miracle.

The next family of groupoids to consider are the transitive ones. We say that a groupoid is transitive if given two units of the groupoid there exists an element of the groupoid that has one of them as source and the other as range. If a groupoid is finite and transitive, then it is of the form $X \times H \times X$, where X is a finite set and H is a group. If the group is trivial, we are in the full equivalence case discussed above, and as we mentioned,

$$A(X \times X) = (\mathbb{C}^n, \|\cdot\|_\infty) \overset{h}{\otimes} (\mathbb{C}^n, \|\cdot\|_\infty).$$

If X has only one element, our groupoid is just a group H, and hence its Fourier algebra is Eymard's algebra $A(H)$. One may ask (as Professor Nico Spronk did) if for a groupoid $X \times H \times X$ the Fourier algebra is

$$A(X \times H \times X) \simeq (\mathbb{C}^n, \|\cdot\|_\infty) \overset{h}{\otimes} A(H) \overset{h}{\otimes} (\mathbb{C}^n, \|\cdot\|_\infty).$$

This is, in fact, true, and operator space properties of tensor products are the tools needed to prove it. (See diagram 5.19 for a "proof".)

Since finite groupoids are unions of transitive groupoids, this gives a complete description of the Fourier algebra of a finite groupoid. This description has the advantage of providing an operator space structure to $A(G)$, since the space on the right hand side of the equation is an operator space. Moreover, $A(G)$ is a completely contractive Banach algebra. In this case, $\mathbb{C}^n$ is the space of continuous functions on G^0 and H is isomorphic to the isotropy group at any unit. Now we wonder if a decomposition of the Fourier algebra of a groupoid of the type

$$C_0(G^0) \overset{h}{\otimes} A(G^u_u) \overset{h}{\otimes} C_0(G^0)$$

makes sense for non-finite groupoids.

There is one question we needed to address first, and this is what should be the definition of the Fourier algebra of a locally compact groupoid. One possible definition is presented in [33]. (See Section 6.2 for this definition as well as some comments on the existing bibliography for Fourier algebras of groupoids.) Up to now, we considered Fourier algebras of finite groupoids, and in this case there is agreement on how we should define it.

In order to propose a good definition of $A(G)$ for G any locally compact groupoid, we need to decide what are the properties we want this algebra to verify. First of all, we want to obtain a commutative Banach algebra of continuous functions. Second, we expect a relationship with $B(G)$[8] analogous to the group case. Finally, we would like to make of $A(G)$ a completely contractive Banach algebra. With these things in mind, the first goal of our project was to propose a definition for $A(G)$. This is done in Section 4.5. The algebra that we obtain is a commutative Banach algebra included in $C_0(G)$. The relationship between this algebra and the Fourier-Stieltjes algebra $B(G)$ is analyzed in that section. Once we decided on this definition, we came back to the question of whether the decomposition of the Fourier algebra of a finite groupoid can be extended to other groupoids. The answer to this question is included in Chapter 5. In order to obtain such a decomposition, we need to restrict ourselves to transitive groupoids that are locally trivial and whose Haar system (this is the equivalent to the Haar measure of a locally compact group) is compatible with the Haar measure of its isotropy groups. As in the finite case, this decomposition provides an operator space structure for $A(G)$. Moreover, this algebra is proven to be a completely contractive Banach algebra. When we started our work we did not know to what kind of groupoids our decomposition would extend. The conditions that we found are the result of trying to "push" the finite case proof as far as we could. We do not know if the result holds for a wider family of groupoids. Other questions that arose during the project and we have not answered yet are included in the last section of the thesis.

I wish to mention here that many times I was surprised by the mathematical concepts encountered. For instance, the extended Haagerup tensor product was a surprise, as well as geometrical concepts like the one of covering spaces and the approximation property of operator spaces.

The paragraphs above aimed to give an idea of the process towards the writing of this thesis and what are the tools and ideas that you will find in this work. For the sake of

[8] A continuous Fourier-Stieltjes algebra for a locally compact groupoid G is defined in [33], and we adopt that definition in our work.

order, we now explain more carefully the contents of each Chapter.

Chapter 2 is called "Basic definitions and examples of groupoids". The first section is devoted to definitions and notation and the second one to examples. The third one deals with locally trivial groupoids, our favourite class of groupoids. In that section we present a result from Seda ([47]). This result shows how a Haar system with some convenient compatibility property with the Haar measure of the isotropy group can be constructed. The main references for this Chapter are [43] and [29].

As the title "Operator spaces and the Haagerup tensor product" says, Chapter 3 is about operator spaces, and in particular, their tensor products. The Haagerup tensor product is defined and many of its properties stated, but also the projective, injective and extended Haagerup tensor product are presented. The main references for this Chapter are [14] and [49].

We begin Chapter 4, "The Fourier and Fourier-Stieltjes algebras of a locally compact groupoid", with a section devoted to the group case. On Section 2 we present the definition and properties of the continuous G-Hilbert bundles. These bundles are the "building blocks" of the Fourier algebras, they correspond to the continuous and unitary representations of the group case. Section 3 deals with the Fourier-Stieltjes algebra of a locally compact groupoid, following Paterson's definition from [33]. A stabilization theorem for proper groupoids from Paterson, see [35], is the topic of Section 4. This result is needed in Section 5, where we present our definition of the Fourier algebra, as well as its relationship with the Fourier-Stieltjes algebra and some of its properties.

The main result of our work is included in Chapter 5. The first section deals with completely contractive products. We need this section because we aim to find a product on $C_0(G^0) \overset{h}{\otimes} A(G^u_u) \overset{h}{\otimes} C_0(G^0)$ that makes $A(G)$ a completely contractive Banach algebra. Section 2 is very short and refers to the Banach algebra $\mathcal{A}_\vee = C_0(X) \overset{\vee}{\otimes} A(H) \overset{\vee}{\otimes} C_0(X)$, for X a topological space and H a group. We prove that the embedding of $\mathcal{A}_h = C_0(X) \overset{h}{\otimes} A(H) \overset{h}{\otimes} C_0(X)$ in $\mathcal{A}_\vee$ is one-to-one. Later on this will help us to prove that $\mathcal{A}_h$ is an algebra. The decomposition of $A(G)$ is proved in Section 3. The exact statement of our theorem is the following:

Theorem 1.0.1. *Let G be a locally trivial, transitive groupoid. Fix $u \in G^0$. Suppose that G has a Haar system $\{\lambda^v\}_{v \in G^0}$ such that $\lambda^u_{|G^u_u}$ is a left Haar measure on the isotropy group at u.*

Then $A(G)$ is isometrically isomorphic to $C_0(G^0) \overset{h}{\otimes} A(G^u_u) \overset{h}{\otimes} C_0(G^0)$ as Banach algebras. Moreover, since the later space is a completely contractive Banach algebra, so is $A(G)$.

The diagram 5.19 gives an outline of the steps of the proof. Section 4 looks upon the non-transitive case. Assume that G is a locally trivial groupoid with transitive components $\{G_i\}_i$. From the local triviality of G it follows that the transitive components are also connected components. Then $A(G)$ and $B(G)$ can be written as sums of the algebras $A(G_i)$ and $B(G_i)$.

The last Chapter is devoted to the conclusions of our work. The first section reviews all the examples considered and the new information obtained thanks to our results. The second section analyzes other definitions of the Fourier algebras considered by Renault ([43]) and Paterson ([33]). On Section 3 we recapitulate the results of our thesis. The final section, "Further questions", presents question we were not able to answer and we hope will be topic of future research.

Chapter 2

Basic definitions and examples of groupoids

We begin the Chapter by introducing the basic definitions and notation related to groupoids. We present different properties that define families of groupoids (transitive, proper, r-discrete, principal, locally trivial). The second section is devoted to examples of groupoids. Since locally trivial groupoids are the main examples we consider in our thesis, we focus on them in the last Section. In particular, we present a result from Seda ([47]) that constructs a left Haar system for locally trivial, unimodular groupoids.

2.1 Definitions and notation

We closely follow the definitions of [42], although the notation may differ.

Definitions 2.1.1. A **groupoid** is a set G together with a subset $G^2 \subseteq G \times G$, an associative product $G^2 \to G$, $(\delta, \gamma) \to \delta\gamma$ and an inverse $G \to G$, $\gamma \to \gamma^{-1}$, such that:

1. if $\gamma \in G$, $(\gamma^{-1})^{-1} = \gamma$; and

2. $(\gamma^{-1}, \gamma) \in G^2, \forall \gamma \in G$, and if $(\delta, \gamma) \in G^2$, then $\delta^{-1}\delta\gamma = \gamma$, $\delta\gamma\gamma^{-1} = \delta$.

Equivalently, a groupoid is a small category with inverses. This means that the class of morphisms is a set and the morphisms are isomorphisms.

We call the elements of the form $\gamma^{-1}\gamma$ **units**, due to the property above, and use the notation $G^0 = \{\gamma^{-1}\gamma : \gamma \in G\}$ for the set of units of G. The **range map** is $r : G \to G^0$, $r(\gamma) = \gamma\gamma^{-1}$, and the **source map** is $s : G \to G^0$, $s(\gamma) = \gamma^{-1}\gamma$.

If $u, v \in G^0$, $G^u := r^{-1}(u)$, $G_v := s^{-1}(v)$ and $G^u_v := G^u \cap G_v$. We say that G is **transitive** if for any two $u, v \in G^0$ there exists $\gamma \in G$ such that $s(\gamma) = u$ and $r(\gamma) = v$. That is, for all $u, v \in G^0$, we require $G^v_u \neq \phi$. Observe that in this case G^u_u is isomorphic as a group to G^v_v, for all $u, v \in G^0$, via $\gamma \to \gamma'\gamma(\gamma')^{-1}$, where $\gamma' \in G$ is such that $r(\gamma') = v$, $s(\gamma') = u$.

If $u \in G$, G^u_u is a group, called the **isotropy group** at u. If $A, B \subseteq G$, $G^A := r^{-1}(A)$, $G_B := s^{-1}(B)$ and $G^A_B := G^A \cap G_B$.

We say that the elements of G^2 are the **composable pairs**. Explicitly, the associativity mentioned before (and used at the definition of groupoid when writing triple products) says that if $(\varepsilon, \delta), (\delta, \gamma)$ are composable pairs, then $(\varepsilon\delta, \gamma)$ and $(\varepsilon, \delta\gamma)$ are composable as well and $(\varepsilon\delta)\gamma = \varepsilon(\delta\gamma)$ (and thus we drop the parenthesis).

If γ is an element of G, we imagine the following picture:

$$\begin{tikzcd} s(\gamma)\bullet \arrow[r, shift left, "\gamma"] & \bullet r(\gamma) \arrow[l, shift left, "\gamma^{-1}"] \end{tikzcd}$$

We identify the elements of G with arrows of a graph (and we called them $\gamma, \delta, \dots$) and the units with its vertices (and we write them $u, v, \dots$).

If G and G' are groupoids, they are **isomorphic** if there is a bijection $\varphi : G \to G'$ that respects the composable pairs and the product.

We are interested in groupoids that have a locally compact, Hausdorff, second countable topology that makes the operations continuous. We say that G is **locally compact** for short, and the same convention applies to topological spaces and groups. On such groupoids, the role of the Haar measure on a locally compact group is played by a family of measures indexed on the unit space, called left Haar system.

Definition 2.1.2. A **left Haar system** for G is a family $\{\lambda^u\}_{u\in G^0}$, where λ^u is a positive regular Borel measure on G^u and:

1. $\operatorname{supp}(\lambda^u) = G^u$.
2. If $f \in C_c(G)$, the map $\lambda(f) : G^0 \to \mathbb{C}$, $\lambda(f)(u) = \int f\, d\lambda^u$ is continuous.
3. $\forall \gamma \in G$ and $f \in C_c(G)$,
$$\int f(\gamma\gamma')\, d\lambda^{s(\gamma)}(\gamma') = \int f(\gamma')\, d\lambda^{r(\gamma)}(\gamma').$$

Not all groupoids have a left Haar system (see [48]), and we will soon see that for the ones that do have, it does not have to be unique. We assume that we are fixing one, unless otherwise is specified.

In [32], Paterson includes a fixed left Haar system as part of the definition of locally compact groupoid.

Groupoids are, as mentioned in the introduction, a very wide family of beasts, with very different characteristics. This is why, most of the time, when studying them, we do not study them in all their generality, but instead restrict ourselves to some family of groupoids satisfying certain conditions. Here we define some of these conditions and refer to works that consider them.

Definitions 2.1.3. We say that a groupoid G is **principal** if the map $(r, s) : G \to G^0 \times G^0$ is one-to-one. In terms of this map, a groupoid is transitive if (r, s) is onto. Principal groupoids are studied, for instance, in [9].

The groupoid G is **proper** if the map (r, s) is proper (that is, the inverse image of a compact set is a compact set as well). Paterson, in his work [35], (we will need this work later on) considers proper groupoids.

Definitions 2.1.4. If G is a locally compact groupoid, we say that G is **r-discrete** if its unit space is an open subset. These groupoids generalize the discrete groups. They are studied, for instance, in [30].

The **étale** groupoids are a very well studied family of r-discrete groupoids. A groupoid G is called étale if $r : G \to G^0$ is a local homeomorphism. See [17], for example. These groupoids always admit a Haar system of counting measures.

Definition 2.1.5. We say that a locally compact groupoid G is **locally trivial** if there is a family $\{U_i, u_i, \nu_i\}_{i\in I}$, indexed on some set I, such that $\{U_i\}$ is an open cover of G^0, $u_i \in U_i$ and $\nu_i : U_i \to G$ is a continuous map such that $\nu_i(u) \in G_u^{u_i}$, for all $i \in I$. If G is transitive and $u_0 \in G^0$, this is equivalent to the existence of a family $\{U_i, \nu_i\}_{i\in I}$ such that $\{U_i\}_{i\in I}$ is an open cover of G^0 and $\nu_i : U_i \to G$ is a continuous map verifying $\nu_i(u) \in G_u^{u_0}$, for all $i \in I$. Moreover, since our groupoids are locally compact and second countable, they are Lindelöf spaces, so we can assume that the family I is countable.

This is the class of groupoid we concentrate on in our thesis. See [16] and [55]. More references to works that consider this kind of groupoids will be mentioned later.

2.2 Examples

Example 2.2.1. Groups and union of groups.

If G is a (locally compact) group, it is a (locally compact) groupoid. In this case, $G^2 = G \times G$ since all the pairs are composable. The only unit is the unit of the group, that is, $G^0 = \{e\}$. The left Haar system has only one measure, the (essentially unique) left Haar measure. It is transitive, and the only case it is principal is when G is trivial. It is proper if and only if it is compact, and it is r-discrete if and only if it is discrete.

If G is a disjoint union of groups $\{G_i\}_{i\in I}$, it is a groupoid as well, called a **group bundle**. Here, each group G_i is a transitive component and the unit space G^0 is $\{e_i : i \in I\}$, the set of the union of the units of each group. Suppose we consider on $G = \cup G_i$ a locally compact topology that makes it a locally compact groupoid and induces the relative topology on each G_i. If a left Haar system exists, it is essentially unique in the sense that each measure λ^{e_i} is the (essentially unique) left Haar measure on the group G_i. If the family I has more than one element, this groupoid is not transitive. It is principal if and only if each G_i is trivial. It is r-discrete when each G_i is discrete. In order to be proper, each G_i needs to be compact.

Example 2.2.2. Topological spaces.

If X is a (locally compact) topological space, it is a (locally compact) groupoid with no non-trivial multiplication. The unit space X^0 is the whole space X. For each $x \in X$, $X^x = X_x = X_x^x = \{x\}$, therefore, if $\{\lambda^x\}_{x\in X}$ is a Haar system on X, each measure λ^x has to be δ_x, the Dirac measure at x, or a multiple of it. This is, in fact, a left Haar system. Each λ^x has support X^x. If $f \in C_c(X)$, the map $x \to \int f\, d\lambda^x = f(x)$ is continuous and there is no left invariance to check.

These groupoids are sometimes called **base groupoids**. They are transitive only if the space is trivial, and they are always principal. They are also étale (since $r : X \to X$ is the identity in X) and proper (the map $r \times s : X \to X \times X$ is $r \times s(x) = (x, x)$).

Example 2.2.3. Equivalence relations.

If R is an **equivalence relation** on a (locally compact) space U, it is a groupoid with the following structure: a pair $((w, v'), (v, u))$ of elements of R is composable if and only if $v = v'$, in this case the product is $(w, v)(v, u) = (w, u)$. For the inverse map, we define $(v, u)^{-1} = (u, v)$. Observe that the source function is $s(v, u) = (u, u) \simeq u$, and the range is $r(v, u) = (v, v) \simeq v$, for all $u, v \in U$. Hence, $R^0 \simeq U$. Also, $R_u \simeq \{v : vRu\}$ and $R^v \simeq \{u : uRv\}$.

If U is a locally compact space, R is a locally compact groupoid with the relative topology on $U \times U$. It is transitive only when the equivalence is **full**. It is always principal and proper. It is r-discrete when U is discrete. If λ is a positive, regular measure on U that is densely supported, the family $\{\delta_u \times \lambda\}_u$ is a left Haar system.

Note that we can use this example to conclude that Haar systems are not unique: it is enough to pick different regular measures on U. For instance, for $U = [0, 1]$ and the full equivalence relation, λ we could consider different measures and obtain different Haar systems.

Example 2.2.4. Trivial groupoids.

Let X be a locally compact space and H a locally compact group. Then $X \times H \times X$ is a groupoid with the following operations:

- two pairs (z, k, y'), (y, h, x) are composable if and only if $y = y'$; in this case, $(z, k, y)(y, h, x) = (z, kh, x)$;
- $(y, h, x)^{-1} = (x, h^{-1}, y)$;
- $s(y, h, x) \simeq x$, $r(y, h, x) \simeq y$;

for all $h, k \in H$, $x, y, z \in X$. Thus, we can see $X \times H \times X$ as a product of a group H and a full equivalence relation on X. Groupoids of this form are called **trivial groupoids** and are transitive. Note that groups and full equivalence relations are particular cases of these. If we consider the product topology on $X \times H \times X$, this groupoid is locally compact. It is principal only when H is trivial (we obtain, in fact, a full equivalence groupoid). The map $(r, s) : X \times H \times X \to X \times X$ has preimage $(r, s)^{-1}(K_1, K_2) = (K_1, H, K_2)$, for $K_i \subseteq X$. Therefore, this groupoid is proper if and only if H is compact. It is r-discrete if and only if X and H are discrete.

If $x \in X$,

$$G_x = \{(y, h, x);\, y \in X, h \in H\} \simeq X \times H, \quad G^y = \{(y, h, x);\, x \in X, h \in H\} \simeq H \times X.$$

If m_H is the left Haar measure in H and μ is a Borel regular measure supported in X, let $\lambda^y = \delta_y \times m_H \times \mu$. This is a measure supported in G^y and $\{\lambda^y\}_{y \in X}$ is a left Haar system.

Example 2.2.5. Transformation group groupoid.

Let H be a locally compact group that acts continuously on a locally compact space X (on the left). On $G = H \times X$ we define the following groupoid structure. Let $x, y \in X$, $h, k \in H$. We say that $((k, y), (h, x))$ is a composable pair if and only if $y = hx$; in this case, $(k, hx)(h, x) = (kh, x)$. The inverse map is given by $(h, x)^{-1} = (h^{-1}, hx)$ and the source and range maps are $s(h, x) = (e, x) \simeq x$, $r(h, x) = (e, hx) \simeq hx$.

These are the **transformation group groupoids**. They are very well studied examples of groupoids (see for instance [38] and the first Chapter of [31]). We will see soon that an important part of the groupoid nomeclature comes from here.

Note that

$$\begin{aligned}
&G^0 = \{(h^{-1}, hx)(h, x) : h \in H, x \in X\} = \{(e, x) : x \in X\} \simeq X,\\
&G_x = \{(h, x) : h \in H\} \simeq H,\\
&G^x = \{(h^{-1}, hx) : h \in H\} \simeq H \text{ and}\\
&G_x^x = \{(h, x) : hx = x\} \text{ is the isotropy group at } x.
\end{aligned}$$

The groupoid G is transitive if and only if the action is transitive. It is principal if and only if the action is free. It is proper if and only if the action is proper. The groupoid is r-discrete if and only if H is discrete.

If λ is a left Haar measure for H, the family $\{\lambda^x\}_{x\in X}$, where $\lambda^x = \lambda \times \delta_x$, is a left Haar system.

We can also define a groupoid associated to a partial action $(\{\theta_t\}_{t\in H}, \{X_t\}_{t\in H})$ of a group H on a set X, see [1].

A **partial action** of a group H on a space X is a pair $(\{\theta_t\}_{t\in H}, \{X_t\}_{t\in H})$, where $X_t \subseteq X$, $\theta_t : X_{t^{-1}} \to X_t$ is a bijection, $X_e = X$ and θ_{st} extends $\theta_s\theta_t$, for all $s, t \in H$ (this means that if $x \in X_{t^{-1}}$ and $\theta_t(x) \in X_{s^{-1}}$, then $x \in X_{(st)^{-1}}$ and $\theta_s\theta_t(x) = \theta_{st}(x)$). If the group G is locally compact and X is a topological space, we also require that X_t is open in X, θ_t is a homeomorphism, the set $D^{-1} = \{(t, x) \in H \times X : t \in H, x \in X_{t^{-1}}\}$ is open in $H \times X$ and the map $\theta : D^{-1} \to X$, $\theta(t, x) = \theta_t(x)$ is continuous.

On $G_\theta = \{(y, t, x) : x \in X_{t^{-1}}, \theta_t(x) = y\}$, we consider the following groupoid structure: $((z, s, y'), (y, r, x)) \in G^2_\theta$ if and only if $y' = y = \theta_r(x)$, in this case, $(z, s, y)(y, r, x) = (z, sr, x)$ and the inverse is given by $(y, t, x)^{-1} = (x, t^{-1}, y)$, for all $x, y, z \in X$, $r, s, t \in H$. It follows that $s(y, t, x) = (x, e, x) \simeq x$, $r(y, t, x) = (y, e, y) \simeq y$. Thus, $G^0_\theta \simeq X$.

We identify G_θ and $\{(t, y) : y \in X_t\}$, via $(y, t, x) \leftrightarrow (t, y)$, if $x \in X_{t^{-1}}$, $y = \theta_t(x)$. Following this identification, if $x, y \in X$,

$$
\begin{aligned}
(G_\theta)_x &= \{(t, \theta_t(x)) : x \in X_{t^{-1}}, t \in H\} \simeq \{t \in H : x \in X_{t^{-1}}\} \subseteq H,\\
G^y_\theta &= \{(t, y) : y \in X_t, t \in H\} \simeq \{t \in H : y \in X_t\} \subseteq H \text{ and}\\
(G_\theta)^x_x &= \{(t, x) : x \in X_{t^{-1}}, \theta_t(x) = x\} \simeq \{t \in H : x \in X_{t^{-1}}, \theta_t(x) = x\}.
\end{aligned}
$$

If λ is a left Haar measure on H and B is a Borel subset of G_θ, let $\lambda^x(B) = (\lambda \times \delta^x)(B)$. Then, $\{\lambda^x\}_{x\in X}$ is a left Haar system on G_θ.

Example 2.2.6. Discrete and transitive groupoids.

Let G be a discrete transitive groupoid. Remember that G^u_u is isomorphic to G^v_v, for all $u, v \in G^0$.

Fix $u \in G^0$. We want to establish an isomorphism of groupoids between G and $G^0 \times G^u_u \times G^0$. Define a map $\delta : G^0 \to G^u$ such that $s(\delta_v) = v$. Since we are assuming that G is transitive and discrete, we can define such a map and it is continuous. For a non-discrete transitive locally compact groupoid, the condition we seek is the possibility of defining a continuous section $\delta : G^0 \to \sqcup_{v\in G^0} G^u_v$. Later we will consider a similar but weaker condition, called local triviality, that refers to the possibility of defining locally such a continuous section.

Consider the map $G \to G^0 \times G^u_u \times G^0$, $\gamma \to (r(\gamma), \delta_{r(\gamma)}\gamma(\delta_{s(\gamma)})^{-1}, s(\gamma))$. It is easy to check that the map is an isomorphism of groupoids. Therefore, a discrete, transitive groupoid is determined by $|G^0|$ and G^u_u.

For this groupoid, $G^x \simeq G_u^u \times G^0$. Counting measures on each of the spaces G^v form a left Haar system.

Example 2.2.7. Directed graph groupoid.

A **directed graph** $\mathcal{G}$ is a 4-uple (V, E, R, S) such that: V is a countable set of vertices, E is the set of edges and S and R are the source and range maps.

We assume that $S : E \to V$ is onto and $S^{-1}(v)$ is finite $\forall v$ (we say that the graph is **row finite**). Let $F(\mathcal{G})$ be the set of finite paths and $P(\mathcal{G})$ the set of infinite ones. If x is a path, denote by x_i its i-step.

We define an equivalence relation on $P(\mathcal{G})$: $x \sim_k y$ if and only if it exists $n \in \mathbb{N}$ such that $x_i = y_{i+k}$, $\forall i \geq n$ and $x \sim y$ if and only if it exists $k \in \mathbb{Z}$ such that $x \sim_k y$.

Let

$$G = \{(y, k, x) \in P(\mathcal{G}) \times \mathbb{Z} \times P(\mathcal{G}) :\, x \sim_k y\}$$

and

$$G^2 = \{\big((z, l, y'), (y, k, x)\big) \in G \times G :\, y' = y\}.$$

Define the product by $(z, l, y)(y, k, x) = (z, lk, x)$ and the inverse by $(y, k, x)^{-1} = (x, -k, y)$, for $(y, k, x), (z, l, y) \in G$. It follows that $s(y, k, x) \simeq x$ and $r(y, k, x) \simeq y$. Also, $G^0 \simeq P(\mathcal{G})$ and $G^x \subseteq \mathbb{Z} \times P(\mathcal{G})$.

We define a basis of compact open sets that makes $P(\mathcal{G})$ into a locally compact space (see [28]). For $\alpha \in F(\mathcal{G})$, let

$$Z(\alpha) = \{x \in P(G) :\, x_1 = \alpha_1, x_2 = \alpha_2, \ldots, x_{|\alpha|} = \alpha_{|\alpha|}\}$$

For $\alpha, \beta \in F(\mathcal{G})$, $R(\alpha) = R(\beta)$, let

$$Z(\alpha, \beta) = \{(x, k, y) :\, x \in Z(\alpha), y \in Z(\beta), k = |\beta| - |\alpha|, x_i = y_{i+k}, i > |\alpha|\}$$

These sets form a basis for a locally compact topology on G. With this topology, G is r-discrete and has a Haar system of counting measures.

Example 2.2.8. Locally trivial groupoids.

Locally trivial groupoids were defined in 2.1.5. Observe that they are locally isomorphic to groupoids as in Example 2.2.4. If $u_0 \in G^0$ and the local triviality of G is given by a family $\{U_i, \nu_i\}_{i \in I}$, for each $i \in I$, the map $U_i \times G_{u_0}^{u_0} \times U_i \to G_{U_i}^{U_i}$, defined by $(v, \alpha, u) \to \nu_i(v)^{-1} \alpha \nu_i(u)$ is an isomorphism of groupoids.

Then, full equivalence relation groupoids $X \times X$ and trivial groupoids $X \times H \times X$, for a locally compact space X and locally compact group H, are locally trivial transitive groupoids.

A transitive transformation group groupoid $H \times X$ is locally trivial if and only if each evaluation map $ev_x : H \to X$, $ev_x(h) = h \cdot x$ is a submersion. This is the case of a groupoid associated to a smooth transitive action of a Lie group. Recall that if X, Y are topological spaces and $f : X \to Y$ is a continuous map, we say that f is a **submersion** if for all $x_0 \in X$ there exists an open neighbourhood V of $f(x_0)$ in Y and a right inverse $\sigma : V \to X$ of f such that $\sigma(f(x_0)) = x_0$. If follows that f is an open map.

If K is a subgroup of a group H, $H \times H/K$ is locally trivial if and only if $H \to H/K$ admits local sections.

The fundamental groupoid $\Pi(X)$ of a space X is locally trivial if X is path connected, locally connected and semi-locally simply connected. Also locally trivial transitive groupoids are in one-to-one correspondence with principal bundles. See [29] for these (and more) geometrical examples.

In [33], Section 6, Paterson considers groupoids that are called "locally a product". Locally trivial groupoids are examples of them.

Since locally trivial groupoids are the main example we consider in our work, in the next section we will further develop the examples mentioned above.

2.3 Locally trivial groupoids

As promised before, we now come back to the (non-trivial) examples of locally trivial groupoids (see Definition 2.1.5) mentioned above. We note that the transitive components of locally trivial groupoids are both open and closed (that is, they are components also in the topological sense), and that will allow us to write the Fourier algebra of the groupoid as the sum of those of the transitive components (this will be done in Chapter 5). Also, we show how to construct a Haar system for transitive, unimodular, locally trivial groupoids.

Example 2.3.1. Locally trivial transformation group groupoids.

Let $X \times H$ be a transformation group groupoid as in Example 2.2.5. Moreover, assume that the action is continuous and transitive, so that $X \times H$ is a locally compact transitive groupoid. We verify that $H \times X$ is a locally trivial groupoid if each evaluation map $ev_x : H \to X$, $ev_x(h) = h \cdot x$ is a submersion. Suppose, therefore, that each evaluation map is a submersion.

For a fixed $u \in G^0$, we want to find a family $\{U_i, \nu_i\}_{i \in I}$ as in the definition of local triviality. Since $ev_u : H \to X$ is a submersion, for each $h \in H$ there exist V_h open neighbourhood of hu on X and a continuous right inverse $\sigma_h : V_h \to H$, $\sigma(hu) = h$. We consider the family $\{V_h, \sigma_h\}_{h \in H}$. Note that $\{V_h\}$ is an open covering of X, since given any $v \in X$, by transitivity, there exists $(h, u) \in H \times X$ such that $hu = v$, thus $v \in V_h$. Moreover, $H \simeq \{(k^{-1}, ku) : k \in H\} \simeq G^u$. Thus, each σ_h can be seen as a continuous map $\sigma_h : V_h \to G^u$, $\sigma_h(ku) = (k^{-1}, ku) \in G^u_{ku}$. Therefore, $H \times X$ is locally trivial.

As mentioned before, groupoids associated to smooth transitive actions of Lie groups are locally trivial.

Example 2.3.2. The fundamental groupoid of a locally compact space X.

References for this example are [29] and [44]

A **path** is a continuous map $c : [0, 1] \to X$. If c and d are two paths with the same endpoints $c(0) = d(0)$, $c(1) = d(1)$, we say that c and d are **homotopically equivalent** if there exists a continuous map $F : [0,1] \times [0,1] \to X$ such that $F(\cdot, 0) = c(0) = d(0)$, $F(\cdot, 1) = c(1) = d(1)$, $F(0, \cdot) = c$ and $F(1, \cdot) = d$ (note that we are fixing the end points). This is an equivalence relation and we denote its classes by $\Pi(X)$.

On $\Pi(X)$ we consider the following groupoid structure. Two classes of paths $[c]$, $[d]$ are composable if and only if $c(1) = d(0)$. On that case, $[d][c] = [dc]$ where dc is the path obtained by concatenating c and d, at a different speed:

$$dc(t) = \begin{cases} c(2t) & \text{if } t \in [0, 1/2]; \\ d(2t-1) & \text{if } t \in [1/2, 1]. \end{cases}$$

The inverse of $[c]$ is $[c^{-1}]$, where $c^{-1}(t) = c(1-t)$, for all $t \in [0,1]$. If $x \in X$, the constant path at x is $\kappa_x : [0,1] \to X$, $\kappa_x(t) = x$. It follows that $r([c]) = [\kappa_{c(1)}]$ and $s([c]) = [\kappa_{c(0)}]$. Hence, the unit space can be identified with X. Also, if $x, y \in X$, $\Pi(X)^x = \{[c] : c(1) = x\}$, $\Pi(X)_y = \{[c] : c(0) = y\}$ and $\Pi(X)_x^x = \{[c] : c(1) = c(0) = x\}$ is the fundamental group of X at x. With this structure, $\Pi(X)$ is the **fundamental groupoid** of the space X.

We say that a space X is **locally path-connected** if for all $x \in X$ and for all neighbourhoods V of x there exist a path-connected neighbourhood U of x such that $U \subseteq V$.

The space X is said to be **semi-locally simply connected** if each $x \in X$ has a neighbourhood U with the property that each loop at U is homotopically equivalent to a constant path within X (we say that U is nullhomotopic within X). Note that here we do not ask the homotopy to be within U (that would be the stronger concept of U being simply connected).

If X is locally path-connected and semi-locally simply connected space, then we can define on $\Pi(X)$ a topology that makes the groupoid topological and locally trivial. If in addition X is path-connected, then $\Pi(X)$ is transitive as well. Path connected manifolds are examples of such spaces.

In order to define the topology on $\Pi(X)$, let $x, y \in X$. Let W_x and W_y be path-connected neighbourhoods of x and y respectively that are nullhomotopic within X. If $[c]$

is an equivalence class in $\Pi(X)$ with $c(0) = x$ and $c(1) = y$, let

$$N_{[c]}(W_x, W_y) = \{[\eta_y c \eta_x] \in \Pi(X) : \eta_x : [0,1] \to W_x, \eta_y : [0,1] \to W_y, \eta_x(1) = x, \eta_y(0) = y\}.$$

The sets $N_{[c]}(W_x, W_y)$ form a basis of open neighbourhoods for $\Pi(X)$ and with this topology $\Pi(X)$ is a topological groupoid. The construction of this topology is closely related to the topology of the universal covering space of a path-connected space. Equivalently, the topology on $\Pi(X)$ can be obtained via a quotient: if we consider the compact-open topology on $C([0,1], X)$, the map

$$C([0,1], X) \to \Pi(X), \quad c \to [c],$$

is an open quotient map (see Proposition 6.2, [29]).

Our next goal is to prove that with this topology $\Pi(X)$ is a locally compact groupoid.

We begin by proving that $\Pi(X)$ is a Hausdorff space. We split this proof in two cases.

First, suppose that c and d are two paths that are not homotopically equivalent and such that at least one of the end points does not coincide. Without lost of generality, suppose $x = c(0) \neq d(0) = x'$. Denote $y = c(1)$ and $y' = d(1)$, not necessarily different. We want to prove that there exist neighbourhoods $N_c = N_{[c]}(W_x, W_y)$ and $N_d = N_{[d]}(W_{x'}, W_{y'})$ such that $N_c \cap N_d = \phi$. Since X is a Hausdorff space, we can find neighbourhoods W_x and $W_{x'}$ of x and x' respectively that are path connected, null-homotopic within X and have empty intersection. Suppose $\eta_x \subseteq W_x$, $\eta_y \subseteq W_y$, $\gamma_{x'} \subseteq W_{x'}$ and $\gamma_{y'} \subseteq W_{y'}$ are paths verifying $\eta_x(1) = x$, $\eta_y(0) = y$, $\gamma_{x'}(1) = x'$ and $\gamma_{y'}(0) = y'$. Then the classes $[\eta_y c \eta_c]$ and $[\gamma_{y'} d \gamma_{x'}]$ are different, since they correspond to paths with different end points. Then $N_{[c]}(W_x, W_y) \cap N_{[d]}(W_{x'}, W_{y'}) = \phi$.

For the second case, suppose that c and d are paths that are not homotopically equivalent but their end points coincide. Let $x = c(0) = d(0)$ and $y = c(1) = d(1)$. Suppose W_x and W_y are path connected, null-homotopic within X neighbourhoods of x and y. If $N_{[c]}(W_x, W_y) \cap N_{[d]}(W_x, W_y) \neq \phi$, we reach a contradiction. In effect, assume that $[\delta] \in N_{[c]}(W_x, W_y) \cap N_{[d]}(W_x, W_y)$. This means that there exist paths $\eta_x, \gamma_x \subseteq W_x$ and $\eta_y, \gamma_y \subseteq W_y$ such that $\eta_x(1) = x = \gamma_x(1)$, $\eta_y(0) = y = \gamma_y(0)$ and $[\delta] = [\eta_y c \eta_x] = [\gamma_y d \gamma_x]$.

Since W_x and W_y are null-homotopic within X, $[\gamma_x^{-1}\eta_x] = [\kappa_x]$ and $[\gamma_y^{-1}\eta_y] = [\kappa_y]$. Then, $[c\eta_x] = [\gamma_y^{-1}][\eta_y c\eta_x] = [\gamma_y^{-1}][\gamma_y d\gamma_x][d\gamma_x]$ and $[c] = [c\eta_x][\eta_x^{-1}] = [d\gamma_x][\eta_x^{-1}] = [d]$. But this is a contradiction.

Therefore, $\Pi(X)$ is a Hausdorff space.

Next, we prove that $\Pi(X)$ is a second countable space. If X is a second countable space and $p : X \to Y$ is a surjective map which is continuous and open, then it follows that Y is second countable. Recall that $C([0,1], X) \to \Pi(X)$, $c \to [c]$, is continuous and open, where on $C([0,1], X)$ the topology considered is the compact-open one. Thus it is enough to prove that $C([0,1], X)$ is second countable.

This is a good place to include the definition of the compact-open topology. The compact-open topology on $C(X,Y)$ is the topology that has as a sub-basis the sets

$$U(K,V) = \{f \in C(X,Y) : \ f(K) \subseteq U\},$$

for $K \subset X$ compact and $U \subset Y$ open. In [2], Proposition 1.2, it is proven that $C(X,Y)$ is second countable if X is locally compact (this meaning, as always, that X is also second countable and Hausdorff) and Y is second countable. Therefore, $\Pi(X)$ is second countable.

We now prove that $\Pi(X)$ is locally compact. We use the fact that $(r,s) : \Pi(X) \to X \times X$ is a covering space (see [44], Proposition 5.23, p. 184, and Proposition 4.40, p.145). If X is a topological space, we say that (C,p) is a covering space of X if C is a topological space and $p : C \to X$ is a continuous and surjective map such that for all $x \in X$ there exists an open neighbourhood U of x verifying that $p^{-1}(U) = \sqcup_\alpha V_\alpha$, where each V_α is open in C and $p_{|V_\alpha} : V_\alpha \to U$ is a homeomorphism.

Since X is locally compact, so is $X \times X$. Given $[c] \in \Pi(X)$, we want to find a compact neighbourhood K of $[c]$ (this is enough to prove the local compactness of $\Pi(X)$, we already know that this space is Hausdorff). Suppose $(r,s)[c] = (y,x)$. Then there exists an open neighbourhood $U = W_y \times W_x \subseteq X \times X$ such that

$$(r,s)^{-1}(W_y, W_x) = \sqcup V_\alpha.$$

Let $V = V_{\alpha_0}$ be such that $[c] \in V$. Thus, $(r,s)_{|V} : V \to W_y \times W_x$ is a homeomorphism. By the local compactness of $X \times X$, we can find $\overline{W'_x} \times \overline{W'_y} \subseteq W_x \times W_y$ such that $\overline{W'_x} \times \overline{W'_y}$ is compact. Thus, $(r,s)^{-1}(\overline{W'_x} \times \overline{W'_y})$ is a compact neighbourhood of $[c]$ and it follows that $\Pi(X)$ is locally compact.

Therefore, $\Pi(X)$ is a locally compact groupoid.

Let U_i be a basis of the topology of X, where each set is path-connected and null-homotopic. Let $x_i \in U_i$. If $y \in U_i$, there exists a path $\nu_i(y)$ such that $\nu_i(y)(0) = y$ and $\nu_i(y)(1) = x_i$. Thus, the sets U_i together with the maps $U_i \to \Pi(X)^{x_i}$, $y \to [\nu_i(y)]$ form a family as in the definition of local triviality. Therefore, the fundamental groupoid $\Pi(X)$ with the above topology is locally trivial.

Remark 2.3.3. Let G be a locally trivial groupoid with transitive components $\{G_i\}_{i\in I}$. Suppose $\{U_j, \nu_j, x_j\}_{j\in J}$ is a family as in the definition of local triviality. We observe that each transitive component G_i is open and closed on G.

Denote

$$J_l = \{j \in J : x_j \in G_l\} \subseteq J.$$

For $j \in J_l$, $U_j \subseteq G_l^0 := G_l \cap G^0$ and therefore $s^{-1}(U_j) \subseteq G_l$. Thus, $\cup_{j\in J_l} s^{-1}(U_j) \subseteq G_l$. Reciprocally, if $\gamma \in G_l$, there exists $j \in J_l$ such that $s(\gamma) \in U_j$. Then, $\gamma \in s^{-1}(U_j)$ and $G_l = \cup_{j\in J_l} s^{-1}(U_j)$ is open.

Since transitive components are disjoint, for all $i \in I$, $G_i = G \backslash \cup_{l\neq i} G_l$ is a closed subset of G.

Also, $G_i^0 = G^0 \cap G_i$ is open and closed for all $i \in I$.

Definition 2.3.4. A groupoid is **unimodular** if all of its isotropy groups are unimodular.

Notation 2.3.5. Let X be a topological space. The family of Borel subset of X is denoted by $\mathcal{B}(X)$.

Let G be a transitive, locally trivial and unimodular groupoid. Following Seda's work [47], we can define a Haar system on G that is compatible with the Haar measure on the isotropy group.

Proposition 2.3.6. *Let G be a locally trivial and transitive groupoid. Let $u_0 \in G^0$ and suppose that $G_{u_0}^{u_0}$ is unimodular. Then we can define a Haar system $\{\lambda^v\}_{v\in G^0}$ on G such that $\lambda^{u_0}_{|_{G_{u_0}^{u_0}}}$ is the left Haar measure on $G_{u_0}^{u_0}$.*

Proof. For the reader's convenience, we make explicit Seda's construction ([47]) in our context. Let $u_0 \in G^0$ and let m be the left Haar measure on $G_{u_0}^{u_0}$. We fix a Borel measure μ on G^0, finite on compact subsets, such that $\mu(\{u_0\}) = 1$ and $\mathrm{supp}(\mu) = G^0$.

We begin by defining a regular, positive measure λ^0 supported on G^{u_0} that is $G_{u_0}^{u_0}$-invariant.

Since G is locally trivial, there exists a collection of open sets and continuous maps $\{U_i, \lambda_i\}_{i\in I}$ as in Definition 2.2.8. Fix $i \in I$ and $x \in U_i$. We define a Borel measure on $G_x^{u_0}$ by $w_{i,x}(E) = m(E\nu_i^{-1}(x))$, for all E Borel subset of $G_x^{u_0}$.

We verify that this is a regular measure. Since G is a locally compact space (meaning also Hausdorff and second countable), so is the closed subspace $G_x^{u_0}$. Thus it is enough to verify that $w_{i,x}(K) < \infty$, for all compact K. In fact, for any compact K, $K\nu_i^{-1}(x)$ is a compact subset of $G_{u_0}^{u_0}$ and by regularity of the Haar measure m, $w_{i,x}(K) = m(K\nu_i(x)) < \infty$.

Observe that if $x \in U_i \cap U_j$, there exists a unique $\alpha \in G_{u_0}^{u_0}$ such that $\nu_i^{-1}(x) = \nu_j^{-1}(x)\alpha$. Hence, if $E \in \mathcal{B}(G_x^{u_0})$,

$$w_{i,x}(E) = m(E\nu_i^{-1}(x)) = m(E\nu_j^{-1}(x)\alpha) = m(E\nu_j^{-1}(x)) = w_{j,x}(E),$$

by the right invariance of m.

On $s^{-1}(U_i) \cap G^{u_0}$, we define, for a Borel subset E,

$$\lambda_i^0(E) = \int_{U_i} w_{i,x}(E \cap G_x)\, d\mu(x).$$

This is a positive Borel measure, and by the second countability of G, to check that it is regular it is again enough to verify that it is finite on compact subsets. If $K \subseteq s^{-1}(U_i)\cap G^{u_0}$

is compact, $s(K) \subseteq U_i$ is compact as well and

$$\lambda_i^0(K) = \int_{s(K)} w_{i,x}(K \cap G_x)\, d\mu(x)$$

is the integral of a finite function on a compact set, and hence it is finite.

Note that, if $U_i \cap U_j \neq \phi$, $\lambda_{i|_{U_i \cap U_j}} = \lambda_{j|_{U_i \cap U_j}}$.

If E is a Borel subset of $G_{u_0}^{u_0}$, there exists an i such that $u_0 \in U_i$. Thus, $E \subseteq s^{-1}(U_i \cap G^{u_0})$ and

$$\lambda_i^0(E) = \int_{U_i} w_{i,x}(E \cap G_x)\, d\mu(x) = w_{i,u_0}(E)\mu(\{u_0\}) = m(E\nu_i^{-1}(x)) = m(E).$$

Hence, $\lambda^0_{i\,|_{G_{u_0}^{u_0}}} = m$.

Also, λ_i^0 is $G_{u_0}^{u_0}$-invariant. If $E \in \mathcal{B}(s^{-1}(U_i) \cap G^{u_0})$ and $\alpha \in G_{u_0}^{u_0}$,

$$\lambda_i^0(\alpha E) = \int_{U_i} m(\alpha(E \cap G_x)\nu_i^{-1}(x))\, d\mu(x) = \int_{U_i} m((E \cap G_x)\nu_i^{-1}(x))\, d\mu(x) = \lambda_i^0(E)$$

It is at this point that we need $G_{u_0}^{u_0}$ to be unimodular.

Next, we define a Borel measure λ^0 on G^{u_0} such that $\lambda_{|_{s^{-1}(U_i)}} = \lambda_i^0$, for all $i \in I$, and λ^0 is $G_{u_0}^{u_0}$-invariant. If E is a Borel subset of G^{u_0}, let $E_i = E \cap s^{-1}(U_i)$, then, $E = \cup_i E_i$ and we can make the union disjoint by considering $E_1' = E_1$, $E_2' = E_2 \backslash E_1'$ and $E_n' = E_n \backslash \cup_1^{n-1} E_i'$. Hence, we can define $\lambda^0(E) = \sum \lambda_i^0(E_i')$. We obtain a Borel measure λ^0 on G^{u_0} such that $\lambda_{|_{s^{-1}(U_i)}} = \lambda_i^0$, for all $i \in I$, and λ^0 is $G_{u_0}^{u_0}$-invariant.

Let K be a compact subset of G^{u_0}. Since $\{s^{-1}(U_i)\}_{i \in I}$ is an open cover of G, K has a finite subcovering $\{s^{-1}(U_i)\}_{i=1}^p$. Therefore,

$$\lambda^0(K) = \lambda^0(\cup_{i=1}^p K \cap s^{-1}(U_i)) \leq \sum_{i=1}^p \lambda_i^0(K \cap s^{-1}(U_i)) < \infty$$

and it follows that λ^0 is a regular measure.

Observe that the support of λ^0 is G^{u_0}. In fact, for $\gamma \in G^{u_0}$ and an open neighborhood V of γ, we show that $\lambda^0(V) > 0$. Since $\gamma \in G$, there exists $i \in I$ such that $\gamma \in s^{-1}(U_i)$. Then,

$V \cap s^{-1}(U_i)$ is an open neighborhood of γ and it is enough to show that $\lambda^0(V \cap s^{-1}(U_i)) > 0$. But

$$\lambda^0(V \cap s^{-1}(U_i)) = \lambda_i^0(V \cap s^{-1}(U_i)) = \int_{U_i \cap s^{-1}(V)} w_{i,x}(V \cap s^{-1}(U_i) \cap G_x)\, d\mu(x) > 0$$

because it is the integral of a positive function on an open set with respect to a measure supported on the whole space.

Finally, we show that by translating λ^0 we obtain a left Haar system on G. If $v \in G^0$, there exists $i \in I$ such that $v \in U_i$. Then, $\nu_i(v) \in G_v^{u_0}$ and for $E \in \mathcal{B}(G^v)$, we let $\lambda^v(E) = \lambda^0(\nu_i(v)E)$. It follows that λ^v is a positive, regular, Borel measure supported on G^v. Since the left invariance of the family of measures $\{\lambda^v\}_{v \in G^0}$ is obvious from the definition, it only remains to check the continuity condition of the definition of a left Haar system.

We want to verify that if $f \in C_c(G)$, the map $G^0 \to \mathbb{C}$, $v \to \int f\, d\lambda^v$, is continuous. For each $v \in G^0$, there exists $i \in I$ such that $v \in U_i$. Therefore it is enough to check the continuity of the restrictions of the map to U_i. But

$$\int f(\gamma)\, d\lambda^v(\gamma) = \int f(\nu_i(v^{-1}\gamma))\, d\lambda^0(\gamma)$$

by the definition of λ^v. Thus, we check the continuity of the map $U_i \to \mathbb{C}$, $u \to \int f(\nu_i(v^{-1}\gamma)\, d\lambda^0(\gamma)$, that is, the composition of the continuous maps $U_i \to C_c(G^0)$, $u \to f(\nu_i^{-1}(u)\cdot)$ and $C_c(G^0) \to \mathbb{C}$, $g \to \int g\, d\lambda^0$.

□

We now come back to the examples of locally trivial groupoids that we have available.

Example 2.3.7. Trivial groupoids $X \times H \times X$.

Let X be a locally compact space, then $X \times X$ is (trivially!) locally trivial. For a fixed $x_0 \in X$, the "family" that gives the local triviality is just $\{X, \nu\}$, where $\nu : X \to \{x_0\} \times X$, $\nu(x) = (x_0, x)$. Note that the isotropy groups of this groupoid are trivial, $G_x^x = \{x\} \times \{x\}$, and hence the Haar measure on G_x^x is $\delta_x \times \delta_x$.

When we presented this groupoid (see 2.2.3), we considered the left Haar system $\{\delta^u \times \lambda\}_u$, for any positive, regular measure λ on X. If we consider the restriction of $\delta^u \times \lambda$ to G_u^u, we obtain a zero measure unless $\lambda(\{u\}) > 0$. For instance, if $X = [0,1]$ and λ is the Lebesgue measure, even though it is very natural to consider the Haar system $\{\delta^u \times \lambda\}_u$, the restriction of its measures to the isotropy groups is not the Haar measure of the isotropy group, and that is going to be an important property for us later on. However, we can apply Seda's construction above and obtain another Haar system. This is what we do now, but for a locally trivial groupoid $X \times H \times X$ (if H is trivial, we are back to the full equivalence relation case).

Again, the groupoid $X \times H \times X$ is trivially locally trivial, with the "family" witnessing the local triviality having only one element: (X, ν), for $\nu : X \to G^{x_0} \simeq H \times X$, $\nu(x) = (e, x)$, for any fixed $x_0 \in X$. Denote by m_H the left Haar measure on H and let l be a regular, Borel measure supported on X. To apply Seda's construction we need to make sure that $l(\{x_0\}) > 0$. In many cases this is not true for the "natural" measure we would like to consider, but we can solve this by taking $\mu = l + \delta_{x_0}$. The first step of the construction is defining a measure λ^0 supported in G^{x_0}, that is $G_{x_0}^{x_0}$-invariant and restricted to $G_{x_0}^{x_0}$ is m_H. For each $x \in X$, $G_x^{x_0} \simeq H$ and if E is a Borel subset of $G_x^{x_0}$, $E = \{x_0\} \times E_H \times \{x\}$, we define , $w_x(E) = m_H(E_H)$. If E is a Borel subset of G^{x_0}, $E = \{x_0\} \times E_H \times E_X$, let

$$\lambda^0(E) = \int_X w_x(E \cap G_x)\, d\mu(x) = m_H(E_H)\mu(E_x).$$

Now we use the measure λ^0 to define the rest of the measures of the Haar system. If $y \in X$ and F is a Borel subset of G^y, let

$$\lambda^y(F) = \lambda^0((x_0, e, y)F) = \lambda^0(\{x_0\} \times F_H \times F_X) = m_H(F_H)(l(F_X) + \delta_{x_0}(F_X)).$$

We will come back to this example later.

Chapter 3

Operator spaces and the Haagerup tensor product

The theory of operator spaces is one of the main tools used in our work. In this Chapter we present the background needed on this topic. The first section contains the basic definitions of the theory. A few examples are presented (we will have opportunities to encounter others in the next Chapters) and the main theorems are stated. Here, we do not include proofs and we closely follow Effros and Ruan's book [14]. Other references for this work are Pisier's book [39] and Blecher and LeMerdy's book [5].

The second section concerns tensor products of operator spaces. We begin by presenting Banach space tensor products (the projective tensor product and the injective tensor product) and then their operator space analogues. We also define the nuclear tensor product. Again, our main reference here is [14].

The third section is dedicated to the Haagerup tensor product. This product does not have a Banach space analogue, and, in the author's opinion, it is best described as a miracle! We will need in the following Chapters many of its properties. In addition to [14], in this section we follow Smith's article [49] and here we opt for presenting proofs of some of the results stated.

The last and very short section refers to the extended Haagerup tensor product of operator spaces, and our reference here is [15].

3.1 Operator spaces

Definition 3.1.1. An **operator space** is a vector space $\mathcal{V}$ together with a family of norms $\{\|\cdot\|_n : M_n(\mathcal{V}) \to \mathbb{R}^{\geq 0}\}$ such that the following properties are satisfied:

OS1 If $v = (v_{ij}) \in M_n(\mathcal{V})$ and $w = (w_{kl}) \in M_m(\mathcal{V})$, for $n, m \in \mathbb{N}$, let

$$v \oplus w = \begin{pmatrix} v_{11} & \cdots & v_{1n} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ v_{n1} & \cdots & v_{nn} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & w_{11} & \cdots & w_{1m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & w_{m1} & \cdots & w_{mm} \end{pmatrix}.$$

Then, $\|v \oplus w\|_{n+m} = \max\{\|v\|_n, \|w\|_m\}$.

OS2 If $v = (v_{ij}) \in M_n(\mathcal{V})$, $\alpha = (\alpha_{ij}) \in M_n(\mathbb{C})$ and $\beta = (\beta_{ij}) \in M_n(\mathbb{C})$, let

$$\alpha v = \Big((\sum_{k=1}^{n} \alpha_{ik} v_{kj})_{ij} \Big) \in M_n(\mathcal{V}) \quad \text{and} \quad v\beta = \Big((\sum_{k=1}^{n} \beta_{kj} v_{ik})_{ij} \Big) \in M_n(\mathcal{V}).$$

Then, $\|\alpha v \beta\|_n \leq \|\alpha\| \|v\|_n \|\beta\|$.

Very often we will drop the subindex n from the norms. Also, we refer to $M_n(\mathbb{C})$ as M_n.

Example 3.1.2. Subspaces of operators acting on Hilbert spaces.

Let $\mathcal{H}$ be a Hilbert space. If $\mathcal{V}$ is a subspace of $\mathcal{B}(\mathcal{H})$, it is an operator space with the following structure. To define the norm on $M_n(\mathcal{V})$, we consider the isomorphism (as algebras) between $M_n(\mathcal{B}(\mathcal{H}))$ and $\mathcal{B}(\mathcal{H}^n)$. This isomorphism associates to a matrix $(x_{ij}) \in M_n(\mathcal{B}(\mathcal{H}))$ the operator

$$\begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} \to \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}, \quad \text{for} \quad \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} \in \mathcal{H}^n.$$

Thus, we have a norm on $M_n(\mathcal{B}(\mathcal{H}))$ and since $M_n(\mathcal{V})$ is a subspace of $M_n(\mathcal{B}(\mathcal{H}))$, we give $M_n(\mathcal{V})$ the subspace norm. These norms verify the properties **OS1** and **OS2** and hence $\mathcal{V}$ is an operator space. Later we will see that all operator spaces are, in fact, of this form.

Example 3.1.3. C^*-algebras.

Let $\mathcal{A}$ be a C^*-algebra. By the Gelfand-Naimark-Segal construction, there exist a Hilbert space $\mathcal{H}$ and a faithful representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$. We use this representation to norm $M_n(\mathcal{A})$. With these norms, $\mathcal{A}$ is an operator space. Note that $\mathcal{A}$ is complete.

Remark 3.1.4. If $\mathcal{U}$ is an operator space and $n \neq m$, on $M_{n,m}(\mathcal{U})$ we can consider unambiguously a norm. In order to do that, we add columns or rows of zeros to embed $M_{n,m}(\mathcal{U})$ into $M_{\max\{n,m\}}(\mathcal{U})$.

Remark 3.1.5. Let $\mathcal{V}$ be an operator space. Then each $M_n(\mathcal{V})$ is complete if and only if $\mathcal{V}$ is complete.

We now define the morphisms of the category of operator spaces.

Definitions 3.1.6. Let $\mathcal{U}$ and $\mathcal{V}$ be operator spaces and $T : \mathcal{U} \to \mathcal{V}$ a linear map.

If $n \in \mathbb{N}$, the n^{th}**-amplification of** T is the map $T^n : M_n(\mathcal{U}) \to M_n(\mathcal{V})$ such that $T^n\big((u_{ij})\big) = \big(T(u_{ij})\big)$.

We denote by $\widetilde{M_\infty}(\mathcal{U})$ the $\mathbb{N} \times \mathbb{N}$ matrices over $\mathcal{U}$. If $n = \infty$, $T^\infty : \widetilde{M_\infty}(\mathcal{U}) \to \widetilde{M_\infty}(\mathcal{U})$ is defined by $T\infty((u_{ij})) = (T(u_{ij}))$.

The map T is **completely bounded** if $\|T\|_{cb} := \sup_{n\in\mathbb{N}} \|T^n\|$ is finite.

We denote by $\mathcal{CB}(\mathcal{U},\mathcal{V})$ the space of completely bounded maps from $\mathcal{U}$ to $\mathcal{V}$. This space is a vector space and $\|\cdot\|_{cb}$ is a norm on it. Moreover, $\mathcal{CB}(\mathcal{U},\mathcal{V})$ is an operator space via the identification $M_n(\mathcal{CB}(\mathcal{U},\mathcal{V})) \simeq \mathcal{CB}(\mathcal{U}, M_n(\mathcal{V}))$.

We say that T is a **complete contraction** if $\|T\|_{cb} \leq 1$, a **complete isometry** if each n^{th}-amplification T^n is an isometry and a **complete isomorphism** if T is an isomorphism and the inverse map $T^{-1} : \mathcal{V} \to \mathcal{U}$ is completely bounded.

A bounded linear map $\varphi : E \to F$ between normed spaces is a **quotient map** if the induced map $\overline{\varphi} : \frac{E}{\text{Ker } \varphi} \to F$ is an isometry. Equivalently, φ is a quotient map if $\varphi(E_{\|\cdot\|\leq 1}) = F_{\|\cdot\|\leq 1}$. We say that T is a **complete quotient map** if each T^n is a quotient map.

Example 3.1.7. Homomorphism of C^*-algebras.

Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras. If $\varphi : \mathcal{A} \to \mathcal{B}$ is a $*$-homomorphism, then it is a complete contraction. If φ is injective, it follows that each φ^n is injective, and thus an isometry. Hence φ is a complete isometry.

Example 3.1.8. An isometry that is not completely bounded.

Let $t : M_n \to M_n$ be the transpose map. This map is an isometry but it is not completely contractive. Let ε_{ij} be the matrix in M_n that has a 1 in the i,j entry and 0 everywhere else. Let μ be the matrix in $M_n(M_n)$ that has ε_{ij} in the entry i,j. The norm of μ is 1, since it is a permutation matrix. However, by reordering $t^n(\mu)$ we get the matrix $1_n \oplus \mathcal{O}_{n^2-n}$[1] that has norm greater than or equal to n.

Proposition 3.1.9. *[[14], Corollary 4.1.9] If $\mathcal{V}$, $\mathcal{W}$ are complete operator spaces, then $\varphi : \mathcal{V} \to \mathcal{W}$ is a complete quotient map if and only if $\varphi^* : \mathcal{W}^* \to \mathcal{V}^*$ is a complete isometry.*

Theorem 3.1.10 ([14], Theorem 2.3.5)**.** *If $\mathcal{V}$ is an operator space, there exist a Hilbert space $\mathcal{H}$ and a subspace $\mathcal{W}$ of $\mathcal{B}(\mathcal{H})$ such that $\mathcal{V}$ is completely isometric to $\mathcal{W}$.*

[1]We denote by 1_n the $n \times n$ matrix that has all the entries 1.

We now consider various constructions that give rise to new operator spaces.

1. If $\mathcal{V}$ is an operator space, then $M_n(\mathcal{V})$ is also an operator space, via the identification $M_m(M_n(\mathcal{V})) = M_{mn}(\mathcal{V})$.

2. The dual $\mathcal{V}^*$ of an operator space $\mathcal{V}$ is an operator space as well, since we can identify $M_n(\mathcal{V}^*)$ and $\mathcal{CB}(\mathcal{V}, M_n)$.

3. If $\mathcal{V}$ is an operator space and $\mathcal{V}_0$ is a closed subspace of $\mathcal{V}$, then $\frac{\mathcal{V}}{\mathcal{V}_0}$ is an operator space thanks to the identification
$$M_n\Big(\frac{\mathcal{V}}{\mathcal{V}_0}\Big) \simeq \frac{M_n(\mathcal{V})}{M_n(\mathcal{V}_0)}.$$

4. Let $\{V_\alpha\}_{\alpha\in\Lambda}$ be a family of operator spaces. The direct product $l^\infty - \oplus_{\alpha\in\Lambda}\mathcal{V}_\alpha$ is an operator space via
$$M_n(l^\infty - \oplus_{\alpha\in\Lambda}\mathcal{V}_\alpha) \simeq l^\infty - \oplus_{\alpha\in\Lambda}M_n(\mathcal{V}_\alpha).$$

5. If E is a normed space, for $x \in M_n(E)$ we define
$$\|x\|_{\min,n} = \sup\{\|(f(x_{ij})\| : \, f \in E^*, \, \|f\| \le 1\},$$
$$\|x\|_{\max,n} = \sup\{\|(T(x_{ij})\| : \, T \in \mathcal{B}(E, M_p), \, \|T\| \le 1, \, p \in \mathbb{N}\}.$$

   Let $\min E = (E, \{\| \cdot \|_{\min,n}\})$ and $\max E = (E, \{\| \cdot \|_{\max,n}\})$. Since we have complete isometries
$$\min E \hookrightarrow l^\infty - \{f \in E^* : \, \|f\| \le 1\},$$
$$\max E \hookrightarrow l^\infty - \oplus_{p\in\mathbb{N}} \oplus \{T \in \mathcal{B}(E, M_P) : \, \|T\| \le 1\},$$

   $\min E$ and $\max E$ are operator spaces. We state two properties of these operator spaces that will be needed later.

   Proposition 3.1.11 ([14], (3.3.8) and (3.3.9)). *Let $\mathcal{V}$ be an operator space and E, F normed spaces. The following are isometrical identifications:*

(a) $\mathcal{B}(\mathcal{V}, E) \simeq \mathcal{CB}(\mathcal{V}, \min E)$*. Moreover,* $\mathcal{B}(E, F) \simeq \mathcal{CB}(\min E, \min F)$*.*

(b) $\mathcal{B}(E, \mathcal{V}) \simeq \mathcal{CB}(\max E, \mathcal{V})$*. Moreover,* $\mathcal{B}(E, F) \simeq \mathcal{CB}(\max E, \max F)$*.*

Proposition 3.1.12 ([14], Proposition 3.3.1)**.** *Let $\mathcal{V}$ be an operator space. Then $\mathcal{V} \simeq \min \mathcal{V}$ completely isometrically if and only if there exists a complete isometry $\pi : \mathcal{V} \to \mathcal{A}$ into a commutative C^*-algebra $\mathcal{A}$.*

6. Let $\mathcal{H}$ be a Hilbert space. We define its column and row operator space structure. Here we use the following identification: if $\mathcal{K}$ is another Hilbert space,

$$M_{n,m}(\mathcal{B}(\mathcal{H}, \mathcal{K})) \simeq \mathcal{B}(\mathcal{H}^m, \mathcal{H}^n).$$

If $\xi \in \mathcal{H}$, define the map $\xi_c : \mathbb{C} \to \mathcal{H}$ by $\xi_c(\alpha) = \alpha\xi$. Then, $\xi_c \in \mathcal{B}(\mathbb{C}, \mathcal{H})$. We identify $\mathcal{H}_c = \mathcal{B}(\mathbb{C}, \mathcal{H})$, and this is the **column operator space** structure determined by $\mathcal{H}$.

Theorem 3.1.13 ([14], Theorem 3.4.1)**.** *If $\mathcal{H}$, $\mathcal{K}$ are Hilbert spaces,*

$$\mathcal{B}(\mathcal{H}, \mathcal{K}) \simeq \mathcal{CB}(\mathcal{H}_c, \mathcal{K}_c)$$

is a complete isometrical identification.

Definitions 3.1.14. If E is a normed space, the space of conjugates $\overline{E}$ [2] is an operator space with norm $\|\overline{x}\| = \|x\|$.

The conjugate $\overline{\mathcal{H}}$ of a Hilbert space $\mathcal{H}$ is a Hilbert space as well, with sesquilinear form $\langle \overline{h}, \overline{k} \rangle = \langle k, h \rangle$.

If $\mathcal{V}$ is an operator space, its **conjugate** $\overline{\mathcal{V}}$ is an operator space with structure $\|(\overline{v_{ij}})\| = \|(v_{ij})\|$.

If $\xi \in \mathcal{H}$, let $\xi_r : \overline{\mathcal{H}} \to \mathbb{C}$, $\xi_r(\overline{\eta}) = \langle \xi, \eta \rangle$. Thus, $\xi_r \in \mathcal{B}(\overline{H}, \mathbb{C})$. We identify $\mathcal{H}_r = \mathcal{B}(\overline{\mathcal{H}}, \mathbb{C})$, and this is the **row operator space** of $\mathcal{H}$.

[2]The space $\overline{E}$ has the same additive structure as E, but the multiplication by a scalar is defined as follows: if $\alpha \in \mathbb{C}$ and $\overline{e} \in \overline{E}$, then $\alpha \cdot \overline{e} = \overline{\alpha}e$.

Proposition 3.1.15 ([14], p. 59)**.** *We have complete isometric identifications*

$$(\mathcal{H}_c)^* \simeq \overline{\mathcal{H}_r} \quad \textit{and} \quad (\mathcal{H}_r)^* \simeq \overline{\mathcal{H}_c}.$$

Moreover,

$$\overline{\mathcal{H}_c} \simeq (\overline{\mathcal{H}})_c \quad \textit{and} \quad \overline{\mathcal{H}_r} \simeq (\overline{\mathcal{H}})_r.$$

Proposition 3.1.16 ([14], p. 59)**.** *If $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces, the identification*

$$\mathcal{B}(\overline{\mathcal{K}}, \overline{\mathcal{H}}) \simeq \mathcal{CB}(\mathcal{H}_r, \mathcal{K}_r)$$

is a complete isometry.

3.2 Tensor products of operator spaces

We first briefly consider tensor products of Banach spaces. We follow sections 7.1 and 8.1 of [14].

If E and F are Banach spaces, we can consider different norms on the algebraic tensor product $E \otimes F$ in such a way that the completion with respect to the norm is a new Banach space. If μ is a norm on the space $E \otimes F$, we denote $E \otimes_\mu F = (E \otimes F, \|\cdot\|_\mu)$, that is, the algebraic tensor product with the norm $\|\cdot\|_\mu$ (before completing) and $E \overset{\mu}{\otimes} F$ is the completion of that space.

Definitions 3.2.1. A norm $\|\cdot\|_\mu$ on $E \otimes F$ is a **subcross norm** (**cross norm**) if

$$\|x \otimes y\|_\mu \leq \|x\|\|y\| \quad (\|x \otimes y\|_\mu = \|x\|\|y\|).$$

Remark 3.2.2. If $\|\cdot\|_\mu$ is a subcross-norm and $\sum_{i=1}^n x_i \otimes y_i \in E \otimes F$, it follows that

$$\|\sum_{i=1}^n x_i \otimes y_i\|_\mu \leq \sum_{i=1}^n \|x_i\|\|y_i\|.$$

Definition 3.2.3. The **projective tensor product norm** on $E \otimes F$ is defined by

$$\|u\|_\gamma = \inf \Big\{ \sum_{i=1}^n \|x_i\|\|y_i\| : \, u = \sum_{i=1}^n x_i \otimes y_i \Big\}.$$

It is a cross norm, and the completion of $E \otimes_\gamma F$ is the **Banach space projective tensor product** $E \overset{\gamma}{\otimes} F$.

Definitions 3.2.4. If E, F and G are Banach spaces and $\varphi : E \times F \to G$ is a bilinear map, the **norm** of φ is

$$\|\varphi\| = \sup\{\|\varphi(v, w)\| : \, \|v\|, \|w\| \leq 1\},$$

and

$$\mathcal{B}(E \times F, G) = \{\varphi : E \times F \to G : \, \varphi \text{ is bilinear and } \|\varphi\| < \infty\}.$$

Proposition 3.2.5. *The following identifications are isometric isomorphisms*

$$\mathcal{B}(E \overset{\gamma}{\otimes} F, G) \simeq \mathcal{B}(E \times F, G) \simeq \mathcal{B}(E, \mathcal{B}(F, G)).$$

Proposition 3.2.6. *If (X, μ) is a measure space, for any Banach space E,*

$$L^1(X, \mu) \overset{\gamma}{\otimes} E \simeq L^1(X, E).$$

Definitions 3.2.7. Let E and F be Banach spaces. If $u \in E \otimes F$, the Banach space **injective tensor product norm** of u is

$$\|u\|_\lambda = \sup\{|(f \otimes g)(u)| : \, f \in E^*, g \in F^*, \|f\|, \|g\| \leq 1\}.$$

The completion of $E \otimes_\lambda F$ is $E \overset{\lambda}{\otimes} F$, the **Banach space injective tensor product**. This space can also be defined by considering the natural injection

$$E \otimes F \hookrightarrow \mathcal{B}(E^*, F), \quad (e \otimes f)(\varphi) = \varphi(e) f,$$

for $e \in E, f \in F, \varphi \in E^*$.

Proposition 3.2.8. *Let E and E_1 be Banach spaces. If there is an isometry $E \hookrightarrow E_1$, then for any Banach space F the map $E \overset{\lambda}{\otimes} F \hookrightarrow E_1 \overset{\lambda}{\otimes} F$ is an isometry.*

Proposition 3.2.9. *If X is a locally compact space and E is a Banach space,*

$$C_0(X) \overset{\lambda}{\otimes} E \simeq C_0(X, E)$$

isometrically.

Remark 3.2.10. Let E and F be Banach spaces. Note that the map $E \times F \to E \overset{\lambda}{\otimes} F$, $(x, y) \to x \otimes y$, is norm decreasing (here we are considering on $E \times F$ the norm from Definition 3.2.4). Then it extends to the canonical map $E \overset{\gamma}{\otimes} F \to E \overset{\lambda}{\otimes} F$. Later we will analyze the importance of the kernel of this map and its analogous in the operator space context.

We now consider tensor products of operator spaces. As in the Banach space case, we have a tensor product called "projective" (here, we follow Chapter 7 of [14]) and one called "injective" (see Chapter 8 of [14]).

Definition 3.2.11. If $\mathcal{V}$ and $\mathcal{W}$ are operator spaces and $\| \cdot \|_\mu$ is a norm on $\mathcal{V} \otimes \mathcal{W}$, we say that μ is a **subcross matrix norm** (**cross matrix norm**) if

$$\|v \otimes w\| \leq \|v\| \|w\| \quad (\|v \otimes w\| = \|v\| \|w\|),$$

for all $v \in M_p(\mathcal{V})$, $w \in M_q(\mathcal{W})$.

Remark 3.2.12. Let $u \in M_n(\mathcal{V} \otimes \mathcal{W})$, where $\mathcal{V}$ and $\mathcal{W}$ are operator spaces. Then, there exist $p, q \in \mathbb{N}$ and matrices α, β, v, w such that:

- $\alpha \in M_{n,pq}$, $\beta \in M_{pq,n}$,
- $v \in M_p(\mathcal{V})$,
- $w \in M_q(\mathcal{W})$ and

- $u = \alpha(v \otimes w)\beta$.

Definition 3.2.13. Let $u \in M_n(\mathcal{V} \otimes \mathcal{W})$. The **operator space projective tensor norm** is defined by

$$\begin{aligned}\|u\|_\wedge = \inf\{\|\alpha\|\|v\|\|w\|\|\beta\| :\ & u = \alpha(v \otimes w)\beta, \\ & v \in M_p(\mathcal{V}), w \in M_q(\mathcal{W}), \alpha \in M_{n,p\times q}, \beta \in M_{p\times q,n}, p, q \in \mathbb{N}\}.\end{aligned}$$

Theorem 3.2.14. *Let $\mathcal{V}$ and $\mathcal{W}$ be operator spaces. The space $\mathcal{V} \otimes \mathcal{W}$ together with the family of norms $\{\|\cdot\|_{\wedge,n}\}$ is an operator space. Moreover, $\|\cdot\|_n$ is a subcross norm and it is the largest we can consider on $\mathcal{V} \otimes \mathcal{W}$.*

In fact, $\|\cdot\|_\wedge$ is a cross norm (we need the injective tensor product of operator spaces - to be defined soon- to prove this fact).

We denote by $\mathcal{V} \overset{\wedge}{\otimes} \mathcal{W}$ the **projective tensor product of the operator spaces** $\mathcal{V}$ and $\mathcal{W}$.

Definition 3.2.15. Suppose that $\mathcal{V}$, $\mathcal{W}$ and $\mathcal{Z}$ are operator spaces. If $T : \mathcal{V} \times \mathcal{W} \to \mathcal{Z}$ is a bilinear map and $n, m \in \mathbb{N}$, we define

$$\begin{aligned} T^{(n;m)} : M_n(\mathcal{V}) \times M_n(\mathcal{W}) &\longrightarrow M_{nm}(\mathcal{Z}) \\ T^{(n;m)}\big((v_{ij}), (w_{kl})\big) &= \big(T(v_{ij}, w_{kl})\big). \end{aligned}$$

We say that T is **jointly completely bounded** if $\|T\|_{jcb} := \sup\{\|T^{(n;m)}\| :\ n, m \in \mathbb{N}\}$ is finite.

With this norm, the space $\mathcal{JCB}(\mathcal{V}, \mathcal{W}; \mathcal{Z})$ of jointly completely bounded maps from $\mathcal{V} \times \mathcal{W}$ to $\mathcal{Z}$ is a normed space. It is, in fact, an operator space via the isometric identification

$$M_n(\mathcal{JCB}(\mathcal{V}, \mathcal{W}; \mathcal{Z})) \simeq \mathcal{JCB}(\mathcal{V}, \mathcal{W}; M_n(\mathcal{Z})).$$

Proposition 3.2.16. *For operator spaces $\mathcal{V}$, $\mathcal{W}$ and $\mathcal{Z}$, there are natural completely isometric identifications*

$$\mathcal{CB}(\mathcal{V} \overset{\wedge}{\otimes} \mathcal{W}, \mathcal{Z}) \simeq \mathcal{JCB}(\mathcal{V}, \mathcal{W}; \mathcal{Z}) \simeq \mathcal{CB}(\mathcal{V}, \mathcal{CB}(\mathcal{W}, \mathcal{Z})).$$

Corollary 3.2.17. *If $\mathcal{V}$ and $\mathcal{W}$ are operator spaces, there is a completely isometric identification $(\mathcal{V} \overset{\wedge}{\otimes} \mathcal{W})^* \simeq \mathcal{CB}(\mathcal{V}, \mathcal{W}^*)$. The identification is given by $T(v \otimes w) = T(v)(w)$, for $T \in \mathcal{CB}(\mathcal{V}, \mathcal{W}^*)$, $v \in V$ and $w \in W$.*

Proposition 3.2.18. *Let $\mathcal{V}$, $\mathcal{V}_1$, $\mathcal{W}$ and $\mathcal{W}_1$ be operator spaces. If $R : \mathcal{V} \to \mathcal{V}_1$ and $S : \mathcal{W} \to \mathcal{W}_1$ are complete contractions, then $R \otimes S : \mathcal{V} \otimes \mathcal{W} \to \mathcal{V}_1 \overset{\wedge}{\otimes} \mathcal{W}_1$ extends to a complete contraction*

$$R \otimes S : \mathcal{V} \overset{\wedge}{\otimes} \mathcal{W} \to \mathcal{V}_1 \overset{\wedge}{\otimes} \mathcal{W}_1.$$

Moreover, if R and S are complete quotient maps, then the map $R \otimes S$ is a complete quotient map as well.

*This property is called **projectivity**. Moreover,*

$$\text{Ker } (R \otimes S) = \overline{\text{Ker } (R) \otimes \mathcal{W} + \mathcal{V} \otimes \text{Ker } (S)}.$$

Proposition 3.2.19. *The projective tensor product of operator spaces is commutative and associative.*

Proposition 3.2.20. *If E is a Banach space and W is an operator space, then there is an isometry*

$$\max E \overset{\wedge}{\otimes} W \simeq E \overset{\gamma}{\otimes} W.$$

If F is also a Banach space, then there is a complete isometry

$$\max E \overset{\wedge}{\otimes} \max F \simeq \max(E \overset{\gamma}{\otimes} F).$$

We state a result about the projective tensor product of preduals of von Neumann algebras. First, we need to define the normal tensor product of dual operator spaces.

Proposition 3.2.21 ([14], Proposition 3.2.4)**.** *Let $\mathcal{V}$ be a complete operator space. Then there exist a Hilbert space $\mathcal{H}$ and a map $i : \mathcal{V} \to \mathcal{B}(\mathcal{H})$ such that i is a weak* homeomorphic completely isometric injection. We say that i is a* **dual realization** *of $\mathcal{V}$ on $\mathcal{H}$.*

Definition 3.2.22. Let $\mathcal{V}$ and $\mathcal{W}$ be complete operator spaces with dual realizations $i_V : \mathcal{V}^* \hookrightarrow \mathcal{B}(\mathcal{H}_V)$ and $i_W : \mathcal{W}^* \hookrightarrow \mathcal{B}(\mathcal{H}_W)$. The normal tensor product of $\mathcal{V}^*$ and $\mathcal{W}^*$ is

$$\mathcal{V}^* \overline{\otimes} \mathcal{W}^* := \overline{\mathcal{V}^* \otimes \mathcal{W}^*}^{w^*} \subseteq \mathcal{B}(\mathcal{H}_V \otimes \mathcal{H}_W).$$

Here the weak*-topology is defined by $\mathcal{B}(\mathcal{H}_V \otimes \mathcal{H}_W)_* = \mathcal{B}(\mathcal{H}_V)_* \overset{\wedge}{\otimes} \mathcal{B}(\mathcal{H}_W)_*$.

Theorem 3.2.23 ([14], Theorem 7.2.4)**.** *Suppose $\mathcal{R} \subseteq \mathcal{B}(\mathcal{H}_R)$ and $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H}_S)$ are von Neumann algebras. Then we have a natural complete isometry*

$$\mathcal{R}_* \overset{\wedge}{\otimes} \mathcal{S}_* \simeq (\mathcal{R} \overline{\otimes} \mathcal{S})_*.$$

We now introduce the injective tensor product of operator spaces.

Let $\mathcal{V}$ and $\mathcal{W}$ be operator spaces. If $u \in M_n(\mathcal{V} \otimes \mathcal{W})$, the **injective matrix norm** is

$$\|u\|_{\vee} := \sup\{\|(f \otimes g)^{(n)}(u)\| : \ f \in M_p(\mathcal{V}^*), g \in M_q(\mathcal{W}^*), \|f\|, \|g\| \leq 1, p, q \in \mathbb{N}\}$$

Here,

$$(f \otimes g)^{(n)}(u) = [f_{k,l} \otimes g_{s,t}(u_{i,j})].$$

Proposition 3.2.24. *If $\mathcal{V}$ and $\mathcal{W}$ are operator spaces, the injective matrix norm defined above is determined by the natural embedding $\theta : \mathcal{V} \otimes \mathcal{W} \hookrightarrow \mathcal{CB}(\mathcal{V}^*, \mathcal{W})$.*

*Hence, the injective matrix norms are operator space norms and $\mathcal{V} \overset{\vee}{\otimes} \mathcal{W}$ is an operator space, called the **injective tensor product**.*

Corollary 3.2.25. *If $\mathcal{V}$ is an operator space and $n \in \mathbb{N}$, then the natural identification $M_n(\mathcal{V}) \simeq M_n \overset{\vee}{\otimes} \mathcal{V}$ is a complete isometry.*

Proposition 3.2.26. *Let $\mathcal{V}$, $\mathcal{V}_1$, $\mathcal{W}$ and $\mathcal{W}_1$ be operator spaces. If $R : \mathcal{V} \to \mathcal{V}_1$ and $S : \mathcal{W} \to \mathcal{W}_1$ are complete contractions, then the corresponding map*

$$R \otimes S : \mathcal{V} \otimes \mathcal{W} \longrightarrow \mathcal{V}_1 \overset{\vee}{\otimes} \mathcal{W}_1$$

extends to a complete contraction

$$R \otimes S : \mathcal{V} \overset{\vee}{\otimes} \mathcal{W} \longrightarrow \mathcal{V}_1 \overset{\vee}{\otimes} \mathcal{W}_1.$$

Moreover, if R and S are completely isometric injections, then $R \otimes S$ is a complete isometry as well. This property is called ***injectivity****.*

Proposition 3.2.27. *If $\mathcal{V} \subseteq \mathcal{B}(\mathcal{H})$ and $\mathcal{W} \subseteq \mathcal{B}(\mathcal{K})$ are operator subspaces, for Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, then the map $\mathcal{V} \overset{\vee}{\otimes} \mathcal{W} \hookrightarrow \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ is completely isometric. This motivates calling $\overset{\vee}{\otimes}$ the "spatial tensor product".*

Proposition 3.2.28. *The injective tensor product of operator spaces is commutative and associative. The injective norms are matrix cross norms.*

Proposition 3.2.29. *If E is a Banach space and W is an operator space, then there is an isometry*

$$\min E \overset{\vee}{\otimes} W \simeq E \overset{\lambda}{\otimes} W.$$

If F is also a Banach space, then there is a complete isometry

$$\min E \overset{\vee}{\otimes} \min F \simeq \min(E \overset{\gamma}{\otimes} F).$$

We present some definitions and notations concerning infinite matrices. We will need one result that involves these along with the injective tensor product.

Definitions 3.2.30. Let $\mathcal{V}$ be an operator space. We denote by $\widetilde{M}_\infty(\mathcal{V})$ the $\mathbb{N} \times \mathbb{N}$ matrices over $\mathcal{V}$ and by $\widetilde{M}_n(\mathcal{V})$ the subspace of infinite matrices on $\mathcal{V}$ that have non-zero entries only in the first n rows and columns.

If $v \in \widetilde{M}_\infty(\mathcal{V})$, let

$$v_n := \begin{pmatrix} v_{11} & \cdots & v_{1n} \\ \vdots & \ddots & \vdots \\ v_{n1} & \cdots & v_{nn} \end{pmatrix} \in M_n(\mathcal{V})$$

be the n**-truncation of** v, and

$$\widetilde{v_n} := \begin{pmatrix} v_n & 0 & \cdots & 0 & \cdots \\ 0 & 0 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \in \widetilde{M_n}(\mathcal{V}).$$

We define

$$M_\infty(\mathcal{V}) := \{v \in \widetilde{M}_\infty(\mathcal{V}) : \|v\| := \sup_n \|v_n\| < \infty\}$$

and

$$M_{fin}(\mathcal{V}) = \cup_{n=1}^{\infty} \widetilde{M}_n(\mathcal{V}).$$

Note that with these notations $M_\infty(\mathbb{C}) = \mathcal{B}(l^2) = M_\infty$.

Proposition 3.2.31 ([14], 10.1.1)**.** *If $\mathcal{V}$ is an operator space, so is $M_\infty(\mathcal{V})$.*

Definition 3.2.32. Let $K_\infty(\mathcal{V})$ be the closure of $M_{fin}(\mathcal{V})$ in $M_\infty(\mathcal{V})$.

Proposition 3.2.33. *If $\mathcal{V}$ is an operator space,*

$$K_\infty(\mathcal{V}) \simeq K_\infty \overset{\vee}{\otimes} \mathcal{V}.$$

Moreover,

$$K_\infty(\mathcal{V}) = \{v \in M_\infty(\mathcal{V}) : \lim_{n\to\infty} \|\widetilde{v_n} - v\| = 0\}.$$

We now define the nuclear tensor product of Banach spaces.

Definition 3.2.34. Let $\mathcal{V}$ and $\mathcal{W}$ be operator spaces. As in the Banach space case (see Remark 3.2.10), the canonical map $\iota : \mathcal{V} \overset{\wedge}{\otimes} \mathcal{W} \to \mathcal{V} \overset{\vee}{\otimes} \mathcal{W}$ is a contraction. It is, in fact, a complete contraction.

The **nuclear tensor product** of $\mathcal{V}$ and $\mathcal{W}$ is

$$\mathcal{V} \overset{nuc}{\otimes} \mathcal{W} := \frac{\mathcal{V} \overset{\wedge}{\otimes} \mathcal{W}}{\mathrm{Ker}\ (\iota)}.$$

Remark 3.2.35. Let $\mathcal{V}$, $\mathcal{V}_1$, $\mathcal{W}$ and $\mathcal{W}_1$ be operator spaces. If $R : \mathcal{V} \to \mathcal{V}_1$ and $S : \mathcal{W} \to \mathcal{W}_1$ are complete contractions, then the corresponding map

$$R \otimes S : \mathcal{V} \otimes \mathcal{W} \longrightarrow \mathcal{V}_1 \overset{nuc}{\otimes} \mathcal{W}_1$$

extends to a complete contraction

$$R \otimes S : \mathcal{V} \overset{nuc}{\otimes} \mathcal{W} \longrightarrow \mathcal{V}_1 \overset{nuc}{\otimes} \mathcal{W}_1.$$

3.3 The Haagerup tensor product

The Haagerup tensor product was first introduced by Haagerup in 1980 ([22]). In contrast with the projective and injective tensor product of operator spaces, this product does not have an analog in the Banach space world. This product, that we denote by $\overset{h}{\otimes}$, is one of the main tools of our work, and that is why we include in this section proofs of various of its properties. The main references that we follow here are [14] and [49]. This last reference, the article "Completely bounded module maps and the Haagerup tensor product", has a point of view somehow different from other works on this product. It also presents a result that is very useful for our work (see [49], Theorem 4.5, here Theorem 3.3.27, used in 5.2.8 and 5.3.2). We feel this work is not acknowledged often enough in the literature.

It may seem at first glance that the Haagerup tensor product is overly complicated and non-intuitive. However, after gaining a fuller understanding of its properties, the author believes that it is quite miraculous.

Definition 3.3.1. Let $\mathcal{V}$ and $\mathcal{W}$ be operator spaces. Suppose that $v \in M_{n,p}(\mathcal{V})$ and $w \in M_{p,m}(\mathcal{W})$. Let $v \odot w$ be the matrix in $M_{n,m}(\mathcal{V} \otimes \mathcal{W})$ that has entries $(v \odot w)_{ij} = \sum_{k=1}^{p} v_{ik} \otimes w_{kj}$.

Lemma 3.3.2. *If $\mathcal{V}$ and $\mathcal{W}$ are operator spaces and $u \in M_{nm}(\mathcal{V} \otimes \mathcal{W})$, then there exist $r \in \mathbb{N}$, $v \in M_{nr}(\mathcal{V})$ and $w \in M_{rm}(\mathcal{W})$ such that $u = v \odot w$.*

Definition 3.3.3. If $\mathcal{V}$ and $\mathcal{W}$ are operator spaces and $u \in M_{nm}(\mathcal{V} \otimes \mathcal{W})$, we define the **Haagerup operator space norm**

$$\|u\|_{n,h} = \inf\{\|v\|\|w\| : u = v \odot w, \, v \in M_{nr}(\mathcal{V}), \, w \in M_{rm}(\mathcal{W}), \, r \in \mathbb{N}\}.$$

Theorem 3.3.4. *Let $\mathcal{V}$ and $\mathcal{W}$ be operator spaces. The family of norms $\{\|\cdot\|_{n,h}\}_n$ form an operator space matrix norm on the tensor product $\mathcal{V} \otimes \mathcal{W}$.*

Moreover, if $u \in M_n(\mathcal{V} \otimes \mathcal{W})$, then $\|u\|_\vee \leq \|u\|_h \leq \|u\|_\wedge$.

The completion $\mathcal{V} \overset{h}{\otimes} \mathcal{W}$ of $\mathcal{V} \otimes_h \mathcal{W}$ is the **Haagerup tensor product** *of $\mathcal{V}$ and $\mathcal{W}$.*

Definitions 3.3.5. Let $\mathcal{V}$, $\mathcal{W}$ and $\mathcal{Z}$ be operator spaces. If $B : \mathcal{V} \times \mathcal{W} \to \mathcal{Z}$ is a bilinear map and $n, p \in \mathbb{N}$, define

$$B^{[n,p,n]} : M_{np}(\mathcal{V}) \times M_{pn}(\mathcal{W}) \to M_n(\mathcal{Z}), \quad B^{[n,p,n]}((v_{ij}), (w_{kl})) = \big(\sum_{k=1}^{p} B(v_{ik}, w_{kl})\big)_{ij}.$$

If $p = n$, we write $B^{[n]}$.

Note that if we define $\widetilde{B} : \mathcal{V} \otimes \mathcal{W} \to \mathcal{Z}$, $\widetilde{B}(e \otimes f) = B(e, f)$, and extend linearly, then $B^{[n,p,n]}((v_{ij}), (w_{kl})) = \widetilde{B}^{(n)}(v \odot w)$.

The **multiplicative norm** of B is

$$\|B\|_{mb} = \sup\{\|B^{[n,p,n]}\| : n, p \in \mathbb{N}\} = \sup\{\|B^{[n]}\| : n \in \mathbb{N}\}.$$

We say that B is **multiplicatively bounded** if $\|B\|_{mb} < \infty$. The space of multiplicatively bounded bilinear maps is $\mathcal{MB}(\mathcal{V}, \mathcal{W}; \mathcal{Z})$. This space together with the norm $\|\cdot\|_{mb}$ is a normed space, and moreover, an operator space via the identification

$$M_n(\mathcal{MB}(\mathcal{V}, \mathcal{W}; \mathcal{Z})) = \mathcal{MB}(\mathcal{V}, \mathcal{W}; M_n(\mathcal{Z})).$$

Proposition 3.3.6. *Suppose $\mathcal{V}$, $\mathcal{W}$ and $\mathcal{Z}$ are operator spaces, and $\mathcal{Z}$ is complete. Then*

$$\mathcal{CB}(\mathcal{V} \overset{h}{\otimes} \mathcal{W}; \mathcal{Z}) \simeq \mathcal{MB}(\mathcal{V}, \mathcal{W}; \mathcal{Z})$$

completely isometrically.

The following two lemmas are needed to prove that the Haagerup tensor product is projective and injective.

Lemma 3.3.7. *Suppose $\mathcal{V}$ and $\mathcal{W}$ are operator spaces and $u \in \mathcal{V} \otimes_h \mathcal{W}$, with $\|u\|_h < 1$. Then there exist $v = (v_1, v_2, \cdots, v_r) \in M_{1r}(\mathcal{V})$ and $w = (w_1, w_2, \cdots, w_r)^t \in M_{r1}(\mathcal{W})$ such that $u = v \odot w$, $\|v\|\|w\| < 1$, $v_1, v_2, \cdots, v_r$ are linearly independent in $\mathcal{V}$ and $w_1, w_2, \cdots, w_r$ are linearly independent in $\mathcal{W}$.*

Lemma 3.3.8. *Suppose that $\mathcal{V}$ and $\mathcal{W}$ are operator spaces. Then there is a natural isometry*

$$M_{mn}(\mathcal{V} \overset{h}{\otimes} \mathcal{W}) \simeq M_{m1}(\mathcal{V}) \overset{h}{\otimes} M_{1n}(\mathcal{W}).$$

Proposition 3.3.9. *Let $\mathcal{V}$, $\mathcal{V}_1$, $\mathcal{W}$ and $\mathcal{W}_1$ be operator spaces. If $R : \mathcal{V} \to \mathcal{V}_1$ and $S : \mathcal{W} \to \mathcal{W}_1$ are complete contractions, then the corresponding map*

$$R \otimes S : \mathcal{V} \otimes \mathcal{W} \longrightarrow \mathcal{V}_1 \overset{h}{\otimes} \mathcal{W}_1$$

extends to a complete contraction

$$R \otimes S : \mathcal{V} \overset{h}{\otimes} \mathcal{W} \longrightarrow \mathcal{V}_1 \overset{h}{\otimes} \mathcal{W}_1.$$

If R and S are complete isometries, so is $R \otimes S$. That is, the Haagerup tensor product is injective.

If R and S are complete quotient maps, the same is true for $R \otimes S$. That is, the Haagerup tensor product is projective.

Proposition 3.3.10. *The Haagerup tensor product of operator spaces is associative.*

We present some computations that involve Haagerup tensor products of row and column Hilbert spaces and the injective and projective tensor products. These results are greatly needed later in our work.

Proposition 3.3.11. *Let $\mathcal{V}$ be an operator space and $\mathcal{H}$ a Hilbert space. Then the following identifications are complete isometries:*

$$\mathcal{H}_c \overset{h}{\otimes} \mathcal{V} \simeq \mathcal{H}_c \overset{\vee}{\otimes} \mathcal{V} \quad \text{and} \quad \mathcal{V} \overset{h}{\otimes} \mathcal{H}_r \simeq \mathcal{V} \overset{\vee}{\otimes} \mathcal{H}_r.$$

Moreover,

$$\mathcal{V} \overset{h}{\otimes} \mathcal{H}_c \simeq \mathcal{V} \overset{\wedge}{\otimes} \mathcal{H}_c \quad \text{and} \quad \mathcal{H}_r \overset{h}{\otimes} \mathcal{V} \simeq \mathcal{H}_r \overset{\wedge}{\otimes} \mathcal{V}.$$

completely isometrically as well.

Proposition 3.3.12. *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. Then we have complete isometries*

$$\mathcal{H}_c \overset{\wedge}{\otimes} \mathcal{K}_c \simeq \mathcal{H}_c \overset{h}{\otimes} \mathcal{K}_c \simeq \mathcal{H}_c \overset{\vee}{\otimes} \mathcal{K}_c \simeq (\mathcal{H} \otimes \mathcal{K})_c$$

and

$$\mathcal{H}_r \overset{\wedge}{\otimes} \mathcal{K}_r \simeq \mathcal{H}_r \overset{h}{\otimes} \mathcal{K}_r \simeq \mathcal{H}_r \overset{\vee}{\otimes} \mathcal{K}_r \simeq (\mathcal{H} \otimes \mathcal{K})_r.$$

Remark 3.3.13. These computations can be used to prove that the Haagerup tensor product is not commutative (see [14], p. 163).

Theorem 3.3.14. *Let $\mathcal{V}$ and $\mathcal{W}$ be operator spaces. The natural imbedding*

$$\mathcal{V}^* \overset{h}{\otimes} \mathcal{W}^* \hookrightarrow (\mathcal{V} \overset{h}{\otimes} \mathcal{W})^*$$

is a complete isometry.

Proposition 3.3.15. *Let $\mathcal{V}$ and $\mathcal{W}$ be operator spaces. Let $n \in \mathbb{N}$. If $u \in M_n(\mathcal{V} \otimes_h \mathcal{W})$, then there exist $v \in M_{nr}(\mathcal{V})$ and $w \in M_{rn}(\mathcal{W})$ such that $u = v \otimes w$ and $\|u\|_h = \|v\|\|w\|$.*

The result we aim to present now, from Smith's paper [49], is a generalization of the last proposition. Namely, for any element $u \in M_n(\mathcal{V} \overset{h}{\otimes} \mathcal{W})$, we can find infinite matrices $v \in M_{n\infty}(\mathcal{V})$ and $w \in M_{\infty n}(\mathcal{W})$ such that $u = v \otimes w$ and $\|u\|_h = \|v\|\|w\|$.

From now on, and until the end of the section, we follow [49]. Here we opt for including proofs of the results presented, for completeness. We feel that the techniques of this work are interesting by themselves and they are not as well-known as they deserve to be. We also state some results slightly differently than in Smith's work (see the "complete" added to the statement of Proposition 3.3.17 and Theorem 3.3.23); these changes do not make the proofs any harder.

The main theorem of this section is 3.3.27. We need not only the result stated but also a property that follows from the proof (see 3.3.28).

Theorem 3.3.16. *Let $\mathcal{H}$ be a separable Hilbert space. Let $\mathcal{K}(\mathcal{H})$ be the compact operators on $\mathcal{H}$. If $\phi : \mathcal{K}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is a completely bounded map, then there exist sequences $\{s_i\}$, $\{t_i\}$ in $\mathcal{B}(\mathcal{H})$ such that $\sum_i s_i s_i^*$ and $\sum_i t_i^* t_i$ belong to $\mathcal{B}(\mathcal{H})$, $\| \sum_i s_i s_i^* \| \| \sum_i t_i^* t_i \| = \|\phi\|_{cb}^2$ and for $k \in \mathcal{K}(\mathcal{H})$, $\phi(k) = \sum_i s_i k t_i$.*

Proposition 3.3.17. *Let $\mathcal{H}$ be a separable Hilbert space. The map*

$$\phi : \mathcal{B}(\mathcal{H}) \otimes_h \mathcal{B}(\mathcal{H}) \to \mathcal{CB}(\mathcal{K}(\mathcal{H})), \quad \phi(\sum_{i=1}^{p} v_i \otimes w_i)(k) = \sum_{i=1}^{p} v_i k w_i$$

extends to a complete contraction on $\mathcal{B}(\mathcal{H}) \overset{h}{\otimes} \mathcal{B}(\mathcal{H})$.

Proof. If $u = \sum_{i=1}^{p} v_i \otimes w_i \in \mathcal{B}(\mathcal{H}) \otimes_h \mathcal{B}(\mathcal{H})$, we write $\phi(u)(k) = \phi_u(k) = v k^{(p)} w \in \mathcal{K}(\mathcal{H})$, for

$$v = (v_1 \cdots v_p) \in M_{1p}(\mathcal{B}(\mathcal{H})), \quad w = \begin{pmatrix} w_1 \\ \vdots \\ w_p \end{pmatrix} \in M_{p1}(\mathcal{B}(\mathcal{H}))$$

and

$$k^{(p)} = \begin{pmatrix} k & 0 & 0 & \cdots & 0 \\ 0 & k & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & k \end{pmatrix}$$

Since $\|\phi_u(k)\| \leq \|v\|\|k\|\|w\|$, it follows that $\phi_u \in \mathcal{B}(\mathcal{K}(\mathcal{H}))$ and $\|\phi_u\| \leq \|u\|_h$. We verify that ϕ_u is a completely bounded map. The n^{th}-amplification of ϕ_u is

$$\phi_u^{(n)} : M_n(\mathcal{K}(\mathcal{H})) \to M_n(\mathcal{K}(\mathcal{H})), \quad \phi_u^{(n)}((k_{ij})) = (\phi_u(k_{ij}))_{ij}.$$

We check that $\varphi_u^{(n)}(k) = \tilde{v}\tilde{k}\tilde{w}$, for some $\tilde{v} \in M_{n(np)}(\mathcal{B}(\mathcal{H}))$, $\tilde{w} \in M_{(np)n}(\mathcal{B}(\mathcal{H}))$ and $\tilde{k} \in M_{np}(\mathcal{K}(\mathcal{H}))$. Let

$$\tilde{v} = \left(\begin{array}{c|c|c|c} v_1 \cdots v_p & 0 \cdots 0 & \cdots & 0 \cdots 0 \\ \hline 0 \cdots 0 & v_1 \cdots v_p & 0 \cdots 0 & 0 \vdots 0 \\ \hline \vdots & \ddots & 0 \ddots 0 & \vdots \\ \hline 0 \cdots 0 & \cdots & 0 \cdots 0 & v_1 \cdots v_p \end{array}\right) = \underbrace{v \oplus v \oplus \cdots \oplus v}_{n} \in M_{n(np)}(\mathcal{B}(\mathcal{H})),$$

$$\tilde{w} = \left(\begin{array}{c|c|c|c|c} w_1 & 0 & & & 0 \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ w_p & 0 & & & 0 \\ \hline 0 & w_1 & 0 & & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & w_p & 0 & & 0 \\ \hline \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \hline 0 & & & 0 & w_1 \\ \vdots & \cdots & \cdots & \vdots & \vdots \\ 0 & & & 0 & w_p \end{array}\right) = \underbrace{w \oplus w \oplus \cdots \oplus w}_{n} \in M_{(np)p}(\mathcal{B}(\mathcal{H}))$$

and

$$\tilde{k} = \begin{pmatrix} k_{11}^{(p)} & k_{12}^{(p)} & \cdots & k_{1n}^{(p)} \\ \vdots & \ddots & \ddots & \vdots \\ k_{n1}^{(p)} & \cdots & \cdots & k_{nn}^{(p)} \end{pmatrix} \in M_{(np)(np)}(\mathcal{K}(\mathcal{H})).$$

Note that $\|\tilde{v}\| = \|v\|$, $\|\tilde{w}\| = \|w\|$ and $\|\tilde{k}\| = \|k\|$ (for this last equality, we need to shuffle the entries of the matrix k). Therefore, $\|\phi_u^{(n)}(k)\| \leq \|v\|\|k\|\|w\|$. It follows that $\|\phi_u^{(n)}\| \leq \|u\|_h$ and hence $\phi_u \in \mathcal{CB}(\mathcal{K}(\mathcal{H}))$.

So far we proved that the map

$$\phi : \mathcal{B}(\mathcal{H}) \overset{h}{\otimes} \mathcal{B}(\mathcal{H}) \to \mathcal{CB}(\mathcal{K}(\mathcal{H})), \quad u \to \phi_u,$$

is a contraction. The n^{th}-amplification of ϕ is

$$\phi : M_n(\mathcal{B}(\mathcal{H}) \overset{h}{\otimes} \mathcal{B}(\mathcal{H})) \to M_n(\mathcal{CB}(\mathcal{K}(\mathcal{H}))), \quad (u_{ij}) \to (\phi_{u_{ij}})_{ij}.$$

From 3.3.8 and since $\mathcal{H}$ is separable,

$$M_n(\mathcal{B}(\mathcal{H}) \overset{h}{\otimes} \mathcal{B}(\mathcal{H})) \simeq M_{n1}(\mathcal{B}(\mathcal{H})) \overset{h}{\otimes} M_{1n}(\mathcal{B}(\mathcal{H})) \simeq \mathcal{B}(\mathcal{H}, \mathcal{H}^n) \overset{h}{\otimes} \mathcal{B}(\mathcal{H}^n, \mathcal{H}) \simeq \mathcal{B}(\mathcal{H}) \overset{h}{\otimes} \mathcal{B}(\mathcal{H}).$$

Also,

$$M_n(\mathcal{CB}(\mathcal{K}(\mathcal{H}))) \simeq \mathcal{CB}(\mathcal{K}(\mathcal{H}), \mathcal{K}(\mathcal{H}^n)) \simeq \mathcal{CB}(\mathcal{K}(\mathcal{H}), \mathcal{K}(\mathcal{H}))$$

Then, the amplifications of ϕ have the same properties of ϕ itself and therefore, ϕ is a complete contraction. $\square$

We want to prove that the map ϕ as above is in fact a complete isometry. To this end, we need to introduce the notion of strong independence for sequences of operators.

Definitions 3.3.18. [49] Let $\mathcal{H}$ be a separable Hilbert space and $\mathcal{W}$ a norm closed subspace of $\mathcal{B}(\mathcal{H})$.

Let $\{\eta_i\}$ be a set of operators in $B(\mathcal{H})$, such that $\sum \eta_i^* \eta_i \in B(\mathcal{H})$. We say that $\{\eta_i\}$ is **strongly independent over** $\mathcal{W}$ if for all $\{c_i\} \in l^2$, $\sum c_i \eta_i = 0$ implies $c_i = 0$ for all i.

If $\mathcal{W}$ is the zero subspace, then we say that $\{\eta_i\}$ is **strongly independent**.

Notation 3.3.19. If $\mathcal{V}$ is a vector space and $\{e_i\} \subseteq \mathcal{V}$, we denote by $[e_i]$ the subspace generated by $\{e_i\}$.

Lemma 3.3.20. *Let $\mathcal{H}$ be a separable Hilbert space. Suppose $t \in \mathcal{B}(\mathcal{H}^\infty, \mathcal{H})$ is a column with entries $t_i \in \mathcal{B}(\mathcal{H})$ (then, $\sum_i t_i^* t_i \in \mathcal{B}(\mathcal{H})$) and $s \in \mathcal{B}(\mathcal{H}, \mathcal{H}^\infty)$ is a row with entries $s_i \in \mathcal{B}(\mathcal{H})$ (then, $\sum_i s_i s_i^* \in \mathcal{B}(\mathcal{H})$). Let $\mathcal{W}$ be a closed subspace of $\mathcal{B}(\mathcal{H})$.*

Then there exist unitaries $u_1, u_2 \in B(l^2)$ and disjoint decompositions $N_1 \sqcup N_2 \sqcup N_3 = \mathbb{N}$ and $M_1 \sqcup M_2 \sqcup M_3 = \mathbb{N}$ such that the components $\tilde{t}_i$, $\tilde{s}_i$ of $\tilde{t} = u_1 t$ and $\tilde{s} = s u_2$ satisfy:

1. $\tilde{t_i} = \tilde{s_j} = 0$ *for* $i \in N_1$ *and* $j \in M_1$*;*

2. *for each* $i \in N_2$*,* $\tilde{t_i} \in \mathcal{W} \cap [t_k]_{k \in N_2}$ *and* $\{\tilde{t_i}\}_{i \in N_2}$ *is strongly independent; also, for each* $j \in M_2$*,* $\tilde{s_j} \in \mathcal{W} \cap [s_l]_{l \in M_2}$ *and* $\{\tilde{s_i}\}_{i \in N_2}$ *is strongly independent;*

3. *for each* $i \in N_3$*,* $\tilde{t_i} \in [t_k]_{k \in N_3}$ *and* $\{\tilde{t_i}\}_{i \in N_3}$ *is strongly independent over* $\mathcal{W}$*; also, for each* $j \in M_3$*,* $\tilde{s_j} \in [s_l]_{l \in M_3}$ *and* $\{\tilde{s_j}\}_{j \in N_3}$ *is strongly independent over* $\mathcal{W}$*;*

4. $\|\tilde{t}\| = \|t\|$ *and* $\|\tilde{s}\| = \|s\|$ *and*

5. *if* $\mathcal{W}$ *is finite dimensional, then the sets* N_2 *and* M_2 *are finite.*

Proof. We begin by decomposing the space l^2 in the following way. Let

$$L_1 = \{\lambda \in l^2 : \lambda \cdot t = 0\}$$

and

$$L_2' = \{\lambda \in l^2 : \lambda \cdot t \in \mathcal{W}\}.$$

We let L_2 be the orthogonal complement of L_1 in L_2' and L_3 the orthogonal complement of $L_1 \oplus L_2$ in l^2. Thus, $l^2 = L_1 \oplus L_2 \oplus L_3$.

We consider an orthonormal basis $\{\alpha_i\}_i$ for l^2 such that $\{\alpha_i\}_{i \in N_j}$ is a basis for L_j, for $j = 1, 2, 3$. Hence $\mathbb{N}$ is the disjoint union $N_1 \sqcup N_2 \sqcup N_3$. Each α_i is a sequence $\{\alpha_n^i\}_{n \in \mathbb{N}}$. Let u_1 be the infinite matrix with complex entries that has α_i as the i^{th} row. It is a unitary matrix. Define $\tilde{t} = u_1 t$. Hence the i^{th} entry of $\tilde{t}$ is $\sum_j \alpha_j^i t_j = \alpha_i \cdot t$ and if $j \in N_k$, $\tilde{t_j} \in [t_l]_{l \in N_K}$.

Suppose $i \in N_1$. Then $\alpha_i \in L_1$, and hence $\alpha_i \cdot t = 0$. Thus, $\tilde{t_i} = 0$. If $i \in N_2$, $\alpha_i \in L_2$ and $\alpha_i \cdot t \in \mathcal{W}$. By definition of L_2, $\{\tilde{t_i}\}_{i \in N_2}$ is strongly independent. In order to prove the strong independence of $\{\tilde{t_i}\}_{i \in N_3}$ over $\mathcal{W}$, let $\lambda \in l^2$ such that $\lambda \cdot \{\tilde{t_i}\}_{i \in N_3} = 0$. Then, $\lambda \cdot \{\alpha_i\}_{i \in N_3} \in L_2' \cap L_3$ and this can only be true if $\lambda = 0$.

If $\dim \mathcal{W} = j$, suppose, to reach a contradiction, that N_2 has at least $j + 1$ elements $i_1, i_2, \cdots, i_{j+1}$. By definition of L_2, we know that $\alpha_{i_r} \cdot t \in \mathcal{W}$ for $r = 1, 2, \cdots, j + 1$. Thus, by linear dependence, there exist $\lambda_1, \lambda_2, \cdots, \lambda_{j+1} \in \mathbb{C}$, not all of them zero, such that $\sum_{r=1}^{j+1} \lambda_r \alpha_{i_r} \cdot t = 0$. Then, $\lambda_r \alpha_{i_r} \in L_1 \cap L_2$, and this is a contradiction. We conclude that N_2 has at most j elements.

In order to complete the proof, we repeat the same strategy, but this time multiplying by the row s on the left. $\square$

Corollary 3.3.21. *Suppose $\mathcal{H}$ is a separable Hilbert space. Let $s \in \mathcal{B}(H^\infty, H)$, $t \in \mathcal{B}(H, H^\infty)$, $e \in \mathcal{B}(H^n, H)$ and $t \in \mathcal{B}(H, H^n)$ such that $\|s\|, \|t\| < 1$ and*

$$\sum_{i=1}^{\infty} s_i k t_i - \sum_{i=1}^{n} e_i k f_i = 0, \quad \forall k \in \mathcal{K}(\mathcal{H}). \tag{3.1}$$

Then there exists $\tilde{s} \in \mathcal{B}(H^m, H)$ and $\tilde{t} \in \mathcal{B}(H, H^m)$ such that $\|\tilde{s}\|, \|\tilde{t}\| < 1$,

$$\sum_{i=1}^{m} \tilde{s}_i k \tilde{t}_i - \sum_{i=1}^{n} e_i k f_i = 0, \quad \forall k \in \mathcal{K}(\mathcal{H}).$$

and $\tilde{s}_i \in [e_j] \cap [s_j]$ and $\tilde{t}_i \in [f_j] \cap [t_j]$.

Proof. We begin the proof by applying the lemma above to t, with $\mathcal{W} = [f_j]_{j=1,\cdots,n}$. Then, there exist a unitary u and a partition $\mathbb{N} = N_1 \sqcup N_2 \sqcup N_3$ as in the lemma. Again, we let $t' = ut$.

Define $s' = su^*$. Note that if $k \in \mathcal{K}(\mathcal{H})$, $k^\infty u^* = u^* k^\infty$ (here we denote by k^∞ the ∞-amplification of k) and thus

$$sk^\infty t = sk^\infty u^* ut = s' k^\infty t'.$$

Hence, $\sum_{i=1}^{\infty} s_i' k t_i' - \sum_{i=1}^{n} e_i k f_i = 0$. Write

$$T_k := \sum_{i \in N_2} s_i' k t_i' + \sum_{i \in N_3} s_i' k t_i' - \sum_{i=1}^{n} e_i k f_i.$$

From the definition of N_1 and equation 3.1, it follows that $T_k = 0$.

Pick vectors $h_i \in \mathcal{H}$, $i = 1, 2, 3, 4$, and let $k = h_1 \otimes h_2$, that is $k(h) = \langle h, h_2 \rangle h_1$. Then,

$$\begin{aligned} 0 = \langle T_k h_3, h_4 \rangle &= \sum_{i \in N_2 \cup N_3} \langle s_i'(h_1 \otimes h_2) t_i'(h_3), h_4 \rangle - \sum_{i=1}^{n} \langle e_i (h_1 \otimes h_2) f_i(h_3), h_4 \rangle \\ &= \sum_{i \in N_2 \cup N_3} \langle t_i'(h_3), h_2 \rangle \langle s_i'(h_1), h_4 \rangle - \sum_{i=1}^{n} \langle f_i(h_3), h_2 \rangle \langle e_i(h_1), h_4 \rangle \\ &= \big\langle \big[\sum_{i \in N_2 \cup N_3} \langle s_i'(h_1), h_4 \rangle t_i' - \sum_{i=1}^{n} \langle e_i(h_1), h_4 \rangle f_i \big](h_3), h_2 \big\rangle. \end{aligned} \tag{3.2}$$

Since this is true for all $h_2, h_3 \in \mathcal{H}$, we conclude that

$$\sum_{i \in N_3} \langle s_i'(h_1), h_4 \rangle t_i' = \sum_{i=1}^{n} \langle e_i(h_1), h_4 \rangle f_i - \sum_{i \in N_2} \langle s_i'(h_1), h_4 \rangle t_i'$$

and the right hand of the equation belongs to $\mathcal{W}$. Since $\{t_i'\}_{i \in N_3}$ is strongly independent over $\mathcal{W}$, we conclude that $\langle s_i' h_1, h_4 \rangle = 0$, for all $i \in N_3$. Since this is true for all $h_1, h_4 \in \mathcal{H}$, it follows that $s_i' = 0$ for all $i \in N_3$. Then,

$$\sum_{i=1}^{n} e_i k f_i - \sum_{i \in N_2} s_i' k t_i' = 0 \quad \forall k \in \mathcal{K}(\mathcal{H}).$$

But since $\mathcal{W}$ is finite dimensional, N_2 is a finite set. Renaming the terms of the sum, for some $m \leq n$ we have that

$$\sum_{i=1}^{n} e_i k f_i - \sum_{i=1}^{m} \tilde{s}_i k \tilde{t}_i = 0 \quad \forall k \in \mathcal{K}(\mathcal{H}).$$

It is clear that $\|\tilde{s}\| \leq \|su^*\| = \|s\| \leq 1$, $\tilde{s}_i \in [s_j]$, $\|\tilde{t}\| \leq \|ut\| = \|t\| \leq 1$ and $\tilde{t}_i \in [t_j] \cap [f_j]$. It only remains to prove that $\tilde{s}_i \in [e_j]$. The strong independence of $\{\tilde{t}_i\}_{i=1,\cdots,m}$ means that this set is linearly independent and then extends to a basis $\{\tilde{t}_i\}_{i=1,\cdots,n}$ for $\mathcal{W}$. Rewriting each f_i in terms of the elements of that basis, we obtain that

$$\sum_{i=1}^{n} e_i' k \tilde{t}_i - \sum_{i=1}^{m} \tilde{s}_i k \tilde{t}_i = 0 \quad \forall k \in \mathcal{K}(\mathcal{H}),$$

where each $e'_i \in [e_j]$. Then,

$$\sum_{i=1}^{m}(e'_i - \tilde{s}_i)k\tilde{t}_i + \sum_{i=m+1}^{n} e'_i k\tilde{t}_i = 0 \quad \forall k \in \mathcal{K}(\mathcal{H}),$$

The same method we used above in 3.2 applies here to prove that $\tilde{s}_i - e'_i = 0$ and hence $\tilde{s}_i \in [e_j]$. $\square$

Corollary 3.3.22. *Suppose $\mathcal{H}$ is a separable Hilbert space. Let $s, c \in \mathcal{B}(\mathcal{H}^\infty, \mathcal{H})$, $t, d \in \mathcal{B}(\mathcal{H}, \mathcal{H}^\infty)$, $e \in \mathcal{B}(\mathcal{H}^n, \mathcal{H})$ and $f \in \mathcal{B}(\mathcal{H}, \mathcal{H}^n)$ be such that*

$$sk^\infty t + ck^\infty d - ek^\infty f = 0 \quad \forall k \in \mathcal{K}(\mathcal{H}). \tag{3.3}$$

Moreover, assume that $\|s\|, \|t\| \leq 1$ and $\|c\|, \|d\| \leq \varepsilon < 1$. Then, there exist $\tilde{s}, \tilde{c} \in \mathcal{B}(\mathcal{H}^m, \mathcal{H})$ and $\tilde{t}, \tilde{d} \in \mathcal{B}(\mathcal{H}, \mathcal{H}^m)$ satisfying

1. *$\tilde{s}_i \in [s_j]$ and $\tilde{t}_i \in [t_j]$,*
2. *$\|\tilde{s}\|, \|\tilde{t}\| \leq 1$ and $\|\tilde{c}\|, \|\tilde{d}\| \leq (3\varepsilon)^{1/2}$,*
3. *$\tilde{s}k^\infty\tilde{t} + \tilde{c}k^\infty\tilde{d} - ek^\infty f = 0$, for all $k \in \mathcal{K}(\mathcal{H})$.*

Theorem 3.3.23. *The map $u \to \phi_u$ from Proposition 3.3.17 is a complete isometry.*

Proof. Recall that ϕ is a completely contractive map from $\mathcal{B}(\mathcal{H}) \overset{h}{\otimes} \mathcal{B}(\mathcal{H})$ to $\mathcal{CB}(\mathcal{K}(\mathcal{H}))$. Applying the identifications

$$M_n(\mathcal{B}(\mathcal{H}) \overset{h}{\otimes} \mathcal{B}(\mathcal{H})) \simeq \mathcal{B}(\mathcal{H}) \overset{h}{\otimes} \mathcal{B}(\mathcal{H}) \text{ and } M_n(\mathcal{CB}(\mathcal{K}(\mathcal{H}))) \simeq \mathcal{CB}(\mathcal{K}(\mathcal{H}))$$

as we did before, we see that it is enough to prove that ϕ is an isometry.

Therefore, we only need to prove that if $u \in \mathcal{B}(\mathcal{H}) \otimes_h \mathcal{B}(\mathcal{H})$ is such that $\|\phi_u\| = 1$, then $\|u\|_h \leq 1$. From the structure theorem 3.3.16, since $\phi_u \in \mathcal{CB}(\mathcal{K}(\mathcal{H}))$, there exist $s \in \mathcal{B}(\mathcal{H}^\infty, \mathcal{H})$, $t \in \mathcal{B}(\mathcal{H}, \mathcal{H}^\infty)$, $\|s\| = \|t\| = 1$ and

$$\phi_v(k) = sk^\infty t \quad \forall k \in \mathcal{K}(\mathcal{H}). \tag{3.4}$$

Suppose $u = \sum_{i=1}^n v_i \otimes w_i$. Then, it follows from equation 3.4 that

$$\sum_{i=1}^{\infty} s_i k t_i - \sum_{i=1}^{n} e_i k f_i = 0 \quad \forall k \in \mathcal{K}(\mathcal{H}).$$

Applying Corollary 3.3.21, there exist $\tilde{s} \in \mathcal{B}(\mathcal{H}^m, \mathcal{H})$ and $\tilde{t} \in \mathcal{B}(\mathcal{H}, \mathcal{H}^m)$ of norm less than or equal to one such that

$$\sum_{i=1}^{m} \tilde{s}_i k \tilde{t}_i - \sum_{i=1}^{n} e_i k f_i = 0 \quad \forall k \in \mathcal{K}(\mathcal{H}).$$

From here it follows that $u = \tilde{s} \odot \tilde{t}$. Therefore, $\|u\|_h \leq \|\tilde{s}\|\|\tilde{t}\| \leq 1$. □

We now define the right and left slice maps and the Fubini tensor product.

Definitions 3.3.24. Suppose $\mathcal{H}$ is a separable Hilbert space and $\psi \in \mathcal{B}(\mathcal{H})^*$.

Let

$$R_\psi : \mathcal{B}(\mathcal{H}) \otimes_h \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \quad R_\psi(\sum_{i=1}^{n} e_i \otimes f_i) = \sum_{i=1}^{n} \psi(e_i) f_i.$$

Since this map is a linear functional, it is completely bounded and it extends to

$$R_\psi : \ \mathcal{B}(\mathcal{H}) \overset{h}{\otimes} \ \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}),$$

the **right slice map** of ψ.

Similarly we define the **left slice map** of ψ,

$$L_\psi : \mathcal{B}(\mathcal{H}) \overset{h}{\otimes} \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \quad L_\psi(\sum_{i=1}^{n} e_i \otimes f_i) = \sum_{i=1}^{n} \psi(f_i) e_i.$$

Suppose that $\mathcal{E}_1 \subseteq \mathcal{E}_2$, $\mathcal{F}_1 \subseteq \mathcal{F}_2$ are subspaces of $\mathcal{B}(\mathcal{H})$. The **Fubini product** of $\mathcal{E}_1$ and $\mathcal{F}_1$ relative to $\mathcal{E}_2 \overset{h}{\otimes} \mathcal{F}_2$ is

$$F(\mathcal{E}_1, \mathcal{F}_1; \mathcal{E}_2 \overset{h}{\otimes} \mathcal{F}_2) := \{v \in \mathcal{E}_2 \overset{h}{\otimes} \mathcal{F}_2 : \ R_\psi(v) \in \mathcal{F}_1, \ L_\psi(v) \in \mathcal{E}_1, \ \forall \psi \in \mathcal{B}(\mathcal{H})^*\}.$$

If $\mathcal{E}_2 = \mathcal{F}_2 = \mathcal{B}(\mathcal{H})$, we denote the Fubini product $F(\mathcal{E}_1, \mathcal{F}_1)$.

Remark 3.3.25. Note that $\mathcal{E}_1 \overset{h}{\otimes} \mathcal{F}_1 \subseteq F(\mathcal{E}_1, \mathcal{F}_1; \mathcal{E}_2 \overset{h}{\otimes} \mathcal{F}_2)$. We will prove that those sets in fact coincide.

Remark 3.3.26. If $\mathcal{H}$ is a separable Hilbert space and h_1, h_2 are elements of $\mathcal{H}$, let $L_{12} = L_{\langle \cdot h_1, h_2 \rangle}$ and $R_{12} = R_{\langle \cdot h_1, h_2 \rangle}$.

If $u = \sum_{i=1}^n e_i \otimes f_i \in \mathcal{B}(\mathcal{H}) \otimes_h \mathcal{B}(\mathcal{H})$, note that, for $h_3, h_4 \in \mathcal{H}$,

$$\begin{aligned}
\langle L_{12}(u)(h_3), h_4 \rangle &= \langle \sum_{i=1}^n L_{12}(e_i \otimes f_i)(h_3), h_4 \rangle \\
&= \langle \sum_{i=1}^n \langle f_i(h_1), h_2 \rangle e_i(h_3), h_4 \rangle \\
&= \langle \sum_{i=1}^n e_i(h_3 \otimes h_2) f_i(h_1), h_4 \rangle \\
&= \langle \phi_u(h_3 \otimes h_2)(h_1), h_4 \rangle.
\end{aligned}$$

By continuity, for any $u \in \mathcal{B}(\mathcal{H}) \overset{h}{\otimes} \mathcal{B}(\mathcal{H})$,

$$\langle L_{12}(u)(h_3), h_4 \rangle = \langle \phi_u(h_3 \otimes h_2)(h_1), h_4 \rangle. \tag{3.5}$$

Similarly, if $u \in \mathcal{B}(\mathcal{H}) \overset{h}{\otimes} \mathcal{B}(\mathcal{H})$,

$$\langle R_{12}(u)(h_3), h_4 \rangle = \langle \phi_u(h_1 \otimes h_4)(h_3), h_2 \rangle. \tag{3.6}$$

We can now present the main result of this section.

Theorem 3.3.27. *Suppose $\mathcal{H}$ is a separable Hilbert space and $\mathcal{V}$, $\mathcal{W}$ are closed subspaces of $\mathcal{B}(\mathcal{H})$. If $u \in \mathcal{B}(\mathcal{H}) \overset{h}{\otimes} \mathcal{B}(\mathcal{H})$, the following are equivalent:*

1. *$u \in \mathcal{V} \overset{h}{\otimes} \mathcal{W}$;*

2. *$R_\psi(u) \in \mathcal{W}$ and $L_\psi(u) \in \mathcal{V}$, for all $\psi \in \mathcal{B}(\mathcal{H})^*$;*

3. *the map* ϕ_u *has a representation* $\phi_u(k) = vk^\infty w$, *for all* $k \in \mathcal{K}(\mathcal{H})$, *where* $v \in \mathcal{B}(\mathcal{H}^\infty, \mathcal{H})$, *with components in* $\mathcal{V}$, $w \in \mathcal{B}(\mathcal{H}, \mathcal{H}^\infty)$, *with components in* $\mathcal{W}$, *and* $\|v\| = \|w\| = \|u\|_h^{1/2}$.

Proof. That 1 implies 2 follows from the definition of the Fubini product.

To prove that 2 implies 3, we assume that $\|u\|_h = 1$. Then $\|\phi_u\| = 1$ as well, and there exist $e \in \mathcal{B}(\mathcal{H}^\infty, \mathcal{H})$ and $f \in \mathcal{B}(\mathcal{H}, \mathcal{H}^\infty)$ such that $\phi_u(k) = ek^\infty f$, for all $k \in \mathcal{K}(\mathcal{H})$. Applying Lemma 3.3.20 to the column f and the space $\mathcal{W}$, we obtain a decomposition $\mathbb{N} = N_1 \sqcup N_2 \sqcup N_3$ and a unitary infinite matrix U. Let $w' = Uf$ and $v' = eU^*$. Then, for all $k \in \mathcal{K}(\mathcal{H})$,

$$\phi_u(k) = v'k^\infty w' = v'k^\infty U^*Uw' = v'k^\infty w'.$$

Also, $w'_i = 0$ for all $i \in N_1$, $w'_i \in \mathcal{W}$ for all $i \in N_2$, $\{w'_i\}_{i \in N_2}$ is strongly independent and $\{w'_i\}_{i \in N_3}$ is strongly independent over $\mathcal{W}$.

Let $h_i \in \mathcal{H}$, for $i = 1, 2, 3, 4$. As in equation (3.6) before,

$$\begin{aligned}
\langle R_{12}(u)h_3, h_4\rangle &= \langle \phi_u(h_1 \otimes h_4)h_3, h_2\rangle \\
&= \sum_{i \in N_2 \cup N_3} \langle [v'(h_1 \otimes h_4)^\infty w']h_3, h_2\rangle \\
&= \sum_{i \in N_2} \langle w'_i h_3, h_4\rangle \langle v'_i h_1, h_2\rangle + \sum_{i \in N_3} \langle w'_i h_3, h_4\rangle \langle v'_i h_1, h_2\rangle
\end{aligned}$$

Then,

$$R_{12}(u) - \sum_{i \in N_2} \langle v'_i h_1, h_2\rangle w'_i = \sum_{i \in N_3} \langle v'_i h_1, h_2\rangle w'_i.$$

Since both left-hand side terms belong to $\mathcal{W}$, so does $\sum_{i \in N_3} \langle v'_i h_1, h_2\rangle w'_i$. But $\{w'_i\}_{i \in N_3}$ is strongly independent over $\mathcal{W}$, then $\langle v'_i h_1, h_2\rangle = 0$, for all $h_1, h_2 \in \mathcal{H}$ and $i \in N_3$. Thus, $v'_i = 0$ for all $i \in N_3$.

Then, $\phi_u(k) = \sum_{i \in N_2} v'_i k w'_i$, for all $k \in \mathcal{K}(\mathcal{H})$, and $\|v'\|, \|w'\| \leq 1$. Also, the components of w belong to $\mathcal{W}$. However, these are not the v, w we are looking for, since we do not know if the components of v belong to $\mathcal{V}$. We repeat the procedure above to make sure this is the case.

We apply Lemma 3.3.20 to the row v' and the space $\mathcal{V}$. Then, there exist an infinite, unitary matrix $\tilde{U}$ and a decomposition $N_2 = M_1 \cup M_2 \cup M_3$. We let $v = v'\tilde{U}$ and $w = \tilde{U}^* w'$. Then, $v_i = 0$ for all $i \in N_1$, $v_i \in \mathcal{W}$ for all $i \in N_2$, $\{v_i\}_{i \in N_2}$ is strongly independent and $\{v_i\}_{i \in N_3}$ is strongly independent over $\mathcal{W}$.

Similarly as before, if h_1, h_2, h_3, h_4 belong to $\mathcal{H}$, we can apply equation (3.5) to prove that $w_i = 0$ for all $i \in M_3$. Thus,

$$\phi_v(k) = \sum_{i \in M_2} v_i k w_i \quad \forall k \in \mathcal{K}(\mathcal{H}),$$

the components of v belong to $\mathcal{V}$, the ones of w to $\mathcal{W}$ and $\|v\|, \|w\| \leq 1$.

Finally we prove that 3 implies 1. Let $u \in \mathcal{B}(\mathcal{H}) \overset{h}{\otimes} \mathcal{B}(\mathcal{H})$ such that $\|u\|_h = 1$ and $\phi_u(k) = vk^\infty w$, for all $k \in \mathcal{K}(\mathcal{H})$, with v, w with components in $\mathcal{V}$ and $\mathcal{W}$ respectively. Pick $0 < \varepsilon < 1$ and choose $u_0 = \sum_{i=1}^n a_i \otimes b_i \in \mathcal{B}(\mathcal{H}) \otimes_h \mathcal{B}(\mathcal{H})$ verifying $\|u - u_0\| \leq \varepsilon^2$. Let $u_1 = u_0 - u$, then $\|u_1\| \leq \varepsilon^2$ and $\|\phi_{u_1}\| \leq \varepsilon^2$. Thus, there exist c, d such that $\phi_{u_1}(k) = ck^\infty d$, for all $k \in \mathcal{K}(\mathcal{H})$ and $\|c\|, \|d\| \leq \varepsilon$.

By definition of u_1, $u + u_1 - u_0 = 0$, therefore

$$\sum_{i=1}^{\infty} v_i k w_i + \sum_{i=1}^{\infty} c_i k d_i - \sum_{i=1}^{n} a_i k b_i = 0.$$

We apply Corollary 3.3.22 to find $\tilde{v} \in \mathcal{B}(\mathcal{H}^m, \mathcal{H})$, $\tilde{w} \in \mathcal{B}(\mathcal{H}, \mathcal{H}^m)$, $\tilde{c} \in \mathcal{B}(\mathcal{H}^\infty, \mathcal{H})$ and $\tilde{d} \in \mathcal{B}(\mathcal{H}, \mathcal{H}^\infty)$ verifying:

1. $\|\tilde{v}\|, \|\tilde{w}\| \leq 1$, $\|\tilde{c}\|, \|\tilde{d}\| \leq (3\varepsilon)^{1/2}$;
2. $\tilde{v}k^m \tilde{f} + \tilde{c}k^\infty \tilde{d} - ak^n b = 0$ and
3. the components of $\tilde{c}$ and $\tilde{d}$ are in $\mathcal{V}$ and $\mathcal{W}$ respectively.

Define $u_2 = \sum_{i=1}^{m} \tilde{v}_i \otimes \tilde{w}_i \in \mathcal{V} \otimes \mathcal{W}$. Thus, it follows that $\phi_{u_2-u_0}(k) = -\tilde{c}k^\infty \tilde{d}$. Then, $\|u_2 - u_0\|_h = \|\phi_{u_2-u_0}\| \leq \|\tilde{c}\|\|\tilde{d}\| \leq 3\varepsilon$. Therefore,

$$\|u - u_2\| \leq \|u - u_0\| + \|u_0 - u_2\| \leq \varepsilon^2 + 3\varepsilon \leq 4\varepsilon$$

and since ε was arbitrary, $v \in \mathcal{V} \overset{h}{\otimes} \mathcal{W}$. ☐

Remark 3.3.28. When proving that 2 implies 3, observe that the components of v and w that "survive" the proof are strongly independent.

Definition 3.3.29. Let $u \in \mathcal{B}(\mathcal{H}) \overset{h}{\otimes} \mathcal{B}(\mathcal{H})$. We define

$$\mathcal{R}_u := \overline{\operatorname{span}}\{R_\psi(u) : \psi \in \mathcal{B}(\mathcal{H})^*\}$$

and similarly $\mathcal{L}_u$.

Corollary 3.3.30. *Let $\mathcal{V}_1 \subseteq \mathcal{V}_2$ and $\mathcal{W}_1 \subseteq \mathcal{W}_2$ be closed subspaces of $\mathcal{B}(\mathcal{H})$. Then*

$$\mathcal{V}_1 \overset{h}{\otimes} \mathcal{W}_1 = F(\mathcal{V}_1, \mathcal{W}_1; \mathcal{V}_2 \overset{h}{\otimes} \mathcal{W}_2\}.$$

Proof. The inclusion that we were missing follows from the equivalence between 1 and 2 in the Theorem above. ☐

Corollary 3.3.31. *If $u \in \mathcal{B}(\mathcal{H}) \overset{h}{\otimes} \mathcal{B}(\mathcal{H})$, then $u \in \mathcal{L}_u \overset{h}{\otimes} \mathcal{R}_u$.*

Proof. The result follows from the fact that $u \in F(\mathcal{L}_u, \mathcal{R}_u)$ and the Corollary above. ☐

3.4 The extended Haagerup tensor product

We finish this Chapter by presenting the extended Haagerup tensor product $\overset{eh}{\otimes}$. This tensor product of operator spaces was introduced by Effros and Ruan in an unpublished manuscript ([13]). In 2003, an article containing most of the material of the manuscript was published ([15]). Here we follow that work.

Definitions 3.4.1. Suppose $\mathcal{V}_1, \mathcal{V}_2, \mathcal{W}_1$ and $\mathcal{W}_2$ are operator spaces.

If $\mathcal{V}_1$ and $\mathcal{W}_1$ are dual spaces, we denote by $\mathcal{CB}^\sigma(\mathcal{V}_1, \mathcal{W}_1)$ the subspace of $\mathcal{CB}(\mathcal{V}_1, \mathcal{W}_1)$ consisting of weak* continuous maps.

Suppose $\mathcal{V}_1, \mathcal{V}_2$ and $\mathcal{W}_1$ are dual spaces. If $B : \mathcal{V}_1 \times \mathcal{V}_2 \to \mathcal{W}_1$ is a multiplicatively bounded map, we say that B is **normal** if it weak* continuous on each variable. We denote by $\mathcal{MB}^\sigma(\mathcal{V}_1 \times \mathcal{V}_2; \mathcal{W}_1)$ the space of normal, multiplicatively bounded maps.

The **extended Haagerup tensor product** $\mathcal{V}_1 \overset{eh}{\otimes} \mathcal{V}_2$ is the space of all normal multiplicatively bounded maps $B : \mathcal{V}_1^* \times \mathcal{V}_2^* \to \mathbb{C}$, that is,

$$\mathcal{V}_1 \overset{eh}{\otimes} \mathcal{V}_2 := \mathcal{MB}^\sigma(\mathcal{V}_1^* \times \mathcal{V}_2^*; \mathbb{C}) = (\mathcal{V}_1^* \overset{h}{\otimes} \mathcal{V}_2^*)_\sigma^*.$$

It is an operator space with matrix norms

$$M_n(\mathcal{V}_1 \overset{eh}{\otimes} \mathcal{V}_2) = M_n(\mathcal{MB}^\sigma(\mathcal{V}_1^* \times \mathcal{V}_2^*; \mathbb{C})) = \mathcal{MB}^\sigma(\mathcal{V}_1^* \times \mathcal{V}_2^*; M_n).$$

We now mention without proof some properties of the extended Haagerup tensor product.

Proposition 3.4.2 ([15], p.143)**.** *Suppose $\mathcal{V}_1, \mathcal{V}_2, \mathcal{W}_1$ and $\mathcal{W}_2$ are operator spaces and $S_i : \mathcal{V}_i \to \mathcal{W}_i$ is a completely bounded map, for $i = 1, 2$. Let*

$$\overline{S} = (S_1^* \otimes S_2^*)^* : (\mathcal{V}_1^* \overset{h}{\otimes} \mathcal{V}_2^*)^* \to (\mathcal{W}_1^* \overset{h}{\otimes} \mathcal{W}_2^*)^*.$$

The map $\overline{S}$ satisfies $\overline{S}((\mathcal{V}_1^* \overset{h}{\otimes} \mathcal{V}_2^*)_\sigma^*) \subseteq (\mathcal{W}_1^* \overset{h}{\otimes} \mathcal{W}_2^*)_\sigma^*$. Then, the restriction of $\overline{S}$ to $\mathcal{V}_1 \overset{eh}{\otimes} \mathcal{V}_2$, that is, the map

$$S_1 \otimes S_2 : \mathcal{V}_1 \overset{eh}{\otimes} \mathcal{V}_2 \to \mathcal{W}_1 \overset{eh}{\otimes} \mathcal{W}_2,$$

is completely bounded. If S_1 and S_2 are completely contractive, $S_1 \otimes S_2$ is completely contractive as well.

Theorem 3.4.3 ([15], Theorem 5.3)**.** *If $\mathcal{V}_1$ and $\mathcal{V}_2$ are operator spaces, we have a complete isometry*

$$(\mathcal{V}_1 \overset{h}{\otimes} \mathcal{V}_2)^* \simeq \mathcal{V}_1^* \overset{eh}{\otimes} \mathcal{V}_2^*.$$

Proposition 3.4.4 ([15], p. 145). *The extended Haagerup tensor product of operator spaces is injective but is not projective.*

Proposition 3.4.5. *Let $\mathcal{V}$ and $\mathcal{W}$ be operator spaces. Then*

$$V \overset{h}{\otimes} W = \{u \in V \overset{eh}{\otimes} W : \ u = v \odot w, v \in M_{1,\infty}(V), w \in M_{\infty,1}(W),$$
$$u = \| \cdot \| - \lim_{p} \sum_{j=1}^{p} v_j \otimes w_j\}$$

Lastly, we present a result that will be needed in Chapter 5.

Proposition 3.4.6 ([15], p. 146). *If $\mathcal{V}_1$ and $\mathcal{V}_2$ are operator spaces, there is a complete isometry $\mathcal{V}_1 \overset{h}{\otimes} \mathcal{V}_2 \to \mathcal{V}_1 \overset{eh}{\otimes} \mathcal{V}_2$.*

Proof. The map $\mathcal{V}_1 \overset{h}{\otimes} \mathcal{V}_2 \to \mathcal{V}_1 \overset{eh}{\otimes} \mathcal{V}_2$ is completely contractive since the definition of the norm $\| \cdot \|_{eh}$ uses more decompositions. In the commutative diagram

$$\begin{tikzcd}
\mathcal{V}_1 \overset{h}{\otimes} \mathcal{V}_2 \arrow[d] \arrow[dr] & \\
\mathcal{V}_1 \overset{eh}{\otimes} \mathcal{V}_2 \arrow[r] & (\mathcal{V}_1^* \overset{h}{\otimes} \mathcal{V}_2^*)^*
\end{tikzcd}$$

the bottom arrow is a complete isometry by definition of the extended Haagerup tensor product and so is the diagonal arrow by the self-duality of the Haagerup tensor product (see 3.3.14). If follows that the map that we are considering is a complete isometry as well. $\square$

Chapter 4

The Fourier and Fourier-Stieltjes algebras of a locally compact groupoid

In this Chapter we introduce the Fourier-Stieltjes and Fourier algebras of a locally compact groupoid. Before doing that, in Section 1, for a locally compact group H we present the definition of these algebras, denoted by $B(H)$ and $A(H)$. These definitions are due to Eymard [18]. For the first of these algebras, we need to consider the coefficient functions of all (equivalence classes of) unitary, continuous representations of H. For the second one, we restrict ourselves to the coefficient functions of one representation, the left regular one. We also state equivalent definitions and some of their properties that are going to be relevant when studying the groupoid case. We also explain how $B(H)$ and $A(H)$ have operator spaces structures and the importance of seeing $A(H)$ as a completely contractive Banach algebra. No proofs are presented in this Sections, and other important references, in addition to Eymard's work, are [4] and [45].

The second Section is devoted to the analog of the continuous, unitary representations for the groupoid case: the continuous G-Hilbert bundles. These bundles, indexed over the unit space of a locally compact groupoid G, consist of separable Hilbert spaces (the fibers), together with a notion of continuity of sections and an "action" of the groupoid on the bundle, given by a family of isomorphisms between the Hilbert spaces. For each locally compact groupoid G together with a left Haar system there is one favorite representation: the left regular G-Hilbert bundle. Here, classical work of Dixmier and Douady ([12]) is used, as well as many results for [6].

In the third Section we introduce the definition of the continuous Fourier-Stieltjes algebra $B(G)$, as presented by Paterson in [33]. This definition is the continuous analog of the Borel case presented by Ramsay and Walter ([41]) and the measurable case presented by Renault ([43]). In all three cases, the Fourier-Stieltjes algebra $B(G)$ is a unital, commutative Banach algebra and we include the proof of this fact for the continuous case.

The fourth Section presents a result from Paterson's article [35], namely a stabilization theorem for proper groupoids. This result is going to be needed in the last Section. The equivalence between the category of $C_0(G^0)$-Hilbert modules and the category of Hilbert bundles over G^0 is explained here.

It is in the last Section that the new material of this Chapter appears. We define a continuous Fourier algebra $A(G)$; this definition is, a priori, different from the one presented by Paterson in [33]. In a similar fashion as it was done for $B(G)$, we show that $A(G)$ is a commutative Banach space of continuous functions, this time vanishing at infinity. If G is a locally compact group, this algebra is the same we considered in Section 1. If G is a topological space with base groupoid structure, $A(G) = C_0(G)$. Some desirable properties for a Fourier algebra are verified: $A(G)$ separates points, and if the groupoid is proper, $A(G)$ is a $B(G)$-module (we cannot claim that it is an ideal since we do not know if the norms that we are considering on both spaces coincide). Also, if the groupoid is transitive and compact, $A(G) = B(G)$. For locally trivial groupoids we obtain an easier to handle description of $A(G)$, that is going to be greatly needed in the next Chapter. Using this description, we study the Fourier algebra of a full equivalence relation.

4.1 The group case

In this section H is a locally compact group with left Haar measure m and modular function Δ. For more details see [20].

The **group algebra** of H is $L^1(H, m) = L^1(H)$. This space, with pointwise addition, convolution and involution defined by $f^*(h) = \Delta(h^{-1})\overline{f(h^{-1})}$, is in fact a $*$-Banach algebra. It is also a closed ideal in the space $M(H)$ of complex-valued Radon measures on H.

The "building blocks" of the Fourier-Stieltjes algebra are the coefficients of the continuous unitary representations of the group. If $\mathcal{H}$ is a Hilbert space, let $\mathcal{U}(\mathcal{H})$ be the group of unitary operators on $\mathcal{H}$. Suppose $\pi : H \to \mathcal{U}(\mathcal{H})$ is a group homomorphism. We say that π is WOT-continuous if for all $\xi, \eta \in \mathcal{H}$ the map

$$(\xi, \eta)_\pi = (\xi, \eta) : H \to \mathbb{C}, \quad (\xi, \eta)(h) = \langle \pi(h)\xi, \eta\rangle$$

is continuous. The WOT- continuous group homomorphisms as above are the **continuous unitary representations** of H and the maps $(\xi, \eta)_\pi$ are the **coefficients** associated to π. If π is a continuous unitary representation it extends to a non-degenerate norm-decreasing representation of $L^1(H)$ on $\mathcal{B}(\mathcal{H})$. We still denote this representation by π. It is given by the following formula: if $\xi, \eta \in \mathcal{H}$ and $f \in L^1(H)$, then

$$\langle \pi(f)\xi, \eta\rangle = \int_G f(h)\langle \pi(h)\xi, \eta\rangle \, dm(h).$$

For $f \in L^1(H)$, define the norm

$$\|f\|_{C^*} = \sup\{\pi(f) : \pi \text{ is a continuous, unitary representation of } H\}.$$

The **C^*-algebra** of H, denoted by $C^*(H)$, is the completion of $L^1(H)$ with respect to the norm $\|\cdot\|_{C^*}$.

The **Fourier-Stieltjes algebra** of H is

$$B(H) := \{(\xi, \eta)_\pi : \pi \text{ is a continuous, unitary representation of } H \text{ on } \mathcal{H},\ \xi, \eta \in \mathcal{H}\}.$$

If H is abelian and $\widehat{H}$ is its dual group, then $B(H)$ can be identified with $M(\widehat{H})$. For an arbitrary locally compact group H, $B(H)$ was defined and studied by Eymard in [18]. Note that this space is included in $C_b(H)$, the space of continuous and bounded functions on H. It is, in fact, an algebra with point-wise product.

We define a norm on $B(H)$: if $\varphi \in B(H)$,

$$\|\varphi\|_{B(H)} = \inf_{\varphi=(\xi,\eta)_\pi} \|\xi\|\|\eta\|,$$

where π varies over the continuous unitary representations of H. With this norm, $B(H)$ is a unital, commutative, Banach algebra. Moreover, the infimum on the definition of $\|\cdot\|_{B(H)}$ is always attained.

Eymard proved that $B(H)$ can also be obtained as the Banach space dual of $C^*(H)$.

There is one more description of $B(H)$ that we wish to mention here. Let $u : H \to \mathbb{C}$ be a continuous function. We say that u is **positive definite** if there exist a continuous unitary representation π of H on $\mathcal{H}_\pi$ and a vector $\xi \in \mathcal{H}_\pi$ such that $u = (\xi, \xi)_\pi$. Then, the elements of $B(H)$ are linear combinations with complex coefficients of positive definite functions.

If π is a continuous, unitary representation of H on $\mathcal{H}_\pi$, let

$$A_\pi(H) := \overline{\operatorname{span}}^{\|\cdot\|_{B(H)}}\{(\xi,\eta)_\pi : \xi, \eta \in \mathcal{H}_\pi\},$$

see [4], Definition 2.1. The spaces $A_\pi(H)$ are left and right invariant closed subspaces of $B(H)$. In particular, for the **left regular representation** λ of H on $L^2(H, m)$,

$$\lambda : H \to \mathcal{U}(L^2(H)), \quad \lambda(h)\xi(h') = \xi(h^{-1}h'),$$

for $h, h' \in H$ and $\xi \in L^2(H)$, the space $A_\lambda(H)$ is called the **Fourier algebra** of H. We denote it by $A(H)$ (without the subscript λ). This algebra, for an arbitrary locally compact group H, was introduced by Eymard in [18], and he proved that

$$A(H) = \{(\xi,\eta)_\lambda : \xi, \eta \in L^2(H)\},$$

without need of considering the span and the norm closure. Note that it is a subspace of $C_0(H)$, the space of continuous functions vanishing at infinity. Moreover, it is an ideal of $B(H)$. If H is abelian, $A(H)$ can be identified with $L^1(\widehat{H})$.

Suppose π is a representation of H on $\mathcal{H}_\pi$. Then $\pi(H)$ is included in $\mathcal{B}(\mathcal{H}_\pi)$. Let $VN_\pi(H) := \overline{\pi(H)}^{w^*}$, the weak-* closure of $\pi(H)$ on $\mathcal{B}(\mathcal{H}_\pi)$. This is the **von Neumann algebra generated by** $\pi(H)$ on $\mathcal{B}(\mathcal{H}_\pi)$. Recall that von Neumann algebras are C^*-algebras. For $\pi = \lambda$, the left regular representation, $VN_\lambda(H)$ is denoted by $VN(H)$ and it is called the **group von Neumann algebra** of H. Each von Neumann algebra V has a unique predual, that is, a Banach space V_* whose dual $(V_*)^*$ is V. The predual is a closed subspace of V^*, but it is, in general, smaller. In particular,

$$\mathrm{VN}(H)_* \simeq A(H). \tag{4.1}$$

In $A(H)$ we consider the norm inherited as a subspace of $B(H)$. There is another norm we could define, and we mention it here since we will consider an analog to it in the groupoid context. If $\varphi \in A(H)$, let

$$\|\varphi\|_{A(H)} := \inf_{\varphi=(\xi,\eta)_{\lambda^p}, p\geq 1} \|\xi\|\|\eta\|,$$

where λ^p is the left regular representation with multiplicity $p \in \mathbb{N}$, that is, $\lambda^p : H \to \mathcal{U}(L^2(H)^p)$. From Remarque 2.6 of [18], it follows that

$$(A(H), \|\cdot\|_{A(H)}) = (A(H), \|\cdot\|_{B(H)}). \tag{4.2}$$

We consider one more characterization of $A(H)$.

Theorem 4.1.1 (Théorème 2.2, [4]). *Suppose $\pi : H \to \mathcal{U}(\mathcal{H}_\pi)$ is a representation. Let*

$$q_0 : \overline{\mathcal{H}_\pi} \overset{\gamma}{\otimes} \mathcal{H}_\pi \to A_\pi(H), \quad q_0(\overline{\xi} \otimes \eta) = (\xi, \eta)_\pi$$

and extend linearly to a bounded map. Then,

$$\Big(\frac{\overline{\mathcal{H}_\pi} \overset{\gamma}{\otimes} \mathcal{H}_\pi}{\mathrm{Ker}\ q_0}\Big)^* \simeq VN_\pi(H)$$

isometrically isomorphic. Therefore,

$$\frac{\overline{L^2(H)} \overset{\gamma}{\otimes} L^2(H)}{\text{Ker } q_0} \simeq A(H). \tag{4.3}$$

We aim to derive an equation similar to 4.3 for the **left regular representation with infinite multiplicity**

$$\lambda_\infty : H \to \mathcal{U}(L^2(H; l^2)).$$

Let $\pi : H \to \mathcal{U}(\mathcal{H}_\pi)$ and $\sigma : H \to \mathcal{U}(\mathcal{H}_\sigma)$ be continuous representations of H. We say that π and σ are **unitarily equivalent** if there is a unitary operator $U : \mathcal{H}_\pi \to \mathcal{H}_\sigma$ such that $U\pi(h) = \sigma(h)U$ for all $h \in H$. We write $\pi \sim \sigma$.

We denote $q_0^\infty = q_0^{\lambda^\infty}$.

If π and σ are two continuous unitary representations of H, we have the following result ([11], Proposition 5.3.1):

$$VN_\pi(H) \simeq VN_\sigma(H) \Longleftrightarrow \text{there exists } \alpha, \beta \text{ such that } \pi^\alpha \sim \sigma^\beta.$$

Therefore, $VN(H) \simeq VN_{\lambda_\infty}(H)$ and hence

$$\frac{\overline{L^2(H; l^2)} \overset{\gamma}{\otimes} L^2(H; l^2)}{\text{Ker } q_0^\infty} \simeq A(H). \tag{4.4}$$

Thanks to the identification of $B(H)$ as the dual of $C^*(H)$ and $A(H)$ as the predual of $VN(H)$, we can consider on the Fourier-Stieltjes and the Fourier algebra an operator space structure. Considering this structure, the result above 4.4 admits an operator space version:

$$\frac{\overline{L^2(H; l^2)_r} \overset{\wedge}{\otimes} L^2(H; l^2)_c}{\text{Ker } q_0^\infty} \simeq A(H) \tag{4.5}$$

completely isomorphically.

To end this section we wish to include a few words on the importance of the operator space structure on $A(H)$. We aim to explain why the operator space category is the right context to study the amenability of H.

It is outside the scope of this work to present an introduction to the multifaceted concept of amenability. We refer the reader to [36] for the equivalent definitions of amenable group.

Let A be a Banach algebra and V an A-bimodule. Note that if V is an A-bimodule, so is its dual V^*, with actions

$$(\varphi \cdot a)(v) = \varphi(a \cdot v) \text{ and } (a \cdot \varphi)(v) = \varphi(v \cdot a), \tag{4.6}$$

for $\varphi \in V^*$, $a \in A$ and $v \in V$.

A **bounded derivation** $\delta : A \to V$ is a bounded linear map verifying

$$\delta(ab) = \delta(a)b + a\delta(b) \quad \forall a, b \in A.$$

If $v \in V$, the map $\delta_v : A \to V$ defined by $\delta_v(a) = a \cdot v - v \cdot a$ is a bounded derivation. The derivations δ_v, for $v \in V$, are called **inner**.

We say that a Banach algebra A is **amenable** if for any A-bimodule V, every bounded derivation from A into V^* is inner. In [24], B.E. Johnson proved that a locally compact group H is amenable if and only if the convolution algebra $L^1(H)$ is amenable as a Banach algebra. For the Banach algebra $A(H)$ (which we know to be a L^1 space when H is commutative), the situation is more complicated. It was also Johnson that proved that there are amenable groups whose Fourier algebras are not amenable as a Banach algebras (see [23]). It turns out that the context of operator spaces is more appropriate to study the relationship between the amenability of $A(H)$ and H.

We know that for a locally compact group H, the Fourier algebra $A(H)$ is a Banach algebra and an operator space. Moreover, there is a nice compatibility between this two structures: $A(H)$ is a completely contractive Banach algebra. This means that the product on $A(H)$ extends to a complete contraction

$$m : A(H) \overset{\wedge}{\otimes} A(H) \to A(H).$$

If A is a completely contractive Banach algebra and V is an A-bimodule, we say that V is an **operator A-bimodule** if V is an operator space and the bimodule actions are completely bounded. If V is an operator A-bimodule, so is V^*, with actions defined as above in 4.6. For a completely contractive Banach algebra A and an operator A-bimodule V, the inner derivations $\delta_v : A \to V$ are completely bounded.

In analogy with the Banach space case, we say that a completely contractive Banach algebra A is **operator amenable** if for every operator A-bimodule V every completely bounded derivation $\delta : A \to V^*$ is inner. With these definitions, the relationship between the amenability of H and $A(H)$ is explained by the following theorem proved by Ruan:

Theorem 4.1.2 ([45], Theorem 3.6)**.** *Let H be a locally compact group. Then H is amenable if and only if the Fourier algebra $A(H)$ is operator amenable.*

4.2 Continuous Hilbert bundles

As we explained at the beginning of this Chapter, if H is a locally compact group, the unitary, continuous representation of H are the building blocks of the Banach algebras $B(H)$ and $A(H)$. In this section, we present the "representations" we are going to use to build the Fourier algebras in the groupoid case.

If G is a groupoid, we consider continuous fields of separable Hilbert spaces over the unit space G^0. This is, roughly speaking, a bundle of Hilbert spaces indexed by the units, together with a notion of continuity of sections. If, in addition, there is an action of the groupoid G over the bundle, that is, in some sense, unitary and continuous, we call these bundles G-Hilbert bundles. These are the representation we will use to build $B(G)$ and $A(G)$.

Also in this case, there is a "favourite" representation, the left regular G-Hilbert bundle. We define it here and prove that is one of the continuous and unitary representations we wish to consider.

Definitions 4.2.1. Let X be a topological space. A **continuous field of Banach spaces** over X (see [11]) is a pair $\mathcal{E} = (\{E_x\}_{x\in X}, \Gamma)$ where each E_x is a Banach space and Γ is a family of sections of $\{E_x\}_{x\in X}$ such that:

1. Γ is a $\mathbb{C}$-subspace of sections;
2. for all $x \in X$, $\{\xi(x) : \xi \in \Gamma\}$ is dense in E_x;
3. for all $\xi \in \Gamma$, the map $x \to \|\xi(x)\|$ is continuous; and
4. let η be a section; suppose that $\forall x \in X$ and $\varepsilon > 0$, there exists $\xi \in \Gamma$ such that $\|\eta(y) - \xi(y)\| < \varepsilon$ for all y in neighbourhood of x. Then, $\eta \in \Gamma$.

The elements of Γ are the **continuous sections** of $\mathcal{E}$.

Note that if X is discrete, Γ consists of all the sections.

Remark 4.2.2. Some authors consider a different version of condition 2 above, namely:

2'. For all $x \in X$, $\{\xi(x) : \xi \in \Gamma\} = E_x$.

This is the case in [35] and [6]. Since we opt for the density condition, when adapting some proofs from the mentioned papers we will need to add an ε-technicality.

Example 4.2.3. Constant field.

If E is Banach space and X a topological space, we consider the pair $\mathcal{E} = (\{E\}_{x\in X}, C(X, E))$. It is a continuous field of Banach spaces and it is called **constant field**.

Definitions 4.2.4. Suppose $\mathcal{E} = (\{E_x\}_{x\in X}, \Gamma)$ is a continuous field of Banach spaces and $y \in X$. We say that a section ξ is **continuous** at y if for all $\varepsilon > 0$ there exists $\eta \in \Gamma$ such that $\|\xi(x) - \eta(x)\| \leq \varepsilon$, for all x in a neighbourhood of y. If ξ is continuous at y for all $y \in X$, by condition 4, $\xi \in \Gamma$.

If $\mathcal{E} = (\{E_x\}_{x\in X}, \Gamma)$ is a continuous field and $\Delta \subseteq \Gamma$, we say that Δ is **total** if for all $x \in X$, $\operatorname{span}\{\xi(x) : \xi \in \Delta\}$ is dense on E_x.

We have the following propositions:

Proposition 4.2.5 ([12], Proposition 1,). *Let $\mathcal{E} = (\{E_x\}_{x\in X}, \Gamma)$ be a continuous field of Banach spaces and Δ a total subset of Γ. Let Δ' be the subspace of Γ generated by Δ. If ξ is a section, the following statements are equivalent:*

1. *$\xi \in \Gamma$;*
2. *if $y \in X$ and $\varepsilon > 0$, there exists $\eta \in \Gamma$ such that $\|\xi - \eta\| < \varepsilon$ in a neighbourhood of y;*
3. *if $y \in X$ and $\varepsilon > 0$, there exists $\eta \in \Delta'$ such that $\|\xi - \eta\| < \varepsilon$ in a neighbourhood of y.*

Proposition 4.2.6 ([12], Proposition 3). *Let X be a topological space and $\{E_x\}_{x\in X}$ a family of Banach spaces. Suppose that Δ is a family of sections of $\{E_x\}$ satisfying conditions 1), 2) and 3) from definition 4.2.1. Then there exist a unique family of sections Γ that satisfies condition 4) as well, that is the family of sections satisfying 3. from the proposition above.*

Lemma 4.2.7 ([12], Chapitre I, 2, p. 231). *If $(\{E_x\}_{x\in X}, \Gamma)$ is a continuous field of Banach spaces, then there is a topology on the total space $E := \sqcup_{x\in X} E_x$ such that Γ is the set of continuous functions.*

If $p : E \to X$ is the projection ($p(h) = x$ if $h \in E_x$), for each $\varepsilon > 0$, $V \subseteq X$ and $\xi \in \Gamma$, the sets

$$U(\varepsilon, V, \xi) := \{h \in E : \|h - \xi(p(h))\| < \varepsilon,\, p(h) \in V\}$$

form a basis for the topology on E.

Proof. We include the proof of this lemma, hoping it would give the reader familiarity with the topology we are working with. Also, we need here one of the "ε-tricks" we mentioned in Remark 4.2.2.

Let $U_i = U(\varepsilon_i, \xi_i, V_i)$ be sets as above, for $i = 1, 2$. Suppose $h_0 \in U_1 \cap U_2$. We want to find $\varepsilon_0, \xi_0, V_0$ such that $h_0 \in U(\varepsilon_0, \xi_0, V_0) \subseteq U(\varepsilon_1, \xi_1, V_1) \cap U(\varepsilon_2, \xi_2, V_2)$. Let $\delta = \min\{\varepsilon_1, \varepsilon_2\}$, $x_0 = p(h_0)$ and $\alpha = \max_{i=1,2}\{\|h_0 - \xi_i(x_0)\|\}$. Note that $\alpha < \delta$.

By the density of $\{\eta(x) : \eta \in \Gamma\}$ on E_x, we can find $\xi_0 \in \Gamma$ such that $\|\xi_0(x_0) - h_0\| < \frac{\delta-\alpha}{2}$. It follows that

$$\|\xi_0(x_0) - \xi_i(x_0)\| \leq \|\xi_0(x_0) - h_0\| + \|h_0 - \xi_i(x_0)\| < \frac{\delta - \alpha}{2} + \delta < \frac{\delta + \alpha}{2}.$$

Let $\varepsilon_0 = \frac{\delta-\alpha}{2}$ and

$$V_0 = \Big\{x \in V_1 \cap V_2 : \|\xi_0(x) - \xi_i(x)\| < \frac{\delta + \alpha}{2},\ i = 1, 2\Big\}.$$

Then, $x_0 \in V_0$ and $h_0 \in U(\varepsilon_0, \xi_0, V_0)$. If $h \in U(\varepsilon_0, \xi_0, V_0)$, then $p(h) \in V_0 \subseteq V_i$ and

$$\|h - \xi_i(p(h))\| \leq \|h - \xi_0(p(h))\| + \|\xi_0(p(h)) - \xi_i(p(h))\| < \frac{\delta - \alpha}{2} + \frac{\delta + \alpha}{2} < \varepsilon_i,$$

for $i = 1, 2$. $\square$

Definitions 4.2.8. Let $(\{E_x^1\}_{x\in X}, \Gamma^1)$ and $(\{E_x^2\}_{x\in X}, \Gamma^2)$ be continuous fields of Banach spaces.

A **morphism** between them is a family of linear bounded maps $\{\psi_x : E_x^1 \to E_x^2\}_{x\in X}$ such that

1. $\{\psi \circ \xi : \xi \in \Gamma^1\} \subseteq \Gamma^2$ and
2. if we denote $\|\psi_x\| := \sup_{\|h\|_{E_x^1} \leq 1} \|\psi_x(h)\|$, then $\sup_x \|\psi_x\| < \infty$.

If Δ^1 is a family of continuous sections of $\{E_x^1\}_{x\in X}$ as in 4.2.6, then condition (1.) above can be replaced by

1'. $\{\psi \circ \xi : \xi \in \Delta^1\} \subseteq \Gamma^2$

A morphism of continuous fields of Banach spaces $\{\psi_x\}_{x\in X}$ is an **isometric isomorphism** if each ψ_x is an isometric isomorphism and $\psi \circ \Delta^1$ is dense in Γ^2.

If a continuous field of Banach spaces is isometrically isomorphic to a constant one, we say that it is **trivial**.

Lemma 4.2.9. *[12] Let $(\{E^1_x\}_{x\in X}, \Gamma^1)$ and $(\{E^2_x\}_{x\in X}, \Gamma^2)$ be continuous fields of Banach spaces. A fiber preserving map $\psi : \sqcup_{x\in X} E^1_x \to \sqcup_{x\in X} E^2_x$ is continuous if and only if ψ is a morphism of continuous fields of Banach spaces.*

Definitions 4.2.10. Let $\mathcal{H} = (\{\mathcal{H}^u\}_{u\in X}, \Gamma)$ be a continuous field of Banach spaces over X, where each $\mathcal{H}^u$ is a Hilbert space. We say that $\mathcal{H}$ is a **Hilbert bundle**.

If $\mathcal{H}^1 = (\{\mathcal{H}^u_1\}_{u\in X}, \Gamma_1)$ and $\mathcal{H}^2 = (\{\mathcal{H}^u_2\}_{u\in X}, \Gamma_2)$ are Hilbert bundles over X, a **morphism** between them is a morphism of continuous fields of Banach spaces $\{\psi_x\}$ which also is such that the family of adjoint maps $\{\psi^*_x\}$ determines a morphism.

Definition 4.2.11. Let G be a groupoid. A continuous field of separable Hilbert spaces $(\{\mathcal{H}^u\}_{u\in G^0}, \Gamma)$ over G^0 is a G**-Hilbert bundle** if for each $\gamma \in G$ there is a unitary isomorphism of Hilbert spaces $L_\gamma : \mathcal{H}^{s(\gamma)} \to \mathcal{H}^{r(\gamma)}$ such that

1. $L_u = Id$, $\forall u \in G^0$,
2. if $\gamma'\gamma$ makes sense, then $L_{\gamma'\gamma} = L_{\gamma'} L_\gamma$ and
3. the map from $G \to \mathbb{C}$, $\gamma \to \langle L_\gamma \xi(s(\gamma)), \eta(r(\gamma))\rangle$ is continuous, for all ξ, η continuous and bounded sections of $\mathcal{H}$. Such maps are called **coefficients** and denoted by (ξ, η).

   We use the notation $\mathcal{H} = (\{\mathcal{H}^u\}, \Gamma, L)$.

Remark 4.2.12. If the locally compact groupoid G is a group, then a G-Hilbert bundle is a continuous, unitary representation of G.

Definitions 4.2.13. Note that we are considering G-Hilbert bundles that are **unitary** and **weakly continuous** (see condition 3 above). We say that the bundle is **strongly continuous** if we substitute condition 3 by

3'. the map from $G \to \mathcal{H} = \sqcup_{u \in G^0} H^u$, $\gamma \to L_\gamma \xi(s(\gamma))$ is continuous for every continuous, bounded section ξ of $\mathcal{H}$.

For unitary G-Hilbert bundles, both notions are equivalent:

Proposition 4.2.14. *([6], Lemma 5.1.6) A unitary G-Hilbert bundle $\mathcal{H}$ is weakly continuous if and only if it is strongly continuous.*

Proof. Later on, we will need one of the directions of this Proposition, namely that strong continuity implies weak continuity. For the reader's convenience, we present here a proof of this fact, based on the reference above, that fits our definitions (see Remark 4.2.2).

Suppose that $\mathcal{H} = (\{\mathcal{H}^u\}, \Gamma, L)$ is strongly continuous, we now show that it is weakly continuous as well. Let ξ, η be continuous and bounded sections of $\mathcal{H}$. Let $\gamma \in G$ and $\varepsilon > 0$. We want to find a neighbourhood V of γ such that

$$|\langle L_\gamma \xi(s(\gamma)), \eta(r(\gamma))\rangle - \langle L_{\gamma'} \xi(s(\gamma')), \eta(r(\gamma'))\rangle| < \varepsilon \quad \text{if } \gamma \in V.$$

Since $\{\zeta(r(\gamma)) : \zeta \in \Gamma\}$ is dense in $\mathcal{H}^{r(\gamma)}$, there exists $\xi' \in \Gamma$ such that

$$\|L_\gamma \xi(s(\gamma)) - \xi'(r(\gamma))\| < \frac{\varepsilon}{3\|\eta\|_\infty}.$$

The map $G^0 \to \mathbb{C}$, $v \to \langle \eta(v), \xi'(v)\rangle$, is continuous. Then, we can select an open neighbourhood $U \subseteq G^0$ of $r(\gamma)$ such that if $v \in U$, then

$$|\langle \eta(v), \xi'(v)\rangle - \langle \eta(r(\gamma)), \xi'(r(\gamma))\rangle| < \frac{\varepsilon}{3}.$$

Moreover, since $\mathcal{H}$ is strongly continuous, if $U(\bar{\varepsilon}, \bar{U}, \mu)$ is a neighbourhood of $L_\gamma \xi'(r(\gamma))$, there exists V neighbourhood of γ such that $L_{\gamma'} \xi'(r(\gamma')) \in U(\bar{\varepsilon}, \bar{U}, \mu)$ for all $\gamma' \in V$. In particular, if $\bar{\varepsilon} = \frac{\varepsilon}{3\|\eta\|_\infty}$, $\bar{U} = U$ as above and $\mu = \xi'$, then

$$\|L_{\gamma'} \xi'(r(\gamma')) - \xi(r(\gamma'))\| < \frac{\varepsilon}{3\|\eta\|_\infty}.$$

Then, if $\gamma' \in V$,

$$
\begin{aligned}
|\langle L_\gamma \xi(s(\gamma)), \eta(r(\gamma))\rangle - \langle L_{\gamma'}\xi(s(\gamma')), \eta(r(\gamma'))\rangle|& \\
&\leq |\langle L_{\gamma'}\xi(s(\gamma')), \eta(r(\gamma'))\rangle - \langle \xi'(r(\gamma')), \eta(r(\gamma'))\rangle| \\
&+ |\langle \xi'(r(\gamma')), \eta(r(\gamma'))\rangle - \langle \xi'(r(\gamma)), \eta(r(\gamma))\rangle| \\
&+ |\langle \xi'(r(\gamma)), \eta(r(\gamma))\rangle - \langle L_\gamma \xi(s(\gamma)), \eta(r(\gamma))\rangle| \\
&< \|L_{\gamma'}\xi(s(\gamma')) - \xi'(r(\gamma'))\| \|\eta(r(\gamma'))\| \\
&+ \frac{\varepsilon}{3} \\
&+ \|\xi'(r(\gamma)) - L_\gamma(\xi(s(\gamma))\| \|\eta(r(\gamma))\| \\
&< \varepsilon.
\end{aligned}
$$

Note that here we did not need to assume that $\mathcal{H}$ is unitary. $\square$

Definition 4.2.15. Let $E = (\{E^x\}_{x\in X}, \Gamma)$ be a continuous field of Banach spaces. If ξ is a continuous section of E, we say that ξ **vanishes at infinity** if for all $\varepsilon > 0$ there exists a compact subset K of X such that

$$\|\xi(x)\|_{E_x} < \varepsilon, \quad \forall x \in K^c.$$

Notation 4.2.16. Let $E = (\{E^x\}_{x\in X}, \Gamma)$ be a continuous field of Banach spaces. We denote by $\mathcal{S}C_0(E)$ the continuous sections of E that vanish at infinity and by $\mathcal{S}C_b(E)$ the continuous and bounded ones.

If ξ is a section, the norm of ξ is

$$\|\xi\|_\infty = \sup_{x\in X} \|\xi(x)\|_{E^x}.$$

With this norm, $\mathcal{S}C_b(E)$ is a Banach space and $\mathcal{S}C_0(E)$ is a closed subspace of it. We will see more properties of this subspace on 4.4.3.

Now suppose that $\mathcal{H}$ is a G-Hilbert bundle. Observe that if $\xi, \eta \in \mathcal{S}C_0(\mathcal{H})$,

$$\|(\xi, \eta)\|_\infty \leq \|\xi\| \|\eta\|.$$

Definition 4.2.17 ([6], 5.1.8)**.** Let $\mathcal{H}^i = (\{\mathcal{H}_i^u\}_{u\in G^0}, \Gamma_i, L^i)$ be continuous and unitary G-Hilbert bundles. Suppose that ψ is a morphism of continuous fields of Hilbert spaces between them. We say that ψ is a **morphism** of G-Hilbert bundles if it intertwines L^i, that is, the diagram

$$\begin{tikzcd} \mathcal{H}_1^{s(\gamma)} \arrow[r, "L_\gamma^1"] \arrow[d, "\psi_{s(\gamma)}"'] & \mathcal{H}_1^{r(\gamma)} \arrow[d, "\psi_{r(\gamma)}"] \\ \mathcal{H}_2^{s(\gamma)} \arrow[r, "L_\gamma^2"'] & H_2^{r(\gamma)} \end{tikzcd}$$

commutes.

If ψ is a morphism of G-Hilbert bundles as above and

$$\langle \psi_u(\xi(u)), \psi_u(\eta(u))\rangle = \langle \xi(u), \eta(u)\rangle,$$

for all ξ, η continuous sections of $\mathcal{H}^1$ and for all $u \in G^0$, we say that $\mathcal{H}^1$ is **isometrically isomorphic to a sub-bundle** of $\mathcal{H}^2$.

If, in addition, $\psi \circ \Delta^1$ is dense in Γ^2, for Δ^1 a subset of continuous sections as in 4.2.6, we say that ψ is an **isometric isomorphism**.

Remark 4.2.18. Let $\mathcal{H}^1$ and $\mathcal{H}^2$ be continuous G-Hilbert bundles as in the definition above. Suppose that $\mathcal{H}^1$ is isometrically isomorphic to a sub-bundle of $\mathcal{H}^2$ via ψ. If $\xi, \eta \in \mathcal{S}C_b(\mathcal{H}^1)$, then there exist $\xi', \eta' \in \mathcal{S}C_b(\mathcal{H}^2)$ such that $(\xi, \eta) = (\xi', \eta')$. In effect, if $\gamma \in G$,

$$\begin{aligned} (\xi,\eta)(\gamma) &= \langle L_\gamma^1 \xi(s(\gamma)), \eta(r(\gamma))\rangle \\ &= \langle \psi_{r(\gamma)}(L_\gamma^1 \xi(s(\gamma))), \psi_{r(\gamma)}(\eta(r(\gamma)))\rangle \\ &= \langle L_\gamma^2 \psi_{s(\gamma)} xi(s(\gamma)), \psi_{r(\gamma)}(\eta(r(\gamma)))\rangle \\ &= \langle L_\gamma^2 \xi'(s(\gamma)), \eta'(r(\gamma))\rangle \\ &= (\xi', \eta')(\gamma). \end{aligned}$$

We now present the most important G-Hilbert bundle for our work, the left regular bundle. It is the analogue of the left regular representation of locally compact groups. As in that case, the Hilbert spaces considered are L^2 spaces, so we need to fix a Haar system (see 2.1.2).

Given a Haar system $\{\lambda_u\}_{u\in G^0}$, we consider the **left regular** G-Hilbert bundle

$$L^2 = (\{L^2(G^u, \lambda^u)\}_{u\in G^0}, C_c(G), L),$$

where

$$L_\gamma : L^2(G^{s(\gamma)}, \lambda^{s(\gamma)}) \to L^2(G^{r(\gamma)}, \lambda^{r(\gamma)}) \quad L_\gamma\xi(\gamma') = \xi(\gamma^{-1}\gamma').$$

Note that we can see the elements of $C_c(G)$ as sections as follows: if $f \in C_c(G)$ and $u \in G^0$, let

$$f(u) = f_{|G^u} \in L^2(G^u).$$

Here, $C_c(G)$ is a family of sections as in Proposition 4.2.6, thus determining a unique family of continuous sections.

If $\xi, \eta \in \mathcal{S}C_0(L^2(G))$,

$$(\xi, \eta)(\gamma) = \langle L_\gamma\xi(s(\gamma)), \eta(r(\gamma))\rangle = \int \overline{\xi(s(\gamma))(\gamma^{-1}\gamma')}\eta(r(\gamma))(\gamma')\, d\lambda^{r(\gamma)}(\gamma').$$

If we denote $\xi^*(\gamma) = \overline{\xi(\gamma^{-1})}$, then for $f, g \in C_c(G)$,

$$\begin{aligned}(f,g)(\gamma) &= \int \overline{f(s(\gamma))^*(\gamma'^{-1}\gamma)}g(r(\gamma))(\gamma')\, d\lambda^{r(\gamma)}(\gamma') \\ &= (g * f^*)(\gamma).\end{aligned}$$

The following lemma will be used on the next proposition.

Lemma 4.2.19 ([10], see also [6], 5.3.2)**.** *If $f \in C_c(G^{(2)})$, the map*

$$\gamma \to \int f(\gamma, \gamma')\, d\lambda^{s(\gamma)}(\gamma')$$

is continuous.

Proposition 4.2.20. *([6], 5.3.1) The left regular G-Hilbert bundle is unitary and continuous.*

Proof. Due to Remark 4.2.2, the proof that we present here slightly differs from the reference above.

To check that the left unitary bundle is unitary, let $\gamma \in G$ and $f, g \in L^2(G^{s(\gamma)})$. Then, by the left invariance of the Haar system,

$$\begin{aligned} < L_\gamma f, L_\gamma g > &= \int \overline{f(\gamma^{-1}\delta)} g(\gamma^{-1}\delta)\, d\lambda^{r(\gamma)}(\delta) \\ &= \int \overline{f(\delta)} g(\delta)\, d\lambda^{s(\gamma)}(\delta) \\ &=< f, g > . \end{aligned}$$

We now prove that it is strongly continuous. For all $\xi \in \mathcal{S}C_0(L^2(G))$, we want to check that the map $G \to \sqcup L^2(G^{r(\gamma)})$, $\gamma \to L_\gamma \xi(s(\gamma))$ is continuous. Fix $\gamma \in G$. Let $U(\varepsilon, V, \eta)$ be a neighbourhood of $L_\gamma \xi(s(\gamma))$, where $\varepsilon > 0$, $V \subseteq G^0$ is open and η is a continuous section. Assume $\|L_\gamma \xi(s(\gamma)) - \eta(r(\gamma))\| = \alpha < \varepsilon$. Let $\alpha' = \varepsilon - \alpha$. By 4.2.5, we can find $\xi', \eta' \in C_c(G)$ such that $\|\xi - \xi'\| < \frac{\alpha'}{6}$ on a neighbourhood U_s of $s(\gamma)$ and $\|\eta - \eta'\| < \frac{\alpha'}{6}$ on a neighbourhood U_r of $r(\gamma)$. Let $U = s^{-1}(U_s) \cap r^{-1}(U_r)$, it is a neighbourhood of γ.

We apply now the lemma above, for $f(\gamma', \delta') := |\xi'((\gamma')^{-1}\delta') - \eta'(\delta')|$. Let

$$F(\gamma') := (\int f(\gamma', \delta')^2\, d\lambda^{r(\gamma')}(\delta'))^{1/2}.$$

By the lemma, this is a continuous function. We compute $F(\gamma)$:

$$\begin{aligned} F(\gamma) &= \Big[\int |L_\gamma \xi'(s(\gamma))(\delta') - \eta'(r(\gamma))(\delta')|^2\, d\lambda^{r(\gamma)}(\delta') \Big]^{1/2} \\ &= \|L_\gamma \xi'(s(\gamma)) - \eta'(r(\gamma))\| \\ &\leq \|L_\gamma \xi'(s(\gamma)) - L_\gamma \xi(s(\gamma))\| + \|L_\gamma \xi(s(\gamma)) - \eta(r(\gamma))\| \\ &\quad + \|\eta(r(\gamma)) - \eta'(r(\gamma)\| \\ &< \alpha + \frac{\alpha'}{3} \end{aligned}$$

Since F is continuous, there exists a neighbourhood $\hat{U}$ of γ such that $F_{|_{\hat{U}}} < \alpha + \frac{\alpha'}{2}$. Hence, if $\delta \in U \cap \hat{U} \cap r^{-1}(V)$, we want to verify that $L_\delta \xi(s(\delta)) \in U(\varepsilon, V, \eta)$. Note that $L_\gamma \xi(s(\gamma)) \in L^2(G^{r(\gamma)})$ and $p(L_\delta \xi(s(\delta))) = r(\delta) \in V$. Moreover,

$$\begin{aligned}
\|L_\delta \xi(s(\delta)) - \eta(r(\delta))\| &\leq \|L_\delta \xi(s(\delta)) - L_\delta \xi'(s(\delta))\| + F(\delta) + \|\eta'(r(\delta)) - \eta(r(\delta))\| \\
&< \|\xi(s(\delta)) - \xi'(s(\delta))\| + (\alpha + \frac{\alpha'}{2}) + \frac{\alpha'}{6} \\
&= \alpha + \frac{5\alpha'}{6} < \varepsilon.
\end{aligned}$$

Hence, the left regular bundle is strongly continuous. □

Remark 4.2.21. Many times it is convenient to consider the left regular bundle with multiplicity. Again, we need to fix a Haar system $\{\lambda^u\}_{u\in G^0}$. The **left regular bundle with multiplicity** $p \in \mathbb{N}$ is

$$L^2(G)^p := \big(\{L^2(G^u, \lambda^u; \mathbb{C}^p)\}_{u\in G^0}, C_c(G; \mathbb{C}^p), L\big),$$

where $L_\gamma(f_1, f_2, \dots, f_p) = (L_\gamma f_1, L_\gamma f_2, \dots, L_\gamma f_p)$.

For $p = \infty$,

$$L^2(G; l^2) := \big(\{L^2(G^u, \lambda^u; l^2)\}_{u\in G^0}, C_c(G; l^2), L\big),$$

where

$$L^2(G^u; l^2) = \{f : G^u \to l^2 \text{ measurable} : \ \|f\|_2 < \infty\}.$$

If we identify $f \in L^2(G^u; l^2)$ with a sequence $\{f_n\}$, where each function f_n is

$$f_n : G^u \to \mathbb{C}, \quad f_n(\gamma) = f(\gamma)_n,$$

the norm is

$$\|f\|_2 := (\sum_n \|f_n\|_2^2)^{1/2}$$

and the inner product

$$\langle f, g\rangle = \sum_n \langle f_n, g_n\rangle.$$

The functions $C_c(G; l^2)$ uniquely determine the continuous sections as in 4.2.6. The definition of the action L is the same as before.

The proof that the left regular bundle is continuous works as well for the infinite multiplicity case, the only modifications that need to be done involve adding $\sum_n$ everywhere.

4.3 The Fourier-Stieltjes algebra

Let G be a locally compact groupoid.

Following Paterson's work ([33]), the **Fourier-Stieltjes algebra** of G is

$$B(G) = \{(\xi, \eta) : \xi, \eta \in \mathcal{S}C_b(\mathcal{H}), \text{for some } G\text{-Hilbert bundle } \mathcal{H}\}$$

In this section we will prove that this set of continuous and bounded sections in G is, in fact, an algebra. Morever, it is a unital, commutative, Banach algebra.

In [43], Renault considers measurable Hilbert bundles over measured groupoids and essentially bounded sections. In [41], Ramsay and Walter take a Borel approach.

Remark 4.3.1. Let $\mathcal{H} = (\{\mathcal{H}^u\}_{u\in G^0}, \Gamma_{\mathcal{H}}, L^{\mathcal{H}})$ and $\mathcal{K} = (\{\mathcal{K}^u\}_{u\in G^0}, \Gamma_{\mathcal{K}}, L^{\mathcal{K}})$ be G-Hilbert bundles. We construct two new G-Hilbert bundles from them, namely $\mathcal{H} \oplus \mathcal{K}$ and $\mathcal{H} \otimes \mathcal{K}$.

For the **sum**, the Hilbert spaces are $\{\mathcal{H}^u \oplus \mathcal{K}^u\}_{u\in G^0}$, the family of continuous sections is defined from $\{\xi \oplus \zeta : \xi \in \Gamma_{\mathcal{H}}, \zeta \in \Gamma_{\mathcal{K}}, \xi \oplus \zeta(u) = \xi(u) \oplus \zeta(u)\}$, applying Proposition 4.2.6, and the action is $L^{\oplus}_\gamma(h \oplus k) = L^{\mathcal{H}}_\gamma \oplus L^{\mathcal{K}}_\gamma$, for all $\gamma \in G$. Observe that if $\xi, \zeta \in \mathcal{S}C_b(\mathcal{H})$ and $\mu, \eta \in \mathcal{S}C_b(\mathcal{K})$, the sum of the coefficients $(\xi, \zeta) + (\mu, \eta)$ is a new coefficient $(\xi \oplus \mu, \zeta \oplus \eta)$, where $\xi \oplus \mu, \zeta \oplus \eta \in \mathcal{S}C_b(\mathcal{H} \oplus \mathcal{K})$.

Similarly, for the **product**, the new G-Hilbert bundle is

$$\mathcal{H} \otimes \mathcal{K} = (\{\mathcal{H}^u \otimes \mathcal{K}^u\}_{u\in G^0}, \Gamma_{\otimes}, L^{\otimes})$$

and if ξ, ζ, μ, η are sections as above, the point-wise product of the coefficients $(\xi, \zeta)(\mu, \eta)$ is $(\xi \otimes \mu, \zeta \otimes \eta)$, with $\xi \otimes \mu, \zeta \otimes \eta \in \mathcal{S}C_b(\mathcal{H} \otimes \mathcal{K})$.

We define the **trivial bundle** $\mathrm{Id} = (\{\mathbb{C}^u\}_{u\in G^0}, C_c(G), \mathrm{Id})$, where $\mathbb{C}^u = \mathbb{C}$ for all $u \in G^0$, this bundle is the identity for the product of bundles.

Definition 4.3.2. Let $\varphi \in C_b(G)$. We say that φ is **positive definite** if for any $p \in \mathbb{N}$ and for any $u \in G^0$, for all $x_k \in G^u$ and for all $\alpha_k \in \mathbb{C}$, $k = 1, 2, \ldots, p$,

$$\sum_{k,l} \alpha_k \overline{\alpha_l} \varphi(x_k^{-1} x_l) \geq 0.$$

We denote the set of continuous positive definite functions on G by $P(G)$.

Proposition 4.3.3 ([33], Thm 1, Proposition 3, [43], Proposition 1.1, [41], Theorem 3.5)**.** *Let $\varphi \in C_b(G)$. Then $\varphi \in P(G)$ if and only if there exists a G-Hilbert bundle $\mathcal{H}$ and $\xi \in \mathcal{S}C_b(\mathcal{H})$ such that $\varphi = (\xi, \xi)$.*

Remark 4.3.4. In a personal communication J. Renault showed us that this result is deeper in the continuous case, since the G-Hilbert bundle associated to a positive definite continuous function is, in some sense that we will not make explicit here, unique.

We denote by I_2 the trivial groupoid on two elements $\{1, 2\} \times \{1, 2\}$. In order to prove that $B(G)$ is a Banach algebra we are going to relate coefficients of G to positive definite functions of $G \times I_2$. This trick is known as the groupoid version of Paulsen's off-diagonalization technique. The groupoid $G \times I_2$ has composable pairs $((\gamma', k, j), (\gamma, j, i))$, for $(\gamma, \gamma') \in G^2$ and $i, j, k \in \{1, 2\}$. The product is given by $(\gamma', k, j), (\gamma, j, i) = (\gamma'\gamma, k, i)$ and the inverse is $(\gamma, j, i)^{-1} = (\gamma^{-1}, i, j)$. The unit space $(G \times I_2)^0$ identifies to $G^0 \times \{1, 2\}$. If F is a function, $F : G \times I_2 \to \mathbb{C}$, we write it as a 2×2 matrix valued function

$$F : G \to M_2, \quad F(\gamma) = \begin{pmatrix} F(\gamma, 1, 1) & F(\gamma, 1, 2) \\ F(\gamma, 2, 1) & F(\gamma, 2, 2) \end{pmatrix}$$

Note that the left regular I_2-Hilbert bundle is $L^2(I_2) = (\{\mathbb{C}_i^2\}_{i=1,2}, \mathbb{C}^2, \mathrm{Id})$, where $\mathbb{C}_i^2 = \mathbb{C}^2$, for $i = 1, 2$.

We include the proofs of the next propositions since they are going to be needed when proving that the Fourier algebra of a groupoid is a Banach algebra.

Proposition 4.3.5 ([43], Proposition 1.3, [33], Proposition 5)**.** *If $\varphi \in C_b(G)$, the following conditions are equivalent:*

1. *φ is a linear combination of elements of $P(G)$.*

2. *There exist a G-Hilbert bundle $\mathcal{H}$ and sections $\xi, \eta \in \mathcal{S}C_b(\mathcal{H})$ such that $\varphi = (\xi, \eta)$.*

3. *There exist $\rho, \tau \in P(G)$ such that $\begin{pmatrix} \rho & \varphi \\ \varphi^* & \tau \end{pmatrix} \in P(G \times I_2)$.*

Proof. We first verify that 1 and 2 are equivalent. The first direction follows from the characterization of positive definite functions in Proposition 4.3.3 and the direct sum of Hilbert bundles defined on Remark 4.3.1. The converse follows by applying the polarization identity:

$$4(\xi, \eta) = (\xi + \eta, \xi + \eta) - (\xi - \eta, \xi - \eta) - i(\xi + i\eta, \xi + i\eta) + i(\xi - i\eta, \xi - i\eta).$$

We now prove that 2 implies 3. If $\varphi = (\xi, \eta)$, for ξ, η continuous and bounded sections of a G-Hilbert bundle $\mathcal{H}$, let $\rho = (\eta, \eta)$ and $\tau = (\xi, \xi)$.

We consider the $G \times I_2$-Hilbert bundle

$$\mathcal{H} \otimes L^2(I_2) = (\{\mathcal{H}^u \otimes \mathbb{C}_i^2\}_{u \in G^0, i \in \{1,2\}}, \Gamma_{\mathcal{H}} \otimes \mathbb{C}^2, L_{\mathcal{H}} \otimes \mathrm{Id}).$$

We define the $G \times I_2$ section ε by

$$\varepsilon(u, i) = \begin{cases} \eta(u) \otimes 1_{\mathbb{C}_1^2} & \text{if } i = 1; \\ \xi(u) \otimes 1_{\mathbb{C}_2^2} & \text{if } i = 2. \end{cases}$$

Since $\eta, \xi \in \mathcal{S}C_b(\mathcal{H})$ and I_2 is discrete, $\varepsilon \in \mathcal{S}C_b(\mathcal{H} \otimes \mathrm{Id})$. Then, $(\varepsilon, \varepsilon) \in P(G \times I_2)$.

Note that

$$\begin{aligned} (\varepsilon, \varepsilon)(\gamma, 1, 2) &= \langle (L \otimes \mathrm{Id})_{(\gamma,1,2)} \varepsilon(s(\gamma, 1, 2)), \varepsilon(r(\gamma, 1, 2)) \rangle \\ &= \langle (L_\gamma \otimes \mathrm{Id})(\xi(s(\gamma)) \otimes 1), \eta(r(\gamma)) \otimes 1 \rangle \\ &= \langle L_\gamma \xi(s(\gamma)), \eta(r(\gamma)) \rangle \cdot 1 \\ &= \varphi(\gamma). \end{aligned}$$

Similarly, we can verify that $(\varepsilon, \varepsilon) = \begin{pmatrix} \rho & \varphi \\ \varphi^* & \tau \end{pmatrix} \in P(G \times I_2)$.

Lastly, we prove that 3 implies 2. Suppose $F \in P(G \times I_2)$. Then, there exists $\mathcal{H} = (\{\mathcal{H}^{u,i}\}, \Gamma, L)$, a $G \times I_2$-bundle, and $\zeta \in \mathcal{S}C_b(\mathcal{H})$ such that $F = (\zeta, \zeta)$.

We define two G-Hilbert bundles $\mathcal{H}_i = (\{H^{u,i}\}, \Gamma_i, L^i)$, for $i = 1, 2$, where $\Gamma_i = \{\xi(\cdot, i) : \xi \in \Gamma\}$ and $L^i_\gamma(h) = L_{\gamma ii}(h)$. We take $\mathcal{H}' = \mathcal{H}_1 \oplus \mathcal{H}_2 = (\{\mathcal{H}^{u,1} \oplus \mathcal{H}^{u,2}\}, \Gamma_1 \oplus \Gamma_2, L')$, where

$$L'_\gamma = \frac{1}{2} \begin{pmatrix} L_{\gamma 11} & L_{\gamma 12} \\ L_{\gamma 21} & L_{\gamma 22} \end{pmatrix}.$$

It is true that $L'_{\gamma'} L'_\gamma = L_{\gamma'\gamma}$ if (γ', γ) is a composable pair. But for a unit $u \in G^0$ we have that

$$L'_u = \frac{1}{2} \begin{pmatrix} L_{u11} & L_{u12} \\ L_{u21} & L_{u22} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \mathrm{Id} & L_{u12} \\ L_{u21} & \mathrm{Id} \end{pmatrix},$$

and that may be different from the identity matrix. Thus this is not yet the Hilbert bundle we are looking for. Following [33], we "cut it down" to obtain a G-Hilbert bundle. In order to do this, the first step is to note that if $u \in G^0$, $L'_{u^2} = L'_u$, therefore L'_u is a projection. Let $P_u = L'_u$ and $\mathcal{K}^u = P_u(\mathcal{H}'^u)$. We define $\mathcal{K} = (\{\mathcal{K}^u\}, P\Gamma, L'P)$, where $P\Gamma = \{P\xi : \xi \in \Gamma\}$, $P\xi(u) = P_u \xi(u)$, and $L'P_\gamma = L'_\gamma P_{s(\gamma)}$. In order to verify that $\mathcal{K}$ is, in fact, a G-Hilbert bundle, we observe that

$$L'_{\gamma^{-1}} P_{r(\gamma)} L'_\gamma P_{s(\gamma)} = L'_{\gamma^{-1} r(\gamma)} L'_{\gamma s(\gamma)} = L'_{\gamma^{-1}} L'_\gamma = L'_{s(\gamma)}$$

and conclude that $L'P_{\gamma^{-1}} L'P_\gamma(h) = h$ for $h \in \mathcal{K}^{s(\gamma)}$.

Finally, let $\zeta_i = \zeta(\cdot, i)$, for $i = 1, 2$, and we define $\xi = P(\xi_1, 0)$ and $\eta = P(0, \xi_2)$. We observe that $\xi, \eta \in \mathcal{SC}_b(\mathcal{K})$. We want to verify that $2(\eta, \xi) = (\zeta, \zeta)(\cdot, 1, 2) = \varphi$. If $\gamma \in G$,

$$\begin{aligned}
(\eta, \xi)(\gamma) =&\langle L' P_\gamma \eta(s(\gamma)), \xi(r(\gamma))\rangle \\
=&\langle L' P_\gamma P_{s(\gamma)}(0, \zeta_2(s(\gamma))), P_{r(\gamma)}(\zeta_1(r(\gamma)), 0)\rangle \\
=&\langle L'_\gamma P_{s(\gamma)} P_{s(\gamma)}(0, \zeta_2(s(\gamma))), P_{r(\gamma)}(\zeta_1(r(\gamma)), 0)\rangle \\
=&\langle L'_\gamma (0, \zeta_2(s(\gamma))), (\zeta_1(r(\gamma)), 0)\rangle \\
=&\langle \frac{1}{2}(L_{\gamma 12}\zeta_2(s(\gamma)), L_{\gamma 22}\zeta_2(s(\gamma))), (\zeta_1(r(\gamma)), 0)\rangle \\
=&\frac{1}{2}\langle L_{\gamma 12}\zeta_2(s(\gamma)), \zeta_1(r(\gamma))\rangle \\
=&\frac{1}{2}\langle L_{\gamma 12}\zeta(s(\gamma), 2), \zeta(r(\gamma), 1)\rangle \\
=&\frac{1}{2}(\zeta, \zeta)(\gamma, 1, 2).
\end{aligned}$$

□

In $B(G)$ we define the (candidate) norm

$$\|\varphi\|_{B(G)} = \inf_{(\xi, \eta) = \varphi} \|\xi\| \|\eta\|.$$

We will omit the sub-index $B(G)$ whenever that will not cause confusion.

Proposition 4.3.6 ([33], Proposition 4)**.** *If $\varphi \in B(G)$, then $\|\varphi\|_\infty \leq \|\varphi\|$. If $\varphi \in P(G)$, then $\|\varphi\| = \|\varphi_{|_{G^0}}\|_\infty$. Moreover, if $\varphi = (\xi, \xi)$, where ξ is a continuous and bounded section of some G-Hilbert bundle, then $\|\varphi\| = \|\xi\|^2$.*

Proof. Suppose $\xi, \eta \in \mathcal{SC}_b(\mathcal{H})$, for some G-Hilbert bundle $\mathcal{H}$, and $\varphi = (\xi, \eta)$. Then, $\|\varphi\|_\infty = \|(\xi, \eta)\|_\infty \leq \|\xi\| \|\eta\|$. Taking the infimum over all possible ξ, η as above, we obtain the first inequality.

We now assume that $\varphi \in P(G)$. Then there exists ξ such that $\varphi = (\xi, \xi)$. If $u \in G^0$, $0 \leq \varphi(u) = (\xi, \xi)(u) = \|\xi(u)\|^2$. It follows that

$$\|\xi\|_\infty^2 = \|\varphi_{|_{G^0}}\|_\infty \leq \|\varphi\| \leq \|\xi\|_\infty^2.$$

Thus, $\|\varphi\| = \|\varphi_{|_{G^0}}\|_\infty = \|\xi\|^2$. □

Lemma 4.3.7 ([43], Lemma 1.1). *If $\varphi \in B(G)$, the following are equivalent:*

1. *There exists a G-Hilbert bundle $\mathcal{H}$ and sections $\xi, \eta \in \mathcal{S}C_b(\mathcal{H})$ of norm less than or equal to 1 such that $\varphi = (\xi, \eta)$.*

2. *There exist $\rho, \tau \in P(G)$ of norm less than or equal to 1 such that*

$$\begin{pmatrix} \rho & \varphi \\ \varphi^* & \tau \end{pmatrix} \in P(G \times I_2). \tag{4.7}$$

Proof. Suppose there exist ξ, η as above. Then, as in Proposition 4.3.5, there exist ρ, τ satisfying 4.7. Moreover, $\rho = (\xi, \xi)$ and by 4.3.6, $\|\rho\| = \|\xi\|^2 \leq 1$. Similarly for $\tau = (\eta, \eta)$.

Let

$$F_\varphi = \begin{pmatrix} \rho & \varphi \\ \varphi^* & \tau \end{pmatrix} \in P(G \times I_2).$$

From Proposition 4.3.6,

$$\|F_\varphi\| = \|F_\varphi|_{(G\times I_2)^0}\|_\infty = \max\{\|\rho\|_\infty, \|\tau\|_\infty\}.$$

Thus, $\|F_\varphi\| \leq 1$.

Recall that there exists $\zeta \in \mathcal{S}C_b(\mathcal{K})$ for some $G \times I_2$-Hilbert bundle $\mathcal{K}$ such that $(\zeta, \zeta) = F_\varphi$. Note that $\|\zeta\|_\infty \leq 1$ as well. Since $\xi = P(\zeta(\cdot, 1), 0)$ and $\eta = P(0, \zeta(\cdot, 2))$, $\|\xi\| \leq 1$ and $\|\eta\| \leq 1$. $\square$

Corollary 4.3.8 ([43], Corollary 1.1). *If $\varphi \in B(G)$, then $\|\varphi\| = \inf \|F\|_\infty$ for $F \in P(G \times I_2)$ such that $\varphi = F(\cdot, 1, 2)$.*

Theorem 4.3.9 ([43], Proposition 1.4, [33], Theorem 2). *With $\| \cdot \|_{B(G)}$ and pointwise operations, $B(G)$ is a unital, involutive, commutative, Banach algebra.*

Proof. We first check that $\| \cdot \|_{B(G)}$ is, in fact, a norm. Since $\|\varphi\|_\infty \leq \|\varphi\|$, if $\|\varphi\| = 0$, then $\varphi = 0$. If $\varphi_i \in B(G)$ and $\|\varphi_i\| < \alpha_i$, there exists $F_i \in P(G \times I_2)$ such that $F_i(\cdot 12) = \varphi_i$

and $\|F_i\| \leq \alpha_i$, for $i = 1, 2$. Then $F_1 + F_2 \in P(G \times I_2)$ as well, $F_1(\cdot 12) + F_2(\cdot 12) = \varphi_1 + \varphi_2$ and $\|F_1 + F_2\| \leq \alpha_1 + \alpha_2$. Thus, $\|\varphi_1 + \varphi_2\| \leq \|\varphi_1\| + \|\varphi_2\|$.

Suppose $\varphi = (\xi, \eta)$, for sections ξ, η of some G-Hilbert bundle. Since $\varphi^*(\gamma) = \overline{\varphi(\gamma^{-1})}$, it follows that $\varphi^* = (\eta, \xi)$ and $\|\varphi\| = \|\varphi^*\|$. Thus $B(G)$ is involutive.

It is clear that $B(G)$ is unital and commutative.

If $\varphi_i = (\xi_i, \eta_i)$, for sections ξ_i, η_i of some G-Hilbert bundle, for $i = 1, 2$, then $\varphi_1\varphi_2 = (\xi_1 \otimes \xi_2, \eta_1 \otimes \eta_2)$ and it follows that $\|\varphi_1\varphi_2\| \leq \|\varphi_1\|\|\varphi_2\|$.

Finally, we verify that $B(G)$ with this norm is complete. Let $\{\varphi_n\}$ be a sequence in $B(G)$ such that $\|\varphi_n\| < \alpha_n$ and $\sum \alpha_n < \infty$. Let $\{F_n\}$ be the corresponding sequence on $P(G \times I_2)$. Then, $\sum F_n = F$ belongs to $C_b(G \times I_2)$ and it is positive definite. Thus, $\varphi = F(\cdot 12)$ exists in $B(G)$. Since

$$\|\varphi - \sum_{i=1}^{n} \varphi_i\| \leq \|F - \sum_{i=1}^{n} F_i\| \to 0,$$

we conclude that $\sum \varphi_n \to \varphi$ in $B(G)$. □

4.4 The stabilization theorem for proper groupoids

In this section, we present a result from Paterson [35] that is an equivariant stabilization theorem for proper groupoids. We do not need the result itself, but one of the claims of its proof. We will present proofs whenever we feel they help to understand the result we are aiming for (with slight modifications, as suggested in Remark 4.2.2, when needed). In his article, Paterson uses different but equivalent definitions of Hilbert bundles and G-Hilbert bundles. We begin by presenting those definitions and showing the equivalence to the ones we have considered earlier on in this Chapter.

From now on, we shall assume that G is a locally compact proper groupoid. As always, we denote its unit space by G^0. We also fix a left Haar system $\{\lambda^u\}_{u \in G^0}$.

Definitions 4.4.1. Let F be a complex vector space and A a C^*-algebra. We say that F is a **pre-A- Hilbert module** if we have a module action $F \times A \to F$ and a pre-inner product $\langle \cdot, \cdot \rangle : F \times F \to A$ satisfying the following properties:

1. $\langle \cdot, \cdot \rangle$ is linear in the second component;
2. if $x, y \in F$ and $a \in A$, then $\langle x, ya \rangle = \langle x, y \rangle a$;
3. if $x, y \in F$, then $\langle x, y \rangle = \langle y, x \rangle^*$; and
4. for all $x \in F$, $\langle x, x \rangle$ is a positive element of the C^*-algebra A; moreover, $\langle x, x \rangle = 0$ if and only if $x = 0$.

On such a space F we can define a norm:

$$\|x\|_F = \|\langle x, x \rangle\|^{1/2}.$$

We say that F is an **A-Hilbert module** if it is complete with respect to this norm.

Now suppose that Y is a locally compact Hausdorff space.

If P and Q are $C_0(Y)$-modules, a map $T : P \to Q$ is a **morphism** if it is an **adjointable** map. That is, if there exists a map $T^* : Q \to P$ such that if $p \in P$ and $q \in Q$, then $\langle T(p), q \rangle = \langle p, T^*(q) \rangle$. The $C_0(Y)$-modules with these morphisms form a category.

We say that two $C_0(Y)$-modules are **equivalent**, denoted by $P \simeq Q$, if there exists a unitary morphism $U : P \to Q$.

Definition 4.4.2. Suppose $\{\mathcal{H}^u\}_{u \in Y}$ is a family of Hilbert spaces. On the disjoint union $E = \sqcup_{u \in Y} \mathcal{H}^u$ we consider a topology that is second countable. Let $\pi : E \to Y$ be the projection map. We denote by $\mathcal{S}C_0(E)$ the continuous sections that vanish at infinity. We say that E is a Hilbert bundle over Y if the following conditions are satisfied:

1. the scalar multiplication and addition on each fiber are continuous;

2. for each $\xi \in \mathcal{S}C_0(E)$, the map $u \to \xi(u)$ is continuous;

3. for each $u \in Y$, $\{\xi(u) : \xi \in \mathcal{S}C(E)\}$ is dense in $\mathcal{H}^u$;

4. if V is an open subset of Y, $\varepsilon > 0$ and $\xi \in \mathcal{S}C_0(E)$, then the sets $U(\varepsilon, V, \xi)$ (defined in 4.2.7) form a basis for the topology on E.

Proposition 4.4.3. *[[35], Proposition 1] Let E be a Hilbert bundle over Y as in the definition above. Then $\mathcal{S}C_0(E)$ is a separable $C_0(Y)$-Hilbert module in the uniform topology.*

Remark 4.4.4. If $\mathcal{H} = (\{\mathcal{H}_u\}_{u\in Y}, \Gamma)$ is a continuous field of separable Hilbert spaces, let E be the disjoint union of the $\mathcal{H}^u$'s. In order to prove that E is a Hilbert bundle over Y with the new definition, we need to check that we can consider a countable family of sets $U(\varepsilon, V, \xi)$. This follows from [27], p.57. The rest of the conditions follow from the definition of continuous field of Banach spaces and Lemma 4.2.7.

On the other hand, suppose that $E = \sqcup_{u\in Y}\mathcal{H}^u$ is a Hilbert bundle over Y, then $\mathcal{H} := (\{\mathcal{H}_u\}_{u\in Y}, \mathcal{S}C_0(E))$ uniquely defines a continuous field of separable Hilbert spaces as in Proposition 4.2.6.

Definition 4.4.5. Let $E = \sqcup_u E^u$ and $F = \sqcup_u F^u$ be Hilbert bundles over a space Y. A continuous map $\varphi : E \to F$ is a morphism if

1. each $\varphi_u : E^u \to F^u$ is a bounded linear map;

2. the norm of φ, $\|\varphi\| = \sup_u \|\varphi_u\|$, is finite and

3. the adjoint map $\varphi^* : F \to E$, $\varphi^*(f_u) = (\varphi_u)^*(f_u)$ is continuous.

With these morphisms, the class of Hilbert bundles is a category.

Proposition 4.4.3 states that given a Hilbert bundle we can associate a $C_0(Y)$-Hilbert module, namely $\mathcal{S}C_0(E)$. We now show that given a $C_0(Y)$-Hilbert module P we can associate a Hilbert bundle E_P.

We need to use the following result:

Remark 4.4.6 ([26], from Theorem 1)**.** If P is a Hilbert A-module and $p \in P$, then

$$p = \lim_{\varepsilon \to 0^+} p \frac{\langle p, p \rangle}{\langle p, p \rangle + \varepsilon}. \tag{4.8}$$

If P is a $C_0(Y)$- Hilbert module, it follows from Cohen's factorization theorem and the equation above that

$$P = \{pf : \, p \in P, \, f \in C_0(Y)\}. \tag{4.9}$$

Remark 4.4.7. Suppose P is a $C_0(Y)$-Hilbert module. If $u \in Y$, define

$$I_u := \{f \in C_0(Y) : \, f(u) = 0\}.$$

This is a closed ideal of $C_0(Y)$. By Cohen's factorization theorem, $I_u P$ is closed in P. Let

$$P_u := \frac{P}{I_u P}, \quad \widehat{p}(u) = p + I_u P.$$

On P_u we consider the inner-product $\langle p + I_u P, q + I_u P \rangle = \langle p, q \rangle(u)$. It is non-degenerate: if $\langle p, p \rangle(u) = 0$, then $\langle p, p \rangle \in I_u$. From (4.8), it follows that $p \in \overline{I_u P} = I_u P$. Let $E_P := \sqcup_{u \in Y} P_u$.

If V is an open subset of Y and $\varepsilon > 0$, we write $U(\varepsilon, V, p) = U(\varepsilon, V, \widehat{p})$.

Proposition 4.4.8. *Let P be a $C_0(Y)$-Hilbert module. The space E_P defined above, with the second countable topology that has basis $\{U(\varepsilon, V, p)\}$, is a Hilbert bundle over Y.*

The map $P \to \mathcal{S}C_0(E_P)$, $p \to \widehat{p}$, is a $C_0(Y)$-Hilbert module unitary. The section $\widehat{p}$ is defined by $\widehat{p}(u) = p + I_u P$.

Moreover, the map $P \to E_P$ is an equivalence between the category of $C_0(Y)$-Hilbert modules and the category of Hilbert bundles over Y.

In this paper, Paterson uses the following definition of G-Hilbert bundle:

Definition 4.4.9. Suppose G is a locally compact groupoid with unit space G^0. Let E be a Hilbert bundle over G^0. Let

$$G * E := \{(\gamma, h) : s(\gamma) = \pi(h)\} \subseteq G \times E,$$

with the relative product topology. We say that E is a G**-Hilbert bundle** if there is a continuous map $L : G * E \to E$, $L(\gamma, h) = L_\gamma(h)$, satisfying:

1. if $(\gamma', \gamma) \in G^{(2)}$ and $(\gamma, h) \in G * E$, then $L(\gamma', L(\gamma, h)) = L(\gamma'\gamma, h)$;
2. if $u \in G^0$ and $\pi(h) = u$, then $L(u, h) = h$, and
3. if we fix $\gamma \in G$, the map $L_\gamma : \mathcal{H}^{s(g)} \to \mathcal{H}^{r(g)}$ is unitary.

In other words, L is an algebraic left groupoid action by unitaries that we also require to be continuous.

The following proposition together with Proposition 4.2.14 show that the new definition of G-Hilbert bundle coincides with the one that we already had.

Proposition 4.4.10 ([35], Proposition 3)**.** *A left groupoid action of G on E is continuous if and only if for all $\xi \in \mathcal{S}C_0(E)$ the map $G \to E$, $\gamma \to L_\gamma(\xi(s(\gamma))$ is continuous.*

Definitions 4.4.11. A $C_0(G^0)$-Hilbert module P is a G**-Hilbert module** if E_P is a G-Hilbert bundle.

If P and Q are G-Hilbert modules, we say that a $C_0(G^0)$ morphism $T : P \to Q$ is G**-equivariant** if the corresponding map $\phi_T : E_P \to E_Q$ is a morphism of G-Hilbert bundles, see definition 4.2.17.

Moreover, if $T : P \to Q$ is unitary, then T^* is G-equivariant as well, and we say that P and Q are **equivalent**. We write $P \simeq Q$.

Definition 4.4.12. Suppose that Q is a pre-Hilbert $C_0(G^0)$-module. We say that Q is a **pre-G-Hilbert module** if $\overline{Q}$ is a G-Hilbert module and the Q^u's are invariant by the action of G on $E_{\overline{Q}}$.

Example 4.4.13. The pre-G-Hilbert module $C_c(G)$.

The space $C_c(G)$ is a pre-$C_0(G^0)$ Hilbert module with action

$$C_0(G^0) \times C_c(G) \to C_c(G), \quad fF = (f \circ r)F$$

and inner product

$$C_c(G) \times C_c(G) \to C_0(G^0), \quad \langle F_1, F_2 \rangle(u) = \langle (F_1)_{|G^u}, (F_2)_{|G^u} \rangle_{L^2(G^u;\lambda^u)}.$$

Denote by P_G the Hilbert $C_0(G^0)$-module completion of $C_c(G)$. The Hilbert bundle determined by P_G is, in fact, our already well-known left regular bundle $L^2(G)$. The Hilbert spaces obtained via this association are naturally identified with the completion of $C_c(G^u)$ with the $L^2(G^u)$ inner product. So $(E_P)^u = L^2(G^u)$, for all $u \in G^0$. The isomorphims from $C_c(G)$ to $\mathcal{S}C_c(E_P)$ takes F to $u \to \widehat{F}(u)$. The left groupoid action that makes it a pre-G-Hilbert module is

$$L : G * L^2(G) \to L^2(G) \quad L(\gamma, \xi)(\gamma') = \xi(\gamma^{-1}\gamma'),\ \gamma' \in G^{r(\gamma)}.$$

Remark 4.4.14. Sums of bundles.

Let E_i be a Hilbert bundle over Y, for $i = 1, 2, \cdots, n$. Each $P_i = \mathcal{S}C_0(E_i)$ is a $C_0(Y)$-Hilbert module. The sum $\oplus_{i=1}^n P_i$ is also a $C_0(Y)$-Hilbert module. We denote by E the Hilbert bundle over Y associated to $\oplus_{i=1}^n P_i$, with the relative topology it has as a subspace of $E_1 \times E_2 \times \cdots E_n$. The elements of $\mathcal{S}C_0(E)$ are of the form $(F_1, F_2, \cdots, F_n)$, for $F_i \in \mathcal{S}C_0(Y)$. This is the sum of bundles we considered before.

We now present an infinite version. If $\{E_i\}_{i \in \mathbb{N}}$ is an infinite family of Hilbert bundles over Y and $P_i = \mathcal{S}C_0(E_i)$, the sum of the $C_0(Y)$-Hilbert bundles is

$$\oplus_{i=1}^\infty P_i := \{\{p_i\} : p_i \in P_i, \sum_{i=1}^\infty \langle p_i, p_i \rangle \text{ converges in } C_0(Y)\}.$$

Then, $E := E_{\oplus_{i=1}^\infty P_i}$ is a Hilbert bundle over Y, with Hilbert spaces $E^u = \oplus_{i=1}^\infty (E_i)^u$. A sequence $\{h_n\}$ in E, where each $h_n = \{h_i^n\}$, converges to h if and only if $h_i^n \to_n h_i$ in E_i, for all i, and $\sum_{i=N}^\infty \|h_i^n\|^2 \to_{N,n} 0$.

If $E_i = E_1$, for all i, then $P = P_1^\infty$ and $E := E_1^\infty$.

If each E_i is a G-Hilbert bundle, then $\oplus_{i=1}^n P_i$ and $\oplus_{i=1}^\infty P_i$ are G-Hilbert modules and $\oplus_{i=1}^n E_i$ and $\oplus_{i=1}^\infty E_i$ are G-Hilbert bundles.

Moreover, if P is a G-Hilbert module, then

$$(P^\infty)^\infty \simeq P^\infty, \tag{4.10}$$

using a Cantor diagonal process.

Remark 4.4.15. Products of bundles.

Let P and Q be pre-Hilbert $C_0(Y)$-modules. We consider the algebraic, balanced tensor product $P \otimes_{C_0(Y)} Q$. This space is a pre-Hilbert $C_0(Y)$-module as well, with module action

$$(p \otimes q)F = p \otimes qF = pF \otimes q,$$

for $p \in P$, $q \in Q$ and $F \in C_0(Y)$. The pre-inner product is defined by

$$\langle p_1 \otimes q_1, p_2 \otimes q_2 \rangle = \langle p_1, q_1 \rangle \langle p_2, q_2 \rangle,$$

for $p_1, p_2 \in P$, $q_1, q_2 \in Q$. We quotient out the null space $\{\langle p_1 \otimes q_1, p_1 \otimes q_1 \rangle = 0\}$ and complete with respect to the norm induced by the inner product, obtaining the Hilbert $C_0(Y)$-module $P \otimes^{C_0(G^0)} Q$.

Note that $\overline{P} \otimes^{C_0(G^0)} \overline{Q} = P \otimes^{C_0(G^0)} Q$.

We write

$$E_{P \otimes^{C_0(G^0)} Q} = E_P \otimes E_Q.$$

The Hilbert spaces of this Hilbert bundle are $P^u \otimes Q^u$, the Hilbert space tensor of the Hilbert spaces P^u and Q^u, for all $u \in Y$.

Proposition 4.4.16 ([35], Proposition 5)**.** *Let P and Q be G-pre-Hilbert $C_0(Y)$-modules. Then $P \otimes^{C_0(G^0)} Q$ is a G-Hilbert module with diagonal G-action.*

Thus, the product presented above, at the G-Hilbert bundle level, coincides with the product we considered before.

Example 4.4.17. $C_0(G^0)$ as a G-Hilbert module.

Since $C_0(G^0)$ is a C^*-algebra, it is a Hilbert module over itself. It determines the trivial bundle $G^0 \times \mathbb{C}$ with trivially continuous action

$$G * (G^0 \times \mathbb{C}) \to (G^0 \times \mathbb{C}), \quad \gamma(s(\gamma), \alpha) = (r(\gamma), \alpha).$$

We now state Kasparov stabilization theorem. Note that this is a non-equivariant result. We will use this theorem for the C^*-algebra $C_0(G^0)$.

Theorem 4.4.18 ([26], Theorem 2)**.** *If $\mathcal{A}$ is a C^*-algebra and P is a Hilbert $\mathcal{A}$-module, then*

$$P \oplus \mathcal{A}^\infty \simeq \mathcal{A}^\infty.$$

Remark 4.4.19. If P is a G-Hilbert module, then $C_0(G^0) \otimes^{C_0(G^0)} P \simeq P$. The map

$$C_0(G^0) \otimes^{C_0(G^0)} P \to P \quad f \otimes p \to fp \tag{4.11}$$

is an equivariant Hilbert module map.

Remark 4.4.20. If P and Q are G-Hilbert modules, then

$$(P^\infty \otimes^{C_0(G^0)} Q) \simeq (P \otimes^{C_0(G^0)} Q^\infty) \simeq (P \otimes^{C_0(G^0)} Q)^\infty. \tag{4.12}$$

We present a proof of the second equivalence, the first one is similar. Let

$$Q_0 = \{(q_1, q_2, \cdots, q_n, 0, \cdots) \in Q^\infty : n \in \mathbb{N}\},$$

it is a dense subspace of Q^∞. Let

$$\alpha : P \otimes_{C_0(G^0)} Q_0 \to (P \otimes^{C_0(G^0)} Q)^\infty$$
$$\alpha(p \otimes (q_1, q_2, \cdots, q_n, 0, \cdots)) = (p \otimes q_1, p \otimes q_2, \cdots, p \otimes q_n, 0, \cdots).$$

The map α is a $C_0(G^0)$-module map. It preserves the inner product, since for $p, p' \in P$, $q = (q_1, q_2, \cdots, q_n, 0, \cdots)$ and $q' = (q'_1, q'_2, \cdots, q'_n, 0, \cdots)$ in Q_0, we have

$$
\begin{aligned}
&\langle \alpha(p \otimes q), \alpha(p' \otimes q') \rangle \\
&\qquad = \langle (p \otimes q_1, p \otimes q_2, \cdots, p \otimes q_n, 0, \cdots), (p' \otimes q'_1, p' \otimes q'_2, \cdots, p' \otimes q'_{n'}, 0, \cdots) \rangle \\
&\qquad = \sum_{i=1}^{\max\{n,n'\}} \langle p \otimes q_i, p' \otimes q'_i \rangle \\
&\qquad = \sum_{i=1}^{\max\{n,n'\}} \langle p, p' \rangle \langle q_i \otimes q'_i \rangle \\
&\qquad = \langle p, p' \rangle \langle q, q' \rangle \\
&\qquad = \langle p \otimes q, p' \otimes q' \rangle.
\end{aligned}
$$

Thus, α extends to an isometric map on $P \otimes^{C_0(G^0)} Q^\infty$. Also, since the range of α is a dense subspace of $(P \otimes^{C_0(G^0)} Q)^\infty$, α is onto $(P \otimes^{C_0(G^0)} Q)^\infty$.

It only remains to check that the map α preserves the G-action, that is, to confirm the commutativity of the following diagram:

$$
\begin{tikzcd}
\left(E_{P \otimes^{C_0(G^0)} Q^\infty}\right)^{s(\gamma)} \arrow[r, "L_\gamma"] \arrow[d, "\phi_\alpha^{s(\gamma)}"'] & \left(E_{P \otimes^{C_0(G^0)} Q^\infty}\right)^{r(\gamma)} \arrow[d, "\phi_\alpha^{r(\gamma)}"] \\
\left(E_{(P \otimes^{C_0(G^0)} Q)^\infty}\right)^{s(\gamma)} \arrow[r, "L_\gamma"'] & \left(E_{(P \otimes^{C_0(G^0)} Q)^\infty}\right)^{r(\gamma)}
\end{tikzcd}
$$

Let $p \in P$ and $q = (q_1, q_2, \cdots, q_n, 0 \cdots) \in Q_0$. The Hilbert bundle map associated to α is

$$
\begin{gathered}
(\phi_\alpha)^u : E_P^u \otimes (\oplus_{i=1}^\infty E_Q^u) \to \oplus_{i=1}^\infty E_P \otimes E_Q \\
(\phi_\alpha)^u(\hat{p}(u) \otimes (\hat{q}_1(u), \cdots, \hat{q}(u), 0, \cdots)) = (\hat{p}(u) \otimes \hat{q}(u), \cdots, \hat{p}(u) \otimes \hat{q}(u), 0, \cdots),
\end{gathered}
$$

for all $u \in G^0$. Then, if $\gamma \in G$,

$$
\begin{aligned}
&(\phi_\alpha)^{r(\gamma)}(L_\gamma(\hat{p}(s(\gamma)) \otimes (\hat{q}_1(s(\gamma)), \cdots, \hat{q}_n(s(\gamma)), 0 \cdots)) \\
&\qquad = (\phi_\alpha)^{r(\gamma)}\big(L_\gamma^P(\hat{p}(s(\gamma)) \otimes (L_\gamma^Q(\hat{q}_1(s(\gamma)), \cdots, L_\gamma^Q(\hat{q}_n(s(\gamma)), 0, \cdots)\big) \\
&\qquad = \big(L_\gamma^P(\hat{p}(s(\gamma)) \otimes L_\gamma^Q(\hat{q}_1(s(\gamma)), \cdots, L_\gamma^P(\hat{p}(s(\gamma)) \otimes L_\gamma^Q(\hat{q}_1(s(\gamma)), 0, \cdots\big)
\end{aligned}
$$

and

$$
\begin{aligned}
&L_\gamma\big((\phi_\alpha)^{r(\gamma)}(\hat{p}(s(\gamma)) \otimes (\hat{q}_1(s(\gamma)), \cdots, \hat{q}_n(s(\gamma)), 0 \cdots))\big) \\
&\quad = L_\gamma\big(\hat{p}(s(\gamma)) \otimes \hat{q}_1(s(\gamma)), \cdots, \hat{p}(s(\gamma)) \otimes \hat{q}_n(s(\gamma)), 0, \cdots\big) \\
&\quad = \big(L_\gamma^P(\hat{p}(s(\gamma))) \otimes L_\gamma^Q(\hat{q}_1(s(\gamma))), \cdots, L_\gamma^P(\hat{p}(s(\gamma))) \otimes L_\gamma^Q(\hat{q}_n(s(\gamma))), 0, \cdots\big).
\end{aligned}
$$

Thus, the diagram is commutative and $(P \otimes^{C_0(G^0)} Q)^\infty \simeq (P \otimes^{C_0(G^0)} Q^\infty)$.

Example 4.4.21. The G-Hilbert module $P_G \otimes^{C_0(G^0)} P$.

Suppose P is a G-Hilbert module, and denote its associated G-Hilbert bundle by E. Then, $P_G \otimes^{C_0(G^0)} P$ is a G-Hilbert module with associated bundle

$$E_{P_G \otimes^{C_0(G^0)} P} = L^2(G) \otimes E.$$

The Hilbert spaces of this bundle are $\{L^2(G^u, E_u)\}_{u \in G^0}$. The G action is the diagonal action,

$$L_\gamma \xi(\delta) = L_\gamma^E(\xi(\gamma^{-1}\delta)),$$

for $\gamma \in G$, $\xi \in L^2(G^{s(\gamma)}, E^{s(\gamma)})$ and $\delta \in G^{r(\gamma)}$. The continuous sections are determined by the span of functions of the form $\hat{F} \otimes \hat{p}$, where

$$\hat{F} \otimes \hat{p}(u) = F_{|G^u} \otimes \hat{p}(u),$$

for $F \in C_c(G)$ and $p \in P$.

Let

$$C_c(G, r^*(E)) = \{\phi \in C_c(G, E) : \phi(\gamma) \in E^{r(\gamma)}, \forall \gamma \in G\}.$$

If $u \in C_0(G^0)$ and $\phi \in C_c(G, r^*(E))$, let $\hat{\phi}(u) = \phi_{|G^u}$. Then,

$$\hat{\phi}(u) \in C_c(G^u, E^u) \subseteq (L^2(G) \otimes E)^u.$$

Thus, $\hat{\phi}$ is a section of $L^2(G) \otimes E$. In fact, we claim that $C_c(\widehat{G, r^*(E)})$ is a dense subspace of $\mathcal{S}C_0(L^2(G) \otimes E)$ that contains all of the functions of the form $\hat{F} \otimes \hat{p}$ as before. First, we note that $\hat{F} \otimes \hat{p} \in C_c(G, r^*(E))$ and the span of those functions is dense in $\mathcal{S}C_0(L^2(G) \otimes E)$. Moreover, we show that if $\phi \in C_c(G, r^*(E))$, then $\hat{\phi}$ is in the closure of this span.

Fix $\phi \in C_c(G, r^*(E))$. Let $H = \mathrm{supp}(\phi)$ and $u \in G^0$. We consider a compact subset of G such that $H \subset W^0$, the interior of W. Fix $\varepsilon > 0$. Let

$$\eta = \frac{\varepsilon}{(\sup_{u \in G^0} \lambda^u(W))^{1/2} + 1}.$$

If $\gamma \in H$, we pick $p_\gamma \in P$ such that $\|\phi(\gamma) - \hat{p}_\gamma(r(\gamma))\| < \eta/2$. We can pick such a p_γ because $\{\hat{p}(r(\gamma))\}$ is dense in $E^{r(\gamma)}$. For the same γ, let $h \in C_c(G)$ such that $h_\gamma(\gamma) = 1$. Then, by continuity, we can find a neighbourhood U_γ of γ in G with the property that $U_\gamma \subseteq W$ and for all $\gamma' \in U_\gamma$,

$$\|\phi(\gamma') - h_\gamma(\gamma')\hat{p}_\gamma(r(\gamma'))\| < \eta.$$

We can use here a partition of unity argument, since H is compact: we find a finite open subcover $U_{\gamma_1}, U_{\gamma_2}, \cdots, U_{\gamma_n}$ of H and functions $f_i \in C_c(U_{\gamma_i})$, $0 \leq f \leq 1$, $\sum_{i=1}^n f_i = 1$. Then

$$\sum_{i=1}^n \widehat{f_i h_{\gamma_i}} \otimes \hat{p}_{\gamma_i}$$

is one of the functions we want to use to approximate ϕ in $\mathcal{S}C_0(L^2(G) \otimes E)$. If $\gamma \in W$, then

$$\|\phi(\gamma') - \sum_{i=1}^n f_i(\gamma')h_{\gamma_i}(\gamma')\hat{p}_{\gamma_i}(r(\gamma'))\| < \eta.$$

If $u \in G^0$,

$$\begin{aligned}\|\phi_u - \sum_{i=1}^{n} \left(f_i h_{\gamma_i} \otimes \hat{p}_{\gamma_i}\right)_u\|_2^2 &= \int_W \|\phi(\gamma') - \sum_{i=1}^{n} f_i(\gamma') h_{\gamma_i}(\gamma') \hat{p}_{\gamma_i}(r(\gamma'))\|^2 \, d\lambda^u(\gamma') \\ &< \eta^2 \lambda^u(W) \\ &\le \varepsilon^2.\end{aligned}$$

Remark 4.4.22. If P, Q and R are G-Hilbert modules, then $P \otimes Q$ is a G-Hilbert module as well and

$$(P \oplus Q) \otimes^{C_0(G^0)} R \simeq (P \otimes^{C_0(G^0)} R) \oplus (Q \otimes^{C_0(G^0)} R). \tag{4.13}$$

The following proposition establishes that the G-Hilbert module tensor of P_G has the property of upgrading equivalence of Hilbert $C_0(G^0)$-modules to equivariance.

Proposition 4.4.23. *Suppose P and Q are G-Hilbert modules that are equivalent as Hilbert $C_0(G^0)$-modules. Then $P \otimes^{C_0(G^0)} P_G \simeq Q \otimes^{C_0(G^0)} P_G$ as G-Hilbert modules, that is, the last equivalence is equivariant.*

Proof. In order to simplify the notation, let $E = E_P$ and $F = E_Q$. Also, we will denote the actions L^P and L^Q just by a dot, hoping this will not create confusion. Remember that we are assuming the existence of a Hilbert module unitary $U : P \to Q$, its associate map on the Hilbert bundle category will be denoted by just $\{U^u\}_{u \in G^0}$.

Given $\phi \in C_c(G, r^*E)$, we define $V_\phi : G \to r^*(F)$ by

$$V_\phi(\gamma) = \gamma \cdot U^{s(\gamma)}(\gamma^{-1} \cdot \phi(\gamma)).$$

Since the map U is continuous, and so are the actions of E and F, it follows that V_ϕ is continuous. Also, V_ϕ is compactly supported if ϕ is, and respects the fibers. It is, in fact, an isomorphism onto $C_c(G, r^*F)$, with inverse

$$V_\chi^{-1}(\gamma) = \gamma \cdot (U^{s(\gamma)})^*(\gamma^{-1} \cdot \chi(\gamma)),$$

for $\chi \in C_c(G, r^*F)$.

We verify that V respects the inner product. Let $\phi, \psi \in C_c(G, r^*E)$ and $u \in G^0$.

$$\begin{aligned}
\langle (V_\phi)_{|G^u}, (V_\psi)_{|G^u} \rangle &= \int \langle V_\phi(\gamma), V_\psi(\gamma) \rangle \, d\lambda^u(\gamma) \\
&= \int \langle \gamma \cdot U^{s(\gamma)}(\gamma^{-1} \cdot \phi(\gamma)), \gamma \cdot U^{s(\gamma)}(\gamma^{-1} \cdot \psi(\gamma)) \rangle \, d\lambda^u(\gamma) \\
&= \int \langle \phi(\gamma), \psi(\gamma) \rangle \, d\lambda^u(\gamma) \\
&= \langle \phi_{|G^u}, \psi_{|G^u} \rangle.
\end{aligned}$$

Since $C_c(G, r^*(E))$ is a dense subspaces of $\mathcal{S}C_0(L^2 \otimes E)$ and $C_c(G, r^*(E))$ is a dense subspace of $\mathcal{S}C_0(L^2(G) \otimes F)$, V extends to a Hilbert module unitary

$$V : \mathcal{S}C_0(L^2(G) \otimes E) \to \mathcal{S}C_0(L^2(G) \otimes F).$$

Finally, we check that V is G-equivariant. The Hilbert module map associated to V is

$$V^u : L^2(G^u, E^u) \to L^2(G^u, F^u), \quad V_u(f)(\gamma) = V_f(\gamma),$$

for $f \in L^2(G^u, E^u)$ and $\gamma \in G^u$. Note that

$$V^{r(\gamma)}(\gamma \cdot f)(\delta) = V_{\gamma \cdot f}(\delta) = \delta \cdot U^{s(\delta)}(\delta^{-1} \cdot (\gamma \cdot f))(\delta) = \delta \cdot U^{s(\delta)}(\delta^{-1}\gamma f(\delta^{-1}\gamma))$$

and

$$\gamma \cdot V^{s(\gamma)}(f)(\delta) = \gamma \cdot (V_f(\gamma^{-1}\delta)) = \gamma\gamma^{-1}\delta U^{s(\gamma^{-1})}\delta)(\delta^{-1}\gamma \cdot f(\gamma^{-1}\delta)) = \delta \cdot U^{s(\delta)}(\delta^{-1}\gamma \cdot f(\gamma^{-1}\delta)),$$

for $\gamma \in G$, $f \in L^2(G^{s(\gamma)}, E^{s(\gamma)})$ and $h \in G^{r(\gamma)}$. □

We now state Paterson's groupoid stabilization theorem. As mentioned before, the result that we need is a claim used to prove the theorem.

Theorem 4.4.24. *Let P be a G-Hilbert module. Then,*

$$P \oplus P_G^\infty \simeq P_G^\infty.$$

The result we need is the following:

Proposition 4.4.25. *If P is a G-Hilbert module, then*

$$P_G^\infty \simeq (P \otimes^{C_0(G^0)} P_G^\infty) \oplus P_G^\infty. \tag{4.14}$$

In particular, if $\mathcal{H}$ is a G-Hilbert bundle, $\mathcal{H} \otimes L^2(G; l^2)$ is a sub-G-Hilbert bundle of $L^2(G; l^2)$.

Proof. The result is proved through the following chain of equivalences of G-Hilbert modules:

$$\begin{aligned}
P_G^\infty &\simeq (C_0(G^0) \otimes^{C_0(G^0)} P_G)^\infty & (4.15)\\
&\simeq \left((C_0(G^0) \otimes^{C_0(G^0)} P_G)^\infty\right)^\infty & (4.16)\\
&\simeq (C_0(G^0)^\infty \otimes^{C_0(G^0)} P_G)^\infty & (4.17)\\
&\simeq C_0(G^0)^\infty \otimes^{C_0(G^0)} P_G^\infty & (4.18)\\
&\simeq (P \oplus C_0(G^0)^\infty) \otimes^{C_0(G^0)} P_G^\infty & (4.19)\\
&\simeq (P \otimes^{C_0(G^0)} P_G^\infty) \oplus (C_0(G^0)^\infty \otimes^{C_0(G^0)} P_G^\infty) & (4.20)\\
&\simeq (P \otimes^{C_0(G^0)} P_G^\infty) \oplus (C_0(G^0) \otimes^{C_0(G^0)} P_G)^\infty & (4.21)\\
&\simeq (P \otimes^{C_0(G^0)} P_G^\infty) \oplus P_G^\infty. & (4.22)
\end{aligned}$$

The equivalences 4.15 and 4.22 are due to 4.11, 4.16 follows from 4.10 and both 4.17 and 4.18 apply 4.12. For 4.19, Kasparov's result 4.4.18 is used to establish that

$$C_0(G^0)^\infty \simeq P \oplus C_0(G^0)^\infty$$

as $C_0(G^0)$-Hilbert modules, and from 4.4.23 it follows that when tensoring with P_G^∞ the equivalence obtained is equivariant. The equivalence 4.20 follows from the distributive property 4.13 and 4.21 reverses what was done in 4.17.

Now that we established 4.14, we know that $P \otimes^{C_0(G^0)} P_G^\infty$ is a sub-G-Hilbert module of P_G^∞, that is, the inclusion map $\iota : P \otimes^{C_0(G^0)} P_G^\infty \hookrightarrow P_G^\infty$ is G-equivariant. If P is the G-Hilbert module associated to a G-Hilbert bundle $\mathcal{H}$, this means that the inclusion $\mathcal{H} \otimes L^2(G; l^2) \hookrightarrow L^2(G; l^2)$ is G-equivariant. This statement will be needed in the next section. $\square$

4.5 The Fourier algebra

We fix a left Haar system $\{\lambda^u\}_{u \in G}$ and we consider the left regular G-Hilbert bundle with infinite multiplicity $L^2(G; l^2)$. We define the set

$$A(G) = \{(\xi, \eta) : \xi, \eta \in \mathcal{S}C_0(L^2(G; l^2))\} \subseteq C_0(G).$$

This is the **Fourier algebra** of G. Our next goals are to prove that it is a commutative Banach algebra and study its relationship with $B(G)$.

The set we just defined is a subset of $B(G)$, but on it we wish to consider a (potentially) different norm:

$$\|\varphi\|_A = \inf \|\xi\| \|\eta\|,$$

where $\xi, \eta \in \mathcal{S}C_0(L^2(G; l^2))$ (instead of considering sections of any G-Hilbert bundle $\mathcal{H}$, see 4.2). We now want to prove that it is, in fact, a norm. To that end, we are going to use Paulsen's off-diagonal technique again. We prove that the new bundles built on 4.3.5 are L^2 bundles if we start with L^2 sections.

Proposition 4.5.1. *Let $\varphi \in C_b(G)$. The following statements are equivalent:*

1. *The function φ belongs to $A(G)$.*
2. *There exist $\rho, \tau \in P(G)$ and $\zeta \in \mathcal{S}C_0(L^2(G \times I_2; l^2))$ such that $(\zeta, \zeta) = \begin{pmatrix} \rho & \varphi \\ \varphi^* & \tau \end{pmatrix}$.*

Proof. We first prove that 1 implies 2. We follow the proof of 4.3.5. The $G \times I_2$-bundle that we consider in this case is

$$L^2(G; l^2) \otimes L^2(I_2) = (\{L^2(G^u; l^2) \otimes \mathbb{C}_i^2\}_{u \in G^0, i \in \{1,2\}}, C_c(G) \otimes \mathbb{C}^4, L \otimes \mathrm{Id}).$$

Here, for $i = 1, 2$, $\mathbb{C}_i^2 = \mathbb{C}^2$. We want to show that this bundle is isometrically isomorphic to

$$L^2(G \times I_2; l^2) = (\{L^2(G^u \times \{(i,1),(i,2)\}; l^2)\}_{u \in G^0, i \in \{1,2\}}, C_c(G \times I_2), L).$$

If $u \in G^0, i \in \{1, 2\}$, let

$$\begin{aligned} \psi_{u,i} : L^2(G^u; l^2) \otimes \mathbb{C}_i^2 &\to L^2(G^u \times \{(i,1),(i,2)\}; l^2) \\ f \otimes (\alpha_1, \alpha_2) &\to \psi_{u,i}(f \otimes (\alpha_1, \alpha_2))(\gamma i j) = \alpha_j f(\gamma). \end{aligned}$$

The inverse of this map is given by

$$(\psi_{u,i})^{-1}(F) = F(\cdot i1) \otimes (1, 0) + F(\cdot i2) \otimes (0, 1).$$

Let $u \in G^0$, $i \in \{1, 2\}$. Suppose $f \otimes (\alpha_1, \alpha_2), g \otimes (\beta_1, \beta_2) \in L^2(G^u; l^2) \otimes \mathbb{C}_i^2$. Then,

$$\begin{aligned} &\langle \psi_{u,i}(f \otimes (\alpha_1, \alpha_2)), \psi_{u,i}(g \otimes (\beta_1, \beta_2)) \rangle \\ &\qquad = \sum_n \sum_{j=1,2} \int_{G^u} \overline{\psi_{u,i}(f \otimes (\alpha_1, \alpha_2))_n(\gamma i j)} \psi_{u,i}(g \otimes (\beta_1, \beta_2))_n(\gamma i j)\, d\lambda^u(\gamma) \\ &\qquad = \sum_n \sum_{j=1,2} \int_{G^u} \overline{\alpha_j f_n(\gamma)} \beta_j g_n(\gamma)\, d\lambda^u(\gamma) \\ &\qquad = (\sum_n \langle f_n, g_n \rangle) \langle (\alpha_1, \alpha_2), (\beta_1, \beta_2) \rangle \\ &\qquad = \langle f \otimes (\alpha_1, \alpha_2), g \otimes (\beta_1, \beta_2) \rangle. \end{aligned}$$

Suppose $f \otimes \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \in C_c(G) \otimes \mathbb{C}^4$. As a section, this is, for $u \in G^0$ and $i \in \{1, 2\}$,

$$f \otimes \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} (uii) = f_{|G^u} \otimes (\alpha_{i1}, \alpha_{i2}) \in L^2(G^u; l^2) \otimes \mathbb{C}_i^2,$$

and those sections form a family as in 4.2.6. We want verify that $\psi \circ (C_c(G) \otimes \mathbb{C}^4)$ give us "enough" continuous sections on $L^2(G \times I_2; l^2)$.

First, we note that $\psi \circ f \otimes \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \in \mathcal{S}C(G \times I_2; l^2)$. If $u \in G^0$ and $i = \{1, 2\}$, $\psi_{ui}\big(f \otimes \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}\big)(\gamma ij) = \alpha_{ij} f_{|G^u}$, and this is a continuous section since I_2 is discrete and f is continuous.

On the other hand, it is easy to see that if $F \in C_c(G \times I_2, l^2)$,

$$\psi_{ui}^{-1}(F_{|G^u} \times \{(i,1),(i,2)\}) = F_{i1|_{G^u}} \otimes (1,0) + F_{i2|_{G^u}} \otimes (0,1).$$

Here F_{ij} is defined by $F_{ij}(\gamma) = F(\gamma ij)$ and it is a function on $C_c(G; l^2)$. Thus,

$$F_{i1|_{G^u}} \otimes (1,0) + F_{i2|_{G^u}} \otimes (0,1)$$

is a function of $C_c(G; l^2) \otimes \mathbb{C}_i^2$ viewed as a section.

It remains to prove that ψ intertwines the bundle actions, that is, the following diagram is commutative,

$$\begin{tikzcd}
L^2(G^{s(\gamma)}; l^2) \otimes \mathbb{C}_i^2 \arrow[r, "L_\gamma \otimes L_{ji}"] \arrow[d, "\psi_{s(\gamma)i}"'] & L^2(G^{r(\gamma)}; l^2) \otimes \mathbb{C}_j^2 \arrow[d, "\psi_{r(\gamma)j}"] \\
L^2(G^{s(\gamma)} \times \{(i,1),(i,2)\}; l^2) \arrow[r, "L_{\gamma ji}"'] & L^2(G^{r(\gamma)} \times \{(j,1),(j,2)\}; l^2)
\end{tikzcd}$$

for $\gamma \in G$ and $i, j \in \{1, 2\}$. Observe that

$$\begin{aligned}
\psi_{r(\gamma)j}\big(L_\gamma \otimes L_{ji}(f \otimes (\alpha_1, \alpha_2))\big)(\gamma' jk) &= \psi_{r(\gamma)j}(L_\gamma f \otimes (\alpha_1, \alpha_2)))(\gamma' jk) \\
&= (L_\gamma f)(\gamma')\alpha_k \\
&= f(\gamma^{-1}\gamma')\alpha_k,
\end{aligned}$$

for $\gamma' \in G^{s(\gamma)}$, $k \in \{1, 2\}$. On the other hand,

$$\begin{aligned}
L_{\gamma ji}(\psi_{s(\gamma)i}(f \otimes (\alpha_1, \alpha_2)))(\gamma' jk) &= (\psi_{s(\gamma)i}(f \otimes (\alpha_1, \alpha_2))(\gamma^{-1}\gamma' ik) \\
&= f(\gamma^{-1}\gamma')\alpha_k.
\end{aligned}$$

Thus, we proved that

$$L^2(G \times I_2; l^2) = L^2(G; l^2) \otimes L^2(I_2).$$

Therefore, following the proof of 4.3.5, the section ζ of $G \times I_2$ is in fact an L^2 section. By definition, it vanishes at infinity, since ξ and η vanish at infinity. Then,

$$(\zeta, \zeta) = \begin{pmatrix} \rho & \varphi \\ \varphi^* & \tau \end{pmatrix} \in P(G \times I_2) \cup A(G \times I_2).$$

We now prove that 2 implies 1. One more time, we will follow the proof of the corresponding result for $B(G)$ and show that the sections obtained are L^2 sections.

We are assuming that $\rho, \tau \in P(G)$ and $\zeta \in \mathcal{S}C_0(L^2(G \times I_2; l^2))$ verifies

$$(\zeta, \zeta) = \begin{pmatrix} \rho & \varphi \\ \varphi^* & \tau \end{pmatrix}.$$

We define two G-Hilbert bundles,

$$L^2(G \times I_2)_i = (\{L^2(G^u \times \{(i,1),(i,2)\}; l^2)\}_{u \in G}, C_c(G \times \{(i,1),(i,2)\}; l^2), L^i),$$

where $L^i_\gamma = L_{\gamma ii}$, for $i = 1, 2$.

We consider the sum of these continuous fields of Hilbert spaces but with a different action:

$$\begin{aligned} &L^2(G \times I_2)_1 \oplus L^2(G \times I_2)_2 = \\ &\qquad (\{L^2(G^u \times \{(1,1),(1,2)\}; l^2) \oplus L^2(G^u \times \{(2,1),(2,2)\}; l^2)\}_{u \in G}, \\ &\qquad C_c(G \times \{(i,1),(i,2)\}; l^2) \oplus C_c(G \times \{(i,1),(i,2)\}; l^2), L'), \end{aligned}$$

where $L' = \frac{1}{2}\begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}$. As before, we cut down the Hilbert spaces to obtain a G-Hilbert bundle. We consider the projections $P_u = L'_u$, for each $u \in G^0$, and define

$$\begin{aligned} \mathcal{K} = &\big(\{P_u\Big(L^2(G^u \times \{(1,1),(1,2)\}; l^2) \oplus L^2(G^u \times \{(2,1),(2,2)\}; l^2)\Big)\}_{u \in G}, \\ &P\big(C_c(G \times \{(i,1),(i,2)\}; l^2) \oplus C_c(G \times \{(i,1),(i,2)\}; l^2)\big), L'P\big). \end{aligned}$$

Here, $L'P_\gamma = L'_\gamma P_{s(\gamma)}$.

If $u \in G^0$, $f \in L^2(G^u \times \{(1,1),(1,2)\}; l^2)$ and $g \in L^2(G^u \times \{(2,1),(2,2)\}; l^2)$, then

$$L'_u(f \oplus g) = \frac{1}{2}(f + L_{u12}g \oplus L_{u21}f + g) \in \mathcal{K}^u.$$

Moreover, if $\gamma \in G$,

$$\begin{aligned}(L'P)_\gamma(L'_{s(\gamma)}(f \oplus g)) &= (L'P)_\gamma(\frac{1}{2}(f + L_{s(\gamma)12}g \oplus L_{s(\gamma)21}f + g)) \\ &= L'_\gamma\big(\frac{1}{2}(f + L_{s(\gamma)12}g \oplus L_{s(\gamma)21}f + g)\big) \\ &= \frac{1}{4}(2L_{\gamma 11}f + 2L_{\gamma 12}g \oplus 2L_{\gamma 21 f} + L_{\gamma 22}g) \\ &= L'_\gamma(f \oplus g).\end{aligned}$$

We want to prove that $\mathcal{K}$ is isometrically isomorphic to a sub-bundle of the left regular bundle with multiplicity 4,

$$L^2(G; l^2)^4 = (\{L^2(G^u; l^2)^4\}_u, C_c(G; l^2)^4, L^4).$$

If $h \in L^2(G^u \times \{(i,1),(i,2)\}; l^2)$ and $j \in \{1,2\}$, let $h_{ij} \in L^2(G^u; l^2)$ such that

$$h_{ij}(\gamma) = h(\gamma ij).$$

Note that

$$L_\gamma h_{ij}(\gamma') = h(\gamma^{-1}\gamma' ij) = h_{ij}(\gamma^{-1}\gamma') = L_{\gamma ii}h(\gamma' ij) = L_{\gamma ii}h_{ij}(\gamma').$$

Also, if h is continuous and compactly supported, so is h_{ij}.

We define $\psi_u : \mathcal{K}^u \to (L^2(G^u; l^2))^4$ by

$$\psi_u(f + L_{u12}g \oplus L_{u21}f + g) = (f_{11} + L_{u12}g_{21}) \oplus (f_{12} + L_{u12}g_{22}) \oplus (L_{u21}f_{11} + g_{21}) \oplus (L_{u21}f_{12} + g_{22}).$$

If $f, f' \in L^2(G^u \times \{(1,1),(1,2)\}; l^2)$ and $g, g' \in L^2(G^u \times \{(2,1),(2,2)\}; l^2)$,

$$\begin{gathered}
\langle \psi_u\big((f + L_{u12}g \oplus L_{u21}f + g)\big), \psi_u\big((f' + L_{u12}g' \oplus L_{u21}f' + g')\big)\rangle = \\
\langle (f_{11} + L_{u12}g_{21}) \oplus (f_{12} + L_{u12}g_{22}) \oplus (L_{u21}f_{11} + g_{21}) \oplus (L_{u21}f_{12} + g_{22}), \\
(f'_{11} + L_{u12}g'_{21}) \oplus (f'_{12} + L_{u12}g'_{22}) \oplus (L_{u21}f'_{11} + g'_{21}) \oplus (L_{u21}f'_{12} + g'_{22})\rangle = \\
2\langle f_{11}, f'_{11}\rangle + 2\langle f_{12}, f'_{12}\rangle + 2\langle L_{u21}f_{11}, g'_{21}\rangle + 2\langle L_{u21}f_{12}, g'_{22}\rangle \\
+2\langle L_{u12}g_{21}, f'_{11}\rangle + 2\langle L_{u12}g_{22}, f'_{12}\rangle + 2\langle g_{21}, g'_{21}\rangle + 2\langle g_{22}, g'_{22}\rangle = \\
2\langle f, f'\rangle + 2\langle L_{u21}f, g'\rangle + 2\langle L_{u12}g, f'\rangle + 2\langle g, g'\rangle = \\
\langle (f + L_{u12}g \oplus L_{u21}f + g), (f' + L_{u12}g' \oplus L_{u21}f' + g')\rangle
\end{gathered}$$

It remains to prove that ψ intertwines $L'P$ and L^4. Let $\gamma \in G$. Suppose

$$f + L_{u12}g \oplus L_{u21}f + g \in \mathcal{K}^{s(\gamma)}.$$

On one hand,

$$\begin{gathered}
\psi_{r(\gamma)}(L'P_\gamma(f + L_{u12}g \oplus L_{u21}f + g)) = \psi_{r(\gamma)}(L_{\gamma 11}f + L_{\gamma 12g} \oplus L_{\gamma 21}f + L_{\gamma 22}g) \\
= L_{\gamma 11}f_{11} + L_{\gamma 12}g_{21} \oplus L_{\gamma 11}f_{12} + L_{\gamma 12}g_{22} + L_{\gamma 21}f_{11} + L_{\gamma 22}g_{21} \oplus L_{\gamma 21}f_{12} + L_{\gamma 22}g_{22}.
\end{gathered}$$

On the other hand,

$$\begin{gathered}
(L_\gamma)^4(\psi_{s(\gamma)}(f + L_{u12}g \oplus L_{u21}f + g)) = \\
(L_\gamma)^4((f_{11} + L_{u12}g_{21}) \oplus (f_{12} + L_{u12}g_{22}) \oplus (L_{u21}f_{11} + g_{21}) \oplus (L_{u21}f_{12} + g_{22})) = \\
L_{\gamma 11}f_{11} + L_{\gamma 12}g_{21} \oplus L_{\gamma 11}f_{12} + L_{\gamma 12}g_{22} \oplus L_{\gamma 21}f_{11} + L_{\gamma 22}g_{21} \oplus L_{\gamma 21}f_{12} + L_{\gamma 22}g_{22},
\end{gathered}$$

using the fact that $L_\gamma h_{ij} = L_{\gamma ii}h_{ij}$.

Thus, we proved that $\mathcal{K}$ is isometrically isomorphic to a sub-bundle of $L^2(G; l^2)^4$. But $L^2(G; l^2)^4 \simeq L^2(G; (l^2)^4) \simeq L^2(G; l^2)$. Therefore, the sections ξ, η constructed as in Proposition 4.3.5 are in fact continuous sections of $L^2(G; l^2)$ and they vanish at infinity since ζ does. Then, $\varphi \in A(G)$. $\square$

Lemma 4.5.2. *If $\varphi \in A(G)$, the following are equivalent:*

1. *There exist sections $\xi, \eta \in \mathcal{S}C_0(L^2(G; l^2))$ of norm less or equal than 1 such that $\varphi = (\xi, \eta)$.*

2. *There exist $\rho, \tau \in P(G)$ of norm less or equal than 1 such that*

$$\begin{pmatrix} \rho & \varphi \\ \varphi^* & \tau \end{pmatrix} \in P(G \ \times I_2) \cap A(G \times I_2). \tag{4.23}$$

Proof. It follows from Proposition 4.5.1 above and Lemma 4.3.7. □

Corollary 4.5.3. *If $\varphi \in A$, then*

$$\|\varphi\|_A = \inf \|F\|_\infty$$

for $F \in P(G \times I_2) \cap A(G \times I_2)$ such that $\varphi = F(\cdot, 1, 2)$.

Proposition 4.5.4. *The function $\|\cdot\|_A$ is a norm on $A(G)$.*

Proof. Let $\varphi \in A(G) \subset B(G)$. Suppose that $\|\varphi\|_A = 0$. Since $\|\varphi\|_B \leq \|\varphi\|_A$, it follows that $\varphi = 0$.

If $\varphi_1, \varphi_2 \in A(G)$, we apply the L^2 version of 4.3.7 to show that

$$\|\varphi_1 + \varphi_2\| \leq \|\varphi_1\| + \|\varphi_2\|.$$

□

Proposition 4.5.5. *$A(G)$ separates points, that is, given $\gamma, \gamma' \in G$, there exists $\varphi \in A(G)$ such that $\varphi(\gamma) \neq \varphi(\gamma')$.*

Proof. Let $\gamma \neq \gamma'$. Denote $y = r(\gamma)$, and remember that $i_y\gamma = \gamma$ and $\gamma\gamma^{-1} = i_y$. Since G is a Hausdorff space, let U, V be neighborhoods of γ and γ' respectly such that $U \cap V = \phi$.

Let $V' = G \setminus V$. We can find O open neighborhood of i_y such that $OU \subseteq V'$. In particular, $\gamma' \notin OU$.

Let $f = \chi_O$ and $g = \chi_{U^{-1}}$. Then $f * g^* \in A(G)$; moreover,

$$f * g^*(\delta) = \int_O \chi_{U^{-1}}(\delta^{-1}\delta')\, d\lambda^{r(\delta)}(\delta') = \lambda^{r(\delta)}(O \cap \delta U^{-1}).$$

Therefore, $f * g^*(\gamma) = \lambda^{r(\gamma)}(O \cap \gamma U^{-1}) > 0$ (since $O \cap \gamma U^{-1}$ is a non-empty open set) and $f * g^*(\gamma') = 0$ (since $\gamma' \notin OU$). $\square$

Note that if we write

$$A_s(G) = \operatorname{span}\{(\xi, \eta) : \xi, \eta \in \mathcal{S}C_0(L^2(G))\},$$

$A_s(G)$ is a subspace of $B(G)$. We can consider the closure of $A_s(G)$ on the $B(G)$-norm, and we call it $A'(G)$. Moreover, in the following proposition we prove that $A(G)$ coincides with $A'(G)$. Thus, $A(G) = A'(G)$ as sets, but the norms we consider are, a priori, different.

Proposition 4.5.6. *If $\xi \in \mathcal{S}C_0(L^2(G; l^2))$ and $\varphi = (\xi, \xi)$, then*

$$\|\xi\|_\infty^2 = \|\varphi_{|_{G^0}}\|_\infty = \|\varphi\|_\infty = \|\varphi\|_B = \|\varphi\|_A.$$

Proof. See the proof of Proposition 4.3.6. $\square$

Proposition 4.5.7 (See [43], Lemma 1.2)**.** *The spaces $A(G)$ and $A'(G)$ are equal.*

Proof. We first prove that $A(G)$ is closed in the $\|\cdot\|_A$ norm. Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be a sequence in $A(G)$ such that $\sum_n \|\varphi_n\|_A < \infty$. We can assume that $\varphi_n \in P(G)$ for all n, otherwise, we consider the sequence $\{F_{\varphi_n}\} \in P(G \times I_2) \cap A(G \times I_2)$, as in Proposition 4.5.1. Let $\xi_n \in \mathcal{S}C_0(L^2(G; l^2))$ be such that $(\xi_n, \xi_n) = \varphi_n$. Since $\|\varphi_n\| = \|\xi_n\|^2$, taking a sub-sequence if needed, we can assume that $\sum_n \|\xi_n\| < \infty$. Thus, the sequence $\mu_n = \xi_1 \oplus \xi_2 \oplus \cdots \oplus \xi_n$ is a Cauchy sequence in $\mathcal{S}C_0(L^2(G; l^2))$. Hence, there is $\xi = \sum_n \xi_n \in \mathcal{S}C_0(L^2((G; l^2))$ and then $\varphi = \sum_n \varphi_n \in A(G)$. Therefore, $A(G)$ is closed in the $\|\cdot\|_A$ norm.

Moreover, any element of $A(G)$ can be approximated by an element of $A_s(G)$ in the $\|\cdot\|_A$ norm. Thus, $A(G) = \overline{A_s(G)}^{\|\cdot\|_A}$.

We can also prove that $A(G)$ is closed in the $\|\cdot\|_B$ norm. Assume $\{\varphi_n\}_{n\in\mathbb{N}}$ is a sequence in $A(G)$ such that $\sum_n \|\varphi_n\|_B < \infty$. As before, we can suppose that $\varphi_n \in P(G)$ for all n. Then, from Proposition 4.5.6, $\|\varphi_n\|_A = \|\varphi_n\|_B = \|\xi_n\|$ and following the proof as above we conclude that $A(G)$ is closed in the $\|\cdot\|_B$ norm.

So far we know that $A'(G) \subseteq A(G)$, since $A'(G) = \overline{A_s(G)}^{\|\cdot\|_B} \subseteq \overline{A(G)}^{\|\cdot\|_B} = A(G)$. But $\|\cdot\|_B \leq \|\cdot\|_A$, then $A(G) \subseteq A'(G)$ as well. Therefore, $A(G)$ and $A'(G)$ coincide as spaces.

It follows from the Open Mapping Theorem that the norms $\|\cdot\|_A$ and $\|\cdot\|_B$ on $A(G)$ are equivalent. $\square$

Example 4.5.8. If G is a locally compact group, the Fourier and Fourier Stieltjes algebras defined above coincide with the ones defined by Eymard. Moreover,

$$(A(G), \|\cdot\|_A) = (A'(G), \|\cdot\|_{B(G)}),$$

see 4.2.

Example 4.5.9. Let X be a locally compact space with trivial groupoid structure and left Haar system $\{\delta^x\}_{x\in X}$, as in the example 2.2.2. In order to compute $A(X)$, we look at the left regular Hilbert bundle $L^2(X) = (\{\mathbb{C}\}_{x\in X}, C_c(X), \mathrm{Id})$. Then, $\mathcal{S}C_0(L^2(X)) = C_0(X)$ and $(f,g)(x) = \overline{f(x)}g(x)$, for $f, g \in C_0(X)$. It is easy to see that

$$C_c(X) \subseteq \operatorname{span}\{(f,g) :\ f, g \in C_0(X)\} \subseteq C_0(X).$$

If $f \in C_c(X)$, let $g \in C_c(X)$ such that $g_{|_{\mathrm{supp}(f)}} = 1$ and $g(X) \subseteq [0,1]$. Then, $(g,f) = f$ and $\|f\|_{A(X)} \leq \|f\|_\infty \|g\|_\infty = \|f\|_\infty$. It follows that $A(X) = C_0(X)$.

For the Fourier-Stieltjes algebra, note that $\mathcal{S}C_b(L^2(X)) = C_b(X)$ and if $f \in C_b(X)$, $(1_X, f) = f$, here $1_X(x) = x$ for all $x \in X$. Hence, $B(X) = C_b(X)$.

The result from the previous Section, concerning the stability theorem for groupoids, is used in the next proposition. Note that we need to add the hypothesis of the groupoid being proper.

Proposition 4.5.10. *Let G be a locally compact proper groupoid. Then, $A(G)$ is an involutive, commutative, Banach algebra that is also a $B(G)$-bimodule and $A'(G)$ is a norm-closed, involutive ideal of $B(G)$.*

Proof. Let $\xi, \eta \in \mathcal{S}C_0(L^2(G))$ and $\zeta, \mu \in \mathcal{S}C_b(\mathcal{H})$, then $(\xi, \eta)(\zeta, \mu) = (\xi \otimes \zeta, \eta \otimes \mu)$ is a coefficient of $L^2(G) \otimes \mathcal{H}$, coming from continuous sections vanishing at infinity. It follows from 4.4.25 that $L^2(G) \otimes \mathcal{H}$ is isomorphic to a subbundle of $L^2(G; l^2)$ and hence

$$(\xi \otimes \zeta, \eta \otimes \mu) \in A(G).$$

Therefore, $A(G)$ is an involutive Banach algebra and $A'(G)$ is an ideal of $B(G)$. $\square$

Proposition 4.5.11. *If G is compact and transitive, the norms $\|\cdot\|_A$ and $\|\cdot\|_B$ coincide in $A(G)$ and $A(G) = B(G)$.*

Proof. Since $A'(G)$ is an ideal of $B(G)$, we want to show that $1_G : G \to \mathbb{C}$, $1_G(\gamma) = 1$ for all $\gamma \in G$, belongs to $A'(G)$. Since the functions of $C_c(G) = C(G)$ are continuous sections of the G-bundle $L^2(G)$, the function 1_G is a continuous section. Note that

$$(1_G, 1_G)(\gamma) = \lambda^{r(\gamma)}(G^{r(\gamma)}) = \alpha^2,$$

a fix positive number α, for all $\gamma \in G$, due to the transitiveness of the groupoid and the properties of the Haar system. Let $\xi : G \to \mathbb{C}$, $\xi = 1_G/\alpha$, then $(\xi, \xi) = 1_G \in A'(G)$. Therefore, the equality of sets holds.

To prove the equality of the norms, given $\varphi \in B(G)$ and $\varepsilon > 0$, we want to approximate $\|\varphi\|$ with L^2 sections. We can find $\zeta, \eta \in \mathcal{S}C(\mathcal{H})$, for some G-Hilbert bundle $\mathcal{H}$, such that $\|\zeta\|\|\eta\| < \|\varphi\| + \varepsilon$. Note that, if ξ is a section as above,

$$\xi \otimes \zeta, \xi \otimes \eta \in \mathcal{S}C(\mathcal{H} \otimes L^2(G)) \subseteq \mathcal{S}C(L^2(G; l^2)),$$

see Proposition 4.5.10 for the last inclusion. Since, $\|\xi \otimes \zeta\| = \|\zeta\|$, $\|\xi \otimes \eta\| = \|\eta\|$, and $(\xi \otimes \zeta, \xi \otimes \eta) = 1_G \varphi = \varphi$, we proved that the norms coincide. $\square$

Remark 4.5.12. For any groupoid G, the Fourier algebra $A(G)$ and the algebra $A'(G)$ are built using some sections of $L^2(G)$. The Hilbert bundle $L^2(G)$ depends on the Haar system considered on G, and hence, we could obtain potentially different Fourier algebras by considering distinct Haar systems. As such, it would be more precise to write $A(G, \{\lambda^u\})$ and $A'(G, \{\lambda^u\})$. On the other hand, the Fourier-Stieltjes algebra does not depend on the selection of the Haar system. Thus, if G is a groupoid such that $A(G, \{\lambda^u\}) = B(G)$, for all possible Haar systems, (for instance, as in Proposition 4.5.11), all the possible Fourier algebras coincide and it may be convenient to fix a particular Haar system to work with.

Proposition 4.5.13. *Suppose G is a locally trivial groupoid. If G is transitive and we fix $u \in G^0$, then*

$$A(G) = \overline{\text{span}}\{(\xi, \eta) :\ \xi, \eta \in C_0(G^0, L^2(G^u))\}.$$

Proof. Fix $u \in G^0$. Recall from 2.2.8 that there is a family $\{U_n, \nu_n\}_{n\in\mathbb{N}}$, where $\{U_n\}_n$ is an open cover of G^0 and $\nu_n : U_n \to G^u$ is a continuous map such that $\nu_n(v) \in G^u_v$. For each unit v we fix the smallest index n_v such that $v \in U_{n_v}$. Then, there is a neighbourhood V of v such that for all $w \in V$, $n_w = n_v$ (take $V \subseteq U_{n_v} \setminus (\cup_{j<n_v} U_j)$).

We consider the constant continuous field of Hilbert spaces

$$L^2(G^u) := (\{L^2(G^u)\}_{v\in G^0}, C_c(G^0, G^u)).$$

We want to show that it is a G-Hilbert bundle with action L^u,

$$L^u_\gamma := L_{\nu_j(r(\gamma))\gamma\nu_i(s(\gamma))^{-1}}$$

for $n_{r(\gamma)} = j$ and $n_{s(\gamma)} = i$. To ease the notation, we write $\beta(\gamma) = \nu_j(r(\gamma))\gamma\nu_i(s(\gamma))^{-1}$. The only condition that is not immediate to check is the continuity. In order to check that this bundle is strongly continuous, that is, that for a fixed $\xi \in \mathcal{S}C_0(L^2(G^u))$, the map

$$G \to \sqcup L^2(G^u), \quad \gamma \to L^u_\gamma \xi(s(\gamma))$$

is continuous, we proceed as in the proof of the continuity of the left regular bundle.

Fix $\gamma \in G$. Suppose that $L^u_\gamma \xi(s(\gamma)) \in U(\varepsilon, V, \eta)$, for $\varepsilon > 0$, $V \subseteq G^0$ open and $\eta \in C_0(G^0, L^2(G^u))$. Let $\alpha = \|L^u_\gamma \xi(s(\gamma)) - \eta(r(\gamma))\|$ and $\alpha' = \varepsilon - \alpha$. We can find $\eta', \xi' \in C_c(G^0, L^2(G^u))$ such that $\|\eta - \eta'\| < \alpha'/6$ and $\|\xi - \xi'\| < \alpha'/6$. Let

$$h(\gamma', \delta') = |\xi'(s(\gamma'))(\beta(\gamma')^{-1}\delta') - \eta'(r(\delta'))(\delta')|^2.$$

Observe that $h : G \times G^u \to \mathbb{C}$ is continuous and compactly supported. Then, it is easy to verify (either applying lemma 4.2.19 or by hand) that

$$\gamma' \to H(\gamma') = \Big(\int h(\gamma', \delta')\, d\lambda^u(\delta')\Big)^2$$

is continuous.

Note that

$$\begin{aligned}
H(\gamma) &= \|L_{\beta(\gamma)}\xi'(s(\gamma)) - \eta'(r(\gamma))\| \\
&\leq \|L_{\beta(\gamma)}\xi'(s(\gamma)) - L_{\beta(\gamma)}\xi(s(\gamma))\| \\
&+ \|L_{\beta(\gamma)}\xi(s(\gamma)) - \eta(r(\gamma))\| + \|\eta(r(\gamma)) - \eta'(r(\gamma))\| \\
&\leq \|\xi'(s(\gamma)) - \xi(s(\gamma))\| + \alpha + \frac{\alpha'}{6} \\
&< \alpha + \frac{\alpha'}{3}.
\end{aligned}$$

Then, there exist a neighbourhood W of γ such that

$$H(\gamma') < (\alpha + \frac{\alpha'}{3}) + \frac{\alpha'}{6} = \alpha + \frac{\alpha'}{2}$$

for all $\gamma' \in W$.

Therefore, if $\delta \in W \cap r^{-1}(V)$, then

$$\begin{aligned}
&\|L_{\beta(\delta)}\xi(s(\delta)) - \eta(r(\delta))\| \\
&\qquad \leq \|L_{\beta(\delta)}\xi'(s(\delta)) - L_{\beta(\delta)}\xi(s(\delta))\| + H(\delta) + \|\eta(r(\delta)) - \eta'(r(\delta))\| \\
&\qquad < \alpha + \frac{5\alpha'}{6} < \varepsilon.
\end{aligned}$$

Thus, $\delta \in U(\varepsilon, V, \eta)$, and the bundle is strongly continuous.

Let $\xi \in \mathcal{S}C_0(L^2(G^u))$. Define $\xi_G : G^0 \to \sqcup L^2(G^v)$ by $\xi_G(v) = L_{\nu_i(s(\gamma))^{-1}}\xi(s(\gamma))$. Note that ξ_G is a section of $L^2(G)$ vanishing at infinity. The proof of the continuity of ξ_G is very similar, again, to the case of the left regular bundle. Note that if $\xi, \eta \in \mathcal{S}C_0(L^2(G^u))$,

$$\begin{aligned}
(\xi,\eta)^u(\gamma) &= < L_{\nu_j(r(\gamma))\gamma\nu_i(s(\gamma))^{-1}}\xi(s(\gamma)), \eta(r(\gamma)) > \\
&= < L_\gamma(L^{-1}_{\nu_i(s(\gamma))}\xi(s(\gamma)), L^{-1}_{\nu_j(r(\gamma))}\eta(r(\gamma)) > \\
&= < L_\gamma \xi_G(s(\gamma)), \eta_G(r(\gamma)) > \\
&= (\xi_G, \eta_G)(\gamma).
\end{aligned}$$

Moreover, for $\xi \in \mathcal{S}C_0(L^2(G))$, we define $\xi^u \in \mathcal{S}C_0(L^2(G^u))$ by $\xi^u(v) = L_{\nu_i(v)}\xi(v)$. The continuity of these sections at a point v, with $n_v = i$, follows from composing ν_i and the left regular representation. Therefore, we have a one to one correspondence between sections of the left regular bundle and the constant left regular bundle. Observe that the section norm is preserved by this correspondence.

Then,

$$\begin{aligned}
\operatorname{span}\{(\xi,\eta) : \xi,\eta \in \mathcal{S}C_0(L^2(G))\} &= \operatorname{span}\{(\zeta,\mu) : \zeta,\mu \in \mathcal{S}C_0(L^2(G^u))\} \\
&= \operatorname{span}\{(\zeta,\mu) : \zeta,\mu \in C_0(G^0, L^2(G^u))\}
\end{aligned}$$

and $A(G) = \overline{\operatorname{span}}\{(\zeta,\mu) : \zeta,\mu \in C_0(G^0, L^2(G^u))\}$. □

Example 4.5.14. Full equivalence groupoid on a finite set.

Let I_n be the full equivalence groupoid on the set $\{1, 2, \cdots, n\}$. In this case,

$$A(G) = A'(G) = B(G) = M_n$$

and the product is the Schur product of matrices. Let ξ be a section, then $\xi : G^0 \to \mathcal{H}$, for some G-Hilbert bundle $\mathcal{H}$. Hence it can be written as $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathcal{H}^n$. We apply the alternative description of $A(G)$ given on 4.5.13. The norm is given by

$$\|\varphi\| = \inf\{\|\xi\|_\infty \|\eta\|_\infty : \xi = (\xi_i)_{i=1}^n,\ \eta = (\eta_j)_{j=1}^n,\ \xi_i, \eta_j \in \mathcal{H}, \varphi_{ij} = \langle \xi_i, \eta_j \rangle\},$$

where $\mathcal{H}$ is a Hilbert space, and this is the Schur multipliers norm. Therefore, the Fourier algebra $A(I_n)$ is the algebra of matrices M_n with Schur product and Schur norm. Thus, $A(I_n) = l_n^\infty \overset{h}{\otimes} l_n^\infty$. See [37], Theorem 8.7 and [51], Section 3.1.

Example 4.5.15. Full equivalence groupoid on a locally compact space.

Let X be a locally compact space and fix $u \in G^0$. Let λ a positive regular Borel measure supported on X such that $\lambda(\{u\}) = 1$. Consider the full equivalence groupoid $G = X \times X$, with product topology and Haar system $\{\lambda^x\}_{x \in X}$, where $\lambda^u = \delta_x \times \lambda$. Remember that this is the Haar system obtained by applying Seda's construction (2.3.6). Our main theorem will allow us to prove that that $A(G) = C_0(X) \overset{h}{\otimes} C_0(X)$.

If X is compact, the Fourier-Stieltjes and the Fourier algebra (for any fixed Haar system) coincide and are $C(X) \overset{h}{\otimes} C(X)$.

Chapter 5

A decomposition of $A(G)$

This chapter contains the main result of this work. It is divided into four sections. The first of these presents two ways of defining a completely contractive product on certain Banach spaces. The first way concerns spaces of the form $C_0(G^0, \overline{L^2(G^u; l^2)_r}) \overset{h}{\otimes} C_0(G^0, L^2(G^u; l^2)_c)$. Here u is a fixed unit of a locally trivial and transitive groupoid G, with Haar system $\{\lambda^v\}_{v \in G^0}$ verifying $\lambda^u_{|G^u_u} = m$, the left Haar measure at G^u_u. The second involves images by completely quotient maps of completely contractive Banach algebras. Recall that at the beginning of Chapter 4 we discussed the importance of having a completely contractive Banach algebra structure for the Fourier algebra of a locally compact group.

The second section is very short and relates the Banach space $C_0(G^0) \overset{h}{\otimes} A(G^u) \overset{h}{\otimes} C_0(G^0)$ (that is going to be of crucial importance in Section 3) to the Banach algebra $C_0(G^0) \overset{\vee}{\otimes} A(G^u) \overset{\vee}{\otimes} C_0(G^0)$. The characterization of the Haagerup tensor product (3.3.27) from Smith is used here.

Our main theorem is included in Section 3 and is the following. Let G be a transitive, locally trivial, locally compact groupoid. Let $u \in G^0$ and suppose $\{\lambda^v\}_{v \in G^0}$ is a left Haar system for G such that $\lambda^u_{|G^u_u}$ is the left Haar measure on the isotropy group G^u_u. Then, we prove that as Banach algebras,

$$A(G) \simeq C_0(G^0) \overset{h}{\otimes} A(G^u_u) \overset{h}{\otimes} C_0(G^0),$$

where the product on the right hand side is given by

$$(a \otimes b \otimes c)(a' \otimes b' \otimes c') = aa' \otimes bb' \otimes cc'.$$

Thus, in $A(G)$ we can consider the operator space structure of $C_0(G^0) \overset{h}{\otimes} A(\tilde{G}) \overset{h}{\otimes} C_0(G^0)$. Moreover, applying the results from the first Section, the last space is a completely contractive Banach algebra, and therefore so is $A(G)$.

The last section is devoted to the non-transitive case. Since we are considering locally trivial groupoids, the transitive components are components also in the topological sense (see 2.3.3). If $G = \cup_i G_i$, where the G_i's are the transitive components, we show here that both $B(G)$ and $A(G)$ can be expressed in terms of sums of the algebras $B(G_i)$ and $A(G_i)$.

5.1 Completely contractive products

This section has two goals. The first is to define a completely contractive product on Banach spaces of the form $C_0(G^0, \overline{L^2(G^u; l^2)_r}) \overset{h}{\otimes} C_0(G^0, L^2(G^u; l^2)_c)$. Here we assume that G is a transitive, locally trivial groupoid, u is a fixed unit of G and the left Haar system $\{\lambda^v\}_{v \in G^0}$ on G is such that $\lambda^u_{|G^u_u}$ is the left Haar measure m on the isotropy group G^u_u. The main tools to prove this are Effros-Ruan's shuffle map (see 5.1.2) and the operator space version of Grothendieck's approximation property (see 5.1.13).

The second goal is to show that if A admits a completely contractive product, B is an operator space that admits a bilinear map and between them there is a complete quotient map $\varphi : A \to B$ that intertwines the product and the bilinear map, then the bilinear map in B can be extended to a completely contractive map on $B \overset{\wedge}{\otimes} B$ (see 5.1.29).

Definitions 5.1.1. Let A be an algebra (not necessarily associative) that is also a complete operator space. A **completely contractive product** on A is an extension of the multiplication $m : A \times A \to A$ to a completely contractive linear mapping on the projective tensor product $A \overset{\wedge}{\otimes} A$.

If A is, in addition, associative, we say that A together with a completely contractive product is a **completely contractive Banach algebra**.

We need to introduce a shorter notation to fit equations. Let

$$C_{0r} = C_0(G^0, \overline{L^2(G^u; l^2)}_r)$$

and

$$C_{0c} = C_0(G^0, L^2(G^u; l^2)_c).$$

Our first goal is to define a completely contractive map

$$(C_{0r} \overset{h}{\otimes} C_{0c}) \overset{\wedge}{\otimes} (C_{0r} \overset{h}{\otimes} C_{0c}) \to C_{0r} \overset{h}{\otimes} C_{0c}.$$

To accomplish this, we use the following result from Effros and Ruan:

Theorem 5.1.2. *[[15], Theorem 6.1] If V_k, W_k, $k = 1, 2$ are operator spaces, then the shuffle map*

$$S : (V_1 \otimes W_1) \otimes (V_2 \otimes W_2) \to (V_1 \otimes V_2) \otimes (W_1 \otimes W_2)$$

defined on elementary tensors by $S[(v_1 \otimes w_1) \otimes (v_2 \otimes w_2)] = (v_1 \otimes v_2) \otimes (w_1 \otimes w_2)$ extends to a complete contraction that we still call S

$$S : (V_1 \overset{eh}{\otimes} W_1) \overset{nuc}{\otimes} (V_2 \overset{eh}{\otimes} W_2) \to (V_1 \overset{nuc}{\otimes} V_2) \overset{eh}{\otimes} (W_1 \overset{nuc}{\otimes} W_2).$$

By the properties of the nuclear tensor product, there is always a complete contraction $\pi : V \overset{\wedge}{\otimes} W \to V \overset{nuc}{\otimes} W$ (see 3.2.35). Also, there is a completely isometric embedding $V \overset{h}{\otimes} W \to V \overset{eh}{\otimes} W$ (see 3.4.6). Then, we obtain a complete contraction that we still call S:

$$S : (V_1 \overset{h}{\otimes} W_1) \overset{\wedge}{\otimes} (V_2 \overset{h}{\otimes} W_2) \to (V_1 \overset{nuc}{\otimes} V_2) \overset{eh}{\otimes} (W_1 \overset{nuc}{\otimes} W_2).$$

Hence, for our spaces C_{0r} and $C_{0c} = C_0(G^0, L^2(G^u; l^2)_c)$, we have a completely contractive map

$$(C_{0r} \overset{h}{\otimes} C_{0c}) \overset{\wedge}{\otimes} (C_{0r} \overset{h}{\otimes} C_{0c}) \to (C_{0r} \overset{nuc}{\otimes} C_{0r}) \overset{eh}{\otimes} (C_{0c} \overset{nuc}{\otimes} C_{0c})$$

We want to show that the nuclear tensor products on the range of the map are, in fact, projective tensor products. Hence, we want to show that the canonical map

$$\iota : C_{0r} \overset{\wedge}{\otimes} C_{0r} \to C_{0r} \overset{\vee}{\otimes} C_{or} \tag{5.1}$$

is one-to-one, and similarly for the column case. This condition is the operator space version of Grothendieck's approximation property. We introduce some equivalent definitions of the approximation property for a Banach space X. The first of these concerns the possibility of approximating the identity operator by a finite rank operator on compact subsets of X.

Proposition 5.1.3 ([46], Proposition 4.1)**.** *If X is a Banach space, the following statements are equivalent:*

1. *Let K be a compact subset of X and $\varepsilon > 0$. Then there exists a finite rank operator $S : X \to X$ such that $\|x - Sx\| \leq \varepsilon$ for all $x \in K$.*
2. *Let Y be a Banach space. If $T : X \to Y$ is an operator, K is a compact subset of X and $\varepsilon > 0$, there there exists a finite rank operator $S : X \to Y$ such that $\|Tx - Sx\| \leq \varepsilon$ for all $x \in K$.*
3. *Let Y be a Banach space. If $T : Y \to X$ is an operator, K is a compact subset of Y and $\varepsilon > 0$, there there exists a finite rank operator $S : Y \to X$ such that $\|Ty - Sy\| \leq \varepsilon$ for all $y \in K$.*

Definition 5.1.4. We said that a Banach space X has the **approximation property** if it satisfies any (and hence all) the properties above.

This property also has formulations in terms of properties of the Banach space injective tensor product $\overset{\lambda}{\otimes}$ and the Banach space projective tensor product $\overset{\gamma}{\otimes}$.

Proposition 5.1.5 ([46], Proposition 4.6)**.** *If X is a Banach space, the following statements are equivalent:*

1. *X has the approximation property.*

2. *Let* $u = \sum_{n=1}^{\infty} \varphi_n \otimes x_n$ *be an element of* $X^* \overset{\gamma}{\otimes} X$*, where* $\{\varphi_n\}_{n\in\mathbb{N}}$ *is a bounded sequence in* X^**,* $\{x_n\}_{n\in\mathbb{N}}$ *is a bounded sequence in* X *and* $\sum_{n=1}^{\infty} \|\varphi_n\|\|x_n\| < \infty$*. If* $\sum_{n=1}^{\infty} \varphi_n(x)x_n = 0$ *for every* $x \in X$*, then* $u = 0$*.*

3. *Let* Y *be a Banach space. Suppose* $u = \sum_{n=1}^{\infty} x_n \otimes y_n$ *is an element of* $X \overset{\gamma}{\otimes} Y$*, where* $\{x_n\}_{n\in\mathbb{N}}$ *is a bounded sequence in* X*,* $\{y_n\}_{n\in\mathbb{N}}$ *is a bounded sequence in* Y *and* $\sum_{n=1}^{\infty} \|x_n\|\|y_n\| < \infty$*. If* $\sum_{n=1}^{\infty} \varphi(x_n)y_n = 0$ *for every* $\varphi \in X^*$*, then* $u = 0$*.*

4. *Let* Y *be a Banach space. Suppose* $u = \sum_{n=1}^{\infty} x_n \otimes y_n$ *is an element of* $X \overset{\gamma}{\otimes} Y$*, where* $\{x_n\}_{n\in\mathbb{N}}$ *is a bounded sequence in* X*,* $\{y_n\}_{n\in\mathbb{N}}$ *is a bounded sequence in* Y *and* $\sum_{n=1}^{\infty} \|x_n\|\|y_n\| < \infty$*. If* $\sum_{n=1}^{\infty} \psi(y_n)x_n = 0$ *for every* $\psi \in Y^*$*, then* $u = 0$*.*

We also consider two stronger versions of the approximation property for Banach spaces.

Definitions 5.1.6 ([46], p. 80)**.** We say that a Banach space X has the **bounded (metric) approximation property** if there is a net $\{\psi_\alpha\}_{\alpha\in}$ of finite rank operators on X, with $\|\psi_\alpha\| \le M$ for some constant M (for $M = 1$), for all α, such that $\psi_\alpha \to \mathrm{Id}_X$ uniformly on compact subsets of X.

Proposition 5.1.7 ([46], Proposition 4.3 and p.80)**.** *Let* X *be a Banach space. Suppose there is a net* $\{T_\alpha\}_{\alpha\in\Lambda}$ *of uniformly bounded finite rank operators on* X *verifying* $T_\alpha x \to x$ *for all* $x \in X$*. Then* X *has the bounded approximation property. If* $\|T_\alpha\| \le 1$*, then* X *has the metric approximation property.*

Example 5.1.8. The spaces $C(K)$ and $C_0(X)$ have the metric approximation property.

Let K be a compact space. Suppose $J \subseteq C(K)$ is compact and $\varepsilon > 0$. By Arzelà-Ascoli Theorem, we can find an open cover $\{U_1, U_2, \cdots, U_n\}$ of K such that, for all $i = 1, 2, \cdots, n$, if $s, t \in U_i$, then $|f(s) - f(t)| < \varepsilon$, for all $f \in J$. Let $\{g_1, g_2, \cdots, g_n\}$ be a partition of the unity subordinated to the covering. For each i, we pick $t_i \in U_i$.

Now we define $S = S_J : C(K) \to C(K)$, $S(f) = \sum_{i=1}^{n} f(t_i)g_i$. Then S is a finite rank operator. Also,

$$\|S\| = \sup_{\|f\|\le 1} \|Sf\| \le \sup_{\|f\|\le 1} \sup_{x\in X} \sum_{i=1}^{n} |f(t_i)|g_i(x) \le 1.$$

Moreover, if $x \in K$ and $f \in J$,

$$|Sf(x) - f(x)| = |\sum_{i=1}^{n} f(t_i)g_i(x) - f(x)g_i(x)| \leq \sum_{i=1}^{n} |f(t_i) - f(x)|g_i(x).$$

There exists l such that $x \in U_l$. Then $|f(t_l) - f(x)| < \varepsilon$ and for $i \neq l$, $g_i(x) = 0$. Thus

$$\|Sf - f\| = \sup_x |Sf(x) - x| < \varepsilon.$$

Consider the net $\{S_J\}$, indexed on the compact subsets of $C(K)$ with reverse inclusion as partial order. It verifies all the conditions of the definition of the metric approximation property. Thus, $C(K)$, for K compact, has this property.

If X is a locally compact space, we now prove that $C_0(X)$ has the metric approximation property.

Let K be a compact subset of $C_0(X)$ and $\varepsilon > 0$. We denote by X_∞ the one point compactification of X. Then, $C_0(X)$ is a closed ideal of $C(X_\infty)$ and $C(X_\infty)$ has the metric approximation property. Also, K is a compact subset of $C(X_\infty)$. Then, there exists a finite rank operator $S_\infty : C(X_\infty) \to C(X_\infty)$, with norm less than one, such that for all $f \in K$, $\|S_\infty f_\infty - f_\infty\| < \varepsilon$, where $f_\infty(x) = f(x)$ if $x \in X$ and $f_\infty(\infty) = 0$. Let $S_K = S_{\infty|_{C_0(X)}}$, it is a finite rank operator of norm at most one and $\|Sf - f\| < \varepsilon$, for all $f \in K$.

As before, we can find a net $\{S_K\}$ indexed on the compact subsets of $C_0(X)$ and hence $C_0(X)$ has the metric approximation property.

Example 5.1.9. [[46], p.73] If $\mathcal{H}$ is a Hilbert space, the it has the metric approximation property. Let $\{e_i\}_{i\in I}$ be an orthonormal basis for $\mathcal{H}$. If $F \subseteq I$ is a finite subset, define $P_F : \mathcal{H} \to \mathcal{H}$ by $P_F(h) = \sum_{i\in F}\langle h, e_i\rangle e_i$. Each P_F is a finite-rank operator on $\mathcal{H}$ of norm less or equal than 1. Since $P_F(h) \to h$ for all $h \in \mathcal{H}$, $\mathcal{H}$ has the metric approximation property.

Before considering the operator space versions of these properties, we present one more equivalent formulation of the approximation property. It is based on the following characterization of compactness for subsets of Banach spaces:

Lemma 5.1.10 ([14], Lemma 11.1.1). *Let X be a Banach space. A subset K of X is compact if and only if K is contained in the closed convex hull of a sequence with limit* 0.

Let $c_0(X)$ be the space of sequences on X with limit 0. On $c_0(X)$ we consider the norm $\|\{x_n\}\| = \sup_n \|x_n\|$. Then, if $\varphi : X \to X$ is a linear contraction, it determines a linear map

$$\varphi^\infty = \mathrm{Id} \otimes \varphi : c_0(X) \to c_0(X), \quad \varphi^\infty(\{x_n\}) = \{\varphi(x_n)\},$$

that is a contraction as well.

Corollary 5.1.11 ([14], Corollary 11.1.2). *If X is a Banach space, it has the approximation property if and only if given $\varepsilon > 0$ and $x = \{x_n\} \in c_0(X)$ there exists $\varphi : X \to X$ of finite rank such that $\|\varphi^\infty(x) - x\| < \varepsilon$.*

In order to formulate the approximation property for an operator space V, the analog to $c_0(X)$ is $K_\infty(V) = K_\infty \overset{\vee}{\otimes} V$ (see 3.2.33).

Definition 5.1.12. An operator space V has the **approximation property** if for each $v \in K_\infty(V)$, there exists a completely bounded map $\varphi : V \to V$ of finite-rank such that $\|\varphi^\infty(v) - v\|_{cb} < \varepsilon$, where $\varphi^\infty = \mathrm{Id} \otimes \varphi$.

Theorem 5.1.13 ([14], Theorem 11.2.5). *Let V be an operator space. The following are equivalent:*

1. *V has the operator space approximation property.*
2. *For any operator space W, the canonical mapping $V \overset{\wedge}{\otimes} W \to V \overset{\vee}{\otimes} W$ is one-to-one.*
3. *The map $V \overset{\wedge}{\otimes} V^* \to V \overset{\vee}{\otimes} V^*$ is one-to-one.*

Also in the operator space case there are stronger versions of the approximation property.

Definitions 5.1.14. We say that V has the **completely bounded (completely contractive) approximation property** if there is a net $\{\psi_\alpha\}_{\alpha\in\Lambda}$ of finite rank maps on V, with $\|\psi_\alpha\|_{cb} \leq M$ for some constant M (for $M = 1$) such that for all $v \in V$, $\|\psi_\alpha(v) - v\| \to 0$.

Note that the completely bounded approximation property asks for approximation of elements of V, whereas the approximation property refers to approximation of elements of $K_\infty(V)$. We may wonder why it was claimed that the completely bounded version is stronger. In fact, the completely bounded operator space approximation property implies the operator space approximation property (see [14], Theorem 11.3.3). If $\{\psi_\alpha\}_{\alpha\in\Lambda}$ is as in the definition of the cb-approximation property, the uniform bound on the completely bounded norm of the elements ψ_α permits one to prove that $\|\psi_\alpha^\infty(v) - v\|_{cb} \to 0$.

Thus, it is enough for us to show that $C_0(X, \overline{\mathcal{H}}_r)$ and $C_0(X, \mathcal{H}_c)$ have the completely bounded approximation property, for all Hilbert space $\mathcal{H}$.

Proposition 5.1.15. *The space $C_0(X)$ has the completely contractive approximation property.*

Proof. Suppose the net $\{\psi_\alpha\}_{\alpha\in\Lambda}$ realizes the metric approximation property of $C_0(X)$. Since $\mathcal{B}(C_0(X)) = \mathcal{CB}(C_0(X))$ and contractions correspond to complete contractions (see 3.1.11 and 3.1.12), the maps ψ_α are complete contractions. Then $C_0(X)$ has the completely contractive approximation property. □

Proposition 5.1.16. *If $\mathcal{H}$ is a Hilbert space, the row and column spaces $\overline{\mathcal{H}_r}$ and $\mathcal{H}_c$ have the completely bounded approximation property.*

Proof. As in 5.1.9, let $\{P_\lambda\}_{\lambda\in\Lambda}$ be a net of projections witnessing the metric approximation property of $\mathcal{H}$. Since $\mathcal{B}(\mathcal{H}) = \mathcal{CB}(\mathcal{H}_c)$ (see 3.1.13), those projections are in fact completely bounded. Moreover, the identification above takes contractions to complete contractions, hence $\mathcal{H}_c$ has the completely contractive approximation property.

The same applies to $\overline{\mathcal{H}}_r$, since $\mathcal{B}(\mathcal{H}) = \mathcal{CB}(\overline{\mathcal{H}}_r)$ (by 3.1.16). □

Proposition 5.1.17. *The spaces C_{0r} and C_{0c} have the operator space approximation property. Then, the map ι of 5.1 is one-to-one.*

Proof. We now combine the nets $\{\psi_\alpha\}_{\alpha\in A}$ and $\{P_\lambda\}_{\lambda\in\Lambda}$ from 5.1.15 and 5.1.16 to construct a net for $C_0(X,\overline{L^2(Y;l^2)_r})$ and similarly for $C_0(X,L^2(Y;l^2)_c)$. Let $\{E_{\alpha,\lambda}\}_{\alpha,\lambda\in A\times\Lambda}$, where $E_{\alpha,\lambda}: C_0(X,\overline{L^2(Y;l^2)}_r)\to C_0(X,\overline{L^2(Y;l^2)}_r)$, $E_{\alpha,\lambda}=\psi_\alpha\otimes P_\lambda$. Note that each of these maps has finite rank and is completely contractive. If $u\in C_0(G^0)\otimes_\vee \overline{L^2(Y;l^2)}_r$, $u=\sum_{i=1}^n f_i\otimes\xi$,

$$\begin{aligned}\|E_{\alpha,\lambda}(u)-u\| &\le \|(\psi_\alpha\otimes P_\lambda)\big(\sum f_i\otimes\xi_i\big)-(\mathrm{Id}\otimes P_\lambda)\big(\sum f_i\otimes\xi_i\big)\|\\ &\quad+\|(\mathrm{Id}\otimes P_\lambda)(\sum f_i\otimes\xi_i)-\sum f_i\otimes\xi_i\|\\ &\le \sum_{i=1}^n\|\psi_\alpha(f_i)-f_i\|\|P_\lambda(\xi_i)\|+\sum_{i=1}^n\|f_i\|\|P_\lambda(\xi_i)-\xi_i\|\\ &\underset{\alpha,\lambda}{\to} 0.\end{aligned}$$

□

Therefore, we have a completely contractive map

$$S:(C_{0r}\overset{h}{\otimes}C_{0c})\overset{\wedge}{\otimes}(C_{0r}\overset{h}{\otimes}C_{0c})\to(C_{0r}\overset{\wedge}{\otimes}C_{0r})\overset{eh}{\otimes}(C_{0c}\overset{\wedge}{\otimes}C_{0c}).$$

Our next goal is to show that the image of $(C_{0r}\overset{h}{\otimes}C_{0c})\overset{\wedge}{\otimes}(C_{0r}\overset{h}{\otimes}C_{0c})$ under S is included in $(C_{0r}\overset{\wedge}{\otimes}C_{0r})\overset{h}{\otimes}(C_{0c}\overset{\wedge}{\otimes}C_{0c})$. To this end, we recall that for operator spaces V and W,

$$\begin{aligned}V\overset{h}{\otimes}W=\{u\in V\overset{eh}{\otimes}W:\ &u=v\odot w, v\in M_{1,\infty}(V), w\in M_{\infty,1}(W),\\ &u=\|\cdot\|\text{-}\lim_p\sum_{j=1}^p v_j\otimes w_j\}\end{aligned}$$

(from 3.4.5) and, by 3.3.8,

$$\begin{aligned} M_n(V \overset{h}{\otimes} W) &= M_{n,1}(V) \overset{h}{\otimes} M_{1,n}(W) \\ &= \{u \in M_{n,1}(V) \overset{eh}{\otimes} M_{1,n}(W) :\ u = v \odot w, v \in M_{n,\infty}(V), w \in M_{\infty,n}(W), \\ &\qquad u = \|\cdot\|\text{-}\lim_p \sum_{j=1}^{p} v_j \otimes w_j\}. \end{aligned}$$

Thus, we want to prove that if u is an element of the algebraic tensor product

$$(C_{0r} \overset{h}{\otimes} C_{0c}) \otimes (C_{0r} \overset{h}{\otimes} C_{0c}),$$

then $S(u)$ is the norm limit of truncated sums of tensors as above. It is enough to verify this for an elementary tensor $v \otimes w$, $v, w \in C_{0r} \overset{h}{\otimes} C_{0c}$. Let $s, x \in M_{1,\infty}(C_{0r})$, $t, y \in M_{\infty,1}(C_{0c})$ such that $v = s \odot t$ and $w = x \odot y$. Hence,

$$v = \|\cdot\|\text{-}\lim_p \sum_{i=1}^{p} s_i \otimes t_i \text{ and } w = \|\cdot\|\text{-}\lim_q \sum_{j=1}^{q} x_j \otimes y_j.$$

Since S is a complete contraction,

$$\begin{aligned} S(v \otimes w) &= \|\cdot\|_{eh}\text{-}\lim_{p,q} \sum_{i,j}^{p,q} (s_i \otimes x_j) \otimes (t_i \otimes y_j) \\ &= \|\cdot\|_{eh}\text{-}\lim_{r} \sum_{k}^{r} a_k \otimes b_k, \end{aligned}$$

by re-naming the indexes.

Then, so far we have defined a complete contractive map

$$S : (C_{0r} \overset{h}{\otimes} C_{0c}) \overset{\wedge}{\otimes} (C_{0r} \overset{h}{\otimes} C_{0c}) \to (C_{0r} \overset{\wedge}{\otimes} C_{0r}) \overset{h}{\otimes} (C_{0c} \overset{\wedge}{\otimes} C_{0c}). \tag{5.2}$$

The diagram below shows the process we went through to define the map S:

$$
\begin{tikzcd}
(C_{0r} \overset{eh}{\otimes} C_{0c}) \overset{nuc}{\otimes} (C_{0r} \overset{eh}{\otimes} C_{0c}) \arrow[r] & (C_{0r} \overset{nuc}{\otimes} C_{0r}) \overset{eh}{\otimes} (C_{0c} \overset{nuc}{\otimes} C_{0c}) \\
(C_{0r} \overset{eh}{\otimes} C_{0c}) \overset{\wedge}{\otimes} (C_{0r} \overset{eh}{\otimes} C_{0c}) \arrow[u] & \\
(C_{0r} \overset{h}{\otimes} C_{0c}) \overset{\wedge}{\otimes} (C_{0r} \overset{h}{\otimes} C_{0c}) \arrow[u, hook] \arrow[uur] \arrow[r, dashed, "S"'] & (C_{0r} \overset{\wedge}{\otimes} C_{0r}) \overset{h}{\otimes} (C_{0c} \overset{\wedge}{\otimes} C_{0c}) \arrow[uu, hook]
\end{tikzcd}
$$

At this point, we have just used Effros and Ruan's shuffle map to reorder the spaces C_{0r} and C_{0c}, but we have not defined the product we aim for yet. In order to gain some intuition as far as this product is concerned, we restrict ourselves to the group case.

If H is a group, we are trying to define a product on $\overline{L^2(H;l^2)_r} \overset{\wedge}{\otimes} L^2(H;l^2)_c$. We denote this space $T(H,l^2)$. The reference for the definition of this product is Spector's work [50], but our context is a bit different: we consider an operator space tensor product version, that is in addition amplified by l^2.

We first consider the non-amplified version, thus we want to define a product on

$$T(H) = \overline{L^2(H)_r} \overset{\wedge}{\otimes} L^2(H)_c.$$

Proposition 5.1.18. *Let $\diamond : T(H) \times T(H) \to T(H)$ be defined as follows. If $\overline{\xi}, \overline{f} \in \overline{L^2(H)}$ and $\eta, g \in L^2(H)$, let*

$$(\overline{\xi} \otimes \eta) \diamond (\overline{f} \otimes g) = \int_H \overline{\xi x f} \otimes \eta x g \, dx, \tag{5.3}$$

where the integral is a Bochner integral ([19], Chapter II, Section 5) and xf is the right translation, that is, $xf(y) = f(yx)$, for all $y \in H$. Then $\diamond$ has range $T(H)$ and is a complete contraction.

Proof. We first check that $(\overline{\xi} \otimes \eta) \diamond (\overline{f} \otimes g)$ belongs to $T(H)$. Note that

$$\begin{aligned}
\|\overline{\xi} \otimes \eta \diamond \overline{f} \otimes g\|^2 &\leq \Big(\int_H \|\xi r f\| \|\eta r g\| \, d\mu(r) \Big)^2 \\
&\leq \int_H \|\xi r f\|^2 \, d\mu(r) \int_H \|\eta r g\|^2 \, d\mu(r),
\end{aligned}$$

Hölder's inequality is applied above. Also, by Tonelli's theorem and the left invariance of the Haar system,

$$\begin{aligned}\int_H \|\xi r f\|^2\, d\mu(r) &= \int_H \int_H |\xi(x) f(xr)|^2\, d\mu(x)\, d\mu(r) \\ &= \int_H |\xi(x)|^2 (\int_H |f(xr)|^2\, d\mu(r))\, d\mu(x) \\ &= \|\xi\|^2 \|f\|^2.\end{aligned}$$

Therefore,

$$\|\overline{\xi} \otimes \eta \diamond \overline{f} \otimes g\|^2 \leq \|\xi\|^2 \|f\|^2 \|\eta\|^2 \|g\|^2.$$

We also need to prove that $\diamond$ is a complete contraction. To this end, we are going to verify that it is the pre-adjoint of a complete isometry (see 3.1.9). We begin by observing that

$$\begin{aligned}T(H)^* = (\overline{L^2(H)}_r \overset{\wedge}{\otimes} L^2(H)_c)^* &\simeq \mathcal{CB}(\overline{L^2(H)}_r, L^2(H)_c^*) \\ &\simeq \mathcal{CB}(\overline{L^2(H)}_r, \overline{L^2(H)}_r) \simeq \mathcal{B}(L^2(H))\end{aligned} \tag{5.4}$$

via the identification $T(\overline{\xi} \otimes \eta) = \langle \xi, T(\eta) \rangle$, for $T \in B(L^2(H))$, $\xi, \eta \in L^2(H)$. See 3.2.17, 3.1.15 and 3.1.16.

Moreover,

$$(T(H) \overset{\wedge}{\otimes} T(H))^* = (\mathcal{B}(L^2(H))_* \overset{\wedge}{\otimes} \mathcal{B}(L^2(H))_*)^* \simeq \mathcal{B}(L^2(H)) \overline{\otimes} \mathcal{B}(L^2(H)) \simeq \mathcal{B}(L^2(H \times H).$$

The identification here is

$$T(\overline{\xi} \otimes \eta \otimes \overline{f} \otimes g) = \langle \xi \otimes f, T(\eta \otimes g) \rangle, \tag{5.5}$$

for $T \in \mathcal{B}(L^2(H \times H)$, $\overline{\xi}, \overline{f} \in \overline{L^2(H)}_r$ and $\eta, g \in L^2(H)_c$, see 3.2.23.

As in the theory of quantum groups (see, for instance, [25] and [40]), we define a fundamental unitary W and an associated normal isometric isomorphism Γ_W. Let

$$\begin{aligned}W : L^2(H \times H) &\to L^2(H \times H) \\ Wh(r, s) &= h(r, r^{-1}s).\end{aligned}$$

This is, in fact, a unitary map with adjoint

$$W^* : L^2(H \times H) \to L^2(H \times H)$$
$$W^* h(r,s) = h(r, rs).$$

Define

$$\Gamma_W : \mathcal{B}(L^2(H) \to \mathcal{B}(L^2(H))\overline{\otimes}\mathcal{B}(L^2(H))$$
$$\Gamma_W(T) = W(T \otimes \mathrm{Id})W^*,$$

it is a normal isometric isomorphism and hence a complete isometry.

We want to prove that $\diamond = (\Gamma_W)_*$, that is,

$$\Gamma_W(T)(\omega \otimes \nu) = T(\omega \diamond \nu)$$

for $T \in \mathcal{B}(L^2(H))$, $\omega \otimes \nu \in T(H) \overset{\wedge}{\otimes} T(H)$. It is enough to consider $\omega = \overline{\xi} \otimes \eta$, $\nu = \overline{f} \otimes g$ and a rank one operator $T = \theta_{\zeta,\mu}$, $\theta_{\zeta,\mu}(f) = \langle \mu, f \rangle \zeta$.

On the left-hand side,

$$\begin{aligned}
\Gamma_W(T)(\overline{\xi} \otimes \eta) \otimes (\overline{f} \otimes g)) &= \langle \xi \otimes f, W(T \otimes \mathrm{Id})W^*(\eta \otimes g) \rangle \\
&= \langle W^* \xi \otimes f, (T \otimes \mathrm{Id})W^*(\eta \otimes g) \rangle \\
&= \int_H \int_H \overline{W^*(\xi \otimes f)(r,s)} (T \otimes \mathrm{Id})W^*(\eta \otimes g)(r,s)\, dr\, ds \\
&= \int_H \int_H \overline{\xi(s) r f(s)} \langle \mu, \eta r g \rangle \zeta(\varepsilon)\, ds\, dr \\
&= \int_H \langle \mu, \eta r g \rangle \langle \xi r f, \zeta \rangle\, dr.
\end{aligned}$$

On the right-hand side,

$$T((\overline{\xi} \otimes \eta) \diamond (\overline{f} \otimes g)) = T(\int_H \overline{\xi r f} \otimes \eta r g\, dr) = \int_H \langle \xi r f, T(\eta r g) \rangle\, dr = \int_H \langle \mu, \eta r g \rangle \langle \xi r f, \zeta \rangle\, dr.$$

by Proposition II.5.7 from [19].

Then, since $\diamond$ is the pre-adjoint of the complete isometry Γ_W, it is a complete quotient map, and in particular, a complete contraction. $\square$

Proposition 5.1.19. *([50]) If in $T(H)$ we consider the product $\diamond$, then the map*

$$q_0 : T(H) \to A(H), \quad q_0(\overline{\xi} \otimes \eta) = (\xi, \eta)$$

from 4.3 is an homomorphism of algebras.

Proof. In effect, let $\overline{\xi} \otimes \eta$ and $\overline{f} \otimes g$ be elementary tensors on $T(H)$ and $x \in H$.

$$\begin{aligned}
q_0(\overline{\xi} \otimes \eta)(x) q_0(\overline{f} \otimes g)(x) &= (\xi, \eta)(x)(f, g)(x) \\
&= \langle \lambda(x)\xi, \eta \rangle \langle \lambda(x) f, g \rangle \\
&= \int_H \overline{\xi(x^{-1}y)}\eta(y)\, d\mu(y) \int_H \overline{f(x^{-1}z)} g(z)\, d\mu(z) \\
&= \int_H \overline{\xi(x^{-1}y)}\eta(y) \int_H \overline{f(x^{-1}z)} g(z)\, d\mu(z)\, d\mu(y) \\
&= \int_H \overline{\xi(x^{-1}y)}\eta(y) \int_H \overline{f(x^{-1}yz)} g(yz)\, d\mu(z)\, d\mu(y) \\
&= \int_H \int_H \overline{\xi(x^{-1}y) f(x^{-1}yz)}\eta(y) g(yz)\, d\mu(z)\, d\mu(y) \\
&= \int_H \int_H \overline{\lambda(x)(\xi z f)(y)}\eta z g(y)\, d\mu(y)\, d\mu(z) \\
&= \int_H \langle \lambda(x)\xi z f, \eta z g \rangle\, d\mu(z) \\
&= \int_H (\xi z f, \eta z g)(x)\, d\mu(z) \\
&= q_0\big(\int_H \overline{\xi z f} \otimes \eta z g\, d\mu(z) \big)(x) \\
&= q_0(\overline{\xi} \otimes \eta \diamond \overline{f} \otimes g)(x).
\end{aligned}$$

Here, the left invariance of the Haar measure μ, Fubini's Theorem and Proposition II.5.7 from [19] of the Bochner integral are applied. $\square$

The next step is to define an amplified version of $\diamond$ on

$$T(H; l^2) = \overline{L^2(H; l^2)_r} \overset{\wedge}{\otimes} L^2(H; l^2)_c.$$

Observe that

$$
\begin{aligned}
T(H;l^2) &= \overline{L^2(H;l^2)}_r \overset{\wedge}{\otimes} L^2(H;l^2)_c \\
&= \overline{(L^2(H)\otimes l^2)}_r \overset{\wedge}{\otimes} (L^2(H)\otimes l^2)_c \\
&= \overline{(L^2(H)}_r \overset{\wedge}{\otimes} \overline{l^2}_r \overset{\wedge}{\otimes} L^2(H)_c \overset{\wedge}{\otimes} l^2_c \\
&= ((\overline{L^2(H)}_r \overset{\wedge}{\otimes} L^2(H)_c) \overset{\wedge}{\otimes} (\overline{l^2}_r \overset{\wedge}{\otimes} l^2_c) \\
&= ((\overline{L^2(H)}_r \overset{\wedge}{\otimes} L^2(H)_c) \overset{\wedge}{\otimes} (\overline{l^2(\mathbb{Z})}_r \overset{\wedge}{\otimes} l^2(\mathbb{Z})_c) \\
&= T(H) \overset{\wedge}{\otimes} T(\mathbb{Z}).
\end{aligned}
$$

The associativity and commutativity of the projective tensor product as well as 3.3.12 are applied here.

Definition 5.1.20. We define the amplified version of $\diamond$, that we denote $\diamond^\infty$, by

$$\diamond^\infty : T(H;l^2)\times T(H;l^2) \to T(H;l^2), \quad \diamond^\infty = \diamond_H \otimes \diamond_{\mathbb{Z}}.$$

Here we see l^2 as $l^2(\mathbb{Z})$. But the group $\mathbb{Z}$ does not play any special role, and could be substituted by any countable group.

Remark 5.1.21. The map $\diamond^\infty$ is a complete contraction, since both $\diamond_H$ and $\diamond_{\mathbb{Z}}$ are complete contractions.

Proposition 5.1.22. *If we consider the product $\diamond^\infty$ on $T(H;l^2)$, then the amplified version of q_0 as in 4.5, $q_0^\infty : T(H;l^2) \to A(H)$, respects the product.*

Proof. Recall that q_0^∞ is defined by $q_0^\infty(\overline{\xi}\otimes\eta) = (\xi,\eta) = \langle \lambda^\infty(\cdot)\xi,\eta\rangle$, for $\xi \in \overline{L^2(H;l^2)_r}$ and $\eta \in L^2(H;l^2)$. If $\overline{\xi}\otimes\{\overline{\zeta_n}\} \in \overline{L^2(H)_r} \overset{\wedge}{\otimes} \overline{l^2_r} = \overline{L^2(H;l^2)_r}$ and $\eta\otimes\{\nu_n\} \in L^2(H)_c \overset{\wedge}{\otimes} l^2_c =$

$L^2(H;l^2)_c$, then

$$\begin{aligned}
q_0^\infty((\overline{\xi}\otimes\{\overline{\zeta_n}\})\otimes(\eta\otimes\{\nu_n\})) &= (\xi\otimes\{\zeta_n\},\eta\otimes\{\nu_n\})\\
&= \sum_n\langle\lambda(\cdot)\xi\otimes\zeta_n,\eta\otimes\nu_n\rangle\\
&= \langle\lambda(\cdot)\xi,\eta\rangle\sum\overline{\zeta}_n\nu_n\\
&= (\xi,\eta)\langle\lambda(0)\{\zeta_n\},\{\nu_n\}\rangle\\
&= q_0(\overline{\xi}\otimes\eta)\,q_0(\overline{\{\zeta_n\}}\otimes\{\nu_n\})(0).
\end{aligned}$$

Note that here we are simultaneously working with the group H and its left regular representation and the group $\mathbb{Z}$ and its left regular representation.

In order to verify that q_0^∞ is an homomorphism of algebras when on $T(H;l^2)$ we consider the product defined above, let

$$\overline{\xi}\otimes\{\overline{\zeta_n}\},\overline{f}\otimes\{\overline{c_n}\}\in\overline{L^2(H)_r}\overset{\wedge}{\otimes}\overline{l^2_r}=\overline{L^2(H;l^2)_r}$$

and

$$\eta\otimes\{\nu_n\},g\otimes\{d_n\}\in L^2(H)_c\overset{\wedge}{\otimes}l^2_c=L^2(H;l^2)_c.$$

Then,

$$\begin{aligned}
&q_0^\infty((\overline{\xi}\otimes\{\overline{\zeta_n}\}\otimes\eta\otimes\{\nu_n\})\diamond^\infty(\overline{f}\otimes\{\overline{c_n}\}\otimes g\otimes\{d_n\}))\\
&\quad= q_0^\infty\big(\sum_r\int_H(\overline{\xi xf}\otimes\overline{\{\zeta_n\}r\{c_n\}})\otimes(\eta xg\otimes\{\nu_n\}r\{d_n\})\,dx\big)\\
&\quad= \sum_r\int_H q_0(\overline{\xi xf}\otimes\eta xg)\langle\{\zeta_n rc_n\},\{\nu_n rd_n\}\rangle\,dx\\
&\quad= q_0(\overline{\xi}\otimes\eta\diamond\overline{f}\otimes g)\sum_r\sum_m\overline{\zeta_m c_{m+r}}\nu_m d_{m+r}.
\end{aligned}$$

Doing the change of variable $n = m + r$,

$$\begin{aligned}
&q_0^\infty((\overline{\xi} \otimes \{\overline{\zeta_n}\}) \otimes (\eta \otimes \{\nu_n\}) \diamond^\infty (\overline{f} \otimes \{\overline{c_n}\} \otimes g \otimes \{d_n\}) \\
&\quad = q_0(\overline{\xi} \otimes \eta \diamond \overline{f} \otimes g) \sum_{m,n} \overline{\zeta_m c_n} \nu_m d_n \\
&\quad = q_0(\overline{\xi} \otimes \eta \diamond \overline{f} \otimes g) \sum_{m} \overline{\zeta_m} \nu_m \sum_{n} \overline{c_n} d_n
\end{aligned}$$

Since q_0 respects the product,

$$\begin{aligned}
&q_0^\infty((\overline{\xi} \otimes \{\overline{\zeta_n}\}) \otimes (\eta \otimes \{\nu_n\}) \diamond^\infty (\overline{f} \otimes \{\overline{c_n}\} \otimes g \otimes \{d_n\}) \\
&\quad = q_0(\overline{\xi} \otimes \eta) q_0(\overline{f} \otimes g) \sum_{m} \overline{\zeta_m} \nu_m \sum_{n} \overline{c_n} d_n \\
&\quad = q_0(\overline{\xi} \otimes \eta) q_0(\overline{f} \otimes g) q_0(\{\overline{\zeta_m}\} \otimes \{\nu_m\})(0) q_0(\{\overline{c_n}\} \otimes \{d_n\}(0) \\
&\quad = q_0(\overline{\xi} \otimes \eta) q_0(\{\overline{\zeta_m}\} \otimes \{\nu_m\})(0) q_0(\overline{f} \otimes g) q_0(\{\overline{c_n}\} \otimes \{d_n\}(0) \\
&\quad = q_0^\infty(\overline{\xi} \otimes \{\overline{\zeta_n}\}) \otimes (\eta \otimes \{\nu_n\}) q_0^\infty(\overline{f} \otimes \{\overline{c_n}\} \otimes g \otimes \{d_n\}).
\end{aligned}$$

Therefore, q_0^∞ respects the product. $\square$

Remark 5.1.23. Note that $T(H; l^2) = T(H \times \mathbb{Z})$ (as mentioned before, instead of $\mathbb{Z}$ we could consider any countable group). We can also see q_0^∞ as the composition of

$$q_0 : T(H \times \mathbb{Z}) \to A(H \times \mathbb{Z}) = A(H) \overset{\wedge}{\otimes} A(\mathbb{Z})$$

and

$$\mathrm{Id} \otimes \mathrm{ev}_0 : A(H) \overset{\wedge}{\otimes} A(\mathbb{Z}) \to A(H) \overset{\wedge}{\otimes} \mathbb{C} = A(H).$$

Remark 5.1.24. We are interested in a groupoid version of $\diamond^\infty$. The new product we are trying to define should coincide with this one if the groupoid is a group. Moreover, note that

$$C_0(G^0, \overline{\mathcal{H}}_r) \overset{h}{\otimes} C_0(G^0, \mathcal{H}_c) \simeq (C_0(G^0) \overset{\vee}{\otimes} \overline{\mathcal{H}}_r) \overset{h}{\otimes} (\mathcal{H}_c \overset{\vee}{\otimes} C_0(G^0)),$$

as operator spaces, for any Hilbert space $\mathcal{H}$ (in particular, for $\mathcal{H} = L^2(G^u; l^2)$). This is due to the facts that $C_0(X, \mathcal{V})$ is completely isomorphic to $C_0(X) \overset{\vee}{\otimes} \mathcal{V}$ (this follows from

3.2.9 and 3.2.29), for any locally compact space X and operator space $\mathcal{V}$, and that $\overset{\vee}{\otimes}$ is commutative.

Also, by 3.3.11, $\mathcal{V} \overset{\vee}{\otimes} \mathcal{H}_r = \mathcal{V} \overset{h}{\otimes} \mathcal{H}_r$, $\mathcal{H}_c \overset{\vee}{\otimes} \mathcal{V} = \mathcal{H}_c \overset{h}{\otimes} \mathcal{V}$ and $\mathcal{H}_r \overset{\wedge}{\otimes} \mathcal{H}_c = \mathcal{H}_r \overset{h}{\otimes} \mathcal{H}_c$, for any Hilbert space $\mathcal{H}$ and any operator space $\mathcal{V}$. These complete identifications together with the associativity of $\overset{h}{\otimes}$ yield

$$(C_0(G^0) \overset{\vee}{\otimes} \overline{\mathcal{H}}_r) \overset{h}{\otimes} (\mathcal{H}_c \overset{\vee}{\otimes} C_0(G^0)) \simeq C_0(G^0) \overset{h}{\otimes} (\overline{\mathcal{H}}_r \overset{\wedge}{\otimes} \mathcal{H}_c) \overset{h}{\otimes} C_0(G^0).$$

Then,

$$C_0(G^0, \overline{L^2(G^u; l^2)}_r) \overset{h}{\otimes} C_0(G^0, L^2(G^u; l^2)_c) \simeq C_0(G^0) \overset{h}{\otimes} (\overline{L^2(G^u; l^2)}_r \overset{\wedge}{\otimes} L^2(G^u; l^2)_c) \overset{h}{\otimes} C_0(G^0). \tag{5.6}$$

The product we aim to define on the left hand side of the equality above should translate on the right hand side to what we expect for a space of the form $A \overset{\delta}{\otimes} B \overset{\delta}{\otimes} C$: if $a, a' \in A, b, b' \in B, c, c' \in C$, then

$$(a \otimes b \otimes c)(a' \otimes b' \otimes c') = aa' \otimes bb' \otimes cc'.$$

This means that we need to define a product on

$$T(G^u; l^2) = \overline{L^2(G^u; l^2)}_r \overset{\wedge}{\otimes} L^2(G^u; l^2)_c.$$

We now prove that defining that product is enough to accomplish our goal.

Up to now we have defined a completely contractive shuffle map

$$S : (C_{0r} \overset{h}{\otimes} C_{0c}) \overset{\wedge}{\otimes} (C_{0r} \overset{h}{\otimes} C_{0c}) \to (C_{0r} \overset{\wedge}{\otimes} C_{0r}) \overset{h}{\otimes} (C_{0c} \overset{\wedge}{\otimes} C_{0c}).$$

We will now define a completely contractive map

$$\tilde{\tau} : (C_{0r} \overset{\wedge}{\otimes} C_{0r}) \overset{h}{\otimes} (C_{0c} \overset{\wedge}{\otimes} C_{0c}) \to C_{0r} \overset{h}{\otimes} C_{0c}.$$

Our product will be the composition $\tilde{\tau} \circ S$.

We begin by observing that, due to the completely contractive approximation property of $C_0(G^0, \mathcal{H})$ and the injectivity of $\overset{h}{\otimes}$,

$$(C_{0r} \overset{\wedge}{\otimes} C_{0r}) \overset{h}{\otimes} (C_{0c} \overset{\wedge}{\otimes} C_{0c}) \hookrightarrow (C_{0r} \overset{\vee}{\otimes} C_{0r}) \overset{h}{\otimes} (C_{0c} \overset{\vee}{\otimes} C_{0c}).$$

To simplify the notation, we write $X = G^0$ and $\mathcal{H} = L^2(G^u; l^2)$. The purpose of the following chain of complete isomorphisms is to "push the C_0's" to the sides:

$$(C_{0r} \overset{\vee}{\otimes} C_{0r}) \overset{h}{\otimes} (C_{0c} \overset{\vee}{\otimes} C_{0c}) \tag{5.7}$$

$$\simeq \big((C_0(X) \overset{\vee}{\otimes} \overline{\mathcal{H}}_r) \overset{\vee}{\otimes} (C_0(X) \overset{\vee}{\otimes} \overline{\mathcal{H}}_r)\big) \overset{h}{\otimes} \big((C_0(X) \overset{\vee}{\otimes} \mathcal{H}_c) \overset{\vee}{\otimes} (C_0(X) \overset{\vee}{\otimes} \mathcal{H}_r)\big) \tag{5.8}$$

$$\simeq \big((C_0(X) \overset{\vee}{\otimes} C_0(X)) \overset{\vee}{\otimes} (\overline{\mathcal{H}}_r \overset{\vee}{\otimes} \overline{\mathcal{H}}_r)\big) \overset{h}{\otimes} \big((\mathcal{H}_c \overset{\vee}{\otimes} \mathcal{H}_c) \overset{\vee}{\otimes} (C_0(X) \overset{\vee}{\otimes} C_0(X))\big) \tag{5.9}$$

$$\simeq \big((C_0(X) \overset{\vee}{\otimes} C_0(X)) \overset{\vee}{\otimes} (\overline{\mathcal{H} \otimes \mathcal{H}})_r\big) \overset{h}{\otimes} \big((\mathcal{H} \otimes \mathcal{H})_c \overset{\vee}{\otimes} (C_0(X) \overset{\vee}{\otimes} C_0(X))\big) \tag{5.10}$$

$$\simeq \big((C_0(X) \overset{\vee}{\otimes} C_0(X)) \overset{h}{\otimes} (\overline{\mathcal{H} \otimes \mathcal{H}})_r\big) \overset{h}{\otimes} \big((\mathcal{H} \otimes \mathcal{H})_c \overset{h}{\otimes} (C_0(X) \overset{\vee}{\otimes} C_0(X))\big) \tag{5.11}$$

$$\simeq ((C_0(X) \overset{\vee}{\otimes} C_0(X)) \overset{h}{\otimes} \big((\overline{\mathcal{H} \otimes \mathcal{H}})_r \overset{h}{\otimes} (\mathcal{H} \otimes \mathcal{H})_c\big) \overset{h}{\otimes} (C_0(X) \overset{\vee}{\otimes} C_0(X))) \tag{5.12}$$

$$\simeq ((C_0(X) \overset{\vee}{\otimes} C_0(X)) \overset{h}{\otimes} \big(\overline{\mathcal{H}}_r \overset{h}{\otimes} (\overline{\mathcal{H}}_r \overset{h}{\otimes} \mathcal{H}_c) \overset{h}{\otimes} \mathcal{H}_c\big) \overset{h}{\otimes} (C_0(X) \overset{\vee}{\otimes} C_0(X))) \tag{5.13}$$

$$\simeq ((C_0(X) \overset{\vee}{\otimes} C_0(X)) \overset{h}{\otimes} \big(\overline{\mathcal{H}}_r \overset{\wedge}{\otimes} (\overline{\mathcal{H}}_r \overset{\wedge}{\otimes} \mathcal{H}_c) \overset{\wedge}{\otimes} \mathcal{H}_c\big) \overset{h}{\otimes} (C_0(X) \overset{\vee}{\otimes} C_0(X))) \tag{5.14}$$

$$\simeq ((C_0(X) \overset{\vee}{\otimes} C_0(X)) \overset{h}{\otimes} \big((\overline{\mathcal{H}}_r \overset{\wedge}{\otimes} \mathcal{H}_c) \overset{\wedge}{\otimes} (\overline{\mathcal{H}}_r \overset{\wedge}{\otimes} \mathcal{H}_c)\big) \overset{h}{\otimes} (C_0(X) \overset{\vee}{\otimes} C_0(X))) \tag{5.15}$$

We explain the isomorphisms above. The step 5.8 is due to $C_0(X, \mathcal{V}) \simeq C_0(X) \overset{\vee}{\otimes} \mathcal{V}$. Steps 5.9 and 5.15 are due to the commutativity and associativity of $\overset{\vee}{\otimes}$ and $\overset{\wedge}{\otimes}$ respectively. Both in 5.10 and 5.13 Proposition 3.3.12 is applied. Proposition 3.3.11 is used in 5.11 and 5.14. The associativity of $\overset{h}{\otimes}$ is applied in 5.12 and 5.13.

Let $D : C_0(X) \overset{\vee}{\otimes} C_0(X) \to C_0(X)$ be defined on elementary tensors by

$$D(f \otimes g)(x) = f(x)g(x).$$

This is a complete contraction. If we succeed at defining a complete contraction

$$\Diamond^\infty : (\overline{\mathcal{H}}_r \overset{\wedge}{\otimes} \mathcal{H}_c) \overset{\wedge}{\otimes} (\overline{\mathcal{H}}_r \overset{\wedge}{\otimes} \mathcal{H}_c) \to \overline{\mathcal{H}}_r \overset{\wedge}{\otimes} \mathcal{H}_c,$$

then we can consider $\tilde{\tau} = D \otimes \lozenge^{\infty} \otimes D$. Thus, it is enough to define $\lozenge^{\infty}$ extending $\diamond^{\infty}$. In order to do so, we need to re-define the right action of 5.3.

Definition 5.1.25. Let G be a locally trivial groupoid. Suppose that δ and ε are elements of G^u. Thus, the pair (δ, ε) is not, in general, composable. However, by the local triviality of G, there exists $\nu_k(s(\delta)) \in G^u_{s(\delta)}$. Since $\delta(\nu_k(s(\delta)))^{-1}$ is a loop on G^u_u, we can define the right action as follows. For $f \in L^2(G^u; l^2)$ and $\varepsilon \in G^u$, we define $\varepsilon \cdot f \in L^2(G^u; l^2)$ by

$$\varepsilon \cdot f(\delta) = f(\delta(\nu_k(s(\delta)))^{-1}\varepsilon).$$

If the groupoid G is a group, the definition of the right action coincides with the one we had before.

As in the group case, we first consider the non-amplified version, that we denote by $\lozenge$.

Proposition 5.1.26. *Let G be a locally trivial groupoid. If $\overline{\xi} \otimes \eta$ and $\overline{f} \otimes g$ are elementary tensors of $\overline{L^2(G^u)_r} \overset{\wedge}{\otimes} L^2(G^u)_c$, we define*

$$(\overline{\xi} \otimes \eta)\lozenge(\overline{f} \otimes g) = \int_{G^u} \overline{\xi\varepsilon \cdot f} \otimes \eta\varepsilon \cdot g \, d\lambda^u(\varepsilon).$$

Then $\lozenge$ extends to a complete contraction

$$\lozenge : \overline{L^2(G^u)_r} \overset{\wedge}{\otimes} L^2(G^u)_c \times \overline{L^2(G^u)_r} \overset{\wedge}{\otimes} L^2(G^u)_c \to \overline{L^2(G^u)_r} \overset{\wedge}{\otimes} L^2(G^u)_c.$$

Proof. As in the group case, we use the properties of the Bochner integral to verify that $\overline{\xi} \otimes \eta \lozenge \overline{f} \otimes g \in T(G^u)$. Note that

$$\begin{aligned}
\|\overline{\xi} \otimes \eta \lozenge \overline{f} \otimes g\|^2 &\leq \Big(\int_{G^u} \|\xi\varepsilon \cdot f\| \|\eta\varepsilon \cdot g\| \, d\lambda^u(\varepsilon)\Big)^2 \\
&\leq \int_{G^u} \|\xi\varepsilon \cdot f\|^2 \, d\lambda^u(\varepsilon) \int_{G^u} \|\eta\varepsilon \cdot g\|^2 \, d\lambda^u(\varepsilon) \\
&= \int_{G^u} |\xi(\delta)|^2 \Big(\int_{G^u} |\varepsilon \cdot f(\delta)|^2 \, d\lambda^u(\varepsilon)\Big) \, d\lambda^u(\delta) \int_{G^u} |\eta(\delta)|^2 \Big(\int_{G^u} |\varepsilon \cdot g(\delta)|^2 \, d\lambda^u(\varepsilon)\Big) \, d\lambda^u(\delta) \\
&\leq \|\xi\|^2 \|f\|^2 \|\eta\|^2 \|g\|^2,
\end{aligned}$$

as before.

We now prove that $\Diamond$ is a complete contraction. This is done in the same way as in the group case, therefore we just list the step we need to take. We verify that $\Diamond$ is the pre-adjoint of a complete isometry . We begin by observing that

$$T(G^u)^* = (\overline{L^2(G^u)}_r \overset{\wedge}{\otimes} L^2(G^u)_c)^* \simeq \mathcal{CB}(\overline{L^2(G^u)}_r, L^2(G^u)_c^*) \simeq \mathcal{CB}(\overline{L^2(G^u)}_r) \simeq \mathcal{B}(L^2(G^u)) \tag{5.16}$$

via the identification $T(\overline{\xi} \otimes \eta) = \langle \xi, T(\eta) \rangle$, for $T \in B(L^2(G^u))$, $\xi, \eta \in L^2(G^u)$. See 3.2.17, 3.1.15 and 3.1.16.

Moreover,

$$(T(G^u) \overset{\wedge}{\otimes} T(G^u))^* \simeq \mathcal{B}(L^2(G^u)) \overline{\otimes} \mathcal{B}(L^2(G^u)) \simeq \mathcal{B}(L^2(G^u \times G^u)).$$

The identification here is

$$T(\overline{\xi} \otimes \eta \otimes \overline{f} \otimes g) = \langle \xi \otimes f, T(\eta \otimes g) \rangle, \tag{5.17}$$

for $T \in \mathcal{B}(L^2(G^u \times G^u)$, $\overline{\xi}, \overline{f} \in \overline{L^2(G^u)}_r$ and $\eta, g \in L^2(G^u)_c$, see 3.2.23.

The definition of the fundamental unitary W and the associated normal isometric isomorphism Γ_W is very similar. Let

$$W : L^2(G^u \times G^u) \to L^2(G^u \times G^u), \quad Wh(\delta, \varepsilon) = h(\delta, \nu_k(s(\delta))\delta^{-1}\varepsilon).$$

This is, in fact, a unitary map with adjoint

$$W^* : L^2(G^u \times G^u) \to L^2(G^u \times G^u), \quad W^*h(\delta, \varepsilon) = h(\delta, \delta\nu_k(s(\delta))^{-1}\varepsilon).$$

Define

$$\Gamma_W : \mathcal{B}(L^2(G^u) \to \mathcal{B}(L^2(G^u)) \overline{\otimes} \mathcal{B}(L^2(G^u)), \quad \Gamma_W(T) = W(T \otimes \mathrm{Id})W^*,$$

it is a normal isometric isomorphism and hence a complete isometry.

We want to prove that $\Diamond = (\Gamma_W)_*$, that is,

$$\Gamma_W(T)(\omega \otimes \nu) = T(\omega \Diamond \nu)$$

for $T \in \mathcal{B}(L^2(G^u; l^2))$, $\omega \otimes \nu \in T(G^u) \overset{\wedge}{\otimes} T(G^u)$. It is enough to consider $\omega = \overline{\xi} \otimes \eta$, $\nu = \overline{f} \otimes g$ and a rank one operator $T = \theta_{\zeta,\mu}$, $\theta_{\zeta,\mu}(f) = \langle \mu, f \rangle \zeta$. The computations are very similar to the group case, the only difficulty added is the definition of the right action. On the right-hand side,

$$\begin{aligned}
T((\overline{\xi} \otimes \eta) \Diamond (\overline{f} \otimes g)) &= T(\int_{G^u} \overline{\xi\delta \cdot f} \otimes \eta\delta \cdot g \, d\lambda^u(\delta)) \\
&= \int_{G^u} \langle \xi\delta \cdot f, T(\eta\delta \cdot g) \rangle \, d\lambda^u(\delta) \\
&= \int_{G^u} \langle \mu, \eta\delta \cdot g \rangle \langle \xi\delta \cdot f, \zeta \rangle \, d\lambda^u(\delta).
\end{aligned}$$

On the left-hand side,

$$\begin{aligned}
\Gamma_W(T)(\overline{\xi} \otimes \eta) \otimes (\overline{f} \otimes g)) &= \langle \xi \otimes f, W(T \otimes \mathrm{Id}) W^*(\eta \otimes g) \rangle \\
&= \langle W^* \xi \otimes f, (T \otimes \mathrm{Id}) W^*(\eta \otimes g) \rangle \\
&= \int_{G^u} \int_{G^u} \overline{W^*(\xi \otimes f)(\varepsilon, \delta)} (T \otimes \mathrm{Id}) W^*(\eta \otimes g)(\varepsilon, \delta) \, d\lambda^u(\varepsilon) \, d\lambda^u(\delta) \\
&= \int_{G^u} \int_{G^u} \overline{\xi(\varepsilon)\delta \cdot f(\varepsilon)} \langle \mu, \eta\delta \cdot g \rangle \zeta(\varepsilon) \, d\lambda^u(\varepsilon) \, d\lambda^u(\delta) \\
&= \int_{G^u} \langle \mu, \eta\delta \cdot g \rangle \langle \xi\delta \cdot f, \zeta \rangle \, d\lambda^u(\delta).
\end{aligned}$$

Then, since $\Diamond$ is the pre-adjoint of the complete isometry Γ_W, it is a complete quotient map, and in particular, a complete contraction. $\square$

Remark 5.1.27. Later on we will define a groupoid version of the map q_0 and we will verify that this map respects the product $\Diamond$.

Definition 5.1.28. We now need to define an amplified version of $\Diamond$, that we denote $\Diamond^\infty$,

$$\Diamond^\infty : (\overline{\mathcal{H}_r} \overset{\wedge}{\otimes} \mathcal{H}_c) \overset{\wedge}{\otimes} (\overline{\mathcal{H}_r} \overset{\wedge}{\otimes} \mathcal{H}_c) \to \overline{\mathcal{H}_r} \overset{\wedge}{\otimes} \mathcal{H}_c,$$

where $\mathcal{H} = L^2(G^u; l^2)$. We use the notation $T(G^u; l^2) = \overline{L^2(G^u; l^2)_r} \overset{\wedge}{\otimes} L^2(G^u; l^2)_c$. As in the group case, $T(G^u; l^2) = T(G^u) \overset{\wedge}{\otimes} T(\mathbb{Z})$. Then, the amplified version of $\Diamond$ is defined by

$$\Diamond^\infty : T(G^u; l^2) \times T(G^u; l^2) \to T(G^u; l^2), \quad \Diamond^\infty = \Diamond \otimes \diamond.$$

This map is a complete contraction.

Going back to 5.15, we can define $\tilde{\tau} = D \otimes \Diamond^\infty \otimes D$ and this is a completely contractive map. Also, by pre-composing with ι we obtain a complete contraction

$$\tilde{\tau} \circ \iota : (C_{0r} \overset{\wedge}{\otimes} C_{0r}) \overset{h}{\otimes} (C_{0c} \overset{\wedge}{\otimes} C_{0c}) \to C_0(X) \overset{h}{\otimes} T(G^u) \overset{h}{\otimes} C_0(X) = C_{0r} \overset{h}{\otimes} C_{0c}.$$

By pre-composing with the shuffle map S we obtain the desired completely contraction $\tau = \tilde{\tau} \circ \iota \circ S$,

$$\tau : (C_{0r} \overset{h}{\otimes} C_{0c}) \overset{\wedge}{\otimes} (C_{0r} \overset{h}{\otimes} C_{0c}) \to C_0(X) \overset{h}{\otimes} T(G^u) \overset{h}{\otimes} C_0(X) = C_{0r} \overset{h}{\otimes} C_{0c},$$
$$\tau\big(((\alpha \otimes \overline{\xi}) \otimes (\beta \otimes \eta)) \otimes ((a \otimes \overline{f}) \otimes (b \otimes g))\big) = \overline{\alpha a} \otimes ((\overline{\xi} \otimes \eta) \Diamond^\infty (\overline{f} \otimes g)) \otimes \beta b. \quad (5.18)$$

If we could prove that $\Diamond^\infty$ is associative, it would follow that so is τ and therefore with this product, $C_0(G^0, \overline{L^2(G^u; l^2)}_r) \overset{h}{\otimes} C_0(G^0, L^2(G^u; l^2)_c)$ would be a completely contractive Banach algebra. We do not attempt to prove this at this point, since the completely contractive product that we have defined is enough for our purposes.

We now concentrate on our second goal for this section.

Proposition 5.1.29. *Let A and B be operator spaces. Suppose that B admits a bilinear map denoted by $\overline{M}$. Let $\varphi : A \to B$ be a complete quotient map. Assume that there exists a completely contractive map $m : A \overset{\wedge}{\otimes} A \to A$ such that $\overline{M} \circ (\varphi \times \varphi) = \varphi \circ m$.*

Then the bilinear map on B extends to a complete contraction $M : B \overset{\wedge}{\otimes} B \to B$ such that

$$M \circ (\varphi \otimes \varphi) = \varphi \circ m.$$

Proof. Since $\varphi : A \to B$ is a complete quotient map and $\overset{\wedge}{\otimes}$ is projective, the map

$$\varphi \otimes \varphi : A \overset{\wedge}{\otimes} A \to B \overset{\wedge}{\otimes} B$$

is a complete quotient map as well. Hence, it induces a complete isomorphism

$$\widetilde{\varphi \otimes \varphi} : \frac{A \overset{\wedge}{\otimes} A}{\text{Ker } (\varphi \otimes \varphi)} \to B \overset{\wedge}{\otimes} B.$$

We have the following situation:

$$\begin{tikzcd}
A \overset{\wedge}{\otimes} A \arrow[r, "\pi"] \arrow[rr, bend left, "\varphi\otimes\varphi"] \arrow[d, "m"] & \frac{A \overset{\wedge}{\otimes} A}{\text{Ker } (\varphi\otimes\varphi)} \arrow[r, equal, "\widetilde{\varphi\otimes\varphi}"] & B \overset{\wedge}{\otimes} B \\
A \arrow[rr, "\varphi"] & & B
\end{tikzcd}$$

and we want to define a completely contractive map from $B \overset{\wedge}{\otimes} B$ to B that extends the product $\bar{M}$.

Let $\beta \in B \overset{\wedge}{\otimes} B$. Then, there exists a unique $[\alpha] \in \frac{A \overset{\wedge}{\otimes} A}{\text{Ker } (\varphi\otimes\varphi)}$ such that $\widetilde{\varphi \otimes \varphi}([\alpha]) = \beta$, $\|[\alpha]\| = \|\beta\|$.

Suppose $\alpha' \in [\alpha]$. We verify that $\varphi(m(\alpha)) = \varphi(m(\alpha'))$, that is $\alpha - \alpha' \in \text{Ker } (\varphi \circ m)$. Since

$$\alpha - \alpha' \in \text{Ker } (\varphi \otimes \varphi) = \overline{\text{Ker } (\varphi) \otimes A + A \otimes \text{Ker } (\varphi)},$$

(see 3.2.18) there exists a sequence $\{\gamma_n\}_{n\in\mathbb{N}} \in \text{Ker } (\varphi) \otimes A + A \otimes \text{Ker } (\varphi)$ converging to $\alpha - \alpha' \in A \overset{\wedge}{\otimes} A$. Let

$$\gamma_n = \sum_{i=1}^{p_n} k_i^n \otimes c_i^n + \sum_{j=1}^{q_n} d_j^n \otimes l_j^n,$$

for $k_i^n, l_j^n \in \text{Ker }(\varphi)$, $c_i^n, d_j^n \in A$. Then,

$$\begin{aligned}\varphi(m(\gamma_n)) &= \varphi(m(\sum_{i=1}^{p_n} k_i^n \otimes c_i^n + \sum_{j=1}^{q_n} d_j^n \otimes l_j^n)) \\ &= \varphi(\sum_i k_i^n c_i^n + \sum_j d_j^n l_j^n) \\ &= \sum_i \varphi(k_i^n)\varphi(c_i^n) + \sum_j \varphi(d_j^n)\varphi(l_j^n) = 0 \quad \forall n \in \mathbb{N}.\end{aligned}$$

Hence, $\varphi(m(\alpha - \alpha')) = \lim_n \varphi(m(\gamma_n)) = 0$ and $\varphi(m(\alpha)) = \varphi(m(\alpha'))$.

Therefore, it makes sense to define $M(\beta) := \varphi(m(\alpha))$, for α such that $\widetilde{\varphi \otimes \varphi}([\alpha]) = \beta$.

Let $\beta = \sum_{i=1}^n b_i \otimes b_i' \in B \otimes B$. We wish to check that $M(\beta) = \sum_{i=1}^n b_i b_i'$. Since $b_i, b_i' \in B$, there exist $a_i, a_i' \in A$ such that $\varphi(a_i) = b_i$, $\varphi(a_i') = b_i'$, for all $i = 1, 2, \ldots, n$. Thus, $\varphi \otimes \varphi(\sum_{i=1}^n a_i \otimes a_i') = \beta$ and it follows that $\widetilde{\varphi \otimes \varphi}([\sum_{i=1}^n a_i \otimes a_i']) = \beta$. By the definition of M,

$$M(\beta) = \varphi(m(\sum_{i=1}^n a_i \otimes a_i')) = \sum_{i=1}^n \varphi(a_i a_i') = \sum_{i=1}^n \varphi(a_i)\varphi(a_i') = \sum_{i=1}^n b_i b_i',$$

and therefore M extends the product on B to $B \overset{\wedge}{\otimes} B$.

Let $a, a' \in A$. Then

$$\varphi(m(a \otimes a')) = \overline{M}(\varphi(a), \varphi(a')) = M(\varphi(a) \otimes \varphi(a')) = M(\varphi \otimes \varphi(a \otimes a')).$$

It remains to prove that M is completely contractive, that is, for all $n \in N$,

$$M^{(n)} : M_n(B \overset{\wedge}{\otimes} B) \to M_n(B), \quad M^{(n)}([\beta_{ij}]) = [\varphi(m(\alpha_{ij}))] = \varphi^{(n)} \circ \alpha^{(n)}([\alpha_{ij}])$$

is contractive, if $\widetilde{\varphi \otimes \varphi}^{(n)}([\alpha_{ij}]) = [\beta_{ij}]$.

Let $[\beta_{ij}]$ and $[\alpha_{ij}]$ be as above. Since $\widetilde{\varphi \otimes \varphi}^{(n)}$ is isometric,

$$\|[\beta_{ij}]\| = \|\Big[[\alpha_{ij}]\Big]\| = \inf\{\|[\alpha_{ij}] - k\| :\, k \in \text{Ker }(\varphi \otimes \varphi)^{(n)}\}$$

and then for each $l \geq 1$ there exists $k_l \in \text{Ker } (\varphi \otimes \varphi)^{(n)}$ such that $\|[\alpha_{ij}] - k_l\| < \|[\beta_{ij}]\| + \frac{1}{l}$. Then,

$$\|M^{(n)}([\beta_{ij}])\| = \lim_l \|\varphi^{(n)} \circ m^{(n)}([\alpha_{ij}] - k_l)\| \leq \|[\alpha_{ij}] - k_l\| \leq \lim_l \|[\beta_{ij}]\| + \frac{1}{l}.$$

Thus, B is a completely contractive Banach algebra. $\square$

This last result is going to be used in Section 3 to prove that the Fourier algebra of a locally compact, locally trivial, transitive groupoid with a "nice" Haar system is a completely contractive Banach algebra.

5.2 The Banach algebra $C_0(X) \overset{\vee}{\otimes} A(H) \overset{\vee}{\otimes} C_0(X)$.

Suppose X is a locally compact space and H is a locally compact group. In this section we consider some properties of the Banach algebra $C_0(X) \overset{\vee}{\otimes} A(H) \overset{\vee}{\otimes} C_0(X)$ and we relate it to the Banach space $C_0(X) \overset{h}{\otimes} A(H) \overset{h}{\otimes} C_0(X)$ (later we will see more properties of this spaces for $X = G^0$ and $H = G_u^u$).

Definition 5.2.1. Let A be a commutative Banach algebra. The **spectrum** of A is

$$\widehat{A} := \{h : A \to \mathbb{C} : \; h \text{ is a homomorphism of algebras and } h \neq 0\}.$$

Remark 5.2.2. Any h as above is bounded, and in particular, verifies $\|h\| \leq 1$. Hence, $\widehat{A} \subseteq A^*$. The spectrum $\widehat{A}$ together with the w^*-topology is a locally compact Hausdorff space.

Examples 5.2.3. If X is a locally compact Hausdorff space, $\widehat{C_0(X)} = X$. Eymard proved in [18] that if H is a locally compact group, then $\widehat{A(H)} = H$.

Remark 5.2.4. Let A be a commutative Banach algebra and $a \in A$. The map

$$\hat{a} : \widehat{A} \to \mathbb{C} \quad \hat{a}(h) = h(a)$$

is continuous and vanishes at infinity.

Proposition 5.2.5. *Let A be a commutative Banach algebra. Define the map*

$$\mathcal{G} : A \to C_0(\widehat{A}) \quad \mathcal{G}(a) = \hat{a}.$$

Then $\mathcal{G}$ is an algebra homomorphism and $\|\mathcal{G}\| \leq 1$. It is called the **Gelfand transform** *of A.*

Definition 5.2.6. We say that a commutative Banach algebra is **semisimple** if its Gelfand transform is injective.

Remark 5.2.7. We denote $\mathcal{A}_\vee = C_0(X) \overset{\vee}{\otimes} A(H) \overset{\vee}{\otimes} C_0(X) = C_0(X \times X, A(H))$. Observe that this is a semisimple Banach algebra. Its spectrum is

$$\begin{aligned}\widehat{\mathcal{A}_\vee} &= \widehat{C_0(X)} \times \widehat{A(H)} \times \widehat{C_0(X)} \\ &= X \times H \times X \\ &= \{ev_{y,h,x} \,:\, x, y \in X, h \in H\}\end{aligned}$$

where $ev_{y,h,x}$ are the evaluation maps, see [52].

Proposition 5.2.8. *The inclusion*

$$\iota : C_0(X) \overset{h}{\otimes} A(H) \overset{h}{\otimes} C_0(X) \to C_0(X) \overset{\vee}{\otimes} A(H) \overset{\vee}{\otimes} C_0(X)$$

is one-to-one.

Proof. We first show that $i : C_0(X) \overset{h}{\otimes} A(H) \to C_0(X) \overset{\vee}{\otimes} A(H)$ is one-to-one. Indeed, if $i(u) = 0$, for some $u \in C_0(X) \overset{h}{\otimes} A(H)$, we can suppose that $u = (\alpha_1, \cdots) \odot (\eta_1, \cdots)^t$, for some $(\alpha_1, \cdots) \in M_{1\infty}(C_0(X))$, $(\eta_1, \dots)^t \in M_{\infty 1}(A(H))$, $\{\eta_i\}_i$ strongly independent (see the proof of Theorem 3.3.27). Then, for all $x \in X$, $0 = i(u)(x) = \sum \alpha_i(x)\eta_i$, and then by strong independence, $\alpha_i(x) = 0$. Therefore, $u = 0$.

Now, if $u \in \mathcal{A}_h$ is such that $\iota(u) = 0$, we can assume that $u = (\mu_1, \cdots) \odot (\beta_1, \cdots)^t$, for some $(\mu_1, \cdots) \in M_{1\infty}(C_0(X) \overset{h}{\otimes} A(H))$ and $(\beta_1, \dots)^t \in M_{\infty 1}(C_0(X))$. Note that $i((\mu_1, \cdots)) \in C_0(X) \overset{\vee}{\otimes} A(H)$ is a non-zero element, so we can suppose that $\{\mu_i\}_i$ is strongly independent. The proof finishes as in the previous step. $\square$

Thus, $\mathcal{A}_h$ is a subspace of $\mathcal{A}_\vee$ and the evaluation maps from Remark 5.2.7 restricted to $\mathcal{A}_h$ are continuous maps.

5.3 A decomposition of $A(G)$.

Suppose G is a transitive, locally trivial, groupoid and u is a unit in G. Assume that $\{\lambda^v\}$ is a left Haar system on G such that $\lambda^u_{|G^u_u}$ is a left Haar measure on the isotropy group G^u_u. On 2.3.6 we saw that if G^u_u is unimodular, we can always construct such a Haar system. For such a G, we want to prove that

$$A(G) \simeq C_0(G^0) \overset{h}{\otimes} A(G^u_u) \overset{h}{\otimes} C_0(G^0)$$

as Banach spaces. Moreover, we are going to prove that the space on the right hand side is a completely contractive Banach algebra and the isometry between the spaces respects the product. Therefore, we can consider an operator space structure on the Fourier algebra $A(G)$ and make it a completely contractive Banach algebra.

For the fixed unit u of the groupoid, since G is locally trivial, we use the description of $A(G)$ from Proposition 4.5.13.

The big picture of what we are going to do can be expressed with a commutative diagram. In order to fit it on the page, we write

$$\mathcal{H} = L^2(G^u, l^2) \quad \text{and} \quad \mathcal{V} = C_0(G^0, \overline{\mathcal{H}}_r) \overset{h}{\otimes} C_0(G^0, \mathcal{H}_c).$$

We will show that the diagram below is commutative, all the spaces involved admit a product, all the maps respect the product, the induced map $\overline{\psi}$ is an isometric isomorphism and φ and the induced map $\overline{Q}$ are complete isometric isomorphisms:

$$\begin{tikzcd}
C_0(G^0, \overline{\mathcal{H}}_r) \overset{h}{\otimes} C_0(G^0, \mathcal{H}_c) \arrow[rr, equal] \arrow[d, "\psi"'] \arrow[dr, "\pi_\psi"] & & (C_0(G^0) \overset{\vee}{\otimes} \overline{\mathcal{H}}_r) \overset{h}{\otimes} (\mathcal{H}_c \overset{\vee}{\otimes} C_0(G^0)) \arrow[rr, equal] & & C_0(G^0) \overset{h}{\otimes} (\overline{\mathcal{H}}_r \overset{\wedge}{\otimes} \mathcal{H}_c) \overset{h}{\otimes} C_0(G^0) \arrow[d, "Q"] \arrow[dl, "\pi_Q"'] \\
A(G) & \frac{\mathcal{V}}{\mathrm{Ker}\ (\psi)} \arrow[l, dotted, "\overline{\psi}"] & & \frac{\mathcal{V}}{\mathrm{Ker}\ (Q)} \arrow[ll, "\varphi"'] \arrow[r, dotted, "\overline{Q}"'] & C_0(G^0) \overset{h}{\otimes} A(G^u_u) \overset{h}{\otimes} C_0(G^0)
\end{tikzcd} \tag{5.19}$$

Remark 5.3.1. We already proved in 5.6 that the first row of the diagram is a complete isomorphism. Recall that the product is defined in terms of elementary tensors of the right end of the equation by

$$\tau : (C_{0r} \overset{h}{\otimes} C_{0c}) \overset{\wedge}{\otimes} (C_{0r} \overset{h}{\otimes} C_{0c}) \to C_0(X) \overset{h}{\otimes} T(G^u) \overset{h}{\otimes} C_0(X) = C_{0r} \overset{h}{\otimes} C_{0c},$$
$$\tau\big(((\alpha \otimes \overline{\xi}) \otimes (\beta \otimes \eta)) \otimes ((a \otimes \overline{f}) \otimes (b \otimes g))\big) = \overline{\alpha a} \otimes ((\overline{\xi} \otimes \eta)\Diamond(\overline{f} \otimes g)) \otimes \beta b,$$

where

$$\overline{\xi} \otimes \eta \Diamond \overline{f} \otimes g = \int_{G^u} \overline{\xi\varepsilon \cdot f} \otimes \eta\varepsilon \cdot g \, d\lambda^u(\varepsilon).$$

We now consider the triangle-like diagram on the left side of the main diagram. The following Proposition is a groupoid generalization of the results 5.1.19 and 5.1.22, see Remark 5.1.27.

Proposition 5.3.2. *Let G be a transitive, locally trivial groupoid. Suppose $u \in G^0$ is fixed. Then the map*

$$\psi : C_0(G^0, \overline{L^2(G^u; l^2)}_r) \otimes^h C_0(G^0, L^2(G^u; l^2)_c) \to A(G), \quad \psi(\overline{\xi} \otimes \eta) = (\xi, \eta)$$

is a surjective quotient map that induces an isometry

$$\overline{\psi} : \frac{C_0(G^0, \overline{L^2(G^u; l^2)}_r) \otimes^h C_0(G^0, L^2(G^u; l^2)_c)}{Ker\ (\psi)} \to A(G),$$

making commutative the left side of the diagram above. Moreover, if we consider ψ as a map with range $B(G)$, it respects the product. It follows that $A(G)$ is an associative algebra.

Proof. Recall from 4.5.13 that for a groupoid G such as the one that we are considering

$$A(G) = \{(\xi, \eta) : \ \xi, \eta \in C_0(G^0, L^2(G^u; l^2))\}.$$

Then, it makes sense to consider

$$\psi : C_0(G^0, \overline{L^2(G^u; l^2)}_r) \otimes_h C_0(G^0, L^2(G^u; l^2)_c) \to \operatorname{span}\{(\xi, \eta) : \ \xi, \eta \in \mathcal{S}C_0(L^2(G))\},$$

defined on elementary tensors by $\psi(\overline{\xi}\otimes\eta) = (\xi,\eta)$ and extended linearly. We want to show that ψ is a contraction.

Let $v \in C_0(G^0, \overline{L^2(G^u;l^2)}_r)\otimes_h C_0(G^0, L^2(G^u;l^2)_c)$. Let $\overline{u}, w$ be such that:

- $\overline{u} \in M_{1,p}(C_0(G^0, \overline{L^2(G^u;l^2)}_r))$,
- $w \in M_{p,1}(C_0(G^0, L^2(G^u;l^2)_c))$,
- $v = \overline{u} \odot w = \sum_{k=1}^{p} \overline{u_k} \otimes w_k$.

We need to find sections ξ, η of G such that $(\xi,\eta) = \psi(v)$ and $\|\xi\|\|\eta\| \leq \|u\|\|w\|$, therefore, we would prove that

$$\|\psi(v)\| \leq \inf_{\xi,\eta \text{ as above}} \|\xi\|\|\eta\| \leq \inf \|u\|\|w\| = \|v\|_h$$

and thus ψ would be a contraction.

Observe that if we define $\tilde{u} : G^0 \to M_p(\overline{L^2(G^u;l^2)}_r)$,

$$\tilde{u}(x) = \begin{pmatrix} \overline{u_1(x)} & \overline{u_2(x)} & \cdots & \overline{u_p(x)} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

then

$$\|\overline{u}\|_{M_{1,p}(C_0(G^0),\overline{\mathcal{H}}_r)} \overset{*}{=} \|\overline{u}\|_{C_0(G^0, M_p(\overline{\mathcal{H}}_r))} = \sup_{j\in G^0} \|\tilde{u}(j)\|_{M_p(\overline{L^2(G^u;l^2)}_r)} = \|\tilde{u}\|_\infty.$$

For the equality $*$, we are applying that

$$M_p(C_0(X,\mathcal{V})) = M_p \overset{\vee}{\otimes} C_0(X) \overset{\vee}{\otimes} \mathcal{V} = C_0(X) \overset{\vee}{\otimes} M_p(\mathcal{V}) = C_0(X, M_p(\mathcal{V})),$$

for any locally compact space X and operator space $\mathcal{V}$.

Similarly, $\|w\|_{M_{p,1}(C^0(G^0,\mathcal{H}_c)} = \|\tilde{w}\|_\infty$, for $\tilde{w}(x) = \begin{pmatrix} w_1(x) & 0 & \cdots & 0 \\ w_2(x) & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ w_p(x) & 0 & \cdots & 0 \end{pmatrix}$.

Consider the left regular bundle with multiplicity p, $(L^2)^p = (L^2(G^u;l^2)^p, L_p)$, where $(L_p)_x : L^2(G^u;l^2)^p \to L^2(G^u;l^2)^p$ is such that $(L_p)_x(\xi_1,\xi_2,\cdots,\xi_p) = (L_x\xi_1, L_x\xi_2,\cdots,L_x\xi_p)$.

Both $\tilde{u}$ and $\tilde{w}$ are sections of $(L^2)^p$, since

$$M_{1,p}(\overline{L^2(G^u;l^2)}_r) \simeq \mathcal{B}(L^2(G^u;l^2)^p,\mathbb{C}) \simeq \overline{L^2(G^u;l^2)}_r^p$$

and

$$M_{p,1}(L^2(G^u;l^2)_c) \simeq \mathcal{B}(\mathbb{C},L^2(G^u;l^2)^p) \simeq L^2(G^u;l^2)_c^p.$$

Moreover,

$$\begin{aligned} (\tilde{u},\tilde{w})(\gamma) &= \langle (L_p)_\gamma \tilde{u}(s(\gamma)), \tilde{w}(r(\gamma))\rangle \\ &= \sum_{k=1}^p \langle L_\gamma u_k(s(\gamma)), w_k(r(\gamma))\rangle = \sum_{k=1}^p (u_k,w_k)(\gamma) \\ &= \sum_{k=1}^p \psi(\overline{u_k}\otimes w_k) = \psi(v). \end{aligned}$$

Therefore the map ψ extends to the whole space and it is a contraction. We call the extension ψ as well.

We now want to prove that ψ is still surjective. We wish to show that for all functions $\zeta,\mu \in C_0(G^0,L^2(G^u;l^2))$, we have that

$$\overline{\zeta}\otimes\mu \in C_0(G^0,\overline{L^2(G^u;l^2)}_r) \overset{h}{\otimes} C_0(G^0,L^2(G^u;l^2)_c)$$

and $\psi(\zeta\otimes\mu) = (\zeta,\mu)$.

Note that, from Theorem 3.3.27, it follows that

$$C_0(G^0,\overline{\mathcal{H}}_r) \overset{h}{\otimes} C_0(G^0,\mathcal{H}_c) = \{\overline{\xi}\odot\eta : \xi \in M_{1,\infty}(C_0(G^0,\overline{\mathcal{H}}_r)), \eta \in M_{\infty,1}(C_0(G^0,\mathcal{H}_c))\}$$

where $\overline{\xi} \odot \eta = \sum_k \overline{\xi_k} \otimes \eta_k$. But

$$M_{1,\infty}(C_0(G^0, \overline{\mathcal{H}}_r)) = C_0(G^0, M_{1,\infty}(\overline{\mathcal{H}}_r))$$

and

$$M_{\infty,1}(C_0(G^0, \mathcal{H}_c)) = C_0(G^0, M_{\infty,1}(\mathcal{H}_c)).$$

Moreover,

$$M_{1,\infty}(\overline{\mathcal{H}}_r) = M_{1,\infty} \overset{\vee}{\otimes} \overline{\mathcal{H}}_r = l^2_r \overset{\vee}{\otimes} \overline{\mathcal{H}}_r = \overline{\mathcal{H}}_r \otimes l^2_r.$$

Since here $\mathcal{H} = L^2(G^u; l^2)$,

$$M_{\infty,1}(L^2(G^u; l^2)_c) = L^2(G^u; l^2)_c \otimes l^2_c = L^2(G^u; l^2)_c$$

and

$$M_{1,\infty}(\overline{L^2(G^u; l^2)}_r) = \overline{L^2(G^u; l^2)}_r.$$

Hence

$$M_{1,\infty}(C_0(G^0, \overline{L^2(G^u; l^2)}_r) = C_0(G^0, \overline{L^2(G^u; l^2)}_r)$$

and

$$M_{\infty,1}(C_0(G^0, L^2(G^u; l^2)_c) = C_0(G^0, L^2(G^u; l^2)_c).$$

Thus, given $\zeta, \mu \in C_0(G^0, L^2(G^u; l^2))$, we verified that

$$\overline{\zeta} \otimes \mu \in C_0(G^0, \overline{L^2(G^u; l^2)}_r) \overset{h}{\otimes} C_0(G^0, L^2(G^u; l^2)_c)$$

and $\psi(\overline{\zeta} \otimes \mu) = (\zeta, \mu)$.

We now prove that ψ is a quotient map. In order to do that, we verify that

$$\psi(C_0(G^0, \overline{L^2(G^u; l^2)}_r) \overset{h}{\otimes} C_0(G^0, L^2(G^u; l^2)_c)_{\|\cdot\|<1})$$

is a dense subspace of $A_s(G)_{\|\cdot\|<1}$. This is true since given $\omega \in A_s(G)_{\|\cdot\|<1}$, there exist sections $\zeta, \mu \in \mathcal{S}C_0((L^2)^p)$ such that $\omega = (\zeta, \mu)$ and $\|\zeta\|\|\mu\| < 1$. Then, $\omega = (\zeta, \mu) = \psi(\overline{\zeta} \otimes \mu)$ and $\|\overline{\zeta} \otimes \mu\|_h \leq \|\zeta\|\|\mu\| < 1$.

Therefore, we can consider the maps

$$\pi_\psi : C_0(G^0, \overline{L^2(G^u; l^2)}_r) \overset{h}{\otimes} C_0(G^0, L^2(G^u; l^2)_c) \longrightarrow \frac{C_0(G^0, \overline{L^2(G^u; l^2)}_r) \overset{h}{\otimes} C_0(G^0, L^2(G^u; l^2)_c)}{\text{Ker } \psi}$$

and

$$\overline{\psi} : \frac{C_0(G^0, \overline{L^2(G^u; l^2)}_r) \overset{h}{\otimes} C_0(G^0, L^2(G^u; l^2)_c)}{\text{Ker } \psi} \longrightarrow A(G),$$

that make commutative the diagram

$$\begin{tikzcd}
C_0(G^0, \overline{L^2(G^u; l^2)}_r) \overset{h}{\otimes} C_0(G^0, L^2(G^u; l^2)_c) \arrow[d, "\psi"'] \arrow[dr, "\pi_\psi"] & \\
A(G) & \frac{C_0(G^0, \overline{L^2(G^u; l^2)}_r) \overset{h}{\otimes} C_0(G^0, L^2(G^u; l^2)_c)}{\text{Ker } \psi} \arrow[l, dotted, "\overline{\psi}"']
\end{tikzcd} .$$

Then, $\overline{\psi}$ is an isometric isomorphism.

It remains to consider the product on

$$C_0(G^0, \overline{L^2(G^u; l^2)}_r) \overset{h}{\otimes} C_0(G^0, L^2(G^u; l^2)_c)$$

(see 5.18) and $B(G)$ (pointwise product) and verify that ψ respects them.

We are trying to prove that if $\alpha \otimes \overline{\xi}, a \otimes \overline{f} \in C_0(G^0, \overline{L^2(G^u; l^2)}_r)$ and $\beta \otimes \eta, b \otimes g \in C_0(G^0, L^2(G^u; l^2)_c)$,

$$\psi\big((\alpha \otimes \overline{\xi}) \otimes (\beta \otimes \eta) \lozenge (a \otimes \overline{f}) \otimes (b \otimes g)\big) = \psi\big((\alpha \otimes \overline{\xi}) \otimes (\beta \otimes \eta)\big)\psi\big((a \otimes \overline{f}) \otimes (b \otimes g)\big).$$

Let $\gamma \in G$. Then,

$$\psi\big((\alpha \otimes \overline{\xi}) \otimes (\beta \otimes \eta)\big)\psi\big((a \otimes \overline{f}) \otimes (b \otimes g)\big)(\gamma) = \overline{(\alpha a)(s(\gamma))}(\beta b)(r(\gamma))\langle L^u_\gamma \xi, \eta\rangle\langle L^u_\gamma f, g\rangle,$$

for $L^u_\gamma = L_{\nu_i(s(\gamma))^{-1}\gamma\nu_j(r(\gamma))}$, as in 4.5.13. To simplify the notation, we write

$$\omega(\gamma) = \nu_i(s(\gamma))^{-1}\gamma\nu_j(r(\gamma)).$$

On the other hand,

$$
\begin{aligned}
\psi\big((\alpha \otimes \overline{\xi}) \otimes (\beta \otimes \eta) \lozenge (a \otimes \overline{f}) \otimes (b \otimes g)\big)(\gamma) \\
&= \psi\Big(\int_{G^u} (\overline{\alpha a} \otimes \overline{\xi\varepsilon \cdot f}) \otimes (\beta b \otimes \eta\varepsilon \cdot g) \, d\lambda^u(\varepsilon)\Big)(\gamma) \\
&= \int_{G^u} (\alpha a \otimes \xi\varepsilon \cdot f, \beta b \otimes \eta\varepsilon \cdot g)(\gamma) \, d\lambda^u(\varepsilon) \\
&= \int_{G^u} \overline{\alpha(s(\gamma))a(s(\gamma))} \langle L^u_\gamma \xi\varepsilon \cdot f, \eta\varepsilon \cdot g\rangle \beta(r(\gamma))b(r(\gamma)) \, d\lambda^u(\varepsilon) \\
&= \overline{\alpha(s(\gamma))a(s(\gamma))}\beta(r(\gamma))b(r(\gamma)) \int_{G^u} \langle L^u_\gamma \xi\varepsilon \cdot f, \eta\varepsilon \cdot g\rangle \, d\lambda^u(\varepsilon).
\end{aligned}
$$

Thus, we need to prove that

$$
\langle L^u_\gamma \xi, \eta\rangle \langle L^u_\gamma f, g\rangle = \int_{G^u} \langle L^u_\gamma \xi\varepsilon \cdot f, \eta\varepsilon \cdot g\rangle \, d\lambda^u(\varepsilon).
$$

Starting by the left-hand side,

$$
\begin{aligned}
\langle L^u_\gamma \xi, \eta\rangle \langle L^u_\gamma f, g\rangle &= \int_{G^u} \overline{L_{\omega(\gamma)}\xi(\delta)}\eta(\delta) \, d\lambda^u(\delta) \int_{G^u} \overline{L_{\omega(\gamma)}f(\varepsilon)} g(\varepsilon) \, d\lambda^u(\varepsilon) \\
&= \int_{G^u} \overline{L_{\omega(\gamma)}\xi(\delta)}\eta(\delta) \int_{G^u} \overline{L_{\omega(\gamma)}f(\varepsilon)} g(\varepsilon) \, d\lambda^u(\varepsilon) \, d\lambda^u(\delta) \\
&= \int_{G^u} \overline{L_{\omega(\gamma)}\xi(\delta)}\eta(\delta) \int_{G^u} \overline{L_{\omega(\gamma)}f(\delta\nu_k(s(\delta))^{-1}\varepsilon)} g(\delta\nu_k(s(\delta))^{-1}\varepsilon) \, d\lambda^u(\varepsilon) \, d\lambda^u(\delta) \\
&= \int_{G^u} \int_{G^u} \overline{L_{\omega(\gamma)}\xi(\delta) L_{\omega(\gamma)}f(\delta\nu_k(s(\delta))^{-1}\varepsilon)} \eta(\delta) g(\delta\nu_k(s(\delta))^{-1}\varepsilon) \, d\lambda^u(\delta) \, d\lambda^u(\varepsilon) \\
&= \int_{G^u} \langle L_{\omega(\gamma)} \xi\varepsilon \cdot f, \eta\varepsilon \cdot g\rangle \, d\lambda^u(\varepsilon),
\end{aligned}
$$

as we wished to show. The left invariance of the Haar system and Fubini's theorem were applied.

It is easy to check that the same holds for finite sums of elementary tensor, as well as for limits of them. Next, we note that since ψ is surjective when considered with range $A(G)$, for $v, w \in A(G)$, there exist V, W in domain of ψ such that $\psi(V) = v$, $\psi(W) = w$. Then, $vw = \psi(V)\psi(W) = \psi(VW) \in A(G)$. Therefore, $A(G)$ is an algebra (it is also associative, since so is $B(G)$).

Therefore, the maps ψ, π_ψ and $\overline{\psi}$ respect the product and the quotient

$$\frac{C_0(G^0, \overline{L^2(G^u;l^2)}_r) \overset{h}{\otimes} C_0(G^0, L^2(G^u;l^2)_c)}{\text{Ker } \psi}$$

is isometrically isomorphic to $A(G)$.

It follows that we can consider an operator space structure on $A(G)$. $\square$

Now, we concentrate on the right triangle-like diagram of 5.19.

Remark 5.3.3. Recall from 4.1 that if H is a locally compact group,

$$\frac{\overline{L^2(H)}_r \overset{\wedge}{\otimes} L^2(H)_c}{\text{Ker } q_0} \simeq A(H),$$

where $q_0(\overline{f} \otimes g) = (f, g)$. Moreover, if we consider the amplified version of q_0, from 4.5

$$\frac{\overline{L^2(H;l^2)}_r \overset{\wedge}{\otimes} L^2(H;l^2)_c}{\text{Ker } q_0} \simeq A(H)$$

as well.

Remember that we fixed $u \in G^0$. For $H = G^u_u$, we wish to extend q_0 to a map with domain $\overline{L^2(G^u;l^2)}_r \overset{\wedge}{\otimes} L^2(G^u;l^2)_c$, still with the same range. If G^u_u is open in G^u, we can let $\mu = \lambda^u_{|_{G^u_u}}$ be the restriction of the measure λ^u on G^u to the isotropy group G^u_u. In this case, μ is the Haar measure on G^u_u and it makes sense to define

$$\begin{aligned} q : \overline{L^2(G^u;l^2)}_r \overset{\wedge}{\otimes} L^2(G^u;l^2)_c &\to A(G^u_u) \\ \overline{\xi} \otimes \eta &\to (\xi_{|_{G^u_u}}, \eta_{|_{G^u_u}}). \end{aligned}$$

In order to be able to define the map q above, and hence relate $L^2(G^u;l^2)$ and $L^2(G^u_u)$, from now on we restrict ourselves to Haar systems $\{\lambda^v\}_v$ such that $\lambda^u_{|_{G^u_u}} = \mu$ is the Haar measure at G^u_u for the fixed unit u. We saw on 2.3.6 that for any transitive locally trivial groupoid with G^u_u unimodular we can construct such a Haar system.

Proposition 5.3.4. *Let G be a locally trivial transitive groupoid. Fix $u \in G^0$. Suppose $\{\lambda^v\}_{v \in G^0}$ is a left Haar system such that $\lambda^u_{|G^u_u}$ is the Haar measure on G^u_u . Then, we can define a map $Q = \mathrm{Id} \otimes q \otimes \mathrm{Id}$, with q extending q_0, such that the following diagram is commutative:*

$$
\begin{tikzcd}
 & C_0(G^0) \overset{h}{\otimes} (\overline{L^2(G^u;l^2)}_r \overset{\wedge}{\otimes} L^2(G^u;l^2)_c) \overset{h}{\otimes} C_0(G^0) \cdot \arrow[dl, two heads, "\pi_Q"'] \arrow[d, "Q"] \\
\frac{C_0(G^0)\overset{h}{\otimes}(\overline{L^2(G^u;l^2)}_r \overset{\wedge}{\otimes} L^2(G^u;l^2)_c)\overset{h}{\otimes}C_0(G^0)}{Ker\ Q} \arrow[r, dotted, "\overline{Q}"'] & C_0(G^0) \overset{h}{\otimes} A(G^u_u) \overset{h}{\otimes} C_0(G^0)
\end{tikzcd}
$$

Also, Q is a complete quotient map and the induced map $\overline{Q}$ is a complete isometric isomorphism. Moreover, $C_0(G^0) \overset{h}{\otimes} A(G^u_u) \overset{h}{\otimes} C_0(G^0)$ is an associative algebra and the map Q respects the product.

Proof. We begin by defining a completely contractive map

$$\rho : \overline{L^2(G^u;l^2)}_r \overset{\wedge}{\otimes} L^2(G^u;l^2)_c \to \overline{L^2(G^u_u;l^2)}_r \overset{\wedge}{\otimes} L^2(G^u_u;l^2)_c.$$

Let $\rho_R : \overline{L^2(G^u)}_r \to \overline{L^2(G^u_u)}_r$, $\rho_R(f) = f_{|G^u_u}$, and $\rho_C : L^2(G^u)_c \to L^2(G^u_u)_c$, $\rho_C(g) = g_{|G^u_u}$. Both these maps are completely contractive. Since $L^2(Y) \otimes l^2 \simeq L^2(Y;l^2)$, from 3.3.12,

$$\overline{L^2(Y)}_r \overset{\wedge}{\otimes} \overline{l^2}_r \simeq \overline{(L^2(Y) \otimes l^2)}_r \simeq \overline{L^2(Y;l^2)}_r$$

and

$$L^2(Y)_c \overset{\wedge}{\otimes} l^2_c \simeq (L^2(Y) \overset{\wedge}{\otimes} l^2)_c \simeq L^2(Y;l^2)_c.$$

Then, the maps $\rho_r = \rho_R \otimes \mathrm{Id} : \overline{L^2(G^u;l^2)}_r \to \overline{L^2(G^u_u;l^2)}_r$ and $\rho_c = \rho_C \otimes \mathrm{Id} : L^2(G^u;l^2)_c \to L^2(G^u_u;l^2)_c$ are complete contractions as well. Thus, the map $\rho = \rho_r \otimes \rho_c$ is completely contractive and it follows that $q = q_0 \circ \rho$ is a complete contraction.

Recall that the product $\Diamond$ on $T(G^u) = \overline{L^2(Y;l^2)}_r \overset{\wedge}{\otimes} L^2(Y;l^2)_c$ was defined on 5.1.26. We wish to verify that for $\overline{\xi} \otimes \eta, \overline{f} \otimes g \in T(G^u)$,

$$q(\overline{\xi} \otimes \eta \,\Diamond\, \overline{f} \otimes g) = q(\overline{\xi} \otimes \eta) q(\overline{f} \otimes g).$$

By the properties of Bochner's integral, the left-hand side of the equality we are trying to prove is

$$\begin{aligned}
q(\overline{\xi} \otimes \eta \Diamond \overline{f} \otimes g) &= q\big(\int_{G^u} \overline{\xi\varepsilon \cdot f} \otimes \eta\varepsilon \cdot g \, d\lambda^u(\varepsilon) \big) \\
&= \int_{G^u} (\overline{(\xi\varepsilon \cdot f)}_{|G_u^u}, (\eta\varepsilon \cdot g)_{|G_u^u}) \, d\lambda^u(\varepsilon) \\
&= \int_{G_u^u} (\overline{(\xi\alpha f)}_{|G_u^u}, (\eta\alpha g)_{|G_u^u}) \, d\lambda^u_{|G_u^u}(\alpha) \\
&= q_0(\overline{\xi}_{|G_u^u} \otimes \eta_{|G_u^u} \diamond \overline{f}_{|G_u^u} \otimes g_{|G_u^u}),
\end{aligned}$$

and since q_0 respects the product,

$$q_0(\overline{\xi}_{|G_u^u} \otimes \eta_{|G_u^u} \diamond \overline{f}_{|G_u^u} \otimes g_{|G_u^u}) = q_0(\overline{\xi}_{|G_u^u} \otimes \eta_{|G_u^u}) q_0(\overline{f}_{|G_u^u} \otimes g_{|G_u^u}) = q(\overline{\xi} \otimes \eta) q(\overline{f} \otimes g).$$

Thus, q respects the product.

We wish to show that q it is a complete quotient map. Note that both ρ_R and ρ_C are completely quotient maps. By the projectivity of $\overset{\wedge}{\otimes}$, ρ is a quotient map as well. Recall that q_0 is a quotient map. Then, so is $q = q_0 \circ \rho$.

We define the map $Q = \mathrm{Id} \otimes q \otimes \mathrm{Id}$,

$$Q : C_0(G^0) \overset{h}{\otimes} (\overline{L^2(G^u; l^2)}_r \overset{\wedge}{\otimes} L^2(G^u; l^2)_c) \overset{h}{\otimes} C_0(G^0) \to C_0(G^0) \overset{h}{\otimes} A(G_u^u) \overset{h}{\otimes} C_0(G^0).$$

Since $\overset{h}{\otimes}$ is projective, it follows that Q is a complete quotient map and induces the commutative diagram

$$\begin{tikzcd}
 & C_0(G^0) \overset{h}{\otimes} (\overline{L^2(G^u; l^2)}_r \overset{\wedge}{\otimes} L^2(G^u; l^2)_c) \overset{h}{\otimes} C_0(G^0) \arrow[dl, "\pi_Q"'] \arrow[d, "Q"] \\
\frac{C_0(G^0) \overset{h}{\otimes} (\overline{L^2(G^u; l^2)}_r \overset{\wedge}{\otimes} L^2(G^u; l^2)_c) \overset{h}{\otimes} C_0(G^0)}{\operatorname{Ker} Q} \arrow[r, dotted, "\overline{Q}"'] & C_0(G^0) \overset{h}{\otimes} A(G_u^u) \overset{h}{\otimes} C_0(G^0)
\end{tikzcd}$$

The map $\overline{Q}$ induced by Q is a complete isometric isomorphism and this finishes the proof of commutativity of the right subdiagram.

We can consider Q as a map with codomain $\mathcal{A}_v$, since

$$\mathrm{Ran}(Q) = C_0(G^0) \overset{h}{\otimes} A(G_u^u) \overset{h}{\otimes} C_0(G^0) \subseteq C_0(G^0) \overset{\vee}{\otimes} A(G_u^u) \overset{\vee}{\otimes} C_0(G^0) = \mathcal{A}_v,$$

see 5.2.8. Thus, it make sense to ask ourselves if Q respects the product (we do not know yet if $\mathcal{A}_h$ is an algebra!). We can see that since $Q = \mathrm{Id} \otimes q \otimes \mathrm{Id}$ and both q and the identity map respect the product and are continuous, Q respects the product as well. Now we can use this to verify that $\mathcal{A}_h$ is, in fact, an algebra. Suppose $h, h' \in \mathcal{A}_h$, by the surjectivity of Q, there exist $a, a' \in C_0(G^0) \overset{h}{\otimes} (\overline{L^2(G^u;l^2)}_r \overset{\wedge}{\otimes} L^2(G^u;l^2)_c) \overset{h}{\otimes} C_0(G^0)$ such that $Q(a) = h$, $Q(a') = h'$. The product $Q(a)Q(a')$ belongs to $\mathcal{A}_v$ and verifies $Q(a)Q(a') = Q(aa')$. Thus, $Q(a)Q(a') \in \mathcal{A}_h$ and this space is an associative algebra. Note that the algebra structure of $\mathcal{A}_v$ is the one we will expect from a space of the form $A \overset{\delta}{\otimes} B \overset{\delta}{\otimes} C$. □

Corollary 5.3.5. *The algebra* $C_0(G^0) \overset{h}{\otimes} A(G_u^u) \overset{h}{\otimes} C_0(G^0)$ *is a completely contractive Banach algebra.*

Proof. If follows from 5.1.29 and the properties of the map Q. □

Finally, we want to establish the commutativity of the middle triangle of the diagram.

Proposition 5.3.6. *The map*

$$\varphi : \frac{C_0(G^0) \overset{h}{\otimes} (\overline{\mathcal{H}}_r \overset{\wedge}{\otimes} \mathcal{H}_c) \overset{h}{\otimes} C_0(G^0)}{Ker\ Q} \longrightarrow \frac{C_0(G^0, \overline{\mathcal{H}}_r) \overset{h}{\otimes} C_0(G^0, \mathcal{H}_c)}{Ker\ \psi}, \quad \varphi([v]_Q) = [v]_\psi$$

for $\mathcal{H} = L^2(G^u; l^2)$, *is a complete isometric isomorphism that preserves the product. Thus, the middle triangle of the main diagram is commutative.*

Proof. Let $\varphi([v]_Q) = [v]_\psi$. We want to show that φ is well-defined. To this end, we prove that Ker $Q \simeq$ Ker ψ via the identification

$$C_0(G^0) \overset{h}{\otimes} (\overline{L^2(G^u;l^2)}_r \overset{\wedge}{\otimes} L^2(G^u;l^2)_c) \overset{h}{\otimes} C_0(G^0) \simeq C_0(G^0, \overline{L^2(G^u;l^2)}_r) \overset{h}{\otimes} C_0(G^0, L^2(G^u;l^2)_c).$$

Fix $\gamma \in G$ and define

$$F_\gamma : C_0(G^0) \overset{h}{\otimes} A(G_u^u) \overset{h}{\otimes} C_0(G^0) \to \mathbb{C}$$

such that

$$F_\gamma(\alpha \otimes q(\lambda) \otimes \beta) = \overline{\alpha(s(\gamma))} q(\lambda)(\omega(\gamma))\beta(r(\gamma)).$$

Here we are using the local triviality of the groupoid: if $\gamma \in G$ and $\{U_i, \nu_i\}$ is the family that expresses the local triviality, $\omega(\gamma)$ is the element of G_u^u defined by $\omega(\gamma) = \nu_j(r(\gamma))\gamma\nu_i(s(\gamma))^{-1}$. Recall (4.5.13) that i, j are uniquely determined, and hence so is $\omega(\gamma)$. We extend the map F_γ linearly. Since

$$F_\gamma = ev_{\nu_j(r(\gamma)),\omega(\gamma),\nu_i(s(\gamma))^{-1}},$$

it is a continuous map as was observed at the end of the section 5.2.

Observe that, using the identification $\mathcal{V} \otimes C_0(X) \to C_0(X, \mathcal{V})$, $v \otimes \alpha \to \alpha(\cdot)v$,

$$\begin{aligned}
\psi(\alpha \otimes \overline{f} \otimes g \otimes \beta)(\gamma) &= \psi(\alpha\overline{f} \otimes \beta g)(\gamma) \\
&= (\alpha f, \beta g)(\gamma) \\
&= \langle L_\gamma^u \alpha f(s(\gamma)), \beta g(r(\gamma))\rangle \\
&= \overline{\alpha(s(\gamma)}\langle L_{\omega(\gamma)} f, g\rangle \beta(r(\gamma)) \\
&= \overline{\alpha(s(\gamma))}(g * f^*)(\omega(\gamma))\beta(r(\gamma)) \\
&= \overline{\alpha(s(\gamma))} q(\overline{f} \otimes g)(\omega(\gamma))\beta(r(\gamma)) \\
&= F_\gamma(\alpha \otimes q(\overline{f} \otimes g) \otimes \beta).
\end{aligned}$$

Hence, by continuity of F_γ and ψ, it follows that $F_\gamma(Q(v)) = \psi(v)(\gamma)$, for all $\gamma \in G$, $v \in C_0(G^0, \overline{L^2(G^u; l^2)}_r) \overset{h}{\otimes} C_0(G^0, L^2(G^u; l^2)_c)$.

If $v \in \text{Ker } Q$, then $\varphi(v)(\gamma) = 0$, for all $\gamma \in G$, and since $A(G)$ separates points (4.5.5), $v \in \text{Ker } \varphi$.

If $v \in \text{Ker } \varphi$, $F_\gamma(Q(v)) = 0$, for all $\gamma \in G$. Equivalently, $ev_{u',g,u}(Q(v)) = 0$, for all $u, u' \in G^0$, $g \in G_u^u$. We denote the Gelfand transform of $\mathcal{A}_\vee$ by $\mathcal{G}$. Since the inclusion $\iota : \mathcal{A}_h \to \mathcal{A}_\vee$ is one-to-one (by Proposition 5.2.8), we have that

$$0 = ev_{u',g,u}(Q(v)) = \mathcal{G} \circ \iota(Q(v))(ev_{u',g,u}), \quad \forall u, u' \in G^0, g \in G_u^u$$

implies that $Q(v) = 0$. Therefore, the linear function $\varphi([v]_Q) = [v]_\psi$ is well-defined and bijective. In order to see that φ is in fact a complete isometry, note that

$$\begin{aligned}
\|[v_{i,j}]_Q\|_n &= \inf \left\{ \|[v_{ij} + w_{ij}]\|_n : \, w_{ij} \in \text{Ker } Q \right\} \\
&= \inf \left\{ \|[v_{ij} + w_{ij}]\|_n : \, w_{ij} \in \text{Ker } \psi \right\} \\
&= \|[v_{i,j}]_\psi\|_n.
\end{aligned}$$

It only remains to prove that φ respects the product. It is easy to check that if $f : A \to B$ is a homomorphism of algebras, on the quotient space $\frac{A}{\text{Ker }(f)}$, we have that $[a][a'] = [aa']$, for all $a, a' \in A$. Let $b, b' \in C_0(G^0) \overset{h}{\otimes} (\overline{L^2(G^u; l^2)}_r \overset{\wedge}{\otimes} L^2(G^u; l^2)_c) \overset{h}{\otimes} C_0(G^0)$. Then,

$$\varphi([b]_Q[b']_Q) = \varphi([bb']_Q) = [bb']_\psi = [b]_\psi [b']_\psi = \varphi([b]_Q)\varphi([b']_Q).$$

$\square$

We now state the main result of our work:

Theorem 5.3.7. *Let G be a locally trivial, transitive groupoid. Fix $u \in G^0$. Suppose that G has a Haar system $\{\lambda^v\}_{v \in G^0}$ such that $\lambda^u_{|G_u^u}$ is a left Haar measure on the isotropy group at u.*

Then $A(G)$ is a Banach algebra isometrically isomorphic to $C_0(G^0) \overset{h}{\otimes} A(G_u^u) \overset{h}{\otimes} C_0(G^0)$. Moreover, since the last space is a completely contractive Banach algebra, so is $A(G)$.

Proof. Note that the only condition we are missing to conclude that $A(G)$ is a Banach algebra is the inequality of the norms of the product: if $v, w \in A(G)$, then $\|vw\| \leq \|v\|\|w\|$. This follows from the corresponding property for $\mathcal{A}_h$ and the fact that those spaces are isometrically isomorphic as algebras. $\square$

5.4 The non-transitive case

If G is a non-transitive groupoid such that each transitive component G_i is open and closed on G, then the transitive components are components also in a topological sense (recall from 2.3.3 that this is the case for locally trivial groupoids). We can establish a one-to-one correspondence between G-Hilbert bundles and families of G_i-Hilbert bundles. Moreover, continuous and bounded sections of G-Hilbert bundles correspond to bounded families of continuous and bounded sections of G_i-Hilbert bundles. If we change the boundedness condition for vanishing at infinity, the correspondence is with c_o families. In the following proposition these correspondences allow us to relate $B(G)$ and $A(G)$ to $\oplus B(G_i)$ and $\oplus A(G_i)$.

Proposition 5.4.1. *Let G be a locally compact groupoid such that each transitive component G_i is open and closed on G. Then*

1. *There is a one-to-one correspondence between G-Hilbert bundles and families of G_i-Hilbert bundles.*
2. *The correspondence above links the left regular G-Hilbert bundle with multiplicity $L^2(G; l^2)$ and the family of left regular G_i-Hilbert bundles with multiplicity $\{L^2(G_i; l^2)\}_i$*
3. *If $\mathcal{H}$ and $\{\mathcal{H}_i\}$ are Hilbert bundles corresponding as above, there is a one-to-one correspondence that respects the norm between bounded (vanishing at infinity) continuous sections of $\mathcal{H}$ and bounded (c_o) families of bounded (vanishing at infinity) continuous sections of $\mathcal{H}_i$.*
4. *As sets,*

$$\begin{aligned} l^\infty\text{-}\oplus_{i\in I} B(G_i) =& \{\{(\xi_i, \eta_i)\}_i : \{\xi\}_i, \{\eta_i\}_i \in l^\infty\text{-}\oplus \mathcal{S}C_b(\mathcal{H}_i), \\ & \{\mathcal{H}_i\}_i \textit{ family of } G_i\textit{-Hilbert bundles}\}. \end{aligned}$$

Here, by $l^\infty\text{-}\oplus_{j\in J} D_j$ we mean the sequences $\{d_j\}_{j\in J}$, with $d_j \in D_j$, such that $\sup_j \|d_j\| < \infty$.

5. *The map $\beta : l^\infty\text{-} \oplus_{i\in I} B(G_i) \to B(G)$, $\beta(\{(\xi_i,\eta_i)\}_i) = (\xi,\eta)$ is an isometric isomorphism that respects the pointwise product.*

6. *The map above restricts to an isometric isomorphism $\alpha : c_o\text{-} \oplus_{i\in I} A(G_i) \to A(G)$.*

Proof. (1) Given a G-Hilbert bundle $\mathcal{H} = (\{\mathcal{H}^u\}_{u\in G^0}, \Gamma, L)$, for each $i \in I$ we consider $\mathcal{H}_i = (\{\mathcal{H}_i^u\}_{u\in G_i^0}, \Gamma_{|i}, L_{|i})$, where $\Gamma_{|i} = \{\xi_{|_{G_i^0}} : \xi \in \Gamma\}$ and $(L_{|i})_\gamma = L_\gamma$ if $\gamma \in G_i$. In order to check that $\mathcal{H}_i$ is in fact a G_i-Hilbert bundle there is only one property that deserves some explanation, namely condition 4 from definition 4.2.1. Let η be a section of $\mathcal{H}_i$. Suppose that for all $u \in G_i^0$ and for all $\varepsilon > 0$ there exists $\eta' \in \Gamma_{|i}$ such that $\|\eta(v) - \eta'(v)\| < \varepsilon$ for all v on a neighborhood V of u. We wish to check that $\eta \in \Gamma_{|i}$ as well. Let ξ be a section of $\mathcal{H}$ such that $\xi_{|_{G_i^0}} = \eta$, $\xi = 0$ otherwise. Fix $u \in G^0$ and $\varepsilon > 0$. If $u \in G_i^0$, by the property of η, there exists $\eta' \in \Gamma_{|i}$ as above. Hence, there exists $\xi' \in \Gamma$ such that $\xi'_{|i} = \eta'$ and $\|\xi(v) - \xi'(v)\| = \|\eta(v) - \eta'(v)\| < \varepsilon$ for v on some neighborhood of u. If $u \notin G_i^0$, the zero section 0 approches (in fact coincides with) ξ on a neighborhood of u. Therefore, since $\mathcal{H}$ is a G-Hilbert bundle, $\xi \in \Gamma$ and thus $\xi_{|_{G_i^0}} = \eta \in \Gamma_{|i}$. Then $\mathcal{H}_i$ is a G_i-Hilbert bundle for all $i \in I$.

On the other hand, suppose that $\{\mathcal{H}_i\}_i$ is a family of G_i-Hilbert bundles. Let $\mathcal{H} = \cup_i \mathcal{H}_i = (\{\mathcal{H}^u\}_{u\in G^0}, \Gamma, L)$, where:

- $\mathcal{H}^u = \mathcal{H}_i^u$ if $u \in G_i^0$.
- $\Gamma = \{\xi \text{ section of } \mathcal{H} : \forall i \in I, \xi_{|_{G_i^0}} \in \Gamma_i\}$
- If $\gamma \in G_i$, $L_\gamma = (L_i)_\gamma$.

It is easy to verify that $\mathcal{H}$ is a G-Hilbert bundle and the correspondence stablished is one-to-one.

(2) Note that each left Haar system $\{\lambda^u\}_{u\in G^0}$ on G determines a family of left Haar systems $\{\, \{\lambda^u\}_{u\in G_i^0}\}_{i\in I}$ and vice-versa.

If $\{\lambda^u\}_{u\in G^0}$ is a Haar system on G and $L^2(G,\lambda)$ is the left regular representation, we claim that the family of G_i-Hilbert bundles obtained by the correspondence above is $\{L^2(G_i,\lambda_i)\}_i$. We only need to verify that the continuous sections considered coincide.

On $L^2(G,\lambda)$, the family of continuous sections Γ is obtained from $C_c(G)$. If $f\in C_c(G)$, $f_{|_{G_i}}\in C_c(G_i)$ and $\{f_{|_{G_i}} : f\in C_c(G)\} = C_c(G_i)$ is the family of sections that defines the notion of continuity for $L^2(G_i,\lambda_i)$. Therefore, $\{L^2(G,\lambda)_i\}_{i\in I} = \{L^2(G_i,\lambda_i)\}_{i\in I}$.

Reciprocally, suppose that $f_i\in C_c(G_i)$, for all $i\in I$. Then, if f is defined by $f_{|_{G_i}} = f_i$, f is a continuous function on G (note that here we are applying that each G_i is open and closed). Also, any compactly supported continuous function g is of this form . Thus, $C_c(G)\subseteq \{\{f_i\} : f_i\in C_c(G_i)\}$. Therefore the notion of continuity coincides and we established the correspondence between the left regular bundles.

If we consider L^2 spaces with multiplicity l^2, the same reasoning applies.

(3) We now look at the correspondence between sections of $\mathcal{H}$ and sections of $\{\mathcal{H}_i\}_i$.

Suppose $\mathcal{H}$ is a G-Hilbert bundle corresponding to the family of G_i-Hilbert bundles $\{\mathcal{H}_i\}$. Let $\xi\in\mathcal{S}C_b(\mathcal{H})$. Then, $\xi_i = \xi_{|_{G_i^0}}\in\mathcal{S}C_b(\mathcal{H}_i)$. Moreover, $\{\xi_i\}\in l^\infty - \oplus_i\mathcal{S}C_b(\mathcal{H})$, since $\|\xi_i\|\le\|\xi\|$, $\forall i\in I$. Reciprocally, if $\{\xi_i\}\in l^\infty - \oplus_i\mathcal{S}C_b(\mathcal{H})$, the section ξ such that $\xi_{|_{G_i^0}} = \xi_i$ belongs to $\mathcal{S}C(\mathcal{H})$ and

$$\|\xi\| = \sup_{u\in G^0}\|\xi(u)\| = \sup_{i\in I}\sup_{u\in G_i^0}\|\xi_i(u)\| = \|\{\xi_i\}_i\| < \infty.$$

If $\xi\in\mathcal{S}C_0(\mathcal{H})$, we want to prove that $\{\xi_i\}_{i\in I}\in c_0 - \oplus_i\mathcal{S}C_0(\mathcal{H}_i)$. Clearly, each ξ_i belongs to $\mathcal{S}C_0(\mathcal{H}_i)$. We wish to check that given $\varepsilon>0$, there exists $F\subset I$ finite such that $\|\xi_i\|<\varepsilon$ if $i\notin F$. Since ξ vanishes at infinity, there exists a compact $K\subseteq G^0$ such that $\|\xi_{|_{K^c}}\|<\varepsilon$. Since $\{G_i^0\}_i$ is an open covering of K, there exists $G_{i_1}^0,\cdots,G_{i_p}^0$ that cover K. Therefore, $\|\xi_i\|\le\|\xi_{|_{K^c}}\|<\varepsilon$ if $i\notin\{i_1,\cdots,i_p\}$.

Conversely, if $\{\xi_i\}_{i\in I}\in c_0 - \oplus_i\mathcal{S}C_0(\mathcal{H}_i)$, we want to verify that the continuous $\mathcal{H}$-section ξ vanishes at infinity. Let $\varepsilon>0$. Let $F=\{i_1,\cdots,i_p\}$ such that $\|\xi_i\|<\varepsilon$ if $i\notin F$. Let

$K_{i_k} \subseteq G^0_{i_k}$ be a compact subset such that $\|\xi_{|_{K^c_{i_k}}}\| < \varepsilon$. Hence, $K = \cup_{k=1}^p K_{i_k}$ is a compact subset of G and $\|\xi_{|_{K^c}}\| < \varepsilon$.

(4) If $\{\varphi_i\}_{i\in I} \in l^\infty\text{-} \oplus_i B(G_i)$, then $\{F_{\varphi_i}\}_{i\in I} \in \infty\text{-} \oplus_i P(G_i \times I_2)$, where I_2 is the trivial groupoid $\{1,2\} \times \{1,2\}$ and we are applying the groupoid version of Paulsen's "off-diagonalization technique" as in Chapter 4. More explicitly, we identify F_{φ_i} with a 2×2 matrix $F_{\varphi_i} = \begin{pmatrix} \rho_i & \varphi_i \\ \varphi_i^* & \tau_i \end{pmatrix}$, where $\rho, \tau \in P(G)$. Since there exists a bound M for the norms $\{\|\varphi_i\|\}_i$, $\|F_{\varphi_i}\| = \|\varphi_i\| < M$ for all i. Since F_{φ_i} is positive definite, there exists $\zeta_i \in \mathcal{S}C_b(\mathcal{H}_i \times \mathbb{C}^2)$ verifying $(\zeta_i, \zeta_i) = F_{\varphi_i}$ and $\|F_{\varphi_i}\| = \|\zeta_i\|$. Therefore, $\sup_i \|\zeta_i\| < M$. If $\varphi_i = (\xi_i, \eta_i)$ for all i, then $\|\xi_i\| \leq \|\zeta_i\| < M$ and $\|\eta_i\| \leq \|\zeta_i\| < M$. Then, $\{\varphi_i\}_i = \{(\xi_i, \eta_i)\}_i$ and $\{\xi_i\}_{i\in I}, \{\eta_i\}_{i\in I} \in l^\infty - \oplus_i \mathcal{S}C_b(\mathcal{H}_i)$.

The other inclusion is clear.

(5) We define $\beta : l^\infty - \oplus_i B(G_i) \to B(G)$ by $\beta(\{(\xi_i, \eta_i)\}_i) = (\xi, \eta)$. We first check that β is well-defined. Suppose $(\xi_i, \eta_i) = (\zeta_i, \mu_i)$, $\forall i$. If $\gamma \in G$, there exists i such that $\gamma \in G_i$. Then, $(\xi, \eta)(\gamma) = (\xi_i, \eta_i)(\gamma) = (\zeta_i, \mu_i)(\gamma) = (\zeta, \mu)(\gamma)$. Similarly, β is one-to-one.

Suppose $\psi \in B(G)$. Then there exists a G-Hilbert bundle $\mathcal{H}$ and $\xi, \eta \in \mathcal{S}C_b(\mathcal{H})$ verifying $\psi = (\xi, \eta)$. Thus, $\{\xi_i\}_i, \{\eta_i\}_i \in l^\infty - \oplus_i \mathcal{S}C_b(\mathcal{H}_i)$ and $\beta(\{(\xi_i, \eta_i)\}_i) = (\xi, \eta) = \psi$.

Let $\{\varphi_i\}_{i\in I} \in l^\infty - \oplus_i B(G_i)$ and $\varphi = \beta(\{\varphi_i\}_{i\in I})$. Write $N = \|\{\varphi_i\}_{i\in I}\| = \sup_i \|\varphi_i\|_{B(G_i)}$. We want to show that for all $\varepsilon > 0$, there exists $\xi, \eta \in \mathcal{S}C_b(\mathcal{H})$ such that $\|\xi\|\|\eta\| < N + \varepsilon$, $(\xi, \eta) = \varphi$ and hence $\|\beta(\{\varphi_i\}_{i\in I})\| \leq \|\{\varphi_i\}_{i\in I}\|$. Fix $\varepsilon > 0$. For all i, since $\|\varphi\| < N + \varepsilon$, there exist $\{\xi_i\}_{i\in I}, \{\eta_i\}_{i\in I} \in \mathcal{S}C_b(\mathcal{H}_i)$ such that $\|\xi_i\|\|\eta_i\| < N+\varepsilon$. We can assume $\|\eta_i\| = 1$, $\|\xi_i\| < N + \varepsilon$. Thus, for ξ, η obtained from $\{\xi_i\}_{i\in I}, \{\eta_i\}_{i\in I}$, we have $\|\xi\|\|\eta\| = \|\xi\| = \sup_i \|\xi_i\| < N + \varepsilon$ and then β is contractive.

Conversely, for $\varphi \in B(G)$, if $\varepsilon > 0$ and $M = \|\varphi\| = \inf_{(\xi,\eta)=\psi} \|\xi\|\|\eta\|$, there exists $\xi, \eta \in \mathcal{S}C_b(\mathcal{H})$, $\|\xi\|\|\eta\| < M + \varepsilon$. Assume $\|\xi\| < M + \varepsilon, \|\eta\| = 1$. If $\{\xi_i\}_{i\in I}, \{\eta_i\}_{i\in I}$ are families of sections corresponding to ξ, η,

$$\|\{(\xi_i, \eta_i)\}_i\| = \sup_i \|(\xi_i, \eta_i)\| \leq_i \|\xi_i\|\|\eta_i\| = \|\xi\|\|\eta\| < M + \varepsilon$$

and therefore $\|\{(\xi_i,\eta_i)\}_i\| \leq \|\beta(\{(\xi_i,\eta_i)\}_i)\|$.

Moreover, β also preserves the pointwise product. To prove this, we first need to look at how the product of sections relates to the correspondence of them. Let $\{\mathcal{H}_i\}_i$ and $\{\mathcal{K}_i\}_i$ be families of G_i-Hilbert bundles. Suppose $\{\xi_i\}_i$ and $\{\zeta_i\}_i$ are families of sections in $\mathcal{S}C_b(\mathcal{H}_i)$ and $\mathcal{S}C_b(\mathcal{K}_i)$ respectively. The section of $\mathcal{H}$ corresponding to $\{\xi_i\}$ is ξ and the one corresponding to $\{\zeta_i\}_i$ is ζ. Each $\xi_i \otimes \zeta_i$ is in $\mathcal{S}C_b(\mathcal{H}_i \otimes \mathcal{K}_i)$ and via the correspondence above, the family $\{\xi_i \otimes \zeta_i\}_i$ is associated to $\xi \otimes \zeta$.

If $\{\eta_i\}_i$ is another family of sections of $\mathcal{H}$ and $\{\mu_i\}_i$ is a family of sections of $\mathcal{K}$, then

$$\begin{aligned}
\beta(\{(\xi_i,\eta_i)\},\{(\zeta_i,\mu_i)\}) &= \beta(\{(\xi_i,\eta_i),(\zeta_i,\mu_i)\}) \\
&= \beta(\{(\xi_i \otimes \zeta_i, \nu_i \otimes \mu_i)\}) \\
&= (\xi \otimes \zeta, \eta \otimes \mu) \\
&= (\xi,\eta)(\zeta,\mu) \\
&= \beta(\{(\xi_i,\eta_i)\})\beta(\{(\zeta_i,\mu_i)\}).
\end{aligned}$$

Then, β respects the product.

(6) Let $\alpha = \beta_{|_{c_0 \oplus_i A(G_i)}} : c_0 \oplus_i A(G_i) \to B(G)$. Note that since the correspondence of sections preserves the vanishing at infinity property and the correspondence of bundles matches the L^2 bundles, $\alpha(c_0 \oplus_i A(G_i)) \subseteq A(G)$. Also, it is clear that α is one-to-one and respects the product. By (3), α is surjective as well. The fact that α is isometric follows similarly to the β case. $\square$

Corollary 5.4.2. *Let G be a locally trivial groupoid. Let $\{G_i\}_i$ be the family of transitive components of G. Suppose for a fixed unit $u_i \in G_i^0$, we can find a Haar system $\{\lambda_i^v\}_{v\in G_i^0}$ such that $\lambda^{u_i}_{|_{G^{u_i}_{u_i}}}$ is the left Haar measure in $G^{u_i}_{u_i}$. Then*

$$A(G) \simeq c_0 - \oplus_i \Big(C_0(G_i^0) \overset{h}{\otimes} A(G^{u_i}_{u_i}) \overset{h}{\otimes} C_0(G_i^0) \Big).$$

as Banach algebras. Moreover, since the right hand side is an operator space, so is the Fourier algebra of G.

Proof. Thanks to Remark 2.3.3, we can apply Proposition 5.4.1 above. $\square$

Chapter 6

Conclusions

This chapter has four sections. In the first section we revisit the examples presented in our work and we analyze the information we obtain thanks to our results. The second section is called "Other Fourier algebras" and there we introduce the Fourier algebras that were defined by Renault ([43]) and Paterson [33]. The third Section is devoted to the conclusions of our work. In the last Section we present some open questions we hope will be matter of study in the future.

6.1 Examples (revisited)

We gather here all the information that we have about the Fourier algebras of the examples considered in our work.

Example 6.1.1. Groups and unions of groups.

If H is a locally compact group, the Fourier-Stieltjes and Fourier algebras that we consider here coincide with the algebras defined by Eymard.

If $H = \sqcup_i H_i$ is a group bundle, suppose there is a topology on H that makes it a locally compact groupoid and such that each group H_i is open and closed on H. If a Haar system exists, it is the union of the Haar mesures on each group. Then, we can apply 5.4.1 and conclude that

$$A(H) = c_0\text{-} \oplus_i A(H_i) \quad \text{and} \quad B(H) = l^\infty\text{-} \oplus_i B(H_i).$$

Note that these equations give us operator space structures on $A(H)$ and $B(H)$.

Example 6.1.2. Locally compact spaces.

Let X be a locally compact space. From the definitions of the Fourier-Stieltjes and Fourier algebra, we observed that (see 4.5.9) $B(X) = C_b(X)$ and $A(X) = C_0(X)$. Again, we have operator space structures on these algebras.

Example 6.1.3. Equivalence relations and trivial groupoids.

We cannot say much about these family of examples, unless we restrict ourselves to full equivalence relations. In this case, we are talking about groupoids of the form $X \times X$. This is a particular case of trivial groupoids $X \times H \times X$. These are the trivial examples of locally trivial groupoids. The intuitive left Haar systems that we can consider on groupoids of this type do not allow us to apply the main result of our work. Instead, we can consider another Haar system (that differs on a Dirac measure on a unit) and conclude that

$$A(X \times H \times X) \simeq C_0(X) \overset{h}{\otimes} A(H) \overset{h}{\otimes} C_0(X)$$

as Banach algebras. We then can see $A(X \times H \times X)$ as a completely contractive Banach algebra.

If $X \times H \times X$ is finite, then the Haar system is a system of counting measures and 5.4.1 also applies. More in general, any finite groupoid is an union of trivial groupoids and we obtained a complete description of its Fourier algebra. Note that in this case, the Fourier algebra and the Fourier-Stieltjes algebra coincide.

If H is trivial, we are back to the full equivalence relation case. Once again, we have information about the Fourier algebra associated to $X \times X$ and the Haar system that is not the most intuitive. We do not know if the Fourier algebra depends on the election of the Haar system. But if X is compact,

$$B(X \times X) = A(X \times X) \simeq C(X) \overset{h}{\otimes} C(X)$$

independently of the Haar system chosen.

For the non-compact case, we do not have a description (other than the definition) of $B(X \times H \times X)$ or $B(X \times X)$.

Example 6.1.4. Locally trivial transformation group groupoids.

Let $X \times H$ be a transformation group groupoid as in Example 2.2.5 In 2.2.8 we mentioned conditions to make sure that $X \times H$ is locally trivial: for instance, if H is a Lie group and the action is transitive and smooth, or if K is a normal subgroup of H and the projection $H \to H/K$ admits local sections. Suppose we know that $X \times H$ is locally trivial and H is unimodular. Then, we can construct a Haar system as in 2.3.6. Our structure theorem tells us in the transitive case that

$$A(X \times H) \simeq C_0(X) \overset{h}{\otimes} A(H) \overset{h}{\otimes} C_0(X)$$

as Banach algebras and in the non-transitive case we obtain a c_0 sum of the Fourier algebras of the transitive components.

Example 6.1.5. Directed Graphs.

Unfortunately, we cannot apply our structure theorem for this much liked family of groupoids. The difficulty comes from the fact that we do not know conditions for these groupoids to be locally trivial. The local triviality is needed to prove the alternative description of $A(G)$ from 4.5.13 that is a key point of our result.

Example 6.1.6. The fundamental groupoid.

Let X be a locally path-connected, semi-locally simply connected space. We know that the fundamental groupoid $\Pi(X)$ is a locally compact, locally trivial, groupoid.

First suppose that X is connected. Let x be a unit. Thanks to the semi-locally simplicity, the isotropy group $\Pi(X, x)$ is discrete, and hence the left (and right) Haar measure is a counting measure. Let μ be Radon measure on X, supported on X and such that $\mu(\{x\}) = 1$. Then, applying 2.3.6 we can define a Haar system $\{\lambda^y\}_{y\in X}$ on $\Pi(X)$ such that $\lambda^x_{|\Pi(X,x)}$ is the counting measure. Then,

$$A(\Pi(X)) \simeq C_0(X) \overset{h}{\otimes} A(\Pi(X,x)) \overset{h}{\otimes} C_0(X).$$

If X is not connected, let $\{X_i\}_{i\in\mathbb{N}}$ be the connected components and let x_i be a unit on each X_i. Again, the isotropy groups $\Pi(X, x_i)$ are discrete, and have counting measures as Haar measures. We want to consider a Radon measure μ on X, supported on X and such that $\mu(\{x_i\}) = 1$, for all $i \in \mathbb{N}$. Thus, we can apply again Seda's construction to define a Haar system $\{\lambda^x\}_{x\in X}$ on $\Pi(X)$ such that $\lambda^{x_i}_{|\Pi(X,x_i)}$ is the counting measure for each i. We apply 5.4.2 to obtain

$$A(\Pi(X)) \simeq c_0 - \oplus_i C_0(X_i) \overset{h}{\otimes} A(\Pi(X,x_i)) \overset{h}{\otimes} C_0(X_i).$$

6.2 Other Fourier algebras

Fourier algebras have been studied in the context of groupoids by Ramsay and Walter (see [41]), Renault (see [43]) and Paterson ([33]). They considered different conditions on the groupoids. Here, we briefly present Renault's and Paterson's constructions, since they both provide a definition of $A(G)$, and their ideas were crucial inspiration for our work. In addition to presenting their definitions, we want to also mention the examples that they considered as well as the main properties that they proved.

In [43] Renault studies Fourier algebras for measurable groupoids. That is, G is a locally compact groupoid with a fixed Haar system $\lambda = \{\lambda^u\}_{u\in G^0}$ and a quasi-invariant measure μ on G^0. Given μ and λ they induce a measure $\nu = \lambda^\mu$ on G defined by

$$\int f\, d\lambda^\mu = \int\int f(\gamma)\, d\lambda^u(\gamma) d\mu(u).$$

The measure μ is **quasi-invariant** if the null sets of ν and ν^{-1} coincide.

The G-Hilbert bundles considered in this case are measurable (on definitions 4.2.1 and 4.2.11 change "continuous" for "measurable"). If $\mathcal{H}$ is a measurable G-Hilbert bundle, the relevant sections are the essentially bounded ones, that we denote $\mathcal{S}L^\infty(\mathcal{H})$. We still denote the coefficient associated to two sections ξ, η by (ξ, η).

The Fourier-Stieltjes algebra of G is

$$B_\mu(G) = \{(\xi, \eta) : \xi, \eta \in \mathcal{S}L^\infty(\mathcal{H}), \text{ for any } G\text{-Hilbert bundle } \mathcal{H}\}.$$

On this space we consider the norm

$$\|\varphi\|_B = \inf_{(\xi,\eta)=\varphi} \|\xi\|\|\eta\|.$$

With this norm and point-wise product, $B_\mu(G)$ is a unital, involutive, commutative Banach algebra, included in $L^\infty(G, \nu)$.

Let $P_\mu(G)$ be the space of positive definite, essentially bounded functions on G. Then $B_\mu(G)$ is spanned by $P_\mu(G)$.

The left regular G-Hilbert bundle has Hilbert spaces $L^2(G^u, \lambda^u)$ and the action is the usual one. The Fourier algebra $A_\mu(G)$ of G is the closure of the linear span in $B_\mu(G)$ of the coefficients of this representation. Our definition, other the being in the continuous, vanishing at infinity context, differs to this one by the norm that we consider on $A(G)$. Also, every function on $A_\mu(G)$ is the coefficient of the left regular bundle with infinite multiplicity. The Fourier algebra is a norm closed ideal of $B_\mu(G)$.

Renault includes the following examples. For a locally compact group H, $A_\mu(H)$ and $B_\mu(H)$ are the algebras defined by Eymard (here the measure μ is just the Dirac measure at the identity of the group). If $G = X$, then $B(X) = L^\infty(X)$. If $G = X \times X$, for a measure space (X, μ), $B(X \times X)$ are the Hilbertian functions defined by Grothendieck in [21].

Very nice duality results are presented in this work. Let $C^*_\mu(G)$ be the C^*-algebra associated to G, obtained as the completion of the convolution space $L^I(G)$ (analogous to the $L^1(H)$ space of a group H) by the largest C^*-norm. Let $C^*_{red}(G)$ be the reduced C^*-algebra of G, defined as in the group case using the left regular representation of $C_c(G)$. Also in the usual way the von Neumann algebra $VN(G)$ is defined.

Both $L^2(G^0)_c$ and $C^*_\mu(G)$ are completely contractive left $L^\infty(G^0)$-modules, and $\overline{L^2(G^0)_r}$ is a completely contractive right $L^\infty(G^0)$-module. In this situation, we can define the **module Haagerup tensor product**. If A is an operator algebra, E a right A-operator module and F a left one, the module Haagerup tensor product of E and F over A is the quotient of $E \overset{h}{\otimes} F$ by the closed subspace that the tensors $ea \otimes f - e \otimes af$ span. It is denoted by $E \overset{h}{\otimes}_A F$. In Renault's case, the space considered is

$$\mathcal{X}(G) = \overline{L^2(G^0)_r} \overset{h}{\otimes}_{L^\infty(G^0)} C^*_\mu(G) \overset{h}{\otimes}_{L^\infty(G^0)} L^2(G^0)_c.$$

He proves that $\mathcal{X}(G)^* = B_\mu(G)$, giving an operator space structure to $B_\mu(G)$ that is inherited by $A_\mu(G)$. He also considers the space

$$\mathcal{Y}(G) = \overline{L^2(G^0)_r} \overset{h}{\otimes}_{L^\infty(G^0)} A_\mu(G) \overset{h}{\otimes}_{L^\infty(G^0)} L^2(G^0)_c$$

and proves that this space is the predual of $VN(G)$. Note that if G is just a locally compact group, then $G^0 = \{e\}$ and the results above are Eymard's dualities.

These dualities are used to study the multipliers of the Fourier algebra. As in the group case, if G is amenable, the multipliers of $A_\mu(G)$ are $B_\mu(G)$. Also, the space of completely bounded multipliers of $A_\mu(G)$ is studied if G is an r-discrete groupoid.

We now concentrate on Paterson's work [33]. This is, as far as we know, the only article devoted to continuous Fourier algebras of groupoids. For a locally compact groupoid G, the definition of the Fourier-Stieltjes algebra $B(G)$ is the one that we adopted in our work. Also, a continuous Fourier algebra is presented. Let $L^2(G)$ be the continuous left regular G-Hilbert bundle that we considered in our work. The functions of $C_c(G)$ are continuous and vanishing at infinity sections of $L^2(G)$. Define

$$A_{cf} = \{(f, g) : \ f, g \in C_c(G)\}$$

and let $A(G)$ be the closure in $B(G)$ of the algebra generated by $A_{cf}(G)$. Then, the definition itself assures that $A(G)$ is a commutative Banach algebra. However, the relationship to $B(G)$ is not obvious. If G is r-discrete, then $A(G)$ is a norm closed ideal of $B(G)$.

A duality result is presented for groupoids that are locally a product (this family includes r-discrete as well as locally trivial groupoids). This duality is stated in terms of the group of bisections of G and multiplicative module maps from $A(G)$ into $C_0(G^0)$. We do not explain this result in detail, but instead we briefly consider a very recent work from Paterson, [34]. In this work Paterson presents a duality result similar in spirit to Renault's result for the Fourier-Stieltjes algebra.[1] The perspective of this work is not the same as [33]. The groupoids consider here are have a fixed quasi-invariant measure μ and the definition of $B(G)$ depends on it (thus we use the notation $B(G, \mu)$). Also, the sections that build the coefficients are asked to be Borel bounded sections. The C^*-algebra used is not Renault's $C^*_\mu(G)$, but a completion of the convolution algebra $C_c(G)$ denoted $C^*(G, \mu)$. The statement of the duality result is the following. If

$$\mathcal{X}_\mu(G) = \overline{L^2(G^0,\mu)}_r \overset{h}{\otimes}_{C_0(G^0)} C^*(G,\mu) \overset{h}{\otimes}_{C_0(G^0)} L^2(G^0)_c,$$

then $\mathcal{X}_\mu(G)^* = B(G, \mu)$. Note that in this case the balanced Haagerup product is over $C_0(G^0)$ instead of $L^\infty(G^0)$. We mention this result here because in this work Paterson adopts a "continuous-measurable-Borel" approach. We ask ourselves if there is a "way around" measurability when studying locally compact groupoids. We will come back to this issue at the end of the chapter.

6.3 Conclusions

The goal of this work is to present a new definition of a continuous Fourier algebra for a locally compact groupoid G and to study some properties of this algebra. We define

[1] It was pointed to us that they may be a gap in the proof of Renault's duality result and that this work would provide complete proofs.

$A(G)$ by considering the span of coefficients of the left regular representation with infinite multiplicity. In order to be able to consider a left regular representation we need a left Haar system on G. Such a system does not even have to exist, but we do need one to define $A(G)$. It could also be the case that there is more than one Haar system available, and hence we should ask ourselves how the selection of the Haar system affects $A(G)$. We will come back to this question soon.

Once the Haar system is fixed and the set $A(G)$ is defined, we opt for considering on it a candidate norm that is, a priori, different from the one on $B(G)$ and that we call $\|\cdot\|_A$. This norm is defined using only L^2 coefficients. Recall that on the group case both norms coincide, but to prove this a duality result is needed, and so far we do not have one in our context. Once $A(G)$ is defined and a choice for the norm is made, we prove that $\|\cdot\|_A$ is in fact a complete norm on $A(G)$. This is done in a very similar fashion as the $B(G)$ case.

We also wish to prove that $A(G)$ is an algebra, of course! If G is proper, this is part of the result 4.5.10, that also states that $A(G)$ is a $B(G)$-module. This result is based on Paterson's stability theorem from [35]. Note that here we cannot say that $A(G)$ is an ideal of $B(G)$, as in the group case, because we are considering a potentially different norm in $A(G)$. If G is compact and transitive, we prove that the Fourier algebras coincide as normed algebras.

On 4.5.13 we present an alternate description of the Fourier algebra for the locally trivial case that basically trivializes the left regular bundle. This description is also needed to define the map ψ from Proposition 5.3.2. Recall that this proposition extends the result that says that if H is a locally compact group

$$\frac{\overline{L^2(H)}_r \overset{h}{\otimes} L^2(H)_c}{\text{Ker } q_0} \simeq A(H)$$

as Banach algebras and as operator spaces.

Another desirable property for a Fourier algebra that we prove in our context is that $A(G)$ separates points, see 4.5.5.

Finally, the main result of the thesis provides a description of the Fourier algebra for a locally trivial, locally compact groupoid G that has a "nice" Haar system. The condition we ask the Haar system to verify is, for a fixed unit u of G, the compatibility between the measure λ^u and the left Haar measure at G_u^u: we want $\lambda^u_{|_{G_u^u}}$ to be the left Haar measure of the isotropy group. Then, for a such a Haar system (that we know we can construct if the groupoid is locally trivial and unimodular), we obtain a decomposition of $A(G)$ as $C_0(G^0) \overset{h}{\otimes} A(\tilde{G}) \overset{h}{\otimes} C_0(G^0)$, if the groupoid is transitive. This decomposition, in addition of providing a description of the Fourier algebra, provides an operator space structure for $A(G)$ and makes this algebra a completely contractive Banach algebra. If G has more than one transitive component, say $G = \sqcup_i G_i$, since these components are also topological components, there is a correspondence between G-Hilbert bundles and families of G_i-Hilbert bundles. Thanks to this correspondence, the Fourier-Stieltjes and Fourier algebra of G can be written as sums of the algebras of the G_i components. Thus, we can "put together" the decompositions of each component to conclude that

$$A(G) \simeq c_0 - \oplus_i C_0(G_i^0) \overset{h}{\otimes} A(G_{u_i}^{u_i}) \overset{h}{\otimes} C_0(G_i^0),$$

if $u_i \in G_i$ for all i.

6.4 Further questions

A number of questions arose during the process of defining and understanding a continuous Fourier algebra for a groupoid. We were not able to answer many of them, and we hope they will be matter of study in the future. We already mentioned some of them, we state them and some more in this last section.

First, suppose G is a locally compact groupoid with two different Haar systems $\lambda = \{\lambda^u\}_{u \in G^0}$ and $\mu = \{\mu^u\}_{u \in G^0}$. A priori, we obtain two possible different Fourier algebras $A(G, \lambda)$ and $A(G, \mu)$. We would like to know if those algebras coincide. We consider the examples we have available: if G is a group, group bundle or a topological space, the Haar system is (if it exists) essentially unique and this question does not apply.

If G is a full equivalence relation on a space X, we do not know the answer, unless the space X is compact. On that case, since all the possible Fourier algebras coincide with the Fourier-Stieljtes algebra, that does not depend on any Haar system, we conclude $A(G,\lambda) = A(G,\mu)$. This same reasoning applies, of course, to any compact, transitive and proper groupoid.

If G is a locally trivial, locally compact groupoid (for instance, the fundamental groupoid of a "nice" space X), and λ and μ are Haar systems, both of them compatible (on the sense of our structure theorem) with the Haar measure at the isotropy group of a fixed unit u, then both $A(G,\lambda)$ and $A(G,\mu)$ are isometrically isomorphic to $C_0(G^0) \overset{h}{\otimes} A(G_u^u) \overset{h}{\otimes} C_0(G^0)$ and hence coincide. If we do not have the condition of compatibility between the Haar system and the Haar measure of the isotropy group, we do not know the answer to our question.

This first question is specially important for us. Our main result applies only to locally trivial groupoids with certain type of Haar system. As we have seen through examples, those Haar system do not have to be the "natural" ones we may wish to consider. To answer this question, we believe we need to find an alternate description of $A(G)$ that does not depend on the Haar system, for instance extending that for a locally compact group H, $A(H) = \overline{C_c(H) \cap B(H)}$.

Another question we already mentioned is the relationship between the norms $\|\cdot\|_A$ and $\|\cdot\|_B$. We could consider any of them on the span of coefficients of the left regular bundle, yet we opted for the first one. Remember that they coincide in the group case, and this is proved using the duality properties of the Fourier algebras. Duality results were presented by Renault ([43]) and Paterson ([34]), but in a measurable context, hence we are not able to apply them for the continuous case. Obtaining duality results in our context should be the goal of some future research.

For the group case, Walter proved in [54] that the Fourier algebra is an invariant. That is, two locally compact groups H and H' are isomorphic as groups if and only if its corresponding Fourier algebras $A(H)$ and $A(H')$ are isomorphic as algebras. We do not know if this result extends to locally compact groupoids. Our main result, the decomposition of the Fourier algebra, hints that this may not be the case. If we could find non-isomorphic groupoids satisfying the hypothesis of our theorem, with isomorphic unit spaces and isotropy groups, their Fourier algebras would be isomorphic, and hence the Fourier algebra will not be an invariant for groupoids.

The main result of our thesis applies only to a family of groupoids, the locally trivial ones, provided that we can find a convenient Haar system. We already discussed if the Fourier algebra depends on the selection of the Haar system or not, and we do not have an answer for this question. The other natural question we ask ourselves is whether such a decomposition could be true for a bigger family of groupoids. We mentioned before that groupoids include such a wide variety of examples that when studying them is usually necessary to restrict ourselves to families of them (proper, r-discrete, principal, etc.). When we started studying the Fourier algebra of groupoids, the first example that we analyzed was the finite and transitive case, and for them we proved the decomposition of $A(G)$. Then, we tried to push that result further and we found that the appropriate family of groupoids for our goal was the locally trivial one. But, could it be that there is a wider family of groupoids whose Fourier algebra admits such a decomposition? We do not have an answer for this question but we have some intuitive idea we would like to share.

When we started our research we made the choice of concentrating on continuous Fourier algebras. It seemed desirable, still in the groupoid case, to obtain continuous functions as the elements of the Fourier algebras. The main reference for the study of continuous Fourier algebras is Paterson's article [33]. Over there, as we discussed before, the definition of $A(G)$ for a locally compact groupoid G "forces" this space to be an algebra. Paterson's definition seems to us less natural than the one presented by Renault ([43]).

We opted for a definition of $A(G)$ that we see as the continuous version of Renault's and we proved that for proper groupoids this space is an algebra. For the family of locally trivial groupoids, we have a nice description of $A(G)$, in terms of coefficients of a bundle that has constant Hilbert space $L^2(G^u)$ for a fixed unit u. The action on this bundle is almost the left regular one, but it has a twist. For each element of the groupoid γ, we associate an element $\beta(\gamma)$ of G_u^u and the new action on γ is the left regular action on $\beta(\gamma)$. Note that we need to make sure that the association $\gamma \to \beta(\gamma)$ is well-defined and that it gives rise to a continuous bundle, and this can be done using a countable family $\{U_i, \nu_i\}_{i\in\mathbb{N}}$ that ensures the local triviality. This (together with all the failed attempts to push further our decomposition!) suggest to us that the locally trivial condition provides the "right context" to study a continuous Fourier algebra: maybe we do need a nice continuity within our groupoid G to obtain a meaningful, continuous $A(G)$. Note that, as mentioned before, in [34] Paterson opted for a non-continuous context. It could be that the continuous context, at least in all its generality, is not the best approach to study Fourier algebras of groupoids.

We believe that the weakest point of our thesis is that our point of view does not include étale groupoids. And this bothers us, not only because they are a very important source of examples, but also because étale groupoids have an obvious choice for a Haar system (a system of counting measures), and this system has the nice compatibility result with the Haar measure that we so much need.

We end presenting a few questions that could be matter of future research. The phenomenon of amenability has been studied also in the groupoid context. In their book "Amenable groupoids" (see [3]), C. Anantharaman-Delaroche and J. Renault present definitions of amenability for both measurable and locally compact groupoids. For measurable groupoids, the analogue of Leptin's condition (amenability is equivalent to the existence of an approximate unit on the Fourier algebra) was proved by J.M. Vallin on [53]. We would like to obtain an analogue for locally compact groupoids, as well as to consider other amenability related questions (for example, computing amenability constants for finite groupoids and relating the amenability of the groupoid to the one of its isotropy groups). For locally compact groups, the operator space structure of its Fourier algebra plays an important role when studying amenability. We expect the operator space structure that we can consider on $A(G)$ thanks to our decomposition could be useful in the groupoid context.

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